Relativistic Wave Equations

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1—Introduction

The classical relativistic connexion between the energy p_t of a free article and its momentum p_x, p_y, p_z, namely,*

$$p_t^2 - p_x^2 - p_y^2 - p_z^2 - m^2 = 0, \tag{1}$$

ads in the quantum theory to the wave equation

$$\{p_t^2 - p_x^2 - p_y^2 - p_z^2 - m^2\} \psi = 0, \tag{2}$$

here the p's are understood as the operators $i\hbar \dfrac{\partial}{\partial t}$, $-i\hbar \dfrac{\partial}{\partial x}$ .. The eneral theory of the physical interpretation of quantum mechanics equires a wave equation of the form

$$\{p_t - \mathrm{H}\} \psi = 0, \tag{3}$$

here H is a Hermitian operator not containing p_t, and is called the Hamiltonian. The obvious equation of the form (3) which one gets rom (2), namely,

$$\{p_t - (p_x^2 + p_y^2 + p_z^2 + m^2)^{\frac{1}{2}}\} \psi = 0,$$

s unsatisfactory on account of the square root, which makes the appli-ation of Lorentz transformations very complicated. By allowing our article to have a spin, we can get wave equations of the form (3) which re consistent with (2) and do not involve square roots. An example, pplying to the case of a spin of half a quantum, namely, the equation

$$\{p_t + \alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \alpha_m m\} \psi = 0, \tag{4}$$

here the four α's are anti-commuting matrices whose squares are unity, s well known, and has been found to give a satisfactory description of he electron and positron. The present paper will be concerned with other examples, applying to spins greater than a half.

* We are taking the velocity of light to be unity.

The elementary particles known to present-day physics, the electron, positron, neutron, and proton, each have a spin of a half, and thus the work of the present paper will have no immediate physical application. All the same, it is desirable to have the equations ready for a possible future discovery of an elementary particle with a spin greater than a half, or for approximate application to composite particles. Further, the underlying theory is of considerable mathematical interest.

2—The Fundamental Matrices

In a relativisitic theory an angular momentum appears as a six-vector. Let the spin angular momentum of our particle expressed in units of $\hbar$ be the six-vector s, with components $s_{jk} = - s_{kj}$. These satisfy the commutation relations,

$$\begin{aligned} s_{xy}s_{yz} - s_{yz}s_{xy} &= is_{zx} \\ s_{xy}s_{xt} - s_{xt}s_{xy} &= is_{yt} \\ s_{xy}s_{yt} - s_{yt}s_{xy} &= - is_{xt} \\ s_{xy}s_{zt} - s_{zt}s_{xy} &= 0 \\ s_{xt}s_{yt} - s_{yt}s_{xt} &= - is_{xy}, \end{aligned}$$

together with the relations obtained from these by cyclic permutations of xyz. The usual procedure for studying the consequences of these commutation relations is to introduce

$$\left.\begin{aligned} \alpha_x &= \tfrac{1}{2}(s_{yz} - is_{xt}) \qquad & \beta_x &= \tfrac{1}{2}(s_{yz} + is_{xt}) \\ \alpha_y &= \tfrac{1}{2}(s_{zx} - is_{yt}) \qquad & \beta_y &= \tfrac{1}{2}(s_{zx} + is_{yt}) \\ \alpha_z &= \tfrac{1}{2}(s_{xy} - is_{zt}) \qquad & \beta_z &= \tfrac{1}{2}(s_{xy} + is_{zt}) \end{aligned}\right\}. \tag{5}$$

Then the α's and β's satisfy

$$\alpha_x\alpha_y - \alpha_y\alpha_x = i\alpha_z \tag{6}$$

$$\beta_x\beta_y - \beta_y\beta_x = i\beta_z, \tag{7}$$

and the relations obtained from these by cyclic permutations of xyz, and further all the α's commute with all the β's. The α's and β's are thus like the components of two independent angular momenta in ordinary three-dimensional space. We introduce the magnitudes k and l of these angular momenta in the usual way by

$$\alpha_x^2 + \alpha_y^2 + \alpha_z^2 = k(k+1) \tag{8}$$

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = l(l+1), \tag{9}$$

and consider k and l as numbers. Each of them may be any non-negative integer or half-odd integer. In this way we see that in general four-dimensional theory a spin is characterized by two numbers. In the special case of the electron spin, $k = l = \frac{1}{2}$. We may consider α_x, α_y, α_z as matrices with $2k + 1$ rows and columns and β_x, β_y, β_z as matrices with $2l + 1$ rows and columns.

We shall find it convenient to pass to spinor notation, following Van der Waerden* and Laporte and Uhlenbeck.† According to these authors, the quantities α_x, α_y, α_z, which determine a self-dual six-vector, can be represented by a spinor $s_{\mu\nu}$ with two undotted suffixes, its components being

$$s_{11} = -(\alpha_x + i\alpha_y), \quad s_{22} = \alpha_x - i\alpha_y, \quad s_{12} = s_{21} = \alpha_z. \tag{10}$$

Similarly, the quantities β_x, β_y, β_z can be represented by a spinor $s_{\dot\mu\dot\nu}$ with two dotted suffixes, its components being

$$s_{\dot1\dot1} = -(\beta_x - i\beta_y), \quad s_{\dot2\dot2} = \beta_x + i\beta_y, \quad s_{\dot1\dot2} = s_{\dot2\dot1} = \beta_z. \tag{11}$$

Let us raise the first suffix in $s_{\mu\nu}$, so as to obtain $s^\mu{}_\nu$, with components

$$\begin{aligned} s^1{}_1 &= s_{21} = \alpha_z & \qquad s^1{}_2 &= s_{22} = \alpha_x - i\alpha_y \\ s^2{}_1 &= -s_{11} = \alpha_x + i\alpha_y & \qquad s^2{}_2 &= -s_{12} = -\alpha_z. \end{aligned}$$

We may arrange these as a matrix A in $2(2k+1)$ rows and columns, thus

$$\mathrm{A} = \begin{pmatrix} s^1{}_1 & s^1{}_2 \\ s^2{}_1 & s^2{}_2 \end{pmatrix} = \begin{pmatrix} \alpha_z & \alpha_x - i\alpha_y \\ \alpha_x + i\alpha_y & -\alpha_z \end{pmatrix}. \tag{12}$$

The matrix A then has some rather remarkable properties. We find

$$\mathrm{A}^2 = \begin{pmatrix} \alpha_z{}^2 + (\alpha_x - i\alpha_y)(\alpha_x + i\alpha_y) & \alpha_z(\alpha_x - i\alpha_y) - (\alpha_x - i\alpha_y)\alpha_z \\ (\alpha_x + i\alpha_y)\alpha_z - \alpha_z(\alpha_x + i\alpha_y) & (\alpha_x + i\alpha_y)(\alpha_x - i\alpha_y) + \alpha_z{}^2 \end{pmatrix},$$

which reduces, with the help of (6) and (8), to

$$\mathrm{A}^2 = \begin{pmatrix} k(k+1) - \alpha_z & -\alpha_x + i\alpha_y \\ -\alpha_x - i\alpha_y & k(k+1) + \alpha_z \end{pmatrix} = k(k+1) - \mathrm{A}$$

and hence

$$\mathrm{A}(\mathrm{A} + 1) = k(k+1). \tag{13}$$

It follows that the eigenvalues of A are k and $-(k+1)$. Let us suppose these eigenvalues occur m and n times respectively, $m + n$ being

* 'Nachr. Ges. Wiss. Gött.,' p. 100 (1929).
† 'Phys. Rev.,' vol. 37, p. 1380 (1931).

$2(2k+1)$. If we assume α_x, α_y, α_z to be Hermitian matrices, then A is a Hermitian matrix and can be transformed into a diagonal matrix by a unitary transformation, thus

$$\mathrm{A} = \mathrm{U}^{-1}\,\mathrm{D}\mathrm{U}, \tag{14}$$

where D is the diagonal matrix whose first m diagonal elements are k and whose remaining n diagonal elements are $-(k+1)$. However, we do not wish to use the condition that α_x, α_y, α_z are Hermitian at present, since this condition is not Lorentz invariant. If we do not have this condition, α_x, α_y, α_z will still be transformable into Hermitian matrices by a canonical transformation, and thus an equation of the form (14) will still hold, but U will no longer be a unitary matrix.

Let us express U in the form

$$\mathrm{U} = (2k+1)^{-\frac{1}{2}}\begin{pmatrix} b_1 & b_2 \\ v_1 & v_2 \end{pmatrix}, \tag{15}$$

where b_1 and b_2 are matrices with m rows and $(2k+1)$ columns, and v_1 and v_2 are matrices with n rows and $(2k+1)$ columns. Thus when we multiply D into U in accordance with (14), b_1 and b_2 will get multiplied by k and v_1 and v_2 will get multiplied by $-(k+1)$. Similarly, let us express U^{-1} in the form

$$\mathrm{U}^{-1} = (2k+1)^{-\frac{1}{2}}\begin{pmatrix} a^1 & u^1 \\ a^2 & u^2 \end{pmatrix}, \tag{16}$$

where a^1 and a^2 are matrices with $(2k+1)$ rows and m columns, and u^1 and u^2 are matrices with $(2k+1)$ rows and n columns. Equations (12) and (14) now give us

$$\left.\begin{aligned} (2k+1)\,\alpha_z &= ka^1b_1 - (k+1)\,u^1v_1, \\ (2k+1)(\alpha_x - i\alpha_y) &= ka^1b_2 - (k+1)\,u^1v_2 \\ (2k+1)(\alpha_x + i\alpha_y) &= ka^2b_1 - (k+1)\,u^2v_1, \\ -(2k+1)\,\alpha_z &= ka^2b_2 - (k+1)\,u^2v_2 \end{aligned}\right\}. \tag{17}$$

Further, the condition $\mathrm{U}^{-1}\,\mathrm{U} = 1$ gives us

$$\left.\begin{aligned} 2k+1 &= a^1b_1 + u^1v_1 & 0 &= a^1b_2 + u^1v_2 \\ 0 &= a^2b_1 + u^2v_1 & 2k+1 &= a^2b_2 + u^2v_2 \end{aligned}\right\}, \tag{18}$$

and the condition $\mathrm{U}\mathrm{U}^{-1} = 1$ gives us

$$\left.\begin{aligned} 2k+1 &= b_1a^1 + b_2a^2 & 0 &= b_1u^1 + b_2u^2 \\ 0 &= v_1a^1 + v_2a^2 & 2k+1 &= v_1u^1 + v_2u^2 \end{aligned}\right\}. \tag{19}$$

Multiplying each equation (18) by (k + 1) and adding to the corresponding equation (17), we obtain, after cancelling the factor $(2k + 1)$,

$$\left.\begin{array}{ll} \alpha_z + k + 1 = a^1 b_1 & \alpha_x - i\alpha_y = a^1 b_2 \\ \alpha_x + i\alpha_y = a^2 b_1 & -\alpha_z + k + 1 = a^2 b_2 \end{array}\right\} . \quad (20)$$

Similarly, multiplying each equation (18) by $-k$ and adding to the corresponding equation (17), we obtain, again cancelling the factor $(2k + 1)$,

$$\left.\begin{array}{ll} \alpha_z - k = -u^1 v_1 & \alpha_x - i\alpha_y = -u^1 v_2 \\ \alpha_x + i\alpha_y = -u^2 v_1 & -\alpha_z - k = -u^2 v_2 \end{array}\right\} . \quad (21)$$

We can now determine m and n, the number of eigenvalues of A equal to k and $-(k + 1)$ respectively. The eigenvalues of α_z form the arithmetical progression, with constant difference unity, extending from k to $-k$. Thus $\alpha_z + k + 1$ has no eigenvalue zero and the determinant of the matrix representing it must not vanish. It follows from the first of equations (20) that the number of columns in the matrix a^1 must not be less than $(2k + 1)$, *i.e.*,

$$m \geqslant 2k + 1.$$

Again, $\alpha_z - k$ has one and only one eigenvalue zero, so that the matrix representing it must be of rank $2k$. It follows from the first of equations (21) that the number of columns in the matrix u^1 must not be less than $2k$, *i.e.*,

$$n \geqslant 2k.$$

Since $m + n = 2(2k + 1)$, we have only two possibilities, namely,

(i) $m = 2k + 1 \qquad n = 2k + 1$

(ii) $m = 2k + 2 \qquad n = 2k.$

The first of these, which makes all the matrices square, can be excluded by the following argument. We have from (6) and (8)

$$(\alpha_x - i\alpha_y)(\alpha_x + i\alpha_y) = k(k + 1) - \alpha_z(\alpha_z + 1),$$

and the right-hand side here has one eigenvalue zero. It follows that the determinant of either $\alpha_x - i\alpha_y$ or $\alpha_x + i\alpha_y$ must vanish. From (20) it now follows that, in the case of possibility (i), one (at least) of the matrices a^1, b_2, a^2, b_1 must have a vanishing determinant. Hence either $\alpha_z + k + 1$ or $-\alpha_z + k + 1$ must have a vanishing determinant, which is not true since these matrices have no eigenvalue zero. It follows that possibility (ii) is the correct one.

Referring to (12), we see that we can write (20) in the form

$$s^{\mu}{}_{\nu} + (k+1)\,\delta^{\mu}_{\nu} = a^{\mu} b_{\nu}, \tag{22}$$

and (21) in the form

$$s^{\mu}{}_{\nu} - k\delta^{\mu}_{\nu} = -\,u^{\mu} v_{\nu}. \tag{23}$$

Further, we can write (19) in the form

$$\left.\begin{aligned} 2k+1 = b_{\mu} a^{\mu} = v_{\mu} u^{\mu} \\ 0 = b_{\mu} u^{\mu} = v_{\mu} a^{\mu} \end{aligned}\right\}. \tag{24}$$

These equations are the only conditions that a^{μ}, b_{ν}, u^{μ}, v_{ν} have to satisfy and since they are all of the correct tensor form in the suffixes μ and ν, we may assume that a^{μ}, b_{ν}, u^{μ}, v_{ν} transform under Lorentz transformations like single-suffix spinors, as the notation implies.

Let us study the algebraic properties of the u's and v's. From (21) and (19) we have

$$u^1 v_1 + u^2 v_2 = 2k \tag{25}$$

$$v_1 u^1 + v_2 u^2 = 2k + 1. \tag{26}$$

Multiplying (26) by v_2 on the right and (25) by v_2 on the left and subtracting, we obtain

$$v_1 u^1 v_2 - v_2 u^1 v_1 = v_2. \tag{27}$$

This is of the nature of a commutation relation. Other commutation relations which may be obtained in a similar way are

$$\left.\begin{aligned} v_1 u^2 v_2 - v_2 u^2 v_1 = -\,v_1 \\ u^1 v_1 u^2 - u^2 v_1 u^1 = -\,u^2 \\ u^1 v_2 u^2 - u^2 v_2 u^1 = u^1 \end{aligned}\right\}. \tag{28}$$

These equations give us all the commutability information which is possible for the non-square matrices with which we are here dealing. A similar set of algebraic properties may be deduced for the a's and b's based on

$$\left.\begin{aligned} a^1 b_1 + a^2 b_2 = 2k + 2 \\ b_1 a^1 + b_2 a^2 = 2k + 1 \end{aligned}\right\}. \tag{29}$$

We can deal in a corresponding way with the spinor $s_{\dot{\mu}\dot{\nu}}$. We raise its second suffix and consider it as forming a matrix B with $2(2l+1)$ rows and columns.

$$\mathrm{B} = \begin{pmatrix} s_{\dot 1}{}^{\dot 1} & s_{\dot 1}{}^{\dot 2} \\ s_{\dot 2}{}^{\dot 1} & s_{\dot 2}{}^{\dot 2} \end{pmatrix} = \begin{pmatrix} s_{\dot 1 \dot 2} & -\,s_{\dot 1 \dot 1} \\ s_{\dot 2 \dot 2} & -\,s_{\dot 2 \dot 1} \end{pmatrix} = \begin{pmatrix} \beta_z & \beta_x - i\beta_y \\ \beta_x + i\beta_y & -\,\beta_z \end{pmatrix}. \tag{30}$$

As before, we have

$$\mathrm{B}\,(\mathrm{B}+1) = l\,(l+1), \tag{31}$$

from which we can deduce, corresponding to (22) and (23),

$$\left.\begin{aligned} s_{\dot\mu}{}^{\dot\nu} + (l+1)\,\delta_{\dot\mu}{}^{\dot\nu} &= a_{\dot\mu} b^{\dot\nu} \\ s_{\dot\mu}{}^{\dot\nu} - l\delta_{\dot\mu}{}^{\dot\nu} &= -\,u_{\dot\mu} v^{\dot\nu} \end{aligned}\right\}, \tag{32}$$

where the a's, b's, u's, v's with dotted suffixes have similar properties to those with undotted suffixes, with the number l replacing the number k.

3—Wave Equations for Zero Rest-Mass

Let us suppose our particle to be in an eigenstate of momentum, so that its momentum and energy have definite numerical values. The spinor components of the momentum-energy four-vector will be

$$\left.\begin{aligned} p_{\dot 1 1} &= p_t + p_z \qquad & p_{\dot 1 2} &= p_x - ip_y \\ p_{\dot 2 1} &= p_x + ip_y \qquad & p_{\dot 2 2} &= p_t - p_z \end{aligned}\right\}. \tag{33}$$

If the particle has zero rest-mass, its momentum and energy will satisfy

$$p_t^2 - p_x^2 - p_y^2 - p_z^2 = 0,$$

which becomes, in spinor notation,

$$p_{\dot 1 1}\, p_{\dot 2 2} - p_{\dot 1 2}\, p_{\dot 2 1} = 0.$$

This equation shows that $p_{\dot\mu\nu}$ can be expressed as a product of two single-suffix spinors, thus

$$p_{\dot\mu\nu} = \eta_{\dot\mu}\xi_\nu,$$

or, since $p_{\dot\mu\nu} = \overline{p_{\dot\nu\mu}}$,

$$p_{\dot\mu\nu} = \bar\xi_{\dot\mu}\xi_\nu. \tag{34}$$

Let us now assume that the momentum and spin of our particle are connected in such a way that

$$v_\nu \xi^\nu \psi = 0. \tag{35}$$

The wave function ψ here must have $2k+1$ components, in order that v_ν may be multiplied into it. We take (35) as our general wave equation for a particle of zero rest-mass. It is obviously relativistically invariant. In order to give a meaning to (35) when ψ is not an eigenstate of momentum, we consider ψ to be expanded in terms of these eigenstates and then require (35) to hold for each term separately in the expansion.

Let us express (35) in a more usual notation in order to see better its physical significance. Multiplying it by $u^{\mu}\bar{\xi}_{\dot{\lambda}}$ on the left, we obtain

$$u^{\mu}v_{\nu}\bar{\xi}_{\dot{\lambda}}\xi^{\nu}\psi = 0,$$

which gives, with the help of (23) and (34),

$$\left.\begin{aligned}(k\delta^{\mu}_{\nu} - s^{\mu}{}_{\nu})\, p_{\dot{\lambda}}{}^{\nu}\psi &= 0\\ \text{or}\qquad (kp_{\dot{\lambda}\mu} - s_{\mu\nu}p_{\dot{\lambda}}{}^{\nu})\,\psi &= 0\end{aligned}\right\}. \tag{36}$$

We have four equations here, corresponding to the two possible values for λ and the two for μ. Putting $\lambda = \mu$ and summing, we obtain as one of these four equations

$$\{k\,(p_{\dot{1}1} + p_{\dot{2}2}) - (s_{11}p_{\dot{1}}{}^{1} + s_{12}p_{\dot{1}}{}^{2} + s_{21}p_{\dot{2}}{}^{1} + s_{22}p_{\dot{2}}{}^{2})\}\,\psi = 0$$

or

$$\{k\,(p_{\dot{1}1} + p_{\dot{2}2}) + (-\,s_{11}p_{\dot{1}2} + s_{12}p_{\dot{1}1} - s_{21}p_{\dot{2}2} + s_{22}p_{\dot{2}1})\}\,\psi = 0,$$

with the rule for lowering spinor suffixes. Written in ordinary vector notation, with the help of (10) and (33), this becomes

$$\{kp_t + \alpha_x p_x + \alpha_y p_y + \alpha_z p_z\}\,\psi = 0. \tag{37}$$

The other three equations, written in ordinary vector notation, become

$$\left.\begin{aligned}\{kp_x + \alpha_x p_t - i\alpha_y p_z + i\alpha_z p_y\}\,\psi &= 0\\ \{kp_y + \alpha_y p_t - i\alpha_z p_x + i\alpha_x p_z\}\,\psi &= 0\\ \{kp_z + \alpha_z p_t - i\alpha_x p_y + i\alpha_y p_x\}\,\psi &= 0\end{aligned}\right\}. \tag{38}$$

These may be taken instead of (35) as the fundamental equations of the present theory for a particle of zero rest-mass. Equation (37) is a wave equation of the usual type. From each of the equations (38) we can eliminate p_t with the help of (37), and we then obtain a supplementary condition which the wave function has to satisfy at each instant of time. Thus the first of equations (38) leads to the supplementary condition

$$\{(k^2 - \alpha_x{}^2)\,p_x + (ik\alpha_z - \alpha_x\alpha_y)\,p_y - (ik\alpha_y + \alpha_x\alpha_z)\,p_z\}\,\psi = 0. \tag{39}$$

The other two may be obtained from this one by cyclic permutation of x, y, and z.

4—Wave Equations for Non-Zero Rest-Mass

To get a satisfactory relativistic wave equation for a particle of non-zero rest-mass, we assume that our wave function consists of two parts, which we call ψ_A and ψ_B, which satisfy the equations

$$\left.\begin{aligned} p^{\dot\mu\nu} v_\nu \psi_A &= m' v^{\dot\mu} \psi_B \\ p_{\dot\mu\nu} v^{\dot\mu} \psi_B &= m'' v_\nu \psi_A \end{aligned}\right\}, \qquad (40)$$

where m' and m'' are two constants whose values in terms of the rest-mass will be determined later. In order that these equations may have a meaning, it is necessary that ψ_A shall have $(2k+1)\,2l$ components, labelled by two parameters, one of which is associated with the columns of the matrices v_ν and takes on $2k+1$ values, while the other is associated with the rows of $v^{\dot\mu}$ and takes on $2l$ values. Similarly, ψ_B must have $2k\,(2l+1)$ components, labelled by two parameters, one of which is associated with the columns of $v^{\dot\mu}$ and takes on $2l+1$ values, while the other is associated with the rows of v_ν and takes on $2k$ values. Thus altogether our wave function has $(2k+1)\,2l + 2k\,(2l+1)$ components. The case $k = l = \frac{1}{2}$ reduces to the ordinary electron equations, for which the wave function has four components.

Equations (40) are evidently relativistic. There is a certain amount of arbitrariness in them, owing to the v's not being uniquely determined by the equations (17), (18), and (19) which define them. We may multiply v_1 and v_2 on the left by any non-singular matrix γ with $2k$ rows and columns, and u^1 and u^2 on the right by γ^{-1}, without changing any of the conditions that they have to satisfy. Similarly we may multiply $v^{\dot 1}$ and $v^{\dot 2}$ on the left by any non-singular matrix δ with $2l$ rows and columns and multiply $u_{\dot 1}$ and $u_{\dot 2}$ on the right by δ^{-1}. We have that γ commutes with $v^{\dot 1}$ and $v^{\dot 2}$ since it acts on a different parameter in the labelling of the components of the wave function, and similarly δ commutes with v_1 and v_2. Thus the wave equations that we get with the new v's, namely,

$$\left.\begin{aligned} p^{\dot\mu\nu} \gamma v_\nu \psi_A &= m' \delta v^{\dot\mu} \psi_B \\ p_{\dot\mu\nu} \delta v^{\dot\mu} \psi_B &= m'' \gamma v_\nu \psi_A \end{aligned}\right\}, \qquad (41)$$

may be written

$$p^{\dot\mu\nu} v_\nu\, \delta^{-1}\psi_A = m' v^{\dot\mu} \gamma^{-1} \psi_B$$

$$p_{\dot\mu\nu} v^{\dot\mu} \gamma^{-1} \psi_B = m'' v_\nu\, \delta^{-1} \psi_A,$$

which are of the same form as (40) with $\delta^{-1}\psi_A$ and $\gamma^{-1}\psi_B$ for ψ_A and ψ_B respectively.

Eliminating ψ_B from the two equations (40), we obtain the second-order equation

$$p_{\dot\mu\lambda} p^{\dot\mu\nu} v_\nu \psi_A = m' m'' v_\lambda \psi_A,$$

which gives us, on passing back to ordinary vector notation with the help of (33),

$$(p_t^2 - p_x^2 - p_y^2 - p_z^2)\, v_\lambda \psi_A = m'm''v_\lambda\psi_A.$$

Multiplying by u^λ on the left and using (25), we obtain

$$(p_t^2 - p_x^2 - p_y^2 - p_z^2)\, \psi_A = m'm''\psi_A.$$

Similarly

$$(p_t^2 - p_x^2 - p_y^2 - p_z^2)\, \psi_B = m'm''\psi_B.$$

Thus, provided we choose

$$m'm'' = m^2, \tag{42}$$

each of the components of our wave function will satisfy the ordinary second-order wave equation (2).

We must now obtain the Hamiltonian of our theory. The first of equations (40) gives us

$$\left.\begin{aligned}(p^{\dot{1}1}v_1 + p^{\dot{1}2}v_2)\,\psi_A &= m'v^{\dot{1}}\psi_B\\ (p^{\dot{2}1}v_1 + p^{\dot{2}2}v_2)\,\psi_A &= m'v^{\dot{2}}\psi_B\end{aligned}\right\}. \tag{43}$$

Multiplying these equations on the left by u^1 and u^2 respectively and adding, we obtain

$$(p^{\dot{1}1}u^1v_1 + p^{\dot{2}2}u^2v_2 + p^{\dot{1}2}u^1v_2 + p^{\dot{2}1}u^2v_1)\,\psi_A = m'\,(u^1v^{\dot{1}} + u^2v^{\dot{2}})\,\psi_B,$$

which leads, with the help of (21) and (33), and the rule for lowering spinor suffixes, to

$$\{(p_t - p_z)(k - \alpha_z) + (p_t + p_z)(k + \alpha_z) + (p_x + ip_y)(\alpha_x - i\alpha_y) + (p_x - ip_y)(\alpha_x + i\alpha_y)\}\,\psi_A = m'\,(u^1v^{\dot{1}} + u^2v^{\dot{2}})\,\psi_B$$

or

$$\{kp_t + \alpha_x p_x + \alpha_y p_y + \alpha_z p_z\}\,\psi_A = \tfrac{1}{2}m'\,(u^1v^{\dot{1}} + u^2v^{\dot{2}})\,\psi_B. \tag{44}$$

Similarly, the second of equations (40) leads to

$$\{lp_t - \beta_x p_x - \beta_y p_y - \beta_z p_z\}\,\psi_B = \tfrac{1}{2}m''\,(u_{\dot{1}}v_1 + u_{\dot{2}}v_2)\,\psi_A.$$

These two equations may be combined and written as the matrix equation

$$\begin{pmatrix} p_t + k^{-1}(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z) & -\tfrac{1}{2}m'k^{-1}(u^1v^{\dot{1}} + u^2v^{\dot{2}}) \\ -\tfrac{1}{2}m''l^{-1}(u_{\dot{1}}v_1 + u_{\dot{2}}v_2) & p_t - l^{-1}(\beta_x p_x + \beta_y p_y + \beta_z p_z) \end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = 0. \tag{45}$$

Thus our Hamiltonian is the matrix

$$\begin{pmatrix} -k^{-1}(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z) & \tfrac{1}{2}m'k^{-1}(u^1v^{\dot{1}} + u^2v^{\dot{2}}) \\ \tfrac{1}{2}m''l^{-1}(u_{\dot{1}}v_1 + u_{\dot{2}}v_2) & l^{-1}(\beta_x p_x + \beta_y p_y + \beta_z p_z) \end{pmatrix}. \tag{46}$$

We must verify that this Hamiltonian is Hermitian in a suitably chosen epresentation. Let us take a representation in which the α's and β's re Hermitian. Then we may choose the matrix U introduced in equation 14) to be unitary. This will result in the matrices u^1 and u^2 being the onjugate Hermitians of the matrices v_1 and v_2. Similarly we can make he matrices $u_{\dot 1}$ and $u_{\dot 2}$ the conjugate Hermitians of the matrices $v^{\dot 1}$ and $^{\dot 2}$. We shall then have $u^1 v^{\dot 1} + u^2 v^{\dot 2}$ the conjugate Hermitian of $u_{\dot 1} v_1 + u_{\dot 2} v_2$, nd the Hamiltonian (46) will then be Hermitian, provided

$$m' k^{-1} = m'' \, l^{-1}.$$

'his condition combined with (42) gives

$$m' = m\,(k/l)^{\frac{1}{2}} \quad m'' = m\,(l/k)^{\frac{1}{2}}. \tag{47}$$

The Hermitian condition reduces the arbitrariness in the v's. If U s unitary, the arbitrary matrix γ appearing in (41) must be unitary, in rder that we may have $u^{\mu}\gamma^{-1}$ conjugate Hermitian to γv_{μ} at the same ime as u^{μ} is conjugate Hermitian to v_{μ}. Similarly δ must be unitary. If we put

$$v_{\nu}\psi_{A} = k^{\frac{1}{2}}\psi_{\nu} \qquad v^{\dot\mu}\psi_{B} = l^{\frac{1}{2}}\psi^{\dot\mu},$$

quations (40) take, with the help of (47), the form

$$p^{\dot\mu\nu}\psi_{\nu} = m\psi^{\dot\mu}$$
$$p_{\dot\mu\nu}\psi^{\dot\mu} = m\psi_{\nu},$$

vhich is just the form of the usual electron equations in spinor notaion. This equivalence between our equations (40) and the usual electron quations persists when there is an electromagnetic field present, provided he effect of the field on equations (40) is the usual one of requiring p to be eplaced by $p + e$A, A being the vector potential. Thus our equation 40) will give us something essentially new only provided other kinds of forces can be put into them, requiring other kinds of terms in the equaions. The possibility of putting other kinds of terms into the equations is very much restricted by the requirement that the equations shall remain nvariant under the introduction of γ and δ as in (41), combined with a suitable transformation of the ψ's. However, there do exist terms that atisfy this requirement, for example, the term $B^{\dot\mu\nu} v_{\lambda} s^{\lambda}{}_{\nu} \psi_{A}$ can be put into the first of equations (40), $B^{\dot\mu\nu}$ being some vector field function.

There exist supplementary conditions in our present theory, as in the theory for zero rest-mass. These are provided by the whole set of equations which one gets when one multiplies the first of equations (40) by u_λ and the second by $u_{\dot\lambda}$. We get an example of a supplementary condition by multiplying equations (43) on the left by u^1 and u^2 respectively and subtracting (instead of adding, as we did to get the Hamiltonian). We obtain in this way

$$(p^{\dot{1}1}u^1v_1 - p^{\dot{2}2}u^2v_2 + p^{\dot{1}2}u^1v_2 - p^{\dot{2}1}u^2v_1)\,\psi_A = m'\,(u^1v^{\dot{1}} - u^2v^{\dot{2}})\,\psi_B,$$

which leads, with the help of (21) and (33), to

$$\{(p_t - p_z)(k - \alpha_z) - (p_t + p_z)(k + \alpha_z) + (p_x + ip_y)(\alpha_x - i\alpha_y) \\ - (p_x - ip_y)(\alpha_x + i\alpha_y)\}\,\psi_A = m'\,(u^1v^{\dot{1}} - u^2v^{\dot{2}})\,\psi_B,$$

or

$$\{kp_z + \alpha_z p_t - i\alpha_x p_y + i\alpha_y p_x\}\,\psi_A = -\tfrac{1}{2}m'\,(u^1v^{\dot{1}} - u^2v^{\dot{2}})\,\psi_B.$$

Eliminating p_t with the help of (44), we obtain

$$\{(ik\alpha_y - \alpha_z\alpha_x)\,p_x - (ik\alpha_x + \alpha_z\alpha_y)\,p_y + (k^2 - \alpha_z^2)\,p_z\}\,\psi_A \\ = -\tfrac{1}{2}m'\,\{(k + \alpha_z)\,u^1v^{\dot{1}} - (k - \alpha_z)\,u^2v^{\dot{2}}\}\,\psi_B,$$

which is one of the supplementary conditions.

5—Wave Functions in Spinor Notation

We shall here consider briefly an alternative way of introducing the u and v matrices and of expressing the wave equations (40). Let F be any spinor containing $2k$ undotted suffixes and symmetrical between all these suffixes. Its general component may be written $F_{r,2k-r}$, r being the number of suffixes equal to 1 and $2k - r$ the number equal to 2. We now define v_1 to be an operator which can operate on F and turns it into a spinor, called v_1F, containing $(2k - 1)$ undotted suffixes and symmetrical between them. The general component of v_1F is assumed to be

$$(v_1F)_{r-1,\,2k-r} = F_{r,\,2k-r}.$$

Similarly, we define v_2 to be the operator which turns F into a symmetrical spinor in $(2k - 1)$ undotted suffixes whose general component is

$$(v_2F)_{r,\,2k-r-1} = F_{r,\,2k-r}.$$

Further, we define u^1 to be the operator which, operating on any symmetrical spinor G in $(2k - 1)$ undotted suffixes, turns it into a symmetrical spinor in $2k$ undotted suffixes whose general component is

$$(u^1G)_{r,\,2k-r} = rG_{r-1,\,2k-r}.$$

The right-hand side here vanishes for $r = 0$, so that we do not have to ive a meaning to $G_{-1,2k}$. Finally, we define u^2 to be the operator which urns G into

$$(u^2G)_{r,2k-r} = (2k-r)\,G_{r,2k-r-1}.$$

With these definitions we obtain

$$(u^1v_1F)_{r,2k-r} = r\,(v_1F)_{r-1,2k-r} = rF_{r,2k-r}$$

$$(u^2v_2F)_{r,2k-r} = (2k-r)\,(v_2F)_{r,2k-r-1} = (2k-r)\,F_{r,2k-r}.$$

hus

$$u^1v_1 + u^2v_2 = 2k.$$

gain

$$(v_1u^1G)_{r-1,2k-r} = (u^1G)_{r,2k-r} = rG_{r-1,2k-r}$$

$$(v_2u^2G)_{r-1,2k-r} = (u^2G)_{r-1,2k-r+1} = (2k-r+1)\,G_{r-1,2k-r}.$$

hus

$$v_1u^1 + v_2u^2 = 2k+1.$$

hese algebraic equations for the u's and v's are the same as (25) and 26). It follows that the commutation relations for the present u's and 's are the same as those for the u's and v's of § 2. Further, by considerng the effect of a Lorentz transformation on the spinor components, it is asily verified that the present u's and v's satisfy equations (21), and thus hat they are completely equivalent to those of § 2.

We may introduce $u^{\dot{\mu}}$, v^{μ} in a corresponding way as operators on pinors with dotted suffixes upstairs, and verify that they are the same as hose of § 2. Thus we have an alternative way of building up the theory.

Consider now ψ_A to be a spinor $A_{\kappa\lambda\mu\ldots}^{\dot{\beta}\dot{\gamma}\ldots}$ with $2k$ undotted suffixes lownstairs and $(2l-1)$ dotted suffixes upstairs, and ψ_B to be a spinor $B_{\lambda\mu\ldots}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\ldots}$ with $(2k-1)$ undotted suffixes downstairs and $2l$ dotted ones ipstairs, both spinors being symmetrical between all the undotted and etween all the dotted suffixes. Equations (40) may then be written,

$$p^{\dot{\alpha}\kappa}A_{\kappa\lambda\mu\ldots}^{\dot{\beta}\dot{\gamma}\ldots} = m'B_{\lambda\mu\ldots}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\ldots}$$

$$p_{\dot{\alpha}\kappa}B_{\lambda\mu\ldots}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\ldots} = m''A_{\kappa\lambda\mu\ldots}^{\dot{\beta}\dot{\gamma}\ldots}.$$

This is an alternative and very simple way of writing our fundamental equations.

Summary

The paper deals with the setting up of relativistic wave equations, linear n the energy operator, for particles of spin greater than half a quantum. t is found that this can be done quite simply when the rest-mass is zero, and somewhat less simply when the rest-mass is not zero.