# Quantum Autoencoders via Quantum Adders with Genetic Algorithms

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The quantum autoencoder is a recent paradigm in the field of quantum machine learning, which may enable an enhanced use of resources in quantum technologies. To this end, quantum neural networks with less nodes in the inner than in the outer layers were considered. Here, we propose a useful connection between approximate quantum adders and quantum autoencoders. Specifically, this link allows us to employ optimized approximate quantum adders, obtained with genetic algorithms, for the implementation of quantum autoencoders for a variety of initial states. Furthermore, we can also directly optimize the quantum autoencoders via genetic algorithms. Our approach opens a different path for the design of quantum autoencoders in controllable quantum platforms.

*Introduction.*— Quantum machine learning is an emerging field that aims at enhancing machine learning methods with quantum technologies [1–4]. The synergy works two-fold: either through genuine quantum effects, such as entanglement, to speed up the calculations of machine learning [1, 2], or to employ classical machine learning to improve quantum processes [5, 6]. In this respect, an advanced protocol has been considered, inspired in the classical autoencoder techniques of deep learning [7], namely, a quantum autoencoder [8, 9]. Other topics related to biomimetic quantum technologies in general, which have emerged in recent years, involve quantum artificial life [10, 11], quantum reinforcement learning in quantum technologies [12], quantum memristors [13–16], quantum Helmholtz and Boltzmann machines [17–19], and quantum machine learning with time-delay equations [20, 21].

A quantum autoencoder, see Fig. 1, is a quantum device which can reorganize the quantum information of a subset of a Hilbert space spanned by a basis of initial $n$-qubit states onto a subset of a Hilbert space spanned by $n' < n$ qubit states. Therefore, in this way, one may encode the information into a smaller amount of resources, which may be useful in different quantum technologies. Quantum autoencoders, at variance with previous results on compression of quantum information [22, 23], aim at reshuffling the information to employ less resources, similarly to defragmentation in classical computing. To achieve this reordering, one may employ a feedforward quantum neural network [8], which requires a smaller amount of qubits in the inner layer than in the input and output layers. Via gradient descent techniques, we can optimize the intermediate transition unitary operations between layers such that, for each input state, the output state has almost perfect overlap with it. Therefore, the first part of the network (encoder), which reorganizes the input states onto the inner layer of qubits, contains the desired solution to the problem due to the discarding of superfluous information. Classical autoencoders usually employ that useful information to make feasible the training of deep neural networks.

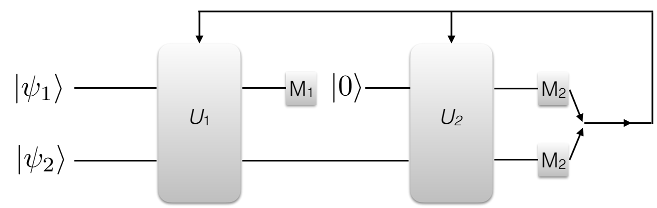



FIG. 1. Scheme of a 2-qubit quantum autoencoder as proposed in Refs. [8, 9]. It is based on a quantum neural network where the outer layers (two-qubit input and output states) contain one more qubit than the inner layer (single-qubit state in between the unitary gates $U_1$ and $U_2$) and the former are connected to the latter via the encoder $U_1$ and the decoder $U_2$. These operations are optimized, via, e.g., gradient descent techniques, with controllable parameters by means of a classical feedback loop, in order that the output state maximizes the overlap with the input state. $M_1$ is a dummy measurement, while $M_2$ is a computational basis measurement on the output two-qubit state that is employed to close the feedback loop. After convergence, $U_1$ is the optimized encoding operation.

In the last years, the design and implementation of approximate quantum adders [24–32] has emerged with unexpected connections. Recently, it was proven that a quantum operation that adds two unknown quantum states is forbidden in general [24, 25]. Since then, important results in optimized approximate or probabilistic quantum adders have been analyzed, and in some cases implemented, which may find applications to different aspects of quantum technologies [24–32].

In this article, we propose and analyze a connection between a quantum autoencoder and an approximate quantum adder, see Fig. 2. The main insight in this respect is the realization that, provided a perfect quantum adder, a perfect quantum autoencoder and perfect quantum compressor would be attained. The reason for this is that, given a hypothetical unitary operation $U$ able to add

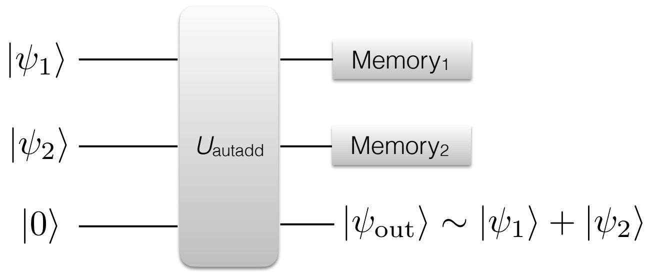



FIG. 2. Scheme of our proposed 2-qubit quantum autoencoder. It is based on a 3-qubit device composed of an optimized approximate quantum adder operation $U_{\text{autadd}}$ that acts on the input states, $|\psi_1\rangle$ and $|\psi_2\rangle$, as well as an ancilla qubit $|0\rangle$. The approximate quantum adding operation produces a state with large overlap to $|\psi_1\rangle+|\psi_2\rangle$, allowing for an encoding of the initial 2-qubit quantum information subspace into a single-qubit subspace, to a certain fidelity. The memory units allow for maintaining entanglement of the three output qubits and, after possible local operations on $|\psi_{\text{out}}\rangle$, under proper circumstances, to invert the encoding via $U^{\dagger}_{\text{autadd}}$.

two unknown quantum states, $U|\psi_1\rangle|\psi_2\rangle \propto |\psi_1\rangle + |\psi_2\rangle$, a trivial encoding would be produced. In this sense, $U$ would allow us to map the quantum information of tensor products of two, or by iteration of $n$, single-qubit states onto a single qubit. Then, in order to decode the state, one would apply the inverse of $U$, i.e., $U^{\dagger}$, retrieving the initial state. By linearity, this would also apply to superpositions of the initial entries. As proved in Ref. [24], an ideal quantum adder is forbidden by the laws of quantum mechanics. Another complementary proof is given by this connection, because if an ideal and universal $n$-qubit quantum adder existed, reorganization, as well as compression of quantum information from $n$-qubit states to a single one would be feasible, which is clearly forbidden in general because of different Hilbert-space dimensionality and because it would violate Schumacher's theorem [22]. Despite this, one can employ the insight obtained on approximate quantum adders optimized with genetic algorithms [30] to propose quantum autoencoders that rely on the previous optimization. A further possibility is to optimize the protocol of quantum autoencoders via genetic algorithms, using similar techniques as in previous works [5, 30]. In this article, we will follow both approaches. We firstly consider a quantum autoencoder based on an approximate quantum adder that adds perfectly the elements of the computational basis, but non-perfectly the superpositions. Later on, we consider quantum autoencoders optimized with genetic algorithms that employ a restricted amount of gates and in this context achieve also high fidelities.

A decomposition of approximate quantum adders in terms of single and two-qubit gates can be obtained according to diverse criteria. Some adders may be obtained by defining them to add optimally the computational basis states, and extended by linearity to superpositions of these. Other quantum adders can be optimized by genetic algorithms in the context of limited available amount of gates. By employing the proposed protocol of Fig. 2, we describe now how to implement a quantum autoencoder based on each of these cases. For simplicity, the unitary gate denoted as $U_{\text{autadd}}$ in the figure will be called $U$ in the following example.

*The basis quantum adder.—* This quantum adder was defined [24, 30] to perfectly add the basis states, and approximately the superposition states, according to

$$\begin{aligned}
&U|000\rangle = |000\rangle, \quad U|010\rangle = |01+\rangle, \quad U|100\rangle = |10+\rangle, \\
&U|110\rangle = |001\rangle, \quad U|001\rangle = |110\rangle, \quad U|011\rangle = |01-\rangle, \\
&U|101\rangle = |10-\rangle, \; U|111\rangle = |111\rangle,
\end{aligned} \tag{1}$$

where the first two qubits in the lhs are the addend states, the third one is an ancillary qubit, and the last qubit in the rhs is the outcome state of the addition. In the protocol the initial ancilla is always the $|0\rangle$ state, but the previous definition of the $U$ quantum adder includes all possible states for univocally defining the gate.

The fidelity of the quantum adder $U$ is defined in terms of the output state $\rho_{\text{out}}$ as

$$\begin{aligned}
&F = \text{Tr}(|\Psi_{\text{id}}\rangle\langle\Psi_{\text{id}}|\,\rho_{\text{out}}), \\
&\rho_{\text{out}} = \text{Tr}_{12}(U|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2| \otimes |0\rangle\langle 0|U^{\dagger})
\end{aligned} \tag{2}$$

where $|\Psi_{\text{id}}\rangle \propto |\psi_1\rangle + |\psi_2\rangle$ is the ideal outcome of the sum, and the trace is taken over the first two qubits. In Ref. [30], a subset of all possible input states were considered in order to make the approximate quantum adder consistent, namely, $|\psi_i\rangle$, $i = 1, 2$ were defined as

$$|\psi_i\rangle = \begin{pmatrix} \cos\,\theta_i \\ \sin\,\theta_i \end{pmatrix}, \tag{3}$$

and the angles $\theta_i \in \{0, \pi/2\}$.

The average fidelity of this region is 94.9%, while the lowest fidelity is 85.4%.

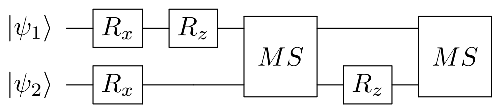
*Quantum autoencoder based on approximate quantum adders.—* We propose a quantum autoencoder based on this approximate quantum adder following the scheme of Fig. 2. For any input, two-qubit product state in the considered region defined by Eq. (3), i) apply the quantum adder given by $U$ onto this state and an ancilla $|0\rangle$ state, ii) store the first two outputs onto quantum memories, and employ the approximate addition of the input states onto $|\psi_{\text{out}}\rangle$ for any desired quantum task, including quantum communication and single-qubit gates, iii) retrieve the decoded, modified, two-qubit state via application of $U^{\dagger}$ onto the memory qubits and the output state. By linearity, the protocol can be extended to superpositions of the input states. A criterion that should hold for consistency is that the two memory qubits should be in the same quantum state for the considered input states after the quantum adding operation. This way, the effect of any local operation on the encoded $|\psi_{\rm out}\rangle_i$ that acts differently on the latter for different $i$, where $i$ refers to the specific input $|\psi_1\rangle_i|\psi_2\rangle_i$ state considered, can be efficiently retrieved onto the corresponding two-qubit gate on $\{|\psi_1\rangle_i|\psi_2\rangle_i\}$ via $U^\dagger$. For cases in which, for different $i$, the memory qubit states differ, the encoding operation will work up to a certain fidelity, which will decrease with the distance between these memory states and also depend on the specific adding operation and input states.

We point out that in the case of a quantum autoencoder based on an approximate quantum adder, the usage of quantum memories for the output first two entries is necessary. The reason is that the inverse operation of the adder, $U^\dagger$, may retrieve the input two-qubit subspace with higher fidelity, as far as the $|\psi_{\rm out}\rangle$ state is controllably modified in the process. In order to map an original two-qubit gate in the initial subspace onto the single-qubit one, one has to encode it according to the specific quantum adder employed. In the case here proposed, for example, if one wants to apply a CPHASE gate onto the inputs $|00\rangle$ and $|11\rangle$, one would employ the mapping of Eq. (1). Realizing that in this case both memory qubits would take the same values for both input states, i.e., $|00\rangle$, one would just need to apply a phase gate on $|\psi_{\rm out}\rangle$ to introduce a minus sign between $|0\rangle$ and $|1\rangle$ single-qubit states. Later on, one would apply the inverse adding operation, $U^\dagger$, retrieving the input states with a minus sign on the $|11\rangle$ state, as corresponds to the correct application of the CPHASE gate.

In general, the encoding of a desired two-qubit gate operation onto a single qubit gate in the compressed space will depend both on the gate and on the subspace on which it acts. Clearly, a two-qubit gate acting on the complete two-qubit Hilbert space cannot be mapped onto a single-qubit gate without losing information. But the previous example can be useful to illustrate that, in some cases, the complexity of a two-qubit gate implementation may be reduced to a single-qubit gate. This may be useful, for example, if we have full availability of a universal quantum computer in some lab, where we can implement the quantum adder and store the memory qubits, but only availability of single-qubit gates in some other lab, to which we send the $|\psi_{\rm out}\rangle$ qubit via a quantum communication channel. Another possibility where this approach can be useful is a situation in which one considers quantum information encoded in qudits but has only access to single-qubit gates. In these cases, the encoding via approximate quantum adders may also prove fruitful. This formalism could also be employed as a useful transducer between a processing and a storage unit.

*Quantum autoencoders via genetic algorithms.—* As pointed out above, approximate quantum adders can be also optimized with genetic algorithms. Following the previous approach for the basis adder, these quantum adders can be also employed to define quantum autoencoders. The formalism of genetic algorithms allows one to define the maximum number of gates in the desired decomposition, as well as its structure of single- and two-qubit gates [5, 30]. Therefore, it can be useful for practical purposes in current or near future quantum implementations with trapped ions, superconducting circuits, and quantum photonics, which employ a limited amount of resources. In order to define a quantum autoencoder via an optimized quantum adder, we propose to follow the scheme on Fig. 2, where now the unitary gate $U_{\rm autadd}$ will be given by the corresponding quantum adding operation.

An alternative approach is to directly optimize the quantum autoencoders with genetic algorithms following the scheme of Fig. 1, instead of previously employed optimization methods, e.g., gradient descent. An advantage of genetic algorithms is that local minima can in principle be better avoided in some situations. The first step is to design a function that relates the "genetic code" of each "individual" in the program to a specific autoencoder, either $U_{\rm autadd}$ or the pair of $U_1$ and $U_2$ in the more general case. The genetic code is represented with a $3 \times g$ matrix where $g$ is the number of gates. Accordingly, each row provides specific information about the gate to implement. The interaction type and phase are encoded in the first two columns, while the third one determines the qubit to act upon. The universal set of gates we are working with is given by single-qubit rotations and the Mølmer-Sørensen gate: $\{R_x(\theta), R_y(\theta), R_z(\theta), e^{-i\sigma_y\otimes\sigma_y\theta/2}\}$. In each loop a new generation is bred, where the individuals are hierarchically recombined, mutated and selected according to the fidelity of the corresponding autoencoder. We have followed this approach for a certain set of cases via numerical simulations, and obtained consistent results. Namely, when compressing a set of two-qubit states which contains at most two linearly-independent states, the compression is optimal and the protocol converges to fidelity one onto a single-qubit subspace. When trying to compress four orthogonal two-qubit states onto two single-qubit ones, a 50% average fidelity is obtained. For three orthogonal states, an average fidelity of 2/3 is obtained. For non orthogonal four-qubit or three-qubit states, the fidelity will be larger than for the corresponding orthogonal ones, the difference depending on the overlap with each other. More precisely, in order to test the validity of our algorithm we have analyzed an optimized autoencoder for a nontrivial case involving a set of three states: $|\psi\rangle_1 = \cos\frac{\pi}{3}|00\rangle + \sin\frac{\pi}{3}|01\rangle$, $|\psi\rangle_2 = |01\rangle$, $|\psi\rangle_3 = \cos\frac{\pi}{8}|10\rangle - i\sin\frac{\pi}{8}|11\rangle$. We have achieved an average fidelity of 87.51% in the general case of $U_1 \neq U_2^\dagger$ for a reduced number of gates. See the following quantum circuit diagrams for the specific decomposition of the autoencoder in quantum gates. We point out that an interesting feature of genetic algorithms is precisely that they already provide the sequence of elementary gates generating the solution.



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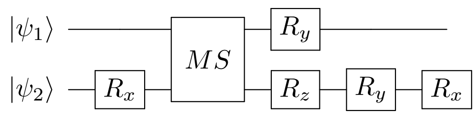

(5)

Here, $R_{x,y,z}$ represent the single qubit rotations, while $MS$ is the Mølmer-Sørensen gate. The diagrams show the type of obtained interactions, not specifying the phases, for $U_1$ and $U_2$ respectively. The algorithm we employed is a variant of a previous program [5, 30, 33].

In the following, we describe possible implementations with trapped ions, superconducting circuits, and quantum photonics.

*Trapped Ions.—* In trapped-ion quantum platforms, linear chains of ions are confined via electromagnetic fields [34]. Both internal electronic states, and quantized motional degrees of freedom, are available for quantum control. The Hamiltonian describing the coupling between an ion and a laser, involving internal states and a motional mode, in the Lamb-Dicke regime, is given by

$$H = \hbar\Omega\sigma^{+}\left[1 + i\eta\left(ae^{-i\nu t} + a^{\dagger}e^{i\nu t}\right)\right]e^{i(\phi-\delta t)} + \text{H.c.} \quad (6)$$

Here, $\sigma^{+}$ is the spin raising operator, $\eta$ the Lamb-Dicke parameter, $\Omega$ is the Rabi frequency, $a$ and $a^{\dagger}$ the motional annihilation and creation operators, $\nu$ the trap frequency, $\phi$ the laser field phase, and $\delta$ the detuning between the laser and qubit frequencies.

In our proposal, each two-qubit state can be encoded either in four metastable levels of a single ion or two levels of a pair of ions, and an additional ion serves as an ancillary qubit in the case of autoencoder with quantum adder. The necessary entangling gates can be implemented with Mølmer-Sørensen gates and sequences of single-qubit gates [35], allowing us to implement our approximate quantum adders and therefore the quantum autoencoders. The memory qubits can be stored in long-lived internal electronic states.

*Superconducting Circuits.—* Superconducting circuits are made of superconducting elements as microwave cavities, as well as nonlinear elements, i.e., Josephson junctions, which allow one to design effective quantum two-level systems [36]. Via standard circuit quantization, one often can obtain a Hamiltonian resembling the Jaynes-Cummings model,

$$H = \omega a^{\dagger}a + \frac{\omega_0}{2}\sigma^{z} + g(a\sigma^{+} + a^{\dagger}\sigma^{-}), \quad (7)$$

where $\omega$ is the microwave mode frequency, $g$ is the photon-qubit coupling constant, and $\omega_0$ is the qubit frequency, which is encoded in the quantum excitations of the effective two-level system.

Transmon qubits have long coherence times, and therefore they can be most appropriate for this kind of encoding. Entangling gates can be carried out with fidelities around 99%, e.g., via capacitive coupling, or via resonators, as in Eq. (7) when considering more than one qubit. In Ref. [30], an optimized approximate quantum adder obtained via genetic algorithms was carried out experimentally in the IBM Quantum Experience processor, obtaining a good agreement with theoretical simulations. Therefore, it is expected that quantum autoencoders based on approximate quantum adders, or on direct optimization via genetic algorithms, may be achieved with current technology.

*Quantum Photonics.—* Quantum photonic platforms can be most appropriate for implementing quantum autoencoders via quantum adders, given that probabilistic quantum adders have been already performed in some photonic laboratories, e.g., Ref. [26]. In this quantum platform, qubits may be encoded in polarization or dual rail states, and gates performed via linear optics elements in combination with ancillary qubits and detectors [37]. The fact that photons are the best quantum information carriers for long quantum communication, makes them interesting for implementing quantum autoencoders via quantum adders and genetic algorithms, given that having to transmit a smaller amount of quantum information through a quantum channel can save resources.

*Conclusions.—* We have proposed and analyzed a connection between approximate quantum adders and quantum autoencoders, which may be useful to reorganize or compress quantum information while employing less resources. This connection allows one to implement optimized approximate quantum adders with genetic algorithms that are mapped onto quantum autoencoders, as well as a direct optimization of quantum autoencoders with genetic algorithms. The emerging field of quantum adders enhances the paradigm of quantum autoencoders, since the former have already been implemented in current quantum technologies [26–28, 30].

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