A new classical theory of electrons

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In the theory of the electromagnetic field without charges, the potentials are not fixed by the field, but are subject to gauge transformations. The theory thus involves more dynamical variables than are physically needed. It is possible by destroying the gauge transformations to make the superfluous variables acquire a physical significance and describe electric charges. One gets in this way a simplified classical theory of electrons, which appears to be more suitable than the usual one as a basis for a passage to the quantum theory.

1. Introduction

Classical electrodynamics is based on Maxwell's equations for the electromagnetic field and Lorentz's equations of motion for electrons. It is an approximate theory, valid only if the accelerations of the electrons are small, and attempts to make it accurate bring one up against the problem of the structure of the electron, which has not yet received any satisfactory solution. If one assumes the charge of the electron to be concentrated at a point, one gets an infinite self-energy, which is physically meaningless, and if one assumes it to be spread through a small volume, as Lorentz himself did, one gets into great complexities when one tries to treat the acceleration of an electron relativistically.

People hoped at one time that quantum mechanics would remove these difficulties, but this hope has not been fulfilled. Models of the electron with the charge not localized have proved too cumbersome for quantization, and the point-charge model leads to infinities which are more troublesome in the quantum theory than in the classical theory. Recent work by Lamb, Schwinger, Feynman and others has been very successful in setting up rules for handling the infinities and subtracting them away, so as to leave finite residues which can be compared with experiment, but the resulting theory is an ugly and incomplete one, and cannot be considered as a satisfactory solution of the problem of the electron.

The troubles of the present quantum electrodynamics should be ascribed primarily, in my opinion, not to a fault in the general principles of quantization, but to our working from a wrong classical theory. To make progress one should therefore re-examine the classical theory of electrons and try to improve on it. One has a much better chance of doing this at the present time than one had in Lorentz's time because of the information, which quantum theory has given us, that *the Hamiltonian form for the equations of motion is all-important.*

In the present paper a radically different theory for the motion of electrons is put forward, which is simpler than Lorentz's from the Hamiltonian standpoint and which provides exact equations of motion without requiring any assumptions about the structure of the electron.

2. The electromagnetic field without charges

In the absence of charges, Maxwell's equations for the electromagnetic field, in relativistic notation, are

$$\partial F_{\mu\nu}/\partial x^{\lambda} + \partial F_{\nu\lambda}/\partial x^{\mu} + \partial F_{\lambda\mu}/\partial x^{\nu} = 0, \tag{1}$$

$$\partial F_{\mu\nu}/\partial x_{\nu} = 0. \tag{2}$$

Equation (1) allows one to express the field quantities $F_{\mu\nu}$ in terms of potentials, according to

$$F_{\mu\nu} = \partial A_{\nu}/\partial x^{\mu} - \partial A_{\mu}/\partial x^{\nu}. \tag{3}$$

The potentials A_{μ} are not completely determined by (3). They can be transformed to A^*_{μ}, where

$$A^*_{\mu} = A_{\mu} - \partial S/\partial x^{\mu} \tag{4}$$

for any function S, without affecting the $F_{\mu\nu}$. Such transformations are called gauge transformations.

One can restrict the arbitrariness in the potentials and the extent of the gauge transformations by introducing the subsidiary condition

$$\partial A_{\mu}/\partial x_{\mu} = 0. \tag{5}$$

People usually do adopt this restriction in electrodynamics, but I believe it brings an undesirable feature into the theory. One gets a more interesting and more powerful mathematical theory if one does not have it, so that one retains the possibility of making gauge transformations with S arbitrary.

The introduction of the potentials A_{μ} automatically satisfies Maxwell's equation (1) and allows one to derive (2) from an action principle

$$\delta \int \mathscr{L}\, dx_0 dx_1 dx_2 dx_3 = 0 \tag{6}$$

with
$$\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = \tfrac{1}{2}(\partial A_{\nu}/\partial x^{\mu} - \partial A_{\mu}/\partial x^{\nu})\, \partial A^{\mu}/\partial x_{\nu}. \tag{7}$$

Thus equation (2) appears as the Lagrangian equations of motion with the Lagrangian function

$$L = \int \mathscr{L}\, dx_1 dx_2 dx_3, \tag{8}$$

in which the A_{μ} and $\partial A_{\mu}/\partial x_0$ at various points at a given time are the dynamical co-ordinates and velocities. This is useful as a first step towards getting the equations in Hamiltonian form, as is necessary for a passage to the quantum theory to be possible.

The usual method of passing from the Lagrangian to the Hamiltonian fails when applied to the Lagrangian (8), (7), because the momenta conjugate to the A_0 dynamical co-ordinates turn out to be zero and so cannot be used. A way out of the difficulty was found by Fermi (1932), who modified the Lagrangian by adding a further term

$$-\frac{1}{2}\frac{\partial A_{\mu}}{\partial x_{\mu}}\frac{\partial A_{\nu}}{\partial x_{\nu}}$$

to $\mathscr{L}$. The modified Lagrangian leads to suitable momentum variables and still gives the correct equations of motion, provided one uses the subsidiary condition (5). Thus *the subsidiary condition* (5) *is essential with Fermi's method of putting the equations in Hamiltonian form.*

I have recently (1950, 1951) given a more general theory of the passage from the Lagrangian to the Hamiltonian, according to which one is not bothered by any of the momenta turning out to be zero. An old method of Rosenfeld (1930) is also adequate in the present case. With either of these methods one can very well work with the unmodified Lagrangian (8), (7) and get from it a Hamiltonian which is suitable for taking over to the quantum theory. Thus *one can build up electrodynamic theory without using the subsidiary condition* (5). One gets a more powerful theory, capable of being transformed into a wide variety of different forms, and so providing better prospects for enabling one to introduce electric charges in a satisfactory way.

The existence of gauge transformations in the theory means that there are more variables present in the mathematics than are physically necessary. With Fermi's theory the extra variables describe longitudinal electromagnetic waves, which have no physical reality and may be eliminated by a transformation of the equations, the gauge transformations disappearing of course in the process. With the theory which does not use the subsidiary condition (5) the extra variables do not describe longitudinal waves, but can vary arbitrarily with the time. They are thus, to a certain extent, *at our disposal*, and we shall see that they can be made to serve in the description of electrons, instead of remaining physically meaningless.

3. The electromagnetic field with charges

The usual way of introducing electric charges into the theory is by bringing into the mathematical scheme further dynamical variables describing electrons and adding suitable terms involving these variables to the action integral. A simpler and more direct way is to use the superfluous variables in the theory without charges for the purpose of describing the charges, and not to bring in any new variables at all. The gauge transformations must then be destroyed, since so long as they remain, the superfluous variables of the theory without charges must continue to have no physical significance.

Let us study the simplest relativistic way of destroying the gauge transformations, namely, by imposing the subsidiary condition

$$A_\mu A^\mu = k^2, \tag{9}$$

where k is a universal constant. We shall see that this condition is the only assumption we need to make to get classical electrons appearing in the theory.

The condition (9) must be imposed on the potentials in the action principle (6), (7). We can take it into account conveniently by adding a term to $\mathscr{L}$, to make it

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\lambda(A_\mu A^\mu - k^2), \tag{10}$$

where λ is an unknown function, which may be treated as a further field quantity.

The field equations which follow from the new action principle (6), (10), are (9) and

$$\partial F_{\mu\nu}/\partial x_\nu = \lambda A_\mu. \tag{11}$$

We see that we must identify the charge-current density j_μ as

$$j_\mu = -\lambda A_\mu, \tag{12}$$

and then (11) gives correctly the generalization of the Maxwell equation (2) for the presence of charges. From (11) and (12) follows the conservation of electric charge

$$\partial j_\mu/\partial x_\mu = 0. \tag{13}$$

Equations (9) and (11) are the fundamental field equations of the new theory. There are five equations to determine the five field quantities A_μ, λ, so the solution of the equations is fixed when suitable initial conditions are prescribed.

Let us examine the physical consequences of the equations. In the first place we see that there exist solutions with $\lambda = 0$. These obey the Maxwell equations in the absence of charges. For these solutions the new theory is equivalent to the usual one, so far as the field quantities $F_{\mu\nu}$ are concerned. The only difference is that the potentials cannot now be chosen arbitrarily to satisfy (3), but must satisfy also (9)—a condition which has no physical effects since the potentials are not observable.

Let us take a solution with $\lambda = 0$, corresponding to no charges, and make a small change in it so as to bring in small charges, and then investigate how these charges move. For this purpose we take λ in (11) to be infinitesimal. Let $F^{(0)}_{\mu\nu}$ be the original field corresponding to no charges. The new field will differ from this by an infinitesimal which we can neglect. Let A^*_μ be any potentials which lead to the field $F^{(0)}_{\mu\nu}$ according to

$$F^{(0)}_{\mu\nu} = \partial A^*_\nu/\partial x^\mu - \partial A^*_\mu/\partial x_\nu.$$

The potentials A_μ of our theory must be connected with the A^*_μ by a gauge transformation (4), with the function S chosen so as to make A_μ satisfy (9). Substituting for A_μ in terms of A^*_μ in (9), we get

$$(\partial S/\partial x^\mu + A^*_\mu)(\partial S/\partial x_\mu + A^{\mu *}) = k^2. \tag{14}$$

From a solution of (14) we get a possible motion of charges satisfying the conditions of the theory by taking

$$j_\mu = -\lambda(\partial S/\partial x^\mu + A^*_\mu). \tag{15}$$

The λ here can be chosen to be an arbitrary infinitesimal at one instant of time, and its value at other times is then fixed by the conservation law (13).

The direction of motion of the charges in space-time is given by the 4-vector $\partial S/\partial x^\mu + A^*_\mu$. Now equation (14) is just the Hamilton-Jacobi equation for an electron moving according to Lorentz's equations in the field of the potentials A^*_μ, provided we take the universal constant k to be connected with the charge and mass of the electron by

$$k = m/e. \tag{16}$$

With equation (14) interpreted in this way, the energy-momentum 4-vector of the electron is

$$p_\mu = e\,\partial S/\partial x^\mu$$

and the velocity 4-vector is

$$(p_\mu + eA^*_\mu)/m = e(\partial S/\partial x^\mu + A^*_\mu)/m.$$

This is in the same direction in space-time as the motion of the charges (15). We can conclude that *the new theory requires that infinitesimal charges in a given field shall move in a way which agrees with Lorentz's equations for electrons moving in this field, with neglect of the influence of the field produced by the electrons.*

To study the motion of charges which are not small, we may build them up by making successive infinitesimal changes in the solution of the equations. Each such infinitesimal change can be treated by the same method as before. We replace $F^{(0)}_{\mu\nu}$ by $F^{(1)}_{\mu\nu}$ say, the field in the presence of the already existing charges, express it in terms of potentials A^*_μ, and arrive again at equation (14). A solution of this equation provides us with the possibility of changing the charges by δj_μ equal to the right-hand side of (15), with λ chosen to be an arbitrary infinitesimal at one instant of time. Equation (14) can still be interpreted as a Hamilton-Jacobi equation, but it now refers to an electron moving in the field of the already existing charges. We can conclude that *any infinitesimal change in the charges must move in a way which agrees with Lorentz's equations for electrons moving in the field of the already existing charges.*

To obtain the Hamiltonian of the new theory, the most convenient method is to take the Lagrangian (8), (7) and use the equation (9) to eliminate the dynamical co-ordinates A_0 in terms of A_1, A_2, A_3. One can then use the ordinary method of passing from the Lagrangian to the Hamiltonian. The momentum B_r conjugate to the dynamical co-ordinate A^r ($r = 1, 2, 3$) is the functional derivative

$$B_r = \frac{\partial L}{\partial(\partial A^r/\partial x_0)} = F_{r0}, \tag{17}$$

so B_r is just the electric field. The Hamiltonian is now

$$\begin{aligned} H &= \int B_r \frac{\partial A^r}{\partial x_0} dx_1 dx_2 dx_3 - L \\ &= \int \left\{ \tfrac{1}{4} F_{rs} F^{rs} - \tfrac{1}{2} B_r B^r - (A_s A^s + k^2)^{\frac{1}{2}} \frac{\partial B_r}{\partial x_r} \right\} dx_1 dx_2 dx_3. \end{aligned} \tag{18}$$

4. Discussion

The new theory seems to be adequate for the description of the motion of electrons when quantum effects are not considered. It describes correctly the motion of a beam of electrons in an electromagnetic field, giving the deflexion of the beam by the field and also, if the beam is a strong one, giving the divergence of different parts of the beam produced by space charges.

It does not provide a detailed description for individual electrons of the beam in the way the usual theory of electrons does. Such a detailed description is not needed when quantum phenomena are not being considered, so it ought not to appear in a classical theory and it is an advantage of the new theory not to give it. This advantage shows itself through the mathematics of the new theory involving fewer dynamical variables, and thus being of an essentially simpler character.

The new theory requires a continuity in the dependence of the velocity of an electron on its position in the beam—a continuity which is described mathematically by the condition that one solution of the Hamilton-Jacobi equation fixes the motion of all the electrons. One can introduce oscillations of short wave-length into the Hamilton-Jacobi function S, to give rise to fluctuations in the velocity of an electron according to its position in space, but such fluctuations are much restricted compared to those allowed by the usual theory, and one cannot have electrons with widely different velocities arbitrarily close together. This limitation is a necessary consequence of the reduced number of dynamical variables in the new theory. As far as I can see it is not in disagreement with classical experiments.

An important feature of the new theory is that it involves only the ratio e/m, not e and m separately. This is what one should expect in a purely classical theory. The existence of e should be looked upon as a quantum effect, and it should appear in a theory only after quantization, and not be a property of classical electrons.

In the usual electrodynamic theory one takes a first approximation by putting $e = 0$, so as to get electrons which do not interact with the electromagnetic field, and one then introduces the interaction as a perturbation. This is not possible in the new theory, where one does not have any e to put equal to zero. The electron of the new theory cannot be considered apart from its interaction with the electromagnetic field.

The theory of the present paper is put forward as a basis for a passage to a quantum theory of electrons. To make this passage one will presumably have to replace the square root in the Hamiltonian (18) by something involving spin variables, and one will also have to bring in the Fermi statistics for the electrons. This may be a difficult problem, but one can hope that its correct solution will lead to the quantization of electric charge and will fix e in terms of h. One can also hope for an improvement in the situation with regard to the infinities, since in the new classical theory questions of the interaction of an electron with itself no longer arise.

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