# THE HAMILTONIAN FORM OF FIELD DYNAMICS

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**1. Introduction.** In classical dynamics one has usually supposed that when one has solved the equations of motion one has done everything worth doing. However, with the further insight into general dynamical theory which has been provided by the discovery of quantum mechanics, one is lead to believe that this is not the case. It seems that there is some further work to be done, namely to group the solutions into families (each family corresponding to one principal function satisfying the Hamilton-Jacobi equation). The family does not have any importance from the point of view of Newtonian mechanics; but it is a family which corresponds to one state of motion in the quantum theory, so presumably the family has some deep significance in nature, not yet properly understood.

The importance of the family is brought out by the Schrödinger form of quantum mechanics and not by the Heisenberg form. The latter is in direct analogy with the classical Hamiltonian equations of motion and does not require any grouping of the solutions. The Schrödinger form goes beyond this in ascribing importance to the concept of a quantum state, subject to the principle of superposition and described by a solution of Schrödinger's wave equation, and this concept requires the introduction of families of solutions for its analogue in classical mechanics, the Schrödinger equation itself being the analogue of the Hamilton-Jacobi equation.

One can build up a relativistic dynamical theory by starting with a Lorentz invariant action function involving field quantities. The requirement that the total action shall be stationary under arbitrary small variations of those field variables that play the role of dynamical coordinates at all points of space-time leads to a relativistic set of field equations as the equations of motion. These equations may be put in the Hamiltonian form, and one can then pass from them to Heisenberg's form of quantum mechanics. This has already been done by Weiss [1]. The present paper is concerned with the further mathematical development, connected with the grouping of the solutions into families, which is needed before one can pass to Schrödinger's form of quantum mechanics.

The dynamical variables of the Hamiltonian equations of motion must be a set of variables that can serve as initial conditions—they must be independent of one another and sufficient to fix the state of motion. In non-relativistic theory one takes them to be physical quantities referring to an instant of time.

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The concept of an instant of time is rather artificial from the relativistic point of view. It is to be pictured as a flat three-dimensional "surface" in four-dimensional space-time, with the direction of its normal lying within the light-cone. It would be more natural in a relativistic theory to replace the flat surface by an arbitrary curved one, subject to the restriction that it is everywhere space-like, i.e. the normal at every point of it lies within the light-cone. One would then work with dynamical variables referring to physical conditions on such a curved surface, as was done by Weiss.

The use of a curved surface instead of a flat one of course increases the complexity of the mathematical equations. In working out practical examples one would always revert to the flat surface to simplify the calculations as much as possible, the flat surface being adequate for describing all experimental results. The curved surface is desirable in a general theory because of the flexibility and mathematical power that it gives. It shows up the transformation properties of the Hamiltonian theory applied to field dynamics. In any problem which involves seeking for a new dynamical system, rather than working out the properties of a given dynamical system, it would be advantageous to use the curved surface, because it brings more conditions into the mathematics and so restricts the region of search.

The curved surface will be described by certain mathematical variables, which we shall call the *surface variables*. (Actually, they will consist of functions, as will be discussed in the next section.) The equations of motion will give the change in the dynamical variables when the surface is moved in space-time. The surface can be subjected to arbitrary changes of direction and deformations during the motion, provided it remains always space-like. Thus the equations of motion give the change in the dynamical variables for any change in the surface variables. One can get these equations in the Hamiltonian form by working from the Lagrangian, as was shown by Weiss, and one then has Poisson bracket relations between the dynamical variables.

Let us now consider what development is needed to make possible the grouping of the solutions of the equations of motion into families. One can infer from analogy with non-relativistic dynamics that the Hamilton-Jacobi equation, whose solutions define the families, is a partial differential equation of the first order in the dynamical coordinates, and also in the surface variables. It thus involves the surface variables in the same way as the dynamical coordinates. This provides the key to the problem. *We must put the dynamical theory into a form in which the surface variables are treated on the same footing as dynamical coordinates.* They must have conjugate momenta and there must be P. b. relations connecting them with other dynamical variables. When we have done this, the setting up of the Hamilton-Jacobi equation is straightforward, and the ground is prepared for Schrödinger's form of quantization.

The equations of motion must now be considered in the first place as making all the dynamical variables, including the surface variables, vary together, and giving them all as functions of some independent variable, $\tau$ say, when suitable initial conditions are prescribed. Actually, the surface variables can vary arbitrarily with $\tau$. This is to be taken into account by supposing the general solution of the equations of motion to involve some arbitrary functions (or functionals). The arbitrariness in the motion of the surface variables is then to be ascribed to the "accidental" appearance of these arbitrary functions in the solution of equations which, in a general way, one would normally expect to fix the motion completely.

The generalization of Hamiltonian dynamics which is needed in order that arbitrary functions may appear in the solution of the equations of motion has been given in a previous paper by the author [2]. The present paper is a direct application of the method given there.

**2. The general space-like surface.** We describe space-time by the four coordinates $x_\mu (\mu = 0, 1, 2, 3)$ of a rectilinear orthogonal system of coordinates. For simplicity we shall write all vectors referred to this coordinate system in the contravariant form, such as $a_\mu$ and will make the convention

$$a_\mu b_\mu = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3,$$

applying whenever one of these contravariant suffixes is repeated in a term.

We can describe a general three-dimensional surface in space-time by giving the four coordinates $x_\mu$ of any point on it as functions of three parameters $u_r (r = 1, 2, 3)$, say

$$x_\mu = y_\mu(u). \tag{1}$$

This involves setting up a parametrization $u$ on the surface. The parametrization is not necessary physically and brings some extra complication into the mathematics. It could be avoided by using a different method of description of the surface, specifying $x_0$ as a function of $x_1, x_2, x_3$. However, this would spoil the relativistic treatment of the four $x$'s. For the sake of relativistic form it seems preferable to use the method (1), the extra complication arising from the parametrization being not very troublesome if properly handled.

The condition that the surface shall be space-like is that there shall exist at every point on it a unit normal vector $l_\mu$, satisfying

$$l_\mu y_\mu{}^r = 1, \qquad l_\mu l_\mu = 1, \qquad l_0 > 0, \tag{2}$$

where $y_\mu{}^r$ is short for the derivative $\partial y_\mu / \partial u_r$. The $l_\mu$ are to be understood as functions of the parameters $u$.

The parameters $u$ form a system of coordinates in the three-dimensional surface. The metric of the surface referred to these coordinates is

$$ds^2 = dy_\mu dy_\mu = y_\mu{}^r du_r y_\mu{}^s du_s = \gamma^{rs} du_r du_s$$

with

$$\gamma^{rs} = \gamma^{sr} = y_\mu{}^r y_\mu{}^s, \tag{3}$$

For a space-like surface this $ds^2$ is negative and thus the determinant of the $\gamma^{rs}$ is negative. Put it equal to $-\Gamma^2$, $\Gamma$ being positive. Then an element of three-dimensional "area" of the surface is

$$\Gamma du_1 du_2 du_3 = \Gamma d^3u$$

say.

Let $\gamma_{rs}$ be the cofactor of $\gamma^{rs}$ in the determinant, divided by the determinant, so that

$$\gamma_{rs}\gamma^{sp} = \delta_s{}^p.$$

The $\gamma^{rs}, \gamma_{rs}$ can be used for raising and lowering the suffixes of vector and tensor quantities in the three-dimensional space of the surface. We shall use generally the notation of adding an upper suffix $^r$ to a quantity to denote its derivative with respect to $u_r$ (ordinary, not covariant derivative). Thus with $\zeta$ any function of the $u$'s, $\zeta^r = \partial\zeta/\partial u_r$, and then $\zeta_r = \gamma_{rs}\zeta^s$. We have $\zeta^{rs} = \zeta^{sr}$, but in general $\zeta_{rs} \neq \zeta_{sr}$.

We have

$$\partial\Gamma^2/\partial u_p = \Gamma^2\gamma_{rs}\partial\gamma^{rs}/\partial u_p$$

and hence

$$\Gamma^p = \tfrac{1}{2}\Gamma\gamma_{rs}\gamma^{rsp}. \tag{4}$$

Let $a_\mu$ be any 4-vector located at a point on the surface. It has a normal component

$$a_\mu l_\mu = a_l$$

say, and a contravariant component in the direction $u_r$ in the surface

$$a_\mu y_\mu{}^r = a^r$$

say. The covariant components in the surface directions may be introduced by $a_r = \gamma_{rs}a^s$. It is now easily seen that

$$(a_\mu - a_l l_\mu - a_r y_\mu{}^r)\ l_\mu = 0$$

$$(a_\mu - a_l l_\mu - a_r y_\mu{}^r)\ y_\mu{}^s = 0,$$

so that the 4-vector $a_\mu - a_l l_\mu - a_r y_\mu{}^r$ has its normal component and its components in the directions $u_r$ all vanishing. This means the vector itself vanishes, i.e.

$$a_\mu = a_l l_\mu + a_r y_\mu{}^r. \tag{5}$$

This equation expresses the resolution of the vector $a_\mu$ into its normal and tangential parts. It is convenient to write the tangential part $a_r y_\mu{}^r$ as $a_{-\mu}$ for brevity.

The scalar product of two 4-vectors $a_\mu$ and $b_\mu$ is

$$\begin{aligned} a_\mu b_\mu &= a_l b_l + a_{-\mu} b_{-\mu} = a_l b_l + a_r y_\mu{}^r b_s y_\mu{}^s \\ &= a_l b_l + \gamma^{rs} a_r b_s \\ &= a_l b_l + a_r b^r. \end{aligned} \tag{6}$$

This result may be written as

$$g_{\mu\nu} a_\mu b_\nu = a_\mu l_\mu b_\nu l_\nu + a_\mu y_{\mu r} b_\nu y_\nu{}^r.$$

Since this holds for arbitrary $a_\mu$ and $b_\nu$, we can infer

$$g_{\mu\nu} = l_\mu l_\nu + y_{\mu r} y_\nu{}^r, \tag{7}$$

a fundamental formula which is frequently useful.

If $V(u)$ is any function defined on the surface, we can obtain by differentiation the 3-vector $\partial V/\partial u_r = V^r$. We may then form $V^r y_{\mu r} = V_{-\mu}$ and consider it as a 4-vector in space-time. If $V$ is a field function, so that it is defined off the surface as well as on the surface, we can form $\partial V/\partial x_\mu$ and, by changing the sign of its spatial components, obtain the contravariant 4-vector $V_\mu$. Then the above $V_{-\mu}$ is just the tangential part of $V_\mu$.

Differentiating equations (2) with respect to $u_s$, we get

$$l_\mu{}^s y_\mu{}^r + l_\mu y_\mu{}^{rs} = 0, \tag{8}$$

$$l_\mu l_\mu{}^s = 0. \tag{9}$$

Equation (8) shows that

$$l_\mu{}^s y_\mu{}^r = l_\mu{}^r y_\mu{}^s.$$

Define

$$\Omega^{rs} = \Omega^{sr} = l_\mu{}^r y_\mu{}^s. \tag{10}$$

It is the curvature tensor. It may be expressed in the four dimensional form

$$\begin{aligned} \Omega_{-\mu-\nu} &= y_{\mu r} y_{\nu s} \Omega^{rs} = y_{\mu r} y_{\nu s} l_\sigma{}^r y_\sigma{}^s \\ &= y_{\mu r} l_\sigma{}^r (g_{\nu\sigma} - l_\nu l_\sigma). \end{aligned}$$

with the help of (7). Using (9) we now get

$$\Omega_{-\mu-\nu} = y_{\mu r} l_\nu{}^r = l_{\nu-\mu}. \tag{11}$$

We can infer that

$$l_{\mu-\nu} = l_{\nu-\mu}. \tag{12}$$

**3. Poisson bracket relations.** The $y_\mu(u)$ of (1), considered as a set of numbers with $\mu$ and the $u$'s taking all permissible values, are the surface variables. They are to be treated as dynamical coordinates. Since the $u$'s take on a continuous range of values, these variables will give us a continuous range of degrees of freedom, instead of the usual discrete set. The formalism of paper [2] was set up for a dynamical system with a discrete set of degrees of freedom, but can be made to apply to a continuous range by replacing sums by integrals and the Kronecker $\delta$-symbol by a $\delta$-function. The $\delta$-function we now need is $\delta(u - u')$, which vanishes for $u \neq u'$ and has its integral

$$\int \delta(u - u') d^3u = 1. \tag{13}$$

Note that there is no factor $\Gamma$ occurring with the $d^3u$, so this $\delta$-function refers to the parametrization of the surface and not to the metric. If one did introduce the factor $\Gamma$ into (13) one would get a $\delta$-function with a more directly physical meaning, but it would not be so suitable for dynamical theory because it would not have zero P.b. with the $w$ variables introduced below.

The dynamical variables $y_\mu(u)$ have conjugate momenta, $w_\mu(u)$ say, satisfying the standard P.b. relations. If for brevity we write $y_\mu$, $y'_\mu$ for $y_\mu(u)$, $y_\mu(u')$ and similarly $w_\mu$, $w'_\mu$ for $w_\mu(u)$, $w_\mu(u')$, these relations are

$$[y_\mu, y'_\nu] = 0, \quad [w_\mu, w'_\nu] = 0, \quad [y_\mu, w'_\nu] = g_{\mu\nu}\delta(u - u'). \tag{14}$$

The momenta may be pictured as associated with displacements and deformations of the parametrized surface, the linear combination $\epsilon \int a_\mu w_\mu d^3u$, where $a_\mu$ is a function of $u$ and $\epsilon$ is small, being associated with the displacement in which $y_\mu$ is changed to $y_\mu + \epsilon a_\mu$ for all $u$ values. Thus the normal component $w_\mu l_\mu = w_l$ is associated with a motion of the surface normal to itself, and the tangential components $w_\mu y_\mu{}^r = w^r$ with merely changes in the parametrization.

From the fundamental P.b. relations (14) one can deduce a number of useful P.b. relations connecting the $w$ variables with functions of the surface variables and with each other. Some of these relations have been obtained previously by the author [3], working from the association of the $w$ variables with deformations, and by Chang [4], using a more direct method similar to the one used below. Note that the $\Pi^r, \Pi^n$ of these two papers are minus the present $w^r$, $w_l$ and the $\gamma^{rs}$, $\gamma_{rs}$ of [3] are minus the present $\gamma^{rs}$, $\gamma_{rs}$.

In working out P.b.'s one should note in the first place that all quantities depending only on the surface and its parametrization are functions of the dynamical coordinates $y_\mu(u)$ only, and so have zero P.b.'s with one another. Thus $y_\mu$, $l_\mu$, $y_\mu{}^r$, $\Omega^{rs}$ for all $u$ values, and all their derivatives with respect to $u$'s, have zero P.b.'s with one another.

We have from the first of equations (2)

$$0 = [l_\mu y_\mu{}^r, w'_\nu] = [l_\mu, w'_\nu] y_\mu{}^r + l_\mu [y_\mu{}^r, w'_\nu].$$

Now

$$[y_\mu{}^r, w'_\nu] = [y_\mu, w'_\nu]^r$$

$$= g_{\mu\nu}\delta^r(u - u'). \tag{15}$$

Hence

$$[l_\mu, w'_\nu] y_\mu{}^r = - l_\nu \delta^r(u - u').$$

Again, from the second of equations (2),

$$0 = \tfrac{1}{2}[l_\mu l_\mu, w'_\nu] = [l_\mu, w'_\nu] l_\mu.$$

Thus, using (7),

$$[l_\lambda, w'_\nu] = [l_\mu, w'_\nu](l_\mu l_\lambda + y_\mu{}^r y_{\lambda r})$$

(16) $$= -\, l_\nu y_{\lambda r}\delta^r(u - u') = -\, l_\nu \delta_{-\lambda}(u - u').$$

We have, using (15),

(17) $$[\gamma^{rs}, w'_\nu] = y_\mu{}^r[y_\mu{}^s, w'_\nu] + y_\mu{}^s[y_\mu{}^r, w'_\nu] = y_\nu{}^r\delta^s(u - u') + y_\nu{}^s\delta^r(u-u').$$

Following the method by which (4) was obtained, we get

$$[\Gamma^2, w'_\nu] = \Gamma^2\gamma_{rs}[\gamma^{rs}, w'_\nu] = 2\Gamma^2\gamma_{rs}y_\nu{}^r\delta^s(u - u')$$

$$= 2\Gamma^2 y_{\nu s}\delta^s(u - u') = 2\Gamma^2\delta_{-\nu}(u - u'),$$

or

(18) $$[\Gamma, w'_\nu] = \Gamma\delta_{-\nu}(u - u').$$

From (17) again

$$[\gamma_{pr}, w'_\nu]\gamma^{rs} = -\, \gamma_{pr}[\gamma^{rs}, w'_\nu] = -\, y_{\nu p}\delta^s(u - u') - y_\nu{}^s\delta_p(u - u'),$$

so

(19) $$[\gamma_{pr}, w'_\nu] = -\, y_{\nu p}\delta_r(u - u') - y_{\nu r}\delta_p(u - u').$$

We shall use the notation of adding a dashed suffix $r'$ or $-\mu'$ to any function $X$ of $u'$ with the following meanings:

(20) $$X^{r'} = \partial X/\partial u'_r, \qquad X_{r'} = \gamma_{rs}(u')X^{s'}, \qquad X_{-\mu'} = y'_{\mu r'}X^{r'}.$$

We get from (16)

$$[l_\lambda, w'_l] = -\, l'_\nu l_\nu \delta_{-\lambda}(u - u')$$

$$= -\left\{l'_\nu l_\nu\delta(u - u')\right\}_{-\lambda} + l'_\nu l_{\nu-\lambda}\delta(u - u')$$

$$= -\left\{l_\nu l_\nu\delta(u - u')\right\}_{-\lambda} + l_\nu l_{\nu-\lambda}\delta(u - u')$$

(21) $$= -\, \delta_{-\lambda}(u - u').$$

Again, from (16)

$$[l_\lambda, w'^r] = -\, y'_\nu{}^{r'} l_\nu\delta_{-\lambda}(u - u')$$

$$= -\left\{y'_\nu{}^{r'} l_\nu\delta(u - u')\right\}_{-\lambda} + y'_\nu{}^{r'} l_{\nu-\lambda}\delta(u - u')$$

$$= y_\nu{}^r l_{\nu-\lambda}\delta(u - u')$$

$$= y_\nu{}^r l_{\lambda-\nu}\delta(u - u')$$

(22) $$= l_\lambda{}^r\delta(u - u').$$

Similarly, we get from (18)

$$[\Gamma, w'_l] = l'_\nu\Gamma\delta_{-\nu}(u - u') = \Gamma\left\{l'_\nu\delta(u - u')\right\}_{-\nu}$$

(23) $$= \Gamma l_{\nu-\nu}\delta(u - u').$$

and again from (18)

$$[\Gamma, w'^r] = y'_\nu{}^{r'} \Gamma \delta_{-\nu}(u - u') = \Gamma\{y'_\nu{}^{r'} \delta(u - u')\}_{-\nu}$$
$$= \Gamma (y_\nu{}^r)_{-\nu} \delta(u - u') + \Gamma y_\nu{}^r \delta_{-\nu}(u - u').$$

Now

$$(y_\nu{}^r)_{-\nu} = y_\nu{}^{rs} y_\nu{}^p \gamma_{ps} = y_\nu{}^{sr} y_\nu{}^p \gamma_{ps}$$
$$= \tfrac{1}{2}(y_\nu{}^s y_\nu{}^p)^r \gamma_{ps} = \tfrac{1}{2}\gamma^{psr}\gamma_{ps}$$
$$= \Gamma^r/\Gamma \tag{24}$$

from (4). Hence

$$[\Gamma, w'^r] = \Gamma^r \delta(u - u') + \Gamma \delta^r(u - u')$$
$$= \{\Gamma \delta(u - u')\}^r. \tag{25}$$

To get the P.b.'s of $w_l$, $w^r$, we have

$$[w_l, w'_l] = [w_\mu l_\mu, w'_\nu l'_\nu]$$
$$= w_\mu l'_\nu [l_\mu, w'_\nu] + l_\mu w'_\nu [w_\mu, l'_\nu]$$
$$= w_\mu [l_\mu, w'_l] + w'_\nu [w_l, l'_\nu].$$

From (21) this equals

$$[w_l, w'_l] = -\, w_\mu \delta_{-\mu}(u - u') + w'_\nu \delta_{-\nu'}(u - u'). \tag{26}$$

Again

$$[w_l, w'^r] = [w_\mu l_\mu, w'_\nu y'_\nu{}^{r'}]$$
$$= w_\mu [l_\mu, w'^r] + l_\mu w'_\nu [w_\mu, y'_\nu{}^{r'}]$$
$$= w_\mu l_\mu{}^r \delta(u - u') + l_\mu w_\mu' \delta^r(u - u')$$

with the help of (22). This gives

$$[w_l, w'^r] = w_\mu l_\mu{}^r \delta(u - u') + l_\mu \{w'_\mu \delta(u - u')\}^r$$
$$= \{w_l \delta(u - u')\}^r. \tag{27}$$

Again

$$[w^r, w'^s] = [w_\mu y_\mu{}^r, w'_\nu y'_\nu{}^{s'}]$$
$$= w_\mu y'_\nu{}^{s'} [y_\mu{}^r, w'_\nu] + y_\mu{}^r w'_\nu [w_\mu, y'_\nu{}^{s'}]$$
$$= w_\mu y'_\mu{}^{s'} \delta^r(u - u') - w'_\mu y_\mu{}^r \delta^{s'}(u - u')$$
$$= w_\mu\{y_\mu{}^{sr}\delta(u-u') + y_\mu{}^s \delta^r(u-u')\} - w_\mu'\{y'_\mu{}^{r's'}\delta(u-u') + y'_\mu{}^{r'}\delta^{s'}(u - u')\}$$
$$= w^s \delta^r(u - u') - w'^r \delta^{s'}(u - u'). \tag{28}$$

Further useful relations are

$$[y_{\mu s}, w'_\nu] = [\gamma_{rs} y_\mu{}^r, w'_\nu]$$
$$= \gamma_{rs} g_{\mu\nu} \delta^r(u - u') - y_\mu{}^r\{y_{\nu r}\delta_s(u - u') + y_{\nu s}\delta_r(u - u')\}$$
$$= l_\mu l_\nu \delta_s(u - u') - y_{\nu s}\delta_{-\mu}(u - u') \tag{29}$$

with the help of (7), and

$$\begin{aligned}[y_{\mu s}, w'^{r}] &= y'_{\nu}{}^{r'}\{l_\mu l_\nu \delta_s(u-u') - y_{\nu s} y_{\mu p} \delta^p(u-u')\} \\ &= y'_{\nu}{}^{r'} l_\mu \{l'_\nu \delta_s(u-u') - l_{\nu s}\delta(u-u')\} - y_{\mu p} y_{\nu s}\{y_\nu{}^{rp}\delta(u-u') \\ &\qquad + y_\nu{}^r \delta^p(u-u')\} \\ &= -l_\mu l_\nu{}^r y_{\nu s}\delta(u-u') - y_{\mu p} y_\nu{}^{pr} y_{\nu s}\delta(u-u') - \delta^r{}_s \delta_{-\mu}(u-u') \\ &= l_\mu l_\nu y_{\nu s}{}^r \delta(u-u') + y_{\mu p} y_\nu{}^p y_{\nu s}{}^r \delta(u-u') - \delta^r{}_s\delta_{-\mu}(u-u') \\ &= y_{\mu s}{}^r \delta(u-u') - \delta_s{}^r \delta_{-\mu}(u-u') \end{aligned} \tag{30}$$

with the help of (7) again.

**4. Changes of parametrization.** Using the homogeneous velocity formulation of the dynamical equations, we have a Hamiltonian of the general form given by the equation (20) of paper [2]. The $\phi$'s here are functions of the dynamical coordinates and momenta and the $v$'s involve the $\tau$ derivatives of quantities that can vary arbitrarily with $\tau$. With our present dynamical system we have the surface variables $y_\mu(u)$ that can vary arbitrarily with $\tau$ and may take their $\tau$ derivatives to be $v$'s. (Alternatively, we could take any complete set of independent functions of the $\dot{y}_\mu$'s to be $v$'s and get an equivalent but less convenient formulation). If these are the only quantities that can vary arbitrarily with $\tau$, their $\tau$ derivatives are the only $v$'s and the Hamiltonian is

$$H \equiv \int \dot{y}_\mu \phi_\mu(u) d^3u, \tag{31}$$

with $\phi_\mu(u)$ some function of the dynamical coordinates and momenta, weakly equal to zero, for each value of $u$ If there are other quantities that can vary arbitrarily with $\tau$, there will be further terms in $H$. These will be left understood for the present, as they will not affect the arguments now being used.

From (6) we may write (31) alternatively in terms of the normal and tangential components of $\dot{y}_\mu$ and $\phi_\mu$,

$$H \equiv \int \dot{y}_l \phi_l d^3u + \int \dot{y}_r \phi^r d^3u. \tag{32}$$

We now have $\phi_l$ and $\phi^r$ functions of the dynamical coordinates and momenta weakly equal to zero.

According to equation (21) of paper [2], the equation of motion for a general dynamical variable $g$ is

$$\dot{g} = \int \dot{y}_\mu [g, \phi_\mu] d^3u, \tag{33}$$

or alternatively

$$\dot{g} = \int \dot{y}_l [g, \phi_l] d^3u + \int \dot{y}_r [g, \phi^r] d^3u. \tag{34}$$

The second term here gives the change in $g$ when $\dot{y}_l = 0$, which means that the surface itself does not move but only its parametrization changes. Thus for a small change in parametrization

$$dg = \int (dy)_r [g, \phi^r] d^3u. \tag{35}$$

A small change in the parametrization is given by

$$u_r \to u_r + \epsilon a_r, \tag{36}$$

meaning that the point on the surface with parameters $u_r$ becomes the point with parameters $u_r + \epsilon a_r$, where $a_r$ is a function of the $u$'s and $\epsilon$ is a small number. This makes $y_\mu(u)$ change by

$$dy_\mu = y_\mu(u + \epsilon a) - y_\mu(u) = \epsilon y_\mu{}^r a_r$$

and gives
$$(dy)_r = y_{\mu r} dy_\mu = \epsilon a_r.$$

The change in $g$ for this change in the parametrization is thus

$$dg = \epsilon \int a_r [g, \phi^r] d^3u \tag{37}$$

$$= \epsilon [g, \int a_r \phi^r d^3u], \tag{37$'$}$$

since $\phi^r = 0$. The result (37) or (37′) holds even if $a_r$ is a function of any of the dynamical variables. We see now the importance of the $\phi^r$'s as the quantities with which one must form the P.b. of any dynamical variable to get its change under a change in the parametrization.

If a quantity refers to a point $u'$ on the surface and is invariant under any change of the parametrization that leaves the point $u'$ invariant, I call it a *$u$-scalar at the point $u'$*. A quantity that is invariant under any change of the parametrization whatever I call a *$u$-invariant.* The concepts of $u$-scalar and $u$-invariant refer only to the dependence on transformations of the $u$'s and not on how the quantity behaves under a Lorentz transformation. A $u$-scalar or $u$-invariant may very well be a component of a vector or tensor so far as Lorentz transformations are concerned.

Now $y_\mu(u')$ for a particular value of $\mu$ is evidently a $u$-scalar at $u'$. So is $l_\mu(u')$. Any function of $u$-scalars at $u'$ is a $u$-scalar at $u'$. If $S(u')$ is any $u$-scalar at $u'$, then $\int S(u') \Gamma' d^3u'$ is a $u$-invariant.

Let $S(u)$ be a $u$-scalar at the point $u$. Under the change of parametrization (36) it will change to the same $u$-scalar at $u + \epsilon a$, namely $S(u + \epsilon a)$, so that

$$dS = S(u + \epsilon a) - S(u) = \epsilon a_r S^r.$$

Thus from (37)

$$a_r S^r = \int a'_r [S, \phi'^r] d^3u'.$$

Since the functions $a_r(u)$ are arbitrary, we can infer

$$[S, \phi'^r] = S^r \, \delta(u - u'). \tag{38}$$

This is an equation expressing the condition for $S$ to be a $u$-scalar at $u$.

Let $Q$ be a $u$-invariant. From (37) we see that we must have

$$[Q, \phi'^r] = 0. \tag{39}$$

for all $u'$. Putting $Q = \int S\Gamma d^3u$, we get

$$\int S[\Gamma,\phi'^r]d^3u = -\int\Gamma[S,\phi'^r]d^3u = -\int\Gamma S^r\delta(u-u')d^3u$$
$$= \int S\{\Gamma\delta(u-u')\}^r d^3u.$$

Since this must hold for any $u$-scalar $S$, we see that

$$[\Gamma,\phi'^r] = \{\Gamma\delta(u-u')\}^r. \tag{40}$$

Suppose we have a field quantity $V(x)$ having a definite value at every point $x$ of space-time after the equations of motion have been integrated. The values of $V(x)$ on the surface will provide an $\infty^3$ of numbers, which may be labelled by the parameters $u$ so as to give the function $V(u)$. The quantities $V(u)$ for all values of the $u$'s will be dynamical variables having zero P.b. with all the $y$ and $w$ variables, thus

$$[y_\mu, V'] = 0, \qquad [w_\mu, V'] = 0. \tag{41}$$

They may be dynamical coordinates, in which case they would have zero P.b. with one another,

$$[V, V'] = 0. \tag{42}$$

There will then be dynamical variables $U(u)$ say, forming the conjugate momenta to the $V(u)$'s, satisfying the P.b. relations

$$\begin{aligned} [y_\mu, U'] &= 0 \qquad [w_\mu, U'] = 0 \\ [U, U'] &= 0 \qquad [V, U'] = \delta(u-u'). \end{aligned} \tag{43}$$

From its physical meaning $V(u)$ must be independent of changes of parametrization that do not change the point $u$, so it must be a $u$-scalar at $u$. The conjugate momentum $U(u)$ is not also a $u$-scalar, on account of the dependence of $\delta(u-u')$ in (43) on the parametrization. In order that $V$ may satisfy the condition (38) for a $u$-scalar, we must have

$$\phi^r \equiv UV^r + \phi^{r+}, \tag{44}$$

where $\phi^{r+}$ has zero P.b. with all the $V$'s. We may assume that $\phi^{r+}$ also has zero P.b. with all the $U$'s, this being the simplest assumption leading to a self-consistent scheme, and then

$$\begin{aligned} [U,\phi'^r] &= U'[U, V'^{r'}] = U'\delta^r(u-u') \\ &= \{U\delta(u-u')\}^r. \end{aligned} \tag{45}$$

From (40) we now get

$$\begin{aligned} [U\Gamma^{-1},\phi'^r] &= \{U\delta(u-u')\}^r\Gamma^{-1} - U\Gamma^{-2}\{\Gamma\delta(u-u')\}^r \\ &= \{U\Gamma^{-1}\}^r\delta(u-u'), \end{aligned}$$

which shows, according to (38), that $U\Gamma^{-1}$ is a $u$-scalar at $u$. We may call $U$

itself a $u$-scalar density. The result that a field quantity that is $u$-scalar has for its dynamical conjugate a $u$-scalar density has been obtained previously by Chang [4].

The dynamical coordinates $y_\mu(u)$, being $u$-scalars, may be treated in the same way as the $V(u)$ above. Corresponding to (44) we can infer that

$$\phi^r \equiv w_\mu y_\mu{}^r + \phi^{r*} \equiv w^r + \phi^{r*}, \tag{46}$$

where $\phi^{r*}$ has zero P.b. with all the $y$'s. We may assume $\phi^{r*}$ also has zero P.b. with all the $w$'s. The consistency of this assumption is easily checked. Thus with $\phi^r$ given by (46), one sees that the P.b. relation (22) leads to the condition, equation (38) with $l_\lambda$ for $S$, for $l_\lambda$ to be a $u$-scalar; the P.b. relation (25) leads to (40); and

$$[w_\nu, \phi'^r] = w'_\mu g_{\mu\nu} \delta^r(u - u') = \left\{ w_\nu \delta(u - u') \right\}^r,$$

which checks that $w_\nu$ is a $u$-scalar density.

For a dynamical system in which the only dynamical coordinates are the $y$'s and a number of field quantities $V_a(u)$, $(a = 1, 2, \ldots)$, there will be the momentum variables $w$ and the conjugates $U_a(u)$ to the $V_a(u)$, and $\phi^r$ will be

$$\phi^r \equiv w^r + \textstyle\sum_a U_a V_a{}^r, \tag{47}$$

neglecting an unimportant term which has zero P.b. with all the dynamical variables.

If $S(u)$ is a $u$-scalar at $u$, we should expect $S_{-\mu} = y_{\mu r} S^r$ to be also a $u$-scalar at $u$, as its formation from $S(u)$ does not depend on the parametrization. A formal proof of this result is as follows. We start from the condition (38) for $S(u)$. Differentiating both sides of this equation with respect to $u_s$, we get

$$[S^s, \phi'^r] = S^{rs}\delta(u - u') + S^r \delta^s(u - u').$$

Thus
$$\begin{aligned} y_{\mu s}[S^s, \phi'^r] &= \left\{ (S^s y_{\mu s})^r - S^s y_{\mu s}{}^r \right\} \delta(u - u') + S^r y_{\mu s} \delta^s(u - u') \} \\ &= \left\{ (S_{-\mu})^r - S^s y_{\mu s}{}^r \right\} \delta(u - u') + S^r \delta_{-\mu}(u - u') \\ &= (S_{-\mu})^r \delta(u - u') - S^s[y_{\mu s}, \phi'^r] \end{aligned}$$

from (30). Hence

$$[S_{-\mu}, \phi'^r] = y_{\mu s}[S^s, \phi'^r] + S^s[y_{\mu s}, \phi'^r] = (S_{-\mu})^r \delta(u - u'),$$

which is the condition for $S_{-\mu}$ to be a $u$-scalar.

**5. Passage from the Lagrangian to the Hamiltonian.** Consider a dynamical system for which the action density $\mathfrak{L}$ in space-time is a function of certain field quantities $V$ and their first derivatives $\partial V / \partial x^\mu = V_\mu$,

$$\mathfrak{L} = \mathfrak{L}(V, V_\mu).$$

Only one $V$ will be written in the equations of this section for brevity. The action is then

$$I \equiv \int \mathfrak{L} d^4x.$$

An element of four-dimensional "volume" of space-time $d^4x$ can be expressed as an element of three-dimensional "area" of the surface, $\Gamma d^3u$, multiplied by an element of distance normal to the surface, $l_\mu \dot{y}_\mu d\tau = \dot{y}_l d\tau$. Thus

$$I \equiv \int\int \mathfrak{L} \dot{y}_l \Gamma d^3u d\tau,$$

and so the Lagrangian is

$$L \equiv dI/d\tau \equiv \int \mathfrak{L} \dot{y}_l \Gamma d^3u. \tag{48}$$

This Lagrangian will now be treated according to the general method of paper [2] and the Hamiltonian obtained from it. The equations of sections **2** and **3** can all be used as strong equations in this work.

We must first express $L$ in terms of the dynamical coordinates and velocities, the $q$'s and $\dot{q}$'s of paper [2]. The variables $y_\mu(u)$, $V(u)$ are the $q$'s. Tangential derivatives of $y_\mu$ and $V$, such as $y_\mu{}^r$, $V^r$, $V_{-\mu}$, are functions of the $q$'s. Derivatives which are not tangential, such as $V_\mu$, are not functions of the $q$'s only, but can be expressed as functions of the $q$'s and $\dot{q}$'s. We have

$$\dot{V} \equiv V_\nu \dot{y}_\nu \equiv (V_l l_\nu + V_{-\nu}) \dot{y}_\nu$$

from (5). Thus

$$V_l \equiv (\dot{V} - V_{-\nu} \dot{y}_\nu)/\dot{y}_l, \tag{49}$$

so that

$$\begin{aligned} V_\mu &\equiv V_{-\mu} + l_\mu V_l \\ &\equiv V_{-\mu} + l_\mu (\dot{V} - V_{-\nu} \dot{y}_\nu)/\dot{y}_l. \end{aligned} \tag{50}$$

Here we have $V_\mu$ expressed in terms of $\dot{V}$ and $\dot{y}_\mu$, which are $\dot{q}$'s, and $V_{-\mu}, l_\mu$, which are functions of the $q$'s. $L$ now becomes a function of the $q$'s and $\dot{q}$'s. Note that $V_\mu$ is homogeneous of degree zero in the velocities, so that $L$ is homogeneous of the first degree in the velocities, as is needed for the homogeneous velocity formulation of the dynamical equations.

If we vary the $\dot{q}$'s keeping the $q$'s constant, we get from (50)

$$\begin{aligned} \delta V_\mu &= l_\mu (\delta \dot{V} - V_{-\nu} \delta \dot{y}_\nu)/\dot{y}_l - l_\mu (\dot{V} - V_{-\nu} \dot{y}_\nu) \delta \dot{y}_l / \dot{y}_l{}^2 \\ &= l_\mu (\delta \dot{V} - V_\nu \delta \dot{y}_\nu)/\dot{y}_l \end{aligned}$$

using (50) again. The variation in $L$ given by (48) is then

$$\begin{aligned} \delta L &= \int \{ \partial \mathfrak{L}/\partial V_\mu \,.\, \delta V_\mu \dot{y}_l + \mathfrak{L} \delta \dot{y}_l \} \Gamma d^3u \\ &= \int \{ \partial \mathfrak{L}/\partial V_\mu \,.\, l_\mu (\delta \dot{V} - V_\nu \delta \dot{y}_\nu) + \mathfrak{L} l_\nu \delta \dot{y}_\nu \} \Gamma d^3u. \end{aligned} \tag{51}$$

From the definition of the momenta $w_\mu(u)$, $U(u)$ conjugate to $y_\mu(u)$, $V(u)$ respectively, equation (2) of paper [2] applied to a continuous range of degrees of freedom, we have

$$\delta L = \int \{ w_\nu \delta \dot{y}_\nu + U \delta \dot{V} \} d^3u. \tag{52}$$

Comparing (51) and (52), we find

$$U = \partial \mathfrak{L} / \partial V_\mu \, . \, l_\mu \Gamma, \tag{53}$$

$$w_\nu = - \partial \mathfrak{L} / \partial V_\mu \, . \, l_\mu V_\nu \Gamma + \mathfrak{L} l_\nu \Gamma = - U V_\nu + \mathfrak{L} l_\nu \Gamma. \tag{54}$$

These are the weak equations giving the momenta in terms of the $q$'s and $\dot{q}$'s.

Following the method of paper [2], we eliminate the $\dot{q}$'s from equations (53) and (54) so as to get weak equations involving only the $p$'s (i.e. the $w$'s and $U$'s) and the $q$'s. These are the $\phi$ equations. Equation (54) is best treated by splitting it into a normal part, obtained by multiplying it by $l_\nu$,

$$w_l + U V_l - \mathfrak{L} \Gamma = 0, \tag{55}$$

and a tangential part, obtained by multiplying it by $y_\nu{}^r$,

$$w^r + U V^r = 0. \tag{56}$$

There are no $\dot{q}$'s in (56), so (56) as it stands is a $\phi$ equation. Its left-hand side is just the $\phi^r$ of (47) associated with changes of parametrization, the summation sign being understood in (56).

Equations (53) and (55) involve the derivatives $V_\mu$ of $V$, which derivatives can be expressed in terms of $V_{-\mu}$ and $V_l$. The $\dot{q}$'s then occur in (53) and (55) only through $V_l$, that is, from (49), only through the combination $(\dot{V} - V_{-\nu}\dot{y}_\nu)/\dot{y}_l$.

For many dynamical systems (see example 1 below) one can solve equations (53)—there is one of these equations for each field quantity $V$—to get all the $V_l$'s expressed as functions of the $U$'s and $q$'s. This case will be referred to as the *standard case* in field dynamics; it corresponds to the case in ordinary dynamics with homogeneous velocities when the ratios of the velocities can all be expressed as function of the $p$'s and $q$'s.

In the standard case one can get no $\phi$ equations from equations (53) alone, but in other cases one can get $\phi$ equations from (53) alone. In the standard case one can get a $\phi$ equation from (55) with the help of (53), namely the equation

$$w_l + \mathfrak{H} = 0, \tag{57}$$

where $\mathfrak{H}$ is the result of substituting for $V_l$ in $U V_l - \mathfrak{L}\Gamma$ its value in terms of the $U$'s and $q$'s given by (53). Equations (57) and (56), taken for all $u$-values, are then the only $\phi$ equations. In other cases one can still get a $\phi$ equation like (57) from (55) with the help of (53), as will be shown later. This equation, together with (56) and the $\phi$ equations which follow from (53) alone, are then the only $\phi$ equations.

The field equations are obtained in the usual way from the variation of the action integral and are

$$\frac{\partial}{\partial x_\mu} \frac{\partial \mathfrak{L}}{\partial V_\mu} = \frac{\partial \mathfrak{L}}{\partial V} \tag{58}$$

These equations must be examined to see whether they lead to any equations between p's and $q$'s only. Such equations would be $\chi$ equations. With the help of (7) and (53), (58) may be written as

$$\frac{l_\mu}{\Gamma}\left\{\frac{\partial U}{\partial x_\mu} - \frac{\partial \mathfrak{L}}{\partial V_\nu}\frac{\partial(l_\nu\Gamma)}{\partial x_\mu}\right\} + y_{\nu r}\frac{\partial}{\partial u_r}\frac{\partial \mathfrak{L}}{\partial x_\nu} = \frac{\partial \mathfrak{L}}{\partial V}, \tag{59}$$

and thus involves the normal derivative of $U$. In the standard case the $U$'s are all independent functions of the velocities and one cannot eliminate their normal derivatives from (59), so there can be no $\chi$ equations. In other cases, however, there may well be $\chi$ equations (see examples 2 and 3 below).

When we have obtained all the $\phi$'s and $\chi$'s we must see which of them are first class, that is, have zero P.b. with all the $\phi$'s and $\chi$'s. This can always be decided by working out the P.b's using the results of section **3**, but we can infer that some $\phi$'s are first class more easily by observing that they occur in the Hamiltonian, giving rise to arbitrary functions in the general solution of the equations of motion.

The Hamiltonian is, from the definition (5) of paper [2],

$$H \equiv \int (w_\mu \dot{y}_\mu + U\dot{V} - \mathfrak{L}\dot{y}_l\Gamma)d^3u. \tag{60}$$

It may be written

$$\begin{aligned} H &\equiv \int\{w^r\dot{y}_r + w_l\dot{y}_l + U(V_l l_\nu + V_{-\nu})\dot{y}_\nu - \mathfrak{L}\dot{y}_l\Gamma\}d^3u \\ &\equiv \int\{\dot{y}_l(w_l + UV_l - \mathfrak{L}\Gamma) + \dot{y}_r(w^r + UV^r)\}d^3u \\ &\equiv \int \dot{y}_l(w_l + UV_l - \mathfrak{L}\Gamma)d^3u + \int \dot{y}_r\phi^r d^3u, \end{aligned} \tag{61}$$

with $\phi^r$ given by (47), the summation sign being now understood. According to the general theory of paper [2], $H$ must strongly equal a linear function of first class $\phi$'s of the form (31) or (32), with extra terms if there are other quantities besides the surface variables $y_\mu(u)$ that can vary arbitrarily with $\tau$. The $\phi^r$ of (61) is the same as the $\phi^r$ of (32), so we can infer that it must be first class. The presence of this first class $\phi$ in $H$ gives rise to arbitrary changes of parametrization during the motion.

In the standard case the only $\phi$ equation, apart from the $\phi^r$ equation (56), is equation (57). Hence (57) must be the same equation as $\phi_l = 0$. We can infer that the left-hand side of (57) must be first class, and also that the equation

$$\mathfrak{H} \equiv UV_l - \mathfrak{L}\Gamma, \tag{62}$$

which is needed to make the first term of (61) go over into the first term of the right-hand side of (32), holds strongly, and not merely weakly, as we knew previously. Further, we can infer that in the standard case there are no extra terms in $H$ besides the ones appearing in (32), so there are no other quantities that can vary arbitrarily with $\tau$ besides the surface variables.

In other cases there must still be a $\phi$ equation (57), deducible from (55) with the help of (53), in order that its left-hand side may be the first class $\phi_l$, whose presence in $H$ gives rise to the arbitrary motion of the surface normal to itself. Equation (62) still holds strongly if there are no extra terms in $H$ (see example 2 below). However if there are extra terms in $H$, as is the case when there are first class $\phi$ equations deducible from (53) alone, the first term of (61) no longer strongly equals the first term of the right-hand side of (32) and so (62) no longer holds strongly. $\mathfrak{H}$ is then not a uniquely defined quantity, as one can add to it any linear function of the first class $\phi$'s that follow from (53) alone. (See example 3 below).

To prepare the theory for quantization we must divide all the $\phi$ and $\chi$ equations into first and second class and then change the second class ones into strong equations by a redefinition of P.b.'s, in the way discussed in section **8** of paper [2]. Further, we must change the first class $\chi$'s into first class $\phi$'s, adding them, with arbitrary coefficients, to the Hamiltonian. This change merely involves an increase in the number of solutions of the equations of motion and does not invalidate the existing solutions. (See example 3 below for a discussion of the physical significance of such a change).

We are left with a set of weak first class $\phi$ equations, from which we can get the Hamilton-Jacobi equations by putting each momentum variable $p$ equal to $\partial S/\partial q$, so that they become first order partial differential equations in $S$. Their mutual consistency follows from the first class condition. Each of their solutions gives a family of solutions of the equations of motion.

The passage to the quantum theory can now be made according to the rules of section **11** of paper [2]. Each of the weak first class $\phi$ equations provides one Schrödinger wave equation.

**Example 1. The scalar meson field.** Some simple examples will now be treated according to the method of the present paper to illustrate various features of the theory. Let us take first the scalar meson field. For this example there is one field quantitiy $V$, a Lorentz scalar, and the action density is

$$\mathfrak{L} \equiv \tfrac{1}{2} V_\mu V_\mu - \tfrac{1}{2} m^2 V^2, \tag{63}$$

$m$ being a constant.

Equation (53) gives for the momentum $U$

$$U = V_\mu l_\mu \Gamma = V_l \Gamma. \tag{64}$$

This can be solved to give $V_l$ in terms of $U$ and the $q$'s, so we have the standard case. Thus there can be no $\chi$ equations, and the only $\phi$ besides $\phi^r$ will be (57). To get the $\mathfrak{H}$ here, we note that (62) and (63) give

$$\mathfrak{H} \equiv U V_l - \tfrac{1}{2}(V_l V_l + V_{-\mu} V_{-\mu} - m^2 V^2)\Gamma.$$

From the weak equation (64) we can infer the strong equation

$$0 \equiv \tfrac{1}{2}(U - V_l \Gamma)^2 \Gamma^{-1}.$$

Adding this to the preceding equation, we obtain

$$\mathfrak{H} \equiv \tfrac{1}{2}U^2\Gamma^{-1} - \tfrac{1}{2}(V_{-\mu}V_{-\mu} - m^2V^2)\Gamma. \tag{65}$$

The normal derivative $V_l$ has disappeared from this expression for $\mathfrak{H}$, so this expression is the correct one, involving only $U$'s and $q$'s, to occur in the $\phi$ equation (57).

The above derivation of $\mathfrak{H}$ shows that one can eliminate $V_l$ from the right-hand side of (62) using only strong equations. This checks that the equation (62) for $\mathfrak{H}$ is a strong equation, as is necessary in the standard case.

**Example 2. The vector meson field.** Let us suppose there are four field quantities $V$ forming a Lorentz vector $A_\mu$ and let us take the action density

$$\mathfrak{L} \equiv -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}m^2A_\mu A_\mu, \tag{66}$$

where $$F_{\mu\nu} \equiv A_{\nu\mu} - A_{\mu\nu}, \qquad A_{\nu\mu} \equiv \partial A_\nu/\partial x^\mu.$$

There are considerable differences in the treatment of this problem according to whether the constant $m$ is zero or not. In the present example we restrict it to be not zero.

Let $B_\mu$ be the momentum conjugate to $A_\mu$, its sign being defined so that

$$[A_\mu, B'_\nu] \equiv g_{\mu\nu}\delta(u - u'). \tag{67}$$

Then (53) gives

$$B_\nu = F_{\nu\mu}l_\mu\Gamma. \tag{68}$$

We can deduce

$$B_l \equiv l_\nu B_\nu = 0. \tag{69}$$

This equation involves only the $p$'s and $q$'s, so it is a $\phi$ equation. It follows that we do not have the standard case. One can easily see that there are no other $\phi$ equations deducible from (68) alone, so the only other $\phi$ equations are (56), which now reads

$$\phi^r \equiv w^r + B_\mu A_\mu{}^r = 0, \tag{70}$$

and (57), which will be discussed later.

The field equations (58) give

$$(F_{\nu\mu})_\mu = m^2A_\nu. \tag{71}$$

We can infer by differentiating with respect to $x_\nu$, using the condition $m \neq 0$,

$$A_{\nu\nu} = 0. \tag{72}$$

Also $$(F_{\nu\mu})_{-\mu} = -l_\mu l_\sigma(F_{\nu\mu})_\sigma + m^2A_\nu,$$

so that $$(l_\nu F_{\nu\mu})_{-\mu} = l_{\nu-\mu}F_{\nu\mu} + l_\nu(F_{\nu\mu})_{-\mu}$$

$$= m^2A_l$$

with the help of (12). Now from (68)

$$l_\nu F_{\nu\mu} = -B_\mu\Gamma^{-1} = -B_{-\mu}\Gamma^{-1},$$

so we get

$$(B_{-\mu}\Gamma^{-1})_{-\mu} + m^2 A_l = 0. \tag{73}$$

This equation involves only the $p$'s and $q$'s, so it is a $\chi$ equation.

It is easily seen that $B_l$ has zero P.b. with the first term of (73) and not with the second, provided $m \neq 0$. Thus *$B_l$ must be a second class $\phi$ and* (73) *a second class $\chi$*. There are thus no first class $\phi$'s besides $\phi^r$ and $\phi_l$, so there are no extra terms in $H$ besides the ones appearing in (32). Hence equation (62) holds strongly, i.e.

$$\mathfrak{H} \equiv B_\mu A_{\mu l} + \tfrac{1}{4} F_{\mu\nu}F_{\mu\nu}\Gamma - \tfrac{1}{2}m^2 A_\mu A_\mu\Gamma. \tag{74}$$

We can now obtain $\mathfrak{H}$ as a function of the $p$'s and $q$'s by eliminating the normal derivative of $A_\mu$ from (74). We must take care to use only strong equations in this work, otherwise we may get extra terms containing $B_l$ as a factor in the expression for $\mathfrak{H}$. With such extra terms in $\mathfrak{H}$, (57) would still be a correct $\phi$ equation, but its left-hand side would not be first class and would not be the $\phi_l$ occurring in $H$.

Let $F_{-\mu-\nu}$ be the tangential part of $F_{\mu\nu}$, given, according to a natural extension of (5), by

$$F_{-\mu-\nu} \equiv F_{\mu\nu} - l_\nu F_{\mu l} - l_\mu F_{l\nu} \tag{75}$$

where

$$F_{\mu l} \equiv -F_{l\mu} \equiv l_\sigma F_{\mu\sigma}.$$

Substituting

$$F_{\mu\nu} \equiv A_{\nu-\mu} - A_{\mu-\nu} + l_\mu A_{\nu l} - l_\nu A_{\mu l},$$

$$F_{\mu l} \equiv l_\sigma A_{\sigma-\mu} + l_\sigma l_\mu A_{\sigma l} - A_{\mu l}$$

in (75), we get after some reduction

$$F_{-\mu-\nu} \equiv A_{\nu-\mu} - A_{\mu-\nu} + l_\sigma(l_\mu A_{\sigma-\nu} - l_\nu A_{\sigma-\mu}). \tag{76}$$

Since the normal derivative of $A_\mu$ does not occur here, $F_{-\mu-\nu}$ is a function of $p$'s and $q$'s only.

We have

$$F_{-\mu-\nu}F_{-\mu-\nu} \equiv F_{\mu\nu}F_{-\mu-\nu} \equiv F_{\mu\nu}F_{\mu\nu} - 2F_{\mu l}F_{\mu l} \tag{77}$$

from (75). Thus (74) becomes

$$\mathfrak{H} \equiv B_\mu A_{\mu l} + \tfrac{1}{2} F_{-\mu-\nu}F_{-\mu-\nu}\Gamma + \tfrac{1}{2} F_{\mu l}F_{\mu l}\Gamma - \tfrac{1}{2}m^2 A_\mu A_\mu\Gamma. \tag{78}$$

From (68) we get the strong equation

$$0 \equiv -\tfrac{1}{4}(B_\mu - F_{\mu l}\Gamma)(B_\mu - F_{\mu l}\Gamma)\Gamma^{-1}.$$

Adding this to (78), we get

$$\mathfrak{H} \equiv B_\mu l_\nu A_{\nu\mu} - \tfrac{1}{2} B_\mu B_\mu\Gamma^{-1} + \tfrac{1}{4} F_{-\mu-\nu}F_{-\mu-\nu}\Gamma - \tfrac{1}{2}m^2 A_\mu A_\mu\Gamma. \tag{79}$$

Equation (72) may be written

(80) $$l_\nu A_{\nu l} + A_{\nu-\nu} = 0.$$

From it we can infer the strong equation

$$B_l(l_\nu A_{\nu l} + A_{\nu-\nu}) \equiv 0$$

and hence

$$B_\mu l_\nu A_{\nu\mu} \equiv B_\mu l_\nu A_{\nu-\mu} + B_l l_\nu A_{\nu l}$$
$$\equiv B_{-\mu} l_\nu A_{\nu-\mu} - B_l A_{\nu-\nu}.$$

Thus (79) becomes

(81) $$\mathfrak{H} \equiv B_{-\mu} l_\nu A_{\nu-\mu} - B_l A_{\nu-\nu} - \tfrac{1}{2} B_\mu B_\mu \Gamma^{-1} + \tfrac{1}{4} F_{-\mu-\nu} F_{-\mu-\nu}\Gamma - \tfrac{1}{2} m^2 A_\mu A_\mu \Gamma.$$

The normal derivative of $A_\mu$ has now disappeared and we have the correct expression for $\mathfrak{H}$ to use in (57).

To adapt the theory for quantization we must redefine P.b.'s by the method of section **8** of paper [2] so as to make the second class $\phi$ and $\chi$ equations (69) and (73) hold strongly. We take as the $\theta$'s of paper [2] the left-hand sides of (69) and (73). There are thus two $\theta$'s for each value of the $u$'s, say

(82) $$\theta \equiv (B_{-\mu}\Gamma^{-1})_{-\mu} + m^2 A_l, \qquad \theta^+ \equiv B_l.$$

Their P.b.'s are

$$[\theta,\theta'] \equiv 0, \qquad [\theta^+,\theta^{+\prime}] \equiv 0, \qquad [\theta,\theta^{+\prime}] \equiv m^2\delta(u - u').$$

The coefficients $c$ must be determined from equation (35) of paper [2], with the sum over $s$ interpreted as a sum over $\theta$ and $\theta^+$ together with an integral over all $u$ values. The solution is that the $c$ associated with $\theta(u)$ and $\theta^+(u')$ is $m^{-2}\delta(u - u')$ and the other $c$'s vanish. Formula (36) of paper [2] then gives, with the sums over $s$ and $s'$ interpreted as above,

(83) $$[\xi,\eta]^* \equiv [\xi,\eta] + m^{-2}\int[\xi,(B_{-\mu}\Gamma^{-1})_{-\mu} + m^2 A_l]\,[B_l,\eta]d^3u$$
$$- m^{-2}\int[\xi,B_l]\,[(B_{-\mu}\Gamma^{-1})_{-\mu} + m^2 A_l,\eta]d^3u.$$

The new definition of P.b.'s makes $B_l$ and $(B_{-\mu}\Gamma^{-1})_{-\mu}+m^2A_l$ have zero P.b. with everything, so that one can put them strongly equal to zero without inconsistency. In working out the new P.b. of two given quantities $\xi$ and $\eta$, it would be convenient first to make them independent of $A_l$ and $B_l$ by substituting

(84) $$A_l \equiv -m^{-2}(B_{-\mu}\Gamma^{-1})_{-\mu}, \qquad B_l \equiv 0$$

in them. If they are then independent of the $w$ variables, we have $[\xi,B_l] = [\eta,B_l] = 0$, and so the new P.b. equals the old one. The formula (83) is thus needed only for evaluating the new P.b.'s of quantities involving the $w$ variables.

When one passes to the quantum theory, the strong equations (84) become equations between operators. The weak $\phi$ equations (70) and (57) with $\mathfrak{H}$ given by (81) provide the Schrödinger wave equations, the dynamical variables in these equations having the new P.b. relationships.

**Example 3. The electromagnetic field.** We get the electromagnetic field by putting $m = 0$ in the vector meson field. This case has some special features which necessitate a separate investigation.

We now have

$$\mathfrak{L} \equiv - \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu}. \tag{85}$$

Equations (67) . . . (70) still hold and (69) is still a $\phi$ equation. Equation (71) becomes

$$(F_{\nu\mu})_\mu = 0. \tag{86}$$

Equation (72) cannot now be deduced. In the usual theory of electrodynamics, which was first given in a Hamiltonian form comparable with the present paper by Fermi [5], equation (72) is assumed as a supplementary condition. It will not be assumed in the present treatment.

Corresponding to (73), we now have the $\chi$ equation

$$\chi \equiv (B_{-\mu}\Gamma^{-1})_{-\mu} = 0. \tag{87}$$

It is easily seen that

$$[B_l, B'_l] = 0, \qquad [\chi, \chi'] = 0, \qquad [B_l, \chi'] = 0. \tag{88}$$

Since $B_l$ and $\chi$ have zero P.b. with the other $\phi$'s, namely $\phi^r$ and $\phi_l$, as can be inferred from $\phi^r$ and $\phi_l$ being first class, we see that $B_l$ and $\chi$ must be first class. Thus *all the $\phi$'s and $\chi$'s are now first class.* This is the essential difference between the present example and the preceding one.

Let us see how to express $H$ linearly in terms of the first class $\phi$'s. By using the analysis which led to (79), we can write (61) as

$$\begin{aligned} H &\equiv \int \dot{y}_l(w_l + l_\nu A_{\nu\mu} B_\mu - \tfrac{1}{2} B_\mu B_\mu \Gamma^{-1} + \tfrac{1}{4} F_{-\mu-\nu} F_{-\mu-\nu}\Gamma) d^3u + \int \dot{y}_r \phi^r d^3u \\ &\equiv \int \dot{y}_l \phi_l d^3u + \int \dot{y}_l l_\nu A_{\nu l} B_l d^3u + \int \dot{y}_r \phi^r d^3u, \end{aligned} \tag{89}$$

with $\phi_l$ defined by

$$\phi_l \equiv w_l + l_\nu A_{\nu-\mu} B_{-\mu} - \tfrac{1}{2} B_\mu B_\mu \Gamma^{-1} + \tfrac{1}{4} F_{-\mu-\nu} F_{-\mu-\nu}\Gamma. \tag{90}$$

This expression for $\phi_l$ does not involve the normal derivative of $A_\mu$, so it is a function of the $p$'s and $q$'s only, and it vanishes weakly, as it differs from the left-hand side of (55) only by a multiple of $B_l$, so it is a $\phi$. It must be first class, as all the $\phi$'s are now first class. Thus (89) expresses $H$ as a linear function of first class $\phi$'s.

The $\phi_l$ introduced above may be considered as the $\phi$ which gives rise to the motion of the surface normal to itself. However, we could take an alternative

$\phi_l$, differing from (90) by any function of the $p$'s and $q$'s which vanishes weakly through being a linear function of $B_l$, and could consider it equally well as the $\phi$ which gives rise to the motion of the surface normal to itself. An example of special interest will here be given.

Putting $A_l$ for $V$ in (49) and writing $A_l{}^\tau$ for $dA_l/d\tau$ to have an unambiguous notation, we get

(91) $$A_l{}^\tau - \dot{y}_r A_l{}^r \equiv \dot{y}_l A_{ll} \equiv \dot{y}_l(l_\nu A_{\nu l} + l_{\nu l} A_\nu).$$

Putting $y_\nu{}^r$ for $V$ in the same formula, we get

$$\dot{y}_l y_\nu{}^r{}_l \equiv \dot{y}_\nu{}^r - \dot{y}_s y_\nu{}^{rs}.$$

Multiplying this by $l_\nu$, we get

$$-\dot{y}_l y_\nu{}^r l_{\nu l} \equiv \dot{y}_\nu{}^r l_\nu + \dot{y}_s y_\nu{}^s l_\nu{}^r \equiv \dot{y}_l{}^r.$$

Multiplying again by $A_r$ and subtracting (91), we get

$$-A_l{}^\tau + \dot{y}_r A_l{}^r \equiv \dot{y}_l{}^r A_r - \dot{y}_l l_\nu A_{\nu l}.$$

With the help of this result the second term of (89) becomes

$$\begin{aligned}\int \dot{y}_l l_\nu A_{\nu l} B_l d^3u &\equiv \int (A_l{}^\tau - \dot{y}_r A_l{}^r + \dot{y}_l{}^r A_r) B_l d^3u \\ &\equiv \int A_l{}^\tau B_l d^3u - \int \dot{y}_r A_l{}^r B_l d^3u - \int \dot{y}_l (A_r B_l)^r d^3u.\end{aligned}$$

So we may write (89) as

(92) $$H \equiv \int \dot{y}_l \phi^+{}_l d^3u + \int A_l{}^\tau B_l d^3u + \int \dot{y}_r \phi^{+r} d^3u,$$

with

(93) $$\phi^+{}_l \equiv \phi_l - (A_r B_l)^r$$

and

(94) $$\phi^{+r} \equiv \phi^r - A_l{}^r B_l.$$

We now have $H$ expressed in terms of the first class $\phi$'s $\phi^+{}_l$, $B_l$, $\phi^{+r}$. We may look upon $\phi^+{}_l$ as an alternative $\phi_l$ giving rise to the motion of the surface normal to itself and $\phi^{+r}$ as an alternative $\phi^r$ giving rise to a change of parametrization. It should be noted that $\phi^+{}_l$ and $\phi^{+r}$ have zero P.b. with $A_l$. We have from (67)

$$[A_l, A'_l{}^{r'} B'_l] = A_l{}^r \delta(u - u'),$$

and since $A_l$ is a $u$-scalar it satisfies the condition (38), which gives

$$[A_l, \phi'^r] \quad = A_l{}^r \delta(u - u'),$$

and hence

(95) $$[A_l, \phi'^{+r}] = 0.$$

Again, with the help of (21)

$$(96) \qquad [A_l, \phi^{+\prime}{}_l] = A_\lambda[l_\lambda, w'_l] - l_\lambda[A_\lambda, (A'_r B'_l)^{r'}]$$
$$= -A_\lambda \delta_{-\lambda}(u - u') - l_\lambda\{A'_r l'_\lambda \delta(u - u')\}^{r'} = 0.$$

The Hamiltonian contains an extra term besides those that give rise to arbitrary changes in the surface and its parametrization, namely the second term in (89) or (92). This extra term gives a further freedom in the motion. It allows $A_l$ to vary arbitrarily with $\tau$, the equation $A_l^\tau = [A_l, H]$ being identically fulfilled, as follows from the expression (92) for $H$, with the help of (95) and (96).

The further freedom corresponds physically to the possibility of changes of gauge taking place while the motion develops. The initial conditions, fixing an initial surface and the potentials and their normal derivatives on it, do not restrict the gauge at points in space-time away from this surface. One can make a gauge transformation

$$(97) \qquad A_\mu \to A_\mu + \partial S/\partial x_\mu$$

with $S$ an arbitrary function of the four $x_\mu$'s. Thus one can choose $S$ so that there is no change in the conditions on the initial surface while there is an arbitrary change of gauge in other regions of space-time. This change will affect the dynamical variables at later $\tau$ values and give rise to arbitrary functions in the solution of the equations of motion, even when the motion of the surface is prescribed.

In the usual theory of electrodynamics one has the supplementary condition (72), which results in the $S$ of (97) being restricted to satisfy

$$(98) \qquad S_{\mu\mu} = 0.$$

One can then no longer make a change of gauge without affecting the potentials or their normal derivatives on the initial surface, so the extra arbtrariness in the motion no longer occurs. The present theory of electrodynamics differs from the usual one through allowing more general gauge transformations, but the two theories are equivalent for all gauge invariant effects, and thus for all effects of physical importance.

The question arises with the present theory whether one can have a motion for which the gauge changes while the surface and its parametrization do not change. Using the form (92) for $H$, it is immediately evident that one can have such a motion, since one can put $\dot{y}_l = \dot{y}_r = 0$ in this expression for $H$ and the second term survives, leaving the rate of change of $A_l$ arbitrary. A general change of gauge involves independent changes in the normal component of $A_\mu$, namely $A_l$, and the three-dimensional divergence of its tangential component, namely $A_{-\mu-\mu}$, on the surface. Thus the change of gauge allowed by our equations of motion when there is no change in the surface is not a general one.

If we require the trajectories of the motion in phase space to form integrable subspaces, in the way discussed on page 142 of paper [2], we must be able to have a general change of gauge with no change in the surface, since such a change could be attained by first moving the surface and making some change in the gauge, and then moving the surface back again and making a further change of gauge, not cancelling the previous one in any way. In order to get equations of motion allowing general changes of gauge with no change in the surface, we must add a further term to $H$, namely

$$\int v\chi d^3u \tag{99}$$

with $\chi$ given by (87) and the coefficient $v$ arbitrary. This means treating $\chi$ as a first class $\phi$. The Hamiltonian modified in this way is not derivable from an action density, but is still a permissible Hamiltonian for a dynamical system, leading to consistent equations of motion, on account of $\chi$ being first class. The modification in $H$ merely adds to the solutions of the equations of motion without altering the previously existing solutions, the latter being just the special case $v = 0$ of the new solutions. Thus one can consider the modification as not a change to a new dynamical system, but merely an extension of the treatment of the original dynamical system.

We can now pass to the quantum theory by making each first class $\phi$, including the first class $\chi$'s that get changed into first class $\phi$'s to satisfy the integrability condition, into a Schrödinger wave equation. Thus we get the wave equations

$$\phi^{+}{}_{l}\psi = 0, \qquad \phi^{+r}\psi = 0, \qquad B_{l}\psi = 0, \qquad (B_{-\mu}\Gamma^{-1})_{-\mu}\psi = 0. \tag{100}$$

The last two of these equations show that the wave function $\psi$, if expressed in terms of the longitudinal and transverse components of the $A$'s on the surface, is independent of the longitudinal components. It thus involves the longitudinal field variables in a different way from the usual quantum electrodynamics.

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