The theory of gravitation in Hamiltonian form

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The author's generalized procedure for putting a theory into Hamiltonian form is applied to Einstein's theory of gravitation. It is shown that one can make a change in the action density, not affecting the equations of motion, which causes four of the ten degrees of freedom associated with the ten $g_{\mu\nu}$ to drop out of the Hamiltonian formalism. This simplification can be achieved only at the expense of abandoning four-dimensional symmetry.

In the weak field approximation one can make a Fourier resolution of the field quantities, and one then gets a clean separation of those degrees of freedom whose variables depend on the system of co-ordinates from those whose variables do not. There are four of the former and two of the latter for each Fourier component. The two latter correspond to gravitational waves with two independent states of polarization. One of the four former is responsible for the Newtonian attraction between masses and also gives a negative gravitational self-energy for each mass.

Introduction

The quantum theory has taught us the importance of the Hamiltonian form for equations of motion. Any dynamical theory must first be put in the Hamiltonian form before one can quantize it. Apart from the question of quantization, it is desirable to obtain physical theories such as the theory of gravitation in the Hamiltonian form, because of the great transforming power associated with this form and because it helps one to distinguish those dynamical variables that are physically important from those that merely describe the co-ordinate system.

The basic concept in Hamiltonian theory is that of a state at a given time. In a relativistic theory this is to be understood as the state on a general three-dimensional space-like surface in space-time. The Hamiltonian equations of motion determine how the dynamical variables that fix this state vary with a variation of the surface. As the surface can vary arbitrarily, subject to the condition that it remains always space-like, there is a great deal of freedom in the motion, in consequence of which the usual procedure for obtaining the Hamiltonian from an action principle does not apply, and it is necessary to use a generalized procedure, such as that given by the author (1958, preceding paper).

In setting up the theory, one may choose the surface to be independent of the system of co-ordinates x^μ. This has the advantage that one can preserve the symmetry between the four x's. But it has the disadvantage of bringing in more variables than are needed for the mathematical treatment of the problem, with the result that the equations are complicated and obscure. One gets a simpler scheme by choosing the system of co-ordinates x^μ such that the surfaces x^0 = constant are all space-like, and then considering only states on these surfaces and using the co-ordinates x^1, x^2, x^3 as parameters to label the points on these surfaces. The symmetry between the four x's is completely destroyed, but the resulting simplification makes this sacrifice well worth while.

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Einstein's theory of gravitation has already been put in the Hamiltonian form, with the surface independent of the system of co-ordinates, by Pirani & Schild (1950), using my generalized Hamiltonian procedure, and by Bergmann, Penfield, Schiller & Zatzkis (1950), using a different procedure. These papers are incomplete in that no account is taken of the χ-equations. A more recent paper by Pirani, Schild & Skinner (1952) passes over to the surface $x^0 = c$ and does bring in the χ-equations.

The present paper goes beyond the previous papers by developing the theory in such a way that some of the degrees of freedom drop out. A considerable simplification in the Hamiltonian then results. For convenience, the work is set out so as to be independent of the previous papers.

Notation

The co-ordinates of space-time are x^μ. (Greek suffixes take on the values 0, 1, 2, 3.) If f is any function of the x's, we write $\partial f/\partial x^\mu = f_\mu$. A suffix always denotes differentiation in this way if the symbol without the suffix has a meaning.

The metric is given by $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$. The determinant of the $g_{\mu\nu}$ is negative and is written $-J^2$. The Christoffel symbols are written

$$\tfrac{1}{2}(g_{\alpha\mu\beta}+g_{\beta\mu\alpha}-g_{\alpha\beta\mu}) = \Gamma_{\alpha\beta\mu}, \quad \Gamma_{\alpha\beta\mu}g^{\mu\nu} = \Gamma_{\alpha\beta}{}^{\nu}.$$

Thus
$$J_\mu/J = \tfrac{1}{2}g^{\alpha\beta}g_{\alpha\beta\mu} = \Gamma_{\mu\nu}{}^{\nu}. \tag{1}$$

Note the formula
$$g^{\alpha\beta}{}_{\rho} = -g^{\alpha\mu}g^{\beta\nu}g_{\mu\nu\rho}. \tag{2}$$

In the surface $x^0 = c$, with c a constant, a point is labelled by x^r. (Small roman suffixes take on the values 1, 2, 3.) The metric in this surface is given by $g_{rs}\,\mathrm{d}x^r\,\mathrm{d}x^s$. The determinant of the g_{rs} is negative and is written $-K^2$. We have

$$g^{00}J^2 = K^2. \tag{3}$$

Define
$$e^{\mu\nu} = e^{\nu\mu} = g^{\mu\nu} - g^{\mu 0}g^{\nu 0}/g^{00}. \tag{4}$$

Note that $e^{\mu\nu}$ is zero unless both μ and ν differ from 0. We have

$$e^{rs}g_{s\alpha} = e^{r\mu}g_{\mu\alpha} = g^r_\alpha - g^{r0}g^0_\alpha/g^{00}.$$

Thus
$$e^{rs}g_{sa} = g^r_a \tag{5}$$

and
$$e^{rs}g_{s0} = -g^{r0}/g^{00}. \tag{6}$$

Equation (5) shows that the matrix e^{rs} is the reciprocal of the matrix g_{rs}. Thus e^{rs} is the fundamental tensor to be used for raising the suffixes of tensors in the three-dimensional space $x^0 = c$. Corresponding to (1), we have

$$K_r/K = \tfrac{1}{2}e^{ab}g_{abr}. \tag{7}$$

THE ϕ EQUATIONS

The Lagrangian is

$$L = \int \mathscr{L} \, \mathrm{d}^3 x,$$

where d^3x denotes $\mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$ and $\mathscr{L}$ is the action density. For the gravitational field alone, the action density in Einstein's theory is

$$\begin{aligned}\mathscr{L}_{\mathrm{G}} &= Jg^{\mu\nu}(\Gamma_{\mu\nu}{}^{\rho}\Gamma_{\rho\sigma}{}^{\sigma} - \Gamma_{\mu\rho}{}^{\sigma}\Gamma_{\nu\sigma}{}^{\rho}) \\ &= \tfrac{1}{4}Jg_{\mu\nu\rho}g_{\alpha\beta\sigma}\{(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})g^{\rho\sigma} + 2(g^{\mu\rho}g^{\alpha\beta} - g^{\mu\alpha}g^{\beta\rho})g^{\nu\sigma}\}.\end{aligned} \tag{8}$$

To allow the gravitational field to interact with other physical things (e.g. particles of matter or the electromagnetic field or other kinds of fields), we introduce a supplementary action density $\mathscr{L}_{\mathrm{M}}$, so that

$$\mathscr{L} = \mathscr{L}_{\mathrm{G}} + \mathscr{L}_{\mathrm{M}}.$$

For the purposes of the present theory $\mathscr{L}_{\mathrm{M}}$ may be any function of the $g_{\mu\nu}$ and of any additional dynamical variables that are needed to describe the other physical things, but must not involve derivatives of the $g_{\mu\nu}$. (This condition is actually fulfilled in the case of particles and of the electromagnetic field.)

The $g_{\mu\nu}$ at all points of the surface $x^0 = c$ are taken to be the dynamical co-ordinates that describe the gravitational field (corresponding to the q's in my previous paper). The $g_{\mu\nu 0}$ at all points of the surface are then the velocities (corresponding to the $\dot{q}$'s in my previous paper). The $g_{\mu\nu r}$ at any point of the surface are fixed by the $g_{\mu\nu}$ at all points of the surface and are thus functions of the dynamical co-ordinates only, independent of the velocities. We now see that expression (8) is of the form

$$\mathscr{L}_{\mathrm{G}} = \mathscr{L}(2) + \mathscr{L}(1) + \mathscr{L}_{\mathrm{G}}(0),$$

where $\mathscr{L}(2)$ is homogeneous quadratic in the velocities, $\mathscr{L}(1)$ is homogeneous linear in the velocities and $\mathscr{L}_{\mathrm{G}}(0)$ is independent of the velocities. We get $\mathscr{L}(2)$ by putting $\rho = \sigma = 0$ in (8), and it may be written

$$\begin{aligned}\mathscr{L}(2) = \tfrac{1}{4}Jg^{00}g_{\mu\nu 0}g_{\alpha\beta 0}\{&(g^{\mu\alpha} - g^{\mu 0}g^{\alpha 0}/g^{00})(g^{\nu\beta} - g^{\nu 0}g^{\beta 0}/g^{00}) \\ &- (g^{\mu\nu} - g^{\mu 0}g^{\nu 0}/g^{00})(g^{\alpha\beta} - g^{\alpha 0}g^{\beta 0}/g^{00})\}.\end{aligned}$$

With the help of (4) it reduces to

$$\mathscr{L}(2) = \tfrac{1}{4}Jg^{00}g_{rs0}g_{ab0}(e^{ra}e^{sb} - e^{rs}e^{ab}). \tag{9}$$

The velocities $g_{\mu 00}$ do not occur here.

To define the momenta $p^{\mu\nu}$ conjugate to $g_{\mu\nu}$, we make arbitrary small variations in the velocities $g_{\mu\nu 0}$ and express the resulting variation in the Lagrangian as

$$\int \delta\mathscr{L} \, \mathrm{d}^3x = \int p^{\mu\nu}\delta g_{\mu\nu 0} \, \mathrm{d}^3x, \tag{10}$$

with the condition $p^{\mu\nu} = p^{\nu\mu}$. The usual P.b. relations between dynamical co-ordinates and their conjugate momenta in field theory must be replaced by

$$[p^{\mu\nu}, g'_{\alpha\beta}] = \tfrac{1}{2}(g^{\mu}_{\alpha}g^{\nu}_{\beta} + g^{\mu}_{\beta}g^{\nu}_{\alpha})\,\delta_3(x - x'), \tag{11}$$

in which the right-hand side has been made symmetrical between μ and ν and also between α and β. The $g'_{\alpha\beta}$ denotes the value of $g_{\alpha\beta}$ at the point x'^1, x'^2, x'^3 on the surface $x^0 = c$.

The $\mathscr{L}_M$ part of $\mathscr{L}$ will not contribute to $p^{\mu\nu}$ in (10). Since none of the terms in $\mathscr{L}_G$ that are quadratic in the velocities involves $g_{\mu 00}$, $p^{\mu 0}$ will not involve any velocities. Thus

$$p^{\mu 0} - f^\mu \approx 0, \tag{12}$$

where f^μ is a function of dynamical co-ordinates only, namely, of the $g_{\alpha\beta}$ and $g_{\alpha\beta r}$. The equations (12) are thus ϕ equations. There are four of them for each point x^r in the surface. They are the only ϕ-equations, as will be confirmed later (see equation (22)). We must find out whether they are first-class.

If we make a small change in the system of co-ordinates, say $x^\mu \to x^\mu + b^\mu$, then

$$\delta g_{\mu\nu} = g_{\mu\rho} b^\rho{}_\nu + g_{\nu\rho} b^\rho{}_\mu + g_{\mu\nu\rho} b^\rho. \tag{13}$$

Take

$$b^\rho = \tfrac{1}{2}(x^0 - c)^2 \beta^\rho$$

with β^ρ an arbitrary function of x^1, x^2, x^3. Then on the surface $x^0 = c$,

$$\delta g_{\mu\nu} = 0, \quad \delta g_{\mu\nu 0} = (g_{\mu\rho} g^0_\nu + g_{\nu\rho} g^0_\mu)\, \beta^\rho,$$

so that

$$\delta g_{rs0} = 0, \quad \delta g_{\mu 00} = \beta_\mu + g^0_\mu \beta_0.$$

Thus we can make an arbitrary change in the velocities $g_{\mu 00}$, while keeping all the dynamical co-ordinates $g_{\mu\nu}$ and the velocities g_{rs0} invariant. Thus, the equations of motion cannot restrict the velocities $g_{\mu 00}$, so the ϕ-equations (12) must be first-class.

The modified Lagrangian

By a suitable change in the Lagrangian which does not affect the equations of motion, the ϕ-equations (12) can be brought into the form

$$p^{\mu 0} \approx 0. \tag{14}$$

We shall make the change in Lagrangian corresponding to the change in action density

$$\mathscr{L}^* - \mathscr{L} = \{(Jg^{00})_v g^{v0}/g^{00}\}_0 - \{(Jg^{00})_0 g^{v0}/g^{00}\}_v, \tag{15}$$

and shall verify that it has the desired effect. This change does not affect the equations of motion, because its contribution to the action can be expressed as a surface integral.

We may write (15)

$$\mathscr{L}^* - \mathscr{L} = (Jg^{00})_v (g^{v0}/g^{00})_0 - (Jg^{00})_0 (g^{v0}/g^{00})_v. \tag{16}$$

From (1) and (2)

$$(Jg^{00})_\rho = J(\tfrac{1}{2} g^{\alpha\beta} g^{00} - g^{\alpha 0} g^{\beta 0})\, g_{\alpha\beta\rho},$$

$$(g^{v0}/g^{00})_\rho = g^{\nu 0}(g^{v0} g^{\mu 0} - g^{\mu v} g^{00})\, g_{\mu\nu\rho}/(g^{00})^2.$$

So

$$\begin{aligned}\mathscr{L}^* - \mathscr{L} &= J g_{\mu\nu 0} g_{\alpha\beta v} \{(\tfrac{1}{2} g^{\alpha\beta} g^{00} - g^{\alpha 0} g^{\beta 0})\, g^{\nu 0}(g^{v0} g^{\mu 0} - g^{\mu v} g^{00}) \\ &\quad - (\tfrac{1}{2} g^{\mu\nu} g^{00} - g^{\mu 0} g^{\nu 0})\, g^{\beta 0}(g^{v0} g^{\alpha 0} - g^{\alpha v} g^{00})\}/(g^{00})^2 \\ &= \tfrac{1}{2} J g_{\mu\nu 0} g_{\alpha\beta v} \{ g^{\mu\nu} g^{\alpha v} g^{\beta 0} - g^{\alpha\beta} g^{\mu v} g^{\nu 0} \\ &\quad + [2(g^{\mu v} g^{\alpha 0} - g^{\alpha v} g^{\mu 0})\, g^{\beta 0} g^{\nu 0} + (g^{\alpha\beta} g^{\mu 0} g^{\nu 0} - g^{\mu\nu} g^{\alpha 0} g^{\beta 0})\, g^{v0}]/g^{00}\}. \end{aligned} \tag{17}$$

This expression is linear homogeneous in the velocities, so $\mathscr{L}(2)$ and $\mathscr{L}_G(0)$ are unchanged, while $\mathscr{L}(1)$ gets changed to $\mathscr{L}^*(1)$ say. By adding on to (17) $\mathscr{L}(1)$, given by the expression (8) with $\rho = 0$, $\sigma = v$ plus this expression with $\rho = v$, $\sigma = 0$, we get

$$\mathscr{L}^*(1) = \tfrac{1}{2}Jg_{\mu v0}g_{\alpha\beta v}\{(g^{\mu\alpha}g^{v\beta} - g^{\mu v}g^{\alpha\beta})g^{v0} + 2(g^{\mu v}g^{\alpha v} - g^{\mu\alpha}g^{vv})g^{\beta 0} + [2(g^{\mu v}g^{\alpha 0} - g^{\alpha v}g^{\mu 0})g^{\beta 0}g^{v0} + (g^{\alpha\beta}g^{\mu 0}g^{v0} - g^{\mu v}g^{\alpha 0}g^{\beta 0})g^{v0}]/g^{00}\},$$

which reduces to

$$\mathscr{L}^*(1) = \tfrac{1}{2}Jg_{\mu v0}g_{\alpha\beta v}\{(e^{\mu\alpha}e^{v\beta} - e^{\mu v}e^{\alpha\beta})g^{v0} + 2(e^{\mu v}e^{\alpha v} - e^{\mu\alpha}e^{vv})g^{\beta 0}\}. \tag{18}$$

All the terms here vanish except those for which μ, v, α all differ from zero. The velocities $g_{\mu 00}$ thus do not occur in $\mathscr{L}^*$, which verifies that $\mathscr{L}^* - \mathscr{L}$ was correctly chosen.

By adding the expressions (9) and (18), we get

$$\begin{aligned}\mathscr{L}(2) + \mathscr{L}^*(1) &= \tfrac{1}{4}J(e^{ra}e^{sb} - e^{rs}e^{ab})\{g_{rs0}g_{ab0}g^{00} + 2g_{rs0}g_{abv}g^{v0} - 4g_{rs0}g_{a\beta b}g^{\beta 0}\}\\ &= \mathscr{L}_X + \tfrac{1}{4}J(e^{ra}e^{sb} - e^{rs}e^{ab})\{g_{rs0}g^{00} + g_{rsu}g^{u0} - (g_{r\alpha s} + g_{s\alpha r})g^{\alpha 0}\}\\ &\qquad \times\{g_{ab0}g^{00} + g_{abv}g^{v0} - (g_{a\beta b} + g_{b\beta a})g^{\beta 0}\}/g^{00},\end{aligned}$$

where

$$\mathscr{L}_X = -\tfrac{1}{4}J(r^{ra}e^{sb} - e^{rs}e^{ab})\{g_{rsu}g^{u0} - (g_{r\alpha s} + g_{s\alpha r})g^{\alpha 0}\}\{g_{abv}g^{v0} - (g_{a\beta b} + g_{b\beta a})g^{\beta 0}\}/g^{00} \tag{19}$$

and is a function only of the dynamical co-ordinates. Now

$$g_{rs0}g^{00} + g_{rsu}g^{u0} - (g_{r\alpha s} + g_{s\alpha r})g^{\alpha 0} = -2\Gamma_{rs}{}^{0},$$

so we get
$$\mathscr{L}(2) + \mathscr{L}^*(1) = \mathscr{L}_X + J(e^{ra}e^{sb} - e^{rs}e^{ab})\Gamma_{rs}{}^{0}\Gamma_{ab}{}^{0}/g^{00}. \tag{20}$$

To obtain the momenta p^{rs} with the modified Lagrangian, we must vary the velocities g_{rs0} in $\mathscr{L}^*$ and substitute in (10) with $\mathscr{L}^*$ for $\mathscr{L}$. All the terms of $\mathscr{L}^*$ that involve g_{rs0} are included in (20) and give

$$\begin{aligned}\delta\mathscr{L}^* &= 2J(e^{ra}e^{sb} - e^{rs}e^{ab})\Gamma_{ab}{}^{0}\delta\Gamma_{rs}{}^{0}/g^{00}\\ &= -J(e^{ra}e^{sb} - e^{rs}e^{ab})\Gamma_{ab}{}^{0}\delta g_{rs0}.\end{aligned}$$

Hence
$$p^{rs} = J(e^{rs}e^{ab} - e^{ra}e^{sb})\Gamma_{ab}{}^{0}. \tag{21}$$

This formula may be inverted to give

$$J\Gamma_{ab}{}^{0} = (\tfrac{1}{2}g_{rs}g_{ab} - g_{ra}g_{sb})p^{rs}. \tag{22}$$

Equations (21) and (22) provide the connexion between the gravitational momenta and velocities. Since the velocities g_{ab0} are all expressible by (22) in terms of the momenta, there can be no ϕ-equations for the gravitational variables other than (14).

The general equation of motion

We have seen that the $g_{\mu 0}$ can vary arbitrarily with time, unrestricted by equations of motion. Let us see how these variables enter into the general equation of motion.

Take a dynamical variable η on the surface $x^0 = c$ such that η depends only on the g_{rs} and on variables needed to describe any matter that is present and fields other than the gravitational field, and does not depend on the $g_{\mu 0}$. η may be localized at one point x'^r of the surface or may be non-localized.

Make a small displacement of the surface by applying to each point of it the displacement $x^\mu \to x^\mu + a^\mu$ with a^μ small and a function of x^1, x^2, x^3. The change in η will be linear in the functions a^μ and thus of the form

$$\delta\eta = \int \xi_\mu a^\mu \, \mathrm{d}^3 x \tag{23}$$

where the ξ_μ are certain functions of x^1, x^2, x^3 independent of the a^μ. We have evidently

$$\dot{\eta} = \int \xi_0 \, \mathrm{d}^3 x. \tag{24}$$

Let l^μ be the unit normal to the surface $x^0 = c$ at any point x^r on it, so that

$$l^\mu g_{\mu r} = 0, \quad l^\mu l_\mu = 1.$$

Thus
$$l^\mu = g^{\mu 0} g^{00^{-\frac{1}{2}}}. \tag{25}$$

Put
$$\xi_{\mathrm{L}} = l^\mu \xi_\mu = g^{\mu 0} g^{00^{-\frac{1}{2}}} \xi_\mu,$$

so that
$$\xi_0 = g^{00^{-\frac{1}{2}}} \xi_{\mathrm{L}} - g^{s0} \xi_s / g^{00}$$
$$= g^{00^{-\frac{1}{2}}} \xi_{\mathrm{L}} + g_{r0} e^{rs} \xi_s \tag{26}$$

with the help of (6). Thus (24) becomes

$$\dot{\eta} = \int (g^{00^{-\frac{1}{2}}} \xi_{\mathrm{L}} + g_{r0} r^{rs} \xi_s) \, \mathrm{d}^3 x. \tag{27}$$

Now ξ_{L} and ξ_s are determined by $\delta\eta$, given by (23), for normal and tangential displacements of the surface, respectively, and are thus independent of the $g_{\mu 0}$. So equation (27) shows explicitly how the $g_{\mu 0}$ enter into the equation of motion for η.

We can infer that the Hamiltonian must be of the form

$$H = \int (g^{00^{-\frac{1}{2}}} \mathscr{H}_{\mathrm{L}} + g_{r0} e^{rs} \mathscr{H}_s) \, \mathrm{d}^3 x, \tag{28}$$

where $\mathscr{H}_{\mathrm{L}}$ and $\mathscr{H}_s$ are independent of the $g_{\mu 0}$, because this makes the equation of motion $\dot{\eta} = [\eta, H]$ become (27) provided

$$[\eta, \mathscr{H}_{\mathrm{L}}] = \xi_{\mathrm{L}}, \quad [\eta, e^{rs} \mathscr{H}_s] = e^{rs} \xi_s. \tag{29}$$

From the requirement $[p^{\mu 0}, H] = 0$ we now get the χ-equations

$$\mathscr{H}_{\mathrm{L}} \approx 0, \quad \mathscr{H}_s \approx 0, \tag{30}$$

which have to be first-class in order that the equations of motion with arbitrary $g_{\mu 0}$ may be consistent.

The above argument leading to the form (28) for the Hamiltonian applies generally for any relativistic dynamical theory in a Riemann space. It follows that the parts of the Hamiltonian due to the gravitational field and to the matter must separately be of the form (28), say

$$H_{\mathrm{G}} = \int (g^{00^{-\frac{1}{2}}} \mathscr{H}_{\mathrm{GL}} + g_{r0} e^{rs} \mathscr{H}_{\mathrm{G}s})\, \mathrm{d}^3x, \tag{31}$$

$$H_{\mathrm{M}} = \int (g^{00^{-\frac{1}{2}}} \mathscr{H}_{\mathrm{ML}} + g_{r0} e^{rs} \mathscr{H}_{\mathrm{M}s})\, \mathrm{d}^3x, \tag{32}$$

with $\mathscr{H}_{\mathrm{GL}}$, $\mathscr{H}_{\mathrm{G}s}$, $\mathscr{H}_{\mathrm{ML}}$, $\mathscr{H}_{\mathrm{M}s}$ all independent of the $g_{\mu 0}$.

The Hamiltonian

The gravitational part of the Hamiltonian is, by definition,

$$\begin{aligned} H_{\mathrm{G}} &= \int (p^{rs} g_{rs0} - \mathscr{L}^*_{\mathrm{G}})\, \mathrm{d}^3x \\ &= \int p^{rs}(g_{rs0} + \Gamma_{rs}{}^0/g^{00})\, \mathrm{d}^3x - \int \{\mathscr{L}_{\mathrm{X}} + \mathscr{L}_{\mathrm{G}}(0)\}\, \mathrm{d}^3x \end{aligned} \tag{33}$$

from (20) and (21). Now

$$g_{rs0} + \Gamma_{rs}{}^0/g^{00} = g_{r0s} + g_{s0r} + (2g^{v0}\Gamma_{rsv} - \Gamma_{rs}{}^0)/g^{00},$$

so the first term of (33) becomes, with the help of (22) and (6),

$$\begin{aligned} &\int \{(g^{00}J)^{-1}(g_{ra}g_{sb} - \tfrac{1}{2}g_{rs}g_{ab})\, p^{rs}p^{ab} - 2p^{rs}{}_{s}g_{r0} - 2g_{u0}e^{uv}p^{rs}\Gamma_{rsv}\}\, \mathrm{d}^3x \\ &\quad = \int \{g^{00-\frac{1}{2}}K^{-1}(g_{ra}g_{sb} - \tfrac{1}{2}g_{rs}g_{ab})\, p^{rs}p^{ab} + g_{u0}e^{uv}[p^{rs}g_{rsv} - 2(p^{rs}g_{rv})_s]\}\, \mathrm{d}^3x, \end{aligned} \tag{34}$$

with the help of (3).

The second term of (33) is very complicated and a great deal of labour would be needed to calculate it directly. We know, however, from (31), that it must be of the form

$$-\int \{\mathscr{L}_{\mathrm{X}} + \mathscr{L}_{\mathrm{G}}(0)\} \mathrm{d}^3x = \int (g^{00^{-\frac{1}{2}}} X_{\mathrm{L}} + g_{r0} X^r)\, \mathrm{d}^3x, \tag{35}$$

where X_{L} and X^r are functions only of the g_{rs} and their spatial derivatives. We see at once that X^r must vanish, because it cannot contain a suffix 0, which would be needed to balance the suffix 0 of its coefficient g_{r0}. We can proceed to calculate X_{L} with the simplifying assumption $g_{r0} = 0$, which makes

$$g^{r0} = 0, \quad g^{rs} = e^{rs}, \quad g^{00} = g_{00}{}^{-1}. \tag{36}$$

We now have $\mathscr{L}_{\mathrm{X}}$, given by (19), vanishing, while $\mathscr{L}_{\mathrm{G}}(0)$, given by (8) with $\rho = u$, $\sigma = v$, becomes

$$\mathscr{L}_{\mathrm{G}}(0) = g^{00^{-\frac{1}{2}}} B + \tfrac{1}{2} J g_{rsu} g_{00v} g^{00} (e^{ru}e^{sv} - e^{rs}e^{uv}), \tag{37}$$

where

$$B = \tfrac{1}{4} K g_{rsu} g_{abv} \{(e^{ra}e^{sb} - e^{rs}e^{ab})\, e^{uv} + 2(e^{ru}e^{ab} - e^{ra}e^{bu})\, e^{sv}\}. \tag{38}$$

With the help of

$$(g^{00^{-\frac{1}{2}}})_v = (g_{00}{}^{\frac{1}{2}})_v = \tfrac{1}{2} g_{00v} g^{00^{\frac{1}{2}}},$$

which is a consequence of (36), the last term of (37) becomes

$$(g^{00^{-\frac{1}{2}}})_v\, K g_{rsu}(e^{ru}e^{sv} - e^{rs}e^{uv}),$$

so we find, on substituting (37) into (35),

$$X_{\mathrm{L}} = -B + \{Kg_{rsu}(e^{ru}e^{sv} - e^{rs}e^{uv})\}_v. \tag{39}$$

As both sides of this equation do not depend on g_{r0} at all, the equation must hold also when g_{r0} does not vanish.

By adding the expressions (34), (35), (32) and using (39), we find that H is given by (28) with

$$\mathscr{H}_{\mathrm{L}} = K^{-1}(g_{ra}g_{sb} - \tfrac{1}{2}g_{rs}g_{ab})\,p^{rs}p^{ab} - B + \{Kg_{rsu}(e^{ru}e^{sv} - e^{rs}e^{uv})\}_v + \mathscr{H}_{\mathrm{ML}} \approx 0, \tag{40}$$

$$\mathscr{H}_v = p^{rs}g_{rsv} - 2(p^{rs}g_{rv})_s + \mathscr{H}_{\mathrm{M}v} \approx 0. \tag{41}$$

These equations provide the Hamiltonian for the gravitational field in interaction with matter. They involve, apart from the matter variables, only six degrees of freedom, described by g_{rs}, p^{rs}, for each point of the surface on which the state is considered. The $g_{\mu0}$ appear in the theory only through the variables g_{r0}, $(g^{00})^{-\frac{1}{2}}$, which occur as arbitrary coefficients in the equations of motion, leading to arbitrary functions in the general solution of the equations of motion.

The weak-field approximation

In practical problems the curvature of space-time is extremely small. This means that, provided the system of co-ordinates is suitably chosen, the $g_{\mu\nu}$ differ from their values in special relativity by small quantities, of order ϵ say, and all derivatives of $g_{\mu\nu}$ are of order ϵ. From (21), the p^{rs} are also of order ϵ.

With neglect of terms of order ϵ^2, the χ-equations (40), (41) become

$$g_{rsrs} - g_{rrss} + \mathscr{H}_{\mathrm{ML}} \approx 0, \tag{42}$$

$$2p^{rs}{}_s + \mathscr{H}_{\mathrm{M}r} \approx 0. \tag{43}$$

With neglect of terms of order ϵ^3, the Hamiltonian given by (28), (40), (41), (38) becomes

$$H = \int \{p^{rs}p^{rs} - \tfrac{1}{2}p^{rr}p^{ss} + \tfrac{1}{4}g_{rsu}g_{rsu} - \tfrac{1}{4}g_{rru}g_{ssu} + \tfrac{1}{2}g_{rsr}g_{uus} - \tfrac{1}{2}g_{rsu}g_{rus} + \mathscr{H}_{\mathrm{ML}}\}\,\mathrm{d}^3x$$
$$+ \int (g^{00^{-\frac{1}{2}}} - 1)\,(g_{rsrs} - g_{rrss} + \mathscr{H}_{\mathrm{ML}})\,\mathrm{d}^3x - \int g_{r0}(2p^{rs}{}_s + \mathscr{H}_{\mathrm{M}r})\,\mathrm{d}^3x. \tag{44}$$

The first term of (44) is an ordinary Hamiltonian, involving only effective dynamical variables and not involving the arbitrary $g_{\mu0}$. The second and third terms consist of arbitrary linear combinations, with coefficients of order ϵ, of the χ-functions (42) and (43). These terms are needed in the Hamiltonian in order to bring into the equations of motion the arbitrariness associated with our being able to make changes of order ϵ in the system of co-ordinates.

Equations (42) and (43) show that $\mathscr{H}_{\mathrm{ML}}$ and $\mathscr{H}_{\mathrm{M}s}$ are of order ϵ. Their smallness arises, of course, from their containing the constant of gravitation. $\mathscr{H}_{\mathrm{ML}}$ is now seen to be the largest term in the first integrand of (44), all the others being of order ϵ^2. This results in the motion being mainly determined by the Hamiltonian density $\mathscr{H}_{\mathrm{ML}}$, with gravitational effects producing only a small perturbation.

Let us obtain the contribution to the Hamiltonian of a particle of rest-mass m. Let z^r be the co-ordinates of the particle and let $g_{Z\mu\nu}$ denote $g_{\mu\nu}$ at the point $x^r = z^r$. Then we have

$$L_M = -m(g_{Z\alpha\beta}\dot{z}^\alpha\dot{z}^\beta)^{\frac{1}{2}}, \tag{45}$$

in which $\dot{z}^0$ is unity and is not to be counted as a dynamical variable. The momentum P_r conjugate to z^r is

$$P_r = -m g_{Z\mu r}\dot{z}^\mu(g_{Z\alpha\beta}\dot{z}^\alpha\dot{z}^\beta)^{-\frac{1}{2}}. \tag{46}$$

This leads to

$$\begin{aligned} e_Z^{rs}P_rP_s &= m^2(g_Z^{\rho\sigma} - g_Z^{\rho 0}g_Z^{\sigma 0}/g_Z^{00})\, g_{Z\mu\rho}g_{Z\nu\sigma}\dot{z}^\mu\dot{z}^\nu(g_{Z\alpha\beta}\dot{z}^\alpha\dot{z}^\beta)^{-1} \\ &= m^2 - m^2(g_Z^{00}g_{Z\alpha\beta}\dot{z}^\alpha\dot{z}^\beta)^{-1}. \end{aligned} \tag{47}$$

The contribution of the particle to the Hamiltonian is

$$\begin{aligned} H_M &= P_r\dot{z}^r - L_M \\ &= m(g_{Z\mu\nu}\dot{z}^\mu\dot{z}^\nu - g_{Z\mu r}\dot{z}^\mu\dot{z}^r)\,(g_{Z\alpha\beta}\dot{z}^\alpha\dot{z}^\beta)^{-\frac{1}{2}} \\ &= m g_{Z\mu 0}\dot{z}^\mu(g_{Z\alpha\beta}\dot{z}^\alpha\dot{z}^\beta)^{-\frac{1}{2}} \\ &= m(e_Z^{rs}g_{Zs0}g_{Z\mu r} + g_\mu^0/g_Z^{00})\,\dot{z}^\mu(g_{Z\alpha\beta}\dot{z}^\alpha\dot{z}^\beta)^{-\frac{1}{2}}, \end{aligned}$$

with the help of (6). So, using (46) and (47), we get

$$H_M = -g_{Zs0}e_Z^{rs}P_r + g_Z^{00^{-\frac{1}{2}}}(m^2 - e_Z^{rs}P_rP_s)^{\frac{1}{2}}. \tag{48}$$

Substituting in (32), we find

$$\mathscr{H}_{ML} = (m^2 - e^{rs}P_rP_s)^{\frac{1}{2}}\,\delta_3(x-z), \tag{49}$$

$$\mathscr{H}_{Mr} = -P_r\delta_3(x-z). \tag{50}$$

These are the contributions of a single particle to $\mathscr{H}_{ML}$, $\mathscr{H}_{Mr}$. For several particles interacting only through the gravitational field, one would merely have to sum the contributions of each particle separately. The results (49), (50) were obtained without the use of the weak field approximation, but they could not be used in the exact theory, because they refer to a point particle, which gives rise to a singularity in the field where the field equations cannot be satisfied accurately.

Gravitational waves

With the weak-field approximation, let us consider gravitational waves in the absence of matter, or in the presence only of matter that is moving slowly. Then from (50), $\mathscr{H}_{Mr}$ is small and may be neglected, so the χ-equation (43) takes on the simple form

$$p^{rs}{}_{,s} \approx 0. \tag{51}$$

Make a Fourier resolution of all field quantities in the three-dimensional space x^1, x^2, x^3, thus

$$\left.\begin{aligned} g_{rs} &= -\delta_{rs} + \int g_{\mathbf{k}rs}\,\mathrm{e}^{\mathrm{i}(\mathbf{kx})}\,\mathrm{d}^3k, \\ p^{rs} &= \int p_{\mathbf{k}}^{rs}\,\mathrm{e}^{\mathrm{i}(\mathbf{kx})}\,\mathrm{d}^3k, \\ \mathscr{H}_{ML} &= \int \rho_{\mathbf{k}}\,\mathrm{e}^{\mathrm{i}(\mathbf{kx})}\,\mathrm{d}^3k. \end{aligned}\right\} \tag{52}$$

In each case the Fourier coefficient with the suffix $-\mathbf{k}$ is the conjugate complex of that with the suffix $\mathbf{k}$.

For the sake of a simple discussion, let us fix our attention on waves travelling in the direction of the axis $\mathbf{x}^3$, so that $\mathbf{k}_1 = \mathbf{k}_2 = 0$. Then equation (51) gives

$$p_{\mathbf{k}}^{13} \approx 0, \quad p_{\mathbf{k}}^{23} \approx 0, \quad p_{\mathbf{k}}^{33} \approx 0, \tag{53}$$

and equation (42) gives

$$\mathbf{k}^2(g_{\mathbf{k}11} + g_{\mathbf{k}22}) + \rho_{\mathbf{k}} \approx 0. \tag{54}$$

It is reasonable, in the case of weak gravitational fields, to assume the total energy to be equal to the Hamiltonian (44). Only the first term of (44) contributes anything, as the other terms vanish weakly. The energy of the gravitational field is then given by the first term of (44) without the term $\mathscr{H}_{\mathrm{ML}}$ in the integrand. When we substitute for g_{rs}, p^{rs} their Fourier expansions (52), this energy becomes the sum of a number of terms each of the form

$$\iiint f_{\mathbf{k}} f_{\mathbf{k}'} \,\mathrm{e}^{\mathrm{i}(\mathbf{kx})}\,\mathrm{e}^{\mathrm{i}(\mathbf{k}'\mathbf{x})}\,\mathrm{d}^3k\,\mathrm{d}^3k'\,\mathrm{d}^3x = 8\pi^3 \int f_{\mathbf{k}} f_{-\mathbf{k}}\,\mathrm{d}^3k.$$

The contribution to this energy of waves moving in the direction x^3 is, with neglect of the factor $8\pi^3$,

$$E_{(3)} = p_{\mathbf{k}}^{rs} p^{rs}_{-\mathbf{k}} - \tfrac{1}{2} p_{\mathbf{k}}^{rr} p^{ss}_{-\mathbf{k}} + \tfrac{1}{4}\mathbf{k}^2(g_{\mathbf{k}rs} g_{-\mathbf{k}rs} - g_{\mathbf{k}rr} g_{-\mathbf{k}ss} + g_{\mathbf{k}33} g_{-\mathbf{k}uu} + g_{\mathbf{k}uu} g_{-\mathbf{k}33} - 2 g_{\mathbf{k}r3} g_{-\mathbf{k}r3}). \tag{55}$$

The last term here reduces to

$$\begin{aligned}
&\tfrac{1}{4}\mathbf{k}^2(2g_{\mathbf{k}12} g_{-\mathbf{k}12} - g_{\mathbf{k}11} g_{-\mathbf{k}22} - g_{\mathbf{k}22} g_{-\mathbf{k}11}) \\
&\quad = \tfrac{1}{4}\mathbf{k}^2\{2g_{\mathbf{k}12} g_{-\mathbf{k}12} + \tfrac{1}{2}(g_{\mathbf{k}11} - g_{\mathbf{k}22})(g_{-\mathbf{k}11} - g_{-\mathbf{k}22}) - \tfrac{1}{2}(g_{\mathbf{k}11} + g_{\mathbf{k}22})(g_{-\mathbf{k}11} + g_{+\mathbf{k}22})\}.
\end{aligned}$$

So (55) becomes, with the help of (53) and (54),

$$\begin{aligned}
E_{(3)} \approx{} & 2p_{\mathbf{k}}^{12} p^{12}_{-\mathbf{k}} + \tfrac{1}{2}(p_{\mathbf{k}}^{11} - p_{\mathbf{k}}^{22})(p^{11}_{-\mathbf{k}} - p^{22}_{-\mathbf{k}}) \\
& + \tfrac{1}{4}\mathbf{k}^2\{2g_{\mathbf{k}12} g_{-\mathbf{k}12} + \tfrac{1}{2}(g_{\mathbf{k}11} - g_{\mathbf{k}22})(g_{-\mathbf{k}11} - g_{-\mathbf{k}22})\} - \tfrac{1}{8}\mathbf{k}^{-2}\rho_{\mathbf{k}}\rho_{-\mathbf{k}}.
\end{aligned} \tag{56}$$

The above analysis enables one to see the significance of the various degrees of freedom of the gravitational field when resolved into its Fourier components. The χ-equations (53) and (54) show that variables in the degrees of freedom 13, 23, 33, $(11+22)$ are affected by changes in the co-ordinate system, those in the degrees of freedom 12, $(11-22)$ are invariant under changes in the co-ordinate system. The expression (56) for the energy splits up into terms each associated with one of these six degrees of freedom, without any cross terms associated with two of them.

The degrees of freedom 13, 23, 33 do not appear at all in (56). The degrees of freedom 12, $(11-22)$ contribute a positive definite amount to (56), of such a form as to represent the energy of gravitational waves. These two degrees of freedom correspond, in the language of quantum theory, to gravitational photons with spin values ± 2 in their direction of motion.

The degree of freedom $(11+22)$ gives rise to the last term of (56). If one takes for $\rho_{\mathbf{k}}$ its value according to (49) for a number of slowly moving particles, namely

$$\rho_{\mathbf{k}} = (2\pi)^{-3}\,\Sigma m\,\mathrm{e}^{-\mathrm{i}(\mathbf{kz})},$$

and transforms this term back to the x-variables, it becomes just the Newtonian potential energy of each pair of particles, plus a self-energy term for each particle, representing the energy of the Newtonian field around it. The last term of (56) is negative definite, showing that the gravitational force between two positive masses is attractive and that the gravitational self-energy of every mass is negative.

The co-ordinate-independent degrees of freedom 12 and $(11-22)$ are the only physically essential ones. The others drop out of the Hamiltonian equations of motion. This is a general feature of the weak field approximation. Even without the restriction to slowly moving particles, when $\mathscr{H}_{\mathrm{M}r}$ is not negligible, one could use the χ-equations (42) and (43) to eliminate the degrees of freedom 13, 23, 33, $(11+22)$ by a contact transformation, analogous to the transformation that eliminates the longitudinal waves in electrodynamics.

Conclusion

The exact Hamiltonian for the theory of gravitation, given by equations (28), (40), (41), turns out to be rather simpler than one might have expected. One starts with ten degrees of freedom for each point in space, corresponding to the ten $g_{\mu\nu}$, but one finds with the method here followed that some drop out, leaving only six, corresponding to the six g_{rs}. *This is a substantial simplification, but it can be obtained only at the expense of giving up four-dimensional symmetry.* I am inclined to believe from this that four-dimensional symmetry is not a fundamental property of the physical world.

Einstein's great achievement was to show that each individual solution of the equations of motion that constitute the laws of nature exhibits four-dimensional symmetry. However, we now know that a physical state does not correspond to an individual solution of the equations of motion, but to a family of solutions all related to the same Hamilton's principal function—it is such a family that corresponds to a wave function in the quantum theory, while the individual solution has no quantum analogue. For dealing with the family one must use Hamiltonian methods. The present paper shows that Hamiltonian methods, if expressed in their simplest form, *force one to abandon the four-dimensional symmetry.*

From the mathematical point of view the loss of four-dimensional symmetry is to be regretted merely because it means a loss of transforming possibilities in the equations. It is amply compensated for by the increase in transforming possibilities arising from one's being able to make contact transformations in the Hamiltonian equations.

It would be permissible to look upon the Hamiltonian form as the fundamental one, and there would then be no fundamental four-dimensional symmetry in the theory. One would have a Hamiltonian built up from four weekly vanishing functions, given by (40) and (41). The usual requirement of four-dimensional symmetry in physical laws would then get replaced by the requirement that the functions have weakly vanishing P.b.'s, so that they can be provided with arbitrary coefficients in the equations of motion, corresponding to an arbitrary motion of the surface on which the state is defined.

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