Proc. Roy. Soc. Lond. A. **322**, 435–445 (1971)
Printed in Great Britain

A positive-energy relativistic wave equation

BY P. A. M. DIRAC, F.R.S.

Physics Department, Florida State University, Tallahassee, Florida

(*Received* 30 *December* 1970)

The relativistic wave equations currently used in physical theory are symmetrical between positive and negative energies. A new relativistic wave equation for particles of non-zero rest-mass is here proposed, allowing only positive values for the energy. There is great formal similarity between it and the usual relativistic wave equation for the electron, but the physical significance is very different. In particular, the new equation gives integral values for the spin.

1. THE WAVE EQUATION

The internal degrees of freedom involve two harmonic oscillators. Let the dynamical variables describing these oscillators be q_1, p_1, q_2, p_2. For economy of letters, it is convenient to put $p_1 = q_3$, $p_2 = q_4$ so the dynamical variables become q_a $(a = 1, 2, 3, 4)$. Their commutation relations are (if we take $\hbar = 1$)

$$[q_a, q_b]_- = q_a q_b - q_b q_a = \mathrm{i}\beta_{ab}, \tag{1.1}$$

where β is the matrix
$$\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

Note that β is skew and $\beta^2 = -1$.

The new wave equation, for a particle with unit rest-mass, is

$$\left\{\frac{\partial}{\partial x_0} + \alpha_r \frac{\partial}{\partial x_r} + \beta\right\} q\psi = 0. \tag{1.3}$$

The suffix r takes on the values 1, 2, 3 and the α_r are 4×4 matrices that anticommute with one another and with β and have their squares equal to unity. The wavefunction ψ has only one component, but it is a function of two commuting q's, say q_1 and q_2, as well as of the four x's. Thus one can give a meaning to each q_a operating on ψ. The q in (3) means the column matrix with the four elements q_1, q_2, q_3, q_4. Thus $q\psi$ is a column matrix with the four elements $q_1\psi, q_2\psi, q_3\psi, q_4\psi$. The 4×4 matrices α_r, β are to be multiplied into this column matrix in the usual way.

Equation (1.2) strongly resembles the usual wave equation. If one drops the factor q and makes ψ have four components to correspond to the 4×4 matrices, it becomes exactly the usual wave equation. However, the resemblance is only superficial. The present wavefunction with just one component and two internal variables q is quite a different thing from the usual wavefunction.

In the present theory it is necessary to impose some further conditions on the α_r matrices that are not needed for the usual wave equation, namely, they must be symmetrical and have real elements. There is no difficulty in satisfying these conditions. For example, we may take

$$\left.\begin{aligned} \alpha_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned}\right\} \tag{1.4}$$

There are many other choices that will do equally well.

We shall use the notation $A^\sim$ to denote the transpose of a matrix A. Thus $q^\sim$ will denote the row-matrix formed by the four q's. For any 4×4 matrix κ we have the formula

$$\begin{aligned} q^\sim(\kappa-\kappa^\sim)q &= q_a(\kappa_{ab}-\kappa_{ba})q_b \\ &= \kappa_{ab}(q_aq_b-q_bq_a) = -\mathrm{i}(\kappa\beta)_{aa}. \end{aligned} \tag{1.5}$$

2. Consistency

Let us put $\partial^\mu \equiv \partial/\partial x_\mu$ and $\alpha_0 = 1$. Then the wave equation (1.3) can be written more concisely

$$(\alpha_\mu\partial^\mu+\beta)q\psi = 0. \tag{2.1}$$

It consists really of four equations, corresponding to the four values of a in

$$(\alpha_\mu\partial^\mu+\beta)_{ab}q_b\psi = 0.$$

With the choice (1.4) for the α's, they are

$$\{(\partial^0+\partial^3)q_1-\partial^1q_3+\partial^2q_4+q_3\}\psi = 0, \tag{2.2}$$

$$\{(\partial^0+\partial^3)q_2+\partial^2q_3+\partial^1q_4+q_4\}\psi = 0, \tag{2.3}$$

$$\{(\partial^0-\partial^3)q_3-\partial^1q_1+\partial^2q_2-q_1\}\psi = 0, \tag{2.4}$$

$$\{(\partial^0-\partial^3)q_4+\partial^2q_1+\partial^1q_2-q_2\}\psi = 0. \tag{2.5}$$

If we take q_2 times (2.2), $-q_1$ times (2.3), $-q_4$ times (2.4), q_3 times (2.5) and add, we get $0 = 0$. Thus the four equations are not independent. However, three of them are independent. Each of them is a differential equation in the four x's and two commuting q's.

In general a function ψ cannot be chosen to satisfy three differential equations. It is necessary for the three equations to satisfy some consistency conditions. Let us write the four equations (2.2) to (2.5) as

$$P_a \psi = 0, \tag{2.6}$$

where $$P_a = (\alpha_\mu \partial^\mu + \beta)_{ab} q_b.$$

The consistency of equations (2.6) requires that

$$[P_a, P_b]_- \psi = 0. \tag{2.7}$$

We have
$$\begin{aligned} [P_a, P_c]_- &= (\alpha_\mu \partial^\mu + \beta)_{ab} (\alpha_\nu \partial^\nu + \beta)_{cd} [q_b, q_d]_- \\ &= \mathrm{i}(\alpha_\mu \partial^\mu + \beta)_{ab} (\alpha_\nu \partial^\nu + \beta)_{cd} \beta_{bd} \\ &= \mathrm{i}\{(\alpha_\mu \partial^\mu + \beta) \beta (\alpha_\nu \partial^\nu - \beta)\}_{ac} \\ &= \mathrm{i}\{(\partial_0 + \alpha_r \partial^r + \beta)(\partial_0 - \alpha_s \partial^s - \beta) \beta\}_{ac} \\ &= \mathrm{i}(\partial_\mu \partial^\mu + 1) \beta_{ac}. \end{aligned}$$

Thus equations (2.7) lead to $$(\partial_\mu \partial^\mu + 1) \psi = 0. \tag{2.8}$$

The wavefunction must satisfy the de Broglie equation for all values of its internal variables.

We see now that equations (2.2) to (2.5), joined with the equation (2.8), form a consistent set.

3. Relativistic invariance

We may write the wave equation (2.1) in a form that looks relativistic by multiplying it by β on the left and introducing the notation $\gamma_\mu = \beta\alpha_\mu$. We then have

$$(\gamma_\mu \partial^\mu - 1) q \psi = 0, \tag{3.1}$$

with $$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2g_{\mu\nu}.$$

However, (3.1) is not a manifestly relativistic equation, since the four γ's cannot be considered as a 4-vector, as γ_0 is skew while γ_1, γ_2 and γ_3 are symmetrical.

To establish the relativistic invariance of the theory we must proceed by a method parallel to that for the usual electron wave equation. Let us work with the original form (2.1), involving α's that satisfy

$$\alpha_\mu \beta \alpha_\nu + \alpha_\nu \beta \alpha_\mu = 2\beta g_{\mu\nu}. \tag{3.2}$$

Apply an infinitesimal Lorentz transformation

$$\left.\begin{aligned} x^*_\mu &= x_\mu + a_\mu{}^\nu x_\nu, \\ \partial^{\mu *} &= \partial^\mu + a^\mu{}_\nu \partial^\nu, \end{aligned}\right\} \tag{3.3}$$

the $a_{\mu\nu}$ being infinitesimal coefficients such that $a_{\mu\nu} = -a_{\nu\mu}$. The equation (2.1) then gives, to the first order,
$$\{\alpha_\mu(\partial^{\mu *} - a^\mu{}_\nu \partial^{\nu *}) + \beta\} q \psi = 0,$$

or $$\{(\alpha_\mu + a_\mu{}^\nu \alpha_\nu) \partial^{\mu *} + \beta\} q \psi = 0. \tag{3.4}$$

Define $$N = \tfrac{1}{4}a^{\rho\sigma}\alpha_\rho\beta\alpha_\sigma.$$

Note that its transpose is $$N^\sim = -\tfrac{1}{4}a^{\rho\sigma}\alpha_\sigma\beta\alpha_\rho = N,$$

so it is a symmetrical matrix.

We have
$$\begin{aligned}\alpha_\mu\beta N - N\beta\alpha_\mu &= \tfrac{1}{4}a^{\rho\sigma}(\alpha_\mu\beta\alpha_\rho\beta\alpha_\sigma - \alpha_\rho\beta\alpha_\sigma\beta\alpha_\mu)\\ &= \tfrac{1}{4}a^{\rho\sigma}\{(2\beta g_{\mu\rho} - \alpha_\rho\beta\alpha_\mu)\beta\alpha_\sigma - \alpha_\rho\beta(2\beta g_{\sigma\mu} - \alpha_\mu\beta\alpha_\sigma)\}\\ &= -a_\mu{}^\sigma\alpha_\sigma. \end{aligned} \tag{3.5}$$

Thus, multiplying (3.4) by $(1-N\beta)$ on the left, we get

$$\{\alpha_\mu(1-\beta N)\partial^{\mu *} + \beta + N\}q\psi = 0,$$

or $$(\alpha_\mu\partial^{\mu *} + \beta)q^*\psi = 0,$$

where $$q^* = (1-\beta N)q. \tag{3.6}$$

Thus the wave equation takes the same form in the new system of coordinates, with the column matrix q^* replacing q. The four q^*'s are linear functions of the four q's with real coefficients (on account of the α's and β having only real elements). They satisfy the commutation relations

$$\begin{aligned}[q_a^*, q_b^*]_- &= [(1-\beta N)_{ac}q_c, (1-\beta N)_{bd}q_d]_-\\ &= \mathrm{i}(1-\beta N)_{ac}(1-\beta N)_{bd}\beta_{cd}\\ &= \mathrm{i}[(1-\beta N)\beta\{1-(\beta N)^\sim\}]_{ab}\\ &= \mathrm{i}\{(1-\beta N)\beta(1+N\beta)\}_{ab} = \mathrm{i}\beta_{ab}\end{aligned}$$

to the first order. The q^*'s thus have the same properties as the q's.

This shows that the form of the wave equation is unchanged by an infinitesimal Lorentz transformation. It is thus unchanged also for finite Lorentz transformations that do not involve reflexions.

There is a unitary transformation connecting the q^*'s with the q's. Let

$$W = q^\sim N q = q_a N_{ab} q_b.$$

Then
$$\begin{aligned}Wq_a - q_a W &= N_{bc}(q_b q_c q_a - q_a q_b q_c)\\ &= \mathrm{i}N_{bc}(q_b\beta_{ca} + \beta_{ba}q_c) = -2\mathrm{i}\beta_{ab}N_{bc}q_c.\end{aligned}$$

We may write this result as $$Wq - qW = -2\mathrm{i}\beta N q, \tag{3.7}$$

an equation in which every term is a column matrix. Equation (3.6) now becomes

$$q^* = (1-\tfrac{1}{2}\mathrm{i}W)q(1+\tfrac{1}{2}\mathrm{i}W)$$

to the first order.

4. An eigenstate of momentum and energy

Let us take a solution of the wave equation that corresponds to an eigenstate of momentum and energy. We put $\mathrm{i}\partial^\mu = p^\mu$ and consider the p^μ to be real numbers. The de Broglie equation (2.8) shows that these numbers must be connected by

$$p^\mu p_\mu = 1. \tag{4.1}$$

The wave equations are now

$$(p_0 - \alpha_r p_r + \mathrm{i}\beta)\, q\psi = 0.$$

With the choice of α's given by (1.4), they are

$$\left.\begin{aligned}
\{(p_0-p_3)q_1+(\mathrm{i}+p_1)q_3-p_2q_4\}\psi &= 0,\\
\{(p_0-p_3)q_2-p_2q_3+(\mathrm{i}-p_1)q_4\}\psi &= 0,\\
\{(p_0+p_3)q_3+(p_1-\mathrm{i})q_1-p_2q_2\}\psi &= 0,\\
\{(p_0+p_3)q_4-p_2q_1-(p_1+\mathrm{i})q_2\}\psi &= 0.
\end{aligned}\right\} \tag{4.2}$$

We may apply a Lorentz transformation so as to make $p_1 = p_2 = p_3 = 0$. Then from (4.1), $p_0 = 1$ or -1. Equations (4.2) now reduce to

$$(p_0q_1+\mathrm{i}q_3)\psi = 0,$$

$$(p_0q_2+\mathrm{i}q_4)\psi = 0,$$

$$(p_0q_3-\mathrm{i}q_1)\psi = 0,$$

$$(p_0q_4-\mathrm{i}q_2) = 0.$$

The last two equations are the same as the first two, in both cases $p_0 = 1$ and $p_0 = -1$.

Take a representation for the q's with q_1 and q_2 diagonal. We have

$$q_3 = -\mathrm{i}\partial/\partial q_1, \quad q_4 = -\mathrm{i}\partial/\partial q_2$$

and we get
$$\left(p_0q_1+\frac{\partial}{\partial q_1}\right)\psi = 0, \quad \left(p_0q_2+\frac{\partial}{\partial q_2}\right)\psi = 0.$$

With $p_0 = 1$, the solution of these equations is

$$\psi = k\exp\{-\tfrac{1}{2}(q_1^2+q_2^2)\}, \tag{4.3}$$

with any numerical coefficient k. With $p = -1$, we should get ψ proportional to $\exp\{\frac{1}{2}(q_1^2+q_2^2)\}$, which is not normalizable and not physically permissible.

We see therefore that p_0 has to be 1. The result would be the same if we used any other representation for the q's. With general values for p_1, p_2, p_3, the energy has to be positive.

The equations (4.2) may be solved for general values of p_1, p_2, p_3, the result being

$$\psi = k\exp\{-\tfrac{1}{2}[q_1^2+q_2^2+\mathrm{i}p_1(q_1^2-q_2^2)-2\mathrm{i}p_2q_1q_2]/(p_0+p_3)\}\exp\{-\mathrm{i}p^\mu x_\mu\}. \tag{4.4}$$

We see here again the need for the energy to be positive, in order that p_0+p_3 may be positive, to get a physically permissible ψ.

5. The spin

If we apply the infinitesimal rotation about the origin (3.3) to ψ, we change it to

$$(1+\tfrac{1}{2}a^{\rho\sigma}M_{\rho\sigma})\psi, \tag{5.1}$$

where
$$M_{\rho\sigma} = x_\rho\partial_\sigma - x_\sigma\partial_\rho - \mathrm{i}s_{\rho\sigma}, \tag{5.2}$$

the $s_{\rho\sigma}$ being spin operators that operate on the q's. We shall evaluate them, using the condition that (5.1) must satisfy the wave equation if ψ does.

We have
$$[(\alpha_\mu\partial^\mu+\beta)q,\, a^{\rho\sigma}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)]_- = 2a^{\mu\sigma}\alpha_\mu\partial_\sigma q.$$

(Every term in this and the following equations is a column matrix.) Again, from (3.7)

$$[(\alpha_\mu\partial^\mu+\beta)q,\, W]_- = 2\mathrm{i}(\alpha_\mu\partial^\mu+\beta)\beta Nq.$$

With the help of (3.5), this expression becomes

$$2\mathrm{i}\{(N\beta\alpha_\mu - a_\mu{}^\sigma\alpha_\sigma)\partial^\mu q - Nq\} = 2\mathrm{i}N\beta(\alpha_\mu\partial^\mu+\beta)q + 2\mathrm{i}a^{\mu\sigma}\alpha_\mu\partial_\sigma q.$$

Thus
$$[(\alpha_\mu\partial^\mu+\beta)q,\, a^{\rho\sigma}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)+\mathrm{i}W]_- = -2N\beta(\alpha_\mu\partial^\mu+\beta)q,$$

which vanishes when applied to a ψ satisfying the wave equation.

We can infer that
$$\begin{aligned} a^{\rho\sigma}s_{\rho\sigma} &= -W = -q^\sim Nq \\ &= -\tfrac{1}{4}a^{\rho\sigma}q^\sim\alpha_\rho\beta\alpha_\sigma q. \end{aligned}$$

We need $s_{\rho\sigma}$ to be antisymmetrical, so we get

$$s_{\rho\sigma} = -\tfrac{1}{8}q^\sim(\alpha_\rho\beta\alpha_\sigma - \alpha_\sigma\beta\alpha_\rho)q.$$

With the help of (3.2) and (1.5), this reduces to

$$\begin{aligned} s_{\rho\sigma} &= -\tfrac{1}{4}q^\sim\alpha_\rho\beta\alpha_\sigma q + \tfrac{1}{4}g_{\rho\sigma}q^\sim\beta q \\ &= -\tfrac{1}{4}q^\sim\alpha_\rho\beta\alpha_\sigma q + \tfrac{1}{2}\mathrm{i}g_{\rho\sigma}. \end{aligned} \tag{5.3}$$

For $\rho, \sigma = 1, 2, 3$, we can interpret $\mathrm{i}M_{\rho\sigma}$ as the total angular momentum. Then $s_{\rho\sigma}$ is the spin angular momentum.

If we evaluate the $s_{\rho\sigma}$ with the choice (1.4) for the α's, we get

$$\left.\begin{aligned} s_{23} &= \tfrac{1}{2}(q_1q_2+q_3q_4), \\ s_{31} &= \tfrac{1}{4}(q_1^2-q_2^2+q_3^2-q_4^2), \\ s_{12} &= \tfrac{1}{2}(q_2q_3-q_1q_4), \\ s_{01} &= \tfrac{1}{4}(q_1^2-q_2^2-q_3^2+q_4^2), \\ s_{02} &= \tfrac{1}{2}(q_3q_4-q_1q_2), \\ s_{03} &= \tfrac{1}{2}(q_1q_3+q_4q_2). \end{aligned}\right\} \tag{5.4}$$

The first three equations give

$$s_{23}^2+s_{31}^2+s_{12}^2 = \tfrac{1}{16}(q_1^2+q_2^2+q_3^2+q_4^2)^2-\tfrac{1}{4}.$$

The magnitude s of the spin is defined, according to quantum mechanics, by

$$s(s+1) = s_{23}^2+s_{31}^2+s_{12}^2.$$

We thus find
$$s = \tfrac{1}{4}(q_1^2+q_2^2+q_3^2+q_4^2)-\tfrac{1}{2}. \tag{5.5}$$

The eigenvalues of $\frac{1}{2}(q_1^2+q_3^2)$ are $n+\frac{1}{2}$, with n a non-negative integer. Similarly, the eigenvalues of $\frac{1}{2}(q_2^2+q_4^2)$ are $n'+\frac{1}{2}$, with n' another non-negative integer. Thus the eigenvalues of s are $\frac{1}{2}(n+n')$.

The value of the spin s depends on the wavefunction, which, according to (4.4), depends on the momentum of the particle. We see that, whatever the momentum, the wavefunction is always an even function of the q's. This corresponds to $n+n'$ being even. Thus the spin is always integral. For zero momentum the wavefunction is (4.3), giving $n = n' = 0$, and then the spin is zero.

The six quantities $s_{\rho\sigma}$ provide a representation of the infinitesimal operators of the Lorentz group. They are associated mathematically with four more quantities

$$s_{\mu 5} = -s_{5\mu} = \tfrac{1}{4}q^{\sim}\alpha_\mu q. \tag{5.6}$$

The ten quantities $s_{ab} = -s_{ba}$ $(a, b = 0, 1, 2, 3, 5)$ then provide a representation of the $3+2$ de Sitter group, as was discussed by the author (1963).

6. Equations quadratic in the q's

We can obtain a number of equations that are quadratic in the q's by multiplying (2.2), (2.3), (2.4) and (2.5) by linear functions of the q's and adding. The general equation of this kind is

$$\{q^{\sim}\lambda\alpha_\mu q\,\partial^\mu+q^{\sim}\lambda\beta q\}\psi = 0, \tag{6.1}$$

with λ any 4×4 matrix. There are 16 independent λ's, and one of them, namely

$$\lambda = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \alpha_1\alpha_2\alpha_3,$$

gives identically zero, as was discussed in § 2.

The others give, with the notation of (5.3) and (5.6) and with the help of formula (1.5) in the case of (6.2),

$$1 \qquad (s_{\nu 5}\partial^\nu+\tfrac{1}{2}\mathrm{i})\psi = 0, \tag{6.2}$$

$$\alpha_r \qquad (s_{r5}\partial^0+s_{05}\partial^r-s_{r0})\psi = 0, \tag{6.3}$$

$$\alpha_\mu\beta \qquad (s_{\mu\nu}\partial^\nu-\tfrac{1}{2}\mathrm{i}\partial_\mu+s_{\mu 5})\psi = 0, \tag{6.4}$$

$\alpha_1\alpha_2\alpha_3\beta$ $$(s_{23}\partial^1+s_{31}\partial^2+s_{12}\partial^3)\psi = 0, \tag{6.5}$$

$\alpha_1\alpha_2$ $$(s_{15}\partial^2-s_{25}\partial^1+s_{12})\psi = 0, \tag{6.6}$$

$\alpha_1\alpha_2\beta$ $$(s_{12}\partial^0+s_{02}\partial^1-s_{01}\partial^2)\psi = 0. \tag{6.7}$$

The corresponding λ values are written on the left. Equations (6.6) and (6.7) must each be supplemented by the two equations obtained from it by cyclic permutation of the suffixes 1, 2, 3.

If we put $\mu = 0$ in (6.4) we get

$$(\partial_0+2is_{0r}\partial^r+2is_{05})\psi = 0. \tag{6.8}$$

This equation fixes the time derivative of ψ in terms of ψ and its spatial derivatives. It shows that the initial value of ψ fixes the solution at all times.

Equations (6.5) and (6.6) do not involve the time derivative of ψ. They provide conditions that the initial wavefunction has to satisfy.

Equations (6.5) and (6.7) may be combined to give

$$\epsilon^{\mu\nu\rho\sigma}s_{\mu\nu}\partial_\rho\psi = 0, \tag{6.9}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is antisymmetric in the four indices.

7. The density-current vector

One may introduce a density-current vector corresponding to that for the usual electron theory. The conjugate complex of $q_a\psi$ is $\overline{\psi}q_a$, where the operator $\partial/\partial q_1$ or $\partial/\partial q_2$ operating to the left means minus the derivative of $\overline{\psi}$, as in ordinary quantum mechanics. Thus if one takes the conjugate complex of (2.1) and transposes the matrices, one gets

$$(\partial^\mu\overline{\psi}q^\sim\alpha_\mu-\overline{\psi}q^\sim\beta) = 0. \tag{7.1}$$

Multiplying (2.1) by $\overline{\psi}q^\sim$ on the left and integrating over the two q-variables in the wavefunction, we get

$$\int\{\overline{\psi}q^\sim\alpha_\mu\partial^\mu q\psi+\overline{\psi}q^\sim\beta q\psi\}\,\mathrm{d}^2q = 0. \tag{7.2}$$

Multiplying (7.1) by $q\psi$ on the right and integrating over the q's, we get

$$\int\{(\partial^\mu\overline{\psi}q^\sim\alpha_\mu)q\psi-\overline{\psi}q^\sim\beta q\psi\}\,\mathrm{d}^2q = 0. \tag{7.3}$$

Adding (7.2) and (7.3), we get

$$\partial^\mu\int\overline{\psi}q^\sim\alpha_\mu q\psi\,\mathrm{d}^2q = 0.$$

Thus we take the density–current vector to be

$$j_\mu = \int\overline{\psi}q^\sim\alpha_\mu q\psi\,\mathrm{d}^2q \tag{7.4}$$

and we have the conservation law $\partial^\mu j_\mu = 0$.

We should check that the four quantities (7.4) transform like the components of a 4-vector. Let us leave the integration over the q's understood. Then with the Lorentz transformation of § 3, they transform to

$$\overline{\psi}q^{*\sim}\alpha_\mu q^*\psi = \overline{\psi}q^\sim(1+N\beta)\alpha_\mu(1-\beta N)q\psi.$$

With the help of (3.5) this becomes

$$\bar{\psi}q^{\sim}(\alpha_\mu + a_\mu{}^\sigma \alpha_\sigma) q\psi = j_\mu + a_\mu{}^\sigma j_\sigma,$$

which is the correct transformation law.

We can get a more general conserved density-current. Let κ be any operator on the q's, represented by a matrix $\langle q_1' q_2' | \kappa | q_1'' q_2'' \rangle$. If we multiply (2.1) by $\bar{\psi}q^{\sim}\kappa$ on the left and integrate over the q's, then multiply (7.1) by $\kappa q \psi$ on the right and integrate over the q's, then add the resulting equations, we get

$$\partial^\mu(\bar{\psi}q^{\sim}\kappa\alpha_\mu q\psi) = 0,$$

with the integration over the q's understood. So we could take

$$j_\mu = \bar{\psi}q^{\sim}\kappa\alpha_\mu q\psi. \tag{7.5}$$

This generalized density-current satisfies the conservation law, but it does not transform like a 4-vector.

According to (7.4) the density is

$$j_0 = \bar{\psi}q^{\sim}q\psi,$$

which is positive definite. It can be used to normalize a wave function, according to the condition

$$\int \bar{\psi}q^{\sim}q\psi \, \mathrm{d}^3x = 1.$$

The eigenfunction (4.4) cannot be normalized to unity like this. We may normalize it to refer to p_0 particles per unit volume, a Lorentz-invariant condition. We find then that

$$k = \{2\pi(p_0 + p_3)\}^{-\frac{1}{2}}. \tag{7.6}$$

8. The Fock representation

The foregoing discussion is based on a representation for the harmonic oscillators in which two commuting q's are diagonal. The theory would be similar if we used a representation in which any two commuting real linear functions of the q's are diagonal. We get an interesting variation if we use instead the Fock representation for the oscillators. This is a representation involving the two complex variables

$$\eta_1 = 2^{-\frac{1}{2}}(q_1 - \mathrm{i}q_3), \quad \eta_2 = 2^{-\frac{1}{2}}(q_2 - \mathrm{i}q_4). \tag{8.1}$$

Their conjugates are

$$\bar{\eta}_1 = 2^{-\frac{1}{2}}(q_1 + \mathrm{i}q_3), \quad \bar{\eta}_2 = 2^{-\frac{1}{2}}(q_2 + \mathrm{i}q_4).$$

They satisfy

$$[\bar{\eta}_1, \eta_1]_- = 1, \quad [\bar{\eta}_2, \eta_2]_- = 1,$$

which means that we can put

$$\bar{\eta}_1 = \partial/\partial\eta_1, \quad \bar{\eta}_2 = \partial/\partial\eta_2.$$

We now have ψ a function of the two η's as well as the four x's. It is a power series in the η's, and terms involving $\eta_1^n \eta_2^{n'}$ refer to the two oscillators being in their nth and n'th excited states.

The wave equations (2.2) to (2.5) may be written in terms of the η's and $\bar{\eta}$'s instead of the q's. Taking (2.2) plus i times (2.4), we get

$$\{(\partial^0 - \mathrm{i})\,\bar{\eta}_1 + (\partial^3 - \mathrm{i}\partial^1)\,\eta_1 + \mathrm{i}\partial^2\eta_2\}\,\psi = 0. \tag{8.2}$$

Taking (2.3) $+\mathrm{i}$ times (2.5) we get

$$\{(\partial^0 - \mathrm{i})\,\bar{\eta}_2 + (\partial^3 + \mathrm{i}\partial^1)\,\eta_2 + \mathrm{i}\partial^2\eta_1\}\,\psi = 0. \tag{8.3}$$

The other two equations involve the conjugate complex operators and are

$$\{(\partial^0 + \mathrm{i})\,\eta_1 + (\partial^3 + \mathrm{i}\partial^1)\,\bar{\eta}_1 - \mathrm{i}\partial^2\bar{\eta}_2\}\,\psi = 0, \tag{8.4}$$

$$\{(\partial^0 + \mathrm{i})\,\eta_2 + (\partial^3 - \mathrm{i}\partial^1)\,\bar{\eta}_2 - \mathrm{i}\partial^2\bar{\eta}_1\}\,\psi = 0. \tag{8.5}$$

We still have the de Broglie equation (2.8), holding for each of the coefficients of the power series.

The solution of (8.2) to (8.5) giving an eigenstate of momentum is

$$\psi = k\exp\{\tfrac{1}{2}[(p_3 - \mathrm{i}p_1)\,\eta_1^2 + 2\mathrm{i}p_2\eta_1\eta_2 + (p_3 + \mathrm{i}p_1)\,\eta_2^2]/(p_0 + 1)\}\exp\{-\mathrm{i}p^\mu x_\mu\}.$$

If we normalize it to represent p_0 particles per unit volume, using the rules for the interpretation of the Fock representation, we get

$$k = (p_0 + 1)^{-\frac{1}{2}}.$$

With any representation, involving say the variables u_1 and u_2, there is a subgroup of the Lorentz group for which the transformation of $\psi(u_1, u_2)$ is specially simple, the variables u_1 and u_2 having to be replaced simply by certain linear combinations of them with ψ getting multiplied by a function of u_1 and u_2. The infinitesimal operators of the subgroup do not contain any terms quadratic in $\partial/\partial u_1$ and $\partial/\partial u_2$.

With $u_1, u_2 = q_1, q_2$, the infinitesimal operators of the subgroup do not contain any terms quadratic in q_3 and q_4. Referring to (5.4), we see that these operators are s_{12}, s_{03}, $s_{01} + s_{31}$, $s_{02} + s_{32}$. The subgroup is thus composed of those Lorentz transformations that leave the null-direction $x_0 + x_3 = 0$, $x_1 = x_2 = 0$, invariant.

With u_1, u_2 = any two commuting real linear functions of the q's, the subgroup leaves some null-direction invariant.

With the Fock representation we have u_1, $u_2 = \eta_1$, η_2. The equations (5.4) give

$$\begin{aligned}
s_{23} &= \tfrac{1}{2}(\eta_1\bar{\eta}_2 + \eta_2\bar{\eta}_1),\\
s_{31} &= \tfrac{1}{2}(\eta\bar{\eta}_1 - \eta_2\bar{\eta}_2),\\
s_{12} &= \tfrac{1}{2}\mathrm{i}(\eta_1\bar{\eta}_2 - \eta_2\bar{\eta}_1),\\
s_{01} &= \tfrac{1}{4}(\eta_1^2 + \bar{\eta}_1^2 + \eta_2^2 + \bar{\eta}_2^2),\\
s_{02} &= \tfrac{1}{2}(\eta_1\eta_2 + \bar{\eta}_1\bar{\eta}_2),\\
s_{03} &= \tfrac{1}{4}\mathrm{i}(\eta_1^2 - \bar{\eta}_1^2 + \eta_2^2 - \bar{\eta}_2^2).
\end{aligned}$$

We see now that s_{23}, s_{31}, s_{12} are not quadratic in $\bar{\eta}_1$ and $\bar{\eta}_2$ or in $\partial/\partial\eta_1$ and $\partial/\partial\eta_2$. Thus the specially simple subgroup is that which leaves the time axis invariant.

For a physicist, those Lorentz transformations that leave the time axis invariant are much more frequently used than the others. This results in the Fock representation, which makes these transformations specially simple, being an advantageous one to use.

9. Conclusion

The foregoing work provides a relativistic theory of a particle. If it is to be applied in physics one must find a way of making the particle interact with other particles or with a field. This is not easy, because any modification one makes in the equations is very liable to spoil their consistency.

One may try the usual way of introducing an electromagnetic field, replacing the p_μ in the wave equation by $p_\mu + eA_\mu$. One finds that the equations are no longer consistent except in the special case $A^\mu = \partial S/\partial x_\mu$, which means no field.

The author's stay in Tallahassee was supported by the National Science Foundation (Science Development Grant no. GU-2612).

Reference

Dirac, P. A. M. 1963 *J. Math. Phys.* 4, 901.