A reformulation of the Born–Infeld electrodynamics

By P. A. M. Dirac, F.R.S.
St John's College, University of Cambridge

(*Received* 21 *March* 1960)

The electrodynamics proposed by Born & Infeld in 1934 differs from the Maxwell electrodynamics for very strong fields and gives a finite total energy for the field around a point charge. At that time not much was known about the dynamical theory of fields, and recent developments of this subject enable one to give a better formulation of the Born–Infeld theory. The basis is a comprehensive action principle that determines both the field equations and the equations of motion of charged particles. From this action principle a Hamiltonian formulation is obtained, the various Hamiltonian constraints being worked out. The classical theory is found to be completely satisfactory, but difficulties arise with the passage to the quantum theory, which appear to be insoluble with present methods of quantization.

1. Introduction

The Maxwell equations for the electromagnetic field have been verified only for fields that are very weak compared with the field at the surface of a classical electron. The basic idea of the Born–Infeld electrodynamics is to change the field equations for strong fields in such a way that the total energy of a singularity in the field becomes finite. One may hope in this way to avoid the difficulties associated with point charges in the Maxwell theory.

One retains the idea that the electromagnetic field is described by four potentials A_μ and the idea that the action density $\mathscr{L}$ is a function only of the field quantities $f_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. In this way gauge invariance is preserved. One replaces the usual expression for the action density

$$\mathscr{L} = -\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu} \tag{1}$$

by a different function of the $f_{\mu\nu}$, equal to (1) for small $f_{\mu\nu}$ (except possibly for an additive constant, which is unimportant) but differing considerably for $f_{\mu\nu}$ not small. The resulting field equations are not linear, except in the limit of weak fields.†

To preserve the relativistic invariance of the theory, $\mathscr{L}$ must involve $f_{\mu\nu}$ only through the two invariants $f_{\mu\nu}f^{\mu\nu}$ and $f_{\mu\nu}f^{*\mu\nu}$ (f^* being the dual of f). Born (1934) first proposed

$$\mathscr{L} = -\{1 + \tfrac{1}{2}f_{\mu\nu}f^{\mu\nu}\}^{\frac{1}{2}}. \tag{2}$$

Later, Born & Infeld (1934) favoured

$$\mathscr{L} = -\{1 + \tfrac{1}{2}f_{\mu\nu}f^{\mu\nu} - (\tfrac{1}{4}f_{\mu\nu}f^{*\mu\nu})^2\}^{\frac{1}{2}}. \tag{3}$$

In each of these expressions $f_{\mu\nu}$ is referred to some absolute unit of field strength provided by the field at the surface of a classical electron.

Born & Infeld showed that either of the expressions (2), (3) makes the total energy around a singularity finite and also makes the potentials and their first derivatives

† A different modification of the Maxwell theory has been proposed by Podolsky & Schwed (1948), in which the linearity of the field equations is preserved at the expense of the appearance of derivatives of $f_{\mu\nu}$ in $\mathscr{L}$.

at the singularity finite. The first derivatives, however, are discontinuous at the singularity.

In order to deal with quantization one must get the theory into the Hamiltonian form, which requires in the first place that one shall have an action principle for the equations of motion. It must be a *comprehensive action principle* giving all the equations of motion, those of the particles as well as those of the field. Such a comprehensive action principle has not previously been given and forms the first problem dealt with here.

With the help of this principle a precise treatment of the equations of motion of a particle becomes possible, and is given in § 3. In § 4 it is shown how the ordinary Lorentz equations of motion appear as an approximation. The work up to this stage is carried through for an arbitrary $\mathscr{L}$ satisfying the requirements of relativistic and gauge invariance and giving singularities of a similar kind to those of (2) and (3).

The later work is concerned with the passage to the Hamiltonian. For this one must work with a particular $\mathscr{L}$. There are not many choices for $\mathscr{L}$ that lead to tractable equations. The simplest one is (3), the one adopted by Born & Infeld, and this one will be used here.

2. The comprehensive action principle

It will be sufficient to take into account just one particle, as no new features arise for more than one particle. Let e be the charge of the particle and M the part of the rest-mass that is not of electromagnetic nature. We may have $M = 0$, which is probably the case for an electron.

Let z^μ denote the co-ordinates of a point on the world-line of the particle. We introduce some arbitrary parameter τ to specify a point on the world-line, so the z^μ become functions of τ and these functions serve to define the world-line.

We now take the action to be the sum of three terms

$$I = I_1 + I_2 + I_3,$$

where

$$I_1 = \int \mathscr{L}\, \mathrm{d}^4x, \tag{4}$$

$$I_2 = -e\int A_\mu(z)\, (\mathrm{d}z^\mu/\mathrm{d}\tau)\, \mathrm{d}\tau, \tag{5}$$

$$I_3 = -M\int \mathrm{d}s. \tag{6}$$

In I_1 the integration must, of course, be taken over the singularity. This gives only a finite contribution for the Born–Infeld type of function $\mathscr{L}$, in distinction to the Maxwell function (1), for which it would be infinite. Again, in I_2 there appear the quantities $A_\mu(z)$, the potentials at the singularity, which are finite and well-defined with the Born–Infeld type of singularity, not with the Maxwell singularity.

For the variation procedure one must choose some independent variables q, make arbitrary small variations δq in them, and work out the resulting variation δI. *The q's must be chosen so that δI depends linearly on the δq's.*

One might think of taking as q's the $A_\mu(x)$ and $z^\mu(\tau)$. But then δI would not depend linearly on the δA_μ and δz^μ. One can see this by taking the special case of $\delta A_\mu(x) = 0$. Then the $\delta A_\mu(z)$ which occurs in δI_2 does not depend linearly on the δz^μ on account of the derivatives of $A_\mu(x)$ being discontinuous at $x = z$.

We can avoid the difficulty in the following way. We refer our action integral to a general set of curvilinear co-ordinates x^μ. We now keep $z^\mu(\tau)$ constant and vary the world-line by varying the co-ordinate system. Such a variation, combined with $\delta A^\mu(x)$ equal to a non-singular function of the x^μ, means a shifting of the world-line together with the singularity in the field around it, and gives a value for δI depending linearly on the parameters that determine the variation of the co-ordinate system and on the $\delta A^\mu(x)$. We get in this way a variation that is physically the most general permissible one and is adequate for fixing all the equations of motion, without any variation being made in $z^\mu(\tau)$. One could, for example, set up a system of co-ordinates such that the particle is at the origin, $z^1 = z^2 = z^3 = 0$, and it would remain at the origin under the variation.

The variation in the co-ordinate system involves a variation in the $g_{\mu\nu}$. We shall not consider a general variation in the $g_{\mu\nu}$ as this would bring in the gravitational field. We shall restrict ourselves to $g_{\mu\nu}$'s that apply to a flat space and consider only variations $\delta g_{\mu\nu}$ that preserve the flatness.

A convenient way to take these restrictions into account is to introduce a set of orthogonal rectilinear co-ordinates y_K, which are kept fixed. The curvilinear co-ordinate system may then be specified by the functions $y_K(x)$ and its variation by the quantities $\delta y_K(x)$. We have $g_{\mu\nu} = y_{K,\mu} y_{K,\nu}$, the summation over K being understood to be combined with the $+$ sign for $K = 0$ and the $-$ sign for $K = 1, 2, 3$. Thus

$$\delta g_{\mu\nu} = y_{K,\mu}\delta y_{K,\nu} + y_{K,\nu}\delta y_{K,\mu}, \tag{7}$$

and we get δI depending linearly on the δA^μ and δy_K for all x.

The type of singularity that occurs with the Born–Infeld electrodynamics is

$$A^\mu(x) = ar\,\mathrm{d}z^\mu/\mathrm{d}s + \text{less singular terms}, \tag{8}$$

where a is a constant and r is the perpendicular distance of the field point x from the world-line of the particle. It involves the contravariant potentials A^μ, and so gets preserved by a variation consisting of a non-singular $\delta y_K(x)$ and a non-singular $\delta A^\mu(x)$. Such a variation leads to a $\delta A_\mu(x)$ with a singularity of the strength of a discontinuity in its first derivatives. In spite of this singularity, δI can be expressed linearly in terms of δA_μ and δy_K, and this expression is a more convenient one to work with than the one in terms of δA^μ and δy_K.

The action density $\mathscr{L}$ in (4) is a function only of the $f_{\mu\nu}$ and $g_{\mu\nu}$. Thus

$$\delta\mathscr{L} = \tfrac{1}{2}p^{\mu\nu}\delta f_{\mu\nu} - \tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}, \tag{9}$$

where

$$p^{\mu\nu} = -p^{\nu\mu} = 2\partial\mathscr{L}/\partial f_{\mu\nu}, \tag{10}$$

$$T^{\mu\nu} = T^{\nu\mu} = -2\partial\mathscr{L}/\partial g_{\mu\nu}. \tag{11}$$

For a relativistic theory $\mathscr{L}$ must be a scalar density. This condition leads to

$$T^\nu_\sigma = p^{\nu\rho}f_{\sigma\rho} - g^\nu_\sigma\mathscr{L}, \tag{12}$$

as follows from a small modification in formula (4·5) of Born & Infeld's paper (1934), or from an application of the general result of Rosenfeld (1940, formula (29)).

3. The exact equations of motion

We shall proceed with the variation of the action $I_1+I_2+I_3$. We have, using (9) and (7),

$$\delta I_1 = -\int\{p^{\mu\nu}\delta A_{\mu,\nu} + y_{K,\mu}T^{\mu\nu}\delta y_{K,\nu}\}\,\mathrm{d}^4x \tag{13}$$

$$= \int\{p^{\mu\nu}{}_{,\nu}\delta A_\mu + (y_{K,\mu}T^{\mu\nu})_{,\nu}\,\delta y_K\}\,\mathrm{d}^4x. \tag{14}$$

Again

$$\delta I_2 = -e\int\delta A_\mu(z)\,(\mathrm{d}z^\mu/\mathrm{d}\tau)\,\mathrm{d}\tau$$

$$= -e\int v^\mu\,\mathrm{d}s\int\delta_4(x-z)\,\delta A_\mu(x)\,\mathrm{d}^4x, \tag{15}$$

where $v^\mu = \mathrm{d}z^\mu/\mathrm{d}s$. Note that the four-dimensional δ-function $\delta_4(x-z)$ is a function of s through z being a function of s. Finally

$$\delta I_3 = -M\int\delta\left(g_{\mu\nu}\frac{\mathrm{d}z^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}z^\nu}{\mathrm{d}\tau}\right)\mathrm{d}\tau = -\tfrac{1}{2}M\int\left(g_{\rho\sigma}\frac{\mathrm{d}z^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}z^\sigma}{\mathrm{d}\tau}\right)^{-\frac{1}{2}}\frac{\mathrm{d}z^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}z^\nu}{\mathrm{d}\tau}\,\delta g_{\mu\nu}\,\mathrm{d}\tau$$

$$= -M\int v^\mu v^\nu y_{K,\mu}\delta y_{K,\nu}\,\mathrm{d}s = -M\int y_{K,\mu}v^\mu(\mathrm{d}\,\delta y_K/\mathrm{d}s)\,\mathrm{d}s$$

$$= M\int\{\mathrm{d}(y_{K,\mu}v^\mu)/\mathrm{d}s\}\,\delta y_K\,\mathrm{d}s. \tag{16}$$

After making the variation, we have passed from τ to s as the parameter specifying the point on the world-line.

Equating to zero the coefficient of δA_μ in (14) + (15) + (16), we get

$$p^{\mu\nu}{}_{,\nu} = e\int v^\mu\,\delta_4(x-z)\,\mathrm{d}s. \tag{17}$$

These are the field equations of Born & Infeld, generalized to take into account a point charge at $x = z$.

Equating to zero the coefficient of δy_K, we get

$$-(y_{K,\mu}T^{\mu\nu})_{,\nu} = M\int\{\mathrm{d}(y_{K,\mu}v^\mu)/\mathrm{d}s\}\,\delta_4(x-z)\,\mathrm{d}s. \tag{18}$$

Now

$$\begin{aligned} y_{K,\sigma}(y_{K,\mu}T^{\mu\nu})_{,\nu} &= (y_{K,\sigma}y_{K,\mu}T^{\mu\nu})_{,\nu} - y_{K,\sigma\nu}y_{K,\mu}T^{\mu\nu} \\ &= T^\nu_{\sigma,\nu} - \tfrac{1}{2}g_{\mu\nu,\sigma}T^{\mu\nu} \\ &= (p^{\nu\rho}f_{\sigma\rho})_{,\nu} - \mathscr{L}_\sigma - \tfrac{1}{2}g_{\mu\nu,\sigma}T^{\mu\nu}, \end{aligned}$$

from (12). With the help of (10) and (11), and also of $f_{\rho\sigma,\nu}+f_{\sigma\nu,\rho}+f_{\nu\rho,\sigma} = 0$, this becomes

$$y_{K,\sigma}(y_{K,\mu}T^{\mu\nu})_{,\nu} = p^{\nu\rho}{}_{,\nu}f_{\sigma\rho}. \tag{19}$$

So (18) multiplied by $y_{K,\sigma}$ becomes, with the help of (17),

$$\begin{aligned} ef_{\sigma\rho}v^\rho &= My_{K,\sigma}\,\mathrm{d}(y_{K,\mu}v^\mu)/\mathrm{d}s \\ &= Mg_{\sigma\rho}(\mathrm{d}v^\rho/\mathrm{d}s + \Gamma_{\mu\nu}{}^\rho v^\mu v^\nu), \end{aligned} \tag{20}$$

where $\Gamma_{\mu\nu}{}^\rho$ is the Christoffel symbol.

We have here an equation of motion for the particle. It is like Lorentz's equation with the field $f_{\sigma\rho}$ acting on the charge e. However, the left-hand side is not well defined, because $f_{\sigma\rho}$ is discontinuous at $x = z$. The equation therefore needs amendment.

The error arose in the passage from (13) to (14), as the partial integration is not valid with the domain of integration extended over the singular line. To correct the work, one must cut off the domain of integration at a small distance ϵ from the singular line. The error introduced by the cut-off tends to zero as $\epsilon \to 0$, as the integrand in (13) has no singularity of the nature of a δ-function. The cut-off brings an extra term into (14), of the form of a surface integral over a three-dimensional tube-like surface surrounding the singular line, namely

$$\int \{p^{\mu\nu}\delta A_\mu + y_{K,\mu} T^{\mu\nu} \delta y_K\} \mathrm{d}S_\nu. \tag{21}$$

Let us attach three parameters to each point of the surface, namely s, u_1, u_2, with s having the same value as for a point on the world-line at a distance of order ϵ. Then

$$\begin{aligned} \mathrm{d}S_\nu &= \epsilon_{\nu\alpha\beta\gamma} \frac{\partial x^\alpha}{\partial s} \frac{\partial x^\beta}{\partial u_1} \frac{\partial x^\gamma}{\partial u_2} \mathrm{d}s\, \mathrm{d}u_1 \mathrm{d}u_2 \\ &= \mathrm{d}s\, \mathrm{d}\sigma_\nu, \end{aligned} \tag{22}$$

say, with
$$\mathrm{d}\sigma_\nu = \epsilon_{\nu\alpha\beta\gamma} \frac{\partial x^\alpha}{\partial s} \frac{\partial x^\beta}{\partial u_1} \frac{\partial x^\gamma}{\partial u_2}.$$

We may neglect the variation of δA_μ, δy_K through distances of order ϵ, and so we can replace these quantities in (21) by their values at the point s on the singular line and get

$$\int \mathrm{d}s \left\{\delta A_\mu \int p^{\mu\nu} \mathrm{d}\sigma_\nu + \delta y_K \int y_{K,\mu} T^{\mu\nu} \mathrm{d}\sigma_\nu\right\}. \tag{23}$$

The expression (23) must be added to the previous expression for δI, namely (14) + (15) + (16). Equating to zero the total coefficient of $\delta A_\mu(s)$, we get

$$\int p^{\mu\nu} \mathrm{d}\sigma_\nu = ev^\mu. \tag{24}$$

This result is in agreement with (17) and is included in it.

Equating to zero the total coefficient of $\delta y_K(s)$, we get

$$-\int y_{K,\mu} T^{\mu\nu} \mathrm{d}\sigma_\nu = M\, \mathrm{d}(y_{K,\mu} v^\mu)/\mathrm{d}s. \tag{25}$$

Multiplying by $y_{K,\sigma}$ and using (12), we get, with neglect of terms of order ϵ,

$$\begin{aligned} \int p^{\rho\nu} f_{\sigma\rho} \mathrm{d}\sigma_\nu &= M y_{K,\sigma} \mathrm{d}(y_{K,\mu} v^\mu)/\mathrm{d}s \\ &= M g_{\sigma\rho}(\mathrm{d}v^\rho/\mathrm{d}s + \Gamma_{\mu\nu}{}^{\rho} v^\mu v^\nu). \end{aligned} \tag{26}$$

The $p^{\rho\nu}$ in the integrand here is of order ϵ^{-2} and to this order its value is the same as it would be if the particle at the point s were moving with uniform velocity. The $f_{\sigma\rho}$ is of order 1, so the integral itself is of order 1.

We can express $f_{\sigma\rho}$ as the sum of two parts

$$f_{\sigma\rho} = f_{\sigma\rho}^{\text{sing.}} + f_{\sigma\rho}^{\text{rem.}},$$

where the main part of the singularity is put into the first term and the remainder is continuous and has a definite value on the singular world-line. We can make $f_{\sigma\rho}^{\text{sing.}}$ determinate on the tube, with neglect of terms of order ϵ, by taking it to equal the $f_{\sigma\rho}$ produced by the particle at s moving with uniform velocity in the absence of any external field. Thus $f_{\sigma\rho}^{\text{rem.}}$ is made determinate. The left-hand side of (26) now becomes

$$\int p^{\rho\nu} f_{\sigma\rho}^{\text{sing.}} \,\mathrm{d}\sigma_\nu + f_{\sigma\rho}^{\text{rem.}} \int p^{\rho\nu} \,\mathrm{d}\sigma_\nu.$$

The first term here is the same as for a particle moving with uniform velocity and vanishes from symmetry. So (26) becomes finally, with the help of (24)

$$e f_{\sigma\rho}^{\text{rem.}} v^\rho = M g_{\sigma\rho}(\mathrm{d}v^\rho/\mathrm{d}s + \Gamma_{\mu\nu}{}^{\rho} v^\mu v^\nu). \tag{27}$$

In this way we get an interpretation for the ambiguous term on the left-hand side of (20) and we have well-defined equations of motion for the particle.

4. The Lorentz approximate equations of motion

The equations of motion (27) are superficially like the Lorentz equations, but differ from them in two respects. First, the mass M is only the non-electromagnetic mass. Secondly, the field $f^{\text{rem.}}$ is not at all equal to the external field. A field, coming from outside and falling on the particle, gets substantially modified in passing through the region close to the singularity, on account of the non-linear character of the field equations, and so it gives rise to a field $f^{\text{rem.}}$ at the centre very different from what the field there would be if there were no charged particle.

The calculation of $f^{\text{rem.}}$ will usually be very complicated and will make the equations (27) difficult to apply to practical examples. One can develop approximate equations of motion not involving $f^{\text{rem.}}$, which are applicable when the external field is weak and involves only waves of long wavelength, so that the acceleration of the particle is small. The first stage of this approximation will now be worked out.

Take a length γ which is small compared with the wavelength of the incident waves and with the radius of curvature of the world-line, but large compared with the classical radius of the particle. Construct a second tube around the world-line, with radius γ.

Integrate equation (25) over a small range of s values, say Δs, so that it becomes, from (22),

$$-\int y_{K,\mu} T^{\mu\nu} \,\mathrm{d}S_\nu = \Delta s M \,\mathrm{d}(y_{K,\mu} v^\mu/\mathrm{d}s), \tag{28}$$

where the left-hand side is integrated over a small length of the tube corresponding to Δs. The integrand here has its divergence vanishing everywhere except on the singular world-line, as is shown by (19) and (17). We may therefore replace the surface of integration by any other surface with the same boundary, provided the region between the two surfaces does not contain any of the singular world-line. Let us take the second surface to be the corresponding length Δs of the tube γ, together with two end surfaces that connect the ends of the bit of the tube ϵ with those of the bit of the tube γ.

The integral over the surface of the tube γ can be handled by the same method as was used for the tube ϵ (the equations being now only approximate since we cannot make $\gamma \to 0$), with the Coulomb field playing the role of $f^{\text{sing.}}$ and the external field $f^{\text{ext.}}$ playing the role of $f^{\text{rem.}}$. Bringing in the multiplying factor $y_{K,\sigma}$, we get for this term

$$\Delta s\, e f^{\text{ext.}}_{\sigma\rho}\, v^{\rho}.$$

We can evaluate the integral over one of the end surfaces, to the first order of accuracy in γ, by neglecting the external field and the acceleration of the particle, and we then find for it $-my_{K,\mu}v^{\mu}$, where m is the mass of the field around a stationary charge e, a mass that is finite with the Born–Infeld type of field. The contribution of both end surfaces is thus

$$-\Delta s\, m\, \mathrm{d}(y_{K,\mu}v^{\mu})/\mathrm{d}s.$$

So equation (28), multiplied by $y_{K,\sigma}$ and divided by Δs, becomes

$$\begin{aligned} e f^{\text{ext.}}_{\sigma\rho}\, v^{\rho} &= (M+m)\, y_{K,\sigma}\mathrm{d}(y_{K,\mu}v^{\mu})/\mathrm{d}s \\ &= (M+m)\, g_{\sigma\rho}(\mathrm{d}v^{\rho}/\mathrm{d}s + \Gamma_{\mu\nu}{}^{\rho} v^{\mu}v^{\nu}). \end{aligned} \tag{29}$$

These are just the Lorentz equations of motion. They have previously been obtained by other authors by other methods, for example Born & Infeld (1934), Frenkel (1934), Feenberg (1935) and Pryce (1936). The present work shows that they are approximate consequences of the comprehensive action principle. If this work were carried through with greater accuracy the Lorentz damping term would have to appear, because there is strict conservation of energy and momentum.

5. The Hamiltonian in the absence of charges

We follow the general method for passing from an action principle to the Hamiltonian formulation of the equations of motion, given by the author (1951) and (1958). We assume the surfaces x^0 = constant to be space-like and deal with states on these surfaces. The quantities y_K, A_μ for all x^1, x^2, x^3 for a given x^0 are the basic dynamical co-ordinates of the theory and their derivatives with respect to x^0 are the velocities.

We shall work with the action density proposed by Born & Infeld (1934), which is extremely simple referred to curvilinear co-ordinates and is favoured for that reason. Put

$$\lambda_{\mu\nu} = g_{\mu\nu} + f_{\mu\nu},$$

and let the determinant of the $\lambda_{\mu\nu}$ be $-\Lambda^2$. Then the action density is $\mathscr{L} = -\Lambda$. It goes over into (3) for rectilinear co-ordinates.

The Lagrangian L is given by

$$I = \int L\,\mathrm{d}x^0$$

and is therefore

$$L = -\int \Lambda\,\mathrm{d}^3x.$$

Making a general variation in the $\lambda_{\mu\nu}$, we get

$$\delta L = -\frac{1}{2}\int \Lambda\kappa^{\nu\mu}\,\delta\lambda_{\mu\nu}, \tag{30}$$

where $\kappa^{\nu\mu}$ is the reciprocal matrix to $\lambda_{\mu\nu}$, so that

$$\kappa^{\nu\mu}\lambda_{\mu\rho} = \kappa^{\mu\nu}\lambda_{\rho\mu} = g^{\nu}_{\rho}. \tag{31}$$

Thus if we vary the velocities $y_{K,0}$, $A_{\mu,0}$ we get, from (7),

$$\delta L = \frac{1}{2}\int \Lambda\{-(\kappa^{0\mu}+\kappa^{\mu0})\, y_{K,\mu}\,\delta y_{K,0} + (\kappa^{0\mu}-\kappa^{\mu0})\,\delta A_{\mu,0}\}\,\mathrm{d}^3x.$$

The momenta w_K and D^μ conjugate to the dynamical co-ordinates y_K and A_μ respectively are given by

$$\delta L = \int\{w_K\,\delta y_{K,0} + D^\mu\,\delta A_{\mu,0}\}\,\mathrm{d}^3x.$$

So comparing coefficients, we find

$$w_K = -\tfrac{1}{2}\Lambda(\kappa^{0\mu}+\kappa^{\mu0})\, y_{K,\mu}, \tag{32}$$

$$D^\mu = \tfrac{1}{2}\Lambda(\kappa^{0\mu}-\kappa^{\mu0}). \tag{33}$$

The equations (32) and (33) lead to some relations between the dynamical co-ordinates and the momenta, independent of the velocities. These are the primary constraints of the Hamiltonian theory. One of them is evidently

$$D^0 = 0. \tag{34}$$

Using Latin suffixes to take on the values 1, 2 and 3, we have

$$\begin{aligned} w_K y_{K,r} &= -\tfrac{1}{4}\Lambda(\kappa^{0\mu}+\kappa^{\mu0})\,(\lambda_{r\mu}+\lambda_{\mu r}) \\ &= -\tfrac{1}{4}\Lambda(\kappa^{0\mu}-\kappa^{\mu0})\,(\lambda_{r\mu}-\lambda_{\mu r}) \end{aligned}$$

from (31). Hence

$$w_K y_{K,r} + f_{rs}D^s = 0, \tag{35}$$

which gives us more primary constraints. Further, from (32)

$$\begin{aligned} w_K w_K &= \tfrac{1}{4}\Lambda^2(\kappa^{0\mu}+\kappa^{\mu0})\,(\kappa^{0\nu}+\kappa^{\nu0})\,g_{\mu\nu} \\ &= \tfrac{1}{4}\Lambda^2(\kappa^{0\mu}+\kappa^{\mu0})\,(\kappa^{0\nu}+\kappa^{\nu0})\,\lambda_{\mu\nu}, \end{aligned}$$

and similarly from (33)

$$g_{rs}D^rD^s = \tfrac{1}{4}\Lambda^2(\kappa^{0\mu}-\kappa^{\mu0})\,(\kappa^{0\nu}-\kappa^{\nu0})\,\lambda_{\mu\nu}.$$

Adding these two equations, we get

$$\begin{aligned} w_K w_K + g_{rs}D^rD^s &= \tfrac{1}{2}\Lambda^2(\kappa^{0\mu}\kappa^{0\nu}+\kappa^{\mu0}\kappa^{\nu0})\,\lambda_{\mu\nu} \\ &= \Lambda^2\kappa^{00} \end{aligned} \tag{36}$$

from (31). The quantity (36) is minus the determinant of the λ_{rs}, and its expansion yields

$$\Lambda^2\kappa^{00} = \Gamma^2 - g_{rs}B^rB^s,$$

where $-\Gamma^2$ is the determinant of the g_{rs} and

$$B^r = \tfrac{1}{2}\epsilon^{rst}f_{st},$$

ϵ^{rst} being antisymmetrical with $\epsilon^{123} = 1$. Thus (36) gives

$$w_K w_K + g_{rs}(D^rD^s + B^rB^s) - \Gamma^2 = 0, \tag{37}$$

which is another primary constraint.

We can put (37) into an alternative form. We resolve the vector w_K into components tangential and normal to the surface $x^0 = \text{constant}$. The tangential components are $w_K y_{K,r} = w_r$, say, and the normal component is $w_K y_{K,\mu} g^{\mu 0}(g^{00})^{-\frac{1}{2}} = w_{\text{nor.}}$, say. Equation (35) gives the tangential components in terms of the other Hamiltonian variables, and (37) can be put into a similar form in which it gives $w_{\text{nor.}}$ in terms of the other Hamiltonian variables. We have

$$w_K w_K = (w_{\text{nor.}})^2 + e^{rs} w_r w_s,$$

where e^{rs} is the reciprocal matrix to g_{rs}. So (37) and (35) lead to

$$w_{\text{nor.}} + \{\Gamma^2 - g_{rs}(D^r D^s + B^r B^s) - e^{rs} f_{rt} D^t f_{su} D^u\}^{\frac{1}{2}} = 0. \tag{38}$$

The physical meaning of various quantities here may be noted. D^r is the p^{0r} of (10) and is the electric induction, while B^r is the magnetic induction. The value of $-w_{\text{nor.}}$ given by (38) is the energy density, and the value of $-w_r$ given by (35) is the momentum density or Poynting vector.

Since the velocity $A_{0,0}$ does not occur in L, we get a Lagrangian equation of motion if we vary A_0 in L and put the coefficient of δA_0 equal to zero. The variation gives, from (30),

$$\delta L = -\frac{1}{2}\int \Lambda(\kappa^{0r} - \kappa^{r0})\,\delta A_{0,r}\,\mathrm{d}^3x$$

$$= -\int D^r \delta A_{0,r}\,\mathrm{d}^3x = \int D^r{}_{,r}\,\delta A_0\,\mathrm{d}^3x,$$

so we get the equation

$$D^r{}_{,r} = 0. \tag{39}$$

This equation does not involve any velocities and is therefore a secondary constraint.

The equations (34), (35), (37) or (38), (39) are the only independent constraints of the theory. There is, of course, one constraint of each kind for each point x^1, x^2, x^3.

The Hamiltonian variables y_K, w_K, A_μ, D^μ satisfy P.b. relations of the standard type, namely

$$\left.\begin{aligned}[y_K, w'_{K'}] &= \eta_{KK'}\delta_3(x-x'),\\ [A_\mu, D'^\nu] &= g^\nu_\mu \delta_3(x-x'),\end{aligned}\right\} \tag{40}$$

with the other P.b.'s vanishing. The $\eta_{KK'}$ here is the metric for the y_K co-ordinate system.

The constraints (34), (35), (37) or (38), (39) are all first-class, that is, the P.b.'s of their left-hand sides all vanish. This result is immediately evident for (34) and (39), while for the others it can be inferred from the requirement that the velocities $y_{K,0}$ shall not be restricted by equations of motion.

The Hamiltonian must be a linear combination of all the first-class primary constraints, with coefficients which are functions of the velocities not restricted by equations of motion. One can easily determine these coefficients by making some of the simple equations of motion fit. Writing the primary constraints (35), (38) for brevity as

$$w_r + \chi_r = 0, \quad w_{\text{nor.}} + \chi_{\text{nor.}} = 0, \tag{41}$$

we find for the Hamiltonian

$$H = \int \{(g^{00})^{-\frac{1}{2}}(w_{\text{nor.}} + \chi_{\text{nor.}}) + g_{r0}e^{rs}(w_s + \chi_s) + A_{0,0}D^0\}\,\mathrm{d}^3x. \tag{42}$$

The last term in the integrand may be ignored, as it refers to degrees of freedom that are not physically important.

6. The Hamiltonian for a particle

With a charged particle present, we must add to the action the term $I_2 + I_3$, given by (5) and (6). For the Hamiltonian treatment we may conveniently take $\tau = z^0$. The z^r are to be considered as given functions of z^0, and not dynamical co-ordinates. The $\mathrm{d}z^r/\mathrm{d}\tau = z^r{}_{,0}$ are likewise given functions of z^0, and not dynamical velocities. The presence of the particle does not bring in any extra dynamical co-ordinates and velocities, and thus no extra momentum variables.

The extra Lagrangian is

$$L_2 + L_3 = -eA_\mu(z)\,z^\mu{}_{,0} - M\{g_{\mu\nu}(z)\,z^\mu{}_{,0}z^\nu{}_{,0}\}^{\frac{1}{2}}. \tag{43}$$

Variation of the velocities $y_{K,0}$, $A_{\mu,0}$ in it leads to

$$\begin{aligned}\delta(L_2 + L_3) &= -M\{g_{\rho\sigma}z^\rho{}_{,0}z^\sigma{}_{,0}\}^{-\frac{1}{2}}\,z^\mu{}_{,0}y_{K,\mu}(z)\,\delta y_{K,0}(z)\\ &= -Mv^\mu \int y_{K,\mu}(x)\,\delta_3(x-z)\,\delta y_{K,0}(x)\,\mathrm{d}^3x.\end{aligned}$$

The equation (33) for D^μ is therefore unchanged, while the equation (32) for w_K gets changed to

$$w_K = -\tfrac{1}{2}\Lambda(\kappa^{0\mu} + \kappa^{\mu 0})\,y_{K,\mu} - Mv^\mu y_{K,\mu}\delta_3(x-z). \tag{44}$$

Variation of A_0 in (43) leads to

$$\delta(L_2 + L_3) = -e\,\delta A_0(z) = -e\int \delta_3(x-z)\,\delta A_0(x)\,\mathrm{d}^3x,$$

so the Lagrangian equation of motion (39) gets changed to

$$D^r{}_{,r} - e\delta_3(x-z) = 0. \tag{45}$$

It is still a secondary constraint.

Corresponding to (35) we now get, with the notation of (41),

$$w_r + \chi_r = -Mg_{r\mu}v^\mu\delta_3(x-z). \tag{46}$$

For $M \neq 0$, the right-hand side involves velocities, so the equation is not a constraint. However, (46) is really a large number of equations, three for each point x^r, and for $x^r \neq z^r$ the right-hand side vanishes and provides us with constraints. The set of constraints given in this way may conveniently be written

$$\int a^r(w_r + \chi_r)\,\mathrm{d}^3x = 0, \tag{47}$$

where the a^r are functions of x^r that are restricted to vanish for $x^r = z^r$. These constraints are first-class, and correspond to the possibility of one's making arbitrary changes in the three-dimensional co-ordinate system x^r leaving the point $x^r = z^r$ invariant.

Similarly, corresponding to the last of equations (41), we can deduce

$$w_{\text{nor.}} + \chi_{\text{nor.}} = -Mv_{\text{nor.}}\,\delta_3(x-z), \tag{48}$$

where $v_{\text{nor.}}$ is the normal component of the vector v^μ. This gives us the constraints

$$\int b(w_{\text{nor.}} + \chi_{\text{nor.}})\,\mathrm{d}^3x = 0, \tag{49}$$

where b is a function of x^r restricted to vanish for $x^r = z^r$. These constraints are first-class and correspond to the possibility of one's making arbitrary deformations of the surface $x^0 =$ constant that leave the point $x^\mu = z^\mu$ invariant.

We need a further first-class constraint to correspond to the possibility of moving the point $x^\mu = z^\mu$ along the assigned world-line for the particle. To get this further constraint, we must eliminate the velocities from the right-hand sides of (46) and (48) at the point $x^r = z^r$. The result is

$$\int (w_{\text{nor.}} + \chi_{\text{nor.}})\,\mathrm{d}^3x \int (w'_{\text{nor.}} + \chi'_{\text{nor.}})\,\mathrm{d}^3x' + e^{rs}(z) \int (w_r + \chi_r)\,\mathrm{d}^3x \int (w'_s + \chi'_s)\,\mathrm{d}^3x' - M^2 = 0, \tag{50}$$

the domain of integration in each case being any domain extending over the point $x^r = z^r$.

For the case $M \neq 0$, the constraints (47), (49), (50), (45) and (34) are the only independent ones, and the Hamiltonian is a linear combination of their left-hand sides. The case $M = 0$ is quite different and needs further investigation.

7. Quantization

We have a classical theory in the Hamiltonian form, with all the constraints first-class. To quantize such a theory, we must make the dynamical variables into operators satisfying commutation relations corresponding to the classical P.b. relations, and we must make each constraint, say $\phi_n = 0$, into a condition on the wave function

$$\phi_n \Psi = 0.$$

In order that these conditions may be consistent with one another, it is necessary that the order of non-commuting factors in the ϕ's be suitably chosen, so that

$$[\phi_n, \phi_m] = \Sigma_k c_{nmk} \phi_k, \tag{51}$$

where the coefficients c_{nmk} are all to the left of their corresponding ϕ_k's.

With the Born–Infeld electrodynamics there does not seem to be any way of arranging the ϕ's to satisfy the conditions (51), even in the absence of charged particles. It we take the form (38) for some of the ϕ's, expressing $w_{\text{nor.}}$ in terms of other dynamical variables, we have square roots appearing, leading to hopeless complications in the commutators. One gets the nearest to satisfying the conditions (51) if one works with the rational form (37) instead of (38), but it does not completely fit.

The Born–Infeld electrodynamics is completely satisfactory in the classical theory. It avoids all the troublesome infinities associated with point charges in the Maxwell theory. Also, it gives an expression for the energy density $-w_{\text{nor.}}$ that is positive definite, as is shown by (38) and (48), the quantities under the square root in (38) being also positive definite. It follows that there cannot be any runaway solutions of the equations of motion for a charged particle (in which the particle continually accelerates in the absence of an external field), such as occur with the Maxwell–Lorentz theory.

But the Born–Infeld electrodynamics cannot be quantized with the present rules of quantization, even in the absence of charges. To get a quantum theory from it one will need to make some fundamental alteration, probably involving the introduction of anticommuting quantities into the description of the field and using them to avoid the square roots in the expression for the energy.

References

Born, M. 1934 *Proc. Roy. Soc.* A, **143**, 410.
Born, M. & Infeld, L. 1934 *Proc. Roy. Soc.* A, **144**, 425.
Dirac, P. A. M. 1951 *Canad. J. Math.* **3**, 1.
Dirac, P. A. M. 1958 *Proc. Roy. Soc.* A, **246**, 326.
Feenberg, E. 1935 *Phys. Rev.* **47**, 148.
Frenkel, J. 1934 *Proc. Roy. Soc.* A, **146**, 930.
Podolsky, B. & Schwed, P. 1948 *Rev. Mod. Phys.* **20**, 40.
Pryce, M. H. L. 1936 *Proc. Roy. Soc.* A, **155**, 597.
Rosenfeld, L. 1940 *Acad. Roy. Belg.* **18**, fasc. 6.