The Adiabatic Invariance of the Quantum Integrals.

By P. A. M. Dirac, St. John's College, Cambridge.

(Communicated by Prof. Sir E. Rutherford, F.R.S.—Received December 19, 1924.)

§ 1. *Introduction.*

The postulate of the existence of stationary states in multiply periodic dynamical systems requires that if the condition of such a system, when quantised, is changed in any way by the application of an external field or by the alteration of one of the internal constraints, the new state of the system must also be correctly quantised. It follows that the laws of classical mechanics cannot in general be true, even approximately, during the transition. There is one kind of change, however, during which one may expect the classical laws to hold, namely, the so-called adiabatic change, which takes place infinitely slowly and regularly, so that the system practically remains multiply periodic all the time. In this case the quantum numbers cannot change, and it should be possible to deduce from the classical laws that the quantum integrals remain invariant. This was attempted by Burgers,* who showed that they are invariant provided there are no linear relations of the type

$$\Sigma_r m_r \omega_r = 0 \tag{1}$$

between the frequencies ω_r of the system, where the m_r are integers. In general, however, the frequencies will alter during the adiabatic change, and in so doing will pass through an infinity of values for which relations such as (1) hold. A closer investigation is therefore necessary, as was pointed out by Burgers himself.

In the following work, conditions which are rigorously sufficient to ensure the invariance of the quantum integrals, are obtained in such a form that it is possible for one to see whether they are satisfied or not without having to integrate the equations of adiabatic motion.

§ 2. *The Equations of Adiabatic Motion.*

Let q_r, p_r ($r = 1, 2 \ldots n$) be a set of Hamiltonian co-ordinates of a multiply periodic dynamical system of n degrees of freedom. The quantum integrals J_r and their conjugate angle variables w_r form another set of Hamiltonian co-ordinates connected with the first by the contact transformation

$$\Sigma\, p_r dq_r = \Sigma\, J_r dw_r + dS. \tag{2}$$

* Burgers, 'Proc. Amsterdam Roy. Acad. of Sciences,' vol. 20, p. 163.

Also the p_r and q_r, when expressed in terms of the J_r and w_r, are multiply periodic in the w_r with periods unity, and the Hamiltonian function H, when similarly expressed, is a function of the J_r only.

Regarding S and the q_r as functions of the J_r and w_r, we have

$$\Sigma\, p_r \frac{\partial q_r}{\partial w_s} = J_s + \frac{\partial S}{\partial w_s}$$

and

$$\Sigma\, p_r \frac{\partial q_r}{\partial J_s} = \frac{\partial S}{\partial J_s}.$$

The first of these equations shows that S is equal to a periodic function of the w_r, plus terms of the form $-\Sigma\,(J_r + \alpha_r)\, w_r$, where the α_r are independent of the w_r. The second now shows that the $(J_r + \alpha_r)$ are independent of the J_r. The $(J_r + \alpha_r)$ are therefore constants, and can be made zero by a suitable choice of the arbitrary constants that may be added to the J_r, as shown by Burgers. This makes S consist entirely of a periodic function of the w_r.

A change in the condition of the system is represented mathematically by the continuous variation (from a_1 to a_2, say) of a parameter (a) occurring in the Hamiltonian function H, the Hamiltonian equations remaining valid, though H may involve $\dot{a}$. An adiabatic change is the limiting case when $\dot{a}$ tends to zero in such a way that

$$\ddot{a}/\dot{a} \to 0,^*$$

which makes

$$d\dot{a}/da \to 0.$$

With (a) varying, equation (2) must be replaced by

$$\Sigma\, p_r \left(dq_r - \frac{\partial q_r}{\partial a}\, \dot{a}\, dt\right) = \Sigma\, J_r dw_r + dS - \frac{\partial S}{\partial a}\, \dot{a}\, dt.$$

This gives

$$\Sigma\, p_r dq_r - H dt = \Sigma\, J_r dw_r - \overline{H} dt + dS$$

where

$$\overline{H} = H - \Sigma\, p_r \frac{\partial q_r}{\partial a}\, \dot{a} + \frac{\partial S}{\partial a}\, \dot{a}.$$

Hence the transformation from p_r, q_r, H to J_r, w_r, $\overline{H}$ is a contact transformation, so that

$$\dot{J}_K = -\frac{\partial \overline{H}}{\partial w_K}, \quad \dot{w}_K = \frac{\partial \overline{H}}{\partial J_K}. \tag{3}$$

For small values of $\dot{a}$, H can be expanded in powers of $\dot{a}$. Thus we can put

$$H = H_0 + \dot{a} H_1.$$

* This relation cannot hold at the ends of the range of (a) from a_1 to a_2, but we may make the intervals of (a) during which it does not hold as small as we please.

Here H_0 is the value of the Hamiltonian function when (a) is constant, so that

$$\frac{\partial H_0}{\partial w_K} = 0, \quad \frac{\partial H_0}{\partial J_K} = \omega_K,$$

where the ω_K may be considered as the instantaneous frequencies.

Let

$$F = \Sigma\, p_r \frac{\partial q_r}{\partial a} - \frac{\partial S}{\partial a} - H_1$$

so that

$$\overline{H} = H_0 - \dot{a} F.$$

H_1, being a function of the p_r and q_r, must be a periodic function of the w_r of period unity. Hence F is also a periodic function of the w_r of period unity, so we can write

$$\begin{aligned} 2\pi F &= \Sigma_{m_1 \ldots m_n} C_{m_1 \ldots m_n} \sin 2\pi\,(m_1 w_1 + m_2 w_2 + \ldots + m_n w_n + \gamma_{m_1 \ldots m_n}) \\ &= \Sigma_m C_m \sin 2\pi\,(W_m + \gamma_m), \end{aligned} \tag{4}$$

say, W_m being equivalent to $\Sigma m_r w_r$, and the summation being taken over all integral values of m that make W_m positive. The coefficients C_m are functions of the J_K, a and $\dot{a}$. We shall require the series $\Sigma m_K C_m$ to be absolutely and uniformly convergent, as is the case in general with continuous forces.

Equations (3) now become

$$\dot{J}_K = \dot{a}\frac{\partial F}{\partial w_K}, \quad \dot{w}_K = \omega_K - \dot{a}\frac{\partial F}{\partial J_K},$$

which may be written, when $\dot{a}$ does not vanish,

$$\frac{dJ_K}{da} = \frac{\partial F}{\partial w_K} \tag{5}$$

$$\frac{dw_K}{da} = \frac{\omega_K}{\dot{a}} - \frac{\partial F}{\partial J_K}. \tag{6}$$

These are the exact equations of adiabatic motion. Burgers derived them on the assumption that H does not contain $\dot{a}$ explicitly, but pointed out that it may be sufficient to assume that H involves $\dot{a}$ only through higher powers than the first.* The present method shows, however, that such assumptions are unnecessary, as $\partial H/\partial \dot{a}$ can be absorbed in F.

§ 3. *The Conditions for Adiabatic Invariance.*

Suppose for definiteness that $\dot{a}$ is positive throughout the range of (a) from a_1 to a_2, except at the ends, where it vanishes. Equations (4) and (5) show that

$$\left|\frac{dJ_K}{da}\right| \leqq \Sigma_m |m_K C_m|,$$

* *Loc. cit.*, p. 169.

so that the dJ_K/da remain bounded as $\dot{a}$ tends to zero. Hence the J_K, and therefore also the C_m, dC_m/da, $d\gamma_m/da$, and $d\omega_K/da$, are also bounded.

Let a' to $a' + \delta a$ be a small interval of the range a_1 to a_2, ultimately to be made to contract to zero about a certain point a_0. Integrating equation (5) through this interval, we get

$$\delta J_K = \int_{a'}^{a'+\delta a} \frac{\partial F}{\partial w_K} da = \sum_m^{\infty} \int_{a'}^{a'+\delta a} m_K C_m \cos 2\pi (W_m + \gamma_m) da$$

$$= \sum_m^{\infty} m_K C_m \int_{a'}^{a'+\delta a} \cos 2\pi (W_m + \gamma_m) da + \delta a . \varepsilon(\delta a), \tag{7}$$

where $\varepsilon(x)$ is used to denote a quantity which tends to zero as x tends to zero. The C_m in (7) are the values of the coefficients for the point $a = a_0$, and the γ_m are also supposed to have their values for this point.

Let M be a large positive number, ultimately to be made to tend to infinity, and let m_0 denote the largest numerical value of the m's in any specified set. We can divide the series on the right-hand side of (7) into two parts, one comprising only those sets of m's whose m_0 is less than M, and the other comprising those sets of m's whose m_0 is greater than or equal to M. This second part tends to zero as M tends to infinity, on account of the absolute convergence of the series $\Sigma m_K C_m$, and so it must be of the form $\delta a\, \varepsilon(1/M)$. Thus

$$\delta J_K = \sum_m^{M} m_K C_m \int_{a'}^{a'+\delta a} \cos 2\pi (W_m + \gamma_m) da + \delta a\, \varepsilon(\delta a) + \delta a\, \varepsilon(1/M),$$

so that we are left with only a finite series with which to deal.

Consider now the contribution to δJ_K of a set of m's for which

$$\left| \frac{dW_m}{da} \right| > \frac{m_0^{\frac{1}{2}}}{\eta} \tag{α}$$

throughout the interval δa, where η is a small positive quantity which will ultimately be made to tend to zero in a certain way. We have

$$m_K C_m \int_{a'}^{a'+\delta a} \cos 2\pi (W_m + \gamma_m) da = m_K C_m \int_{a'}^{a'+\delta a} \left[\cos 2\pi (W_m + \gamma_m) dW_m \Big/ \frac{dW_m}{da} \right].$$

Now

$$\left| \frac{d}{da} \left(1 \Big/ \frac{dW_m}{da} \right) \right| = \left| \frac{d^2W_m}{da} \Big/ \left(\frac{dW_m}{da^2} \right)^2 \right|$$

$$< \frac{\eta^2}{m_0} \left| \Sigma_K m_K \frac{d}{da} \left(\frac{\omega_K}{\dot{a}} - \frac{\partial F}{\partial J_K} \right) \right| \qquad \text{from (6) and } (\alpha)$$

$$= \eta^2 \left| \Sigma_K \frac{m_K}{m_0} \left(\frac{1}{\dot{a}} \frac{d\omega_K}{da} - \frac{\omega_K}{\dot{a}^3} \ddot{a} - \frac{d}{da} \frac{\partial F}{\partial J_K} \right) \right|.$$

This may be put equal to

$$\varepsilon_m(\eta^2/\dot{a}) + \varepsilon_m(\eta^2\ddot{a}/\dot{a}^3)$$

as

$$\frac{d}{da}\frac{\partial F}{\partial J_K} = \frac{\partial^2 F}{\partial a \partial J_K} + \frac{\partial^2 F}{\partial \dot{a} \partial J_K}\frac{d\dot{a}}{da} + \Sigma_i \left\{ \frac{\partial^2 F}{\partial J_i \partial J_K}\frac{dJ_i}{da} + \frac{\partial^2 F}{\partial w_i \partial J_K}\left(\frac{\omega_i}{\dot{a}} - \frac{\partial F}{\partial J_i}\right)\right\}.$$

Suppose (a) increases by Δa while W_m increases by $\frac{1}{2}$. During this time $1 \left/ \frac{dW_m}{da} \right.$ cannot change by more than $[\varepsilon_m(\eta^2/\dot{a}) + \varepsilon_m(\eta^2\ddot{a}/\dot{a}^3)]\,\Delta a$, while $\cos 2\pi(W_m + \gamma_m)$ changes sign. Hence the net contribution of two successive half-cycles to the integral we are evaluating must certainly be numerically less than

$$m_K C_m [\varepsilon_m(\eta^2/\dot{a}) + \varepsilon_m(\eta^2\ddot{a}/\dot{a}^3)]\,\Delta a.$$

Also, the fractions of complete cycles left over at the ends of the interval of integration contribute terms that cannot numerically exceed $m_K C_m \left/ \frac{dW_m}{da} \right.$, which is of the form $m_K C_m \varepsilon_m(\eta/\delta a)\,\delta a$, from (α). This gives

$$\left| m_K C_m \int_{a'}^{a'+\delta a} \cos 2\pi (W_m + \gamma_m)\, da \right| < m_K C_m \delta a \\ [\varepsilon_m(\eta^2/\dot{a}) + \varepsilon_m(\eta^2\ddot{a}/\dot{a}^3) + \varepsilon_m(\eta/\delta a)].$$

The sum of the right-hand side for all sets of m's satisfying (α) and having $m_0 < M$ is of the form

$$\delta a\,[\varepsilon(\eta^2/\dot{a}) + \varepsilon(\eta^2\ddot{a}/\dot{a}^3) + \varepsilon(\eta/\delta a)],$$

owing to the absolute convergence of the series $\Sigma\, m_K C_m$, and to the fact that the ε_m's tend to their zero values uniformly with respect to the m's. Hence we have

$$\frac{\delta J_K}{\delta a} = \Sigma_m\, m_K C_m \frac{1}{\delta a}\int_{a'}^{a'+\delta a} \cos 2\pi (W_m + \gamma_m)\, da \\ + \varepsilon(\delta a) + \varepsilon(1/M) + \varepsilon(\eta^2/\dot{a}) + \varepsilon(\eta^2\ddot{a}/\dot{a}^3) + \varepsilon(\eta/\delta a) \quad (8)$$

where the summation need include only those sets of m's for which $m_0 < M$ and for which, at some point in the interval δa,

$$\left|\frac{dW_m}{da}\right| = \left|\Sigma\, m_r\left(\frac{\omega_r}{\dot{a}} - \frac{\partial F}{\partial J_r}\right)\right| \leqq \frac{m_0^{\frac{1}{2}}}{\eta}$$

or for which

$$|\Sigma\, m_r \omega_r| \leqq m_0^{\frac{1}{2}}\dot{a}/\eta < M^{\frac{1}{2}}\dot{a}/\eta, \quad (\beta)$$

the terms involving $\partial F/\partial J_r$ being negligible.

We can choose η to tend to zero with $\dot{a}$ in such a way that each of the quantities $\dot{a}/\eta$, $\eta^2/\dot{a}$ and $\eta^2\ddot{a}/\dot{a}^3$ tends to zero, this being possible since $\ddot{a}/\dot{a}$ tends to zero. We can now make δa and $1/M$ tend to zero so slowly that $\eta/\delta a$, $\dot{a}/\eta\delta a$ and

$M^{\frac{1}{2}}\dot{a}/\eta$ tend to zero. In this way all the ε terms in (8) (whose sum we shall denote simply by ε) tend to zero, and also $\dot{a}/\eta\delta a$ and the right-hand side of (β) tend to zero.

We may assume that the J_K tend to certain limiting values J_{0K} at each point $a = a_0$.* Throughout the interval δa the ω_K must lie within certain regions surrounding the values ω_{0K} that they take for $J_K = J_{0K}$, $a = a_0$, which regions ultimately contract to zero. There may be sets of m's which make $\Sigma m_K \omega_{0K} = 0$, and these, if they exist, must be included in the summation in (8). Any other particular set of m's originally satisfying (β), $m_K{}^t$ say, must cease to do so sooner or later during the limiting process, since

$$|\Sigma_K m_K{}^t \omega_K| \to |\Sigma_K m_K{}^t \omega_{0K}|$$

at all points in the interval δa, and this will ultimately be greater than $M^{\frac{1}{2}}\dot{a}/\eta$. Hence we may choose M to be always sufficiently small to exclude all such sets of m's from the summation in (8), while at the same time M tends to infinity during the limiting process. So the only sets of m's that need be included are those which actually make $\Sigma m_r \omega_r = 0$ when $a = a_0$ and $J_K = J_{0K}$.

For any of these included sets, $m_r{}^s$ say, we have

$$\delta\Sigma_r m_r{}^s \omega_r = \Sigma_r m_r{}^s \frac{\partial \omega_r}{\partial a}\delta a + \Sigma_{rK} m_r{}^s \frac{\delta \omega_r}{\partial J_K}\delta J_K + \delta a \,.\, \varepsilon(\delta a)$$

$$= \Sigma_r m_r{}^s \frac{\partial \omega_r}{\partial a}\delta a + \Sigma_{rK} m_r{}^s \frac{\partial \omega_r}{\partial J_K}\Sigma_m m_K C_m \int_{a'}^{a'+\delta a}\cos 2\pi (W_m + \gamma_m)\, da + \varepsilon \,.\, \delta a.$$

Now $\frac{1}{\delta a}\int_{a'}^{a'+\delta a}\cos 2\pi (W_m + \gamma_m)\, da$ cannot numerically exceed unity, and may therefore be put equal to $\cos\sigma_m$. Thus

$$\frac{\delta\Sigma_r m_r{}^s \omega_r}{\delta a} = \Sigma_r m_r{}^s \frac{\partial \omega_r}{\partial a} + \Sigma_{rK} m_r{}^s \frac{\partial \omega_r}{\partial J_K}\Sigma_m m_K C_m \cos\sigma_m + \varepsilon. \tag{9}$$

Suppose
$$\left|\Sigma_r m_r{}^s \frac{\partial \omega_r}{\partial a}\right| > \Sigma_m \left| C_m \Sigma_{rK} m_K m_r{}^s \frac{\partial \omega_r}{\partial J_K}\right| \tag{10}$$

* If they do not, then we could always choose a subsequence of the $\dot{a}$'s which make the J_K tend to certain limiting values J_K' (functions of a) at an infinite enumerable everywhere-dense set of points, in which case it must also make $J_K \to J_K'$ for any intermediate point, on account of the continuity (uniform with respect to $\dot{a}$) of J_K. The method of the text would now show that these J_K' are constants. In the same way we could choose another subsequence of $\dot{a}$'s which makes $J_K \to J_K''$ not always equal to J_K', and the J_K'' would also have to be constants. This is impossible since, initially, $J_K' = J_K''$.

when $a = a_0$ and $J_K = J_{0K}$. With this condition there is a lower bound b_s to the value of $|\delta\Sigma_r m_r^s \omega_r / m_0^s \delta a|$ independent of the σ_m. This makes

$$|\delta' \Sigma_r m_r^s \omega_r| > m_0^s b_s \delta' a,$$

where $\delta' a$ denotes any interval for which

$$\delta' a \to 0 \quad \text{and} \quad \eta/\delta' a \to 0,$$

these being the only restrictions to which δa in equation (9) is subject. Thus $\Sigma_r m_r^s \omega_r$ satisfies the inequality (β) only during a small sub-interval whose extent cannot exceed any $\delta' a$ for which

$$m_0^s b_s \delta' a > 2\,(m_0^s)^{\frac{1}{2}} \dot{a}/\eta,$$

i.e., whose extent cannot exceed $2\dot{a}/(m_0^s)^{\frac{1}{2}} b_s \eta$, as this value for $\delta' a$ satisfies the above restrictions. It is only during this sub-interval that

$$\int \cos 2\pi\,[\mathrm{W}\,(m^s) + \gamma\,(m^s)]\,da$$

is of the same order of magnitude as the range of (a) through which it is taken. Hence,

$$m_K^s \mathrm{C}\,(m^s) \int_{a'}^{a'+\delta a} \cos 2\pi\,[\mathrm{W}\,(m^s) + \gamma\,(m^s)\,]\,da = \frac{m_K^s \mathrm{C}\,(m^s)}{b_s}\,\varepsilon_s\,(\dot{a}/\eta\delta a)\,\delta a$$

plus other ε terms like those already included in (8).

Summing this expression for all the sets of m's left in (8), on the assumption that the inequality (10) holds for each of them, we get

$$\delta a\,\varepsilon\,(\dot{a}/\eta\;\delta a) \sum^{\mathrm{M}} m_K \mathrm{C}_m / b_m.$$

This series may be divergent, since the b_m may tend to zero as the m's tend to infinity. We can avoid this difficulty by using the fact that there is no limitation on how slowly M tends to infinity. We may make the finite series $\sum^{\mathrm{M}} m_K \mathrm{C}_m/b_m$ tend to infinity more slowly than $\varepsilon\,(\dot{a}/\eta\;\delta a)$ tends to zero (however slowly that may be), so that their product tends to zero, and may be put equal to $\varepsilon\,[\dot{a}\,f\,(\mathrm{M})/\eta\;\delta a]$.

Hence

$$\frac{\delta J_K}{\delta a} = \varepsilon\,(\delta a) + \varepsilon\,(1/\mathrm{M}) + \varepsilon\,(\eta^2/\dot{a}) + \varepsilon\,(\eta^2 \ddot{a}/\dot{a}^3) + \varepsilon\,(\eta/\delta a) + \varepsilon\,[\dot{a}\,f\,(\mathrm{M})/\eta\delta a]$$

and this tends to zero if first $\dot{a}$ and then δa are made to tend to zero, the arbitrary quantities η and $1/\mathrm{M}$ tending to zero in a suitable manner. So the J_K are constants in the limit provided, whenever a relation of the type

$$\Sigma_r\, m_r^s\, \omega_r = 0$$

holds for $a = a_0$ with integral m_r^s, and the corresponding coefficient $C(m^s)$ does not vanish, the inequality

$$\left| \frac{\partial}{\partial a} \Sigma_r\, m_r^s\, \omega_r \right| > \Sigma_{m_1 \ldots m_n} \left| C_{m_1 \ldots m_n}\, \Sigma_{r\mathrm{K}}\, m_\mathrm{K}\, m_r^s \frac{\partial \omega_r}{\partial \mathrm{J}_\mathrm{K}} \right|$$

is satisfied, where the summation with respect to the m's refers to all those sets of m's for which

$$\Sigma m_r\, \omega_r = 0$$

when $a = a_0$.

The maximum value of

$$\left| \Sigma_{r\mathrm{K}}\, m_r^s \frac{\partial \omega_r}{\partial \mathrm{J}_\mathrm{K}}\, \Sigma_m\, m_\mathrm{K}\, \mathrm{C}_m \cos 2\pi\, (\mathrm{W}_m + \gamma_m) \right|$$

for all values of the w's will in general be less than

$$\Sigma_m \left| \mathrm{C}_m\, \Sigma_{r\mathrm{K}}\, m_\mathrm{K}\, m_r^s \frac{\partial \omega_r}{\partial \mathrm{J}_\mathrm{K}} \right| .$$

If this maximum is evaluated for any particular dynamical system (it cannot easily be done in the general case) and substituted for the right-hand side of (10), less stringent conditions for the adiabatic invariance will be obtained. If the improved conditions are not satisfied, the J_K can vary with suitable initial values of the phases of the motion, and will do so in such a way as to make $\Sigma m_r^s\, \omega_r$ remain equal to zero while (a) changes by a finite amount. As this contradicts the postulate of the existence of stationary states, one must conclude that in these cases the motion cannot be completely described by the use of classical mechanics.

If, however, there are relations of the type $\Sigma m_r \omega_r = 0$ holding identically for all values of the J's, it is possible to reduce the number of J's and w's necessary to describe the system and to replace them by new conjugate variables α_r and β_r which do not enter into H. Equations (5) and (6) are still true, the only difference being that the C's are now functions also of the α's and β's. It is easily verified that the $d\alpha/da$, $d\beta/da$ are bounded, and hence the same method applies to show that the reduced number of J's are adiabatically invariant.

§ 4. *The Application of the Conditions.*

As there is an infinite number of inequalities to be satisfied for any variation in (a), however small, the conditions would not be very useful unless it were possible to prove all of them except, perhaps, a finite number by general arguments. This does not seem to be possible for the general case of systems of more than two degrees of freedom.

For systems of two degrees of freedom, however, there is only a finite number of values of (a) for which

$$\omega_1 + x\,\omega_2 = 0 \quad \text{and} \quad \frac{\partial}{\partial a}(\omega_1 + x\,\omega_2) = 0$$

at the same time, x being a rational or irrational number, since there are two equations to determine the two unknowns (a) and x. In any closed interval of (a) which does not contain any of these values, there must be a lower limit to the value of

$$\left|\frac{\partial}{\partial a}(m_1\omega_1 + m_2\omega_2)/m_0\right|,$$

where (a) and m_1/m_2 have values which make $m_1\omega_1 + m_2\omega_2 = 0$; otherwise there would be an infinite sequence of values of (a) and m_1/m_2 which make

$$m_1\omega_1 + m_2\omega_2 = 0 \quad \text{and} \quad \frac{\partial}{\partial a}(m_1\omega_1 + m_2\omega_2)/m_0 \to 0,$$

and the limiting values would give one of the points we have excluded. Hence, whenever

$$m_1{}^s\omega_1 + m_2{}^s\omega_2 = 0,$$

we have

$$\left|\frac{\partial}{\partial a}(m_1{}^s\omega_1 + m_2{}^s\omega_2)/m_0{}^s\right| > \xi$$

say, so that the condition

$$\left|\frac{\partial}{\partial a}(m_1{}^s\omega_1 + m_2{}^s\omega_2)\right| > \Sigma_m\left|\mathrm{C}_m\Sigma_{r\mathrm{K}}m_\mathrm{K}m_r{}^s\frac{\partial\omega_r}{\partial \mathrm{J}_\mathrm{K}}\right|$$

is satisfied if

$$\xi > \Sigma_m\left|\mathrm{C}_m\Sigma_{r\mathrm{K}}m_\mathrm{K}\frac{m_r{}^s}{m_0{}^s}\frac{\partial\omega_r}{\partial \mathrm{J}_\mathrm{K}}\right|. \tag{11}$$

For any given ξ there can be only a finite number of values for $m_1{}^s$, $m_2{}^s$ which do not satisfy the inequality (11), since as $|m_1{}^s|$, $|m_2{}^s|$ tend to infinity the m's included in the first summation in (11) also tend to infinity, so that the right-hand side of (11) tends to zero. Hence in any interval of (a) which does not contain any points at which

$$\omega_1/\omega_2 = \frac{\partial\omega_1}{\partial a}\Big/\frac{\partial\omega_2}{\partial a},$$

there is only a finite number of points at which the J's may not be constant.

The same result is true for a system of n degrees of freedom provided the coefficients $\mathrm{C}_{m_1 \ldots m_n}$ of the Fourier series vanish for all but a finite number of values of the suffixes $m_3, m_4 \ldots m_n$. The proof is the same as before, except that the factor $(m_1\omega_1 + m_2\omega_2)$ must be replaced by $\Sigma m_r\omega_r$, where $m_3 \ldots m_n$ are restricted to have only values for which $\mathrm{C}_{m_1 \ldots m_n}$ does not vanish.

This case includes all the systems usually met with in the Quantum Theory, the vanishing of the C's for certain values of the suffixes manifesting itself by selection principles which allow only a finite number of changes for each of the quantum numbers except at most two of them. The quantum integrals of such systems are invariant under any adiabatic change except at a finite set of points where

$$\omega_r/\omega_s = \frac{\partial \omega_r}{\partial a} \bigg/ \frac{\partial \omega_s}{\partial a},$$

ω and ω_s being the frequencies corresponding to the quantum numbers whose changes are unrestricted, and at another finite set (or possibly infinite enumerable set tending to points of the previous set) where relations of the type $\Sigma m_r \omega_r = 0$ hold.

The writer is much obliged to Mr. R. H. Fowler for suggesting this investigation, and for his help during its progress.

On the Theory of Elastic Stability.

By W. R. Dean, B.A., Fellow of Trinity College, Cambridge.

(Communicated by Prof. G. I. Taylor, F.R.S.—Received December 20, 1924.)

The object of the present paper is to derive equations that are adequate to decide questions of the stability under stress of thin shells of isotropic elastic material. Equations for the same purpose have been given by R. V. Southwell,* who used a method that is closely followed in a part of this paper.

Such equations must contain terms that may be, and are, neglected in applications of the theory of elasticity to problems in which the stability of configurations is not considered. The truth of Kirchhoff's uniqueness theorem,† which has reference to the ordinary equations of elasticity, in which powers of the displacement co-ordinates above the first are neglected, is sufficient proof of this statement. In practice it is generally sufficient to retain only the first and second order terms,‡ and no terms of higher order are considered

* "On the General Theory of Elastic Stability," 'Phil. Trans.,' A, vol. 213, p. 187.

† A. E. H. Love, 'Mathematical Theory of Elasticity' (3rd Edition), § 118.

‡ There are exceptions to this. *Cf.* a paper by J. Prescott, 'Phil. Mag.,' vol. 43, p. 97 (1922), which, though not immediately concerned with elastic stability, obtains equations which can be applied to this theory. See also § 9 below.