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# The Large Numbers hypothesis and the Einstein theory of gravitation

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A study of the relations between large dimensionless numbers leads one to believe that $G$, expressed in atomic units, varies with the epoch. The Einstein theory requires $G$ to be constant. One can reconcile these two requirements by supposing that the Einstein theory applies with a metric that differs from the atomic metric.

The theory can be developed with conservation of mass by supposing that the continual increase in the mass of the observable universe arises from a continual slowing down of the velocity of recession of the galaxies. One is led to a model of the Universe that was first proposed jointly by Einstein & de Sitter (the E.S. model). The observations of the microwave radiation fit in with this model.

The static Schwarzchild metric has to be modified to fit in with the E.S. model for large $r$. The modification is worked out, and also the motion of planets with the new metric. It is found that there is a difference between ephemeris time and atomic time, and also that there should be an inward spiralling of the planets, referred to atomic units, superposed on the motion given by ordinary gravitational theory. These are effects that can be checked by observation, but there is no conclusive evidence up to the present.

## The Large Numbers hypothesis

From data provided by atomic physics and astronomy one can construct some large dimensionless numbers. The most important ones are: (i) the ratio of the electric to the gravitational force between an electron and a proton, $e^2/Gm_e m_p$. It has the value about $7 \times 10^{39}$; (ii) the age of the Universe $t$, expressed in terms of a unit of time provided by atomic constants, say $e^2/m_e c^3$. It has the value about $2 \times 10^{39}$; (iii) the mass of that part of the Universe that is receding from us with a velocity $< \frac{1}{2}c$, expressed in units of the proton mass. Call it $N$. Its value is rather uncertain, because one does not know how much matter there is that is not seen by our telescopes, but it is of the order $10^{78}$.

The Large Numbers hypothesis (L.N.h.) asserts that these numbers are related by equations in which the coefficients are close to unity. Since the number (ii) varies with the age of the Universe, the L.N.h. requires that the others must also vary. We get

$$e^2/Gm_e m_p \propto t, \tag{1}$$

and $$N \propto t^2. \tag{2}$$

One can formulate the relation (1) by saying that $G \propto t^{-1}$ when expressed in atomic units.

The relation (2) requires a continual increase in the amount of matter in the observable part of the Universe. To take it into account I have been assuming continuous creation of matter. However, there are grave difficulties with this assumption, both conceptually and with regard to fitting it in with observation, and I believe it should be abandoned.

One can reconcile the relation (2) with conservation of mass by assuming that the velocity of recession of a galaxy is continually decreasing, so that more and more galaxies are continually appearing with velocity of recession $< \frac{1}{2}c$. This was the picture adopted in my first paper on the subject (1938). I could not develop it at the time because it disagreed violently with the value of the Hubble constant that was then accepted.

## The two metrics

We have the problem of fitting in the L.N.h. with the Einstein theory of gravitation. The difficulty is that the L.N.h. gives $G \propto t^{-1}$ while the Einstein theory requires $G$ to be constant. The difficulty can be met if we assume that the Einstein theory is valid in a different system of units from those provided by the atomic constants. We use the suffixes E and A to refer to quantities in the two systems of units. Thus we have $$G_{\mathrm{A}} \propto t^{-1}, \quad G_{\mathrm{E}} = \text{const.} \tag{3}$$

With no continuous creation of matter, the mass of a body in atomic units is constant. In the Einstein theory the mass is also constant. Thus the two units of mass are the same, and we have $$m_{\mathrm{E}} = m_{\mathrm{A}}. \tag{4}$$

We shall keep the velocity of light to be unity in both the atomic and the Einstein units. Thus the units of time and of distance are both changed in the same ratio. This ratio may be worked out from (3) with dimensional arguments. The dimensions of $G$ are [distance]$^3$ [time]$^{-2}$ [mass]$^{-1}$. Thus from (4) we must have
$$\mathrm{d}s_{\mathrm{E}} = t\,\mathrm{d}s_{\mathrm{A}}. \tag{5}$$
The relations (4) and (5) are fundamental for the reconciliation of the L.N.h. and the Einstein theory.

We shall continue to use $t$ to denote the epoch in atomic units. Let $\tau$ denote the epoch in Einstein units. Then $$\mathrm{d}\tau = t\,\mathrm{d}t \tag{6}$$
or $$\tau = \tfrac{1}{2}t^2.$$

## The law of expansion

In the present theory the Universe at a particular time is infinite, so we cannot talk about the radius of the Universe. But we can introduce a variable $R$ which gives the distance of a particular galaxy and discuss how $R$ varies with the epoch. This will provide the law of expansion of the Universe. We may take any not too distant galaxy, so that we are not disturbed by questions of the curvature of space.

The variation of $R$ can be worked out from the condition (2). We use atomic units, so $R$ becomes $R_A$. The average density in atomic units is, with conservation of mass,

$$\rho_A \propto R_A^{-3}. \tag{7}$$

Let us take a general law of expansion

$$R_A \propto t^n. \tag{8}$$

Then the velocity of recession of the particular galaxy we are concerned with is

$$\mathrm{d}R_A/\mathrm{d}t = nR_A/t.$$

The distance of a galaxy whose velocity of recession equals $\frac{1}{2}c$ is now $\frac{1}{2}t/n$. The total mass within this distance is $\propto \rho_A t^3$. According to (2) this must be $\propto t^2$. So

$$\rho_A \propto t^{-1} \tag{9}$$

and (7) gives

$$R_A \propto t^{\frac{1}{3}}. \tag{10}$$

This is a much slower rate of expansion than in the usually accepted picture, for which $R_A \propto t$ rather closely.

With the new law of expansion we must revise our estimate of the age of the Universe. For the general law of expansion (8) the Hubble constant is

$$H = (\mathrm{d}R_A/\mathrm{d}t)/R_A = n/t.$$

With the usual value $n = 1$ we get $H = t^{-1}$. With the present theory $n = \frac{1}{3}$ and $H = \frac{1}{3}t^{-1}$. Thus with the same value for $H$, the age of the Universe is reduced by a factor 3.

The value of the Hubble constant is still not known with great accuracy, but is believed to correspond to an age of the Universe of about $18 \times 10^9$ years with the law of expansion $R_A \propto t$. It thus corresponds to an age of about $6 \times 10^9$ years with the new law of expansion. This is rather less than the age of the Universe that is usually believed, but still it is greater than the age of the Solar System, namely $4.5 \times 10^9$ years, so it is not impossible.

## The curvature of three-dimensional space

Let us consider the three-dimensional space corresponding to a particular epoch $t$. If we neglect local irregularities it will be a uniform space of constant curvature (constant in space, not necessarily constant in time). The curvature is either positive, zero or negative.

The case of positive curvature can be immediately ruled out. It would require the Universe to have a finite total mass which, expressed in proton units, would give a large number that is constant, and thus contradicts the L.N.h.

The case of negative curvature can also be ruled out, though not quite so directly. Introduce coordinates $\xi^1$, $\xi^2$, $\xi^3$ in the three-dimensional space such that they are constant for any particular galaxy. Then, corresponding to a definite $d\xi^1$, $d\xi^2$, $d\xi^3$, we shall have a distance $ds_A$ which varies as $t^{\frac{1}{3}}$. The radius of curvature, $\mathscr{R}$ say, is determined by such distances and therefore also varies as $t^{\frac{1}{3}}$. The total mass within a sphere of radius $\mathscr{R}$ is $\mathscr{R}^3$ times the density and is thus from (9) independent of $t$. It gives a constant large number, which is not allowed.

We are thus left with the case of zero curvature, or flat three-dimensional space, as the only one consistent with the L.N.h. We may then use Cartesian coordinates $x^1, x^2, x^3$ in three-dimensional space.

## The model of the Universe

The L.N.h. requires that *every* large dimensionless number that appears in fundamental theory shall vary according to a certain power of the epoch, depending on the value of the number. Thus there must not be any large dimensionless number that is constant.

We consider the Universe with local irregularities smoothed out. There are several possible models, which have been worked out by various authors. A comprehensive list of them has been given by Robertson (1933). Nearly all of them contradict the L.N.h. as they provide a constant large number. For example, those models that give the Universe growing to a maximum size and then contracting are to be ruled out, because the maximum size, expressed in atomic units, gives a constant large number.

Let us discuss the model in terms of the Einstein metric. We have

$$ds_E^2 = d\tau^2 - R_E^2\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\}. \tag{11}$$

Here $R_E$ is a function of $\tau$ only. We shall use a dot to denote $d/d\tau$. This metric corresponds to a uniform density $\rho_E$ and a uniform pressure $p_E$,

$$\rho_E = 3(\dot{R}_E/R_E)^2, \quad p_E = -(\dot{R}_E/R_E)^2 - 2\ddot{R}_E/R_E.$$

(See, for example, Robertson, eqns (3.2) with $\lambda = 0$, $k = 0$.) To correspond to the actual Universe we must take the pressure to be zero. This leads to

$$R_E \propto \tau^{\frac{2}{3}},$$

in agreement with the law of expansion provided by the L.N.h., namely from (10)

$$R_E = tR_A \propto t^{\frac{4}{3}} \propto \tau^{\frac{2}{3}}.$$

We now get $$\rho_E = \tfrac{4}{3}\tau^{-2} = \tfrac{16}{3}t^{-4}.$$

So $$\rho_A = \tfrac{16}{3}t^{-1},$$

agreeing with (9).

This model was proposed in a joint paper by Einstein & de Sitter (1932). They pointed out that it automatically gives the correct order of magnitude for the density. We shall refer to it as the E.S. model. It seems to be a very good model for describing the actual Universe. It is the simplest non-static model.

There is an arbitrary constant at our disposal in connection with the coordinates $x^1, x^2, x^3$ and we may choose it so as to make $R_{\mathrm{E}} = \tau^{\frac{2}{3}}$, so that (11) becomes

$$ds_{\mathrm{E}}^2 = d\tau^2 - \tau^{\frac{4}{3}}\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\}. \tag{12}$$

### The natural microwave radiation

Is there any confirmation for the law of expansion (10)? There is very good confirmation, provided by the observation of the natural microwave radiation.

The microwave radiation is believed to be of cosmological origin because of its uniformity and isotropy. So far as it can be observed it appears to be black body radiation, satisfying Planck's law. Now black body radiation in an expanding universe remains black body radiation (provided there is no creation of photons, such as might occur in a theory with continuous creation of matter). Each spectral component of the radiation is red-shifted according to the same law as the distance of a galaxy, thus from (10)

$$\lambda \propto t^{\frac{1}{3}}.$$

The temperature $T$ of the radiation decreases according to the same law as the frequency of one of its components, thus

$$T \propto \nu \propto \lambda^{-1} \propto t^{-\frac{1}{3}}. \tag{13}$$

The rate of cooling is much slower than in the usual theory, according to which the distance of a galaxy is roughly proportional to $t$, so that $\lambda \propto t$. This would make

$$T \propto \lambda^{-1} \propto t^{-1}. \tag{14}$$

The observed value of $T$ is about 2.8 K. This gives an energy $kT$, which may be compared to the rest-energy of a proton to give a dimensionless number

$$kT/m_{\mathrm{p}}c^2 = 2.5 \times 10^{-13}.$$

According to the L.N.h. we should expect this to vary with the epoch according to the law

$$kT/m_{\mathrm{p}}c^2 \propto t^{-\frac{1}{3}}.$$

Thus $T \propto t^{-\frac{1}{3}}$, in agreement with (13) above, but disagreeing strongly with (14).

The microwave radiation thus provides confirmation of our present picture. The radiation has been cooling according to the $t^{-\frac{1}{3}}$ law since a time close to the Big Bang. According to the usual views it has been cooling according to the $t^{-1}$ law since a certain decoupling time, when it became decoupled from matter. This decoupling time must have been around $t = 10^{26}$, when $T$ was $10^{13}$ times greater than now, so that $kT$ was approximately $m_{\mathrm{p}}c^2$. The existence of such a decoupling time, playing a fundamental rôle in cosmology, would contradict the L.N.h.

### The modified Schwarzschild metric

For dealing with the motion of the planets we must use the Einstein metric and refer it to polar coordinates $r$, $\theta$, $\phi$. The usual expression for this metric, the Schwarzschild solution, has to be modified so as to fit the metric (12) of the E.S. model for large values of $r$. It ceases to be static.

We can set up a general expression for the metric,

$$\mathrm{d}s_{\mathrm{E}}^2 = \alpha^2\,\mathrm{d}\tau^2 - \beta^2\,\mathrm{d}r^2 - \gamma^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{15}$$

where $\alpha$, $\beta$, $\gamma$ are functions of $r$ and $\tau$. Thus

$$g_{00} = \alpha^2, \quad g_{11} = -\beta^2, \quad g_{22} = -\gamma^2, \quad g_{33} = -\gamma^2\sin^2\theta,$$

and

$$g_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu.$$

We also find

$$g^{00} = \alpha^{-2}, \quad g^{11} = -\beta^{-2}, \quad g^{22} = -\gamma^{-2}, \quad g^{33} = -\gamma^{-2}\sin^{-2}\theta.$$

We calculate all the Christoffel symbols. The non-vanishing ones are, with a dot denoting $\partial/\partial\tau$ and a prime denoting $\partial/\partial r$,

$$\begin{array}{ll}
\Gamma^0_{00} = \dot{\alpha}/\alpha & \Gamma^0_{11} = \beta\dot{\beta}/\alpha^2 \\
\Gamma^0_{22} = \gamma\dot{\gamma}/\alpha^2 & \Gamma^0_{33} = (\gamma\dot{\gamma}/\alpha^2)\sin^2\theta \\
\Gamma^1_{00} = \alpha\alpha'/\beta^2 & \Gamma^1_{11} = \beta'/\beta^2 \\
\Gamma^1_{22} = -\gamma\gamma'/\beta^2 & \Gamma^1_{33} = -(\gamma\gamma'/\beta^2)\sin^2\theta \\
\Gamma^0_{01} = \alpha'/\alpha & \Gamma^1_{01} = \dot{\beta}/\beta \\
\Gamma^2_{20} = \dot{\gamma}/\gamma & \Gamma^3_{30} = \dot{\gamma}/\gamma \\
\Gamma^2_{21} = \gamma'/\gamma & \Gamma^3_{31} = \gamma'/\gamma \\
\Gamma^2_{22} = -\sin\theta\cos\theta & \Gamma^3_{32} = \cot\theta.
\end{array}$$

We get

$$\begin{aligned}
\Gamma^\sigma_{0\sigma} &= \dot{\alpha}/\alpha + \dot{\beta}/\beta + 2\dot{\gamma}/\gamma \\
\Gamma^\sigma_{1\sigma} &= \alpha'/\alpha + \beta'/\beta + 2\gamma'/\gamma \\
\Gamma^\sigma_{2\sigma} &= \cot\theta \quad \Gamma^\sigma_{3\sigma} = 0.
\end{aligned}$$

We now calculate the Ricci tensor

$$R_{\mu\nu} = \Gamma^\sigma_{\mu\sigma,\nu} - \Gamma^\sigma_{\mu\nu,\sigma} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho}.$$

We see that it is of the form

$$R_{\mu\nu} = R_{\mu\nu}(0) + R_{\mu\nu}(1) + R_{\mu\nu}(2),$$

where $R_{\mu\nu}(0)$ is independent of $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, $R_{\mu\nu}(1)$ is linear in $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ and $R_{\mu\nu}(2)$ consists of quadratic terms in $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ plus linear terms in $\ddot{\alpha}$, $\ddot{\beta}$, $\ddot{\gamma}$. We shall have $\alpha$, $\beta$, $\gamma$ varying only slowly with $\tau$, so $R_{\mu\nu}(1)$ will be small of the first order, and $R_{\mu\nu}(2)$ of the second order.

We can secure that the $R_{\mu\nu}(0)$ shall all vanish by choosing $\alpha$, $\beta$, $\gamma$ to satisfy the conditions for the Schwarzschild solution when their dependence on $\tau$ is neglected. There is then no need to work out the $R_{\mu\nu}(0)$. We must, however, work out the $R_{\mu\nu}(1)$ and the $R_{\mu\nu}(2)$. We find

$$R_{01}(1) = 2\gamma^{-1}(\dot{\gamma}' - \dot{\gamma}\alpha'/\alpha - \gamma'\dot{\beta}/\beta). \tag{16}$$

All the other components of $R_{\mu\nu}(1)$ vanish. We find further

$$R_{00}(2) = 2\frac{\dot{\gamma}^2}{\gamma^2}+\frac{\dot{\beta}^2}{\beta^2}-\frac{\dot{\alpha}\dot{\beta}}{\alpha\beta}-2\frac{\dot{\alpha}\dot{\gamma}}{\alpha\gamma}+\left(\frac{\dot{\beta}}{\beta}\right)^{\cdot}+2\left(\frac{\dot{\gamma}}{\gamma}\right)^{\cdot}, \tag{17}$$

$$R_{11}(2) = -\frac{\beta\dot{\beta}\dot{\alpha}}{\alpha^3}-2\frac{\beta\dot{\beta}\dot{\gamma}}{\alpha^2\gamma}+\frac{\dot{\beta}^2}{\alpha^2}-\left(\frac{\beta\dot{\beta}}{\alpha^2}\right)^{\cdot}, \tag{18}$$

$$R_{22}(2) = -\frac{\gamma\dot{\gamma}\dot{\alpha}}{\alpha^3}-\frac{\gamma\dot{\gamma}\dot{\beta}}{\alpha^2\beta}-\left(\frac{\gamma\dot{\gamma}}{\alpha^2}\right)^{\cdot}, \tag{19}$$

$$R_{33}(2) = R_{22}(2)\sin^2\theta, \tag{20}$$

$$R_{\mu\nu}(2) = 0 \quad \mu \neq \nu.$$

We cannot impose the usual Einstein field equations $R_{\mu\nu} = 0$. This would not be consistent with the E.S. model for large $r$. We assume as much as we can of the Einstein equations without getting an inconsistency, namely

$$R_{\mu\nu} = 0 \quad \mu \neq \nu, \tag{21}$$

and

$$R_1^1 = R_2^2 = R_3^3. \tag{22}$$

Equation (21) requires the vanishing of the expression (16) for $R_{01}(1)$. To apply (22) we use

$$-R_1^1 = \beta^{-2}R_{11}, \quad -R_2^2 = \gamma^{-2}R_{22}, \quad -R_3^3 = \gamma^{-2}\sin^{-2}\theta R_{33}.$$

We see at once that $R_2^2 = R_3^3$. Further, we see that $-R_1^1$, which is $\beta^{-2}$ times expression (18), equals $-R_2^2$, which is $\gamma^{-2}$ times expression (19), provided that

$$\gamma = \beta f,$$

where $f$ is a function of $r$ only, so that $\dot{\gamma} = \dot{\beta}f$.

We may now introduce a new radial coordinate just equal to $f$. We must not introduce a new radial coordinate that is a function of both $r$ and $\tau$, because that would spoil the orthogonal character of the metric (15) and would lead to the appearance of $g_{01}$. But with $f$ a function of $r$ only, we may very well take it as a new $r$. Then

$$\gamma = \beta r. \tag{23}$$

The metric (15) now appears as

$$ds_E^2 = \alpha^2\,d\tau^2-\beta^2(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2).$$

It is isotropic with respect to the three spatial coordinates. It is to be compared with the isotropic form of the Schwarzschild solution, according to which

$$ds_E^2 = (1-m/2r)^2(1+m/2r)^{-2}\,d\tau^2-(1+m/2r)^4(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2).$$

This would make

$$\alpha = (1-m/2r)(1+m/2r)^{-1}, \quad \beta = (1+m/2r)^2.$$

The static $\alpha$ and $\beta$ here can be generalized to be time dependent in three ways.

(i) We may multiply $\alpha$ by any function of $\tau$.

(ii) We may multiply $\beta$ by any function of $\tau$.

(iii) We may have $m$ depending on $\tau$.

With these generalizations we still have $R_{\mu\nu}(0) = 0$.

The functions of $\tau$ that are to be multiplied into the static $\alpha$ and $\beta$ are fixed by the requirement that the metric shall go over into the E.S. metric (12) for large $r$. Thus we get

$$\alpha = (1-m/2r)(1+m/2r)^{-1}, \quad \beta = \tau^{\frac{2}{3}}(1+m/2r)^2. \tag{24}$$

Finally, the dependence of $m$ on $\tau$ is fixed by the requirement $R_{01} = 0$.

With the help of the relation $\dot\beta/\beta = \dot\gamma/\gamma$, the vanishing of the expression (16) for $R_{01}$ gives

$$\dot\gamma'/\dot\gamma - \alpha'/\alpha - \gamma'/\gamma = 0.$$

This can be integrated to give

$$\dot\gamma/\alpha\gamma = F(\tau),$$

or

$$\dot\beta = \alpha\beta F(\tau),$$

with $F(\tau)$ some function of $\tau$ alone. Substituting for $\alpha$ and $\beta$ their values given by (24), we find

$$\tfrac{2}{3}\tau^{-1}(1+m/2r)^2 + (1+m/2r)\dot m/r = (1-m^2/4r^2)F,$$

or

$$\tfrac{2}{3}\tau^{-1}(1+m/2r) + \dot m/r = (1-m/2r)F.$$

Equating the coefficients of the different powers of $r$, we get

$$\tfrac{2}{3}\tau^{-1} = F,$$

and

$$\tfrac{1}{3}m\tau^{-1} + \dot m = -\tfrac{1}{2}mF.$$

They give

$$\dot m/m = -\tfrac{2}{3}\tau^{-1}. \tag{25}$$

Thus

$$m \propto \tau^{-\frac{2}{3}}. \tag{26}$$

The metric is now completely determined.

Let us evaluate the surviving components of $R_{\mu\nu}$. We use

$$\dot\beta/\beta = \dot\gamma/\gamma = \tfrac{2}{3}\alpha\tau^{-1}. \tag{27}$$

We get from (17)

$$\begin{aligned} R_{00} &= 3\left(\frac{\dot\gamma}{\gamma}\right)^2 - 3\frac{\dot\alpha\dot\gamma}{\alpha\gamma} + 3\left(\frac{\dot\gamma}{\gamma}\right)^{\cdot}, \\ &= \tfrac{4}{3}\alpha^2/\tau^2 - (2/\tau)\dot\alpha + 2(\alpha/\tau)^{\cdot}, \\ &= \tfrac{4}{3}\alpha^2/\tau^2 - 2\alpha/\tau^2. \end{aligned}$$

Again, from (18)

$$\begin{aligned} R_{11} &= -\frac{\beta\dot\beta\dot\alpha}{\alpha^3} - \left(\frac{\beta\dot\beta}{\alpha^2}\right)^{\cdot} - \frac{\dot\beta^2}{\alpha^2}, \\ &= -\frac{1}{\alpha}\left(\frac{\beta\dot\beta}{\alpha^2}\alpha\right)^{\cdot} - \frac{\dot\beta^2}{\alpha^2}, \\ &= -\tfrac{2}{3}\alpha^{-1}(\beta^2/\tau)^{\cdot} - \tfrac{4}{9}\beta^2/\tau^2, \\ &= \tfrac{2}{3}\beta^2/\tau^2\alpha - \tfrac{4}{3}\beta^2/\tau^2. \end{aligned}$$

Thus we have

$$\begin{aligned} R_0^0 &= \tfrac{4}{3}\tau^{-2}(1-\tfrac{3}{2}\alpha^{-1}), \\ R_1^1 = R_2^2 = R_3^3 &= \tfrac{4}{3}\tau^{-2}(1-\tfrac{1}{2}\alpha^{-1}). \end{aligned}$$

So $$R^\lambda_\lambda = \tfrac{4}{3}\tau^{-2}(4-3\alpha^{-1}).$$

The stress tensor $T_{\mu\nu}$ is defined by

$$T_{\mu\nu} = \tfrac{1}{2}g_{\mu\nu}R^\lambda_\lambda - R_{\mu\nu}.$$

It has the components

$$T^0_0 = \tfrac{4}{3}\tau^{-2},$$
$$T^1_1 = T^2_2 = T^3_3 = \tfrac{4}{3}\tau^{-2}(1-\alpha^{-1}).$$

These can be expressed in terms of a density $\rho_{\mathrm{E}}$ and a pressure $p_{\mathrm{E}}$ by

$$T_{\mu\nu} = \rho_{\mathrm{E}}\, g_{\mu 0}\, g^0_\nu + p_{\mathrm{E}}\, g_{\mu\nu}.$$

We find $$p_{\mathrm{E}} = \tfrac{4}{3}\tau^{-2}(1-\alpha^{-1}), \tag{28}$$

$$\rho_{\mathrm{E}} = \tfrac{4}{3}\tau^{-2}\alpha^{-1}. \tag{29}$$

For space in the absence of matter there is a cosmological density $\rho_{\mathrm{E}} = \tfrac{4}{3}\tau^{-2}$, coming from the E.S. model, and there is no cosmological pressure. In the neighbourhood of a stationary mass the cosmological density gets modified by the factor $\alpha^{-1}$ and a cosmological pressure appears, given by (28). Note that it is negative. These effects are to be considered as the necessary modifications that have to be made in the Einstein field equations $R_{\mu\nu} = 0$ to apply to the E.S. cosmological model.

## Planetary orbits

We consider a planet moving along a geodesic in the modified Schwarzschild metric. Let $z^u$ denote its coordinates. Consider them as functions of $z^0$ and use a dot to denote $\mathrm{d}/\mathrm{d}z^0$. Thus $\dot{z}^0 = 1$. Let $s$ denote proper time, measured along the world-line, in Einstein units. Then the geodesic equations are

$$\ddot{z}^\mu + \Gamma^\mu_{\nu\rho}\,\dot{z}^\nu\dot{z}^\rho - \dot{z}^\mu\ddot{s}/\dot{s} = 0. \tag{30}$$

Introduce the velocity $v$ of the planet by the definition

$$g_{00}v^2 = -g_{ab}\dot{z}^a\dot{z}^b \quad (a, b = 1, 2, 3). \tag{31}$$

Then $$\dot{s}^2 = g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu = g_{00}(1-v^2).$$

Thus $$2\dot{s}\ddot{s} = (g_{00,\,0} + g_{00,\,a}\dot{z}^a)(1-v^2) - 2g_{00}v\dot{v}.$$

Apply equation (30) with $\mu = 0$. It gives

$$\Gamma^0_{\nu\rho}\dot{z}^\nu\dot{z}^\rho = \ddot{s}/\dot{s},$$
$$= g^{00}\,\tfrac{1}{2}(g_{00,\,0} + g_{00,\,a}\dot{z}^a) - v\dot{v}/(1-v^2).$$

But $$\Gamma^0_{\nu\rho}\dot{z}^\nu\dot{z}^\rho = g^{00}(\tfrac{1}{2}g_{00,\,0} + g_{00,\,a}\dot{z}^a - \tfrac{1}{2}g_{ab,\,0}\dot{z}^a\dot{z}^b), \tag{32}$$

so that we get

$$\tfrac{1}{2}g_{00,\,a}\dot{z}^a - \tfrac{1}{2}g_{ab,\,0}\dot{z}^a\dot{z}^b + g_{00}v\dot{v}/(1-v^2) = 0.$$

Take the case of an orbit that is circular except for the cosmological variation of the radius coordinate $z^1$. Then $\dot{z}^1$ is small and we may neglect its square. We get

$$\alpha\alpha'\dot{z}^1+\gamma\dot{\gamma}\{(\dot{z}^2)^2+\sin^2\theta(\dot{z}^3)^2\}+\alpha^2 v\dot{v}/(1-v^2)=0,$$

while (31) reduces to

$$\alpha^2 v^2=\gamma^2\{(\dot{z}^2)^2+\sin^2\theta(\dot{z}^3)^2\}.$$

Thus
$$\frac{\alpha'}{\alpha}\dot{z}^1+\frac{\dot{\gamma}}{\gamma}v^2+\frac{v\dot{v}}{1-v^2}=0.$$

Writing $r$ for $z^1$, we get with the help of (27)

$$\frac{m}{r^2}\frac{1}{1-m^2/4r^2}\dot{r}+\frac{2v^2}{3\tau}\frac{1-m/2r}{1+m/2r}+\frac{v\dot{v}}{1-v^2}=0. \tag{33}$$

Now apply equation (30) with $\mu=1$ and neglect $\dot{z}^1$. We get

$$\Gamma^1_{\nu\rho}\dot{z}^\nu\dot{z}^\rho=0,$$

or
$$\alpha\alpha'-\gamma\gamma'\{(\dot{z}^2)^2+\sin^2\theta(\dot{z}^3)^2\}=0,$$

or
$$\alpha\alpha'-(\gamma'/\gamma)\alpha^2v^2=0,$$

or
$$\frac{m}{r^2}\frac{1}{1-m^2/4r^2}-\frac{1}{r}\frac{1-m/2r}{1+m/2r}v^2=0,$$

or
$$m=rv^2(1-m/2r)^2. \tag{34}$$

Substituting this value for $m$ into (33) we get

$$v\left(\frac{\dot{r}}{r}+\tfrac{2}{3}\tau^{-1}\right)\frac{1-m/2r}{1+m/2r}+\frac{\dot{v}}{1-v^2}=0. \tag{35}$$

The solution of the two equations (34) and (35) is evidently

$$r\propto\tau^{-\frac{2}{3}},$$

so that
$$r/m=\text{const.},$$

and
$$v=\text{const.}$$

The $r$ appearing here is just the radial coordinate of the planet. The distance of the planet from the Sun is, in Einstein units,

$$r_{\mathrm{E}}=\beta r=\text{const.}$$

The planet thus moves with constant velocity at a constant distance from the Sun, just as with the static Schwarzschild metric.

Ephemeris time is defined as the time marked out by the motion of a planet. We now see that *ephemeris time is the same as Einstein time*. Thus ephemeris time differs from atomic time, as marked out by atomic clocks. The relation (6) shows that atomic clocks are slowing down with respect to ephemeris time.

In atomic units the distance of the planet from the Sun is

$$r_A = t^{-1} r_E \propto t^{-1}.$$

So the distance in atomic units is continually getting less. The planet is spiralling inwards. This is a cosmological effect, which is superposed on the motion of the planet due to all other physical causes.

### Comparison with observation

The discussion of the motion of planets led to two theoretical effects which are capable of being compared with observation. These effects are (i) the discrepancy between ephemeris time and atomic time, and (ii) the inward spiralling of the planets referred to an atomic unit of distance.

To check on (i) we need accurate observations of the motion of some heavenly body both with ephemeris time and with atomic time. The Moon is the body most suitable, as it has been the most carefully observed. For some centuries the Moon's angular motion against the background of stars has been observed accurately in ephemeris time. Since 1955 it has been observed with atomic clocks.

The lunar motion has been studied for several years by T. C. van Flandern. The problem is complicated because tidal disturbances are large compared with the accuracy needed to check the theory, and also the perturbations produced by the other planets cannot be neglected. van Flandern believed that he had evidence showing that atomic clocks are speeding up with respect to ephemeris time, the opposite effect to that required by the present theory. He reported on his results at a conference on the variation of $G$ held in Tallahassee in 1975.

I have heard recently from van Flandern that, with more recent data obtained since 1975 and with revised calculations, he now believes his earlier results may be inaccurate and it could even be that the sign of the effect will have to be changed.

The inward spiralling effect should apply also to the motion of the Moon around the Earth, but it would be difficult to observe because it is masked by much larger tidal effects. The best chance of observing the effect is from radar observations of the nearer planets, Mercury, Venus and Mars. One can send radar waves to one of these planets and observe the time taken for the reflected waves to get back. If one uses an atomic clock to measure the time, one gets the distance of the planet in atomic units.

I. I. Shapiro and R. D. Reasenberg have been working on this method and Reasenberg reported on their results at the 1975 meeting. For each of the planets Mercury, Venus, and Mars they find an effect indicating inward spiralling, but the probable errors are in each case about as large as the observed effect, so there is as yet no real confirmation of the theory.

The results given at the 1975 Conference are available in a book *On the measurement of cosmological variations of the gravitational constant*, published by the University Presses of Florida, Gainesville, Florida in 1978.

With the Viking expedition to Mars in 1976 a transponder was landed on the surface of Mars and is now being used to monitor the distance of Mars with very great accuracy. The results are not yet available, but one can hope in a short time to have a good check on the theory.

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