Generalized Hamiltonian dynamics

By P. A. M. Dirac, F.R.S.
St John's College, Cambridge

(*Received* 13 *March* 1958)

The author's procedure for passing from the Lagrangian to the Hamiltonian when the momenta are not independent functions of the velocities is put into a simpler and more practical form, the main results being obtained by a direct solution of the equations provided by the consistency requirements. It is shown how, under certain conditions, one can eliminate some of the degrees of freedom and so make a substantial simplification in the Hamiltonian formalism.

The usual procedure for passing from the Lagrangian form of the equations of motion to the Hamiltonian form requires the momenta to be independent functions of the velocities. There are important practical cases where this condition is not fulfilled, e.g. with a relativistic field theory, and it becomes necessary to generalize the procedure. A method of doing so has been given by the author (1950). The present paper is concerned with putting the method in a more direct and practical form.

An alternative treatment of the problem has been given by Anderson & Bergmann (1951). Their method is less general than the present one, because it is applicable only when the Lagrangian is quadratic in the velocities. With the present method the Lagrangian may be any function of the velocities and co-ordinates, subject only to the restriction that the Lagrangian equations of motion shall not lead to an inconsistency.

The ϕ equations

We consider a dynamical system described in terms of co-ordinates q_n $(n = 1, 2, \ldots, \mathrm{N})$ and velocities $\dot{q}_n$, with a Lagrangian $L = L(q, \dot{q})$. We define the momenta in the usual way

$$p_n = \partial L/\partial \dot{q}_n. \tag{1}$$

It may be that the p's are not independent functions of the $\dot{q}$'s. If the p's involve only $\mathrm{N}-\mathrm{M}$ independent functions of the $\dot{q}$'s, there will be M independent relations

$$\phi_m(q, p) = 0 \quad (m = 1, 2, \ldots, \mathrm{M}). \tag{2}$$

M may be anything from 0 to N.

The Lagrangian equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_n} = \frac{\partial L}{\partial q_n} \tag{3}$$

now fix $\mathrm{N}-\mathrm{M}$ functions of the accelerations $\ddot{q}_n$ and give M equations between co-ordinates and velocities only. It may be that by time differentiation of these M equations (once or possibly more than once), we can get some further independent equations involving accelerations. If there are not enough such equations to fix all the accelerations, the general solution of the equations of motion, starting from given initial values for the q's and $\dot{q}$'s, will contain a number of arbitrary functions of the time.

Let us make arbitrary small variations δq_n, $\delta\dot{q}_n$ in the co-ordinates and velocities. They will give rise to variations δp_n which preserve equations (1). These variations must preserve equations (2), which are consequences of (1), so

$$\frac{\partial\phi_m}{\partial q_n}\delta q_n+\frac{\partial\phi_m}{\partial p_n}\delta p_n = 0. \tag{4}$$

Equations (4) will be the only restrictions on the variations δp_n, provided equations (2) are written in such a way that first-order independent variations in the p's and q's make first-order variations in the ϕ's.

We have

$$\begin{aligned}\delta(p_n\dot{q}_n-L) &= p_n\delta\dot{q}_n+\dot{q}_n\delta p_n-\frac{\partial L}{\partial q_n}\delta q_n-\frac{\partial L}{\partial\dot{q}_n}\delta\dot{q}_n\\ &= \dot{q}_n\delta p_n-\frac{\partial L}{\partial q_n}\delta q_n.\end{aligned} \tag{5}$$

Since the terms in $\delta\dot{q}$ have cancelled, a variation in the $\dot{q}$'s which preserves equations (1) without any variation in the q's and p's leaves $p_n\dot{q}_n-L$ unchanged. This means that $p_n\dot{q}_n-L$ is a function of the q's and p's only, so we can put

$$p_n\dot{q}_n-L = H(q,p). \tag{6}$$

The function $H(q,p)$ is, of course, not uniquely determined. We may change it by

$$H\to H+c_m\phi_m \tag{7}$$

with the c_m any functions of the q's and p's. In the case when L is homogeneous of the first degree in the $\dot{q}$'s we may take $H = 0$.

Equation (5) now gives

$$\frac{\partial H}{\partial p_n}\delta p_n+\frac{\partial H}{\partial q_n}\delta q_n = \dot{q}_n\delta p_n-\frac{\partial L}{\partial q_n}\delta q_n$$

for variations δq_n, δp_n which are restricted by (4) but are otherwise arbitrary. It follows that

$$\dot{q}_n = \frac{\partial H}{\partial p_n}+u_m\frac{\partial\phi_m}{\partial p_n}, \tag{8}$$

$$-\frac{\partial L}{\partial q_n} = \frac{\partial H}{\partial q_n}+u_m\frac{\partial\phi_m}{\partial q_n}, \tag{9}$$

for suitable coefficients u_m. Under the transformation (7) the u_m change by functions of the q's and p's only, namely minus the c_m.

Equations (8) show that the $\dot{q}$'s are fixed by the q's together with the N $-$ M independent variables p and the M new variables u. We may thus take as our basic dynamical variables, instead of the q's and $\dot{q}$'s, the q's, p's and u's. They are the Hamiltonian variables.

The equations of motion (3) become, with the help of (9),

$$\dot{p}_n = -\frac{\partial H}{\partial q_n}-u_m\frac{\partial\phi_m}{\partial q_n}. \tag{10}$$

With Poisson brackets defined in the usual way

$$[A, B] = \frac{\partial A}{\partial q_n}\frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n}\frac{\partial B}{\partial q_n}, \tag{11}$$

we have, for g any function of the q's and p's,

$$\dot{g} = [g, H] + u_m[g, \phi_m]. \tag{12}$$

This single equation comprises all the equations (8) and (10). It is the general Hamiltonian equation of motion.

The definition (11) of P.b.'s requires the q's and p's to be considered as independent variables. Any relations which restrict the independence of the q's and p's, such as equations (2), must not be used before one works out P.b.'s, or the P.b.'s would cease to be well-defined quantities. To remind us of this limitation in the use of some of our equations, it is convenient to call such equations *weak equations* and to write them

$$\phi_m \approx 0.$$

The χ equations

By differentiating (2) with respect to the time and using (12), we get

$$[\phi_{m'}, H] + u_m[\phi_{m'}, \phi_m] = 0. \tag{13}$$

These equations, unless they all reduce to $0 = 0$, will diminish the number of independent Hamiltonian variables q, p, u, by providing some relations between them.

It may be that equations (13) lead to some relations between the q's and p's only, independent of the ϕ equations. They must be weak equations, so we write them

$$\chi_k(q, p) \approx 0 \quad (k = 1, 2, \ldots). \tag{14}$$

By differentiating each equation (14) with respect to the time, we get

$$[\chi_k, H] + u_m[\chi_k, \phi_m] = 0. \tag{15}$$

These equations may lead to further relations between the q's and p's only, which means further equations (14), leading in turn to further equations (15). We continue this procedure as far as it goes, and so obtain all the equations (14) and (15) that are consequences of (2) and the general equation of motion (12). Let us suppose the complete set is described by $k = 1, 2, \ldots, \mathrm{K}$. We thus have the number of independent p's and q's reduced to $2\mathrm{N} - \mathrm{M} - \mathrm{K}$, and we have the u's restricted by equations (13) and (15), in so far as these equations do not reduce to $0 = 0$ or to χ equations.

Let us look upon these equations (13) and (15) as equations in the unknowns u_m, with coefficients that are given functions of the q's and p's. They must have a solution

$$u_m = U_m(q, p), \tag{16}$$

because if they did not, it would imply that the Lagrangian equations of motion (3) are inconsistent. The solution (16) means that the equations

$$\left.\begin{aligned}[\phi_{m'}, H] + U_m[\phi_{m'}, \phi_m] \approx 0, \\ [\chi_k, H] + U_m[\chi_k, \phi_m] \approx 0\end{aligned}\right\} \tag{17}$$

hold as consequences of (2) and (14).

The solution (16) is in general not unique. We can add to U_m any solution $V_m = V_m(q, p)$ of the equations

$$V_m[\phi_{m'}, \phi_m] \approx 0, \quad V_m[\chi_k, \phi_m] \approx 0. \tag{18}$$

Let all the independent solutions of (18) be V_{am} ($a = 1, 2, \ldots, \text{A}$). Then the general solution of (13) and (15) is

$$u_m = U_m + v_a V_{am} \tag{19}$$

with arbitrary coefficients v_a.

We can use equations (19) to eliminate the variables u_m, so that we have as basic Hamiltonian variables the $2\text{N} - \text{M} - \text{K}$ independent q's and p's left by the equations (2) and (14), and the A variables v_a. The total number of these variables may well be less than 2N, the initial number of independent variables, because the equations of motion can cut them down.

Anderson & Bergmann (1951) call the ϕ equations primary constraints and the χ equations secondary constraints. For many purposes the ϕ's and the χ's are treated on the same footing, and it is then convenient to call them all χ_j ($j = 1, 2, \ldots, \text{M} + \text{K}$). The essential difference between them, from the Hamiltonian point of view, is that the ϕ's occur in the general equation of motion (12) while the χ_k do not.

The first class condition

A function of the q's and p's is defined to be *first-class* if its P.b.'s with H and the χ_j's all vanish. It is sufficient for these P.b.'s to vanish weakly, i.e. with the help of equations (2) and (14). A function of the q's and p's that does not satisfy these conditions is called *second-class*.

Theorem. *The P.b. of two first-class quantities is first-class.* To prove it, let X and Y be first-class, so that

$$[X, \chi_j] \approx 0, \quad [Y, \chi_j] \approx 0.$$

These weak equations mean

$$[X, \chi_j] = x_{jj'}\chi_{j'}, \quad [Y, \chi_j] = y_{jj'}\chi_{j'},$$

with suitable coefficients $x_{jj'}$ and $y_{jj'}$. Hence

$$\begin{aligned}[[X, Y], \chi_j] &= [[X, \chi_j], Y] - [[Y, \chi_j], X] \\ &\approx x_{jj'}[\chi_{j'}, Y] - y_{jj'}[\chi_{j'}, X] \approx 0.\end{aligned}$$

One can replace χ_j by H in the above argument and it remains valid, showing that $[[X, Y], H] \approx 0$. It follows that $[X, Y]$ is first-class.

Put

$$H + U_m\phi_m = H'. \tag{20}$$

Equations (17) show that the P.b. of H' with any χ_j vanishes weakly. Further,

$$[H', H] \approx U_{m'}[\phi_{m'}, H] \approx 0$$

from the first of equations (17) multiplied by $U_{m'}$. Thus H' is first-class. Note that H' is a Hamiltonian obtainable from H by a transformation (7).

Any linear combination of the ϕ's, with coefficients that are functions of the q's and p's, may be considered as another ϕ. Put

$$V_{am}\phi_m = \phi_a. \tag{21}$$

Equations (18) show that the P.b. of ϕ_a with any χ_j vanishes weakly. We have just seen that the P.b. of ϕ_a with H' vanishes, so its P.b. with H must also vanish. It follows that ϕ_a is first-class.

The general equation of motion (12) becomes, with the help of (19)

$$\dot{g} = [g, H'] + v_a[g, \phi_a]. \tag{22}$$

It now involves the first-class Hamiltonian H' and the first-class ϕ's ϕ_a. The coefficients v_a associated with these first-class ϕ's are not restricted in any way by the equations of motion. Each of them thus leads to an arbitrary function of the time in the general solution of the equations of motion with given initial conditions.

Every first-class ϕ is of the form $U_m\phi_m$ with U_m satisfying (17). Thus every independent first-class ϕ must appear in (22). It follows that the number of arbitrary functions of the time in the general solution of the equations of motion equals the number of independent first-class ϕ's. Different solutions of the equations of motion, obtained by different choices of the arbitrary functions of the time with given initial conditions, should be looked upon as all corresponding to the same physical state of motion, described in various way by different choices of some mathematical variables that are not of physical significance (e.g. by different choices of the gauge in electrodynamics or of the co-ordinate system in a relativistic theory).

In practical applications one usually knows what arbitrary functions of the time there are in the general solution of the equations of motion, from the invariance properties of the action integral, This knowledge enables one to see which of the ϕ's are first-class without going through the labour of evaluating all their P.b.'s. Any velocity variable that can be altered without affecting the physical state must appear in the Hamiltonian equation of motion (22) as a coefficient v_a associated with a first-class ϕ.

Reduction in the number of degrees of freedom

Let us suppose that some of the first-class ϕ's involve the momentum variables only linearly with numerical coefficients. Although this is a very special case mathematically, it often occurs in practical applications and is of importance.

By a trivial change of variables, we can arrange that these first-class ϕ's take the form

$$p_r - f_r \approx 0 \quad (r = 1, 2, \ldots, \mathrm{R}), \tag{23}$$

with f_r a function of the q's only. The first-class condition requires that the quantities $p_r - f_r$ shall have their P.b.'s vanishing weakly. These P.b.'s can involve only the q's.

We shall assume there are no χ_j involving only q's, and then these P.b.'s must vanish strongly. It follows that

$$f_r = \partial F/\partial q_r$$

for some function F of the q's. Now add to the Lagrangian L the term

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_n}\dot{q}_n, \tag{24}$$

which will not affect the equations of motion. The p_r will be increased by $\partial F/\partial q_r$, so the ϕ equations (23) get brought into the form

$$p_r \approx 0. \tag{25}$$

We shall continue to work with the new Lagrangian. Any of the χ_j's which are not included in (25) we shall call χ_i ($i = 1, 2, \ldots, \mathrm{M}+\mathrm{K}-\mathrm{R}$). The χ_i's may be either first- or second-class. We may assume without loss of generality that the χ_i do not involve the variables p_r. We may also assume that H' does not involve the p_r, because if it does we can make a transformation (7) to another first-class H' which does not.

Since the p_r are first-class, we have

$$[\chi_i, p_r] \approx 0, \quad [[\chi_i, p_r], p_{r'}] \approx 0, \tag{26}$$

and so on. It follows that the χ_i, if they involve the variables q_r at all, can do so only through being of the form

$$\chi_i = \beta_{ii'}\chi^*, \tag{27}$$

where the χ^*'s vanish weakly and are independent of the q_r, so that the q_r occur only in the coefficients $\beta_{ii'}$. This means that the conditions $\chi_i \approx 0$ are equivalent to the conditions $\chi^* \approx 0$, which do not involve the q_r variables. The number of χ^*'s must equal the number of χ_i's. (There cannot be more χ^*'s than χ_i's, because the $\chi_i^* \approx 0$ conditions are all consequences of the conditions $\chi_i \approx 0$ together with the conditions (26), and the latter are themselves consequences of $\chi_i \approx 0$.)

If χ_i is first-class we see, by applying the theorem of the preceding section to χ_i and p_r, that χ_i is expressible in terms of first-class χ^*'s, i.e. we can arrange that in (27) the coefficient $\beta_{ii'}$ is zero unless $\chi_{i'}^*$ is first-class.

We can go through the above work with χ_i replaced by H' and we find that

$$H' = H'' + \gamma_i \chi_i^*, \tag{28}$$

where H'', like the χ^*'s, does not involve the q_r. Since H' is first-class, we can infer that H'' is first-class and that any χ^*'s that occur in (28) are first-class.

Let us now see what becomes of the equation of motion (22). For g equal to one of the q_r, we find that $\dot{q}_r$ is arbitrary, so q_r varies arbitrarily. For g a function of the q_s, p_s variables ($s = \mathrm{R}+1, \ldots, \mathrm{N}$) we get an equation of the form

$$\dot{g} = [g, H''] + w_a[g, \chi_a^*], \tag{29}$$

where the χ_a^* are first-class χ^*'s. (They may or may not include all the first-class χ^*'s.) The variables q_r, p_r do not occur in this equation, except in that the q_r may occur in the coefficients w_α.

Let us assume that we can get arbitrary variations in the w_α by making variations in the q_r and in those coefficients v_a of (22) that are associated with first-class ϕ's other than the p_r. This assumption usually holds in practice. Then we can look upon the w_α in (29) as arbitrary coefficients, which together with the q_s and p_s make up the basic Hamiltonian variables. The q_r and p_r no longer appear in the general equation of motion (29). This equation has the same fundamental form as (22), but refers only to the degrees of freedom q_s, p_s. The degress of freedom q_r, p_r thus drop out of the theory.

It may be that some of the χ_α^*'s occurring in (29) involve the momentum variables only linearly, with numerical coefficients. We can then repeat the whole procedure and get a further reduction in the number of effective degrees of freedom.

References

Anderson, J. L. & Bergman, P. G. 1951 *Phys. Rev.* **83**, 1018.
Dirac, P. A. M. 1950 *Canad. J. Math.* **2**, 129.