PHYSICS OLYMPIAD

(ΠΗΨΣΙΧΣ ΟΛΨΜΠΙΑΔ)

1993 MULTIPLE CHOICE SCREENING TEST

30 QUESTIONS—40 MINUTES

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

This test contains 30 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes

Select the single answer that provides the best response to each question. Please be sure to

an answer, the previous mark must be completely erased.

programs. Calculators may not be shared.

Your grade on this multiple choice test will equal your number of correct answers. There is no penalty for guessing. It is to your advantage to answer every question.

The values of some possibly useful constants are given below:

mass of electron	m_e	$= 9.1 \times 10^{-31}$ kg
mass of proton	m_p	$= 1.7 \times 10^{-27}$ kg
electronic charge	e	$= 1.6 \times 10^{-19}$ C
speed of light	c	$= 3.0 \times 10^{8}$ m/s
Coulomb's constant	k	$= 9.0 \times 10^{9}$ Nm^2/C^2
permittivity constant	ε_0	$= 8.9 \times 10^{-12}$ C^2/Nm^2
permeability constant	μ_0	$= 4\pi \times 10^{-7}$ Tm/A
gravitational constant	G	$= 6.7 \times 10^{-11}$ Nm^2/kg^2
acceleration due to gravity	g	$= 10$ m/s^2
speed of sound (20 ^{o}C)	v_s	$= 340$ m/s

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1. A ball of mass *m* is thrown vertically upward. Air resistance is <u>not</u> negligible. Assume the force of air resistance has magnitude proportional to the velocity, and direction opposite to the velocity's. At the highest point, the ball's acceleration is:

(A) 0 (B) less than g (C) g (D) greater than g (E) upward

2. A train is moving forward at a velocity of 2.0 m/s. At the instant the train begins to accelerate at 0.80 m/s^2, a passenger drops a quarter which takes 0.50 s to fall to the floor. Relative to a spot on the floor directly under the quarter at release, it lands:

(A) 1.1 m toward the rear of the train.

(B) 1.0 m toward the rear of the train.

(C) 0.10 m toward the rear of the train.

(D) directly on the spot.

(E) 0.90 m toward the front of the train.

3. The dropped quarter in the preceding question (#2) is viewed by an observer standing next to the tracks. Relative to this observer, the quarter moves_____before landing.

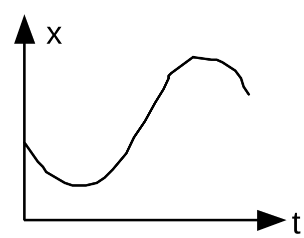
(A) forward 1.1 m

(B) forward 1.0 m

(C) forward 0.10 m

(D) straight down

(E) backward 0.10 m



4. The accompanying graph of position *x* versus time *t* represents the motion of a particle. If *b* and *c* are both positive constants, which of the following expressions best describes the acceleration *a* of the particle?

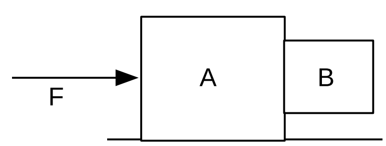
(A) a = 0

(B) a = +b

(C) a = -c

(D) a = b + ct

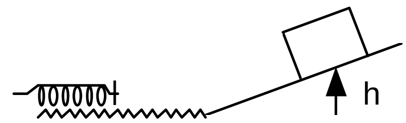
(E) a = b - ct



5. In the system shown to the right, a force *F* pushes on block A, giving the system an acceleration *a*. The coefficient of static friction between the blocks is *μ*. The correct equation for block B <u>not</u> to slip is:

(A) a > μg (B) a < μg (C) a > g (D) a > g/μ (E) a < g/μ

6. A block of mass *m* starts at rest at height *h* on a frictionless inclined plane. The block slides down the plane, travels a total distance *d* across a rough surface with coefficient of kinetic friction μ, and compresses a spring with force constant *k* a distance *x* before momentarily coming to rest. Then the spring extends and the block travels back across the rough surface, sliding up the plane. The correct expression for the maximum height *h'* that the block reaches on its return is:



(A) mgh' = mgh - 2μmgd

(B) mgh' = mgh + 2μmgd

(C) mgh' = mgh + 2μmgd + kx^2

(D) mgh' = mgh - 2μmgd- kx^2

(E) mgh' = mgh - 2μmgd- kx^2 - 1/2 mv^2

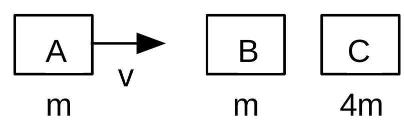
7. Air track car A has mass *m* and velocity *v*. Air track car B has mass *2m* and velocity *3v*. The same constant force *F* is applied to each car until it stops. Car A is brought to rest in time *t*. The time required to stop car B is:

(A) 2t (B) 3t (C) 6t (D) 9t (E) 18t

8. In the preceding question (#7), car A travels a distance *d* before coming to rest. The distance traveled by car B before coming to rest is:

(A) 2d (B) 3d (C) 6d (D) 9d (E) 18d

9. Three air track cars are initially placed as shown in the accompanying figure. Car A has mass *m* and initial velocity *v* to the right. Car B with mass *m* and car C with mass *4m* are both initially at rest. Car A collides elastically with car B, which in turn collides elastically with car C. After the collision, car C has a velocity of *0.4v* to the right. The final velocities of cars A and B are:



(A) Car A: 0.6v to the left Car B: at rest

(B) Car A: 1.4v to the left Car B: at rest

(C) Car A: v to the left Car B: 0.6v to the left

(D) Car A: 0.4v to the left Car B: v to the left

(E) Car A: 1.6v to the left Car B: v to the right

10. Three cylinders, all of mass M, roll without slipping down an inclined plane of height H. The cylinders are described as follows:

I. hollow of radius R

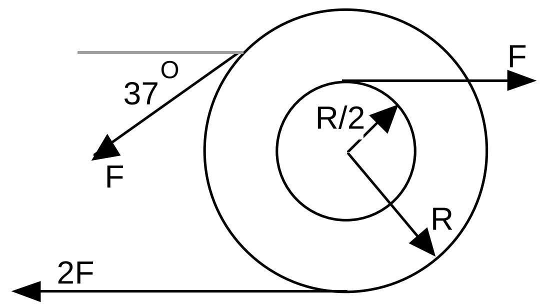
II. solid of radius $R/2$

III. solid of radius R

If all cylinders are released simultaneously from the same height, the cylinder (or cylinders) reaching the bottom first is (are):

(A) I (B) II (C) III (D) I & II (E) II & III

11. The system shown to the right is free to rotate about a frictionless axis through its center and perpendicular to the page. All three forces are exerted tangent to their respective rims. The magnitude of the net torque acting on the system is:



(A) 1.5 FR

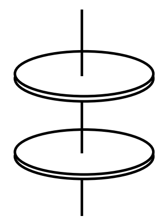
(B) 1.9 FR

(C) 2.3 FR

(D) 2.5 FR

(E) 3.5 FR

12. Two identical disks are positioned on a vertical axis. The bottom disk is rotating at angular velocity ω_0 and has rotational kinetic energy K_0. The top disk is initially at rest. It is allowed to fall, and sticks to the bottom disk. What is the rotational kinetic energy of the system after the collision?



(A) 1/4 K_0 (B) 1/2 K_0 (C) K_0 (D) 2 K_0 (E) 4 K_0

13. If the sun were suddenly replaced by a black hole of one solar mass, what would happen to the earth's orbit immediately after the replacement?

(A) The earth would spiral into the black hole.

(B) The earth would spiral out away from the black hole.

(C) The radius of the earth's orbit would be unchanged, but the period of the earth's motion would increase.

(D) Neither the radius of the orbit nor the period would change.

(E) The radius of the earth's orbit would be unchanged, but the period of the earth's motion would decrease.

14. A hypothetical planet has density ρ, radius R, and surface gravitational acceleration g. If the radius of the planet were doubled, but the planetary density stayed the same, the acceleration due to gravity at the planet's surface would be:

(A) 4g (B) 2g (C) g (D) g/2 (E) g/4

15. An ideal organ pipe resonates at frequencies of 50 Hz, 150 Hz, 250 Hz,.... The pipe is:

(A) open at both ends and of length 1.7 m.

(B) open at both ends and of length 3.4 m.

(C) open at both ends and of length 6.8 m.

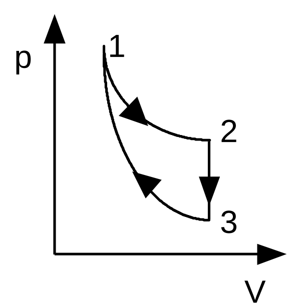
(D) closed at one end, open at the other, and of length 1.7 m.

(E) closed at one end, open at the other, and of length 3.4 m.

16. A porpoise, whistle-clicking at a frequency f_0, swims toward an underwater vertical cliff at a velocity that is 1.0% of the velocity of sound in sea water. The reflected frequency experienced by the swimming porpoise is:

(A) 0.98 f_0 (B) 0.99 f_0 (C) f_0 (D) 1.01 f_0 (E) 1.02 f_0

17. Three processes compose a thermodynamic cycle shown in the accompanying pV diagram. Process 1→2 takes place at constant temperature. Process 2→3 takes place at constant volume, and process 3→1 is adiabatic. During the complete cycle, the total amount of work done is 10 J. During process 2→3, the internal energy decreases by 20 J; and during process 3→1, 20 J of work is done on the system. How much heat is added to the system during process 1→2?



	Q	W	ΔU
1Ý 2			
2Ý 3			-20J
3Ý 1		-20J	
cycle		+10J	

(A) 0

(B) 10 J

(C) 20 J

(D) 30 J

(E) 40 J

18. The root mean square velocity of oxygen gas is v at room temperature. What is the root mean square velocity of hydrogen gas at the same temperature?

(A) 16 v (B) 4 v (C) v (D) v/4 (E) v/16

19. Monochromatic light of wavelength λ is shone on a grating consisting of six equally spaced slits. The first order interference maximum occurs at an angle of 0.00100 radians. If the outer two slits are covered, the first order maximum will occur at ______ radians.

(A) 0.00025 (B) 0.00067 (C) 0.00100 (D) 0.00150 (E) 0.00400

20. You are given two lenses, a converging lens with focal length +10 cm and a diverging lens with focal length -20 cm. Which of the following would produce a virtual image that is larger than the object?

(A) Placing the object 5 cm from the converging lens.

(B) Placing the object 15 cm from the converging lens.

(C) Placing the object 25 cm from the converging lens.

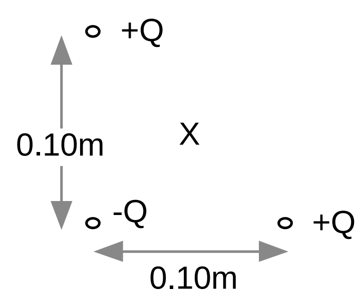
(D) Placing the object 15 cm from the diverging lens.

(E) Placing the object 25 cm from the diverging lens.

21. Two small identical conducting spheres are separated by a distance much larger than their diameter. They are initially given charges of -2.00×10^{-6} C and $+4.00 \times 10^{-6}$ C, and found to exert a force on each other of magnitude 1.000 N. Without changing their position, they are connected by a conducting wire. When the wire is removed, what is the magnitude of the force between them?

(A) 0 (B) 0.125 N (C) 0.250 N (D) 1.000 N (E) 1.125 N

22. Charges of $\pm$Q, with $Q = 2.0 \times 10^{-7}$ C, are placed at three corners of a square whose sides are 0.10 m. The magnitude of the total field at the center of the square is:



(A) 5.1×10^3 V/m

(B) 2.5×10^4 V/m

(C) 7.5×10^4 V/m

(D) 3.6×10^5 V/m

(E) 1.08×10^6 V/m

23. Three 60 W light bulbs are mistakenly wired in series and connected to a 120 V power supply. Assume the light bulbs are rated for single connection to 120 V. With the mistaken connection, the power dissipated by each bulb is:

(A) 6.7 W (B) 13.3 W (C) 20 W (D) 40 W (E) 60 W

24. Two thin spherical conducting shells are centered on the same point. The inner sphere has radius *b* and total charge *q*. The outer sphere has radius *B* and total charge *Q*. At a point a distance *R* from the common center, where $b < R < B$, and assuming the electric potential is zero an infinite distance away, the total potential due to the two spheres is:

(A) kq/b + kQ/B

(B) kq/b + kQ/R

(C) kq/R

(D) kq/R + kQ/B

(E) kq/R + kQ/R

25. What is the magnitude of the total electric field for the two spheres at the point described in the preceding question (#24)?

(A) $kq/b^2 + kQ/B^2$

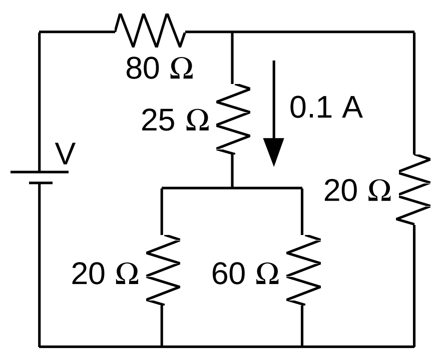
(B) $kq/b^2 + kQ/R^2$

(C) kq/R^2

(D) $kq/R^2 + kQ/B^2$

(E) $kq/R^2 + kQ/R^2$

26. A current of 0.10 A flows through the 25 ½ resistor represented in the diagram to the right. The current through the 80 ½ resistor is:



(A) 0.10 A

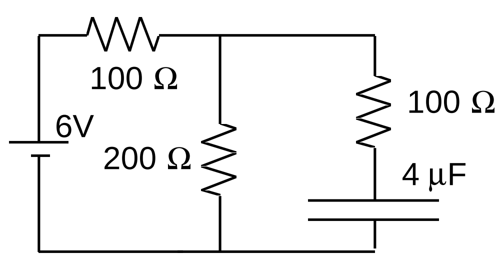
(B) 0.20 A

(C) 0.30 A

(D) 0.40 A

(E) 0.60 A

27. The circuit shown in the accompanying figure has been connected for a very long time. The voltage across the capacitor is:



(A) 1.2 V

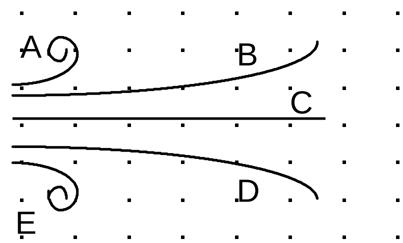
(B) 2.0 V

(C) 2.4 V

(D) 4.0 V

(E) 6.0 V

28. The accompanying sketch represents a bubble chamber photograph. The magnetic field is directed out of the plane of the sketch. All particles have the same velocity initially from left to right. The one that is most likely an electron is:



(A) A

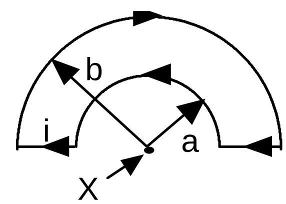
(B) B

(C) C

(D) D

(E) E

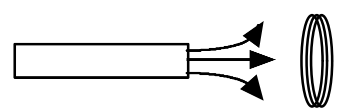
29. What is the correct expression for the magnetic field at point X in the diagram to the right? HINT: The magnitude of the magnetic field at the center of a circular current loop of radius *R* is $\mu_0 i/(2R)$.



(A) $(\mu_0 i/4)(1/a + 1/b)$	into the page
(B) $(\mu_0 i/4)(1/a - 1/b)$	out of the page
(C) $(\mu_0 i/4)(1/a - 1/b) - \mu_0 i/(2\pi a)$	out of the page
(D) $(\mu_0 i/2)(1/a - 1/b)$	out of the page
(E) $(\mu_0 i/2)(1/a + 1/b) + \mu_0 i/(2\pi a)$	into the page

30. You are given a bar magnet and a looped coil of wire. Which of the following would induce an emf in the coil?

I. Moving the magnet away from the coil.
II. Moving the coil toward the magnet.
III. Turning the coil about a vertical axis.



(A) I only (B) II only (C) I & II (D) I & III (E) I, II, III

1993 MULTIPLE CHOICE SCREENING TEST
ANSWER KEY

1. C
2. C
3. B
4. E
5. D
6. A
7. C
8. E
9. A
10. E
11. A
12. B
13. D
14. B
15. D
16. E
17. D
18. B
19. C
20. A
21. B
22. D
23. A
24. D
25. C
26. C
27. D
28. A
29. B
30. E

Physics Olympiad

Entia non multiplicanda sunt praeter necessitatem

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INSTRUCTIONS

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speed of sound (20 ^{o}C)	v_s	$= 340$ m/s

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1. An object is projected straight upward from ground level with a velocity of 50 m/s. Ignoring air resistance, it will return to ground level in approximately:

A. 2.5 s B. 5.0 s C. 7.5 s D. 10 s E. 15 s

2. A jogger runs with constant speed v through a forest of pine trees. A pine cone starts to fall from a height h when the jogger is directly below it. How far behind the jogger will the pine cone land?

A. $\sqrt{\frac{2hv^2}{g}}$ B. $\sqrt{\frac{hv^2}{2g}}$ C. $\frac{gh^2}{2v^2}$ D. $\frac{2gh^2}{v^2}$ E. $\frac{v^2}{2g}$

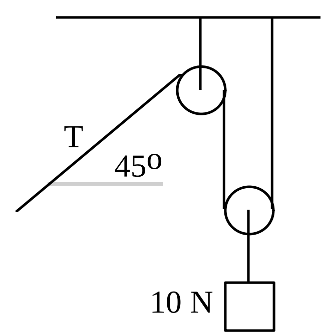
3. A ball is thrown downward with speed 15 m/s from the roof of a 30 m building. At the same instant a ball is thrown upward with speed 15 m/s from ground level. Relative to ground level, the two balls pass each other at a height of:

A. 0 B. 5.0 m C. 10 m D. 15 m E. 20 m

4. A swimmer can swim with a velocity of 1.0 m/s in still water. The swimmer wishes to swim directly across a river with a current of 0.50 m/s directed from upstream to downstream. To end up directly across the river the swimmer must head at an angle of:

A. $\tan^{-1}(1/2)$ upstream.
B. $\sin^{-1}(1/2)$ upstream.
C. directly across the river.
D. $\sin^{-1}(1/2)$ downstream
E. $\tan^{-1}(1/2)$ downstream

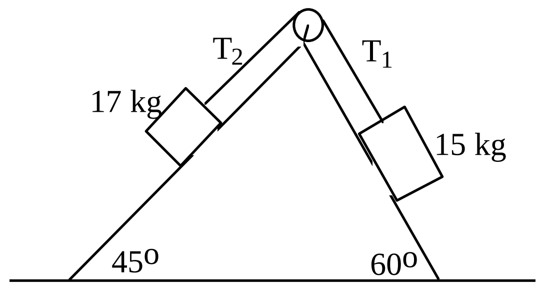
5. What is the tension *T* in the rope if the 10-N weight is moving upward with constant velocity?
A. 3.5 N
B. 5.0 N
C. 7.1 N
D. 10 N
E. 14 N



6. As shown to the right, two blocks with masses m and M ($M > m$) are pushed by a force F in both Case I and Case II. The surface is horizontal and frictionless. Let R_I be the force that m exerts on M in case I and R_{II} be the force that m exerts on M in case II. Which of the following statements is true?

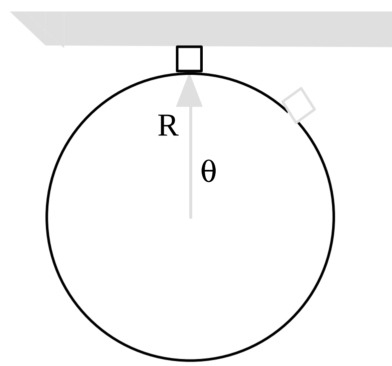
A. $R_I = R_{II} = 0$
B. $R_I = R_{II}$ and is not equal to zero or F.
C. $R_I = R_{II} = F$
D. $R_I < R_{II}$
E. $R_I > R_{II}$

7. Two blocks, with masses 17 kg and 15 kg, are connected by a light string that passes over a frictionless pulley of negligible mass as shown to the right. The surfaces of the planes are frictionless. The blocks are released from rest. T_1 and T_2 are the tensions in the strings. Which of the following statements is correct?



A. The 15-kg block accelerates down the plane.
B. The 17-kg block accelerates down the plane.
C. Both blocks remain at rest.
D. $T_1 > T_2$
E. $T_1 < T_2$

8. A small block of mass m starts from rest at the top of a globe of radius R. At what angle θ does it slide off the surface of the globe? Assume the system is frictionless.



A. $\theta = 0^o$
B. $\theta = \cos^{-1}(1/3)$
C. $\theta = \cos^{-1}(2/3)$
D. $\theta = 60^o$
E. $\theta = 90^o$

9. An object with mass m and initial velocity v is brought to rest by a constant force F acting for a time t and through a distance d. Possible expressions for the magnitude of the force F are:

I. $(mv^2)/(2d)$
II. $(2md)/t^2$
III. $(mv)/t$

Which of these give(s) the correct expression for the magnitude of the force F?

A. II only B. III only C. I and II only D. II and III only E. I, II, and III

10. A small sphere is moving at a constant speed in a vertical circle. Below is a list of quantities that could be used to describe some aspect of the motion of the sphere.

I – kinetic energy
II – potential energy
III – momentum

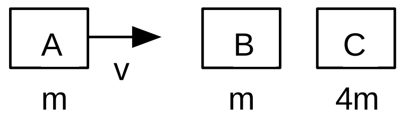
Which of these quantities will change as this sphere moves around the circle?

A. I and II only B. I and III only C. II only D. III only E. II and III only

11. A roller coaster travels with speed v_A at point A. Point B is a height H above point A. Assuming no frictional losses and no work done by a motor, what is the speed at point B?

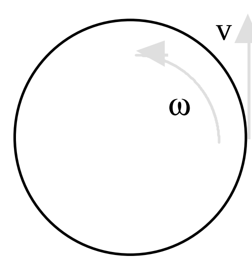
A. $\sqrt{v_A^2 - 2gH}$ B. $v_A - \sqrt{2gH}$ C. $v_A - 2gH$ D. $v_A + \sqrt{2gH}$ E. $\sqrt{v_A^2 + 2gH}$

12. Three air track cars are initially placed as shown in the accompanying figure. Car A has mass m and initial velocity v to the right. Car B with mass m and car C with mass 4m are both initially at rest. Car A collides elastically with car B, which in turn collides elastically with car C. After the collision, car B is at rest. The final velocities of cars A and C are:



A. Car A: 0.6v to the left Car C: 0.4v to the right
B. Car A: 2.6v to the left Car C: 0.4v to the right
C. Car A: at rest Car C: 0.5v to the right
D. Car A: at rest Car C: 0.25v to the right
E. Car A: at rest Car C: v to the right

13. A child with mass m is standing at the edge of a playground merry-go-round with moment of inertia I, radius R, and initial angular velocity ω. See figure to the right. The child jumps off the edge of the merry-go-round with tangential velocity v with respect to the ground. The new angular velocity of the merry-go-round is:



A. w

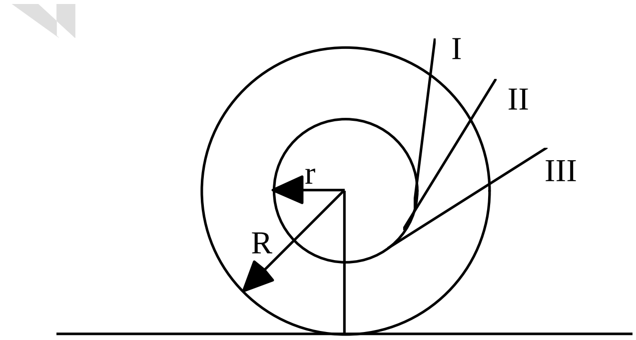
B. $\sqrt{\dfrac{I\omega^2 - mv^2}{I}}$

C. $\sqrt{\dfrac{(I + mR^2)\omega^2 - mv^2}{I}}$

D. $\dfrac{I\omega - mvR}{I}$

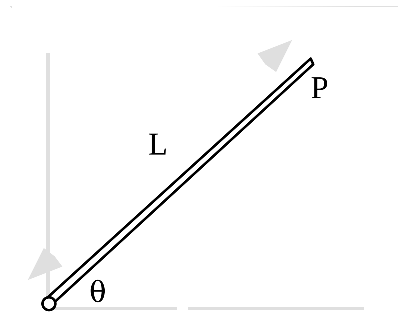
E. $\dfrac{(I + mR^2)\omega - mvR}{I}$

14. As shown in the figure to the right, a spool has outer radius R and axle radius r. A string is wrapped around the axle of the spool and can be pulled in any of the directions labeled by I, II, or III. The spool will slide to the right without rolling on the horizontal surface if it is pulled in direction(s):



A. I only
B. II only
C. III only
D. I and II only
E. II and III only

15. A uniform flag pole of length L and mass M is pivoted on the ground with a frictionless hinge. The flag pole makes an angle θ with the horizontal. The moment of inertia of the flag pole about one end is $(1/3)ML^2$. If it starts falling from the position shown in the accompanying figure, the linear acceleration of the free end of the flag pole – labeled P – would be:

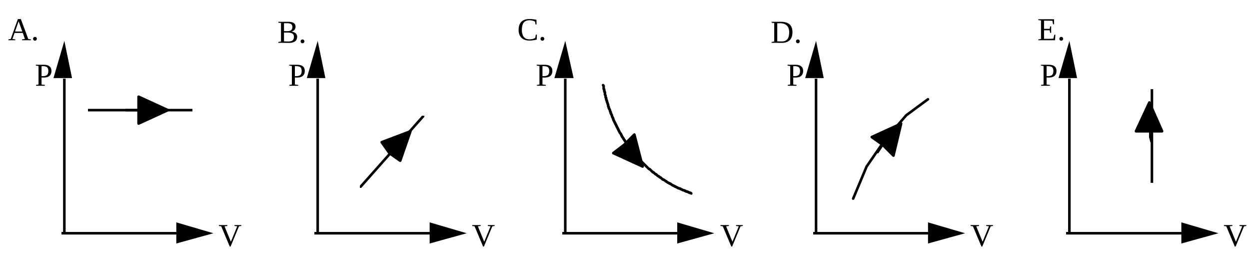


A. (2/3) g cos θ
B. (2/3) g
C. g
D. (3/2) g cos θ
E. (3/2) g

16. The root mean square velocity of oxygen gas (atomic mass 16) is v at room temperature. What is the root mean square velocity of helium (atomic mass 4) at the same temperature?

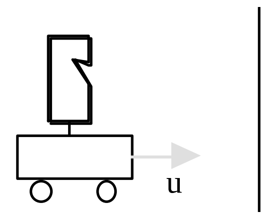
A. 4 v   B. 2 v   C. v   D. v/2   E. v/4

17. Which of the accompanying PV diagrams best represents an adiabatic process (process where no heat enters or leaves the system)?



18. String A and String B have the same mass and length. String A is under tension T and string B is under tension 2T. The speed of waves in B is _____ times the speed of waves in A.
A. 0.50   B. 0.71   C. 1.00   D. 1.4   E. 2.0

19. On a day when the velocity of sound in air is v, a whistle moves with velocity u toward a stationary wall. The whistle emits sound with frequency f. What frequency of reflected sound will be heard by an observer traveling along with the whistle?



A. $f\left(\frac{v-u}{v+u}\right)$

B. $f\left(\frac{v}{v+u}\right)$

C. f

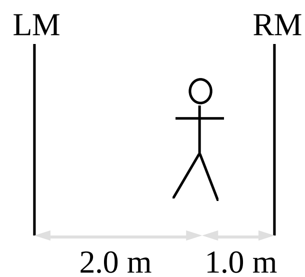
D. $f\left(\frac{v}{v-u}\right)$

E. $f\left(\frac{v+u}{v-u}\right)$

20. You are given two lenses, a converging lens with focal length + 10 cm and a diverging lens with focal length - 20 cm. Which of the following would produce a real image that is larger than the object?

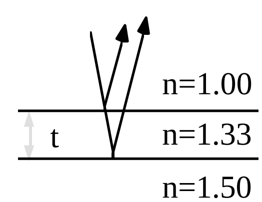
A. Placing the object 5 cm from the converging lens.
B. Placing the object 15 cm from the converging lens.
C. Placing the object 25 cm from the converging lens.
D. Placing the object 15 cm from the diverging lens.
E. Placing the object 25 cm from the diverging lens.

21. Two mirrors, labeled LM for left mirror and RM for right mirror in the accompanying figure, are parallel to each other and 3.0 m apart. A person standing 1.0 m from the right mirror (RM) looks into this mirror and sees a series of images. How far from the person is the second closest image seen in the right mirror (RM)?



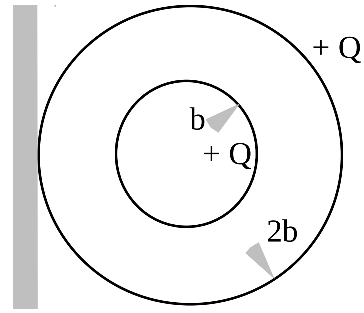
A. 2.0 m
B. 4.0 m
C. 6.0 m
D. 8.0 m
E. 10.0 m

22. A thin film of thickness t and index of refraction 1.33 coats a glass with index of refraction 1.50. What is the least thickness t that will strongly reflect light with wavelength 600 nm? Hint: Light undergoes a 180º phase shift when it is reflected off a material with a higher index of refraction.



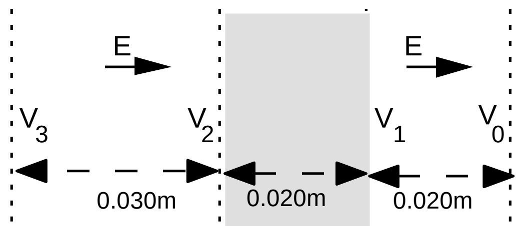
A. 225 nm B. 300 nm C. 400 nm D. 450 nm E. 600 nm

23. The accompanying figure shows two concentric spherical shells isolated from each other. The smaller shell has radius b and net charge +Q. The larger shell has radius 2b and an equal net charge +Q. If R is the distance from the common center, the highest electric field magnitude E occurs:



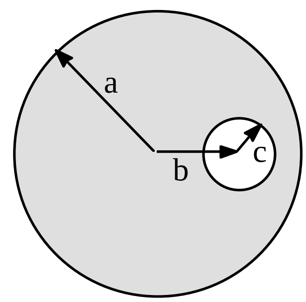
A. only at R = 0, where E is infinite.
B. anywhere R < b, where E is constant
C. immediately outside the smaller (R = b) shell.
D. immediately outside the larger (R = 2b) shell.
E. far away from the shells; E increases with distance.

24. An infinite conducting plate of thickness 0.0200 m is surrounded by a uniform field E = 400 V/m directed left to right. See the figure to the right. Let the potential $V_o = 0$ a distance 0.0200 m to the right of the plate. What is V_3, the potential 0.0300 m to the left of the plate?



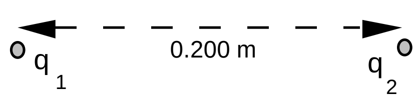
A. - 28 V
B. - 20 V
C. + 12 V
D. + 20 V
E. + 28 V

25. A sphere of radius a has uniform charge density ρ. A spherical cavity of radius c is formed in the sphere. The cavity is centered a distance b (b>c) from the center of the sphere. What is the magnitude of the electric field at the center of the cavity?



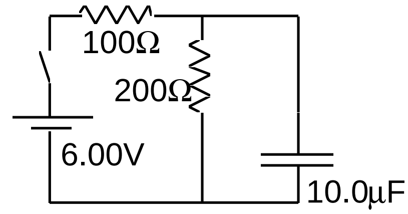
A. $\dfrac{r\,b}{3e_o}$ B. $\dfrac{r\,a^3}{3e_o b^2}$ C. $\dfrac{r\left(a^3 - c^3\right)}{3e_o b^2}$ D. $\dfrac{r\left(b - \dfrac{c^3}{2b^2}\right)}{3e_o}$ E. $\dfrac{r\left(b - \dfrac{c^3}{b^2}\right)}{3e_o}$

26. As shown in the diagram to the right, two fixed charges, q_1 = +1.00 μC and q_2 = - 4.00 μC, are 0.200 m apart. Where is the total field zero?



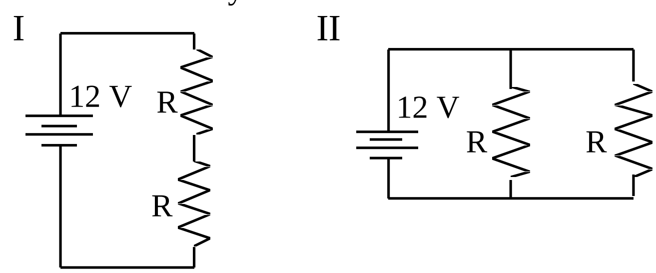
A. 0.40 m to the right of q_1
B. 0.13 m to the right of q_1
C. 0.1m to the right of q_1
D. 0.067 m to the left of q_1
E. 0.20 m to the left of q_1

27. The switch is closed in the circuit shown to the right. What is the charge on the capacitor when it is fully charged?



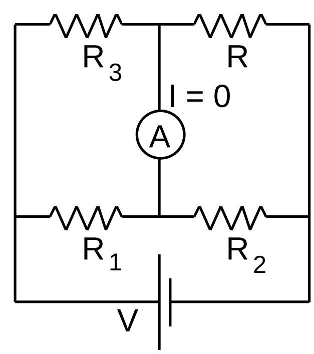
A. 5.0 µC
B. 10 µC
C. 20 µC
D. 40 µC
E. 60 µC

28. Two identical resistors with resistance R are connected in the two circuits drawn below. The battery in both circuits is a 12 volt battery. Which statement is correct?



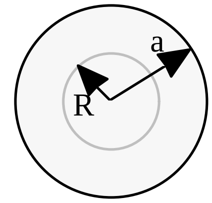
A. More current will flow through each R in circuit I than in circuit II.
B. More total power will be delivered by the battery in circuit II than in circuit I.
C. The potential drop across each R will be greater in circuit I than in circuit II.
D. The equivalent resistance will be greater in circuit II than in circuit I.
E. The power dissipated in each R will be greater in circuit I than in circuit II.

29. The resistors – R_1, R_2, and R_3 – have been adjusted so that the current in the ammeter (labeled A in the accompanying circuit diagram) is zero. What is R?



A. R_2
B. R_3
C. R_1R_2/R_3
D. R_1R_3/R_2
E. R_2R_3/R_1

30. A long cylindrical conducting wire – shown in cross section to the right – carries a conventional current out of the page. The wire has uniform current density J and radius a. What is the magnetic field inside the wire, a distance R (R<a) from the wire's center?



A. $(1/2)\ \mu_o Ja$ clockwise
B. $(1/2)\ \mu_o Ja^2/R$ clockwise
C. $(1/2)\ \mu_o JR$ counterclockwise
D. $(1/2)\ \mu_o Ja^2/R$ counterclockwise
E. $(1/2)\ \mu_o Ja$ counterclockwise

1996 MULTIPLE CHOICE SCREENING TEST
ANSWER KEY

1. D	11. A	21. C
2. A	12. A	22. A
3. C	13. E	23. C
4. B	14. B	24. D
5. B	15. D	25. A
6. D	16. B	26. E
7. A	17. C	27. D
8. C	18. D	28. B
9. E	19. E	29. E
10. E	20. B	30. C

Physics Olympiad

(Πηψσιχσ Ολψμπιαδ)

1995 MULTIPLE CHOICE SCREENING TEST
30 QUESTIONS—40 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

This test contains 30 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 30 are to be used on the answer sheet.

Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.

A hand-held calculator may be used. However, any memory must be cleared of data and programs. Calculators may not be shared.

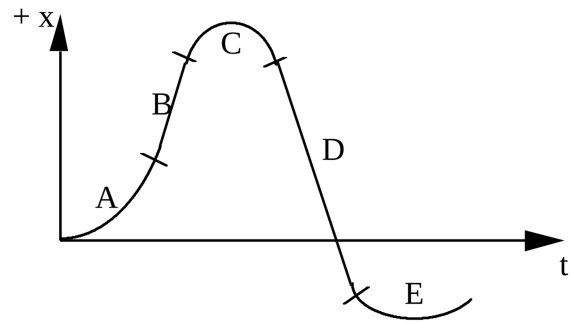
Your grade on this multiple choice test will be your number of correct answers. There is no penalty for guessing. It is to your advantage to answer every question.

The values of some possibly useful constants are given below:

mass of electron	m_e	= 9.1 x 10^{-31} kg
mass of proton	m_p	= 1.7 x 10^{-27} kg
electronic charge	e	= 1.6 x 10^{-19} C
speed of light	c	= 3.0 x 10^{8} m/s
Coulomb's constant	k	= 9.0 x 10^{9} N·m^2/C^2
permittivity constant	ε_0	= 8.9 x 10^{-12} C^2/N·m^2
permeability constant	μ_0	= 4^1 x 10^{-7} T·m/A
gravitational constant	G	= 6.7 x 10^{-11} N·m^2/kg^2
acceleration due to gravity	g	= 10 m/s^2
speed of sound (20 ^{o}C)	v_s	= 340 m/s

DO NOT OPEN THIS TEST UNTIL YOU ARE INSTRUCTED TO BEGIN

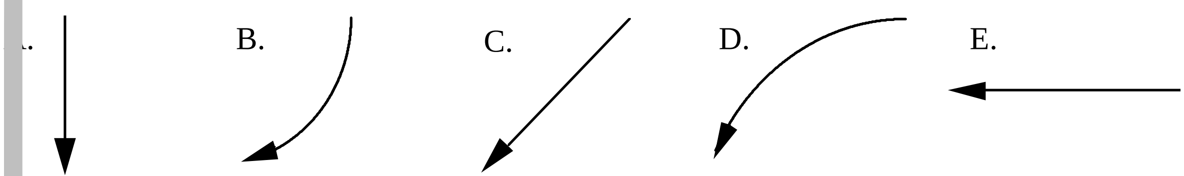
. The accompanying figure is a graph of an
bject's position as a function of time. Which
ettered segment corresponds to a time when the
bject has a negative acceleration?



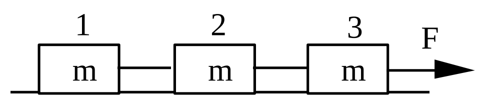
.. A
. B
. C
). D
. E

. A car is moving at a constant velocity to the right along a straight level highway. Just as the
ar passes a cliff, a rock falls straight down in the cliff's reference system. Which of the
ccompanying curves best depicts the path the rock takes in the car's reference system?

.. B. C. D. E.



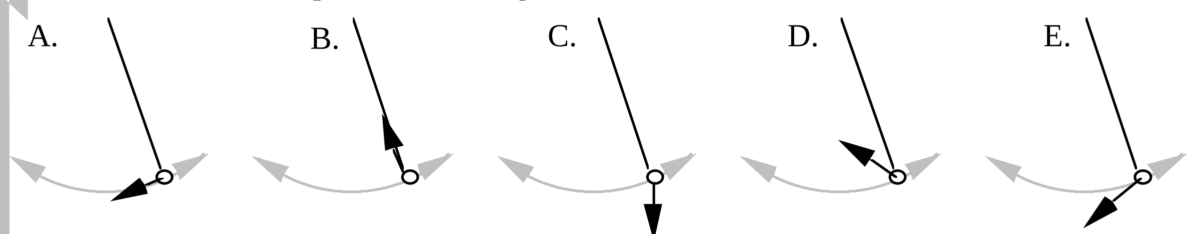
. Three blocks – 1, 2, and 3 – rest on a horizontal
rictionless surface, as shown in the accompanying figure.
ach block has a mass *m*, and the blocks are connected by
assless strings. Block 3 is pulled to the right by a force *F*. The resultant force on block 2 is:



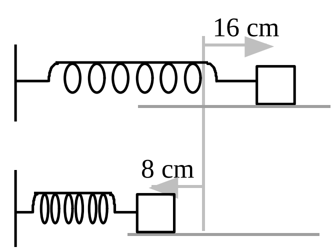
.. Zero B. (1/3) F C. (1/2) F D. (2/3) F E. F

. Which solid vector in the accompanying figures best represents the acceleration of a pendulum
ass at an intermediate point in its swing?

A. B. C. D. E.

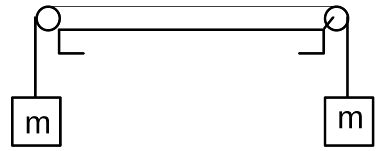


5. A mass attached to a horizontal massless spring is displaced 16 cm
to the right of its equilibrium position (see accompanying figure) and
released from rest. At its 16 cm extension, the spring-mass system has
1.28 joules of potential energy. Upon release, it slides across a rough
surface and comes momentarily to rest 8 cm to the left of its
equilibrium position. How much mechanical energy was dissipated by friction?



A. 0.16 J B. 0.32 J C. 0.64 J D. 0.96 J E. 1.12 J

6. Two identical masses m are connected to a massless string which is hung over two frictionless pulleys as shown on the right. If everything is at rest, what is the tension in the cord?



A. Less than mg.
B. Exactly mg.
C. More than mg but less than $2mg$.
D. Exactly $2mg$.
E. More than $2mg$.

7. Car A and car B are both traveling down a straight highway at 25 m/s (about 56 mph). Car A is only 6.0 m behind car B. The driver of car B brakes, slowing down with a constant acceleration of 2.0 m/s^2. After a time 1.2 s, the driver of car A begins to brake, also at 2.0 m/s^2. What is the relative velocity of the two cars when they collide? Hint: Both cars are still moving forward when they collide.

A. 2.4 m/s B. 5.0 m/s C. 9.5 m/s D. 21 m/s E. 24 m/s

8. During the collision between car A and car B described in Problem # 7 above, which car experiences the greater change in momentum?

A. The more massive car.
B. The less massive car.
C. Car A because its velocity at the start of the collision is greater.
D. Car B because its velocity at the start of the collision is less.
E. Both cars experience the same magnitude of momentum change.

9. A 70 kg hunter ropes a 350 kg polar bear. Both are initially at rest, 30 m apart on a frictionless and level ice surface. When the hunter pulls the polar bear to him, the polar bear will move:

A. 5 m B. 6 m C. 15 m D. 24 m E. 25 m

10. A bullet of mass m is fired at a block of mass M initially at rest. The bullet, moving at an initial speed v, embeds itself in the block. The speed of the block after the collision is:

A. $\frac{Mv}{M+m}$ B. $\frac{(M+m)v}{m}$ C. $\left(\sqrt{\frac{m}{M+m}}\right)v$ D. $\left(\sqrt{\frac{M+m}{m}}\right)v$ E. $\frac{mv}{M+m}$

11. The driver of a 1000 kg car tries to turn through a circle of radius 100 m on an unbanked curve at a speed of 10 m/s. The maximum frictional force between the tires and the slippery road is 900 N. The car will:

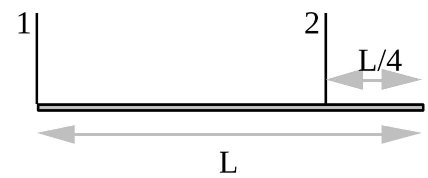
A. Slide into the inside of the curve.
B. Make the turn.
C. Slow down due to the centrifugal force.
D. Make the turn only if it speeds up.
E. Slide off to the outside of the curve.

12. A solid cylinder weighing 200 N has a fixed axis and a string wrapped around it. The string is pulled with a force equal to the weight of the cylinder. The acceleration of the string is approximately:

(A) 10 m/s^2 (B) 20 m/s^2 (C) 30 m/s^2 (D) 40 m/s^2 (E) 50 m/s^2

13. A spinning ice skater has an initial kinetic energy $(1/2)I\omega^2$. She pulls in her outstretched arms, decreasing her moment of inertia to (1/4)I. Her new angular speed is:

A. $\omega/4$ B. $\omega/2$ C. ω D. 2ω E. 4ω



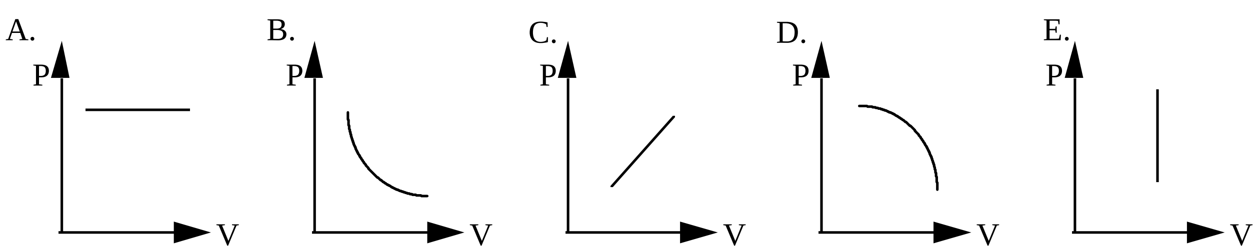
14. A heavy rod of length *L* and weight *W* is suspended horizontally by two vertical ropes as shown on the right. The first rope is attached to the left end of the rod while the second rope is attached a distance *L*/4 from the right end. The tension in the second rope is:

A. (1/2)W B. (1/4)W C. (1/3)W D. (2/3)W E. W

15. A block of ice with mass *m* falls into a lake. After impact, a mass of ice *m/5* melts. Both the block of ice and the lake have a temperature of 0^{o}C. If *L* represents the heat of fusion, the minimum distance the ice fell is:

A. $\frac{L}{5g}$ B. $\frac{5L}{g}$ C. $\frac{gL}{5m}$ D. $\frac{mL}{5g}$ E. $\frac{5gL}{m}$

16. Which of the accompanying PV diagrams best represents an isothermal (constant temperature) process?



17. If the heat is added at constant volume, 6300 joules of heat are required to raise the temperature of an ideal gas by 150 K. If instead, the heat is added at constant pressure, 8800 joules are needed for the same temperature change. When the temperature of the gas changes by 150 K, the internal energy of the gas changes by:

A. 2500 J B. 6300 J C. 8800 J D. 11,300 J E. 15,100 J

18. A firecracker exploding on the surface of a lake is heard as two sounds – a time interval *t* apart – by the paddler of a nearby canoe. Sound travels with a speed *u* in water and a speed *v* in air. The distance from the exploding firecracker to the canoe is:

A. $\frac{uvt}{u+v}$ B. $\frac{t(u+v)}{uv}$ C. $\frac{t(u-v)}{uv}$ D. vt E. $\frac{uvt}{u-v}$

19. Two interfering waves have the same wavelength, frequency, and amplitude. They are traveling in the same direction but are 90^o out of phase. Compared to the individual waves, the resultant wave will have the same:

A. amplitude and velocity but different wavelength.
B. amplitude and wavelength but different velocity.
C. wavelength and velocity but different amplitude.
D. amplitude and frequency but different velocity.
E. frequency and velocity but different wavelength.

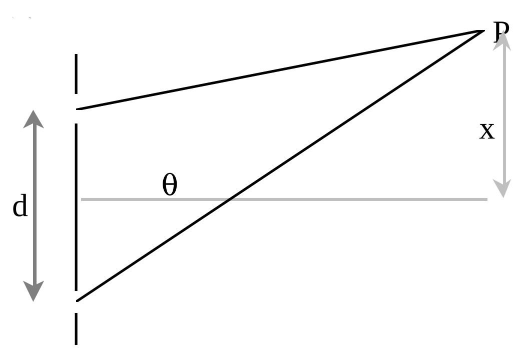
20. An organ pipe which is open at both ends resonates with fundamental frequency 300 Hz. If one end of the pipe is closed, it will resonate with a fundamental frequency of:

A. 75 Hz B. 150 Hz C. 300 Hz D. 600 Hz E. 1200 Hz

21. You are given two lenses, a converging lens with focal length + 10 cm and a diverging lens with focal length - 20 cm. Which of the following would produce a virtual image that is larger than the object?

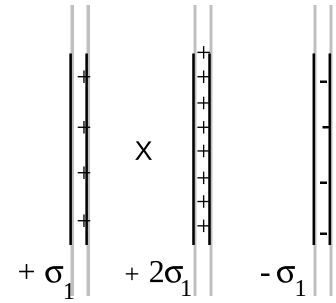
A. Placing the object 5 cm from the converging lens.
B. Placing the object 15 cm from the converging lens.
C. Placing the object 25 cm from the converging lens.
D. Placing the object 15 cm from the diverging lens.
E. Placing the object 25 cm from the diverging lens.

22. Light shining through two very narrow slits produces an interference maximum at point P when the entire apparatus is in air (see accompanying figure). For the interference maximum represented, light through the bottom slit travels one wavelength further than light through the top slit before reaching point P. If the entire apparatus is immersed in water, the angle θ to the interference maximum:



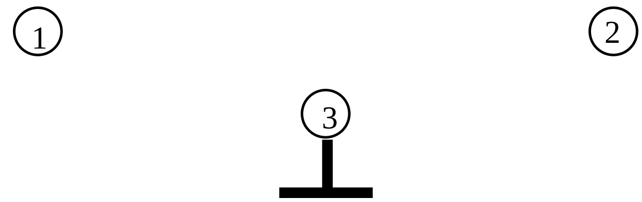
A. is unchanged
B. decreases because the frequency decreases.
C. decreases because the wavelength decreases.
D. increases because the frequency increases.
E. increases because the wavelength increases.

23. The magnitude of the field due to an infinite plate of charge is $\sigma/(2\varepsilon_o)$, where σ is the charge per unit area and ε_o is the vacuum permittivity. The figure to the right depicts three infinite plates of charge perpendicular to the plane of the page with charge per unit area $+\sigma_1$, $+2\sigma_1$, and $-\sigma_1$. The total field at the point labeled X is:



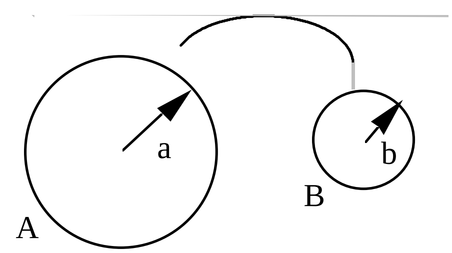
A. $4\sigma_1/(2\varepsilon_o)$ to the left.
B. $\sigma_1/(2\varepsilon_o)$ to the left.
C. 0
D. $\sigma_1/(2\varepsilon_o)$ to the right.
E. $4\sigma_1/(2\varepsilon_o)$ to the right.

24. There are three conducting spheres of identical diameter, shown to the right. Spheres 1 and 2 are separated by a distance that is large compared to their diameter. They have equal charges and repel each other with an electrostatic force *F*. Sphere 3 is initially uncharged and has an insulated handle. It is touched first to sphere 1, then to sphere 2, and then removed. If the distance between spheres 1 and 2 has not changed, the force between these two spheres is:



A. 0 B. (1/16)F C. (1/4)F D. (3/8)F E. (1/2)F

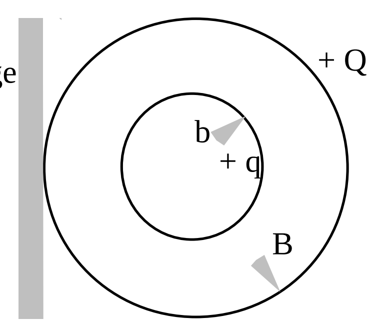
25. Two conducting spheres, A of radius *a* and B of radius *b*, are shown in the accompanying figure. Both are positively charged and isolated from their surroundings. They had been connected by a conducting wire but the wire has been removed. Assuming the potential an infinite distance away is zero, which of the following statements about the spheres are true:



I. Sphere A is at the higher potential.
II. Sphere B is at the higher potential.
III. Both spheres have the same potential.
IV. Sphere A has the larger charge.
V. Sphere B has the larger charge.
VI. Both spheres have the same charge.

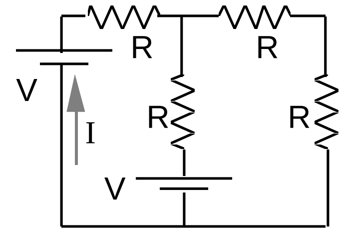
A. I and IV B. I and VI C. II and VI D. III and IV E. III and V

26. The accompanying figure shows two concentric spherical shells isolated from each other. The smaller shell has radius *b* and net charge + *q*. The larger shell has radius *B* and net charge + *Q*. Assume that the potential is zero at an infinite distance from the shells. If *R* is the distance from the common center, the highest electric potential *V* occurs:



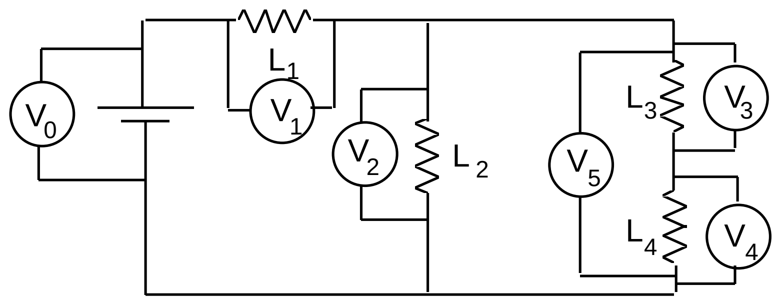
A. only at $R = 0$, where V is infinite.
B. anywhere $R \le b$, where *V* is constant
C. in the region $b < R < B$.
D. immediately outside the larger shell; *V* is zero everywhere within it.
E. far away from the shells; *V* increases with distance.

27. In the circuit represented to the right, the current I equals:



A. $V/5R$
B. $V/4R$
C. $2V/5R$
D. $2V/4R$
E. $2V/R$

Use the circuit below to answer questions 28 and 29. L_1, L_2, L_3, and L_4 are identical light bulbs. There are six voltmeters connected to the circuit as shown. Assume that the voltmeters do not effect the circuit.



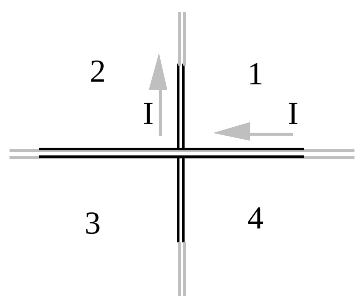
28. Which of the following combinations are equal to V_0?

A. $V_1 + V_3$ B. $V_2 + V_3 + V_4$ C. $V_1 + V_5$ D. $V_3 + V_4$ E. $V_1 + V_4$

29. If L_3 were to burn out, opening the circuit, which voltmeter(s) would read zero volts?

A. None would read zero.
B. only V_3
C. only V_4
D. only V_3, V_4, and V_5
E. they would all read zero

30. Identical currents flow in two perpendicular wires, as shown in the accompanying figure. The wires are very close but do not touch. The magnetic field can be zero:



A. at a point in region 1 only
B. at a point in region 2 only
C. at points in both regions 1 and 2
D. at points in both regions 1 and 4
E. at points in both regions 2 and 4

1995 MULTIPLE CHOICE SCREENING TEST
ANSWER KEY

1.	C	11.	E	21.	A
2.	D	12.	B	22.	C
3.	B	13.	E	23.	C
4.	D	14.	D	24.	D
5.	D	15.	A	25.	D
6.	B	16.	B	26.	B
7.	A	17.	B	27.	A
8.	E	18.	E	28.	C
9.	A	19.	C	29.	C
10.	E	20.	B	30.	E

Physics Olympiad

(Πηψσιχσ Ολψμπιαδ)

1994 MULTIPLE CHOICE SCREENING TEST
30 QUESTIONS—40 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

This test contains 30 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 30 are to be used on the answer sheet.

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Your grade on this multiple choice test will be your number of correct answers. There is no penalty for guessing. It is to your advantage to answer every question.

The values of some possibly useful constants are given below:

mass of electron	m_e	$= 9.1 \times 10^{-31}$ kg
mass of proton	m_p	$= 1.7 \times 10^{-27}$ kg
electronic charge	e	$= 1.6 \times 10^{-19}$ C
speed of light	c	$= 3.0 \times 10^{8}$ m/s
Coulomb's constant	k	$= 9.0 \times 10^{9}$ N·m^2/C^2
permittivity constant	ε_0	$= 8.9 \times 10^{-12}$ C^2/N·m^2
permeability constant	μ_0	$= 4\pi \times 10^{-7}$ T·m/A
gravitational constant	G	$= 6.7 \times 10^{-11}$ N·m^2/kg^2
acceleration due to gravity	g	$= 10$ m/s^2
speed of sound (20 °C)	v_s	$= 340$ m/s

DO NOT OPEN THIS TEST UNTIL YOU ARE INSTRUCTED TO BEGIN

1. A motorist travels 320 km at 80 km/h and then 320 km at 100 km/h. What is the average speed of the motorist for the entire trip?

A. 84 km/h B. 89 km/h C. 90 km/h D. 91 km/h E. 95 km/h

2. A sports car is stopped at a light. At $t = 0$ the light changes and the sports car accelerates at a constant 2.0 m/s^2. At t = (10/3) s a station wagon traveling at a constant 15 m/s in the same direction passes the stop light. When does the station wagon catch up to the sports car?

A. never B. t = 5.0 s C. t = 7.5 s D. t = 12 s E. t = 21 s

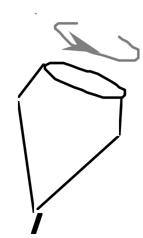
3. A ball is thrown from ground level with an initial velocity of v_o in the upward direction. It reaches a maximum height y and returns to ground level at time t seconds after it was thrown. If its initial velocity is doubled, it will be in the air _____ seconds and reach a maximum height _____.

A. t, 4y B. 2t, y C. 2t, 2y D. 2t, 4y E. 4t, 2y

4. The ball in the preceding question (#3) is taken to Mars where the acceleration due to gravity is approximately 4 m/s^2 (0.4 g_e). It is thrown from ground level with the same initial velocity as it originally had on Earth, v_o in the upward direction. On Mars, it will be in the air _____ seconds and reach a maximum height _____.

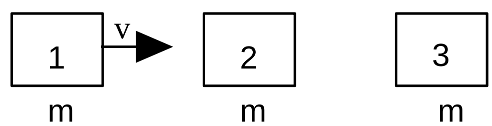
A. 0.4 t, 2.5 y B. 0.4 t, 6.25 y C. 2.5 t, 2.5y D. 6.25 t, 2.5 y E. 6.25 t, 6.25 y

5. A top is spinning in the direction shown in the accompanying figure. Its axis of rotation makes an angle of 15° with the vertical. Assume friction can be neglected. The magnitude of the top's angular momentum will _____ while its direction will _____.



A. increase, precess in the counterclockwise direction when seen from above.
B. increase, not precess
C. remain the same, not precess
D. remain the same, precess in the clockwise direction when seen from above.
E. remain the same, precess in the counterclockwise direction when seen from above.

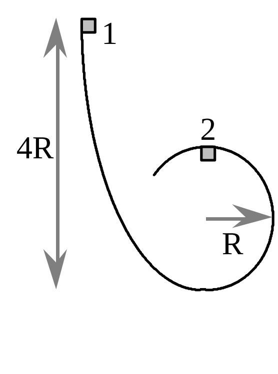
6. Three air track cars, shown in the accompanying figure, all have the same mass m. Cars 2 and 3 are initially at rest. Car 1 is moving to the right with speed v. Car 1 collides with car 2 and sticks to it. The 1-2 combination collides elastically with car 3. Which of the following is most nearly the final speed of the 1-2 combination?



A. 0.17 v B. 0.50 v C. 0.67 v D. 0.80 v E. 1.0 v

7. A cube with mass M starts at rest at point 1 at a height $4R$, where R is the radius of the circular part of the track. The cube slides down the frictionless track and around the loop. The force that the track exerts on the cube at point 2 is most nearly _____ times the cube's weight Mg.

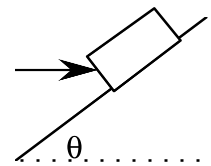
A. 1 B. 2 C. 3 D. 4 E. 5



8. An astronaut with weight *W* on Earth lands on a planet with mass 0.1 times the mass of Earth and radius 0.5 times the radius of Earth. The astronaut's weight is _____ on the planet.

A. 0.02 W B. 0.04 W C. 0.2 W D. 0.4 W E. W

9. A horizontal force *F*, represented by the arrow in the figure to the right, is used to push a block of weight *mg* up an inclined plane making an angle of θ with the horizontal. The coefficient of friction between the plane and the block is μ. The magnitude of the frictional force acting on the block is:



A. μ mg cosθ

B. μ mg/(cosθ)

C. μ (mg cosθ + F sinθ)

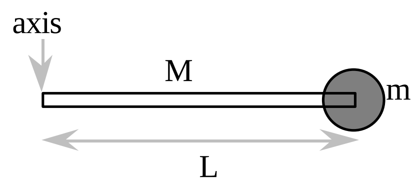
D. μ (F cosθ – mg sinθ)

E. μ F cosθ

10. A child of mass *M* stands on the edge of a merry-go-round of radius *R* and moment of inertia *I*. Both the merry-go-round and child are initially at rest. The child walks around the circumference with speed *v* with respect to the ground. What is the magnitude of the angular velocity of the merry-go-round with respect to the ground?

A. 0 B. $\omega = \frac{MRv}{I}$ C. $\omega = \frac{v}{R}$ D. $\omega = \frac{MRv}{I - MR^2}$ E. $\omega = \frac{MRv}{I + MR^2}$

11. A rigid rod of mass *M* and length *L* has moment of inertia $\frac{1}{12}ML^2$ about its center of mass. A sphere of mass *m* and radius *R* has moment of inertia $\frac{2}{5}mR^2$ about its center of mass. A combined system is formed by centering the sphere at one end of the rod and placing an axis at the other (see accompanying figure). What is the moment of inertia of the combined system about the axis shown?



A. $I = \frac{1}{12}ML^2 + \frac{2}{5}mR^2$

B. $I = \frac{1}{12}ML^2 + \frac{2}{5}mR^2 + mL^2$

C. $I = \frac{1}{3}ML^2 + \frac{2}{5}mR^2 + mL^2$

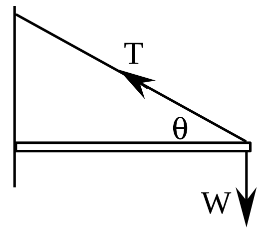
D. $I = \frac{1}{12}ML^2 + mL^2$

E. $I = \frac{1}{3}ML^2 + mL^2$

12. A rocket is launched from the surface of a planet with mass *M* and radius *R*. What is the minimum velocity the rocket must be given to completely escape from the planet's gravitational field?

A. $v = \sqrt{\frac{2GM}{R^2}}$ B. $v = \sqrt{\frac{2GM}{R}}$ C. $v = \sqrt{\frac{GM}{R^2}}$ D. $v = \sqrt{\frac{GM}{R}}$ E. $v = \sqrt{GM}$

13. A uniform rod of length L and weight W_R is suspended as shown in the accompanying figure. A weight W is added to the end of the rod. The support wire is at an angle θ to the rod. What is the tension T in the wire?



A. $T = \dfrac{W}{\sin\theta}$

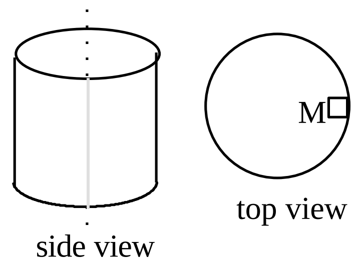
B. $T = W + W_R$

C. $T = W + \frac{1}{2}W_R$

D. $T = \dfrac{W + \frac{1}{2}W_R}{\sin\theta}$

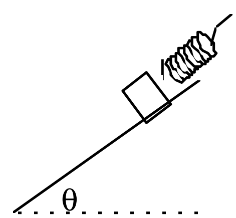
E. $T = \dfrac{W + W_R}{\sin\theta}$

14. A hollow vertical cylinder of radius R is rotated with angular velocity ω about an axis through its center. What is the minimum coefficient of static friction necessary to keep the mass M suspended on the inside of the cylinder as it rotates?



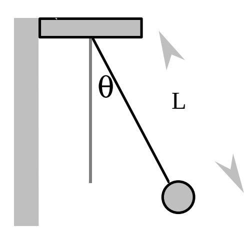
A. $\mu = \dfrac{gR}{\omega^2}$ B. $\mu = \dfrac{\omega^2 g}{R}$ C. $\mu = \dfrac{\omega^2 R}{g}$ D. $\mu = \dfrac{\omega^2}{gR}$ E. $\mu = \dfrac{g}{\omega^2 R}$

15. A block of mass M is attached to a relaxed spring with force constant k, placed on a frictionless inclined plane as shown in the accompanying figure, and released. What is the maximum extension of the spring?



A. $x = \dfrac{2Mg\sin\theta}{k}$ B. $x = \dfrac{Mg\sin\theta}{k}$ C. $x = \dfrac{2Mg}{k}$ D. $x = \dfrac{Mg}{k}$ E. $x = \sqrt{2gM}$

16. A simple pendulum of length L and mass m is attached to a moving support. In order for the pendulum string to make a constant angle θ with the vertical, the support must be moving to the:



A. right with constant acceleration $a = g\tan\theta$.

B. left with constant acceleration $a = g\tan\theta$.

C. right with constant acceleration $a = g\sin\theta$.

D. right with constant velocity $v = \sqrt{Lg}\tan\theta$.

E. left with constant velocity $v = \sqrt{Lg}\tan\theta$.

17. A Carnot cycle takes in 1000 J of heat at a high temperature of 400 K. How much heat is expelled at the cooler temperature of 300 K?

A. 0 B. 250 J C. 500 J D. 750 J E. 1000 J

18. An ideal gas is expanded at constant pressure from initial volume V_i and temperature T_i to final volume V_f and temperature T_f . The gas has molar heat capacity C_P at constant pressure. The amount of work done by n moles of the gas during the process can be expressed:

A. 0 B. $nRT_i \ln(V_f/V_i)$ C. $C_P n(T_f - T_i)$ D. $nk(V_f - V_i)$ E. $nR(T_f - T_i)$

19. The average translational kinetic energy of any ideal gas depends only on:

A. the absolute (Kelvin) temperature.

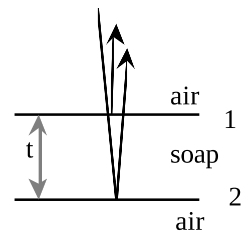
B. the mass of the gas.

C. the pressure of the gas.

D. the amount of the gas.

E. whether the gas is monatomic or diatomic.

20. A soap film of thickness t is surrounded by air (see accompanying figure). It is illuminated at near normal incidence by monochromatic light which has wavelength λ in the film. A film thickness _____ will produce maximum brightness of the reflected light.



A. 1/4 λ B. 1/2 λ C. 1 λ D. 2 λ E. 4 λ

21. You are given two lenses, a converging lens with focal length +10 cm and a diverging lens with focal length –20 cm. Which of the following would produce a real image that is smaller than the object?

A. Placing the object 5 cm from the converging lens.

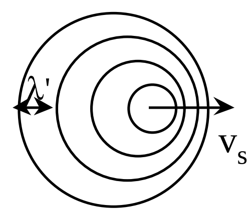
B. Placing the object 15 cm from the converging lens.

C. Placing the object 25 cm from the converging lens.

D. Placing the object 15 cm from the diverging lens.

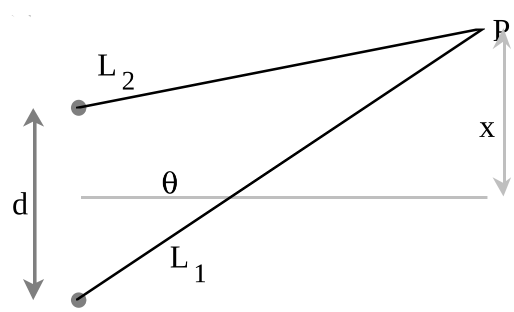
E. Placing the object 25 cm from the diverging lens.

22. A source at rest emits waves with wavelength λ in a medium with velocity v. If the source moves to the right with velocity v_s(see accompanying figure), the distance between adjacent crests λ' directly behind the source is:



A. $\dfrac{\lambda v}{v + v_s}$ B. $\dfrac{\lambda v}{v - v_s}$ C. $\lambda\left(1 + \dfrac{v}{v_s}\right)$ D. $\lambda\left(1 + \dfrac{v_s}{v}\right)$ E. $\lambda\left(1 - \dfrac{v_s}{v}\right)$

23. Two sources, in phase and a distance *d* apart, each emit a wave of wavelength λ. See accompanying figure. Which of the choices for the path difference $\Delta L = L_1 - L_2$ will always produce constructive interference at point P?



A. d sinθ

B. x/L_1

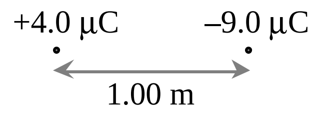
C. $(x/L_2)d$

D. $\lambda/2$

E. 2 λ

Questions 24 and 25 are based on the following description: Two point charges of +4.00 μC and –9.00 μC are placed 1.00 m apart, as shown in the accompanying figure. Assume the potential goes to zero as *R* goes to infinity.

24. The total electric field due to the two charges is zero at a point:

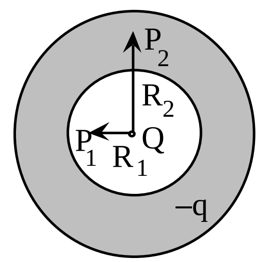


A. 3.00 m to the right of the –9.00 μC charge.
B. 0.40 m to the right of the +4.00 μC charge.
C. 0.31 m to the right of the +4.00 μC charge.
D. 0.80 m to the left of the +4.00 μC charge.
E. 2.00 m to the left of the +4.00 μC charge.

25. How much work is done moving the –9.00 μC charge from its original position to a new position 2.00 m from the +4.00 μC charge?

A. – 0.324 J B. –0.081 J C. +0.162 J D. +0.243 J E. +0.486 J

26. A point charge *Q* is placed at the center of a spherical conducting shell, the shaded part of the accompanying figure. A total charge of –*q* is placed on the shell. The magnitude of the electric field at point P_1 a distance R_1 from the center is _____. The magnitude of the electric field at point P_2 a distance R_2 from the center is _____.



A. 0, 0

B. $k\dfrac{Q}{R_1^{\,2}}$, 0

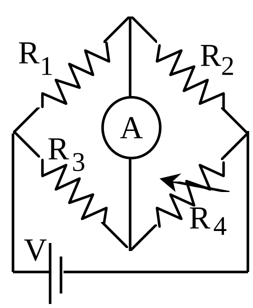
C. $k\dfrac{Q-q}{R_1^{\,2}}$, 0

D. 0, $k\dfrac{Q-q}{R_2^{\,2}}$

E. $k\dfrac{Q}{R_1^{\,2}}$, $k\dfrac{Q-q}{R_2^{\,2}}$

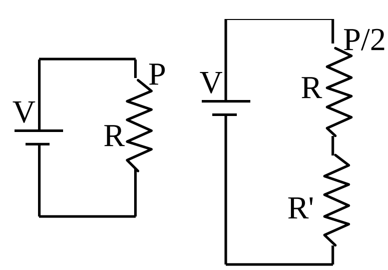
27. R_4, shown in the figure to the right, is a variable resistor. In order for there to be no current through the ammeter, R_4 must be equal to:

A. R_2 B. R_3 C. R_1R_2/R_3 D. R_1R_3/R_2 E. R_2R_3/R_1

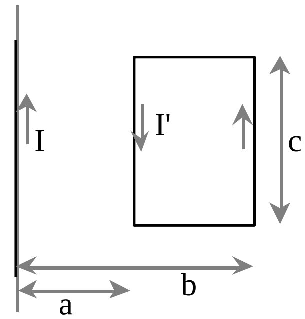


28. A resistor *R* dissipates power *P* when connected directly to a voltage source V, as shown in the accompanying figures. What resistance *R′* must be connected in series with *R* to decrease the power dissipated in *R* to *P/2*?

A. $\dfrac{R}{2}$ B. $\dfrac{R}{\sqrt{2}}$ C. R D. $R\left(\sqrt{2}-1\right)$ E. $R\sqrt{2}$



29. The infinitely long straight wire carries a conventional current I as shown in the accompanying figure. The rectangular loop carries a conventional current I' in the counterclockwise direction. The net force on the rectangular loop is:



A. $\frac{\mu_o I I' c}{2\pi}\left(\frac{1}{a}-\frac{1}{b}\right)$ to the right

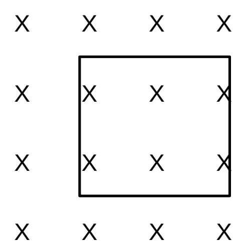
B. $\frac{\mu_o I I' c}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$ to the left

C. $\frac{\mu_o I I'}{2\pi}\left(\frac{c}{a}+\frac{c}{b}+\frac{2(b-a)}{c}\right)$ to the left

D. $\frac{\mu_o I I'}{2\pi}\frac{2(b-a)}{c}$ to the right

E. 0

30. A spatially uniform magnetic field of 0.080 T is directed into the plane of the page and perpendicular to it, as shown in the accompanying figure. A wire loop in the plane of the page has constant area 0.010 m^2. The magnitude of the magnetic field decreases at a constant rate of 3.0×10^{-4} T/s. What is the magnitude and direction of the induced emf?



A. 3.0×10^{-6} V clockwise

B. 3.0×10^{-6} V counterclockwise

C. 2.4×10^{-5} V counterclockwise

D. 8.0×10^{-4} V counterclockwise

E. 8.0×10^{-4} V clockwise

1994 MULTIPLE CHOICE SCREENING TEST ANSWER KEY

1.	B	11.	C	21.	C
2.	B	12.	B	22.	D
3.	D	13.	D	23.	E
4.	C	14.	E	24.	E
5.	E	15.	A	25.	C
6.	A	16.	B	26.	B
7.	C	17.	D	27.	E
8.	D	18.	E	28.	D
9.	C	19.	A	29.	A
10.	B	20.	A	30.	A

Physics Olympiad

Entia non multiplicanda sunt praeter necessitatem

1997 MULTIPLE CHOICE SCREENING TEST

30 QUESTIONS—40 MINUTES

INSTRUCTIONS

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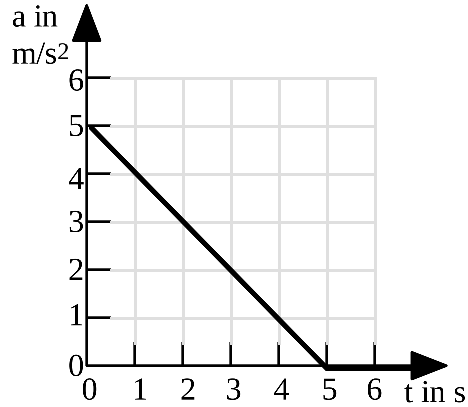
The values of some possibly useful constants are given below:

mass of electron	m_e	$= 9.1 \times 10^{-31}$ kg
mass of proton	m_p	$= 1.7 \times 10^{-27}$ kg
electronic charge	e	$= 1.6 \times 10^{-19}$ C
speed of light	c	$= 3.0 \times 10^{8}$ m/s
Coulomb's constant	k	$= 9.0 \times 10^{9}$ N·m^2/C^2
permittivity constant	ε_0	$= 8.9 \times 10$ C / ^{2}
	μ	$= \pi$ $^{-7}$ T·m/A
	G	$^{-11}$ N·m /kg
mass of Earth	$_E$	$= 6.0 \times 10^{24}$ kg
radius of Earth	R_E	$= 6.4 \times 10^{6}$ m
gravitational field at Earth's surface	g	$= 9.8$ m/s^2
speed of sound (20 °C)	v_S	$= 340$ m/s

DO NOT OPEN THIS TEST UNTIL YOU ARE INSTRUCTED TO BEGIN

1. Starting from rest at time t = 0, a car moves in a straight line with an acceleration given by the accompanying graph. What is the speed of the car at t = 3 s?



A. 1.0 m/s
B. 2.0 m/s
C. 6.0 m/s
D. 10.5 m/s
E. 12.5 m/s

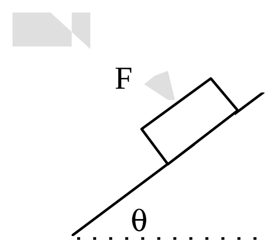
2. A flare is dropped from a plane flying over level ground at a velocity of 70 m/s in the horizontal direction. At the instant the flare is released, the plane begins to accelerate horizontally at 0.75 m/s^2. The flare takes 4.0 s to reach the ground. Assume air resistance is negligible. Relative to a spot directly under the flare at release, the flare lands

A. directly on the spot.
B. 6.0 m in front of the spot.
C. 274 m in front of the spot.
D. 280 m in front of the spot.
E. 286 m in front of the spot.

3. As seen by the pilot of the plane (in question #2) and measured relative to a spot directly under the plane when the flare lands, the flare lands

A. 286 m behind the plane.
B. 6.0 m behind the plane.
C. directly under the plane.
D. 12 m in front of the plane.
E. 274 m in front of the plane

4. A force *F* is used to hold a block of mass *m* on an incline as shown in the diagram. The plane makes an angle of θ with the horizontal and *F* is perpendicular to the plane. The coefficient of friction between the plane and the block is μ. What is the minimum force, *F*, necessary to keep the block at rest?



A. μ mg
B. mg cosθ
C. mg sinθ
D. (mg/μ) sinθ
E. (mg/μ) (sinθ - μ cosθ)

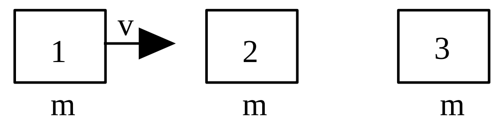
5. You hold a rubber ball in your hand. The Newton’s third law companion force to the force of gravity on the ball is the force exerted by the

A. ball on the Earth.
B. ball on the hand.
C. hand on the ball.
D. Earth on the ball.
E. Earth on your hand.

6. A ball of mass m is fastened to a string. The ball swings in a vertical circle of radius R with the other end of the string held fixed. Neglecting air resistance, the difference between the string's tension at the bottom of the circle and at the top of the circle is

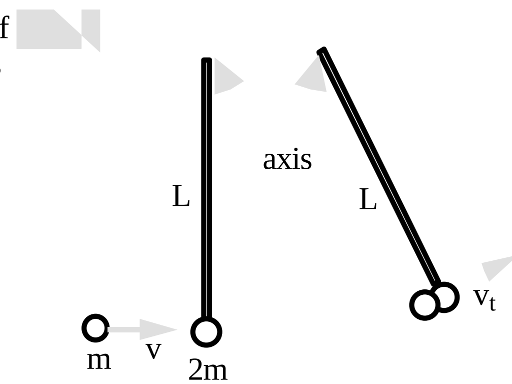
A. mg B. 2 mg C. 4 mg D. 6 mg E. 8 mg

7. Three air track cars, shown in the accompanying figure, all have the same mass m. Cars *2* and *3* are initially at rest. Car *1* is moving to the right with speed v. Car *1* collides with car *2* and sticks to it. The *1-2* combination collides elastically with car *3*. Which of the following is most nearly the final speed of car *3*?



A. 0.17 v B. 0.50 v C. 0.67 v D. 0.80 v E. 1.0 v

8. A point object of mass *2m* is attached to one end of a rigid rod of negligible mass and length L. The rod is initially at rest but free to rotate about a fixed axis perpendicular to the rod and passing through its other end (see diagram to the right). A second point object with mass m and initial speed v collides and sticks to the $2m$ object. What is the tangential speed v_t of the object immediately after the collision?



A. $v/3$

B. $v/2$

C. $v/\sqrt{3}$

D. $v/\sqrt{2}$

E. $2v/\sqrt{3}$

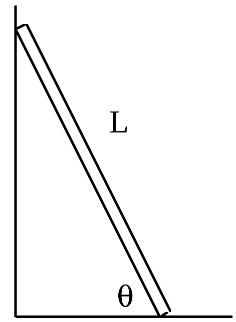
9. Two artificial satellites I and II have circular orbits of radii R and $2R$, respectively, about the same planet. The orbital velocity of satellite I is v. What is the orbital velocity of satellite II?

A. $\frac{v}{2}$ B. $\frac{v}{\sqrt{2}}$ C. v D. $v\sqrt{2}$ E. 2v

10. The gravitational acceleration on the surface of the moon is 1.6 m/s^2. The radius of the moon is 1.7×10^6 m. The period of a satellite placed in a low circular orbit about the moon is most nearly

A. 1.0×10^3 s B. 6.5×10^3 s C. 1.1×10^6 s D. 5.0×10^6 s E. 7.1×10^{12} s

11. A uniform ladder of length L rests against a smooth frictionless wall. The floor is rough and the coefficient of static friction between the floor and ladder is μ. When the ladder is positioned at angle θ, as shown in the accompanying diagram, it is just about to slip. What is θ?



A. $\theta = \dfrac{\mu}{L}$ B. $\tan\theta = 2\mu$ C. $\tan\theta = \dfrac{1}{2\mu}$ D. $\sin\theta = \dfrac{1}{\mu}$ E. $\cos\theta = \mu$

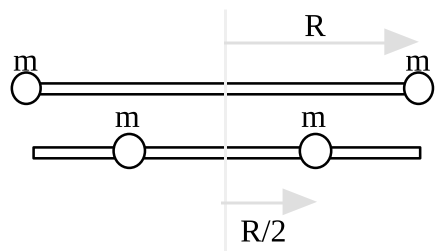
12. Three objects, all of mass M, are released simultaneously from the top of an inclined plane of height H. The objects are described as follows

I. a cube of side R.
II. a solid cylinder of radius R
III. a hollow cylinder of radius R

Assume the cylinders roll down the plane without slipping and the cube slides down the plane without friction. Which object(s) reach(es) the bottom of the plane first?

A. I B. II C. III D. I & II E. II & III

13. A massless rod of length $2R$ can rotate about a vertical axis through its center as shown in the diagram. The system rotates at an angular velocity ω when the two masses m are a distance R from the axis. The masses are simultaneously pulled to a distance of $R/2$ from the axis by a force directed along the rod. What is the new angular velocity of the system?



A. $\omega/4$ B. $\omega/2$ C. ω D. 2ω E. 4ω

14. A meter stick moves with a velocity of 0.60 c relative to an observer. The observer measures the length of the meter stick to be L. Which of the following statements is always true?

A. L = 0.60 m B. L = 0.80 m C. 0.80 m ^{2} L ^{2} 1.00 m D. L = 1.00 m E. L ^{3} 1.00m

15. A glowing ember (hot piece of charcoal) radiates power P in watts at an absolute temperature T. When the temperature of the ember has decreased to $T/2$, the power it radiates is most nearly

A. P B. $P/2$ C. $P/4$ D. $P/8$ E. $P/16$

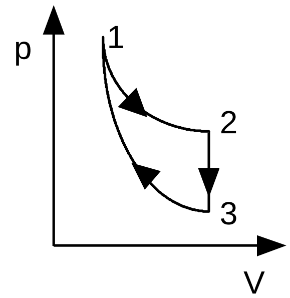
16. Three processes compose a thermodynamic cycle shown in the accompanying pV diagram of an ideal gas.

Process 1→2 takes place at constant temperature (300 K). During this process 60 J of heat enters the system.

Process 2→3 takes place at constant volume. During this process 40 J of heat leaves the system.

Process 3→1 is adiabatic. T_3 is 275 K.

What is the change in internal energy of the system during process 3→1?



A. -40 J B. -20 J C. 0 D. +20 J E. +40 J

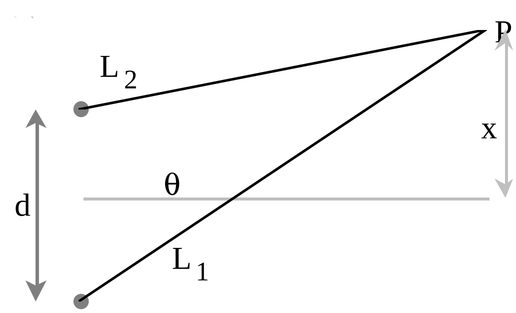
17. What is the change in entropy of the system described in Question # 16 during the process 3→1?

A. +5.0 K/J B. +0.20 J/K C. 0 D. -1.6 J/K E. -6.9 K/J

18. A wave is described by the equation: $y(x,t) = 0.030 \cdot \sin(5\pi x + 4\pi t)$ where x and y are in meters and t is in seconds. The +x direction is to the right. What is the velocity of the wave?

A. 0.80 m/s to the left
B. 1.25 m/s to the left
C. 0.12 π m/s to the right
D. 0.80 m/s to the right
E. 1.25 m/s to the right

19. Two sources, in phase and a distance *d* apart, each emit a wave of wavelength l . See accompanying figure. Which of the choices for the path difference $\Delta L = L_1 - L_2$ will *always* produce destructive interference at point P?

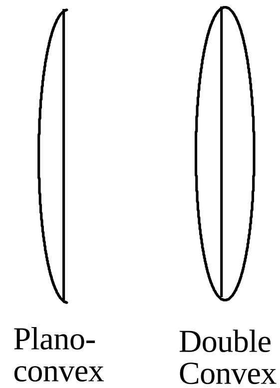


A. d sinθ
B. x/L_1
C. $(x/L_2)d$
D. $\lambda/2$
E. 2λ

20. You are given two lenses, a converging lens with focal length + 10 cm and a diverging lens with focal length - 20 cm. Which of the following would produce a virtual image that is larger than the object?

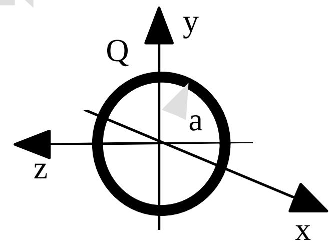
A. Placing the object 5 cm from the converging lens.
B. Placing the object 15 cm from the converging lens.
C. Placing the object 25 cm from the converging lens.
D. Placing the object 15 cm from the diverging lens.
E. Placing the object 25 cm from the diverging lens.

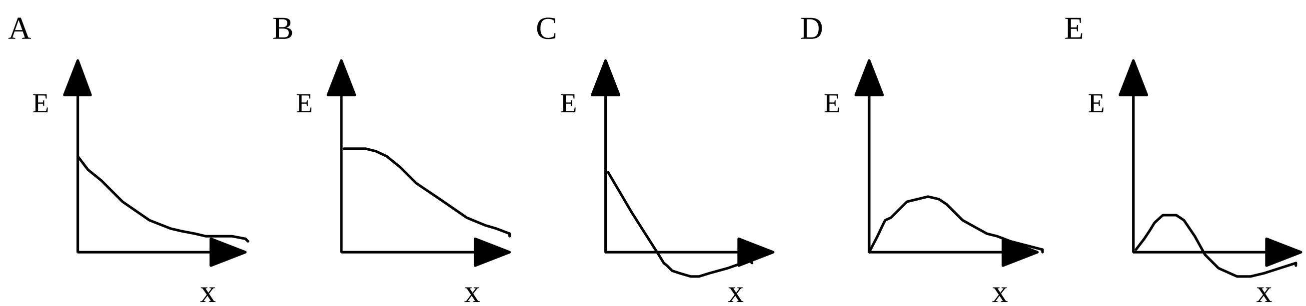
21. You are given two identical plano-convex lenses, one of which is shown to the right. When you place an object 20 cm to the left of a single plano-convex lens, the image appears 40 cm to the right of the lens. You then arrange the two plano-convex lenses back to back to form a double convex lens. If the object is 20 cm to the left of this new lens, what is the approximate location of the image?



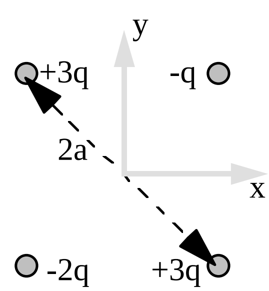
A. 6.7 cm to the right of the lens.
B. 10 cm to the right of the lens.
C. 20 cm to the right of the lens.
D. 80 cm to the right of the lens.
E. 80 cm to the left of the lens.

22. Positive charge *Q* is uniformly distributed over a ring of radius *a* that lies in the y-z plane as shown in the diagram. The ring is centered at the origin. Which of the following graphs best represents the value of the electric field *E* as a function of *x*, the distance along the positive x axis?



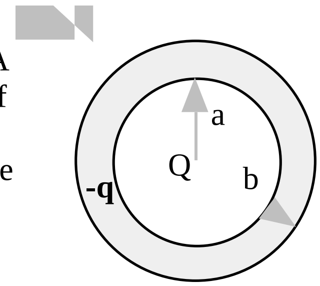


23. Four point charges are placed at the corners of a square with diagonal *2a* as shown in the diagram. What is the total electric field at the center of the square?



A. kq/a^2 at an angle 45° above the +x axis.
B. kq/a^2 at an angle 45° below the -x axis.
C. $3\ kq/a^2$ at an angle 45° above the -x axis.
D. $3\ kq/a^2$ at an angle 45° below the +x axis.
E. $9\ kq/a^2$ at an angle 45° above the +x axis.

Both questions 24 and 25 refer to the system shown in the diagram. A spherical shell with an inner surface of radius *a* and an outer surface of radius *b* is made of conducting material. A point charge +*Q* is placed at the center of the spherical shell and a total charge -*q* is placed on the shell.



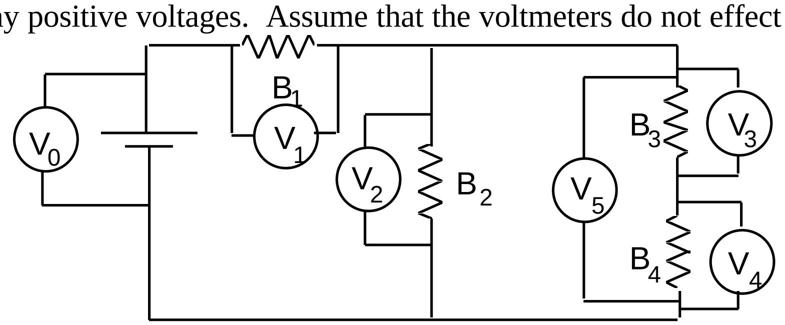
24. How is the charge -*q* distributed after it has reached equilibrium?

A. Zero charge on the inner surface, -q on the outer surface.
B. -Q on the inner surface, -q on the outer surface.
C. -Q on the inner surface, -q+Q on the outer surface.
D. +Q on the inner surface, -q-Q on the outer surface.
E. The charge -*q* is spread uniformly between the inner and outer surface.

25. Assume that the electrostatic potential is zero at an infinite distance from the spherical shell. What is the electrostatic potential at a distance *R* from the center of the shell, where *b* ³ *R* ³ *a*?

A. 0 B. $k\frac{Q}{a}$ C. $k\frac{Q}{R}$ D. $k\frac{Q-q}{R}$ E. $k\frac{Q-q}{b}$

Use the circuit below to answer questions 26 and 27. B_1, B_2, B_3, and B_4 are **identical** light bulbs. There are six voltmeters connected to the circuit as shown. All voltmeters are connected so that they display positive voltages. Assume that the voltmeters do not effect the circuit.



26. If B_2 were to burn out, opening the circuit, which voltmeter(s) would read zero volts?

A. none would read zero.
B. only V_2
C. only V_3 and V_4
D. only V_3, V_4, and V_5
E. they would all read zero

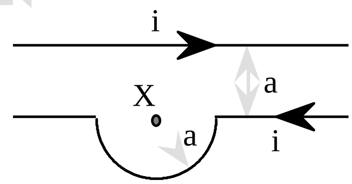
27. If B_2 were to burn out, opening the circuit, what would happen to the reading of V_1? Let V be its original reading when all bulbs are functioning and let V' be its reading when B_2 is burnt out.

A. V' ³ 2V B. 2V > V' > V C. V' = V D. V > V' > V/2 E. V/2 ³ V'

28. A particle with positive charge *q* and mass *m* travels along a path perpendicular to a magnetic field. The particle moves in a circle of radius *R* with frequency f. What is the magnitude of the magnetic field?

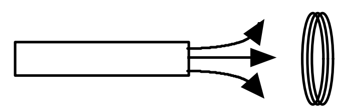
A. $\dfrac{mf}{q}$ B. $\dfrac{2\pi fm}{q}$ C. $\dfrac{m}{2\pi fq}$ D. $\dfrac{mc}{qR}$ E. $\dfrac{\mu qf}{2\pi R}$

29. Two wires, each carrying a current *i*, are shown in the diagram to the right. Both wires extend in a straight line for a very long distance on both the right and the left. One wire contains a semi-circular loop of radius *a* centered on point X. What is the correct expression for the magnetic field at point X? HINT: The magnitude of the magnetic field at the center of a circular current loop of radius *R* is $\mu_0 i/(2R)$.



A. $\mu_0 i/(4a) + \mu_0 i/(2\pi a)$ out of the page
B. $\mu_0 i/(2a) - \mu_0 i/(2\pi a) + \mu_0 i/(2\pi a)$ out of the page
C. $\mu_0 i/(4a) + \mu_0 i/(2\pi a)$ into the page
D. $\mu_0 i/(4a) + \mu_0 i/(2\pi a) + \mu_0 i/(2\pi a)$ into the page
E. $\mu_0 i/(2a) - \mu_0 i/(2\pi a)$ into the page

30. You are given a bar magnet and a looped coil of wire. Which of the following would induce an emf in the coil?



I. Moving the magnet toward the coil.
II. Moving the coil away from the magnet.
III. Turning the coil about a vertical axis.

A. I only B. II only C. I & II D. I & III E. I, II, III

1997 MULTIPLE CHOICE SCREENING TEST
ANSWER KEY

1.	D	11.	C	21.	B
2.	D	12.	A	22.	D
3.	B	13.	E	23.	B
4.	E	14.	C	24.	C
5.	A	15.	E	25.	E
6.	D	16.	E	26.	A
7.	C	17.	C	27.	D
8.	A	18.	A	28.	B
9.	B	19.	D	29.	C
10.	B	20.	A	30.	E

1999 AAPT PHYSICS OLYMPIAD

Entia non multiplicanda sunt praeter necessitatem

1999 MULTIPLE CHOICE SCREENING TEST
30 QUESTIONS - 40 MINUTES

INSTRUCTIONS
DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

This test contains 30 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 30 are to be used on the answer sheet.

Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.

A hand-held calculator may be used. However, any memory must be cleared of data and programs. Calculators are not to be shared.

Your score on this multiple choice test will be your number of correct answers. There is no penalty for guessing. It is to your advantage to answer every question.

The values of some possibly useful constants are given below:

mass of electron	m_e	$= 9.1 \times 10^{-31}$ kg
mass of proton	m_p	$= 1.7 \times 10^{-27}$ kg
electronic charge	e	$= 1.6 \times 10^{-19}$ C
speed of light	c	$= 3.0 \times 10^{8}$ m/s
Coulomb's constant	k	$= 9.0 \times 10^{9}$ $N{\cdot}m^2/C^2$
permittivity constant	e_0	$= 8.9 \times 10^{-12}$ $C^2/N{\cdot}m^2$
permeability constant	m_0	$= 4\pi \times 10^{-7}$ $T{\cdot}m/A$
gravitational constant	G	$= 6.7 \times 10^{-11}$ $N{\cdot}m^2/kg^2$
mass of Earth	M_E	$= 6.0 \times 10^{24}$ kg
radius of Earth	R_E	$= 6.4 \times 10^{6}$ m
gravitational field	g	$= 9.8$ N/kg
speed of sound (20^0 C)	v_s	$= 340$ m/s

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

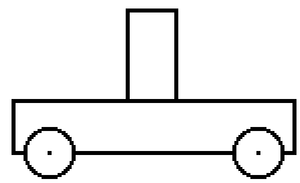
1. A truck driver travels three-fourths the distance of his run at one velocity (v) and then completes his run at one half his original velocity (½v). What was the trucker's average speed for the trip?

[A] 0.85v [B] 0.80v [C] 0.75v [D] 0.70v [E] 0.65v

2. Hercules and Ajax horizontally push in the same direction on a 1200 kg crate. Hercules pushes with a force of 500 newtons and Ajax pushes with a force of 300 newtons. If a frictional force provides 200 newtons of resistance, what is the acceleration of the crate?

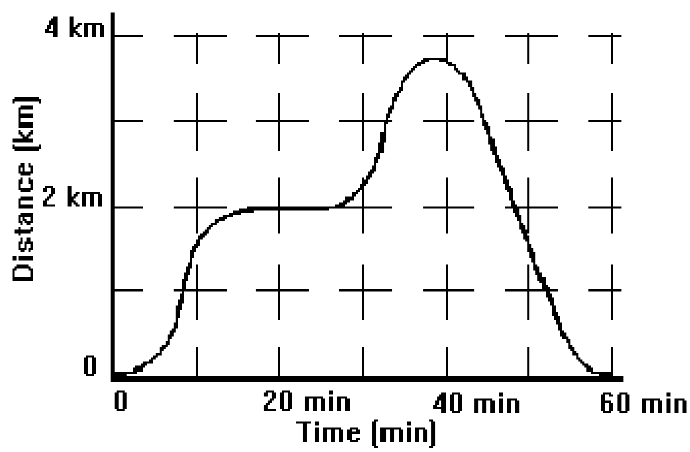
[A] 1.3 m/s^2 [B] 1.0 m/s^2 [C] 0.87 m/s^2 [D] 0.75 m/s^2 [E] 0.5 m/s^2

3. A uniform 2 kg cylinder rests on a laboratory cart as shown. The coefficient of static friction between the cylinder and the cart is 0.5. If the cylinder is 4 cm in diameter and 10 cm in height, which of the following is closest to the minimum acceleration of the cart needed to cause the cylinder to tip over?



[A] 2 m/s^2 [B] 4 m/s^2 [C] 5 m/s^2 [D] 6 m/s^2 [E] The cylinder would slide at all of these accelerations.

4. At right is a graph of the distance vs. time for car moving along a road. According the graph, at which of the following times would the automobile have been accelerating positively?



[A] 0, 20, 38, & 60 min.
[B] 5, 12, 29, & 35 min.
[C] 5, 29, & 57 min.
[D] 12, 35, & 41 min.
[E] at all times from 0 to 60 min.

5. A ball which is thrown upward near the surface of the earth with a velocity of 50 m/s will come to rest about 5 seconds later. If the ball were thrown up with the same velocity on Planet X, after 5 seconds it would still be moving upwards at nearly 31 m/s. The magnitude of the gravitational field near the surface of Planet X is what fraction of the gravitational field near the surface of the earth?

[A] 0.16 [B] 0.39 [C] 0.53 [D] 0.63 [E] 1.59

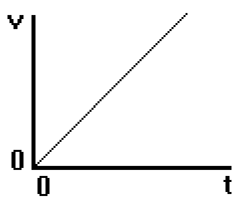
6. Two light plastic shopping bags of negligible mass are placed 2 meters apart. Each bag contains 15 oranges. If 10 oranges were moved from one bag to the other, the gravitational force between the two bags would

[A] increase to 3/2 the original value.
[B] decrease to 2/5 the original value.
[C] increase to 5/3 the original value.
[D] decrease to 5/9 the original value.
[E]   not change.

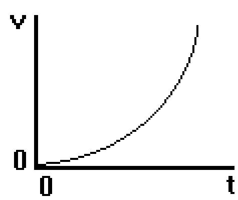
7.   If a net force F applied to an object of mass m will produce an acceleration of a, what is the mass of a second object which accelerates at $5a$ when acted upon by a net force of $2F$?

[A] $(2/5)m$ [B] $2m$ [C] $(5/2)m$ [D] $5m$ [E] $10m$

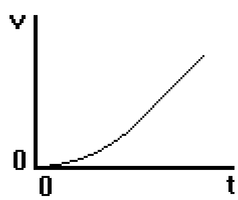
8. A large beach ball is dropped from the ceiling of a school gymnasium to the floor about 10 meters below. Which of the following graphs would best represent its velocity as a function of time?



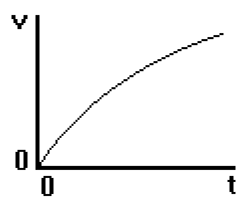
[A]



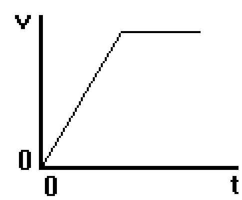
[B]



[C]



[D]



[E]

9. A driver in a 1500 kg sports car wishes to pass a slow moving truck on a two lane road. What is the average power in watts required to accelerate the sports car from 20 m/s to 40 m/s in 3 seconds?

[A] 10,000 [B] 20,000 [C] 100,000 [D] 300,000 [E] 400,000

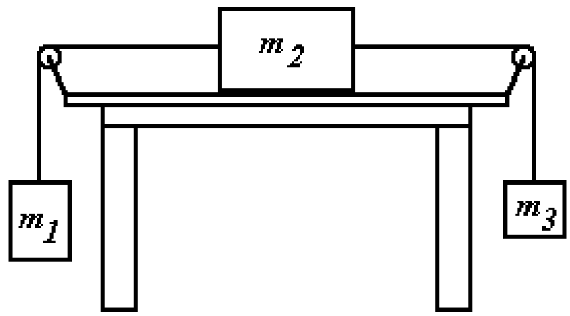
10. A 40 kg mass is attached to a horizontal spring with a constant of 500 N/m. If the mass rests on a frictionlless horizontal surface, what is the total energy of this system when set into simple harmonic motion by an original displacement of 0.2 meters?

[A] 10 J [B] 20 J [C] 50 J [D] 4000 J [E] 100,000 J

11. Two skaters on a frictionless pond push apart from one another. One skater has a mass M much greater than the mass m of the second skater. After some time the two skaters are a distance d apart. How far has the lighter skater moved from her original position?

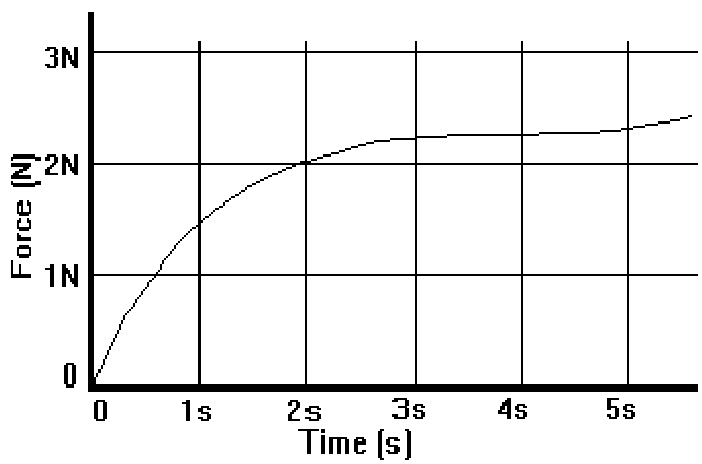
[A] d [B] $d\left(\frac{M}{m}\right)$ [C] $d\left(\frac{m}{M}\right)$ [D] $d\left(\frac{m}{M+m}\right)$ [E] $d\left(\frac{M}{m+M}\right)$

12. Given the three masses as shown in the diagram, if the coefficient of kinetic friction between the large mass (m_2) and the table is μ, what would be the upward acceleration of the small mass (m_3)? The mass and friction of the cords and pulleys are small enough to produce a negligible effect on the system.



[A] $\frac{m_1 g}{(m_1+m_2+m_3)}$ [B] $\frac{g(m_1+m_2 m)}{(m_1+m_2+m_3)}$ [C] $\frac{gm(m_1+m_2+m_3)}{(m_1-m_2-m_3)}$ [D] $\frac{gm(m_1-m_2-m_3)}{(m_1+m_2+m_3)}$ [E] $\frac{g(m_1-m_2 m-m_3)}{(m_1+m_2+m_3)}$

13. An object with a mass of 2 kilograms is accelerated from rest. The graph at right shows the magnitude of the net force in newtons as a function of time in seconds. At t = 4 seconds the object's velocity would have been closest to which of the following:



[A] 2.2 m/s
[B] 3.5 m/s
[C] 5.8 m/s
[D] 7.0 m/s
[E] 11.5 m/s

14. A thin ring of mass m and radius r rolls across the floor with a velocity v. Which of the following would be the best estimate of the ring's total kinetic energy as it rolls across the floor?

[A] mv^2 [B] $\frac{1}{2}mv^2$ [C] $\frac{1}{4}mv^2$ [D] $\frac{1}{2}mv^2+\frac{mv^2}{r}$ [E] $\frac{1}{2}mv^2+m\frac{r^2}{t^2}$

15.   A 4.0 kg mass is attached to one end of a rope 2 m long. If the mass is swung in a vertical circle from the free end of the rope, what is the tension in the rope when the mass is at its highest point if it is moving with a speed of 5 m/s?

[A] 5.4 N   [B] 10.8 N   [C] 21.6 N   [D] 50 N   [E] 65.4 N

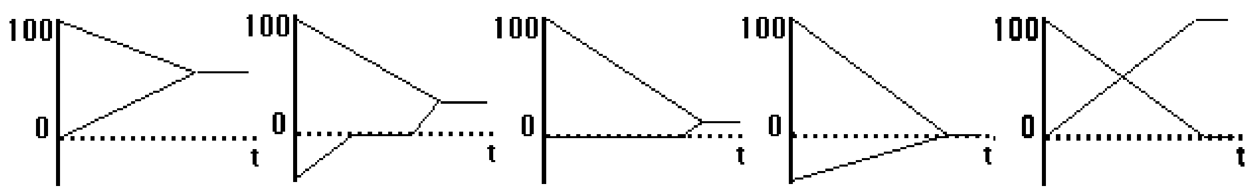
16. A mole of ideal gas at STP is heated in an insulated constant volume container until the average velocity of its molecules doubled. Its pressure would therefore increase by what factor?

[A] 0.5   [B] 1   [C] 2   [D] 4   [E] 8

17. An ideal heat engine takes in heat energy at a high temperature and exhausts energy at a lower temperature. If the amount of energy exhausted at the low temperature is 3 times the amount of work done by the heat engine, what is its efficiency?

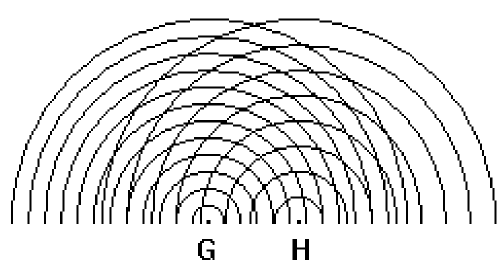
[A] 0.25   [B] 0.33   [C] 0.67   [D] 0.9   [E] 1.33

18 . If 100 g of ice at 0^o C is mixed with 100 g of boiling water at 100^o C, which of the following graphs would best represent the temperature vs time of the two components of the mixture?



[A]   [B]   [C]   [D]   [E]

19. Two wave sources, G and H, produce waves of different wavelengths as shown in the diagram. Lines shown in the diagram represent wave crests. Which of the following would be true of the resulting interference pattern?



[A] The perpendicular bisector of GH would be an antinodal line.
[B] The interference pattern will be a stable pattern but not symetrical left to right.
[C] The pattern of nodal and antinodal lines will sweep from left to right.
[D] If G and H were moved farther apart, the fewer the number of nodal and antinodal lines.
[E] There will not be an organized interference pattern because the sources are not producing the same frequency.

20. Given a wave produced by a simple harmonic oscillator whose displacement in meters is given by the equation: $y = .3 \sin (3\pi x + 24\pi t)$, what is the frequency of the wave in hertz?

[A] 3 Hz [B] 7.2 Hz [C] 8 Hz [D] 12 Hz [E] 24 Hz

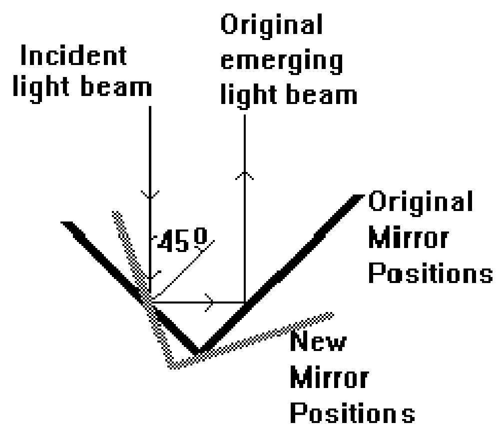
21. Which of the following wave properties cannot be demonstrated by all kinds of waves?

[A] Polarization [B] Diffraction [C] Superposition [D] Refraction [E] Frequency

22. A source when at rest in a medium produces waves with a velocity *v* and a wavelength of l. If the source is set in motion to the left with a velocity v_s, what would be the length of the wavelengths produced directly in front of the source?

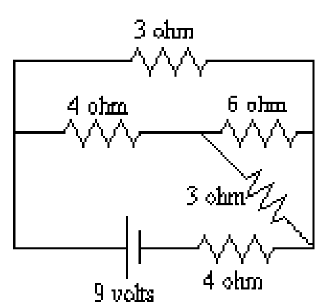
[A] $l\left(1-\frac{v_s}{v}\right)$ [B] $l\left(1+\frac{v_s}{v}\right)$ [C] $l\left(1+\frac{v}{v_s}\right)$ [D] $\frac{l-v}{v-v_s}$ [E] $\frac{lv}{v+v_s}$

23. A beam of light strikes one mirror of a pair of right angle mirrors at an angle of incidence of 45^o as shown in the diagram. If the right angle mirror assembly is rotated such that the angle of incidence is now 60^o, what will happen to the angle of the beam that emerges from the right angle mirror assembly?



[A] It will move through an angle of 15^o with respect to the original emerging beam.
[B] It will move through an angle of 30^o with respect to the original emerging beam.
[C] It will move through an angle of 45^o with respect to the original emerging beam.
[D] It will move through an angle of 60^o with respect to the original emerging beam.
[E] It will emerge parallel to the original emerging beam.

24. What would be the total current being supplied by the battery in the circuit shown?

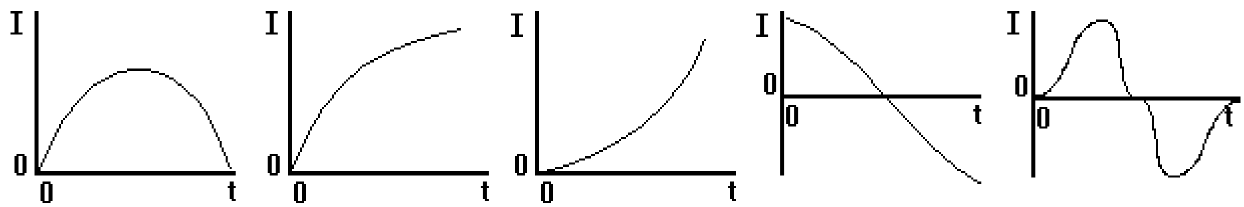


[A] 3.0 amperes
[B] 2.25 amperes
[C] 2.0 amperes
[D] 1.5 amperes
[E] 1.0 amperes

25. Consider a simple circuit containing a battery and three light bulbs. Bulb A is wired in parallel with bulb B and this combination is wired in series with bulb C. What would happen to the brightness of the other two bulbs if bulb A were to burn out?

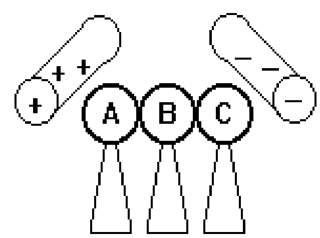
[A] only bulb B would get brighter.
[B] both would get brighter.
[C] bulb B would get brighter and bulb C would get dimmer.
[D] bulb B would get dimmer and bulb C would get brighter.
[E] There would be no change in the brightness of either bulb B or bulb C.

26. A coil of wire is moved vertically at a constant velocity through a horizontal magnetic field. Assume the plane of the coil is perpendicular to the magnetic field. Which of the following graphs would best represent the electric current induced in the coil if it started somewhat above the magnetic field and ended equally as far below the magnetic field ?



[A] [B] [C] [D] [E]

27. Three metal spheres A, B, and C are mounted on insulating stands. The spheres are touching one another, as shown in the diagram. A strong positively charged object is brought near sphere A and a strong negative charge is brought near sphere C. While the charged objects remain near spheres A and C, sphere B is removed by means of its insulating stand. After the charged objects are removed, sphere B is first touched to sphere A and then to sphere C. The resulting charge on B would be:



[A] the same sign but ½ the magnitude as originally on sphere A.
[B] the opposite sign but ½ the magnitude as originally on sphere A.
[C] the opposite sign but ¼ the magnitude as originally on sphere A.
[D] the same sign but ½ the magnitude as originally on sphere C.
[E] neutrally charged.

28. Which of the following graphs would best represent the electric field of a hollow Van de Graff sphere as a function of distance from its center when it is charged to a potential of 400,000 volts?

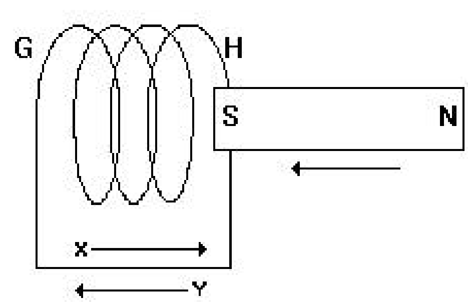
[A] [B] [C] [D] [E]

29. A mass spectrograph separates ions by weight using simple concepts from physics. Charged ions are given a specific kinetic energy by accelerating them through a potential difference. The ions then move through a perpendicular magnetic field where they are deflected into circular paths with differing radii. How would the radius of a singularly ionized common helium atom $\left({}^{4}_{2}He^{1+}\right)$ compare to the radius of a doubly ionized common oxygen atom $\left({}^{16}_{8}O^{2+}\right)$ if they were accelerated through the same potential difference and were deflected by the same magnetic field?

[A] The radius of the He ion path is 4 times the radius of the O ion path.
[B] The radius of the O ion path is 2 times the radius of the He ion path.
[C] The radius of the O ion path is 4 times the radius of the He ion path.
[D] The radius of the O ion path is 8 times the radius of the O ion path.
[E] The radius of the He ion path is equal to the radius of the O ion path.

30. A bar magnet is thrust into a coil of wire as indicated in the diagram. Which of the following statements concerning this experiment is correct?

[A] The polarity of the coil at H is North, at G is South, and the induced current is indicated by arrow X.
[B] The polarity of the coil at H is South, at G is North, and the induced current is indicated by arrow Y.
[C] The polarity of the coil at H is North, at G is South, and the induced current is indicated by arrow Y.
[D] The polarity of the coil at H is South, at G is North, and the induced current is indicated by arrow X.
[E] There is no polarity to the coil and no current in the coil.



1999 AAPT PHYSICS OLYMPIAD
MULTIPLE CHOICE SCREENING TEST
ANSWER KEY

1. B	11. E	21. A
2. E	12. E	22. A
3. B	13. B	23. E
4. C	14. A	24. D
5. B	15. B	25. C
6. D	16. D	26. E
7. A	17. A	27. C
8. D	18. C	28. C
9. D	19. C	29. E
10. A	20. D	30. B

Physics Olympiad

Entia non multiplicanda sunt praeter necessitatem

1998 MULTIPLE CHOICE SCREENING TEST

30 QUESTIONS—40 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

This test contains 30 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 30 are to be used on the answer sheet.

Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.

A hand-held calculator may be used. However, any memory must be cleared of data and programs. Calculators may not be shared.

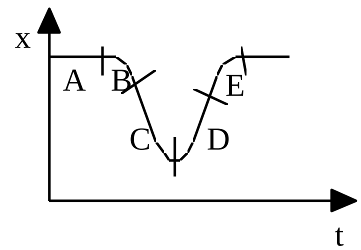
Your grade on this multiple choice test will be your number of correct answers. There is no penalty for guessing. It is to your advantage to answer every question.

The values of some possibly useful constants are given below:

mass of electron	m_e	$= 9.1 \times 10^{-31}$ kg
mass of proton	m_p	$= 1.7 \times 10^{-27}$ kg
electronic charge	e	$= 1.6 \times 10^{-19}$ C
speed of light	c	$= 3.0 \times 10^{8}$ m/s
Coulomb's constant	k	$= 9.0 \times 10^{9}$ N·m²/C²
permittivity constant	ε_0	$= 8.9 \times 10^{-12}$ C²/N·m²
permeability constant	μ_0	$= 4^{1} \times 10^{-7}$ T·m/A
gravitational constant	G	$= 6.7 \times 10^{-11}$ N·m²/kg²
mass of Earth	M_E	$= 6.0 \times 10^{24}$ kg
radius of Earth	R_E	$= 6.4 \times 10^{6}$ m
gravitational field at Earth's surface	g	$= 9.8$ N/kg
speed of sound (20°C)	v_S	$= 340$ m/s

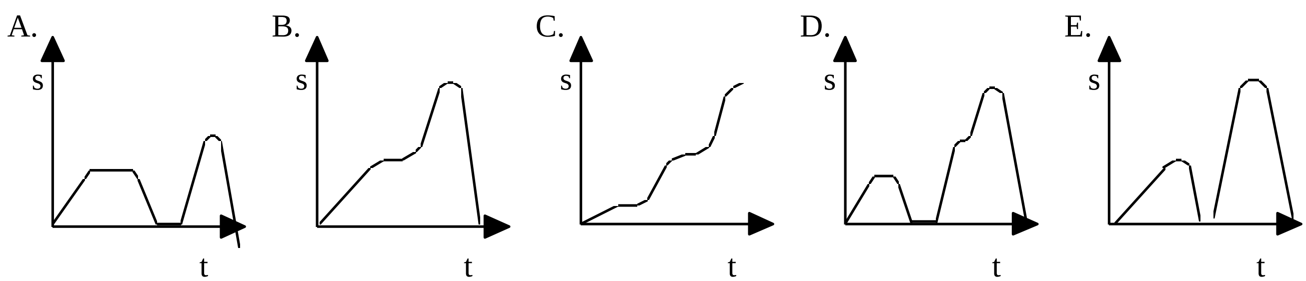
DO NOT OPEN THIS TEST UNTIL YOU ARE INSTRUCTED TO BEGIN

1. The graph to the right is a plot of position versus time. For which labeled region is the velocity positive and the acceleration negative?



A. A B. B C. C D. D E. E

2. A child left her home and started walking at a constant velocity. After a time she stopped for a while and then continued on with a velocity greater than she originally had. All of a sudden she turned around and walked very quickly back home. Which of the following graphs best represents the distance versus time graph for her walk?



3. In a rescue attempt, a hovering helicopter drops a life preserver to a swimmer being swept downstream by a river current of constant velocity *v*. The helicopter is at a height of 9.8 m. The swimmer is 6.0 m upstream from a point directly under the helicopter when the life preserver is released. It lands 2.0 m in front of the swimmer. How fast is the current flowing? Neglect air resistance.

A. 13.7 m/s B. 9.8 m/s C. 6.3 m/s D. 2.8 m/s E. 2.4 m/s

4. A child tosses a ball directly upward. Its total time in the air is T. Its maximum height is H. What is its height after it has been in the air a time $T/4$? Neglect air resistance.

A. (1/4) H B. (1/3) H C. (1/2) H D. (2/3) H E. (3/4) H

5. A whiffle ball is tossed straight up, reaches a highest point, and falls back down. Air resistance is not negligible. Which of the following statements are true?

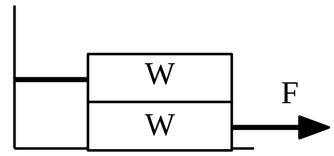
I. The ball's speed is zero at the highest point.
II. The ball's acceleration is zero at the highest point.
III. The ball takes a longer time to travel up to the highest point than to fall back down.

A. I only B. II only C. I & II only D. I & III only E. I, II, & III

6. A pendulum is attached to the ceiling of an elevator car. When the car is parked, the pendulum exhibits a period of 1.00 s. The car now begins to travel upward with an upward acceleration of 2.3 m/s^2. During this part of the motion, what will be the approximate period of the pendulum?

A. 0.80 s   B. 0.90 s   C. 1.00 s   D. 1.10 s   E. 1.20 s

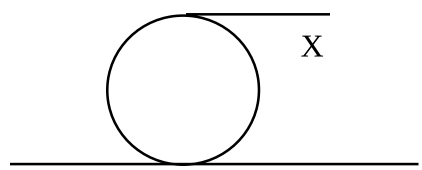
7. Two identical blocks of weight *W* are placed one on top of the other as shown to the right. The upper block is tied to the wall. The lower block is pulled to the right with a force *F*. The coefficient of static friction between all surfaces in contact is μ. What is the largest force *F* that can be exerted before the lower block starts to slip?



A. μW   B. (3/2) μW   C. 2 μW   D. (5/2) μW   E. 3 μW

8. An object placed on an equal arm balance requires 12 kg to balance it. When placed on a spring scale, the scale reads 120 N. Everything (balance, scale, set of masses, and the object) is now transported to the moon where the gravitational force is one-sixth that on Earth. The new readings of the balance and the spring scale (respectively) are:

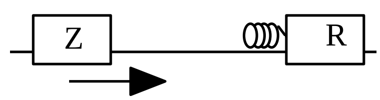
A. 12 kg, 20 N   B. 12 kg, 120 N   C. 12kg, 720 N   D. 2 kg, 20 N   E. 2 kg, 120 N



9. A large spool of rope lies on the ground. See the diagram to the right. The end, labeled X, is pulled a distance *S* in the horizontal direction. The spool rolls without slipping. The distance the spool's center of mass moves is

A. 2S   B. S   C. S/2   D. S/3   E. S/4

10. Air track car Z of mass 1.5 kg approaches and collides with air track car R of mass 2.0 kg. See the accompanying diagram. Car R has a spring attached to it and is initially at rest. When the separation between the cars has reached a minimum, then:

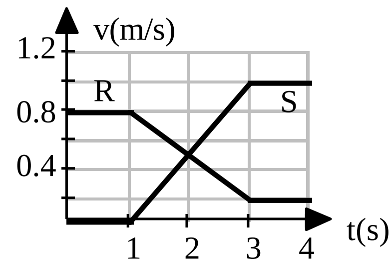


A. car R is still at rest.
B. car Z has come to rest.
C. both cars have the same kinetic energy.
D. both cars have the same momentum.
E. the kinetic energy of the system has reached a minimum.

11. Two ice skaters, a 200 lb man and a 120 lb woman, are initially hugging on a frictionless level ice surface. Ten seconds after they push off from each other, they are 8.0 m apart. How far has the woman moved in that time?

A. 8.0 m B. 6.5 m C. 5.0 m D. 4.0 m E. 3.0 m

12. The diagram to the right shows the velocity-time graph for two masses R and S that collided elastically. Which of the following statements is true?



I. R and S moved in the same direction after the collision.
II. The velocities of R and S were equal at the mid-time of the collision.
III. The mass of S was greater than the mass of R.

A. I only B. II only C. I & II only D. II & III only E. I, II, & III

Questions 13 and 14 refer to the motion of two blocks along a frictionless level track. Block #1 (mass m_1) is initially moving with speed v_o. It collides with and sticks to an initially stationary block (#2) of mass $m_2 = 9\ m_1$.

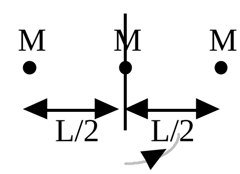
13. What is the speed of the two blocks after the collision?

A. v_o B. $(9/10)\ v_o$ C. $(8/9)\ v_o$ D. $(1/9)\ v_o$ E. $(1/10)\ v_o$

14. What fraction of the initial kinetic energy of the system is converted to other forms (heat, sound, ...) as a result of the collision?

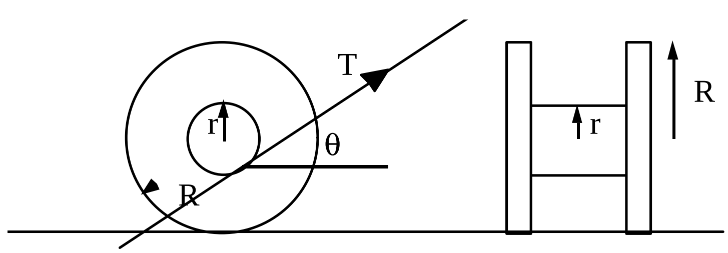
A. 1 % B. 10 % C. 50 % D. 90 % E. 99 %

15. Three identical objects of mass M are fastened to a massless rod of length L as shown. The array rotates about the center of the rod. Its rotational inertia is



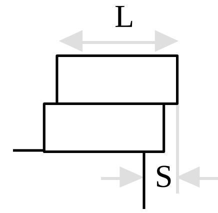
A. $(1/2)\ ML^2$ B. ML^2 C. $(5/4)\ ML^2$ D. $(3/2)\ ML^2$ E. $3\ ML^2$

16 A length of rope is wrapped around a spool of weight W with inner radius r and outer radius R as shown in the accompanying diagram. The rope is pulled with a tension T at an angle θ. Which of the following conditions must be satisfied for the spool to slide uniformly without rolling?



A. $\cos\theta = r/R$ B. $\sin\theta = r/R$ C. $T = W$ D. $T = W \sin\theta$ E. $T = W \cos\theta$

17. Two identical bricks of length L are piled one on top of the other on a table. See the diagram to the right. What is the maximum distance S the top brick can overlap the table with the system still balanced?



A. (1/2) L B. (2/3) L C. (3/4) L D. (7/8) L E. L

18. A gas contains a mixture of He^4 and Ne^{20} atoms. If the average speed of the He^4 atoms is v_o, what is the average speed of the Ne^{20} atoms?

A. (1/5) v_o B. $(1/\sqrt{5})\ v_o$ C. v_o D. $\sqrt{5}\ v_o$ E. 5 v_o

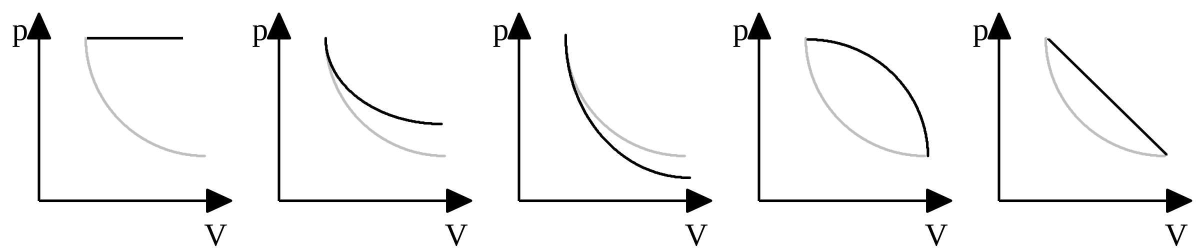
19. One end of a metal rod of length L and cross-sectional area A is held at a constant temperature T_1. The other end is held at a constant T_2. Which of the following statements about the amount of heat transferred through the rod per unit time are true?

I. The rate of heat transfer is proportional to $1/(T_1 - T_2)$.
II. The rate of heat transfer is proportional to A.
III. The rate of heat transfer is proportional to L.

A. II only B. III only C. I and II only D. I and III only E. II and III only

20. On all the pV diagrams shown below the lighter curve represents an isothermal process, a process for which the temperature remains constant. Which dark curve best represents an adiabatic process, a process for which no heat enters or leaves the system?

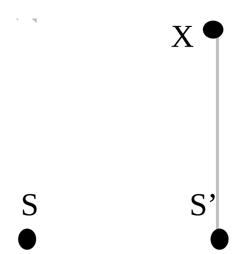
A. B. C. D. E.



21. The longest wavelength photon in the visible Balmer series for the hydrogen atom is

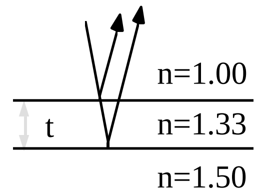
A. 0.66 nm B. 6.56 nm C. 65.6 nm D. 656 nm E. 6560 nm

22. Waves are produced by two point sources S and S' vibrating in phase. See the accompanying diagram. X is a point on the second nodal line. The path difference SX – S'X is 4.5 cm. The wavelength of the waves is approximately



A. 1.5 cm B. 1.8 cm C. 2.3 cm D. 3.0 cm E. 4.5 cm

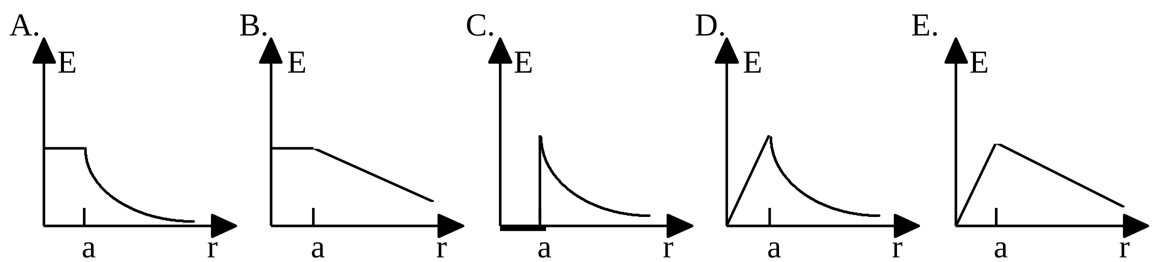
23. A thin film of thickness *t* and index of refraction 1.33 coats a glass with index of refraction 1.50 as shown to the right. Which of the following thicknesses *t* will not reflect light with wavelength 640 nm in air? Hint: Light undergoes a 180º phase shift when it is reflected off a material with a higher index of refraction.



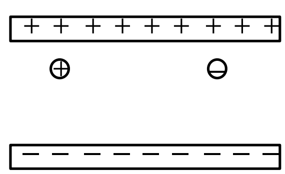
A. 160 nm B. 240 nm C. 360 nm D. 480 nm E. 640 nm

24. An object is placed 12 cm in front of a spherical mirror. The image is right side up and is two times bigger than the object. The image is:
A. 6 cm in front of the mirror and real.
B. 6 cm behind the mirror and virtual.
C. 12 cm in front of the mirror and virtual.
D. 24 cm in front of the mirror and virtual.
E. 24 cm behind the mirror and virtual.

25. A charge is uniformly distributed through a volume of radius *a*. Which of the graphs below best represents the magnitude of the electric field as a function of distance from the center of the sphere?



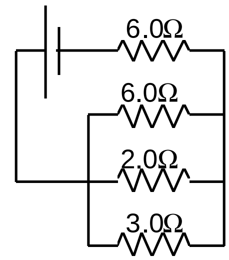
26. A free electron and a free proton are placed between two oppositely charged parallel plates. Both are closer to the positive plate than the negative plate. See diagram to the right. Which of the following statements is true?



I. The force on the proton is greater than the force on the electron.
II. The potential energy of the proton is greater than that of the electron.
III. The potential energy of the proton and the electron is the same.

A. I only B. II only C. III only D. I & II only E. I & III only

27. In the electrical circuit shown to the right, the current through the 2.0 Ω resistor is 3.0 A. The emf of the battery is about



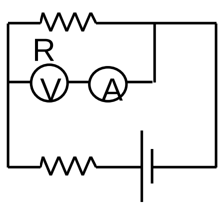
A. 51 V B. 42 V C. 36 V D. 24 V E. 21 V

28. An ion with a charge *q*, mass *m*, and speed *v* enters a magnetic field *B* and is deflected into a path with a radius of curvature *R*. If an ion with charge *q*, mass *2m*, and speed *2v* enters the same magnetic field, it will be deflected into a path with a radius of curvature

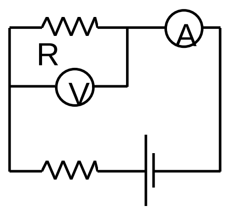
A. 4 R B. 2 R C. R D. (1/2) R E. (1/4) R

29. Which of the following wiring diagrams could be used to experimentally determine *R* using Ohm's Law? Assume an ideal voltmeter and an ideal ammeter.

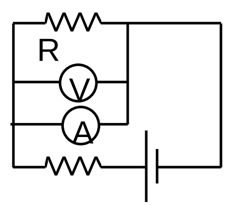
A.



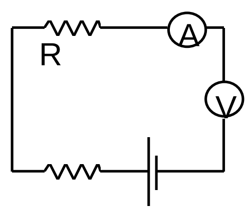
B.



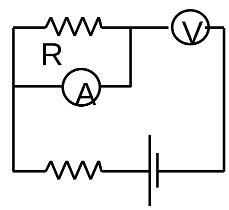
C.



D.



E.



30. A vertical wire carries a current upward through a magnetic field directed to the north. The magnetic force on the wire points

A. south B. north C. east D. west E. downward

1998 MULTIPLE CHOICE SCREENING TEST
ANSWER KEY

1. E	11. C	21. D
2. B	12. C	22. D
3. D	13. E	23. C
4. E	14. D	24. E
5. A	15. A	25. D
6. B	16. A	26. B
7. E	17. C	27. B
8. A	18. B	28. A
9. C	19. A	29. B
10. E	20. C	30. D

1994-1995 Physics Olympiad Preparation Program

– University of Toronto –

Problem Set 1: General Physics

Due October 24, 1994

1. Your physics professor (you're in university now) has babbled all year long in lectures and you couldn't understand a word he said. It's now exam time and, since you didn't bring a cheat sheet (good for you), you decide to derive the formulas for various physical quantities using dimensional analysis in the hopes of partial marks. Your babbling professor may give you partial marks but we'll give you full ones if you get all these (as long as you show your assumptions and your work!).

   (a) You remember a 2π but can't remember the rest of the expression for the period of a simple pendulum. What is the full expression?

   (b) A particle of mass m rotates in a circle of radius r with speed v. The particle has an acceleration a_c called centripetal (centre seeking) acceleration. What is the form of a_c?

   (c) A gas bubble from a deep explosion under water oscillates with a period T. Known variables are p, the static pressure, ρ, the water density, and e, the total energy of the explosion. Find the dependence of the period on the other variables.

2. Being the hip physics-dude/dudette you are, you often go to parties with your non-physics friends. You like to be the centre of attention, telling them about the physics of quarks and gluons, but that usually ends up killing the party. You decide, then to pull a different beast out of your physics hat: *rapid estimation*. You know that sometimes we don't need exact answers but only order-of-magnitude estimations. At a recent party, your friends asked you the following:

   (a) Jake asked: if I beat everyone on the planet into a pulpy liquid, roughly how deep would the liquid be on the Earth's surface? Would I need Doc's, rubber boots, hip waders, or a boat to a avoid getting my new socks dirty? What if I wanted to fill up containers the size of the Skydome$^{\mathrm{TM}}$ in downtown Toronto with this vile mess; how many containers would I need?

   (b) Jane asked: if the entire debt of the Government of Canada were paid off in loonies, how much would it weigh? If I stacked them, how high would it reach? If each loonie was turned into a pound of flesh, how would that affect the answer to Jakes question?

   Remember to justify any non-obvious estimations you make; your physics teacher's daughter is at the party and she'll call your bluff on bad estimations, making you look like a fool in front of your friends.

3. Your Great-Aunt Edna is in town visiting. She's really rich and is about to kick the bucket, so she will be drawing up her will soon. You want to be in it, you materialistic !@?$%#!, you. Now, Auntie Edna, as you like to call her, made her fortune as a high-school physics and english teacher, so she really gets P-O'd when people use poor grammar and physically incorrect phrases. You decide to impress her with the following:

   (a) Metro Toronto Police as well as local newscasters on TV often describe the latest traffic fatality using the phrase "the car was traveling at a high rate of speed." What is wrong with this?

   (b) People describe their watch as being "five minutes fast" or "two minutes slow". What is wrong with this? What do they really mean (or what should they really say)? Is there a correct way of describing a watch's inaccuracy using the words "fast" or "slow"?

   (c) Weather announcers and even meteorologists use the phrase "normal high" and "normal low". What do they mean by "normal"? Is that normal? What might be a better word? Discuss.

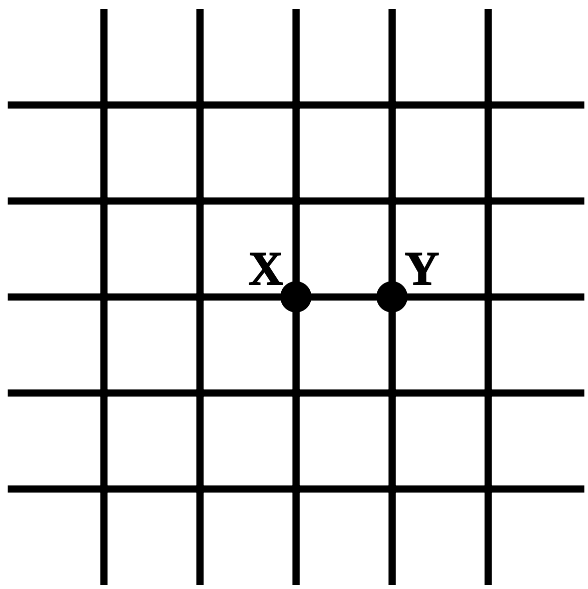


Figure 1: Rectangular wire mesh of infinite extent.

4. Your Uncle Sal from Italy was in North America for the World Cup. He was terribly disappointed with their loss, and is in a blue funk (the blue that is found on the Brazilian flag, as a matter of fact). You want to make him a pesto pizza to cheer him up. While pouring the olive oil, you accidentally spill some into a glass of water, and it spreads out on the surface. Sal, who is watching, is a physics teacher in Italy and he is reminded of an experiment done by Lord Rayleigh. He tells you that he would be happy if only you would solve the following.

   Rayleigh[1] found that 0.81 mg of olive oil on a water surface produced a mono-molecular layer 84 cm in diameter. What value of Avogadro's Number results?

   Note: The approximate composition of olive oil is $H(CH_2)_{18}COOH$, in a linear chain with one end (which?) hydrophilic and the other hydrophobic. Its density is $0.8\,\mathrm{g/cm^3}$

5. Ever the consummate student, you are at Woodbine doing some ~~betting~~ studying. Of course, you brought your $500 camera with a telephoto lens. Watching the thoroughbred racers Eric's Idle and Paul's Bunyon thunder down the home straight, you decide to take a picture. You are looking head on through the telephoto lens and you notice that the horses seem strangely foreshortened (not as long from front to back) as they gallop towards the camera. Explain this observation.

6. Your little brother has been playing with the wire screens and batteries again. Will he ever learn? He asks you the following.

   (a) A rectangular wire mesh of infinite extent in a plane has 1 A of current fed into it at point X, as in the diagram (Figure 1), and 1 A of current taken from it at point Y. Find the current in the wire XY.

   (b) Suppose the wire mesh was made of equilateral triangles joined in the obvious way. What is the current where X and Y are separated by one wire?

[1] Rayleigh, Proc. Roy. Soc., **47**, 364 (1890); Holy smokes, that's 104 years ago! And you're only learning about it now? Bonus points to anyone who can get me a photocopy of the article.

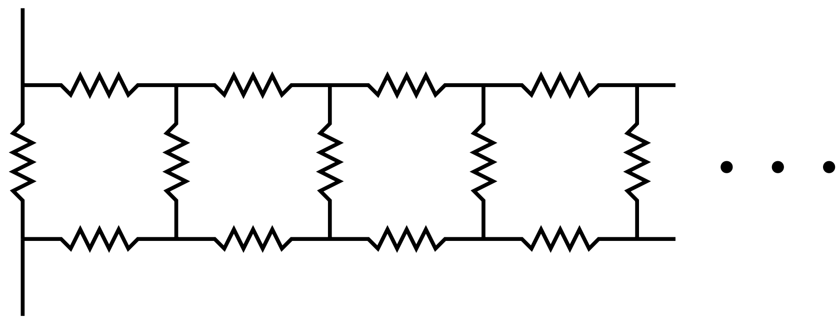


Figure 2: Infinite resistor ladder.

(c) The screen has been cut up and now resembles the circuit in the diagram. What is the resistance of the circuit shown, an *infinite* ladder.

1994-1995 Physics Olympiad Preparation Program

– **University of Toronto** –

Solution Set 1: General Physics

1. Dimensional analysis can be used to check to see if an expression is dimensionally correct or to get the form of an expression if we don't know it. We are going to do the latter.

   (a) Hmm... What could the period of a simple pendulum possibly depend on? There's the mass of the (ideal point) bob (or is that Bob, founder of the Church of the Sub-Genius?), m, the length of the (essentially massless ideal) string, l, the angle of the swing, θ, and the acceleration of gravity, g. We might think to include things like frictional forces, but they're small compared to gravity and besides, that would be too complicated. Let's assume the period T is a function of these four variables, each raised to some power:

   $$T = Cm^w l^x \theta^y g^z.$$

   C is a dimensionless constant, and w, x, y, and z are exponents for which we wish to solve (no dangling prepositions here). The dimensional equation for this relationship is

   $$[T] = [M]^w [L]^x [L/T^2]^z;$$

   the angle θ (measured in radians) has no dimensions. Simplifying,

   $$[T] = [M]^w [L]^{x+z} [T]^{-2z}.$$

   To have dimensional consistency, we must have

   $$\begin{aligned} 0 &= w \\ 0 &= x + z \\ 1 &= -2z \end{aligned}$$

   which gives us $w = 0$, $z = -1/2$, and $x = 1/2$. Thus the desired equation is

   $$T = C\sqrt{l/g}f(\theta)$$

   It turns out that $C = 2\pi$ and $f(\theta) \simeq 1$ for small angles, but we can't get that from this analysis.

   (b) The general equation for this problem is $a_c = Cm^x r^y v^z$, which gives us the dimensional equation $[L/T^2] = [M]^x [L]^y [L/T]^z$. Simplifying, $[L][T]^{-2} = [M]^x [L]^{y+z} [T]^{-z}$, which gives us the three equations $0 = x$, $1 = y + z$, and $-2 = -z$, which are solved by $x = 0$, $z = 2$, and $y = -1$. Thus $a_c = Cv^2/r$. It turns out that $C = 1$.

   (c) The general equation for this problem is $T = Cp^x \rho^y e^z$, which gives us the dimensional equation $[T] = [M/LT^2]^x [M/L^3]^y [ML^2/T^2]^z$. Simplifying, $[T] = [M]^{x+y+z} [L]^{-x-3y+2z} [T]^{-2x-2z}$, which gives us the three equations $0 = x + y + z$, $0 = -x - 3y + 2z$, and $1 = -2x - 2z$, which are solved by $x = -5/6$, $y = 1/2$, and $z = 1/3$. Thus $T = C\left(\frac{\rho^3 e^2}{p^5}\right)^{1/6}$.

2. For these ones, we quote numerical values of things from memory or make good guesses, multiply numbers in our head, rounding them off, and in general not worry about being too precise. It may also help to do the problem algebraically first, then put in the numbers, since this will allow us to plug in more precise numbers if we wish.

(a) In the spirit of rapid estimation, we will give the result as a series of approximate calculations. This way we only have to remember a few numbers at a time.

Presumably the depth of liquid, d, will not be large compared to the radius of the Earth, R_E, so we can use the formula for the volume of a thin shell[1]; $V = 4\pi R_E^2 d$. Thus $d = V/4\pi R_E^2$. What is V? Well, there are nearly 6 billion people on the planet. Their average mass is, say, about 50 kilograms thus the total mass of people is about 300 billion kilograms. The density of a human body is close to that of water[2], about a tonne per cubic metre, so the total volume is about 300 million cubic metres.

The radius of the Earth is about 4 thousand miles (a good number to remember) or over six thousand kilometres. Squared, we have about 40 trillion square metres. Multiplying by $4\pi \simeq 12.5$ we have the surface of the Earth being about 500 trillion square metres. Dividing the volume by this we get a depth of about a half a micron (μm). I don't even know if this would get your socks dirty, but you could easily get by without the rubber boots and just wearing shoes.

The politically-correct restatement of the problem is: if every person on the planet decided to go skinny-dipping in the ocean, then by how much would the ocean level rise? The answer is about 4/3 the answer to the non-politically correct question since the oceans cover about 3/4 of the planet.

What about the number of Skydome™'s? We just divide the volume found above by the volume of a building. The rough area of the football field is 60 m by 130 m or about 8 thousand square metres. The total area is maybe four times that, and the total height is, perhaps, 30 m. Thus the total volume is on the order of one million cubic metres. Dividing the volume of liquid by this we get something on the order of several hundred Skydome™'s.

(b) You may have heard that the federal debt per capita is about \$20,000. If you hadn't heard that, then you should read more newspapers! With 27 million people in Canada, this works out to about \$550 billion. Wow! A loonie has a mass of about 10 grams, I believe. On Earth, 454 grams weighs a pound, so lets say that 50 loonies weigh one pound. Thus the federal debt weighs about 11 billion pounds in loonies.

Stacking them, they are about 2 mm think so that makes about 1.1 trillion metres or a million kilometres. That's to the moon and back easily!

Converting them to a pound of flesh, we have that a pound is about half a kilogram and the density is a tonne per cubic metre. This works out to 270 billion kilograms or 270 million cubic metres. This nearly doubles the volume of liquid for the previous problem, thus it nearly doubles the depth.

3. (a) What is wrong with the phrase "the car was traveling at a high rate of speed"? Well, the word rate usually refers to a change in a quantity measured with respect to an interval of time. The definition of speed is the distance traveled per unit of time, i.e. the rate of change of position with respect to time. Thus the word rate in the above phrase is superfluous. One could say "at a high speed" or one could say "at a high rate" where it is implicit that the rate is position with respect to time, i.e. speed. Note also that we can't really interpret "at a high rate of speed" as acceleration since in that case it would be "at a high rate of change of speed".

(b) As in the previous question, the fast and slow refer to rates as well. What most people mean is "five minutes ahead" or "two minutes behind". One can use rate-implying words when talking about clocks, but in the following context. Suppose we set a watch to coincide with an atomic

[1] You can derive this in the following way. Let r be the radius of a sphere, and $r + \Delta r$ be the radius of a slightly larger sphere. The volume of the shell between the two spheres is the difference in volumes which is $4\pi((r + \Delta r)^3 - r^3)/3 = 4\pi(r^3 + 3r^2\Delta r + 3r\Delta r^2 + \Delta r^3 - r^3)/3$. This is then equal to $4\pi(3r^2\Delta r + 3r\Delta r^2 + \Delta r^3)/3 = 4\pi r^2\Delta r(1 + \Delta r/r + \Delta r^2/3r^2)$. We then use the fact that $\Delta r/r << 1$ to get the volume $V \simeq 4\pi r^2 \Delta r$.

[2] People float with the aid of air in their lungs; without that they would sink. Thus people are denser than water, but not by much.

clock. Thirty days later we check and the watch is ahead one minute. We have gained 1 minute over a period of 30 days. Thus the watch is about one minute per month *fast*. Strictly speaking the units can be canceled to get a unitless number, but it is more meaningful to people to use the above form. Also note that I have seen people with degrees in physics make the mistake of saying fast when they mean ahead!

(c) The normal high and normal low are, in more precise language, the average high and low. That is, they are the averages of the daily highs and lows for a particular day of the year, averaged over the past 100 years or so for which there are recorded temperatures. What is left out is the variance of the highs and lows or rather the distribution of the temperatures.

It is possible that the average temperature on August 28 is 25 ℃ but there may be just as many August 28's which were 23, 24, 26, and 27. Does it then make sense to use the word normal to mean average? In such a case is 25 normal? Probably not.

It is really a case of experts not wanting to use precise language because it may end up confusing people, or so they think.

4. We know that $N/M = N_A/M_A$ where N is the number of molecules in the sample and M is the mass of the sample and the subscript refers to a mole of the substance. Thus Avogadro's Number is $N_A = NM_A/M$. To get M_A, we take the formula and the molecular masses and find the total; $(38 \times 1 + 19 \times 12 + 2 \times 16) = 298\,\mathrm{g}$. To get N, we take the volume of the sample, V, and divide it by the volume of a single molecule, v. Getting v is the tricky part.

The $COOH$ end of the molecule is hydrophilic while the rest of the molecule is hydrophobic. Thus the molecules form a layer standing side-by-side vertically on the surface of the water. The diagram shows approximately what the molecule looks like if we assume that the carbons are at the centres of tetrahedrons with the neighbouring atoms on the corners. There is an end-on view and a side view. The distance between two atoms joined to a common carbon is x—this is the length of an edge of the tetrahedron. It turns out that the end-on area is nearly square—the zig-zag in the carbon chain compensates for the difference between the edge length in one direction and the distance from edge-centre to opposite edge-centre in the other direction. In all three directions a distance of x is added to account for the fact that neighbouring molecules will not be butted up against one-another. The x puts the hydrogen atoms on one molecule a distance x from those on the neighbouring molecules. Thus the volume of each molecule is about $v \simeq 4t^3/121$ where t is the layer thickness. There is a measure of uncertainty in this last equation since we don't know how much space there is between molecules.

The total volume is $V = At$, thus $1/v \simeq 121A^3/4V^3$ and $N \simeq 121A^3/4V^2$. Now $A = \pi(d/2)^2$ and $V = M/\rho$; with a little algebra we get

$$N_A \simeq \frac{121\pi^3\rho^2 d^6 M_A}{256M^3}.$$

Putting in the numbers,

$$\begin{aligned} N_A &\simeq \frac{121\pi^3(0.8\,\mathrm{g/cm^3})^2(84\,\mathrm{cm})^6(298\,\mathrm{g})}{256(0.81\times10^{-3}\,\mathrm{g})^3} \\ &\doteq 2\times10^{24}. \end{aligned}$$

So Avogadro's number is about 10^{24} molecules per mole. We are off by a factor of 3 from the accepted value, which is ok considering how many uncertain numbers there were.

5. A telephoto lens, like a telescope, makes far objects appear near by giving rise to an angular magnification. Let us approximate a horse (call him Boxy) by a rectangular body with a vertical leg at each corner. Suppose the separation of the forelegs and of the back legs is 0.5 m, and the distance between the front and back legs is 2.0 m. If we observe the horse head-on when it is 10 m away, the gap between

its forelegs will subtend an angle of 1/20 rad at the eye, while the gap between the hind legs subtends 1/24 rad. This difference gives most of the impression of the length of the horse.

If the same horse is 100 m away the same two angles become 1/200 rad and 1/204 rad, which are not very different. However, if we use a telescope with an angular magnification of $10\times$, the angles become 1/20 rad and 1/20.4 rad. The apparent size of the front is the same as that of a horse standing 10 m away, but now its hind legs appear to be only 0.2 m further back. A *very* short horse!

6. (a) Break the problem into two parts. Consider the mesh with 1 A going in at X and 1 A coming out at infinity. By symmetry, the wire XY must carry 1/4 A of current. Alternately, consider the mesh with 1 A going in at infinity and 1 A coming out at Y. Again, by symmetry, 1/4 A runs through the wire XY. Add the two cases together and you get the stated problem, with the solution that 1/2 A flows along wire XY.

   (b) Similarly, the two cases each have 1/6 A flowing along XY, giving a solution of 1/3 A.

   (c) Referring to the diagram, you can see that because the circuit is infinite, it can be written as a circuit with itself being one of the components. If R' is the resistance of the entire circuit, then, using the rules for combining resistors in series and parallel, we have

$$\frac{1}{R'} = \frac{1}{R} + \frac{1}{2R + R'}.$$

   Multiplying by all three denominators and rearranging we end up with a quadratic equation for R', ${R'}^2 + 2RR' - 2R^2 = 0$. We use the quadratic formula to solve, with the negative root thrown out. The result is $R' = (\sqrt{3} - 1)R$.

1994-1995 Physics Olympiad Preparation Program

– University of Toronto –

Problem Set 2: Mechanics

Due November 28, 1994

1. A crude estimate of the depth of a well can be obtained by dropping a stone down it and listening for the splash, then using $s = gt^2/2$. You are asked to make first-order estimates of the corrections due to

   (a) the finite speed of sound ($v_s = 340\,\mathrm{m/s}$), and

   (b) air drag.

   Which of these is the bigger correction for $s = 30\,\mathrm{m}$ and a stone of radius $r = 2\,\mathrm{cm}$ and density $3 \times 10^3\,\mathrm{kg/m^3}$? Would your conclusion be significantly different if a table tennis ball were dropped in the same way ($r = 2\,\mathrm{cm}$, $m = 1.7\,\mathrm{g}$)? You may assume that the splash would still be audible!

   Note: the drag for a sphere of radius a depends on the dimensionless Reynolds number $R = \rho v a/\eta$, where ρ is the fluid density and η is its viscosity. For $R < 10$, Stokes' law $F = 6\pi r\eta v$ is valid, but for $R > 100$ the drag force is given by $F = C_D \rho \pi r^2 v^2$; the variation of C_D with $\log_{10} R$ is given in the diagram. For air at 20 ℃, $\rho = 1.3\,\mathrm{kg/m^3}$ and $\eta = 1.8 \times 10^{-5}\,\mathrm{kg/ms}$.

2. Your phys-ed teacher is a student of the physics of human motion, and asks you the following questions:

   (a) When told that the world record for the pole vault was about 5.5 metres, the fast rising athlete Rod Fibreglass told the press, "Give me a pole long enough, and I will raise the record to 9 metres." Could he manage it? How high might he raise it if he tried hard?

   (b) I obtained the following data one Sunday:

      i. There are 238 steps from the basement to the 10^{th} floor of our building.

      ii. After measuring four steps, I believe the step height is $18.2 \pm 0.3\,\mathrm{cm}$.

      iii. I jogged up the steps in 1 minute and 40 seconds, give or take 2 seconds. Well, actually started off jogging and ended up walking up with a not-so-steady pace.

      iv. My mass is $82.07 \pm 0.2\,\mathrm{kg}$. The uncertainty is due to the fact that I drank some water, ate a banana, sweated and urinated in the time between doing the climb and measuring my mass.

      v. The value of g measured by some geophysicists at the base of the north stairwell is $9.804253\,\mathrm{m/s^2}$. They don't give an uncertainty estimate. Note that they say that g varies with height as $\Delta g/g = -2\Delta R/R$, where[1] $R = 6371\,\mathrm{km}$.

      How much (useful) work did I do? What was the average power output? Calculate the uncertainties in these values as well.

   (c) It is proposed to make a human-powered helicopter with a rotor 10 m in diameter. Assuming that the rotor blows a cylindrical column of air uniformly downwards, the cylinder diameter being the same as the rotor diameter, and that the mass of the pilot plus machine is 200 kg, calculate the minimum mechanical power (in Watts) that is necessary for the pilot to generate to remain airborne. Is the system practicable? Compare with my power output above. The density of air is $1.23\,\mathrm{kg/m^3}$.

3. NASA has just contracted you to make the following calculations, you lucky dog. As with many NASA contracts, you are being paid a handsome sum with bonuses for correctness. No $10,000 toilet seats, please.

[1] I don't know why this number, posted in our undergraduate labs, disagrees with others quoted here (6.38×10^3 km), except to say that the Earth isn't a sphere.

(a) A small moon of mass m and radius a orbits a planet of mass M while keeping the same face towards the planet. Show that if the moon approaches the planet closer than $r_c = a(3M/m)^{1/3}$, then loose rocks lying on the surface of the moon will be lifted off.

(b) It is well known that in an orbiting space vehicle the occupants can drift around freely in "zero-gravity" conditions. Assume that you are in a (strong) spaceship, 100 m long and fairly narrow, which is in a circular orbit of 1000 km radius around a neutron star, with its long axis always pointing towards the centre of the star. There is an inspection tunnel running down the axis of the ship.

  i. What would happen if an astronaut attempted to float down it?
  ii. Calculate likely values of the acceleration observed, assuming that the mass of the star is 3×10^5 Earth masses and that the radius of the Earth is 6×10^6 m.
  iii. Is an orbit such that the axis of the craft always points towards the star a stable condition? Explain briefly.

(c) Two stars in a binary system have a separation $2r$ and equal masses m, and move in circular orbits about their centre of mass. One star explodes by expelling a small fraction of its mass very rapidly, and immediately after its recoil speed is v_f. What is the largest value of v_f for which the two stars will remain gravitationally bound?

4. So, Minnesota, you think you know your billiards? Try to make this trick shot. A billiard ball of radius a rests on a table. It is hit with a cue in such a way that it starts out with speed u_0 and backspin ω_0 about a horizontal axis perpendicular to the direction of motion. How does the subsequent motion depend on the ratio $u_0/a\omega_0$? Discuss.

5. It is sometimes stated that it would be possible to construct a spaceship using a photon drive, which would be able to travel away from Earth with a speed close to that of light. Assuming that the fuel consists of equal masses of protons and antiprotons, does one get a greater final velocity

   (a) by allowing half the fuel to annihilate and using the energy released to eject the remaining half, or

   (b) by allowing all the fuel to annihilate and ejecting photons?

   You may use the relativistic relationship between energy and momentum for a particle of rest mass m_0, $E^2 = (pc)^2 + (m_0c^2)^2$ and for a massless photon, $E = pc$, where E is energy, p is momentum, and c is the speed of light.

6. You are on board the USS Enterprise when an artificial structure known as a "ringworld" is discovered circling a star similar to Sol. It is a large circular ribbon-shaped object with the following characteristics. The flat side is illuminated by the star; its radius is that of Earth's orbit, $R_{SE} = 1.50 \times 10^{11}$ m. The width is 1.00×10^9 m and the thickness is 100 km. The edges of the interior "floor" are lined with walls which have a triangular cross-section, are 1000 km high and have a base width of 100 km. This keeps in an earth-like atmosphere. Data is off-duty now so Captain Picard asks you the following.

   (a) If the floor is to have the same "gravity" as the earth, what rotational period should it have? Before calculating, give some limits on the period based on simple facts about the Earth.

   (b) Calculate the tensile strength needed for the material in order for it to remain intact. You may assume the density of the material is that of aluminum, $2.70 \times 10^3\,\mathrm{kg/m^3}$. For comparison, aluminum's tensile strength is about $2 \times 10^8\,\mathrm{N/m^2}$.

   (c) Remember, there is a star in the centre of the ring. Is the spinning ring stable with respect to the star? That is, if we were to push the ring a bit, would it return to its original position (like a marble in a bowl) or would it move further out of position (like a marble on a beach ball)? You may answer qualitatively.

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Solution Set 2: Mechanics

1. (a) If the real depth is s and the velocity of sound is v_s, then the measured time would be $t_m = (2s/g)^{1/2} + s/v_s$ and the measured depth would be $s_m = gt_m^2/2$. Neglecting squares of small terms, we obtain

$$\frac{s_m - s}{s} \simeq \sqrt{\frac{2gs}{v_s^2}} = \sqrt{\frac{2(9.8)(30)}{340^2}} \doteq 7.2\%.$$

That is, the depth calculated without taking into account the finite speed of sound would be about 7 percent too large.

(b) An approximate value for the final velocity is[1] $(2gs)^{1/2} \simeq 24\,\mathrm{m/s}$, showing that the Reynolds number is of the order of 10^4 most of the time; the drag force is therefore $C_D \pi \rho r^2 v^2$, with $C_D \simeq 0.4$. The equation of motion is thus $ma = mg - \beta v^2$, with $\beta = C_D \pi \rho r^2$. For the zeroth approximation, we neglect the last term so that $a = g$ and thus $v = gt$. For the first order approximation, we insert this value of v into the (small) last term to give $ma = mg - \beta g^2 t^2$, or $a = g - \beta g^2 t^2/m$. Now, if you knew calculus, you would immediately integrate this to obtain $s = gt^2/2 - \beta g^2 t^4/12m$. You might also just know that whenever you have a term At^n in the acceleration, it gives you an $At^{(n+1)}/(n+1)$ term in the velocity and an $At^{(n+2)}/(n+1)(n+2)$ term in the displacement. You should really learn calculus.

Anyway, neglecting the air drag thus introduces an error equal to the last term, $\beta g^2 t^4/12m$. The relative error is thus $\beta g t^2/6m \simeq \beta s/3m$. From the data given, we find that this is about 6%, thus the air drag and the finite speed of sound give approximately equal errors. Both errors make the calculated depth too large.

(c) For the table tennis ball, the drag force is much more important relative to the weight. The velocities are smaller, but still large enough to make $R > 10^3$, i.e. C_D is still about 0.4. In this case, the terminal velocity is given approximately by[2] $(mg/\beta)^{1/2} \doteq 5.3\,\mathrm{m/s}$. The ball will reach this speed after a few meters, so that the zeroth approximation, $s_m = gt^2/2$, is a gross overestimate of the depth—by a factor of about 2.6 for $s = 30\,\mathrm{m}$. A better first approximation is to assume that it travels at its terminal velocity for the whole distance. In this approximation, the error due to the finite speed of sound is clearly given by the ratio of the terminal velocity to the speed of sound. The error due to the initial period of acceleration cannot be evaluated without using the exact solution of the equation of motion. This is too complicated to be evaluated in the present context.

2. (a) The dominant consideration is the conversion of the kinetic energy of the running man to gravitational potential energy with something less than 100% efficiency. If, in his approach, he attains a speed of 10 m/s, the corresponding rise is 5 m. Smaller terms arise from

   i. the fact that his centre of mass is already 1 m above the ground when he starts,

   ii. the work done by his legs on take-off and by his arms in climbing up the pole (? = 0.5 m; consider the "vertical" of a standing person),

   iii. the fact that his centre of mass actually passes *below* the bar (? = 10 cm).

Adding these terms together gives approximately 6.6 m. The difference between this and the observed 5.5 m is due to an efficiency of less than 100%—or to errors in the estimated quantities. In any case, there is clearly no hope of Mr. Fibreglass making good on his boast of 9 m since 10 m/s is an excellent sprint speed and the other terms can not be improved by much.

[1] Equate initial potential energy with final kinetic energy and solve for speed.

[2] Write force equation for terminal body, including gravity and drag force but neglecting the small buoyant force, set acceleration to zero, then solve for speed.

(b) Work done is the change in potential energy of my body, $W = mgh$. Uncertainty is

$$\Delta W/W = \sqrt{(\Delta m/m)^2 + (\Delta g/g)^2 + (\Delta h/h)^2}.$$

The $(\Delta g/g)^2$ is negligible. The result is 34.85 ± 0.4 kJ. Note that the uncertainty in the stair height dominates. A measurement of the actual distance would improve things immensely.

The average power output is $P = W/t$, while the uncertainty is

$$\begin{aligned}\Delta P/P &= \sqrt{(\Delta W/W)^2 + (\Delta t/t)^2} \\ &= \sqrt{(\Delta m/m)^2 + (\Delta g/g)^2 + (\Delta h/h)^2 + (\Delta t/t)^2}.\end{aligned}$$

The result is 348.5 ± 8 W. This is nearly half a horsepower. Don't be impressed. I only did this for a couple of minutes. A horsepower is the average power a horse can put out over a work day. I've been told that for longer times (15 minutes to 2 or 3 hours) for people of average mass (65 to 75 kg), useful power output is roughly 275-300 W. By that criteria, given how winded I was, I am out of shape!

(c) If the rotor, of radius R, gives a downward velocity v to a column of air initially at rest, the momentum transferred per second will be μv where μ, the mass of the air accelerated per unit time, is $\pi R^2 v \rho$. If the helicopter is to be airborne, this must equal Mg, where M is the mass of the helicopter. Inserting the values given, we obtain $v \doteq 4.5$ m/s. Neglecting energy losses, the energy transferred per second to the bulk motion of the air is $\mu v^2/2$ which equals 4.5 kW in this case. For the system to be practical, the pilot must thus generate about 5 kW continuously. From the answer to the previous question, it appears that the maximum rate of energy production by a human is of the order of 1/3 kW. Thus we are a far cry from being able to run a helicopter by human power.

3. (a) For the moon in an orbit of radius r, we have[3] $GMm/r^2 = mr\omega^2$, assuming $M \gg m$, where G is the universal gravitational constant and $\omega = 2\pi/T$ is the angular frequency with T being the period. For the rock of mass μ, in orbit of radius $r - a$, Newton's Law gives us

$$\frac{GM\mu}{(r-a)^2} - \frac{Gm\mu}{a^2} + F = \mu(r-a)\omega^2$$

where F is the force of contact between the rock and the moon. If the rock is to be lifted off, $F = 0$. Eliminating ω between the two equations, we obtain

$$\frac{M}{m} = \frac{r^3}{a^3}\frac{(r-a)^2}{3r^2 - 3ra + a^2} \simeq \frac{1}{3}\left(\frac{r}{a}\right)^3$$

as required.

(b) For any spacecraft in a circular orbit, $GMm/r^2 = mr\omega^2$. For a body on the surface of the Earth, $GM_E/r_E^2 = g \doteq 10\,\mathrm{m/s^2}$. With the numbers given, we find $\omega = 10.4$ rad/s, i.e. the period is approximately $\pi/5$ seconds—somewhat uncomfortable for the intrepid astronaut! We can also deduce that the linear velocity in orbit is approximately 10^7 m/s $\simeq 0.03c$—high, but not relativistic.

i. For an astronaut of mass m_a at the mass centre of the craft, $GMm_a/r^2 = m_a r\omega^2$ is also true, i.e. she will, indeed, float.

ii. If r changes to $r - x$, the gravitational force becomes $GMm_a/(r-x)^2$. If sideways motion across the vehicle is prevented, and if the vehicle is large compared with the mass of the person, then ω will remain constant, that is the ship will force the astronaut to orbit with

[3] The gravitational force provides the centripetal force which is set equal to the mass times centripetal acceleration.

the same period. The force needed to maintain the astronaut in the new, smaller orbit will be $m_a(r-x)\omega^2$. This is no longer equal to the gravitational force. The difference, approximately $3m_a\omega^2 x$, will accelerate the person towards the centre at a rate $3\omega^2 x$. This will give rise to a velocity[4] $\sqrt{3}\omega x$. Starting from the end of the tunnel ($x = 50$ m), this gives rise to a maximum speed of 900 m/s (3240 km/h in more homely units). On the way to this unfortunate conclusion there will be a Coriolis acceleration sideways across the tunnel equal to $2\sqrt{3}\omega^2 x$. This will reach the value g before the victim has moved more than a few centimetres.

iii. The orbit as described is indeed stable. The differential radial force calculated above—the so-called tide-raising force—will provide a restoring couple if the attitude is disturbed, as can be seen by drawing a diagram. Also, one can appeal to tides themselves; they are at a maximum when the graviational source is overhead.

(c) Initially Newton's Law gives

$$mv^2/r = Gm^2/4r^2. \tag{1}$$

The final system is most likely to break up if the explosion increases the velocity of one of the one star in the direction in which it is moving at the time of the explosion (consider the diagram where before and after denote the situation an instant before and after the explosion). Call the increased velocity v_f. The centre of mass now has a velocity $(v - v_f)/2$ and the velocity of either star with respect to the centre of mass is, say,

$$(v + v_f)/2 = v'. \tag{2}$$

The system will break up if the new kinetic energy with respect to the centre of mass coordinates is greater than the work needed to separate the components to infinity, i.e. if $mv'^2 > GM^2/2r$. Using relations (1) and (2), the required condition becomes

$$v_f = (2\sqrt{2}-1)v = (2\sqrt{2}-1)(Gm/4r)^{1/2}.$$

The above argument assumes that the mass lost as a result of the explosion is so small that it has no effect on the result. If this quantity is denoted by Δm, then an analysis along exactly the same lines gives the condition $v_f/v = 2(2 - \Delta m/m)^{1/2} - 1$, thus showing that the assumption was valid.

4. We choose the positive sense of ω so that the velocity of the point of contact of the ball with the table, v_c, is given by $u + a\omega$. The frictional force $F = \mu mg$ is assumed constant. The motion of the centre is thus given by $u = u_0 - \mu gt$ and the rotation about the centre by $\omega = \omega_0 - a\mu mgt/I$. Since $I = 2ma^2/5$ for a sphere, the latter expression becomes $\omega = \omega_0 - 5\mu gt/2a$. These results combine to give $v_c = u_0 + a\omega_0 - 7\mu gt/2$. This is equal to zero (i.e. slipping stops and rolling begins) when $t = 2(u_0 + a\omega_0)/7\mu g = \tau$, say. The value of u at $t = \tau$ is $(5u_0 - 2a\omega_0)/7$; this gives the speed of rolling after slipping has stopped. If $u_0/a\omega_0 < 2/5$, the value is negative and the ball will roll backwards towards the cue. This corresponds to the imposition of "backspin" as specified. If $u_0/a\omega_0 > 2/5$, then the ball will roll forwards.

To enquire whether both motions are possible, we consider an impulse Q given to the ball at a point $a - x$ above the table in the direction making an angle θ with the horizontal (see the diagram). Then $mu_0 = Q\cos\theta$ and $I\omega_0 = Qp$, with $p = a\sin(\theta + \phi)$ and $\sin\phi = x/a$. If we use the relation $5u_0 = 2a\omega_0$ to determine the boundary condition between the two modes of behaviour, then, after a little algebra, we get $\cos\theta = \sin(\theta+\phi)$, i.e.

$$\tan\theta = \left(\frac{1 - x/a}{1 + x/a}\right)^{1/2}.$$

Real values of θ can be found for any x/a—except that $x \to a$ would be a little difficult, in practice!

[4] Think of simple harmonic motion with a spring constant over mass $\omega_0^2 = k/m_a = 3\omega^2$; the maximum speed, obtained while passing through the centre, will be $x\omega_0$.

5. (a) The annihilation energy of a particle of rest mass m_1 will make available an energy $\Delta E = m_1 c^2$. If this energy is given to another particle of rest mass m_2, which thereby acquires momentum p, we have the relation[5]

$$(\Delta E + m_2 c^2)^2 = p^2 c^2 + (m_2 c^2)^2.$$

In the particular case envisaged in the problem, $m_1 = m_2 = m$ is the mass of a proton or antiproton. This leads to the result $p = \sqrt{3}mc$. If there are initially N protons and N antiprotons, then this process can be repeated N times[6]. If all N ejected particles can be persuaded to leave the spaceship in the same direction, the effect will be to give a total momentum $\sqrt{3}Nmc$ to the ship.

(b) If we consider the mutual annihilation of two particles, each of rest mass m, a similar argument shows that the (scalar) total of the momentum thereby produced is $2mc$. There will, in fact, be two photons proceeding in opposite directions (since the vector total must be zero). With the aid of a deep paraboloidal mirror made of a material that can reflect high-energy gamma rays (!) all such photons can be collimated into a beam carrying a momentum $2Nmc$. The spaceship will have an equal momentum in the opposite direction. This is larger than the other result.

The above arguments are strictly valid only if the increment in velocity of the spaceship is much less than c. If this is not the case, the total number N can be conceptually subdivided into small groups for which it is true, and the total effect of the momentum increments between successive inertial frames can then be calculated as an integral (sum). The advantage of (5b) over (5a) applies at every stage and therefore also overall.

6. (a) As a first guess, one thing we can do is to get a maximum on the period. We don't feel the effect of the centripetal acceleration of our orbiting the sun very easily on Earth, so it must be less than, say, a percent of g. Since the acceleration goes as the inverse square of the period, the period must be less than 10 percent the Earth's year, that is less than a month. It is difficult to say anything more since we don't have an indication of how restrictive our maximum is.

More quantitatively, the centripetal acceleration of a particle moving in a circle of radius r at a constant speed v is

$$a_c = \frac{v^2}{r} = \left(\frac{2\pi r}{T}\right)^2 \frac{1}{r}$$

where T is the period. Thus $T = 2\pi\sqrt{r/a_c}$. Using $a_c = g$ we get $T \doteq 7.77 \times 10^5\,\mathrm{s} \doteq 9.0$ days. That seems like an awfully short time to go around the sun! Note that we've ignored the gravitation attraction of the star, which is small compared to g.

(b) Consider a narrow section of the ring as shown in the diagram. The net force on the section is $2F_T \sin(\Delta\theta)$ towards the centre. This is the centripetal force and must equal $ma_c = mg$ where m is the mass of the section. This is equal to $2r\Delta\theta M/2\pi r$ where r is the radius of the ring and M is the total mass. The total mass is the volume times the density, $M = 2\pi r A\rho$, where A is the cross-sectional area. Letting $\sin(\Delta\theta) \rightarrow \Delta\theta$ for small angles, we find that the tension is given by $F_T = rA\rho g$. This is just the "weight" of the ringworld on Earth (if we could weigh it) divided by 2π! To get the tensile strength we just divide by the cross-sectional area; $S = r\rho g$. Notice how we don't need to know A. If we take the density of aluminum as a possible density for the material, then the required tensile strength is $4 \times 10^{15}\,\mathrm{N/m^2}$. This is seven orders of magnitude greater than most metals.

[5] The left-hand side is the square of the total energy before, while the right-hand side is the square of the total energy after, using the relationship between energy and relativistic momentum.

[6] Actually, the more complicated process of annihilating a proton-antiproton pair giving the energy to another proton and antiproton pair which doesn't annihilate but rather is ejected in the same direction is performed $N/2$ times, but the effect is the same.

(c) The rotation of the ring will give it orientational stability, but not positional. The positional stability is depends only on the net gravitational attraction of the star.

If the ring is pushed so that it moves perpendicularly to its plane, then one can see a net gravitational force attracting the ring back to equilibrium. If the ring is pushed in its plane, the situation is not as easy to see. Consider the force due to four pieces as shown in the diagram. The size of the force goes as $1/r^2$ which depends on the position of the piece on the circle. The "vertical" component (see figure) also depends on the position but in a opposing way. The result is that one can't find out whether there is a net gravitational force in the plane by simple arguments. One must use math and it turns out there is a net restoring force.

Remarkably enough, there are cases where one can have an instability. A Dyson Sphere (a spherical shell of similar dimensions surrounding a star) would have no net force on it due to a star inside, thus it would be unstable. The argument for this is mathematical as well.

I got the idea for the ringworld question from a book by Larry Niven called, curiously enough, *Ringworld*. It is followed by *Ringworld Engineers*, and may be found in the science fiction section of libraries and bookstores. The book describes the material as "Very dense, with a tensile strength on the order of the force that holds nuclei together." Anyone care to check on whether that would be enough?

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Problem Set 3: Thermodynamics

Due January 9^{th}*, 1995*

1. Helga Helmholtz has a "cool" summer job working for the Acme refrigeration company. Her neurotic boss has asked her to test out the latest Carnot-Deluxe model, one with a 50 W motor in it. She was told to fill the fridge with ice water, the liquid part of which weighed 2 kg, and wheel the unit outside where the air temperature was 27 °C. Her task was to find the smallest possible time necessary to freeze the rest of the water. Now, Helga also happens to be a bright physics student, and she doesn't get paid by the hour either. She instead decides to stay in her air conditioned office, calculate the answer, and go home early. Wouldn't you like to work for a fridge company next summer? Well, even if you don't, you can still try answering this problem. (Note: take the latent heat of fusion for water to be 6.0 kJ$\cdot$mol^{-1}. Be sure to mention any assumptions you make.)

2. Imagine that there are two kinds of *E. coli* bacteria, which are in every way identical except that one kind is "beige" in colour and the other is "fuchsia" in colour. Each type reproduces (they have no sex) by splitting into halves, beige→beige+beige or fuchsia→fuchsia+fuchsia, with a reproduction time of 1 hour. A colony of 5000 beige and 5000 fuchsia *E. coli* is permitted to eat and reproduce. In order to keep the colony size down, a Pacman© predator is introduced that keeps the colony size at 10,000 by devouring the bacteria at random.

   (a) Derive an expression for the probability distribution of the number of "beige" bacteria existing after a very long time.

   (b) Estimate how long you would need to wait for the answer in (a) to be true.

   (c) Suppose the Pacman© predator has a 1% preference for eating "fuchsia" bacteria. How would this affect your answers to (a) and (b) above?

   (d) What do *E. coli* like to eat?

3. It is safe to say that the most popular drug among physicists is, you guessed it, caffeine. If you make a cup of hot coffee in which you would like to have cold milk added, and you are called away and therefore prevented from drinking it for 10 minutes, when should the cold milk be added in order that the cup's contents are as hot as possible when you return after the 10 minutes are up? Discuss.

4. Now, how about some chemical thermodynamics? "Doc" (played by Christopher Lloyd in the movie *Back to the Future*) is building his latest intergalactic space-time instantaneous travel machine. The temporal blaster circuit, in order for it to function, needs to be inserted into a vat of hydrogen iodide. Doc makes his hydrogen iodide by reacting 2.94 g-mol of molecular iodine with 8.10 g-mol of molecular hydrogen at a constant temperature. How much hydrogen iodide can he expect to produce at the given temperature if the equilibrium constant for this reaction is $K_c = 0.02$?

5. One day while contemplating your morning toast, imagine a long, thin grain of dust (shaped like a needle) floating in a box of laughing gas at a fixed temperature T. On average, is the angular momentum vector nearly parallel or perpendicular to the long axis of the dust grain? Explain your answer.

6. You are a NASA scientist charged with the design of the mission abort circuitry for a remote terrestrial volcano exploration module. Once conditions get too hot for the module, a bimetallic strip thermostat trips the abort circuitry and the device is jettisoned from the volcano. The bimetallic strip has a thickness x and is straight at temperature T. What is the radius of curvature of the strip, R, when it is heated to temperature $T + \Delta T$? Take the coefficients of linear expansion of the two metals to be α_1 and α_2, respectively, with $\alpha_1 < \alpha_2$. You can assume that each metal has thickness $x/2$, and that $x \ll R$.

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Solution Set 3: Thermodynamics

1. First, calculate the heat released in freezing 2 kg of water at 0°C, Q_c:

$$Q_c = \frac{(2 \times 10^3 \text{ g}) \cdot (6.0 \text{ kJ/mol})}{(18 \text{ g/mol})} = 667 \text{ kJ}$$

The maximum coefficient of performance, η_{max}, is given by:

$$\eta_{max} \equiv \frac{Q_c}{W_{min}} = \frac{T_c}{T_h - T_c},$$

where W_{min} is the minimum work needed to effect the cooling, and T_c and T_h are the temperatures of the cold (inside fridge) and hot (outer air) reservoirs, respectively.

Thus,

$$W_{min} = \frac{T_h - T_c}{T_c} Q_c.$$

The minimum time, τ_{min}, is given by (P is the power, and note that absolute temperatures must be used):

$$\tau_{min} = \frac{W_{min}}{P} = \frac{(300 \text{ K} - 273 \text{ K})}{(273 \text{ K})(50 \text{ W})} \cdot (667 \times 10^3 \text{ J}) \doteq 1.3 \times 10^3 \text{ s}.$$

Assumptions can include:

(a) The refrigerator, as suggested by the model name, is taken to be a Carnot heat engine, where $Q_c/Q_h = T_c/T_h$.

(b) The ice and water are in equilibrium; therefore, the liquid is at 0°C.

(c) *Only* the water is being cooled (i.e. no container material or air).

(d) If the cold reservoir (interior of the fridge) were completely filled with ice water, then it would need to be expandable in some way, since water expands as it freezes.

2. (a) After a long enough time, and if there were no predator, the bacteria would amount to a huge number, $N \gg 10,000$. Since the predator eats either colour at random to keep the population at $10,000$, then it's the same thing as selecting $n = 10,000$ surviving bacteria out of N total bacteria. Since $N \gg n$, in every selection the probabilities of surviving beige or fuchsia bacteria are the same, namely $\frac{1}{2}$. In other words, there are 2^n ways of selecting the n survivors. There are C_b^n combinations of b beige bacteria to survive out of the n total surviving. The probability distribution of the number of beige *E. coli* is therefore:

$$\frac{C_b^n}{2^n} = \frac{1}{2^n} \cdot \frac{n!}{b!(n-b)!}, b = 0, 1, \ldots, n.$$

(b) To answer this, you basically need to determine how long it takes for $N \gg n$. Let us require $N/n \approx 10^2$. Since $N = 2^t n$, where t is the number of reproductive cycles, $t = 6$ or 7 hours would be sufficient.

(c) Let p be the degree of preference of eating fuchsia bacteria over that of eating beige bacteria. The probability of eating beige bacteria is then $\left(\frac{1}{2}-p\right)$, and that of eating fuchsia bacteria is $\left(\frac{1}{2}+p\right)$. The result in (a) then becomes:

$$\left(\frac{1}{2}-p\right)^b \cdot \left(\frac{1}{2}+p\right)^{n-b} \cdot C_b^n = \left(\frac{1}{2}-p\right)^b \cdot \left(\frac{1}{2}+p\right)^{n-b} \cdot \frac{n!}{b!(n-b)!}$$

The result in (b) is unaffected by the predator preference p.

(d) Sewage.

3. First of all, we assume that the room temperature falls between that of the milk and the black coffee. There are 2 principal competing effects:

   (a) The addition of the milk cools the coffee directly by a simple law of mixtures.

   (b) "Newton's Law of Cooling", which basically says that the rate of heat loss is greater when the temperature difference between the two heat reservoirs (i.e. the coffee or milk/coffee mixture and the surrounding air) is greater.

   For the simple case where the milk is colder than room temperature, it is in fact better to add the milk at the *beginning* of the 10 minutes in order that the milk/coffee mixture be as hot as possible at the end of the 10 minutes. By cooling with milk the cup's contents right at the beginning, the temperature difference between the two reservoirs is lessened, and hence the rate of cooling slows. The "blackness" of coffee without milk also enables the coffee to be a better radiator of heat, but this effect is rather smaller than those above. (**Note:** For a good article on this subject, see Rees, W.G., and C. Viney, *Am. J. Phys.* **56** (5) 434, May 1988).

4. We are dealing with the chemical equation:

$$H_2 + I_2 \rightleftharpoons 2HI.$$

   We can write this in the form $\sum_i \nu_i A_i = 0$, where A_i are the reacting substances and reaction products, and ν_i are the coefficients of the chemical equation. Hence,

$$2(HI) - 1(H_2) - 1(I_2) = 0.$$

   Reaction equilibria can be determined by the Law of Mass Action,

$$\prod_i C_i^{\nu_i} = K_c,$$

   where C_i are the equilibrium concentrations of the various constituents of the reaction.

   Let x be the number of g-mol of H_2 (or I_2) used to produce HI. Then:

$$C_{(H_2)} = \frac{8.1-x}{11.04}$$

$$C_{(I_2)} = \frac{2.94-x}{11.04}$$

$$C_{(HI)} = \frac{2x}{11.04}$$

   (11.04 is the total number of g-mol present).

Now, using the Law of Mass Action gives:

$$\left(\frac{2.94-x}{11.04}\right)^{-1}\cdot\left(\frac{8.10-x}{11.04}\right)^{-1}\cdot\left(\frac{2x}{11.04}\right)^{2}=0.02.$$

Solving the quadratic equation, and choosing the root that doesn't violate the conservation laws, we get $x = 0.319$. Therefore, there will be 0.638 g-mol of HI.

5. Let the long axis of the dust grain be parallel with the z-axis. From the shape of the grain, we know that the principal moments of inertia satisfy:

$$I_z < I_x, I_y.$$

At thermal equilibrium:

$$\frac{1}{2}I_z\omega_z^2=\frac{1}{2}I_x\omega_x^2=\frac{1}{2}I_y\omega_y^2,$$

where $w \equiv$ the average angular velocity. This is due to the *energy equipartition theorem* of thermodynamics, which states that in thermal equilibrium each degree of freedom contributes an equal amount of kinetic energy to the total energy of a molecule. (Here the dust grain is being treated like a very big molecule.)

Therefore,

$$|\omega_z|=\sqrt{\frac{I_x}{I_z}}\,|\omega_x|=\sqrt{\frac{I_y}{I_z}}\,|\omega_y|,$$

and

$$|I_z\omega_z|=\sqrt{\frac{I_z}{I_x}}\,|I_x\omega_x|<|I_x\omega_x|.$$

Similarly,

$$|I_z\omega_z|<|I_y\omega_y|.$$

So, the angular momentum vector is nearly perpendicular to the long axis of the dust grain.

6. Assume the initial length is l_o. After heating, the average lengths of the 2 metallic strips are:

$$l_1 = l_o(1+\alpha_1\Delta T)$$

and

$$l_2 = l_o(1+\alpha_2\Delta T).$$

Assuming that the radius of curvature is R, the subtending angle of the strip is θ, and the change of thickness is negligible, then

$$l_2 = (R + x/4)\theta$$

and

$$l_1 = (R - x/4)\theta.$$

Therefore,

$$l_2 - l_1 = \frac{x}{2}\theta = \frac{x}{2}\frac{l_1+l_2}{2R} = \frac{x}{4R}l_o\left(2+(\alpha_1+\alpha_2)\Delta T\right).$$

But, we know that

$$l_2 - l_1 = l_o \Delta T \left(\alpha_2 - \alpha_1 \right).$$

Making the substitution, we have

$$l_o \Delta T \left(\alpha_2 - \alpha_1 \right) = \frac{x}{4R} l_o \left(2 + \left(\alpha_1 + \alpha_2 \right) \Delta T \right)$$

giving

$$R = \frac{x \left(2 + \left(\alpha_1 + \alpha_2 \right) \Delta T \right)}{4 \Delta T \left(\alpha_2 - \alpha_1 \right)}.$$

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Problem Set 4: Waves and Optics

Due February 6^{th}*, 1995*

1. One dark and stormy night you find yourself lost in the forest when you come upon a small hut. Entering it you see a crooked old woman in the corner hunched over a crystal ball. You are about to make a hasty exit when you hear the howl of wolves outside. Taking another look at the gypsy you decide to take your chances with the wolves, but the door is jammed shut. Resigned to a bad situation you approach her slowly, wondering just what is the focal length of that nifty crystal ball...

   (a) If the crystal ball is 20 cm in diameter with $n = 1.5$, the gypsy lady is 1.2 m from the centre of the ball, where is the image of the gypsy in focus as you walk towards her?

   (b) Describe the image of the crooked old lady, and be more quantitative than just telling me she is hideous.

   (c) The old lady moves the crystal ball closer to her wrinkled old face. At some point you can no longer get an image of her, which at this point is no small blessing. At what object distance will there be no image of the gypsy formed?

2. If any of you people end up going into fysix, you will study about Young's two slit experiment until you are just about ready to yak. Just to magnify the extent of any future yakking, here we go... The idea of the experiment is to take a wave and pass it through two slits, and then observe an interference pattern on a screen (see figure #1). The figure uses a laser but the experiment can be done with any wave-like entity such as electrons or water waves.

   (a) What pattern is observed on the screen if the laser has a wavelength λ? The answer should be expressed as the intensity of the interference pattern at the screen in terms of the variables x, h, D and λ, where x is the distance from the middle of the screen. You will need to make two assumptions to solve this problem, namely $D >> x$ and $x >> h$. Start with an electric field at the two slits that looks like

   $$E = E_0 e^{i\phi}$$

   where $i = \sqrt{-1}$ and $\phi = \text{pathlength}/\lambda$ is the phase of the wave at the slits.

   I will give some quick hints on how to handle complex numbers. We introduce complex numbers in problems like this to make the math easier to handle. At the end of the problem we get rid of the complex parts of the equations to give us a real answer. Some rules when handling complex numbers:

      i. $e^{i\phi} = Cos(\phi) + iSin(\phi)$
      ii. $e^{i\phi} \times e^{i\delta} = e^{i(\phi+\delta)}$
      iii. Intensity $= E \times E^*$
      iv. $E^* = E_0 e^{-i\phi}$ is the complex conjugate

   (b) In the diagram the laser source is centred with respect to the two slits. What changes in the pattern would occur if the laser was shifted such that the waves going into the two holes were π radians out of phase?

   (c) This theoretical pattern goes on forever in both directions along the screen, whereas the experimental pattern fades. What causes this fading of the experimental interference pattern?

3. You are conducting a feasibility study into the SDI (Strategic Defense Initiative) missile defense system. You are to consider the viability of a ground based laser defense against ICBM's, which will use powerful CO_2 lasers to bore holes in the incoming missiles, thus destroying them. The lasers have an output power of 50 W in a beam diameter of 1 mm. The laser beam is fired at the missile when it is 10 km away and the beam loses 3% of its intensity for every kilometer travelled. The outer skin of the missile is aluminum that is 3 cm thick. When the laser fires the skin temperature of the missile is $-50\,^{\circ}C$ and must be heated to its boiling point at $2500\,^{\circ}C$. The density of aluminum is $2.34\,\mathrm{g/cm^3}$ and heat capacity is $0.9\,\mathrm{J/g\,^{\circ}C}$.

   (a) How long will it take the laser to burn through the outer skin of the missile, thus destroying it? Assume that all of the laser power that reaches the missile goes into heating the 1 mm diameter spot.

   (b) What angular accuracy must the aiming system of the laser have in order for the laser not to drift away from the target spot?

   (c) Suggest three ways that one could use to protect a missile from this laser defense system.

4. We have all seen those really cool holographic images of our favourite cartoon characters on the cereal boxes, not that I still watch cartoons anymore, well occasionally I see the odd one, actually on Saturday... anyway, back to the problem. Being a physics nerd, and proud of it by the way, I could never just enjoy those pictures without trying to figure out how they work. I will now inflict my geeky inquisitiveness on all of you, heh heh heh.

   (a) Holograms are 3-D images that change depending on the viewer's perspective. How is the dimensionality of the image captured on the holographic plate? How is this different than normal 2-D film?

   (b) Sketch an experimental apparatus that could be used to produce a hologram. Be sure to explain what is going on.

   (c) When normal film is exposed to light and developed, a negative is produced. It is called a negative because the dark and light areas are interchanged by the developing process. When light is shone through the negative onto photographic paper a positive image is produced, which then ends up in your photo album. In holography it makes no difference whether the holographic plate ends up being a negative or positive. Why is this?

5. Now for a really dull question, for which I humbly apologize; it won't happen again. There are two thin lenses of focal lengths f_1 and f_2 separated by a distance d. In general, the focal length of a system of lenses will be different depending on which side the image is on. For systems with more than one lens we define a back focal length and a front focal length, which are just the two focal lengths for these two possibilities.

   (a) Find the back focal length and the front focal length of this system of lenses.

   (b) Something nifty happens when $d = f_1 + f_2$. What is it?

   (c) Find the focal length of the system for $d = 0$. Does it matter which side of the system the image is on in this limit?

6. Now for a few small questions that will really get you wondering why you are doing these Olympics in the first place. Just give qualitative answers.

   (a) I'm sure most of you have been on trips up into northern Ontario and seen the Northern Lights first hand. What causes the Aurora? Why are they strongest in Northern Canada?

   (b) We were all kids at one time or another (except maybe Pekka) and probably sung Twinkle Twinkle Little Star. What makes stars twinkle? Do the planets and moon twinkle also?

(c) If you are walking on the beach one day and find a seashell, and just happen to hold it up to your ear, you will be able to hear the roar of the ocean. What causes this?

(d) If lightning strikes a tree, it can either be left unharmed or be completely shattered. Why will some trees be destroyed and others be fine? It turns out that Oak trees are preferentially destroyed over other species. Why?

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Solution Set 4: Waves and Optics

1. (a) To solve this problem we need the equation which describes how a curved surface bends light

$$\frac{n_1}{S_o} + \frac{n_2}{S_i} = \frac{n_2 - n_1}{R}$$

where n_1 and n_2 are the indices of refraction of the two media, R is the radius of curvature of the surface, S_i is the image distance and S_o is the object distance. Solving for the image distance

$$S_i = \frac{n_2 R S_o}{(n_2 - n_1)S_o - n_1 R}.$$

At the second surface we have

$$\frac{n_2}{2R - S_i} + \frac{n_1}{x} = \frac{-(n_2 - n_1)}{-R}$$

where x is the image distance we are trying to find and all of the other variables have the same meaning as above. Here, the second object distance $S_o = 2R - S_i$ because a virtual image is formed by the first surface. Solving for x gives

$$x = \frac{n_1 R(2R - S_i)}{(n_2 - n_1)(2R - S_i) - n_2 R}.$$

Putting $n_1 = 1.0$, $n_2 = 1.5$, $S_o = 1.2m$ and $R = 10cm$ gives $S_i = 36cm$ and $x = 6.9cm$. Therefore the image is $6.9cm$ from the crystal ball.

(b) The image is real, inverted and minified.

(c) For the gypsy lady to produce no image means that $x \to \infty$. This corresponds to

$$(n_2 - n_1)(2R - S_i) = n_2 R$$

or $S_i = -10cm$ and $S_o = 5cm$. In other words, that gypsy's mug is awfully close to the crystal ball.

2. (a) This problem is solved by looking at the interference that the two waves undergo at a particular point on the screen. The first thing to do is work out the path length that each wave undergoes on its way to the screen (see figure #1). Consider first r_1

$$r_1 = \sqrt{D^2 + (x - \frac{h}{2})^2}.$$

Using $h << x$ gives

$$r_1 = D\sqrt{1 + \frac{x^2 - hx}{D^2}}$$

and using $D >> x$ gives

$$r_1 = D(1 + \frac{x^2 - hx}{2D^2}).$$

Similarly

$$r_2 = D(1 + \frac{x^2 + hx}{2D^2}).$$

The path difference between the two waves will be $r_2 - r_1 = hx/D$ and phase difference $\Delta = 2\pi xh/\lambda D$. The electric field at the screen will be the superposition of the waves that went through the slits

$$E_{tot} = \frac{E}{2}e^{i\phi} + \frac{E}{2}\exp i(\Delta + \phi)$$

where the factors of two appear because the wave splits in wave in half when it hits the slits. Notice that one wave is phase shifted Δ radians with respect to the other. To find the intensity we multiply E_{tot} by its complex conjugate, which gives

$$I_{screen} = \frac{E^2}{2}(1 + \cos 2\pi\frac{xh}{\lambda D}).$$

(b) If the laser is shifted so that the wave coming from one slit is out phase by π *radians* with respect to the other then the electric field becomes

$$E_{tot} = \frac{E}{2}\exp i(\phi + \pi) + \frac{E}{2}\exp i(\phi + \Delta)$$

Multiplying this by its complex conjugate gives

$$I_{screen} = \frac{E^2}{2}(1 - \cos 2\pi\frac{xh}{\lambda D}).$$

This new formula has its maximum off centre with respect to the two slits, which is the major difference between the interference patterns.

(c) The reason the experimental pattern fades is because the same amount of light doesn't hit all parts of the screen. Most of the light passes straight through the slits, with far less scattered at large angles. This means that the interference pattern is brightest opposite the two slits (ie at $x \simeq 0$).

3. (a) To solve this question we must find out how much energy is needed to evaporate the aluminum, and then how fast the laser can supply this energy. The volume of aluminum to evaporate is $V = (\pi/4)d^2h$ where d is the spot diameter and h is the thickness, which means that the mass of aluminun is $M = V\rho$ with ρ. The heat needed to evaporate this amount of aluminum (neglecting the heat of vapourization) is $H = CM\Delta T$ where C is the heat capacity and ΔT is the temperature difference. Plugging in the numbers gives $H = 126.5\,\mathrm{J}$. The laser beam is attenuated to $P = 50(0.97)^{10} = 36.9\,\mathrm{W}$ by the time it arrives at the missle so it will take 3.4 sec for the laser beam to burn through the missle's skin.

(b) The angular accuracy needs to be enough to keep the laser aimed at the same spot on the missle for the entire 3.4sec. Assuming the beam can't move by more than its radius gives an accuracy of

$$tan\theta < \frac{5 \times 10^{-4}}{10^4}$$

or the drift in θ must be less than $8.4e - 7$ degrees/sec.

(c) To defend against this sort of attack one could: rotate the missle, thicken the skin, make the skin reflecting, increase heat transfer over the skin, have dummy warheads, send up chaffe, or as some of you suggested, not fire the missles, use a cloaking device, blast laser system and finally, don't piss the guys at ICBM control off.

4. (a) The thing that makes holograms different from normal pictures is that holograms record phase information, in addition to amplitude information on the film. A picture only records how intense the light is, which gives no information about depth. A hologram on the other hand is just a recorded interference pattern of the image. The depth information about an image is preserved on the plate in the form of an interference pattern.

(b) To create a hologram one needs to produce an interference pattern of the desired object. This is done by splitting the laser beam into a reference beam and an object beam (see figure #2). The probe beam is reflected off the object, and then interferes with the reference beam. Because the object has depth to it, the reflected beam will not have the same phase across it when it reaches the holographic plate. It will then interfere with the uniform reference beam, thus producing an interferomic record of the object.

(c) It makes no difference wether the holographic plate is a negative or a positive. Since the information is just an interference pattern, the difference between positives and negatives will only effect the overall phase of the pattern and will not effect the final image.

5. (a) I will first work out the back focal length using the Lensmakers equation. At the first lens we can write

$$\frac{1}{f_1} = \frac{1}{S_{i1}} + \frac{1}{S_{o1}}$$

or

$$S_{i1} = \frac{S_{o1} f_1}{S_{o1} - f_1}.$$

This will be the position of the image after the first lens. The same can be done for the second lens, which gives

$$S_{i2} = \frac{S_{o2} f_2}{S_{o2} - f_2}$$

Since the lenses are located a distance d away from each other we can write $S_{o2} = d - S_{i1}$. Using these three equations to solve for S_{i2} while eliminating S_{o2} and S_{i1} gives

$$\text{back focal length} = S_{i2} = \frac{f_2(d - f_1)}{d - f_1 - f_2}$$

Doing the same analysis but in the other direction gives

$$\text{front focal length} = \frac{f_1(d - f_2)}{d - f_1 - f_2}.$$

(b) If we set $d = f_1 + f_2$ then the system acts like it isn't there. Putting plane waves in one end produces plane waves at the other. Note that the magnification of the system will not likely be unity.

(c) If $d = 0$ in the above equations then

$$\text{back focal length} = \text{front focal length} = \frac{f_1 f_2}{f_1 + f_2}$$

or in other words it doesn't matter which side the image is placed on. We can write

$$\frac{1}{f_{eff}} = \frac{1}{f_1} + \frac{1}{f_2}.$$

This is just a modified version of the thin lens equation.

6. (a) The Aurora are caused by low energy solar electrons that are caught by the Earth's magnetic field. They are then sped up and enter the atmosphere at the magnetic North Pole. These electrons excite nitrogen and oxygen which produce the characteristic blue of the Aurora. The reason it is strongest over Canada is that the magnetic North Pole is centred over ??? Island in the Canadian Arctic.

(b) Stars twinkle because the atmosphere that the light passes through to get to our eyes is turbulent. This is due to uneven heating of the atmosphere by the sun. This turbulence causes random refracting of the light which we see as a twinkle. This twinkle also appears for larger bodies (eg. the moon) but is much less noticable.

(c) The sound from the seashells is actually produced by air currents that pass by the shell and excite the shell's air volume at its natural resonances. Since these air currents are not constant, the resonances will be excited sporadically giving the impression of ocean waves.

(d) When lightning strikes a tree, the current in the lighning stroke needs a path to ground. If the bark of the tree is wet, then the current will pass over the outside of the tree leaving it undamaged. If, on the other hand, the tree isn't wet then the current will pass into the tree and descend through the sap. When this happens the sap is superheated and expands, which ends up turning the tree into toothpicks. The Oak tree has especially rough bark which makes it difficult for the current to find a path to ground along the bark. It is therefore not that Oak trees get struck more by lightning, just that they are more likely to be destroyed when hit.

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Problem Set 5: Electricity and Magnetism

Due March 6^{th}, *1995*

1. Okay, all you mental gymnasts – it's time for the parallel conducting bar problem, an Olympian event designed to exercise your upper bodies. Consider two long parallel conductors, each carrying currents in the same direction, as shown in Figure 1. Conductor A carries a current of 100 A and is held firmly in position. Conductor B carries 147 A and is allowed to slide freely up and down (parallel to A) between a set of non-conducting guides. If the equilibrium distance between the two conductors is 2 cm, what is the linear mass density of conductor B?

2. Pretend for a moment that you're an engineer working on the design of a resistor that is to have a temperature coefficient of expansion of zero at 20°C. The design is a composite of two bars having octagonal cross sections (each side of the octagon having a length $d = 0.25$ mm) as shown in Figure 2. The ratio of the resistivities of the two materials is $\rho_1/\rho_2 = 3.2$, and the ratio of the lengths is $l_1/l_2 = 2.6$. The cross sectional geometry and dimensions are uniform throughout the resistor. Assuming that the temperature of the two sections remains equal, calculate α_1/α_2, the required ratio of the temperature coefficients of resistivity of the two materials.

3. It is the year 2236, and the Federation has just banned all telecommunication optical fibre in your sector for some unknown reason. The replacement configuration, consisting of two coaxial dielectrics, is depicted in Figure 3. The permittivities are ϵ_1 and ϵ_2, respectively, the conductor has potential V_0, and the outer shield is grounded.

   (a) Derive an expression for the capacitance per unit length for the cable.

   (b) Given that $V_0 = 1.2$ kV, $\epsilon_{r1} = (4.5/36\pi) \times 10^{-9}$ F/m, $\epsilon_{r2} = (3.0/36\pi) \times 10^{-9}$ F/m, and $r_3 = 2r_2 = 4r_1 = 40$ mm, determine the maximum electric field intensity in each dielectric.

   Note that, for an infinite single dielectric coaxial cable, the capacitance per unit length is $c = 2\pi\epsilon/\ln(b/a)$ where ϵ is the permittivity of the dielectric with outer radius b, and a is the radius of the conductor ($a < b$). By Gauss' Law, the electric field intensity for such a coaxial cable is given by $E = q/2\pi r\epsilon$, where $a < r < b$ and q is the charge per unit length in the region enclosed by r.

4. Consider two conducting spheres, one of radius 6.0 cm and the other of radius 12 cm, where each has 3×10^{-8} C of charge and is positioned very far apart from the other. The spheres are then connected by a platinum wire maintained at a temperature of 312.5 K. Find:

   (a) the direction of motion and magnitude of the charge transferred, and

   (b) the final charge on and electric potential of each sphere.

5. As an accelerator physicist, you have been asked to design a fixed target machine to produce antiprotons, $\bar{p}$, the antimatter partner of protons, p. The reaction is $p + p \longrightarrow p + p + p + \bar{p}$, where one high energy proton hits another proton that is at rest (since it's part of the target). In addition to the original two reactant particles, a proton-antiproton pair is created. What is the minimum energy required by the incident proton for this reaction to occur? Express your answer in terms of the proton mass m and the speed of light c. Hint: consider the centre of mass reference frame.

6. Imagine a long copper conductor with rhombic cross section having sides of length $\sqrt{5}$ mm and diagonal dimensions of 2 mm and 4 mm. The conductor is carrying a current of 10 A. Each second, what percentage of the conduction electrons must leave (to be replaced by others) a 100 mm length?

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Solution Set 5: Electricity and Magnetism

1. All forces mentioned in the following are with respect to conductor B (refer to Figure 1). The magnetic force on a conductor in a uniform external magnetic field is given by

$$F_{\text{mag}} = I_B \vec{l} \times \vec{B}_A,$$

where $\vec{l}$ is a displacement vector along the conductor B, and $\vec{B}_A$ is the magnetic field from conductor A. The above expression can be simplified to

$$F_{\text{mag}} = I_B l B_A \sin\theta = I_B l B_A,$$

since $\vec{B}_A$ is perpendicular to $\vec{l}$ ($\theta = 90°$).

¿From the Biot-Savart Law, the magnetic field $\vec{B}_A$ at a distance r from a wire carrying a current I_A is given by

$$B_A = \frac{\mu_o I_A}{2\pi r},$$

where μ_o is the permeability of free space.

The magnetic force per unit length on conductor B is therefore

$$\frac{F_{\text{mag}}}{l} = \frac{\mu_o I_A I_B}{2\pi r}.$$

The gravitational force per unit length on conductor B, F_g, neglecting the gravitational fields of conductor A and the non-conducting guides, is

$$\frac{F_g}{l} = \lambda_B g,$$

where $\lambda_B \equiv$ the linear density of conductor B and g is the acceleration due to gravity.

At equilibrium, we set the forces to be equal, which gives:

$$\begin{aligned}
\frac{F_{\text{mag}}}{l} &= \frac{F_g}{l} \\
\frac{\mu_o I_A I_B}{2\pi r} &= \lambda_B g \\
\lambda_B &= \frac{\mu_o I_A I_B}{2\pi r g} \\
&= \frac{(12.566 \times 10^{-7}\ \text{N/A}^2)(100\ \text{A})(147\ \text{A})}{2\pi(0.02\ \text{m})(9.8\ \text{m/s}^2)} \\
&= 1.5 \times 10^{-2}\ \text{kg/m} = 0.15\ \text{g/cm}.
\end{aligned}$$

2. Calculate the change in *total* resistance with respect to the change in temperature, ΔR_t:

$$\begin{aligned} R(T) &= R_o\,(1+\alpha_t \Delta T) \\ \Longrightarrow \frac{R(T)-R_o}{\Delta T} &= R_o \alpha_t \\ \Longrightarrow \frac{\Delta R_t}{\Delta T} &= R_o \alpha_t . \end{aligned}$$

Given that $\alpha_t = 0$ at 20°C,

$$\frac{\Delta R_t}{\Delta T} = 0.$$

For each of the two individual resistance regions:

$$\frac{\Delta R_1}{\Delta T_1} = (R_o)_1 \alpha_1$$

$$(R_o)_1 = \frac{\rho_1 l_1}{A},$$

where A is the cross sectional area of the octagonal bars.
Therefore,

$$\frac{\Delta R_1}{\Delta T_1} = \frac{\rho_1 l_1 \alpha_1}{A}.$$

Similarly,

$$\frac{\Delta R_2}{\Delta T_2} = \frac{\rho_2 l_2 \alpha_2}{A}.$$

Since the two materials are in a series configuration:

$$\frac{\Delta R_t}{\Delta T} = \frac{\Delta R_1}{\Delta T_1} + \frac{\Delta R_2}{\Delta T_2} = 0.$$

Since $\Delta T_1 = \Delta T_2$, $\Delta R_1 = -\Delta R_2$ and

$$\rho_1 l_1 \alpha_1 = -\rho_2 l_2 \alpha_2$$

$$\Longrightarrow \frac{\alpha_1}{\alpha_2} = -\frac{\rho_2}{\rho_1} \cdot \frac{l_2}{l_1} = \frac{-1}{(3.2)(2.6)} = -0.12.$$

3. (a) The capacitances per unit length due to each dielectric are given by:

$$c_1 = \frac{2\pi \epsilon_1}{\ln(r_2/r_1)}$$

and

$$c_2 = \frac{2\pi\epsilon_2}{\ln(r_3/r_2)}.$$

The total capacitance per unit length is calculated by treating the above two capacitances in series. The total capacitance per unit length is therefore given by:

$$c = \frac{c_1 c_2}{c_1 + c_2},$$

$$c = \frac{2\pi\epsilon_1\epsilon_2}{\epsilon_2 \ln(r_2/r_1) + \epsilon_1 \ln(r_3/r_2)}.$$

(b)

$$c_1 = \frac{2\pi\frac{4.5}{36\pi} \times 10^{-9} \text{ F/m}}{\ln 2} \doteq 0.36 \text{ nF/m},$$

$$c_2 = \frac{2\pi\frac{3}{36\pi} \times 10^{-9} \text{ F/m}}{\ln 2} \doteq 0.24 \text{ nF/m},$$

$$\frac{V_2}{V_1} = \frac{c_1}{c_2} = \frac{0.36}{0.24} = 1.5,$$

$$V_1 + V_2 = 1200 \text{ V}.$$

Solving for V_1 gives $V_1 = 480$ V. Therefore

$$q = c_1 V_1 = 172.8 \text{ nC/m}.$$

The maximum electric field intensity occurs at the inner surface of either dielectric. Hence, at $r = r_1$:

$$E_{\max}(r_1) = \frac{172.8 \times 10^{-9} \text{ C/m}}{2\pi(0.010 \text{ m})(4.5)(10^{-9}/36\pi \text{ F/m})} \doteq 69.1 \text{ kV/m}.$$

Similarly for $r = r_2$:

$$E_{\max}(r_2) = \frac{172.8 \times 10^{-9} \text{ C/m}}{2\pi(0.020 \text{ m})(3.0)(10^{-9}/36\pi \text{ F/m})} \doteq 51.8 \text{ kV/m}.$$

4. First calculate the initial potentials on the sphere surfaces. For the smaller sphere:

$$V_1 = \frac{1}{4\pi\epsilon}\frac{q}{r} = \frac{(9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)(3 \times 10^{-8} \text{ C})}{(0.06 \text{ m})} = 4.5 \times 10^3 \text{ V}.$$

For the larger sphere:

$$V_2 = \frac{(9 \times 10^9\ \mathrm{N \cdot m^2/C^2})(3 \times 10^{-8}\ \mathrm{C})}{(0.12\ \mathrm{m})} = 2.25 \times 10^3\ \mathrm{V}.$$

Since $V_1 > V_2$, the direction of charge motion is towards the 12 cm radius ball.

When the conducting wire is connected, the spheres seek to assume equal potentials. Therefore,

$$\frac{1}{4\pi\epsilon}\frac{q_1}{r_1} = \frac{1}{4\pi\epsilon}\frac{q_2}{r_2},$$

$$\frac{q_1}{r_1} = \frac{q_2}{r_2},$$

$$q_1 = \frac{q_2 r_1}{r_2}.$$

Since $q_1 + q_2 = 6 \times 10^{-8}$ C,

$$q_2\left(\frac{r_1}{r_2} + 1\right) = 6 \times 10^{-8}\mathrm{C},$$

and the final charge on the larger sphere becomes

$$q_2 = 4 \times 10^{-8}\ \mathrm{C},$$

whereas the final charge on the smaller sphere becomes

$$q_1 = 6 \times 10^{-8}\ \mathrm{C} - 4 \times 10^{-8}\ \mathrm{C} = 2 \times 10^{-8}\ \mathrm{C}.$$

The magnitude of transferred charge is therefore 1×10^{-8} C.

The final potential of each sphere is given by:

$$V_{\mathrm{smaller}} = \frac{1}{4\pi\epsilon}\frac{q}{r} = \frac{(9 \times 10^9\ \mathrm{N \cdot m^2/C^2})(2 \times 10^{-8}\ \mathrm{C})}{(0.06\ \mathrm{m})} = 3 \times 10^3\ \mathrm{V}$$

and

$$V_{\mathrm{larger}} = \frac{(9 \times 10^9\ \mathrm{N \cdot m^2/C^2})(4 \times 10^{-8}\ \mathrm{C})}{(0.12\ \mathrm{m})} = 3 \times 10^3\ \mathrm{V}.$$

5. In the frame of the lab, the total energy is $E + mc^2$, where E is the energy of the incident proton and mc^2 is that for the proton at rest. Let the incident proton have momentum p.

The invariant mass, M, of this lab system is then given by

$$M_{\mathrm{lab}}^2 = \frac{E_{\mathrm{total}}^2}{c^4} - \frac{p^2}{c^2}.$$

In the centre of mass (CM) frame, the 4 reaction products are at *rest* for this problem, since we are only calculating the minimum, or threshold energy, for this reaction. The total energy in this system is therefore $4mc^2$ and the invariant mass is

$$M_{\text{CM}}^2 = \frac{\left(4mc^2\right)^2}{c^4}.$$

Since $M_{\text{lab}}^2 = M_{\text{CM}}^2$ and $E^2 = p^2c^2 + m^2c^4$, we get

$$\frac{(E+mc^2)^2}{c^2} - p^2 = \frac{(4mc^2)^2}{c^2},$$

$$(E+mc^2)^2 - E^2 + m^2c^4 = (4mc^2)^2,$$

$$E^2 + m^2c^4 + 2Emc^2 - E^2 + m^2c^4 = 16m^2c^4,$$

$$2Emc^2 = 14m^2c^4,$$

and

$$E = 7mc^2.$$

6. First calculate the number of electrons per unit volume. Take Avogadro's number to be $N = 6.02 \times 10^{26}$ atoms/kmol, the density of copper to be 8.96×10^3 kg/m^3, and the atomic weight of copper to be 63.54 g/mol. Assuming one conduction electron per atom, the number of electrons per unit volume is

$$N_e = \left(6.02 \times 10^{26}\ \frac{\text{atoms}}{\text{kmol}}\right) \cdot \left(\frac{1\ \text{kmol}}{63.54\ \text{kg}}\right) \cdot \left(8.96 \times 10^3\ \frac{\text{kg}}{\text{m}^3}\right) \cdot \left(1\ \frac{\text{electron}}{\text{atom}}\right)$$

$$N_e = 8.49 \times 10^{28}\ \text{electrons/m}^3.$$

The rhombic cross sectional area is $\frac{1}{2}(2\ \text{mm})(4\ \text{mm}) = 4 \times 10^{-6}\ \text{m}^2$.
The number of electrons in a 100 mm length is

$$N = (4 \times 10^{-6}\ \text{m}^2)(0.100\ \text{m})(8.49 \times 10^{28}\ \text{electrons/m}^3) = 3.40 \times 10^{22}\ \text{electrons}.$$

A 10 A current requires that

$$(10\ \text{C/s})(1.6 \times 10^{-19}\ \text{C/electron})^{-1} = 6.25 \times 10^{19}\ \text{electrons/s}$$

pass a fixed point.
The percentage leaving the 100 mm length per second is

$$\frac{6.25 \times 10^{19}}{3.40 \times 10^{22}} \cdot (100) = 0.184\%.$$

1994-1995 Physics Olympiad Preparation Program

– University of Toronto –

Problem Set 6: Electronics

Due April 3^{rd}*, 1995*

1. You just gotta love electronics. All of those little resistors and capacitors to play with. And just when you think there can be nothing as exciting as that, out come those digital circuits with all of the chips that can be used. It's even better than having... sorry about that, just got rambling...on to the business at hand. This problem concerns itself with a circuit called a tank circuit (see figure #1). I have no idea why it is called that, but whatta name.

   (a) Find the absolute value of the impedance of the tank circuit as a function of frequency. The impedance of this circuit is called the transfer function because it describes how signals are transferred across the circuit. At what frequency is the impedance of the circuit infinite? To do this you will need the resistance of both a capacitor $Z_c = 1/i\omega C$ and an inductor $Z_l = i\omega L$, where $i = \sqrt{-1}$. The absolute value of a complex number is defined as

   $$Z = \sqrt{Z \times Z^*}$$

   where Z^* is the complex conjugate of Z.

   (b) Sketch this impedance function as a function of frequency.

   (c) How will the transfer function change if the inductor has some resistance? Just determine the complex transfer function.

2. Often in electronics one uses filter circuits to remove unwanted frequencies in signals. Most of them work by using the frequency dependent resistance of capacitors and inductors. If a certain frequency signal is put into a circuit it will either be allowed to pass through or get blocked by the circuit depending on the frequency dependent resistance of the components. Filter circuits fall into the following categories:

   (a) High pass filters only allow high frequencies through them.

   (b) Low pass filters do the opposite.

   (c) Band rejection filters block an intermediate range of frequencies.

   (d) Not a filter at all.

   Classify the five circuits given in figure #2.

3. You're out with your friends one day and you stumble across an ornately carved lamp lying on the ground. One of your friends says that it must be a magical lamp with a genie inside. You all tease him mercilessly until, sobbing uncontrollably, he rubs the lamp and out comes a genie. The ethereal entity speaks, and I quote:"For teasing my master so ruthlessly, I will ask you one question which you must get right to live. What is the resistance between opposite vertices of an octahedron, if each of the twelve edges has the same resistance." You begin to sweat and really wish you hadn't skipped your electronics class all last term......

4. "Could the Romulin Warbird be hiding in that interstellar gas?" Captain Picard said from the bridge of the Enterprise.

   "Our sensors cannot penetrate the gaseous cloud, but I think it quite likely Captain," says Data in his usual monotone.

   Then Wesley piped up, "The homogeneous material of the cloud has known conductivity σ_c, whereas the Warbird has a conductivity of σ_w. By measuring the resistance of the cloud we could determine if the Warbird is in there." He sits in his chair with a stupid smile on his face.

Picard's face turns red as he shouts, "Will you shut your insolent little face. Worf, stun Wesley and throw him in the brig. After Wesley is unconscious, the captain remarks to Data, "Make it so."

(a) Data assumes that the cloud is cylindrical with radius b and length L, and that the input and output currents are uniformly distributed across the ends of the cloud. Show that Data computes the resistance to be

$$R = \frac{L}{\pi \sigma_c b^2}.$$

(b) Data decides to model the Romulin Warbird as a metallic cylinder of radius a, length a and conductivity σ_w in the centre of the cloud. Estimate the relative change in resistance of the cloud to first order in $\sigma_c - \sigma_w$ if $\sigma_c \simeq \sigma_w$. Assume that $b >> a$. Model the various parts of the cloud as resistors in series or parallel with each other.

5. This problem concerns itself with the rectification of AC voltage. The circuit shown in figure #3 is called a full wave rectifier and is used to turn an AC signal into a DC signal. The funny looking circuit elements are called diodes and only allow current to pass in the direction of the arrow.

   (a) If the voltage source produces a sinusoidal signal, what signal is expected across the resistor? What will be seen at the resistor for a square wave signal as input? Give reasons for your answer and sketch the output and input waveforms. Ignore the capacitor for this part.

   (b) If I add a large capacitor in parallel with the resistor, how will it affect the voltage signal at the resistor? To be more specific, why must the capacitor be large to produce an almost DC signal at the resistor?

6. This just wouldn't be an electronics problem set without at least one question on digital circuits. See figure #4 for a type of circuit that shows up when designing computers and such.

   (a) Find the truth table of the circuit. What is this circuit actually doing and how could it be useful?

   (b) Using the circuit from (a), design a full adder that can add three one digit binary numbers. Give the truth table for your circuit.

1995-1996 Physics Olympiad Preparation Program

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Problem Set 1: General Physics

Due October 16, 1995

1. Joe "build-em tall" Girders is designing a construction crane (did you know he learned his stuff working with Tinker Toys?). He uses two beams with mass $m_1 = 9000$ kg and $m_2 = 3000$ kg with lengths $l_1 = 45$ m and $l_2 = 15$ m, attached to the vertical "trunk" of the crane and supported by cables, as shown in Fig 1.
   (a) How much mass, m_c, should there be in the counterweight located at the end of the second beam to ensure that the crane is perfectly balanced when the crane is not carrying a load.
   (b) The cable support has a height $h = 15$ m. What are the tensions on the two cables supporting the beams?

2. In the original movie *Batman*, our masked avenger has a cool get-away planned when he and Vicky Vale, the hot-shot reporter, are trying to escape the Joker after the debacle in the museum (check out the movie if you folks forget this scene). He fires up one of his awesome cables and attaches it to a girder above the street. With his super-cool winch located on his belt, he begins to lift the two of them out of harm's way. But wait! He only gets part way up. His winch appears to run out of juice. As he and Vicky twist in the wind, he ponders his potentially fatal mistake.
   (a) What is the minimum energy required to lift the two of them up 15 m, if we assume Batman tips the scale at a fit 75 kg and Vicky really does weigh 47 kg (as she claims)?
   (b) They have to be rising at least 1 m/s in order to avoid getting plastered by Joker's henchmen. What is the power required to execute this daring move?
   (c) In the movie, after Batman hooks the winch to Vicky and lets go, she is winched up the rest of the way to the girder. Can you give a plausible explanation as to what really went wrong with the winch? Did it really run out of energy?

3. Sven is tooling down Highway 11/17 with a tractor-trailer load of pulp wood to be fed into the maw of the MacBlo mill in Fort Frances. He is doing 120 km/h, easy. He comes around a corner and finds – what else? – the icon of the northern Ontario bush, a moose, standing in the middle of the road. The old pro that he is, Sven hits the brakes. He is about 150 m from the moose when his size 13 work boot hammers the brakes.
   (a) How heavily does he have to brake to avoid bagging a moose out of season (and making a mess of his front end)?
   (b) He is still in the curve when he starts braking, so his trailer filled with cord wood makes a small angle with the tractor unit. He realises as he hits the brakes that he forgot to connect his trailer brakes, so that all braking is performed by the tractor unit. It has been snowing (what else can go wrong?) so the coefficient of static friction between his wheels and the asphalt is $\mu_s = 0.2$. What is the maximum angle his trailer unit can make with the tractor unit before it starts to jacknife (the truly awesome maneuver when the trailer starts sliding and swings around)?

4. Rob Hanks was a wild and crazy man. By day he worked as an electrician for General Motors, but by night he assumed a secret identity, one that no soul knew a thing about – he became a *physics problem solver*. Now Rob often works with AWG #12 copper wire with an 80.8 mil diameter. He already knows that a 50 ft length can carry a current of 20 A, and that copper has a conductivity of $\sigma = 5.8 \times 10^7\ \Omega^{-1}\mathrm{m}^{-1}$ and an electron mobility of $\mu = 0.0032\ \mathrm{m}^2\mathrm{V}^{-1}\mathrm{s}^{-1}$ at 20 °C. Can you help Rob calculate the electric field intensity E, the voltage drop and resistance across the 50 ft length, and the time it would take an electron to move a distance of 1 cm in the conductor?

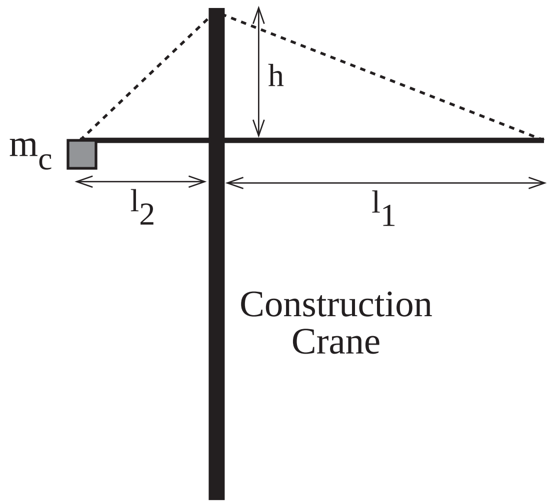

Figure 1: The construction crane in Problem 1.

5. Wanda Steerbutton was a world-class magazine photographer. She earned her living taking quality shots of supermodels, rock bands, movie stars, and politicians for the covers of those glossy magazines that people only seem to buy in airports and train stations. It was a good life, but she wanted more. Understanding the secrets of the universe sounded like a cool idea; besides, it would be a nice change from looking at people's pores through a viewfinder. So Wanda decided to take up astrophysics. Since she already owned a fancy camera, and since one of her lab courses involved photographing stars in the night sky, Wanda wondered whether she could use one of her precision engineered fixed focus lenses for the job. The lens in question had a focal length of 60 mm and was focussed for objects at a distance of 15 m. The problem was that stars are at a distance that is a little more than 15 m, like infinity maybe. Her lens did, however, have an adjustable aperture. For what aperture diameter would the diffraction blur of visible light ($\lambda = 550$ nm) be more or less the same as the defocus blur for a distant star?

6. The starship *Compromise*, the lesser known sister ship to the Federation's *Enterprise*, is to undergo an "upgrade" to its transporter control system under the supervision of your aging systems engineering professor at Starfleet Academy. She has asked you to design logic circuitry that will enable the transporter to function for the following three Away Team scenarios:

   - An android (A), the captain (C), and another senior officer (S) are all part of the Away Team;
   - An android (A), the captain (C), and another senior officer (S) are **NOT** part of the Away Team;
   - The captain (C) is a member of the Away Team.

   (a) Without thinking too deeply about the reasonableness of the above scenarios (remember, your prof is starting to lose it and ought to have retired to a quiet moon at the edge of the Alpha quadrant years ago), express the process state as a digital logic function in terms of A, C, and S. Sketch the circuit.

   (b) Algebraically simplify the circuit above so that you use a minimum number of logic devices. (Besides, if you ever were to mass produce this super intelligent control system, you'd surely want to minimize the manufacturing costs.) Sketch the simplified circuit.

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Solution Set 1: General Physics

1. (a) We require that there is no net torque about the axis through the point where the two beams connect to the vertical support (recall that torque is $\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{r}$ is the vector separating the axis about which you are calculating the torque and the point where the force is applied, and $\vec{F}$ is the force). The torque due to gravity on beam 1 is centred through the midpoint of beam 1. Ditto for beam 2. The torque due to the counterweight is the cross product of the gravitational force and the vector between the axis and the end of beam 2. The force diagram for the crane is shown in Fig. 1a). Thus the net torque is

$$\begin{aligned} \sum \tau &= -m_1 g \frac{l_1}{2} + m_2 g \frac{l_2}{2} + m_c g l_2 = 0 && (1) \\ \Rightarrow m_c &= \frac{1}{2}\left(m_1 \frac{l_1}{l_2} - m_2\right) && (2) \\ &= 12,000 \text{ kg}, \end{aligned}$$

where g is the acceleration of gravity ($g = 9.8$ m/s^2). Note that we can verify that the equation for m_c makes sense. For example, if $m_1 = m_2$ and $l_1 = l_2$, then we would expect $m_c = 0$, which is the case. Can you think of other sanity checks? What about units? Note that we do not have to take into account the torque due to the support cable tension on each beam. Can you understand why not?

(b) We require that the net torque on each arm to be equal. In this case, the tensions on the cable supporting the two beams are forces, T_1 and T_2 that act on the ends of the beams and on the vertical support. Let θ_1 be the opening angle between beam 1 and the cable (as shown in Fig. 1b)). Then the torque acting on beam 1 is

$$\begin{aligned} \sum \tau = 0 &= l_1 T_1 \sin\theta_1 - \frac{l_1}{2} m_1 g && (3) \\ \Rightarrow T_1 &= \frac{m_1 g}{2\sin\theta_1} && (4) \\ &= \frac{m_1 g \sqrt{l_1^2 + h^2}}{2h} && (5) \\ &= 139,000 \text{ N}, \end{aligned}$$

where h is the height of the vertical support above the beams. Similarily, for beam 2, the requirement that the net torque on the beam is zero gives

$$\begin{aligned} \sum \tau = 0 &= -T_2 \sin\theta_2 + m_2 g \frac{l_2}{2} + m_c g && (6) \\ \Rightarrow T_2 &= \left(\frac{m_2 g}{2} + m_c g\right) \frac{\sqrt{l_2^2 + h^2}}{h} && (7) \\ &= 187,000 \text{ N}. \end{aligned}$$

Its interesting that the tension is highest on the cable supporting the counterweight. Them cables are pretty strong. They have to be able to support almost ten tons, and that's without any load being carried.

2. (a) To lift Batman and Ms. Vale up $h = 15$ m, one has to at least expend the work equivalent to raising them that distance in a gravitational field (it could be more–think about the work required

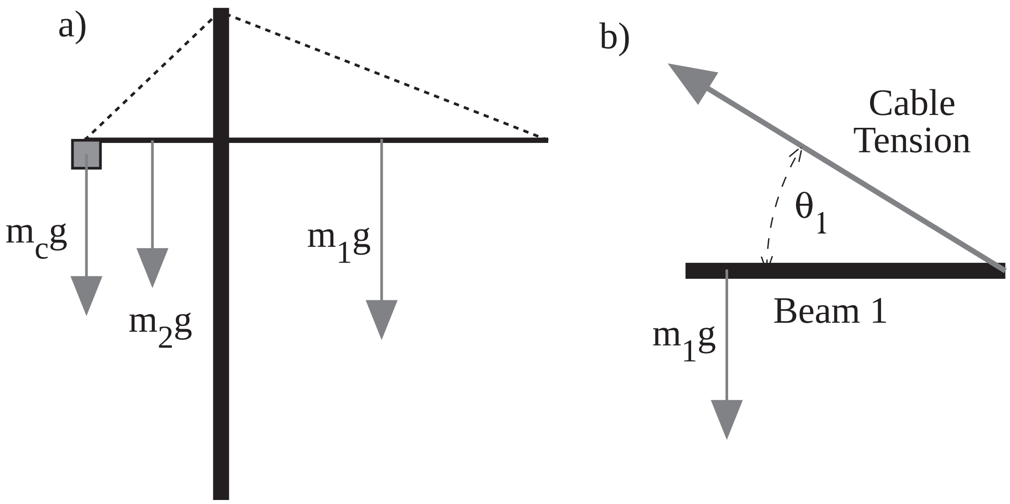

Figure 1: The forces acting on the two beams around the point at shich they are connected to the vertical support are shown in a). Note that we do not consider the the force of tension on the two beams from the support cables. (Why not?) The forces acting on the end of beam 1 due to the cable and the weight of gravity are shown in b).

to accelerate them into motion). This is equal to

$$W = h\,(m_B + m_V)\,g = 17,931 \text{ Joules}, \tag{8}$$

where g is our faithful acceleration due to gravity at Earth's surface.

(b) Let's assume they accelerate to $v_h = 1$ m/s and then rise steadily the rest of the way. The power expended (ignorning the power required for acceleration) is just the force $\times$ velocity, which is

$$P = (m_B + m_V)\,g \tag{9}$$
$$= 1,196 \text{ Watts } = 1.6 \text{ hp}. \tag{10}$$

Recall that 1 horsepower is equal to 746 Watts. That's a pretty hefty little motor he has there.

(c) Let's consider why the winch stopped in the first place. There are only a few likely scenarios (given that it wasn't really broken):

i. motor overheated or otherwise failed,
ii. the energy source couldn't continue to supply the needed power, or
iii. the winch is based on some funky technology that I know nothing about that gives out at exactly the right moment in the script.

The fact that it continued to operate and yank Vicky up when Batman let go and the tension dropped indicates that i) is unlikely; the drop in tension itself would not have been correlated with the moment when the motor became operative. Scenario ii) is plausible, as the power level required to haul Vicky up would be less than that required to haul both of them up. Of course, I would go for iii). Hollywood rarely cares about conforming to reality.

3. (a) We can safely assume that Sven and the pulpwood are going to decelerate uniformly, with deceleration a. This is an approximation, as brakes usually fade as they heat up, but we have to start from somewhere, and it isn't such a bad assumption. We can then employ our usual equations for an object of mass m decelerating uniformly over a distance d. If $v_0 = 120$ km/h is Sven's initial speed, then the time required to decelerate is $t = v_o/a$. Thus,

$$d = \frac{1}{2}a\left(\frac{v_o}{a}\right)^2 = \frac{v_o^2}{2a} \tag{11}$$
$$\Rightarrow a = \frac{v_0^2}{2d} = 5.8 \text{ m/s}^2, \tag{12}$$

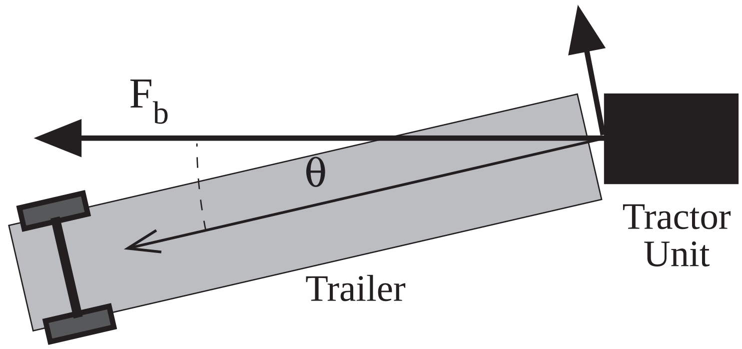

Figure 2: The braking force acting on the trailer.

which is over half a g. That's real heavy braking. I'm guessing Sven ended up with the meat.

(b) The force diagram on the trailer is shown in Fig. 2. The braking force F_b is shown, along with the component of that force perpendicular to the axis of the trailer, $F_p = F_b \sin\theta$, where θ is the angle between the axis of the trailer and the braking force. There is one other force in the problem; the reaction force of the pavement acting on the trailer through its tires to keep the angle θ constant (note that it is not shown in Fig. 2). This reaction force would normally be equal to F_p, so that there is no increase in angular momentum of the trailer (note that the trailer is swinging in uniform motion as it turns the corner). The maximum frictional force is

$$F_f = \mu_s m_w g, \tag{13}$$

where m_w is the weight of the trailer resting on the back axle, which we approximate as half the total trailer weight m_t. Since the braking force is $F_b = m_t a$, the maximum angle between the trailer and tractor unit is given by the requirement

$$\begin{aligned} F_f &= F_p \qquad (14)\\ \Rightarrow \frac{\mu_s m_t g}{2} &= m_t a \sin\theta \qquad (15)\\ \Rightarrow \theta &= \sin^{-1}\left(\frac{\mu_s g}{2a}\right) = 9.7^\circ. \qquad (16)\end{aligned}$$

Not a big angle in this case. Note that it matters not a whit what the trailer weighs.

4. Since a mil is $\frac{1}{1000}$ inch, the cross-sectional area of the conductor is:

$$A = \pi\left[\left(\frac{0.0808\text{ in}}{2}\right)\left(\frac{2.54\times10^{-2}\text{ m}}{1\text{ in}}\right)\right]^2 = 3.31\times10^{-6}\text{ m}^2.$$

The current density therefore is:

$$J = \frac{I}{A} = \frac{20\text{ A}}{3.31\times10^{-6}} = 6.04\times10^{6}\text{ A}\cdot\text{m}^{-2}.$$

The electric field intensity is:

$$E = \frac{J}{\sigma} = \frac{6.04\times10^{6}\text{ A}\cdot\text{m}^{-2}}{5.8\times10^{7}\ \Omega^{-1}\cdot\text{m}^{-1}} = 1.04\times10^{-1}\text{ V}\cdot\text{m}^{-1}.$$

The voltage drop is given by:

$$V = E\ell = (1.04\times10^{-1}\text{ V}\cdot\text{m}^{-1})(50\text{ ft})(12\text{ in}\cdot\text{ft}^{-1})(0.0254\text{ m}\cdot\text{in}^{-1}) = 1.59\text{ V}.$$

The resistance is given by:

$$R = \frac{V}{I} = \frac{1.59 \text{ V}}{20 \text{ A}} = 7.95 \times 10^{-2} \ \Omega.$$

Since $\sigma = \rho\mu$, where ρ is the charge density:

$$\rho = \frac{\sigma}{\mu} = \frac{5.8 \times 10^7 \ \Omega^{-1} \cdot \text{m}^{-1}}{0.0032 \text{ m}^2 \cdot \text{V}^{-1} \cdot \text{s}^{-1}} = 1.81 \times 10^{10} \text{ C} \cdot \text{m}^{-3}.$$

Since $J = \rho U$, where U is the drift velocity:

$$U = \frac{J}{\rho} = \frac{6.04 \times 10^6 \text{ A} \cdot \text{m}^{-2}}{1.81 \times 10^{10} \text{ C} \cdot \text{m}^{-3}} = 3.34 \times 10^{-4} \text{ m} \cdot \text{s}^{-1}.$$

With the above drift velocity, an electron will require approximately 30 seconds to move a distance of 1 cm in the #12 copper conductor.

5. The image distance, v, can be calculated from the lens makers' equation:

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f},$$

which yields a distance of:

$$v = \frac{uf}{u - f},$$

where f is the focal length and u is the object distance.

The Rayleigh criterion, $\theta = \frac{1.22\lambda}{D}$, where θ is the limiting angle of resolution, λ is the wavelength, and D is the diameter of the aperture, can be used to calculate the blur diameter:

$$d = \theta v = \frac{1.22\lambda v}{D}.$$

The defocus blur is calculated by similar triangles as:

$$d = \frac{(v - f)D}{f}.$$

Equating the diffraction blur and the defocussing blur gives:

$$D = \sqrt{\frac{1.22\lambda v f}{(v - f)}} = \sqrt{1.22\lambda u}.$$

For $\lambda = 550$ nm and $u = 15$ m, we end up with $D = 3.2$ mm.

6. (a) The digital logic function is an OR expression encompassing the combinations that lead to a correct process, namely:

$$X = A \cdot C \cdot S + \bar{A} \cdot \bar{C} \cdot \bar{S} + C.$$

The circuit is sketched in Fig. 3a).

(b) Operating on the second term of the above expression with one of DeMorgan's laws of Boolean algebra ($\overline{A \cdot B} = \bar{A} + \bar{B}$) gives:

$$X = A \cdot C \cdot S + \overline{A + C + S} + C.$$

Applying the redundancy law of Boolean algebra ($A + A \cdot B = A$) yields:

$$X = (C + A \cdot C \cdot S) + \overline{A + C + S} = C + \overline{A + C + S}.$$

The simplified circuit is sketched in Fig. 3b).

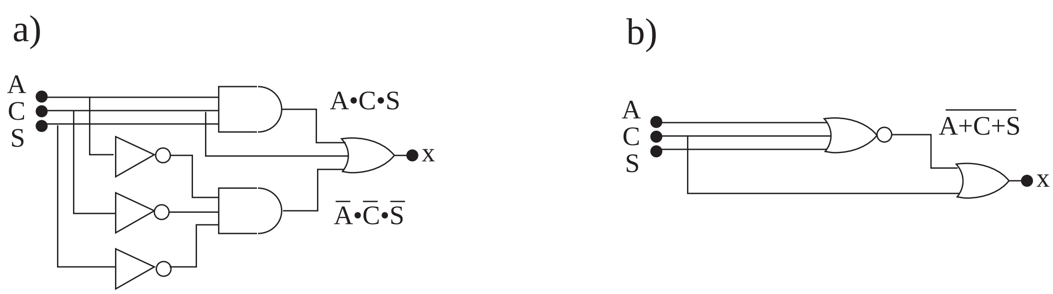

Figure 3: The circuit to handle the logic of the starship *Compromise* transporter control logic is shown in a). A simplified circuit is shown in b).

1995-1996 Physics Olympiad Preparation Program

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Problem Set 2: Mechanics

Due November 13th*, 1995*

1. A fire broke out in your building and you are trapped on the roof. Upon looking around, you find a bungee cord and a harness for it. Being a person who likes to show off, you decided to bungee-jump off flagpole at the top of the building of this and release the harness at the bottom.

   The initial length of the bungee cord is 30 m and $k = 77$ kg/sec^2. The height of the building+flagpole is 60 m (See Figure 1).

   (a) How much bungee cord do you need if you want to have a perfect landing (i.e. $v = 0$)? Assume that you are a point particle with mass of 55 kg.

   (b) So you jumped, but it hit you that the tensile strength times the cross section of the bolt that holds the flagpole that the bungee cord is tied to is T. What's the minimum T you need in order not to be squashed on the ground?

2. Suppose you are a high school student (well, you are) who enjoys working on physics problems. One day your physics teacher asks you to try physics olympiad preparation problems. And you see the following dull questions prepared by a desperately dull guy.

   (a) A yoyo is on a level surface (Figure 2 (a)). A gentle horizontal pull is exerted on the cord so that the yoyo rolls without slipping. The yoyo will roll toward the pull. Why?

   (b) Now a yoyo is sitting on a wedge (Figure 2 (b)). Calculate the force needed on a cord to have the yoyo stationary when it is released. What is the minimum friction coefficient needed to make the yoyo stationary in this condition? Assume that the yoyo is a cylindrical object and the width of its groove is negligible.

   (The moment of inertia of a cylindrical object that has a rotational axis parallel to the axis passing through the centre of mass, which is perpendicular to the flat surface is $I_R = I_M + aM$ where a is the distance of the rotational axis from the principal axis, M is the mass of the cylinder, and I_R and I_M are the moments of inertia about the axis of rotation and about the axis passing through the centre of mass, respectively.)

3. For all your life you wanted to be a geek. Now your life long dream is about to come true. You've been noticed and invited to a prestigious geek party. But you have to answer questions correctly to be accepted as one of the geeks.

   (a) A marble is resting on the top of an overturned popcorn bowl shaped like a hemisphere (See Figure 3 (a)). The radii of the popcorn bowl and the marble are R and a, respectively. At what height, in terms of R, will the marble be airborne if the marble rolls down without slipping?

   (b) Now, the bowl is put upright (Figure 3 (b)). If you let go of the marble near the bottom of the bowl it will oscillate. Compared to a simple pendulum of length $(R - a)$, will the period of oscillation be shorter, the same, or longer? Why? A *big* brownie point will be awarded if you can give a quantitative answer.

4. You are an agent of Her Majesty's Secret Service and are being followed by goons from an organization whose plan is to conquer the world by taking control of the world's gummybear production. You decide to shake the pursuit. Your car sensor indicates that a turn with a radius 30 m lies ahead. Your car

weighs 1500 kg and its centre of mass is 40 cm above the ground. You can approximate your car to have the dimensions $500 \times 200 \times 80$ cm^3 ($length \times width \times height$).

(a) Your car is equipped with a set of SuperSticky tires[tm] (which means the friction coefficient is very, very big). But you realise that you have a cup of coffee without a lid in your cup holder and you do not want to spill the coffee on the rather expensive leather seat but you don't want to throw it away either. (The car pool department is going to raise hell if they find coffee stains on the leather seat.) How fast can you go without spilling your coffee? The coffee cup is 8 cm in diameter and 15 cm tall and 4/5 full. Assume that all four wheels of the car stay on the road without slipping.

(b) Oww, bloody hell, they start shooting at you and, to make it worse, the sensor indicates that the pavement is wet and the friction coefficient of the tire/pavement has dropped to $\mu = .23$ Good thing your car is equipped with an aerodynamic device that produces a downward force of $150 \times v$ N (v in km/h). Forget about ruining leather seats. Your life is on the line. How fast can you go without slipping (and falling from a cliff) or raising inside wheels (probably resulting in loss of control)?

5. Eeck!!! Computers at the satellite control centre are down and you are asked to transfer a $\$1.0 \times 10^9$ communication satellite from a parking orbit ($h = 200$ km) to a circular geosynchronous orbit manually. Due to engine controller malfunction you can use only two short thruster bursts. You are to use an elliptical transfer orbit whose perigee meets the lower parking orbit and apogee meets the geosynchronous orbit. Assume that the thruster expended is negligible. If you succeed, you will get your boss' eternal gratitude for saving his/her/its butt; but if you fail, you will lose your job and more.

   (a) What is the Δv required to put the satellite to a transfer orbit from the parking orbit? Also, what is the Δv needed to put the satellite from the transfer orbit to the geosynchronous orbit?

   (b) How long does the transition take?

6. Pat and Chris bought two identical pendulum clocks at X, which is on the equator. Their next destination is Y, which lies exactly on opposite side of the earth. But their travel agency screwed up and Pat has to take a westbound plane and Chris has to take an eastbound plane, where both planes' routes follow the equator. They arrive at Y exactly 12 hours later at X local time.

   (a) Do the two clocks agree with each other? (Well, obviously not.) If not, what are the effects that would make them not agree?

   (b) Find the time difference between Pat's and Chris' clocks at Y. Consider only the most important physical effect.

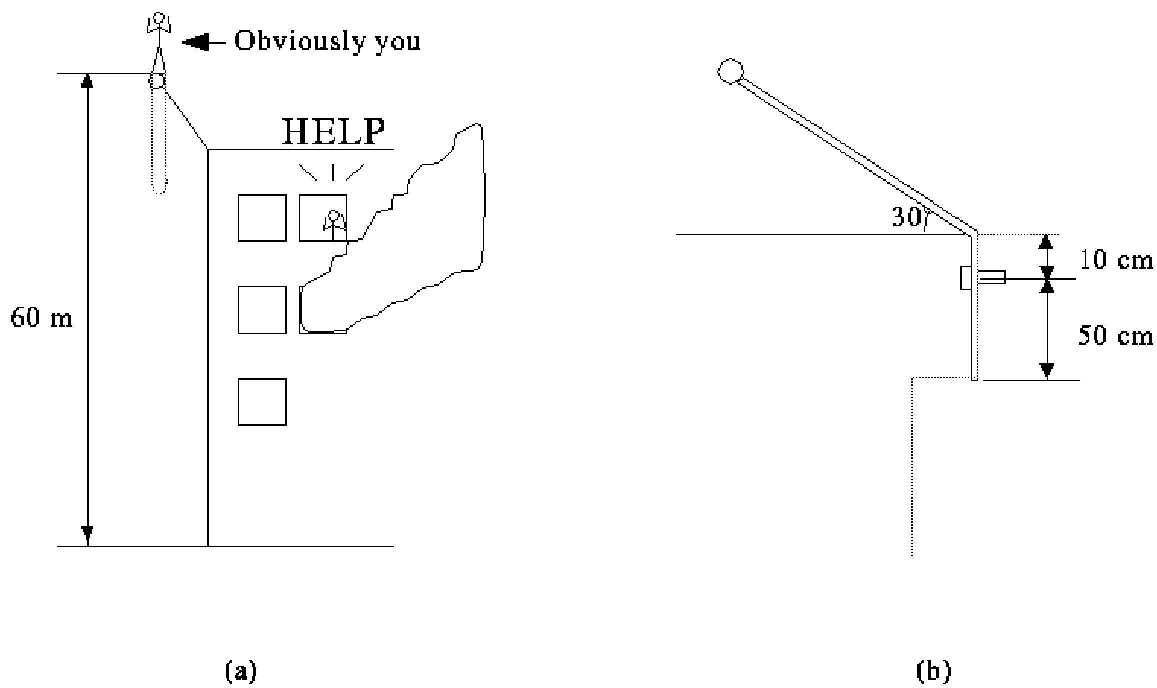

Figure 1: Figure for Problem 1.

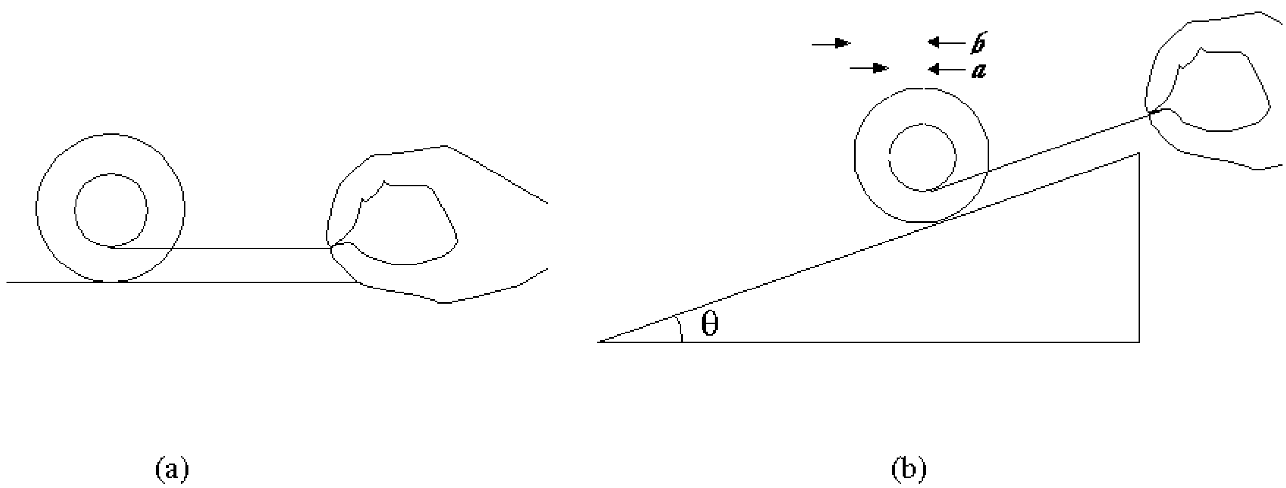

Figure 2: Figure for Problem 2.

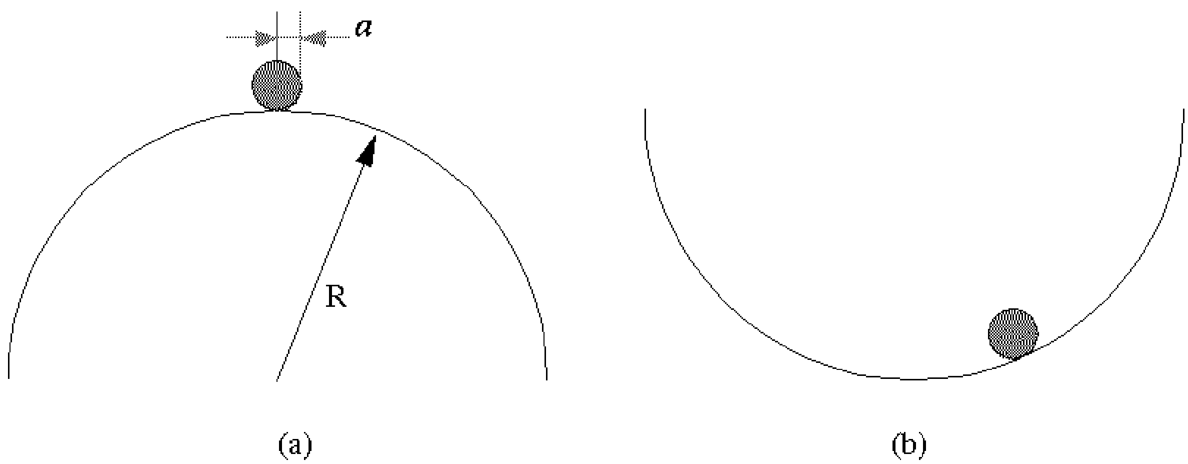

Figure 3: Figure for Problem 3.

1995-1996 Physics Olympiad Preparation Program

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Solution Set 2: Mechanics

1. (a) Assume that the bungee cord obeys Hooke's law,

$$\frac{F}{A} = Y\frac{\Delta l}{l}$$

$$F = (\frac{YA}{l})\Delta l = k\Delta l,$$

where A is the cross section of the elastic cord, Y is Young's Modulus, l is the length of the cord, Δl is the change in the length of the cord, and k is the elastic constant of the cord. Since Y and A are constants, kl is constant, *i.e.* $kl = k'l'$.

At $h = 0$ the velocity of the jumper has to be zero, i.e. $v = 0$, where h is the vertical distance of the jumper from the ground. Thus, using energy conservation

$$\begin{aligned} mgh_\circ &= \frac{1}{2}k'(h_\circ - l')^2 \\ &= \frac{1}{2}k(l/l')(h_\circ - l')^2 \end{aligned}$$

where $h_\circ$ is the initial vertical distance. This can be rearranged as a quadratic equation in l'.

$$l'^2 - 2h_\circ\left(1 + \frac{mg}{kl}\right)l' + h_\circ^2 = 0.$$

Solving for l' then,

$$l' = h_\circ\left(\left(1 + \frac{mg}{kl}\right) \pm \sqrt{\left(1 + \frac{mg}{kl}\right)^2 - 1}\right)$$

Using $g = 9.8$ m/s^2, $m = 55$ kg, $k = 77$ kg/s^2, $h_\circ = 60$ m, and $l = 30$ m, one gets the only physical solution, $l' = 21.4$ m. (Note: In order for l' to be physical, it must be $0 < l' \leq 30$ m.)

(b) The force exerted on the flagpole at $h = 0$ is:

$$\begin{aligned} F &= mg + k'(h_\circ - l') \\ &= mg + k(l/l')(h_\circ - l'). \end{aligned}$$

The torque exerted on the flagpole around the pivot point P is $\tau = FL'\cos\phi$. But from Figure 1, it is obvious that $L'\cos\phi = L\cos\theta$. Thus

$$\tau = FL\cos\theta.$$

The maximum torque the bolt has to withstand is

$$\tau = bT,$$

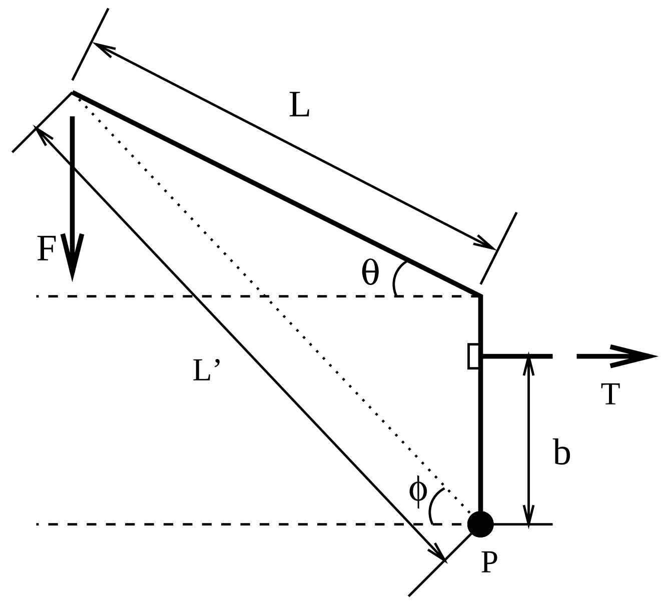

Figure 1: Figure for problem 1

where b is the distance between P and the bolt and T is the tensile strength times the cross section of the bolt.

Thus equating the above two equations,

$$\begin{aligned} T &= FL\cos\theta/b \\ &= (55\times 9.8 + 77\times(30/21.4)\times(60-21.4))\times L/0.5 \text{ m}\cdot\text{kg/s}^2 \\ &= 1.16\times 10^4\times L \text{ m}\cdot\text{kg/s}. \end{aligned}$$

2. (a) When rolling without slipping, the bottom point P of the yoyo has zero instantaneous velocity. Rotation around a fixed axis passing through$P$ is determined by the torque about that axis, but only F has non vanishing torque. Thus the rotation is clockwise (Figure 2(a)) and the centre of mass moves to the direction of the pull.

   (b) In order for the yoyo to stand still, the net torque of the system should be zero about the contact point P (Figure 2 (b)). Thus $mgb\sin\theta = F(b-a)$. Then

$$F = \frac{b}{(b-a)} mg\sin\theta.$$

When the yoyo is about to slip, the sum of the torques around the pivot point P', τ_{tot} is

$$\begin{aligned} \tau_{tot} &= 0 \\ &= \tau_1 + \tau_2 \\ &= mga\sin\theta - \mu mg(b-a)\cos\theta, \end{aligned}$$

where τ_1 is the torque due to the centre of mass of the yoyo being pulled down by the gravitational force and τ_2 is the torque due to the friction between the yoyo and the inclined surface.

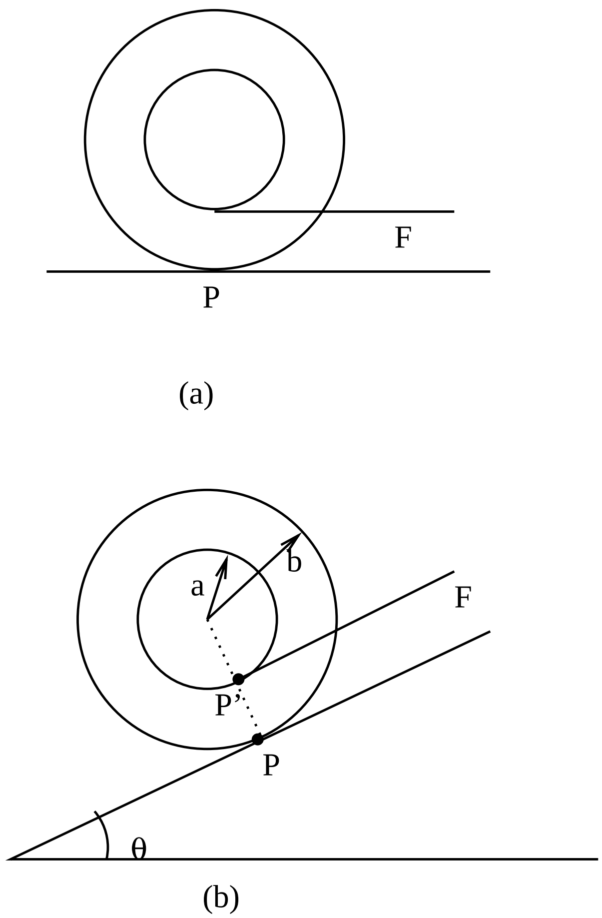

Figure 2: Figure for problem 2

Thus

$$\begin{aligned}\mu &= \frac{a}{(b-a)}\frac{\sin\theta}{\cos\theta} \\ &= \frac{a}{(b-a)}\tan\theta.\end{aligned}$$

(Note: The moment of inertia of a cylinder whose rotational axis is perpendicular to the flat surface of the cylinder and lies a distance a away from the centre is $I_R = I_M + Ma^2$. There was a typo in the problem set. Not that it would have made any difference in solving this problem...)

3. (a) From energy conservation the energy at $h = (R+a)\cos\theta$ is

$$\begin{aligned}mg(R+a)(1-\cos\theta) &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega_{sph}^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{5}mv^2 \\ &= \frac{7}{10}mv^2\end{aligned}$$

where we used $I_{sph} = (2/5)ma^2$ and $v = a\omega_{sph}$.

(Note: The trajectory of the centre of mass of the marble rolled to an angle θ from the top of the bowl is $(a+R)\theta$ (Figure 3 (a)). And $R\theta = a\phi$. Combining these equations, one gets

$$\begin{aligned}(a+R)\theta &= a\theta + R(a/R)\phi \\ &= a(\theta+\phi) \\ &= a\theta_{sph},\end{aligned}$$

which means $v = (a+R)\omega_\theta = a(\omega_\theta + \omega_\phi) = a\omega_{sph}$.)

For a marble about to take off from the surface of the bowl the force perpendicular to the surface of the bowl has the same magnitude as the centripetal force on the marble. Thus,

$$mg\cos\theta = mv^2/(a+R).$$

From the above two equations

$$(1-\cos\theta) = (7/10)\cos\theta$$

Solving for $\cos\theta$ gives $\cos\theta = 10/17$. Thus the marble leaves the surface of the bowl at the height of $h_c = R\cos\theta = (10/17)R$ (or $h_c = (10/17)(R+a)$, depending on what your definition of h_c is.)

(b) For a "rolling pendulum" (due to the lack of a better name) the potential energy has to be distributed between the rotational motion of the marble and the translational motion, whereas the "simple pendulum" potential energy is converted only to the translational motion. Therefore, the simple pendulum has a higher translational speed than the rolling pendulum at the same angle, *i.e.* it takes more time for the rolling pendulum to complete an oscillation than the the simple pendulum.

Brownie points:

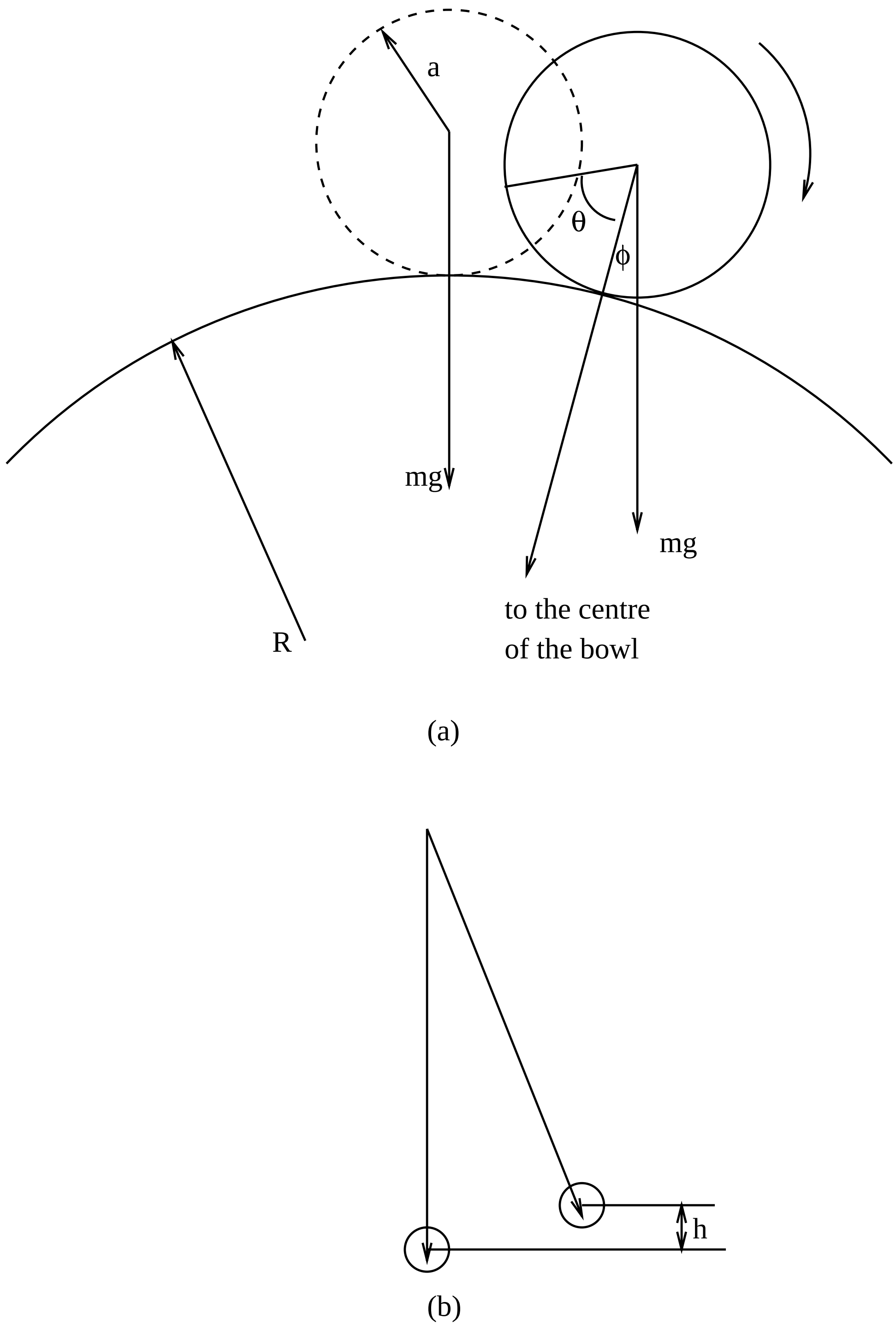

Figure 3: Figure for problem 3

For a simple pendulum, the energy conservation equation at h looks like

$$mg(h_\circ - h) = \frac{1}{2}m{v_s}^2$$

. Then

$$v_s = \sqrt{2g(h_\circ - h)}$$

, where $h_\circ$ is the release height of the pendulum.
For a rolling pendulum,

$$\begin{aligned} mg(h_\circ - h) &= \frac{1}{2}mv_r^2 + \frac{1}{2}I\omega_r^2 \\ &= \frac{7}{10}mv_r^2 \end{aligned}$$

$$v_r = \sqrt{(10/7)g(h_\circ - h)}$$

.
Thus $v_r/v_s = \sqrt{5/7}$ for any given height. Which means $T_s/T_r = v_r/v_s = \sqrt{5/7}$

4. (a) The surface of the liquid is perpendicular to the direction of the sum of the forces exerted, *i.e.* centripetal force and gravitational force (Figure 4). Thus

$$\tan\theta = \frac{v^2/r}{g}$$

.
The liquid touches the rim of the the cup when $\tan\theta$ satifies the following equation:

$$\tan\theta = \frac{2}{d}(H - h_\circ)$$

.
Using the above two equations, one can obtain

$$\begin{aligned} v &= \sqrt{\frac{2gr}{d}(H - h_\circ)} \\ &= \sqrt{\frac{2 \times 9.8 \times 30}{0.08}(0.15 - 0.12)} \\ &= 14.8 \text{ m/s.} \end{aligned}$$

(b) When the inside wheels of the car are about to leave the ground (provided that the car does not slip):

$$\frac{W}{2}F_d = hF_l$$

where

$$\begin{aligned} W &= \text{the track width of the car (or the width of the car for our purpose),} \\ F_d &= \text{downward force,} \\ h &= \text{the height of the centre of mass of the car,} \end{aligned}$$

and

$$F_l = \text{lateral force.}$$

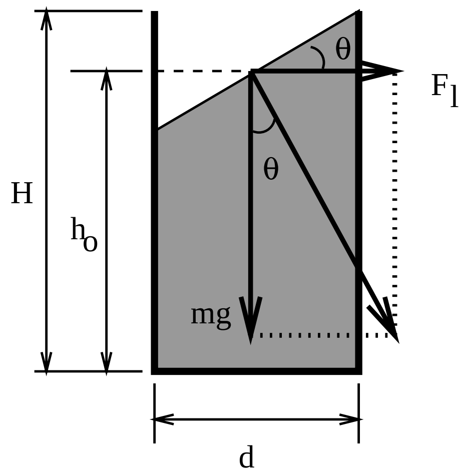

Figure 4: Figure for problem 4

The downward force is the sum of the gravitational force and the force generated by the aerodynamic device, *i.e.* $F_d = mg + cv$. Here c has units of N/(km/hr). Converting it into MKS units, it becomes N/(km/hr) = N/(1000 m/3600 s) = 3.6 N/(m/s). The lateral force is centripetal force, *i.e.* $F_l = mv^2/r$.

Rearrange it as a quadratic equation in v:

$$v^2 - \frac{rW}{2h}\frac{c}{m}v - \frac{rW}{2h}g = 0.$$

Solving for v

$$v = \frac{ac/m + \sqrt{(ac/m)^2 + 4ag}}{2}$$

,

where $a = rW/2h$.

Using $r = 30$ m, $W = 2$ m, $h = 0.4$ m, $m = 1500$ kg, and $c = 150 \times 3.6$ N/(m/s) = 540 N/(m/s), one gets

$$v_r = 43.8 \text{ m/s}.$$

However, if $\mu(mg + cv) = mv^2/r$, the car is about to slide. Rearrange it as a quadratic equation in v:

$$v^2 - \frac{\mu rc}{m}v - \mu rg = 0.$$

Solving for v

$$v = \frac{bc/m + \sqrt{(bc/m)^2 + 4bg}}{2}.$$

Using $\mu = 0.23$ one gets

$$v_s = 9.6 \text{ m/s}.$$

Since $v_r > v_s$, the car will slide before it lifts its inside wheels. Thus the maximum speed you can go through the turn is at 9.6 m/s.

5. (a) For a satellite on a circular orbit of radius r,

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

and

$$v = \sqrt{GM/r}.$$

The velocity of the satellite at the parking orbit ($h = 200$ km) is then $v_p = 7.772 \times 10^3$ m/s, where $G = 6.673 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$ and $M = 5.974 \times 10^{24}$kg are used.
The angular velocity of the satellite on the geosynchronous orbit is $\omega = 2\pi/(24 \times 3600 \text{ s}) = 7.272 \times 10^{-5} \text{ s}^{-1}$. The radius of the geosynchronous orbit, r_g, can be obtained from the first equation using $v = r\omega$,

$$\begin{aligned} r_g &= \left(\frac{GM}{\omega^2}\right)^{1/3} \\ &= 4.224 \times 10^7 \text{ m}. \end{aligned}$$

Thus the velocity of the satellite on the the geosynchronous orbit is $v_g = r_g\omega = 3.072 \times 10^3$ m/s.
From energy conservation

$$\frac{1}{2}mv_{pt} - \frac{GMm}{r_p} = \frac{1}{2}mv_{gt} - \frac{GMm}{r_g},$$

where

$$\begin{aligned} v_{pt} &= \text{The velocity at the perigee of the transfer orbit,} \\ v_{gt} &= \text{The velocity at the apogee of the transfer orbit,} \\ r_p &= \text{The distance of the perigee from the centre of the earth} \\ &= \text{The radius of the parking orbit,} \\ \text{and} & \\ r_a &= \text{The distance of the apogee from the centre of the earth} \\ &= r_g. \end{aligned}$$

From angular momentum conservation

$$\vec{r}_p \times \vec{v}_{pt} = \vec{r}_a \times \vec{v}_{at}$$

,

and at the perigee and the apogee, $\vec{r}$ and $\vec{v}$ are perpendicular to each other, so

$$r_p v_{pt} = r_a v_{at}$$

or

$$v_{at} = (r_p/r_a)v_{pt}.$$

Thus the energy conservation equation now looks like

$$\frac{1}{2}mv_{pt}^2\left(1-\left(\frac{r_p}{r_a}\right)^2\right) = \frac{GMm}{r_a}\left(1-\left(\frac{r_p}{r_a}\right)\right)$$

giving

$$\begin{aligned} v_{pt} &= \left(\frac{2GM}{r_a(1+r_p/r_a)}\right)^{1/2} \\ &= 1.022\times 10^5 \text{ m/s.} \end{aligned}$$

So $v_{pt} - v_p = 2.248 \times 10^3$ m/s.
The velocity at the apogee of the transfer orbit is then

$$\begin{aligned} v_{at} &= \left(\frac{r_p}{r_a}\right)v_{pt} \\ &= 1.597\times 10^3 \text{ m/s.} \end{aligned}$$

Thus the velocity change the satellite has to achieve in order to get into the geosynchronous orbit from the transfer orbit at the perigee of the transfer orbit is $v_g - v_{at} = 1.475 \times 10^3$ m/s.

(b) The transition takes half the time for the satellite to complete the elliptical orbit. So using the Kepler's law

$$\begin{aligned} T_{1/2} &= \frac{T}{2} \\ &= \frac{1}{2}\left(\frac{4\pi^2((r_p+r_a)/2)^3}{GM}\right)^{1/2} \\ &= 1.900\times 10^4 \text{ sec} \\ &\simeq 5 \text{ hrs } 17 \text{ min} \end{aligned}$$

It takes 5 hours and 17 minutes to complete the transition.

6. (a) Of course not, silly. The possible physical effects are
   i. Coriolis force. $2m\vec{\omega}\times\vec{v}$
   ii. Centripetal force. mv^2/r.
   iii. Decrease of gravitational force due to the increase in r.
   iv. Relativistic effect (time dilation). $t = \tau/\sqrt{1-v^2/c^2}$, etc.

   International flight normally flies at an altitude of about 10 km. Thus the change in the graviational acceleration at an altitude h is $\Delta g = g(1-(R/(R+h))^2) = 3.1\times 10^{-3}\times g$. The change in T is $\Delta T \simeq \sqrt{l/(g-\Delta g)} - \sqrt{l/g} \simeq 1.6\times 10^{-4}\sqrt{l/g}$, where l is the length of the pendulum. But it affects the same way regardless of the direction of the flight.
   The speed of the east bound plane that Chris is on from the fixed reference frame is $v = \pi/12\times 3600\times 6.4\times 10^6 \simeq 460$ m/s. Relativistic effects are then on the order of $\Delta T \sim T(1/\sqrt{1-(v/c)^2} - 1) \sim 10^{-11}T \sim 10^{-7}$ sec.

(b) If we use the inertial system with the origin of the coordinates sitting at the centre of the earth as the reference system, the Coriolis force vanishes in this system. The effective gravitational acceleration at the earth's surface for an object with angular velocity $\vec{\omega}$ against an inertial system is

$$\vec{g}_{eff} = -\frac{GM}{R^2}\hat{R} - \vec{\omega} \times (\vec{\omega} \times \vec{R}).$$

Since Pat and Chris are moving along the equator, their angular momentum vectors $\vec{\omega}$ are perpendicular to $\vec{R}$. Thus $|\vec{\omega} \times (\vec{\omega} \times \vec{R})| = |\omega^2 R|$. Then the effective gravitational acceleration looks like

$$g_{eff} = \left(\frac{GM}{R^2} - \omega^2 R\right)$$

,
and if we replace $\frac{GM}{R^2}$ with $g_\circ$

$$g_{eff} = g_\circ - \omega^2 R.$$

The period of a simple pendulum with the length of the pendulum L is

$$T_\circ = 2\pi\sqrt{\frac{L}{g}}.$$

Since Pat is moving westward, the angular velocity is zero. Thus

$$\begin{aligned}
T_{Pat} &= 2\pi\sqrt{\frac{L}{g_\circ}} \\
&= 2\pi\sqrt{\frac{L}{g + \omega^2 R}} \\
&= 2\pi\sqrt{\frac{L}{g}}\sqrt{\frac{1}{1 + \omega^2 R/g}} \\
&= T_\circ\sqrt{\frac{1}{1 + \omega^2 R/g}}.
\end{aligned}$$

Whereas, Chris is moving eastward, so the angular velocity of Chris is 2ω. Thus

$$\begin{aligned}
T_{Chris} &= 2\pi\sqrt{\frac{L}{g_\circ - 4\omega^2 R}} \\
&= 2\pi\sqrt{\frac{L}{g - 3\omega^2 R}} \\
&= 2\pi\sqrt{\frac{L}{g}}\sqrt{\frac{1}{1 - 3\omega^2 R/g}} \\
&= T_\circ\sqrt{\frac{1}{1 - 3\omega^2 R/g}}.
\end{aligned}$$

$$T_{Chris} - T_{Pat} = T_\circ \left(\sqrt{\frac{1}{1 - 3\omega^2 R/g}} - \sqrt{\frac{1}{1 + \omega^2 R/g}} \right) = T_\circ \left(\frac{2\omega^2 R}{g} \right)$$

Using $g = g_\circ - \omega^2 = 9.8\ \mathrm{m/s}^2$, $R = 6.4 \times 10^3\ \mathrm{km}$, $T_\circ = 12\ \mathrm{hrs}$, and $\omega = 2\pi/T_\circ$ one gets $T_{Chris} - T_{Pat} \simeq 298$ seconds. Since $T_{Chris} - T_{Pat} > 0$, Pat's clock goes faster than Chris' clock. (Pat's clock would be 75 seconds faster and Chris' clock would be 224 seconds slower than the local clock at Y.)

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Problem Set 3: Thermodynamics

Due December 18^{th}*, 1996*

1. Let's say you begin a chain e-mail that says "send a copy of this message to someone or you will step on a pile of crap within 24 hours" to a completely random user on a computer network.

   Here are the rules/facts about the chain e-mail.

   i. There are N users on the net, including yourself.

   ii. These users are a unusually superstitious bunch. Thus when one receives this apocalyptic message, s/he will send it to a random user (excluding the immediate sender) without failure.

   iii. If any user receives the message twice, s/he ignores the second message.

   iv. Only the directly immediate sender is known to the user that receives the message.

   v. The network failure rate is p and when it happens, the chain e-mail dies.

   Now you know the rules/facts of the chain e-mail. Here are the questions.

   (a) What is the probability of the e-mail reaching all the users?

   (b) What is the probability of you, the originator of the e-mail, getting the e-mail back before n users receives the message?

   (c) What is the probability of you stepping on a pile of crap within 48 hours?

2. Jacques Cousteau, a fearless marine explorer, prepares for a dive into the cold arctic sea. The sea water temperature is -3°C.

   (a) Jacques dons his rubber suit and scuba gear and dives into the sea. He remains underwater for 20 minutes. How much heat does he lose during the diving? Assume that his body surface is 2 m^2 and his rubber suit is 0.5 cm thick. Also, assume that Jacques skin maintains a temperature of 33°C throughout the dive.

   (b) Jacques then boards his *le Submersible* with his *le Cat*. She, the submersible, not the cat, is essentially spherical in shape 3 m in diameter and is constructed with 1 cm thick iron lined with 1 cm of rubber and then 0.5 cm thick aluminium. *le Cat* gets irritated when the ambient temperature goes down below 15°C. Jacques is an avid cat lover and he would not want to subject his cat in such harsh conditions. *le Submersible* carries 15 kW-h of batteries reserved for heating. How long can Jacques and *le Cat* remain underwater comfortably? (Note: State any assumption and/or approximation you make.)

3. After four years of physics education in university, you ditched the idea of having a career in physics and decided to do the next best thing. Well, now you are working as a road crew for a famous rock'n roll band "Killer Pumpkins and Dead Pumpkins". The subject of your service asks you to make him "a jugful of iced coffee made with 4 scoops of ground coffee brewed at 95°C. No sugar or cream, please. Oh, and I want it to be served at exactly 3°C. Make it fast, physics dude." So you decide to use ice cubes to cool the coffee down to 3°C. The fridge at the kitchen has ice cubes at -5°C. The volume of the jug is 600 ml. How much water (for coffee brewing) and ice cubes do you need?

4. An inventor claimed that he invented a perpetual motion engine. Your uncle Moe, an investor, came to you asking about the plausibility of the claim. The following is what the inventor says about the engine.

You know when water turns into ice, the volume it occupies increases. Now here is a heat engine to exploit that. Figure ?? shows the sequence of how the engine works.

(a) Initial state: water is confined in a cylinder with a piston and under 1 atm.

(b) A weight is placed on the piston.

(c) The cylinder is cooled and the water freezes. The expansion on freezing raises the weight, doing work.

(d) The weight is moved laterally. No work is done in moving the weight laterally.

(e) The temperature is returned to the initial value and the ice melts. The state of the system is now identical to the initial state.

Since ice melts and water freezes at 0°C, which implies that this heat engine can operate at a single temperature, which in turn implies that this engine can run forever.

Now you know that there is no such thing as "perpetual motion machine". What is wrong with this picture? Uncle Moe won't take answers like "yeah, you cannot violate the second law of thermodynamics. that's why."

5. You are a member of a team developing an autonomous volcano exploring vehicle. You are in charge of the cooling system design. The ambient temperature of the volcano is 530°C. The electronics bay of the vehicle is insulated and should be maintained at 30°C.

   The electronics in the bay produces heat at a rate of 720 W-h and the the heat flow rate into the bay through insulation is $0.002 \times (T_{out} - T_{in})$ W.

   (a) What's the maximum coefficiency of performance of the cooling system?

   (b) What's the minimum power requirement for the cooling system?

   (c) The coefficient of performance of the cooling system onboard the vehicle is rated at 0.2 at the conditions above. The typical mission lasts about 100 hours. What is the minimum energy requirement for the dedicated power source of the cooling system?

6. You work for CSA (Canadian Space Agency) and your team is involved with the Mercury Observer, a long waited Mercury probe spacecraft, project. Your supervisor asks you to do the following preliminary calculations. By the way, the funding for the project was approved on the condition that the spacecraft be painted black (bureaucracy, you see).

   Note: The average radius of the orbit of mercury is 5.79×10^6 km. The radius of the sun is 6.96×10^5 km and the temperature on the surface is 6000 K.

   (a) What would be the maximum radiation pressure that the spacecraft experiences from the sun's radiation when the spacecraft is on the mercury orbit.

   (b) What would be the surface temperature of the craft a long time after it arrives at the mercury orbit? The spacecraft is spherical in shape, has a 3 m diameter, and its cooling system expels 4 W from the interior to the exterior of the spacecraft.

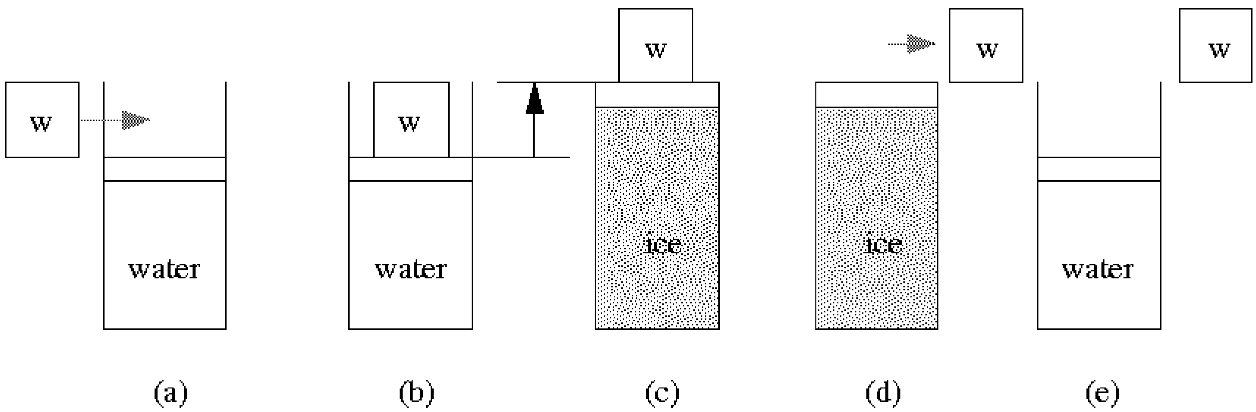

Figure 1: Figure for Problem 4.

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Problem Set 4: Wave and Optics

Due January 22$^{\text{th}}$, 1996

1. As a physics-illiterate rock concert organizer on Lilliput island (remember Gulliver's Travels), Donald Armen is still puzzled by the complaints from the rock fans. In fact, one of the two speakers was destroyed. He just cannot believe it. He bought a new speaker to increase the loudness of this mono sound system to the one he had. He placed them 8.00 m apart. The closest audience member, Christie, was 10.0 m away from each speaker, respectively (or 9.17 m from the midst of the both). Can you help him understand what was going on? Assume the average frequency is 690 Hz (being tiny, these Lilliputs have high pitch). The speed of sound is 344 m/s.

   (a) Could Christie hear the concert?

   (b) Albion, standing on Christie's left, was among the ones who got angry and destroyed the speaker. He just barely listen to the concert. Where did he stand?

   (c) This fiction is far from reality. Explain!

2. A group of police cars moving at 80 km/hr are chasing O'Samson's white Bronco, which is travelling at the same speed. They are afraid that this former famous hockey player will commit suicide. It seems that 3000-Hz sirens from the police car do not bother him much.

   (a) What is the wavelengh of the sirens in the air between O'Samson and his chasers?

   (b) What frequency does he actually hear?

   The speed of sound is 344 m/s.

3. Learning from Donald Armen's story (Problem 1), you decide to study more about loudspeakers. You place two small loudspeakers facing each other at the two ends of a 5.0m-bench. The first loudspeaker emits a sound of pressure amplitude twice as much as the second one. Both sounds are at 1334 Hz and are in phase at the speakers.

   (a) At what point on the line joining the two speakers is the intensity a minimum?

   (b) What is the intensity compared to one from the second speaker alone?

   (c) At what other points on the line joining the speakers will destructive interference occur?

   (d) At what point on the line joining the speakers is the intensity a maximum?

   (e) Express the maximum intensity terms of the intensity from the second speaker!

4. Your professor told you that diffraction gratings are better than prisms for atomic and molecular spectroscopy. You don't believe it; you are sceptical .... So you perform experiments using both and you finally agree with him. Why is your professor right? To support your arguments compare your calculations for both prism and diffraction grating for a source light which emits 434 nm (blue), 486 (green blue), 589 nm (yellow) and 656 nm (red). Take the indices of refraction of heavy flint prisms $n = 1.675$, 1.664, 1.650, and 1.644, which correspond to the wavelengths above, respectively. The angle of the prism is $A = 60^o$ and the diffraction grating has 8000 lines per centimeter.

5. Kids love to play with soap bubbles and are always fascinated by them. Ok, ... you are not kids anymore but you need to guess what colour a bubble of 400 nm thickness appears to be. Need some hints? Alright.

   (a) What colours do not appear in the reflected light?

(b) For what colours does constructive interference occur?

(c) So the colour of the soap bubble is predominantly ....

The index of refraction of the soap film is 1.36.

6. Ok, let us have a break now with a few small questions.

   (a) Most large astronomical telescopes in use today are reflectors rather than refractors. Why is this so?

   (b) You are able to find a very sharp focal length of a crystal ball. Right or wrong? Explain.

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Problem Set 5: Electricity and Magnetism

Due February 19^{th}*, 1996*

1. Pretend that you are an electrical engineer. You supervise the construction of an electric generating plant system at a rate of 1 GW and a voltage of 12,000 V.
   (a) It is desired to transmit the power at 756 kV. What must be the turns ratio of the step-up transformer?
   (b) What current would be sent out over the power lines if the transmission were at 12,000 V.
   (c) What is the actual current you expect?

2. Many environmentalists are disturbed by the radiation from power lines. They may not be concerned with the electric field because they can buy the argument that human body is almost a perfect conductor. (Will you buy it ... ?) Anyway, but they worry so much with the effect of magnetic field in the high voltage transmission lines like your design in the previous problem. As an engineer you try to convince them that the powerful plant like yours (1 GW and 756 kV transmission lines) is still safe because you already put the transmission line high enough. Otherwise, you will lose your job because they are very persuasive these days. What is the minimum height you have to put your lines so the effect of magnetic field on the ground is only 10% of the natural one? The magnetic field at Earth's surface is 0.5 gauss in contrast to the one in the lab which can go up to 1.0 T.

3. You are trapped in a dungeon to assist an old physicist of 1950's who still has his own ancient accelerator. It is a boring job but he pays you very well, better than your previous job as an electrical engineer. He asks you to accelerate a beam of electrons at a voltage of 10,000 V and this beam enters a uniform magnetic field of 0.50 T perpendicular to the motion of the electrons.
   (a) What is the force on each electron due to the magnetic field?
   (b) What will the speed of each electron be after it has been in the field for 10 seconds?

4. After working for years as a research assistant, sometimes you have nightmares. One day, you dreamt that you were an electron travelling parallel to a very long,thin wire at a distance of 20 cm. Suddenly, you felt you were pushed away from the wire by a strong force (well, you are very tiny now) because your boss turned on the electric current of 2.0 A on the wire. How much force did the magnetic field of the current exert on you and what was the direction of the force with respect to the current? After you woke up, you told your boss about your dream and he laughed, "Why didn't you tell me to place the wire between two large plates of a capacitor so I would not have bothered you?" You seemed puzzled. Now, after many years of enlightment, you know the reason why he said this. What was the voltage of the capacitor to keep you travelling happily along the wire and what was the direction of the electric field?

5. Nuclear fusion is like a long term commitment for a blind mouse couple. Sometimes you need to interfere by pushing one or both toward the other. But you can attract them by placing a piece of cheese between them so they will smell it. This kind of cheese is called a catalyst in the fusion reaction.

   To make the story simple, you would like to have fusion of two protons.
   (a) If the catalyst is an electron, what is the total kinetic energy of the system at its minimum potential energy where the electron sits collinearly between these two protons? Assume that all three are far apart from each other and the minimal distance between the protons is about 0.1 nm.

(b) Instead of an electron, you can also use a muon which is heavier by a factor of 207. This heaviness makes the distance between the protons smaller by a factor of 207, too. What is the total kinetic energy of the system now?

Of course, the story does not end here because the two protons will transform into a deuteron, and emit a positron and a neutrino.

NOTE: This minimal distance between the protons is determined by the Heisenberg uncertainty principle of quantum mechanics.

6. Here are a few questions that do not require any calculations.

   (a) You have been told that wave requires a medium to propagate like sound requires air. However, light, which is electromagnetic wave, can travel in vacuum. How do you explain this phenomenon?

   (b) Electromagnetic wave, like other waves, can transmit energy from one place to another, say E. How much momentum does it transmit if it can?

   (c) Why do we all have an AC-, instead of DC-, power in the houses and buildings?

1995-1996 Physics Olympiad Preparation Program

– University of Toronto –

Problem Set 6: Electronics

Due March 18^{th}*, 1996*

1. The circuit in Figure 1 is one that you can't easily escape from. Whether you're taking in tunage from *Offspring*, *Metallica*, or a Bach fugue for harpsichord, this little gem will most likely come into play (oops, no pun intended). It's a – wait for it – noninverting operational amplifier circuit.

   (a) Find an exact expression for the voltage gain ratio, $A_v = \frac{v_o}{v_2}$, in terms of R_1, R_2, R_d (the effective resistance between the "+" and "-" terminals), and $A_{\text{OL}} \equiv \frac{v_o}{v_d}$ (the open loop voltage gain).

   (b) Assume that i_{in} is zero, so that $v_d \approx 0$ and $v_1 \approx v_2$. Derive an expression for the voltage gain ratio for this simplified case.

2. Gill Bates is the billionaire founder of Sicromoft Corp., the creator of *ScreenDoors 96* (a soon-to-be popular operating system). It is a little known fact that Gill spent her teen-age summer months designing power supplies for a small Ontario electronics firm. One year, she dreamed up the $5/2\pi$ kHz circuit shown in Figure 2. Her boss wanted it put on the market immediately, and asked her to write a specification sheet for it, including the magnitude and phase of (a) the open circuit voltage across the terminals AB and (b) the driving point impedance at AB (with the internal voltage source set to zero). Solve for these quantities and you too may be a billionaire someday.

3. Let me tell you about a nightmare I once had. It was a dark and stormy night and I had somehow become trapped in a dripping dungeon guarded by a band of simian subterranean gypsies. Their leader, Murd, handed me a circuit breadboard and a baggy containing half a dozen NOR gates. He told me that if I could realize the logic function $f = A \cdot B + \overline{A} \cdot B$ using the devices provided, then I would be free to leave his dungeon. How did I do it in my dream, using all the given gates? Of course, I then awoke in a cold sweat and realized how stupid I (and Murd's band for that matter) had been. Why?

4. Aren't CDs just too cool, with their glittering colours and cute size? In the early 80s they were heralded as the "perfect sound" medium, and thanks to the power of advertising many folks still think this to be true. The shortcomings of the CD format, which are rooted in the fact that the standard was established way back in the late 70s, could form the basis of many POPTOR problems. CDs are here to stay for a while, anyway, so let's be positive and examine the medium a little. First of all, the data words on a CD are 16 bits long. The digital sampling rate is 44.1 kHz, and the nominal frequency limits of human hearing are $\sim$20 Hz to $\sim$20 kHz.

   (a) How many different levels of sound pressure can be represented with this medium?

   (b) Discuss why the digital sampling rate was chosen to be 44.1 kHz.

   (c) What is the maximum possible dynamic range (in decibels)?

   (d) What is the data density (in bytes/m^2) on a CD if the angular speed is 500 rpm when the laser pickup is at the innermost point and 200 rpm when it's at the outermost point. Assume the outer diameter to be 120 mm, that a CD plays for 74 minutes, and that there's no data redundancy or error-checking information.

5. The configuration in Figure 3 contains two AND gates and two NOR gates. The input marked CLK is connected to a square wave generator that oscillates between the "1" and "0" logic states.

   (a) Give a truth table for this logic circuit.

   (b) How do you think this configuration could be useful?

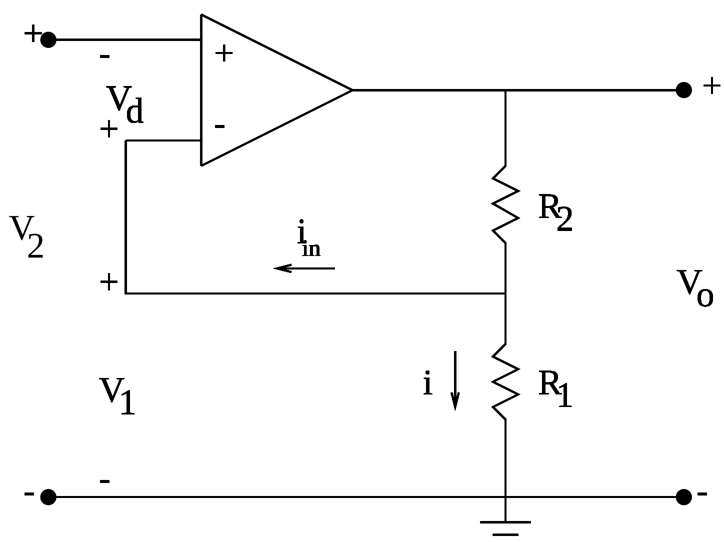

Figure 1: The noninverting op-amp circuit in Problem 1.

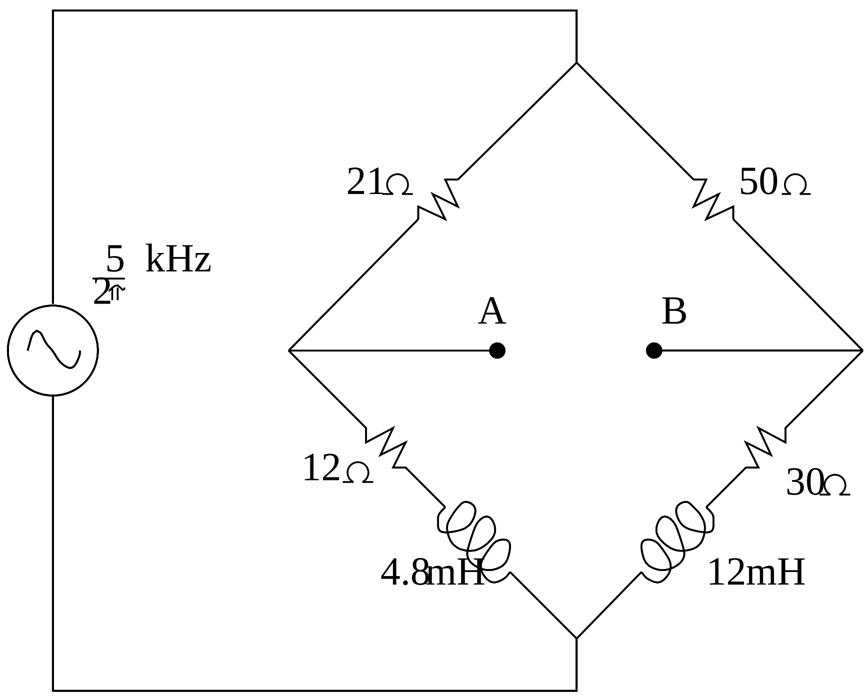

Figure 2: Gill's circuit in Problem 2.

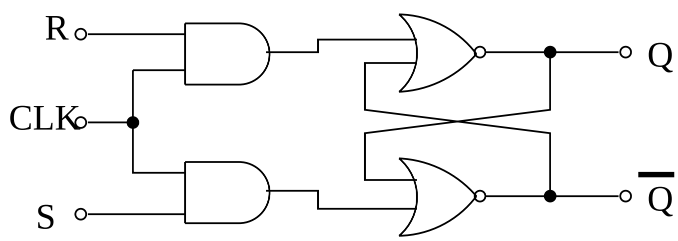

Figure 3: The digital circuit in Problem 5.

(c) Can Figure 3 be replaced by a system of NAND gates? If so, how? If not, why not?

6. Your fairy godmother gives you a black box with four terminals on it (two inputs, A and B, and two outputs, C and D). She tells you that the resistance between terminals B and C is equal to the resistance between terminals C and D, and that there is negligible resistance between terminals B and D. She also informs you that, when a sinusoidal voltage with a frequency f_o is applied across A and B, the output voltage across C and D is also sinusoidal, but has an amplitude that is 20% less than the input amplitude. If the input frequency, f, is varied either higher or lower, the output voltage amplitude is attenuated symmetrically about f_o, roughly as a function of $1/|f - f_o|$. What does the circuit inside the black box look like and what can you say about the values of the devices inside? *Note:* You may assume that there's no active electronics inside – only passive devices are used.

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Solution Set 6: Electronics

1. (a) The feedback voltage, i.e. the voltage at the "-" terminal of the op amp, can be written as:

$$v_1 = v_0 - (i + i_{\text{in}})R_2.$$

Since $i = v_1/R_1$ and $i_{\text{in}} = v_d/R_d = v_0/A_{OL}R_d$:

$$v_1 = \left[\frac{1 - R_2/A_{OL}R_d}{1 + R_2/R_1}\right] v_0.$$

But

$$v_d = v_1 - v_2 = \frac{v_0}{A_{OL}},$$

or

$$v_2 = v_1 - \frac{v_0}{A_{OL}}.$$

Combining and rearranging the above expressions gives:

$$A_v = \frac{v_0}{v_2} = \frac{R_1 + R_2}{R_1 - \frac{R_1 R_2}{A_{OL}R_d} - \frac{R_1 + R_2}{A_{OL}}}.$$

(b) If $i_{\text{in}} = 0$, then the currents through R_2 and R_1 are equivalent and:

$$\frac{v_0 - v_1}{R_2} = \frac{v_1}{R_1},$$

and

$$A_v \equiv \frac{v_0}{v_2} \approx \frac{v_0}{v_1} = 1 + \frac{R_2}{R_1}.$$

2. Gill's circuit is a *Bridge circuit* and the problem is solved by calculating the *Thevenin* equivalent voltage and impedance. The impedance of a resistor and inductor in series is given by the complex expression $Z = R + i\omega L$, where R is the resistance, ω is the angular frequency, and L is the impedance. The quantity $X_L = \omega L$ is called the inductive reactance.

The equivalent impedance at terminals AB with the source set equal to zero is:

$$Z' = \left[\left(\frac{1}{12\ \Omega + i(5\ \text{kHz} \cdot 4.8\ \text{mH})}\right) + \frac{1}{21\ \Omega}\right]^{-1} + \left[\left(\frac{1}{30\ \Omega + i(5\ \text{kHz} \cdot 12\ \text{mH})}\right) + \frac{1}{50\ \Omega}\right]^{-1},$$

which simplifies to:

$$Z' = 47.3\ \Omega \text{ at } 26.8°.$$

With the circuit open at AB, the current on the left side of the bridge is:

$$I_1 = \frac{20\ \text{V}}{21\ \Omega + 12\ \Omega + i(5\ \text{kHz} \cdot 4.8\ \text{mH})},$$

and the current on the right side of the bridge is:

$$I_2 = \frac{20 \text{ V}}{50\ \Omega + 30\ \Omega + i(5 \text{ kHz} \cdot 12 \text{ mH})}.$$

If we assume that point A is at a higher potential than point B, then we have:

$$V' = V_{AB} = I_1(12\ \Omega + i(5 \text{ kHz} \cdot 4.8 \text{ mH})) - I_2(30\ \Omega + i(5 \text{ kHz} \cdot 12 \text{ mH})).$$

Substituting and simplifying gives:

$$V' = 329 \text{ mV at } 170.5°.$$

3. Just like in many other nightmares, things here didn't proceed optimally. The function, $f = A \cdot B + \bar{A} \cdot B$, can be simplified to B, so no NOR gates are needed and the output can simply be wired to the input B. The question, however, insists that we design a logic circuit using NOR gates, so here goes.

   Since $f = A \cdot B + \bar{A} \cdot B$, then:

$$\bar{f} = \overline{\overline{\overline{A \cdot B + \bar{A} \cdot B}}}.$$

   Applying deMorgan's Laws ($\overline{A + B} = \bar{A} \cdot \bar{B}$, and $\overline{A \cdot B} = \bar{A} + \bar{B}$) gives:

$$\bar{f} = \overline{\overline{\overline{(A \cdot B)} \cdot \overline{(\bar{A} \cdot B)}}},$$

   and:

$$\bar{f} = \overline{\overline{(\bar{A} + \bar{B}) \cdot (A + \bar{B})}},$$

   and:

$$\bar{f} = \overline{\overline{(\bar{A} + \bar{B})} + \overline{(A + \bar{B})}}.$$

   Therefore:

$$f = \overline{\overline{\overline{(\bar{A} + \bar{B})} + \overline{(A + \bar{B})}}}.$$

   The circuit of NOR gates is given in Figure 1.

4. (a) Since levels of sound pressure, or intensity, are related to signal (voltage) levels in the electronics, and the data word length for a single sample is 16 bits, the number of levels of sound pressure is:

$$2^{16} = 65,536 \text{ levels}.$$

   (b) The Danish mathematician, Sven Nyquist, proved what is now called *Nyquist's Theorem*, which states that a sampling rate of at least **double** the highest recorded frequency is required for the reproduction of sine waves of the highest recorded frequency, and of any frequencies below that point. Since any waveform can be decomposed into a superposition of sinusoidal waves, the sampling rate for frequencies detectable by the human ear needs to be at least 40 kHz ($2\times$ 20 kHz).

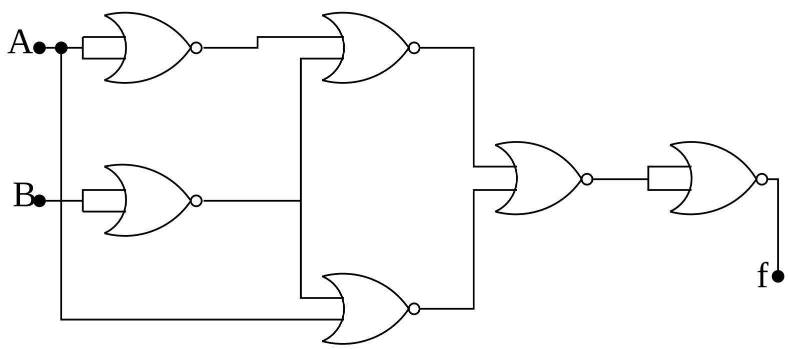

Figure 1: The NOR gate configuration in Problem 3.

S	R	CLK	Q	$\overline{Q}$
0	0	0	Q	$\overline{Q}$
0	0	1	Q	$\overline{Q}$
0	1	0	Q	$\overline{Q}$
0	1	1	0	1
1	0	0	Q	$\overline{Q}$
1	0	1	1	0
1	1	0	Q	$\overline{Q}$
1	1	1	?	?

Table 1: The truth table for the gated R-S flip-flop in Problem 5.

(c) The dynamic range of CDs is nominally said to be at least 90 dB. In this case, the decibel is a logarithmic measure of the relative highest and lowest power levels possible with the CD format. The decibel, when used to compare two power levels, is defined as $10 \log_{10} \frac{P_2}{P_1}$, where P represents power. We know that CDs offer 65,536 *voltage* levels, and thus, since $P = V^2/R$ (V is voltage and R is resistance), the maximum dynamic range is:

$$20 \log_{10} 65,536 \doteq 96.3 \text{ dB}.$$

(d) For vinyl long playing records, the stylus spirals along a groove from the outer radius to the inner one while the turntable rotates at a fixed angular velocity. For CDs, the laser pickup instead tracks a spiral from the inside radius to the outer one, and the disc is rotated with a variable angular velocity to provide a constant (linear) speed difference between the pickup and the track for any given radius on the disc. The linear relative speed at the outer radius is:

$$\left(\frac{120 \text{ mm}}{2}\right)(200 \text{ rpm}) = 12 \text{ m/s}.$$

Equating this with the linear relative speed where the magnitude of the angular velocity is 500 rpm gives an inner radius of 24 mm. Now that we have the inner and outer radii, we can calculate the active area of a CD as:

$$\pi\left((60 \text{ mm})^2 - (24 \text{ mm})^2\right) = 9.5 \times 10^{-3} \text{ m}^2.$$

The storage capacity is approximated by using the maximum playtime and the digital sampling rate:

$$(74 \text{ min})(60 \text{ sec/min})(44.1 \times 10^3 \text{ words/sec})(2 \text{ bytes/word}) = 3.92 \times 10^8 \text{ bytes}.$$

The data density is therefore:

$$\frac{3.92 \times 10^8 \text{ bytes}}{9.5 \times 10^{-3} \text{ m}^2} = 4.1 \times 10^{10} \text{ bytes/m}^2.$$

5. (a) The truth table for this sequential logic circuit, which is a gated R-S flip-flop, is given in Table 1.

(b) The principle of this R-S flip-flop can be used to store information in a clocked computer memory register.

(c) Refer to Figure 2 for an equivalent configuration made up of NAND gates.

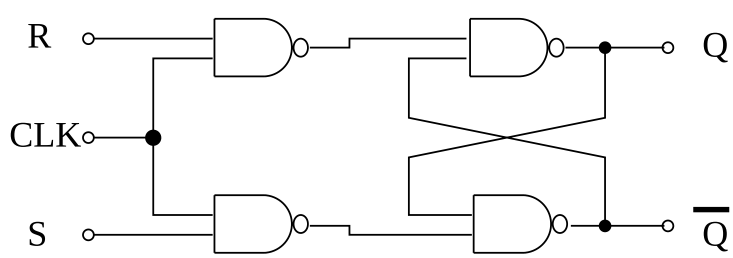

Figure 2: The NAND gate equivalent to Problem 5.

6. Combining the the given attenuation and interterminal resistance information suggests that there's a passive bandpass filter in the black box. Refer to Figure 3 to see the circuit. Since we're only concerned with amplitudes here (we can ignore the phase), we can deal with the equation $V = I|Z|$, where Z is the impedance. The ratio of the output over input voltage amplitudes is therefore given by:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R}{\sqrt{(2\pi f L)^2 + r^2 - \left(\frac{1}{2\pi f C}\right)^2 + R^2}}.$$

The resonant frequency from the above equation, f_o, is calculated to be:

$$f_o = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}.$$

We know that the resistance r exists because of the nonzero attenuation (20%) at the resonance frequency f_o. At resonance:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = 80\% = \frac{R}{\sqrt{R^2 + r^2}}.$$

The relationship between r and R is therefore:

$$r = 0.75R.$$

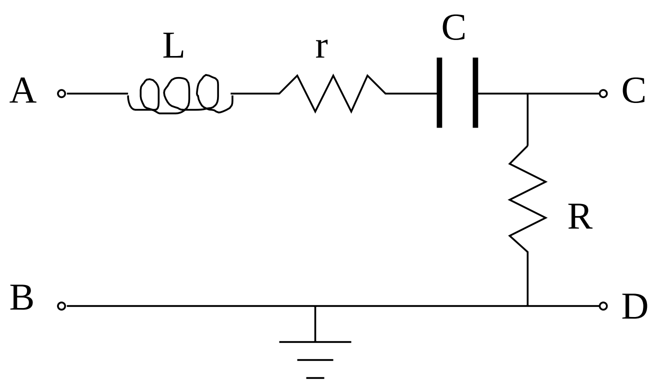

Figure 3: The passive LRC circuit in the black box of Problem 6.

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1996-1997 Physics Olympiad Preparation Program

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Problem Set 1: General

Due November 1^{st}*, 1996*

1. While playing a game of *Laser-Tag*™, Daniel is trying to target his pal Yiorgos with his laser beam. Daniel (from his sonar display) knows that Yiorgos is hiding in the next room separated from him by an opaque wall and a thin window. Daniel wants to fire at Yiorgos, but he can't aim without exposing himself. Suddenly he gets a great idea! He pulls a Gas-Bomb from his belt and pulls the trigger. The invisible gas fills the room, slowing the speed of light to one half of its normal value in air.

Figure 1: For problem 1

   (a) What is the smallest lateral distance (i.e. parallel to the window) Daniel must move to target Yiorgos? (Remember that for physicists, 'thin', 'small', 'minuscule', and similar adjectives generally mean 'negligible'.)

   (b) Daniel, delighted with his ingenuity, moves this distance and fires his laser gun through the window. He is surprised to see the beam exit the window and continue straight. What went wrong?

2. An Ontario apple of mass $M = 0.2$ kg rests on fence post of height $h = 3$ m. A bullet with mass $m = 10$ g, traveling at $v_0 = 500$ m/s, passes horizontally through the centre of the fruit. The apple reaches the ground at a distance of $s = 10$ m. Where does the bullet land? What part of the kinetic energy of the bullet was lost as heat when the arrow passes through the apple?

3. The prototype spaceship *Stealth* (mass: 250 000 kg) is in a solar orbit(r=1.5×10^8km), testing out its new cloaking mechanism. The cloaking device makes the ship perfectly absorbing of all electromagnetic radiation.

   (a) Assuming the ship has a cross-sectional area facing the sun of 20 000 m^2 and knowing that the sun produces 3.8×10^{26}W of power , what is the energy the *Stealth* absorbs after being cloaked for ten minutes?

   (b) Assume that the energy the ship radiates is negligible. Before the test period, the average temperature in the *Stealth* is 18 °C. What will be the temperature after the ten minute test period? Assume that all the absorbed energy is going into heat, and the average specific heat of the *Stealth* and its crew is 2 J/g°C.

(c) In reality the ship does radiate energy. Taking this into account, what will be its final temperature if the ship stays cloaked? (you might want to use Stefan's constant: $\sigma = 5.67 \times 10^{-8} \mathrm{Wm}^{-2}\mathrm{K}^{-4}$). What is the peak wavelength of the spectrum that the ship emits?

4. Wesley (of *Star Trek: Next Generations* fame) is given a box with three electric terminals protruding from it labelled A, B, and C. He is told that the box contains three resistors and, using an ohm-meter, finds out that the resistances across AB, AC, and BC are exactly 8 Ω each. He is asked to describe the configuration and values of resistances in the box, knowing that if he gives the wrong answer, phasers on "kill" will automatically wipe him out. He's under a little stress. Find the resistances. Is the answer unique?

5. In her laboratory, Marie-Eve finds an unknown component with two electrical leads. To determine what it is, she applies different voltages across the leads and measures the following current going through the device:

Figure 2: Device for problem 5

(a) What is the effective resistance of the device at 10V? at 4V?

(b) Marie-Eve then uses the unknown device in the following simple circuit:

Figure 3: Circuit for problem 5

Using the I-V graph, circle the point (or points) that represents the voltage and current that could be passing through the unknown component in the above circuit.

(c) Marie_Eve replaces the 10V supply with a variable power supply and an ammeter (current measuring device with negligible resistance). Starting a 0 V, she increases the power supply voltage to 15 V and then decreases it slowly back to 0 V. Roughly sketch a graph of the current she would observe as a function of this voltage.

6. For some roll dispensers of toilet tissue, it is possible to drag the end of the roll over and flush the entire roll down the loo like spaghetti, with the tissue spinning freely off the roll. Presumably to interrupt such physics experiments, manufacturers often construct the holder to have a spring-loaded pressure plate at either end of the holder, which press flat against the ends of the cardboard core. The friction is enough to keep the paper from spinning off the roll, but as the roll becomes smaller, it soon becomes impossible to pull the paper directly off the roll without breaking.

Figure 4: Roll dispenser for problem 6

(a) Consider a cardboard core 4 cm in diameter and with walls 1mm thick, and take that the pressure plates push only on circular ends of this core with a force of 13 N on either end, and that the core has a static coefficient of friction $\mu_s = 0.27$. Empirical studies (!) show that most toilet paper segments will burst at the perforations for a typical tension of $T = 3.5N$. If the toilet paper can be pulled off the roll without breaking at the perforations, what is the minimum diameter of the toilet paper for this to be possible?

(b) The kinetic coefficient of friction μ_k is measured to be 0.18. If the paper is pulled off starting with a full roll (diameter 11.5 cm) and the motion never stops, what now will be the diameter of the roll of paper at the point where the paper tears?

(c) Take that the paper on the roll wraps at 10 layers per millimetre. What length of paper remains on the roll for the critical diameter you found in (a) and (b) above?

(d) Suppose that the pressure plates pushed on the whole side of the roll, cardboard core and rolled paper together, so that the contact area changed as the roll got smaller. How might the whole problem be changed? Assume you can use the same static coefficient of friction μ_s.

<u>PROBLEM SET #1 – GENERAL</u>

1. **Laser–Tag**

This is a standard refraction-type question. Since Daniel is in a medium with a different (i.e. higher) refractive index than air, the laser beam will 'bend' around the corner.

Let x be the minimum distance Daniel must move to target Yiorgos. The only other variable to worry about is at what angle Daniel will aim (i.e. where will the beam hit the window). For minimum x, Daniel will aim to hit the very edge of the window.

From Snell's law

$$n_i \sin\theta_i = n_r \sin\theta_r$$

We know that after the Gas-Bomb is used, $n_i = 2n_r$

$$\sin\theta_r = \frac{4}{\sqrt{4^2 + 1^2}} = \frac{4}{\sqrt{17}}$$

Also

$$\sin\theta_i = \frac{x}{\sqrt{x^2 + 2^2}} = \frac{x}{\sqrt{x^2 + 4}}$$

$$\therefore \quad 2n_r \frac{x}{\sqrt{x^2 + 4}} = n_r \frac{4}{\sqrt{17}}$$

$$17x^2 = 4(x^2 + 4)$$

$$13x^2 = 16$$

$$x = 1.1 \text{ m} \quad \text{(Negative root extraneous).}$$

Daniel must move at least 1.1 m to the right to target Yiorgos.

(Common error: In Snell's law, the incident and refracted angle are measured between the ray and the *normal* to the interface, not the ray and the interface.)

b) Nothing went wrong, if Daniel was looking along the beam with the laser-gun held high to aim. Daniel sees his laser beam due to light scattering back to him from dust or other particles in the air. This scattered light, along nearly the same path, but backwards, will also obey Snell's law, and as such the light that reaches his eye will follow a similar path as the laser beam. Therefore, to Daniel it looks straight.

(Many of you came up with ingenious ways that would result in the laser beam not bending, and, if it was physically possible, got part-marks for your efforts. As you can see from the solution, none of these were necessary.)

2. Apple shoot

The total momentum of the system remains constant in a collision. Thus,

$$mv_0 = mv + MV \tag{2.1}$$

Here, v is the velocity of the bullet and V that of the apple after the collison. Both objects fall a distance h to the ground; their time of flight is $t = \sqrt{2h/g} = 0.78$ s. During time t, the apple covers a horizontal distance of s = 10 m, so its horizontal velocity is V = s/t = 12.8 m/s. Thus, the conservation of momentum equation yields v = 104 m/s for the horizontal velocity of the bullet immediately after the collision. Since the bullet's time of flight is also t, it travels a distance vt = 105 m horizontally from the column.

Initially, the kinetic energy of the system is $E_i = mv_0^2/2 = 1250$ J. After the collision, the kinetic energy is $E_f = mv^2/2 + MV^2/2 = 93.2$ J. Thus,

$$1250 - 93.2 = 1156.8 \text{ J} \tag{2.2}$$

is lost as heat (92.5% of the original energy). Note that the collision is *not* completely inelastic. In a completely elastic collision, the kinetic energy is conserved. In a completely inelastic collision, the bullet would remain inside the ball.

3. Stealth

In the world of science-fiction, a 'cloaked' ship has to be perfectly transparent; or, next-best, absorbing all radiation that strikes it but emitting only the same radiation as the molecules it displaced would have. Our current knowledge of physics doesn't allow this, in steady state, but let's assume the second case.

a) We can assume that the sun radiates uniformly over a sphere.

What is the energy the Stealth absorbs per second?

The sun radiates $P = 3.8 \times 10^{26}$ W over it's entire sphere. At $r = 1.5 \times 10^8$ km, this corresponds to a flux of $P/(4\pi r^2) = 1.3 \times 10^3$ W m^{-2}.

$\therefore$ Stealth absorbs a total power of 1.3×10^3 W m^{-2} • 20 000 m^2 = 2.6×10^7 W.

Over 10 minutes, Stealth absorbs 2.6×10^7 J s^{-1} • 10 min • 60 s min^{-1} = 1.6×10^{10} J.

b) With no radiation losses: $\Delta t = 1.6 \times 10^{10}$ J / { (2.5×10^8 g)(2 J g^{-1} °C^{-1})} = 32° C,
$\therefore$ t_{final} = (18 + 32) = 50°C)

c) Unfortunately, the question does not give enough information to really answer this problem. We need the *total* surface area A_{surf}, not just the cross-sectional area. We can make some guesses at this, though. Since the cross-sectional area = 20 000 m^2, the surface area must be at least 40 000 m^2. There is no upper limit to this number.

(Common error: many people just plugged in the only area value they were given, i.e. 20 000. A ship that has a cross-sectional area of 20 000 cannot have a surface area of 20 000, this answer is impossible.)

At steady-state, *power absorbed = power radiated*. Using Stefan's law: $H = e\sigma A_{surf} T^4$ (T in Kelvin), and since ship is perfectly absorbing, emissivity $e = 1$, ship absorbs 1.3×10^3 W/m^2, $\therefore$ at steady-state $1.3 \times 10^3 \text{ W m}^{-2} \bullet 20\,000 \text{ m}^2 = \sigma A_{surf} T^4$,

$$T = \left(\frac{2.6\times10^7}{\sigma A_{surf}}\right)^{1/4}$$

Since we do not know A_{surf}, we can determine only a rough range of T.

Maximum for $A_{surf} = 40\,000 \text{ m}^2$ (the minimum):

$$T = \left(\frac{2.6\times10^7}{\sigma A_{surf}}\right)^{1/4} = 389 \text{ K} = 116° \text{ C}$$

Since A_{surf} can be any larger surface area, T_{min} can approach zero K.

A possible ship shape would be a sphere. The radius of such a sphere is

$\pi r^2 = 20\,000 \text{ m}^2$
$\Rightarrow r = 80 \text{ m}$
$A_{surf} = 4\pi r^2 = 80\,000 \text{ m}^2$
giving $T = 275 \text{ K} \rightarrow 2°\text{C}$

No matter what your answer for T, you can determine the corresponding peak wavelength of the radiation distribution. Wien's Displacement law is the quickest way to the answer. It states:

$$\lambda_{max} T = 0.002898 \text{ m} \bullet \text{K}$$

Substitute for T (in Kelvin) and solve for wavelength in meters. 275K gives λ_{max} = 20.5 μm.

4. Wesley's test

A typical first guess at the contents of the black box is a triangular configuration of resistors of resistances R_{ab}, R_{bc} and R_{ca} shown in Fig. 4.1(a). Since Wesley is such a stereotypical thinker, he immediately grasps at this solution. When the resistance across terminals *ab* is measured, the resistance of the parallel circuit formed by resistors R_{ab} and $R_{bc} + R_{ca}$ is

$$R_{ab\text{-total}} = \left(\frac{1}{R_{ab}} + \frac{1}{R_{ca} + R_{bc}}\right)^{-1} \qquad [4.1]$$

Similarly, the resistances measured across terminals bc and ca are

$$R_{bc\text{-total}} = \left(\frac{1}{R_{bc}} + \frac{1}{R_{ca} + R_{ab}}\right)^{-1} \qquad [4.2]$$

and

$$R_{ca\text{-total}} = \left(\frac{1}{R_{ca}} + \frac{1}{R_{ab} + R_{bc}}\right)^{-1} \qquad [4.3]$$

respectively.

Since all three measurements give the same result, $R_{ab\text{-total}} = R_{bc\text{-total}} = R_{ca\text{-total}}$. From the similar equations (4.1), (4.2) and (4.3) it can easily be shown that

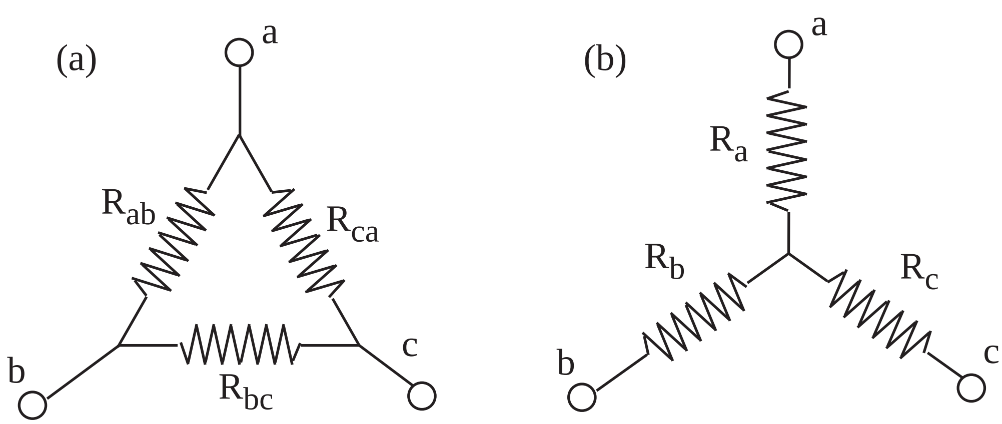
$$R_{ab} = R_{bc} = R_{ca} \equiv R_p \qquad [4.4]$$

The total resistance for each pair of terminals, measured in terms of R_p, is

$$R^{total} = \left(\frac{1}{R_p} + \frac{1}{2R_p}\right)^{-1} = \frac{2}{3} R_p \qquad [4.5]$$

and so, if R^{total} is to be 8Ω, then $R_p = 12\ \Omega$.

<u>Figure 4.1: Two black box circuits.</u>



Is the answer unique? This is the tough part of the problem (assuming you got past the 'guessing' part). Well, if the configuration above were the only solution the answer would certainly be yes. However, there is another possibility, which is shown in Fig. 4.1(b). The resistors are connected together and to each terminal is associated a single resistor named R_a, R_b, and R_c. In this circuit, a measurement across terminals ab will give the resistance of the series circuit formed by R_a and R_b. As above, it can be shown that equal measurements across ab, bc, and ca must require $R_a = R_b = R_c = R_s$ and for $R^{total} = 8\ \Omega$, $R_s = 4\ \Omega$.

Wesley, that impetuous fellow, immediately thought that the answer was unique. Too bad. Shows that one should put one's mind into gear before thinking. Would you have been able to confidently answer this question?

5. Nonlinear electronics

a) For a linear device, $R = V / I$. But this is not a linear device, so this relationship is not useful. A more general definition is $R_{effective} = dV/dI$, the local rate of change of voltage with current — the slope of the tangent. When the device *is* linear, V vs. I is a straight line through the origin, so this definition comes back to $R=V/I$ then.

Even if you have no calculus experience, you can find an intuitive solution for $V = 10$ V and $V = 4$ V. At both these points, an increase or decrease in voltage corresponds to *no* change in current, *i.e.*, R_{eff} is huge, approaches ∞.

b) We do not know the voltage drop across either component or the current. Start by listing the unknowns.

V_R :	Voltage accross resistor
$V_?$:	Voltage across unknown device
i_R:	Current through resistor
$i_?$:	Current through unknown device

Right away, we know

$i_R = I_?$ *the current flows through one then the other*

$V_R + V_? = 10$ Volts

$V_R = i_R \bullet R$ *the resistor is a linear device*

$I_? = F(V_?)$ where F() is the function shown in fig. 2, PS#1.

So we have 4 equations with 4 unknowns. Thus we can find a solution(s).

Start with $V_R = i_R R$

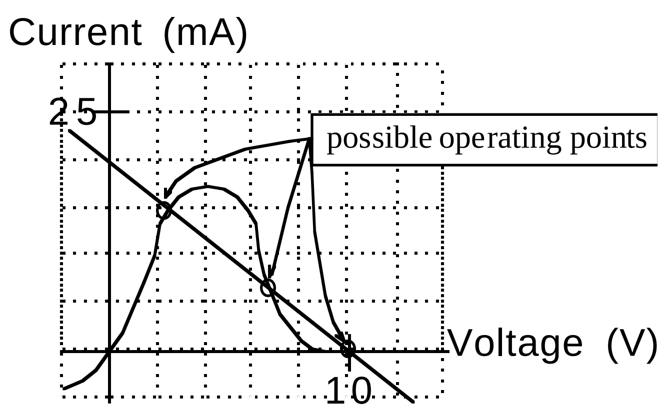
but $V_R = 10 - V_?$

$\therefore\ 10 - V_? = i_R R$

but $i_R = I_?$

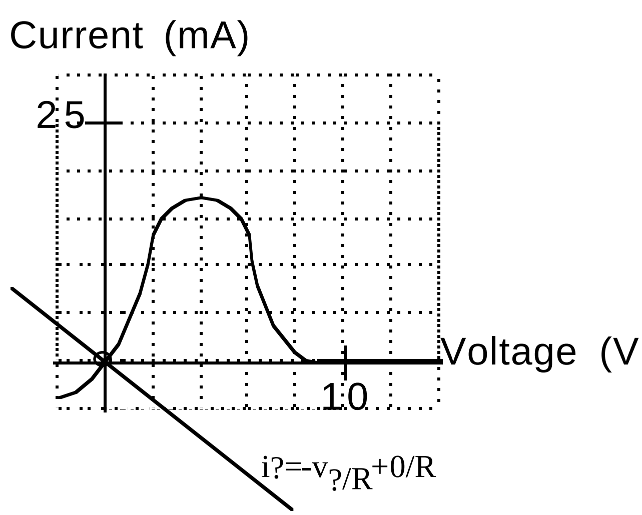
$\therefore\ 10 - V_? = I_? R \quad \Rightarrow \quad i_? = - V_?/R + 10/R$

Now we have 2 equations with 2 unknowns, but we must solve it graphically. The intersections between the two lines are possible operating points.

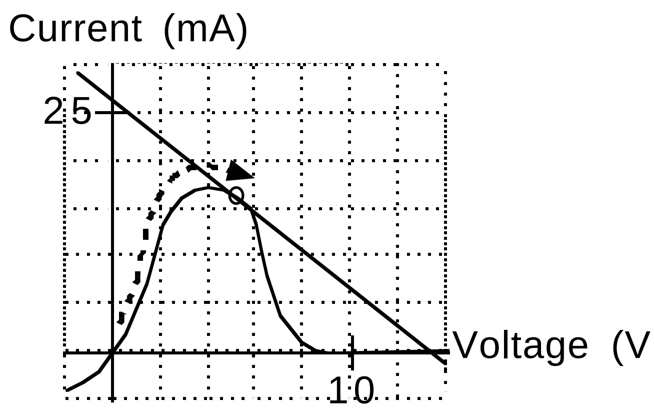


The actual voltage/current across/through the unknown device depends on how Marie-Eve started the circuit (as shall be discussed in the section below).

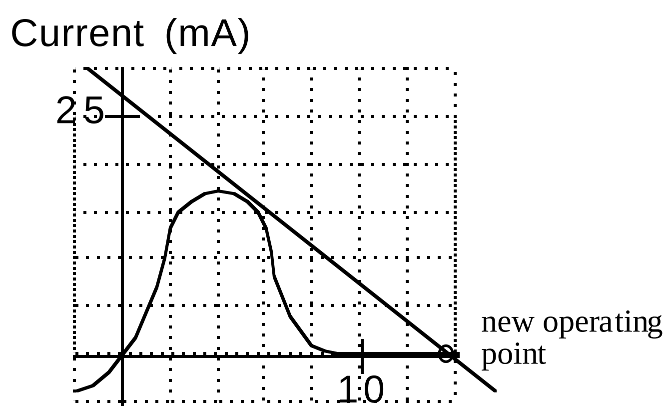
c) At V = 0 V



This line moves as Marie-Eve increases the voltage. Since we assume there is no noise in the system, the operating point moves up the curve:

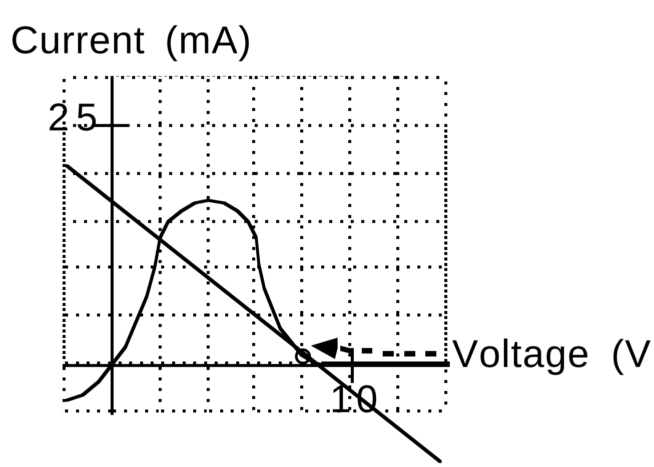


At this point ($V_{var.\ P.s.} = 13$ V), the operating point jumps to

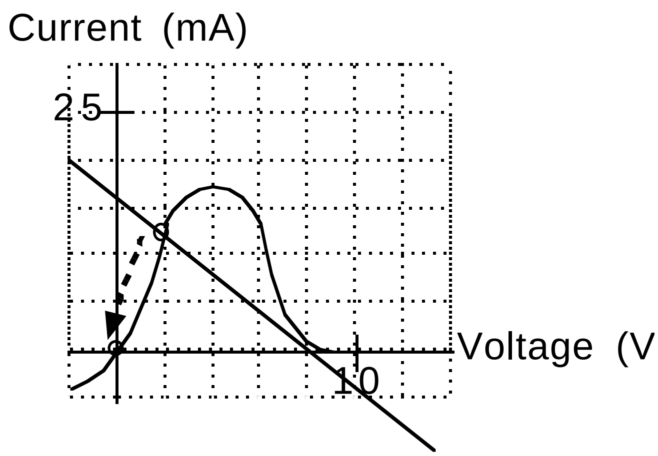


$V_{var\ p.s.}$ continues up to 15 V with $I_{?} = 0$A

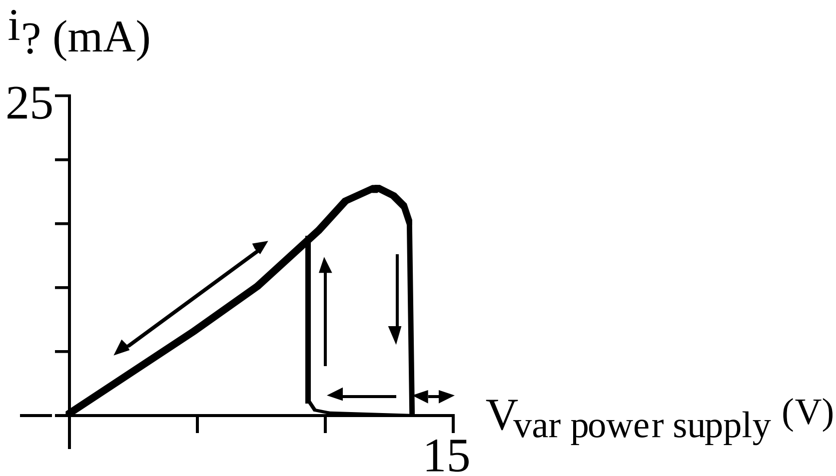
As Marie-Eve decreases from 15 V:



At this point ($V_{var\ p.s.} = 9V$), the operating point jumps to:



Let's put this all together.



This is an excellent case of hysteresis!

6. TP Pull

Comment: see the POPTOR web pages, under Problem Set Extras for a photograph of such a roll-dispenser. Also a friction-lock, using stacks of many plates to increase the drag torque, in the quick-release adjustments of an adjustable office-chair.

a) Knowing the normal force on each of two faces of the roll, we can find the total frictional (drag) force:

$$\left.\begin{array}{r} F_n = 13N \\ \mu_s = 0.27 \end{array}\right\} \Rightarrow F_D = \mu_s F_N = 0.27 \times 13N$$

$$= 3.51N \text{ @ each end}$$

The question is properly a matter of **torques**, since it is *rotational* forces

Torque from drag: $T_D = 2 \cdot F_D \cdot r_o = 2.351 \text{ N} \cdot 2 \text{ cm}$
$= 14.0 \text{ N}\cdot\text{cm} \qquad (0.14 \text{ N}\cdot\text{m})$

Torque from pulling: $T_{pull} = F_{pull} \cdot b$ where $F_{pull} \leq 3.5$ N. $T_{pull} = T_{drag}$ while pulling, and T_{drag} is constant in the static case (i.e. testing from rest at different 'b')

So, $14.0 \text{ N}\cdot\text{cm} = F_{pull} \cdot b$ unless it breaks b; b is minimized where F_{pull} is *maximized;* the last b for not breaking is

$$\boxed{b = \frac{14.0 \text{N}\cdot\text{cm}}{3.5\text{N}} = 4 \text{ cm}}$$

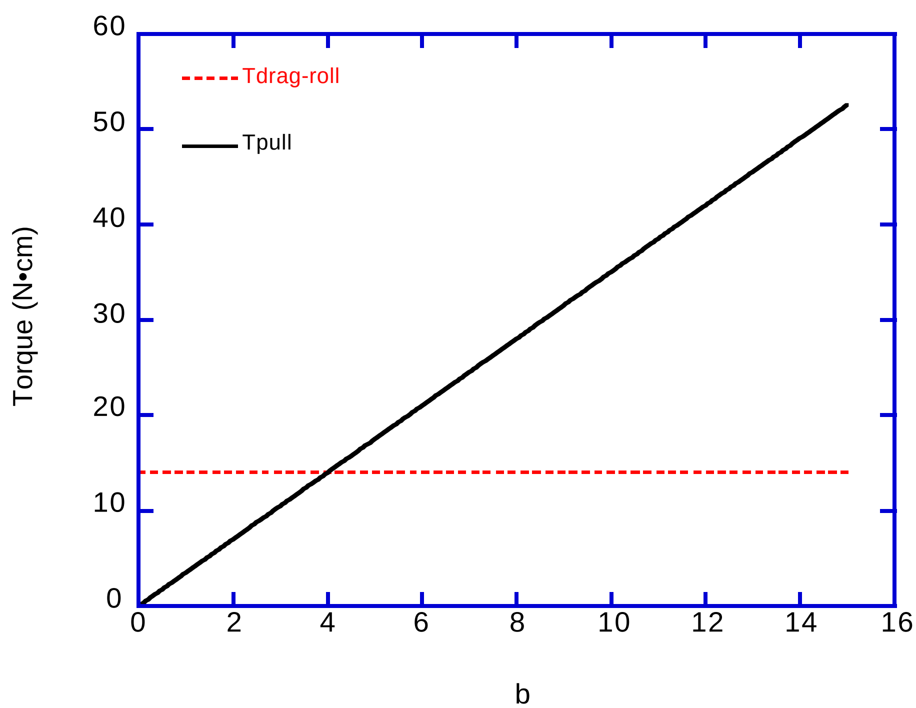


Figure 6.1: Comparing the constant drag-torque from the friction plates on the cardboard roll (red dashed line) to the maximum-deliverable pulling-torque, as the roll size increases (black solid line). The paper is pulled only with as much force as is necessary to overcome the drag-torque, but as the roll becomes smaller, even the maximum tension on the paper (burst-strength) is not enough, and the paper breaks at b = 4 cm, as the lines cross.

b) $T_D = 2\, \mu_S F_N \cdot r_o = 2\,(0.18)\, 13\, (2) = 9.4 \text{ N} \cdot \text{cm} \; (0.094 \text{ N} \cdot \text{m})$

$T_{pull} = F_{pull} \cdot b$

$$b = \frac{9.4 N \bullet cm}{3.5 N} = 2.7 \text{ cm}$$

$$\boxed{T_D = T_{pull} \Rightarrow \text{ 2.7 cm, i.e. 0.7 cm left on roll.}}$$

c) Approximate, rather than finding the spiral length from an integral: consider separate layers, starting from the core (not a perfect spiral anyway ...)

length layer = $2\pi r$ 2 cm $\leq r \leq$ 5.75 cm (d/2)
increment in r is Δr = 1mm/(10 layers) = 0.1 mm layer^{-1}
= 0.01 layer^{-1}
Thus we have (4 cm – 2 cm) * 100 layers cm^{-1} = 200 layers

$$\sum_{n=0}^{200} 2\pi(r_o + n\,\Delta r)$$

where Δr = 0.01 cm. Using $\sum_{n=1}^{j} n = \frac{1}{2} j(j+1)$ to simplify, we get

$$= \sum_{n=0}^{200} 2\pi_o + 2\pi\,\Delta r \sum_{n=0}^{200} n$$

$= 200(2\pi r_o) + 2\pi\,\Delta r\,(200(201)/2)$
$= 200\ 2\pi\ 2$ cm $+ 2\pi\ 0.01\ (100)\ (201)$ cm = 3776 cm
= 37.76 m

Likewise for b = 2.7 cm (2.7 - 2) * 100 = 70 layers

length = $70\ (2\pi r_o) + 2\pi\,\Delta r\ (70(71)/2)$
= $70\ (2\pi)\ 2$ cm + $2\pi\ 0.01 \bullet 35 \bullet 71$ cm = 1036 cm
= 10.36 m

Compare this with the answer you can get by dividing the area of the side of the roll by the thickness of a single layer — giving a length. A = ($\pi b^2 - \pi r_o^2$), Δr = 0.01 cm; for b = 4 cm, A = 37.70 cm^2, so A/Δr = 37.70 m; for b = 2.7 cm, A = 10.34 cm^2, so A/Δr = 10.34 m. This is probably just as good as the approximation above (thanks, Gordon Cook!)

d) In this case the frictional force is the same (area increases but normal **pressure** decreases) BUT the torque changes. The torque changes because the frictional force is applied at a changing **effective** radius.

To best do the problem requires adding up all the torques contributed by all the **similar** regions (r = const).

Plan 'A': If you can **integrate**, it is not hard:

Consider a roll of diameter b, and core size r_o. With a normal force F_N evenly distributed over this area, the force per unit area is $P_N = F_N/(\pi b^2 - \pi r_o^2)$. Then for an annulus (ring) very thin of width dr set at radius r, the force on the annulus is

$$dF_{drag} = \mu_s P_N dA = \mu_s P_N 2\pi r dr$$

then the torque-contribution due to this annulus is

$$dT_{drag} = r \cdot dF_{drag} = \mu_s P_N 2\pi r^2 dr$$

The whole drag-torque on each side is then the sum of torques over all annuli

$$\frac{1}{2} T_{drag} = \int_{T(r_o)}^{T(b)} d\,T_{drag} = \int_{r_o}^{b} \mu_s \; P_N$$

$$2\pi^2 \; dr$$

$$= \mu_S \; P_N \; 2\pi \int_{r_o}^{b} r^2 \; dr = \mu_S \; P_N$$

$$2\pi \left[\frac{1}{3} r^3\right]_{r_o}^{b}$$

$$= \mu_s \; P_N \frac{2\pi}{3}(b^3 - r_o^3)$$

So, since $F_n = P_n \; \pi(b^2 - r\;o^2) = P_n \; \pi(b - r_o)(b + r_o)$, then we can write $\pi P_n(b - r_o)$ $= \dfrac{F_n}{(b + r_o)}$ and substitute for P_N

$$\boxed{\begin{aligned} T_{drag} &= \frac{4}{3} \; \mu_S \; F_N \frac{(b^3 - r_o^3)}{b^2 - r_o^3} \\ &= \frac{4}{3} \; \mu_S \; F_N \frac{(b^2 + br_o + r_o^2)}{(b + r_o)} \end{aligned}}$$

now as you can see, the drag torque is not constant but scales with the outer diameter of the roll, b. To find out if the paper breaks — ever, never, or always — you have to compare this drag torque to the pulling torque you can produce from the maximum (breaking) tension of the paper applied as a force at the radius b of the roll.

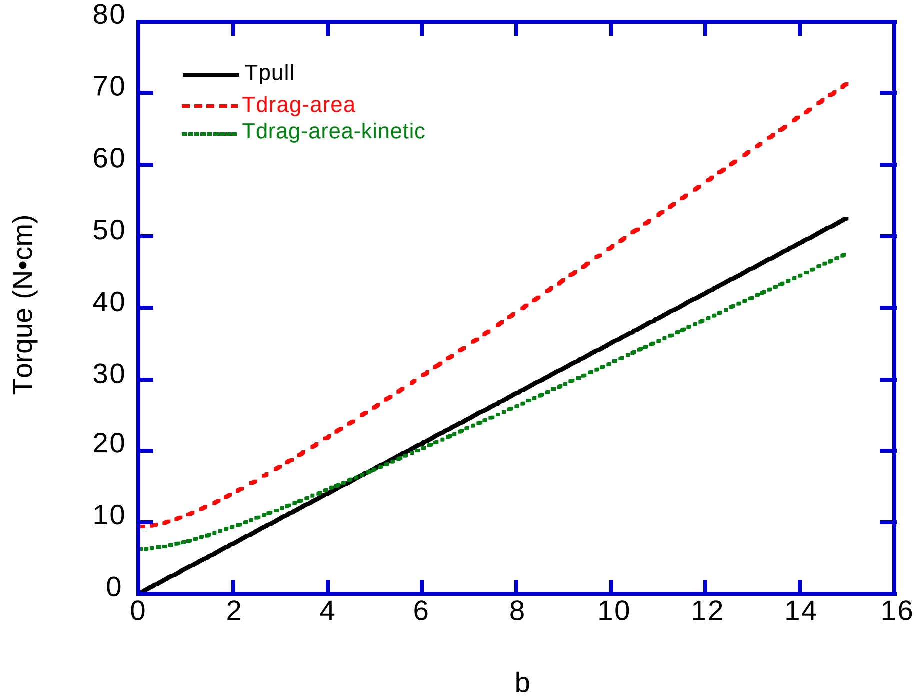


Figure 6.2: Comparing the *changing* drag-torque from the friction plates on the decreasing roll (dashed lines: red–μ_{static}, green–$\mu_{kinetic}$) to the maximum-deliverable pulling-torque (solid black line), as the roll size increases. In the static case, the maximum pulling torque is never enough — the paper always breaks; with a rolling start (the kinetic case), there can be barely a solution for a little while until the paper roll becomes too small.

Plan 'B': If you do not yet integrate, you can also do it by **series summation**:

$$\Delta F_{drag}(r) = \mu_s \; P_N \, \Delta(\text{Area}) = \mu_s P_N 2\pi r \Delta r$$

the torque about the cylinder axis from just this ring is then

$$\Delta T_{drag}(r) = r \cdot \Delta \, F_{drag} = \mu_s P_N \, 2\pi r^2 \Delta r$$

we can take each ring of width Δr to be an additional layer, then $\Delta r = 0.01$ cm as in part (c) above. Then for a diameter b there are $(b - r_o)/\Delta r$ layers, each contributing a torque from the drag at its particular radius; adding all the torques up on **one** side gives us **half** the **total** drag torque, so

$$\frac{1}{2}T_{drag} = \Delta T_{drag}(r_o) + \Delta T_{drag}(r_o + \Delta r) + \Delta T_{drag}(r_o + 2\Delta r) + \ldots$$

$$= \sum_{n=1}^{(b-r_o)/\Delta r} \Delta T_{drag}(r_o + n\Delta r),$$

$$= \sum_{n=1}^{(b-r_o)/\Delta r} \mu_s \; P_N \; 2\pi(r_o + n\Delta r)^2 \; \Delta r,$$

$$= \mu_s \; P_N \; 2\pi\Delta r \sum_{n=1}^{(b-r_o)/\Delta r} (r_o + n\Delta r)^2,$$

$$= \mu_s \; P_N \; 2\pi\Delta r \sum_{n=1}^{(b-r_o)/\Delta r} (r_o^2 + 2r_o \Delta r \bullet n + (\Delta r)^2 n^2),$$

$$= \mu_S \; P_N \; 2\pi\Delta r \sum_{n=1}^{(b-r_o)/\Delta r} r_o^2 + \sum 2r_o \Delta r \bullet n + \sum(\Delta r)^2 \; n^2$$

$$= \mu_S \; P_N \; 2\pi\Delta r \{ r_o^2 \sum 1 + 2r_o \Delta r \sum n + (\Delta r)^2 \sum n^2 \}$$

Now

$$\boxed{\begin{aligned} \sum_{n=1}^{N} 1 &= N \\ \sum_{n=1}^{N} n &= \frac{N(N+1)}{2} \\ \sum_{n=1}^{N} n^2 &= \frac{N(N+1)(2N+1)}{6} \end{aligned}}$$

where N is number of layers = 100 (b - r_o), n is layer index-number. So

$$T_{drag} = 4\pi\mu_s P_N \Delta r \{r_o^2 N + 2 r_o \Delta r \frac{N(N+1)}{2} + (\Delta r)^2 \frac{N(N+1)(2N+1)}{6}\}$$

where $N = (b - r_o)/\Delta r$
$\Delta r = 0.01$ cm
b = diameter of roll (cm)
r_o diameter of core (cm)

If now you let the layers become very thin, so that $\Delta r \to 0$ and $N \to \infty$, this becomes

$$T_{drag} = 4\pi\mu_S \; P_N \; r_o^2 (N\Delta r) + r_o(N\Delta r)^2 + \frac{1}{6} \bullet 2(N\Delta r)^3$$

with $N\Delta r = (b - r_o)$,

$$= 4\pi\mu_S \quad P \quad n\, r_o^2\,(b - r_o) \quad + r_o(b - r_o)^2 \quad + \frac{1}{3}(b - r_o)^3$$

$$= \frac{4}{3}\,\pi\mu_s \quad P_N(b - r_o)^3 \quad + 3r_o(b - r_o)\ \ 2 + 3r_o^2(b - r_o)$$

$$= \frac{4}{3}\,\pi\mu_s \quad P_n\,(b - r_o)\,(b - r_o)^2 \quad + 3r_o(b - r_o) \quad + 3r_o^2$$

Then with $\pi P_n(b - r_o) \;= \dfrac{F_n}{(b + r_o)}$ as above,

$$\boxed{\begin{aligned} T_{drag} &= \frac{4}{3}\mu_s\,F_n\,\frac{(b - r_o)^2 + 3r_o(b - r_o) + 3r_o^2}{(b + r_o)} \\ &= \frac{4}{3}\mu_s\,F_n\,\frac{(b^2 + br_o + r_o^2)}{(b + r_o)} \end{aligned}}$$

The **same answer** as the integral, because this **is** in fact doing the integral from first principles.

1996-1997 Physics Olympiad Preparation Program

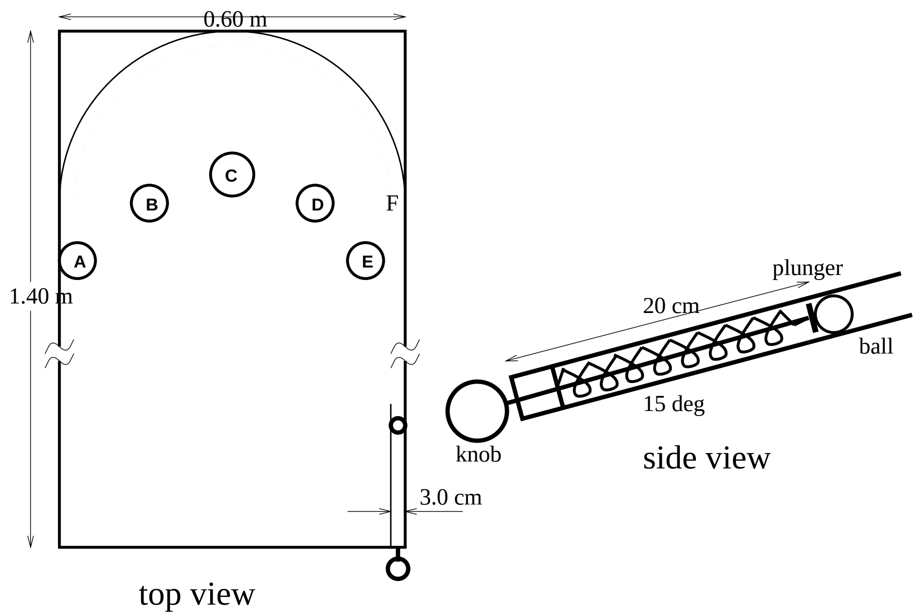
— University of Toronto —

Problem Set 2: Mechanics

Due December 13, 1996

1) Pinball pastimes

A pinball machine is one of the ancient 'arcade' games (see figure). As a good pinballer, you do not need to pull the spring knob too hard. In this machine, the table is tilted at 15°, and the ball has a diameter of 3.0 cm and a mass of 100 g. The pinball-plunger spring constant is 200 N•m^{-1}. The ball is assumed *not* to roll and the contact between the ball with the table is frictionless. Assume that all thicknesses not specifically indicated are negligibly small.



a) What is the minimum distance from the equilibrium point of the spring that one has to pull the knob so that the ball could reach the top of the semi-circle?

b) After it reaches the top, where will the ball go: bumper A, *B*, *C*, *D*, or *E*? Explain.

c) Pinball semi-wizard Elton does not pull the spring hard enough, and the ball does not even get into the semi-circle (it barely touches F) after he releases it. How far did he pull the knob? *[Chairul]*

2) Net result

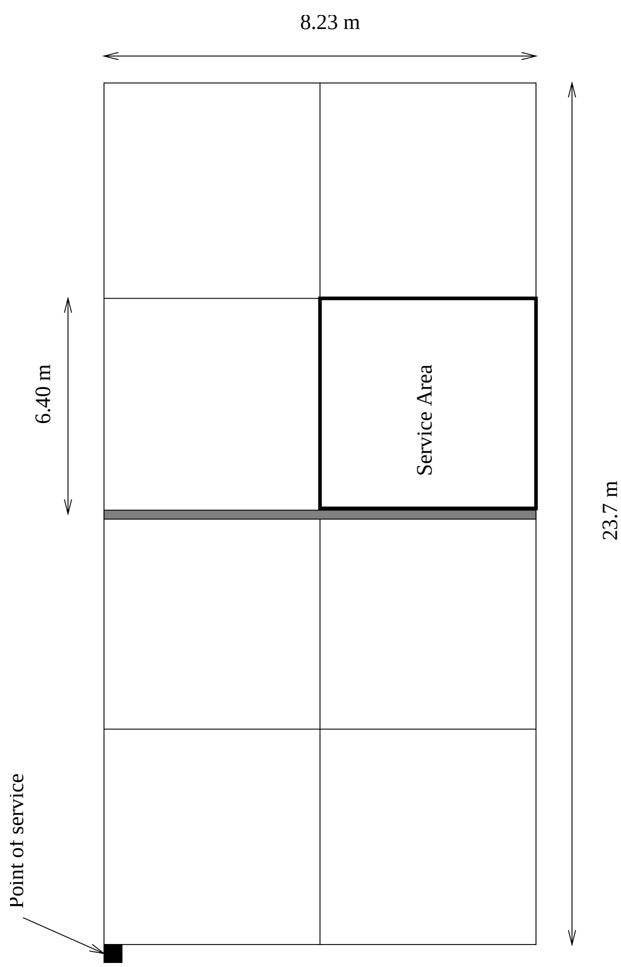
Martina is 6 feet tall and is a very good tennis player. Amanda is also a good tennis player, at 5′ 3″ tall. In tennis, a good serve can make the difference between a mediocre player and a very good one. For a serve to be a legal one, it must bounce in the service area after clearing the net (see figure). We model the service stroke of both players as a circular arc with the arm and racquet forming a straight line from the shoulder to the tip of the racquet.

Assume that they each can generate the same racquet-head speed, and further that the angular velocity remains constant throughout the stroke. Take it that the ball speed after leaving the racquet is 1.8× the racquet-head speed.

Here are Martina and Amanda's vital statistics:

Statistic	Martina	Amanda
Height	1.82m	1.60m
Height from ground to shoulder	1.57m	1.35m
Arm length	0.71m	0.56m

length of racquet from handle to middle of racquet head: 0.46m. Height of the net 0.91 m.



a) Serves must land in the service area, which includes the lines. The fastest way to get the ball past your opponent is to maximize the it's velocity component parallel to the ground, v_p. At what point in the stroke should each player hit the ball to maximize v_p? Now assume both players hit the ball at this point. If they want to hit a power serve touching the centre service line, what is the maximum v_p with which each player can hit a legal serve?

b) Contacting the ball at the top of their stroke, what is the minimum v_p with which each player can hit the ball and still clear the net? (Ignore air resistance.)

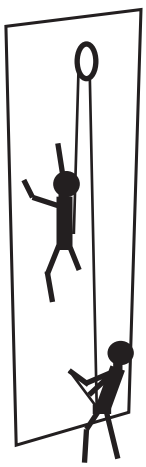
c) What is the maximum speed with which each player can hit the ball at the top of their stroke and still have it land in the service area? (Ignore air resistance.)

d) In a match, Amanda tried a trick serve in which she struck the ball early — she hit the ball with her arm at an angle of 50° to the ground. What would the angular velocity of her arm need to be in order that the ball would make it over the net into the service area? How high in the air would the ball go? (Ignore air resistance.)

e) If Amanda wants to increase the speed v_p with which she can serve the ball into the service area, explain qualitatively how she could apply spin to the ball and use it to her advantage. *[Nipun]*

3) Mind for rocks, or rocks for brains?

Some of us students here at the University of Toronto have taken up gym climbing (indoor rock climbing). For safety reasons, a rope is attached to the harness of the climber. It runs through a metal loop bolted to the top of the wall, and down to a device attached to the harness of a partner (or 'belayer') standing on the floor. A good belayer never allows any slack on the rope. For the sake of this problem assume that climbing ropes don't stretch.



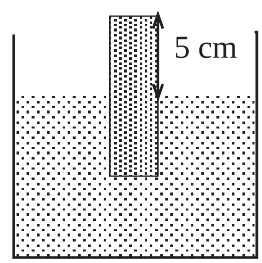
a) Assume no friction in the system, what is the maximum mass of climber with whom you would want to be partnered? Why?

b) When we started climbing, we were told that the general rule for relative masses between partners was "More than double, you're in trouble". If this is the case, what would be the coefficient of friction between the rope and the metal loop if this is the only friction in the system?

c) There is also friction between the floor and the belayer. For simplicity ignore the friction between the metal loop and the rope for this part of the question. Does standing farther away from the wall allow you to support more mass than in part (a)? Is there an optimum angle? Assume the floor friction coefficient is 1.0. *[James]*

4) Dense and denser

Diane walked into her physics lab to find her demonstrator Erica standing in front of a cylinder floating upright in a tank of a clear fluid. Diane asked her what she was doing. "I'm measuring the density of the fluid. I know that the 10 cm long cylinder has a density of 0.63 g•cm^{-3} ".



a) Diane, trying to impress her demonstrator, remarked that since the cylinder was half submerged, she knew what the fluid density was. What was her answer?

b) "Good", responded Erica," now try something more challenging. Cause the cylinder to sink farther into the fluid without touching it." Diane looked around the lab and spotted a compression chamber. "I need a force, and pressure is force per unit area, so I'll let air pressure do my work for me." She put the tank with the cylinder in the chamber and increased the air pressure to 100 atm while keeping the temperature constant. What happened to the floating cylinder? Assume that the fluid is incompressible, and that air is an ideal gas (density= 1.29 kg•m^{-3} at 1 atm).

c) What would Diane observe if she could increase the air pressure in the chamber to 1000 atm? (Keep the totally silly assumption that the gas remains ideal.) *[James]*

5) 'I crush your moon — crush crush!'

a) The moon takes about 29 1/2 days to orbit the earth, as related to the earth (the *synodic* period) or about 27 1/3 days as related to the stars (the *sidereal* period). Knowing the constant of acceleration on the surface of the earth, and the earth's radius of 6.38×10^6 m, show how to estimate how far the moon is from the earth.

b) Knowing the result of part (a), the neighbourhood tyke, O. Bray, takes a ruler out at night and uses it to figure out the approximate diameter of the moon. Explain how he might best have made his measurement. *[Robin]*

- For Hubble Space Telescope info & pictures, try *http://www.stsci.edu/public.html*
- For NASA info, pictures and teaching materials, try:

http://spacelink.msfc.nasa.gov:80/Instructional.Materials/Curriculum.Materials/Sciences/Astronomy/Our.Solar.System/

6) What's the buzz?

PART I: Sepsis is a housefly. Interestingly, when houseflies land on a ceiling, it goes something like this: they fly up close to the surface while still upright, put up their forelegs up above their head, and stick 'em to the ceiling — then they swing around that way, doing a sort of flip, until they smack all their legs onto the ceiling and they are upside down.

Take it that Sepsis weighs 0.3 gm, and that he is about 15 mm long from 'tail' to compound eyes. We need a simplified model of a fly, so take Sepsis to have legs that reach 1 cm from his centre-of-mass, and take that his main legs let him bounce about 1.5 mm on landing.

Say that Sepsis is cruising along at a good 1 m s^{-1}, when he decides to alight on the spot of the last ceiling spaghetti-test. He sticks his forelegs to the ceiling when his centre-of-mass is 7 mm below the ceiling, and pivots rigidly around that sticking-spot to a landing.

a) If you ignore any slowing-down by drag in the air, how quickly does he land — how much time between sticking his forelegs and touching down all his legs?

b) How fast is Sepsis going toward the ceiling when he finally 'sticks' his landing?

c) If he has only 1.5 mm to bounce, then he has only 1.5 mm to stop — what is the minimum acceleration Sepsis suffers at the end?

d) Given his small weight, what is the minimum force to stop Sepsis at the ceiling? What would be the force to similarly decelerate a 70 kg human in the same distance? Is that a practical force — would the ceiling hold up? Would the human?

PART II: Automobiles are made nowadays to be energy-absorbing, in the event of a crash. It means that components of the body are designed to be anything but elastic, and to be sacrificed while dissipating kinetic energy. Sometimes this is described loosely by the size of the *crumple-zone* ahead or behind the passengers of the car.

A Datsun I once met had a crumple zone of roughly 1.5 m, and about 1 m of it had been used up when it hit a concrete wall at about 30 km/hour.

e) What is the minimum acceleration needed to bring a car to rest in this distance? If the car weighs 1,000 kg, what is the minimum force required?

f) At about 50 km/hour, your lap and shoulder belts stretch considerably in a hard collision. Take a uniform deceleration, thanks to a 1.5-m crumple zone. If the belts stretch like a spring so that a passenger's centre of mass moves an additional 25 cm, then what is the overall peak force on the passenger? With a belt 5 cm wide, and contact over about 35 cm of length, what is the average pressure of the belt on the skin? *[Robin]*

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Solution Set 2: Mechanics

1. **Pinball pastimes**

We can consider the ball as a point concentrated in its centre of mass. This assumption is valid since the ball is not rolling and the contact between the ball and the table is frictionless.

a) When the ball reaches the top of the semi–circle, it already has speed v; thus, it also experiences a centripetal force:

$$mv^2/(0.285\text{ m}) = mg \sin 15° \qquad [1.1]$$

or

$$v = \sqrt{(0.285)(9.79) \sin 15°} = 0.850 \text{ ms}^{-1} \qquad [1.2]$$

Let x be the distance between the ball with the equilibrium point of the spring. Using the conservation of energy (with the equilibrium of the spring as the reference point, x = 0):

$$\tfrac{1}{2}kx^2 - mgx \sin 15° = \tfrac{1}{2}mv^2 + mg(1.170) \sin 15° \qquad [1.3]$$

or

$$100x^2 - 0.253x - 0.333 = 0 \qquad [1.4]$$

Solving this equation, one obtains x = 0.0590 m or 5.90 cm. (The other root is not used for obvious reason).

b) After it reaches the top, the ball continues to pass A since it has momentum. Due to the inertia, the ball tends to go straight but the curve bends the path. One can also get the same answer using the symmetry argument.

c) Since the ball barely touches F, its speed v = 0 and its position from the equilibrium is s = 0.870 m.

$$\tfrac{1}{2}kx^2 - mgx \sin 15° = mg(0.870) \sin 15° \qquad [1.5]$$

or

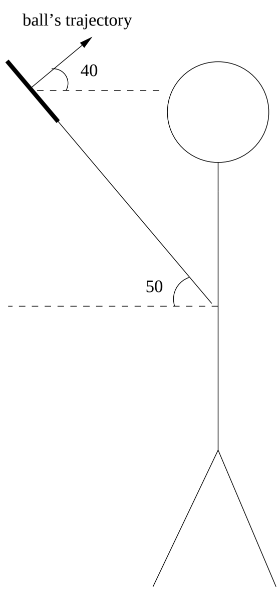
$$100x^2 - 0.253x - 0.220 = 0 \qquad [1.6]$$

Solving this equation, one obtains x = 0.0482 m or 4.82 cm.

2. **Net Result**

To maximize v_p, want to maximize the velocity component of the ball parallel to the ground. Therefore, one should hit the ball at the top of the stroke, i.e., when the racquet head is perpendicular to the ground.

a) The "power serve" hitting the centre service line with the maximum v_p (and still being a legal serve) would land at the back of the service area.



Horizontal distance travelled by ball:

$$x^2 = a^2 + b^2 = (4.12)^2 + (18.25)^2 \quad \Rightarrow x = 18.7 \text{ m}$$

vertical distance travelled by ball:

Martina: 1.57 m + 0.71 m + 0.46 m = 2.74 m
Amanda: 1.35 m + 0.56 m + 0.46 m = 2.37 m

time ball is in the air: $d = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2d}{g}}$

Martina: $t_M = 0.75$ s
Amanda: $t_A = 0.70$ s

$\therefore$ maximum v_p for:

Martina: $v = x/t_M$ = 18.7m / 0.75s = 25m/s or 90 Km/h
Amanda: $v = x/t_A$ = 18.7m / 0.70s = 26.9m/s or 97 Km/h

We, of course, have to make sure that balls with these velocities will clear the net! (see part b)

b) (Note that a more precise solution to this problem can be obtained by the methods of calculus. We sketch here an approximate solution.)

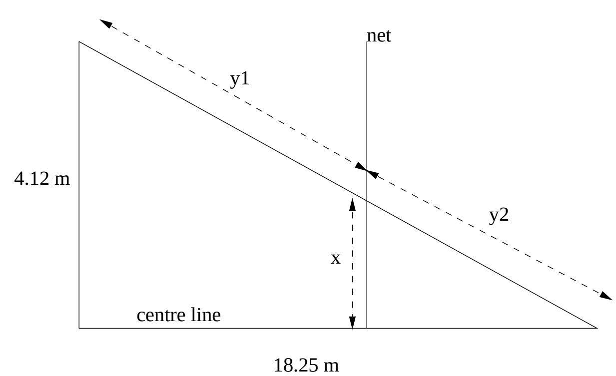
Using the same trajectory for the ball as in part (a), we need to find the distance from the server to the net.

By similar triangles:

$$\frac{4.12}{18.25} = \frac{x}{6.40} \quad \Rightarrow x = 1.44\text{m}$$

$$\therefore\ y_2^2 = (6.40)^2 + (1.44)^2 \quad \Rightarrow y_2 = 6.56\text{m}$$

$$\therefore\ y_1 = 18.7 \text{ m} - 6.56 \text{ m} = 12.14 \text{ m}$$



To find the minimum v_p for each player, time for ball to fall from point of stroke to height of the net:

$$t_{m'} = \sqrt{\frac{2(d_m - h_{net})}{g}} = \sqrt{\frac{2\ (274 - 0.91)}{g}} = 0.61\ s$$

$$t_{a'} = \sqrt{\frac{2(d_a - h_{net})}{g}} = \sqrt{\frac{2\bullet(2.73 - 0.91)}{g}} = 0.55\ s$$

v_{min}:

Martina: 12.14m / 0.61 s = 19.9 m/s or 71.5 km/h
Amanda: 12.14m / 0.55 s = 22.1 m/s or 79.5 km/h

So, the serves in part (A) are fast enough to clear the net.

c) maximum speed is for longest-possible horizontal trajectory, as

$$v_p = \frac{\text{horizontal distance}}{\text{time for ball to fall to ground}}$$

so we want the ball to land in for rear corner of service area:

Horizontal distance travelled by ball = $((8.23)^2 + (18.25)^2)^{1/2}$ = 20.0m

Maximum speed:

Martina: 20.0m / 0.75s = 26.7 m/s, or 96.0 km/h
Amanda: 20.0m / 0.70s = 28.6 m/s, or 103 km/h

d) We need to know:

1) how high above the ground the ball is when Amanda hits it
2) how much vertical distance between the release point of the ball and the top of the net — gives time available to clear net
3) the horizontal velocity component of the ball needed to reach the net at the closest distance to the point of service such that the ball will land in the service area. (See part b)

1) ball height at release point: 1.35 + (0.56 + 0.46) sin 50° = 2.13 m
2) d = (2.13 m – h_{net}) = 1.22 m
∴ Time available to clear net = $(2d/g)^{1/2}$ = 0.50 s

3) From part (b): distance = 12.14 m
$v_{horizontal}$ = 12.14m / 0.50s = 24.3 m/s
v_{total} (tangential to the arc of the racquet): $v_{horizontal} = v_{total} \cos 40°$
v_{total} = 31.7 m

Racquet head speed × 1.8 = ball speed
$v_{racquet}$ = 17.6 m/s

angular velocity of arm = ω = v / r = 17.6 / (0.56+0.46) = 17.3 rad/s

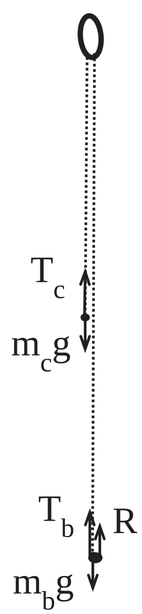
vertical velocity of ball: $v_y = v_{total} \sin 40^\circ = 20.37$ m/s
So, the maximum height above the ball's release point is

$$2gy = v_y^2 \quad \Rightarrow y = 21.1 \text{ m}$$

Max. height of ball = 1.22 m + 21.1 m = 22.4 m

3. **Mind for rocks, or rocks for brains?**

The system has been simplified since ropes don't stretch and so no slack is allowed on the line. This of course is not physically reasonable (climber would almost always be supported by belayer and so could never actually 'climb'!) but is a good first approximation.



a) Let 'c' indicate climber
Let 'b' indicate belayer
Let R be force by floor on belayer

Since belayer does not want to accelerate off the ground:

$$\Sigma F_b = 0$$

Since there is no friction $T_c = T_b$

$$\therefore \; T_b + R = m_b g$$
$$T_c + R = m_b g$$

Climber is also not accelerating.

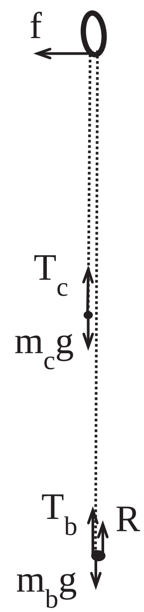
$$\therefore \; T_c = m_b g$$

Maximum m_c for a given m_b occurs for R = 0, i.e., belayer barely touching floor.
$\therefore$ as expected, max. mass of climber is your mass, as belayer
(Of course in real life, you would not want to be right on the edge of stability, but within the assumptions given the correct answer is 'maximum mass = your mass', not '< your mass' which some people stated. This is a minor point.)

b) Same pix as before except for friction on metal loop. Friction force opposes motion.
Again we are trying to find a max. m_c for a given m_b. Start by setting R = 0.

Note that forces on rope must balance
$\therefore \; m_c g = f + m_b g$
(direction of f choosen since we are opposing motion of climber dropping, not belayer dropping).

Frictional force is standard static friction, i.e.



$$f \leq f_{max}$$
$$\leq \mu_s (m_c g + m_b g)$$

For maximum m_c, set $f = \mu_s (m_c g + m_b g)$

$$\therefore \; m_c g = \mu_s (m_c g + m_b g) + m_b g$$
$$m_c g (1 - \mu_s) = m_b g (1 + \mu_s)$$
$$\therefore \; m_c = m_b (1+ \mu_s) / (1- \mu_s)$$

but question states that $m_c = 2m_b$ for maximum m_c

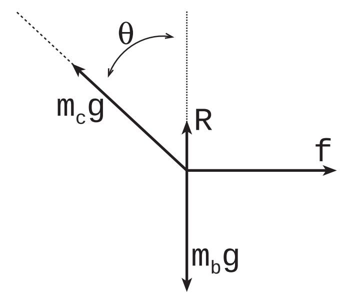
$$\therefore \; 2 = (1+ \mu_s) / (1- \mu_s)$$

$\Rightarrow$ $\mu_s = 1/3$ <— cofficient of friction between loop and rope.

c) Since there is no friction in the rope, $T_c = T_b$. Since climber does not fall $T_c = m_c g$, $\therefore$ the forces on the belayer look like:

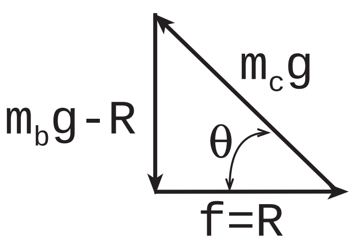
$$f \leq \mu_s R$$
$$\leq 1.0 \; R$$

For maximum m_c, $f = 1.0$ R



Since there is no acceleration, Σ F = 0. Standard way would be to break this up into horizontal and vertical forces but more elegant way is to consider vector sum of forces: (since Σ F = 0, can draw triangle)

And write: $(m_b g - R)^2 + (1.0 \; R)^2 = (m_c g)^2$



but we know: $(a + b)^2 \geq a^2 + b^2$

$$\therefore \; (m_b g - R)^2 + R^2 \leq (m_b g - R + R)^2 \leq (m_b g)^2$$
$$\therefore \; (m_b g)^2 \geq (m_c g)^2$$

$\therefore$ max. m_c is m_b

$\therefore$ friction of floor does not help!

4. Dense and denser

a) Effective buoyancy force = g ($m_{\text{displaced air}} + m_{\text{displaced fluid}}$), but the masss of the displaced air is very small, so can be ignored.

Net force = 0, $\therefore$ $g V_c / 2\rho_{fluid} = g V_c \rho_c$

$$\therefore \; \rho_{fluid} = 2 \rho_c$$
$$= 1.26 \text{ g/cm}^3$$

b) This problem explores what happens when you take a model and go to a silly extreme. You are required to make one simplification. The question states only the pressure for one point in the chamber. Of course, as the gas becomes more dense the pressure increase due to gravitational forces (ρgh) becomes more important. It is this very pressure gradient that causes the cylinder to rise. For this part of the question, since the pressure is increased only a small amount one can assume that the gas it displaces would have a minor pressure gradient across the length of the cylinder. Thus one can assume a constant density and that Diane's measurement was taken at a height similar to the height of the cylinder.

Let x be the distance that the cylinder is submerged.

Net force = 0

$$\therefore \; g\, V_c\, \rho_c = g\,(x\, V_c\, \rho_{fluid} + (1-x) V_c\, \rho_{air})$$

You cannot ignore the displaced air.

$$\rho_c = (l - x)\,\rho_{air} + x\,\rho_{fluid}$$

ρ_{air} is calculated from PV=nRT .

$$\therefore \; x = 4.4 \text{ cm}$$

Cylinder moves up 6 mm.

Note: cylinder floats **higher** in fluid. Diane should have **reduced** air pressure, not increased it.

c) Cylinder + fluid 'float' to top of chamber.

5. **'I crush your moon – crush crush!'**

a) Knowing the period of the moon's orbit, we can figure out its distance from the earth, because the period depends on the orbit radius *r* and orbital speed v:

$$T_{orbit} = \frac{distance}{speed} = \frac{2\pi r_{M,m}}{\mathrm{v}} \qquad r_{M,m} = \text{earth-moon separation, v = speed of moon} \qquad [5.1]$$

The orbital speed and radius are also related by the centripetal-force formula:

$$F_{Mm} = \frac{m\mathrm{v}^2}{r_{M,m}} \qquad m = \text{mass of moon} \qquad [5.2]$$

Where the centripetal force is gravitational attraction:

$$F_{Mm} = \frac{GMm}{r_{M,m}^2} \qquad M = \text{mass of earth, } m = \text{mass of moon, } G = \text{gravitational const.} \qquad [5.3]$$

Equations [5.2] and [5.3] both give the force, so with right-hand-sides equal, we can eliminate the moon mass m, to get:

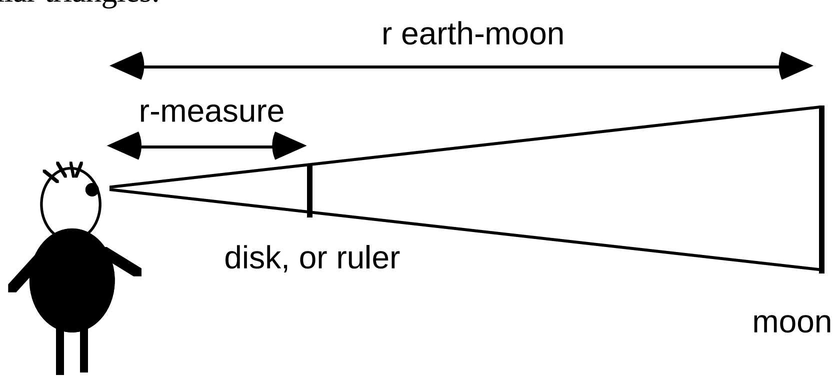
$$\frac{GM}{r_{M,m}} = v^2 = \left(\frac{2\pi r_{M,m}}{T_{orbit}}\right)^2,$$ where the last equality uses [5.1].

Then we have

$$r_{M,m} = \left(\frac{GM\,T_{orbit}^2}{4\pi^2}\right)^{1/3}$$

For G = 6.67×10^{-11} N m^2 kg^{-2}, M = 5.98×10^{24} kg, $T_{orbital}$ = 27.333 days * (8.64×10^4 seconds/day) = 2.36×10^6 s, this gives $r_{M,m}$ = 3.83×10^8 m (*cf.* 3.844×10^8 m).

b) Ah, now that we have the distance $r_{M,m}$ we can figure out the diameter of the moon by constructing similar triangles:



By holding a small paper disk, and moving it toward and away from the eye until it exactly covers the moon, then recording both the disk size d_{disk} and the distance held away from the eye $r_{measure}$, the earth-moon distance from (a) can be used to figure out the moon's size:

$$\frac{d_{moon}}{r_{M,m}} = \frac{d_{disk}}{r_{measure}}$$ or finally

$$\boxed{d_{moon} = d_{disk}\frac{r_{M,m}}{r_{measure}}}$$

When I did this, I found roughly d_{disk} = 5 mm at $r_{measure}$ = 50 cm, and so d_{moon} = (0.5/50)3.83×10^8 m = 3.83×10^6 m. Compare this with a reference-table value of 3.476×10^6 m.

<u>Small Shortcut:</u> Instead of using the values of G and of M, you might note that at the earth's surface ($r_{earth,m} = r_{earth}$) we must have $F_{Mm} = m\,g$, where g = 9.8 kg m^{-1} s^{-2}, so:

$$F_{Mm} = \frac{GMm}{r_{M,m}^2} = mg\frac{r_{earth}^2}{r_{M,m}^2},\ i.e.,\ GM = gr_{earth}^2, \text{ constants easier to remember} \qquad [5.4]$$

Synodic vs. sideral: How *should* you measure a complete rotation of something in space? Suppose the earth did not rotate at all. If it made one orbit of the sun, every spot on earth would have daytime and night-time, both, so the earth would see one revolution or one day in terms of the sun, though in terms of the 'fixed' stars it had not rotated (no sidereal days at all). So the *synodic* and *sidereal* periods for the earth are not the same. The same holds for the *moon* and its orbit of the earth.

As the moon orbits the earth, the earth orbits the sun. If the moon 'start-finish line' for an orbit is an imaginary line between the earth and some very distant star, then it will orbit and come back to this line again in a *sideral period*. If the start-finish line is a line between the earth and the sun, then while the moon is orbiting, this line will change along with the *earth's* orbiting, so the finishing-line also will move through some angle in space. With a retreating finish-line, a complete revolution in these terms will take longer.

(Of course, during the time it takes to catch up, the line will have moved a little bit again, making a bit more to catch up, and so on, and so on. The problem ends up like figuring out from 8:00 pm exactly when the minute hand ends up opposite the hour hand: in 10 minutes it is opposite where it *used* to be, but the hour hand has moved an angle 1/12th as much as the minute hand just moved. So the minute hand reaches the *corrected* place 1/12 of 10 minutes later, but then the hour hand in that time moved 1/12th of 1/12th of 10 minutes... The answer can be found using the sum of the series $(1/12)^n$, which has a well-known formula.)

6. What's the buzz?

a) Need some approximation

$$\text{angle } \theta\text{: } \frac{0.7\text{cm}}{1.0\text{cm}} = \sin\theta = 0.7$$

$$\Rightarrow \theta = 0.775 \text{ radians} = 44.4^\circ$$

Momentum: 0.3 g = 3×10^{-4} kg; mv = 2×10^{-4} kg · 1 m s^{-1} = 3×10^{-4} kg · m s^{-1}. The momentum resolves *along* the rigid rod and along the ⊥ to the rigid rod.

$$\frac{p_\perp}{p} = \frac{0.7\text{cm}}{1\text{cm}} = p_\perp = 0.7\text{ p}$$

Angular momentum is conserved; with the free-pivot action, it sees no torque and so is conserved **without** contribution from the ceiling, earth, etc..

$$L = P_\perp \bullet r = 0.7\text{ p} \bullet 10^{-2}\text{ m},$$

$$= 2.1 \times 10^{-6}\text{ kg m}^2\text{ s}^{-1}$$

It's undefined what the final angle is when he reaches the other side — he could swing around until he is parallel to the ceiling, or maybe until his 1 cm-long legs, halfway down his body, hit the roof. Either is good for the sort of rough estimates we are making on behalf of flies! You figure.

Say, take 1st case: the final angle is π radians. Then he swings through $(\pi - 0.775)$ radians = 2.37 radians or 135.6°

Arc length: $\Delta\theta \cdot r = 2.37 \text{ rads} \cdot 1 \text{ cm} = 2.37 \text{ cm} = 2.37\times 10^{-2}$ m

Speed: $v_\perp = 0.7 \cdot v = 0.7\times 1 \text{ m s}^{-1} = 0.7 \text{ m s}^{-1}$

$$t = \frac{d}{v} = \frac{2.37 \times 10^{-2}\text{m}}{0.7 \text{ ms}^{-1}} = 3.39 \times 10^{-2} \text{ s} = 33.9 \text{ ms}$$

we'll accept 15 – 35 ms, about one frame of a video camera.

b) Again depends on final angle, leg length, etc., but approx $v_\perp = 0.7 \text{ m s}^{-1}$; we'll accept answers 0.35 – 0.7 depending on final angle

$$v = 0 \Rightarrow -\frac{u^2}{2d} = a$$

c) $v^2 = u^2 + 2ad$

$$a = -\frac{(0.7)^2}{21.5 \times 10^{-3}} = \frac{0.49}{3 \times 10^{-3}} = 0.16 \times 10^3 = 1.63 \times 10^2 = 163 \text{ m s}^{-2}$$

d) $F = M_{fly} \bullet a$
$= 0.3 \times 10^{-3}$ kg (-163 m s^{-2})
$= -49 \times 10^{-3}$ kg m s^{-2}
$= -4.9 \times 10^{-2}$ kg m s^{-2}
$= -4.9 \times 10^{-2}$ N

$F = M_{human} \bullet a$
$= 70$ kg (-163 m s^{-2})
$= -1.14 \times 10^4$ kg m s^{-2}
$= -1.14 \times 10^4$ N

<u>PART II</u>

30 km/hr = 30×10^3 m / 3600 s = 8.3 ms^{-1}

e) $v^2 = u^2 + 2ad$

$v = 0 \Rightarrow$

$$a = -\frac{(8.3\text{ms}^{-1})^2}{21.5\text{m}} = -23 \text{ m s}^{-2},$$ much less than for the fly

$F = m \bullet a = (1{,}000 \text{ kg})(-23 \text{ m s}^{-2})$
$= -2.3 \times 10^4$ kg m s^{-2}

f) not very significant that belts stretch a little, if assume constant acceleration force (unless you let the passenger bounce) total distance is 1.75 m

$\Delta x = 1.75$ m

50 km/h = 50×10^3 m / 3600 s = 13.9 m s^{-1}

$$a = -\frac{u^2}{2d} = -\frac{(13.9\ \mathrm{m\ s^{-1}})^2}{2(1.75\ \mathrm{m})} = -\ 55.2\ \mathrm{m\ s^{-2}}$$

$F = m \bullet a = 80\ \mathrm{kg} \bullet (-55.2\ \mathrm{m\ s^{-2}})$
$= -\ 4.416$ kN

Then 0.35 m • 0.05 m = 0.0175 m^2 area for belt. Pressure is
(4.416 kN / 0.0175 m^2) = 2.52×10^5 Pa
[1 ATM = 1.01325×10^5 Pa]

– about 2.5× atmospheric pressure; can bruise heavily where the belt cuts in
– or, say my fist is 7 cm × 9 cm = 63 cm^2 = 0.0063 m^2, so to produce the same pressure I would need a force of about 1.6 kN, the weight of about 162 kg — much more than I weigh, and probably a really decent heavyweight prizefighter punch if the followthrough is strong.

Or, to compare another way, the whole force on the person is F = 4.416 kN corresponds to the weight of ~450 kg – almost half the weight of a small car, so this is a serious force. How long does it last?

$$t = -\left(\frac{13.9\ \mathrm{m\ s^{-1}}}{2(-55.2)\ \mathrm{m\ s^{-2}}}\right) = 0.126\ \mathrm{s} = 126\ \mathrm{ms}$$

Perhaps this is something like being directly hit by a motorcycle? You can hit the belt, or you can hit the windshield, but much better the belt! Buckle up — it saves lives.

1996-1997 Physics Olympiad Preparation Program

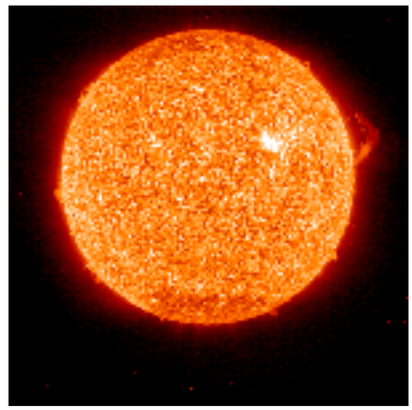
— University of Toronto —

Problem Set 3: Thermodynamics

Due January 13, 1997

1) Great balls o' fire!

The sun may be modelled as a ball of hydrogen gas. Assume its main energy source is the conversion of hydrogen (atomic weight = 1.0078 amu) to helium (atomic weight = 4.0026 amu) roughly by the process 4H → He (the real process is much more intricate than this). The energy emitted is given by the famous Einstein equation, $E = \Delta m\, c^2$, where c is the speed of light.



EIT image in the He II emission line at 304Å, upper chromosphere/lower transition region, ~60,000 K. *http://sohowww.nascom.nasa.gov /gallery/EIT/index.html*

a) How much energy is given off during the life of the sun if only 10% of its hydrogen is converted to helium?

b) The power radiated from the sun and arriving at the earth has a density today equal to 1400 W•m^{-2}. If we assume that this rate is constant throughout the sun's life, estimate the life expectancy of the sun. Is this value plausible? *[Nipun]*

2) Running *Amok*

The *Amok IV* space probe's initial design included no insulation or shielding. The electronics require a minimum operating temperature of 200K. Since it is a deep-space probe the engineers had to provide it with a heat source.

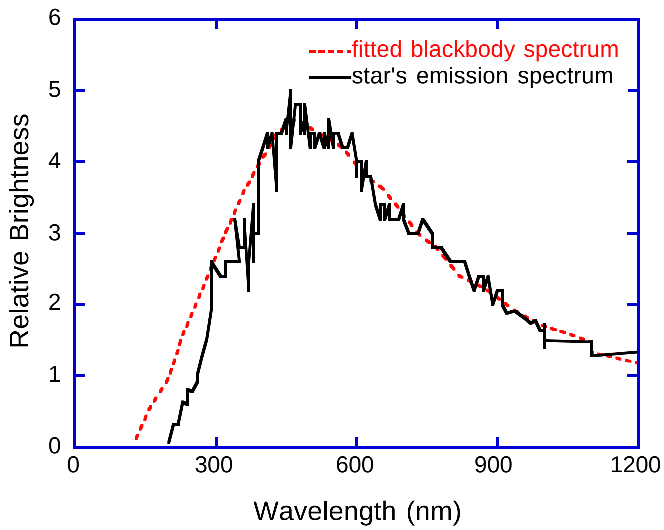
a) What would be the minimum power requirements for the heat source for a black spherical satellite of radius 2 m? Assume that outer space is really,really cold.

b) To reduce heating bills, the engineers decided to put a nice flat black outer layer around the satellite and stuff it with *Fibreglass Fuchsia* insulation. This insulation has an R value in vacuum of 18.8 ft^2•F°•h•BTU^{-1}. (Believe it or not, this is the unit often used for insulation, even in our SI-enlightened society.) What is the new power requirement to keep the electronics working properly? The radius of the shield is only slightly larger than the radius of the probe. (1 BTU=1.054 kJ, 1ft = 30.48 cm, 1F° = 5/9 K)

c) Unfortunately, Alain the co-op student forgot to pick up the insulation before the launch, and somehow also the outer layer never got painted. Assuming that the emissivity of the silver shield was 0.2, what was the steady-state temperature of the interior of the satellite? *[James]*

3) Too blue to be true?

Light from a star has the broad spectrum pictured below, and it fits a blackbody distribution pretty well over much of the frequency range recorded.



a) From the distribution, what is the temperature of this star?

b) Travelling in a spacecraft toward this star at 0.1 c, all the frequencies of the light from the star will be Doppler shifted, and so the peak of the spectrum shifts. By Wien's displacement law, λ_{max} [nm] = 2.8978×10^6 / T [K], what is the new recorded temperature of the star? Present your arguments about whether or not this is really a new temperature for the star, as measured in the frame of reference of the traveller. *[Robin]*

4) Huddle up!

Edna Hillary and her team have set up a research station in the antarctic. Their base is a structure of dimensions, 20×20×3 metres. While they have brought along with them many of the conveniences of home, they have overlooked one very important detail — they forgot to insulate the walls of their cabin from the cold! As soon as they set up, they powered up their generator to heat the room to a cozy 20° C and then shut the generator off, only to find that the temperature then dropped at a rate of 1° C every 5 minutes!

a) Assume that the research team is well-insulated (they're all wearing parkas), so that the heat capacity inside the cabin is due only to the air. Furthermore, treat the air as a diatomic ideal gas. If the air pressure is 1 atmosphere, how much power would the generator have to emit to stabilize the temperature?

b) Bicycle pumps get hot by adiabatic compression of air: if they wanted to try and stabilize the temperature by compressing the volume of the cabin, at what rate would they have to 'shrink' the cabin. Express your answer in $m^3 \bullet s^{-1}$. *[Nipun]*

5) They chilled out and got it together

The first realization of Bose-Einstein condensation was achieved last year by researchers in Colorado. By cooling rubidium atoms to extremely low temperatures, they underwent a phase change resulting in the group of atoms condensing into one single state — a kind of 'superatom'. Much work has been done in the quest to create this new form of matter since it was originally proposed by Albert Einstein and Satyendra Nath Bose. A rough understanding of this new state can be understood in the following way: de Broglie proposed that all matter can be viewed as waves, with wavelength: $\lambda = h / mv$. Bose-Einstein condensation occurs when the de Broglie wavelength exceeds the interatomic spacing. Effectively, the atoms overlap in space.

a) To begin, estimate the wavelength for a car speeding down a highway. What about a nitrogen molecule at room temperature and pressure ($v_{avg} = 400$ m•s^{-1})? While you are sitting in your chair doing this problem set, is your wavelength infinitely long?

b) By recalling the equipartition theory (each degree of freedom contributes on the average $(1/2)kT$ of energy), estimate the temperature one must cool a group of rubidium atoms (ideal gas) down to achieve Bose-Einstein condensation. Assume that you start at room temperature with a pressure of 1 atm and that you maintain constant volume. You should realize that the equipartition theory does not apply well to these low energy atoms, but it is adequate to get a rough idea of the required temperature.

c) If you could reduce the atomic spacing, you wouldn't have to go to as low temperatures. Why do you think the first experiments didn't use very high pressures?

d) Obviously this new state of matter is much more ordered than your original state. Have you broken the second law of thermodynamics, which says that entropy (and randomness) must always increase? *[James]*

6) The curious case of the cold calculating counterfeiter

Svend just arrived in Toronto. When he got a 'tooney' in his change from the newstand at the Pearson airport, he got an idea, "It is a real challenge for me, I will duplicate it." Svend was very good at counterfeiting coins. So he bought two plates of metal: brass and nickel with the same thickness, 2.00 mm. Cooling them to liquid nitrogen temperature (77 K), he drilled a hole in the nickel plate in and fitted it with a brass centre-plug. The diameter of the hole was 12.00 mm and the outer diameter of the nickel is 28.00 mm. He made sure that the brass could barely fit into the nickel while cold.

a) What is the approximate pressure between the two metals at room temperature (293 K)? Assume the hole expands in the same proportion as the rest of the plate.

b) Would police catch Svend if he used this 'tooney' based on its weight? Compare with an actual one. (Measure one yourself!)

(Note: the thermal coefficients of linear expansion brass and nickel are 19×10^{-6} K^{-1} and 11×10^{-6} K^{-1}; their densities at room temperature 8.87 g•cm^{-3} and 8.85 g•cm^{-3}; their Young's moduli 9.1×10^{10} N • m^{-2} and 19.0×10^{10} N • m^{-2}). *[Chairul]*

1996-1997 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 3: Thermodynamics

1. **Great balls o' fire!**

a) Total mass of the sun = 1.99×10^{30} kg

If sun is made of hydrogen, then amount of hydrogen converted to helium = 10% of (1.99×10^{30} kg) = 1.99×10^{29} kg

For reaction $4H \to He$, mass loss is:

$$\Delta m = 4 \times \text{mass of H} - \text{mass He} = 4(1.0078 \text{ amu}) - 4.0026 \text{ amu}$$
$$= 0.0286 \text{ amu} = 4.75 \times 10^{-29} \text{ kg}$$

Energy given off by $4H \to He$:

$$E = \Delta mc^2 = (4.75 \times 10^{-29}) \times (3.00 \times 10^8 \text{ m/s})^2$$
$$= 4.27 \times 10^{-12} \text{ J}$$

Total # of hydrogen atoms in reaction:

$$\frac{1.99 \times 10^{29}\, kg}{1.0078\, amu\,/\,atom \times 1.6606 \times 10^{-27}\, kg\,/\,amu} = 1.19 \times 10^{56}\, atoms$$

4 hydrogen per reaction = 2.97×10^{55} groups of 4 H

∴ total energy: (2.97×10^{55}) x (4.27×10^{12} J per reaction) = 1.27×10^{44} J

b) mean radius of earth's orbit = 1.49597×10^{11} m = r_e

∴ "surface area" of orbital sphere = $4Br_e^2 = 2.8 \times 10^{23}$ m^2

∴ energy flux from sun: 1400 W/m^2 (at earth's distance)

∴ total power = $1400 \times 2.81 \times 10^{23} = 3.94 \times 10^{26}$ W

∴ life of sun = $\dfrac{1.27 \times 10^{44}\text{ J}}{3.94 \times 10^{26}\text{ W}} = 3.23 \times 10^{17}\text{ s} \quad = 1.02 \times 10^{10}\text{ years}$

The life expectancy of the sun is ~10 billion years, so the calculated value is pretty good!

2. **Running Amok**

a) The probe will lose heat due to radiation. The net rate of heat loss depends on its temperature, emissivity (e.g., black, white or grey), surface area, and the background temperature.

Power radiated from a surface follows *Stefan's Law*:

$P = \sigma A e T^4$ where σ = Stefan-Boltzmann constant
A = area
e = emissivity (between 0 (white) and 1 (black)
T = temperature (Kelvin)

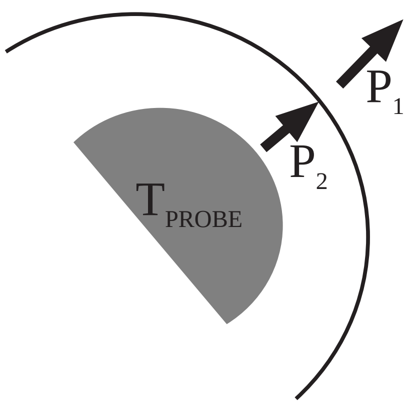
(we are neglecting background temperature since question states "outer space is really, really cold," i.e., $T_{o.s.} = 0$).

$$A = 4\pi r^2 = 50.3\ m^2$$

Since it is black, e = 1.
Minimum power radiated corresponds to minimum temperature, i.e. T_{min} = 200K

$\therefore P_{min} = 4.6$ kW

b) We must find the temperature of the shield. It is radiating power as in part (a).
Energy is being transferred from probe to shield at a rate determined by R value, T_{probe}, T_{shield}, area



At steady–state $P_1 = P_2$

$P_1 = \sigma A_{shield} \varepsilon_{shield} T^4$

$$P_2 = A \frac{\Delta T}{R} = A \frac{(T_{probe} - T_{shield})}{R}$$

$$\text{Thus } \sigma A_{shield} e_{shield} T_s^4 = \frac{A(T_{probe} - T_{shield})}{R}$$

Question states $A_{shield} = A_{probe}$
$e_{shield} = 1$

$$\text{Thus, } R = 18.8 \frac{ft^{2\circ}F\,h}{BTU} = 3.31 \frac{m^2 \cdot {}^\circ C \cdot s}{J}$$

We want minimum power requirement, thus T_{probe} = 200K
Need to solve: $T_{shield} = T_{probe} - R\,\sigma\,T_{shield}^4$ — cannot isolate T_{shield} easily...
Many ways to solve this: 1) computer or fancy calculator (HP28s or better)
2) graphically
3) by iteration, substituting each successive approximation back in

I will demonstrate method (3): numbers [x] indicate the iteration being made

1) Start by assuming $T_{shield} << T_{probe}$, thus take $T_{probe} = R\,\sigma\,T_{shield}^4$ and solve for T_{shield}:

$$T_{shield} = \left(\frac{T_{probe}}{R\sigma}\right)^{1/4} = 181\ K$$

Plug this in and find new T_{shield}, then repeat as it converges and a self-consistent value emerges:

2) $T_{shield}^{\ new} = \left(\dfrac{200-181}{R\sigma}\right)^{1/4} = 100\text{K}$

3) $T_{shield}^{\ new} = \left(\dfrac{200-100}{R\sigma}\right)^{1/4} = 152\text{K}$

4) $T_{shield}^{\ new} = \left(\dfrac{200-152}{R\sigma}\right)^{1/4} = 126\text{K}$

5) $T_{shield}^{\ new} = \left(\dfrac{200-126}{R\sigma}\right)^{1/4} = 141\text{K}$

6) $T_{shield}^{\ new} = \left(\dfrac{200-141}{R\sigma}\right)^{1/4} = 133\text{K}$

7) $T_{shield}^{\ new} = \left(\dfrac{200-133}{R\sigma}\right)^{1/4} = 137\text{K}$

8) $T_{shield}^{\ new} = \left(\dfrac{200-137}{R\sigma}\right)^{1/4} = 135\text{K}$

9) $T_{shield}^{\ new} = \left(\dfrac{200-135}{R\sigma}\right)^{1/4} = 136\text{K}$

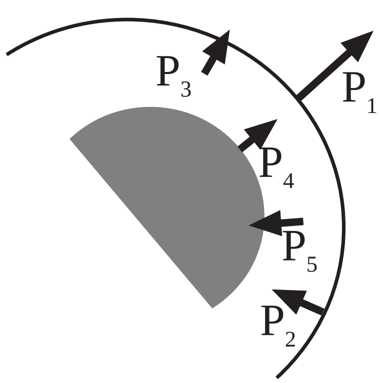
10) $T_{shield}^{\ new} = \left(\dfrac{200-136}{R\sigma}\right)^{1/4} = 136\text{K}$

$\therefore\ T_{shield} = 136\text{ K}$

$$\begin{aligned}\therefore\ P &= \sigma A e T^4 \\ &= \sigma\,(4\pi\,(2)^2)\,(1)\,(136)^4 \\ &= 0.97\text{ kW}\end{aligned}$$

This is larger than part (a) by more than a factor of 4.

c) All heat transfer here is radiative. Consider all possible radiated power processes:



P_1: from outer shield to outer environment
P_2: from inner shield to inner environment
P_3: from inner environment to shield
P_4: from probe to inner environment
P_5: from inner environment to probe

Right away we can write (‘p’ = ‘probe’, ‘s’ = ‘shield’):

$$P_1 = \sigma A e_s T_s^4$$
$$P_2 = \sigma A e_s T_s^4 = P_1$$
$$P_3 = e_s P_4$$
$$P_4 = \sigma A e_p T_p^4$$
$$P_5 = P_2 + P_4 - P_3$$

(since we are at steady state and $e_{probe} = 1$)

For shell, at steady–state:

$$P_1 + P_2 = P_3$$
$$2\sigma A e_{shield} T_{shield}^4 = e_{shield} \sigma A e_{probe} T_{probe}^4$$
$$\therefore\ 2T_{shield}^4 = T_{probe}^4$$

Power radiated out by cabin

$$= P_4 - P_5$$
$$= P_4 - P_2 - P_4 + P_3$$
$$= P_3 - P_2$$
$$= e_{shield}\, \sigma A T_{probe}^4 - \sigma A e_{shield} T_{shield}^4$$
$$= e_{shield}\, \sigma A (T_{probe}^4 - T_{shield}^4)$$
$$= e_{shield}\, \sigma A (T_{probe}^4 - 1/2 T_{probe}^4)$$

$\therefore$ power radiated by cabin: $\dfrac{e_s \sigma A}{2} T_p^4$

Power source is still the same

$$\therefore\ 0.97\ \text{kW} = \frac{e_{shield}}{2}\ \sigma\ A\ T_{probe}^4$$

$$\therefore\ T_{probe} = 242\ \text{K}$$

3. **Too blue to be true?**

a) For a black–body distribution, Wien's Law gives the peak of the distribution, according to:

$$\lambda_{max}\ T = 2.9 \times 10^{-3}\ \text{m} \cdot \text{K}$$

In the figure the peak is around $\lambda_{max} = 460$ nm, so

$$T = \frac{2.9 \times 10^{-3}\,\text{m} \cdot \text{K}}{460 \times 10^{-9}\,\text{m}} = 6{,}300\,\text{K}$$

b) The relativistic Doppler–shift means a change in frequencies:

$$\nu' = \nu \frac{1}{\sqrt{1 - v^2 / c^2}} (1 - \frac{v}{c} \cos \phi)$$

where N is the angle between the direction of the photon and the spacecraft motion in the star's frame.

Here N = 0 so ν'

$$= \nu \frac{1 - v/c}{\sqrt{1 - v^2/c^2}} = v \frac{1 - v/c}{\sqrt{(1 - v/c)(1 + v/c)}}$$
$$= \nu \sqrt{\frac{1 - v/c}{1 + v/c}}$$

with v = –0.1 c this gives $\nu' = 1.11\ \nu$

$$\nu\lambda = c \qquad \Rightarrow \lambda = \frac{c}{\nu}$$

$$\text{so} \quad \lambda' = \frac{c}{\nu'} = \frac{c}{1.11} = 0.90\frac{c}{\lambda} = 0.90\lambda$$

This means a shift in the peak to shorter wavelengths, and so, nonimally, from Wien's law, a higher temperature T' = 1.11 T = 6965 K

Is this a 'real' new temperature? Well, does the blackbody distribution still fit?

$$u_\nu = \frac{8\pi h\nu^3}{c^3[\exp(h\nu / kT - 1)]}$$

if ν' = 1.11 ν and we take T' = 1.11 then $\frac{h\nu'}{kT'} = \frac{h\nu}{kT}$, and $(\nu')^3 = 1.37\ \nu'$, so a distribution of this type still fits, with an overall factor of 1.37.

Since the overall *amplitude* of the distribution u_ν changes, it is no longer normalized properly. However, this is not really an issue in a typical real measurement — it may simply seem like a larger star radiating more power. *If* it were not a perfect blackbody but included some spectral lines, the patterns of these lines would be shifted from their usual positions, helping you guess what had happened.

In thermal terms, fitting a Maxwell-Boltzmann distribution, adding a constant velocity to all particles does not change its kinetic temperature.

4. **Huddle Up!**

a) Temperature drop: 1°C / 5 mins C_V = 5/2 R for a diatomic ideal gas ($J \cdot K^{-1} \cdot mole^{-1}$)
In finding total # of moles, let's assume we deal with the gas only, and that the volume of the researchers is negligible (probably not true)

$$\therefore \text{volume} = 20 \times 20 \times 3 = \frac{1200\text{m}^3}{1\times10^{-3}} = 1.2\times10^6\,\ell$$

$$(1\ell = 1000\ \text{cm}^3 = 1 \times 10^{-3}\ \text{m}^3)$$

at S.T.P., 1 mole of ideal gas occupies 22.4 ℓ

$$\therefore \text{at } 20\ ^\circ\text{C:} \quad 22.4\ell\left(\frac{293\text{K}}{273\text{K}}\right) = 24.04\ \ell \cdot \text{mole}^{-1}$$

Or use PV = NRT which is right for an ideal gas and often OK for a diatomic gas too. (Diatomic nature of gas doesn't change the equation of state, as diatomic molecules add to the molecules' internal energy but don't affect the entropy so the pressure isn't affected. Therefore we have the same equation of state.

$$\therefore \text{Total \# of moles:} \quad \frac{1.2\times10^6\,\ell}{24.04\ell\ /\ \text{mole}} = 4.99\times10^4\ \text{moles}$$

For a diatomic ideal gas, $C_V = \frac{5}{2}R = \frac{5}{2}k_B = 20.7\ J \cdot mole^{-1} \cdot K^{-1}$

$1°C / 300\ s = 3.33 \times 10^{-3}\ °C \cdot s^{-1}$

$\therefore$ Power $= (3.33 \times 10^{-3}\ °C \cdot s^{-1}) \times (4.99 \times 10^{4}\ mole) \times 20.7\ J \cdot mole \cdot\ °C$

$= 3.46 \times 10^{3}\ J \cdot s^{-1}$

$= 3.46 \times 10^{3}\ W$

b) **Adiabatic compression:**

For an ideal gas $pV = nRT$,

$\therefore$ at *constant temperature and particle number*,

$pV = \text{constant}$

Instead, assume the gas is thermally insulated from its surroundings (*adiabatic* conditions) Changing the volume of the gas will do work on the gas and the internal energy of gas will change, so the temperature will change. We need the relationship between volume and temperature.

From 1st law of thermodynamics, heat-energy transfer is

$$\Delta Q = \Delta E + p\Delta V \qquad (1)$$

In terms of heat capacity at constant volume, $\Delta E = n\, C_V\, \Delta T$, where n = # of moles. For an adiabatic process, there is no heat transfer, so $\Delta Q = 0$, and thus from (1)

$$0 = n\, C_V\, \Delta T + p\Delta V \qquad (2)$$

We can write the differential of pV:

$$\Delta(pV) = p(\Delta V) + (\Delta p)V \qquad (3)$$

so, $\Delta T = \dfrac{p\Delta V + V\Delta p}{nR}$

into (2): $(C_V + R)\, p\Delta V + C_V V\Delta p = 0$

Dividing by $C_V pV$:

$$\frac{(C_V + R)}{C_V}\frac{\Delta V}{V} + \frac{\Delta p}{p} = 0 \qquad (4)$$

$$\Rightarrow \quad \gamma\frac{\Delta V}{V} + \frac{\Delta p}{p} = 0 \qquad (5)$$

where $\gamma = \dfrac{C_V + R}{C_V} = \dfrac{C_p}{C_V}$

For diatomic ideal gas, $\dfrac{C_p}{C_V} = \dfrac{\frac{7}{2}R}{\frac{5}{2}R} = \dfrac{7}{5}$

Integrating equation (5), with ΔV as a differential dV:

$\gamma \ln(V) + \ln(p) = \text{constant}$

$\Rightarrow \quad pV^{\gamma} = \text{constant}$

$\Rightarrow \quad V^{\gamma-1}\, T = \text{constant}$

$\Rightarrow \quad V^{2/5}\, T = \text{constant}$ ($\gamma = 7/5$ for diatomic ideal gas)

For a change in temperature of 3.33×10^{-3} °C from 20 °C

$$V_1^{2/5} T_1 = V_2^{2/5} T_2$$

$$\therefore \frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{2/5} \Rightarrow \frac{20^\circ C}{20 - 3.33 \times 10^{-3}} = \frac{V_2^{\,2/5}}{(1200 m^3)^{2/5}}$$

Take the difference: $|V_2 - 1200 \text{ m}^3|$
$\therefore$ Need to compress at a rate of $0.5 \text{ m}^3 \bullet \text{s}^{-1}$ [0.034?]

5. **They chilled out and got it together**

a) DeBroglie wavelength, $\lambda = h/p = h/mv$

assume car $v = 100 \text{ km} \bullet \text{h}^{-1} = 28 \text{ m} \bullet \text{s}^{-1}$
assume car mass = 1000 kg

$$\therefore \lambda = 2.4 \times 10^{-38} \text{ m}$$

N_2 mass is 14 g/mol, $\therefore$ for one molecule,

$m = 2.3 \times 10^{-26}$ kg
$v = 400 \text{ m} \bullet \text{s}^{-1}$

$$\therefore \lambda = 7.1 \times 10^{-11} \text{ m}$$

My net velocity over time might be zero, but my instantaneous velocity is not
$\therefore$ my λ does not approach ∞.

b) Since these are *atoms*, they have 3 degrees of freedom (corresponding to 3 directions of motion), ignoring internal degrees of freedom for now.

$$\therefore E_{kin} = \frac{3}{2} kT$$

But this also equals:

$$E_{kin} = \frac{1}{2} mv^2$$

$$= \frac{3}{2} kT$$

$$\therefore v = \sqrt{\frac{3kT}{m}}$$

Mass of a rubidium atom: $85.5 \text{ g} \bullet \text{mole}^{-1} \Rightarrow 1.42 \times 10^{-25} \text{ kg atom}^{-1}$

$$\lambda = \frac{h}{mv} = \frac{h}{m}\sqrt{\frac{m}{3kT}}$$

$$= \frac{h}{\sqrt{3 \, kT \, m}}$$

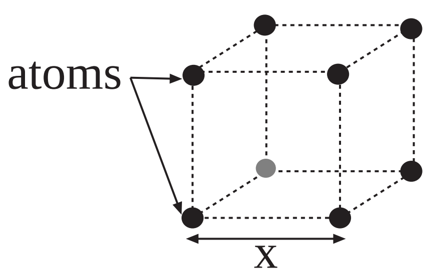
What is the interatomic spacing as a function of T? Since question states that we start at room temperature, and maintain a constant volume as we cool, spacing does not change.

From ideal gas law: $PV = nRT$

$$P = \frac{n}{V}RT$$

$$\frac{n}{V} = \frac{P}{RT} \Rightarrow \frac{N}{V} = \frac{P}{kT}$$

where now N = number of atoms and k = Boltzmann's constant. Assuming atoms are evenly distributed in space, x distance apart from each other:



density is 1 atom/x^3

$$\therefore \frac{1}{x^3} = \frac{N}{V} = \frac{P}{kT}$$

with $P = 1 \text{ atm} = 1.01 \times 10^5 \text{ N} \bullet \text{m}^{-2}$
$k = 1.38 \times 10^{-23} \text{ J} \bullet \text{K}^{-1}$
$T = 300 \text{ K}$

$\therefore$ $x = 3.4 \times 10^{-9}$ m, independent of T for this question

For condensation, $x = \lambda$ (spacing between particles is about one de Broglie wavelength)

$$\therefore \; x = \frac{h}{\sqrt{3kTm}}$$

$$T = \frac{h^2}{x^2 3k\,m}$$

$$T = 6.5 \text{ mK}$$

This is actually much **warmer** than what is really required. (Our problem is that the equipartition theory breaks down for low temperatures.)

c) Relatively vague question, possible answers:
1) increasing P often increases T_1 which is undesirable
2) methods to cool atoms lose the warmest atoms, thus decreasing pressure

d) Overall entropy of system does go up.

6. The curious case of the cold calculating counterfeiter

The assumption that the hole expands in the same proportion as the metal is valid. From the microscopic point of view, when one considers that the plate is composed of atoms, the inter–atomic distance increases in all direction for each atom. (It is similar to scaling up the grid points). One can also look at it from macroscopic view point.

Mathematically, it is a scale transformation, like a photocopier set to enlarge to, say, 110%.

Suppose we have a situation that the brass is NOT inside the nickel. The brass will expand according to thermal expansion:

$r_o \rightarrow r_o + \Delta_B$

where $\Delta_B = r_o \, \alpha_B \, (T - T_o)$

Similarly the hole (of the nickel) will expand

$r_o \rightarrow r_o + \Delta_N$

where $\Delta_N = r_o \, \Delta_N \, (T - T_o)$

a) Now, the brass is inside the nickel. Since $\Delta_B > \Delta_N$, there is a contact force (consequently a pressure) between the two, as action = reaction forces, so that both the brass and the hole expand with the same expansion:

$r_o \rightarrow r_o + \Delta$

Of course $\Delta_N < \Delta < \Delta_B$, and thus using different materials creates pressure. Since the pressure forces act perpendicular to the contact surface (we are dealing with the geometry of annulus), the pressure (J_{press}) is primarily determined by Young moduli Y_B, Y_N. The pressure which inhibits the thermal expansion of the brass:

$$p_B = Y_B \frac{\Delta_B - \Delta}{r_o + \Delta^{\mathrm{I}}}$$

where $r_o + \Delta^{\mathrm{I}}$ is the radius of the brass after an equilibrium temperature T (room temperature).

Similarly for the nickel,

$$p_N = Y_N \frac{\Delta - \Delta_N}{r + \Delta^{II}}$$

where $r_o + \Delta^{\mathrm{II}}$ is the radius of the hole. Both Δ^{I} and Δ^{II} are negligible compared to r_o and

$p_B = p_N = p$

due to action = reaction. So we have two unknowns and two equations:

$(\Delta_B - \Delta)/r_o = p/Y_B$

$(\Delta - \Delta_N)/r_o = p/Y_N$

$$Y\,(\Delta_B - \Delta_N)/r_o = p\left(\frac{1}{Y_B} + \frac{1}{Y_N}\right)$$

which we solve for p:

$$p = \frac{(\Delta_B - \Delta_N)/r_o}{\left(\frac{1}{Y_B} + \frac{1}{Y_N}\right)} = \frac{(\alpha_B - \alpha_N)(T - T_o)}{\left(\frac{1}{Y_B} + \frac{1}{Y_N}\right)}$$

Thus
$$p = \frac{(19 - 11)\times 10^{-6}\,(293 - 77)}{\left(\frac{1}{9.1} + \frac{1}{19.0}\right)\times 10^{-10}} = 1.06 \times 10^{8}\ Pa$$

$$= 1050 \text{ atm}$$

b) The expansion of the brass at T = 293 K

$$\Delta = \Delta_B - \frac{p}{Y_B} r_o$$

$$= [\alpha_B (T - T_o) - p / Y_B] r_o$$

$$= \left[19 \times 10^{-6} (293 - 77) - \frac{1.06 \times 10^8}{9.1 \times 10^{10}} \right] \times 6.00\ mm$$

= 17.65 µm

Thus the radius of the brass r = 6.0177 mm, and so is the inner radius of the nickel (the hole).

The volume of the brass, where r is radius, t is thickness:

$V_B = \pi r^2 t = \pi (6.0177)^2\, 2$

$= 227.53\ mm^3 = 0.228\ cm^3$

The outer radius of the nickel also expands

$R = R_o [1 + \alpha_{Ni} (T - T_o)]$

$= 14.00\ mm\ [1 + 11 \times 10^{-6} (293 - 77)]$

$= 14.0333\ mm$

The volume of the nickel annulus

$V_N = \pi (R^2 - r^2)t = \pi [(14.0333)^2 - (6.0177)^2]\, 2$

$= 1009.84\ mm^3 = 1.010\ cm^3$

Hence the mass of tooney:

$m = (V_B \bullet 8.87) + (V_N \bullet 8.85)$

$= 10.95\ g$

compared to the one I measured: 7.31 g. This is about a 30% difference in weight — easy enough to tell the coin.

In fact, Svend would NOT get away with his coin even by its rough appearance. The brass centre piece is actually *16 mm wide* in a real tooney, not 12 mm!

1996-1997 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 4: Optics and Waves

Due February 14♥, 1997

1) Jesse's 'anti-music'

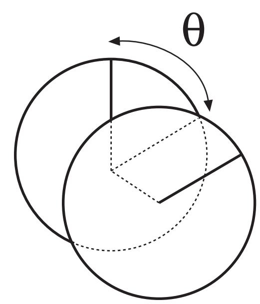
Jesse has a problem with his roommate Bongo. Bongo's stereo is way TOO LOUD! In fact, Bongo has already blown his right speaker, and is pumping his Too-Much Music over the left one. So Jesse decides to fight sound with sound. He places an identical speaker of his own, on the left channel of the *same* station, face-to-face with Bongo's and 6.0 m away. Jesse then sits with his one (remaining) good ear 1.5 m from his own speaker, on a direct line between the two. For the following, you can assume that Jesse and Bongo are pumping out exactly the same signal, in amplitude and in phase, and that Jesse's head doesn't interfere with the sound. The speed of sound in his apartment is 331.45 m s^{-1}.

a) At what frequency(ies) in the music would Jesse hear minimum sound at his position?

b) Are there points along the line joining the speakers at which Jesse cannot hear sound of *any* frequency?

c) What ways might Jesse set up his speaker, better to cancel Bongo's racket? *[Chairul]*

2) A new twist on light



Like any vector, the electric field of linearly polarized light can be considered as the sum of two components. A linear polarizer allows through only the component that is aligned to its 'axis', and absorbs the other component. This results in Malus' law, which states that the light transmitted by a polarizer is: $I = I_o \cos^2\theta$, where I is the transmitted light intensity, I_o is the incident light intensity, and θ is the angle between the axis of the polarizer and the input $\vec{E}$ field direction.

a) Why does Malus' law involve $\cos^2\theta$, not $\cos\theta$?

b) Kimberly wanted to verify Malus' Law. She shone a flashlight on a photodetector and measured 1.0 mA of current generated by the detector due to the light intensity. She inserted a linear polarizer between the flashlight and the detector and measured the current. What was this value?

c) She inserted a second polarizer and rotated it to minimize the current generated by the photodetector. What was this current, and what was the angle between the polarizers' axes?

d) On a whim, she placed a third polarizer after the first two and aligned it with its axis 45° to the first one. Without changing the orientation of the first two polarizers, she measured the current. What current did she measure? She exchanged positions between the second and third polarizers. What was the new current generated?

e) As a result of her findings in section (d), she decided to extend her studies. With the first and last polarizers aligned perpendicularly, she inserted two polarizers aligned at 30° and 60° to the first one. What current did she measure? Write down the general expression for the amount of current generated for a total of *N* polarizers, all aligned at 90°/(*N*-1) to each other. For really big *N*, what do you expect this value to be?

f) For large *N*, Kimberly effectively rotated the polarization of the beam by 90°. What problems would you face if you attempted part e) in real life? What is a much easier way to produce vertically polarized light from a horizontally polarized beam, using only a few standard mirrors? *[James]*

3) People see through Claire

Claire (*not* her real name) plans to smuggle a tiny glowing radioactive pellet out of Ukraine. She safely implants the pellet in the centre of a solid transparent cube of index of refraction $n = 1.75$, but realizes the glow of the pellet will give her away. She decides to paint the cube black, but knows that Customs officers will be suspicious of such an obvious concealment.. Therefore she wants to paint only some parts of the surface.

a) What is the least surface she must paint — and in what shape?

b) Unfortunately, at Customs she accidentally drops the cube in a glass of water and the inspectors catch her. What is the smallest surface she should have painted? (The index of refraction of water is 1.33.) *[Chairul]*

4) Nothin' but blue sky...

A plane wave of light has a wavelength λ. It scatters off a small sphere of radius $a \ll \lambda$. The light scattering off of the sphere causes the charges in it to oscillate and thus induces an electric dipole moment p in the sphere. p increases as a function of the volume of the sphere, $p \propto a^3$. Oscillating dipoles radiate electromagnetic waves, so that the intensity of scattered radiation I_{scatt} from such a sphere scales as the square of p: $I_{scatt} \propto p^2$.

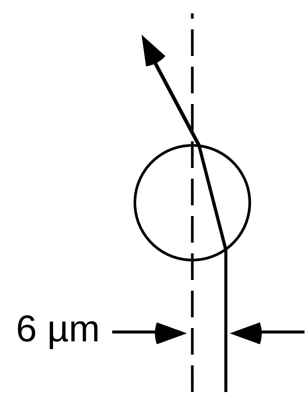
If we know the intensity of the incident wave, I_o, we should be able to calculate the intensity of the scattered wave I_{scatt} as a function of the distance R from the sphere. In particular, we want to know how the scattered intensity varies with the wavelength of the incident radiation. Note that the three length-dependent variables in the problem are λ, R, and a.

a) By conservation of energy, the amount of radiation flowing through any spherical shell at any distance R from the sphere must be a constant. How does I_{scatt} scale with the distance R from the scattering sphere?

b) The ratio I_{scatt} / I_o is dimensionless. As a result, how must I_{scatt} depend on λ? Use the dependence of I_{scatt} on the other length-dependent variables (which you know from above) to obtain your answer.

c) The situation above can be used to model the sky, where visible sunlight of all wavelengths scatters off of water molecules in the air. The human eye has evolved so that it is most sensitive to that part of the spectrum for which the intensity of light given off by the sun is maximal. As a result, we are most sensitive to green and this sensitivity falls off as we move to either the red or violet ends of the spectrum. Use this knowledge and the result of part (b) to explain why the sky appears blue, rather than, say, green. *[Nipun]*

5) Just tweezing...



Focussed laser light can be used to push tiny spheres around. It is possible to coat these tiny spheres with biological antigens and make them glue themselves to particular polymer molecules, even the end of DNA strands. Then in moving the spheres with a laser, the polymer molecules can be manipulated, stretched, moved. Sometimes this is referred to as *optical tweezers*

The force doesn't come from light pressure, exactly, but through refraction. Consider a plastic sphere 25 μm in diameter and with an index of refraction n = 1.4. Visible laser light can be focussed to a spot a few micrometers across, through a microscope, so let's approximate the light by a single ray as illustrated, incident on the sphere at 6 μm off the normal-incidence axis.

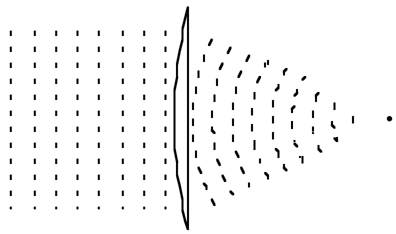
a) Find the approximate path of refraction through the sphere, and analyze the change in momentum for a photon of wavelength λ = 530 nm.

b) For a laser with power 1 W, what is the net force exerted on the sphere as it changes the momentum of all the photons in the beam? You can assume the sphere doesn't move *[Robin]*

Web reference: *http://www-leland.stanford.edu/dept/news/relaged/940509Arc4280.html*
(a sort of press-release on related work at Stanford University); perhaps also see 'Optical Levitation by Radiation Pressure,' A. Ashkin and J.M. Dziedzic, Appl. Phys. Lett. 19, p.182 (1971)

6) Photons get the bends

In one kind of description, lenses work by changing the curvature of the wavefronts of an incoming beam. When a plane wave passes through a positive lens, the flat wavefronts come out curved concave-forward. Then by Huygen's construction, it is not hard to see that the beam must collapse to a focus.



The speed of the wavefronts is the *phase velocity* $v_\phi = c/n$, where n is the index of refraction.

a) Consider a simple lens, flat on one side and with a spherical surface on the other, 50 mm in diameter, and 5 mm thick in the middle (see figure). If the glass has an index of refraction $n = 1.66$, find out how much the wavefronts in the middle of the beam are delayed in time relative to the part of the same wavefront that passes through the very edge of the lens.

b) What is the radius of curvature of the new wavefronts, and where do they come to a focus?

Something similar can happen in an ordinary flat block of glass when *very intense* light is incident on it. It turns out that the index of refraction depends also on intensity I, as:

$$n = n_o + n_2 I$$

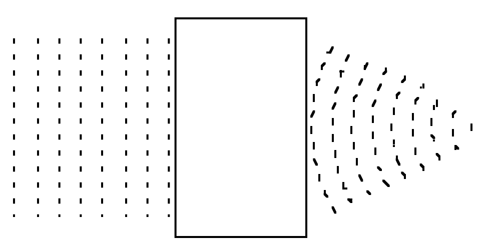
where n is the actual index of refraction, n_o is the ordinary index of refraction for low-intensity light, I is the intensity of light in W cm^{-2}, and n_2 is a constant equal to 5×10^{-15} W cm^{-2} for glass. Where light is very intense, it increases the index of refraction it encounters.

c) Consider light which has flat wavefronts as before, but is more *intense* in the middle than at the edges, with the distribution:

$$I \;\; [W\,cm^{-2}] \;=\; \begin{cases} 2 \times 10^{13} \left(-b + \sqrt{10^6 - x^2}\right) & \textit{for } |x| \le 2.5 \\ 0 & \textit{else} \end{cases}$$

$$\textit{where } b \;\equiv\; \sqrt{10^6 - (2.5)^2} \;=\; 999.996875$$

Because the light is more intense in the middle, the wavefronts are delayed in the middle of the beam relative to the edge, and again the beam will come to a focus. For this intensity and for a block of glass 10 cm thick, find the approximate focal point of the beam. *[Robin]*



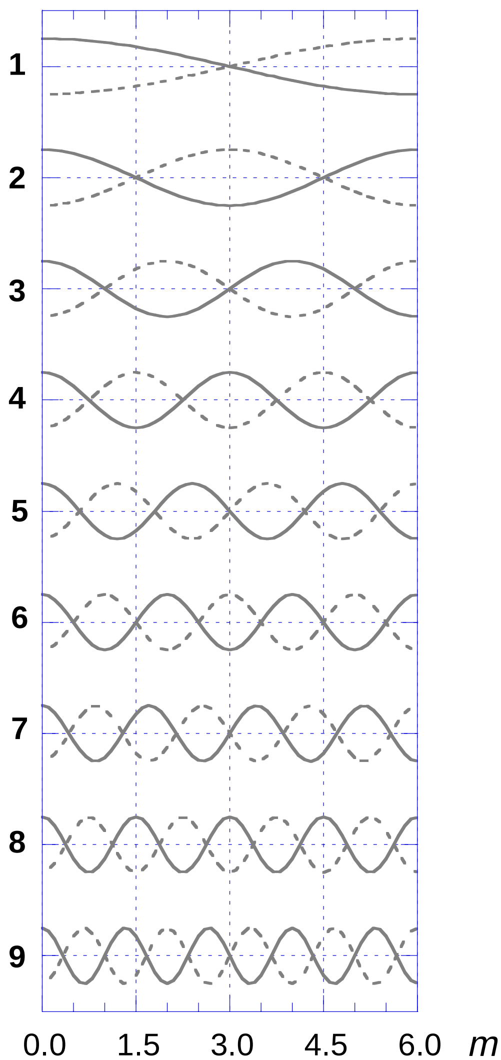
1996-1997 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 4: Optics and Waves

1 . Jesse's anti-music

This problem here is a simple standing-wave problem. Each speaker is at the antinode position. To have minimum sound, Jesse has to be at one of the nodes of the standing wave pattern.

Wave patterns	Mode	n	
	fundamental	1	L = 1/2 λ
		$\boxed{2}$	$\boxed{L = \lambda}$
		3	L =3/2 λ
		4	L = 2 λ
		5	L = 5/2 λ
		$\boxed{6}$	$\boxed{L = 3 \lambda}$
		7	L = 7/2 λ
		8	L = 4 λ
		9	L = 9/2 λ

where L=6.0 m is the distance between the two speakers and λ the wavelength.

a) Analytically, the appropriate wavelengths can be obtained by imposing the antinode boundary conditions:

$$\cos([2\pi/\lambda]\, L) = \pm 1, \text{ thus } \quad [2\pi/\lambda]\, L = n\,\pi \text{ for } n = 1, 2, 3, \ldots$$

At position x =1.5 m, the intensity must be a minimum, meaning

$$\cos([2\pi/\lambda]\, x) = 0 \;\Rightarrow \cos([2\pi/\lambda]\, Lx\ / L) = 0 \;\Rightarrow \cos([2\pi/\lambda]\, L\, [x\ / L\,]) = 0$$

or

$$0 = \cos([n\,\pi]x\ /L\,) = \cos(0.25\, n\,\pi) \quad \text{for the acceptable } n.$$

Thus n = 2, 6, …, and

$$f = \mathrm{v} / \lambda = \mathrm{v}\,(2\,m - 1) / L \qquad \text{for} \qquad m = 1, 2, 3, \ldots.$$

So the frequencies would be minimum for f = 55.24 Hz, 165.73 Hz, 276.21 Hz, ….

b) No. He, in fact, would hear some frequencies louder (at maximum with intensity four times larger than its original without his own speaker).

c) If Jesse has only one speaker, he could not do any better no matter what direction his speaker faces. If he has more speakers he could arrange them as such that more frequencies can be cancelled out. Nevertheless, there is some complication due to the interference between his own speakers.

2. A new twist on light

This problem can be answered with different levels of rigour. To try and be as 'correct' as possible, while avoiding integration. I will define intensity as the time average of the square of the electric field:

$$\mathrm{I} = \langle\, |\vec{E}\,(t)|^2 \rangle$$

This is necessary since for visible light fields, the electric field is oscillating very fast ($\approx 6 \times 10^{14}$ Hz), much faster than a detector can follow. This method can also be used for a slowly-varying field by taking the time average over a length of time corresponding to detector response ($\approx 10^{-12}$ – 10^{-6} s). We do not require 'I' as a function of time for this question.

PLEASE NOTE: Many of you may have taken a simpler approach (more likely to be found in an OAC-level course) by writing:

$$\mathrm{I} = |\vec{E}|^2 \qquad \text{(ignore oscillating nature of } \vec{E}(t)\text{)}$$

This is o.k., but not as rigourous as my solution.

a) Consider a polarizer with a set axis and a general (instantaneous) electric field vector. where $|\vec{E}_{||}| = |\vec{E}_o|$. Transmitted electric field is $\vec{E}_{||}$. Thus transmitted intensity is:

$$\mathrm{I_T} = \langle\, |\vec{E}_{||}|^2 \rangle$$

$$= <\left|\vec{E}_o\right|^2 \cos^2\theta >$$

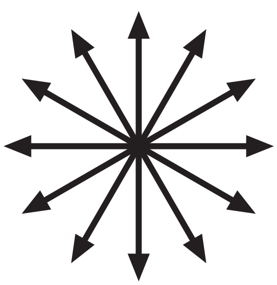
$$= <\left|\vec{E}_o\right|^2 > \cos^2\theta$$

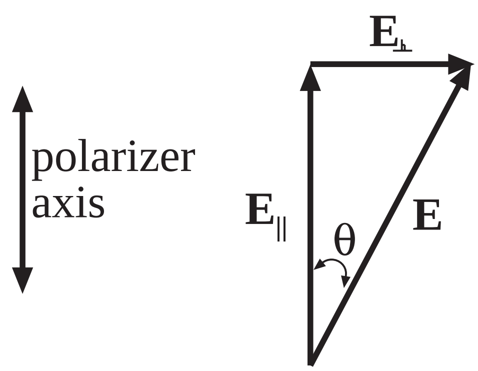
$$\text{but } I_o = <\left|\vec{E}_o\right|^2 >$$

$$\therefore I_T = I_o \cos^2\theta$$

b) Required assumption: current on photodetector $\propto$ intensity of light.

A flashlight is an unpolarized source. This means the polarization of the EM (light) field does not have a preferred direction. A really rough answer to this would be that since there is no preferred direction, 1/2 of the intensity is polarized parallel to the polarizer, and 1/2 is orthogonal.
$\therefore$ she measured 0.5 mA.





Let's do this a little more rigourously. We can model the flashlight as a very large number of linearly polarized sources, all with the same amplitude, and with their polarizations pointing in all directions.

What does Kimberley measure without the polarizer?

$$I_{TOT} = <|\sum_{n=0}^{N} \vec{E}_n|^2 >$$

where N is very large.

This can be written as:

$$I_{TOT} = <|\sum_{n=0}^{N} \vec{E}_{n\|} + \sum_{n=0}^{N} \vec{E}_{n\perp}|^2 > \qquad (1)$$

where $\vec{E}_{n\|}$: is a parallel component of the nth source.
N: a very very big number (I'm working hard to avoid calculus here. If using calculus, N $\rightarrow \infty$ and all the sums become integrals).

With the polarizer she oberves

$$I_{pol} = \left\langle \left| \sum_{n=0}^{N} \vec{E}_{n||} \right|^2 \right\rangle \tag{2}$$

We need $\frac{I_{pol}}{I_{TOT}}$

Take (1) $\Rightarrow$

$$I_{TOT} = \left\langle \left| \sum_{n=0}^{N} \vec{E}_{n||} \right|^2 + \left| \sum_{n=0}^{N} \vec{E}_{n\perp} \right|^2 + 2\left| \left(\sum_{n=0}^{N} \vec{E}_{n||} \right) \cdot \left(\sum_{n=0}^{N} \vec{E}_{n\perp} \right) \right| \right\rangle$$

Due to the absolute value brackets, we can write.

$$= \left\langle \left| \sum_{n=0}^{N} \vec{E}_{n||} \right|^2 \right\rangle + \left\langle \left| \sum_{n=0}^{N} \vec{E}_{n\perp} \right|^2 \right\rangle + 2 \left\langle \left| \left(\sum_{n=0}^{N} \vec{E}_{n||} \right) \cdot \left(\sum_{n=0}^{N} \vec{E}_{n\perp} \right) \right| \right\rangle$$

We know that the flashlight polarization had no preferred direction. Therefore no matter how Kimberly placed the polarizer, she would get the same result. Thus 'parallel' and 'orthogonal' are arbitrary and the first two terms must be equal.

To simplify the third term, consider a case with only two sources. $\vec{A}, \vec{B}$

The third term looks like

$$\left\langle \left| \left(\vec{A}_{||} + \vec{B}_{||} \right) \cdot \left(\vec{A}_{\perp} + \vec{B}_{\perp} \right) \right| \right\rangle$$

$$= 0 \qquad \text{since} \qquad \begin{aligned} \vec{A}_{||} \cdot \vec{A}_{\perp} &= 0 \\ \vec{A}_{||} \cdot B_{\perp} &= 0 \\ \vec{B}_{||} \cdot \vec{A}_{\perp} &= 0 \\ \vec{B}_{||} \cdot B_{\perp} &= 0 \end{aligned}$$

Therefore (1) $\Rightarrow$

$$I_{TOT} = 2 \left\langle \left| \sum_{n=0}^{N} \vec{E}_{n||} \right|^2 \right\rangle \text{ (since first two terms are equal).}$$

$$\therefore\ I_{pol} = \frac{I_{TOT}}{2}$$

She observes 0.5 mA of current.

c) Only linearly polarized light is transmitted by the first polarizer. Therefore the second polarizer transmits

$$I_2 = I_1 \cos^2\theta$$

For minimum I_2, chose $\theta = \frac{\pi}{2}$, $\therefore I_2 = 0$ and current = 0.0 mA. The angle between the axes is 90°.

d) Let I_1 be intensity of light passing through first polarizer. Let I_2 be intensity of light passing through second polarizer.

$$I_2 = I_1 \cos^2 90° = 0$$

And the final polarizer is aligned at 90° – 45° = 45° to the second one:

$$I_3 = I_2 \cos^2 45° = 0$$

$\therefore$ she measured 0 mA.

After exchanging:

$$I_2 = I_1 \cos^2 45° = \frac{I_1}{2}$$

$$I_3 = I_2 \cos^2 (90° - 45°) = \frac{I_1}{4}$$

$\therefore$ she measured 0.13 mA.

e) Output intensity is

$$I_{final} = I_1 \cos^2(30°) \cos^2(30°) \cos^2(30°)$$

$$= I_1\left(\frac{\sqrt{3}}{2}\right)^6$$

$$\approx 0.42\ I_1$$

Therefore she measures 0.21 mA. By adding more polarizers, she is outputting more light!
For *N* polarizers:

$$I_{FINAL} = I_1\left(\cos^2\left(\frac{90^\circ}{(N-1)}\right)\right)^N$$

By 'subbing' in increasing values for N, we see this coefficient is approaching 1.
Slightly more rigourously

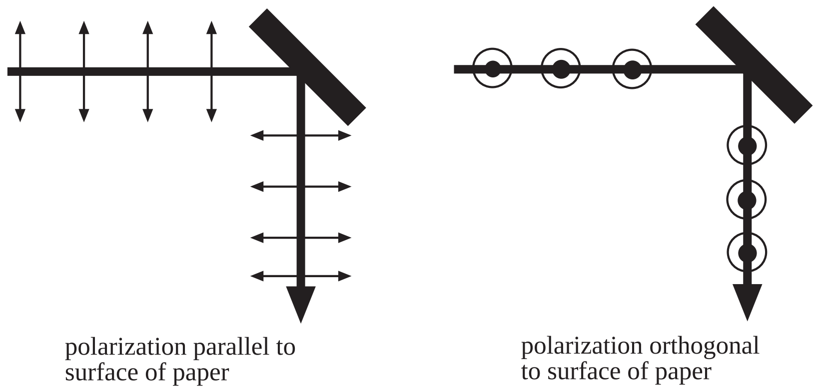
$$\lim_{N\to\infty} \frac{90}{N-1} = 0$$

$$\lim_{N\to\infty} \cos^{2N} 0 = 1$$

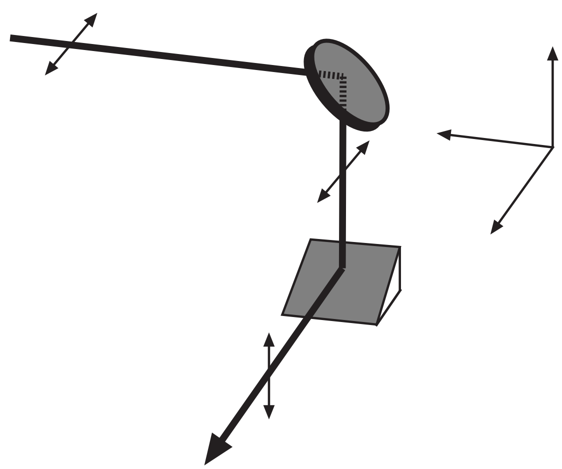
Therefore she measures 0.5 mA. (Recall I_1 is intensity of light passed by one polarizer).

f) Real polarizers are not perfect. They still absorb some of the EM wave polarized parallel to their axes. Thus very little light would make it through N polarizers, where N is a large number.

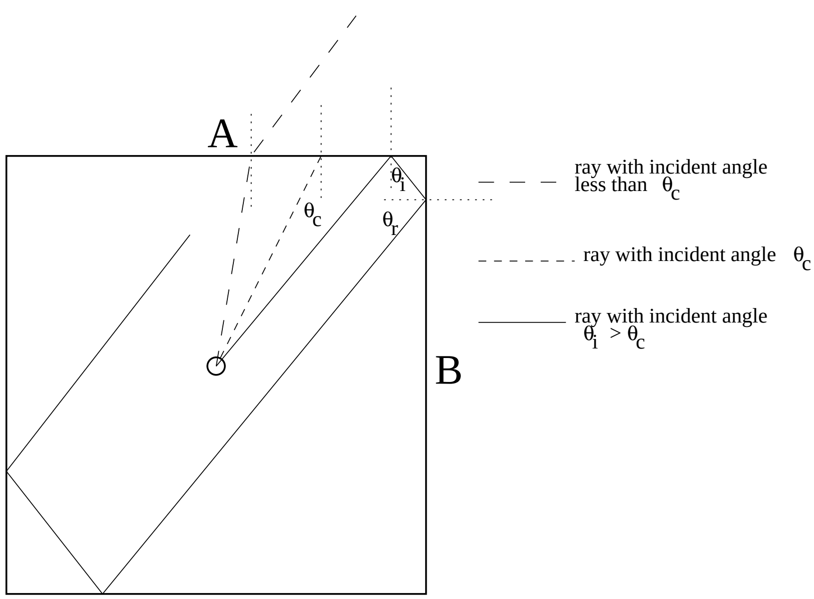
To change the polarization direction using two standard mirrors, recall what happens to a polarized beam in reflection:



So arrange two mirrors as follows:



3 . People see through Claire



The light-ray from the pellet will be refracted out if the angle of incidence (θ_I), measured from the normal, is less than the critical angle (θ_c). Due the geometry of the cube (shown as a square in the figure at left), θ_c has to be less than 45°. Otherwise the whole surface has to be covered. Any direct rays fall with θ_i > 45° in one side will fall on the adjacent one.

If $\theta_c < \theta_i < 45°$, the ray will be totally reflected and it will be reflected forever! To see this effect refer to the figure. Coming from the source, the light will be reflected by side A with θ_i. The second reflection will be $\theta_r = 90° - \theta_i > 45°$ which is certainly greater than θ_c, and the third will be θ_i, and so on alternately. In this way a cube is a good container for hiding an object.

The light rays can also run in a direction not parallel to the edges of the faces of the cube: the extreme case is one in which the rays are directed along the planar diagonal of the cube. In this case θ_i can be as large as 54.73° (i.e., $\tan^{-1}$ (2)), with $\theta_r = 35.26°$. To contain this ray also, we need $\theta_c < \theta_r$, i.e., $\theta_c < 35.26°$.

a) A point source will emit light spherically. So Claire must paint in a circular shape on each side. The area she covers must be such that $\theta_i < \theta_c$ cannot penetrate. The radius of each circle will be determined from θ_c.

$$\theta_c = \sin^{-1} (n_{fluid} / n_{cube})$$

For air $n_{fluid} = 1.00$, thus $\theta_c = 34.85°$ and the radius of the circle $r = \frac{1}{2} d \tan \theta_c = 0.349\, d$, where d is the vertex of the cube. So the area she must cover is

$$R = \pi r^2 / d^2 = \tfrac{1}{4} \pi \tan^2 \theta_c = 0.3808$$

i.e., 38.08% of the total surface.

b) First we have to check whether in water, $\theta_c < 35.26°$. Using the same formula, $\theta_c = 49.46°$ so Claire has to paint it all!

4. Nothin' but blue sky ...

a) Assume that the radiation is emitted symmetrically for 2 radii, R_1 and R_2. We must have same radiation flux through surfaces $4\pi R_1^2$ and $4\pi R_2^2$, so

$$I \propto \frac{1}{R^2}$$

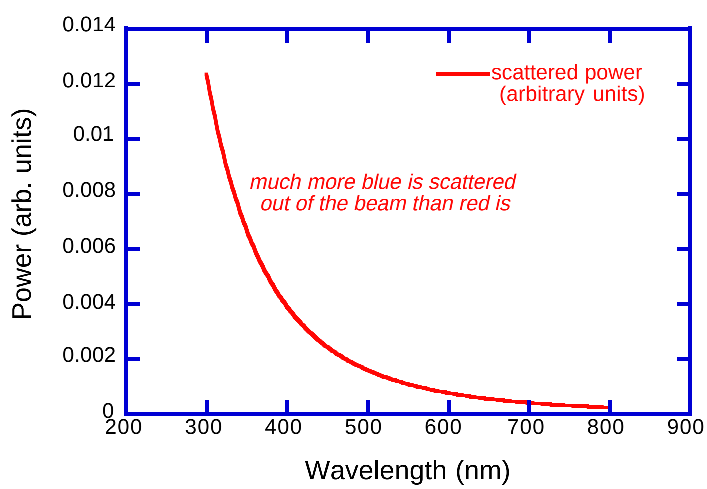
b) $$\frac{I_{scatt}}{I_o} \propto p^2 \propto (a^3)^2$$

and

$$\frac{I_{scatt}}{I_o} \propto \frac{1}{R}$$

So for $\frac{I_{scatt}}{I_o}$ to be dimensionless requires a $\frac{1}{(length)^4}$ dependence. As ∞ is the only remaining length–dependent variable, we have

$$\boxed{\frac{I_{scatt}}{I_o} \propto \frac{1}{\lambda^4}}.$$

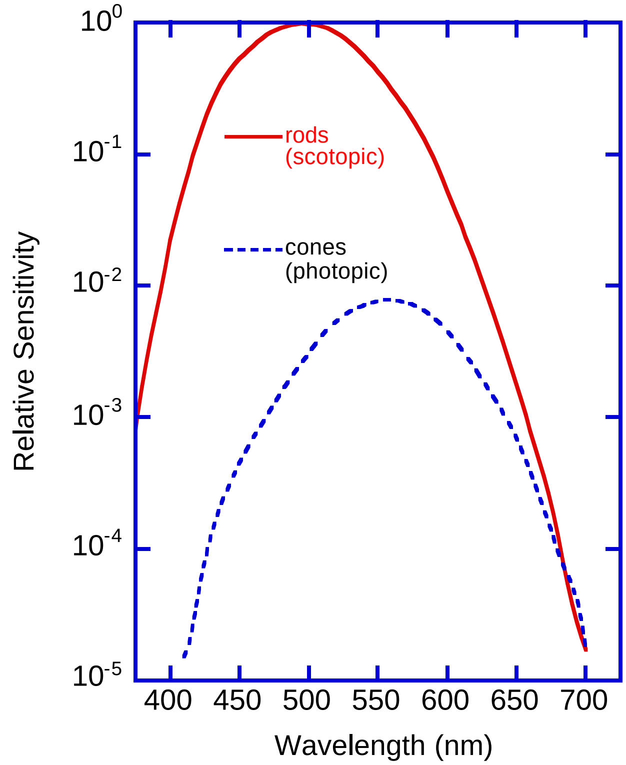


This $\frac{1}{\lambda^4}$ dependence is known as 'Rayleigh scattering'.

c) From part (b), longer wavelengths experience smaller scattering. Thus especially at sunset (when the sunlight passes through the longest stretch of atmosphere), looking at the sun, the blue end of spectrum is preferentially scattered away by water molecules and the sun appears reddish or red-orange. The shorter-wavelength, bluer light — sent sideways from the transmitted beam of sunset light — has showered sideways to become someone else's blue sky.

Our eyes' sensitivity peaks near green and sensitivity falls off towards red and violet.

So, superposing the eye's sensitivity and the Rayleigh scattering law, we see a peak near the blues.



5. Just tweezing ...

As the beam enters the sphere, its direction deviates, due to refraction at the interface

$$\sin\theta_1 = n \sin\theta_2 \qquad \text{(Snell's Law)}$$

The amount the beam **deviates** is then $\Delta\theta$

$$\alpha = \theta_2 + \Delta\theta = \theta_1 \qquad \text{(opp. } \angle \text{ th'm)}$$

$$\Rightarrow \quad \Delta\theta = \theta_1 - \theta_2$$

when the beam leaves the sphere the angle of incidence at the exit is again θ_2 and then θ_1 at the exit, and again the angular deviation is $\Delta\theta$.

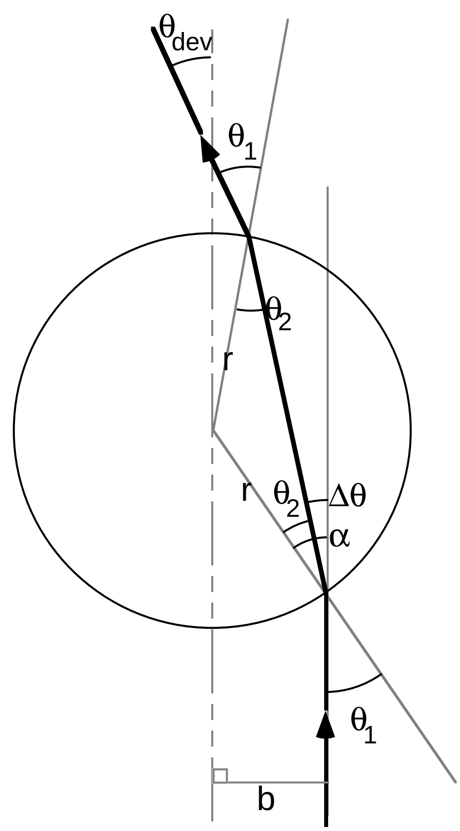
Thus the overall deviation in angle is

$$\theta_{dev} = 2\Delta\theta = 2(\theta_1 - \theta_2)$$

$$\sin\theta_1 = \sin\alpha = \frac{b}{r} \rightarrow \theta_1 = \sin^{-1}\left(\frac{b}{r}\right)$$

$$\sin\theta_2 = \frac{1}{n}\sin\theta_1 = \frac{b}{nr} \rightarrow \theta_2 = \sin^{-1}\left(\frac{b}{nr}\right)$$

Thus $\quad \theta_{dev} = 2\left(\sin^{-1}\left(\frac{b}{r}\right) - \sin^{-1}\left(\frac{b}{nr}\right)\right)$



$$\left.\begin{array}{l} b = 6\mu\text{m} \\ n = 1.4 \end{array}\right\} \quad \Rightarrow \quad \theta_{dev} = 2\left(\sin^{-1}\left(\frac{6\mu\text{m}}{25\mu\text{m}}\right) - \sin\left(\frac{6\mu\text{m}}{1.4\ \ 25\mu\text{m}}\right)\right)$$

$$= 2\,(13.88° - 9.87°)$$
$$= 8.03°$$

This new direction means photons have a new x-component of momentum. If the photon momentum is p_γ originally, then

$$p_x = p_\gamma \sin(\theta_{dev})$$
$$= p_\gamma \sin\left(2\left\{\sin^{-1}\left(\frac{b}{r}\right) - \sin^{-1}\left(\frac{b}{nr}\right)\right\}\right)$$

here

$$p_x = p_\gamma \sin(8.03°)$$
$$= p_\gamma\,(0.140)$$

For a photon: $p = \dfrac{h\nu}{c} = \dfrac{h}{\lambda}$, where h = 6.62×10^{-34} J • s. Here λ = 530 nm = 5.30×10^{-7} m:

$$p = \frac{6.62\times10^{-34}\ \text{J s}}{5.30\times10^{-7}\text{m}} \qquad [\text{but J: kg•m}^2\text{•s}^{-2}]$$
$$= 1.25 \times 10^{-27}\ \text{kg• m•s}^{-1}$$

so $\quad \Delta p_x = 1.75 \times 10^{-28}$ kg•m•s^{-1}

b) Laser 1W = 1 J/s

1 photon has energy $h\nu = \dfrac{hc}{\lambda} = \dfrac{(6.62\times10^{-34}\text{J s})\ 3\times10^{8}\text{ms}^{-1}}{5.3\times10^{-7}\text{m}}$

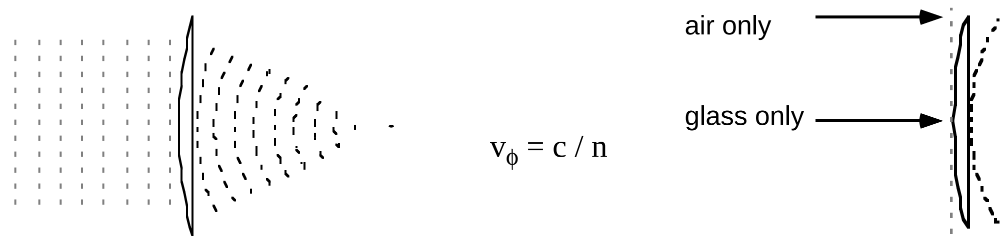
$$= 3.75\times10^{-19}\text{J}$$

so $\quad$ 1 J = 2.67×10^{18} photons per second

$$F_{TOT} = \frac{dp_x}{dt} = N_x • \Delta p_x = 2.67 \times 10^{18} \cdot 1.75 \times 10^{-28}\ \text{kg• m•s}^{-1}$$
$$= 4.67 \times 10^{-10}\ \text{kg•m•s}^{-2}$$
$$= 4.67 \times 10^{-10}\ \text{N}$$

6. **Photons get the bends**

a)



phase fronts move slower in glass where n = 1.66

• time through 5 mm *glass* t = d / v$_\phi$ = $\dfrac{nd}{c} = \dfrac{1.66 \;\; 5\times10^{-3}\text{m}}{3\times10^{8}\text{ms}^{-1}}$

$= 2.77 \times 10^{11}$ s

= 27.7 ps (*picoseconds*)

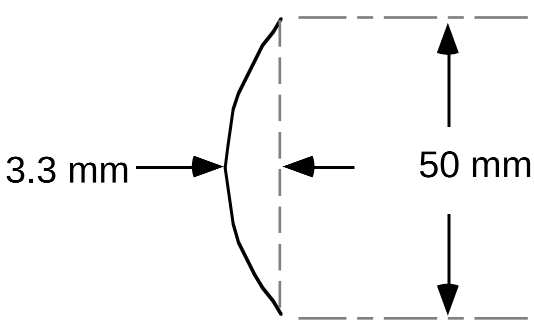
• time through 5 mm *air* t = $\dfrac{d}{v_\phi} = \dfrac{nd}{c} = \dfrac{1.00 \;\; 5\times10^{-3}\text{mm}}{3\times10^{8}\text{ms}^{-1}}$

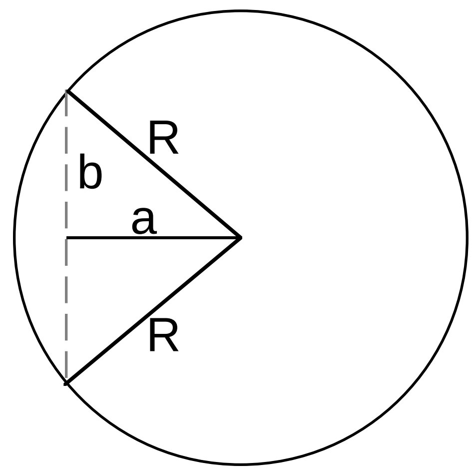
$= 1.67 \times 10^{-11}$ s

= 16.7 ps

⇒ time *difference* is (27.7 – 16.7) ps = 11 ps

b) In one way of describing it, lenses focus by putting a spherical curvature on phase fronts, then Huyghen's Principle shows the wavefronts converge to a point — the focus
Right after lens, the 11 ps relative delay means distance ct = 3×10^{8} ms^{-1} • 11×10^{-12} s = 3.3 mm behind edges:





This is a chord of a circle. What is the radius of the circle?

$$\left.\begin{aligned} b &= 25 \text{ mm} \\ a &= R - 3.3 \text{ mm} \end{aligned}\right\} a^2 + b^2 = R^2$$

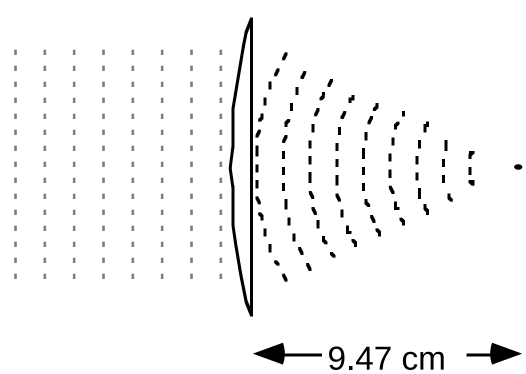
so $R^2 = (R - 3.3 \text{ mm})^2 + (25 \text{ mm})^2$

$R^2 = R^2 - 6.6 \text{ mm}\cdot R + 625 \text{ mm}^2$

$R = \dfrac{625 \text{ mm}^2}{6.6 \text{ mm}} = 94.7$ mm

= 9.47 cm

So the radius of curvature is 9.47 cm, and this is where the beam comes to a focus 9.47 cm after the lens!



c) The form of I looks complicated. That is because it is pre-cooked to give an easier answer!

$$n = n_o + n_2 I$$

$\Delta n = n_2 I$ across the beam, making wavefronts curved following intensity changes across beam. Let's find wavefront delay according to x

$$\Delta t = \frac{n(x)d}{c}$$

$n(x) = n_o + n_2 I(x)$
$d = 10 \text{ cm} = 0.1 \text{ m}$
$c = 3 \times 10^8 \text{ ms}^{-1}$

$$\Delta t = \underbrace{\frac{n_o d}{c}}_{\text{constant for all}} + \underbrace{\frac{n_2 I(x) d}{c}}_{\text{causes curvatures in wavefronts}}$$

and the physical displacement of the wavefronts is:

$$c\Delta t = \underbrace{n_o d}_{\text{constant (0.17 m)}} + \underbrace{n_2 I(x) d}_{\text{depends on x}}$$

So all the curvature in the beam comes from the second term:

$$\Delta z = n_2 I(x)\, d = 5 \times 10^{-15} \cdot 2 \times 10^{13} \text{ W} \cdot \text{cm}^{-2} \cdot (-b + \sqrt{10^6 - x^2})$$
$$= 0.01\,(-b + \sqrt{10^6 - x^2})$$
$$= 0.01\,(\sqrt{10^6 - x^2} - b)$$

Note the units of n_2: cm^2W^{-1} — there was an error in the question! (*eek!*)

Then
$$(100\,\Delta z + b) = \sqrt{10^6 - x^2}$$
$$(100\,\Delta z + b)^2 = 10^6 - x^2$$

Take $y = 100\,\Delta z$, then

$$x^2 + (y + b)^2 = 10^6 \text{ cm}^2$$

the equation of a circle, offset by $-b$ along y, with $R = 10^3$

What is the y-displacement?

$y = y_o$ where $x = 0$
$(y + b)^2 = 10^6$
$y + b = 10^3$
$y = 10^3 - b$
$\quad = 1000 - 999.996875$
$y = 3.13 \times 10^{-3}$

so

$$\Delta z = y/100 = 3.13 \times 10^{-5} \text{ m}$$

Use the same method as in (a) to get the curvature

$a^2 + b^2 = R^2$ $\qquad$ $a = 2.5$ cm

$b = 999.996875$ cm

$R^2 = 10^6\ \text{cm}^2$

$R = 10^3\ \text{cm} = 10\ \text{m}$

The beam focuses in 10 m.

1996-1997 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 5: Electricity and Magnetism

Due March 7, 1997

1) Gauss visits Planesville...

The (very obscure) town of Planesville, Manitoba is much like any small Canadian town, except that the town is two-dimensional — it is lacking the third dimension that we experience everywhere else on earth. We want to see how Gauss's law might be different in Planesville. For an arbitrary charge distribution, Gauss's law takes the form of an integral, but recall that for a point charge Q, we can draw a gaussian sphere at radius R from the charge and use the radial symmetry to write Gauss's law as:

$$E_{sphere} \bullet \text{(area of sphere)} = Q/\varepsilon_o$$

a) If Coulomb's law applies in Planesville in the same form it does everywhere else, would Gauss's law still apply in its usual form? (Hint: Derive the form of Gauss' law in a 2-D world for a point charge.)

b) How could we 'fix' Coulomb's law so that Gauss's law would apply in Planesville? Write the expression for the new force law.

c) For the force law of part (b), consider a solid disc of charge of radius 10 cm and charge density 0.5 mC m^{-2}. What is the electric field of such a disc in Planesville as a function of the radial distance from the centre of the disc? Sketch a graph of this field as a function of the radial distance from the centre of the disc. *[Nipun]*

2) It's not just a good idea... it's Ohm's law!

Ohm's law for conductors can be written in the form $V = IR$ or as $\vec{J} = \sigma \vec{E}$, where $\vec{J}$ is the current per unit cross-sectional area flowing in a wire and σ is the electrical conductivity of the material. For our purposes, the important thing is that the current is proportional to the electric field $\vec{E}$, so a constant electric field produces a constant current. Note also that the current in a wire is proportional to the average velocity with which the charges are moving.

a) A single point charge Q is moving freely under the influence of a constant electric field $\vec{E}$. Is such a charge moving with a constant velocity, or is it accelerating? Does this charge obey Ohm's law?

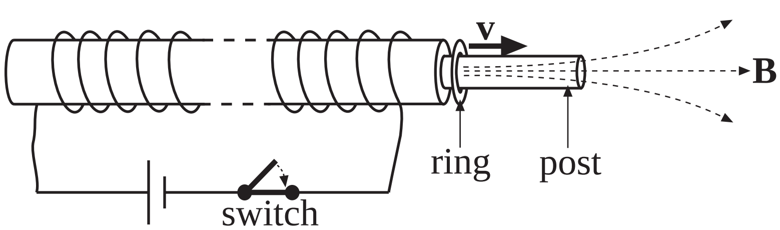
b) Now, instead of free point charges, we consider electrons flowing through a wire. Such electrons undergo occasional collisions with ions making up the metal. On average, an electron travels a distance d before colliding with an ion and stopping altogether. For a constant applied field $\vec{E}$, what is the average time between collisions? What is the average velocity of an electron? Does such a result agree with Ohm's law?

c) In part (b), we neglected the fact that the electrons (at room temperature) have a lot of thermal energy. Therefore, they are constantly jiggling about in random directions with a thermal velocity v_t. If this thermal velocity determines the time between electron-ion collisions (because v_t is much greater than the electron velocity due to any applied field), what is the new time between collisions? If we then apply an electric field $\vec{E}$ to the wire, what is the average electron velocity? Does this result agree with Ohm's law?

d) If the model of conduction in part (c) was accurate, what would happen to the current at very high and very low temperatures for a constant applied field $\vec{E}$? *[Nipun]*

3) Total *Recoil*

After seeing a recent Schwarzenegger film *ERASER*, Noah was impressed by Arnie's nasty rail-gun type weapon. The hand-held gun fired metal projectiles at near-light speed with rather destructive results. Noah decided to try to build one himself. He recalled a physics demonstration, in which a metal ring is placed on a metal post, attached to the top of a solenoid. The solenoid consists of a wire wrapped many times around a cylindrical metal core. When a sudden voltage was applied across the ends of the wire, the metal ring was launched away from the solenoid. Noah decided to make use of this effect to build his own rail gun.



Noah's first design contained the following components: solenoid (core wrapped with 2000 turns of wire, length = 50 cm, $\mu = 500\ \mu_o$), metal post (length = 5 cm, $\mu = 500\ \mu_o$), metal ring (inner diameter = 1 cm, outer diameter = 1.1 cm, thickness = 0.5 cm, resistance = $10^{-3}\ \Omega$, mass = 0.5 g).

a) With the metal ring anchored to the post so it could not slide, Noah turned on the voltage source and found that the current did not reach its maximum value immediately. He measured the current (in amperes) as a function of time and determined that it satisfied the following equation:

$$I = 1.0\,(1 - exp(-t\ /\ T))$$

with T = 0.04 s. What caused this time lag?

b) From a physics text, Noah determined that the magnitude of the magnetic field at the end of a solenoid is:

$$B = \mu N I / 2l$$

where N is the number of turns of the solenoid, l is the length of the solenoid, and I is the current in the wire. What value of magnetic field does Noah generate, as a function of time?

c) Recalling Faraday's law of induction, what is the magnitude and direction of the current induced in the anchored metal ring as a function of time? You can assume that the current response of the ring is instantaneous.

d) If the solenoid creates a uniform magnetic field, pointing exactly parallel to the metal ring's axis, what is the Lorentz force ($\vec{F} = q\ \vec{v} \times \vec{B}$, where $\vec{v}$ = velocity of particle of charge q) the ring feels as a function of time and in what direction? If the ring weren't anchored to the post, what would be its resulting motion?

e) The magnetic field is of course *not* uniformly pointing parallel to the solenoid axis but starts to diverge as it exits the solenoid (as shown in the diagram). For simplicity, assume that at the fixed position of the ring, the field lines leave the post at 15° to its surface. What is the new net force on the anchored metal ring, as a function of time?

f) Noah turned off the voltage supply and removed the anchors so that the metal ring could slide freely on the post. He closed the switch again and the ring was projected forward. To get an approximate value for the resultant speed, approximate the force applied to the metal ring for arbitrary time t to be the same as calculated in part e) at time $t = T$. Use this constant force to get a rough idea of the velocity of the ring as it left the metal post.

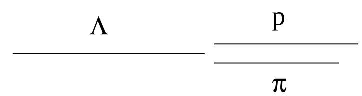
g) Noah decided to improve his design. In the movie, Arnie's gun was supposed to fire at near-light speed, but Noah didn't want to be greedy. He redesigned his device to launch the projectile with a velocity of 0.01 c (c = speed of light, 3×10^8 m•s^{-1}). Somehow Noah succeeded in building it and though it worked perfectly, Noah was seriously injured. What had he forgotten to take into consideration? *[James]*

4) The Lambda particle: outstanding in its ($\vec{B}$) field.

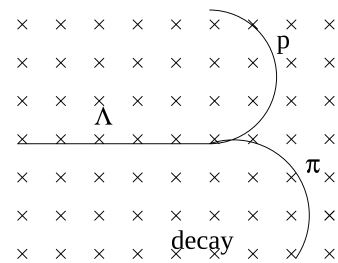
In the dawn of the field of particle physics, the discovery of the Λ^- particle played a significant role. It is relatively long-lived. One of the most important decay modes of this particle is the decay to proton and pion:

$$\Lambda \rightarrow p^+ + \pi^-$$

The minus sign for π denotes that the charge of this pion (π) is –e. Suppose the *free* kinematics of this decay is such that all particles before and after the decay move in the same direction (see figure). Now a magnetic field of 1 T, perpendicular to the motion of the particles, is applied to separate the moving proton and the pion. If the radius of the proton path is 3 cm, what is that of the pion? (rest-masses: $M_\Lambda = 1115$ MeV • c^{-2}, $M_p = 938$ MeV • c^{-2}, $M_\pi = 140$ MeV • c^{-2}, where c is the speed of light) This problem can be solved in a non-relativistic way (even though it will not be accurate), however, one should take into consideration the mass missing after decay, which will contribute to the kinetic energies of the proton and the pion. *[Chairul]*



decay
(no magnetic field)



decay
(under magnetic field)

5) Bohring after the truth

In the Bohr theory of the hydrogen atom an electron circles the proton in an orbit of radius 0.053 nm. The electrostatic attraction of the proton for the electron furnishes the centripetal force needed to hold the lectron in its circular orbit. Find, in this classical model:

(a) The force of electrical attraction between the proton and the electron.

(b) The speed of electron. What is this speed as a fraction of the speed of light c? *[Chairul]*

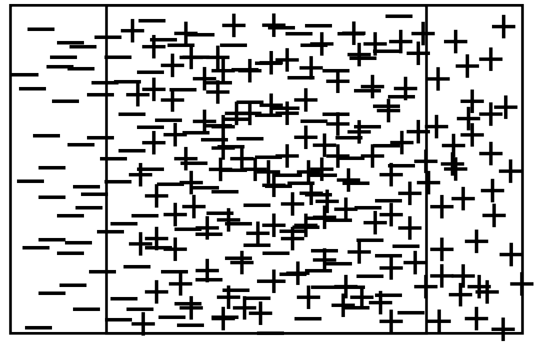
6) Plasma, plasma, on the wall

Practically all of the mass of the universe is not solid, not liquid, and not quite gaseous, but in the form of *plasma*. Plasma is any ionized matter, whether ionized by heating (as in the sun), by electric discharge (as inside a fluorescent light tube), or by dissociation in solution (as in blood).

When spacecraft re-enter the earth's atmosphere, the heat of re-entry produces an envelope or sheath of ionized air and ablating heat-shield surrounding the craft. The density of this plasma increases as the craft enters thicker atmosphere, and later dissipates as the vehicle slows down and cools. This plasma sheath can actually act like a mirror for radio-wave communications between vehicle and ground-control, reflecting the signals and blocking communication for a period during re-entry. A similar kind of reflection takes place for short-wave radio waves reflecting off the right part of the ionisphere. This happens because plasma is a good conductor and because plasma of a certain density has a characteristic resonant frequency, the *plasma frequency*.

Consider a hydrogen plasma, made of a mix of electrons and protons only, with equal density of electrons and protons. If in a region of plasma (as illustrated) we pull the electron component to the left, away from the proton component, it leaves two slabs of excess charge, like a parallel-plate capacitor.

a) Find the amount of charge in each charged slab, assuming a displacement x of the electrons relative to the protons, and a particle density of N cm^{-3} for the protons and the the electrons, each.



b) Find the force on the electrons and ions within this parallel-plate capacitor.

c) Show that this is a *restoring force* $F = - kx$, like a mass on a spring. Under the assumption that the protons have so much more mass that they hardly move, show that the force constant k leads to an oscillation frequency for the electrons sloshing back and forth given by:

$$\nu_p = \frac{1}{2\pi}\omega_p = \frac{1}{2\pi}\sqrt{\frac{N e^2}{\varepsilon_o m_e}}$$

where N is the particle density (electrons or ions), e is the electron charge, and m the electron mass. This is the frequency of best reflection of electromagnetic waves by the plasma — any frequency lower than this is reflected.

d) Find the minimum density N of plasma needed to reflect shortwave radio at λ = 30 m. *[Robin]*

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Solution Set 5: Electricity and Magnetism

1. **Gauss visits Planesville ...**

a) for a pt. charge in 2-D:

GAUSS' LAW: $E \bullet$ (circumference of circle) $= \dfrac{Q}{4\pi\varepsilon_o R^2} \, 2\pi R = \dfrac{Q}{2\varepsilon_o} \cdot \dfrac{1}{R} \neq \dfrac{Q}{\varepsilon_o}$

which is what one would expect from Gauss's law

b) to 'fix' this, make Coulomb's law $\propto \dfrac{1}{R}$

, i.e.,

$$E = \frac{Q}{2\pi\varepsilon_o R} \cdot \frac{1}{R} \qquad \text{(point charge)}$$

then $\quad E \cdot (\text{circumference of circle}) = \dfrac{Q}{2\pi\varepsilon_o} \cdot \dfrac{1}{R} \cdot 2\pi R = \dfrac{Q}{\varepsilon_o}$ (satisfies Gauss's law)

c) <u>inside disc</u>:

$$E \bullet 2\pi \bullet R = \sigma \bullet \pi R^2$$

where $\sigma = 0.5 \text{ mC} \bullet \text{m}^{-2}$

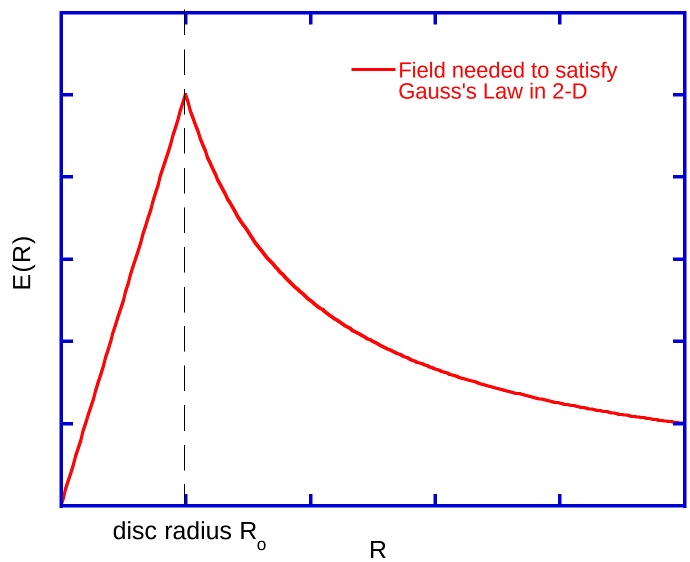
$$\Rightarrow E = \frac{\sigma}{2} R, \quad R < R_o, \quad (R_o = 10 \text{ cm})$$

<u>outside disc</u>:

$$E \bullet 2\pi R = \sigma \bullet \pi R_o^2$$

where $\sigma = 0.5 \text{ mC} \bullet \text{m}^{-2}$

$$\Rightarrow E = \frac{\sigma}{2} \frac{R_o^2}{R}$$



2. **It's not just a good idea ... it's Ohm's law!**

a) $F = ma = QE, \quad \therefore a = \dfrac{qE}{m}$ $\quad \therefore$ charge experiences constant acceleration

$J \propto V_{avg} = \dfrac{qEt}{m}$ $\quad \therefore$ current should increase linearly with time! ('non-Ohmic')

b) $d = \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2d}{a}} \propto \sqrt{\frac{2d}{E}}$

$$\therefore \; v_{avg} = \frac{1}{2}at = \frac{1}{2}a\sqrt{\frac{2d}{a}} \propto \sqrt{a} \propto \sqrt{E}$$

$\therefore$ it is non-ohmic

c) if $v_{thermal} >> v_{due\ to\ field}$, $t = \frac{d}{v_{thermal}}$

then, $v_{avg} = \frac{1}{2}at = \frac{1}{2}a\frac{d}{v_{thermal}} \propto E$

$\therefore$ it is ohmic

d) $\sigma \propto \frac{1}{v_{thermal}}$ from part (c)

so, as T increases, $v_{thermal}$ increases so the resistance is greater. If T decreases, $v_{thermal}$ decreases, so the resistance decreases.

3. **Total Recoil**

a) The voltage increase causes an increasing current in the wire of the solenoid. This increasing current creates an increasing magnetic field through the solenoid. This increasing magnetic field acts back on the solenoid wires to create an 'induced' voltage which *opposes* the original voltage change, and will continue until the system reaches steady-state. In brief, the effect is the 'induction' of a 'reverse' electric field (and current) in the solenoid.

b)
$$B = \frac{\mu NI}{2\ell} = \frac{\mu N}{2\ell}(1 - e^{-t/T})$$
$$= \frac{500\mu_o}{2(0.5)}(2000)(1 - e^{-t/0.04})$$
$$= 1.3(1 - e^{-t/0.04}) \text{ Tesla}$$

c) Faraday's law of induction
$$E = -\frac{d\Phi}{dt}$$
where ε is the induced electric field

M is the flux = $\vec{B} \cdot \vec{A}$

$\vec{B}$ is the magnetic field

$\vec{A}$ is orthogonal to the loop area

$|\vec{A}|$ = area of loop

Since the $\vec{B}$ direction is effectively orthogonal to the loop

$$\Phi = \vec{B}\cdot\vec{A} = A|B|$$

so

$$\varepsilon = -A\frac{\mathrm{d}B}{\mathrm{dt}}$$

$$= -A\ \frac{\mu N}{2\ell}\frac{\mathrm{d}}{\mathrm{dt}}(1 - \mathrm{e}^{-t/T}),\ \ \mathrm{a} = \left(\frac{0.01}{2}\right)^2 \pi\,\mathrm{m}^2$$

$$= 2.5 \times 10^{-3}\ \mathrm{e}^{-t/T}\ V$$

$$\therefore \mathrm{i} = \frac{\varepsilon}{R}$$

$$= 2.5\ \mathrm{e}^{-t/-0.4} \bullet A$$

(in opposite direction to current in solenoid.)

(We can ignore back-induction from the ring since it is very small.)

d) $F = q\vec{v} \times \vec{B}$

Let n be the charged particle density per unit length. Thus q = **–**neR

$$F = -\mathrm{ne}\ell\vec{v} \times \vec{B}$$

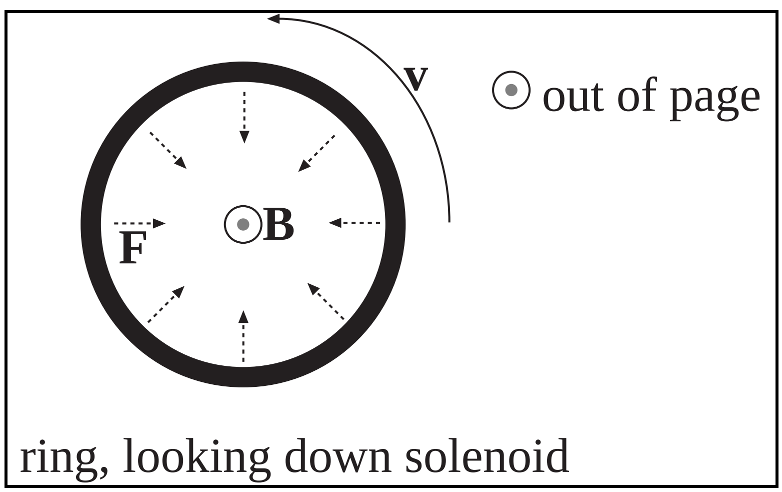
$$\frac{\vec{F}}{\ell} = -\mathrm{ne}\vec{v} \times \vec{B}$$

But within this model $\vec{B}$ is orthogonal to $\vec{v}$ and –ne |v| = i (current).

$$\frac{|\vec{F}|}{\ell} = \mathrm{i}\ |\vec{B}|$$

$$= 2.5\ \mathrm{e}^{-t/0.04}\ \frac{\mu N}{2\ell}(1 - \mathrm{e}^{-t/0.04})$$

$$= 3.3\ \mathrm{e}^{-t/0.04}\ (1 - \mathrm{e}^{-t/0.04})$$

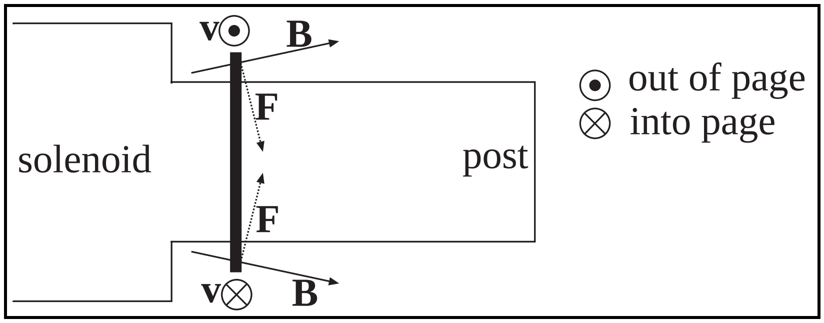


Direction of $\vec{F}$ is parallel to $\vec{v}\times\vec{B}$; $\vec{v}$ is parallel to the ring; $\vec{B}$ is orthogonal to the ring. Thus $\vec{F}$ is on the surface enclosed by the ring, pointing towards the centre of the ring.

This results in **NO** motion (unless the ring implodes).

e) Even in this case $\vec{v}$ is still orthogonal to $\vec{B}$, but now $\vec{F}$ is not on the surface enclosed by the ring.

The vertical components of $\vec{F}$ cancel, and we are left with a horizontal component facing to the right.



$$F_{NET} = \left|\frac{\vec{F}}{\ell}\right| \ell \sin$$

$$= 3.3^{-t/.04} (1 - e^{-t/.04}) (.01)\ \pi \sin 15^{\circ}$$

$$= 2.7 \times 10^{-2}\ e^{-t/.04} (1 - e^{-t/.04})\ \text{N}$$

f) $F(t = 0.04\ \text{s}) = 2.7 \times 10^{-2}\ e^{-1} (1 - e^{-1})$

$$= 6.3 \times 10^{-3}\ \text{N}$$

but $F = ma$

$$\therefore a = \frac{F}{m}$$

$$= 13\ \text{m/s}^2$$

For a constant acceleration:

$$v_2^{\,2} - v_1^{\,2} = 2ad \qquad \text{but } v_1 = 0$$

$$\Rightarrow \qquad v_2 = \sqrt{2ad}$$

$$= 1.1\ \text{m/s}$$

The ring would leave the post going 1.1 m/s.

g) Noah forgot about the conservation of linear momentum.

Consider m_1 to be the mass of the ring
m_2 to be the mass of the gun and Noah
v_1 to be the final velocity of the ring
v_2 to be the final velocity of the gun and Noah

Since the system starts at rest:

$$m_1 v_1 + m_2 v_2 = 0$$

$$v_1 = -v_2 \frac{m_2}{m_1}$$

Guess that Noah and gun mass is on the order of = 100 kg. Their final velocity is

$$v_1 = -3 \times 10^6 \quad \frac{0.0005}{100}$$
$$= 15\, m/s$$
$$= 54\, km/hr$$

Being accelerated to > 50 km/hr in such a short time could hurt quite a bit. Even if Noah wasn't hurt in the acceleration, the deceleration would probably be quite disastrous.

4. The Lambda particle: outstanding in its (B) field

We can determine the speed of proton from the radius of its circular path

$R_p = p_p/e\,B = M_p\, v_p / e\,B$ (for non-relativistic case).

So, its speed

$$v_p = R_p\, e\, B / M_p = (3 \times 10^{-2}\ \text{m})\, e\, (1\ \text{T}) / (938 \cdot 10^6\ \text{eV} \cdot c^{-2})$$
$$= 2.88 \times 10^6\ \text{m} \cdot \text{s}^{-1} = 0.00959\ c. \qquad (4.1)$$

The assumption of non-relativistic proton is applicable here.

We apply the conservation law of energy and momentum:

$$E_\Lambda = E_p + E_\pi$$
$$p_\Lambda = p_p + p_\pi$$

which can be written non-relativistically as

$$M_\Lambda c^2 + 1/2\, M_\Lambda v_\Lambda^2 = M_p c^2 + 1/2\, M_p v_p^2 + M_\pi c^2 + 1/2\, M_\pi v_\pi^2, \qquad (4.2)$$

$$M_\Lambda v_\Lambda = M_p v_p + M_\pi v_\pi. \qquad (4.3)$$

v_Λ can be eliminated by taking the square of (4.3) and substitute the term in (4.2)

$$M_\Lambda c^2 + 1/2\, (M_\Lambda)\, (M_p v_p + M_\pi v_\pi)^2 = M_p c^2 + 1/2\, M_p v_p^2 + M_\pi c^2 + 1/2\, M_\pi v_\pi^2$$

$$(1115 - 938 - 140)\ \text{MeV} + (938 \times 0.00959 + 140\, v_\pi/c)^2/(2 \times 1115)\ \text{MeV} - 938/2\ \text{MeV}\, (0.00959)^2 - 140/2\ \text{MeV}\, (v_\pi/c)^2 = 0$$

$$61.21\, (v_\pi / c)^2 - 1.13\, (v_\pi / c) - 36.99 = 0$$

So we can obtain $v_\pi = 0.787\ c = 2.36 \times 10^6\ \text{m} \cdot \text{s}^{-1}$. This is a relativistic problem and the assumption of non-relativistic pion is incorrect. Nevertheless, let us find out the radius of the pion path.

$$R_\pi = p_\pi / e\,B = M_\pi v_\pi / e\,B = (140 \times 10^6\ \text{eV} \cdot c^{-1} \cdot 0.787)/ e\, (1\ \text{T})$$
$$= 0.367\ \text{m} = 36.7\ \text{cm}$$

The full relativistic treatment (especially for pion) gives the radius of 40.0 cm.

Use the following formulae:

$M_\Lambda c^2 + 1/2\ M_\Lambda v_\Lambda{}^2 = M_p c^2 + 1/2\ M_p v_p{}^2 + E_\pi$,

$M_\Lambda v_\Lambda = M_p v_p + p_\pi$,

$E_\pi{}^2 = M_\pi c^4 + p_\pi c^2$,

to solve for p_π and $R_\pi = p_\pi / e B$.]

5. **Bohring after the truth**

a) Applying Coulomb's law:

$$F = k\, e^2/r^2 = (9.00 \times 10^9) \cdot (1.602 \times 10^{-19})^2 / (0.053 \times 10^{-9})^2$$
$$= 8.22 \times 10^{-8}\ \mathrm{N}$$

b) The centripetal force $F = m v^2 / r$. Hence the speed of the electron is

$$v = \sqrt{(F r / m)} = \{(8.22 \times 10^{-8}) \cdot (0.053 \times 10^{-9}) / (9.31 \times 10^{-31})\}^{1/2}$$
$$= 2.19 \times 10^6\ \mathrm{m \cdot s^{-1}} = 1/137\ c$$

6. **Plasma, plasma, on the wall**

a) Consider a slab of plasma, as in the question, which measures $x \times y \times z$. If the negative electrons are pushed off the positive protons to one side, by a tiny amount Δx, then there will be a thin slab of excess charge on either side — one positively charged and one negatively charged. The volume of each little excess slab will be:

$$(\Delta x) \times y \times z \qquad [1]$$

If the density of electrons is N [cm^{-3}], then the *amount* of excess charge will be

$$N\, e\ (\Delta x) \times y \times z = N\, e\ (\Delta x) \times A \qquad [2]$$

where A is the area of the side of the slab

In other words, what we have is something like a parallel-plate capacitor, with the two thin excess-charge slabs as the two charged plates of the capacitor.

b) For this, we can find the field between the two plates of a parallel-plate capacitor. In the middle, where there are both electrons and ions, the charges of each cancel each other out (unless you give them time to move around and redistribute themselves), so the net field in the middle is just what is produced by the thin slabs of excess (unbalanced) charge.

$$E = 4\,\pi\, k_c\, \sigma \qquad [3]$$

where $k_c = 9 \times 10^{-9}\ \mathrm{N\ m^2\ C^{-2}}$
σ = charge per unit area on each plate

[The easiest way to see this is to use Gauss's Law (if you know it), drawing little rectangular boxes with sides parallel to the plates, and having one side between the plates and the opposing side outside the plates. The field is perpendicular to the plates, by the symmetry of the situation, and it is quick to find the contribution each parallel plate makes to the overall field.]

Then from the second part of [2], we can find the charge *per unit area* of the two slabs of excess:

$$\sigma = (N\, e\, (\Delta x) \times A) / A = N\, e\, (\Delta x) \qquad [4]$$

and the field between the plates is:

$$E = 4\, \pi\, k_c\, N\, e\, (\Delta x) \qquad [5]$$

Now the charges in the middle see the field produced by the excess charges, and experience a force:

$$F = q\, E \qquad \text{which depends on q, +ve or –ve} \qquad [6]$$

This then gives us the force on each charge within the thin slab:

$$F = q\, E = q\, 4\, \pi\, k_c\, \sigma = e\, 4\, \pi\, k_c\, \sigma = e\, 4\, \pi\, k_c\, (N\, e\, (\Delta x)) \qquad [9]$$

$$= 4\pi\, k_c\, N\, e^2\, (\Delta x)$$

c) The whole block of electrons in the diagram of the question is free to move (the ions, being more massive, have a certain 'right of weigh') in the electric field they see. Most of them see the electric field between the two plates, and this pulls them back onto the ion background. So the force on an electron between the charge-excess thin slabs is a *restoring force* against the separation of charge.

$$F_r = -\, k\, (\Delta x) \qquad \text{where } k = 4\pi\, k_c\, N\, e^2 \qquad [10]$$

This is the restoring force of a simple harmonic oscillator (SHO) — it means the electrons mostly will slosh back and forth past the ions, barring collisions, once they are released from their initial displacement. Since the ions are relatively massive, they hardly move, but the electrons oscillate sinusoidally, as does a SHO, with a frequency which depends on the electron mass:

$$\omega_p = \qquad \text{where } \varepsilon_o = \frac{1}{4\pi\, k_c} \qquad [11]$$

Then $\nu_p = \omega_p / 2\pi$ to get the result shown. This is called an *electron plasma wave*.

d) When an electromagnetic wave is incident on a plasma, the E-field can begin to drive electrons back and forth, causing something like the excess charge in the model above. The electrons then oscillate at their own frequency, given just above in [11].

If the EM wave is at exactly the same frequency, it is in resonance with the electron plasma wave, and oscillations can grow very large. This means that a large AC current flows back and forth in the plasma, and that current can radiate EM waves itself, cancelling the light that comes in and sending it back as a reflection. At what frequency does this happen for shortwave radio at λ = 30m?

$$c = \lambda \nu \;\Rightarrow\; \nu = c / \lambda = 3 \times 10^8 \text{ cm•s}^{-1} / 30 \text{ m} = 1 \times 10^{7} \text{ s}^{-1} \qquad [12]$$

$$v = 1 \times 10^7 = \frac{1}{2\pi}\sqrt{\frac{Ne^2}{\varepsilon_o m_e}} \qquad v_p = \frac{1}{2\pi}\omega_p \frac{1}{2\pi}\sqrt{\frac{Ne^2}{\varepsilon_o m_e}}$$

$$\Rightarrow N = \frac{(2\pi v)^2 \varepsilon_o m_e}{e^2} = 1.24 \times 10^{12} \text{ m}^{-3} = 1.24 \times 10^6 \text{ cm}^{-3}$$

Compare this to gases at sea-level: 1 mole of ideal gas occupies 22.4 *l* @ STP,

$$22.4\ l = 22.4 \times 10^3 \text{ cm}^3$$

$$\frac{6.022 \times 10^{23}}{22.4 \times 10^3} \text{cm}^{-3} = 2.69 \times 10^{19} \text{ cm}^{-3}$$

Much less dense than this — shortwave radio is usually reflected not from fully ionized gas at atmospheric pressure, but from a layer of *partially ionized highly rarefied* gas high up, in the ionosphere.

1996-1997 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 6: AC Circuits and Electronics

Due March 28, 1997

1) What's a gigawatt? Thirty of 'em??

The province of Ontario requires something like 3×10^{10} watts of electrical power (a plausible guess?: a kilowatt or so for every person, and then twice as much again for all of industry), at 110 volts AC (*r.m.s.*). A heavy power cable might be an inch in diameter.

Say the whole province was supplied by a single cable a metre in diameter — what would happen? Take it that the cable has a resistance of 0.15 $\mu\Omega\ \mathrm{m}^{-1}$. Calculate:

a) the power lost per metre from 'I^2R losses,'

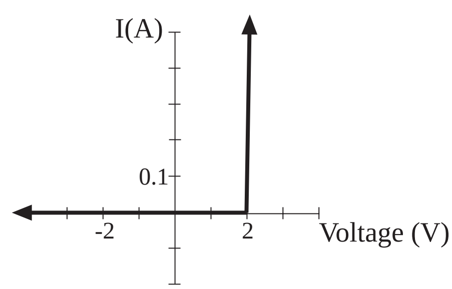
b) the length of cable over which the whole 3×10^{10} watts would be lost, and

c) how hot the cable will get, if it loses all this power by blackbody radiation.

Your answer ought to be preposterous — why is our electrical power system not ridiculous in this way? *[Robin]*

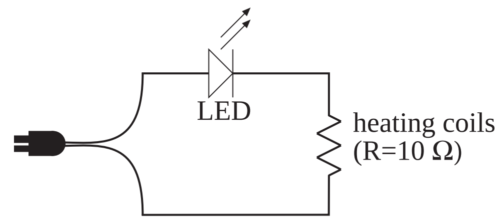
2) Waffling on the issues... the issue of waffles

A diode is an electronic device that allows current to flow only one way. An ideal diode has zero resistance for positive voltage and huge resistance for negative voltage. A slightly more realistic diode has a turn-on voltage, as shown in the figure at right. One typical use of a diode is as a light source. These light-emitting diodes (LED) produce visible light when a voltage greater than the turn-on voltage is applied across the device.



a) After many serious burns, Anton *(the Waffle Czar)* Duzzleeater pledged to improve current waffle-maker technology. He decided to incorporate an LED as an indicator that would light up whenever the waffle iron is plugged in. To begin, Anton decided to characterize his voltage source — normal household AC — so he measured the voltage as a function of time. Graph what Anton measured (2 cycles is sufficient). Don't forget to indicate your axis and scales.

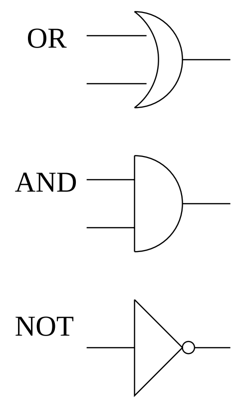
b) Without bothering to think about it, Anton wired the LED in series with the waffle heating element (schematic at right). If the diode had a similar *I–V* curve to the one shown in figure 1, what would be the voltage drop across it as a function of time? Does the LED emit light? When?



c) Unfortunately for Anton as soon as he plugged the circuit in, the diode flashed a brief burst of light and began smoking. The diode couldn't take the excessive current. Draw a more intelligent circuit that would avoid this problem. *[James]*

3) 2B ∨ (~2B) ← Q! ***[...to be or not to be, that is the question!]***

Wendy, a Physics Olympiad student, is given a bunch of digital gates. Unfortunately, she only has three types of gates: *AND*, *OR*, and *NOT*.



Her supervisor said to her that she can make any combination. Lo and behold, she finds

((NOT A) AND B) OR ((NOT B) AND A)

circuit is a very useful combination. Draw the circuit for that combination. Make a Boolean table for that circuit. Can you name that simple circuit? *[Chairul]*

4) Connect the dots!

Who said that physics wasn't all fun and games? See the figure on a separate page: the challenge is to connect the dots using the components given, to simultaneously achieve the desired voltage and current. Note that you may not need to use all the dots (or components) and you may want to attach more than two components to the same dot. The symbols 'A' and 'V' correspond to ideal ammeters and voltmeters. *[James]*

5) Impeding one's reactance

In a DC circuit, the total resistance is a sum of the individual resistances in the circuit, the nature of the sum depending on whether the resistors are in series or in parallel. In an AC circuit, the impedance, Z, plays the role of resistance and sums in the same way as resistance, however it has both real and imaginary parts. For ideal resistors, inductors and capacitors, the contributions to the total impedance are as follows:

resistors:	$Z_R = R$
inductive reactance:	$Z_L = i\omega L$
capacitive reactance:	$Z_C = (i\omega C)^{-1}$

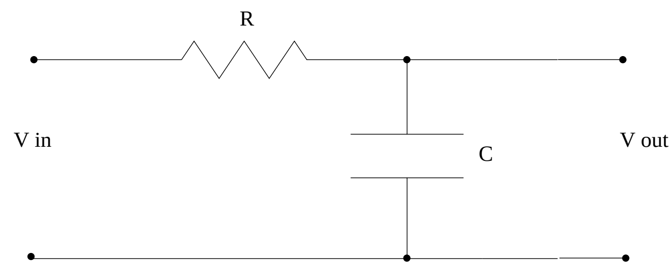
Here, $\omega = 2\pi f$, where f is the frequency of the input AC signal.

a) What is the total impedance of the circuit in the figure below?

b) We are interested in only the amplitude of V_{out}, which is given by

$$|V_{out}| = (V_{out} \bullet V_{out}^*)^{1/2}$$

where V_{out}^* is the complex conjugate of V_{out}. What is $|V_{out}|$ for this circuit? Sketch $|V_{out}| / |V_{in}|$ as a function of $\log_{10} \omega$.

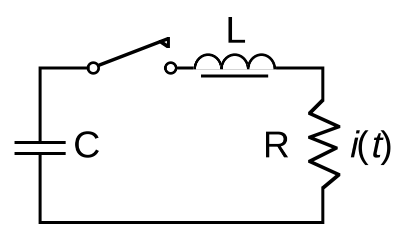


c) How would such a circuit affect a high-frequency input signal? A low-frequency signal? What do you conclude it might be used for? *[Nipun]*

6) Say '*cheese*'! (...Zap !!)

In high-power lasers pumped by a *flashlamp discharge*, large capacitors are charged to several kilovolts and then a fast, high-voltage switch closes to discharge them through the flashlamp. It is very much like an electronic camera flash, but hugely larger. These are the facts about the flashlamp circuit:

- the electric current pulse through the flashlamp must last for a certain amount of time τ in order best to 'pump' the laser
- if only a flashlamp and capacitor are connected in series, the current will oscillate back and forth through the lamp before dying away, and this can damage the lamp or cause it to explode
- the best shape results when the capacitor voltage is a simple decaying exponential
- the exponential should be the fastest-decaying one possible



To make these things possible, an inductor like a coil of heavy wire is added to the circuit, which can then be represented as shown at right: C – capacitor; L – inductor; R – resistance of flashlamp; $i(t)$ – time-dependent current through flashlamp.

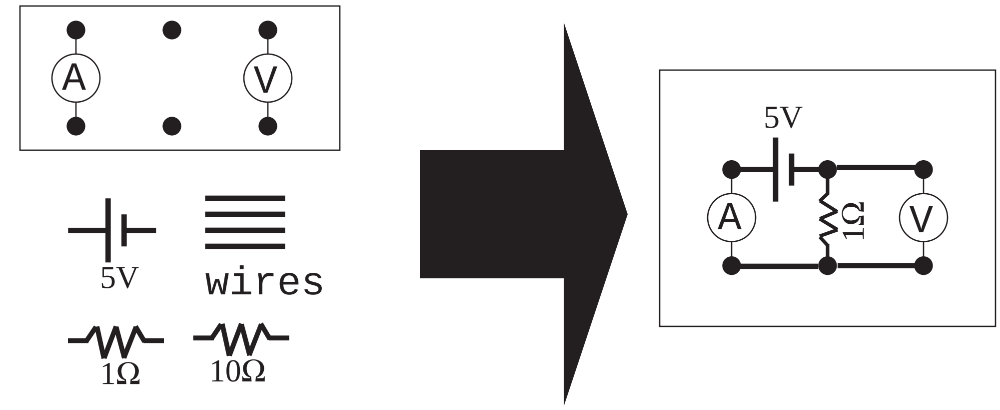
a) What is the equation for the charge q on the capacitor? What is the voltage across the capacitor in terms of the charge q on the capacitor? Show your approach.

b) Assuming the voltage is the fastest-possible decaying simple exponential, and that C and R are fixed, what must be the value of L? What is the exponential decay time, given $R = 1\Omega$, $C = 200\ \mu\text{F}$?

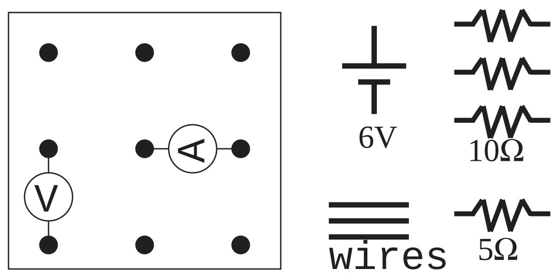
c) Find the current $i(t)$ through the resistor R. If the capacitor is charged to 2.5 kV, how much energy is discharged through the lamp? *[Robin]*

Figure for 'Connect the Dots!'

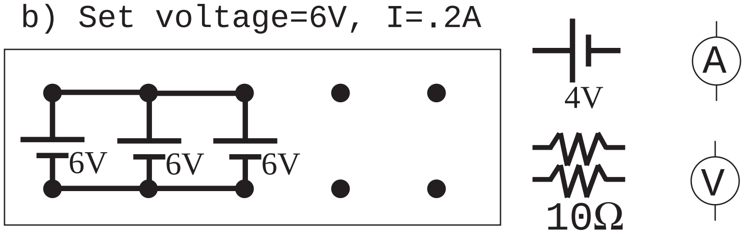
e.g.) Set voltage=5V, I=5A



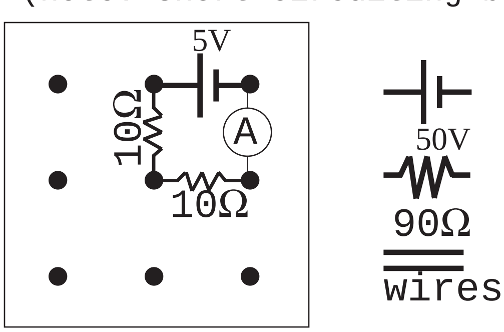
a) Set voltage=4V, I=.2A



b) Set voltage=6V, I=.2A



c) Set current I=0A
(note: short-circuiting batteries is baaaaaad!)



1996-1997 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 6: AC Circuits and Electronics

1. What's a gigawatt? Thirty of 'em??

a) Power lost per meter, with $P = IV$ (i.e., power and voltage are pre-set)

$$P_{loss} = I^2 R = (P/V_{rms})^2 R = (3 \times 10^{10} \text{ W} / 110 \text{ V})^2 \bullet (1.5 \times 10^{-7} \text{ ohm} \bullet \text{m}^{-1})$$
$$= 1.1 \times 10^{10} \text{ W} \bullet \text{m}^{-1}$$

b) length for loss: 3×10^{10} W / 1.1×10^{10} W • m^{-1} = 1.1×10^{10} W • m = 2.7 m (a pretty pathetic distance!)

c) power loss by Stefan-Boltzmann law:

$$P = \sigma A T^4$$

trivial power gain from 20°C room temperature, i.e., likewise:

$$P = \sigma A (293 \text{ K})^4$$

so that net:

$$P = \sigma A [(T^4 - (293 \text{ K})^4] \approx \sigma A T^4$$

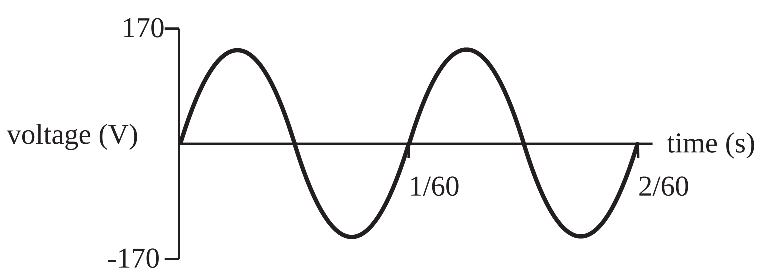
Area of surface of 2.7 m of cable is

$$A = 2\pi r L = 2 \bullet 3.14 (0.5 \text{ m}) 2.7 \text{ m} = 8.48 \text{ m}^2$$
$$T = (P/(\sigma A))^{1/4}$$
$$= (3 \times 10^{10} \text{ W} / (8.48 \text{ m}^2 \bullet 5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}))^{1/4}$$
$$= 15{,}800 \text{ K}$$

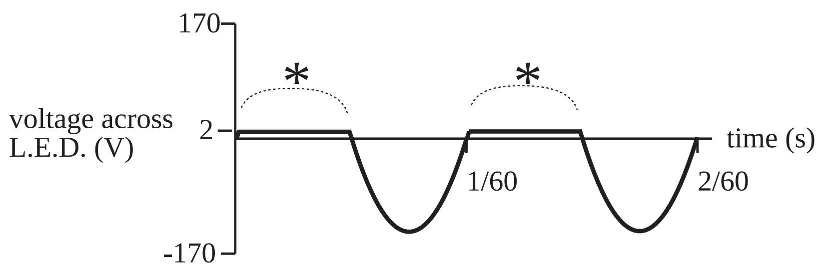
This doesn’t happen, basically because of Nikolai Tesla: with AC current we can easily manage step-up transformers, so the power is delivered at much higher voltages (like 250,000 V, I recall) with smaller losses, and then stepped-down several times, the last time quite near the delivery site. As well, the total diameter of cables is far in excess of 1m.

2. Waffling on the issues ... the issue of waffles

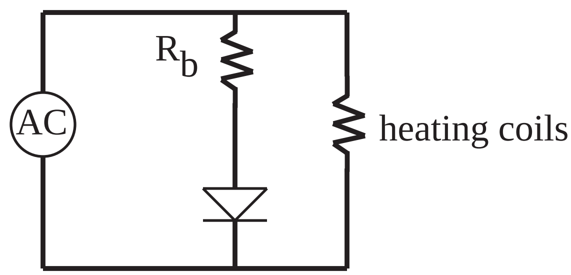
a) Normal household voltage in about 120 V (*rms*) which corresponds to 170 V peak-to-peak



b) The LED emits light only when it is forward-biased, as indicated by the asterisks in the graph below. Thus it flashes at 60 Hz.

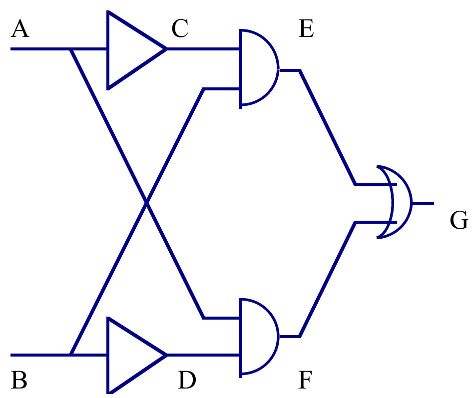


c) One possible circuit is shown at right, where
R_b is a large resistor. This reduces the amount of current that flows through the LED.



3. **2B ∨ (~2B) → Q! [... to be or not to be, that is the question!]**

The circuit of this combination is



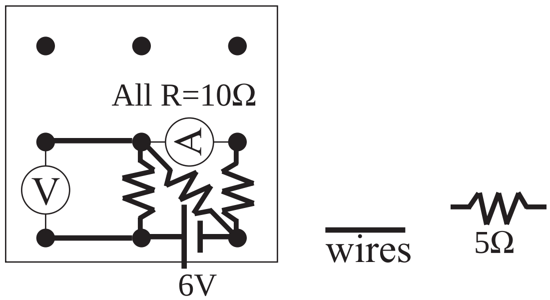
Boolean table

Input		intermediate nodes				**Output**
A	**B**	**C**	**D**	**E**	**F**	**G**
T	**T**	F	F	F	F	**F**
T	**F**	F	T	T	F	**T**
F	**T**	T	F	F	T	**T**
F	**F**	T	T	F	F	**F**

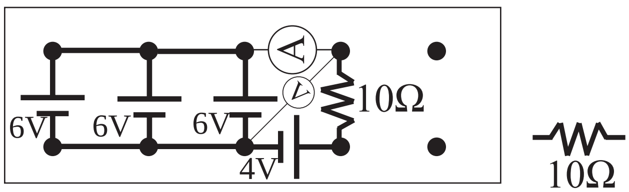
Clearly, this circuit is an exclusive OR gate (XOR).

4. **Connect the dots!**

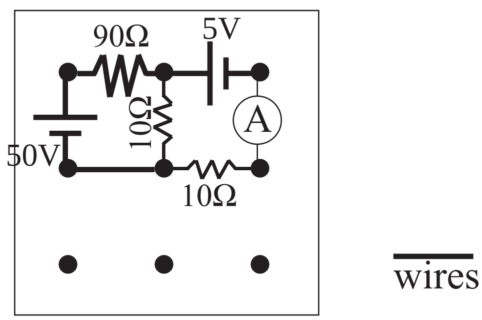
a) Set voltage=4V, I=.2A



b) Set voltage=6V, I=.2A



c) Set current I=0A
(note: short-circuiting batteries is baaaaaad!)



5. **Impeding one's reactance**

a) As they are seen by V_{IN}, the impedances are in series. Therefore, the total impedance is

$$Z_T = R + (i\omega\, C)^{-1}$$

Similarily, as seen by V_{OUT}

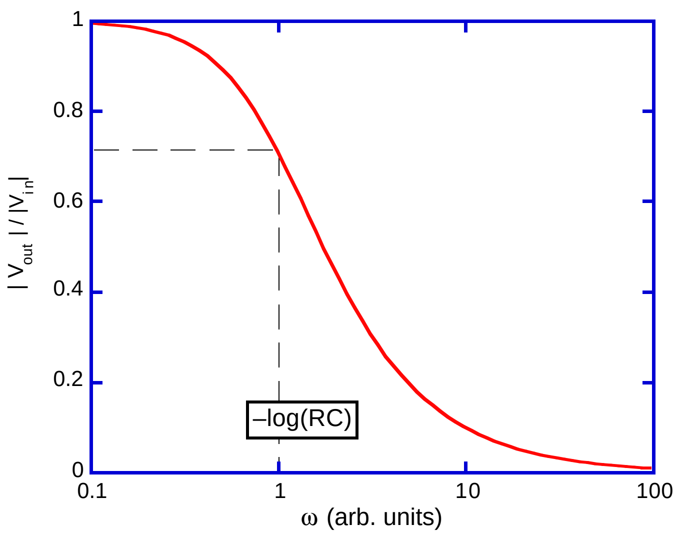
$$Z = \frac{1}{i\omega C}$$

b) The ratio of V_{OUT} to V_{IN} is given by the ratio of the impedances.

$$\frac{V_{OUT}}{V_{IN}} = \frac{Z_T}{Z} = \frac{1}{i\omega C}\left(R + \frac{1}{i\omega C}\right)^{-1} = \frac{1}{1+i\omega CR}$$

$$\frac{|V_{OUT}|}{|V_{IN}|} = \left[\frac{V_{OUT}}{V_{IN}} \times \left(\frac{V_{OUT}}{V_{IN}}\right)^{*}\right]^{\frac{1}{2}} = \left[\frac{1}{1+i\omega CR} \times \frac{1}{1-i\omega CR}\right]^{\frac{1}{2}} = \left[\frac{1}{1+\omega^2 C^2 R^2}\right]^{\frac{1}{2}}$$

so, $|V_{OUT}| = |V_{IN}|\left[\frac{1}{1+\omega^2 C^2 R^2}\right]^{\frac{1}{2}}$



c) for $\omega << \frac{1}{RC}$, all signal is passed.

for $\omega >> \frac{1}{RC}$, all voltage is dropped

So this is a filter which allows only low frequency *AC* signals to pass. The cutoff frequency can be changed by adjusting *R* and *C*

6. **Say 'cheese'! (... Zap !!)**

a) Each element has a charge-dependent voltage associated across it:

resistor: $\Delta V_R = IR$

capacitor: $\Delta V_C = q/c$

inductance: $\Delta V_R = -L\,\frac{dI}{dt}$

Since $I = \frac{dq}{dt}$, and since the potential drops around the circuit must sum to zero, we can establish a relation in terms of change of *charge*. Be careful about the signs:

- choose allocation of +q/–q on capacitor
- +q plate is at higher potential, so sketch current direction (flow of +ve charge) consistent with dropping potential

Then the current carries charge **away** from +q

so $I = \frac{dq}{dt}$

$$q/c + L\,\frac{d^2q}{dt^2} + \quad \frac{dq}{R} \; = 0$$

or

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=0$$ a ‘second-order ordinary differential equation’

Then

$$\Delta V_c \textit{ (rise)} + \Delta V_L \textit{ (drop)} + \Delta V_R \textit{ (drop)} = 0$$

$$q/c - L\frac{dI}{dt} - IR = 0$$

b) assume the form $V(t) = V_o e^{-t/\tau}$

so $q(t) = CV_o e^{-t/\tau}$

and substitute into differential equation

$$\{LCV_o\left(\frac{1}{\tau}\right)^2 + RCV_o\left(-\frac{1}{\tau}\right) + V_o\}e^{-t/\tau} = 0$$

$e^{-t/\tau}$ 0 must mean

$$LCV_o\left(\frac{1}{\tau}\right)^2 - RCV_o\left(\frac{1}{\tau}\right) + V_o = 0$$

or multiplying by τ^2 both sides

$$\tau^2 - RC\,\tau + LC = 0$$

then

$$\tau=\frac{-b\pm\sqrt{b^2-4ac}}{2ac}$$ quadratic formula

$$=\frac{+RC\pm\sqrt{R^2C^2-4LC}}{2}$$

$$= RC\left(\frac{1\pm\sqrt{1-\frac{4L}{R^2C}}}{2}\right)$$

The solution should be a **simple exponential**, meaning that the discrimant (root here) should be non-negative (else get imaginary numbers, which lead to cosine/sine terms — oscillations)

$$1-\frac{4L}{R^2C}>0 \quad \text{or} \quad 4L < R^2C$$

$$L > \frac{R^2 C}{4}$$

Then there are two solutions because of ±

$$q(t) = CV_o\,(a_1 e^{-t/\tau 1} + a_2 e^{-t/\tau 2})$$

$$a_1 + a_2 = 1$$

$$\tau_1 = RC\left(\frac{1+\sqrt{1-\frac{4L}{R^2C}}}{2}\right)$$

$$\tau_2 = RC\left(\frac{1-\sqrt{1-\frac{4L}{R^2C}}}{2}\right)$$

One of these τ_1, τ_2 is always the longer time — the fastest decay is as $\tau_1 \to \tau_2$, i.e. as

$$1-\frac{4L}{R^2C} \to 0^+ \qquad \textit{i.e.}\text{, from above}$$

then $\tau_1, \tau_2 \to \dfrac{RC}{2}$

$R = 1\Omega$, $C = 200\mu F \quad \Rightarrow \; T = 100\mu s$

c) This was rather (too?) tricky: one way to see the solution is to start with τ_1, τ_2 and the condition

$$0 = I(t) = -\frac{dq(\text{t})}{dt} = -CV_o\left(-a_1\left(\frac{1}{\tau_1}\right)e^{-t/\tau_1} - a_2\,\frac{1}{\tau_2}\,e^{-t/\tau_2}\right)$$

so $\quad I(\text{t} = 0) = -CV_o\left(-\dfrac{a_1}{\tau_1} - \dfrac{a_2}{\tau_2}\right)$

$a_1\tau_2 = -a_2\tau_1$

but $\quad a_1 + a_2 = 1$,

$\Rightarrow a_1 = \dfrac{\tau_1}{\tau_1 - \tau_2}$

$a_2 = \dfrac{-\tau_2}{\tau_1 - \tau_2}$

$$I(t) = +CV_o\left(\frac{1}{\tau_1 - \tau_2}e^{-t/\tau_1} - \frac{1}{\tau_1 - \tau_2}e^{-t/\tau_2}\right)$$

$$= \frac{CV_o}{(\tau_1 - \tau_2)}(e^{-t/\tau_1} - e^{-t/\tau_2})$$

as $L \to \frac{R^2C}{4}$ and $\tau_1 \to \tau_2$ this gives a $\frac{0}{0}$ indeterminacy, which we can work with l'Hôpital's rule:

numerator $\qquad \frac{\mathrm{d}}{\mathrm{d}\tau_1}(e^{-t/\tau_1} - e^{-t/\tau_2}) = e^{-t/\tau_1}\left(-t\left(-\frac{1}{\tau_1}^2\right)\right)$

demominator $\qquad \frac{\mathrm{d}}{\mathrm{d}\tau_1}(\tau_1 - \tau_2) = 1$

So the answer

$$I(t) = CV_o\frac{t}{\tau_2}e^{-t/\tau}$$

$$= \frac{CV_o}{(RC)^2}\ te^{-t/\tau} \qquad \text{(close enough)}$$

$$I(t) = \frac{V_o}{R^2C}\ te^{-\frac{t^2}{RC}}$$

$$E = \frac{1}{2}CV^2$$

$$= \frac{1}{2}200\mu\text{F}\ (2.5 \times 10^3\ \text{V})^2$$

= 625 J of energy.

1997-1998 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 1: General

Due November 3, 1997

1) Connect the dots!

Who said that physics wasn't all fun and games? See the figure on a separate page at the end of this set: your mission, should you decide to accept it, is to connect the dots using the components given, to simultaneously achieve the desired voltage *and* current. Note that you may not need to use *all* the dots (or components) and you may want to attach more than two components to the same dot. The symbols 'A' and 'V' correspond to ideal ammeters and voltmeters. *[James]*

2) Out with the bad air, in with the good...

In homes which meet the R-2000 standard of energy conservation, there are virtually no drafts which let in (cold) fresh air, and there might arise a risk of pollution-buildup from building materials, from cooking and cleaning, or from the occupants themselves. So for such houses, designers may include a heat-recovery ventilation system — basically a kind of heat-exchanger in which stale air exits the house but first passes much of its heat on to the incoming cooler air. Because the outgoing and incoming air streams are kept separate as they pass each other, but still allowed to exchange heat, fresh air enters the house fairly warm and stale air exits the house fairly cool. In fact, the efficiency is surprising.

It is a bit too complicated for POPTOR to calculate efficiency for a heat-recovery ventilation system, but consider this problem:

Say you have 1L of hot water (say, dyed red), of temperature 100ºC, and 1L of water (say, dyed blue), of temperature 0° C, presenting the warm and cool air above. Say also that you have many specially-insulated containers, any sizes you wish. These containers prevent heat loss to the air, but exchange heat perfectly when two (or more) touch each other. The question is like warming the incoming air in the house: without mixing the two together, to what temperature can you heat the initially cold (blue) water, and to what temperature can you cool the initially hot (red) water, by arrangements of letting them swap heat? *[Peter]*

[Canada Mortgage and Housing Corporation (CMHC) is a remarkable Canadian gold-mine of housing research and public information of every imaginable sort, including energy-efficient R-2000 houses, heat recovery ventilators, indoor pollution, etc. To order publications, check out: www.cmhc-schl.gc.ca/InfoCMHC/contact.html#Publications]

[Wonder what a heat-exchanger looks like? There are some photographs of heat-exchangers for industrial processes at: www.souheat.com/gallery.htm]

3) Waves à la mode

If you have ever sung in the shower, or maybe a marble washroom stall, you may have noticed that certain frequencies really pick up: they resonate within the space. For sound, the air molecules moving cannot move into the wall, so the walls are 'nodes' or zero-points for the molecular motion. Only certain-frequency waves naturally meet this condition, so they can be resonant within the space, and their reinforced amplitudes can be larger than that for other frequencies. These are 'standing waves,' and sometimes called 'modes.'

In making a laser, several items are brought together: a *laser cavity* (consisting of two mirrors facing one another) and an optical amplifying *laser medium* are the bare minimum. The laser cavity provides 'feedback' — a way to recirculate the light many times through the amplifying medium to make it more intense. Metallic mirrors also set a condition on the electric field in light: the transverse electric field must vanish at the surface of a conductor (see question below). Therefore, standing electromagnetic waves are formed in the laser cavity, which is for that reason sometimes called a 'laser resonator'.

a) Luciano Pavarotten is an operatic shower-singer, with a three-octave range starting at C below middle C (middle C is 256 Hz). You may not know that a note an octave above another has twice the frequency. If his shower stall is 1.5 m × 1 m × 2 m, how many standing wave modes in the shower can his voice excite? (Alas, only the standing waves are excited by his voice) How many different frequencies is this? You should ignore the issue of how his (large) body interferes with the sound in the shower.

b) Krystal has a Nd:glass ('neodymium glass') laser, which typically operates at about 1 micrometer (1 μm = 10^{-6} m) wavelength. It can amplify any wavelength within ±5 nanometers (5 nm = 5×10^{-9} m) of this wavelength. How many standing-wave modes of light are amplified in a laser resonator which is 80 cm long? *[Robin]*

4) Opposites attract, but a narcissist always loves himself...

Consider a grounded cylindrical conductor, of great length compared to its radius r. In orbit around this fat wire is a small charged particle of mass m and charge q. The distance from the point to the surface of the cylinder is $d << r$. Why should a charged particle orbit a grounded conductor? Explain the phenomenon, and find the period of the orbit. Neglect gravity, ignore air particles, and pay no attention to the man behind the curtain. *[Peter]*

<u>HINT:</u> For this quite-neat question, you may need to know these two things that you may *not* already know:

a) A perfect electrical conductor has *no electrical field inside*.

Not obvious? Say that there were a field inside: then the free charges in the conductor would see that field (and so a force), and would begin to move — any positive charges in one direction and any negative ones in the exactly opposite direction. If the conductor is of finite size, the charges can't keep on forever, but when would this whole thing stop? It would stop when the conductor ran out of free charges to move (phenomenally unlikely, for real-sized fields) or until the separated + and – charges made their own field which exactly cancelled the original field. The net result is zero DC field.

b) Therefore the surface of a perfect electrical conductor is a *surface of constant electric potential*.

If the field everywhere in the conductor is zero, then a small 'test' charge could be moved anywhere inside without seeing a force. Therefore the potential energy of the test charge is the same anywhere in the conductor— no force, no work, no potential energy change. So a charge a tiny distance inside the surface of the conductor is everywhere at the same potential energy: this means, then, that the surface of a conductor is an electrostatic equipotential.

5) Fermi Questions

The famous and fabulous Italian physicist Enrico Fermi was known for a curious type of question he sometimes posed his students. One of them was 'how many piano tuners are there in Chicago?'. The students were expected to work out the answer without resorting to any reference materials. The questions were an exercise in a very important problem-solving talent: developing the judgement that lets one make reasonable assumptions. With some notion of how many people are in Chicago, one might estimate roughly what percentage would have pianos, how much a piano-tuning costs, or how long it takes, etc. Here are some 'Fermi questions' to try. The idea is to use only what is already in your head, and to make plausible guesses for the rest. For each, give in brief point form the steps you followed in your reasoning. Estimates within a factor of 3 might be considered good, for some of these.

a) about how many piano tuners are there in Toronto? (Don't just check the Yellow Pages)

b) Claudio Chiapucci's bicycle has reflectors attached to the spokes of his wheels. He is fussy, and starts out each ride with the wheels set so each reflector is in the 12 o'clock position, at the top of the wheel, but they never stay that way. Roughly how long does he ride before they end up opposite each other, e.g., 10 o'clock and 4 o'clock?

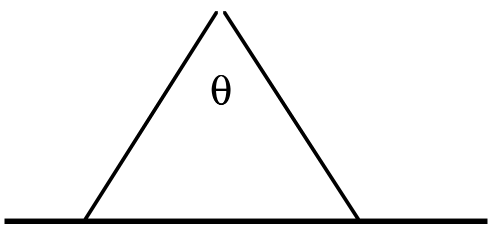
c) Jacques Villeneuve went shopping with his car one recent Saturday, but found his favorite small lot at Eaton's was full. He parked at the top of the lot, and from there he could see, and quickly get to, about fifty parking spaces. On average, about how long would he have to wait for someone to leave?

d) About how many pounds of cigarette-butts are dropped on sidewalks in Canada each year?

e) Half-time during an exciting Grey Cup football game: trips to the kitchen and washroom. How much, roughly, are the sudden increases in electrical power and water demand? *[Robin]*

6) 'Stacking the Deck — Friction in the House of Cards,' *by Kitty Kelly*

<u>PART I</u> – Consider the problem of making a structure out of cards. It's pretty difficult. Step One: consider the simplest problem of leaning two cards against each other on a flat surface. Let θ be the angle between the cards (we're assuming symmetrical stacking), μ_1 be the static coefficient of friction between a card and the smooth table-top, and μ_2 the coefficient between the two cards. I'm using a brand new Bicycle deck, so the cards are slippery and $\mu_2 < \mu_1$.



When two cards are stacked together at some angle θ, there is some minimum force F_{min} required in order to collapse them. Find out whether there is some optimal angle, θ_{best} — an angle at which the cards will be most stable (i.e., the *largest* F_{min}) — and determine that angle.

(θ = 180°, with the cards lying flat on the ground is stable, alright, but useless — that special case does NOT count!)

<u>PART II</u> – Suppose that you want actually to predict the value of the angle θ_{best}, using results from PART I. How do you get μ_1 and μ_2 — just look them up in the CRC Handbook of Playing Card Constants? *Doh!* You could measure it, you know.

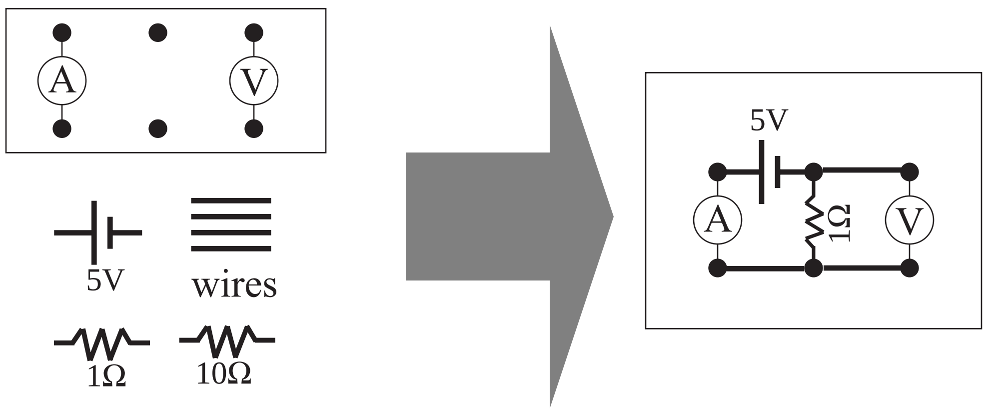
a) Find a deck of playing cards — a couple of baseball or hockey cards (or perhaps even computer diskettes) will do. For your cards, and your smooth table-top, measure μ_1 and μ_2. Describe what you do, and why, and figure out estimates of uncertainties about your measurement process. Give an error-range for your final answer; this is essential because it represents to others how closely they can trust your exact answer.

Note that everyone will (most likely) get a different answer — it's how you do it that matters for POPTOR.

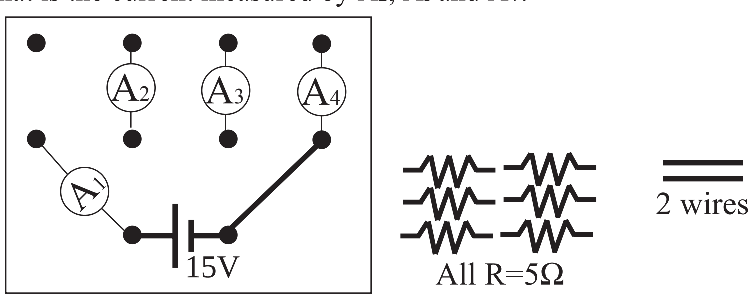
b) Now knowing μ_1 and μ_2, predict θ_{best}. Does it seem reasonable? Explain why. *[Peter]*

For Question #1:

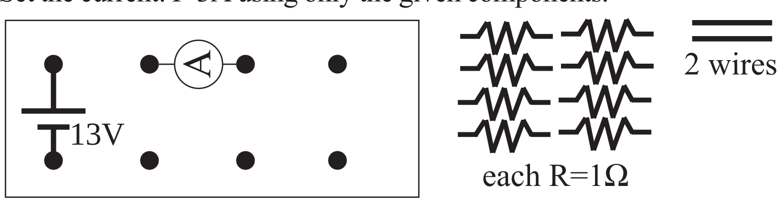
e.g.) Set the voltage: V=5V, current: I=5A using only the given components.



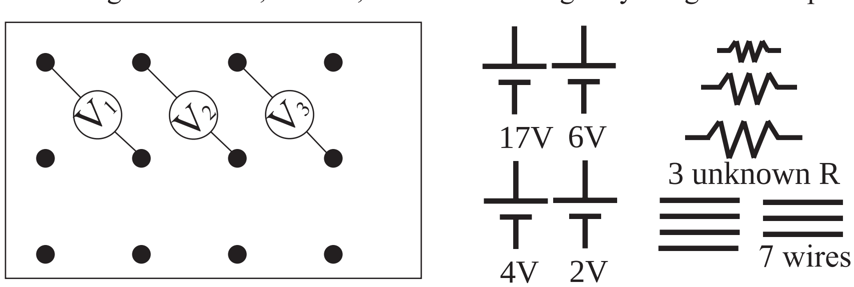
a) Set the current measured by A_1 to be I=2A, using only the given components.
What is the current measured by A_2, A_3 and A_4?



b) Set the current: I=3A using only the given components.



c) Set voltages: V_1=11V, V_2=7V, and V_3=5V using only the given components.



1997-1998 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 2: Mechanics

1) The Pluto Problem

Kepler's Second Law results from the conservation of angular momentum of the planet about the sun. According to this law, the straight line joining the sun and a given planet sweeps out equal areas in equal intervals of time. Thus, the ratio of the time interval t, during which the Pluto's elliptical orbit is situated inside the Neptune's circular orbit, to the Pluto's orbital period T is equal to

$$t/T = S/(\pi ab)$$

The denominator of the right side is the area of the ellipse representing the Pluto's orbit, and the numerator S is the area of the sector of subtended angle 50 degrees. We can estimate the value of S as the area of a circular sector $(50/360)\pi R^2$, where for more accuracy we can substitute R by $(r_{min} + R)/2$. Therefore,

$$t = (50/360) \times (((r_{min} + R)/2)^2 \times T)/(ab) = (5/36) \times ((4.4 + 4.25)/2)^2 \times$$

$$(248)/(5.9 \times 5.73) = (5/36) \times (18.71 \times 248)/(5.9 \times 5.73) = \text{approx. } 20 \text{ years.}$$

It turns out that the question was in error, and Pluto became closer to the sun in **1979**, not 1969. Therefore, Pluto will be again the ninth planet from the sun in 1999. *[http://seds.lpl.arizona.edu/nineplanets/nineplanets/pluto.html]*

2) Free falling, at $4.50 a throw

a) Let l be the length of a non-stretched rope ($l + h < H$), k is its elasticity constant, m the mass of a person who is jumping down and v is a value we are looking for. It is obvious that at the equilibrium height h your velocity had the maximum value. We can write the following three equations:

1) the equilibrium condition at the height h :

$mg = k\ (H - l - h)$ [2.1]

2) the energy conservation law at the height h :

$mgH = mgh + (k/2)(H - l - h)^2 + (1/2)mv^2$ [2.2]

3) the energy conservation law on the ground level :

$mgH = (1/2)\ k\ (H - l)^2$ [2.3]

Divide [3] by [1] and obtain

$l = \sqrt{H\ (H - 2h)}$ [2.4]

Then rewrite [2] using [1] in the following form :

$mg\ (H - h) = (mg/2)\ (H - l - h) + (mv^2/2)$

and get

$v = \sqrt{g\ (H - h + l)}$ [2.5]

or using [4],

$v = \sqrt{g\ (H - h + \sqrt{H\ (H - 2h)})}$ [2.6]

If $H = 25$ m, from [6] we have $v = 16.2$ m/sec or 58 km/hour. If $H = 50$ m, then $v = 28$ m/sec or 101 km/hour.

b) Only the third equation will be different compared to case a) :

$mg\ (H - h_1) = (1/2)\ k\ (H - l - h_1)^2$ [2.3′]

Hence,

$l = \sqrt{H\ (H - 2h) + h_1\ (2h - h_1)}$ [2.4′]

Use [4′] in [5] and get the new value for v. If $H = 25$ m, then $v = 17.1$ m/sec or 61.4 km/hour. If $H = 50$ m, then $v = 28.2$ m/sec of 101.6 km/hour.

c) Elasticity constant can be found from [1] : $k = (mg)/(H - l - h)$.

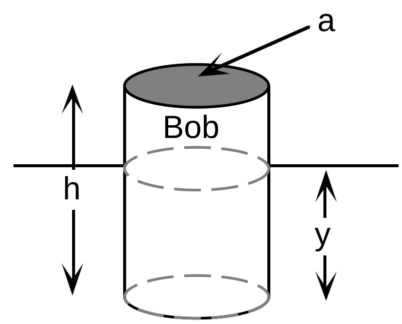
For $H = 25$ m, $h = 10$ m, $m = 100$ kg and $l = 11.18$ m (from [4]), we have $k = 261.8$ H/m. For $H = 50$ m, $h = 10$ m, $m = 100$ kg and $l = 38.73$ m (from [4]), we have $k = 787.4$ H/m (the rope must be 3 times stronger for the height $H = 50$ m compared to the one for the height $H = 25$ m).

3) 'Bob's your uncle', or sometimes he's a *simple harmonic oscillator (SHO)*

a) submerged length of the cylinder — Y
force of gravity on the cylinder: — $F = m_{cyl}g$
volume of water displaced by the cylinder — Ya

So the force of gravity on the cylinder = $Ya\rho_{water}g$

At equilibrium, the two forces are equal, so:



$$m_{cyl}g = Ya\rho_{water}\, g$$

$$Y = \frac{m_{cyl}}{\rho_{water}\pi a}$$

b) For a small displacement from equilibrium, y, there is a small difference between the buoyant force and gravity. Assume we push the cylinder slightly deeper into the water than the equilibrium point. Then the *increase* in the buoyant force is an excess force, in the amount:

$$F = -mg = -ay\rho_{water}\, g$$

and likewise if we pull the cylinder up, the excess is in the opposite direction.

c) (Note: in this section, A is acceleration, while a is the cross sectional area of Bob)

$$F = m_{cyl}A = m_{cyl}\frac{d^2 y}{dt^2} = -\rho_{water}ayg \qquad [3.1]$$

so,

$$\frac{d^2 y}{dt^2} = -\frac{\rho_{water}ag}{m_{cyl}}y. \qquad [3.2]$$

d) Just by substituting, it is easy to show that $y(t) = y_0 \cos(\omega t+\phi)$ is a solution of *any* equation of the form

$$m(d^2y/dt^2) = -ky,$$

as in part (c). We will use this same form in future POPTOR problems!

The maximum/minimum value that y can take occurs when $\cos(\omega t+\phi) = \pm 1$. So, y_0 is the maximum amplitude of Bob, measured from the equilibrium point.

If we let go of Bob at an amplitude y_0, at time t=0, then this should be his maximum amplitude. Then

$$y(t=0) = y_0 \cos\phi = y_0 \qquad [3.3]$$

holds, whether we start off by lifting Bob or by pushing him down a little. This requires $\cos(\phi) = 1$, which is true as long as we make $\phi = n\,2\pi$, where n is any integer.

For convenience, we simply set $\phi = 0$. (If however t=0 is chosen at some other point in Bob's oscillation, we will need ϕ to take on some other value.)

$$\frac{dy}{dt} = -\omega y_0 \sin(\omega t + \phi) \qquad [3.4]$$

To determine ω for Bob, note that,

$$\frac{d^2 y}{dt^2} = -\omega^2 y_0 \cos(\omega t + \phi) = -\omega^2 y = -\frac{k}{m} y \qquad [3.5]$$

so, $\omega = (k/m)^{1/2}$. From part (c),

$$\frac{k}{m_{cyl}} = \frac{\rho_{water} a g}{m_{cyl}} \qquad [3.6]$$

therefore,

$$\omega = \sqrt{\frac{\rho_{water} a g}{m_{cyl}}} \qquad [3.7]$$

e) F = buoyancy + gravity

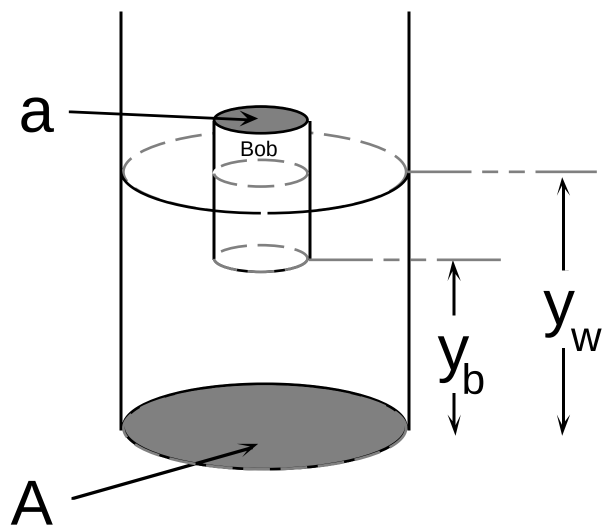
$= \rho_w g a(y_w - y_B) - m_B g$

at equilibrium F = 0, so

$0 = \rho_w g a(y_{wo} - y_{Bo}) - m_B g$

$m_B g = \rho_w g a(y_{wo} - y_{Bo})$

where y_{wo} and y_{Bo} are equilibrium (rest) values.



In general,

$m_B \bullet$ accel'n = F,

so,

$m_B \ddot{y}_B = \rho_w g a(y_w - y_B) - m_B g$, but we substitute for $m_B g$ the value from equilibrium:

$m_B \ddot{y}_B = \rho_w g a(y_w - y_B) - \rho_w g a(y_{wo} - y_{Bo})$

$= \rho_w g a((y_w - y_{wo}) - (y_B - y_{Bo}))$

$= \rho_w g a(\Delta y_w - \Delta y_B)$; $\qquad \Delta y_w \equiv y_w - y_{wo}$

since $(\ddot{y}_{Bo}) = 0$, we have

$\ddot{y}_B = (\Delta\ddot{y}_B)$, (where by $(\Delta\ddot{y}_B)$ I mean the second derivative in time of (Δy_B)).

Thus,

$$m_B(\Delta\ddot{y}_B) = \rho_w g a(\Delta y_w - \Delta y_B) \quad [3.9]$$

The overall volume of water in the tank is constant, and this is the same value at equilibrium:

$$V = A\cdot y_B + (A - a)\,(y_w - y_B) = \text{const}$$

$$= A\, y_{Bo} + (A - a)\,(y_{wo} - y_{Bo}) \quad [3.10]$$

thus

$$A(\Delta y_B) + (A - a)\,(\Delta y_w - \Delta y_B) = 0$$

or

$$(A - a)\,(\Delta y_w) + a(\Delta y_B) = 0 \quad [3.11]$$

Substituting this into $(A - a) \times$ I we get

$$m_B(A - a)\,(\Delta\ddot{y}_B) = \rho_w a g[(A - a)\,\Delta y_w - (A - a)\,\Delta y_B]$$

$$= \rho_w a g[-a\Delta y_B - (A - a)\,\Delta y_B]$$

$$= \rho_w a g[-A\Delta y_B\,]$$

So

$$m_B(\Delta y_B) + \frac{aA}{(A-a)} g\rho_w \Delta y_B = 0 \quad [3.12]$$

Thus

$$\omega_o^2 = \frac{aA}{(A-a)}\frac{g\rho_w}{m_B} \quad [3.13]$$

Note: as $A \to \infty$, i.e., as the cylinder becomes huge, $\omega_o^2 \to \dfrac{a g\rho_w}{m_B}$ as above!

$$T = 2\pi\sqrt{\frac{(A-a)}{aA}\frac{m_B}{g\rho_w}} \quad [3.14]$$

i.e., the period is smaller if the water level also rises and falls.

4) Full of fury, and signifying nothing...

a) The system will annihilate. There is a net force of

$$\frac{3kee}{2a^2} \text{ [towards centre of square]} \qquad (k = \frac{1}{4\pi\varepsilon_o}).$$

on each particle

To see this, consider any particle — the net force is, by symmetry, directed towards the centre of the square. It has a magnitude of:

$$F_{NET} = \frac{kee}{a^2} + \frac{kee}{a^2} - \frac{kee}{2a^2} = \frac{3kee}{2a^2}$$

Since this result holds for any a, and the masses of each particle are equal, and they start from rest, they will accelerate toward the centre of the square at the same rate. Since they all start from rest, their velocities along the diagonals will be equal, and hence their positions will be equal. So the formation will be preserved, and they will keep accelerating inwards.

Their final velocities [same number] could be found from:

$$potential = \frac{1}{2}\sum_{i \neq j} \frac{kq_i q_j}{r_{ij}} = \frac{4kee}{a} + \frac{2kee}{\sqrt{2}a}$$

Hence,

$$\frac{4kee}{b} + \frac{2kee}{\sqrt{2}b} - \frac{4kee}{a} - \frac{2kee}{\sqrt{2}a} = 4\left(\frac{m_e V^2}{2}\right)$$

where b is the final size of the box. Note that this gives infinity for b = 0.

b) This is not as easy as it looks.

The particles will obviously be repelled to some large distance, where their interactions will be virtually zero. By symmetry, the final velocity of each of the positrons will be equal and opposite; the same is true for the protons.

Conserving energy

$$E = \frac{kqq}{\sqrt{2}a} + \frac{kqq}{\sqrt{2}a} + 4\frac{kqq}{a} = 2\frac{mV^2}{2} + 2\frac{MU^2}{2}$$

where V is the final speed of the positrons, U is the final speed of the protons and q is the magnitude of the charge of an electron ($k = 1/(4\pi\varepsilon_0)$). The terms on the left are due to: the energy of the positron-positron pair, the proton-proton pair, and the positrons' with their neighbouring protons (respectively).

Conserving momentum yields no new information — it is conserved because of the symmetry of the system.

Note, however, that because M >> m, the accelerations of the protons = 1/2000 of accelerations of positrons — thus, the positrons will escape before the protons will have moved. I.e., the protons are virtually stationary. This gives

$$E = \frac{kqq}{\sqrt{2}a} + \frac{kqq}{\sqrt{2}a} + 4\frac{kqq}{a} = 2\frac{mV^2}{2} + \frac{kqq}{\sqrt{2}a}$$

Solving,

$$mV^2 = \frac{kqq}{\sqrt{2}a} + 4\frac{kqq}{a} \qquad [1]$$

Now, that the positrons have escaped, we can consider the protons. We get:

$$MU^2 = \frac{kqq}{\sqrt{2}a} \qquad [2]$$

Dividing [1] by [2] gives

$$\frac{mV^2}{MU^2} + 1 = 4\sqrt{2} + 2$$

$$\frac{V}{U} = \sqrt{\frac{M}{m}\left(4\sqrt{2}+1\right)}$$

Or, 115 plugging in M/m = 2000.

5) Sikorsky meets Newton

From Newton's 2nd law, the force is dp/dt.

Now, we'll put ourselves in a frame rotating with the blades.

At a distance r from the origin, an oncoming air particle has a horizontal velocity of r*ω m/s.

We'll assume that the collision is partially elastic with a parameter k. I.e., after the collision the air particle has a velocity k*r*ω (for perfect elasticity, k = 1).

After a collision, the particles will move downward with some speed U and backward with some speed V. Note that the magnitude of their velocity is $kr\varpi = \sqrt{U^2 + V^2}$

By Newton’s second law there must be equal an opposite forces acting on the blade — thus there is a vertical component (lift) mU, and a horizontal one mV, slowing the rotation of the blade. This would make the problem complicated, but luckily we assumed that the rotation rate is constant (the engine compensates). (m = mass of one air particle)

Now, after rotating through an angle dα, the area swept out by a blade is:

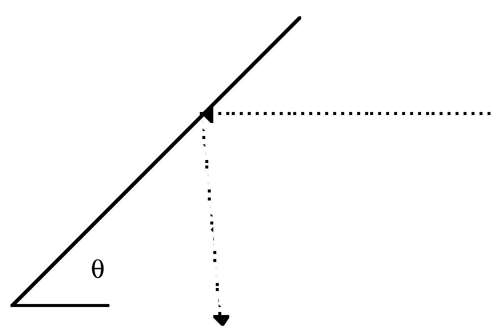
$$area = 1/2 * d\alpha * a^2$$

The volume thus swept out is:

$$volume = 1/2 * d\alpha * a^2 * b\sin(\theta)$$

And the mass hit is:

$$mass = \rho * 1/2 * d\alpha * a^2 * b\sin(\theta)$$



The downward component of k*r*ω (U) is: $kr\varpi\cos(\theta)\sin(\theta)$

Note that this varies this r — technically speaking, we should integrate here, but we'll make life easy and use the average r — that is, R/2.

Thus, for 1 blade:

$$F = \left(\rho * 1/2 * d\alpha / dt * a^2 * b\sin(\theta)\right) * \left(kR/2\varpi * \cos(\theta) * \sin(\theta)\right)$$

For N blades:

$$F = N\left(\rho * 1/2 * \varpi * a^2 * b\sin(\theta)\right) * \left(kR/2\varpi * \cos(\theta) * \sin(\theta)\right)$$

BONUS:

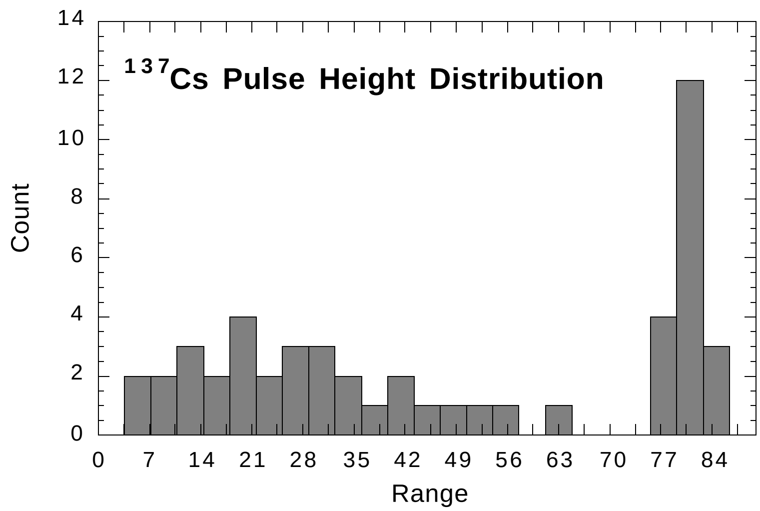
Well, tiliting the rotor axis seems like a good idea (so part of the lift is sideways) but I suppose too hard to build. What is done, instead, is something much more interesting — during a part of the cycle (say, when passing over the front) each blade's pitch (θ) is changed — this adds an extra force, as necessary.

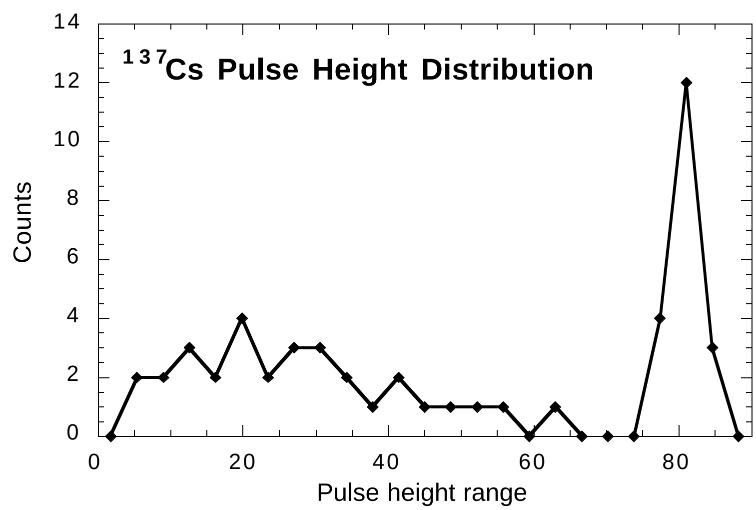
Electronic circuits control that ω is steady and the helicopter does not spin around.

6) Pulse-height analysis

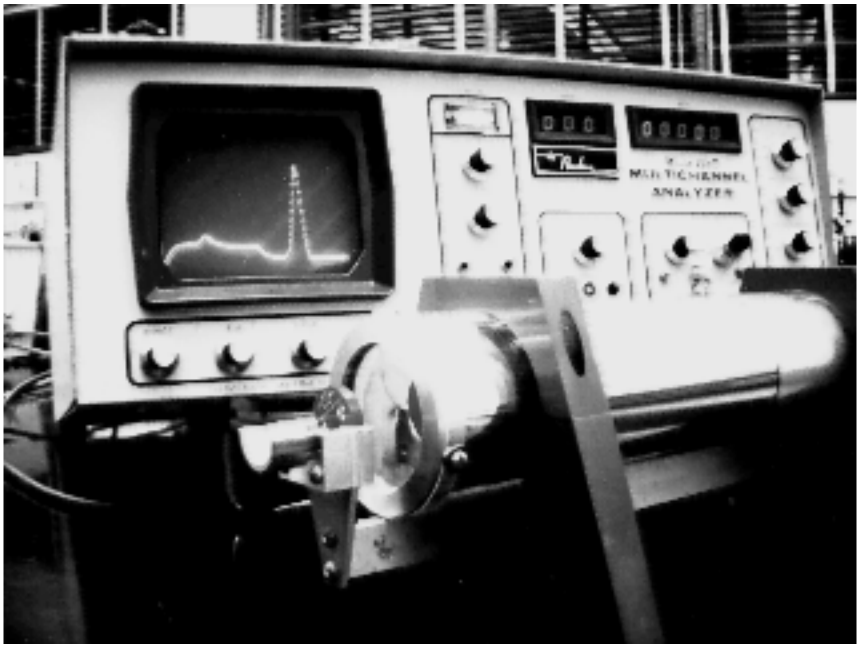
a) The data in the table can be plotted as a histogram — a range of values of the output, and then a count of how many times the values fell into that range. Here is such a table, and two different ways of plotting it:

X_o	N
1.80	0
5.40	2
9.00	2
12.6	3
16.2	2
19.8	4
23.4	2
27.0	3
30.6	3
34.2	2
37.8	1
41.4	2
45.0	1
48.6	1
52.2	1
55.8	1
59.4	0
63.0	1
66.6	0
70.2	0
73.8	0
77.4	4
81.0	12
84.6	3
88.2	0





The first plot better shows how there are bins, and that the columns marked indicate how many count data-points fell into the range of each box of pulse-heights.



For comparison, here is how a 'pulse-height analyzer' package does the job:

In the foreground is the Cs gamma-radioactive source, and the package which holds the NaI scintillator and the photomultiplier tube. In the background is the electronic apparatus which sorts pulses as they happen, and adds them to bins much as you did by hand.

b) The raindrops will have a *terminal velocity*, i.e., some maximum constant velocity. If the velocity is terminal (constant) there must be no net force on the droplet. So we can find the terminal velocity by figuring out the balance of forces: gravity and wind resistance.

Working from:

g	acceleration due to gravity:	$980\ \text{cm s}^{-1}$
ρ_w	density of water:	$1\ \text{g cm}^{-3}$
ρ_f	density of air:	$1.2928 \times 10^{-3}\ \text{g cm}^{-3}$ at NTP
r	droplet radius	

we need to subtract the buoyant force of the air — a small correction given the density of water compared to air.

$$F_g = m_{drop}g - m_{air}g = (\rho_w - \rho_f)Vg = (\rho_w - \rho_f)\frac{4\pi}{3}r^3 g \qquad [6.1]$$

For wind resistance, with density of air ρ_f, drag coefficient C_D, speed v and cross-sectional area S:

$$F_d = \frac{1}{2}\rho_f C_D v^2 S = \frac{1}{2}\rho_f C_D v_{term}^2 \pi r^2 \qquad [6.2]$$

Thus we start with $F_g = F_d$ so that:

$$\frac{1}{2}\rho_f C_D v_{term}{}^2 \pi r^2 = (\rho_w - \rho_f)\frac{4\pi}{3} r^3 g \quad \textit{or, cancelling,}$$

$$\frac{1}{2}\rho_f C_D v_{term}{}^2 = (\rho_w - \rho_f)\frac{4\pi}{3} g r \qquad [6.3]$$

If we assume that we have a sphere, we can consider our two limiting cases for C_D: we assume each case to start, and then see from the results what it will take to justify the assumption.

Case I: $R_e < 80$; $\quad C_D \sim 24/R_e$.

First we find v_{term}:

$$\frac{1}{2}\rho_f \frac{24}{R_e} v_{term}{}^2 = (\rho_w - \rho_f)\frac{4}{3} g r \qquad \textit{and substituting for } R_e$$

$$\frac{6\eta v_{term}}{r} = (\rho_w - \rho_f)\frac{4}{3} g r \qquad \textit{thus, solving for } v_{term}, \qquad [6.4]$$

$$v_{term} = \frac{2(\rho_w - \rho_f)g}{9\eta} r^2$$

Then this goes into the formula for the Reynolds number:

$$R_e = \frac{\rho_f v_{term} 2r}{\eta}$$

$$= \rho_f \left[\frac{(\rho_w - \rho_f)2g r^2}{9\eta}\right]\cdot\frac{2r}{\eta} \qquad [6.5]$$

$$= \frac{\rho_f(\rho_w - \rho_f)4g r^3}{9\eta^2}$$

Now, using this we require that $\boldsymbol{R_e < 80}$, letting us find the corresponding condition on r:

$$80 > R_e = \frac{\rho_f(\rho_w - \rho_f)4g r^3}{9\eta^2} \quad \textit{then solving for } r,$$

$$r^3 < 80 \bullet \frac{9\eta^2}{\rho_f(\rho_w - \rho_f)4g r^3} = \frac{180\eta^2}{\rho_f(\rho_w - \rho_f)g} \qquad [6.6]$$

$$r < \sqrt[3]{\frac{180\eta^2}{\rho_f(\rho_w - \rho_f)g}}$$

$$\boxed{\Rightarrow \quad r < 0.0166\,\text{cm}; \quad v_{\text{term}} = 333.8\ \text{cm s}^{-1}}$$

So this approach works for droplets smaller than 0.17 mm, for which the terminal velocity will be around 3.3 m s^{-1}.

<u>Case II:</u> $R_e > 1000; \quad C_D \cong 0.4$

First we find v_{term}:

$$\frac{1}{2}\rho_f\, 0.4\, v_{term}{}^2 = (\rho_w - \rho_f)\frac{4}{3}gr \qquad \textit{thus, solving directly for } v_{term},$$

$$v_{term} = \sqrt{\frac{(\rho_w - \rho_f)\frac{4}{3}gr}{\frac{1}{2}\rho_f\, 0.4}} = \sqrt{\frac{(\rho_w - \rho_f)}{\rho_f}\frac{20}{3}gr} \qquad [6.7]$$

Then this goes into the formula for the Reynolds number:

$$R_e = \frac{\rho_f\, v_{term}\, 2r}{\eta}$$

$$= \sqrt{\left(\frac{\rho_f\, v_{term}\, 2r}{\eta}\right)^2 \frac{(\rho_w - \rho_f)}{\rho_f}\frac{20}{3}gr} \qquad [6.8]$$

$$= \sqrt{\frac{\rho_f\, (\rho_w - \rho_f)}{\eta^2}\frac{80}{3}gr^3}$$

Now from this we require that $R_e > 1000$, letting us find the corresponding r:

$$R_e^2 > 10^6$$

$$\frac{\rho_f\, (\rho_w - \rho_f)}{\eta^2}\frac{80}{3}gr^3 > 10^6 \qquad \textit{then solving for } r,$$

$$r^3 > \frac{3\eta^2 10^6}{80\, g\rho_f(\rho_w - \rho_f)} \qquad [6.9]$$

$$r > \sqrt[3]{\frac{3\eta^2 10^6}{80\, g\rho_f(\rho_w - \rho_f)}}$$

$$\boxed{\Rightarrow \quad r > 0.098\,\text{cm}; \quad v_{\text{term}} = 703.4\ \text{cm}\,\text{s}^{-1}}$$

So this approach works for droplets larger than 0.98 mm, for which the terminal velocity will be around 7 m s^{-1}.

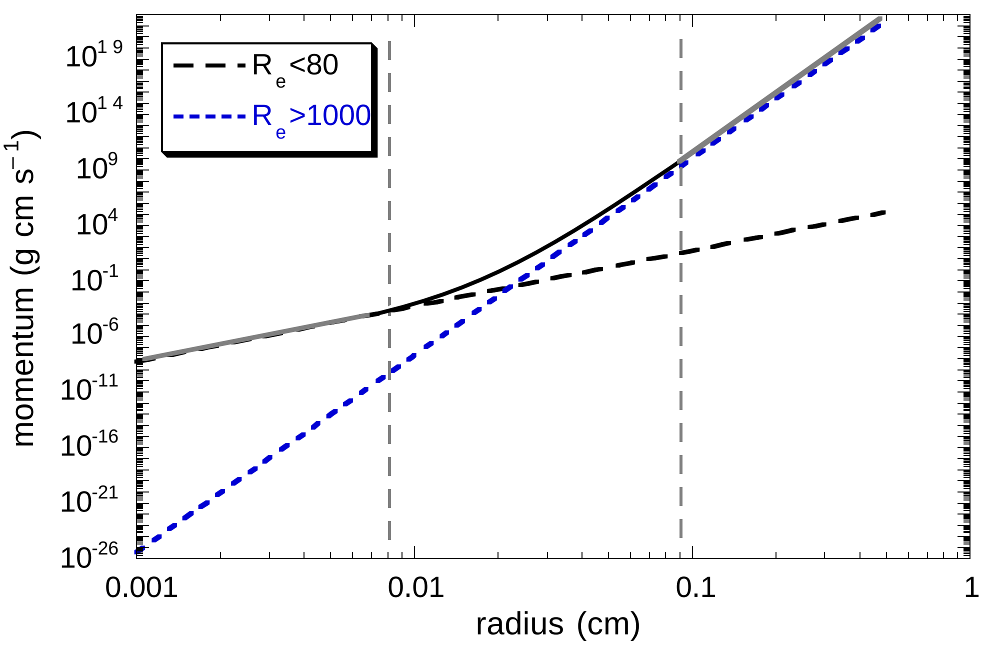
<u>Momentum</u>

The microphone will approximately record the impulse, or momentum deposited by the droplets, so it is the scaling between terminal *momentum* $m_{drop}\bullet v_{term}$ and droplet radius which is needed:

$$\Delta p = m_{drop}\, v_{term} = \rho_w V v_{term} = \rho_w \frac{4\pi}{3} r^3 v_{term}$$

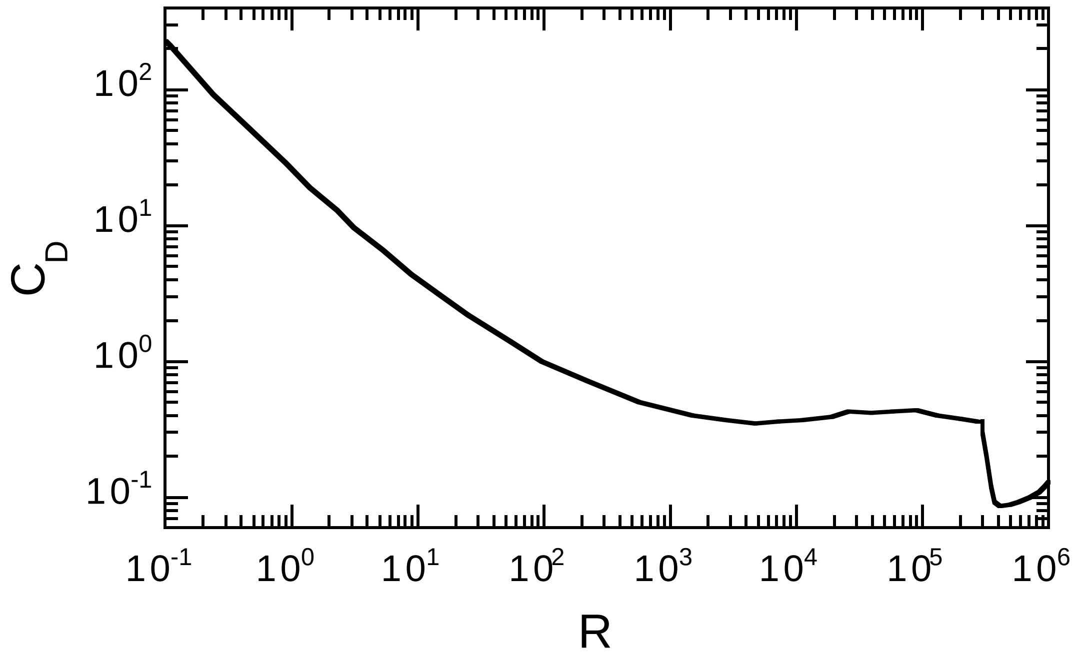
$$= \begin{cases} \rho_w \dfrac{4\pi}{3} r^3 \bullet \dfrac{2(\rho_w - \rho_f)g}{9\eta} r^2 = \dfrac{8\pi}{27\eta}\rho_w(\rho_w - \rho_f) g r^5 \\ \qquad = 5.07\times 10^6\, r^5 \quad (r < 0.016 \text{ cm}) \\ \rho_w \dfrac{4\pi}{3} r^3 \bullet \sqrt{\dfrac{(\rho_w - \rho_f)}{\rho_f}\dfrac{20}{3} g r} = \dfrac{4\pi}{3} \bullet \sqrt{\dfrac{\rho_w(\rho_w - \rho_f)}{\rho_f}\dfrac{20}{3} g}\; r^{7/2} \\ \qquad = 9.39\times 10^3\, r^{3.5} \quad (r > 0.98 \text{ cm}) \end{cases} \qquad [6.10]$$

We can plot this; for points in-between, there is a plausible smooth curve to sketch the transition between the two regimes.



In order to find *radius* from *experimental momentum* data, we start on the y-axis (ordinate) and look up the radius on the x-axis (abscissa).

For your interest, a fuller set of data for C_Ds for spheres — found by experiments — looks like this:



adapted from: A Physicist's Desk Reference, H.L. Anderson, ed., American Institute of Physics, New York (1989).

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Solution Set 3: Thermodynamics

NB: in thermodynamics, some of these questions draw on the relatively advanced international syllabus of physics used in the International Physics Olympiad (IphO) — not all of which is taught in North America.

1) The heat is on at the Pentagon

a)There was an error in one of the given constants. Stefan's constant is 5.6696E^{-8} W/(m^2 K^4). All I can say is that you didn't notice the sign error in the constant during the question, you would have found that your face would be radiating more power than Pickering Nuclear Power Station can generate.

Even though the numbers were wrong, the given constants were meant to help you if you have never seen this type of question before. To find the peak in the black-body radiation curve, you could actual plot the black body radiation curve as a function of wavelength for each temperature, but this is too much work. A little calculus provides Wien's displacement law, which you can use directly:

$$\lambda_{max} = 2898/\,T$$

where T is temperature measured in Kelvin, and λ_{max} is the peak wavelength in the black body emission curve, measured in micrometers.

37°C human emits at 9.3um.
75°C car peak emission at 8.3um.
5°C terrain peak emission at 10.4um.

What is the power/area emitted? We look to Stefan's law for that:

$$P/A = \sigma\, eT^4$$

where σ is Stefan's constant = 5.6696E^{-8} W/(m^2 K^4)
e is the emissivity (for a black object, e=1)
and T is measured in Kelvin

37°C human emits 520 W/m^2.
75°C car radiates 830 W/m^2.
5°C terrain radiates 340 W/m^2.

How about a flashlight? Assume the flashlight has a an excellent lens system that provides a uniform beam of radius 5cm. A guess of a bulb wattage of 0.1W is not bad, thus the power per unit area is 13 W/m^2. Surprising eh? Remember also that the vast majority of this power is also emitted in the infrared (i.e. as heat).

b) Missile time!

In a very short time the missile surface heats up and starts emitting. How much power is it radiating at equilibrium? The same as it is absorbing: 5MW (ow!). Let's use the same expression as used above, assuming that the missile is painted a nice dull black so that emissivity =1:

$$P/A=\sigma\, e\, T^4$$

For spot size of diameter 1m: T=3300K or 3000°C.
For a spot size of diameter 1mm: $T=10^5$K ! Kappoey!!

2. Planet X

a) Let us find first the value of the gravitational field constant for the new planet. From the universal gravitation law we have for any mass m

$$mg = G\ (mM/r^2) \tag{1}$$

$$g = GM/r^2 = ((6.7)(10^{-11})(10^{25}))/((9)(10^{14}))=0.74\ N/kg$$

The atmospheric pressure on the surface of the planet is equal to

$$p = \rho\, gh \tag{2}$$

where ρ is the atmospheric density. Assume that g is constant since h << r.

From the equation of state for ideal gas

$$pV = nRT = (M/(\mu))RT \tag{3}$$

we find

$$\rho = M/V=(p(\mu))/(RT) \tag{4}$$

and from (4) and (2)

$$T = (\mu gh)/R = (10)(0.74)(5)/(8.3) = 4.5\ K = -268°C \tag{5}$$

Unfortunately, it is too cold to live on this planet !!!

b) For the molar mass of our atmosphere (78% of nitrogen and 21% of oxygen), we have μ = (0.78)(28) + (0.21)(32) = 28.56 g/mol. The Earth's atmospheric density decreases at the same rate as the atmospheric pressure (1/2 per each h_0 = 5.6 km). We can express this dependence as the function $\rho(x) = \rho_0\, 2^{-x/h_0} = \rho_0 \exp\{(-\ln 2)(x)/h_0\}$, where x is the distance from the surface of the Earth. The good approximation for the height of the atmosphere equivalent to ours but of constant density can be found from the following condition:

$$\int_0^\infty \rho(x)\,dx = r_0 h$$

From this condition we obtain the value of $h = (h_0)/(\ln 2)=(5.6)/(0.693)=8.1$ km. Now we can use the formula (5) to calculate the temperature on the Earth's surface:

$$T = (\mu)(g)(h)/(R) = (28.56)(8.1)(9.8)/(8.3) = 273\ K = 0°C.$$

3. Blow ye winds and crack your, er, skin ...

a) From the equation of state for ideal gas

$$pV = nRT = (m/\mu)RT,$$

where n is the number of moles and μ is the molar mass, we obtain

$$\rho = m/V = (\mu p)/(RT) = (\mu \alpha p_s)/(RT), \qquad (1)$$

where p_s is the saturation vapor pressure for given temperature T and α is the relative humidity. The molar mass of water μ = 18 g/mol.

In November,

$$\rho_1 = ((\mu)(\alpha_1)(p_{s1}))/((R)(T_1)) = ((18)(0.95)(600))/((8.3)(273)) = 4.5\ g/m^3.$$

In July,

$$\rho_2 = ((\mu)(\alpha_2)(p_{s2}))/(R)(T_2)) = ((18)(0.4)(5500))/((8.3)(308)) = 15.5\ g/m^3.$$

Thus, in dry July (α_2 = 40 %), the air contains 3.4 times more vapor than in humid November (α_1 = 95 %).

b) The mass of water vapor in 1 m^3 of outside air is equal to (0.25)(3.48) = 0.87 g. When the outside air is heated from –4°C to 20°C, its volume increases. Therefore,

$$pV_1/T_1 = pV_2/T_2,$$

$$V_2 = V_1\ (T_2/T_1) = 1.089\ m^3.$$

The density of water vapor in the heated air is equal to 0.87 g / 1.089 m^3 = 0.8 g/m^3, so the relative humidity will be (0.8 g/m^3) / (17.3 g/m^3) = 0.046 = 4.6 %. The air will be uncomfortably dry. When the Santa Ana winds blow about this dry into Los Angeles, some people get nosebleeds.

4. Jessica cycles heat

a) The gas presses on the pump handle with a force = P A where A is the cross-sectional area of the pump and P is normal atmospheric pressure. The distance the handle moves is δx. The definition of work is the force vector dot product with the displacement. In this situation, this gives that the work done by the gas is P A (δx). You should be careful about signs here: I have implicitly defined that compression yields negative δx and expansion corresponds to positive δx.

The work done on the gas is the opposite of this, thus:

$$W = -P\ \delta V, \text{ where } \delta V = (\delta x\ A)\ ,$$

where A is the cross-sectional area of the bike pump.

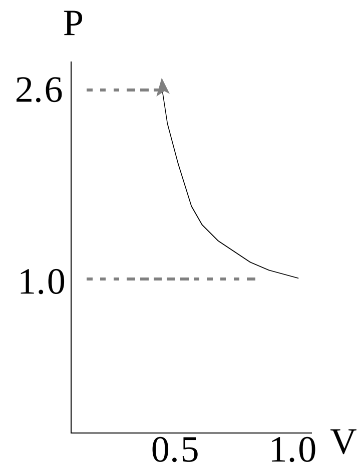
b) Given : P′ = 1.0 atm (at steady-state with the room)
V′ = 1 l
V″ = 0.5 l

Using adiabatic compression expression we find that:

$$P'/P'' = (V''/V')^{1.4}$$

so

$$P'' = 2.6 \text{ atm}$$



Using the ideal gas law: PV=nRT or for a constant number of molecules: PV/T = constant. Thus we can say:

$$P'V'/(P''V'') = T'/T''$$

so that the final temperature is: T″ = 381K or about 110 °C . This assumes that the system started at room temperature (293K).

c)Using part (a), the work corresponds to the area under the graph give in part (b). Integration of this function will provide an accurate value, but you can approximate the curve as a straight line connecting the beginning and final states. Thus the area is a square + a triangle:

$$Work = -Area = -(-0.5\ l*(1\ atm)+(1/2)*(-0.5\ l)*(2.6\ atm - 1\ atm)) = 0.9\ atm\ l$$

or in the more usual units for work:

$$Work = 91\ J$$

d) Using the ideal gas law again:

$$P'V'/(P'''V''') = T'/T'''$$

but here

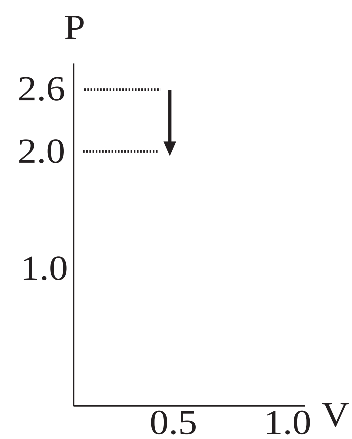
$$T' = T'''$$

so

$$P'V' = P'''V'''$$

giving

$$P''' = 2 \text{ atm}$$

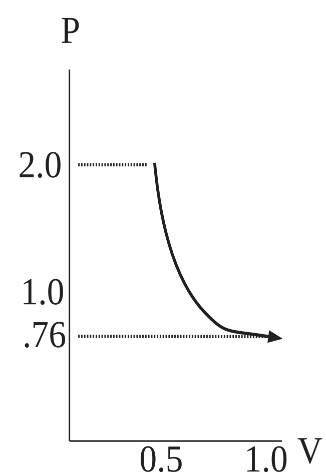


There is no work done on the gas, as can be seen in the PV graph (area under the curve is 0).

e) Given : P‴ = 2.0 atm (at steady-state with the room)
V‴ = 0.5 l
V⁗ = 1.0 l

Using adiabatic compression expression again we find that:

$$P'''' = 0.76 \text{ atm}$$



Using the ideal gas law again:

$$PV/(P''''V'''') = T/T''''$$

so T'''' = 223K or –50°C Wow eh!

Using a linear approximation:

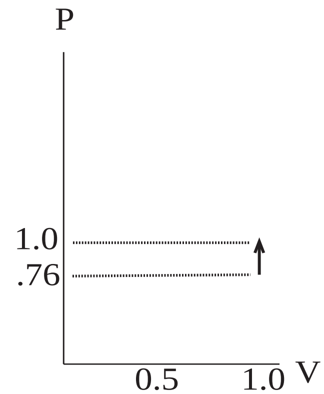
$$Work = -Area = -.69 \text{ atm } l = -70 \text{ J}$$

Thus the gas does work of 70 J (the minus sign indicates that the gas is doing work *on* something, not that work is being done on the gas).

f) Easy one:

final pressure:	1 atm
final T:	room temperature (293K)

No work done on the gas, since the area under the curve on the PV diagram at right is again 0 (as in section d).



g) Net work on the gas is 91 J + 0 J – 70 J + 0 J = 21 J

h) No heat is transferred in the adiabatic compression and expansion. It is only transferred when the gas is held at constant volume.

δT for the cooling stage (*stage 2*):	381K–293K=88K
δT for the warming stage (*stage 4*):	223K–293K=–70K

The total number of moles in the gas is: n=PV/(RT)=0.041 mol
Thus the total heat transferred from the gas to the surroundings is:

total heat = 18K * 2.98 cal/(mol K) * 4.19J/cal * 0.041 mol =9 J

We would expect this to equal the work done on the gas (calculated above as 21 J). The reason for the discrepancy is due to the linear approximation. If the area was calculated accurately, these two numbers would agree.

5. Batboy chills

a) Heat transfer: Q = 0 (No heat flows into the system, as it is adiabatically insulated.)

Work: W = 0 (No net work done by the system)

The energy change in the system after opening the valve is

$$U_2 - U_1 = \Delta U = -W + Q \quad \text{(1st law of thermodynamics)} = 0$$

So

$$U_2 = U_1, \text{ or } U(T_1,V_1) = U(T_2,V_2).$$

For an ideal gas, U is independent of volume, so $U(T_1) = U(T_2)$, meaning that $T_1 = T_2$.

Unless the gas in V_1 was initially very cold (which would've required fancy refrigeration), Batboy would have nothing to fear.

b) As for part (a), $U(T_1,V_1) = U(T_2,V_2)$
For our interacting gas, $\Delta U = c_v \Delta T - a/v^2 \Delta V$, where here the Δ now refers to the difference in T and V in each compartment before and after the valve was opened.

Relative to some arbitrary initial temperature T_o and volume V_o,

$$U_1 = \int_{T_o}^{T_1} C_v(T^1)\,dT^1 + a\int_{V_o}^{V} \frac{dV^1}{(V^1)^2}$$

$$U_2 = \int_{T_o}^{T_2} C_v(T^1)dT^1 + a\int_{V_o}^{V_2} \frac{dV^1}{(V^1)^2},$$ where C_V (T) = heat capacity

$$U_1 = U_2 \Rightarrow \int_{T_o}^{T_1} C_v(T^1)\,dT^1 + a\int_{V_o}^{V} \frac{dV^1}{(V^1)^2} = \int_{T_o}^{T_2} C_v(T^1)dT^1 + a\int_{V_o}^{V_2} \frac{dV^1}{(V^1)^2}$$

Assuming C_v doesn't change with temperature,

$$C_v T_1 - a/V_1 = C_v T_2 - a/V_2$$

so

$$(T_2 - T_1) = a/c_v(1/V_2 - 1/V_1)$$

With $T_2 = -5^oC$ $\quad T_1 = 15^oC$
$a = 0.15$ $\quad C_v = 21$ J/(mol K)

$V_2/V_1 = 2800\,V_2 + 1$, so V_2 would have to be very large!

6. The 'thermal physics' (statistical mechanics) of computing

a) For a system of 2 magnets, $U = -m_{l_1}\, l_2$. The possible configurations of the magnets are:

Both up:	↑ $l = +1$	↑ $l = +1$	**U = –m**
first up, second down:	↑ $l = +1$	↓ $l = -1$	**U = +m**
first down, second up:	↓ $l = -1$	↑ $l = +1$	**U = +m**
both down:	↓ $l = -1$	↓ $l = -1$	**U = –m**

So we want spins to be parallel to one another, to minimize the internal energy U.

For N magnets, all nearest neighbours have the same interaction as for 2 magnets:

Ex.

↑	↓	↓	...	↑	↓	↑	...	↑	↑	↓
1	2	3		(i–1)	i	(i–1)		(N–2)	(N–1)	N

Interaction energy for the i^{th} magnet:

$$U_l = -m\,(l_{i-1}\, l_I + l_I\, l_{I+1})$$

Interaction energy for the $(i+1)^{th}$ magnet:

$$U_{l+1} = -m(l_I\, l_{I+1} + l_{I+1}\, l_{I+2})$$

Note that $U_l + U_{l+1}$ counts the interaction between l and l+1 twice, which is incorrect. If we follow this through for the whole chain, we find:

$$U_{TOT} = -M \sum_{i....=1}^{N-1} l_i\, l_{i+l}$$

So U_{TOT} is maximal if all magnets are aligned in the same direction. ∴ (all spins up or all spins down.

b) Energy difference = (energy before flipping a spin) – (energy after flipping a spin)

If the flipped magnet was at the end of a chain, the change in energy is due to 1 bond.

∴ energy increase = +m

Otherwise, a flipped spin affects 2 bonds.

∴ energy increase = +2m

So, flipping a spin from the parallel state increases the internal energy.

c) For 3 magnets:

i)	U = –m(1+1) = –2m:	↑ ↑ ↑ ↓ ↓ ↓	**2 configurations**
ii)	U = –m(1–1) = 0:	↑ ↓ ↓ ↓ ↑ ↑ ↑ ↑ ↓ ↓ ↓ ↑	**4 configurations**
iii)	U = –m(–1–1) = 2m:	↑ ↓ ↑ ↓ ↑ ↓	**2 configurations**

i) → lowest energy, low entropy
ii) → moderate energy, high entropy
iii) → high energy, high entropy

d) F = U –TS

i) If U > TS, a state with lower internal energy, which has a low entropy, is preferred. This means that internal energy determines the state of the system and so all the spins will tend to align in parallel. This is often called the “ordered state”.

ii) If TS > U, a state of higher entropy is preferred; in fact, the system will want to be in the energy state with the most number of configurations available to it. This is typically the most "random" state of the system.

iii) U = TS defines the "order-disorder" transition

e) This depends on temperature. If U > TS, then the configurations

$$\begin{array}{ccccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\ 1 & 1 & 1 & 1 & 1 & = & \mathbf{31} \end{array} \qquad \text{and} \qquad \begin{array}{ccccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ 0 & 0 & 0 & 0 & 0 & = & \mathbf{0} \end{array}$$

are the easiest to code. Conversely, if TS > U, then the "random" states, such as

$$\begin{array}{ccccccc} \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & & \\ 1 & 0 & 1 & 0 & 1 & = & \mathbf{21} \end{array} \qquad \text{or} \qquad \begin{array}{ccccccc} \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & & \\ 0 & 1 & 0 & 1 & 0 & = & \mathbf{10} \end{array}$$

are easiest. Other values cost energy and entropy – you can see how much of a free energy cost there is in constructing other states by evaluating U and S for a given T.

BONUS:

For 1 spin, state possible = $\uparrow + \downarrow$
2 spins, state possible = $(\uparrow + \downarrow)(\uparrow + \downarrow) = \uparrow\uparrow + \downarrow\downarrow + \uparrow\downarrow + \downarrow\uparrow$
3 spins, state possible = $(\uparrow + \downarrow)^3$
N spins, state possible = $(\uparrow + \downarrow)^N$

Let x = $\uparrow$ and y = $\downarrow$

By binomial theorem, we would know right away for x and y:

$$(x+y)^N = x^N + Nx^{N-1}y + \frac{1}{2}N(N-1)x^{N-2}y^2 + \ldots + y^N = \sum_{t=0}^{N} \frac{N!}{(N-t)!\,t!}x^{N-t}y^t$$

Let $s = N_{up} - N_{down}$. Replace t by $\dfrac{N-s}{2}$ and sum over s

$$(x+y)^N = \sum_{s=-N}^{N} \frac{N!}{\left(\frac{N+s}{2}\right)!\left(\frac{N-s}{2}\right)!} x^{\frac{1}{2}(N+s)} y^{\frac{1}{2}(N-s)}$$

well, the same form also applies to writing out our combinations of $\uparrow + \downarrow$

$$(\uparrow+\downarrow)^N = \sum_{s=-N}^{N} \frac{N!}{\left(\frac{N+s}{2}\right)!\left(\frac{N-s}{2}\right)!} \uparrow^{\frac{1}{2}(N+s)} \downarrow^{\frac{1}{2}(N-s)}$$

The number of states with given energy is determined by # of up and down spins. This is given by the coefficient of the binomial term with value s.

$$\therefore\ g(N,s) = \frac{N!}{\left(\frac{N+s}{2}\right)!\left(\frac{N-s}{2}\right)!}$$

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Problem Set 5 Solutions: Electricity and Magnetism

1) The Earth as Cosmic Doorknob

The charge accumulated by the Earth during the time interval t is equal to $q = aSq_p t$, where $a = 1$ proton/cm^2.sec is the flow density; $S = 4\pi R^2$ is the area of the Earth's surface; $R = 6.4 \times 10^6$ m is the radius of the Earth; $q_p = 1.6 \times 10^{-19}$ C is the charge of proton. The critical charge q of the Earth, which will prevent protons with the kinetic energy $E < 4 \times 10^9$ eV from reaching the Earth, can be found from the following equation:

$$qq_p / (4\pi\varepsilon_0 R) = aStq_p / (4\pi\varepsilon_0 R) = E,$$

where

$$(4\pi\varepsilon_0)^{-1} = 9 \bullet 10^9 \text{ Nm}^2\text{C}^{-2}$$

From this equation we have:

$$t = (4\pi\varepsilon_0 ER)/(aSq_p^2) = (E\varepsilon_0)/(aq_p^2 R) = 10^8/((3.14)(0.9)(1.6)(6.4)) = 3.45 \bullet 10^6 \text{ sec} = 40 \text{ days}$$

This value is tiny compared to the age of the Earth, which is equal to 5×10^9 years. Protons from the cosmic rays continue to reach the Earth because their charge is compensated by the charge of electrons also reaching the Earth. Since their energy is extremely small, they are not considered as a component of cosmic rays.

2) C.C.? No! No!

a) At a distance of 5 cm from a large sphere, we must treat only that section of the sphere near where you are standing (your closest point to the sphere). This is for the same reason that we can treat the earth as being flat locally at its surface even though it is really a sphere.

E field of a spherical conductor with charge *Q*.

NOTE: the charge on a conductor distributes itself uniformly on the conductor's surface.

Gauss' Law

$$\int \underline{E} \bullet \underline{ds} = \frac{\text{charge enclosed}}{\varepsilon_0}$$

$$E \bullet 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\therefore E = \frac{Q}{4\pi r^2 \varepsilon_o}$$

$\therefore$ at 5 cm:

$$0.8 \times 10^3 \frac{V}{m} = \frac{Q}{4\pi\varepsilon_o (.05)^2} \Rightarrow Q = 2.2 \times 10^{-10}\, C$$

b) For the large sphere,

$$V_1 = \frac{1}{4\pi\varepsilon_o} \frac{q_1}{r_1}$$

Small sphere,

$$V_2 = \frac{1}{4\pi\varepsilon_o} \frac{q_2}{r_2}$$

But since the spheres are part of the same conductor, the potentials of the spheres must be equal:

$$V_1 = V_2 \quad \Rightarrow \quad \frac{1}{4\pi\varepsilon_o} \frac{q_1}{r_1} = \frac{1}{4\pi\varepsilon_o} \frac{q_2}{r_2}$$

$$\therefore \frac{q_1}{r_1} = \frac{q_2}{r_2}$$

The electric field at the surface of each sphere is given by

$$E_1 \propto \frac{q_1}{r_1^2},\ E_2 \propto \frac{q_2}{r_1^2}$$

With the same constant of proportionality for each.

$\therefore$The ratio of the electric fields is

$$\frac{E_1}{E_2} = \frac{q_1 / r_1^2}{q_2 / r_2^2} = \frac{\left(\frac{q_1}{r_1}\right)\frac{1}{r_1}}{\left(\frac{q_2}{r_2}\right)\frac{1}{r_2}}$$

but

$$\frac{q_1}{r_1} = \frac{q_2}{r_2}$$

$$\therefore \frac{E_1}{E_2} = \frac{r_2}{r_1}$$

So the field is higher on the surface of the smaller sphere. A more sharply curved or "pointy" surface on a conductor has a higher electric field and therefore reaches spark discharge for a lower applied charge.

I have heard that there was an ongoing disagreement between Benjamin Franklin in the U.S. and European investigators as to what the best shape for a lightning rod should be: sharp spike or small ball. The argument went like this: all this, above, is true, but the strong field around a sharp spike might cause air to ionize as the rod began to charge up. If so, there would be glob of ionized air (plasma) near the rod, and plasma is a conductor — so the *effective* conductor would have a larger radius than if a lightning rod had a ball on the top to start with!

3) Goats & Sheep – the Hall Effect in plasma

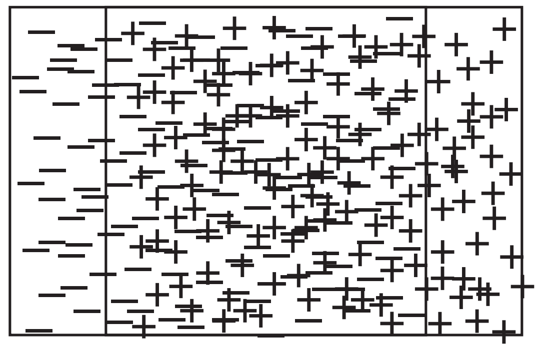
The point here is that moving charges in a magnetic field see a force — the Lorentz force:

$$\vec{F}_{Lorentz} = q\,\vec{v} \times \vec{B}$$

The Lorentz force, on each particle, will be:

$$1.6 \times 10^{-19}\ \text{C} \bullet 0.3\ \text{m s}^{-1} \bullet 0.3\ \text{T}$$

$$= 1.44 \times 10^{-19}\ \text{N} \quad \text{(opposite directions for electrons, ions)}$$



A block of plasma, with the electrons pulled aside from the ions…

So when the flowing gas becomes ionized, the electrons and ions will separate from each other in a B-field — the Lorentz force pushes opposite ways for the opposite charges.

Do they end up going in circles in the B-field? Probably not, because as the charges separate, they are still attracted to one another by the Coulomb force:

$$\vec{F}_{Coulomb} = q\,\vec{E}$$

Then the net force is going to be:

$$\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B})$$

Basically, stream of electrons will move aside from the gas stream, and the stream of positive charges will also move a little in the opposite direction. The separated charges lead to an E-field, and to an attraction between the two streams. The two will balance, so that F = 0

What will be the electric field to cancel this?

$$1.44 \times 10^{-19}\ \text{N} = \text{q E, so}$$

$$\text{E} = 0.9\ \text{V m}^{-1}$$

How much will the charges separate from each other, if we assume they keep their original density, and just separate as a stream? We can figure out the details: air at STP is 22.4 l per mole, and 1 mole = 6.022×10^{23} particles. From this we find the number of particles per unit volume:

$$n = \frac{N}{V} = \frac{6.022 \times 10^{23}}{22.4 \times 10^{3}\,\text{cm}^{3}} = 2.69 \times 10^{19}\,\text{cm}^{-3}$$

With 1% ionization, this means we have 2.69×10^{17} electrons per cm^3, or 2.69×10^{23} electrons per m^3, and the same density of positively charged ions. If the electrons are just pushed somewhat off the ions (the ions are much more massive, and move less from the forces), what we have is something roughly like a parallel-plate capacitor, with the two thin excess-charge slabs as the two charged plates of the capacitor.

For this, we can find the field — the same as between the two plates of a parallel-plate capacitor. In the middle, where there are both electrons and ions, the charges of each cancel each other out (unless you give them time to move around and redistribute themselves), so the net field in the middle is just what is produced by the thin slabs of excess (unbalanced) charge.

$$E = 4\,\pi\,k_c\,\sigma \qquad \text{where } k_c = 9 \times 10^9 \text{ N m}^2\text{ C}^{-2}$$
$$\sigma = \text{charge per unit area on each plate}$$

The charge *per unit area* of the two slabs of excess comes from the charge density, and the distance the electrons are pushed off the ions:

$$\sigma = (n\,e\;(\Delta x) \times A)\;/\;A$$
$$= n\,e\;(\Delta x)$$

and the field between the plates is:

$$E = 4\,\pi\,k_c\,n\,e\;(\Delta x) \qquad [5]$$

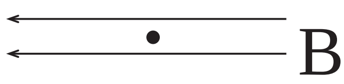
or

$$\Delta x = E/(4\,\pi\,k_c\,N\,e)$$
$$= 1.84 \times 10^{-16}\text{ m}$$

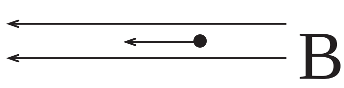
A very tiny displacement (actually nonsensically tiny, given the size of an atom!). Consider, though, that if the jet is 1mm across, the potential measured across the jet from one side to the other would be 0.9 mV.

4) Shepherding fields

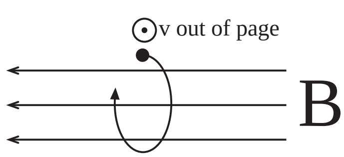
a)For the four cases suggested in the problem:



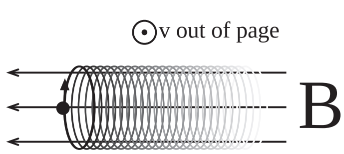
i) v=0 There is no force acting on the particle and therefore no acceleration.



ii) v_{parallel} non-zero: The cross-product is zero, therefore same result as first case.



iii) $v_{\text{perpendicular}}$ non-zero: The cross-product simplifies to a multiplication and the resulting force is always acting perpendicular to the motion. The particle moves in a circle. From the given equation and knowing that for perfect circular motion (such as a satellite in orbit) the acting force is $F = mv^2/r$ we find the radius of the circle is: $r = mv/qB$



iv) We combine the results from sections iii) and ii) and get helical motion.

Since in all cases the velocity of the particle is always orthogonal to the acting force, the work done on the particle is 0.

b) As given in the question, kinetic energy is conserved. Thus the square of the magnitude of the total velocity at any given point in time is constant:

$$v(z)^2 = v_{per}(z)^2 + v_z(z)^2 = \text{constant}$$

where v is the total velocity, v_{per} is the component perpendicular to B (i.e. in the X-Y plane) and v_z is the component parallel to B (i.e., parallel to the z axis). We want to find $v_z(z)$:

$$v_z(z)^2 = v_{per}(0)^2 + v_z(0)^2 - v_{per}(z)^2$$

We also know that flux is conserved:

$$B(0)\ A(0) = B(z)A(z)$$

where B is the magnitude of the magnetic field and A is area enclosed by the loop that is orthogonal to the field. Combining this with the result from part a):

$$v_{per}(z)^2 / B(z) = v_{per}(0)^2 / B_0$$

This gives:

$$v_z(z)^2 = v_z(0)^2 - v_{per}(0)^2\ (b_0\ z^2)\ /\ B_0$$

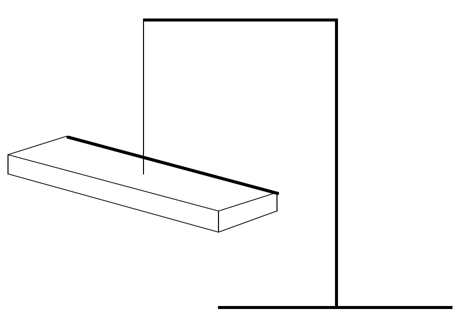
c) Shades of Bob, it's simple harmonic oscillator! *KE* is transferring from parallel motion to perpendicular circular motion. The particle swings back and forth along the Z axis, in a similar way to a weight on a spring, with the added circular motion around the Z axis

d) $v_z(z)=0$ when $z^2 = (B_0\ v_z(0)^2)\ /\ (b_0\ v_{per}(0)^2)$

Total *KE* is conserved. Energy from translational *KE* transfers to rotational *KE*.

5) **Science on a shoestring**

First we suspend the magnet from the stand (using the string). We tilt it by a small angle and let it oscillate.



It undergoes oscillations (much like a torsional pendulum) such that:

$$I\ddot{\theta} = -\mu_o m B_z \sin(\theta) \approx -\mu_o m B_z \theta$$

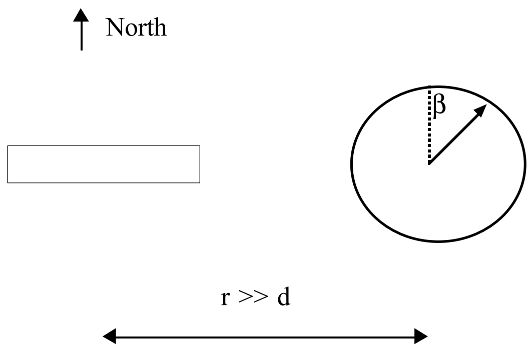
here, B_z is the horizontal component of the Earth's magnetic field, m is the magnetic moment, and I is the moment of inertia of the magnet. Treating it as a thin stick,

$$I = \frac{M d^2}{12}$$

Now, the motion is simple-harmonic with a period of

$$T = 2\pi\sqrt{\frac{I}{m B_z \mu_o}}$$

Now, we align the compass to the north. Then, we place the magnet some distance $r >> d$ away from the compass and measure the deflection angle β.



Clearly, then,

$$\tan(\beta) = \frac{B}{B_z} = \frac{2m}{4\pi r^3 B_z}$$

Combining, we get

$$B_z = \sqrt{\frac{2\pi I}{\mu_o T^2 r^3 \tan(\beta)}}$$

Ideally, one would collect a set of βs by varying r, and plot the equation (on log-log paper), and get Bz that way, but that is a lot of work (when done by hand).

And thus, plugging in one set of numbers will do.

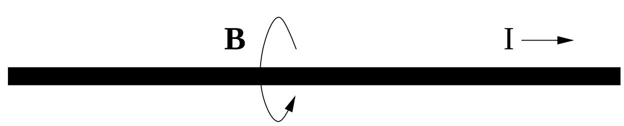
To measure T, it is best to time say 10 periods, and divide the result by 10 (as long as the time does not decay too much). r is simply measures with a ruler. Measuring β might be difficult, and this might be the biggest error here. In fact, it might be best to set β at, say 45^o, and then change r to obtain that deflection (to make measuring it easier) — but one has to be careful that $r >> d$!

To get the error, we note that $\delta Bz/Bz \approx \delta X/X$, where $\delta X/X$ is the biggest error found (most likely for β or r; δX means error in variable X).

The accepted value is about 0.15×10^{-4} Tesla, BUT it varies a lot because of interference, etc. Doing the experiment outside would thus be ideal.

6) **How much do you charge for a free ride?**

a) B circles around a straight wire in a direction dictated by the right-hand rule (which comes from the Biot-Savart law).



b) For a time-varying magnetic field in the z direction, the induced electric field circles the magnetic field direction, like the magnetic field around a wire. The direction of the induced E field is such that it resists the change in the magnetic field (Lenz' Law). *(figure next page)*

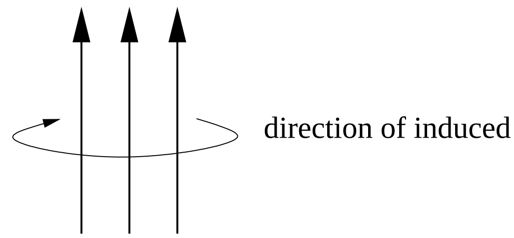
c) $\int(\boldsymbol{E}\bullet d\boldsymbol{l})$ around the loop

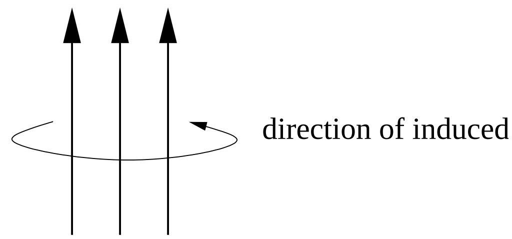
$= -\int ((d\boldsymbol{B}/dt)\bullet d\boldsymbol{s}$ on a surface bounded by the loop

$= -d\Phi/dt$, where $\Phi(t)$ is the magnetic flux through the loop at time t.

In our case, making our loop of radius a, centred at the centre of the wheel,

$$\int(\boldsymbol{E}\bullet d\boldsymbol{l}) = -\pi a^2 (dB(t)/dt)$$





d) The total torque on the wheel about its axis:

$$N = r \times F \text{ over the whole wheel.}$$

For a piece of the wheel of length δl, r and F are perpendicular and $F = qE = (\lambda\delta l)E$. Therefore

$$\delta N = r\lambda\delta l E$$

For the total torque, we integrate around the whole wheel

$$N = r\lambda\int Edl = r\lambda\left(-\pi a^2 \frac{dB(t)}{dt}\right)$$

where we have used the result of part (c).

The total angular momentum is

$$L = \int N dt$$

where the integral is from an initial time t_o just before the B field is changed to the final time t just after the B field has stopped changing.

$$L = -\rho\lambda\, \pi a^2 \int ((dB(t)/dt)^* dt)) \text{ (from } t_o \text{ to } t)$$
$$= -\rho\lambda\, \pi a^2 \int (dB) \quad \text{(from } B_o \text{ to } 0)$$

as initially B=B_o, and finally B=0.

so $L = \rho\lambda\, \pi a^2 B_o$

e) From part (d), we see that the value of the final angular momentum is independent of how fast the field is switched off; it depends only on the initial field value, or, more generally, it depends on the difference between the initial and final field strengths

BONUS

f) Marks were given for any interesting answer — the question is relatively sophisticated.

The torque on the Merry-Go-Round wheel comes as a result of the E-field produced as the B-field produced by the solenoid vanishes (dB/dt ≠ 0). This induced E-field accelerates the charges into circular motion, taking the wheel to which they are bound along too.

XX: Some might argue that this new circulating current produces a B-field of its own; the charges (already moving) in the solenoid see this changing B-field from the Merry-Go-Round disk ($dB/dt \neq 0$) as it induces an E-field. However, this is a bit of a red-herring — the solenoid has equal numbers of positive and negative charges, so it doesn't pick up momentum this way. Someone might argue that the B-field pushes the electrons to one side of the wire (the Hall effect, see Q. 3) so that the angular momentum they pick up doesn't match the positive charges', but there's no hope here!

√√*:* Basically, the wheel will accelerate when it sees the B-field changes, even if the solenoid is a long, long distance *D* away. Conservation of angular momentum can't 'hold off', or wait, until the solenoid 'learns' that it is pushing on the wheel — a time *D*/*c* at the earliest, where *c* is the speed of light. So the angular momentum must be in the electromagnetic field; this changing electromagnetic field must have angular momentum which is 'left' with the wheel. The changing fields the wheel's motion creates will reduce the angular momentum of the overall electromagnetic field, as the (kinetic) angular momentum of the wheel increases.

1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 1: General

Due November 6, 1998

1) Pushy photons

In an episode of *Star Trek: Deep Space 9*, Capt. Benjamin Sisco makes a hobby out of recreating an early spacecraft, from alien archaeological records. This spacecraft uses huge sails to move the ship along on the solar wind (a stream of charged particles) or by light pressure from the sun.

Consider such a craft the same distance from the sun as is the earth, and only consider light pressure.

a) What is the best colour for the sail?

b) For a sail area the size of a soccer field on a small ship of 2,000 kg, how fast will the ship be moving after 24 hours? After 1 year?

In my research laboratory we have a laser which produces a terawatt of power (10^{12} W) in a pulse of light about one picosecond (10^{-12} s) duration. This pulse can be focussed to a spot about 10 μm across.

c) How much force from light pressure does such a pulse exert? Roughly how much momentum is transferred? What is the pressure exerted by the light? *[Robin]*

2) Out of sight, hope you don't mind

In electronics, a 'black box' is a mystery package with electrical terminals. You try to figure out what is inside it by making measurements across the terminals.

a) Is the following black box possible, if only passive elements are used (i.e., resistors, capacitors and inductors, only)?

Two pairs of terminals come from the box. If a battery of voltage V is connected across the terminal-pair 'A', the voltage measured across terminal-pair 'B' is V/2. But — reversing things — when the battery is connected across terminal-pair 'B', the voltage measured across 'A' is V. If it *is* possible, what is inside? *[123 Tricky]*

3) Leonardo’s *camera obscura*

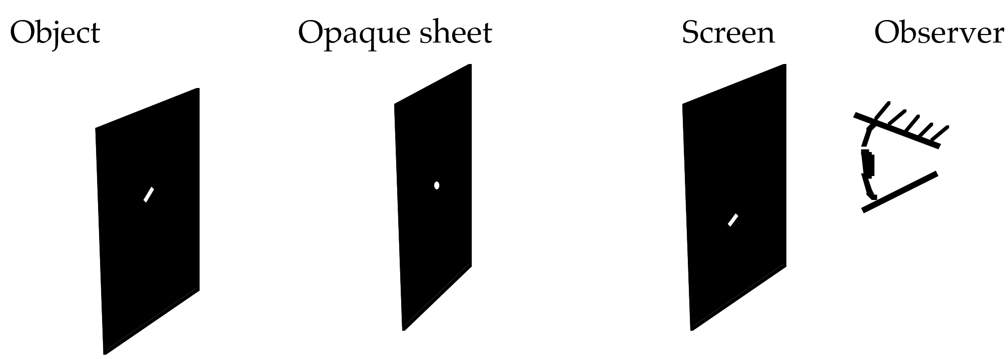
Simple ray optics can be demonstrated using a *pinhole camera*. This consists of an opaque sheet with a small hole in it, and a viewing screen (see layout below). Rays from an illuminated object travel through the pinhole and form an image on the viewing screen. [You can make a pinhole camera yourself: take a shoebox and make a small hole in the centre of one of the small sides. Cut out the side directly opposite it and replace it with wax paper. In a darkened room put a candle in front of the pinhole. You will be able to observe the inverted image of the candle on the wax paper.]

If you use something other than a small hole, the image you see will be distorted. In the following five cases, you are shown two of the three components (object, opaque sheet with hole in it, and image) and your task is to draw the missing component that is consistent with the other two. To make things simpler, I have drawn the objects from the observer's point of view (see below). Don't worry about the exact proportions or intensity, just draw the shape that is consistent with the other two shapes.

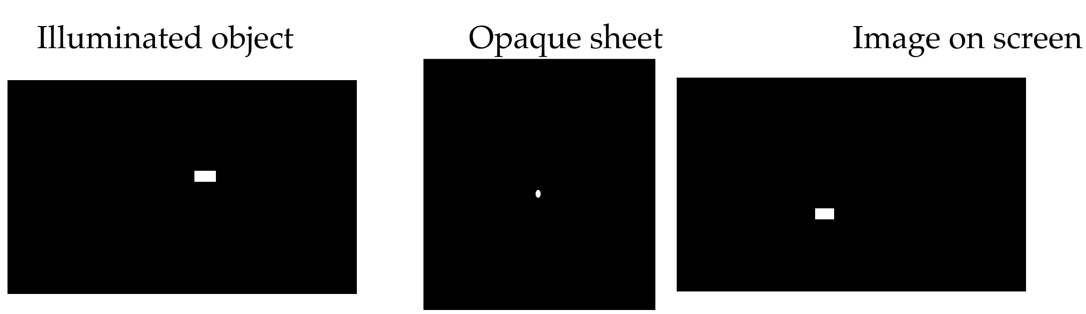
(For you fancy-pants out there, ignore diffraction and anything other than simple ray optics, i.e., take it that light travels in straight lines).

Example of imaging with a pinhole camera:

i) View from side:

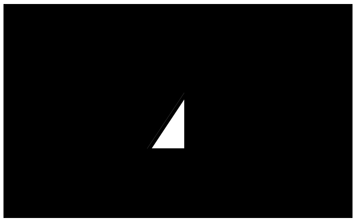
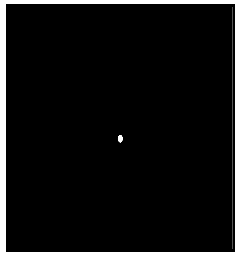


ii) From observer's point of view:

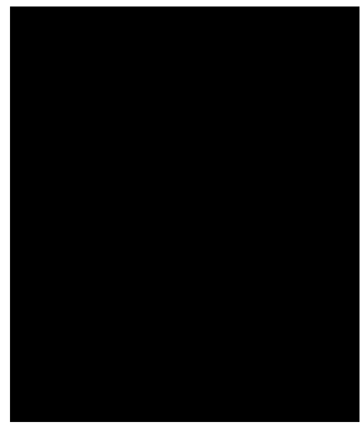


All the following are from the observer's view:

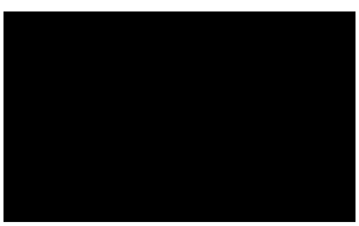
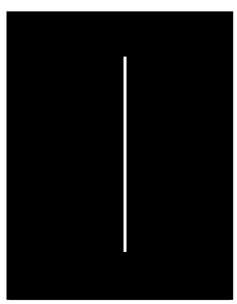
a) Illuminated object | Opaque sheet | Image on screen

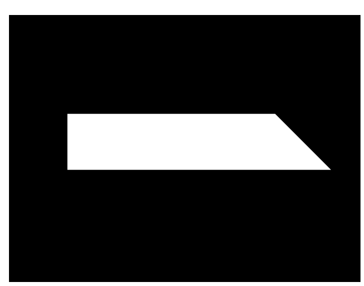
b) Illuminated object | Opaque sheet | Image on screen

c) Illuminated object | Opaque sheet | Image on screen

d) Illuminated object | Opaque sheet | Image on screen

e) Illuminated object Opaque sheet Image on screen

[James]

[Artists used pinhole cameras in their invention of perspective drawing. But their cameras were entire rooms, and the hole was in a wall. They could sketch on the opposite wall. This was the *camera obscura.*]

4) Heat works

Consider two identical spheres made of aluminum. One of the spheres lies on a heat-insulating plate, while the other hangs on a heat-insulating thread. The same amount of heat is transferred to the two spheres. Which one will have the higher temperature?

For spheres of 100g each, and a heat of 250kJ, can you estimate the difference in temperature? *[123 Tricky]*

5) Death of an Atom

Before quantum mechanics saved the day, it was noticed that there was a pretty basic problem with the atom. If one thinks of the atom as an electron orbiting the positively charged nucleus — as a planet orbits the sun, but with electric fields instead of gravity — it means that the electron is accelerating because of centripetal force. But it is well-known that an *accelerating* charge radiates electromagnetic waves: light, x-rays, etc. This means the electron will lose energy, and eventually be drawn into the nucleus, so atoms cannot survive by classical physics alone. How long does it survive?

The power radiated by an accelerating charge is given by:

$$\frac{dE}{dt} = \frac{Kq^2}{c^3} a^2$$

where $K = 6 \times 10^9\ \mathrm{N{\bullet}m^2{\bullet}C^{-3}}$, c = speed of light $[\mathrm{m{\bullet}s^{-1}}]$, q = electron charge [C], a = instantaneous acceleration $[\mathrm{m{\bullet}s^{-2}}]$.

Assuming the electron's orbit is a circle (atom is Hydrogen)

a) Find the energy radiated during 1 cycle (frequency is f)

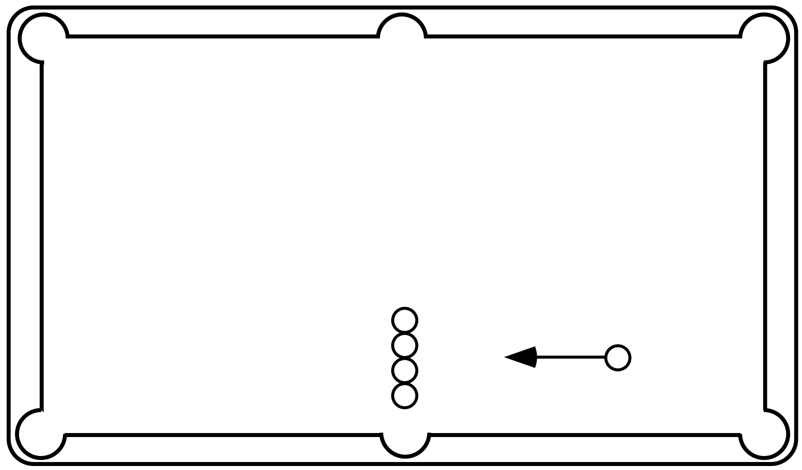
b) At this rate, how many cycles before the radius is down to practically 0?

c) Therefore estimate the classical-physics lifetime of a hydrogen atom. *[Peter & Robin]*

[This radiation by accelerating charges is the principle of a broadcasting antenna for radio, TV or microwaves.]

6) Stupid bet tricks

Consider the following arrangement of balls on a pool table: 4 balls are lined up (touching each other) next to a pocket. Next, the cue ball is aimed exactly between the two middle balls. My friend claims that he can sink all the balls this way. Is he right? Assume an ideal table (no rolling friction) and slippery balls.



Calculate the final velocities of all balls, given the initial velocity of the cue ball is V $m \bullet s^{-1}$, that all collisions are perfectly elastic (energy-conserving), and that all balls are identical.

As a matter of fact, if the balls are hit sufficiently hard they <u>will</u> all sink. If you found this to be impossible, above, then explain how it actually happens. [HINT: what if the slippery assumption doesn't hold?] *[Peter]*

[Try www.physics.utoronto.ca/~POPTOR, to discover if we succeeded in putting a movie of this there!]

<u>Useful Bits of Information</u>

Al:

linear expansion coefficient = 2.31×10^{-5} K^{-1}

heat capacity = 24.4 J K $mole^{-1}$

density = 2.7×10^{3} kg m^{-3}

Light intensity from sun at earth's orbit

Solar constant = 1.353 kW m^{-2}

1997-1998 Physics Olympiad Preparation Program

– University of Toronto –

Problem Set 6 Solutions: AC Circuits and Electronics

1) Misha's Mom's Medallions

Faraday law of electrolysis states that the mass of substance liberated in electrolysis is proportional to the charge passed:

$m = kq = kIt = (m_0/F)It$,

Where k is the electrochemical equivalent of the substance; m_0 is a molar mass; $F = N_A q_e$ is the Faraday constant which is equal to the product of the Avogadro's number and the electron charge. The mass of medallion $m = Vd = Ahd = m_0 It/F$.

Thus the period of time for silver-plating the medallion $t = FdhA/m_0 I$ = $((9.65)(1.05)(5)(5.7)(10))/((1.1)(1.8))$ = 1458 sec = 24.3 min. Therefore, Misha has to pay 4 dollars for his mom's birthday present.

2) Tesla Yes! Edison No!

The electrical power delivered to the factory $P = VI = 55\ kW$. The transmission line conductors have a resistance of 3Ω ⇒ the voltage drop along the line $U = 1{,}500\ V$. Therefore, the output voltage must be $1{,}500\ V + 110\ V = 1{,}610\ V$ to deliver 110 V to the factory. The power loss is $P_1 = 2\ UI = 1{,}500\ kW$, which is 27 times greater than the delivered power.

If we use transformers, we can claculate the turns ratio $n_2/n_1 = V_2/V_1 = 500$ and the current in transmission line $I_2 = I_1 \cdot (n_1/n_2) = 1A$. The voltage drop along the line $U = 3\ V$, which is only $5 \bullet 10^{-3}\%$ of $V_2 = 55\ kV$. The power loss in the line now is equal to $P_1 = 6\ W$, which is only $1 \bullet 10^{-2}\%$ of the value of the delivered power.

3) Analog differentiation and integration

a)we have

$$\frac{q}{C} + R\frac{dq}{dt} = V_{in} \qquad [1]$$

$$R\frac{dq}{dt} = V_{out} \qquad [2]$$

Now, for low frequencies, the capacitor is charging/discharging all the time, and most of the voltage drop occurs across it [$R_{\text{capacitor}} >> R$, if you like]. Hence, we can drop the Rdq/dt in eq. [1]. Differentiating eq. [1] and plugging into [2] gives:

$$\frac{q}{C} \approx V_{in}$$

$$\frac{d}{dt}\left(\frac{q}{C}\right) = \frac{dV_{in}}{dt}$$

$$\frac{dq}{dt} = C\frac{dV_{in}}{dt} = \frac{V_{out}}{R}$$

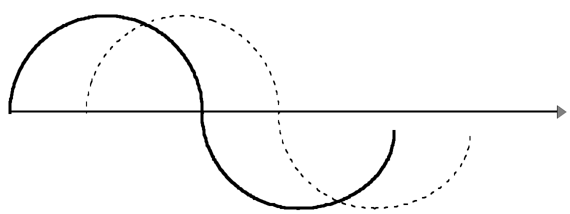
Or,

$$V_{out} = RC\frac{dV_{in}}{dt}$$

[condition: low frequencies; input frequency << $1/RC$]

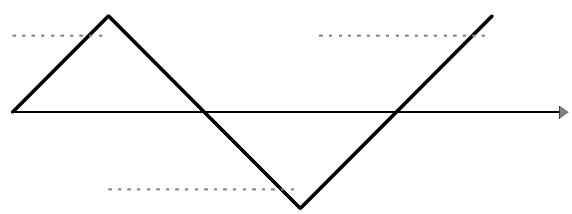
Plots

i)



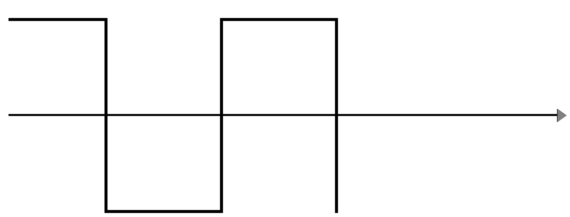
dotted line: output (cosine)
normal line: input (sine)

iii)



dotted line: output
normal line: input

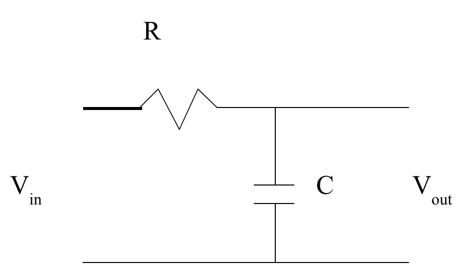
ii)



0-line: output
normal line: input

NOTE: plots are not to scale!

b)Circuit:



we have

$$\frac{q}{C} + R\frac{dq}{dt} = V_{in} \qquad [1]$$

$$\frac{q}{C} = V_{out} \qquad [2]$$

Now, for high frequencies, there is not enough time for the capacitor to be

charged, and most of the voltage drop occurs across the resistor [$R_{\text{capacitor}} << R$, if you like]. Hence, we can drop the q/C in eq. [1].
Differentiating eq. [2] and plugging into [1] gives:

$$R\frac{dq}{dt} \approx V_{in}$$

$$\frac{d}{dt}\left(\frac{q}{C}\right) = \frac{dV_{out}}{dt}$$

$$\frac{dq}{dt} = C\frac{dV_{out}}{dt} = \frac{V_{in}}{R}$$

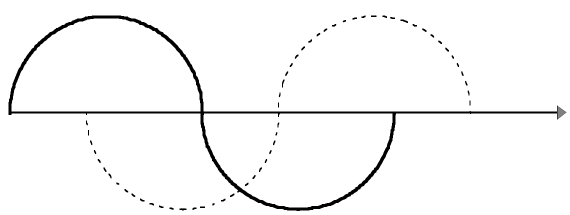
Or,

$$V_{out} = \frac{1}{RC}\int V_{in}dt$$

[condition is: high frequencies; angular frequency of input signal >> $1/RC$]

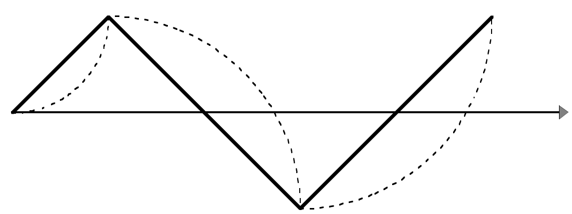
<u>Plots</u>

i)



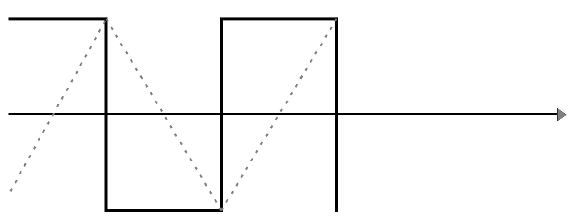
dotted line: output (-cosine)
normal line: input (sine)

iii)



dotted line: output (parabolas)
normal line: input

ii)

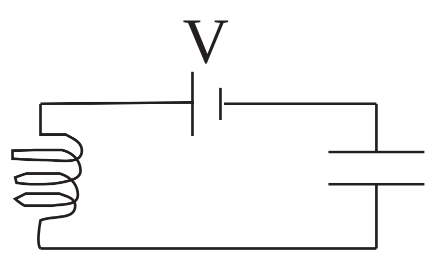


dotted lines: output
normal line: input

NOTE: Plots are not to scale!

4) What goes around, comes around

a) If the voltage across the battery is V, then the voltage across the capacitor is –V in steady state.



b) From the rather direct hints in the question, the voltage across the capacitor looks like: $dVc/dt \propto I(t)$, where $\propto$ indicates proportional, I(t) is the total current running through the circuit as a function of time, and V_c is the voltage across the capacitor

Again, from the oh-so-subtle hints, the voltage across the inductor is:

$$V_i \propto dI(t) \,/\, dt$$

From Kirchhoff's voltage law, the sum of voltages in a circuit must be zero. For this to be true for all time, the change in voltage across one device must equal the opposite change in voltage across the other device. This allows us to write:

$$dV_c/dt = -dV_i/dt$$

$$I(t) \propto -d^2\, I(t) \,/\, dt^2$$

This is the same as

$$x(t) \propto\ -d^2\, x(t) \,/\, dt^2$$

which is the equation for a simple harmonic oscillator, as seen in a previous problem set featuring everyone's favorite oscillator, Bob (no relation to the author).

The solution to this is $I(t) \propto \cos(kt)$ where is k is some constant dependent on the inductance and capacitance in the circuit. If you are not sure if this is a proper solution, substitute it into the equation and verify that the LHS=RHS.

(I am being a little lax here about the boundary conditions, i.e. what is the current at I(0). By writing the solution as cos(kt), I am assuming that the initial current is non-zero, so that we are starting the clock just after the battery has been shorted and removed, not before.)

c) Energy transfers from the electric field in the capacitor to the magnetic field around the inductor and back again.

d) As soon as Frido tries to do something with his circuit, he will no longer have a perfect LC circuit. He must put a resistor in series with it:

As you know, electrical energy is turned into heat in a resistor according to RI^2 where R is the resistance and I is the current flowing through the circuit. Without even solving the circuit equation, you can show that energy is leaving the circuit. Alas, poor Frido has been boondoggled.

5) Operation: 'Amplifier'

a) i) $0V$

ii) $-15V$

iii) $15V$

b) With any op-amp circuit there are two possible results: either the inputs are different and the op-amp is in saturation, or else the inputs are driven so that they are at the same voltage. Feedback allows the the second result in each of the given cases.

i) The current flowing through the feed back loop must be the same current flowing out the V- line. We try to find a self-consistent situation were $V- = 0$. For this to happen, the current must be $1V/100$ Ohms. Thus $V_{\text{out}} = -1V$. This circuit inverts the input voltage

ii) Same reasoning as in part (i). $V_{\text{out}} = -5V$. This circuit inverts and amplifies with a gain of five, the input voltage.

iii) $V_{\text{out}} = -1.9V$. This circuits adds and inverts the input voltages.

iv) $V_{\text{out}} = 2V$. This circuit doubles the input voltage (no inversion)

6) Flip-Flopping on a counter proposal

a) If R and S are both set to 1, there are 2 equally stable output configurations: Q=0 and Q_{bar}=1, or Q=1 and Q_{bar}=0. If we switch R to 0 and back to 1, Q and Q_{bar} will switch from the stable configuration they started in to the other stable configuration. Thus a pulse in R causes the outputs to "flip flop" between the two stable configurations.

b)

pulse	*A*	*B*	*C*
0	0	0	0
1	1	0	0
2	0	1	0
3	1	1	0
4	0	0	1

We can see that if we let the output of A be the 2^0 binary digit, B be the 2^1 digit, and C the 2^2, this counter gives the *binary value* for the number of pulses that have been fed into the input (e.g., 4 = 100 base2)

1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 1: General

1) Pushy Photons

a) Black is good — it would absorb all the light, and therefore transfer all the momentum of the light. But *silver* would be better — when the light reflects back to where it came from, it reverses its momentum. So this would mean (p– (–p)) = 2p for the momentum transferred to the sail.

b) From the useful bits of info at the bottom of the question sheets, the solar constant is 1.353 kW m^{-2}. For light, E = pc, so by taking derivatives we also have that:

$$\frac{dE}{dt} = \frac{dp}{dt} \cdot c$$

However, *power* P is defined as the rate of change of energy with time, on the left-hand side, and *force* F is defined as the rate of change of momentum with time, on the right-hand side. So we can write:

$$P = \frac{dE}{dt} = \frac{dp}{dt} \cdot c = F \cdot c$$

or

$$F = \frac{P}{c}$$

or actually F = 2P/c if we let the light be reflected.

A soccer field isn't always the same size, but has to be in a certain allowed range of sizes; roughly 100m × 50m might be reasonable. The power P incident on the field is intensity times area: P = IA = 1.353 kW m^{-2} • 5000 m^2 = 6.765 MW (quite a bit of power), so the force on the whole large area would be

$$F = 2 \bullet 6.765 \text{ MW} \;/\; (3 \times 10^8 \text{ m s}^{-1}) = 4.51 \times 10^{-2} \text{ N}$$

Also, F = ma, so the acceleration of the 2,000 kg spacecraft would be:

$$a = 2.25 \times 10^{-5} \text{ m s}^{-2}$$

The velocity after constant acceleration for a time t depends on initial velocity (which is zero, here):

$$v = u + at$$

A day of seconds is (24 hr) • (60 min/hr) • (60 sec/min) = 86,400 s. So the velocity after a day would be 1.94 m s^{-1}, which is jogging speed. After a year, it would be 365 times this value, about 710 m s^{-1}, which is roughly 10% of the orbital speed of a low-orbit satellite.

c) Again, if the light is reflected, $F = 2P/c = 2 \bullet 10^{12}\ W\ /(3 \times 10^8\ m\ s^{-1}) = 7 \times 10^3$ N, which is the gravitational force of the smallest automobile. The force is only applied for a trillionth of a second, so the momentum transfer (the impulse) is:

$$p \bullet \Delta t = (7 \times 10^3\ N) \bullet (10^{-12}\ s) = 7 \times 10^{-9}\ kg\ m\ s^{-1}$$

which is small, like a big dust-mote moving in a sunbeam, or a raindrop in a very misty rain.

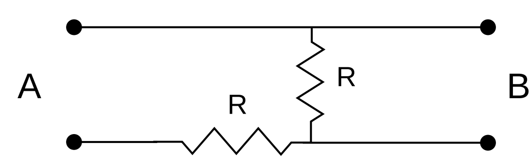
The *pressure* is a different story, though, because this force is applied over a tiny spot:

$$P = F\ /\ A = (7 \times 10^3\ N)\ /\ (\ (10 \times 10^{-6}\ m)^2\ ) = 7 \times 10^{13}\ \ N\ m^{-2}$$

100 kPa = 100 × 10^3 N m^{-2} is roughly one atmosphere of pressure. So this pressure, briefly exerted by the focussed laser pulse, is something like 700 million atmospheres of pressure. *[Robin]*

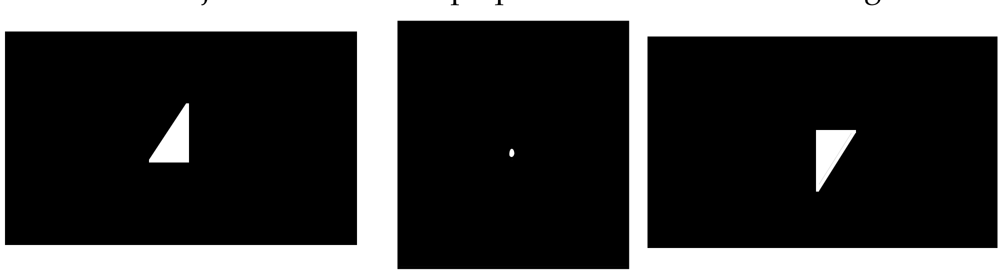
2) Out of sight, hope you don't mind

This simple circuit works, using any two identical resistors. Worth noting: since the terminal-pairs 'A' and 'B' don't behave the same, the circuit must not be symmetric for the pairs of terminals. *[Gnädig/Honyek]*

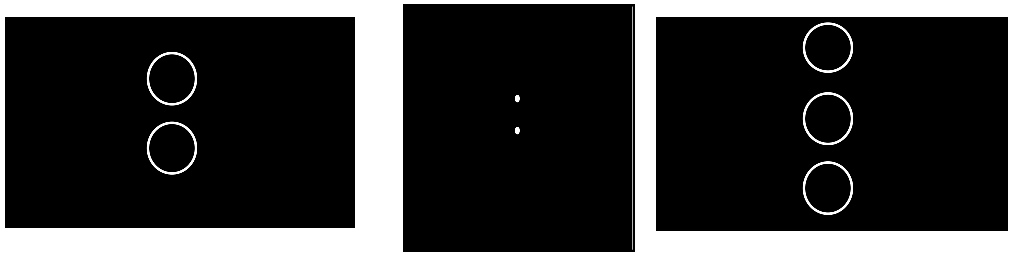


3) Leonardo's *camera obscura*

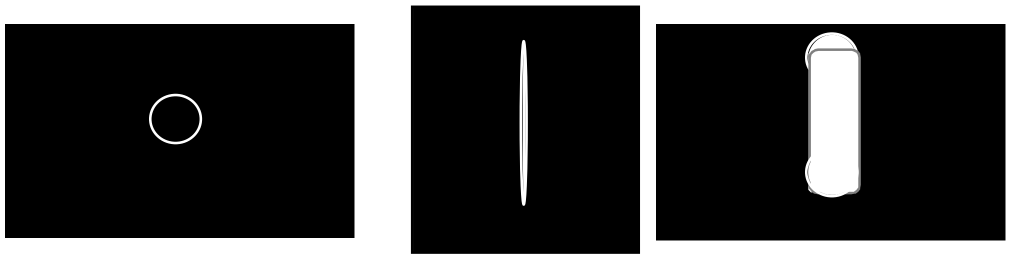
a) Illuminated object Opaque sheet Image on screen



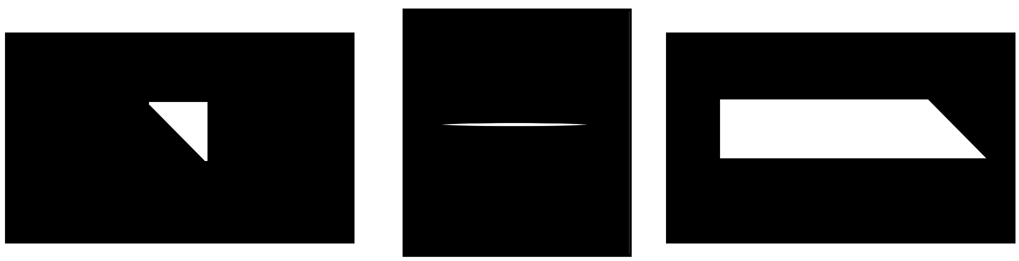
b) Illuminated object Opaque sheet Image on screen



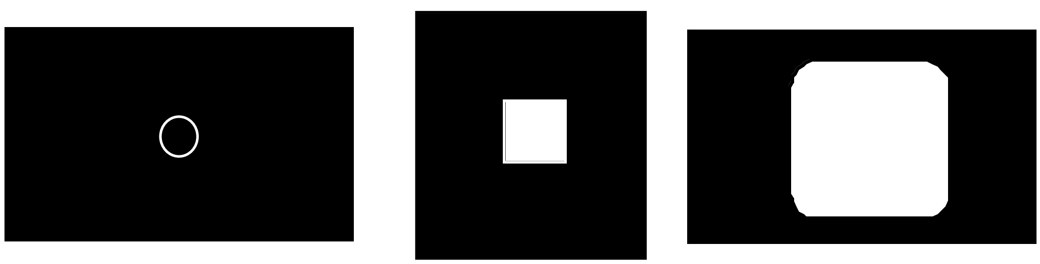
c) Illuminated object Opaque sheet Image on screen



d) Illuminated object Opaque sheet Image on screen



e) Illuminated object Opaque sheet Image on screen



[James]

4) Heat works

We begin with two identical spheres at some uniform temperature. One rests on an insulating table and the other is suspended by an insulating thread. To each sphere the same amount of heat energy, Q Joules say, is transferred.

In the absence of external forces we would expect each sphere to have an equal temperature change. However, since gravity is present and the volumes of the sphere will increase therefore their *centres of mass will move*.

Since the ball on the table will have its centre of mass *rise*, some heat energy will be used to do work against gravity. For the hanging ball, however, the situation is reversed. Therefore the resting ball will have a lower final temperature than that of the hanging ball.

Suppose each sphere is 100 g, initially at some temperature T_I K (the density of aluminum is 2.7×10^3 kg m^{-3}) and 250 KJ of heat is added to each. Let Δr_A and ΔT_A denote the change in radius of the sitting sphere, sphere A and Δr_B and ΔT_B denote that for the hanging sphere, sphere B.

The initial radius of each sphere can be calculated with

$$\frac{4}{3}\pi r_i^3 = \frac{[\text{mass of aluminum}]}{[\text{density of aluminum}]} = \frac{\frac{1}{10}\text{kg}}{2.7 \times 10^3 \text{ kg m}^{-3}}$$

then $$r_i = \left[\frac{3 \times 10^{-3}}{27\,(4\pi)}\right]^{\frac{1}{3}} \approx 0.0207 \text{ m}$$

Assuming a linear expansion coefficient to apply over the entire uniform heating we have (to first approx.),

$$\Delta r_A = [2.31 \times 10^{-5}\, \Delta T_A] r_i = 4.777$$

and

$$\Delta r_B = [2.31 \times 10^{-5}\, \Delta T_B] r_i = 4.777$$

Hence

$$\Delta T_A = \frac{1}{2.31 \times 10^{-5}} \quad \frac{\Delta r_A}{r_i}$$

and

$$\Delta T_B = \frac{1}{2.31 \times 10^{-5}} \quad \frac{\Delta r_B}{r_i}$$

The heat capacity, C, of 100 g of aluminum is simply

$$\begin{aligned} C &= 24.4\frac{\text{J}}{\text{K}\cdot\text{mol}} \cdot \left(\frac{27}{27\text{ g/mol}}\right) \cdot (100\text{ g}) \\ &= 90.3704\frac{\text{J}}{\text{K}} \end{aligned}$$

Where we have used the molar mass of aluminum to be 27 g/mol.

Clearly the centre of mass of sphere A will rise by a distance equal to Δr_A and the centre of mass of sphere B will fall a distance equal to Δr_B.

Since the heat energy transferred must balance the change in potential energy plus the energy due to the temperature change we have

$$250\,000 = \mathrm{mg}\ \Delta \mathrm{r_B} + \mathrm{C}\Delta \mathrm{T_A}$$

$$= \frac{9.81}{10}(4.777 \times 10^{-7})\Delta \mathrm{T_A} + 90.3704\ \Delta \mathrm{T_B}$$

Thus

$$250\,000 = [4.777 \times 10^{-7} + 90.3704]\ \Delta \mathrm{T_A}$$

Similarly

$$250\,000 = [4.777 \times 10^{-7} + 90.3704]\ \Delta \mathrm{T_B}$$

Therefore, the final difference in temperatures is

$$\Delta T = \Delta T_B - \Delta T_A$$

$$= 250\,000\left[\frac{1}{90.3704 - 4.777 \times 10^{-7}} - \frac{1}{90.3704 + 4.777 \times 10^{-7}}\right]$$

$$\approx 250\,000\left[1.170 \times 10^{-10}\right]$$

$$\approx 2.925 \times 10^{-5}$$

Therefore the difference in temperature is approx 2.87×10^{-5} K. Pretty small! *[Gnädig/Honyek & Peter]*

5) Death of an Atom

The energy given off in a time Δt is roughly

$$\Delta E = \frac{kq^2 a^2 \Delta t}{c^3}$$

In 1 orbit (time T),

$$\Delta E = \frac{kq^2 a^2 T}{c^3}.$$

We will assume the radius of an orbit stays constant for any one cycle (the electron is actually spiralling in, so this is not really true). The radius is R ≈ 0.5 Angstroms. The electron is kept in orbit by the attraction between it and the nucleus. We have $m\frac{V^2}{R} = \frac{kqq}{R^2} = ma$, where m is the mass of the electron and V is the orbital speed. The charge of the nucleus is +q for Hydrogen. Plugging all this in yields the nice expression:

$$E_{cycle} = \frac{2\pi kq^2}{R}\left(\frac{V}{c}\right)^3.$$

Here, this equals about $4.4*10^{-24}$ J / cycle [this should be negative, but we will remember that this is the energy lost]

The total energy of the orbit is: $-\frac{kqq}{2R}$

compared to that for a gravitationally orbiting object:

$$-\frac{Gmm}{R}.$$

The electron will presumably "die" when all this energy is exhausted. If we assume that the energy loss per cycle is as in i) (not really true), we will have:

$$\frac{kqq}{2R} = n\left(\frac{2\pi kq^2}{R}\left(\frac{V}{c}\right)^3\right),$$

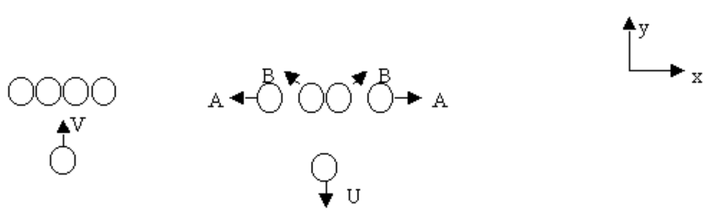
where n is the number of cycles. Plugging in the stuff, n = $3.4*10^5$.

The time, then, for this to happen is simply nT (T is the period) which comes out to about $6*10^{-11}$ s. It is not really meaningful to carry the first digit because of all the approximations we made. Surprisingly, doing a more accurate analysis leads a very similar answer. Either way, this is a very short time, and we have to conclude that the classical answer cannot be right. *[Peter]*

6) Stupid bet tricks

i) First, notice that when the balls collide all forces between them (and hence the momentum transferred) will be normal to the surfaces (that is the only point of contact between them). This is because there are supposed to be no frictional forces.

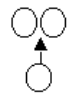
Let the labeling be as follows:



If we tried to do the question directly, we couldn't – we have 4 unknowns (U, B, A, direction of B) and only 2 equations (energy, momentum).

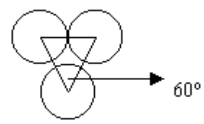
However, since the collisions last a very small amount of time we can split the problem into the two middle balls interacting with the cue ball and THEN the two middle balls interacting with the outside balls.

Doing this we get:



where the final speeds of the pair of balls are C (symmetric).

Additionally, we see that the momentum of each ball must be at a 30° angle to the normal:



So, $\frac{C_x}{C_y} = \tan(30) = \frac{1}{\sqrt{3}}$.

Solving, $C = \frac{2\sqrt{3}}{5}V$ (direction is 30° to y-axis), $U = \frac{1}{5}V$ [negative y direction].

Now, the two remaining collisions (which are symmetric, so we only solve one – the right hand side one).

We could write down all equations and solve this by brute force, or we could simply notice that Cy will be unchanged because there are no tangential forces (the balls are slippery – they don't "stick"). Hence, $B_y = C_y$.

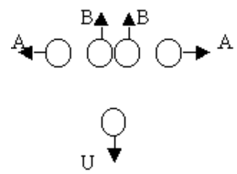
Solving the remaining equations yields

$$A = \frac{C}{2} = \frac{\sqrt{3}}{5}V$$

$$B = \frac{3}{5}V$$

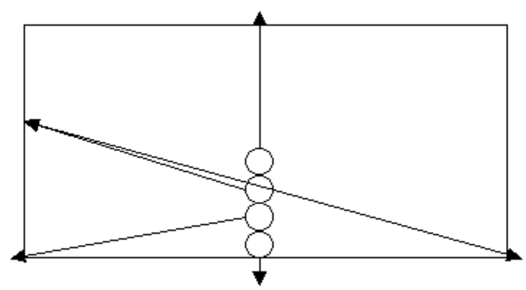
(the direction of B is [positive y], that of A is [positive x-axis]).

So we have:



And it seems the trick shot didn’t work – only the outside balls have been sunk.

ii) When you hit the balls hard they deform slightly and do “stick” to one another. That means that there is a tangential force between the outside most and middle balls (remember in the solution above we assumed no tangential forces to solve this collision). Numbering the balls 1 to 4 from left to right [see picture above] yields the following: 2 has a counterclockwise spin and 3 has a clockwise spin; moreover, the 2 centre balls now also have a horizontal velocity component [why ?]. Upon hitting a bank, which is fairly soft, the spin of ball #3 will make it rebound at more or less the same angle as it came in at — so 3 will go into another corner pocket. Now that you are completely confused look at the picture to make some sense of this:



[Robin is still trying to get a video picture of this pool-shot onto the website,at www.physics.utoronto.ca/~poptor]

[Peter]

1998-1999 Physics Olympiad Preparation Program

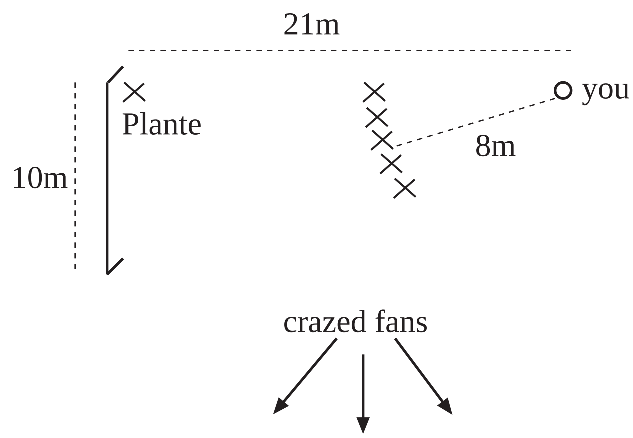
— University of Toronto —

Problem Set 2: Mechanics

Due December 11, 1998

1) Football fizix

In the dying minutes of the World Cup finals, your team is awarded a direct free kick near your opponents' goal (see figure). The defense has set up an impenetrable wall of defenders 8 m from the ball such that the only way to score is to kick it over their heads and into the 3m-high net. Also facing you is Pele Plante, the superb goalie that has kept your team off the score board all game. You want to keep the ball as far away from him as possible. What initial velocity (speed and direction) do you give the ball? Remember the highlight tapes you viewed before the game indicated that the defense can jump to a head height of 2.5 m and your hardest kick corresponds to an initial ball speed of 30 m s^{-1}. (Ignore air effects for the purpose of this question.)



Also remember that if you do not score, your team loses the Cup *and* Plante takes over your Nikke promotional contract. The pressure is on...

BONUS: Describe how your answer would change when you also consider the effects of air on the ball's trajectory. *[James]*

2) Chow, baby

Cat food is what Kit buys at the corner store — in 175 g flat cans. She knows from sad experience that the plastic carry-bags that the store provides will hold well enough for 15 cans, but will tear through right away if she puts in 16 cans.

Coming home with 15 cans in one plastic carry-bag, Kit realizes she has a problem. Between her and her apartment lies The Elevator. Kit knows from physics class that the acceleration in the elevator will add to the acceleration due to gravity, and that the cans will weigh more as the elevator accelerates (though their mass remains constant).

This elevator starts with constant acceleration over two floors between the ground floor and the second floor, and then rises at a constant speed of three seconds per floor. All floors are 4 m apart. Will the bag split open — should Kit have double-bagged?

What should be the maximum number of cans that Kit can load up with? *[Robin]*

3) Sink, sank, sunk

At some point in the movie "Titanic" it is said that the pressure at its sunken depth is 5500 pounds/inch^2. Given this, find the depth at which the *Titanic* supposedly came to rest. The 'real' answer is 12600 feet, density of water is 10^3 kg m^{-3} (ignore air pressure). Comment on the results, i.e., truth in film-making.

At another point in the movie someone estimates that the *Titanic* will sink in a time of roughly half an hour. Approximating the *Titanic* as a rectangular box of cross-sectional area A = 882 × 34 feet^2, mass M = 40 tons (loaded), total height 80 m, and a horizontal hole at the bottom of a side of the box of area A′ = 100*0.5 m^2, estimate the time it takes sink — that is, when the top of the box is even with the water line. Ignore air.

[HINT: assume the flow stays practically at equilibrium as water goes in; you *may* use Bernoulli's equation, but if you do, you have to prove it.] *[Peter]*

4) When bowling, always signal your lane-change…

Consider the problem of throwing a bowling ball with an initial spin, so that its path curves. Take the x-axis as the centre-line down the bowling alley, and y as the horizontal sideways direction. Say that when the ball is released, it has an initial spin ω with the top of the ball moving sideways (i.e., $\vec{\omega}$ points along the x-axis), and a velocity $\vec{v}$ at an angle θ to the centre-line.

Find the path of the bowling ball, by giving the x-coördinate and y-coördinate as time goes by, i.e., x(t) and y(t), where t is time (this is called 'parametric form'). Assume the point of contact between the ball and the floor is very tiny. The moment of inertia of the ball will be that of a sphere.

Plot some nice-looking representative paths ($\omega = 0$, $\omega > 0$, $\omega < 0$) *[Peter]*

5) A matter of some gravity

i) The moon rose in the east today at 7pm. If the lunar cycle (number of days between full moons) is 29 days, when will the moon rise tomorrow? (One other subtly 'hintful' piece of information: if the earth were moving more *slowly*, in orbit around the sun, the lunar cycle would be shorter.)

ii) The spaceship *Galileo's* mission is to study Jupiter and its surrounding moons. To get the vessel into a Jovian orbit, NASA had the vessel fly-by both Venus and Earth to increase its speed. This type of trajectory which provides acceleration without the burning of expensive fuel is called a *sling-shot maneuver*. Consider one of these interactions in which the Earth's large velocity was used to accelerate the vessel. What is the maximum speed that the vessel could have attained after this interaction, if its speed before the encounter was 5 km s^{-1} ?

Galileo (mass: 1000 kg) actually did this type of maneuver *twice* with Earth on its way to Jupiter. How many such sling-shots would have to occur for the length of our year to decrease by one second?. *[James]*

[Check out www.jpl.nasa.gov/galileo/index.html for some great pictures and information on this billion dollar space mission]

6) Balloon tug-o-war experiment

Here's a neat experiment you are welcome to do with a friend (put both your names on, for POPTOR marking). It's not too hard, but it's full of physics. You will need:

2 good-quality round balloons
1 thick drinking straw
some scotch tape

Have your friend blow up one balloon about 1/4 full and the other about 1/2 full. Pinch the neck of each balloon so that no air escapes. Carefully insert one end of the straw into the neck of each balloon and tape it air-tight, while still pinching. It will look like a balloon bar-bell for weightlifting.

Before actually trying it, what do you *think* will happen when your friend stops pinching the balloons so that air can flow between the balloons? Argue for your predictions on the basis of what physics you can figure out or guess. *Then*, hold the straw gently and have your friend release the balloons. Were your predictions right? Try squeezing the balloons, one at a time, and see what happens.

To examine this interesting phenomenon further, do the following measurement: Try to determine the *air pressure* inside the balloon, as a function of the *size* of the balloon.

You will need:

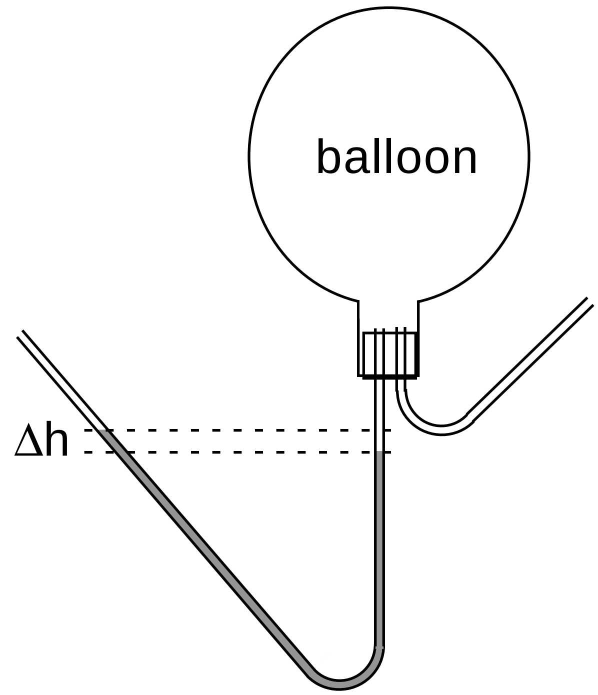
clear flexible tubing about 3 to 4 metres in length (see INFOBITS, below)
1 clear flexible tube about 1/2 metre in length
1 two-hole stopper with holes to fit the tubing snugly
some duct tape
2 metre sticks or long rulers

You might want also to use (optional)

1 500 ml (or larger) kitchen measuring cup, or beaker with volume markings
two large soup cans
1 large bucket or plastic laundry tub (a nearby sink or bathtub will do fine)

To set up:

1. Seal one end of each piece of tubing into the stopper
2. Make the long tube into a rough 'U' shape, and tape each side onto a meter stick or long ruler. Make the end farthest from the stopper to be sloping, as pictured. Be sure your setup is stable.
3. Half fill the long tube with water. This will be a *manometer* for you to measure pressure with. You may need more adjustments once you begin.
4. Now fit the balloon onto the stopper. Gently fill up the balloon, blowing air into it through the short tube. If you notice air leaking anywhere you might use some Vaseline or duct tape to seal it up.



With the right amount of water in the long tube, you can measure the *difference* in height of the water on each side of the 'U'. With the ruler or metre stick at a sloping angle you can measure along the tube, and then use trigonometry to figure out the height. What is the relationship between air pressure in the balloon and difference of water levels in the manometer?

Now make a series of measurements of air pressure, for the balloon inflated to different sizes (including *not* inflated). How can you best measure the size of the balloon: Cast a shadow onto paper, and measure the shadow-size? Use a tape measure to find the circumference? Use the measuring cup and tub to trap air released from the balloon, and read its volume? We're interested in *your* best ideas — change anything you want.

Finally make a graph of your data with height *vs.* balloon size, and connect the points with a smooth line. Be sure to label your axes with dimensions. From your results, can you explain the outcome of the first 2 balloon experiments?

[Bonus-mark 'sequel' experiments: It would be interesting to see if the pressure in the balloon, as a function of size, varied *differently* depending on whether you were inflating or deflating the balloon. Do you get a different answer if you measure volume of air and if you measure diameter of balloon? Would it make a difference if you filled the balloon not with air but something incompressible, like water? (How could you change the manometer in order to use water in the balloon?)] *[Simal and Robin]*

INFOBITS™ — Useful Bits of POPTOR Information

Mass of Earth: $6. \times 10^{24}$ kg

Mass of sun: 2×10^{30} kg

Distance between Earth and Sun: 1.5×10^{11} m

$G = 6.7 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}}$

Where to get parts for experiments:

there is plastic tubing at Canadian Tire, in the Plumbing section, for 17¢ per foot. DO NOT attempt to use *glass* tubing, unless you have teacher supervision. It is pretty easy to break it and have it go right through your hand, if your technique is wrong — especially when putting it into rubber stoppers.

- polyethylene, 0.25″ O.D. (outer diameter); translucent, not quite clear
  (stock no. 7 78380 022022 1; SKU: 039166056347)
- vinyl, 3/16″ O.D.; clear, but soft, and hard to stuff into a rubber stopper
  (stock no. 7 78380 02032 0; SKU: 039166056224)

2-hole rubber stoppers: from chemistry class, we thought

CHECK THE POPTOR WEB PAGE for other hints, and any corrections we might post:

www.physics.utoronto.ca/~poptor

1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Marker's Comments on Set 2: Mechanics

General Comments from POPTOR:: Our apologies on how late we are in getting material back to you. We hope that you find the solutions and these comments are so good that they partially make up for the delays.

We have had a change in our team: **Carolyn MacTavish** joins us, to help us mark, make up solution sets, and run the POPTOR invitational weekend. We're rushing to catch up, and you should have #3 – Thermodynamics returned with solutions pretty quickly this time. If you haven't bothered with #3 because we're so late, please jump right back in for #4 – Waves and Optics! Lots of people miss handing in one set.

Peter sez: With problem set 2 finished, I am now 2 for 2 in getting the wording of my questions mixed up. The wrong revisions of questions going out, me not being able to multiply, etc. have contributed to this. Either way, you all have my apologies. In all cases, however, the mistakes have not changed the core of the questions (yet). The marking, and my comments, follow questions 3, 4 and 6 as you had them (not necessarily as I planned).

Robin adds: Er, well, actually that was my fault. Sorry, Peter & all POPTOR participants.

1) Football fizix

Carrie sez: Good job! This problem was done fairly well by all (avg. 6.7/10). The most common mistake was in assuming an initial velocity of 30 m/s. However, this is not the best choice since it does not yield the maximum horizontal velocity (given the constraints of the defence wall and that you are aiming for the top corner of the net-furthest from Plante).

2) Chow, baby

Carrie sez: Excellent job! The majority did very well on this problem (avg. 8.2/10). The most common error was in the assumption that the bag would break if it contained *exactly* the weight of 16 cans (at rest).

3) Sink, sank, sunk

Peter sez: Question 3 (“Titanic”) was done remarkably well – we at POPTOR have spent weeks trying to figure out the best approach for this, and I thought this question would be a killer (how it ended up a ‘moderate’ #3 I have no clue). I solved the question using a force approach, and that’s where I was trying to lead you with my hint. What happened, of course, was that no one solved the question the way I intended, but instead found much more elegant (and shorter) ways. One person even tried to correct for the fact that the flow of water is initially non-steady – very impressive! All mistakes made on this question were minor.

4) When bowling, always signal your lane-change

Peter sez: Question 4 (“Bowling”) was a bit disappointing, probably because of the mis-wording. In the “real” version of this question I actually gave numbers, which I wanted you to use to figure out the path of the ball – this way you would have to do some algebra. What happened instead was that nobody calculated the paths, they just sketched some graphs. This is fine in principle, but you must explain how you arrived at your answer! Also, most people missed the fact that even though a ball thrown with some spin will turn initially, it will eventually stop slipping and follow a straight line. You can check that the next time you’re in a bowling alley (I recommend 5 pin, unless you work out on a regular basis) – or just watch bowling on TSN, which is what I do… The included solution shows the path I wanted you to obtain (which is the case $\omega > 0$. Other cases are similar. The data I used follows: $\omega = 5$ rev/s, $V_x = 1$ m/s, $V_y = 5$ m/s, μ_k = coefficient of kinetic friction = 0.1, R = 40 cm. The bowling lane is about 10 meters long and 2 meters wide – the bowl was thrown in the middle of the lane).

5) A matter of some gravity

Carrie sez: Bravo to all those who attempted this question!! (avg. 2.2/10) Most seemed to avoid this one. For part i) the common error was not taking into account the earth's axial rotation and/or realizing that the moon rises later every day. For part ii) most tried to solve using conservation of energy — which was good — but then forgot about momentum comservation — which was not so good.

6) Balloon tug-o-war experiment

Peter sez: Question 6 (“Experimental”) was done extremely well, by those who attempted it. I was particularly impressed by the people who plotted their stuff in Excel (or something similar) and typed the whole thing up. Most did not attempt the experimental question – which is perfectly understandable, since experimental questions don't have a long tradition at POPTOR (more like none at all). Also, they do take extra time an effort.

On the other hand, I would argue that experimental questions teach you the most physics, because to do well you not only have to know your equations, but also understand what they mean. This is a purely philosophical argument, so if you're not into that kind of stuff you should realize that at both the national and international levels experiments are a major component of your mark (50%).

To encourage people to do the experimental questions (which we will have more of) they will be weighted more than the other questions – each will be equivalent to two theoretical questions (plus a bonus for any interesting ideas / set-ups you come up with).

Final Comments:

Peter sez: Concluding I once again want to stress how impressed I was by people's solutions to these questions considering how hard they really were. I hope you keep trying, and speaking of trying *we finally got the movie of the trick shot (problem set #1) on-line* (digitizing it was harder than making the shot!). Check it out, at:

www.physics.utoronto.ca/~poptor/Com98/98GenXtra.html

Proof positive that physics works… (rules!)

1998-1999 Physics Olympiad Preparation Program

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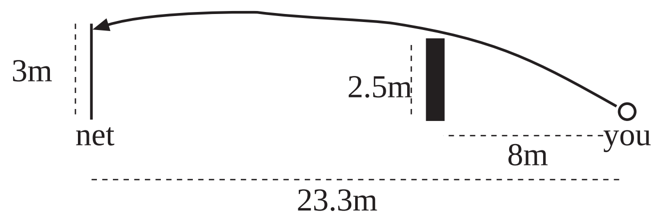
Solution Set 2: Mechanics

1) Football fizix

Your best bet is to go for the corner of the net farthest from Plante (see figure given in question). We can reduce this to a two-dimensional problem by looking at the cross-section along the ball's trajectory. Since there are no air effects, the ball heads straight from your foot, over the heads of the defenders and into the upper corner of the net, as in the figure at right. (Yay! Cheers!)



If you kick it at too high an angle, it will easily clear the heads of the defenders but will take longer than necessary to make it into the net. This gives Plante more diving time to intercept the ball. To minimize the flight time, you want the ball to have the maximum horizontal speed (covering the distance to the net the fastest), but a corresponding vertical speed that will allow it to barely miss the defenders and still go into the top corner of the net. As always in these projectile motion problems: *separate the problem into horizontal and vertical components*.

<u>Constraints:</u>

1) At $x = 8$ m, $y > 2.5$ m
2) At $x = 23.3$ m, $y < 3$ m.
3) At $t = 0$ s, you kick the ball from $x = 0$ m, $y = 0$ m.
4) Max speed $= 30 \text{ m s}^{-1}$

The equations of motion are horizontal at constant speed, and vertical motion with gravity:

$v_x(t) = v_x(t{=}0) = v_x$ (note how I am defining v_x and v_y)
$v_y(t) = v_y(t{=}0) - g * t = v_y - g * t$

So the ball's path is:

$x(t) = v_x * t$
$y(t) = (v_y - (g/2) * t) * t$

Constraint 3 is satisfied by how I set up the equations.

Constraint 1:

Let $t = t_1$ when $x = 8$ m.

$8 = v_x * t_1$.

$y(t_1) = (v_y - (g/2) * (8/v_x)) * (8/v_x)$ but $y(t_1) > 2.5$

Thus $2.5 / (8/v_x) < v_y - (g/2) * (8/v_x)$

$v_y > 2.5 / (8/v_x) + (g/2) * (8/v_x)$

$v_y > v_x * (2.5/8) + (4*g) / v_x$ **<=Eqn 1**

Constraint 2:

Let $t = t_2$ when $x = 23.3$.

$23.3 = v_x * t_2$.

$y(t_2) = (v_y - (g/2) * (23.3/v_x)) * (23.3/v_x)$ but $y(t_2)<3$

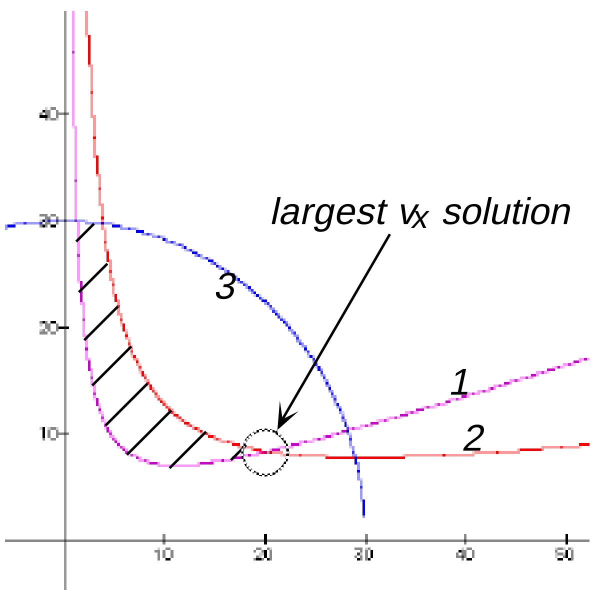
Thus $3 / (23.3/v_x) > v_y - (g/2) * (23.3/v_x)$

$v_y < 3 / (23.3/v_x) + (g/2) * (23.3/v_x)$

$v_y < v_x * (3/23.3) + (23.3/2) * g / v_x$ **<=Eqn 2**

Constraint 4:

$v_x^2 + v_y^2 \le 30^2$ **<=Eqn 3**



Graph the three inequalities, to get a clear picture of how they can be satisfied, in the region above curve '1', below curve '2' *and* below curve '3': see that the maximum horizontal speed v_x occurs very close to the intersection of eqn 1 and eqn 2.

The intersection is at:

$v_x * (3/23.3) + (23.3/2) * g / v_x$

$= v_x * (2.5/8) + (4*g) / v_x$

or

$v_x = 20.2 \text{ m s}^{-1}$

This corresponds to $v_y = 8.25 \text{ m s}^{-1}$.

To make sure that you are not hitting the crossbar, reduce v_x slightly to 20.1 m s^{-1}, say. This will still meet the constraints imposed by equations 1 and 2.

Therefore you must kick it with an initial speed of 21.7 m s^{-1} at an angle of 22 degrees above the horizontal. (Soccer is a science!). (I am using g = 9.8 m s^{-2} throughout this problem.)

<u>BONUS:</u>

Air effects can help you. By giving the ball the proper spin, you can cause the ball to deflect downward from its normal parabolic path. This will allow you to increase the initial horizontal velocity but still have the ball go in the net. If you are really fancy, you might be able to curve the ball enough so you go around the defenders instead of over ('curl' or 'swing' the ball left or right). *[James]*

2) Chow, baby

<u>Find acceleration *a:*</u>

– elevator has constant acceleration over two floors,	$d = 2 \bullet 4\text{m} = 8$ m
– final velocity is one floor per 3 seconds,	$v = 4\text{m}/3\text{s} = 1.33$ m s^{-1}
– initial velocity is zero,	$u = 0$

$$v^2 = u^2 + 2ad$$
$$(1.33)^2 = (0)^2 + 2a(8\text{ m})$$
$$a = 0.111 \text{ m s}^{-2}$$

Adding this to the acceleration due to gravity, the *total* acceleration will be:

$$g' = 9.80 \text{ m s}^{-2} + 0.11 \text{ m s}^{-2}$$
$$= 9.91 \text{ m s}^{-2}$$

The ratio of the acceleration in the elevator to that standing on the ground is therefore: 9.91/9.80 = 1.01. The acceleration is only 1% greater.

<u>Find weight of cans with this *additional* acceleration:</u>

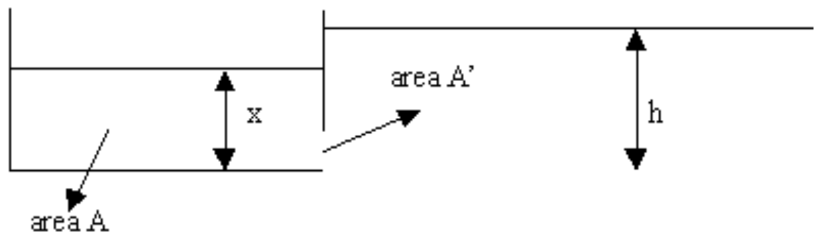
Then 15 cans of cat food accelerated in the elevator would weigh on the bag the way that 15 • (1.01) = 15.15 cans would at rest. This is less than 16.

Does it mean the bag would not break? No! We only know that 16 *will* break the bag, but we *don't know* whether a little bit over 15 would. That's because cans come in units of 1 can — there was no way to test whether 15.15 cans would break the bag. Until now: Kit can be the first person to test out the bags at this weight, without having to cut up a can of cat food in the band-saw.

Recommendations? Double bag, or else carry only 14 cans of cat food. The 14 would weigh in the accelerating elevator the same as 14.14 would on the ground.

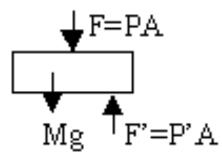
And any weight 15.00 or under in the elevator is definitely safe, because 15 cans under *normal* acceleration was!   *[Robin]*

3) Sink, sank, sunk



Consider a pill box of water of area A and height h.

For equilibrium,



$$PA + mg = P'A$$
$$PA + Ah\rho g = P'A$$
$$P' = h\rho g + P$$

Now extend the pill box to be a tall cylinder of height h (depth h to water level). Ignoring air pressure, P = 0, and we get:

$$h = \frac{P}{\rho g}$$

Here, h ≈ 12600 feet [roughly] for 5500 punds/inch^2, so there's good agreement.

As the system starts without any water inside, the water must be accelerated (that's why you cannot use Bernoulli's equation **inside** the ship, which only applies to steady flows). Later, it reaches steady state.

Let us use the hint and use a force approach:

First note that outside of the hole the water will hardly be moving — the water comes in from all directions, so that the velocity at any one point is almost 0 (but because the ocean is so big this adds up to give a large flow in the hole).

Now, we will consider the system in a small time interval Δt. Since we are in equilibrium, the water inside is stationary at first. For a small time Δt an external force (due to external pressure) and a negative force (due to pressure of water in the ship) acts at the hole. The net force will force in a small amount of water Δm. Additionally, a part of the force will push some of the water upwards, but as Δt –> 0, Δx –> 0 and this is negligible.

Δm is hence pushed an amount

$$\Delta s = 0 + \frac{1}{2}a(\Delta t)^2$$

$$\Delta s = \frac{1}{2}\frac{(\rho g h - \rho g x)A'}{\Delta m}(\Delta t)^2$$

Since the pressure due to a height z of water is ρgz (from i))

$$\Delta s = \frac{1}{2}\frac{\rho g A'(h-x)}{\rho \Delta s A'}(\Delta t)^2 = \frac{1}{2}\frac{g(h-x)}{\Delta s}(\Delta t)^2$$

From conservation of mass, and amount Δm of water at the hole must be equal to the amount of water that causes a rise of the level in the ship, i.e., $\Delta m = \rho \Delta s A' = \rho \Delta x A$

Using this gives:

$$(\Delta s)^2 = \left(\frac{A}{A'}\right)^2 (\Delta x)^2 = \frac{1}{2}g(h-x)(\Delta t)^2$$

$$\frac{\Delta x}{\Delta t} = \frac{A'}{A}\sqrt{\frac{1}{2}g(h-x)} = \frac{dx}{dt} \qquad \text{(in the limit)}$$

As the water flows in, the system comes to equilibrium. To compensate for an increase Δm in mass the ship must have moved (down) an amount

$$\Delta h = -\Delta x$$

Note that this does not imply that h = x! This simply says that x and h change by the same amount. In fact, at time = 0,

x = 0

And h = M/(ρA)

(M is the mass of the ship — 40, 000 tons; this is just Archimedes's principle)

And so

h(t) = x(t) + M/(ρA)

Plugging this into the expression for dx/dt yields:

$$\frac{dx}{dt} = \frac{A'}{A}\sqrt{\frac{1}{2}\frac{gM}{\rho A}}$$

So the water level rises at a constant rate.

From this we find:

$$\frac{dh}{dt} = \frac{A'}{A}\sqrt{\frac{1}{2}\frac{gM}{\rho A}}$$

$$h(t) = \left(\frac{A'}{A}\sqrt{\frac{1}{2}\frac{gM}{\rho A}}\right)t + \frac{M}{\rho A}$$

Putting h = 80 m, we find t ≈ 40 minutes. We didn't catch them this time…

This answer is pretty good — only the "1/2" factor above may have to change [I did an experiment sinking a shoe box, and it looked more like it should be a "1", which is fairly close]. Using Bernoulli's equation gives nonsense for the time it takes the ship to sink (something on the order of 1 minute).

Note that you can use Bernoulli's equation, if you set it up the following way: take a streamline from the top of the water to the hole in the ship — this flow is steady and you get something like:

$P = \frac{Mg}{A} = \frac{1}{2}\rho V^2$ (where V is the speed of the water coming in, P the pressure at the hole's depth).

Using conservation of mass we now get:

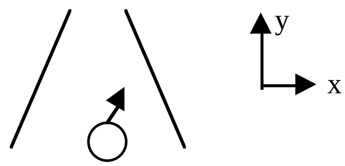
$$\frac{dx}{dt} = \frac{A'}{A}\sqrt{2\frac{gM}{\rho A}}$$

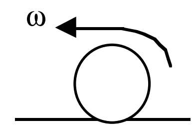
which is the same as the equation above, except for the factor of 2 — which is just as good because we used approximations in either solution. Note also that the flow of the water is different while the hole is not completely covered with water — that's why the height of the hole (0.5 m) is so small, to make this effect negligible.

Speaking of Bernoulli's equation, the proof may be found in virtually any textbook (which is too bad) — see e.g. Halliday & Resnick. *[Peter]*

4) When bowling, always signal your lane-change

The math here isn't too hard, but one has to think about what's going on or it's very easy to be off be a minus sign (the International Olympiad judges kill (well, almost) for errors like this). First, since we know there is friction, we see that it will oppose the ball's motion. Now we resolve the motion to two directions — x and y — and two motions — translational and rotational.





<u>Y:</u>

Translational: (friction slows motion)

$$M\,a = -\mu_k(Normal) = -\mu_k\,Mg$$
$$a = -\mu_k\,g$$
$$V = V_y - \mu_k\,g\,t$$

Rotational: (friction causes ball to spin faster, until

$$I\,\alpha = \tau$$
$$\frac{2}{5}M\,R^2\alpha = \mu_k(Normal)R$$
$$\alpha = \frac{5\mu_k g}{2R}$$
$$\varpi = 0 + \frac{5\mu_k g}{2R}t$$

The ball will continue to skid until it's rotating fast enough so that it can roll. Then, V = ωR. This will occur at a time given by:

$$V_y - \mu_k\,g\,t^* = \frac{5\mu_k g}{2}t\,^*$$
$$t^* = \frac{2\,V_y}{7\mu_k g}$$

The motion thus consists of two phases — with skidding and without:

$$y(t) = V_y t - \frac{1}{2}\mu_k g t^2, \qquad t \le t\,^*$$
$$y@(t) = y(t^*) + V(t^*)t = \frac{12 V_y{}^2}{49\mu_k g} + \frac{5V_y}{7}t, \qquad t \ge t\,^*$$

<u>X:</u>

This is very similar:

$$V = V_x - \mu_k\,g\,t$$
$$\varpi_y = \varpi - \frac{5\mu_k g}{2R}t$$

When skidding stops – V = ωR. But we have to be careful! In this case, V changes sign, and so the above should actually read | V | = | ω | R!

[How do we know it changes sign? Well, you can do what I did — go through the whole calculation carrying the wrong sign. Then do the plot I asked for and get nonsense. Or you can be a lot smarter — assume that V changes sign (or not), and then go back and *see* if your assumption holds]

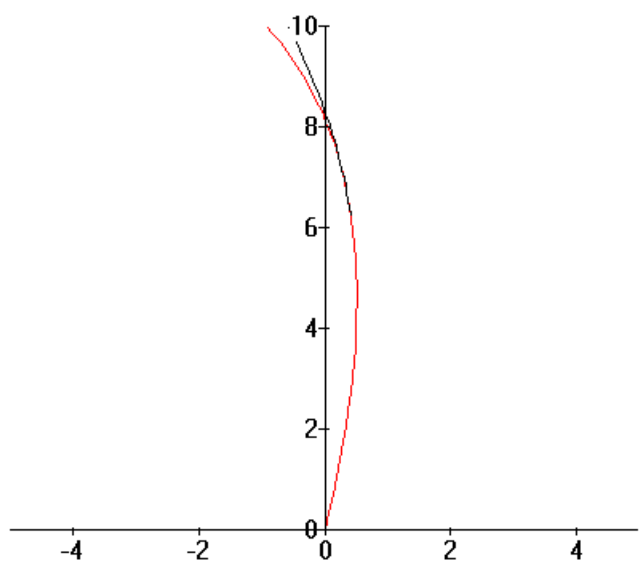
$$t_2{}^* = \frac{2(\varpi R + V_x)}{7\mu_k g}$$

$$x(t) = V_x t - \frac{1}{2}\mu_k g t^2, \qquad t \le t_2{}^*$$

$$x'(t) = x(t_2{}^*) + V(t_2{}^*)t = \frac{(\varpi R + V_x)}{49\mu_k g}\left(12V_x - 2\varpi R\right) + \frac{5V_x - 2\varpi R}{7}t, \quad t \ge t_2{}^*$$

Now we can finally proceed to plot this. We have to calculate t* and t_2* to know which motion occurs when. It turns out that the ball stops slipping after about 3 seconds in the x-direction, and after about 8 in the y, but by that time the ball has long left the bowling alley…

Plotting yields the figure at right.

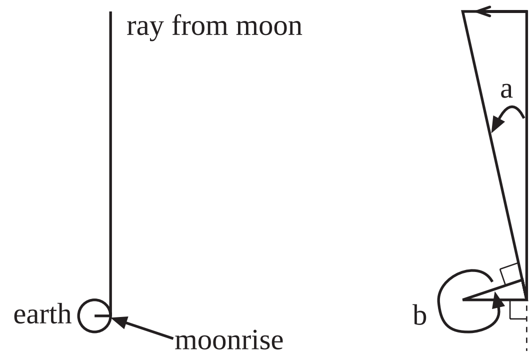


[Peter]

5) A matter of some gravity

i) From the information given we know that if the earth was rotating more slowly, the lunar cycle would be *shorter*. This means the direction of rotation of the earth is the *same* as the direction of the moon around the earth — it has to 'chase' it instead of meeting it 'head-on'. Thus, the moon rises later every day. By how much?

In the time the moon has moved by angle a, the Earth has moved by angle b. The two triangles are set so that the moon rays are parallel to the surface of the earth. They are similar triangles so a = b – 360 °



Also (where t is measured in hours):

$$a = t / (29*24) *360°$$

$$b = t / 24 *360 °.$$

Thus the time between moon rises is: 24.86 hours. Thus if the moon rose today at 7PM, tomorrow it will rise at 7:52PM.

ii) This is a standard two-particle collision problem. The fact that the interaction is through gravitational forces does not change how one solves the problem. Both energy and momentum of the entire system must be conserved. We consider the system before the interaction (far from the earth) and after the interaction (far from the earth).

Variables: v_e, v_e+dv_e: velocity of the earth before and after the interaction
v_s,v_s+dv_s: velocity of the ship before and after interaction
m_e: mass of earth
m_s: mass of the ship

Conservation of energy gives:

$$m_e * v_e^2 + m_s * v_s^2 = m_e * (v_e + dv_e)^2 + m_s * (v_s + dv_s)^2$$

For maximum increase in speed for the ship, the initial velocities should be parallel. We can drop the vector notation:

$$m_e * v_e + m_s * v_s = m_e * (v_e + dv_e) + m_s * (v_s + dv_s)$$

$$==> – m_e * dv_e = m_s * dv_s$$

Substituting this into the above gives and solving for dv_s gives:

$$dv_s = 2 * (v_e – v_s) / ((m_s/m_e) +1)$$

We need to know the speed of the earth relative to the sun. We can write:

$$Fg = m_e * v_e^2 / r = G * m_{sun} * m_e / (r^2)$$

This gives $v_e = 3 \times 10^4$ m s^{-1}. Thus the final speed of the ship is: 5.5×10^4 m s^{-1}.

There are 3.1×10^7 s in each year. For this to change by 1 s, then we must change the momentum of the earth (i.e., its speed) by $1/(3.1 \times 10^7)$ of its total. One slingshot maneuver changes the momentum by $m_s * dv_s$. The total initial momentum is $m_e * v_e$. Thus the total number of sling-shots, N, would have to be:

$$N * m_s * dv_s = 1/(3.1\times 10^7) * m_e * v_e$$

$N \cong 1\times 10^{14}$ Huge. The earth is not in peril from this, yet! *[James]*

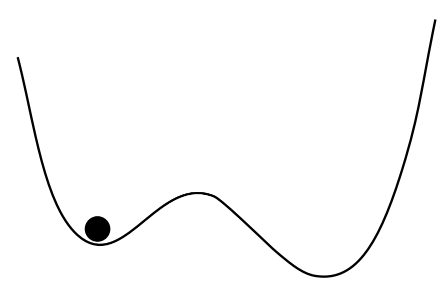
6) Balloon tug-o-war experiment

This experiment is great — it shows all kinds of different physics relationships, even though it is pretty simple. It is even a little bit easier to do than to describe!

What did you think would happen between the two balloons fitted to opposite ends of a straw? The first time I thought of this, I thought perhaps the balloons would end up being the same size, because the balloons are identical and used the same way, so I figured they should do the same thing — and that means they would keep the same amount of air in them. *Wrong!*

Instead, what happens is that one balloon will be bigger and the other smaller. And if you squeeze the bigger one, and make it a bit smaller than the other one, it will almost collapse, and become the smaller one while the other is the bigger one. So, they each do the same thing, but not at the same time! The balloon you squeeze to be the smaller one always expells most of its air. So there are two 'stable' or fixed arrangements: the left balloon small, or the right balloon small. These will stay that way, but you can switch between them. This is technically called 'bistability', and it also describes the kind of room-light switch that you can snap on or snap off — it can be one or the other, but not much in-between, and it won't just change by itself.

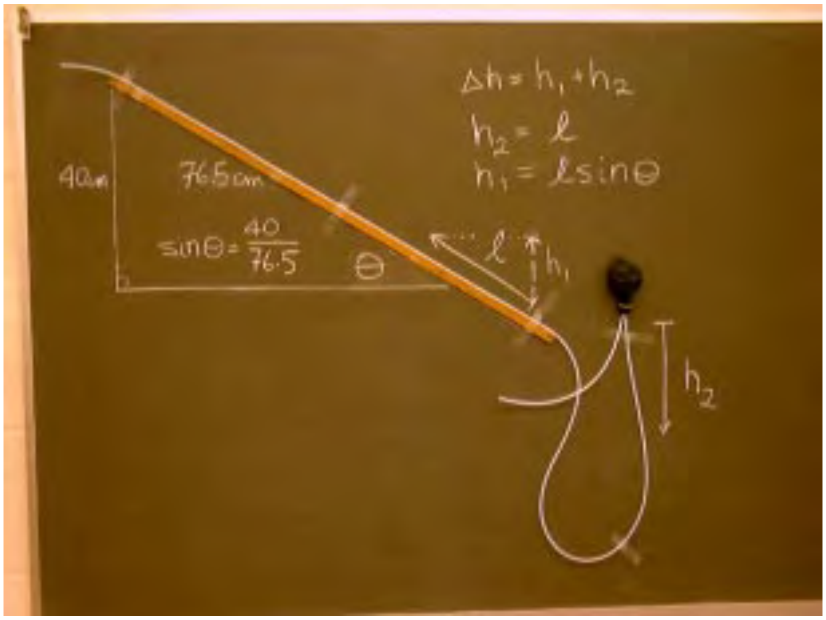
Bistability can be described with a potential-energy diagram that has two local minima: there is equilibrium in one minimum or in the other, but it will cost a little energy to switch between them. That's where squeezing the balloon comes in!



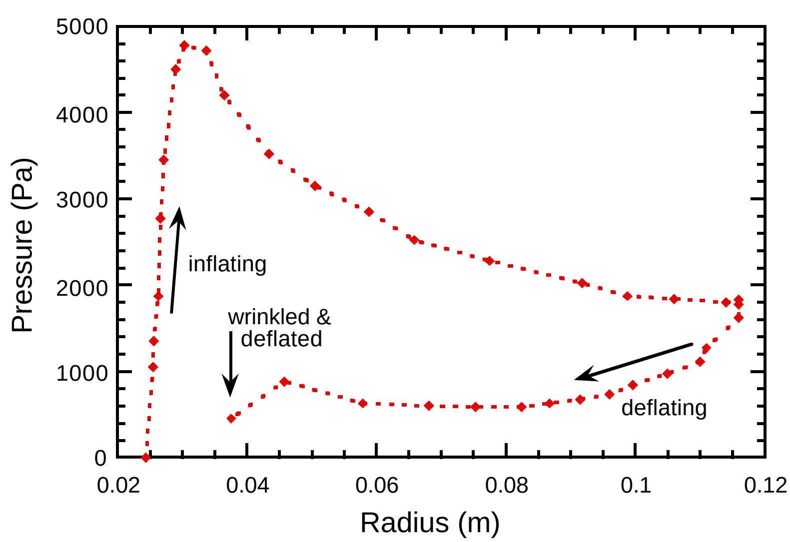
A ball can come to rest at either of two places on a curve like this. Therefore it is 'bistable' (bi = two). It is the same for any system with a similar potential energy curve — like the two balloons on a straw.

The rest of the experiment is about showing why there are two minima, or maybe even showing what the potential energy curve looks like for two balloons on a straw.

The picture below shows my own setup for this experiment. I guessed it would be a bit hard to read the fairly small pressure changes of a balloon, so I tilted the manometer arm, taped it to a meter stick, and measured the surface of the water as it moved along. Since the height difference gives me the pressure in the balloon, over atmospheric pressure, tilting the tube made the water move farther along, ΔL, for a given height change Δy (i.e., $\Delta L/\Delta y = 1/\sin\theta$).



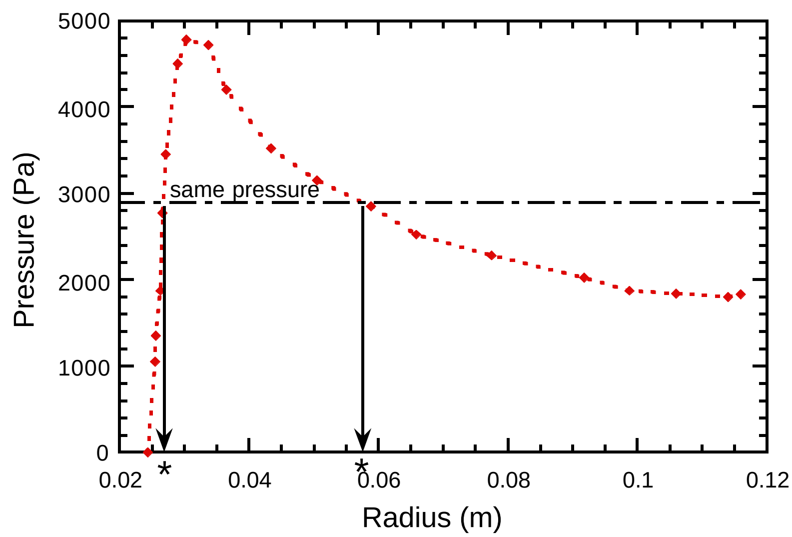
I blew up the balloon, through the extra smaller tube, and measured the difference in height Δh of the water surfaces (meniscus). Actually, I measured ΔL, and converted it to a height raised, adding it to the height ΔL by which the water level in the other arm dropped. This gave me the pressure inside the balloon. To measure balloon size, I tried two methods: deflating the baloon by letting air out into a beaker turned upside down underwater, then measuring its volume; also I used a tape measure to measure the balloon's circumference. The second method was good — it was sensitive, it was pretty easy, I could use it as I inflated the balloon or deflated the balloon, either way, and I didn't get as wet. Below is the curve I graphed from the data I took. Each symbol marked on the graph is a measurement I took.



It's interesting that the curve for the balloon as it is being inflated is not identical to the curve for the same balloon as it is being deflated. This is because the balloon is not perfectly elastic — it has a sort of 'memory' because it takes a little while to recover it's shape after being stretched. So the deflated balloon ends up wrinkly, since it has not gone back to its original shape. This means that even though it returns to zero 'gauge pressure' (pressure above atmospheric), the balloon is larger after being inflated. The balloon may recover its shape after a half-hour or so.

This kind of general effect — having a curve which is different going up than it is coming down, for the same values of radius — is called *hysteresis*. You may see it in the future in things like magnetism (say, a nail used in an electromagnet, which stays a bit magnetized afterward), and it comes up in many places in physics.

The balloon tug-of-war doesn't depend on hysteresis though — it is due mostly to the curve of the inflating balloon, which shows a *maximum pressure* as the balloon is being inflated.



When connected together by the straw, the two balloons must have the same pressure. Draw a line across the graph for this pressure, and you'll find that *two different sizes* of balloon can have that same pressure — these are the sizes of the two balloons. When you squeeze the larger balloon, the pressure of both balloons increases (they are still connected together!), so the line of the shared pressure rises, on the graph above. The large balloon is squeezed smaller, and the small balloon inflates because there is a (nearly) constant volume of air for them to share — they approach each other in size. With a perfect setup, when the line touches the maximum, the balloons have the same size, and can switch which one will be smaller, then the balloon collapses under your hand to become the small balloon. Since the balloon becomes small, you aren't squeezing it much anymore. The pressure drops back to what it was originally — only now the large balloon has become the small one, and the small one the large one!

MORE FOR THE VERY INTERESTED:

You can also describe all of this in terms of the potential energy of the two-balloon system, which is to be *minimimized*. This potential energy has the same type of curve as shown above for *bistability*, and the system rests in one of the local minima of the curve. You can see how the size-changeover works, in the following way:

The work done when squeezing, to compress a balloon is force × distance moved ($W = F \bullet \Delta r$), and the force is the pressure × area ($F = P \bullet A$). So the potential energy *change*, for the balloon as it changes size, is the *work done on it*:

$$\Delta U = -W = -F \bullet \Delta r = -P \bullet A \bullet \Delta r = -P\,\Delta V,$$

since $A \bullet \Delta r$ is the change in volume of the balloon, for a small change Δr. This potential energy change is positive, because $\Delta V < 0$.

You do work, in squeezing the balloon and increasing the pressure, so the potential energy of the system increases. You need to increase it to get it over the hump in the potential energy, and then the system of two balloons can go over to the other minimum. Because each local minimum is 'stable', and there are two of them, the system is *bistable* — it can be at equilibrium in two different states.

[Robin]

1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 2: Mechanics

Erratum

There are minor errors in the version of Question #3 in this problem set, as we rushed to distribute it. We will mark student solutions based on the question we distributed, but you may wish to know that the data, especially, should look more like this!

The change in the hint is interesting, but does not very much alter the answer.

3) Sink, sank, sunk

At some point in the movie "Titanic" it is said that the pressure at its sunken depth is 5500 pounds/inch^2. Given this, find the depth at which the *Titanic* supposedly came to rest. The 'real' answer is 12600 feet, density of water is 10^3 kg m^{-3} (ignore air pressure). Comment on the results, i.e., truth in film-making.

At another point in the movie someone estimates that the *Titanic* will sink in a time of roughly half an hour. Approximating the *Titanic* as a rectangular box of cross-sectional area A = **882*80** feet^2, mass M = **40,000** tons (loaded), total height 80 m, and a horizontal hole at the bottom of a side of the box of area A′ = 100*0.5 m^2 , estimate the time it takes sink — that is, when the top of the box is even with the water line. Ignore air.

[HINT: assume the system comes to equilibrium every time a bit of water goes in; you may assume that the flow is steady (i.e. has a constant velocity) – you may then use conservation theorems (e.g., Bernoulli's equation) – but be careful, as this does NOT hold initially, when the water is accelerating!] *[Peter]*

1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 3: Thermodynamics

1) Newton's Law of Cooling

a) $\dfrac{dQ}{dt} = \kappa_i A \cdot \Delta T$

With ΔT kept constant by the heater system ((90 – 70) °F = 20 °F = 11 K), the heat loss is constant, too. There are different heat conductivities — κ_1 for the (bottom + sides), and κ_2 for the for the top — and they each contribute to heat-flow out.

Top, bottom area (each): $5' \times 7' = 35\ \text{ft}^2 = 3.25\ \text{m}^2$

Sides area (each): $7' \times 9'' = 5.25\ \text{ft}^2 = 0.49\ \text{m}^2$

Ends area (each): $5' \times 9'' = 3.75\ \text{ft}^2 = 0.35\ \text{m}^2$

$$\begin{aligned}
\frac{dQ}{dt} &= (\kappa_1 A_1 \ + \cdot\ \kappa_2 A_2) \cdot \Delta T \\
&= \{\ 0.04\ \text{J K}^{-1}\ \text{m}^{-2}\ \text{s}^{-1} \bullet (3.25\ \text{m}^2 + 2 \bullet 0.49\ \text{m}^2 + 2 \bullet 0.35\ \text{m}^2\ ) + \ldots \\
&\qquad + 0.4\ \text{J K}^{-1}\ \text{m}^{-2}\ \text{s}^{-1} \bullet 3.25\ \text{m}^2\ \} \bullet 11\text{K} \\
&= \{\ (\ 0.20\ + 1.3\ )\ \text{J K}^{-1}\ \text{s}^{-1}\ \} \bullet 11\text{K}\ = 16\ \ \text{J s}^{-1}\ = \mathbf{16\ W}
\end{aligned}$$

The heater must be delivering an average of 16 W, then. Doesn't sound like too much, but this is day and night — like leaving the oven light on. Or like leaving a reading lamp on for five hours a day all month. In a month, then (which is 24 • 30 = 720 hrs), this amounts to 11.5 kW-hrs. At a cost of about \$0.125 /kW-hr, for me, this amounts to roughly \$1.44 per month — much less than a refrigerator. (Hmm, I thought it cost me about \$10 per month for this waterbed!)

b) Newton's Law of cooling tells us the *rate* at which heat will leave the water bed — proportional to the temperature difference ΔT with the room. As the water bed loses heat, it will cool down, and ΔT will decrease, and so the *rate* of cooling will, too. So what we need, to start, is the relationship between temperature and heat content, to go with what we can figure out for heat flow vs. time.

This relationship between heat content and temperature is what we call the *specific heat:* how much heat do we have to add or remove in order to make a kilogram of water go up by one degree? From INFOBITS™ at the bottom of the problem set, we see that this value is:

specific heat of water: $C = 4.2\ \text{kJ kg}^{-1}\ \text{K}^{-1}$

And $\Delta Q = C \bullet M \bullet \Delta T$. The mass of water in the waterbed is:

$$M = 1\ \mathrm{g\ cm^{-3}} \bullet (5\ \mathrm{ft} \times 7\ \mathrm{ft} \times 9/12\ \mathrm{ft}) \bullet (0.305\ \mathrm{m\ ft^{-1}})^3 = 10^3\ \mathrm{kg\ m^{-3}}\ \ 0.74\ \mathrm{m^3} = 740\ \mathrm{kg}$$

and so the heat capacity of the whole waterbed is

$$C \bullet M = 4.2\ \mathrm{kJ\ kg^{-1}\ K^{-1}}\ \ 740\ \mathrm{kg} = 3.1\ \mathrm{MJ\ K^{-1}}\ .$$

Therefore, we can write $\Delta T = \Delta Q/(C \bullet M)$, for ΔQ a change in heat energy of the water in the waterbed. This lets us rewrite Newton's Law of Cooling as:

$$\frac{dT}{dt} = -\frac{dQ}{dt}/(C \bullet M) = -\Delta T \bullet (\kappa_1 A_1 + \kappa_2 A_2)/(C \bullet M)$$
$$= -\Delta T \bullet 1.5\ \mathrm{J\ K^{-1}\ s^{-1}}\ /(3.1\ \mathrm{MJ\ K^{-1}})$$
$$= -\Delta T \bullet 4.8 \times 10^{-7}\ \mathrm{s^{-1}} = -\Delta T\ /\ (574\ \mathrm{hrs})$$

where ΔT is the temperature difference between bed and room air, $\Delta T = \{T - T_{room}\}$. (Heat flowing out of the bed is a drop in the energy in the bed; it gives a drop in temperature, so we have a minus sign.)

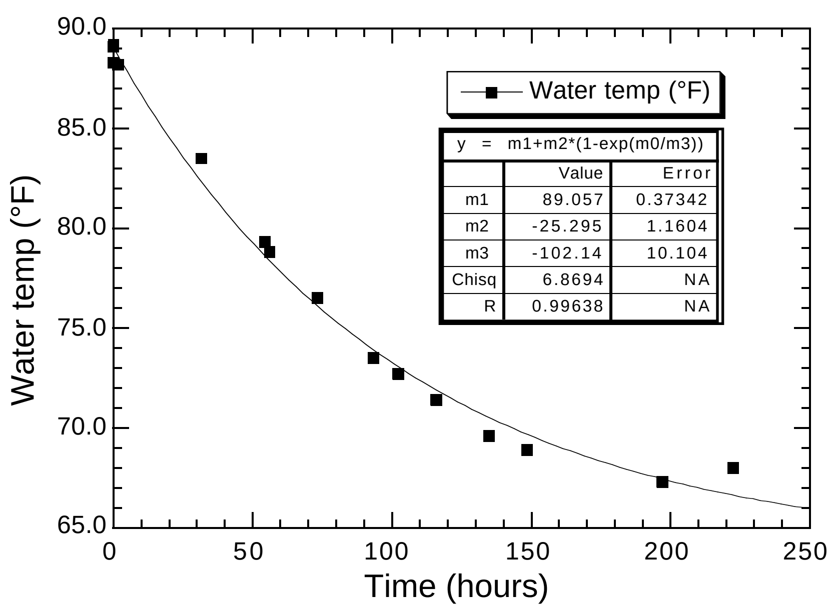
Now, we have that T_{room} is constant so:

$$\frac{d\,\Delta T}{dt} = -(\,4.8 \times 10^{-7}\ \mathrm{s^{-1}}) \bullet \Delta T = -\Delta T\ /\ (574\ \mathrm{hrs.})$$

and you may know that the *exponential* function $A \bullet \exp\{-\alpha t\}$ is a function with just such a derivative. In fact, this is a good solution:

$$\Delta T = (\Delta T)_o \exp\{-t\ /\ (574\ \mathrm{hrs.})\}$$

where $(\Delta T)_o$ indicates the initial value of the temperature difference. In our case, $(\Delta T)_o$ = (90 – 70) °F = 20 °F.



So temperature differences decay, as the energy of the hotter object will flow over to the cooler one, until they are each the same temperature ($\Delta T = 0$).

I turned the heat off on my own waterbed at home, and measured the temperature as time went by. The exponential curve at right shows Newtons's Law of Cooling!

(Actually, I made the

numbers κ up from this data, so that the answer would work out to match the curve. But I messed up, and confused a calorie with a Joule, so all the numbers are 4.2× too large or too small…)

According to the numbers given, after two weeks (336 hrs.) the waterbed will be warmer than the room by:

$$\Delta T = (\Delta T)_0 \exp\{-t \;/\; (574 \text{ hrs.})\}$$

$$= 20\ °F \bullet \exp\{-(336) \;/\; (574 \text{ hrs.})\} = 20\ °F \bullet 0.58$$

$$= 11\ °F$$

so, it will have dropped by 9 °F, or 5 K. To reheat the bed will then take:

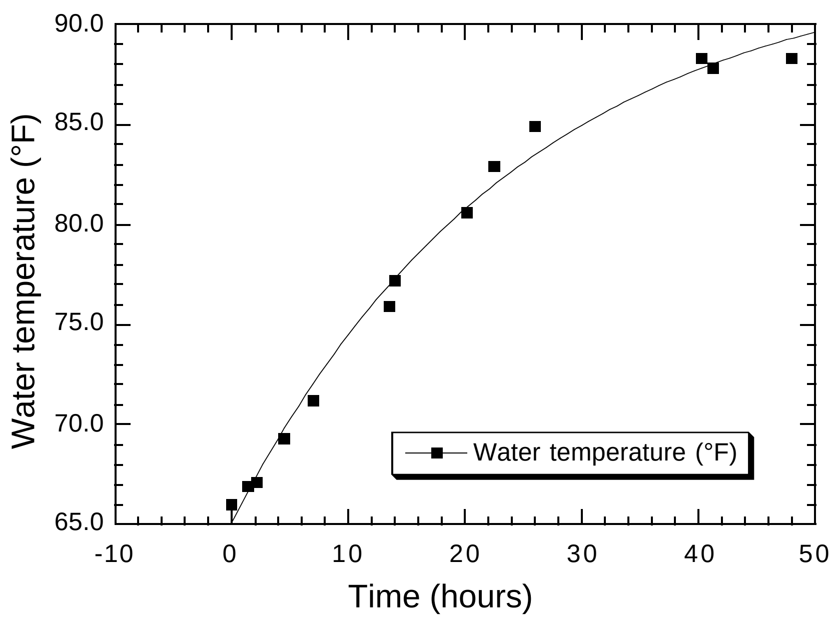
$$5\text{ K} \bullet\ 3.1\text{ MJ K}^{-1} = 15.5\text{ MJ}$$

which is **4.3 kW-hr**.

On the other hand, 16 W average for two weeks to keep the bed warm would have cost **5.4 kW-hr**, or 19.4 MJ. The difference is about 25 cents worth of electricity — maybe not worth the trouble of unplugging the bed.

The increased energy, if the bed is left plugged in, is lost because in keeping the bed warmer, over those two weeks, the average rate of heat lost is *higher*.

In reality, the cooling is faster than these numbers I mis-invented — instead of 574 hrs., the number is more like 135 hrs., in the formulae above. As it works out, the bed would be at room temperature within the two weeks, and it would cost a little more than twice as much to reheat the bed — about 50 cents. The plot at right shows how my waterbed reheated, after I plugged it back in again. I have to remember to plug it in about **2 days** before my guests arrive! *[Robin]*



2) Spears into pruning hooks

a) In order to ignite the ship part we'll need a temperature above a 'dull red heat', about 500 °C or 773 K. Using Stefan's constant σ_B we acquire the rate of radiation

$$I = \sigma_B T^4 \simeq 20\,\mathrm{kW \cdot m^{-2}}$$

Without knowledge of the thermal constants of the wood we can only estimate how much should be added to this to account for heat losses by conduction and convection. We also shouldn't assume that the shields are perfect reflectors and that the wood is a perfect absorber. A factor of 2 should give a reasonable margin of safety and)hopefully) account for all this. So an intensity greater than 40 $\mathrm{kW \cdot m^{-2}}$ is required in order to ignite the ship part.

We are given that the shields are curved, so lets consider the effect of focussing one of the shields on a ship part. Assume the round shields are each 1$\mathrm{m^2}$, thus each has a diameter, D, of about 1.13 m. The focal length, f, is taken to be 100 m (distance to ships) and assume a wavelength of incident light, λ, on the shield of 500 nm (about where sun's energy peaks). Since light diffracts any time it goes through a slit, the worse the narrower the slit, light cannot be focussed to a tight spot either, without diffraction. The limit of a focal spot is given by:

$$y = 2.44\, f\lambda / D$$

(derived from principles of light diffraction) yields a value $y = 1.08 \times 10^{-4}$ m. Where y is the radius of the bright central spot, created by the shield, on the ship part. So, the area of the spot is approximately $3.66 \times 10^{-8}\ \mathrm{m^2}$. The power on the tight spot is the whole power the shield intercepts, 1 kW. Thus, the intensity of light produced from focussing one shield is,

$$I_{\text{shield}} = \frac{1\,\mathrm{kW}}{3.66 \times 10^{-8}\,\mathrm{m^2}} = 2.7 \times 10^{7}\ \mathrm{kW \cdot m^{-2}}$$

This is much, much more than the 40 $\mathrm{kW{\bullet}m^{-2}}$ loss.

To determine the time consider a surface layer about 1 cm thick. Now to calculate the rate of heating lets assume that wood has a density and a specific heat equal, respectively, to 1/2 and 1/10 the corresponding values for water (since the ship is floating!) Then the thermal capacity, C_{total}, of the wood is,

$$\begin{aligned} C_{\text{total}} &= mC = (500\,\mathrm{kg \cdot m^{-3}})(0.01\,\mathrm{m})(0.42\,\mathrm{kJ\ kg^{-1}K^{-1}}) \\ &= 2.1\ \mathrm{kJ\ m^{-2}K^{-1}} \end{aligned}$$

$\therefore$ The rate of heating is $2.7 \times 10^{7}\ \mathrm{kW{\bullet}m^{-2}} \div 2.1\ \mathrm{kJ\ m^{-2}K^{-1}}$ or $1.3 \times 10^{7}\ \mathrm{K\ s^{-1}}$

Assuming a temperature change of 480 K it will take about 4×10^{-5} sec to set fire to a ship part of area 3.66×10^{-8} m^2 and thickness of 1 cm. If the shores are lined with soldiers holding these shields, those Romans are toast!

b) First, lets look at the power required to replace that radiated from the front of the disk at its melting point.

$$\begin{aligned} P_{rad} &= \sigma_B A T^4 \\ &= (5.7 \times 10^{-8}\,\mathrm{W \cdot m^{-2} K^{-4}})(7.85 \times 10^{-5}\,\mathrm{m^2})(1811\,\mathrm{K})^4 \\ &= 48\ \mathrm{W} \end{aligned}$$

(We gave the wrong sign in the exponent, in the value for Stefan's constant at the back of the problem set. Sorry about that, and hope you found it by the crazy wrong answer it gives!)

A solar-collector farm: The mirrors on the ground are adjustable, and redirect sunlight to the focussing mirror in the tower.

The power produced by just one mirror is 2 m^2 $\times$ 1 kW/m^2 = 2 kW. This should be more than enough to make up for the radiation losses.

To melt the disk, first heat energy, Q_1, is required to increase the temperature of the disk to the melting point and then the latent heat, Q_2, must be supplied in order to produce melting, where

$$Q_1 = mC\Delta T \qquad (\text{assume } \Delta T = (1811 - 239) = 1518\ \mathrm{K})$$

$$= (6.18 \times 10^{-3}\ \mathrm{kg})\ (0.444\ \mathrm{kJ{\cdot}kg^{-1}K^{-1}})(1518\ \mathrm{K})$$

$$= 1.53\ \mathrm{kJ}$$

and

$$Q_2 = (\text{moles iron})\ E = 1.53\ \mathrm{kJ}$$

$$\therefore\ Q_{tot} = 5.7\ \mathrm{kJ}$$

The power supplied by 100 mirrors is 200 kW and the power radiated at the melting point is 48 W. So we can estimate,

$$t = \frac{Q_{tot}}{P_{tot}} = \frac{5.7\,\mathrm{kJ}}{(200\,\mathrm{kW} - 4.8 \times 10^{-2}\ \mathrm{kW})} = 0.03\ \mathrm{s}$$

The above assumes that P_{rad} is constant, when it actually changes as the temperature of the disk changes. Initially radiation losses are unimportant compared with input power. But as the temperature of the disk increases the radiation losses become more important. However, in the case of our disk, even at the maximum temperature, the melting point, radiation losses are negligible compared to input power. We have also assumed that the mirrors are perfect reflectors and that the iron is a perfect absorber — which may be close to true for the mirrors but won't be true for the disk. Finally we have also ignored convection losses, while these are not negligible, they are difficult to calculate. *[Carrie]*

3) Sounds cool…

i) We tried to write the question so as to lead you through this standard derivation. You can find it in first year physics and introductory thermodynamic books. For an ideal gas with no heat transfer, we can write:

ii) $-PdV = CdT$ (1)

From the ideal gas law: $PV = nRT$, and taking the differential (like the derivative):

$$PdV + VdP = nR\,dT$$

Using (1)

$$VdP = (nR + C)\,dT \qquad (2)$$

Divide (1) by (2) to get rid of the temperature variable:

$$\frac{PdV}{(VdP)} = \frac{-C}{(nR+C)}$$

$$\frac{P}{dP} = -\left(\frac{V}{dV}\right)\frac{C}{(nR+C)}$$

To *confirm* that $P \bullet V^{\gamma}$=constant is a solution, we substitute it into both sides and verify they are equal, if:

$$\gamma = \frac{(nR+C)}{C}$$

This is fine since these are all constants.

ii) Using the above expression and $B = -V\,\dfrac{dP}{dV}$ we get:

$B = \gamma P$ and thus

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

$\rho = \dfrac{nM}{V}$ where M is the molar mass of air. So,

$$v = \sqrt{\frac{\gamma\, PV}{nM}}$$

Using the ideal gas law: PV = nRT we get finally

$$v=\sqrt{\frac{T\gamma R}{M}}$$

Given that $v=\frac{300m}{s}$ at S.T.P. (*i.e.*, T ≅ 300 K), then

$$\frac{\gamma R}{M}=300$$

so we can say that $v=17\sqrt{T}$. Therefore the colder the air, the slower the speed of sound.

iii) From personal experience, I can not observe a time lag with a friend who is 20 m away, but I do notice it with sound source that is at least ~50 m away (*i.e.*, the firing of a gun at a track meet). So I can observe a time-lag of around 0.17 s. For me to notice a time lag with a friend 20 m away, the speed would have to decrease by a factor of 2.5, corresponding to a temperature of 48 K. That is pretty cold. In fact, all the nitrogen/oxygen/carbon dioxide/etc would be frozen, so, not only would sound not travel (something the Star Trek/Wars folks haven't figured out yet) but *you* physically wouldn't last too long. *[James]*

4) Ice ice baby

i) Temperature of the vapour is the same as the surface of the water. Before you turn on the pump it is 20°C . Just before ice is formed the temperature is 0°C.

The boiling process requires heat thus cooling the remaining liquid until it freezes. Consider x kg freeze while y kg are changed into vapour. Consider that only the x kg had to be cooled from 20°C to just above 0° C.

Conservation of energy:

$$2.3\times 10^{6}\ y = 3\times 10^{5}\ x + 4200 \bullet 20 \bullet x$$

But there is 1 l of water, which has a mass of 1 kg so:

$$y = 1 - x$$

and thus x = 0.86 kg

Now consider that all the water (1 kg) had to be cooled from 20°C to 0°C:

$$2.3\times 10^{6}\ y = 3\times 10^{5}\ x + 4200 \bullet 20 \bullet (x+y)$$

and thus x = 0.85 kg.

Of course neither case is correct, but they are some sort of upper and lower limits for the correct answer. Since we require the answer to only one significant digit, it is 0.9 kg.

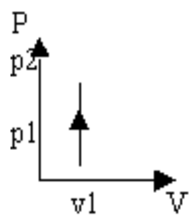
iii) Water evaporation is used in many instances to cool things, from wineskins to your own perspiration. For a glass of water, this process does cause the water to cool but there are other factors that do not allow it to get very cold. There is heat transfer with the rest of the room so you cannot cool the water without also cooling the room. Within the vacuum system, there is very little heat transfer from the vessel to the chamber since vacuum is a superb insulator *[James]*

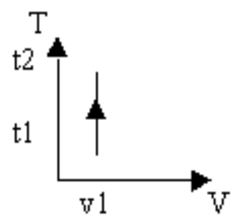
5) Reduce, re-use, recycle…

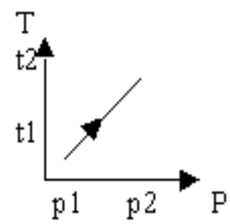
We have:

$$\Delta E = Q - W$$
$$W = \int p dV$$

i)







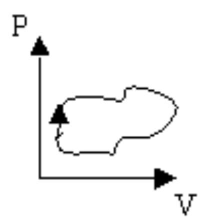
W = 0

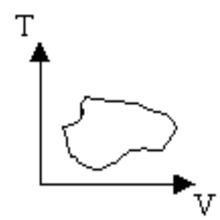
$$Q = \Delta E = \frac{3}{2}V_1(P_2 - P_1)$$

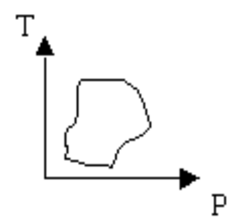
$$\Delta E = \frac{3}{2}nR\Delta T = \frac{3}{2}V_1(P_2 - P_1)$$

(NOTES: all results are obtained from PV = nRT = RT here. Slope of the line in T–P graph is R/v1)

ii)





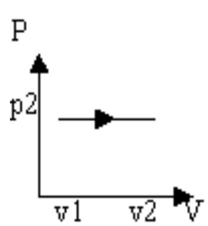


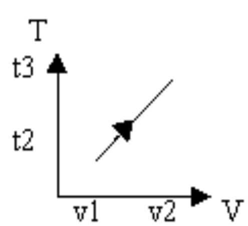
W = 1 J

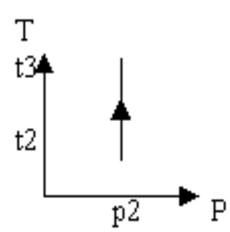
Q = –W = –1 J *(ed's. note: I think this should be +W, i.e., +1J)*

ΔE = 0 (cyclic process)

iii)





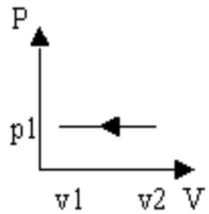


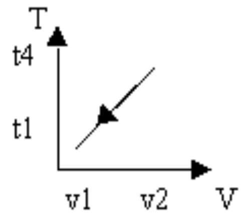
$$W = P_2(V_2 - V_1)$$

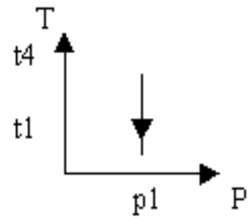
$$Q = \Delta E + W = \frac{5}{2} P_2 (V_2 - V_1)$$

$$\Delta E = \frac{3}{2} P_2 (V_2 - V_1)$$

iv)





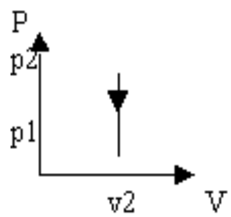


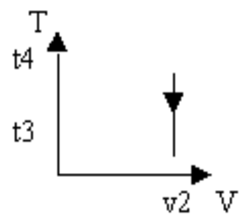
$$W = -P_1 (V_2 - V_1)$$

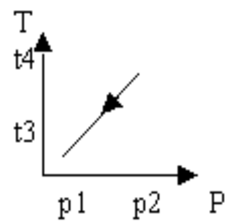
$$Q = -\frac{5}{2} P_1 (V_2 - V_1)$$

$$\Delta E = -\frac{3}{2} P_1 (V_2 - V_1)$$

v)





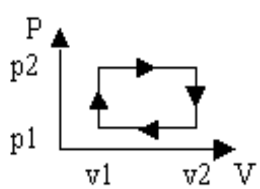


$$W = 0$$

$$Q = -\frac{3}{2} V_2 (P_2 - P_1)$$

$$\Delta E = -\frac{3}{2} V_2 (P_2 - P_1)$$

vi)



The net work is just the area inside the box (only on a P–V graph!) – or you could just add up the work from the previous parts of the question.

$$W = (P_2 - P_1)(V_2 - V_1)$$

The heat is positive during the first 2 processes, so

$$Q_{\text{incoming}} = \frac{3}{2} V_1 (P_2 - P_1) + \frac{5}{2} P_2 (V_2 - V_1)$$

The efficiency is

$$e = \frac{W_{net}}{Q_{inco\min g}} = \frac{2}{3\dfrac{V_1}{V_2 - V_1} + 5\dfrac{P_2}{P_2 - P_1}} = \frac{2}{3\dfrac{x}{1-x} + 5\dfrac{1}{1-y}}$$

Here, $V_1 = xV_2$, $P_1 = yP_2$, so $0 \le x, y \le 1$.

To make *e* big, we have to make the denominator small. To do this, we have to make the denominator of the x and y expressions big. Clearly, the maximum occurs for x = y = 0. Then,

$e = 2/5 =$ or 40 %.

For a Carnot engine we would get

$e = 1 - \dfrac{T_c}{T_h}$ where T_c and T_h are the coldest / hottest temperatures (respectively) occurring during the cycle. Here,

$$e = 1 - \frac{P_1 V_1}{P_2 V_2} = 1 - xy$$

The maximum occurs for either x or y = 0, and is equal to 1.

This is much higher than our cycle, and so I wouldn't recommend rebuilding your car engine, or anything like that. *[Peter]*

6) Keeping your cool

a) We're given: X moles of air inside fridge at temperature T_K

y moles s^{-1} escape

molar heat capacity of air is c_p

(which we shall assume constant)

the room temperature is T_R

In a small finite time Δt $y\Delta t$ moles of air escapes. Therefore $y\Delta t$ moles of warm air at temperature T_R enters the fridge.

The warm air and cold air will mix and we model the final temperature by supposing a reversible flow of heat. Since the heat gained by cold air must equal the heat lost by the warm air we have

$$c_p\,(X - y\Delta t)\,(T_{\Delta t} - T) = c_p y \Delta t\,(T_R - T_{\Delta t})$$

Where $T_{\Delta T}$ is the final temperature of the mixed air.

Therefore

$$T_{\Delta t} - T = \frac{y\Delta t}{X}(T_R - T_{\Delta t})$$

$$\Rightarrow \frac{T_{\Delta t} - T}{\Delta t} = \frac{y}{X}(T_R - T_{\Delta t})$$

Then, as $\Delta t \to 0$, we have, by definition of the derivative

$$\frac{dT}{dt} = \lim_{\Delta t \to 0} \frac{T_{\Delta t} - T}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{y}{X}(T_R - T)$$

$$= \frac{y}{X}(T_R - T)$$

Thus the rate of change of temperature inside the fridge is

$$\frac{dT}{dt} = \frac{y}{X}(T_R - T)$$

b) Verifying $T(t) = T_R + (T(0) - T_R)e^{-\frac{y}{x}t}$ is a solution of 1.

The L.H.S. is

$$\frac{dT}{dt} = (T(0) - T_R)\left(-\frac{y}{X}\right)e^{-\frac{y}{X}t}$$

$$= \frac{y}{X}(T_R - T(0))\, e^{-\frac{y}{X}t}$$

The R.H.S. is

$$\frac{y}{X}\left[T_R - \left\{T_R + \left(T(0) - T_R\right)e^{-\frac{y}{X}t}\right\}\right]$$

$$\frac{y}{X}(T_R - T(0))e^{-\frac{y}{X}t}$$

Thus the L.H.S. equals the R.H.S. and so T(t) = TR + (T(0) – TR) $e^{-\frac{y}{x}t}$ is a solution

c) Since the second term above decays to zero exponentially as t increases, T(t) approaches the asymptote $T = T_R$. This is very reasonable for the unplugged fridge — it goes to room temperature!

d) In a small finite time interval Δt the temperature change is

$$T_{\Delta t} - T = \frac{y\Delta t}{X}(T_R - T_{\Delta t})$$

To make this change zero the refrigerator must do work and remove the heat responsible for this change:

$$\Delta Q = c_p X \frac{y\Delta t}{X}(T_R - T_{\Delta t})$$
$$= c_p\, y\Delta t\,(T_R - T(0))$$

Where $T_{\Delta t} = T(0)$, since the temperature inside is constant.

For a Carnot refrigerator working between the cold air inside at T(0) and to warm air at room temperature T_R the 'coefficient of performance' is given by

$$\eta = \frac{\Delta Q}{W} = \frac{T(0)}{T_R - T(0)}$$

Hence

$$W = \left(\frac{T_R - T(0)}{T(0)}\right)\Delta Q$$
$$= \frac{T_R - T(0)}{T(0)} c_p y\Delta t\,(T_R - T(0))$$

Thus, by definition, the power required is

$$P = \lim_{\Delta t \to 0} \frac{W}{\Delta t} = \lim_{\Delta t \to 0} \frac{(T_R - T(0))^2}{T(0)} c_p y \frac{\Delta t}{\Delta t}$$
$$= \frac{(T_R - T(0))^2}{T(0)} c_p y$$

Since the Carnot refrigerator is the most efficient, therefore we conclude that this value is a minimum for the required power. *[Simal]*

1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 4: Optics and Waves

Due February 12, 1999

1) The land of the midnight sun (is where, exactly?)

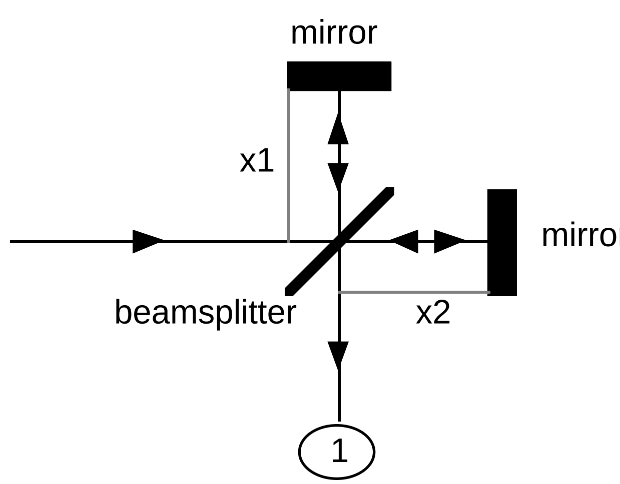
While travelling through the Yukon this past summer, we travelled across the Arctic Circle (latitude 66.7 degrees). If the earth was a simple sphere (like a billard ball) circling the sun while rotating on its own axis, a line at this latitude would experience exactly one day a year of unending daylight. North of this line would experience more than one day of continuous daylight and south of this line would have a sunrise and sunset every day of the year.

a) Using this information, find the tilt of our sphere's axis relative to the plane it sweeps out as it circles the sun.

b) In reality, it is possible to see the midnight sun at points *south* of the Arctic Circle. In each of the following cases, calculate how much farther south than the Arctic Circle you could position yourself and still be able to observe the midnight sun, by taking into consideration:

i) you are standing on a hill (height: 300 m, rising from a perfectly spherical Earth).
ii) the Earth has an atmosphere. For the purpose of this problem, assume that the atmosphere is 3 km high and at standard pressure and 0° C, and that the Earth is a perfect sphere. *[James]*

2) Making light of photons



Since light is a wave, two identical beams can add together constructively or destructively depending on their relative phases. Interferometers, whether large such as found in a satellite array suspended high above the Earth or else small like the one on my laboratory table, make use of this fact to measure very small distances. Consider my simple interferometer: the beamsplitter reflects 50% of the incident intensity and transmits the rest. All reflections involve a 180° phase change.

i) What is the electric field at the output (1) as a function of the relative displacement ($\Delta x = x1 - x2$). Consider the input field to be monochromatic and to have an electric field of $E = E_0 \cos(2\pi\ \nu(x/c - t))$ where E_o is the amplitude, t is time, ν is frequency (Greek 'nu'), and c is the speed of light.

ii) You will notice for certain values of Δx, there is <u>zero</u> electric field at (1). Where has the energy from the light gone?!

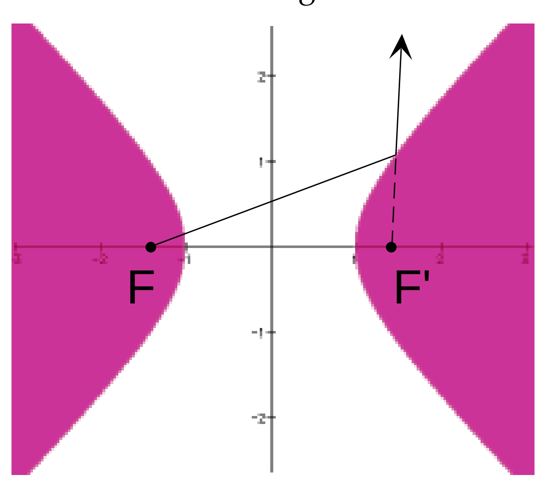
iii) Put a screen at (1). For a beam from my HeNe laser (frequency $\nu = c/632\text{nm}$), what is the smallest Δx you could clearly distinguish?

iv) If you travelled back a century, you would find that many believed that light moved in a medium, just as water waves travel in water or sound waves travel in air. This supposed medium was dubbed the 'ether'. If the ether were also moving, this could change the effective speed of light, by a version of the Doppler shift. Pretend that you are a scientist at this time and you want to establish the *existence* of the ether. How might you use your interferometer to measure the speed of the ether relative to your lab? What limit you can place on the ether's speed? (For simplicity, assume you still had a monochromatic source of the same wavelength as in part iii). *[James]*

3) Imagining imaging

When you look at yourself in a mirror, you see yourself behind the mirror by the same distance as you are in fact in front of it. Your mirror image might even appear far enough from you to be in the next room! No light really passes from the place your image seems to be, so the image is called 'virtual'. On the other hand, when you take a photograph, you need real photons to hit the film; this kind of image is called 'real'.

i) Prove that any point *A* a distance *d* from a flat mirror has a virtual image *A'* which is the same distance *d* on the other side of the mirror. What is the magnification of this virtual image? [Hint: magnification *M* can be defined as the apparent size of an image divided by the size of the original object; it can also be defined using the angle between two rays of light leaving a point on the object, and the corresponding angle the rays seem to form when traced back to the virtual image.]



ii) Any hyperbola has two *foci*, which we can label F and F'. Show that one is the virtual image of the other, reflected in the hyperbola. Can you find a well-defined magnification?

[Robin]

4) Correct time and temperature, at the tone…

Consider a pendulum clock:

i) If the pendulum arm is made of a material with a linear coefficient of thermal expansion α, determine an expression for the ratio of the final period over the initial period, T'/T, if the change in temperature is ΔT.

ii) If the suspension system is brass and the change in temperature is 20° C how much time does the clock gain/lose in one day if $T = 1.000$ s?

A nickel-steel guitar string, initially of length lo, is stretched to a length L such that it has a frequency f for its fundamental frequency.

iii) If the temperature is increased by ΔT, determine an expression for the new fundamental frequency in terms of the old one, i.e., f'/f.

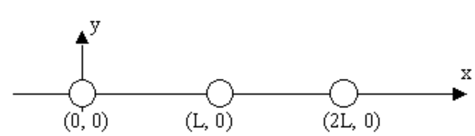
Assume Young's modulus, Y, and the linear expansion coefficient, α, to be constant.

iv) If the initial frequency is 440 Hz ('A' above middle 'C'), how 'out of tune' will the guitar be, due to an increase in temperature of 20° C? Take $lo = 0.795$ m and $L = 0.8$ m. *[Simal]*

5) Toys, Toys, Toys

What are the physics rules for oscillating molecules? In this question we will consider different physical models of a linear molecule (like O_2, N_2, CO_2, or some long-chain polymers) which has just been hit by a single-atom molecule (like He or Ne). What we'll change is the model of the force between atoms, and what we'll examine is their motion after the collision.

Model a linear molecule as consisting of evenly spaced particles (atoms, balls or whatever) of mass m each. All collisions are perfectly elastic, etc. Initially, the first (left-most) particle of the chain is hit by an object of mass m, and velocity $\boldsymbol{V}$ [to the right].



Consider first a chain consisting of just 1 particle (a *monomer*) — i.e., we have an ordinary 2-particle collision. What is the speed and position, as a function of time, of the 1-atom chain afterwards?

<u>Part I</u> – stringy bonds

Consider now a 2-atom chain (a *dimer*) having an inter-particle spacing L. The particles are connected by an initially extended ideal (massless, non-dissipative and non-stretchable) string. The left ball [at (0,0)] is hit by the incoming object. Find the motion of the centre of mass of the lattice.

Generalize to an n-particle chain (a *polymer*); keep it a linear (straight-line) molecule!

Describe (in words and equations) also the motion of the two balls as seen from <u>both</u> the centre of mass frame, and an outside (laboratory) frame of reference. Sketch the corresponding position-time graphs for each ball (use one graph for each of the two reference frames).

<u>Part II</u> – springy bonds

This is same as Part I, except that now the two atoms are connected by a spring (with spring constant *k*). Describe (in words) the ensuing motion from both frames (laboratory and centre of mass) of reference; find the motion of the centre of mass and sketch the position-time graph of the left ball in both frames. [HINT: consider the centre of mass frame first].

Discuss the plausibility of the models used in Part I and Part II.

<u>BONUS:</u> Consider that there still is a spring, as in Part II, but there is now also a *dissipation* mechanism. That is, there is some dissipative force of size $-\gamma^*$ (instantaneous drag force on object) $= -\gamma(dx/dt)$. What is the motion of the centre of mass now? Find and draw a sketch of the position (as a function of time) of the left ball in both the centre of mass and outside frame.

[HINT: a solution to the differential equation: $md^2x/dt^2 = -kx - \gamma(dx/dt)$ is

$$\frac{2Ue^{-\sqrt{\frac{\gamma t}{m}}}\sin\left(\frac{1}{2}\sqrt{4\frac{k}{m}-\left(\frac{\gamma}{m}\right)^2}\,t\right)}{\sqrt{4\frac{k}{m}-\left(\frac{\gamma}{m}\right)^2}}$$

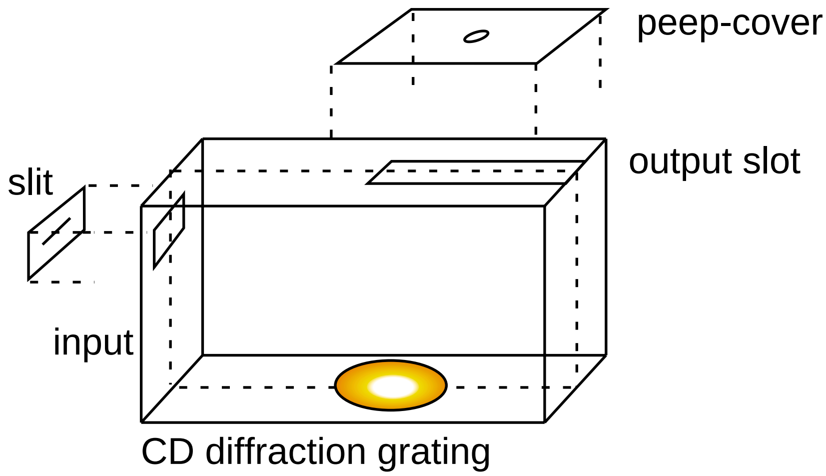
where the initial conditions are: $x(0) = 0$, $dx/dt(0) = V(0) = U$; also, $4k > \gamma^2$]

Can you suggest a realistic dissipation mechanism for molecules? *[Peter]*

6) Making a shoebox spectrograph, using a compact disc

Have you ever noticed the rainbow colours reflected from a compact disc? The tracks on the CD are arranged with such mathematical regularity that a CD makes a nice *diffraction grating*. This pretty easy experiment lets you make a spectrograph from a shoebox and a CD, one which is good enough to reveal the individual lines contained in the spectrum of light from a fluorescent light tube, streetlights, or other light sources you may have.

<u>You will need</u>:



- any compact disc (my favorites are the gold-coloured CD-R <u>blank</u> used for recording CD-ROMs; about $2-3. It saves my music collection, too!)
- a cardboard box the size of a box for hiking boots; a shoebox or small corrugated cardboard box might do
- aluminum foil
- some duct tape, or other reasonably opaque tape
- scissors
- a pen or marker
- a utility razor-knife, or Exacto™ knife (CAUTION: danger to fingers!)
- a ruler

To set up, read all these instructions before beginning. Refer to the illustration, too.

1. lay the box on its side, as sketched in the picture above. Measure and mark a mid-line around the outside of the box, to use for alignment (up, across, and down the middle of each face — see dotted line in figure). Put a similar line around the inside of the box, also at exactly the midpoint.
2. use scissors or razor-knife to cut a lengthwise slot about 1 cm wide along the top of the box, centred on the alignment line. Cut from about the midpoint lengthwise, and go all the way to the end of the topmost panel.
3. cut a hole through the box, on the small end-panel of the box farthest from the slot you cut in (2). It should be roughly 2 cm high by 3 cm wide, and positioned toward the top of the box-end. It must be centred on your alignment line.
4. cut a piece of aluminum foil about twice the size of this hole in (3). Put it on a firm pad of paper to cut it, and along the middle of this foil make a 3cm-long cut with your razor-knife (hold the knife at a shallow angle, to pull the cutting edge, and not the blade-tip, across the foil). You're making the input slit — you want the narrowest slit that will be sure to pass enough light: 0.1 mm is pretty good. When done, tape the foil over the rectangular hole in the end-face, with the slit horizontal and centred.
5. tape a compact disc, label-side down, inside the box on the side box-panel. Set it up so that the hole through the CD's middle is centred on the alignment

line you drew inside the box. Cover the front half of the disc with opaque tape, so only the half farthest from the input slit will be used. The placement of this disc depends on your placement of the input slit: a ray from the input slit to the middle of the exposed section of the CD should make an angle of about 30° from the CD surface. You may have to try moving the disc forward and back in the box to get the input and output right, relative to the disc.

The spectrograph is now complete, except for figuring out how to use it! You'll need to look down at the CD through the long slot in the top, and try different places along the slot. *But* you want the only light inside the box to have to come through the slit, so try cutting a piece of cardboard or dark cloth to cover the slot, with a 1 cm peep-hole to look through. Then you can peep at different points along the slot, and still keep stray light out by sliding the peep-hole card.

To use your spectrograph, you should point it so that light will pass *through* the slit and onto the compact disc in the bottom. When this is exactly right, you can look at different places along the output slot to find the best spread-out rainbow, from an incandescent lamp, or many spectral lines from a fluorescent light, streetlamp, or neon light.

<u>The experimental assignment</u>:

i) Write down your own description of how you made this spectrograph, and how you made it *work*.

ii) Describe what you see, for different light-sources such as fluorescent lights (different types?), incandescent lights, candle-flames, neon lights, streetlights of different types, a laser-pointer (<u>caution:</u> never stare directly into a laser pointer, or HeNe laser, or the sun. Even these simple sources can cause damage to your eyes, though you may not recognize the damage immediately).

iii) Can you use the formula above, and your own observations, to figure out how many total tracks there are on a CD? *[Robin]*

INFOBITS™ — Useful Bits of POPTOR Information

> Remember to check the POPTOR web-page for hints and any necessary corrections!
> www.physics.utoronto.ca/~poptor

index of refraction for air at standard pressure and 0 deg C: n = 1.0003

radius of the earth: $R = 6 \times 10^6$ m

nickel steel

Young's modulus: $Y = 18.2 \times 10^{10}$ Pa
coefficient of thermal expansion: $\alpha \approx 1.4 \times 10^{-5}\ K^{-1}$

brass

Young's modulus: $Y = 9.0 \times 10^{10}$ Pa
coefficient of thermal expansion: $\alpha \approx 2.0 \times 10^{-5}\ K^{-1}$

MINI-TUTORIAL: *The physics of diffraction-grating spectrographs:*

The input and output angles of light diffracted from a diffraction-grating depend on wavelength, and on the diffraction grating itself:

$$n\lambda = 2d \bullet (\sin\theta_i + \sin\theta_d)$$

where θ_i is the input angle, measured from the perpendicular to the surface (the surface *normal*), and θ_d is the angle of the diffracted light, measured similarly; d is the spacing between grooves of the diffraction-grating, or tracks of the CD. The n is an integer which gives the *order* of diffraction, which you can investigate.

The input light is scattered from very many regularly spaced grooves or tracks in the compact disc. Only at very special angles are the conditions right for constructive interference for waves coming from *all* of the grooves. At these special angles the light of a certain wavelength is bright; at other angles there is only random or destructive interference, and practically no intensity of light is sent on.

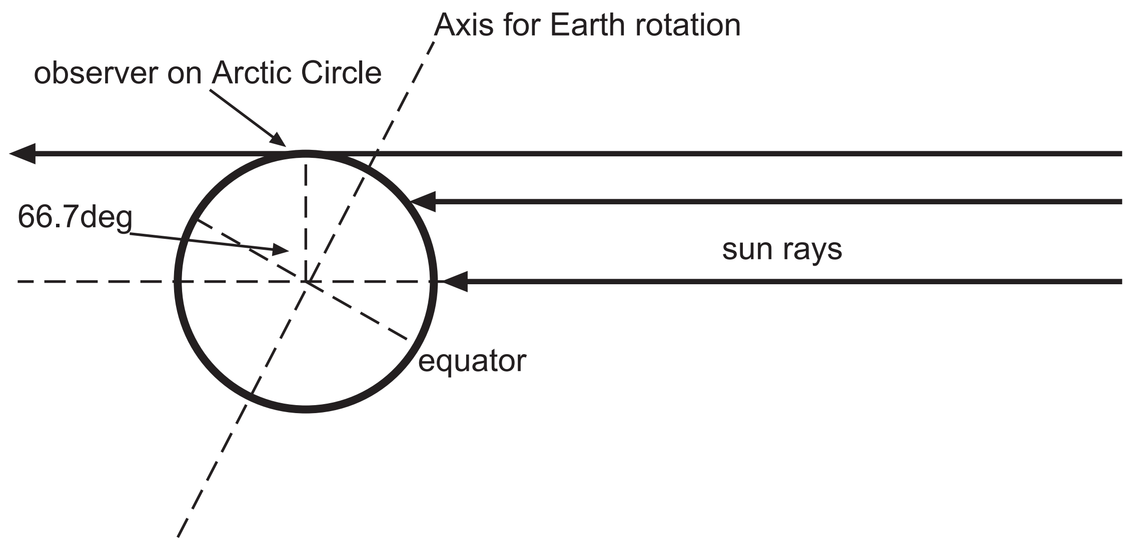
1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 4: Optics and Waves

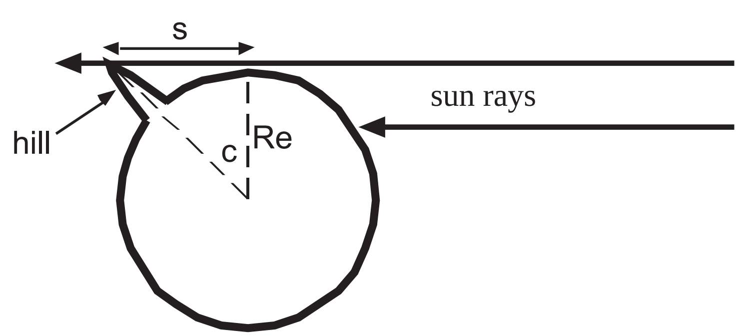
1) The land of the midnight sun (is where, exactly?)

a) At midnight on the summer solstice, the sun rays are tangent to the perfect sphere at the Arctic Circle. The 'side view' looks like:



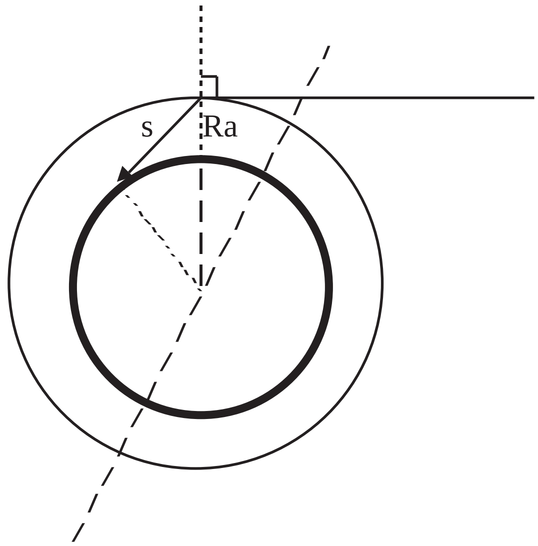
Once the picture is drawn it is obvious that the tilt of the sphere's axis to the plane it sweeps out around the sun is 66.7°.

b) i) Exaggerated for clarity:

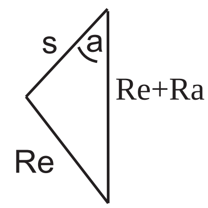


Thus c=arccos(Re/(Re+300m)) and the subtended arc opposite from angle c is 60 km. Thus you are 60 km south of the Arctic Circle.

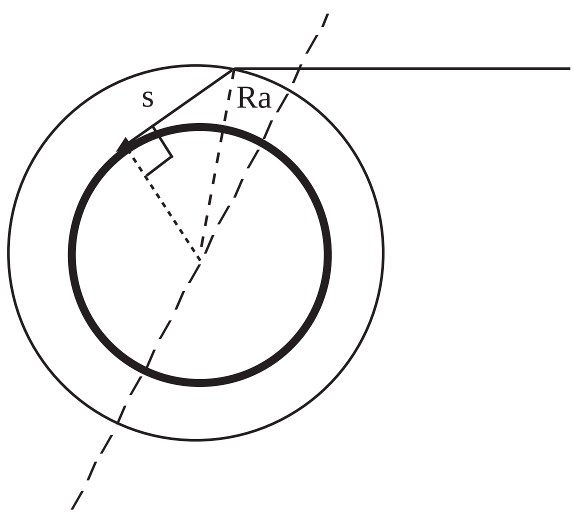
ii) Refraction causes the rays to bend at the air/vacuum interface. Modelling this as a single interface isn't exactly right (the interface is continuous and still causes refraction due to the index gradient). Snell's law: sin(90°)=n sin(a) as defined in the cartoons below:

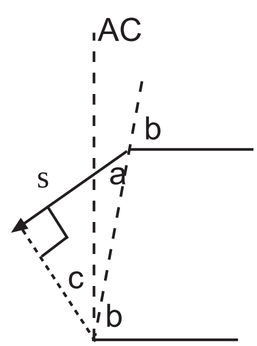


So:



Thus a=88.6°. Solving for s gives an imaginary number (i.e., with $\sqrt{-1}$). What went wrong? We assumed that the refracted ray that intersected the interface exactly over the Arctic Circle would eventually intersect the ground, but it does not. If the interface were lower, the index difference larger, or the Earth bigger, it could intersect the Earth's surface. Thus the most southern ray to intersect the ground intersects the atmosphere at a point towards the sun:





As shown on the diagram, this refracted ray is tangent to the earth surface. We find s=190 km. We still have to find out how far south we are. For this triangle, the refracted angle of the ray in the atmosphere (angle: a) is 88.19° to the normal. The incident angle (angle: b) is 88.85° to the normal. Thus we find that angle c is (90–88.19–(90–88.85)) = 0.66°. Using the value of Re, the corresponding arc length is 69 km. This is the distance you are south of the Arctic Circle. *[James]*

2) Making light of photons

a) The output at (1) is the sum of the two beams. Remember that though the intensity after one reflection followed by transmission through the beam splitter is 0.25, the electric field changes by 0.5 since $I \sim E^2$.

The math in this problem is easier if you replace 2*cos(a) by exp(ia)+c.c. where c.c. indicates complex conjugate (*i.e.*, in this case c.c.=exp(–ia)). Convince yourself that this relationship is true, using the fact (you may not know) that exp(ia) = cos(a) + i sin(a), where i = $\sqrt{-1}$. The problem still can be done by writing cos() and sin() everywhere, but this *phasor notation* is easier and extremely useful!

Thus the sum of the two beams at position (1) is:

$$E_{total} = \frac{E_0}{4}\left[\exp(i\pi)\left\{\exp\left(i2\pi v\left(\frac{2\cdot x1 + x0}{c} - t\right)\right) + \exp\left(i2\pi\left(\frac{2\cdot x2 + x0}{c} - t\right)\right)\right\}\right] + c.c.$$

where x0 is the distance from the beam splitter to position (1).

Rewriting:

$$E_{total} = \frac{E_0}{4}\exp\left(i\left\{2\pi v\left(\frac{2\cdot x2 + x0}{c} - t\right) + \pi + \frac{2\pi\cdot v\cdot x0}{c}\right\}\right)\left\{\exp(i2\pi\cdot v2\cdot Dx) + 1\right\} + c.c.$$

where Dx is (x1–x2).

This can be rewritten in terms of cosines but for the purposes of the rest of this question, this form is easier to deal with.

ii) From the above expression, you can see that there is no electric field at position (1) when $\exp(i2\pi v2Dx) + 1 = 0$ ie. when $Dx = n\frac{c}{(4v)}$ where n=..,–3, –1,1,3,.. Thus, if the difference in distance is 1/4 of the wavelength (recall wavelength=c/ν), no electric field is found at (1) and we have total destructive interference. Where does the energy go? Consider the electric field that is reflected back in the direction of the original beam. Due to fact that one beam undergoes three reflections and the other only one (instead of two and two), you can show using the same method as above that the MAXIMUM reflected occurs when $Dx = n\frac{c}{(4v)}$ where n=..,–3, –1,1,3,.. . Thus if the beam does not exist at position (1), it is reflected back the way it came. Energy is not lost.

iii) There is some subjectivity to what it means to 'clearly distinguish' but I would argue that you can clearly distinguish between a maximum and a minimum output. Maxima at (1) occur when $Dx = n\frac{c}{(4v)}$ where n=..,–2,0,2,.. So we can write

$(Dx_{\max} - Dx_{\min}) = \dfrac{0 - c}{(4v)}$, i.e., 1/4 of the wavelength.

The wavelength is 632 nm, so the smallest distance change you could clearly measure is 158 nm. That is pretty small. To contrast, atomic spacing in a solid is on the order of 5×10^{-10} m, so you could measure a distance corresponding to ~300 atoms.

iv) First assume the ether exists and is moving relative to your laboratory. This seems to make sense unless the ether is held to the surface of the Earth due to gravity. That would require the ether to have sizable mass. The ether must exist in space though since light travels in space, and then you would get some interesting light bending effects due to the concentration of ether around massive objects. Also you would still have ether existing in vacuum, thus you would have something with mass existing in a vacuum and now my head is starting to hurt. Even if the ether is not moving , you could put your apparatus in a vehicle, such that you would be moving relative to the ether.

The trick is to rotate your interferometer and see if you can see a change in the light output at (1). If one arm is parallel to the ether motion and the other is perpendicular to it, you find that destructive interference occurs if

$$\left(\exp\left(i2\pi v\left(\frac{x1}{(c+V)}+\frac{x1}{(c-V)}-\frac{2x2}{c}\right)\right)+1\right)=0$$

where V is the speed of the ether relative to your interferometer. Rotate the interferometer by 90° and this term changes to:

$$\left(\exp\left(i2\pi v\left(\frac{2x1}{c}-\frac{x2}{c+V}-\frac{x2}{(c-V)}\right)\right)+1\right)$$

Define the minimum V such that by rotating your interferometer, you go from a case of destructive interference to a constructive interference. (You might argue that this is too much to require; the output intensity only needs to change by a small amount. This is really determined by the noise in your system, which is a subject we are not going to discuss here, so we take the strongest case required.). Thus:

$$\left(\frac{x1}{(c+V)}+\frac{x1}{(c-V)}-\frac{2x2}{c}\right)=\frac{1}{(2v)} \qquad \text{(Destructive)}$$

$$\left(\frac{2x1}{c}-\frac{x2}{c+V}-\frac{x2}{(c-V)}\right)=0 \qquad \text{(Constructive)}$$

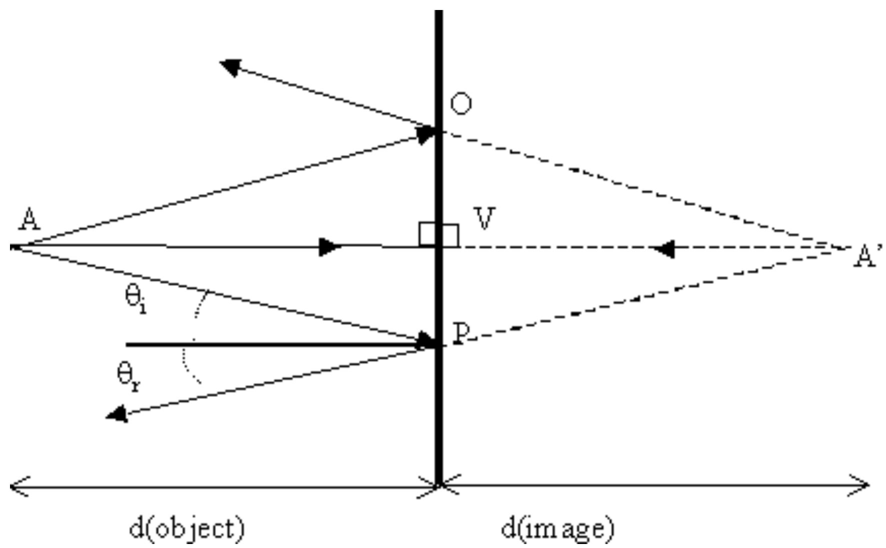
Simplifying this gives:

$$V^2=\frac{\frac{c^2}{(4v)}}{\left(x1+x2+\frac{c}{4v}\right)}$$

To make your scheme as sensitive as possible, you want to make x1+x2 as large as possible. Say 19th century technology limits you to x1+x2=1m. For a source of wavelength 632 nm, you would require $V \sim 1\times10^5$ m/s. Increasing x1+x2 to 100m reduces minimum V required to be observed to $\sim 1\times10^4$ m/s. *[James]*

3) Imagining imaging

i)



From the law of reflection,

$$\theta_i = \theta_r$$

Also $\theta_i + \theta_r$ is the exterior angle of the triangle AA'P, and is therefore equal to the sum of the alternate interior angles ∠VAP and ∠VA'P.

But ∠VAP = θ_i , and therefore ∠VAP = ∠VA'P. This makes the triangles VPA and VA'P congruent, in which case d(object) = d(image).

As for the magnification, this can be determined using the angle between two rays of light leaving a point on the object, and the corresponding angle the rays seem to form when traced back to the virtual image. We have already shown that ∠VAP = ∠VA'P, which is true of all object/image points. Thus the apparent size of an image divided by the size of the original object, or the magnification is +1. The virtual image is life-size and erect.

ii) To demonstrate that one focus is the virtual image of the other we will prove that the angles the incident ray and reflected ray make with the tangent are equal. This is most easily done with the following geometry in mind.

On the diagram given in the question label the following:

the focus F $\rightarrow$ object point with coordinates (–c, o)

the focus F′ $\rightarrow$ image point with coordinates (c, o)

the incident ray (leaving F) $\rightarrow$ line ℓ_2

the reflected ray (`leaving' F′) $\rightarrow$ line ℓ_1

the point of reflection on the surface of the F′ branch of the hyperbola $\rightarrow$ point A with coordinates (x_o, y_o)

Also draw in the tangent line at point A $\rightarrow$ label it line ℓ

Then label

the angle between ℓ_2 and ℓ $\rightarrow$ angle α

the angle between ℓ and ℓ_1 $\rightarrow$ angle β

where both α and β are counter clockwise angles.

So we want to prove $\alpha = \beta$. Suppose the curve is in `standard position' so that the equation is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore \frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \rightarrow y' = \frac{b^2}{a^2}\frac{x}{y}$$

Substituting (x_o, y_o) yields the slope of the tangent line,

$m = \frac{b^2}{a^2}\frac{x_o}{y_o}$ $\rightarrow$ gives the equation of the tangent $\rightarrow$ $\frac{x_o x}{a^2} - \frac{y_o y}{b^2} = 1$

Now we can use the formula for the tangent of the counterclockwise angle from one line ℓ_1 to another ℓ in terms of their respective slopes m_1 and m, namely

$$\tan\alpha = \frac{m - m_1}{1 + mm_1}$$

where the slope of the line ℓ_1 is $m_1 = \dfrac{y_o - 0}{x_o - c}$

plugging in the values for the slope you get something not as bad as it seems,

$$\tan\alpha = \frac{\dfrac{b^2 x_o}{a^2 y_o} - \dfrac{y_o}{x_o - c}}{1 + \left(\dfrac{b^2 x_o}{a^2 y_o}\right)\left(\dfrac{y_o}{x_o - c}\right)} = \frac{b^2 x_o^2 - b^2 c x_o - a^2 y_o^2}{a^2 y_o (x_o - c) + b^2 x_o y_o}$$

$$= \frac{-b^2 x_o c + (b^2 x_o^2 - a^2 y_o^2)}{(a^2 + b^2) x_o y_o - a^2 c y_o}$$

$$= \frac{-b^2 x_o c + b^2 a^2}{c^2 x_o y_o - a^2 c y_o} = \frac{b^2(-x_o c + a^2)}{c y_o (c x_o - a^2)} = \frac{-b^2}{c y_o}$$

where we have used the fact that $a^2 + b^2 = c^2$ and $b^2 x_o^2 - a^2 y_o^2 = a^2 b^2$ for a hyperbola.

The same calculation with –c replacing c gives

$$\tan(-\beta) = \frac{b^2}{c y_o} \qquad \text{so } \tan(\beta) = \frac{-b^2}{c y_o}$$

so tan(α) = tan(β), and therefore α = β.

In other words reflected rays appear to come from F′ or the focus F′ is the virtual image of F. The same argument can be made for the reverse situation (F′ the object point and F the virtual image). Thus each focus is the virtual image of the other.

The magnification is *not* well defined. This is clear when you compare the angle between two rays of light leaving a point on the object with the corresponding angle the rays seem to form when traced back to the virtual image.

If for example you consider the rays drawn in the diagram (given in the question) and the corresponding angles they make with the x-axis (our second ray reflected back along the x-axis) the magnification, M

$$M = \frac{\tan(F')}{\tan(F)} \to \infty$$

Since the angle $F' \to 90^\circ$

If another two rays are taken an entirely different value for M can be obtained. Thus the magnification depends on which pair of rays you are tracing out and thus cannot be well defined. A proper image isn't really formed. *[Carrie]*

4) Correct time and temperature, at the tone…

i) Let the initial and final lengths, due to a temperature change of $\Delta T K$, be L and L' respectively. The respective periods are then:

$$T = \sqrt{\frac{g}{L}} \qquad \text{and} \qquad T' = \sqrt{\frac{g}{L'}}$$

Since $L' = L(1 + \alpha\Delta T)$, we have

$$T' = \sqrt{\frac{g}{L(1+\alpha\Delta T)}}$$
$$= (1+\alpha\Delta T)^{-\frac{1}{2}} T$$

Thus the required ratio is $\frac{T'}{T} = (1+\alpha\Delta T)^{-\frac{1}{2}}$

ii) Now for brass we have $\alpha = 1.9\times10^{-5} K^{-1}$ and we're told $\Delta T = 20°C = 20K$

Since $T = 1.000$ the new period is

$$T' = (1+1.9\times10^{-5}(10))^{-\frac{1}{2}}$$
$$= 0.9998$$

Since the new period is shorter the clock will measure time faster and so gain time.

Since the now warmer clock measure is in just 0.9998 s. Therefore the clock gains time by a factor of $\frac{1}{T} = 1.0002$ per unit of real time.

Thus after one day (as measured by the clock initially cold) the warm clock gains 0.0002 days = 0.0002(24) (60) (60) s = 17.28 s.

Therefore the clock when warmer by 20 K gains approximately 17.28 s per day.

iii) In the case of a stretched string, it isn't exactly the change in lengths which change the frequency. The string is stretched to a length *L*, no matter what. Instead, as the string expands, it becomes less taut — the tension in the string isn't as great, since the string doesn't need to be stretched as much.

For transverse waves in a stretched string we use the equation $c = f\lambda$ where c is the speed of the wave, f the frequency and λ the wavelength. The first harmonic vibrates with the only nodes at the end points of the string. This distance L is then equal to one half the wavelength λ. Thus $f = \frac{c}{2L}$.

The wave speed c is given by $\sqrt{\tau/\mu}$ where τ is the tension in the string and μ the mass density per unit length. Thus $f = \frac{1}{2L}\sqrt{\frac{\tau}{\mu}}$.

We can determine τ from the formula for Young's modulus, namely $\frac{\tau/A}{\Delta l/l} = Y$, where A is the cross sectional area Δl is the change in length due to the applied tension, l is the initial length and Y is 'Young's modulus,' a constant.

So we have $f = \dfrac{1}{2L}\sqrt{\dfrac{Y\,A\,\Delta l}{l\,\mu}}$

Initially we have string-length l_{o}, and the string is stretched out by an amount

$$\Delta l = (L - l_o)$$

and $\mu = \rho A$

where ρ is the density of steel at the initial temperature. After the temperature increase ΔT we have

length l' prior to being stretched, and stretch amount $\Delta l'$

$$l' = l_o(1 + \alpha\,\Delta T)$$

$$\Delta l' = L - l'$$

$$= L - l_o(1 + \alpha\,\Delta T)$$

The linear mass density doesn't change, because we always stretch to length L. The cross-sectional area A *will* increase as the metal expands, but our Young's modulus, assumed to be constant, is defined in terms of the initial A. (We would have been better off to assume that Y•A was constant).

Therefore

$$f = \frac{1}{2L}\sqrt{\frac{YA\ \Delta l}{\mu\ \ l_o}} = \frac{1}{2L}\sqrt{\frac{YA}{\mu}\frac{\{L - l_o\}}{l_o}} = \frac{1}{2L}\sqrt{\frac{YA}{\mu}\left\{\frac{L}{l_o} - 1\right\}} \quad \textit{and}$$

$$f' = \frac{1}{2L}\sqrt{\frac{YA\ \Delta l\copyright}{\mu\ \ l\copyright}} = \frac{1}{2L}\sqrt{\frac{YA}{\mu}\frac{\{L - l\copyright\}}{l\copyright}} = \frac{1}{2L}\sqrt{\frac{YA}{\mu}\left\{\frac{L}{l\copyright} - 1\right\}} = \frac{1}{2L}\sqrt{\frac{YA}{\mu}\left\{\frac{L}{l_o(1 + \alpha\Delta T)} - 1\right\}}$$

It follows:

$$\frac{f'}{f} = \sqrt{\frac{L - l_o(1 + \alpha\,\Delta T)}{(L - l_o)\ (1 + \alpha\,\Delta T)}}$$

and this is the required ratio.

Now, using $\alpha = 1.1\times 10^{-5}\,K^{-1}$, $\Delta T + 20K$, $f = 440\,HZ$, $l_o = 0.795\,m$ *and* $L = 0.8\,m$

we have:

$$\frac{f'}{440} = \sqrt{\frac{0.8 - 0.795(1 + 0.00022)}{(0.8 - 0.795)\cdot(1 + 0.00022)}} = 0.9822$$

$$f' = 432.2\ Hz,$$

Therefore the final frequency of the now-warmer guitar is 432.2 Hz, which is out of tune by about 8 Hz. *[Simal, James & Robin]*

5) Toys, Toys, Toys

The masses are equal, so after the collision the "lattice" will move at **V** m/s [right]; x(t) = Vt is the position as a function of time.

First, note that momentum will be conserved, so that at any point in time we will

have:

$$\begin{aligned} mV &= mL + mR \\ V &= L + R \end{aligned} \qquad [1]$$

where L and R are the velocities of the left and right balls, respectively (they clearly point in the x-direction, so we drop the vector sign).

The position of the centre of mass is given by:

$$x_{c.m.} = \frac{mx_L + mx_R}{2m} = \frac{1}{2}(x_L + x_R) \qquad [2]$$

where x_L and x_R are the positions of the left and right balls (in the laboratory frame), respectively.

Hence,

$$V_{c.m.} = \frac{dx_{c.m.}}{dt} = \frac{1}{2}(L + R) = \frac{V}{2} \quad \text{using [1].}$$

In the centre of mass frame then, which moves at V/2 m/s [right], the velocities of the balls become:

$$L' = V - \frac{V}{2} = \frac{V}{2}; \qquad R' = 0 - \frac{V}{2} = -\frac{V}{2}$$

at t = 0. (the balls are thus approaching each other)

The situation is perfectly symmetric and we can right away conclude that after the two balls collide, which will take a time of L/2/(V/2) = L/V their velocities will be:

$$L'' = -\frac{V}{2}; \qquad R'' = \frac{V}{2}$$

(the balls are going away from each other)

Next, the balls will be pulled by the string, and conserving momentum and energy yields that after the collision we will have:

$$L''' = L' = \frac{V}{2};\; R''' = R' = -\frac{V}{2}$$

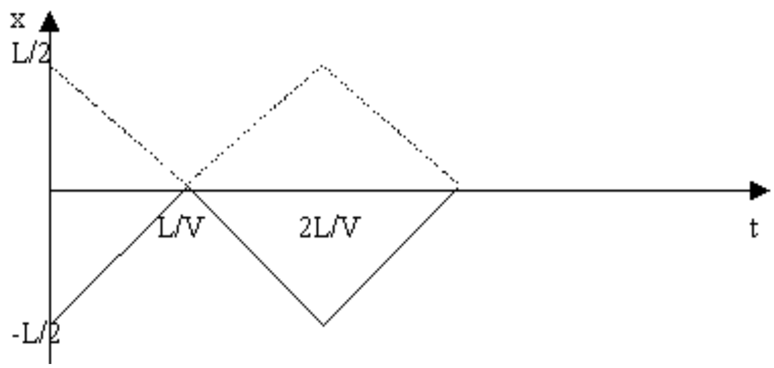
The balls therefore oscillate with a period of 2(L/V). Note that the motion is not simple harmonic, as acceleration $= 0 \neq -kx$.

The motion may be written as

$$L'(t) = \frac{V}{2}, \quad t < \frac{L}{V} \left(\mathrm{mod}\frac{2L}{V}\right) \qquad x_L{}'(t) = \frac{V}{2}t, \quad t < \frac{L}{V} \left(\mathrm{mod}\frac{2L}{V}\right)$$

$$L'(t) = -\frac{V}{2}, \quad t > \frac{L}{V} \left(\mathrm{mod}\frac{2L}{V}\right) \qquad x_L{}'(t) = -\frac{V}{2}t, \quad t > \frac{L}{V} \left(\mathrm{mod}\frac{2L}{V}\right)$$

the primes denote centre of mass frame; "mod" gives the remainder after division, so using it we can figure out what part of the period t is in. If you have never seen "mod" before, don't sweat it. I think it is (more or less) clear the graph looks something like this:



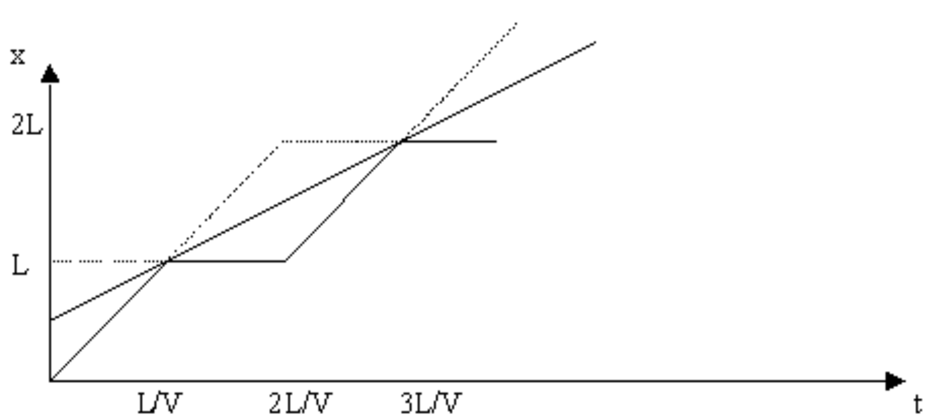
(the dashed line corresponds to the right ball, the solid corresponds to the left one)

Since we have the motion in the centre of mass frame, and we know its velocity, we simply transform to the laboratory frame. This gives:

$$L(t) = V, t\,\mathrm{mod}\frac{2L}{V} < \frac{L}{V} \qquad x_L(t) = Vt, t\,\mathrm{mod}\frac{2L}{V} < \frac{L}{V}$$

$$L(t) = 0, t\,\mathrm{mod}\frac{2L}{V} > \frac{L}{V} \qquad x_L(t) = const, t\,\mathrm{mod}\frac{2L}{V} > \frac{L}{V}$$

Plotting this gives:



The solid line is the motion of the left ball; the dashed that of the right; the thick line is the average of the two, which is also the motion of the centre of mass.

The system looks somewhat like a centipede – first the back moves, with the front being stationary, then the front moves, and so on.

Note that if we had a body of mass 2m colliding with one of mass m we wouldn't get the same results as here – our system loses energy (it's a soft object, like jello).

In an n-particle lattice, the speed of the centre of mass would be V/n.

It is instructive to try this by brute force – keeping the same symbols as in i) (x_L, L

for left object, x_R, R for right)

$$L + R = \frac{dx_L}{dt} + \frac{dx_R}{dt} = V$$
$$mV^2 = m\left(\frac{dx_L}{dt}\right)^2 + m\left(\frac{dx_R}{dt}\right)^2 + k(L - (x_R - x_L))^2$$

This is a system of differential (momentum and energy) equations that just screams out "go away" [NOTE: if you rearrange it – it takes <u>lots</u> of work – you end up with something workable, but not at the high school level]

Let's take the hint and go to the centre of mass frame. From ii), the centre of mass still moves at V/2 m/s [right]. In that frame we once again get the balls approaching at equal speeds (V/2). Since the compression on each half of the spring will be equal, we can treat the system as two springs of length L/2 (k' = 2k).

The motion is simple harmonic with a period:

$$T = 2\pi\sqrt{\frac{m}{2k}}$$

The motion of the left ball is

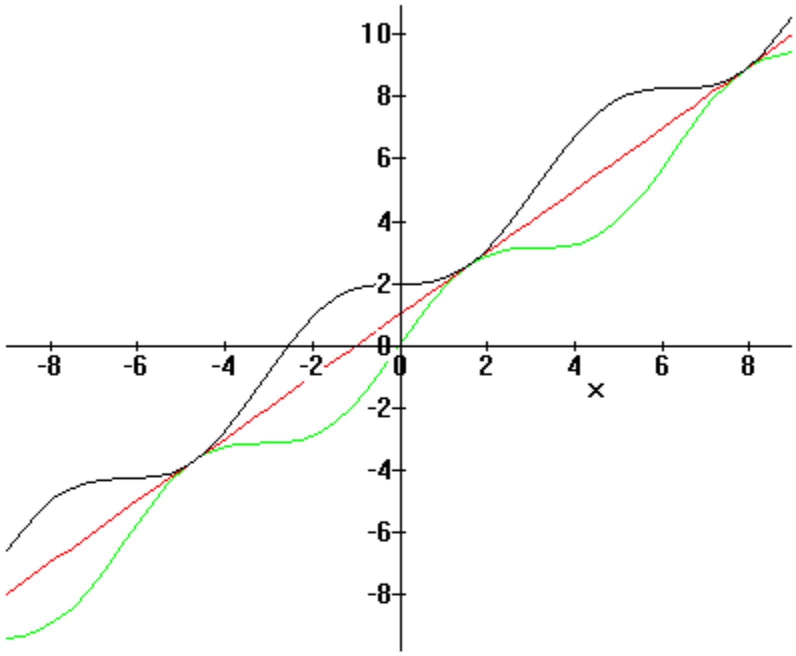
$$x_L{}'(t) = \frac{V}{2}\sqrt{\frac{m}{2k}}\sin\left(\sqrt{\frac{2k}{m}}t\right)$$

[this gives the correct initial conditions – $x_L'(0) = 0$, L(0) = V/2]

Thus, in the outside frame, the motion looks like this:

$$x_L(t) = \frac{V}{2}t + \frac{V}{2}\sqrt{\frac{m}{2k}}\sin\left(\sqrt{\frac{2k}{m}}t\right)$$

Plotting gives (not to scale!):

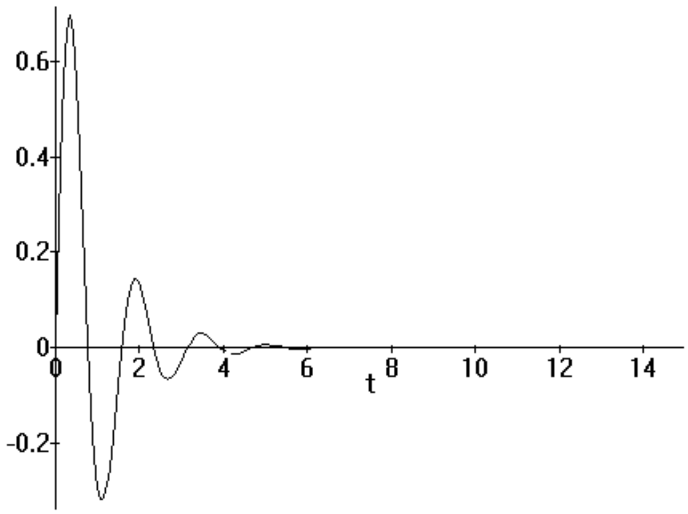


The black line is the right ball, green line is the left ball, straight line is the motion of centre of mass. The motion looks once again like a centipede, but kind of weird (non-uniform).

This is very similar to ii), except that the solution is now

$$x_L{}'(t) = \frac{2\frac{V}{2}e^{-\sqrt{\frac{\gamma t}{m}}}\sin\left(\frac{1}{2}\sqrt{4\frac{k}{m}-\left(\frac{\gamma}{m}\right)^2}\,t\right)}{\sqrt{4\frac{k}{m}-\left(\frac{\gamma}{m}\right)^2}}$$

Plotting gives something like this:

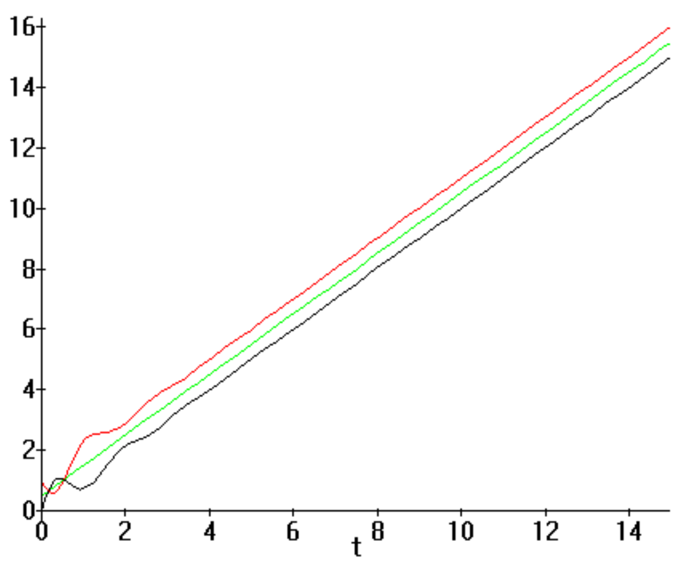


Note that the particle's motion eventually decays and remains at x = 0.

In the outside frame, then:

$$x_L(t) = \frac{V}{2}t + \frac{V e^{-\sqrt{\frac{\gamma t}{m}}} \sin\left(\frac{1}{2}\sqrt{4\frac{k}{m} - \left(\frac{\gamma}{m}\right)^2}\, t\right)}{\sqrt{4\frac{k}{m} - \left(\frac{\gamma}{m}\right)^2}}$$

which gives (not to scale!):



Here, the left ball is black, right ball red and centre of mass is green (straight line).

Clearly, the last model is most realistic. After a body collides with something, we might expect its molecules to vibrate for a little bit, but not forever as in ii) or iii). ii) is most unrealistic as it experiences infinitely long vibrations and the bond between molecules is most likely not a rigid string (this implies that the force acting on the molecules occurs over an infinitely short amount of time, leading to infinite accelerations, etc.).

The dissipation mechanism is due to radiation – an accelerating charge radiates away its energy [see problem set 1]. *[Peter]*

6) Making a shoebox spectrograph, using a compact disc

As given in the 'mini tutorial' under InfoBits™, the input and output angles of light diffracted from a diffraction-grating depend on wavelength, and on the diffraction grating itself:

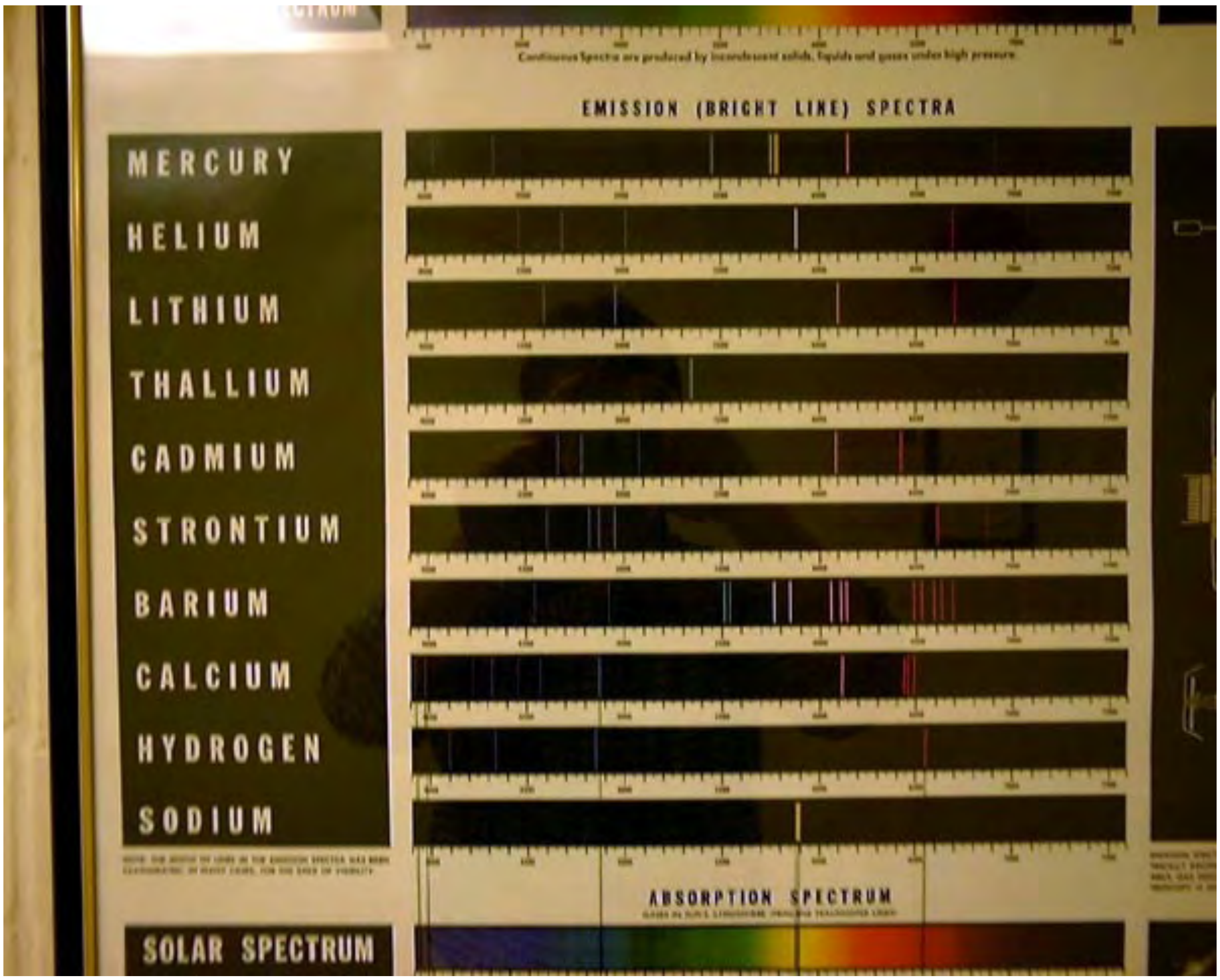
$$n\lambda = 2d \bullet (\sin\theta_i + \sin\theta_d)$$

where θ_i is the input angle, measured from the perpendicular to the surface (the surface *normal*), and θ_d is the angle of the diffracted light, measured similarly. *2d* is the spacing between grooves of the diffraction-grating, or tracks of the CD (rather than d, as given — this is something of a convention). The n is an integer which gives the *order* of diffraction, which you can investigate.

Diffraction orders: This figure was made using a CD and a laser-pointer. You can see the laser-pointer, held at the right. The brightest spot of light is the undiffracted specular reflection (sometimes called the 'zeroth order,' and then to the right are three orders or diffracted light, labelled n = –1, –2, –3. Try the equation yourself to see why they're negative.

We used a HeNe laser from the lab, and a CD that was kicking around, and found that with normal incidence (θ_i =0), the 1st order diffracted angle was found to be 22 deg. Since the HeNe wavelength is 632.8 nm, the spacing between adjacent grooves on the CD works out to be approximately 1.7 μm.

If you look at fluorescent lights, you can easily see the different spectral lines, of different colours, that make the fluorescent light look white. The colours come from different phosphors coated on the inside of the glass tube — each spectral line comes from a quantum transition in the atoms making up the phosphor. The atoms are excited by absorbing ultraviolet light, which comes from mercury atoms which are themselves excited by electrons accelerated up and down the tube. So fluorescent lights are *doubly* fluorescent — first the mercury vapour inside, and then the phosphor coating. Your school may have a wall-chart that shows characteristic spectral lines of different materials (these are available through Edmund Scientific, among other places); a picture of part of one of ours is below.



If you look at street lights, you may find that there are different kinds. Incandescent lamps give a whole or continuous spectrum, but low-pressure and high-pressure sodium lamps are different: the low-pressure lamps give spectral lines, but in high-pressure lamps the much more frequent collisions between atoms make the spectral lines spread out or broaden. Astronomical observatories care very much which sort of lights a city installs, since too much scattered light (light pollution) can wreck observations. If low-pressure sodium lamps are used, the scientists can use the frequencies between spectral lines to peer through to the deep sky. *[Robin]*

1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 5: Electricity and Magnetism

Due March 5, 1999

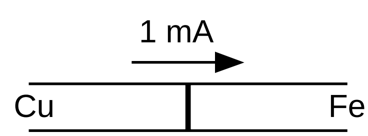
1) Bully for you!

i) Back in kindergarten, the big bully challenges you to a game of 'No-mercy pink leg wrestling'. You decide that this endeavour is not in your best interest so you challenge him back to a 'real tough' game of point-charge placement. "Huh?" says the bully so you explain: "Pretend you are given a set of electrically charged balls, and you must arrange them to be free, but static — so that they do not fly together or apart." "Hey, what about gravity or friction or redistribution of the charges on the balls?" the bully demands. Unfortunately, you seem to have challenged the only bully in advanced-placement physics in the entire kindergarten class. "Hmm, ignore them and pretend it's a perfect insulator, so the charges cannot move" you respond, hoping that if this doesn't go well, the recess bell will ring before leg-wrestling becomes an option again. "Okay, but I go first" says the bully. "Three balls: two with charge +4q, the other with charge –q. Hurry up twerp!" What is your answer?

ii) You got him on that one but now it is your turn. You ask him "Three balls, all with charge q, placed on the points of an equilateral triangle. Two other balls with identical charges on them, but you get to pick the amount of charge. What is the charge and in what arrangement?" He screws up his face to think, exclaiming "There better be a real answer for this, tweek!" You assure him there is one. Better think quick, what is it? *[James]*

2) Charged with resisting…

A current of 1 mA flows through a conductor made of two wires, one copper and the other iron. The cross-sections are identical, and the wires are butt-welded as shown in the figure. What electric charge naturally accumulates at the boundary between the two metals? How many elementary charges does that correspond to? [Hint: Gauss's law] *[Gnädig/Honyek]*



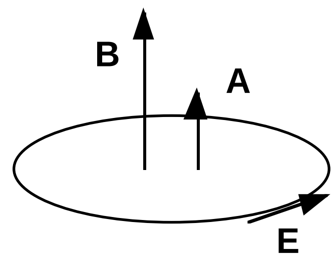
3) 'Ascending and Descending Voltages': a circular argument

"Psst!", a man in a dark trench coat whispers to you, "Psst buddy, ya interested in an almost-new physics equation?" Being the foolhardy type, you decide to check out his wares. “Okay buddy, I’ll even throw in one for free to show you that I’m legit. Say ya got a wire carrying a current. That current induces a magnetic field. Take your right hand like this ya see. Point your thumb out in the direction of the current and the magnetic field curls around the wire like your fingers do.”

You are not impressed. Heck you learned the right-hand rule in Grade 3!

“Okay, waitasecond. I got a real good one. Real hot!” Whipping out a small backboard from within the long folds of his coat, he draws the following diagram while saying, ” Say ya got a closed circular circuit, with a magnetic field going through it. If that field changes in time, there is going to be an electric field induced in the circuit, according to:

$$\vec{E} = \frac{d\phi}{dt} \quad where \quad \phi = \vec{B} \cdot \vec{A}$$



here, **E** is the induced electric field around the circuit, **B** is the magnetic field, and **A** is the area enclosed by the circuit (with its unit vector pointing perpendicular to the circle, in the direction defined by the right hand rule). So waddya think of them apples?”

Wow! This is cool stuff, but something about it sounds fishy. “Okay, I’ll show you. Take a look a this.” He motions you into the dimly-lit alley behind him. Sure enough, your new friend has an electromagnet stashed back there. Above the end of the magnet is a circular loop of five 100 ohm light bulbs all wired in series. Even though there is no battery in the circuit, the bulbs are all glowing weakly. You whip out your magnetometer, and verify that yes, there is a uniform magnetic field running perpendicular to the plane of the circuit (area=0.001 m^2) and it seems to be ramping up in time: $B(t) = B_0 \bullet t$.

a) Using your friend’s equation, what do you conjecture is the total current in the circuit? And what is the voltage across one of the bulbs?

b) *Darn*, you think, *I wish I had brought my voltmeter*. So you say to your new friend:

“Let me get this straight. There is current running through each bulb. That means that voltage must be decreasing across the bulb. Therefore as you go around the circuit the voltage keeps dropping after each bulb. You end up back at where you started but now it seems you are at a lower voltage. Doesn’t this remind you of a print by some Dutch guy named M.C. Eaker, Esther, or something like that?”

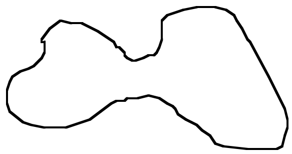
“No, no, no, buddy. You got it wrong!”. What is wrong with your logic? Draw an equivalent circuit. And what is title of the picture that you are mumbling about?

c) “Okay, now I understand”, you tell the huckster, “but there is something still wrong with your equation. If your equation was really right, those bulbs would blow up.” You got him there. Explain. *[James]*

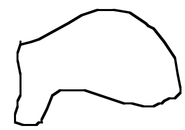
4) Flux compression — putting the squeeze on a B-field

This is an actual experimental problem in practice — a way in which very high magnetic fields can be produced, using a current-carrying circuit and high explosives.

i) Consider a circuit of total area A with some current I passing through it. This causes a uniform magnetic field B perpendicular to the plane of the device. The sides of the device are suddenly imploded so as to cause the area to shrink to A′. What is the resulting magnetic field B′ ?



(total area A)



(total area A′)

ii) If the object is a solenoid shrinking from a radius 10r′ to a radius r′, with I = 1A and n = number of turns per unit length = 100 / 1 cm, what is the resulting field?

[HINT: one of Maxwell's equations will be useful] *[Peter and Bryan]*

5) Scratch and dent sale on capacitors

i) What is the capacitance of the earth, taking it as a perfect sphere?

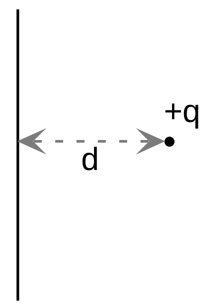
ii) Consider the earth *not* as a perfect sphere — say a comet smacks into the earth and dents it, changing its volume by 3%. By what percentage does the earth's capacitance change?

[BONUS: extra points awarded for deriving the formula for capacitance of a sphere, given in InfoBits below] *[Robin & Gnädig/Honyek]*

6) Mocking mirrors

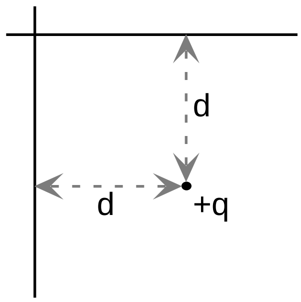
The *method of images* is very useful for solving a variety of electrostatics problems. Consider a point charge +q a distance d away from an infinite conducting plane. Because the plane is a conductor, the charges on it will move so that the potential on the surface of the plane is constant (until this happens, there is a field in the conductor; this field makes the charges move). But you could also make an infinite constant-potential surface if you had *two opposite* charges (+q and – q) a distance 2d apart – the potential on their perpendicular bisector is a constant.

This wouldn't be very important, except there is a theorem which states that the solutions to electrostatic problems are *unique:* if you find *some* electric field that satisfies the boundary conditions, it is the *only* solution. The twinned charges give the right potential for the infinite plane, so their field around +q is the same as the field of a single +q charge above a conducting plane.

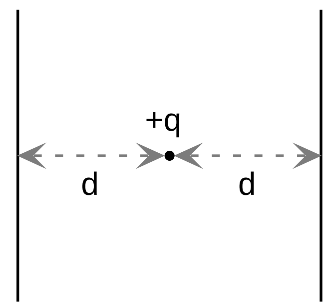


i) So, what is the field at any point y just outside our infinite conducting plane?

ii) Consider *two* infinite conducting planes, crossing at a 90° angle. A charge is placed symmetrically inside one quadrant, a distance d from either plane. What is the field just outside the conductors now? What image charges can you use to produce the same field?



iii) Consider now two infinite, conducting, parallel planes separated by a distance 2d, with a charge +q midway. Where do you have to place image charges so that the planes will be equipotential surfaces?



[HINT: in this case you can think of the planes as mirrors, the charge +q as an object, and the image charges as images of the object formed by the two mirrors] *[Peter]*

INFOBITS™ — Useful Bits of POPTOR Information

Remember to check the POPTOR web-page for hints and any necessary corrections!

www.physics.utoronto.ca/~poptor

$$C_{sphere} = \frac{R}{k}, \qquad C = \text{capacitance}, \; R = \text{radius}$$

$$k = 8.9875 \times 10^9 \; N\,m^2 C^{-2}$$

$$= 8.9875 \times 10^9 \; F\,m^{-1}$$

$$E_{capacitor} = \frac{Q^2}{2C}, \quad E = \text{energy}, \; Q = \text{charge}, \; C = \text{capacitance}$$

1998-1999 Physics Olympiad Preparation Program

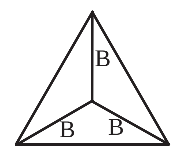
— University of Toronto —

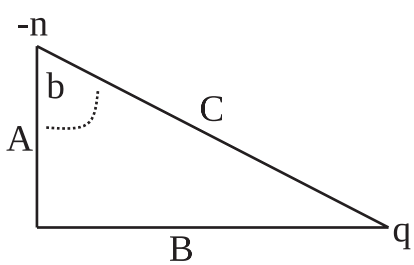
Solution Set 5: Electricity and Magnetism

1) Bully for you!

i) The symmetry gives it away. Put the negative ball in the middle, and place the other two to balance the Coulomb force. No matter how far apart are the positive charges, the net force on each of the three balls will be zero.

ii) This is a classic lateral thinking problem. Trying to balance three balls in a triangle, with two balls sounds impossible, UNLESS you think in 3D. Place each ball above and below the plane holding the three balls. Symmetry says that each negative ball must be an equal distance from each +q ball. Now balance the Coulomb forces. Define the equilateral triangle to have sides of length 1 unit. From the top this looks like the figure at right, from which we can find that $B=\frac{1}{\sqrt{3}}$.





From the side, this looks like the figure at left, where angle b is opposite side B. We find that we have to satisfy two equations:

$$\frac{n^2}{4A^2}=\frac{3nq\cos(b)}{C^2}$$ (vertical forces acting on ball –n)

$$\frac{2nq}{C^2}\sin(b)=\sqrt{3}\ q^2$$ (horizontal on ball q)

Since A = C * cos(b), the first equation simplifies to:

$$\frac{n}{q}=12\left(\frac{A}{C}\right)^3$$

The second equation simplfies to

$$\frac{n}{q}=\frac{3}{2}C^3$$

So $2A=C^2$

Recall that $A^2+B^2=C^2$. These two equations then give:

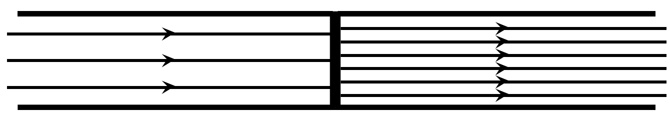
$$A=1\pm\sqrt{\frac{2}{3}}$$

and the value for n=10.4q or 0.33q. You win! *[James]*

2) Charged with resisting…

Let I denote the current flowing in the wire, A the cross-section of the wire and ρ_1 and ρ_2 the resistivities of the materials. Ohm's law for a wire of length ℓ gives $U = I\,\rho\ell\,/A$ which yields $E = U\,/\,\ell = \rho I/A$ for the electric field strength in the wire.

The resistivity of copper is lower than that of iron. Therefore to give the same current in each part, the electric field strength has to be smaller in the copper than in the iron.



According to Gauss's law, the difference in the electric field strengths implies an accumulation of charge at the boundary of the two metals (see figure). The total charge accumulated at the interface is

$$Q = \varepsilon_0 E A = \varepsilon_0 I(\rho_{Fe} - \rho_{Cu})$$

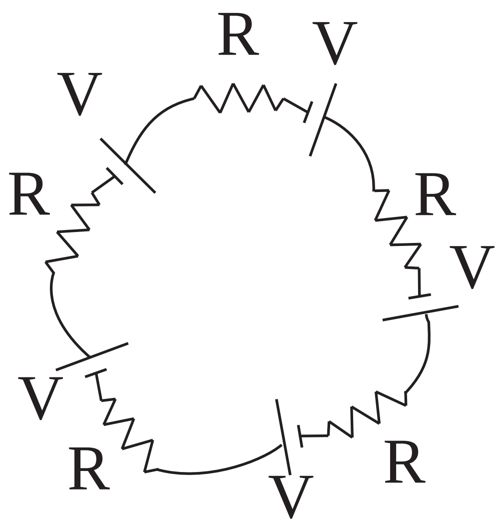
It is interesting that this quantity depends purely on the current and material constants, but not on the cross-sectional area of the wire (i.e., of the interface).

Substituting the known data, the charge is found to be $Q \cong 5.10^{-21}$ C, which is only 1/30th of an elementary charge! Though a measurable macroscopic current flows through the wire, the accumulated charge is only a small proportion of the microscopic elementary charge. This strange result shows that classical electrodynamics (imagining charge carriers as small balls) cannot always correctly describe electrical phenomena. Only application of the more sophisticated laws of quantum theory and statistical physics can give a more accurate description. [You might be interested to note that the 1998 Nobel Prize in Physics was shared by Robert Laughlin for his theory of the fractional quantum Hall effect, and fractional charge excitations.] *[Gnädig/Honyek]*

3) 'Ascending and Descending Voltages': a circular argument

a) Obviously, I did not miss my true calling as a screenwriter. The total induced electric field, if the given equation is correct, is simply $0.001B_o$, over a distance of $2\sqrt{0.001\pi}$, so the total voltage $(E \bullet d)$ is: $B_o \cdot 0.001 \cdot 2\sqrt{0.001\pi}$, and the total current is this value divided by 500 ohm. The voltage across one bulb is 1/5 of the total.

b) The picture you are mumbling about is called "Ascending and descending" (note hint in the title of the question). M.C. Escher makes use of a lovely illusion to draw a staircase with no beginning or end, and with people on it constantly descending or ascending. Actually, my colleague Julian (who is a fine violinist I might add) comments that there is an additional Escher print on the same theme, with a water fall linked in a similar way, endlessly producing power.

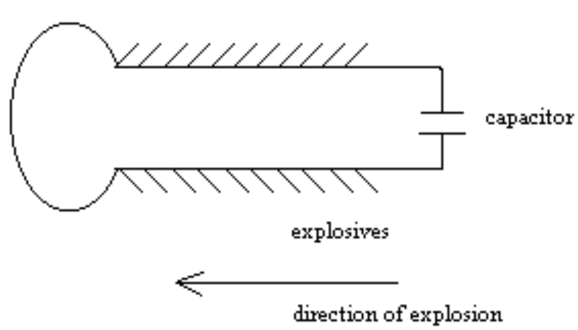


Of course, we have no such 'tricks' in physics, so what is going on in the circuit? The voltage "source" is distributed through every bit of wire, so the voltage drop across each resistor matches the voltage increase across 1/5 of the length of the wire. Thus you start out at the same voltage you began with. This is similar to the circuit at left.

c) Consider the right-hand rule in comparison to the equation your new friend is trying to sell you (another hint that I included in the equation). The induced current will create a magnetic field that will increase the flux in the loop, this will increase the induced current that will create more flux, you have a runaway effect! Any small perturbations in any small magnetic field will cause it to increase dramatically and without end. Of course this is not realistic so there must be a problem with your buddy's model. The real equation has a negative sign. The induced current acts *against* changes in the magnetic field. Mother Nature is truly a conservative force (no political insinuations intended) *[James]*

4) Flux compression — putting the squeeze on a B-field

This experiment is apparently done at Los Alamos National Laboratory, in New Mexico, USA. The two sides of the device have explosives attached to them, so that they start touching each other at the far end and finally end up being connected at the low end. The area of the loop is actually much smaller than that of the sides. The current is caused by a capacitor discharging.



As hinted, the Maxwell equation we will use is that for induced emf. We have:

$$\frac{d\Phi_B}{dt} = -Emf$$

$$\Phi_B = \oint \vec{B} \bullet \vec{dA}$$

So the change in the magnetic flux causes and induced emf. In our case, the initial flux is just B•A, since the field is uniform and perpendicular to the plane of the device. As the area of the device is shrinking, the flux is decreasing. However, we see from above that this change induces an emf in the circuit, which in turn produces a current. Which direction is this current? From the equation we see that it will oppose the decrease in the flux – i.e., the current will flow in a direction to oppose the decrease in flux and will thus increase it. How much does the flux grow by? It will keep growing until it equals the original flux, because then $\Delta B = 0$, emf induced = 0, and we have reached a steady state.

Thus, we see that flux will be conserved in this process and we will get

$$\Phi_B = const = BA = B'A'$$

$$B' = B\frac{A}{A'}$$

Field of solenoid is $B = \mu_0 In$ using Ampere's Law. A direct application of i) gives $B' = B\left(\frac{r}{r'}\right)^2 = \mu_0 In\left(\frac{r}{r'}\right)^2$ which gives 1.26 T in this case. Note that the field grows as the square of the radius, which means that we can achieve very large fields (~ 1000 T) with this technique. *[Peter and Bryan]*

5) Scratch and dent sale on capacitors

i) Capacitance C is defined such that $Q = C\,V$, where Q is the charge on the capacitor and V its electrostatic potential. Thus we can write $C = Q/V$. For a spherical distribution of charge, the field and potential outside the sphere is exactly as it would be if the entire charge were concentrated at a point at the centre of the sphere. Thus at the sphere's surface or a vanishingly small distance beyond, $V = k\,Q/R$, where R is the sphere's radius. From this it follows that the capacitance of an isolated sphere is:

$$C_{sphere} = Q/V = Q/\,(k\,Q/R,) = R/k$$

(the 'grounded' capacitor plate is at infinite distance, where the electrostatic potential approaches zero)

The earth's radius is 6.378×10^6 m, and $k = 8.9875 \times 10^9$ N m^2 C^{-2}; from this it follows that the capacitance of the earth is 7.1×10^{-4} C^2 N^{-1} m^{-1}, or 710 μF. Seems kind of small for such a big planet!

ii) The energy of a capacitor of charge Q and capacitance C is $Q^2/(2C)$. If the change in energy of the capacitor can be found the change in its capacitance can also be calculated.

The energy of the capacitor is higher when it is indented, since the surface charges move in a direction opposite to the force acting on them. Also an electrostatic field of field strength **E** has an energy $\varepsilon E^2/2$ per unit volume and when the capacitor is indented the electric field penetrates a volume where it was not previously present.

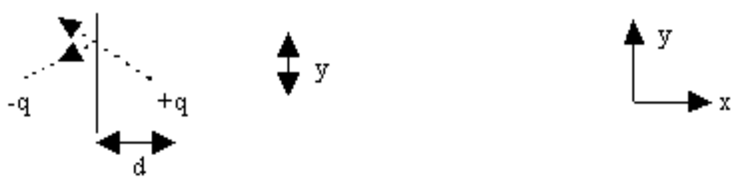
If the surface of the capacitor is only changed a little the electric field can be considered as identical to the original one near the surface. Thus, the change in energy depends purely on the change in volume and not on the actual shape of the indent.

Imagine that the capacitor is hammered so that its volume decreases by 3%, but its shape remains spherical. Its radius is therefore reduced by 1% (as the volume of the sphere is proportional to the cube of its radius). The ratio of the energy of such a capacitor to the energy of the original one is the same as the ratio of the energy of the indented capacitor of the problem to that of the original one. Thus, the relative change in their capacitance is identical as well.

Further, the capacitance of a spherical capacitor is proportional to its radius. The capacitance of the new capacitor (of reduced size) is therefore 1% smaller, and the capacitance of the capacitor of the original question decreases by the same amount. *[Robin & Gnädig/Honyek]*

6) Mocking mirrors

i) We have:

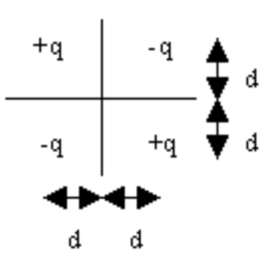


So that the field at a point (0, y) is given by:

$$\vec{E}(0,y) = -\frac{2\cos(\theta)q}{4\pi\varepsilon_o R^2}\hat{x} = -\frac{2dq}{4\pi\varepsilon_o\left(y^2+d^2\right)^{3/2}}\hat{x}$$

And it has no y-component. Hence, the potential, which is just the integral of the field (up to a constant) gives zero (since $\int 0dy = 0$), so the potential is indeed constant in the planes.

ii) By comparing this to i), or noticing that the image charges correspond to images of object as produced by a mirror, a self-consistent solution is:

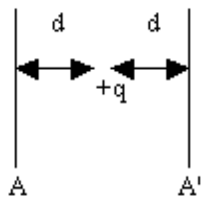


The field at any point (0, y) is

$$\vec{E}(0,y) = -\frac{2dq}{4\pi\varepsilon_o\left((y+d)^2+d^2\right)^{3/2}}\hat{x} + \frac{2dq}{4\pi\varepsilon_o\left((y-d)^2+d^2\right)^{3/2}}\hat{x}$$

which is also the field at a point (–x, 0), by symmetry.

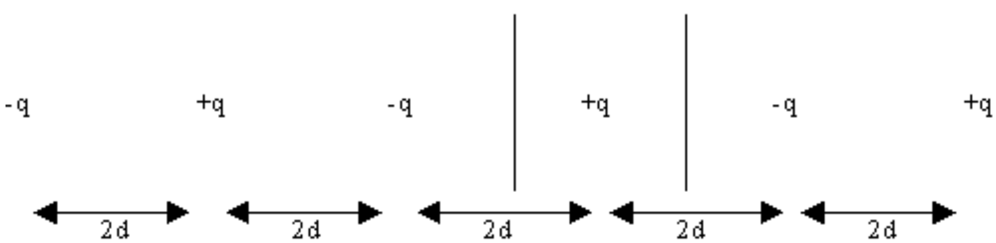
iii)



Consider this reasoning: to make A equipotential we have to put a charge –q a distance d left of it. To make A' equipotential, we put –q a distance d to the right of A'. But now the two –q charges cause A and A' not to be equipotential again. So we put a charge +q a distance 3d left / right of A / A' (respectively). And so on, ad infinitum.

This is equivalent to the forming of infinitely many images when you stand between two mirrors — the 1st seems a distance d away, the 2nd, a distance 3d = d + the distance of the light ray bouncing from the opposite mirror.

We have:



[Peter]

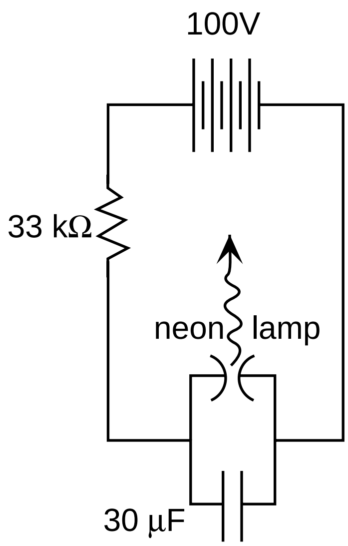
1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 6: AC Circuits and Electronics

Due April 2, 1999

1) Thief of time



A chum of mine once upon a time had built this circuit, and set it by the basement windows of his house as a burglar deterrent — it makes a flashing light, and gives the impression it might be part of a burglar alarm.

The flashing light comes from a small neon-filled bulb, with closely spaced electrodes. The bulb can support about 80 V across it without problems, but at a higher voltage the gas inside ionizes in the strong electric field, the resistance drops drastically, and a discharge takes place between the electrodes. The discharge electrons excite the neon atoms, which produce the light. Once electrons stop flowing, the ionized gas recovers and the resistance rises again to the original value.

i) How frequently does this particular design flash?

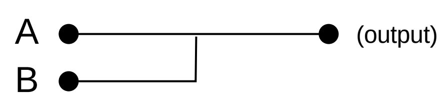
ii) If the neon lamp goes from 220 kΩ in its normal state to about 0.5 Ω during the discharge, roughly how long does each flash of light last?

iii) During a flash, roughly how much electrical energy goes into the lamp?

iv) How could you easily modify this circuit to make the flash last longer? How long would the flash then be? *[Robin]*

2) Logic rules (!)

For the questions below assume that a *logical high* ('1') means a voltage of about 5 V, and a *logical low* ('0') means a voltage around 0 V. Straight lines are wires and things labeled A, B, C… are inputs.

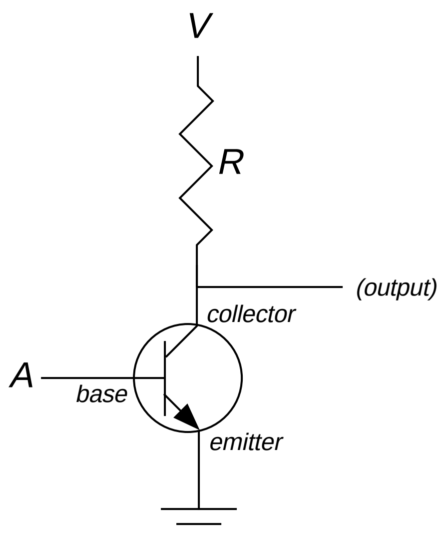


i) Consider this attempt at a logic gate. Make a *truth-table* for this setup: show the output for each possible combination of inputs (use 0s and 1s). What kind of gate is it? Name at least one major problem this gate would have, if used in some circuit. [HINT: does the gate have, say, directionality?]

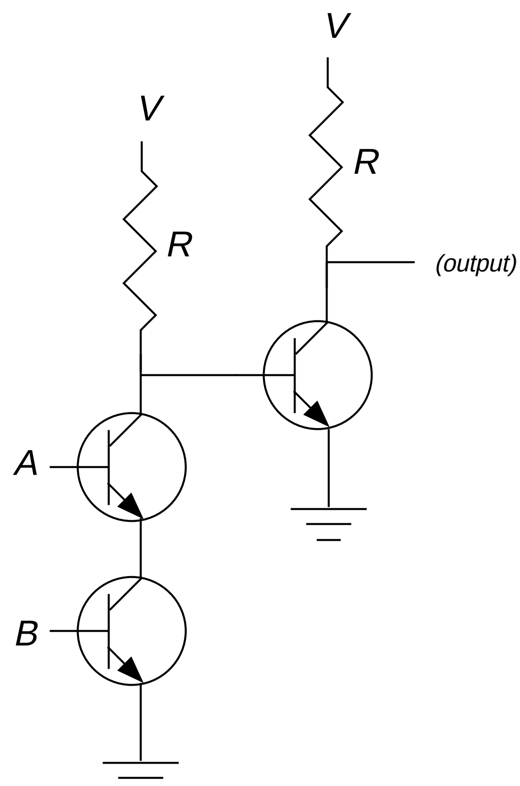
To avoid the problem you may have found above, we can use a wonderful device known as a *transistor*. A transistor essentially functions as a switch. It has 3 electrical connection points – the *base*, the *collector* and the *emitter*. When the base is at a higher voltage than the emitter, the *conduction band* of the semiconductor inside shifts toward the collector and then current can flow from the collector to the emitter. Otherwise, the band is near the base and the device does not conduct between the collector and the base. Semiconductor — get it? Sometimes it's like a conductor and sometimes it isn't.

ii) Knowing this, can you figure out what the following circuits do? Give a basic description of each and figure out the truth tables (as described above).

a)



b)

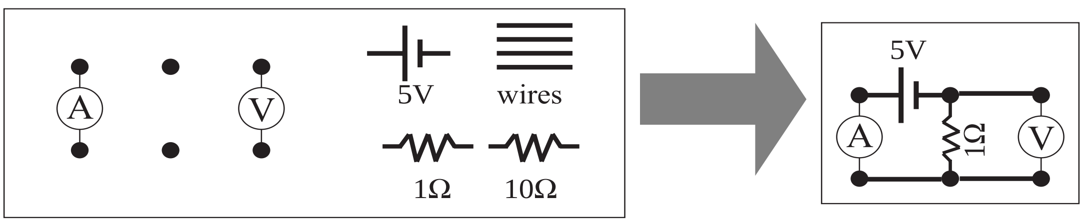


[NOTE: in all circuits the resistance of the load on the output (e.g., voltmeter) is much higher than R] *[Peter]*

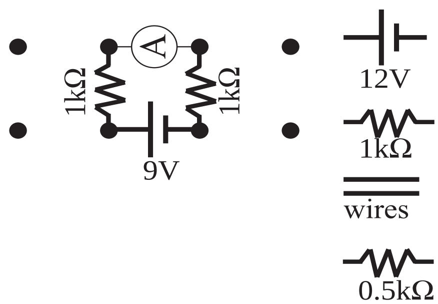
3) Small, but wirey…

Your mission Jim, if you chose to accept it, is to connect the dots using the given components to obtain the required currents or voltages (see figure on next page). The wires and components are delicate so do not bend them. Connecting diagonal dots does work though. Remember Jim, all the components are ideal (wires have no resistance, there is no voltage drop across an ammter, etc.). You have 15 seconds before this unit self-destructs. 14, 13....

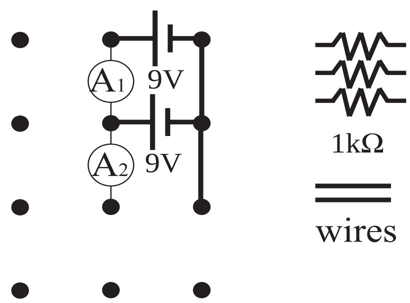
e.g.) Set voltage=5V, current: I=5A.



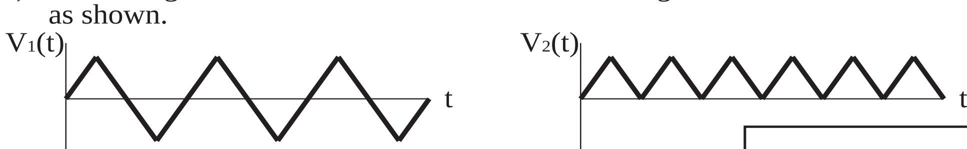
a) Set I=2mA.

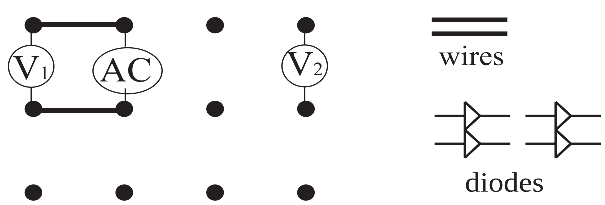


b) Set I_2=18mA. What is I_1?

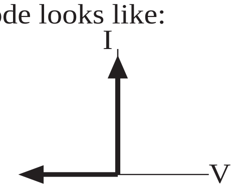


c) The voltage across the unknown element AC changes as a function of time (see $V_1(t)$). Set $V_2(t)$ as shown.





Recall Jim that the current across a diode looks like:



d) Set I=47.5 mA. (Be careful Jim, we think this one might be harder than it looks.)

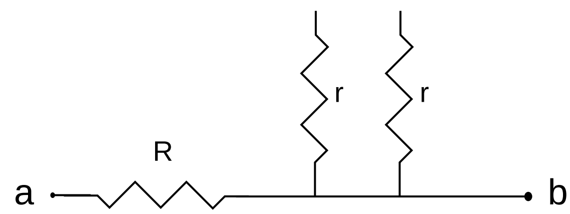
unlimited supply of all three components

[James]

4) Is resistance futile?

Given a resistance R (made of n smaller resistors of value r connected together any way you want), is it possible to add two more r's so that the total resistance remains R? Note that all the resistors must have both their ends connected – i.e., the following is illegal:

The resistance from a to b *obviously* remains at a value R, after connecting two more resistors r, which makes the question far too boring… *[Peter]*



5) *'E pluribus unum'*

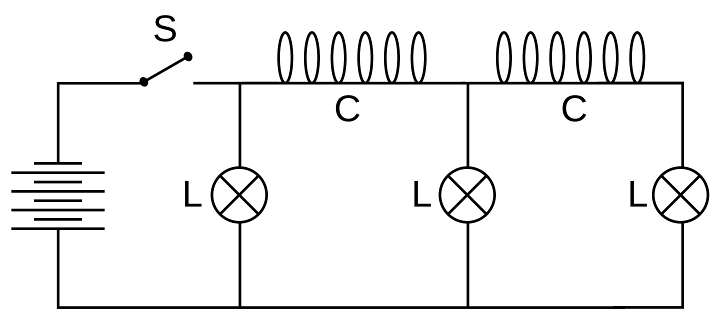
Given a large number of exactly 1 Ω resistors, describe using circuit diagrams how to make a resistance of any (arbitrary) given value.

[HINT: the value of the new resistance is arbitrary, but it can only be measured to a finite number of digits] *[Peter]*

6) Inductive reasoning

The circuit shown in the figure — composed of three identical lamps and two coils — is connected to a DC power supply. Some long time later, the switch S is opened. In order of brightness, how do the lamps rank? Explain.

[Gnädig/Honyek]



Remember to check the POPTOR web-page for hints and any necessary corrections!

www.physics.utoronto.ca/~poptor

1998-1999 Physics Olympiad Preparation Program

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Solution Set 6: AC Circuits and Electronics

1) Thief of time

i) Let's use these symbols for the various bits:

$R = 33\text{k}\Omega$, the resistor after the battery
$R_B = 220\text{k}\Omega$, the normal resistance of the bulb
$C = 30\ \mu\text{F}$, the capacitance of the capacitor in parallel with the bulb

Our basic equations are:

$V_C = V_B$ (in parallel)
$V_R + V_B = 100\,V$ (the whole voltage drop around a closed loop)

$I_R = I_B + I_C$ (the current is conserved as it splits over the bulb and capacitor paths)

$V_R = I_R \cdot R$ (Ohm's law)
$V_B = I_B \cdot R_B$

$Q_C = C\ V_C$ (charge on a capacitor), which gives:

$$\Rightarrow I_C \equiv \dot{Q}_C = C\dot{V}_C \text{ (just taking the derivative)}$$
$$= C\dot{V}_B$$

Then this can go into Ohm's law to give a voltage,

$$V_R = I_R \cdot R = \left(I_B + I_C\right)\cdot R$$
$$= \left(I_B + C\dot{V}_B\right)\cdot R$$
$$= \left(\frac{V_B}{R_B} + C\dot{V}_B\right)\cdot R$$

and then this can be used in the voltage-sum expression

$V_R + V_B = 100\,V$, to reduce just to an expression for the bulb-voltage:

$$\left(\frac{V_B}{R_B} + C\dot{V}_B\right)\cdot R + V_B = 100\,V$$

$$\dot{V}_B + V_B\frac{1}{C}\left(\frac{1}{R_B} + \frac{1}{R}\right) - \frac{100\,V}{RC} = 0$$

which we can write in a simpler form for the constants,

$$\dot{V}_B + V_B = b + c = 0$$

$$b = \frac{1}{C}\left(\frac{1}{R_B} + \frac{1}{R}\right)$$

$$c = -\frac{100\,V}{RC}$$

We can solve this several ways – the most straightforward is just to integrate, using $\dot{V}_B = \frac{dV_B}{dt}$, so that you get an expression $d\,V_B$ on one side and dt on the other, to integrate.

A neater answer is to think about what this means — what is going on. After a long time, we can guess that if the bulb hasn't broken down, the capacitor will be all charged up and that nothing will be changing (*i.e.*, will have a 'steady state'). Then there could be no changes in time, so $\dot{V}_B(t) = 0$, and therefore:

$$V_B(\infty)\cdot b + c = 0$$

$$\Rightarrow V_B(\infty) = -\frac{c}{b} = (100\ V)\cdot\frac{R_B}{(R + R_B)}$$

Thinking about the start, when the battery is connected but no charge has yet flowed through R, we need to have $V_B(0) = 0$. So the right solution starts at zero and asymptotically (over a long time) goes to a constant as a limit. If you already know RC circuits, or if you know 1st order differential equations, or if you integrate the equations (or if you're lucky!), you might guess that the change is an exponential relaxation, which if we start at zero and go to $V_B(\infty)$ would have the form

$$V_B(t) = V_B(\infty)\cdot\left(1 - e^{-at}\right)$$

If you put this into the equation above, you'll see easily that it actually works:

$$(-a)V_B(\infty)e^{-at} + V_B(\infty)\left(1 - e^{-at}\right)\cdot b + c = 0$$

$$\left\{(a-b)\cdot V_B(\infty)e^{-at}\right\} + \left\{V_B(\infty)\cdot b + c\right\} = 0$$

This has to be true for *all* values of t, which means *each* parenthesis term must be zero independently. Taking the second term to be zero gives $V_B(\infty)$, if we didn't already guess it above. The first term to be zero gives us a=b, so the solution is:

$$V_B(t) = 100\,V\cdot\frac{R_B}{R + R_B}\cdot\left(1 - e^{-\frac{1}{C}\left(\frac{1}{R_B} + \frac{1}{R}\right)t}\right)$$

or with our exact values put in:

$$V_B(t) = 87.0\,V \cdot \left(1 - e^{-1.16t}\right)$$

$V_B(t)$ reaches breakdown for the bulb (80V) when $80 = 87 \bullet \left(1 - e^{-1.16t}\right) \quad \Rightarrow t = 2.18\text{s}$

ii) At breakdown, the capacitor discharges through the bulb, until the bulb 'recovers'.

$$V_C = V_B$$

$$I_B + I_C = 0$$

$$Q_C = C V_C$$

$$\Rightarrow I_C = C \dot{V}_C$$

and with

$$V_B = I_B R_B$$

and also (with current left-to-right taken positive)

$$0 = I_B + I_C$$

This then gives

$$\frac{V_B}{R_B} + C \dot{V}_B = 0$$

$$\Rightarrow \dot{V}_B + \frac{1}{RC} V_B = 0$$

which has a solution $V_B(t) = V_B(0) \cdot e^{-\frac{1}{RC}t}$

So a good characteristic time of the bulb's flash is $RC = 0.5\Omega \cdot 30\,\mu\text{F} = 15\mu\text{s}$

iii) The energy stored in a capacitor charged to voltage V is: $\frac{1}{2}CV^2 = \frac{1}{2}\,30\mu\text{F} \cdot (80\,V)^2 = 96\,\text{mJ}$ (not enough to make the bulb explode, I hope!) It is all dumped through the neon lamp.

iv) going back to (ii) we can make the flash last longer by adding a resistor R' series with the bulb. Then $\tau = RC = (0.5\Omega + R') \cdot 30\,\mu\text{F}$ which you can control by the value of R' and we can make the bulb light up for longer. But since

$$I_B = I_C = -C\dot{V}_C = -C\dot{V}_B$$

and $\dot{V}_B(t)$ will be smaller with R' added, then the current through the bulb will be smaller and it won't be as bright. *[Robin]*

2) Logic rules (!)

i) This is an OR gate. When all of A, B, C are low (i.e., voltage is, say, 0 V), the output is naturally low. But when either A or B or C are high, so will the output be.

Problem: when either A or B or C conduct, some current flows through the circuit. But when more than one input is high, the currents actually add. If we have something fairly sensitive connected to the output, we could fry it if the current is too large. What's worse, in a computer many gates are connected together – but if we connect a few of these gates to the inputs of another gate, the currents will add (if we have n inputs, final current is nI. If we feed n of such gates into another similar gate, the current is now n^2I) – this is clearly a problem…

ii) a) This is a NOT gate. When A is low (ca. 0 V), the base and the emitter are both at 0 Volts. The transistor is thus cut-off – no current flows from the collector to the emitter (it acts like an open circuit). If we now connect a device with a large resistance (>> *R*), the voltage drop across *R* is negligible and almost all voltage (Vo) goes through the device. So the output is high.

[Voltage across *R* would be *R*I and that across the device rI, but since r >> *R* the first may be ignored. Note that the current flowing through both is the same]

When A is high, the transistor conducts (with almost no resistance) – thus, all the voltage drop will be across *R* [since it is smaller than r – the resistance of the output device]. Hence, the output is low.

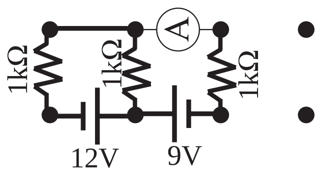
So this is a NOT gate.

b) The second gate (from the left) is a NOT, from above. The first gate is a NAND, by a similar argument as above. Thus, the gate is a NOT (NAND) = AND.

Note that using transistors the output and current voltage are constant. There is still, however another problem, but I'll let you figure it out… *[Peter]*

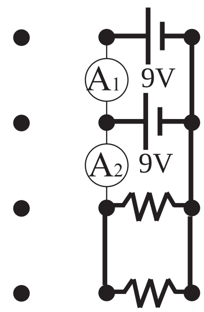
3) Small, but wirey…

a) Set I=2mA.



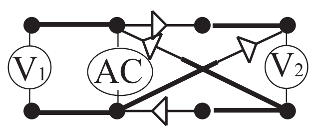
Be careful about the polarity of the 12V source!

b) Set I_2=18mA. What is I_1?



So what is I_1? The normal voltage/current laws do not help you. The only answer that I can argue would be using symmetry, and thus I_1=9mA.

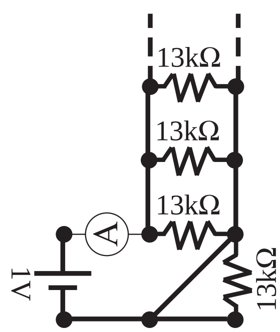
c)



This is known as a full-wave rectifier and can be used to chance AC voltage into DC voltage (with an extra smoothing filter)

d) Set I=47.5 mA.

total: 618 resistors



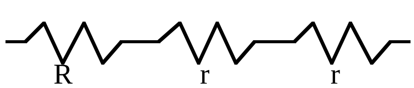
I had a more complicated answer but this one, handed in by a few students, is more straightforward.

[James]

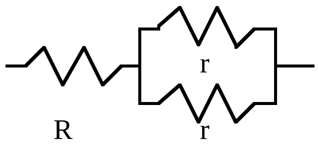
4) Is resistance futile?

Let's try all 6 possibilities:

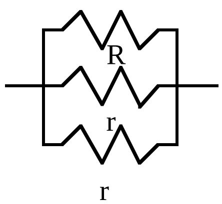
a)



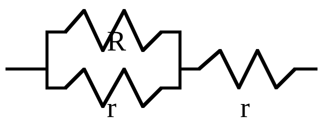
b)



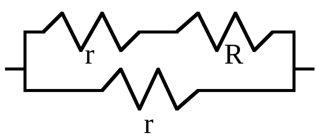
c)



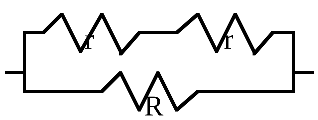
d)



e)



f)



From a) we get: $R = R + 2r$, which is impossible (r > 0)

From b) we get: $R = r/2 + R$, which is again impossible

From c) we get: $r = r + 2R$ – impossible.

From d) we get: $R = r\left(\frac{1+\sqrt{5}}{2}\right)$

From e) we get: $R = r\left(\frac{-1+\sqrt{5}}{2}\right)$

From f) we get: $2r = 2r + R$ – impossible.

So it is possible, using (d) or (e).

A sophisticated aside for question 4:

Note that we have a way of making resistances r that are rational. But note that both the solutions (d) and (e) in question 4 are irrational — if our arbitrary r is rational then it follows that *R* is irrational. But to make *R* we only used *r*'s in some combinations – we only used rational numbers.

Let's recall what kinds of circuits we can make using resistors:

We could have resistors in parallel and series. But the equivalence formulas involve additions (series), or divisions (parallel), so there is no way to obtain a rational number.

In the general case, we need to solve Kirchoff's Laws, which are linear and create linear systems of equations. Solving such a system involves adding / subtracting rows, and multiplying and/or dividing numbers – once again there is no way to obtain an irrational number.

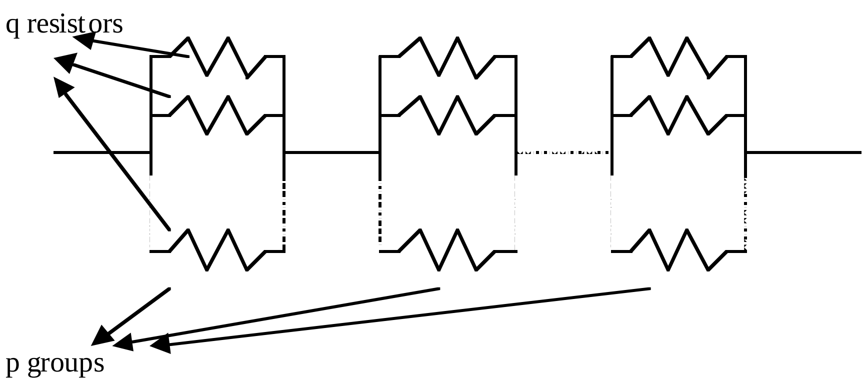
And thus we must conclude that theoretically it is <u>not</u> possible to make the circuit from question 4, unless perhaps we could make *R* from an *infinite* number of resistors *r*.

Experimentally, on the other hand it would be possible. For in experiments we can only measure things to a finite precision. Thus, there are no irrational values – since we only measure a certain amount (say 10) digits, every number is finite and hence rational. And hence we could achieve the circuits from either (d) or (e). *[Peter]*

5) ***'E pluribus unum'***

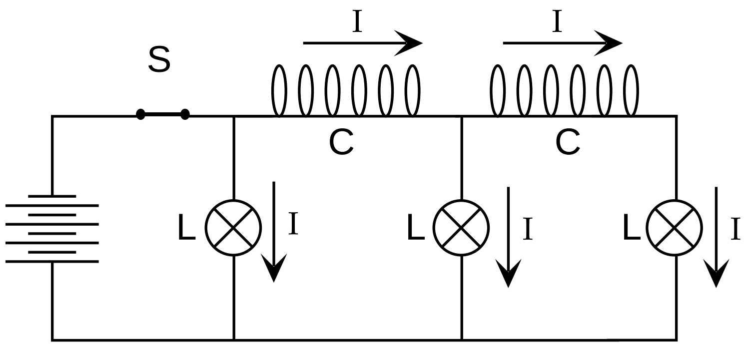
Within experimental error, the other resistor (call it *R*) can only be measured to a finite number of digits. Given any such number, we can express it as a rational number – i.e., in the form p/q where both p and q are integers.

To create a circuit to express p/q we build the following circuit – we connect in series p groups of q groups containing 1 Ω resistors in parallel.

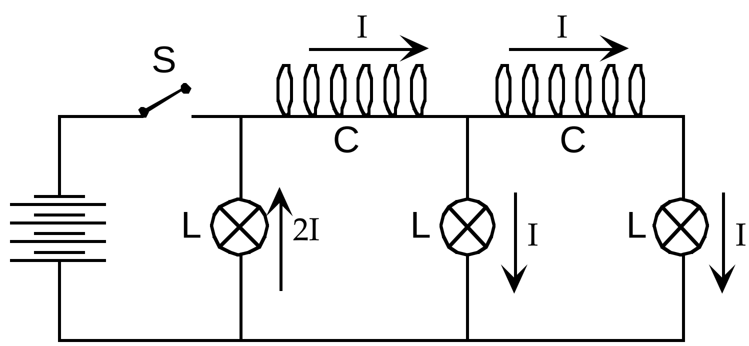


[solution from a Polish Olympiad book "Olimpiady Fizyczne XXI I XXII"] *[Peter]*

6) Inductive reasoning



When the switch is closed, currents as shown at left flow round the circuit. (The value of the current is determined by the voltage of the battery and the resistances of the lamps.) A very short time after turning the switch off the current flowing in the coils is practically unchanged. (If this were not the case, there would be a rapid change in the magnetic flux, which would induce a very high voltage in the coil.) Currents of $2I$ and I continue to flow therefore in the coils, and these determine the currents flowing through the lamps (figure at right).



This means that the lamp closest to the switch suddenly flashes and the brightness of the two other lamps does not change. (This is only true for a short time, later all three lamps fade and go out.) *[Gnädig/Honyek]*

1999-2000 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 1: General

1) Pumpkin paradox

a) $M_{big} = 10000 * M_{small}$

Therefore, for the same relative strength we need a cross sectional area 10000 times larger than the poles in the small model. So,

$$A_{big} = 10000\ A_{small}$$

$$(1/4)\ \pi\ (D_{big})^2 = 10000\ (1/4)\ \pi\ (D_{small})^2 \text{ or}$$

$$D_{big} = 100\ D_{small}$$

So Farmer Joe is going to need poles that are 100 times thicker.

b) Mass is proportional to volume which in turn is proportional to area$^{(3/2)}$. Thus, mass is proportional to area$^{(3/2)}$. So,

$$(M_{big})^{(2/3)}/(M_{small})^{(2/3)} = A_{big}/A_{small} = (1000\text{kg}/0.1\text{kg})^{(2/3)}$$

Therefore, $A_{big} = 464\ A_{small}$

Since the big pumpkin surface area is 464 times greater than the small,

Joe is going to need 4.64 L of paint.

c) $M_{weight}/M_{pumpkin} = 450/1000$ thus,

$$M_{weight} = 9/20$$

The sum of the moments is zero when balanced, thus,

$R_{weight}\ M_{weight}\ g - R_{pumpkin}\ M_{pumpkin}\ g = 0$

Therefore, $R_{weight} = (20\ R_{pumpkin})/9$ *[Carrie]*

2) Cubic quandry

a) There are 6×4 = 24 equal cubes that can perfectly surround the lamp.so we can say that the fraction of power heating one cube is 1/24.

$$\frac{1}{24}Qt = mc\Delta\theta$$

$$? \quad t = \frac{24mc}{Q} \text{ [seconds]}$$

b) A unit area of a blackbody radiator in temperature T, emits energy by the power of σT^4 Watts. so our cube is emitting $6L^2 \sigma T^4$ J/s by the assumption that our cube is in thermal equilibrium each time. Consider our case that any small change in T, will affect the emission rate and the time needed to increase the temperature accordingly. But, in this case, we are looking at changing the temperature by 1K at room temperature, i.e., from 300K to 301K. The relative change is on the order of 1/300. and the change in T^4 is about $4 \times (1/300)$.

$$301^4 = (300 + 1)^4 = 300^4 (1+ \frac{1}{300})^4 \cong 300^4 (1+ 4 \cdot \frac{1}{300})$$

To a good approximation, we can ignore the changes in dT/dt due to a small change in T, and we can substitute the room temperature instead.

$$- 6L^2\sigma T^4 t + \frac{1}{24}Q t = mc \Delta\theta$$

$$\Rightarrow t = \{mc\}/ \left\{\frac{Q}{24} - 6L^2\sigma T^4\right\}$$

c) In thermal equilibrium, the emission rate and heating rate are equal. so we have :

$$- 6L^2\sigma T^4 = \frac{1}{24}Q$$

$$\Rightarrow T = \left\{\frac{Q}{144\sigma L^2}\right\}^{1/4}$$

[Amir]

3) Poles apart, telling

First of all let us note that the North-South (N-S) designations on magnets are a little messed up. You see, the N pole is actually the North-*seeking* pole, meaning that it is actually the South pole (since opposites attract). We will follow these conventions in this document (it doesn't matter how you approach this, so long as you're clear what a North pole of a magnet means to you).

After this lengthy introduction, let us get right to the point. The magnetic field of the magnet will bend the electron orbits into circles. We assume here that the field is uniform throughout the face of the monitor (clearly not true, as the field gets smaller with distance; this solution is thus only valid for electron trajectories not far from the magnet).

i) Suppose we put the N-pole (really the South pole) of the magnet on the right side of the screen. The electrons will be bent according to $\vec{F} = -e\vec{v} \times \vec{B}$ (e is the size of the electron's charge). From this we see that the direction of deflection will be "DOWN" (looking directly at the monitor, $\vec{v}$ is out of the screen, $\vec{B}$ is from left to right).

ii) From the voltage we can find the speed of each electron $Energy = eV = \frac{1}{2}mv^2$, where V = 15 kV. From this, v = 7.3×10^7 m/s (Note that this is actually relativistic… we will ignore this, as the correction is small. Bonus marks will be given to people who realize this).

We also have $\frac{mv^2}{R} = evB$, where R is the radius of rotation of the electron.

We find R = 0.41 cm. Now, each electron will actually only be deflected when it's in the field of the magnet. In this case, this would be roughly the size of the magnet, which we cleverly enough didn't tell you. (we assume that the magnetic field drops off really quickly at the sides of the magnet). Seeing that the turning radius is only 4.1 mm, the magnet really can't be bigger, or there would be no picture whatsoever (all electrons would turn back). Let's assume that the width of the magnet is indeed 4.1 mm. Then, electrons will turn by 90 degrees, and the vertical distance traveled is clearly **4.1 mm**.

DISCUSSION: This somewhat makes sense, but I would expect something bigger. This is mostly due to the fact that the magnetic field actually decays slowly with distance (and doesn't become abruptly zero), whereby the force acts over a larger distance and deflects the electrons more.

In the general case, we can resolve the motion into 2 directions: x (towards the screen), y (down the screen). We have:

$$x(t) = -R\sin(\varpi t) \qquad \text{where } v = \varpi R.$$
$$y(t) = R\cos(\varpi t)$$

We want to find a time τ when the electron reaches the screen (solid line) so $-x(\tau)$ = width of magnet, and calculate what the deflection $y(\tau)$ will be.

[Peter]

4) Physics Rocks!

a) To determine the critical height one must consider the equation of motion of the stone in the water. The forces acting on the stone in water are,

(i) the downward force of gravity = $-m*g$

(ii) the upward drag on the stone = $0.5\ b\ p\ A\ v^2$

(b = drag coefficient, p = density of water)

and from Archimedes' principle,

(iii) the upward buoyant force = $p\ V\ g$

(V = volume of fluid displaced)

Summing the forces we get,

$$p\ V\ g + 0.5\ b\ p\ A\ v^2 - m*g = 0$$

The sum of the forces is zero at equilibrium, when the velocity is the terminal velocity. Given that the diameter of the stone is 5 cm and that the density of quartz is 2600 kg/m^3 it is easy to find the following,

$$A = 1.96 \times 10^{-3}\ m^2$$

$$V = 6.54 \times 10^{-5}\ m^3$$

$$m = 0.17\ kg$$

Plugging in the values yields a terminal velocity of 1.6 m/s (of the stone in water). Thus if the stone is dropped in air with V_o = 0 m/s then under the acceleration of gravity the distance it travels before reaching a velocity of 1.6 m/s is,

$$D = Vt^2/(2\ g) = 0.13\ m$$

So the stone should be dropped from a height of 0.13 m above the surface of the water.

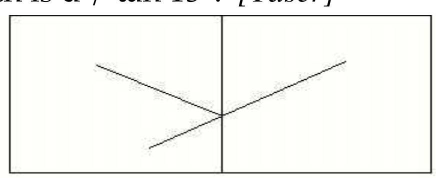
b) At terminal velocity the water-drag problem is trivial! After 6 seconds the rock has fallen, 6*(1.6 m/s) = 9.6 m. *[Carrie]*

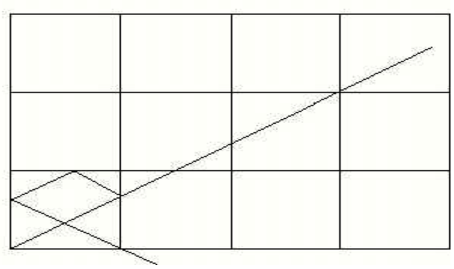
5) Light diversions

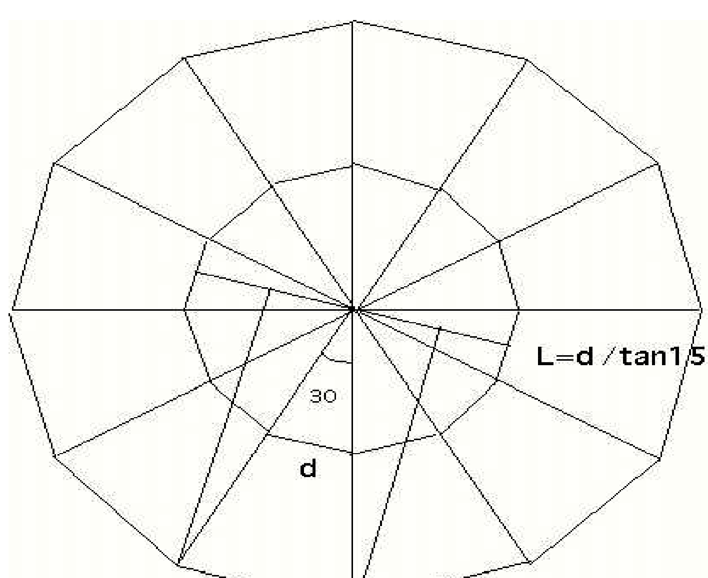
Consider a ray of light reflected from one of the mirrors. The reflected ray is a mirror image of the continuation of the incident ray. Therefore, an easy way to trace the ray path is to make another square beside it and follow the continuation of the incident ray in that square.

For more than one reflection , more squares should be drawn as shown. It is observed from the figure that as the continuation of the initial ray pass through any corners, it means that the light have gone out from the square. Therefore, the necessary condition would be $\tan(\theta) = m / n$ which m and n are two arbitrary integers.

Doing the first part, makes solving this part much easier. We have to build up some triangles which are image of one another. Since the cone angle is 30°, last triangle fir exactly the first one. Two regular polygons with 12 sides is produced. Analogous to condition that the rays leave through the upper hole after many reflections is that the continuation of the incident ray pass through the smaller polygon. After some simple mathematics, we figure out that D must be smaller than the width of the polygon which is d / tan 15°. *[Yaser]*







6) Physicists' pipe dreams

a) The water column will get narrower and narrower because the flow is constant and the speed is increasing, therefore the surface area of the water will decrease until it breaks up into droplets.

b) If we consider low viscosity for water, molecules of water are freely falling down.

$$v^2 - v_0^{\,2} = 2gh$$

$$v\pi r^2 = v_0 \pi R^2$$

$$r = R\sqrt{\frac{\nu}{\nu}}$$

$$r = R\,\{\,v_0^{\,2}/(\,v_0^{\,2} + 2gh)\,\}^{1/4}$$

c) The stream will break up into droplets. It cannot get arbitrarily skinny, the way the formula gives, because:

1 . The radius can't be less than the radius of a water molecule!

2 . Before you reach the limit (1) above, *surface tension* of water won't let you make it as skinny as possible. In fact, surface tension could affect the result which we obtained , but we are neglecting that!

Consider the droplets have radius *a* . What is the height of a same-volume cylinder of radius *r*? The energy of surface tension for these two area (as you could see in many books & specially at the end off this problem set :)

$$\frac{4}{3}\pi\, a^3 = \pi r^2 \Delta x$$

$$? \quad \Delta x = \frac{4}{3} a^3/r^2$$

The surface-tension energy of these two states (water tube & droplet) are :

$$\sigma\, 2\pi r\, \Delta x \quad \textit{and} \quad \sigma\, 4\pi a^2$$

substituting delta-x, we have and comparing them:

$$\sigma\, 8\pi \frac{a^3}{3r} \quad \text{and} \quad \sigma\, 8\pi \frac{a^3}{3r} \langle \frac{3r}{2a}$$

it means that, if 3r/2a is less than 1 , then the surface energy of droplets is less than that of a water cylinder, so it's energetically favourable for the water to break into droplets instead of staying in a cylindrical shape. What does determine the actual radius of a droplet? Several factors may affect that — for example the circular motion of water flow. But the most important one is density of water. It will break into droplets when the weight of the droplet has the same order of magnitude as the surface tension. It means:

$$\rho = 4\pi a^3 / 3\,g = 2\pi r \sigma$$

using previous result, we will get:

$$r = \sqrt{\frac{4\sigma}{4\rho g}} = 2 \leftrightarrow 10^{-3}\ \text{m} = 2\ \text{mm}$$

and then the height at which we reach this redious is:

$$h = v_0^2 / 2gr^4 \quad (R^4 - r^4)$$

d) For the experimental part, I just used a tap with diameter of 6 mm and tried to see at what height the water breaks up. After that, I fixed the height and measured the volume of the poured water and the time interval.

Here are my results:

Height of break up (cm)	**Volume (ml)**	**Time needed (s)**	$\mathbf{v_0 = V/(\pi R_0^2 T)}$
5-6 cm	100 ml	78	4.5 cm/s
7-10 cm	100 ml	62	5.7 cm/s
16-20 cm	200 ml	50	14.7 cm/s
26-30 cm	300 ml	61	17.4 cm/s

Due to large error I have, the experimental behaviour is not exactly the same as our expectation, but `h' is still proportional to v_0^2. *[Amir]*

1999-2000 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 2: Mechanics

Due December 17, 1999 (revised date)

1) Going for a spin

Cally is playing on the whirly-ride (like a merry-go-round, but non-motorised) in her neighbourhood park. To make it go faster she pushes against the ground using one of her feet.

a) If she accelerates the whirly-ride from rest to a speed of 15 r.p.m. in 3 seconds, what is the corresponding angular acceleration?

b) By what fraction does the angular velocity change when her 20 kg kid brother Matt leaps radially onto the edge of the whirly-ride? Assume the initial moment of inertia (just Cally and the whirly-ride) is 2000 kg m^2 and that the whirly-ride has diameter 5 m.

c) Since one of her all-time-favorite past times is tormenting her little brother, Cally decides to speed up the merry-go-round in an effort to fling her little bother from the ride. Assume that Matt is sitting at a radius of 4.5 m, that the coefficient of static friction between him and the floor of the whirly-ride is 0.7 and that he is naively not holding on! At what speed will Cally's little brother start to slip? . *[Carrie]*

2) Bubble bonanza

a) What is the presure inside a soap bubble of radius 'r' if the surface tension of the soap solution is σ?

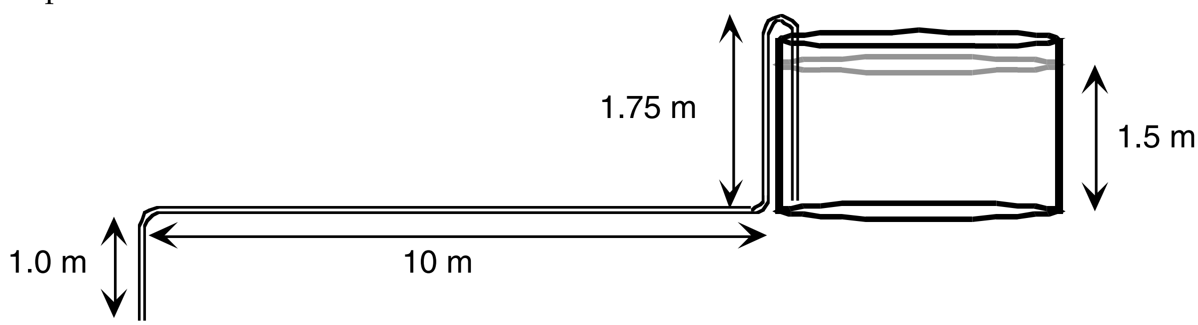
b) What will happen if two bubbles are brought to touch each other? (assume that one of them is bigger than the other, and there is still a bubble-film between them

c) What will be the end result if the wall between the the two touching bubbles breaks, and the two bubbles merge?

d) What if those two bubbles were attached to opposite ends of a tube (e.g., a drinking straw) like a weightlifter's dumbell? *[Amir]*

3) Siphoning cellar

My outside water faucets in my new house don't work, and I'm reduced to running a hose from the laundry tubs out my basement window to fill my above-ground swimming pool. If I forget and leave the hose-end in the pool, then when I disconnect my hose from the tap, water from the pool will siphon back into my basement. The setup looks like this:



Water in a 2 cm-diameter hose flows with a drag force F_{drag} on the hose which is proportional to the length of the hose and to the speed of water flow. This drag results in a pressure drop along the hose. This pressure drop, per unit length, is proportional to v by the formula:

$$\nabla p = \frac{\Delta p}{\Delta x} = \frac{32U\eta}{d^2}$$

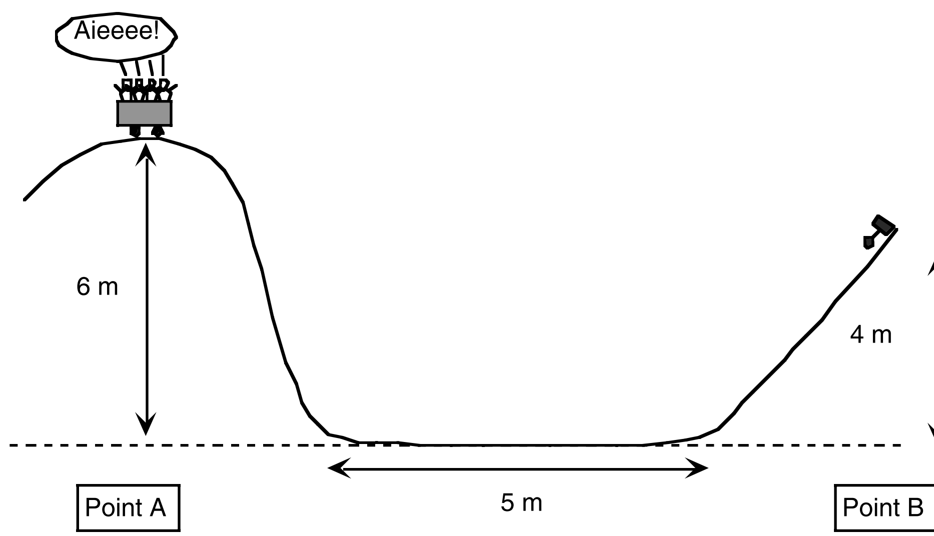
which is described more fully in INFOBITS™ below.

a) If I forget the hose when I leave for work, and it siphons for 12 hours while I'm gone, how much will the water level in the pool drop? Estimate, or calculate exactly.

b) When the last of the water drains out of the pool, then the water in the hose becomes of shorter and shorter length. This changes the drag on the water. See if you can approximately calculate the flow rate of water out of the hose as it empties. *[Robin]*

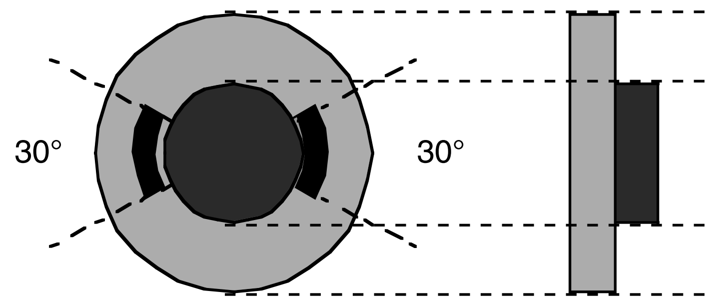
4) Gimme a brake...

The Bumble Bee ride at Super Fun Fun World consists of carts (painted yellow and black of course), which seat 4 people, that traverse a windy, hilly track. The figure below illustrates one section of the ride. On a particularly unusual Tuesday there is a massive power failure at Super Fun Fun World (which consequently isn't so fun fun). The power fails just as one of the carts nears point A with a velocity of 0.5 m/s.



a) If no emergency brakes are applied with what speed will the cart hit the bumper at point B?

Take it that you have *drum brakes*, which press on a hub which is half the wheel diameter. Each drum presses onto a 30° arc of the hub. The thickness of the hub is the same as the wheel. For steel wheels on steel rails, the coefficient of static friction is 0.78, and the coefficient of kinetic friction is 0.42. You can assume that only about 3 mm of the rail is in contact with the wheel.



b) At the bottom of the first hill the emergency brakes are applied radially to either side of the wheels. If the brake pads have a static coefficient of friction 0.85 and kinetic coefficient 0.65, what is the optimum force to apply to the brake pads such that the braking force is a maximum?

c) With the brakes applied (assume with optimal constant force) how far up the second hill will the cart go?

d) With the brakes held on, will the cart go back down the hill? *[Carrie]*

5) It's a ball? What a gas!

This question is the beginning of a kinetic model for an ideal gas. It can lead you to figure out the *ideal gas law* PV = nRT.

a) A ball with mass m and velocity v collides with a stationary wall. If friction is negligible, and bouncing is perfectly elastic, the ball bounces back with the same speed. But what happens if the wall, itself, moves toward the ball with velocity u? Consider all velocities be in the same direction, perpendicular to the wall.

b) Going through (a) you can find out that the momentum has been changed. This means that the wall did some work on the ball. Can you show explicity that the difference in kinetic energy of the ball is exactly the same as the work done by the wall?

BONUS: Can you use this model to show how much the temperature of a gas of atoms behaving like such elastic balls goes up as it is compressed — as long as no heat is allowed in or out? *[Yaser]*

6) Dimensional thinking

A cylindrical vessel of water is leaking via a small hole of radius 'r'. We want to keep the height of the water constant. The best way is to add water at some rate R (kg per second).

a) Can you guess what should be R in terms of ρ (the density of water), H (height of the water level in the vessel), r (radius of the hole), g (acceleration due to gravity) and A (the cross-sectional area of the vessel). [HINT: Use your intuition, or common sense, and watch that you keep the correct units in each stage (i.e., acceleration has units $L\ T^{-2}$, where L is distance and T is time)]

b) Can you prove your formula with physics theorems?. *[Amir]*

INFOBITS™ — Useful Bits of POPTOR Information

Coefficient of resistance for pipe flow:

There is a drag on the inside of a pipe, for a liquid flow through it. This drag means that there's a pressure drop as you go along the pipe — it is why small plumping-pipes lead to poor 'water pressure' for a shower or similar. The pressure gradient (change in pressure per unit length) is:

$$\nabla p = \frac{\Delta p}{\Delta x} = \frac{\lambda}{d}\frac{1}{2}\rho U^2$$

where p is pressure, ρ is fluid density, d is pipe diameter, U the average speed of liquid flow, and λ is a dimensionless constant which is equal to:

$$\lambda = \frac{64}{R_e} \quad \text{for } \textit{laminar} \text{ (smooth) flow}$$

R_e is called the *Reynold's number*. (usually a value of a few 1,000 for laminar flow) The Reynolds number for a pipe can be found from:

$$R_e = \frac{\rho\, U\, d}{\eta}$$

putting all this together, we see that:

$$\nabla p = \frac{\Delta p}{\Delta x} = \frac{32 U \eta}{d^2}$$

CHECK THE POPTOR WEB PAGE for other hints, and any corrections we might post:
www.physics.utoronto.ca/~poptor

1999-2000 Physics Olympiad Preparation Program

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Solution Set 2: Mechanics

1) Going for a spin

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{15 \cdot 2 \cdot \pi / 60}{3} = 0.5 rad/s^2$$

Angular momentum is conserved (but energy is not!!)

$$I_o \omega_o = (I_o + mr^2)\omega$$

$$\frac{\omega}{\omega_o} = \frac{2000}{2000 + 20(2.5)^2} = 0.9\%$$

Ignoring the fact that Matt is sitting off the whirly-ride, what happens is that the force of friction is providing the centripetal acceleration necessary to keep him on the ride. The force is $\leq \mu mg = 137.2N$.

Thus, at maximum,

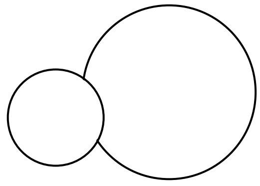
$$\frac{v^2}{4.5} = 0.7 \cdot 9.8,\ v = 6\ \ m s^{\pm 1} \text{ and } \omega = \frac{v}{4.5} = 1\ \ rad s^{\pm 1}$$ *[Carrie]*

2) Bubble bonanza

a) Consider 2 hemispheres of a bubble of radius r. They are repelled by a force of $F = \Delta PA = \Delta P\pi r^2$, where A is the area of the projection of half a sphere and $\Delta P = P_o - P$ (difference between outer (atmospheric) and inner pressure). The hemispheres are pulled together by surface tension (which is given as force per unit length); moreover, there are 2 surfaces to a bubble, so the total force here is $F = 2(2\pi r\sigma)$. Solving yields

$$P = P_o + \frac{4\sigma}{r}$$

b) Note that pressure is inversely proportional to the radius. Thus, the smaller bubble will actually have a higher pressure inside! This means that the film between the 2 bubbles will be pushed into the larger bubble.



If the film breaks, the bubbles will either break (but that's so boring), or merge. Using the fact that the numbers of moles inside the bubble stays constant, assuming that temperature stays constant and using the ideal gas law, one gets

$$PV = P_1V_1 + P_2V_2$$

$$\left(\frac{4\sigma}{r} + P_o\right)r^3 = \left(\frac{4\sigma}{r_1} + P_o\right)r_1^{\,3} + \left(\frac{4\sigma}{r_2} + P_o\right)r_2^{\,3}$$

(P, V, r refer to the final bubble, the remaining variables refer to the 2 initial bubbles)

The equation could in theory be solved; note that for $P_o = 0$, the final bubble will be bigger than either of the 2 initial bubbles…

The system is obviously not in equilibrium. Since the smaller bubble is at a higher pressure, it will in fact contract and force air into the bigger bubble. Thus, we will end up with a membrane at one end and a big bubble at the other. *[Peter, Amir]*

3) Siphoning cellar

Solutions for this problem will be provided with Set #3 *[Robin]*

4) Gimme a break...

First I have to comment that this question was so hard that nobody got it right, including some of us here at POPTOR…

a) Ok, that was easy. Conserving energy yields

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m(0.5)^2 + mg(6-4)$$

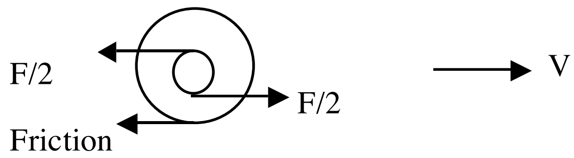
From this, v = 6 m/s.

b) This is where things get difficult. Clearly we want to stop the cart using static friction, as these coefficients are higher than those for kinetic friction, and thus the force will also be larger. Something has to slip, however (you can't just stop the cart dead) – since the coefficient of kinetic friction between the wheels and the rail is lower than that between the brakes and the wheel, we want to avoid letting the wheels slip (it will yield the lowest force).

Now, if we have an object of mass m lying on a level plane, where the coefficient of static friction is μ it certainly is true that the maximum force we can apply so that the object doesn't start moving (and kinetic friction doesn't kick in) is $F_{MAX} = F_{friction} = \mu mg$.

But a rolling object is different!! For one, it is not stationary, it is already moving. So what force can we apply? Well, we have to ensure that the cart keeps on rolling, i.e. V(t) = ω(t)R, and not slipping…

I am also going to assume that there are 2 forces acting on the wheel – each of magnitude F/2 - pointing in different directions.



We have to analyze the linear and rolling motions (about the axle) while the brakes are applied, knowing we start with the cart rolling (m is the mass of the cart, M the mass of the wheel, I the moment of inertia of, F the applied force, R the radius of the wheel and μ the coefficient of static friction between the wheel and the track).

$$ma = -\mu mg$$

$$I\alpha = \mu mgR - FR$$

From this we find

$$v(t) = v_o + at = v_o - \mu gt$$

$$\omega(t) = \omega_o + \alpha t = \omega_o - \frac{F/2 \cdot Rt}{I} + \frac{\mu mgRt}{I}$$

We have that $v_o = \omega_o R$ at t=0, but to ensure that this is true at later times, we clearly need that

$$\mu g = \left(\frac{F/2 \cdot R}{I} - \frac{\mu mgR}{I}\right)R$$

The moment of inertia is that of the wheel, *i.e.*, of 2 cylinders, one of radius R and the other of radius R/2. Thus, $I = \frac{1}{2}MR^2 + \frac{1}{2}M\left(\frac{R}{2}\right)^2 = \frac{5}{8}MR^2$ (I've assumed for simplicity that both sections of the wheel weigh the same amount; it would probably make more sense to assume that their densities are the same, but oh well…)

$$F/2 = \frac{5}{8}\mu Mg + \frac{8}{5}\mu mg \approx \frac{8}{5}(0.78)mg$$

(assuming the wheels are not very heavy)

This is the force felt by the wheel; the actual applied force is

$F = F_a(0.65)$ (the brakes have to be slipping)

Solving for the applied force gives

$$F_a = 2 \cdot 18.8m = 38m$$

NOTE: most people put that $F = 2\mu mg$, which is less than the above. Plugging this into the equations above we can see that $V(t) \neq \omega(t)R$, and so it seems that the cart slips… but you can do some experiments at home to convince yourself otherwise. What's going on here? Well, the force of friction is actually $F_{friction} \le \mu mg$, and what will in fact happen is that it will adjust itself to a lower value so that the cart doesn't slip… The above force is the maximum that can be applied so that the cart doesn't slip; a bigger force and it's all over…

c) The above force does work ($W = F_a d$) to slow the cart down. It starts with an energy $\frac{1}{2}m(0.5)^2 + mg6$. After a distance d it will stop; here, d turns out to be 1.6 m. Thus, the cart never even makes it to the hill…

d) If the cart was on the hill, we need it to be in static equilibrium. From the above diagram we see that this amounts to $mg\sin\theta \le 0.78mg\cos\theta$, or $\theta \le 38^o$. Eyeballing the diagram, this seems to be false, and thus the cart will probably fall… *[Peter]*

5) It's a ball? What a gas!

a) Doing the question in the reference frame of the wall really simplifies things. In the rest frame of the wall, the ball velocity is the same before and after the collision but opposite in the direction. In this system, the ball's initial velocity is v + u. So the ball bounces back with v + u in reference to the wall. This velocity is v + 2u in reference to the laboratory.

b) The change in kinetic energy = $\frac{1}{2}m(v+2u)^2 - \frac{1}{2}mv^2 = 2mu(u+v)$

The work done by the wall = Fx

But $F = \dfrac{\Delta p}{\Delta t} = \dfrac{m(v+2u-(-v))}{t} = \dfrac{2m(u+v)}{t}$; $x = ut$

Therefore work = $Fx = 2mu(u+v)$, same as the change in kinetic energy above…

<u>BONUS</u>: (solution based on Feynman's); "< >" denotes an averaging

Let us work in 1D (results in 3D are very similar); let ρ be the density (#) of atoms per unit volume.

Assume the compression is slow (so the gas stays in equilibrium). Then, the wall is practically stationary. The force exerted by the wall = change in momentum / Δt is

$$F = 2\rho mv^2 A$$

We want to average this over all the speeds, v. Notice that only _ of all atoms (balls) are moving toward the wall – we do not want to average over them, as they do not contribute to the force on the wall. This yields

$$P = F/A = \rho < mv^2 >$$

Multiplying by the volume, V gives,

$$PV = N < mv^2 >$$

In 1D we have that $\frac{1}{2}kT = \frac{1}{2} < mv^2 >$ and thus

$$PV = NkT$$

If you want to know how temperature changes with volume, you can note that this is an adiabatic process and look up the formula in any book (or derive it using the above). *[Peter, Amir]*

6) Dimensional thinking

a) First of all, figuring out how much water to add is equivalent to figuring how much water is actually leaving. There a couple of ways to do this, but dimensional analysis is perhaps the most instructive. First, since the hole is small, the amount of water flowing out in a short time will change the volume in the vessel by very little; thus, the area A is irrelevant.

Now, we need to combine the remaining variables into a flow rate R that has units [kg/s]. We now assume that the formula is of the form

$$R = C\rho^x H^y r^z g^w$$

The units must combine to [kg/s]. It's pretty easy to see that x=1 and w=1/2. Now, it would seem logical that the flow is proportional to the area of the hole, *i.e.,* r^2. Since the flow also grows as H to some power (not as inverse of H), the only logical choice is to set z=2 and H=1/2.

Thus, $R = C\rho r^2 \sqrt{gH}$, where C is some constant.

b) Using Bernoulli's Law to find the flow rate of the water leaving the vessel leads one to

$$\tfrac{1}{2}\rho v^2 = \rho g H + \tfrac{1}{2}\rho U^2$$

Here, v is the speed of flow in the pipe and U is the speed of water flow in the tank. For a large area A (compared to the area of the hole), the U term is negligible. However, we don't have to make this assumption. Since the flow is steady, the continuity equation applies and we have $UA = v\pi r^2$. The rate R is $R = \rho v \pi r^2$. Solving gives

$$R = \rho\pi r^2 \frac{\sqrt{2gH}}{\sqrt{1-\left(\frac{\pi r^2}{A}\right)^2}}.$$

(for $A >> \pi r^2$, this is indeed equivalent to a), if $C = \pi\sqrt{2}$) *[Peter, Yaser]*

1999-2000 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 3: Thermodynamics

Due January 14, 2000

1) Clocks a' Rockin'

A friend of mine knows someone who *overclocked* his computer, that is, he made it run faster than it was originally designed to. The entire system remains the same, except now twice as much data is transferred per unit time, as before. Thus, in effect, the current has doubled. My friend claims that the new system is so hot, that the cooling system's temperature has to be around freezing.

Consider a resistor R, placed in surroundings with room temperature 20 °C, which has been overclocked as described above. If the original system causes a temperature rise of 13 °C, what temperature will the new system heat up to? *[Peter]*

2) Literally linear

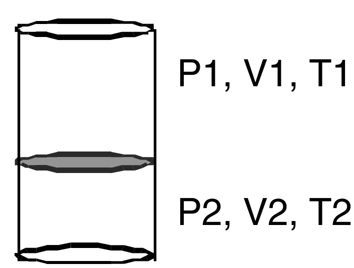
Consider a rod with the *linear coefficient of expansion* λ and initial length of L. It starts out at temperature T_1.

a) Heat the rod to a temperature T_2 — what is its new length? Then cool it back to its initial temperature T_1.— what is the length of the rod after this cycle, assuming the usual formula for linear expansion and contraction? Go through the calculation in a little detail — don't just guess, or use common sense.

b) Do this heating and cooling, again and again for *n* times. What is the length of the rod now? Does your answer seem strange? (say 'yes!') Can you explain (or guess!) what has made the answer weird? You may think that this happens because we assumed that the linear heat expansion is the same in all temperatures. This is not the issue we're looking for. Try to find another reason. *[Yaser]*

3) We will break dividing walls

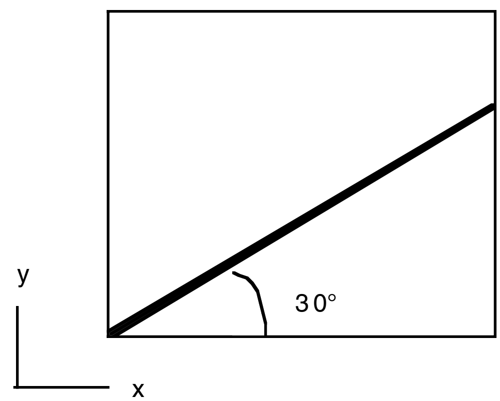
A thermally insulated cylinder, like a vacuum flask, contains a single species of ideal gas in two separate compartments, divided by an insulating wall. The pressure, volume and temperature of gas in each part are : P1, V1, T1 and P2, V2, T2.



What is the final P, V and T of the system, if we remove or break the separating wall? *[Amir]*

4) The nonaligned axis

There are special crystals which have different coefficients of linear heat expansion along two different directions, x and y. Say that the coefficients have values Le_1 and Le_2 in the directions x and y. Such crystals do exist, and are termed *anisotropic*.



Consider a square-sided rod cut from a block of such a crystal, with the rod's longitudinal axis making an angle of 30° with the x axis. What is the linear heat expansion of this rod along its length? (Ignore the third dimension) Does this rod expand properly, or does it somehow bend? *[Yaser]*

5) Patent, or pshaw?

Here's an idea: venetian blinds which are painted black on one side and silver on the other. Assume that the blinds are installed between two panes of glass, with vacuum in between (so air and its convection has no effect in this question), and assume that the reflectivity of the silver side is 90%, whereas that of the black side is 10%.

a) How well does it work to turn the silver side of the blinds outward when the room is already too hot, so as to reflect the light, while turning the black side outward to absorb heat when the room is too cold? Take it that sunlight is 1kW m^{-2}, and find the steady-state temperatures due just to thermal radiation (Stefan-Boltzmann law).

b) Is there any change you can see that would improve the blinds' function to control radiation of heat into the room? *[Robin]*

6) Tea? For who?

Berto used an electrical heater to heat 3 kg of an unknown liquid in a container (don't ever try heating any unknown liquids at home, though!). The heater delivers 1000 watts of power. The initial temperature of the liquid is 20 °C. As the liquid warmed up, Berto wrote down its temperature in different times. In Tables 1 & 2 below, some of the data collected by him are shown.

a) Neglecting the heat capacity of the container, find the specific heat of the liquid.

b) Berto turned off the electrical heater when its temperature reached 60 °C. Then, he started writing down the liquid temperature as it was cooling . You can see in the table, below, the liquid temperature for some different times. Using these data, try to find the heat capacity of the liquid to a better approximation.

Hint: You should consider the loss which is the heat transfer between liquid and room in your calculation in part (a).

c) Sir Isaac Newton, in the 17th century, showed that objects cool at a rate proportional to the *difference* between their temperature and the ambient temperature, such as room temperature. Try to find the proportionality constant for (b).

Table 1: *Liquid temperature during heating*

time(min)	0	3	4	5	6
temperature (°C)	20	34.1	38.5	42.9	47.2

Table 2: *Liquid temperature during cooling*

time(min)	0	8	13	19	26
temperature (°C)	60	50.2	45.1	39.8	35.2

[Yaser & Carolyn]

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Specific heat of water: 4.2 kJ kg^{-1} K^{-1}
Stefan-Boltzmann constant $\sigma = 5.7 \times 10^{-8}$ W m^{-2} K^{-4}
Solar constant = 1.353 kW m^{-2} ; at surface of the earth, about 1 kW m^{-2}

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Solution Set 3: Thermodynamics

1) Clocks a' Rockin'

First, I have to say that this was not an easy question — probably way too hard for #1. The key is to realize that the resistor is radiating energy with a temperature T, with a background temperature 20°C. Most radiating objects behave more or less like black bodies, and I will assume this to be the case here as well …

Running at input power P, the computer's resistor reaches a steady state temperature T at which the input power is balanced by thermal radiation power (from the Stefan-Boltzmann law). Since all objects radiate thermally (unless they're at zero temperature), the resistor also absorbs a little bit of thermal radiation from the room-temperature surroundings at $T_o = 20°$ C or 293 K. So the power balance is:

$$\mathrm{P} = \sigma\{(T)^4 - (T_o)^4\}$$

When the resistor is "overloaded", the power (P′) goes up by a factor of $I^2 = 4$ (*i.e.*, $P = I^2R$, where R is constant). The new temperature T' will radiate more power to balance this.

$$\mathrm{P}' = 4 \cdot \mathrm{P} = \sigma\{(T')^4 - (T_o)^4\}$$

thus, dividing these equations:

$$\frac{(T')^4 - (T_o)^4}{(T)^4 - (T_o)^4} = 4$$

The question tells us that the temperature rises by 13° C, so we know:

$$T' - T = 13 \text{ K}$$

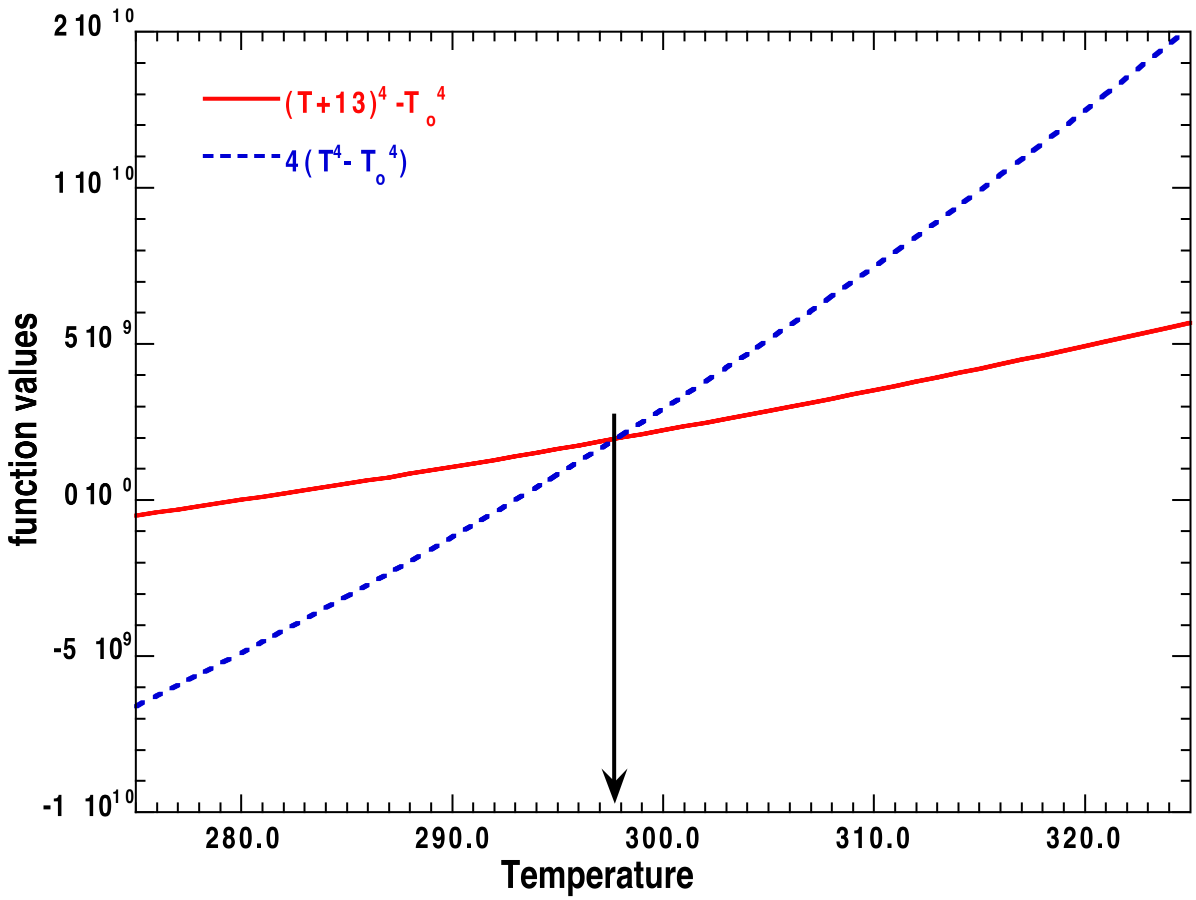
therefore,

$$\frac{(T+13)^4 - (293)^4}{(T)^4 - (293)^4} = 4$$

One way to solve this is to do it graphically, plotting the left and right side of:

$$(T+13)^4 - (293)^4 = 4\left\{(T)^4 - (293)^4\right\}$$

and seeing where they cross. This isn't hard to do, with a computer. Or you can use a spreadsheet program to try a whole lot of guesses.



The solution is T = 297.8 K, or normally about 5° C above room temperature. With the overclocking, the temperature is therefore about 18° C above room temperature, or 38° C.

If we wish to cool the system down to ~20°C (room temperature), and our cooling system is the same size as the resistor, the cooling system all by itself has to be at $T_c \sim 0°C$, for then

$$\frac{0+38}{2} \approx 20^\circ\mathrm{C}$$

so the cooling system is indeed at 0° C when the resistor (actually the processor) is off.

In case you are curious, this is a true story — you may even verify it yourself (I would suggest using someone else's computer, though …).

2. Literally linear

a) Let $T_2 - T_1 = \Delta T > 0$

(1st) heating:

$$L_1 = L(1 + \lambda\Delta T)$$

(2^{nd}) cooling:

$$L_2 = \left[L(1 + \lambda\Delta T)\right]\left[1 - \lambda\Delta T\right] = L(1 - \lambda^2\Delta T^2)$$

Every cycle (heating and then cooling the length of the rod is multiplied by $(1 - \lambda^2\Delta T^2)$

$\therefore$ after n cycles $L_{2n} = L(1 - \lambda^2\Delta T^2)^n$

(this can be proved by induction, if you like)

$\lambda\Delta T$ better be less than 1 (otherwise the length of the rod becomes negative), and hence:

$$\lim_{n \to \infty} (L_{2n}) = 0$$

This clearly isn't right - whatever happened to conservation of mass???

The explanation is that this is actually a differential statement ($dL = L \cdot \lambda \cdot dT$) valid only for tiny ΔL and ΔT. (otherwise "L" in the equation is not actually constant).

Because of this, only 1^{st} order terms (i.e., ΔL and ΔT) are important — 2^{nd} order (ΔL^2, ΔT^2) should be ignored. Hence, $1 - \lambda^2\Delta T^2 \approx 1$ and there is no problem.

If λ were constant, one could write:

$$\frac{dL}{dT} = L\lambda$$

and solve this:

$$\frac{dL}{L} = \lambda\, dT$$

$$\int \frac{dL}{L} = \int \lambda dT$$

$$\ln(L) = \lambda(\Delta T) + C$$

$$L = L_o\, e^{\lambda\Delta T}$$

(you can check for yourselves that the length of the rod is indeed conserved after a cycle when using this formula)

Expanding this yields:

$$L \approx L_o\,(1 + \lambda\Delta T + (\lambda\Delta T)^2 + \ldots)$$

and ignoring 2^{nd} order terms and higher gives:

$$L \approx L_o\,(1 + \lambda\Delta T)$$

Unfortunately, λ varies with T (among other things), and this is not really true …

3) We will break dividing walls

Clearly, $V = V_1 + V_2$ (1) ("conservation of volume")

and $n = n_1 + n_2$ (2) ("conservation of mass")

The vessel is insulated, and hence energy has to be conserved.

$$
\begin{aligned}
E &= \alpha nkT \\
&= E_1 + E_2 \\
&= \alpha n_1 kT_1 + \alpha n_2 kT_2 \\
&\Rightarrow nT = n_1T_1 + n_2T_2 \qquad (3)
\end{aligned}
$$

since $PV = nRT$, we get from (3):

$$PV = P_1V_1 + P_2V_2$$

$$\text{and } P = \frac{P_1V_1 + P_2V_2}{V_1 + V_2}$$

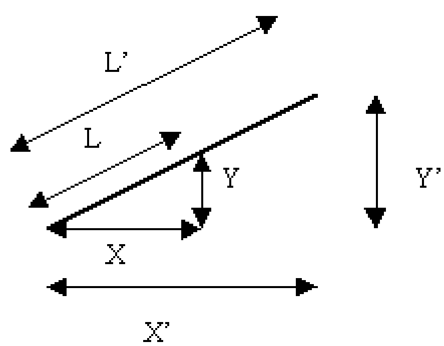
using 2 yields:

$$\frac{PV}{T} = \frac{P_1V_1}{T_1} + \frac{P_2V_2}{T_2}$$

$$\frac{(P_1V_1 + P_2V_2)}{\frac{P_1V_1}{T_1} + \frac{P_2V_2}{T_2}} = T$$

4) The nonaligned axis

The x and y directions expand independently.



$x' = x(1 + Le_1 \Delta T)$

$y' = y(1 + Le_2 \Delta T)$

$x = L \cos(30°)$

$y = L \sin(30°)$

Assuming the rod expands straight:

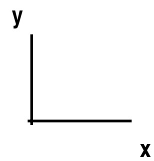
$L' - L = L\alpha\Delta T$

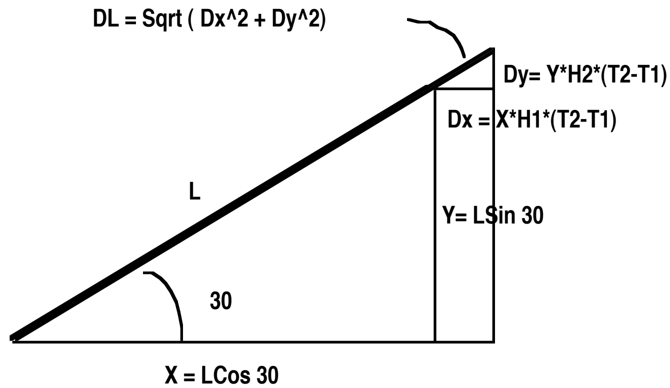
and $(L' - L)^2 = (x' - x)^2 + (y' - y)^2$

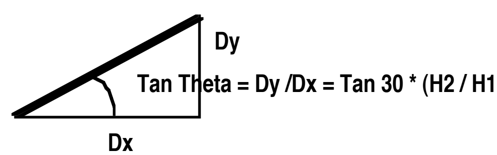
$\therefore\ (L' - L)^2 = L^2 \, (Le_1^{\,2} \cos^2(30) + Le_2^{\,2} \sin^2(30)) \, (\Delta T)$

And hence $\alpha^2 = Le_1^{\,2} \cos^2(30) + Le_2^{\,2} \sin^2(30)$

In fact, from the diagram we can see that the expanded part of the rod is at a different angle to the horizontal, and thus the rod must bend to compensate.

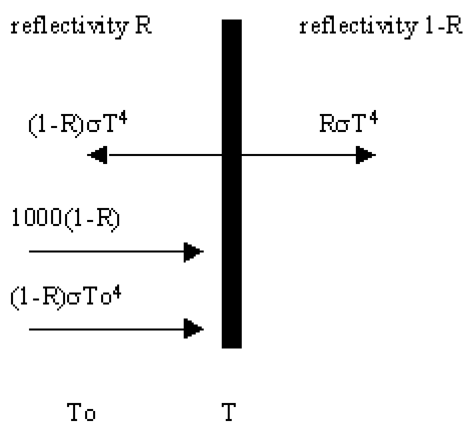






5) Patent, or pshaw?

Here's the situation:



In steady state, nothing is either heating up or cooling down. So the net power being absorbed as radiated heat must equal the power being emitted as radiated heat (we're ignoring convection and conduction here).

The blinds absorb the sunlight, certainly. More subtly, though, there is radiated energy from the room-temperature room inside, and also radiated energy from outdoors — because neither has zero temperature! So all of these go into the absorbed power, keeping track of what the absorption coefficient is on each face.

Let the 'outside reflectivity' be written as R, regardless of whether it's the silver or black side. Likewise, let the 'inside reflectivity' be r, regardless. The absorption is then $(1-R)$ or $(1-r)$.

outside reflectivity = R outside temperature = T_{out}
inside reflectivity = r inside temperature = T_{in}
incident sun power = 1000 W cm^{-2}
area of each side of blinds = A

Absorbed power = absorbed sunlight + absorbed outdoor radiation + absorbed indoor radiation

$$= 1000\,(1-R)\ \mathrm{W\,cm^{-2}} \bullet A + (1-R)\,\sigma\, T_{out}^{\ 4} \bullet A + (1-r)\,\sigma\, T_{in}^{\ 4} \bullet A$$

Radiated power

$$= e_{out}\,\sigma\, T_{blinds}^{\ 4} \bullet A + e_{in}\,\sigma\, T_{blinds}^{\ 4} \bullet A = (1-R)\,\sigma\, T_{blinds}^{\ 4} \bullet A + (1-r)\,\sigma\, T_{blinds}^{\ 4} \bullet A$$

where e is the *emissivity* of the object, and equals the absorption a ($= 1-R$ or $1-r$). Both sides of the blind radiate, of course.

These two powers must balance:

$$1000\,(1-R)\ \mathrm{W\,cm^{-2}} \bullet A + (1-R)\,\sigma\, T_{out}^{\ 4} \bullet A + (1-r)\,\sigma\, T_{in}^{\ 4} \bullet A$$
$$= (1-R)\,\sigma\, T_{blinds}^{\ 4} \bullet A + (1-r)\,\sigma\, T_{blinds}^{\ 4} \bullet A$$

and we solve for the temperature of the blinds (when they're hotter, they're sending more heat to the room, of course).

$$T_{blinds} = \left(\frac{1000(1-R)/\sigma + (1-R)T_{out}^4 + (1-r)T_{in}^4}{(2 - \mathrm{R} - \mathrm{r})} \right)^{1/4}$$

Then when we have the black side outward $R = 0.1$, and $r = 0.9$; when the silver side is outward, $R = 0.9$, and $r = 0.1$. So we get two different temperatures for the blinds. If we take the indoor temperature to be 20° C, and outdoors to be 0° C, we find:

silver side out: $T_{blinds} = 307.5$ K or 34.4° C

black side out: $T_{blinds} = 383.0$ K or 109.9° C

The ratio of absolute temperatures is 1.25 — 25% hotter (Kelvin) with black out.

Interestingly, if we ignore the room and outside temperature, we get the answers we'd have if both were *zero* — the answers we'd have basically in deep space!

silver side out: $T_{blinds} = 204.7$ K or –68.5° C

black side out: $T_{blinds} = 354.5$ K or 81.3° C

The ratio of absolute temperatures is 1.7 — 70% hotter (Kelvin) with black out. However, the temperature is actually much lower in this case, just because the blinds are sitting in 'deep space at zero degrees'.

This isn't the whole story, though! Even if the blinds are hotter with the black side outwards, how efficient are the blinds at radiating into the room from the *silver* side of them?

silver side out (black side in): power/m^2 into room:

$$= (1\text{–}0.1)\ \sigma T_{blinds} = 0.9 \bullet 5.7 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}} \bullet (307.5\ \mathrm{K})^4 = 459\ \mathrm{W\ m^{-2}}$$

black side out (silver side in): $T_{blinds} = 383.0$ K or 109.9° C

$$= (1\text{–}0.9)\ \sigma T_{blinds} = 0.1 \bullet 5.7 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}} \bullet (383.0\ \mathrm{K})^4 = 122\ \mathrm{W\ m^{-2}}$$

Though the temperature is higher, the temperature factor of 1.25 leads to an increase of 2.44 in T^4. The emissivity factor, however, is 1/9th — more important than the temperature. So, really, the *silver side out* leads to more power radiated into the room! This offers plenty of control over the temperature – so the patent is indeed working, just not the way I thought it might!

b) To make them work even better: you could see what happens when you make the black even blacker and the silver more reflective; you could use another set of blinds set up parallel, too — with both silver sides facing outwards, the effect would be even greater (can you calculate this? What then do you do when you put the outside black blinds outwards? Do you do the same with the inside blinds or do you leave them open, neither side outwards?) You could also use a fan, so you could convectively cool the blinds when they are black side out, and put the heat into the room more efficiently. Lots of answers are possible… *[Robin]*

6) Tea? For who?

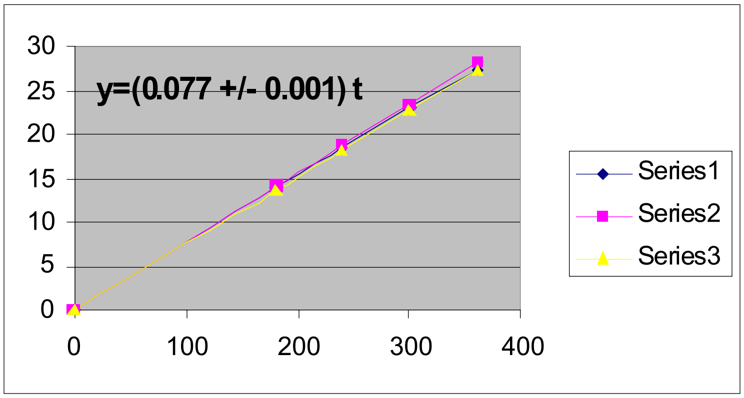
The best way to analyze data is to plot it, whenever possible.

a) Assume no power loss. Then,

$$Power = \frac{dQ}{dt} = 1000 = mc\frac{dT}{dt}$$

Solving, $1000t = mc(T(t) - 20)$. Plotting T(t)-20 versus t (the time in seconds) should yield a straight line with slope 1000/(mc). Note that the line <u>has</u> to pass through (0, 0), as when t=0 we know T(t) is exactly 20°C!

Many commercial packages (e.g., Excel) can fit data, but they will not give you an error-estimate on the slope. Instead, it's best to plot the data yourself, and then fit the steepest and shallowest lines that still fit the data. The best fit is then the average of the two slopes, and the error is the slopes of the steepest line minus the average slope. I've done this in Excel; see below

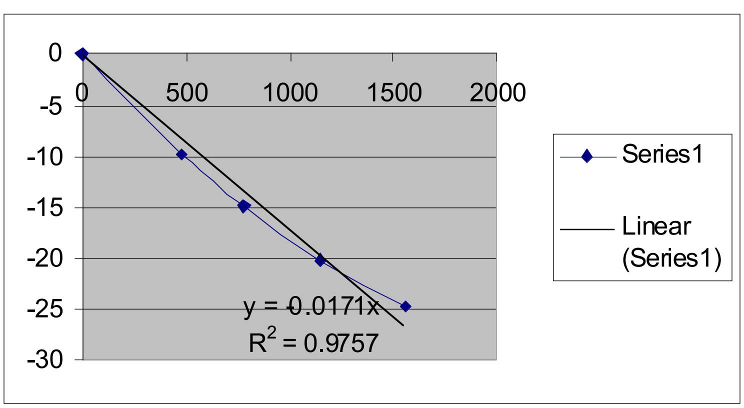


We thus have $\frac{1000}{mc} = (0.077 \pm 0.001)$ and hence $c = (4330 \pm 60) J/(kg^oC)$. We have used the fact that $\frac{\Delta c}{c} = \frac{0.001}{0.077}$ where Δc is the error in c. Note that all errors are rounded off to 1 significant figure, and all results are rounded off to the same decimal place as their errors.

b) The system clearly loses power. This means that our value for c is too high, as we assumed that the liquid absorbed all 1000 W of power, when it in fact did not. Let's assume that the system loses power at a steady rate. Our equations become

$$(1000 - L)t = mc(T(t) - 20)$$

and $-Lt = mc(T'(t) - 60)$, where T'(t) is the temperature during cooling. Plotting this data shows that it is <u>not</u> in fact a straight line (so L is not a constant), but we won't worry about this here. Plotting and fitting the data as in a) yields



(I have only plotted the average slope above).

This yields a value c of $(3600 \pm 100) J/(kg\,^{o}C)$

c) We have that $\frac{\Delta T}{\Delta t} = -k(T'(t) - 20)$. This may be solved, but calculus is required. The easiest way to proceed is probably to plug in all values for T'(t), find a bunch of k's (4, in fact) and then average them. Doing this gives $k \approx 5 \cdot 10^{-4}\ \ 1/s$. This value could be used again to obtain a revised value for c; it will not be much different from the current value, however.

For completeness, let us solve the above equation.

Let $x = T - 20$; hence, $\frac{dT}{dt} = \frac{dx}{dt}$ and our equation reads $\frac{dx}{dt} = -kx$. We also know that x = 40° C at t = 0. Rewriting the above and integrating both sides gives

$$\int \frac{dx}{x} = -k \int dt \text{ or } \ln(x) = -kt + C$$

To make C fit our initial condition we finally arrive at

$$T - 20 = 40 \exp(-kt)$$

This can be fitted e.g. by taking ln() of both sides – plotting ln(T-20) versus t yields a straight line with slope 0.0006 1/s…

Since the Carnot refrigerator is the most efficient, therefore we conclude that this value is a minimum for the required power. *[Peter & Yaser]*

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Problem Set 4: Optics and Waves

Due February 25, 2000 (revised date)

1) Bob-bob-bobbin' along

A *terrifically* important concept in waves and oscillations is that of the *harmonic oscillator*. We want to use some basic principles of the harmonic oscillator in later POPTOR questions, so have a look here at the basics.

The 'generic' harmonic oscillator that you'll see in university, and forever after that on into graduate school and beyond, is a mass resting on a frictionless table and attached to a spring. A property of springs, for small stretches or compressions, is that the force exerted by the spring is proportional to the amount of stretch. This is called a *linear restoring force* and the force is represented $F = -k\,x$, because the force tends always to restore the position of the mass to its original neutral position (unstretched spring).

a) Write the force equation for such a mass-on-a-spring (inertial force and restoring force). Since acceleration is the time-derivative of velocity, and velocity in turn the time-derivative of position, this actually is a 'second-order linear differential equation'.

b) You can do pretty simple derivatives. Suppose that $\sin(\omega t)$ is a solution of this particular differential equation. For that to work, what must be necessary?

c) So, what is the rate of oscillation of a 100g mass on a spring that has a spring constant of 80 N m^{-1}? *[Robin]*

2) Keeping up an image

The distance between a screen and a light source lined up on a table is 120 cm. Moving a lens between them, sharp images can be obtained at two different positions, producing two different images. The ratio of *sizes* of these two images is 1:9.

a) What is the focal length of the lens?

b) Which image is the brighter one? Determine the ratio of the brightness of these two images. *[Gnädig/Honyek]*

3) Shifty radar

Officer Smith and Officer Wesson are on patrol monitoring speeding along Hwy 401. They are using a hand-held radar gun which is set to detect the speed of approaching cars, sending out waves at a frequency of 1000 MHz. After several hours of speed monitoring Wesson turns to Smith and asks, "How does this thing work anyway?" To which Smith replies, "Well now, are you familiar with the Doppler effect?"

a) Finish Smith's explanation.

Suddenly the two officers are distracted from the conversation, coffee and donuts by the sounds of tires screeching and horns honking. They look up to see a milk truck racing down the 401. Smith gives Wesson the eye and Wesson fires the radar gun.

b) If the observed frequency difference is 330 Hz, how fast is the milk truck going? *[Carrie]*

4) To air is human

A cylinder of radius r contains n moles of a monatomic ideal gas. A tight-fitting piston of total mass M is fitted into the cylinder, sealing it. The room temperature is T. Consider that the cylinder is really well thermally insulated, so all changes are 'adiabatic'.

a) What is the new resting height h (height of the piston from the base of cylinder)?

Now, we attach or add a small mass m to the piston at rest. As you may guess, it'll start to oscillate.

b) If the piston's oscillations damp out (e.g.., due to friction) what would the final h be?

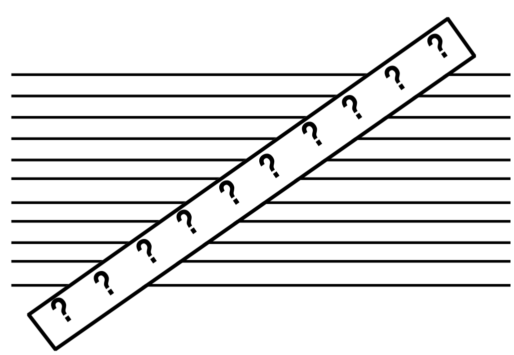
Suppose the motion is *not* damped, and that height from (b) is the equilibrium height of piston and the piston will oscillate around that equilibrium point.

c) Each oscillator can be modelled by a system of mass and spring. What is the corresponding mass and spring coefficient for this example?

d) Find the minimum amount of h in this oscillation. *[Amir]*

5) Bending the truth

Here's an actual test used to find the index of refraction of a material: Consider a piece of paper with a drawing of a group of parallel lines. Consider also a polished cylindrical rod made of some unknown glass (for which we want to find the index of refraction). Set the rod down on the paper, at some angle θ such that $0 < \theta < \pi/2$.

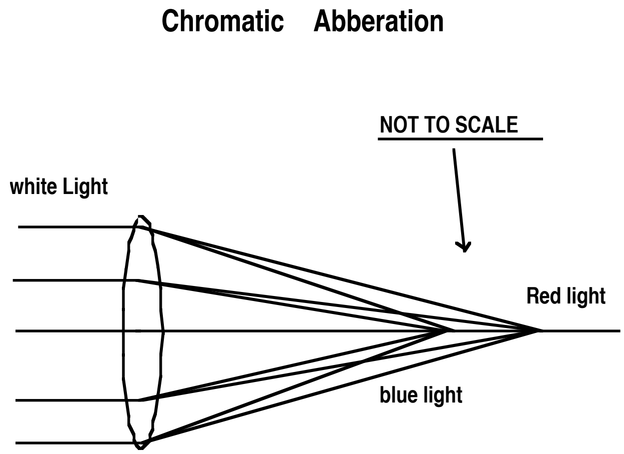


a) How will the lines appear as viewed through the glass rod? Make a sketch.

b) Find the index of refraction of the glass, using details from what you see in part (a). *[Amir]*

6) Cool abrrrations!

Consider a beam of light incident on a convex lens. We expect all the parallel rays to converge at a focal point. But, in reality, the visible beam of light consists of different wavelengths from violet to red. Since the index of refraction of the lens *depends on the wavelength* we are faced with a defect called *chromatic aberration* — different wavelengths will focus at different places (see figure at right).



a) Using the lens-maker's formula, find a formula that gives the change in the focal distance of a lens due to a small change in the index of refraction.

b) One way to minimize the chromatic aberration is to put two lenses of *different* materials together, to make one new lens. Suppose one part is made of Light Crown Glass and the other from Heavy Crown Glass. Below is a table showing the index of refraction vs. wavelength for these two different kinds of glass. We are looking for a lens with f = 10 cm. What should be the focal lengths of the two Light and Heavy Crown glass lenses so as to minimize the chromatic aberration?

Index of refraction of two kinds of glass in terms of wavelength

Wavelength (nanometer)	488	520	568	632	694	890
Light Crown Glass	1.4877	1.4859	1.4836	1.4813	1.4796	1.4758
Heavy Crown Glass	1.5793	1.5766	1.5735	1.5704	1.5681	1.5634

Hint : Lens-maker formula $1/f = (n-1) * (1/R1 + 1/R2)$
f : focal length of the lens
n : index of refraction of the lens material
R1 & R2 : The radii of curvature of the two sides of the lens *[Yaser]*

Remember to check the POPTOR web-page for hints and any possible corrections!
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1999-2000 Physics Olympiad Preparation Program

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Solution Set 4: Optics and Waves

1) Bob-bob-bobbin' along

This question was a sneaky way to get you solve a second order differential equation quite painlessly.

a) The forces involved included the restoring force, –kx and the inertial force, ma, where $a = \frac{d^2x}{dt^2}$. These two balance each other (the net force is zero). Thus the force equation for our system is,

$$m\left(\frac{d^2x}{dt^2}\right) = -kx$$

Which is a second-order ordinary differential equation.

b) We are given x = sin(ωt) as a solution to our force equation. Taking the first time derivative we obtain,

$$\frac{dx}{dt} = \omega\cos(\omega t)$$

Differentiating again with respect to time yields,

$$\frac{d^2x}{dt^2} = -\omega^2\sin(\omega t)$$

Plugging this expression for the acceleration into our force equation, along with x=sin(ωt) yields

$$m\left(-\omega^2\sin(\omega t)\right) = -kx = -k\sin(\omega t)$$

Therefore, solving for ω we get $\omega^2 = \frac{k}{m}$.

So this oscillation x = sin(ωt) isn't always a solution — only for a certain ω will there be a solution. So we find that the frequency of oscillation depends this way on both the spring constant of the spring (its strength) and the mass of the bob attached.

c) Putting the values k = 80 N/m and m = 100 g into our equation for ω, gives us a value

$$\omega = \sqrt{\frac{80}{0.1}} = 28/\text{sec}, \text{ or } \mathbf{4.5\ Hz} \qquad \textit{[Carrie]}$$

2) Keeping up an image

Since the object distance u and the image distance v can be exchanged in the lens law and their ratio shows the magnification to be $(v/u)^2 = 9$ (*or* 1/9), then $v/u = 3$ (or 1/3). Thus, the object distance is 30 cm (or 90 cm) and the new image distance is 90 cm (or 30 cm). The focal length can be calculated from the lens law: $f = 22.5$ cm.

If the same amount of light passed through the lens in both cases then the 9 times smaller image would be 81 times brighter, as the smaller image occupies a surface 81 times smaller on the screen than the larger one. However, the lens placed at a greater distance receives only *one ninth* of the light reaching the nearby lens, therefore the small image is only nine times brighter that the large one.

It can be shown in general that in such cases the small image is as many times *brighter* than the large one as the large one is *larger*. *[Gnädig/Honyek]*

3) Shifty radar

a) The radar gun uses the *Doppler shift* to detect the speed of cars. For light the correct doppler effect involves relativistic considerations. For cars, which have relatively small source velocities, we skip those. Thus we have for the officers viewing at rest,

$$f_L \approx (1 + v/c)f_S$$

Where f_L is the frequency perceived by the officers (or listeners at rest), v is the velocity of the source, c the speed of light, and f_S the source frequency (1000 MHz in this case). If waves are reflected from a moving car the reflected frequency, f_R, is that of the a source moving with twice the car velocity. So,

$$f_R \approx (1 + 2v/c)f_S.$$

The waves which are reflected from the moving cars are beat against the transmitted waves, yielding a frequency difference given by,

$$F = f_R - f_S \approx 2vf_S/c$$

Solving for the velocity of the car we get, $v \approx Fc/2f_S$.

b) Plugging in the numbers to our velocity equation we get,

$$v = [(330\text{ Hz})(3 \times 10^8\text{ m/s}) \;/\; (2\,(10^9\text{ Hz}))](1\text{ km}/1000\text{m})(3600\text{ s}\;/1\text{ hr})$$

$$= 178\text{ km}\;/\text{hr !!!!}$$

Book him, Dano! *[Carrie]*

4) To air is human

a) We know that in an adiabatic process PV^{γ} is constant.

$$P_0 = \frac{nRT_0}{V_0} = \frac{nRT_0}{Ah_0}$$

$$P_1 = P_0 + \frac{Mg}{A}$$

$$P_1 V_1^{\gamma} = P_0 V_0^{\gamma} \quad \Rightarrow \quad P_1 h_1^{\gamma} = P_0 h_0^{\gamma}$$

$$\Rightarrow \ h_1 = h_0 \left(\frac{P_0}{P_1}\right)^{1/\gamma} = h_0 \left(\frac{P_0 A}{P_0 A + Mg}\right)^{1/\gamma}$$

$$h_1 = \frac{nRT_0}{P_0 A} \left(\frac{P_0 A}{P_0 A + Mg}\right)^{1/\gamma}$$

b) Exactly the same as part (a) you can see that:

$$h_2 = \frac{nRT_0}{P_0 A} \left(\frac{P_0 A}{P_0 A + (M + m)g}\right)^{1/\gamma}$$

$$P_2 = P_0 + \frac{(M + m)g}{A}$$

c) The force acting in a mass-spring system is $-k\Delta x$. Consider that we change the equilibrium state by moving the piston x away from equilibrium. The force acting on the piston to push it back will be:

$$F = P_0 A - PA + (M + m)\, g$$

Since $P(h_2 + x)^{\gamma} = P_2 h_2^{\gamma} = P_1 h_1^{\gamma} = P_0 h_0^{\gamma}$

$$P = P_2 \frac{h_2^{\gamma}}{(h_2 + x)^{\gamma}}$$

and since x is quite small compared to h_2,

$$P = P_2 \left(1 + \frac{x}{h_2}\right)^{-\gamma} \approx P_2 \left(1 - \frac{\gamma x}{h_2}\right)$$

$$\Rightarrow F = P_0 A + (M + m)g - P_2 A \left(1 - \frac{\gamma x}{h_2}\right)$$

$$\Rightarrow F = \gamma\, P_2 \frac{A}{h_2} x = \ k\, x$$

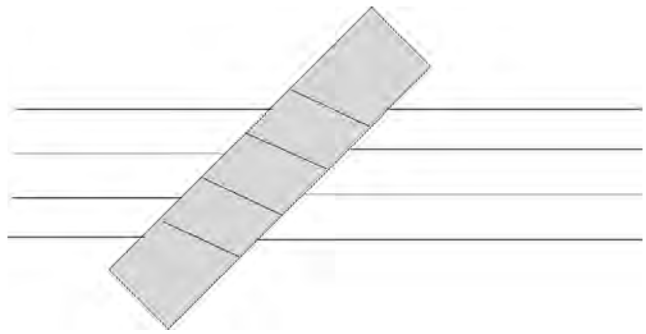
the spring coefficient of the system will be $\frac{\gamma P_2 A}{h_2}$ and the corresponding mass will be $(M + m)$.

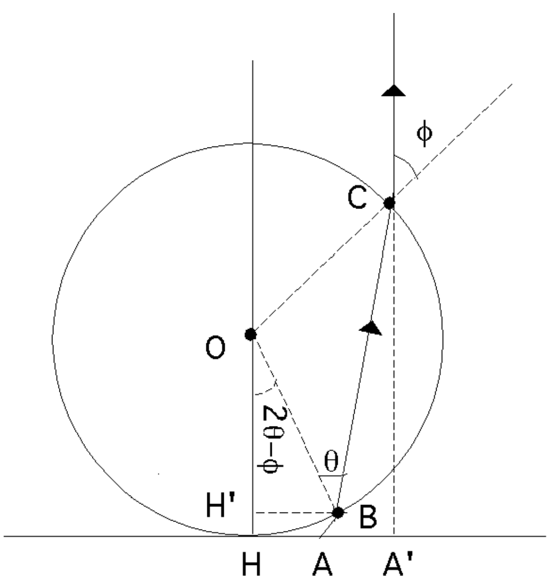
d) Since the harmonic oscillator goes up and down with the same distance, and the upper level of this oscillator is $(h_2 - h_1)$ above the equilibrium, the lower level will be $h_2 - h_1$ under the equilibrium.

$$h_{min} = h_1 - (h_2 - h_1) = 2h_1 - h_2$$

5) Bending the truth

a) A rod is a cylindrical lens which basically magnifies in one direction only, and leaves the other direction unchanged, so the line will be magnified in the direction perpendicular to the rod, and won't change in the direction of parallel to the rod.





b) Since we are looking far from the rod, the outgoing beams (going to your eyes) are parallel. The point A is emitting light in all directions, but we are concerned about the beam that will go out vertically. The beams are sketched at right.

The point A will be seen as A′ and we can find the ratio of $\frac{AH'}{AH}$ by trigonometry. First we assume that AH is approximately equal to the BH' line:

$$HA = BH' = R\sin(\theta - (\varphi - \theta)) = R\sin(2\theta - \varphi)$$
$$HA' = R\sin\varphi$$

$$\text{Magnification} = \frac{HA'}{HA} = \frac{\sin\varphi}{\sin(2\theta - \varphi)}$$

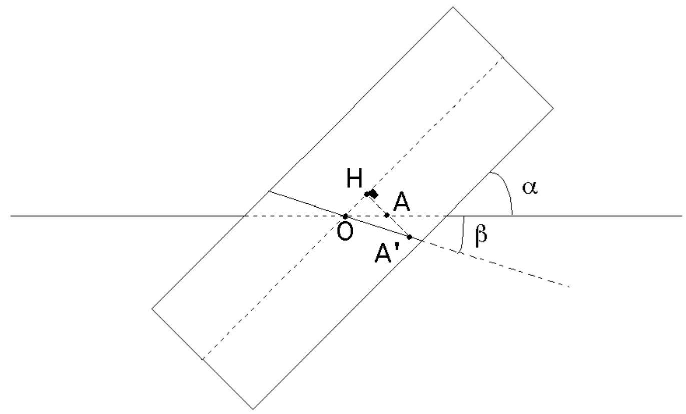
We neglect rays which pass either far from the centre, or at large angles from the axis (the *paraxial approximation*). In this case, θ and φ are small and $\begin{cases}\sin\theta \cong \theta \\ \sin\varphi \cong \varphi\end{cases}$.

$$\sin\varphi = n\sin\theta \quad \Rightarrow \quad \varphi = n\theta$$

$$\Rightarrow \text{Magnification} = \frac{n\theta}{2\theta - n\theta} = \frac{n}{2 - n}$$

In practice:

The rod is magnifying in the direction of the line perpendicular to the axis of rod. The point A will be seen as A' .



$$\frac{HA'}{HA} = m \text{ (Magnification)}$$

$$HA' = OH\ \tan(\alpha + \beta)$$

$$HA = OH \tan(\alpha)$$

$$\Rightarrow \frac{HA'}{HA} = \frac{\tan(\alpha+\beta)}{\tan(\alpha)} = \frac{n}{2-n} \quad \Rightarrow \quad n = \frac{2\tan(\alpha+\beta)}{\tan(\alpha)+\tan(\alpha+\beta)}$$

Where α is the angle between rod and parallel lines and β is the angle between bent lines and parallel lines. One can find n by measuring α and β. *[Amir]*

6) Cool abrrrrations!

a) $$\frac{1}{f} = (n-1)\bullet\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

$$\Delta\left(\frac{1}{f}\right) = \Delta(n)\bullet\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

$$= \left(\frac{1}{f}\right)\bullet\frac{\Delta(n)}{(n-1)}$$

b) When two different lenses are placed together, their inverse focal lengths add up:

$$\frac{1}{f} = \frac{1}{f_1}+\frac{1}{f_2}$$

We want the aberration to be minimum. So

$$\Delta\left(\frac{1}{f}\right) = 0$$

=> $$\frac{1}{f_1}\bullet\frac{\Delta(n_1)}{n_1-1}+\frac{1}{f_2}\bullet\frac{\Delta(n_2)}{n_2-1} = 0$$

Consider two wavelength 488 nm and 890 nm (*nanometers*).

$$\Delta(n_1) = 1.4877 - 1.4758 = 0.0119$$

$$\Delta(n_2) = 1.5793 - 1.5634 = 0.0159$$

Now we have two equations, two unknowns

$$\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{f}=\frac{1}{10}$$

$$\frac{1}{f_1}\bullet\frac{0.0119}{(1.48 - 1)}+\frac{1}{f_2}\bullet\frac{0.0159}{(1.57 - 1)}=0$$

Therefore

$$f_1 = 2.5 \text{ cm}$$

$$f_2 = -3.3 \text{ cm}$$ *[Yaser]*

1999-2000 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 5: Electricity and Magnetism

Due March 17, 2000 (revised date)

1) Vile away the hours

To complete his giant, super-evil-destructor ray gun the great, evil scientist Dr. Vile requires a copper sphere with a charge of exactly +3 μC. He has a copper sphere of mass 2 g.

a) What fraction of the electrons must be removed from his copper sphere to give it a charge of 3 μC ?

Having removed the correct number of electrons Dr. Vile places the 3 μC copper sphere near two other spheres of charges +7 μC and +5 μC. The three spheres are arranged in an equilateral triangle formation, with sides of 9 cm.

b) What is the resultant force (and its direction) on the 3 μC copper sphere? *[Carrie]*

2) Capacity for thought

Two parallel-plate capacitors with equal capacitance C are charged to voltage V. The distance between the plates is d and their area is A. There is no dielectric between the plates.

Connect the two capacitors by two wires so as to make a circuit. Obviously, there is no current in the circuit. Now, imagine that the plates of one capacitor move apart with velocity v and the plates of the other capacitor move toward each other with the same velocity v. At this time, there must be some current in the circuit. Find its value. *[Yaser]*

3) Dielectric City

Consider two parallel-plate capacitors C1 and C2 which carry corresponding charges Q1 and Q2.

a) What is the total energy of the two capacitors, taken as a system?

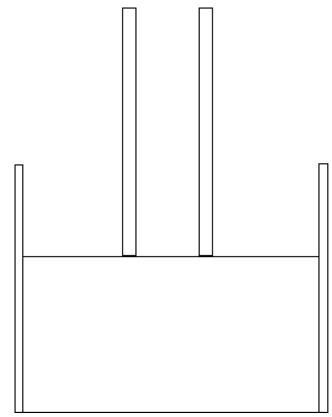
Now, we connect the two positive plates to each other and two negative plates to each other.

b) What will be the new energy of the system?

c) What happened to the energy of the system when the plates were connected?

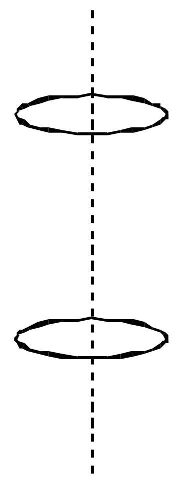
d) What is the energy difference between an empty capacitor and a capacitor full of water, if each carry the same charge?

e) What will happen if I put a charged-up capacitor on the surface of water, as shown in the figure? Why? How high will the column of water be when it rises into the space between capacitor plates?..... *[Amir]*



4) Super loopy

Two identical superconducting circular loops are positioned coaxially (a line passes through both their centres, perpendicular to each) and far apart. The loops have self-inductance L. Equal currents I are passing through each of these loops in the same direction. Now, bring the loops together.



a) What is the current in each loop?

b) Find the difference between initial and final energy of the system? *[Yaser]*

5) Keeping it together...

A narrow beam of electrons, of radius r and all moving at the same velocity $\boldsymbol{v}$ much less than speed of light (c), produces a charge current $\boldsymbol{I}$. We assume that the beam has cylindrical symmetry.

a) What is the electric field at the edge of the beam?

b) What is the magnetic field at the edge of the beam?

c) What is the radial (diverging) velocity of electrons at the border of the beam, after the beam has traveled a longitudinal distance 100 times r?

d) What is the diverging angle for $I = 1$ mA and $r = 1$ cm and $v = 1000$ m/s?

[BONUS: Since the B field depends on electron speed, can you find the speed for which the force due to B-field entirely *cancels* the force due to E-field? Hint: think 'frame of reference'] *[Amir]*

6) A feel for fields

a) Prove that it's *impossible* to have a magnetic field which increases along the z-axis but which has only a z-component. This field must have a radial component! By choosing a short cylinder, show that for a cylindrically symmetric B-field:

$$B_r = \frac{r}{2}\frac{dB}{dz}$$

A loop with electrical resistance R is falling in a cylindrically symmetric magnetic field. The centre of the loop as it falls is exactly aligned with the axis of the cylindrically symmetric field, and perpendicular to it. Along this axis, the field z-component is changing as dB_z/dz. The radius of the loop is r and its mass is m.

b) Write the equation of motion governing the fall of the loop, and sketch the graph of its velocity *vs.* time.

c) What is the terminal velocity of the loop? [*Yaser*]

[Hint: you may want to use the fact that magnetic field lines are always closed loops — they cannot start or end at a point, as E-fields do...]

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1999-2000 Physics Olympiad Preparation Program

— University of Toronto —

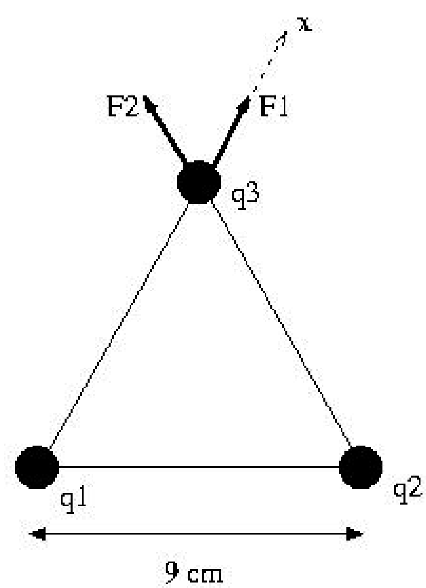
Solution Set 5: Electricity and Magnetism

1) Vile away the hours

i) The 2.0 g sphere contains 1.99×10^{22} atoms. The charge on the nucleus of each atom is 29 e. Thus the sphere contains $(1.99 \times 10^{22}) \bullet 29 = 5.77 \times 10^{23}$ electrons. One electron has a charge of 1.6×10^{-19} C. The electrons removed = $(3 \times 10^{-6}\text{ C})/(1.6 \times 10^{-19}$ C per electron) = 1.875×10^{13} electrons. So the fraction of electrons removed is

$$(1.875 \times 10^{13}) \;/\; (5.77 \times 10^{23}) = 3.25 \times 10^{-11}$$

b) The figure at right illustrates the configuration of the charged spheres with q1 = 7 μC, q2 = 5 μC and q3 = 3 μC. Let the positive x direction be as shown, and the y direction perpendicular to it, towards the upper left. The net force on q3 is the vector sum of the repulsive forces F_1 and F_2. The magnitude of F_1 is given by:



$$\begin{aligned} F_1 &= \frac{k\,q_1\,q_3}{r^2} \\ &= \frac{\left(9\times10^9\,N\,m^2\,C^{-2}\right)\bullet 7\times10^{-6}\,C \bullet 3\times10^{-6}\,C}{(0.09\,m)^2} \\ &= 23.3\,N \end{aligned}$$

Similarly F_2 is:

$$\begin{aligned} F_2 &= \frac{k\,q_2\,q_3}{r^2} \\ &= \frac{\left(9\times10^9\,N\,m^2\,C^{-2}\right)\bullet 5\times10^{-6}\,C \bullet 3\times10^{-6}\,C}{(0.09\,m)^2} \\ &= 16.7\,N \end{aligned}$$

The vector sum $\vec{F} = \vec{F}_1 + \vec{F}_2$ has components along x and along y. In the x direction:

$$\begin{aligned} F_x &= F_{1x} + F_{2x} \\ &= 23.3\,N + 16.7\,N \bullet \cos(60°) \\ &= 31.7\,N \end{aligned}$$

and in the y direction,

$$F_y = F_{1y} + F_{2y}$$
$$= 0\,N + 16.7\,N \bullet \sin(60°)$$
$$= 14.5\,N$$

Thus the magnitude of the resultant force is:

$$F = \sqrt{(31.7\,N)^2 + (14.5\,N)^2}$$
$$= 34.9\,N$$

The direction is at an angle

arctan(14.5/31.7) = 25° from the x axis. *[Carrie]*

2) Capacity for thought

The charge of each capacitor is $Q=CV$. When the capacitors are connected to each other and the separation between the plates change, the total charge is still conserved so $2Q=q_1+q_2$. Since the capacitors are parallel to each other, the voltage across them is equal to $V = \frac{q_1}{C_1} = \frac{q_2}{C_2}$. We know that the capacitance is inversely proportional to the separation of the plates. Therefore $C_1 d_1 = C_2 d_2$. Since the plates are moving together or apart by speed v for the two capacitors, we have

$$d_1 = d + vt \qquad d_2 = d - vt$$

Therefore $\frac{C_1}{C_2} = \frac{d_2}{d_1} = \frac{d - vt}{d + vt}$

On the other hand, $q_1 = \frac{C_1}{C_2} q_2 = \frac{d - vt}{d + vt} q_2$.

Substitute the value of $q_2 = 2Q - q_1$ in the above formula we have

$$q_1 = Q\frac{d - vt}{d} \quad \text{and} \quad q_1 = Q\frac{d + vt}{d}$$

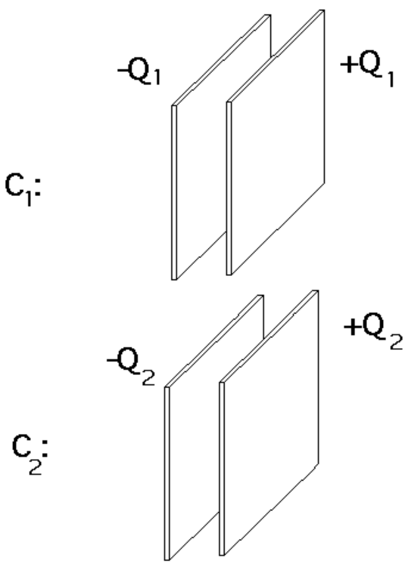
The current in the circuit is the rate of change of charge, which is equal to

$$I = \frac{dq_2}{dt} = -\frac{dq_1}{dt} = \frac{Qv}{d}.$$ *[Yaser]*

3) Dielectric City

a) The energy stored in a capacitor with capacitance C and charge Q is $\frac{Q^2}{2C}$. So the TOTAL energy of a system consist of two capacitors is:

$$E_1 = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2}$$

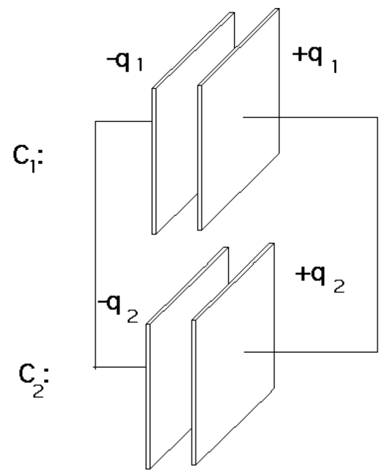


b) At first, I'll find the equivalent capacitor and the total charge on it.

Those two capacitors are connected as parallel capacitors and the total charge on the right-hand-side plates is conserved (there is no external current) we have $q_1 + q_2 = Q_1 + Q_2$ (or similarly for the left plates: $(-q_1) + (-q_2) = (-Q_1) + (-Q_2)$)

$$C_{equivalent} = C_1 + C_2$$

$$Q_{total} = Q_1 + Q_2$$



Now I can find the new energy of the system of one equivalent capacitor.

$$E_2 = \frac{{Q_{total}}^2}{2C_{equivalent}} = \frac{(Q_1 + Q_2)^2}{2(C_1 + C_2)}$$

c) One may find the difference between E_2 and E_1 as:

$$E_1 - E_2 = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} - \frac{(Q_1 + Q_2)^2}{2(C_1 + C_2)}$$

$$= \frac{C_2(C_1 + C_2)Q_1^2 + C_1(C_1 + C_2)Q_2^2 - C_1C_2(Q_1^2 + Q_2^2 + 2Q_1Q_2)}{2C_1C_2(C_1 + C_2)}$$

$$= \frac{C_2^2Q_1^2 + C_1^2Q_2^2 - 2(C_1Q_2)(C_2Q_1)}{2C_1C_2(C_1 + C_2)}$$

$$= \frac{(C_2Q_1 - C_1Q_2)^2}{2C_1C_2(C_1 + C_2)} > 0$$

which is always equal or larger than zero.

The energy loss is because of the emission of accelerated particles due to connection of two wires with different voltages. When you connect the first two wires (say negative plates) nothing will happen and you will just have one ground to measure voltage on each part of the two circuits. You can then calculate the voltage difference of two positive plates.

Connecting the other two wires will create a large electric field between two wires (when they are separated by a small distance) and then the electrons will be accelerated. If we could put a resistor between two capacitors, this energy could have been changed into heat. Often, the energy loss can be found in the form of electromagnetic waves in visual range (you'll see it as a spark!).

d) Capacitance of a capacitor full of water is ε times capacitance of an empty capacitor. (Recall the formula for capacitance $C = \varepsilon\varepsilon_o \frac{A}{c}$, where ε is susceptibility of water.)

Energy of full and empty capacitor can be found by using that formula:

$$E_{empty} = \frac{Q^2}{2C_{empty}}$$

$$E_{full} = \frac{Q^2}{2C_{full}} = \frac{Q^2}{2\varepsilon C_{empty}}$$

and the energy difference is:

$$\Delta E = E_{empty} - E_{full} = \frac{Q^2}{C_e}(1 - \frac{1}{\varepsilon})$$

$$C_f = \varepsilon\varepsilon_o \frac{A_1}{d}\ ; \quad C_e = \varepsilon\varepsilon_o \frac{A_2}{d}$$

$$E_{capacitor} = \frac{(\text{total charge})^2}{2(\text{equivalent capacitance})}$$

in which I've assumed that an equivalent capacitor have the total charge (initial charge).

$$\Rightarrow E = \frac{Q^2}{2\frac{\varepsilon_0}{d}(\varepsilon\, A_1 + A_2)}$$

Since $A_1 = wh$ and $A_2 = w(H\text{-}h)$ we can write:

$$E_c = \frac{Q^2 d}{2\varepsilon_0\, w(\varepsilon h + H - h)}$$

On the other hand, gravitational energy of water is:

$$E_g = mg \times \frac{h}{2} = \rho g w h d \times \frac{h}{2} = \frac{\rho w d g}{2} h^2$$

And total energy of the system is:

$$E_{\text{total}} = E_{\text{c}} + E_{\text{g}} = \frac{Q^2 d}{2\varepsilon_o\, w(H + (\varepsilon - 1)h)} + \frac{\rho w d g}{2} h^2$$

Since the energy tends to be minimized in physics world (and defines the equilibrium state):

$$\frac{\mathrm{d}E_{total}}{\mathrm{d}h} = 0 \Rightarrow \frac{-Q^2 d(\varepsilon - 1)}{2\varepsilon_o w(H + (\varepsilon - 1)\mathrm{h})^2} + \rho w d g h = 0$$

$$\Rightarrow Q^2(\varepsilon - 1) = 2\varepsilon_o \rho w^2 g h(H + (\varepsilon - 1)h)^2$$

$$\Rightarrow \left(\frac{h}{H}\right)^3 + \frac{2}{\varepsilon - 1}\left(\frac{h}{H}\right)^2 + \frac{1}{(\varepsilon - 1)^2}\left(\frac{h}{H}\right) - \frac{Q^2}{2\varepsilon_o(\varepsilon - 1)\rho w^2 g H^3} = 0$$

*Imaginary-valued solutions have no physical meaning, here.

*Answers with $\frac{h}{H} > 1$ mean that the capacitor will be completely filled.

*Answers with $\frac{h}{H} < 0$ mean that the water won't rise at all. (in our equation this will not happen, but if one could find a medium with $\varepsilon < 1$, $\frac{h}{H}$ could be negative).

The condition that capacitor is completely filled with water is:

$$1 + \frac{2}{\varepsilon - 1} + \frac{1}{(\varepsilon - 1)^2} - \frac{Q^2}{2\varepsilon_0(\varepsilon - 1)\rho w^2 g H^3} \geq 0$$

[Amir]

4) Super loopy

The main point of this question is that the loops are superconductors. Superconductors have *no* resistance. Therefore, when the loops are brought together, the magnetic flux, which passes through each of them, must not change. Otherwise there would be a voltage across each loop, which would cause an infinite current! To avoid infinite current, the *currents* in the loops change so as to keep the *flux fixed*.

When the loops are far apart the flux passing through each loop is equal to

$$\Phi_i = LI_i$$

which is because of the self-inductance of each loop. When the two loops are brought close together, the mutual inductance is equal to the self-inductance. The flux passing through each loop is combined in equal parts. One is from the self-inductance and the other is from the mutual inductance. Therefore

$$\Phi_f = LI_f + LI_f = 2LI_f$$

To avoid infinite current,

$$\Phi_i = \Phi_f \Rightarrow I_f = \frac{I_i}{2}$$

The energy of an inductance is equal to

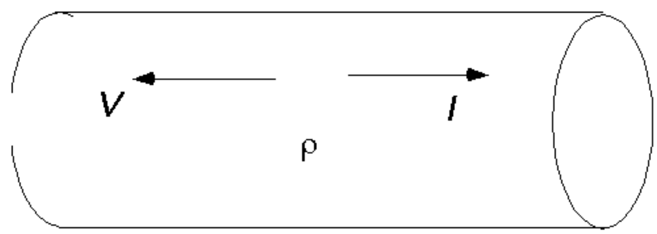
$$U = \frac{1}{2} L I^2$$

Therefore the difference in the energies of the system is equal to

$$\Delta U = U_f - U_i = \frac{1}{2} L I_f^2 + \frac{1}{2} L I_f^2 - \frac{1}{2} L I_i^2 - \frac{1}{2} L I_i^2 = -\frac{3}{4} L I_i^2$$

It can be shown that this is equal to the work needed to bring the loops together. Here the work is negative because the loops attract each other.. *[Yaser]*

5) Keeping it together...

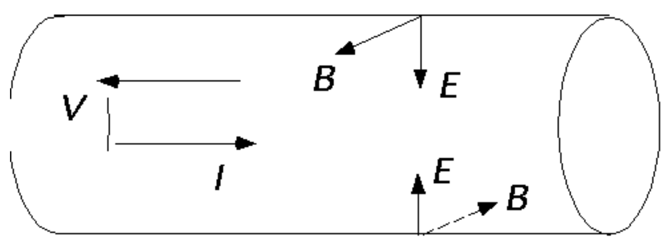


i) I'll find the density of electrons inside the beam.

$$\mathrm{I} = \rho v (\pi r^2) \;\Rightarrow\; \rho = \frac{-I}{\pi r^2 v}$$

The electric field at the edge of beam can be found by using Gauss's law.

$$\vec{E} = \frac{\lambda}{2\pi r \varepsilon_o} = \frac{\rho \cdot \pi r^2}{2\pi \varepsilon_o r} = \frac{\rho r}{2\varepsilon_o} = \frac{-I}{2\pi \varepsilon_o r v} \hat{r}$$



Using Ampere's law, I can find B:

$$B = \frac{\mu_o I}{2\pi r}$$

The radial force on electrons on the edge is

$$\vec{F} = -e(\vec{E} + \vec{v} \times \vec{B})$$

$$\Rightarrow \vec{F} = e\left(\frac{I}{2\pi \varepsilon_o r v} - \frac{\mu_o I v}{2\pi r}\right) \hat{r}$$

$$\Rightarrow \vec{F} = \frac{I e}{2\pi \varepsilon_o r v} (1 - \mu_o \varepsilon_o v^2)$$

as you may know, the quantity $\mu_o \varepsilon_o = \frac{1}{c^2}$ where *c* is the speed of light.

$$\Rightarrow \vec{F} = \frac{I e}{2\pi r \varepsilon_o v} (1 - \frac{v^2}{c^2}) \approx \frac{I e}{2\pi r \varepsilon_o v}$$

The change in radial momentum can be found by using

$$\Delta P_r = F_r\,\Delta t = \frac{I\,e}{2\pi\varepsilon_0\, r\, v} \bullet \frac{100\,r}{v}$$

$$\Delta P_r = \frac{100\,I\,e}{2\pi\varepsilon_0\, v^2}$$

$$\Delta v_r = \frac{\Delta P_r}{m_e} \quad \frac{100\,I\,e}{2\pi\varepsilon_0\, m_e v^2}$$

in which I've assumed that F_r, r and v won't change too much during this process. One may find the ratio:

$$\frac{\Delta v_r}{v} = \frac{100\;I\;e}{2\pi\varepsilon_0\; m_e v^3} = \frac{100 \bullet 10^{-3} \bullet 1.6\times 10^{-19}}{2\pi \bullet 8.85\times 10^{-12} \bullet 9.1\times 10^{-31} \bullet (10^7)^3}$$

$$\tan\vartheta = \frac{\Delta v_r}{v} \approx 0.3$$

$$\Rightarrow \vartheta = 0.3 \text{ radians} \approx 17°$$

(The speed in the problem set was wrongly written as 1000 m/s. It is 10,000 km/s. Sorry!).

BONUS:

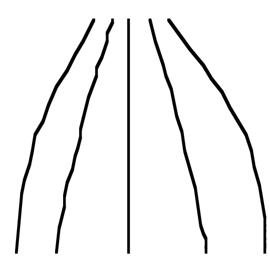
In part (c) we found that the force acting on electrons at the border is

$$\vec{F} = \frac{I\,e}{2\pi\varepsilon_0\, v\, r}\left(1 - \frac{v^2}{c^2}\right)$$

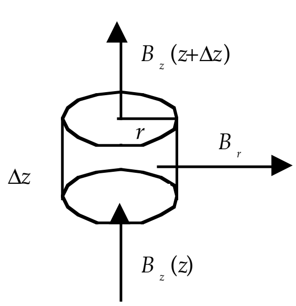
putting this force to be zero will result:

$$1 - \frac{v^2}{c^2} = 0 \;\Rightarrow\; v = c \qquad [Amir]$$

6) A feel for fields



a) Try to draw field lines, which have to be denser as the z changes. The field lines must bend, which shows the presence of a radial component. Use Gauss's Law for this part. The magnetic field lines that enter a cylinder must go out so as to make a closed loop.



The flux from the top and bottom surfaces is

$$\Phi_1 = \pi r^2 (B(z+\Delta z) - B(z) = \pi r^2 \frac{dB_z}{dz}\Delta z$$

The flux from the sides is

$$\Phi_2 = 2\pi r \Delta z B_r$$

For the total flux to be zero we should have

$$\Phi_1 = \Phi_2 \quad \Rightarrow \quad B_r = \frac{r}{2}\frac{dB_z}{dz}$$

b) There are two forces acting on the loop, gravitational force and the magnetic force. The rate of change of the magnetic flux respect to time is

$$\frac{d\Phi}{dt} = \pi r^2 \frac{dB_z}{dt} = \frac{\pi r^2}{v}\frac{dB_z}{dz}$$

where v is the velocity of the loop falling down. For the current in the loop we have

$$i = \frac{V}{R} = \frac{d\Phi / dt}{R} = \frac{\pi r^2 v}{R}\frac{dB_z}{dt}$$

The force exerted by the magnetic field, which is due to radial component of the magnetic field and upward, is equal to

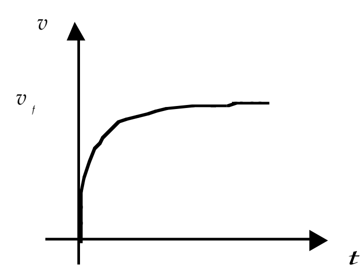
$$F = i l B_r = i(2\pi r)B_r$$

Substitute the value of current and magnetic field we already derived we have

$$F = \frac{\pi^2 r^4}{R}\left(\frac{dB_z}{dz}\right)^2 v$$

Therefore the equation of the motion of the loop is

$$m\frac{d^2 v}{dt^2} = mg - \frac{\pi^2 r^4}{R}\left(\frac{dB_z}{dz}\right)^2 v$$



The magnetic force is negative because of Lenz's law. The magnetic force is proportional to v and negative, causes damping. Initially, the velocity increases but reaches a terminal velocity which is when the acceleration is zero.

$$\frac{d^2 v}{dt^2} = 0 \Rightarrow v_f = \frac{mgR}{\pi^2 r^4 (dB_z / dz)^2}$$

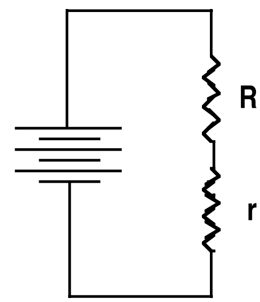
The velocity of the loop versus time is plotted. *[Yaser]*

— University of Toronto —

Problem Set 6: AC Circuits and Electronics

Due April 3, 2000

1) Ohm my!

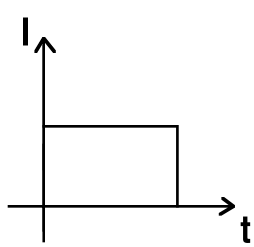


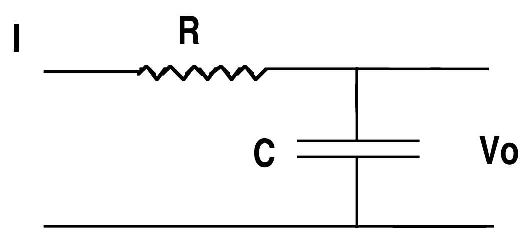
V R

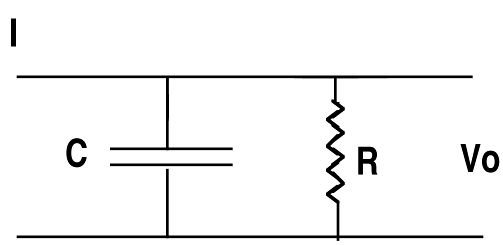
R r

r *parallel* R *[Robin]*

2) A current affair







[Yaser]

3) Pandora's box o' electronics

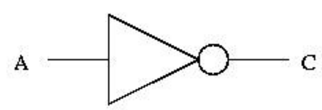
Voltage:

Resistance:

Current:

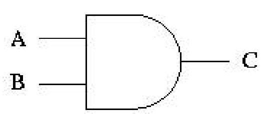
[Peter]

4) Only logical

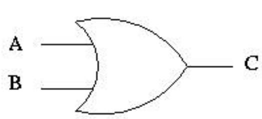


The NOT gate

vice versa



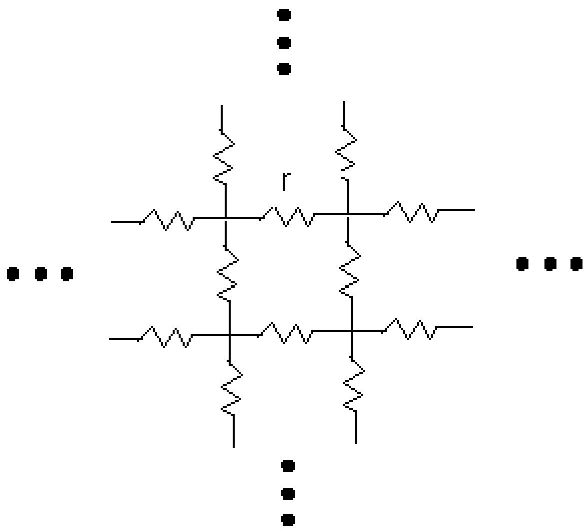
The AND gate



The OR gate

[Carrie]

5) Net worth



r

r

r

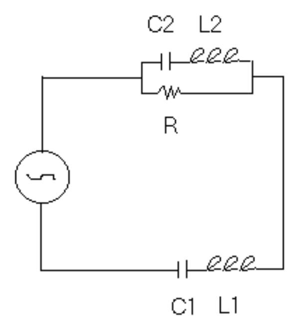
r

r

r

[Peter]

6) My current analysis...



ω

$$\frac{\quad}{L\,C} = \alpha \;\neq\; \frac{\quad}{L\,C} = \beta$$

[Peter]

1999-2000 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 6: AC Circuits and Electronics

1) Ohm my!

You might be able to do this one by inspection: if r → ∞ then *no current* flows in the circuit, so no power is dissipated through r. On the other hand, as r → ∞ power dissipation increases for a constant current. So a few equations are in order.

In the series circuit

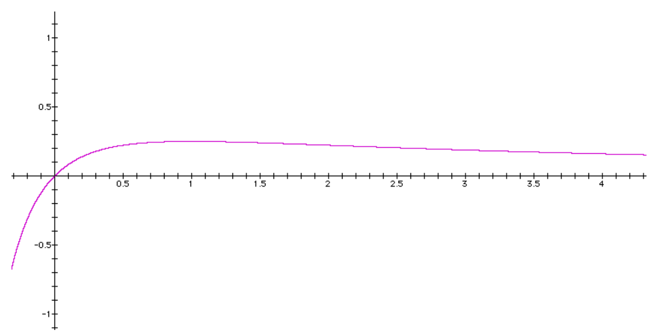
$V = I\,(R + r)$ *(voltage V across resistors in series, current I through resistors)*

$P = I \bullet V_r = I^2\, r$ *(power P dissipated in r, current I through r same as R)*

$$= \left(\frac{V}{R+r}\right)^2 r$$

For an extremum:

$$0 = \frac{dP}{dr} = \left(\frac{V}{R+r}\right)^2 \left(1 - \frac{2r}{R+r}\right)$$



which happens at $r = R$. However, this is a *maximum*, not a minimum.

This is easy to see by graphing it, by finding the second derivative, or by simply noting:

$$\lim_{r \to 0} \left\{ \left(\frac{V}{R+r}\right)^2 r \right\} = 0$$

$$\lim_{r \to \infty} \left\{ \left(\frac{V}{R+r}\right)^2 r \right\} = \lim_{r \to \infty} \left\{ V^2 \frac{r}{(R+r)^2} \right\} = V^2 \lim_{r \to \infty} \left\{ \frac{1}{2(R+r)} \right\} = 0$$

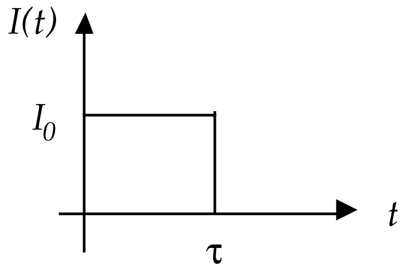
…so if it is zero at the origin and at infinity, has one extremum, and cannot be negative-valued, it must be a maximum. The minima you're asked for correspond to r = 0 and r = ∞.

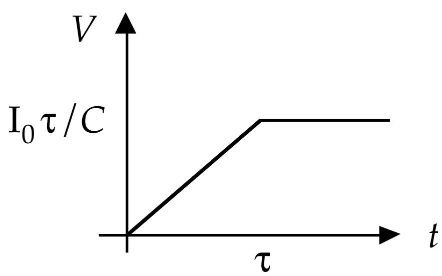
How about in parallel? It's even easier to see: $P_r = V_r\, I_r = (V_r)^2/r$. The voltage $V_r = V_o$ the battery voltage. Clearly this power decreases monotonically as r increases — the miminum is at r = ∞. *[Robin]*

2) A current affair

Usually questions like this one are given with a specified *voltage* applied — which would be easier! Here it is a *current* that is specified (without telling you how they made sure such a current was produced, at whatever voltage needed). This makes some parts tricky, unfortunately.

For the first case, all the current must go through the resistor, and onward to charge the capacitor. Therefore, the capacitor charge goes up linearly respect to time, and the voltage across it would be $V = \frac{q}{C} = \frac{I_0 t}{C}$. After time τ no current flows, so the charge remains on the the capacitor (except perhaps for *leakage* due to some large resistance through the capacitor which ideally should be infinite. Leakage can happen through moisture in the air, too).





For the case where the capacitor and resistor are connected in parallel, the current I_0 is divided through both of them. Initially most of the current will pass into the capacitor and increase the capacitor voltage as it charges up (since we've assumed a constant current flow, this the source voltage will have to increase to match the capacitor voltage, to keep current constant). As this voltage increases, the current through the resistor in parallel also will increase, according to $V = IR$, and this will make the current through the resistor increase — a bigger share of the current flows through the resistor.

Even if the current were constant for all time, the capacitor cannot charge up forever: at some voltage the current drain through the resistor would be equal to the whole current supply, and the capacitor will not take any of the current. Then the circuit will reach 'steady state'. This helps us figure out how to solve the following equation which describes the circuit (a famous physicist once said: "Never write down an equation until you've figured out what the answer should be.")

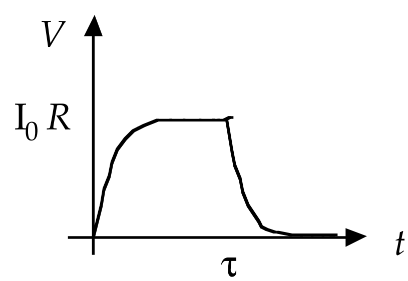
$$
\begin{aligned}
I_o &= I_{capacitor} + I_{resistor} \\
&= \frac{dQ}{dt} + \frac{V}{R} \\
&= C\frac{dV}{dt} + \frac{1}{R}V
\end{aligned}
$$

Where V is the voltage across both the capacitor and the resistor (in parallel), and Q is the charge on the capacitor. The current Io is constant, so we can solve this equation either systematically or by guessing from what we figured out about voltages, above.

We try a solution which has exponentials in it (because the derivative of the exponential is again an exponential, which maybe would cancel with the original). In steady-state, the dV/dt term will be zero (i.e., nothing is changing), so finally $V(t \rightarrow \infty) = V_o = I_o R$. If the voltage approaches this value asymptotically, then maybe it should have the form:

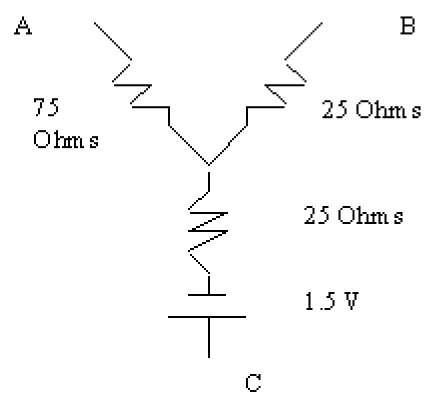
$$V(t) = V_o\left(1 - e^{\alpha t}\right)$$

Try it in the differential equation: it works, as long as $\alpha = -1/RC$. Sounds OK, because if R → 0 (short circuit) or C → 0 (no capacitor), then it takes no time at all to reach the final state.



In fact what happens is that the voltage follows this curve *V(t)* until the current is turned off. Then the charge on the capacitor leaks away through the resistor until it goes back to zero, exponentially with the same time-constant $\alpha = 1/RC$. *[Yaser & Robin]*

3) Pandora's box o' electronics



There is no unique solution to this problem (for example, a 100 Ω resistor could also be drawn as two 50 Ω resistors, etc.).

Trial and error always works, and here's a possible solution. The battery has a value of 1.5V, resistance at A is 75 Ω, resistance at B is 25 Ω and the resistance at C is also 25 Ω.

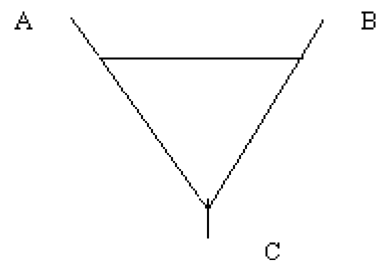
Using these values we first of all see that the resistance from B to C is 100 Ω (within error of the 99 Ω given); measuring resistance from A to C or B to C yields an error, as we're going across the battery [an aside: the way a multimeter measures resistance is to send a small current across the circuit, measure the potential difference created and find the resistance from Ohm's Law. However, when a battery is in the way it will change the current, and perhaps even reverse it, which will yield in very strange values shown by the multimeter].

The current from A to B is 0, and the voltages from A to C and B to C will be 1.5 V (within error of 1.6V), as given [the internal resistance of the multimeter is extremely high when measuring voltage]. The current from A to C (remember that the resistance of the multimeter is very small when measuring current) is 1.5 / 100 = 0.015 A and the current from B to C will be 1.5 / 50 = 0.030 A, again within error of the given values.

If you like a more systematic approach, try this:

- First, try to keep things as simple as possible (i.e., start with the simplest components, like resistors and batteries, adding more complicated ones later).
- Next, the most complex circuit possible is when all terminals are interconnected.

Now, it may be shown that such an 'outer' triangle is equivalent to an 'inner' triangle, just as I've drawn before. So, we re-draw the diagram to the one I had first (this is not necessary; I'm only doing it so that this solution is the same as the one above). Now, it is pretty clear that there has to be at least 1 battery between A-C and B-C. The easiest way to satisfy this is to put one at C.



- As to the remaining resistors, you can either guess the values or solve 3 equations and 3 unknowns, as below:

$$R_A + R_B = 100$$
$$R_A + R_C = 1.5/0.015 = 100$$
$$R_B + R_C = 1.5/0.030 = 50$$

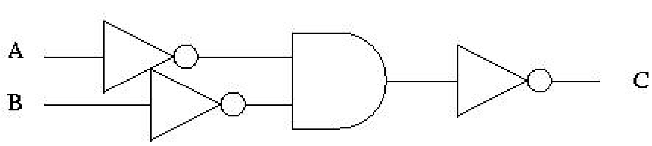
(RA, RB and RC are the resistances in the arms A, B and C, respectively, in the 1st diagram). And this indeed does it… *[Peter]*

4) Only logical

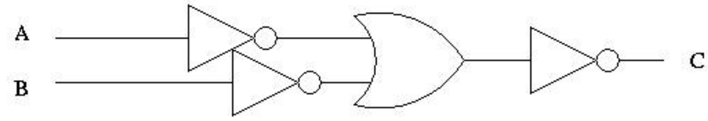
The truth table for the OR gate is shown at right: If either A *or* B is true (1) then the output is true (1).

A	B	C
1	1	1
1	0	1
0	1	1
0	0	0

b) Using NOT and AND gates the following OR gate can be constructed:

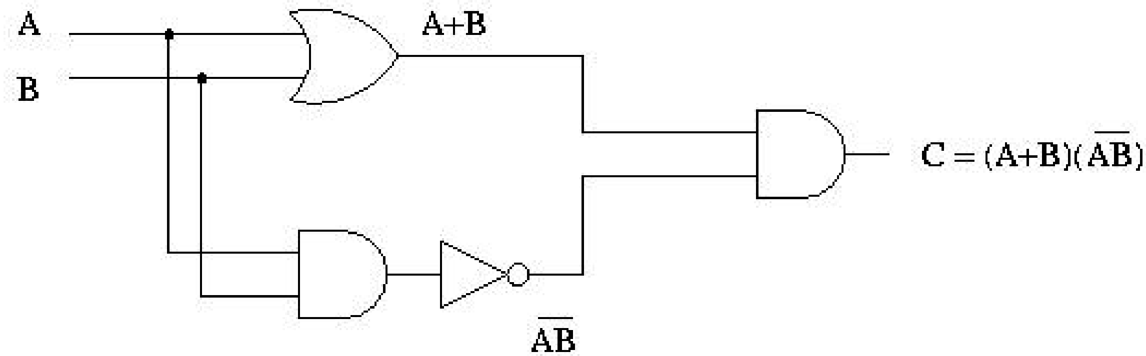


c) Using NOT and OR gates the following AND gate can be constructed:



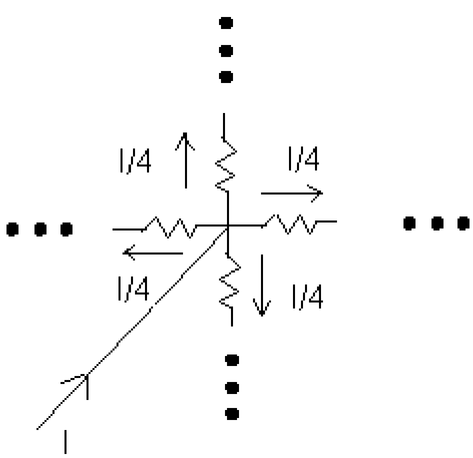
BONUS

d) One possible EXCLUSIVE OR gate built out of AND, OR and NOT gates:



[Carrie]

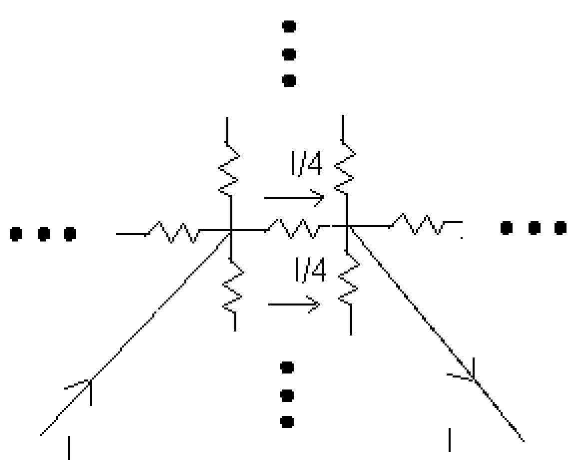
5) Net worth



iv) Let me do the general problem first. Consider an infinite, homogenous grid where at any junction N resistors are connected together. Note that a voltage source is really equivalent to a current source, *i.e.*, something which injects & draws current.

Consider injecting a current I into the junction. Because the whole system is N-symmetric, the current will split evenly and I/N will flow into every resistor. See the example, above left.

Consider now drawing a current I *from* a junction. Again, a current I/N will be drawn from every resistor. Now *superimpose* the 2 current flows, offset by 1 arm (see right). We



will have a current I coming in, and a current I coming out one arm away — equivalent to a battery. The current in the given arm will be $I/N + I/N = 2I/N$. The resistance is r. Thus, the voltage drop is $2Ir/N$ which must also equal RI, where R is the equivalent resistance. From this we see that $R = 2r/N$. This also solves the remainder of the questions…

This question is inspired by a Polish Olympiad question, from way back when…

Answers:

i) $r/2$

ii) $r/3$

iii) $2r/3$

iv) $2r/N$

BONUS: it is *not possible* to draw such a figure in 2D; in 2 dimensions, the only possible figures are those for which N = 3, 4 or 6. To convince yourself of this you can note that at each junction the sum of all the internal angles has to be 360 degrees (of course, you don't have to be able to draw this figure for part (iv) to hold…). *[Peter]*

6) My current analysis...

Here's what you have to know:

i) when the driving frequency ω is small, the current is almost steady (DC). The inductor behaves like a piece of wire, with zero resistance. The capacitor, on the other hand, doesn't let any current through and behaves like a break in the circuit.

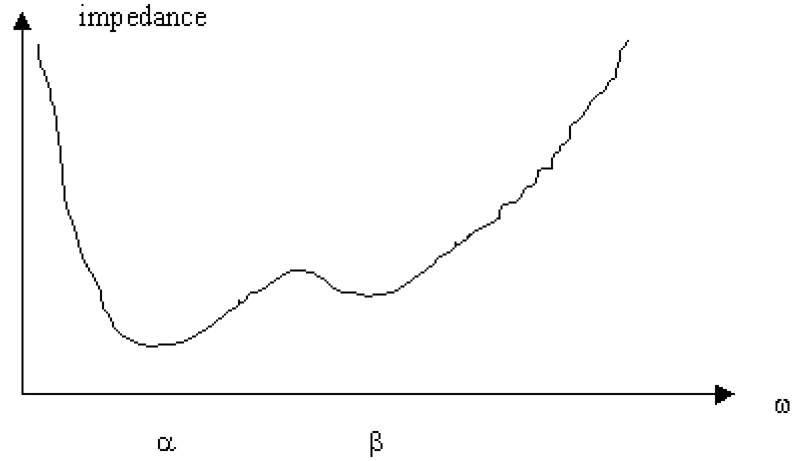
ii) when ω is very large, the opposite is true: the capacitor lets current through but the inductor doesn't (the e.m.f. that builds up inside is very large)

iii) an LC or RLC circuit will resonate at a frequency of $\omega^2 = \frac{1}{LC}$. At this point the power dissipated is maxi-mum, and the impedance is minimum.

Now, the circuit in question contains 2 circuits (the R, L_2, C_2 circuit and the C_1, L_1 circuit), each with their own resonant frequencies. We can thus expect that the overall impedance will have minima near the resonant frequencies. In fact, the impedance of the R, L_2, C_2 circuit will be higher than that of the other circuit, due to the resistor R. Also, the impedance will never quite reach zero, as the two resonant frequencies are not equal ($\alpha \neq \beta$).

From all this, we 'guesstimate' the curve on the following page (note that I have arbitrarily assumed $a < b$).

For an analytical treatment I will employ the complex number technique. What happens here is that a sinusoidal driving source (e.g., $\cos(\omega t)$) can be conveniently represented as a complex number ($\cos(\omega t) = \mathrm{Re}(\exp(i\omega t))$, where Re denotes the real part). This is useful as derivatives of the exponential are really easy to calculate. This, in the end, is useful to solve the differential equations present when dealing with capacitors and/or inductors.



The gist of all this is the fact that the resistor, capacitor and inductor, in a circuit driven by a sinusoidal source, behave as resistors whose generalized 'resistance' (called *impedance*) is a complex number (the number has to be complex because capacitors and inductors change the phase of the current; the phase is represented by the angle the impedance makes with the real axis in the complex plane).

The values turn out to be:

Resistor R: $Z = R$

Capacitor C: $Z = \dfrac{1}{i\omega C}$

Inductor L: $Z = i\omega L$

(Z is the impedance; $i^2 = -1$)

Finally, the standard Kirchoff rules apply to impedances (*i.e.*, they're added when in series and the reciprocals of the sum of their reciprocals is taken in parallel).

After this crash course, let's apply this to our circuit. The total impedance will be

$$Z = (i\omega L_1) + \left(\frac{1}{i\omega C_1}\right) + \frac{1}{\left(\dfrac{1}{R}\right) + \left(\dfrac{1}{i\omega L_2 + \dfrac{1}{i\omega C_2}}\right)}$$

$$= i\omega L_1\left(1 - \frac{\alpha^2}{\omega^2}\right) + \frac{1}{\dfrac{1}{R} + \dfrac{1}{i\omega L_2\left(1 - \dfrac{\beta^2}{\omega^2}\right)}}$$

Real nasty… now, what we really want to plot is the magnitude of this thing, $|Z|$. This yields some pretty ugly expressions, and not much can be done so simplify them.

You can take my word for the fact that this yields very similar results to the graph above; the second minimum isn't really a minimum, more of an inflection point, but it's close enough for this rough analysis. *[Peter]*

2000-2001 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 1: General

Due October 30, 2000

1) The path of least resistance (...and highest voltage)

You are given 32 identical 3-volt batteries, each of which has an internal resistance 2 ohms (i.e., even if you short-circuit them with a wire you still have 2 ohms appearing in such a circuit). How can you connect all these batteries together so as to get *highest possible current* through a 4-ohm resistor?

[HINT: Think of the batteries set up in n rows and m columns...] *[Yaser]*

2) The last straw...

a) When you suck on a straw, you create a partial vacuum in your mouth. If a person could make a partial vacuum in their mouth equal to 50% of atmospheric pressure, what's the farthest possible vertical distance above their drink of water?

b) Is it reasonable that your lungs might produce a vacuum 50% that of the atmosphere, by inhaling? What else is at work?

c) Does the size of the straw (its diameter) have any effect on the maximum height?

To answer parts b) and c) it is suggested that you test the effects on height using different-sized straws. Once you have collected the data you can determine the approximate pressure — this is the *empirical* approach.

Materials:
– a bucket or pitcher of water
– a few ~1.5 m lengths of plastic tubing of different diameters
(Canadian Tire stores sell this, for example)
– a measuring tape

The plan — If you suck up water from the bucket up the tube, like a long straw, at some point you can no longer raise the water level within the tubing. Measure the height of the water level within the tubing above the water level in the bucket. Try it also by inhaling, to draw up the water, instead of sucking.

Record your result and repeat the experiment for tubes of different diameter. Once you have collected the data you can determine the approximate vacuum pressure your lungs can achieve by themselves, and compare that to the best overall partial vacuum you are able to produce otherwise. *[Sal]*

3) Fermi, fer you

Enrico Fermi, a famous physicist of the 20th century, was well-known for asking peculiar questions of his students, intended to develop their practical ability to figure things out from reasonable assumptions and common knowledge (or good guesses!). In one of these 'Fermi Questions' he's reported to have asked "How many piano tuners are there in Chicago?" The point was to figure it out on the spot, not to contact the Tuners' Guild, and the reasoning might have started with a guess of what fraction of households even owned a piano — based, for instance, on thinking about your own friends' households. Then for a typical piano-owner, how often might one hire a tuner — maybe every year or two on average? In the end, one could figure out how many worker-hours of piano tuning per year there would be in a city the size of Chicago, and thereby figure out how many tuners could be supported by that amount of work.

Try these yourselves! You'll have to make a number of assumptions or approximations — please describe each on a separate line, with the numerical estimate or guess you make. Any final answer within a factor of three of the 'real' answer is considered an excellent success, for others a factor of ten is good. Your *thinking* is the main thing of interest to us — please don't go and measure anything, or look anything up.

a) How many piano tuners do you figure there are in Toronto?

b) I saw lots of meteors streak across the sky this summer — they moved through an arc of about 45 degrees of the sky in a little less than a second. How fast do meteors go?

c) How many kilograms of toothbrushes do the people of Canada throw away each year?

d) If icebergs never broke off from the ice-caps at the poles of the earth, roughly how long would it take to tie up all the water on earth as ice at the poles?

e) How many atoms scuff off your sneakers onto the sidewalk with each step you take?

[Robin]

4) Number four with a bullet

A 2 g spherical bullet 9 mm in diameter is fired from a gun at 250 m s^{-1} into one end of a thermally insulated cylinder. The cylinder, 15 m long and with a diameter of 1 m, is pressurized to 20 atm with Argon gas at room temperature. While passing through the cylinder, the bullet experiences a drag force which slows the bullet:

$$F_{drag} = -\frac{1}{2}\varepsilon\rho A v^2$$

where ε = coefficient of drag (a constant)
ρ = density of gas
A = cross-sectional area (area of an object's shadow)
v = speed

Ignore gravity, and assume ideal gas behaviour.

Eventually, the bullet hits the opposite end of the cylinder and stops.

a) How long does it take for the bullet to reach the opposite end of the cylinder?

b) Calculate the bullet's velocity just before hitting the cylinder wall.

c) Calculate the change in temperature of the Ar gas in the cylinder due to the bullet's motion before impact.

[HINT: If you can integrate, you can do this; if you get stuck, check the POPTOR webpage for ways to solve the quite simple *differential equation* you will get]. *[Brian]*

5) Bubbling ideas

Suppose we have a perfectly round soap bubble of initial radius R_o, with internal air pressure p, and external air pressure p_o.

a) What is the change of pressure inside the bubble if the bubble size is increased by a small amount?

Now put a charge Q on the bubble, spread evenly over the surface.

b) What is the new radius of the bubble after charging it up?

[HINT: perhaps you know or can find the field *outside* a spherical charge distribution, and figure out the field *inside* too. But what about the field right on the bubble wall, *neither* inside nor outside? The right answer is a kind of averaging... *[Peter]*

6) Magnus matters

What does it take to pitch a curveball in baseball? Besides some talent, it takes just a bit of physics — the *Magnus* effect.

The drag force on a ball depends on how quickly it is moving through the air. When a ball is moving through the air and also spinning, the airspeed on one side of the ball is greater than on the other. So it isn't strange that the force on one side of a spinning ball can be greater than on the other, and the ball can be pushed sideways — curve in its path.

Here's a simple experiment to find out about this. You need:

a cardboard tube of the type that is found at the centre of a roll of paper towels
a skinny elastic band, at least 5 cm long
a sticky note, like Post-It™ brand

We want to drop the tube through the air while it is spinning. To do this, cut the skinny elastic band so that it's now just an elastic string. Attach one end of the elastic to the sticky-note, using tape (duct tape works well for me). Then stick the sticky note onto the cardboard tube right in the middle (as a kind of anchor) and wind the elastic around the tube. Now hold the tube horizontally while holding the free end of the elastic. Let go of the tube while keeping hold of the elastic — before it really falls, the elastic will pull on the tube and spin it up, and the sticky-note will peel off at the last.. It works best if you can release it over a balcony or from a ladder.

a) What happens to the tube as it falls? Does it make a difference which *way* the tube is spinning as you drop it? How does this compare to the way the tube falls if it isn't spinning at all? How do *you* explain the effect? How do you imagine the spinning motion changes the forces on the tube?

b) Measure how far the tube falls, and how much it moves sideways. You may need to make several trials, measuring each time, and average your answers together to get a reliable result. If you can figure out the error in your measurement, that's worth extra points. Can you tell what *path* or trajectory the tube takes as it falls — curved or straight? You might want to use a video camera and then look at the tape frame-by-frame to analyse things.

c) Use the results from (b) to figure out what the angle of descent is for the tube, relative to the vertical. You should probably find that the falling tube pretty quickly reaches *terminal velocity*, when the air drag-forces exactly match the force of gravity (for a parachutist in free-fall this is roughly 200 km per hour!). At what speed does the cylinder fall? Can you use this to figure out roughly how quickly the tube moves sideways?

d) *Theory* — this just takes some algebra, though it looks a little hairy. Bonus points! Use the F_{drag} formula from question 4; for anything shaped like a *cylinder*, the coefficient of drag is $\varepsilon = 1.0$. Can you use this to predict the terminal velocity of the falling spinning cylinder (you may also have to weigh your tube)? How does this compare to what you measured? You can likewise look at the sideways-motion terminal velocity. What must be the sideways force, to give the sideways terminal velocity you found above?

e) BIG BONUS: For a given cylinder and spin-rate, can you roughly predict the angle, relative to the vertical, at which the spinning tube should fall? *[Robin]*

POPBits™ — Useful bits of information

Drag force:

$$F_{drag} = \varepsilon \rho A v^2$$

where ε = coefficient of drag (a constant)
ρ = density of fluid
A = cross-sectional area (area of an object's shadow)
v = speed

$\varepsilon_{sphere} = 0.5$

$\varepsilon_{cylinder} = 1.0$

C_v (Ar) = 12.5 J mole^{-1} K^{-1} (heat capacity at constant volume)

M_m (Ar) = 39.948 g mole^{-1} (density of argon, per mole; *molar density*)

ρ_{air} = 1.2928 g L^{-1}

2000-2001 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 1: General

1) The path of least resistance (...and highest voltage)

If two identical batteries are connected to each other in series, the equivalent resistance is the sum of their individual resistances. The voltage across them would also be the sum of the individual electromotive forces.

When they are connected in parallel, the total resistance is half the individual resistance and the voltage would be the same as individual voltages.

If we connect all the batteries in series, we could get a high voltage but at the same time the total resistance is big and lowers the current. We could reduce the resistance by connecting all in parallel but then the total voltage is small.

The optimum case is to set the batteries in n rows and m columns. If the voltage of each battery is V and its resistance is r then the voltage in each row is mV and the resistance is mr. So we have n equivalent batteries each having voltage mV and internal resistance mr. Therefore the total votage is mV and the total resistance is mr/n. Then the current that passes through a resistor R is equal to

$$I = \frac{mV}{R + \frac{mr}{n}} = \frac{V}{\frac{R}{m} + \frac{r}{n}}$$

We have to maximize the above equation with the condition $N=nm$. Therefore we should minimize the denominator, which is the sum of two terms having the product of Rr/mn or, equivalently, Rr/N. We know that the sum of two numbers which have a constant product is minimized when they are equal.

Therefore $R/m = r/n$. It means $mn= 32$ and $m/n=2$. So $m=8$ and $n=4$. *[Yaser]*

2) The last straw...

a) By creating a partial vacuum in your mouth, atmospheric pressure forces the liquid up the straw. The difference in pressures will work only to a certain height h until the weight of the water in the straw will balance the differential-pressure force:

$$P/P_{atm} = 0.5, \quad P = 0.5P_{atm} = 5.05 \times 10^4 \text{Pa}$$
$$P_{atm} = P + \rho g h$$
$$h = (P_{atm} - P) / \rho g$$
$$= (1.01 \times 10^5 \text{Pa} - 5.01 \times 10^4 \text{Pa}) / \{(1000 \text{ kg/m}^3)(9.8 \text{ m/s}^2)\}$$

h = 5.15 m

<u>Experimental Results</u>

Note: Inhaling is different than sucking By sucking, resting, then continuing to suck, your tongue acting as a piston can produce a vacuum effect capable of raising the liquid in the tubing higher than a height your lungs are capable of sustaining. I did the experiment both ways.

Experiment by inhalation

At the hardware store I picked up 3 1.5 meter tubes with diameters 1.58 cm, 1.90 cm, 2.54 cm. The tubing was taped along side my washing machine so as to get the tubing as straight as possible. One end of the tubing was submerged in a bucket of water . The other end was brought to my mouth. I exhaled, and then I inhaled as much as possible. As the water level rose in the tubing, it became more and more difficult to breathe in . This was not the result of me running out of lung capacity, because I was able to continue breathing in if I took my mouth away from the tubing.

Once the water level stopped rising I quickly plugged the hole of the tubing with my tongue. The level of the water in the tubing was then recorded. My lungs, I felt, where getting tired after a few attempts so I rested every 3 to 4 attempts. 5 attempts for each tube were performed. Below are the results.

Diameter of tubing (cm):	1.58	1.9	2.54
Attempt 1	112	99	106
Attempt 2	105	114	105
Attempt 3	112	115	111
Attempt 4	106	108	117
Attempt 5	100	109	103
Average	**107 ± 2**	**109 ± 3**	**108 ± 2**

Table 1. The heights of water within a tube achieved by inhaling on different-sized tubing.

b) This result would suggest that the you could *not* raise the water level within the tubing to 5m simply by inhalation.

The results above show that the different average heights for each tube-diameter are within each other's error. The diameter of the tube had *no bearing* on the height attainable, as one expects. The weight of the water within a larger tube will exert greater force against the atmosphere, but this force is distributed over a greater surface area due to the increased tube diameter. The resulting pressure exerted by the weight of the water, Force/Area, is therefore not affected.

These results suggest that your lungs are only capable of producing a partial vacuum of 0.895 atmospheres — 0.105 ATM below atmospheric:

$$1.08 = (P - P_{atm}) \,/\, \rho g = (1.01 \times 10^5 \text{Pa} \,(1 - C) \,/\, \{(1000\text{kg/m}^3)(9.8\text{m/s}^2)\}$$

solving for C,

$$C = 0.895$$

Why does this differ from our earlier calculations? Because experiment shows that a typical person's lungs cannot deliver 0.5 atmospheres of partial vacuum!

Experiment by sucking on tube

It would seem that the vacuum your mouth is capable of producing relies on another mechanism besides lungs capabilities. As mentioned earlier your tongue acting as a piston may be able to produce a greater partial vacuum in your mouth. To test this idea we repeated the experiment by sucking. (Unfortunately we initially misguided you by suggesting a short length of the tubing to use — sorry about that!)

What I did was I found a tube longer than 5 m and repeated the experiment, in a stairwell this time. With a bucket of water on the ground floor I placed the tubing in the bucket and walked up a few flights of stairs. I then began to suck on the tubing, pausing occasionally to catch my breath. By placing my thumb over the opening of the tubing I maintained the water level in the tubing while resting. I continued this pattern of sucking and resting until I could no longer raise the water level. Measurements are shown below.

1.3 cm diameter tubing	Height in meters
Attempt 1	4.5
Attempt 2	4.2

The results are close to those calculated earlier! With each suck I was able to draw the water higher and higher. Not being restricted to one breath I was able to take advantage of the mechanisms my tongue is responsible for. There finally reached a point where the partial vacuum, created in my mouth, could no longer raise the water level beyond the 4.5 meter. This is the height that determines the vacuum capabilities of my mouth. Lungs & diaphragm muscles, in fact, are not nearly as sturdy as mouth muscles, though a little baby crying might leave you wondering... *[Sal]*

3) Fermi, fer you

Folks don't have to get the following answers – it only matters that you reason well, and work with whatever scraps of information you already know. It isn't OK to look up significant answers for things. You're supposed to work like on a desert island, with what's in their heads. Clever resourcefulness, inventiveness but mostly reasoning from common ideas and facts is what gets marks here.

a) *How many piano tuners do you figure there are in Toronto?*

- about 3.5 million people in Toronto — people: 3.5 M
- about 20% of households have a piano — pianos: 700 k
- about 10% of these get their pianos tuned regularly, say 2x per year, others maybe once every two years, (2 x 70 k + 0.5 x 630 k) = — tuning vists/year: 455 k
- say ~1.5 hours to tune a piano + 0.5 hours driving time ⇒ in an 8-hour day maybe 4 visits. If a tuner works 280 days/year — tuning visits/year/worker 1,120

Total tuners in whole Toronto area: **about 400**

Concert pianos probably are the specialty of a small handful of tuners; likewise some people have relatives who make a hobby of it, or have their pianos tuned by people 'off the books'. So I think this number is probably good within a factor of two.

How else can one get an idea? Try the Yellow Pages, which lists the fairly large tuning services, some of which employ more than one person; then figure that a certain number don't pay for ads, or list in regional Yellow Pages, and it probably agrees with this number.

b) *I saw lots of meteors streak across the sky this summer — they moved through an arc of about 45 degrees of the sky in a little less than a second. How fast do meteors go?*

Well, angular speed is only good if we can guess the height of the meteors. That can't be any higher than the height of the atmosphere, which isn't of course a sharp edge. I think the International Space Station orbits at something like 240 miles, if I recollect right. Also that the Russian space station Mir is going to be in trouble when it drops to about 120 miles. Now, a person can still breathe on top of Everest at 10,000 m, about the same height as airliners fly through the air, so there's a reasonable amount of air at 6 miles up (maybe roughly 50-70% of an atmosphere?) A meteor would burn in a lot less air than that. I think the space shuttle is in a fireball, with protection from its heat-tiles, at 40 miles up, so I think meteor showers must show up at heights of roughly 20-40 miles up.

In that case, 45 degrees per second is 2 π (30) / 8 miles per second, about **25 miles per second**. [is that reasonable? I remember that the space shuttle orbits at about 90 minutes per orbit, and the earth radius is about 8,000 miles, so about 25,000 miles in 90 minutes, or 17.000 mph, or almost 5 miles per second, so this seems possible].

Well, meteor showers come often (like the Leonid shower at just this time of year) from comet dust, as the earth moves in its orbit. So if the earth orbits through the dust, the speed above ought to be roughly the orbital speed: 93 million miles radius, in one year, and the earth is moving at almost 20 miles per second. So I think this probably is a pretty good fit, for a sight at night!

c) *How many kilograms of toothbrushes do the people of Canada throw away each year?*

If we followed the dentist's advice, it would be a new brush each 3-6 months, if I remember right. Do you do that? Let's say we buy 1.5 toothbrushes per year for each Canadian (about 30 million); it might only be one, but some people buy new toothbrushes for guests, and others treat their toothbrush like an old friend... Say 5 million are either kids or folks with dentures who soak 'em. That would then mean 37.5 million toothbrushes per year that get bought. Unless we store up toothbrushes, they all eventually get thrown out (even after being used for cleaning bathroom grout or polishing fussy silverware), so we ditch 37.5 million toothbrushes per year. How much do they each weigh? I've never weighed one, but I think it would take at least 20 toothbrushes to weight the same as a quarter-stick of butter — 0.25 lbs /(2.2 kg/lb) / 20 = 10 g. That might be too low: my toothbrush is about 20 cm long, and very roughly 1 cm across, in average width, about 20 cm^3. If it has about the same density as water, then that should be about 20 g.

So 37.5 million x 0.020 kg = **750,000 kg of toothbrushes each year**.

I wonder if any of this plastic could be made recyclable...

d) *If icebergs never broke off from the ice-caps at the poles of the earth, roughly how long would it take to tie up all the water on earth as ice at the poles?*

This is the toughest one. It's like the problem in winter-time that the air moisture in your room ends up frozen on your windows – it's a kind of dehumidifier or moisture-pump, and your room can get a bit drier. Mostly winter dryness is because the frigid outside air, even if it's at 80% relative humidity, holds little moisture; if you only warm up outside air, the *relative* humidity goes down quickly, because your warm house-air could have held a lot more water, but it still has only the little amount of water in frigid air.

Let's see: air doesn't circulate very well north and south, but it would eventually pass over the polar ice-caps. So, if it falls as snow on the ice-caps and never again leaves, eventually all moisture will get trapped in the ice-caps. People suspect this may be an issue for places like Mars. However, the polar regions actually get very little snow – you may remember that people take ice-cores in order to look back at pollen and spores that fell hundreds or thousands of years ago. It's one way that people test for global warming, because that biological record, frozen at different depths, indicates ancient temperatures. Now, to go back 1,000 years in the Antarctic ice cap, one doesn't have to drill 1 km – it's much less than 1 m of snow per year.

In fact, the poles are both desert regions, which by definition means less that 10 inches of precipitation falls (measured as water). So, say that exactly 10 inches falls each year over the area of each icecap. How big is each icecap? Smaller than the arctic/antarctic circles which are at 22.5 degrees, which is the tilt of the earth on its axis. You could figure out the area of a sphere within 20 degrees of the pole, but that's too picky for such a rough approximation. Good enough (with 20% probably!) to take the area of a circle at the radius of the earth, 8000 miles, and say at 15 degrees away from the pole. That means a radius about 15/360 th of the earth circumference, 25,000*(15/360) = about 1,000 miles. That area is then about 6 million square miles at each pole, for 12 million square miles total.

If 10 inches falls each year there and is locked up, and the total surface area of the earth is $4 \pi r^2$ = 780 million square miles, then the rest of the earth loses 10 * (12/780) = 0.15 inches per year. If 70% of the earth's surface is water, then the oceans and lakes would lose about 0.2 inches per year as an equivalent amount is locked up each year in the ice caps. If the ice-caps never calved icebergs, or evaporated or whatever, then in a lifetime the oceans would drop by about 16 inches, which would be noticeable. In about 300,000 years the oceans might drop by a mile. If the oceans are in only a few places a few miles deep, then the average depth might even be less than this – **in less than a million years the oceans might all be sitting as ice at the polar caps**.

Except for icebergs.

e) *How many atoms scuff off your sneakers onto the sidewalk with each step you take?*

Let's see: my sneakers pretty much wear through through the sole in about 18 months, if I wear them to walk on sidewalks outdoors and such. That sole is about 1 cm thick, but only about 1/3 the area of the sole has to wear down before they wear through under the balls of my feet. So, for my size-12s, that's like 1 foot long and perhaps 4 inches wide, or 25.4 cm x 10 cm for a net wear of 1 cm x (25.4 cm x 10 cm) / 3 = roughly 80 cm^3 of rubber & plastic. Plastic is a hydrocarbon, so it's mostly C and H in very roughly even proportions, with an average mass per atom of about 6.5 a.m.u. It has a density a little above water, I think, so say 1.5 g/cm^3. So that's about 120 g of rubber.

At 6.5 a.m.u. average, for the atoms, that means 6.5 g per *mole of sole* (per mole of atoms in the soles of my shoes) – so I lose a little less than 20 moles of atoms from my soles in one year.

And for one step? Well, how many steps in a year? Hmmm, well I walk about 600 m to the streetcar from home, and about 500 m at the other end – 2.2 km just getting to work. My steps are about 1 m per pace (I'm tall), so that's 2,200 steps then. I walk for about 1 hour net, between classes and offices, during the day while I'm at work, and another 20 minutes at lunch. I figure I take about 1 step per second (very roughly), so that's 1 1/3 hours x 3600 seconds/hour x 1 step/second = 4,800 steps. I take my sneakers off at home. So I walk about 7,000 sneaker-steps each day, roughly, 300 days per year because I don't always wear sneakers. So a little over 3 million steps in 18 months to remove about 20 moles ($20 \times 6 \times 10^{22}$) of atoms.

So with each step I must be leaving behind roughly 4×10^{17} atoms! Yikes, one day people ought to be able to figure out where I walked! They just need to make a combination upright vacuum cleaner and mass spectrometer!

REMINDER: all these values are very rough – within a factor of 2 or 3 either way is reasonable. The important thing is the reasoning, as Sherlock Holmes might once have said. *[Robin]*

4) Number four with a bullet

The Physics – The key to this problem is realizing that the acceleration of the bullet is *not constant* while traveling though the cylinder of gas: From the drag equation $F_{\mathrm{drag}} = -\frac{1}{2}\varepsilon\rho A v^2$ we know that bullet experiences a force that is a function of its velocity. Since F = ma and the mass of the bullet is constant, the acceleration has to be a *function* of velocity. Therefore, the following constant acceleration equations *cannot* be used:

$$x(t) = x(0) + v(0)t + \tfrac{1}{2}at^2 \qquad (1)$$

$$v(t) = v(0) + at \qquad (2)$$

including the variations on equations (1) and (2):

$$v(t)^2 = v(0)^2 + 2a(x(t) - x(0)) \qquad (3)$$

$$x(t) = x(0) + \tfrac{1}{2}(v(t) + v(0)) \qquad (4)$$

$$x(t) = x(0) + v(t)t - \tfrac{1}{2}at^2 \qquad (5)$$

In this question, only the following fundamental definitions apply:

$$v = \frac{dx}{dt} \qquad (6)$$

$$a = \frac{dv}{dt} \qquad (7)$$

$$F = ma \qquad (8)$$

The Solution – Given quantities converted to SI units where appropriate:

Variable Name	Definition	Value
v(0)	Initial velocity of bullet	250 m/s
m	Mass of Bullet	2.00 x 10^{-3} kg
r	Radius of bullet	4.500 x 10^{-3} m
L	Length of cylinder	15.0 m
R	Radius of cylinder	0.500 m
T	Initial temperature of gas	300 K
P	Pressure of argon gas in cylinder	2.027 x 10^{6} Pa
ε	Drag coefficient for a sphere	0.500
Mm	Molar mass of argon gas	39.948 g/mol
C	Specific heat of argon gas	312.5 J/kg/K

Quantities to be calculated:

Variable Name	Definition
v(t)	Final velocity of bullet after traversing the cylinder
t	Time it takes for the bullet to traverse the cylinder
V	Volume of cylinder
A	Maximum cross-sectional area of bullet
ΔT	Change in gas temperature
M	Mass of argon gas in cylinder

- Calculate maximum cross-sectional area of bullet:
  $A = \pi r^{2} = \pi (4.50 \times 10^{-3})^2 = 6.362 \times 10^{-5}$ m/s
- Calculate cylinder volume:
  $V = \pi L R^2 = \pi (15.0)(0.500)^2 = 11.78 \text{ m}^3$
- Calculate density of argon gas in cylinder:

  $$PV = nRT \;\; (Ideal\ Gas\ Law) \;\Rightarrow\; \frac{n}{V} = \frac{P}{RT}$$

  $$\rho = \frac{n}{V} Mm = \frac{PMm}{RT} = \frac{(2.027 \times 10^6)(39.948)}{(8.31451)(300)} = 3.246 \times 10^4 \text{ g/m}^3 = 32.46 \text{ kg/m}^3$$
- Calculate the mass of argon gas in the cylinder:
  $M = \rho V = (32.46)(11.78) = 382.4$ kg

<u>Part A</u>

- Calculate the time it takes for the bullet to transverse the cylinder:

  $$ma = F_{drag} = -\frac{1}{2}\varepsilon\rho A v^2$$

  since $a = \frac{dv}{dt}$, then

$$m\frac{dv}{dt} = -\frac{1}{2}\varepsilon\rho A v^2$$

$$\frac{dv}{dt} = -\frac{\varepsilon\rho A}{2m}v^2 = -\beta\, v^2$$

this is a differential equation – what function v(t) can be differentiated once to equal itself-squared, multiplied by a constant –β ?

where $\beta = \dfrac{\varepsilon\,\rho\,A}{2m} = \dfrac{(0.5)(32.46)(6.362\times 10^{-5})}{2(2.00\times 10^{-3})} = 0.2581\ \mathrm{m}^{-1}$

(see <u>*A Guide to Solving Simple Ordinary Differential Equations (ODE's)*</u> on the POPTOR webpage for hints etc. for this problem set to understand the following steps)

$$\frac{dv}{v^2} = -\beta\, dt$$

$$\int_{v(0)}^{v(t)} \frac{dv}{v^2} = -\beta\int_0^t dt$$

$$\frac{1}{v(t)} - \frac{1}{v(0)} = \beta t$$

$$\frac{1}{v(t)} = \beta\, t + \frac{1}{v(0)} = \frac{v(0)\,\beta\, t + 1}{v(0)}$$

$$\boxed{v(t) = \frac{v(0)}{v(0)\,\beta\, t + 1}}$$

equation for the velocity of the bullet as a function of time

We can't solve for time *t* since we don't know what *v(t)* is. However, we do know what the distance is so use the definition for velocity:

since $v = \dfrac{dx}{dt}$, then

$$\frac{dx}{dt} = \frac{v(0)}{v(0)\,\beta\, t + 1}$$

this is another differential equation

(again see <u>*A Guide to Solving Simple Ordinary Differential Equations (ODE's)*</u>)

$$dx = v(t)dt = \frac{v(0)}{v(0)\,\beta\, t + 1}dt$$

$$\int_0^L dx = v(0)\int_0^t \frac{1}{v(0)\beta\, t + 1}dt$$

$$L = \frac{v(0)}{v(0)\beta}\ln\left(v(0)\beta\, t + 1\right) = \frac{1}{\beta}\ln\left(v(0)\beta\, t + 1\right)$$

$$\ln\left(v(0)\beta\, t + 1\right) = L\beta$$

$v(0)\beta\, t + 1 = e^{L\beta}$

$\boxed{t = \dfrac{e^{L\beta} - 1}{v(0)\beta}}$ equation for the time it takes the bullet to transverse the cylinder

substitute in the numbers:

$$t = \frac{e^{(0.2581)(15.0)} - 1}{(250)(0.2581)} = 0.726 \text{ s}$$

Part B

Using the equation for velocity and the value for time found in Part A:

$$v(t) = \frac{v(0)}{v(0)\beta\, t + 1} = \frac{(250)}{(250)(0.2581)(0.726) + 1} = 5.23 \text{ m/s}$$

Part C

the change in the bullet's kinetic energy is converted to heat in the gas:

$$\Delta E_K = \Delta Q$$

$$\frac{1}{2} m\, v(t)^2 - \frac{1}{2} m\, v(0)^2 = \frac{1}{2} m\left(v(t)^2 - v(0)^2\right) = MC\Delta T$$

$\boxed{\Delta T = \dfrac{m\left(v(t)^2 - v(0)^2\right)}{2MC}}$ equation for change in temperature of argon gas

substitute in the numbers:

$$\Delta T = \frac{(2.00 \times 10^{-3})((250)^2 - (5.23)^2)}{2(382.4)(312.5)} = 5.23 \times 10^{-4} \text{ K}$$

FINAL ANSWERS

Question	Answer
Part A	$t = 0.726$ s
Part B	$v(t) = 5.23$ m/s
Part C	$\Delta T = 5.23 \times 10^{-4}$ K

[Brian]

5) Bubbling ideas

a) The temperature of the air inside the bubble is always the ambient temperature. Therefore, pV = constant, where V is the volume of the bubble. So if the volume changes by a small amount ΔV then (using the product rule) we have

$$\Delta(pV) = \Delta p \cdot V + p \cdot \Delta V = 0 \quad \Rightarrow \quad \Delta p = -p\frac{\Delta V}{V}$$

b) If we put a charge on the bubble, the bubble size increases as each part of the bubble is repelled from every other part. This will reduce the internal pressure.

The electric field *outside* the bubble at the surface is $Q/(4\pi\varepsilon R_0^2)$ where R_0 is the radius of the bubble. The field *inside* is zero. We could say that the field exactly *on* the bubble is the average of these two fields $Q/(8\pi\varepsilon R_0^2)$. A full-theory calculation gives exactly this answer.

Therefore, the electrostatic pressure is

$$P_E = \frac{E\sigma\Delta A}{\Delta A} = \frac{Q}{8\pi\varepsilon R_0^2}\frac{Q}{4\pi R_0^2} = \frac{Q}{32\varepsilon\pi^2 R_0^4}$$

where σ is the surface charge density, E is the electric field on the bubble and ΔA is the area of a small piece on the bubble.

This pressure should be equal to the change in the internal gas pressure, following its change in size

$$P_E = -\Delta p \to \Delta V = \frac{V}{P}\frac{Q}{32\varepsilon\pi^2 R_0^4} \to \Delta R = \frac{Q}{96\varepsilon\pi^2 R_0^3 P} \qquad \textit{[Peter \& Yaser]}$$

6) Magnus matters

a) If the tube is not spinning, it just falls straight down. If it is spinning, then it doesn't fall straight, it scoots out to one side or the other, depending on which way it was spinning. If the tube is spinning so one side moves backwards with the passing air, and the other side moves forwards against the passing air, the tube swings *away* from the side that moves forward against the passing air (greatest relative speed).

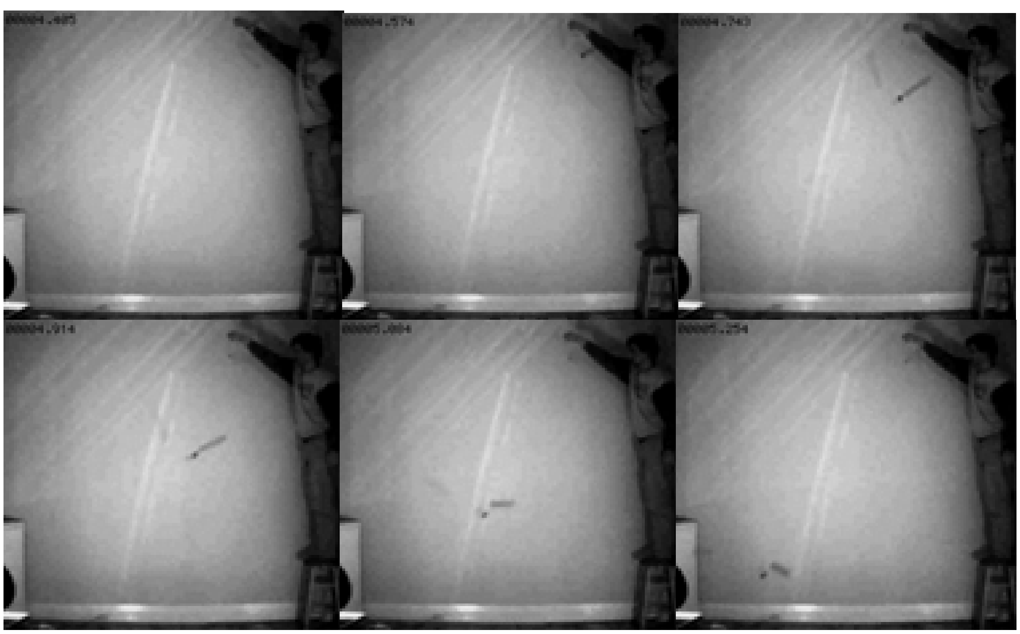
When the tube falls relatively slowly, as it does for us, the flow around the tube is pretty well-behaved — no very serious turbulence (which would make the problem tougher). We call the flow *laminar*, which means it flows in smooth layers. Now, the air is very slightly *viscous* — as syrup is more viscous than water, and water is more viscous than alcohol. That means that as the tube moves, it carries a layer of air along with it, close to its surface.

It also means that when the tube rotates, it carries the air with it then. Imagine the tube holding still, and the air rushing past (as in a wind tunnel). The side of the tube rotating forward *against* the oncoming air will actually slow the air slightly; the side of the tube rotation *with* the oncoming air will carry the airflow a little faster. This means the air is slower on the side of the tube that rotates against the flow, and faster on the other side. Bernouilli's principle (which really is just a conservation of energy formula for fluids!) means that the air pressure will be higher on the side moving against the airflow, and less on the other side. The net pressure difference is a force, which pushes the tube sideways — the *Magnus Force*.

Bernouilli's principle is a big part of how an airplane wing generates lift out of the forward thrust of the engines. It's also the reason why a strip of paper, held below your lower lip, will rise as you blow air across the top surface.

[EXTRAS: When the object moves very quickly (like a fast baseball, or many real airplane wings), it isn't any more exactly the Bernouilli effect, but a bit more complicated. What happens there has to do with the transition that airflow makes when it isn't smooth, but becomes turbulent and begins to detach from the flow around the tube. When the relative airspeed is higher, the flow detaches earlier in the stream around an object; where the surface moves back with the airflow, it delays this *boundary-layer separation*. The result is that the air flowing past a spinning body is tilted, in the stream past the object. So the air in the wake of the spinning tube actually kicks over to one side. The momentum change of that air results in the force pulshing sideways. (Look for a streaming-video description of this on the archive of the Science of Baseball website, from a show on the Discovery Canada: www.exn.ca)]

b) I recorded this on my computer, using a cheap CCD web-camera and shareware NIH Image 1.62 [http://rsb.info.nih.gov/nih-image], which can put a timestamp on each frame to help analyse motion. Here are a few frames in the sequence:



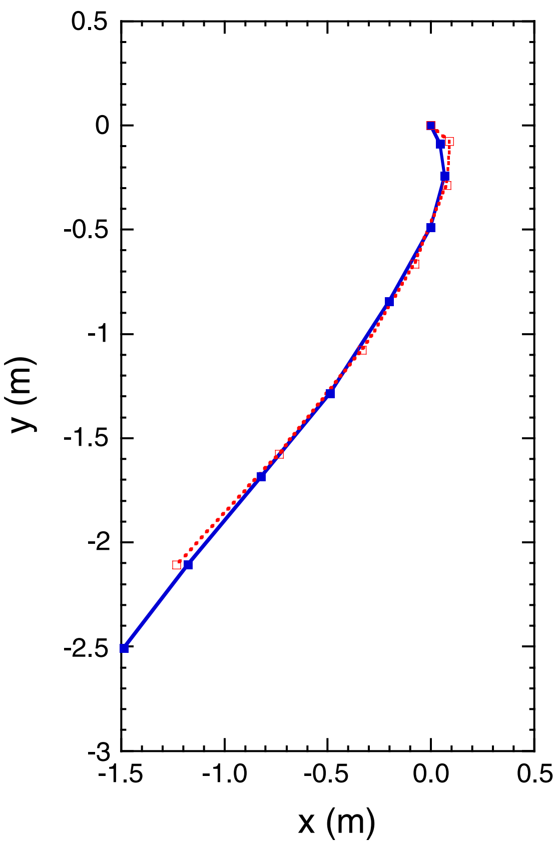
You don't have to do this, though — you can get very nearly as much by finding where it hits the ground, measured from where you let it go, and using a stopwatch to time it.

b) I measured from the video frames on my computer. If you use a video camera, you can measure from the TV screen. You might want to use a dry-erase felt pen (but remember to erase it afterwards!); but be careful about using any markers at all on a computer monitor, since they are sometimes anti-reflection coated with a material that dissolves (they used to often dissolve with ammonia-based cleaners like Windex too! I'm not sure this still is true, so check the manual for your display for warnings!)

In the graph on this page, each marker on the trajectory is a measurement at a different time (the blue ones are a little more than 0.1 second apart). The path *is* curved: it starts off going to the right, because of the elastic band I used to spin it, I think. Then it drops, but as it gains speed it begins to push to the left.

c) The angle of descent was about 45°, so the tube moves at the same speeds in the downwards direction and the sideways direction. It almost reaches a constant speed (*terminal velocity*) after 1 second, when it hits the floor, but doesn't quite get there. If you used a balcony in your school, you probably did better than I did. I *was* able to extrapolate that the terminal velocity would be around 3.8 m s^{-1}.

d) At terminal velocity, there is no acceleration — all forces balance. Then the 45° trajectory indicates that the net force is at 45° down and to the left, so in this case the Magnus force is about the same as gravity, at terminal velocity. People have patented 'sail'-steamships with tall rotating cylinders for sails; they could use wind-power to make the ship tack into the wind at 45° this way.



From POPBits™, at the end of question-set #1:

$$F_{drag} = \varepsilon \rho A v^2$$

The mass of my cardboard tube was 25 g, and its cross-sectional area was 125 cm^2 (see comments below). Balancing forces, and therefore setting ma = F_{drag} at terminal velocity, we can solve simply for $v_{terminal}$:

$$m\,a = \varepsilon \rho A v_{terminal}^2$$

$$0.025 \text{ kg} \bullet 9.8 \text{ m s}^{-2}$$

$$= 1.0 \bullet 1.2928 \text{ g L}^{-1} \bullet (10^3 \text{ L m}^{-3}) \bullet 0.0125 \text{ m}^2 \bullet v_{terminal}^2$$

Thus $v_{terminal} = 3.9$ m s^{-1}

as compared to 3.8 m s^{-1} measured above (this worked unusually well for such a formula). The sideways Magnus force is comparable to the force of gravity, so the x-component of velocity is about the same as this value, and the final terminal velocity is the resultant, about √2 times larger.

e) BIG BONUS: You can come fairly close to the right answer with this slightly hand-waving reasoning (you can also have a look in The Physics of Baseball (2nd Ed.), Robert K. Adair, Harper Collins, NY, 1994, ISBN 0-06—95047-1).

The drag force is:

$$F_{drag} = \varepsilon \rho A v^2$$

which can be described as a pressure-differential ΔP (front – rear) multiplied by the area presented to the wind or rushing air. Since this pressure-differential depends on the relative speed, if an object spins there will be a difference in the force from side-to-side across the ball. Then

$$F_{Magnus} = \Delta P_{left\text{-}right} A = \varepsilon \rho A (v_{right}^2 - v_{left}^2)$$

now,

$$v_{right} = v + 2\pi r \bullet f$$

$$v_{left} = v - 2\pi r \bullet f$$

where f is the rotation frequency (revolutions per second) of the object. Substituting:

$$(v_{right}^2 - v_{left}^2) = (v + 2\pi r \bullet f)^2 - (v - 2\pi r \bullet f)^2$$

$$= 8\pi\, r\, v\, f$$

so

$$F_{Magnus} = \varepsilon \rho A (8\pi\, r\, v\, f)$$

For our case at 45° falling angle, $F_{drag} = F_{Magnus}$ so we can figure out the spinning frequency f:

$$(8\pi\, r\, v\, f) = v^2$$

which we solve to find f = 6.3 rev s^{-1}

I do think the elastic band setup I used could provide this spin-rate. But, in fact, this hand-waving argument fudged a little on the appropriate use of areas and direction of airflow. Even so, any error is by a constant factor: the formula shows the appropriate dependence on speed v and spin-rate f.

Comments on my own setup

Cardboard tube:

circumference 14 cm --> diameter d = 4.5 cm
length 28 cm
cross-sectional area presented to wind (rectangle) A = 125 cm^2
mass 25 g

(I didn't have a weigh-scale at home, so I took a 25 cm rod, a handle from a cat-toy, and hung a plastic grocery bag tied to a string on one end, and just a piece of string on the other. The rod had a piece of string tied around the middle to suspend the whole thing to tilt freely. I slid the string along the middle of the rod until the grocery bag and loose string balanced each other evenly. Then I taped the cardboard tube to the loose string, which tipped the balance over. I then dripped water drop-by-drop into the plastic bag until it balanced the cardboard tube again — it took 25 ml, which weighs 25 g.) *[Robin]*

2000–2001 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 2: Mechanics

Due November 27, 2000

1) Take your best shot

Evangeline, an Olympic biathalon competitor-in-training (who hopes to use her keen grasp of kinematics to win gold), keeps a water supply on her training course: on top of a 10 meter-high water tower platform sits a 10 meter-tall cylindrical drum of water, 3 meters in diameter. Unfortunately, the pipes to the drinking fountain below have frozen solid. Evangeline skis up to a point 5 meters away from the base of the tower, and with her last round she uses her target rifle to shoot a small hole in the wall of the water drum.

Where, precisely, on the drum should the hole be so that she can have a drink without moving from the place where she stands? *[Sal]*

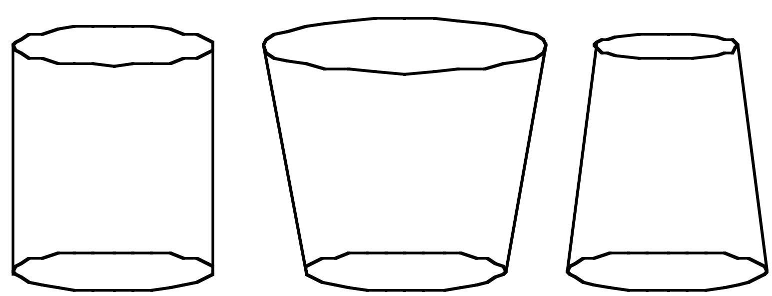
2) The fountain of (misspent) youth

Sukumar has accidentally backed his pickup truck into a fire hydrant, and broken it off at ground level (oops!). Water gushes straight upward at a rate of 150 kg s^{-1}, and with a speed of 20 m s^{-1}. Suku tries to fix things by holding a nearby garbage can over the flow, but finds himself and the can lofted into the air on the water flow.

If Suku weighs 60 kg, and the garbage can weighs 2.5 kg, what is the height that he's waving to his friends from, when the police get there? *[Yaser]*

3) Huh? Hmmm…

Consider three containers, as illustrated at right, each with the same area A at its base, and each with height H. The first one is a cylinder, and the other two are tapered. The area of the tops of the other two vessels are 2A and A/2.



a) If all three containers are filled with water, what's the water pressure at the bottom of each? What total force does this then exert over the whole bottom plate of the container?

b) If the containers of water are

each put on a weigh-scale, individually, what weight will the scale show for each? Treat the containers as being very thin-walled, and essentially massless. Only the bottom plate touches the weigh-scale, so why is the measured weight different from the force found as area × pressure you calculated in part (a) ?

Remember that balance shows the force which is exerted on it. *[Yaser]*

4) All aboard!

a) A freight train traveling with an initial speed of 65 m s^{-1}, cuts its engine and coasts 8 kilometers along a flat track before finally coming to a stop. Calculate the train's coefficient of rolling friction. Is this a value for the whole train, or for each railroad car?

b) A locomotive is pulling an empty 15.0-m-long hopper-loader car at a constant speed of 1.00 m s^{-1}. The mass of the locomotive and empty car is about 75,000 kg. As the hopper-car moves along the track it passes under a grain dispenser that pours grain into the car, at a rate of 1000 kg s^{-1}. The grain dispenser only operates when the car is beneath it. How much work does the locomotive do while the hopper-car is being filled? Assume that the coefficient of rolling friction is the same as calculated in part (a). *[Brian]*

5) Java jump-up (eeyow!)

A few years ago McDonald's™ was sued by a woman who was badly scalded when the cup of hot coffee she was holding between her thighs — while driving — spilled. The woman won the suit, and since then their coffee cups include a printed warning on them that the presumably hot coffee inside is really hot.

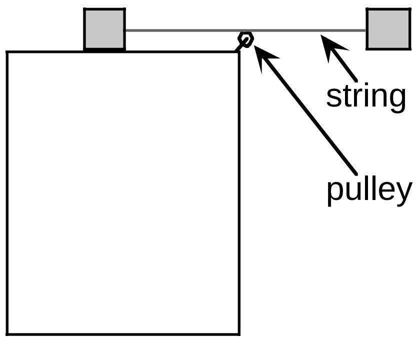
So when the makers of the *Java Jumpstart Commuter Mug** recently came out with a new model, we at POPTOR suggested they have it evaluated by *Deepee-Deetee Consultants Inc*., a group of OAC Physics students, who evaluated the mug's stability on the dashboard of a moving car.

The physics-consultants found that the mug's stability depends significantly on how much coffee is inside. Find the height or coffee level in the mug for which the mug of coffee is least likely to tip when the car accelerates. The mug is 20 cm tall, has a radius of 6 cm, and weighs 200 grams (1/4 of which is the weight contribution of the bottom of the mug) *[Sal]*

*[*a fictitious company. If you find the first true storey unlikely, compare with other true stories of human fallibility at the irreverant Darwin Awards homepage, http://www.darwinawards.com]*

6) Weight! Bob!

In science, either experiment or theory can lead the way in discovery. But it pretty much isn't *finished* science until both experiment and theory come together. Here's one example where a simple experiment can tell you the answer you might not be able to guess theoretically — and steer you in the right way to think, in order to understand.



Two identical weights are connected by a 1 m string. To begin, one weight is set on a frictionless table, and the second is held as in the picture, with the midpoint of the string between them draped over a massless frictionless pulley.

If the second weight is dropped, which happens first:

- does the first weight slide along until it hits the pulley, or
- does the second one swing around until it hits the side of the table?

Try this in an experiment, to find the true answer, and then see how well you can explain it by theory. Try whatever you like to make things practically frictionless as given — but tell us what *you* did in *your* experiment, to approximate these ideal conditions, and then carefully describe what you measured.

HINTS: To make the first weight practically frictionless on the table, you might try two child's wooden cars with wheels. Or you might be able to use an air-rail at school for the table and tie together two identical air-rail cars for the weights. Or perhaps a dry-ice puck on the table would be practically frictionless, and you could tie it to an equal weight to drop at the side. The frictionless pulley can be nearly any small lightweight wheel (no inertia!), or perhaps teflon tubing rotating on a nail — or you might use teflon-coated electronics wire to connect the weights, and then a smooth round edge on the table to slide over with little friction. *[Yaser, Robin]*

CHECK THE POPTOR WEB PAGE for other hints, and any corrections we might post:

www.physics.utoronto.ca/~poptor

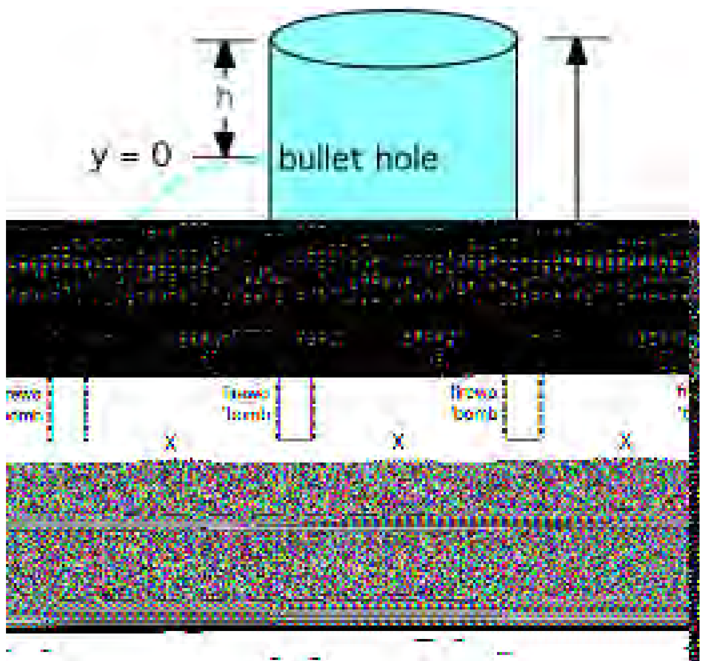
2000–2001 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 2: Mechanics

1) Take your best shot

Due to water pressure in the tank, water squirts out horizontally from the bullet hole at some speed, depending on the 'head' or height of water above where the hole is made. The water then falls a certain vertical distance. The horizontal distance the water reaches depends on the squirt-speed and the time it takes the water to fall — a time that only depends on height above the ground.



Let the hole the rifle makes in the tank be the origin of a vertical axis which we will set to zero. Let h be the height above the hole to the top of the tank.

As water flows out through the bullet-hole, the height of the water in the tank must drop. In a time Δt, the volume of water flowing through the bullet-hole is:

$$A_{flow} \times \text{distance-flowed} = A_{flow}\,(v_{flow} \times \Delta t)$$

where A_{flow} is the area of the hole, and v_{flow} is the flow-speed of water at the hole.

Likewise the water in the tank will drop by a volume:

$$A_{tank} \times \text{distance-dropped} = A_{tank}\,(v_{tank} \times \Delta t)$$

Equating these, we get:

$$v_{tank} = v_{flow} \times (A_{flow} / A_{tank})$$

As the water drains from the tank due to the bullet hole, the kinetic energy put into the squirting water simply equals the potential energy of lowering all the water in the tank from the hole up. In a small time Δt this is:

$$1/2\, M_{flow}\, v_{flow}^{\;2} = M_{tank}\, g\, h$$

$$½\,(\rho\, A_{flow}\,(v_{flow}\,\Delta t))\, v_{flow}^{\;2} = (\rho\, A_{tank}\, h)\, g\, (v_{tank}\,\Delta t)$$

Where ρ is the density of water. We can cancel terms, and use v_{tank} we found above, to get:

$$v_{flow}^{\;2} = 2\, g\, h$$

This is the initial horizontal speed. The initial vertical speed is zero. The time of flight for the water leaving the tank before it hits the ground 5 meters away from the base of the tank is given by:

$$t^2 = 2\,(20 - h)/g$$

This time needs to be exactly the value that lets the water squirt 5 m sideways, at constant speed v_{flow} , before hitting the ground.

$$v_{flow}\; t = 5\text{ m}$$

therefore,

$$4\, h\,(20 - h) = 25$$

Solving for h we get the result that in order for the water to reach the athlete 5 m away from the base of the tank, ***she must shoot a hole approximately 32 cm down from the top of the tank.*** *[Sal]*

2) The fountain of (misspent) youth

Assume that Suku is at height H. The speed of the water at this height, *before* the collision with the garbage can, is

$$v^2 = v_o^{\;2} - 2\, g\, H$$

where v_o = 20 m s^{-1} is the initial speed. After the collision, the water stops its motion, since the collision between a liquid and any surface is always inelastic. The change in momentum of the water during the collision acts provides the a force and compensates for gravity in order to hold the garbage can and Suku up. This force is equal to:

$$F = \Delta m * (v - 0)/\Delta t = 150\ (\text{kg/s}) * v$$

where Δm is the mass of the water which has a collision at a certain time. Δt is the collision time. Both Δm and Δt are supposed to be infinitessimals (vanishingly small numbers) but their *ratio* is actually equal to the rate of the water flow, which is 150 kg/s. This force is equal to the weight of the garbage can plus Suku.

$$F = (60 + 2.5) \bullet 9.8 = 150\, v = 150 \bullet \sqrt{400 - 2 \bullet 9.8 \bullet H}$$

From which we can find ***H = 19.5 m.*** Yikes! *[Yaser]*

3) Huh? Hmmm…

a) The water pressure at the bottom of all the containers is the same and equal to (ρ*g* H) where ρ is the density of the water. The pressure is dependent on the height of the water and not on the amount of the water. The total force which is exerted by the water on the bottom plate, by this pressure, is equal to the product of the pressure and area of the plate and is the same for all.

b) The weigh-balance shows three different values for the containers full of water. For the non-cylindrical containers, this value is *not* equal to the product of the water pressure and the bottom-surface area. For the cylindrical container, the water pressure on the side walls is everywhere *sideways*, and contributes nothing to the weight. This isn't true for the other containers.

For the tapered containers, not only water pressure exerts a force over the bottom plate: the rigid *walls of the container* also exert force. For the one in which the area of the top is bigger, this force is downward and therefore the total force on the bottom plate is *more* than the product of pressure and area. For the other container, in which the area of the top is smaller, this force is upward, and therefore the total force is *less* than than the product of the pressure and the area.

The easiest way to find the right answer is to imagine the sloped wall in a series of stair-steps, and then take the limit of the steps becoming vanishingly tiny.

Each step has a vertical part and a horizontal part. Pressure on the vertical part, which is a tiny cylinder, gives an outward force everywhere; these radial forces add up to zero net force. Pressure on the horizontal part, like a shelf around the bottom of the little cylinder, gives a net downward force, and these side-wall forces add up to exactly the weight of the water which lies above the sloped walls (i.e., outside the cylinder lying above the circular base).

The net force can be found by adding all the contributions. Each horizontal step is actually an *annulus* — a flat ring lying between two circles of radius r and $r + dr$.

For a wall sloped at an angle θ from the vertical, the radius r of the vessel as a function of vertical position z from the base is:

$$r = r_o + z \cdot \tan\theta$$

It's easy to find dr then (as the *differential* of r):

$$dr = dz \cdot \tan\theta$$

The area of the annulus is then:

$$\begin{aligned} A &= 2\pi r \cdot dr = 2\pi r \cdot dz \cdot \tan\theta \\ &= 2\pi(r_o + z\tan\theta) \cdot dz \cdot \tan\theta \end{aligned}$$

The downward force on this small annulus is then:

$$dF_y = p(z) \cdot A$$

where *p(z)* is the pressure, which is a function of z because z gives the depth. The pressure is simply the weight per unit area of all the water lying above, and so

$$p(z) = \rho g(H - z)$$

Thus

$$\begin{aligned} dF_y &= p(z) \cdot A \\ &= \rho g(H - z) \cdot 2\pi(r_o + z\tan\theta) \cdot dz \cdot \tan\theta \end{aligned}$$

The total force down on the side walls is then the sum of all these tiny contributions, for all values of z from z=0 to z=H. This is just the integral (siimpler than it seems — the integrand is only a polynomial, actually):

$$\begin{aligned} F_y &= \int_{z=0}^{z=H} dF_y \\ &= \int_{z=0}^{z=H} \rho g(H - z) \cdot 2\pi(r_o + z\tan\theta) \cdot dz \cdot \tan\theta \\ &= 2\pi\rho g \tan\theta \int_{z=0}^{z=H} (H - z) \cdot (r_o + z\tan\theta) \cdot dz \\ &= 2\pi\rho g \tan\theta \int_{z=0}^{z=H} \left\{ Hr_o + (H\tan\theta - r_o) \cdot z - \tan\theta \cdot z^2 \right\} \cdot dz \\ &= 2\pi\rho g \tan\theta \left\{ Hr_o z\Big|_0^H + \tfrac{1}{2}(H\tan\theta - r_o) \cdot z^2\Big|_0^H - \tfrac{1}{3}\tan\theta \cdot z^3\Big|_0^H \right\} \\ &= 2\pi\rho g \tan\theta \left\{ \tfrac{1}{2}H^2 r_o + \tfrac{1}{6}H^3 \tan\theta \cdot \right\} \end{aligned}$$

Naturally, as θ goes to zero (as for a cylinder), this vertical wall-force contribution goes to zero.

It is obvious that this down-force on the sidewalls is the same as the weight of the water in the ‘extra’ region outside the cylinder — going back to the term for pressure, all we have done is simply to add the weight of the water lying above every little horizontal annular step. Since we considered every little horizontal step, we’ve considered the whole volume of water in the extra region. *[Yaser & Robin]*

4) All aboard!

a) Assume the train experiences constant acceleration (deceleration).

Known quantities converted to SI units where appropriate:

Variable Symbol	Description	Numerical Value
X(0)	Initial displacement of train	0.00 m
X(t)	Final displacement of train	8000 m
V(0)	Initial velocity of train	65.0 m/s
V(t)	Final velocity of train	0.00 m/s

Unknown quantities:

Variable Symbol	Description
μ	Coefficient of rolling friction

the train experiences a rolling frictional force (proportional to its normal force) that is opposite to its direction of motion:

$$F = -\mu mg$$

$$F = ma$$

$$ma = -\mu mg$$

$$\mu = -\frac{a}{g} \qquad (1)$$

constant acceleration displacement equation:

$$X(t) = X(0) + V(0)t + \frac{1}{2}at^2$$

since *X(0)* = 0.00 m, then

$$X(t) = V(0)t + \frac{1}{2}at^2 \qquad (2)$$

constant acceleration velocity equation:

$$V(t) = V(0) + at$$

since *v(t)* = 0.00 m/s, then

$$V(0) + at = 0$$

$$t = \frac{-V(0)}{a} \qquad (3)$$

substitute equation (3) into equation (2) to find the train's constant acceleration:

$$X(t) = -\frac{V(0)^2}{a} + \frac{V(0)^2}{2a}$$

$$2aX(t) = V(0)^2 - 2V(0)^2$$

$$a = -\frac{V(0)^2}{2X(t)} \qquad (4)$$

substitute equation (4) into equation (1) to find the coefficient of rolling friction:

$$\mu = \frac{V(0)^2}{2gX(t)} = \frac{(65.0)^2}{2(9.81)(8000)}$$

$$\boxed{\mu = 0.0269}$$

SOLUTION FOR PART (B):

Note that although the rolling frictional acceleration μg is constant, since $F = \mu gm$, the rolling frictional force will not be constant if the train's mass is changing – that is: $F(t) = \mu gm(t)$.

Known quantities:

Variable Symbol	Description	Numerical Value
V	Constant velocity of train	1.00 m/s
m	Initial mass of train and hopper car	75000 kg
l	Length of hopper car	15.0 m
r	Mass transfer rate of grain	1000 kg/s
μ	Coefficient of rolling friction	0.0269

Unknown quantities:

Variable Symbol	Description
$m(t)$	Mass of train and hopper car as a function of time
$m(x)$	Mass of train and hopper car as a function of displacement
W	Work done by the train

mass of train and hopper car as a function of time:

$$m(t) = m + r\,t$$

convert *m(t)* to *m(x)* using the fact that $t = \frac{x}{V}$:

$$m(x) = m + r\frac{x}{V}$$

in order to maintain a constant velocity the train must exert a force that balances the rolling frictional force:

$$\sum F = F_{friction} + F_{train} = 0$$

$$F_{train} = -F_{friction} = -(-\mu\, mg) = \mu\, mg$$

since m is a function of displacement:

$$F(x) = F_{train} = \mu\, g\, m(x) = \mu\, g\left(m + r\frac{x}{V}\right)$$

the definition of work is:

$$W = \int_0^l F(x)dx$$

$$W = \mu g\int_0^l m(x)dx = \mu g\int_0^l \left(m + r\frac{x}{V}\right)dx$$

$$W = \mu g\left(mx + r\frac{x^2}{2V}\right)\Bigg|_0^l$$

$$W = \mu g\left(ml + r\frac{l^2}{2V}\right) = (0.0269)(9.81)\left((75000)(15.0) + (1000)\frac{(15.0)^2}{2(1.00)}\right)$$

$$\boxed{W = 3.27\times 10^5 \text{ J}}$$ *[Brian]*

5) Java jump-up (eeyow!)

Given that the weight of the mug is M and that the weight of the bottom of the mug is $\frac{M}{4}$ the height of the centre of mass ($y_{c.m.}$) of the mug with coffee of density ρ (~1 g cm^{-3}) filled to height h is given by,

$$y_{c.m.} = \frac{\left(\frac{3}{4}\frac{MH}{2} + \rho\pi R^2\frac{h^2}{2}\right)}{(M + \rho\pi R^2 h)}$$

Define a constant $a = \rho\pi R^2$ such that the mass of the coffee contained is $m = \rho\pi R^2 h \equiv a\,h$

$$\therefore \; y_{c.m.} = \frac{\left(\frac{3}{8}MH + \frac{ah^2}{2}\right)}{(M + ah)}$$

The maximum stability of the mug of coffee is achieved for the lowest centre of mass. To find the minimum of $y_{c.m.}$ we need

$$\frac{d(y_{c.m.})}{dh} = \frac{ah(M + ah) - (a)\left(\frac{3}{8}MH + \frac{ah^2}{2}\right)}{(M + ah)^2} = 0$$

$$\therefore \; h(M + ah) = \frac{3}{8}MH + \frac{ah^2}{2}$$

$$ah^2 + Mh - \frac{3}{8}MH = 0$$

Solving for h we get

$$h = \frac{1}{a}\left(-M \pm \left(M^2 + \frac{3}{4}MHa\right)^{\frac{1}{2}}\right)$$

the negative solution can be ignored.

$$h = \frac{1}{a}\left(-M + \left(M^2 + \frac{3}{4}MHa\right)^{\frac{1}{2}}\right)$$

Given:

$M = 200$ g, $\quad H = 20$ cm $\quad a = \rho \pi R^2 = 36\pi \, g/cm$

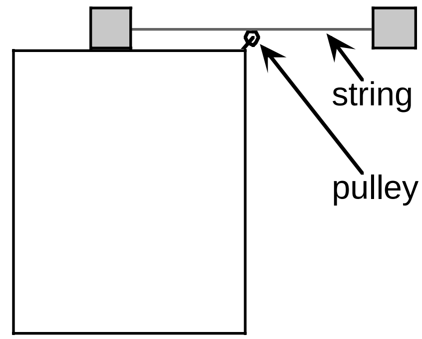
Solving for h we get,

$$h = \frac{1}{36\pi}\left(-200 + \left(40000 + \frac{3}{4}(200)(20)(36\pi)\right)\right)^{\frac{1}{2}}$$

$$h = 3.6\,cm$$

The mug is most stable with coffee filled to a level h = 3.6 cm. *[Sal]*

6) Weight! Bob!



In an experiment, the table-based weight hits the pulley before the free one smacks into the side of the table.

How to explain this?

- It's easiest to analyse the forces in x (horizontal) and y (vertical) components.
- The force on either block comes from tension in the string
- There's only one tension, so the forces on the block are equal in magnitude, though they may change in time
- The difference is that the force on the left-hand block is always directed along the string, towards the pulley. The force on the right-hand block is also always directed along the string, but this vector changes as the angle of the string changes.
- The left-hand block moves horizontally towards the pulley with an acceleration:

  $a_{left} = T/m$ where m is the mass of the block, and T is the tension in the string

- The right-hand block moves horizontally towards the wall with an acceleration:

  $a_{right} = T \cos(\theta) / m$

where m: mass of the block, T: tension in the string, θ: the angle the string makes with the horizontal

- Before the right-hand block falls, there is no tension in the string. Once it does begin to move, then it makes an angle $\theta > 0$, so $a_{right} < a_{left}$
- The left-hand block is always accelerated to its collision with the pulley faster than the right hand block to its collision with the wall.

The left-hand block hits first. *[Yaser, Robin]*

2000-2001 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 3: Thermodynamics

Due January 15, 2001

1) Bye, metallic bar!

Mechanical thermostats for home heating, and flasher units for car turn-signals (and Christmas tree lights!) often use this principle: Two strips, copper and iron, are bonded together, side by side. They share the same elastic properties (e.g., Young's modulus), and the same length l, and diameter d. They *don't* have the same linear thermal-expansion coefficient, though — if we heat them both to a higher temperature, one expands more than the other, and they bend. If the change in temperature is ΔT and the linear thermal-expansion coefficients are λ_{Fe} and λ_{Cu}, find the angle of deflection. *[Yaser]*

2) Maxing Maxwell

Have you ever wondered how fast molecules are flying about, maybe air molecules in your lungs? The atmosphere we breathe is composed of particles of different gases, the kinetic energy of these particles being related to their thermal motion. These gas particles collide with one another constantly, affecting their velocity. To speak of *the* velocity of a particle is therefore not very practical. What we can speak of though is a *probability distribution* of velocities — how likely it is to have one velocity or another.

A result known as *Maxwell's Distribution* describes how particles are distributed over different speed-ranges:

$$P(v) = 4\pi\left(\frac{M}{2\pi RT}\right)^{3/2} v^2\, e^{-Mv^2/2RT}$$

where M is the molar mass of the gas, T is the absolute temperature of the gas and R is a constant. The distribution gives the *probability* (maximum 1.0 = 100%) that a particle in the gas will have a velocity that falls between v and $v+dv$, where dv is an incremental.

a) Calculate the *most probable* velocity for molecules of oxygen at T = 300 Kelvin. Oxygen has a molar mass $M = 0.0320$ kg mole^{-1}. $R = 8.31$ J mL^{-1} K^{-1}.

b) Now graph the *Maxwell Distribution*. *[Sal]*

3) A pressing concern

There is cylinder filled by a gas, with area A. In the cylinder, there is piston with mass m on top which makes the height of the piston from the bottom of the cylinder a distance h. How much weight should we add onto the piston to reduce the height to half of its initial value? Assume the air pressure to be P_0 and the temperature of the gas is always room temperature. *[Yaser]*

4) Ringing the sphere

An aluminum sphere is heated to 100 °C. The diameter of the sphere, at this temperature, is 25.4508 mm. The sphere rests on top of a 20 g copper ring which has been cooled to 0 °C. At this temperature the inside diameter of the ring is 25.4 mm, so the sphere cannot pass through the hole.

A long time later, when the system reaches equilibrium (when the temperature of the sphere and ring are equal), the aluminum sphere is barely able to slip through the copper ring. Assuming there's no loss of heat to the surroundings, only conduction between the sphere and ring, calculate the mass of the aluminum sphere.

Linear coefficients of expansion

Cu = 431.8 nm K^{-1}
Al = 584.2 nm K^{-1}

Specific Heats

Cu = 386 J kg^{-1} K^{-1}
Al = 900 J kg^{-1} K^{-1} *[Sal]*

5) Reality Check

A 1-liter thin-walled stainless steel container is filled with helium gas at standard temperature and pressure. The container sits on a hotplate so that temperature of the container and gas can be controlled.

a) Not only the gas, but the container too, will expand when heated! Calculate the maximum pressure of the gas in the container, and the temperature at which it occurs, assuming ideal gas behaviour and linear thermal expansion of steel. What is the volume of the container at this temperature? The equation for thermal expansion in one dimension is:

$$\Delta x = \alpha\, x\, \Delta T$$

In this equation Δx is the amount of expansion, α is the expansion coefficient, x is the initial length and ΔT is the change in temperature.

$$\alpha(\text{steel}) = 1.1\text{x}10^{-5}\ \text{K}^{-1}$$

b) Do the values you calculated in part (a) make sense? If not, explain possible reasons why not. *[Brian]*

6) Chillin'

Sam likes to enjoy a tall glass of cold lemonade on a hot summer day. He would like to know in advance *how much ice* he'll need to put into a pitcher of lemonade in order to lower its temperature by the desired amount.

a) Calculate the volume of water Sam needs to freeze in order to make ice enough to lower the temperature of a litre of room-temperature lemonade by 10 °C. The temperature of the freezer Sam uses to make the ice is –10 °C. Assume the lemonade is initially at room temperature (20 °C), and is basically water, and that all the ice has melted before the temperature of the lemonade is measured.

b) Now *try it!* As an experiment, measure the change in temperature of a litre of water caused by a small ice cube. You may want to use a vacuum flask (e.g., Thermos™ bottle) to do this experiment, so that heat energy is conserved. Compare your measurements to theory, using the equations from part (a). Comment on your results.

<u>density</u>	$\rho(\text{water}) = 998 \text{ kg m}^{-3}$
	$\rho(\text{ice}) = 917 \text{ kg m}^{-3}$
<u>specific heat</u>	$C(\text{water}) = 4190 \text{ J kg}^{-1} \text{ K}^{-1}$
	$C(\text{ice}) = 2220 \text{ J kg}^{-1} \text{ K}^{-1}$
<u>latent heat</u>	$H_f(\text{ice}) = 333000 \text{ J kg}^{-1}$ *[Brian]*

Remember to check the POPTOR web-page for hints and any necessary corrections!

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Solution Set 3: Thermodynamics

1) Bye, metallic bar!

1) If the change in temperature is ΔT then the final lengths of the two strips are

$$l_{Fe} = l(1 + \lambda_{Fe}\Delta T)$$
$$l_{Cu} = l(1 + \lambda_{Cu}\Delta T)$$

Since they are bonded together, they have to bend in order to have different length. If they are assumed as an arc in which has a (subtended) angle of α, the angle of deflection from a straight line, measured at either end, would be $\alpha/2$.

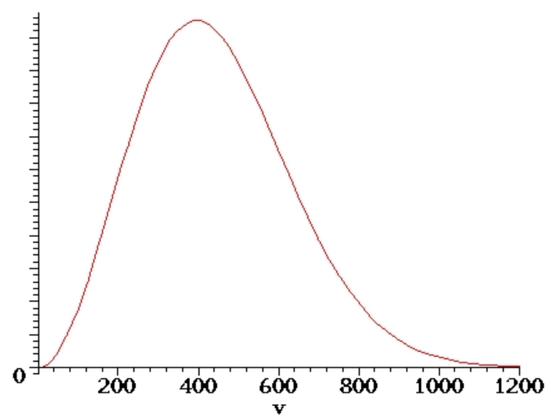
If the radius of the curvature, measured to the midpoint between the two strips, is R then we have (α in *radians*):

$$l_{Fe} = \left(R + \frac{d}{2}\right)\alpha$$
$$l_{Cu} = \left(R - \frac{d}{2}\right)\alpha$$

Combining all the formulas above, and eliminating R from them, we obtain the angle of deflection in radians to be

$$\alpha/2 = \frac{l\Delta T(\lambda_{Fe} - \lambda_{Cu})}{2d}$$ *[Yaser]*

2) Maxing Maxwell



The most probable velocity is the one for which there is a *peak* in the *velocity distribution* curve. To find the maximum of this distribution we find the local maximum: take its derivative with respect to velocity and set that equal to 0.

Solving for the velocity we get,

$$v = (2RT/M)^{1/2}$$

Oxygen gas at a temperature of T = 300 Kelvin with a molar mass of 0.0320 kg mol^{-1} has a *most probable* velocity of,

$$v \sim 395 \text{ m s}^{-1}$$

A plot of the velocity distribution of oxygen at a temperature T = 300 Kelvin is shown above. *[Sal]*

3) A pressing concern

If the initial pressure inside the cylinder is P_i, and the piston is at its equilibrium position, we must have a balance of forces:

$$P_0 A + mg = P_i A$$

When we put an additional mass M on top of the piston then we have

$$P_0 A + (m+M)g = P_f A$$

where P_f is the final pressure.

Since the temperature inside the cylinder is constant and equal to room temperature, the product PV of pressure and volume must finally turn out the same before and after putting the weight. Knowing that the height is reduced to half its initial value, we get

$$PV = P_i \cdot h\,A = P_f \cdot \frac{h}{2} A \qquad \Rightarrow \qquad P_f = 2P_i$$

Substituting P_f in above formulas we get the weight needed to be

$$Mg = P_0 A + mg$$ *[Yaser]*

4) Ringing the sphere

The relative change (per degree Celsius) in the diameter of the copper ring and aluminum sphere is associated with their respective linear coefficients of expansion. After some time the copper ring, initially at a temperature of 0 °C, will reach equilibrium with the aluminum sphere at some mutual temperature T_{eq}. The changes in temperature for each will result in changes ΔD in diameter D:

$$\Delta D = (T_f - T_i)\ D\ \alpha$$

where T_f and T_i are the final and initial temperatures of the copper ring or aluminum sphere. α is the linear coefficient of expansion of each metal.

We are told that when the system reaches equilibrium the diameter of the Al sphere equals the inner diameter of the Cu ring.

$$D_{Cu} = D_{Al}$$

Therefore,

$$D_{Cu}(\text{initial}) + \Delta D_{Cu} = D_{Al}(\text{initial}) + \Delta D_{Al}$$

Solving for the shared temperature T_f, we find that the system reaches equilibrium at a temperature $T_f = 50.38$ °C.

We are also given that the system reaches equilibrium with no loss of energy to the surroundings. This means that heat is only transferred between the metals — the heat lost by the aluminum sphere as it cools is absorbed by the copper ring as it warms.

Therefore,

$$C_{Cu}\,M_{Cu}\,(\Delta T_{Cu}) = C_{Al}\,M_{Al}\,(\Delta T_{Al})$$

Where C_{Cu} , C_{Al} and M_{cu} , M_{Al} are the specific heats and masses of the two metals.

We are given the mass of the copper ring, and the changes in temperature of both metals were calculated above. We can therefore solve the equation above for the mass of the aluminum sphere: $M_{Al} = 8.71 \times 10^{-3}$ kg. *[Sal]*

5) Reality Check

It is very important to *fully understand* where the equations we use come from. Sometimes students are tempted to use the "equation hunting" method when solving problems. This is a very dangerous practice since many of the equations we use (such as those found in textbooks) can be:

(1) A limited case of a more general equation
(2) An approximation to a more complicated expression
(3) Derived under certain assumptions
(4) All of the above

The "equation hunting" method always generates interesting solutions, but not always correct ones – in fact, some pretty crazy answers can result as you will soon find out! *Always* ask yourself if the answers you calculate *make sense*.

Known quantities converted to SI units where appropriate:

Variable Name	Variable Definition	Variable Value
T_o	Initial temperature of gas	300 K
P_o	Initial pressure of gas	101325 Pa
V_o	Initial volume of gas	1.00×10^{-3} m^3
α	Linear expansion coefficient for steel	1.10×10^{-5} K^{-1}

Unknown quantities:

Variable Name	Variable Definition
T	Temperature of gas at maximum pressure
P	Maximum pressure of gas
n	Number of moles of gas in the container

In this question we will assume ideal gas behavior (the "ideal" in ideal gas should raise some red flags here) and linear thermal expansion of an object in one dimension.

- Generate an expression for the volume of the container as a function of temperature:

  change of length in one dimension as a function of the change in temperature

  $$\Delta x = \alpha x_o \Delta T$$

  total length in one dimension as a function of the change in temperature

  $$x = x_o + \Delta x = x_o(1 + \alpha\Delta T) = x_o\left[1 + \alpha(T - T_0)\right] \tag{1}$$

  definition of volume

  $$V = x^3 \tag{2}$$

  substitute (1) into (2):

  $$V = x^3 = x_o^3\left[1 + \alpha(T - T_o)\right]^3 = V_o\left[1 + \alpha(T - T_o)\right]^3 \tag{3}$$

  where

  $$V_o = x_o^3$$

- Generate an expression for the pressure inside the container as a function of temperature:

  ideal gas law

  $$P = \frac{nRT}{V} \tag{4}$$

  substitute (3) into (4)

  $$P = \frac{nRT}{V_o\left[1 + \alpha(T - T_o)\right]^3} \tag{5}$$

- Solve for the temperature of the gas at maximum pressure:

  at maximum pressure $\frac{dP}{dT} = 0$

  $$\frac{dP}{dT} = \frac{nR}{V_o\left[1 + \alpha(T - T_o)\right]^3} - \frac{3\alpha nRT}{V_o\left[1 + \alpha(T - T_o)\right]^4} = \frac{nR\left[1 - \alpha(2T + T_o)\right]}{V_o\left[1 + \alpha(T - T_o)\right]^4}$$

  solve for *T*:

  $$\frac{nR\left[1 - \alpha(2T + T_o)\right]}{V_o\left[1 + \alpha(T - T_o)\right]^4} = 0$$

$$T = \frac{1}{2}\left(\frac{1}{\alpha} - T_o\right) = \frac{1}{2}\left(\frac{1}{(1.10\times10^{-5})} - 300\right) = 4.53\times10^4 \text{ K} \tag{6}$$

- Calculate the number of moles of gas in the container (this is a constant):

$$n = \frac{P_o V_o}{RT_o} = \frac{(101325)(1.00\times10^{-3})}{(8.31451)(300)} = 0.0406 \text{ mol}$$

- Calculate the maximum pressure – substitute (6) into (5):

$$P = \frac{nRT}{V_o\left[1+\alpha(T-T_o)\right]^3} = \frac{(0.0406)(8.31451)(4.53\times10^4)}{(1.00\times10^{-3})\left[1+(1.10\times10^{-5})\left(4.53\times10^4 - 300\right)\right]^3}$$

$$P = 4.58\times10^6 \text{ Pa} = 45.2 \text{ atm}$$

- Calculate the volume at maximum pressure – from (3)

$$V = V_o\left[1+\alpha(T-T_o)\right]^3 = (1.00\times10^{-3})\left[1+(1.10\times10^{-5})\left(4.53\times10^4 - 300\right)\right]^3$$

$$V = 0.00334 \text{ m}^3 = 3.34 \text{ L}$$

So the temperature of the gas and container would be roughly 40,000K, pressure is around 40 atmospheres and the volume of the container has tripled. Note that the *melting point* of steel is about 2,000K. Clearly the container would either melt or rupture before reaching the temperature, pressure and volume calculated above. Note, too, that for large changes in temperature, it is usually incorrect to assume a linear change in length. *[Brian]*

6) Chillin'

The physics: the ice absorbs heat (thermal energy) from the lemonade until the temperatures of the ice and lemonade become equal.

There are two types of energy involved in solving this problem: (1) *thermal-motion energy* and (2) *crystallization energy*. The first is the kinetic energy associated with the movement of molecules or atoms in liquids, solids and gases. Crystallization energy is the bond energy required to take apart a crystalline solid molecule by molecule or atom by atom – in this case, it is the energy required to convert ice (a solid) to water at constant temperature. Crystalline energy is also called *latent heat of melting*.

Known quantities converted to SI units where appropriate:

Variable Name	Variable Definition	Variable Value
$C(water)$	Specific heat of water	$4190\ \mathrm{J/kg/K}$
$C(ice)$	Specific heat of ice	$2220\ \mathrm{J/kg/K}$
$H_f(H_20)$	Heat of formation for H_20	$333000\ \mathrm{J/kg}$
$V(water)$	Volume of lemonade	$1.00\times10^{-3}\ \mathrm{m^3}$
$\rho(water)$	Density of water	$998\ \mathrm{kg/m^3}$
$T(water)$	Initial temperature of lemonade	293 K
$T(ice)$	Initial temperature of ice	263 K
T_f	Final temperature of lemonade and "ice"	283 K

Unknown quantities:

Variable Name	Variable Definition
$m(water)$	Mass of lemonade
$m(ice)$	Mass of ice
V	Volume of water needed to freeze in order to make ice

Solution:

- Calculate mass of lemonade:

$$m(water) = \rho(water)V(water) = (998)(1.00\times10^{-3}) = 0.998\ \mathrm{kg}$$

- Calculate mass of ice:

  As with all thermodynamic problems we need to begin our solution with a conservation of energy equation (energy cannot be created or destroyed). If we consider the ice and lemonade as a closed system (that is, no heat can get in or out of the container holding the lemonade and ice), then the change in internal energy of the ice plus the change in internal energy of the lemonade must equal zero:

$$\Delta E(water) + \Delta E(ice) = 0 \qquad (1)$$

- The heat absorbed by the ice from the lemonade takes place in three steps:

(1) Heat is absorbed to take the ice from its initial temperature up to 0 °C (273 K)

(2) Heat is absorbed to convert the ice to water at constant temperature 0 °C (this is the crystalline energy)

(3) Heat is absorbed to raise the resulting icewater at 0 °C to its final temperature

from equation (1)

$$m(water)C(water)\left(T_f - T(water)\right) + \begin{bmatrix} m(ice)C(ice)(273 - T(ice)) + \\ m(ice)H_f(H_20) + \\ m(ice)C(water)\left(T_f - 273\right) \end{bmatrix} = 0$$

solving for *m(ice)*:

$$m(ice) = \frac{m(water)C(water)\left(T_f - T(water)\right)}{C(ice)(T(ice) - 273) - H_f(H_20) - C(water)\left(273 - T_f\right)}$$

$$m(ice) = \frac{(0.998)(4190)(283 - 293)}{(2220)(263 - 273) - (333000) + (4190)(273 - 283)} = 0.105 \text{ kg}$$

- Calculate the volume of water Sam needs to freeze in order to make enough ice to lower the temperature of the lemonade by 10 degrees:

$$V = \frac{m(ice)}{\rho(water)}$$

$$V = \frac{(0.105)}{(998)} = 1.05 \times 10^{-4} \text{ m}^3 = 0.105 \text{ L}.$$ *[Brian]*

2000-2001 Physics Olympiad Preparation Program

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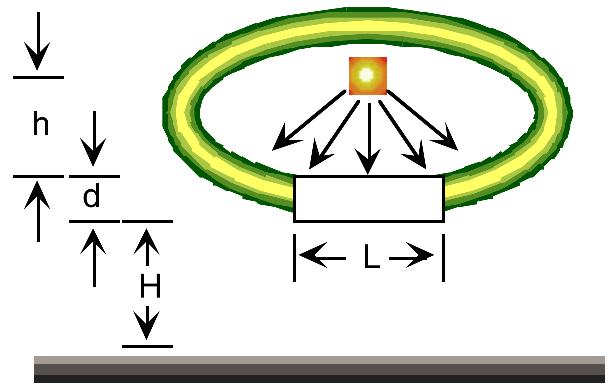
Problem Set 4: Optics and Waves

Due February 16, 2001 (revised date)

1) Sub hubbub

In our deep-sea research submarine *POPSub* there is a frosted (not clear) light bulb of radius R. With you in it, the sub is at a height H from the bottom of the sea, examining life on the ocean floor. The light bulb is at a height h from the square ($L \times L$) window which is d thick and made of a glass with index of refraction n.



a) Calculate the area at the bottom of the sea which is illuminated.

b) As you look out, how big does the spot *look*? *[Yaser]*

2) "Out, out damned spot!"

Each time there's a solar eclipse, a certain number of people suffer eye damage, or even are blinded, because they look directly at the sun. It isn't even necessarily safe at maximum, when the sun is blocked. To be safe, one needs to use the sort of goggles that prevent the same kind of eye-damage in welders, whose torches can be very very bright; there are other special filters and glasses, but some you might think are good aren't actually enough, since, *e.g.*, invisible infrared or ultraviolet may pass through and cause damage .

When you think about using a magnifying glass, in the sun, to burn wood or set fire to paper, it really isn't very surprising your eyes are in danger. Direct sunlight is about 1 kW m^{-2}, at the earth — how about calculating the intensity of light *focussed* from the sun using a lens?

Your eyeball is roughly the size of the circle you make with thumb and forefinger placed tip-to-tip, as when you make an 'OK' sign. Make a guess, from that, of roughly what the focal length is of the lens-system of your eye, and let's figure the numbers:

a) Estimate the size of your eye's *pupil* when it's at its smallest, and figure out the power in watts that would shine through if you looked straight at the sun (don't do this by experimenting!). Now, using the focal length of the lens of your eye, determine the size of the image of the sun formed at the back of your eye. Then find the intensity of light that this gives.

It could be worse: if the sun gave the same intensity of light but was much smaller or much farther away, the focussed image would be much smaller and the intensity much higher. That's because all the rays of sunlight would be practically parallel, and focus nearly in the same place. But lasers give nearly parallel rays, so they *will* focus nearly to a spot…

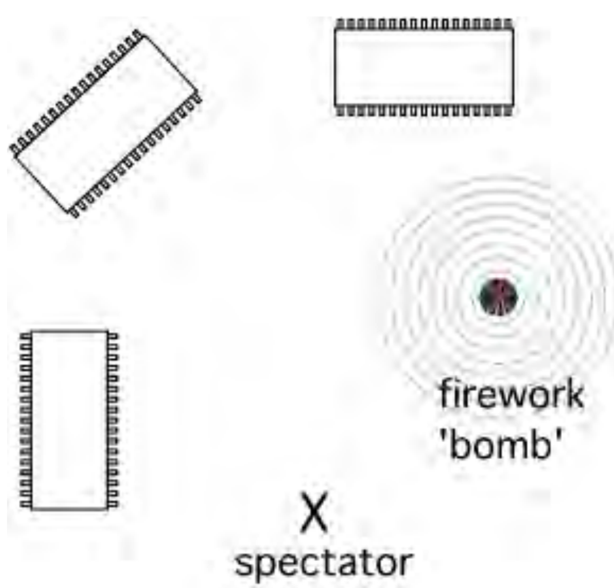
b) Consider a laser-pointer with 5 mW total output and a beam about 1 mm across. It doesn't focus to a point, quite, because of *diffraction*. So, if all the light goes into your eye, what is the focussed intensity at the back of your eye?

This value is about the maximum intensity for a Class IIIa laser. This can cause temporary loss of vision, like a camera-flash going off in your face; beyond this power eye-damage can result. *[Robin]*

Laser Safety: http://www.adm.uwaterloo.ca/infohs/lasermanual/documents/tblcont.html
Classifications: http://www.adm.uwaterloo.ca/infohs/lasermanual/documents/section8.html

3) Musical buildings

I used to enjoy watching fireworks on the 4th of July in Rochester, NY, where I went to graduate school. I noticed something quite interesting, though: when the loud 'bomb' fireworks went off, making a noise like a cannon, the buildings would play music. Reflected from several different buildings, the noise sounded something like the grating on a bridge when your car drives over — a brief rough note of music.



This was because several buildings in Rochester had long vertical ribs in their architecture, running the height of the buildings, separating sets of windows. The 'bang' sound would reflect off each rib, and the reflections coming to me would make a note like running a pencil along a comb.

a) What frequency should you hear reflected? Consider a building with rib-spacing D at a distance L_1 from the 'bomb' and L_2 from the spectator. You may need to know also the angles θ_1 and θ_2 from the building of these two paths. You may need to make L_1 and L_2 large, to simplify.

b) There's a different 'picture' of how this works: that the single bang of a firework 'bomb' is a sound with many many different frequencies in it, rather than a single note — something like white light, as opposed to coloured light. What frequency would you expect to hear, standing in the same place as (a), if all the different wavelengths of sound simply diffracted from the 'grating' made of building-ribs? *[Robin]*

4) A high-tech diet, high in optical fiber...

The internet and most telephone communication depends most heavily not on copper wire, nowadays, but on *optical fiber*. Rather than electrical pulses in wire, for signals, optical fiber communications use laser pulses propagating in tiny flexible glass strands.

Think of the following model of a laser pulse propagating in an optical fiber. Take a glass rod with a circular cross-section of radius r. Say that light is launched into the fiber through the end, and that all rays lie within a cone of angle θ measured from the axis of the rod. The index of refraction of the fiber is n.

a) If the input pulse is very short-duration, what is the duration of the pulse of light coming out at the end of the fiber? Assume the length of the fiber to be L.

b) If the input light is a *train* (series) of pulses with duration τ, and time-separation Δt, what should be the *maximum* length of the fiber in order that the output pulses are still distinguishable. This is a real issue for communications, since we need to know the fastest digital signal that can be sent.

c) If the peak intensity of a single input pulse is I, what is its output peak intensity. *[Yaser]*

[check the POPTOR web-site for an example of sound in a tube doing the same thing as this...]

5) Wave goodbye

Hong and her brother Liang have a clothesline in their back-yard. They mentioned to me their experiments with their giant one-string guitar... If they hit the cable sharply with a stick, what they see is two waves that run away from the impact-place, one in either direction. They want to know why — can you explain for them?

What they see is called a *travelling wave*. The shape of the wave stays the same, as it simply moves along the wire as if it were sliding. A travelling wave has the form:

$$f(x,t) = f(x+ct)$$

$$or$$

$$f(x,t) = f(x-ct)$$

If you look closely, you'll see that the top one is a wave with shape *f(x)* travelling to the right (*increasing x* with time), and the lower one is the same shape, but travelling to the left (*decreasing x* with time). In fact, almost any shape will work.

There are some very interesting properties of such travelling waves:

1) if you have a shape that works as a wave, then the same shape would work as a wave if it were bigger or smaller by any factor *a.*

2) if you have two different shapes that work as travelling waves, then if you add the two together, the sum will also work as a wave.

These are properties of what we call a *linear* system. They're true, for instance, for sound waves in air (except for explosions and such). If they weren't true, we couldn't listen to music (think about that...).

a) Say that Hong and Liang suddenly put a 'dent' $f(x)$ in the shape of the clothesline by quickly striking with the stick. How do you figure out the two waves that run away in either direction? Look for a trick or idea that uses properties (1) and (2) above to help prove what happens; you don't have to use much math.

b) Say that the clothesline is tied firmly at both ends. Then the position of the cable at the end cannot move. What happens when a wave travels right to the end of the cable? Try the same kind of ideas that work for (a).

c) Say that at one end the clothesline is tied to a ring that slides up and down an upright pole; say that this slides perfectly without friction., and the ring has negligible mass. This is a different *boundary condition*. What happens when a wave travels right to the end of this cable? *[Sal/Robin]*

6) Ten yards for interference…

Many people have home theatres these days and although technology makes having a home theatre relatively simple, in order to get the maximum effect one still has to take into account the geometry of the setup. Consider a basic home theatre setup, with distance $2d$ between the speakers, and a distance L from the speakers to the opposite wall of the room. Assume the speakers are point sources, that each puts out exactly the same single-frequency pure sine wave (this must be a pretty boring movie, or a test of the Emergency Measures Systems…) and that the walls do not reflect any sound.

a) Find a general expression giving the position(s) P of maximum intensity along the opposite wall as a function of speaker spacing d and room length L. Give the position(s) P in terms of distance y from the centreline between the speakers. Do not assume $d \ll L$.

b) If $L = 5$ m, $d = 3$ m and both speakers are producing an in-phase 320 Hz sound wave, calculate the position in one direction of the first three points of maximum intensity along the opposite wall, as measured from the centreline between the speakers. The speed of sound in air is 331 m/s.

HINT: This question involves solutions to an equation that can't easily be written as a tidy equation $y = f(x)$, but more like $0 = f(y) + g(y)$. You're welcome to solve the equations graphically — plotting, and then looking for roots. You're also allowed to solve by hand iteratively, or get a computer to solve the equation numerically. *[Brian]*

INFOBITS™ — Useful Bits of POPTOR Information

You don't *need* this to solve question 5, but it's interesting.

Waves satisfy a differential equation (an equation of derivatives) called *the wave equation*:

$$\frac{\partial^2 f(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f(x,t)}{\partial t^2} = 0$$

It is actually a very simple equation — you just need to take the derivative twice. The unusual notation is because these are *partial derivatives*. The *partial* derviative with respect to x means that you take the derivative with respect to the x-variable (totally ignoring the t-variable). So, for instance,

$$\frac{\partial \sin(x+3t)}{\partial x} = \cos(x+3t)$$

but

$$\frac{\partial \sin(x+3t)}{\partial t} = 3\cos(x+3t) \qquad \text{(using the chain rule)}$$

You just pretend the other variable doesn't even exist.

A *travelling wave* can always be described by $f(x,t) = f(x \pm ct)$, if f is any function that can be differentiated properly. It gives the shape f(x) in space, travelling in the direction of *increasing* x for $f(x - ct)$ and *decreasing* x for $f(x + ct)$.

Remember to check the POPTOR web-page for hints and any possible corrections!

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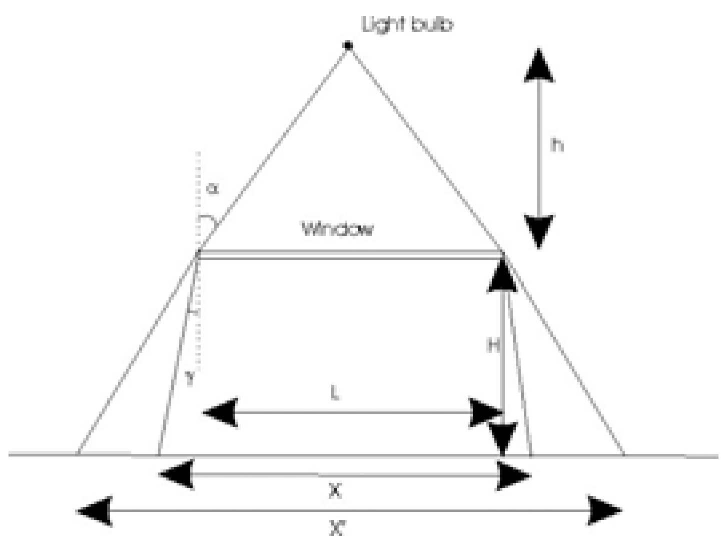
Solution Set 4: Optics and Waves

1) Sub hubbub

This question is supposed to be the easiest one in the set. If you want to consider the size of the window and the light bulb, you'll find the geometry a bit fussy without teaching very much. The reasonable approximation is that the light bulb is tiny compared to other sizes, like the distance from window and therefore its size should be neglected. A thick window also can be solved as we do below (with an extra refraction surface), but let's take the window to be thin compared to other distances, like h and the height H from the bottom of the sea.



The area at the bottom of the sea which is illuminated, is square in shape, each side is equal to an unknown, X. The angles that the rays traveling in the sub and sea makes with the normal are α and γ. These ray which hit the edge of the window determine the size of the illuminated area at the bottom of the sea. X is equal to

$$X = L + 2(H + d)\tan\gamma$$

From Snell's law

$$\sin\gamma = \frac{\sin\alpha}{n_w}$$

$$\Rightarrow \tan\gamma = \frac{\sin\alpha}{\sqrt{n_w^2 - \sin^2\alpha}}$$

where n_w is the index of refraction of the water. From the geometry of the system we have

$$\sin\alpha = \frac{L}{\sqrt{L^2 + 4h^2}}$$

$$\Rightarrow \tan\gamma = \frac{L}{\sqrt{(n_w^2 - 1)L^2 + 4n_w^2 h^2}}$$

Thus,

$$X = L\left(1 + \frac{2(H+d)}{\sqrt{(n_w^2 - 1)L^2 + 4n_w^2 h^2}}\right)$$

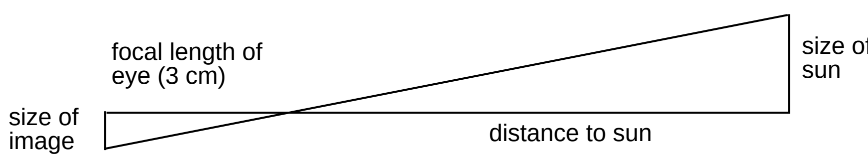
If you were in the sub you would see the illuminated area formed by the continuation of the ray, which is in the sub. Therefore, the length of any side of the square is

$$X' = 2(H+d+h)\tan\alpha = \frac{L(H+d+h)}{h}$$ *[Yaser]*

2) "Out, out damned spot!"

Most of the focussing comes from the curved shape of your eyeball at the *cornea*, and the lens of the eye mostly 'touches up' the focussing. Makes little difference for this question.

a) The focal plane of the eye is located approximately 22 mm behind the cornea or front of the eye. The iris controls the size of the pupil and on a bright day a typical pupil size is 2 mm in diameter. Approximating the eye lens as a "thin lens" we can use Newton's form of the thin lens equation (a similar-triangles law, actually):



$$H_i \,/\, H_o = f \,/\, x$$

Where:
- H_i height of the image;
- H_o height of the object (radius of Sun $\cong 7 \times 10^8$ m)
- f focal length of the lens
- x distance of the object (distance Sun-Earth $\cong 1.5 \times 10^{11}$ m)

The size of the image on the focal plane of the eye is;

$$H_i = (H_o \bullet f) \,/\, x \cong 10^{-4}\ \mathrm{m}$$

The area of the image on the focal plane is therefore;

$$A \cong \pi \bullet 10^{-8}\ \mathrm{m}^2$$

The pupil of the eye, approximately 2 mm in diameter under bright conditions will allow into the eye approximately;

$$P \cong (3 \bullet 10^3\ \mathrm{W/m^2}) \bullet (\pi \bullet 10^{-8}\ \mathrm{m}^2) \cong 3 \bullet 10^{-3}\mathrm{W}$$

This is about the power of a high-school HeNe laser. The resulting intensity in the focus on the retina of the eye is:

$$I \cong P / A \cong 3 \bullet 10^5 \, W/m^2$$

About 300 times concentrated from direct sunlight. If your pupil has not closed down fast enough, this can be much higher — more like 1000 kW m^{-2}.

b) In the problem above, the image size was determined by geometry — actually by the angle the sun subtends at your eye, which is the angular size, responsible for *parallax*. When you consider a laser with practically parallel light rays, the focus should be essentially a *point*, except for the fact that there is *diffraction* of the light, which makes the final focal spot slightly blurry.

The formula for this is:

$$d = 2 \lambda f / a$$

where d is the focal spot diameter, λ is the wavelength, f is the focal length, and a is the beam diameter at the lens. Often the quantity (f/a) is called the *f-number*. For our case, $\lambda \sim 700$ nm (red light), f = 3 cm, a = 1 mm; then d = 42 μm. The laser power of 5 mW all goes into this spot, for an average intensity of 3.6×10^6 W m^{-2}.

[EXTRAS: How do you understand this formula for diffraction? Only an infinite plane wave propagates without any change at all — but it extends infinitely! For any real wave, there is a general relationship:

$$\Delta x \, \Delta k \sim 1$$

where Δx is the *uncertainty in the position*, and Δk is the *uncertainty in the wavevector*, related to the momentum; it's direction is the direction of propagation. For a plane wave, Δx is infinite, so Δk can be zero — you know exactly where the wave is going (but not at all any special place where it is). When a plane wave passes through a slit, or the pupil of your eye, the hole size sets a limit on Δx, and so Δk cannot be zero. The wave doesn't lose any momentum, going through the hole, so what happens is the *direction* of k spreads out — the wave propagates not just in a straight line, but it diffracts from the aperture. The same thing applies for the focal spot, if you trace backwards: light from a tiny focal spot spreads out faster than from a big focal spot. So tiny focal spots go together with smaller f-numbers.

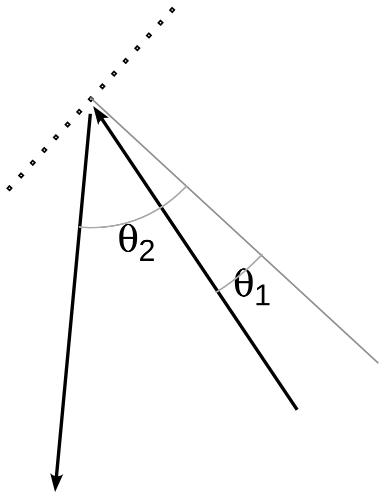
$$\tan\theta = (a/2)/f$$

where θ is the half-angle of the spreading out of the light. Then the formula for d can be written as

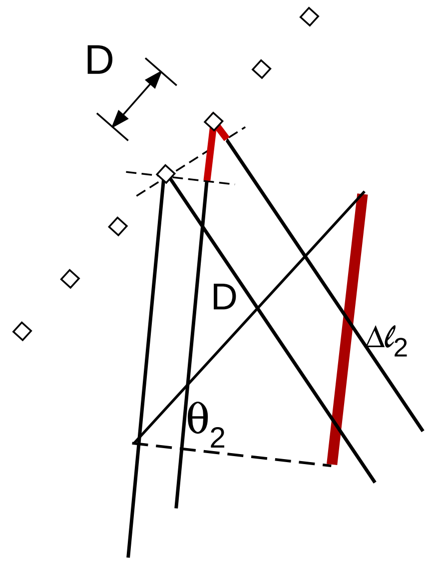
$$d \bullet \tan\theta \sim 1$$

It makes sense, when you realize that d = Δx (the width) and $\tan\theta = \Delta k$ (the spreading).]
[Sal & Robin]

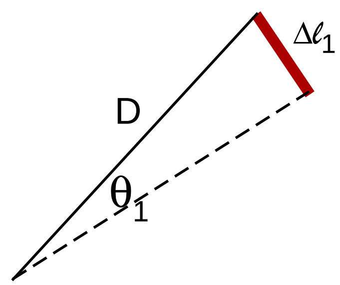
3) Musical buildings



a) The question is pretty easy if you let the distances be large. Then the rays are parallel, and the solution is much like the way you solve for interference in a thin layer, like oil on water — you need to examine the path-differences.



The figure at left shows the basic setup of angles for incoming and outgoing sound waves; I draw them as rays here. The figure at right illustrates the pathlength differences: the straight lines across the path mark where paths are equal, and the extra distance one ray must travel is marked by the fat line (in red, if you have a colour version of this). This extra pathlength is from two parts: extra distance coming in, and extra distance coming out from the building-grating.



The figure at left shows the details for the incoming, and the figure at right shows the detail for the outgoing distance. The total extra path-length is $\Delta\ell_1 + \Delta\ell_2$:

$$\frac{\Delta\ell_1}{D} = \sin\theta_1 \quad \textit{and} \quad \frac{\Delta\ell_2}{D} = \sin\theta_2$$

$$\Delta\ell = \Delta\ell_1 + \Delta\ell_2 = D\cdot(\sin\theta_1 + \sin\theta_2)$$

At the speed of sound C_s, one reflection will be later than the other by a time T:

$$T = \frac{\Delta\ell}{C_s} = \frac{D}{C_s}\cdot(\sin\theta_1 + \sin\theta_2)$$

In fact, all reflections are separated in time by this — the 'bang' of the fireworks burst comes to the listener as a series of pulses, and T is the *period* of this repetitive signal. The frequency is then:

$$\nu = \frac{1}{T} = \frac{C_s}{D\cdot(\sin\theta_1 + \sin\theta_2)}$$

For a building with ribs about 2 m apart (roughly the size of a window), with $C_s = 300$ m s^{-1}, and $\theta_1 = 22°$ and $\theta_2 = 45°$, this gives $\nu = 140$ Hz. That's a fairly low note, but not as low as the famous 40 Hz organ note in Bach's Tocatta and Fugue in D-minor which isn't actually present on many recordings…

It isn't necessary to assume that the distances L_1 and L_2 are very large. If they aren't, and interesting thing happens: not all the periods T between notes are exactly the same, but they increase or decrease as the 'bang' ripples off all the ribs of the building. (This is not difficult to show — just re-draw the diagrams above, with rays coming from a point.) As a result, the note that you hear actually shifts a little while you're listening. The technical term for this is 'frequency chirp', like the whistle of a bird may rise or fall.

b) The fireworks 'bang' has many frequencies in it, so another way to imagine this same problem is to compare it to white light hitting a diffraction grating and spreading out — like a room light diffracting from a compact disc. In that case, it's easy to find what frequency you'll hear, in a given place. It's just the same wavelength formula as for a diffraction grating:

$$n\lambda = D \cdot (\sin\theta_1 + \sin\theta_2)$$

where n is the *order* of diffraction (check this out on a CD, and you'll see several rainbows in the colours from a point-source light).

In terms of sound, the note you hear is usually determined by the lowest frequency, or longest wavelenth — the *fundamental*. The other wavelengths are shorter; these higher frequencies are then multiples of the fundamental, called the *overtones*. These overtones determine the *timbre* or basic sound-quality of an instrument, like flute vs. oboe.

For the fireworks, the fundamental frequency is:

$$\nu = \frac{C_s}{\lambda} = \frac{C_s}{D \cdot (\sin\theta_1 + \sin\theta_2)}$$

which is the same formula we found before. *[Robin]*

4) A high-tech diet, high in optical fiber...

a) There are many paths light can take down the fiber. That ray of light which moves in a straight line parallel to the axis of the fiber reaches the other end of the fiber earlier that some ray which has an angle θ with respect to the axis, and which travels more in a zig-zag pattern to get to the end of the fiber. The time it takes for the straight ray to travel a distance L along the fiber is

$$t_1 = \frac{nL}{c}$$

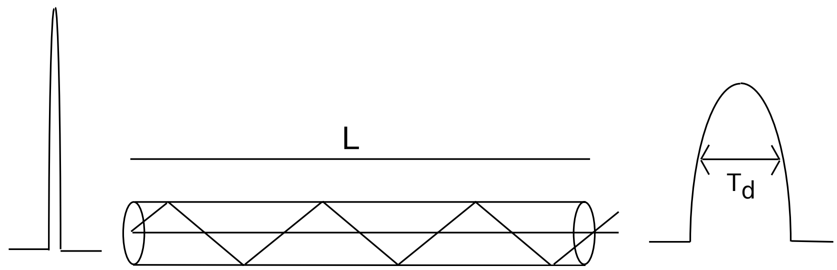
where *c* is the speed of light. The other ray has to travel a longer distance:

$$L/\cos\theta$$

and it gets to the other end at a time

$$t_2 = \frac{nL}{c\cos\theta}$$

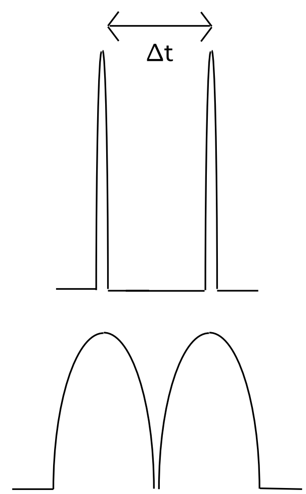
The duration of the pulse at the output of the fiber is usually measured as the width of the pulse measured halfway down from the peak of the pulse — the full-width at half-max (FWHM). To find this exactly, one needs to know how the input light goes into the different rays, but it will be roughly half of the time delay between the first rays to arrive and the last rays to arrive:



$$\tau_d = \frac{t_2 - t_1}{2} = \frac{nL}{2c}\left(\frac{1}{\cos\theta} - 1\right)$$

where τ_d is the pulse duration. This effect is usually called *modal dispersion*. We assumed in our calculation that the initial duration of the input pulse is much smaller that the output pulse, *i.e.*, much smaller than this spreading–out by different rays. Otherwise the time-stretching adds to the original pulse duration the way that two sides of a right-angle triangle give the hypotenuse:

$$\tau = \sqrt{\tau_p{}^2 + \tau_d{}^2}$$



b) Here we again assume that the initial pulse duration is much smaller than the dispersion broadening. In order that we can tell the pulses apart, after they broaden, they shouldn't overlap with each other much. Therefore, the duration of the output pulses should be smaller than roughly half of the time separation between them.

$$\tau_d \le \Delta t/2 \quad \Rightarrow \quad L \le \frac{c\Delta t}{n(1/\cos\theta - 1)}$$

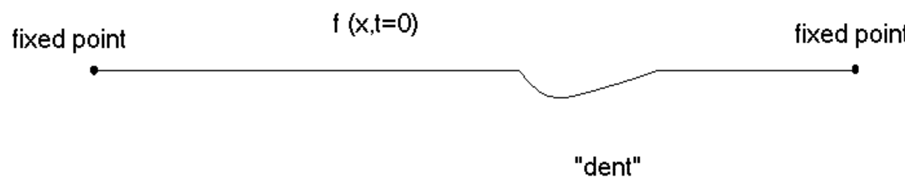
c) The energy of a pulse is equal to the integral of the intensity in terms of the time. If the pulse looks similar to a Gaussian pulse, we can approximate the energy of the pulse by the product of the peak intensity and the pulse duration. If the fiber is lossless, we expect the energy of the output pulse to be equal to the input pulse. Therefore,

$$I_{peak}^{in}\tau = I_{peak}^{out}\tau_d \Rightarrow I_{peak}^{out} = \frac{I\tau}{\Delta t}$$

where $I_{peak}^{in} = I$ and I_{peak}^{out} is the peak intensity of the input and output pulses. We assume that the length of the fiber is the maximum length such that the output pulses are distinguishable. *[Yaser]*

5) Wave goodbye

a) Let us assume that the shape of the "dent" after the stick strikes the clothesline is given by f(x, t=0).



An example of such a "dent," frozen in time, is shown in the figure above.

From the properties of waves given in the question, such a f(x, t=0) can be the instantaneous result of the addition of *two* identically shaped pulses, each *half* the size of the original "dent" — one pulse travelling to the left and the other to the right with the same speed.

$$f(x, t=0) = 1/2\, f_1(x) + 1/2\, f_2(x)$$

where $f_1(x, t=0) = f(x, t=0)$ and $f_1(x, t) = f_1(x+vt)$

$f_2(x, t=0) = f(x, t=0)$ and $f_2(x, t) = f_2(x-vt)$

Once a solution, always a solution: the dent created instantaneously immediately decomposes into these two travelling waves after the line is struck. This explains the two waves you will see emerge, after impact, travelling in opposite directions.

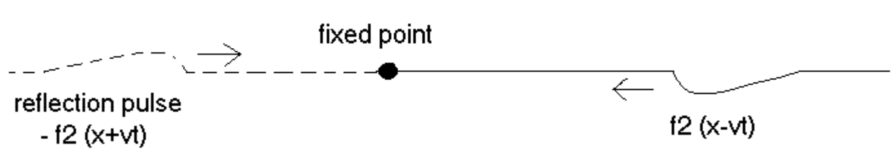
b) The fixed ends of the clothesline impose some conditions on the evolution of the waves. For all times t, the actual wave functions f_1, f_2 must be constant at the fixed ends.

$$f_1(x = L_1, t) = 0 \qquad [1]$$

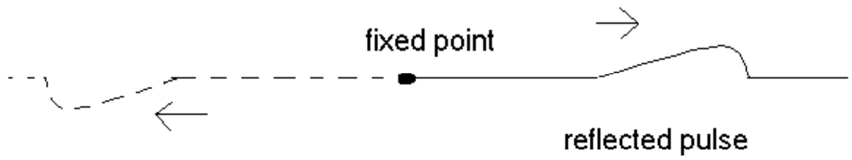
$$f_2(x = -L_2, t) = 0 \qquad [2]$$

where L_1 and $-L_2$ are the positions of the fixed ends relative to the position of impact. The following trick can satisfy conditions (1) and (2) while revealing something familiar from experience.

For any wave f_2, imagine a matching wave which is a negative version, and travelling in

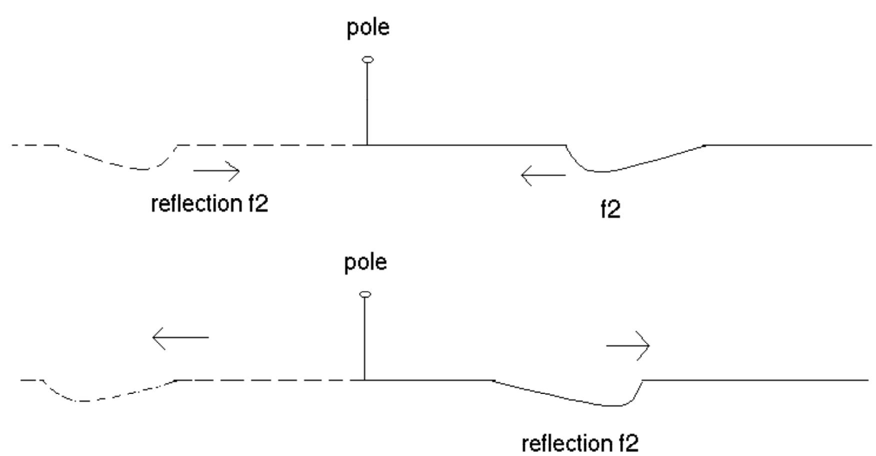


the opposite direction toward the fixed end in some imaginary extension of the wire (see the figure below). There is no fixed point in this picture. As the wave and its negative counterpart approach the spot which *ought* to be fixed, the waves will always add up to zero — satisfying the condition for the fixed point!



What you observe *mathematically* is the two waves passing each other and leaving the tied-point fixed. Since the point in the middle never moved, we could have nailed it down — it's our fixed end. So when a *real* wave goes into a fixed end, the resulting wave is the negative wave coming back from the fixed point, just the solution you'd have from the doppleganger.

c) Using a similar trick as in part (b), we consider superimposing a waves f_2 with a positive clone coming the other way. This will 'explain' or model for us reflections at a sliding ring on a pole. No longer do we have restrictions (1) and (2): at the sliding ring, they're replaced by the condition that there can be *no vertical forces* — only horizontal ones (or the massless ring would immediately move in response and the vertical force would vanish).

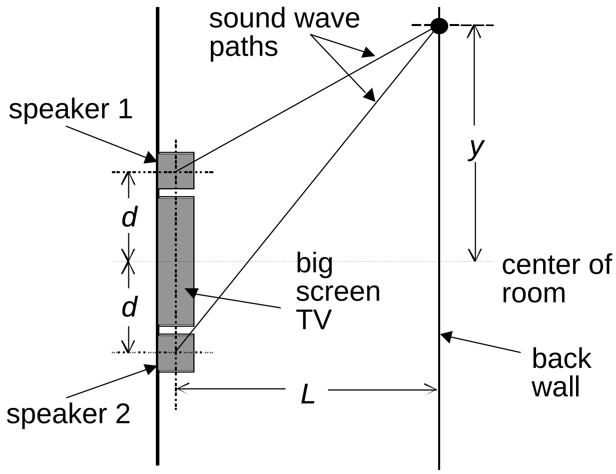


With only a *horizontal* force possible, the tension in the string, for the right solution, must give a horizontal force: so the string itself must be horizontal. That's fine — our doppleganger wave now is always a mirror image of the real one, and therefore *symmetric*. When these two add, the tangent to the string at the place of the pole is always horizontal; this is true for any smooth wave adding with its mirror image.

The waves, as they add, rise to a height of twice the amplitude of each. So the ring at the pole rises to twice the height of the wave that travels to it, and then the wave coming back will be identical to the one going in. *[Sal]*

6) Ten yards for interference…

The Physics: In this question, we do not assume the spacing $2d$ between the speakers is very small when compared to the distance L from the speakers to the back wall. If the difference in path length from speaker 1 to point P and speaker 2 to point P is an integral multiple of the sound wavelength, there will be constructive interference.



a) Path lengths from speaker 1 to point P and speaker 2 to point P can be determined using the Pythagorean theorem:

Speaker 1 to point P: $d_1 = \sqrt{(y-d)^2 + L^2}$

Speaker 2 to point P: $d_2 = \sqrt{(y+d)^2 + L^2}$

Therefore, the difference in path length is:

$y \geq 0$: $\delta = d_2 - d_1 = \sqrt{(y+d)^2 + L^2} - \sqrt{(y-d)^2 + L^2}$

$y \leq 0$: $\delta = d_1 - d_2 = \sqrt{(y-d)^2 + L^2} - \sqrt{(y+d)^2 + L^2}$

For constructive interference, the difference in path length must be an integral multiple of the wavelength ($\delta = m\lambda$):

$$y \geq 0: \quad \sqrt{(y+d)^2 + L^2} - \sqrt{(y-d)^2 + L^2} = m\lambda = m\frac{v}{f} \qquad (1)$$

$$y \leq 0: \quad \sqrt{(y-d)^2 + L^2} - \sqrt{(y+d)^2 + L^2} = m\lambda = m\frac{v}{f} \qquad (2)$$

where $m = (0,1,2,\ldots,\infty)$

In equations (1) and (2), m is the order of the interference (*i.e.*, n=2 means they interfere by being *two* waves out of phase, f is the frequency of the sound wave, v is the speed of sound the medium and λ is the wavelength of sound in the medium.

b) We need to solve for y using equations (1) and (2) when L = 5 m, d = 3 m, f = 320 Hz, v = 331 m/s:

$$y \geq 0: \quad \sqrt{(y+3)^2 + 5^2} - \sqrt{(y-3)^2 + 5^2} = m\frac{331}{320}$$

$$y \leq 0: \quad \sqrt{(y-3)^2 + 5^2} - \sqrt{(y+3)^2 + 5^2} = m\frac{331}{320}$$

The first maximum occurs when $m = 0$:

$$y \geq 0: \qquad \sqrt{(y+3)^2+5^2} - \sqrt{(y-3)^2+5^2} = (0)\frac{331}{320} = 0$$

$$y \leq 0: \qquad \sqrt{(y-3)^2+5^2} - \sqrt{(y+3)^2+5^2} = (0)\frac{331}{320} = 0$$

by inspection, both equation yield $y = 0$ at the first maximum. The second and third maximum occur when $m = 1$:

$$y \geq 0: \qquad \sqrt{(y+3)^2+5^2} - \sqrt{(y-3)^2+5^2} = (1)\frac{331}{320} = \frac{331}{320} \tag{3}$$

$$y \leq 0: \qquad \sqrt{(y-3)^2+5^2} - \sqrt{(y+3)^2+5^2} = (1)\frac{331}{320} = \frac{331}{320} \tag{4}$$

Not everyone will find it easy to determine a regular *closed form* solution for y in equation (3) and equation (4) (some did!). There are, however, a number of easy ways to solve these equations:

- plug in values for y in the equation and iteratively converge to zero in on a solution
- graph the left hand side and right hand side the equation to see where the curves intersect
- use a math program to solve the equation numerically.

I have solved equation (3) and equation (4) numerically using a math program called Maple. The answers I got are:

$y \geq 0$: $\quad y = 1.02$ m

$y \leq 0$: $\quad y = -1.02$ m

Therefore, the three points of maximum intensity closest to the centerline are:

y = {–1.02 m, 0 m, 1.02 m} *[Brian]*

2000-2001 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 5: Electricity and Magnetism

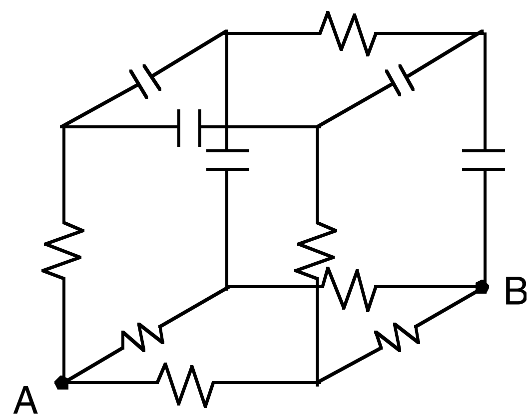
Due *March 9, 2001 (revised date)*

1) Bridging the gaps

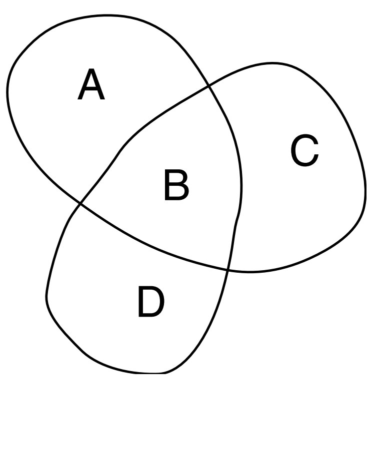
Consider an insulated wire of length L wound on a metal cylinder. The insulation breaks at some hidden place, and the wire inside touches the cylinder. Using a battery, an ammeter and two variable resistors, figure out where along the wire is the break. *[Peter]*

2) Roadkill this question

In the circuit drawn at right, all circuit elements lie on the edges of a cube. All resistors have identical resistance R, all capacitors have identical capacitance C. The voltage between points A and B is V. Figure out what charge is produced on the capacitor next to B. *[Yaser]*



3) Loopy circuits

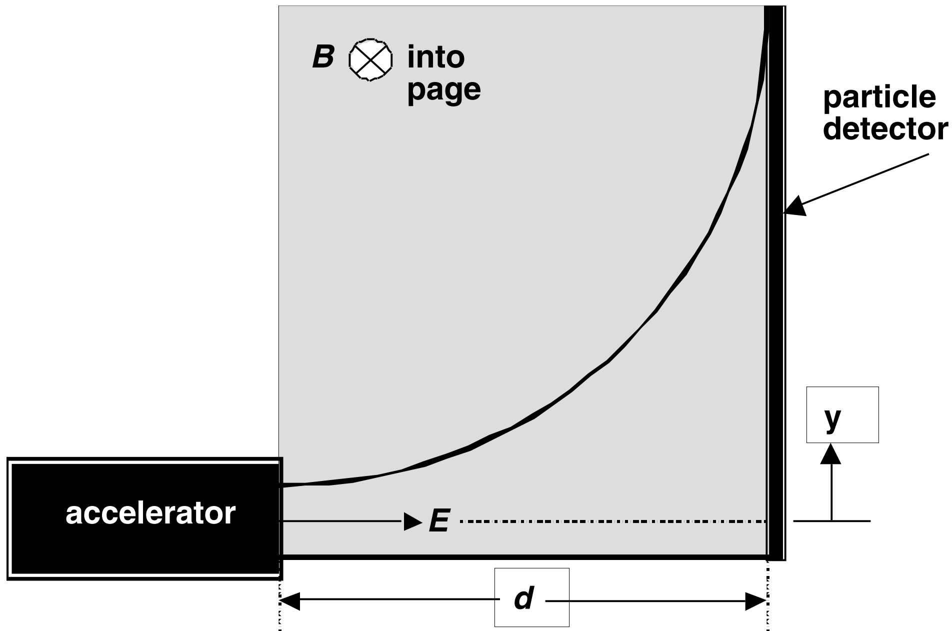


A wire loop is laid out on a plane to make the closed circuit shown at right. A uniform magnetic field B is oriented perpendicular to this plane — sticking directly out of the page. This magnetic field changes at the rate $\mathrm{d}B/\mathrm{d}t$.

What is the induced electromotive force in the wire? Find the answer in terms of the area of each section, A, B, C, and D. *[Yaser]*

4) Use the Force, Luke!

A *mass spectrometer* is a device that measures the mass of charged particles (called ions) by steering particles with the same mass to the same point in space. Mass spectrometers are analogous to prisms, for light: a prism spreads a beam of white light into its component colors (or wavelengths); in a mass spectrometer a beam of ions with different masses is spread out according to mass. There are many uses for mass spectrometers such as in radiocarbon dating, where mass spectrometers measure the ratio of numbers of carbon isotopes ^{13}C and ^{14}C. The figure at right shows the geometry of a simple mass spectrometer: A given ion is accelerated by an electric field to a specific energy — and hence velocity. As the ion leaves the accelerator, it enters a region containing a uniform magnetic field that is perpendicular to its direction of motion. As the ion moves through the magnetic field, it experiences a centripetal force that causes its path to curve along a circle. A detector set along the y-axis, in this figure, can determine the intercept of the ion-path and that axis.



a) Derive an expression relating the energy, charge and mass of the ion to its location (along the y-axis) on the detector. Assume non-relativistic energies.

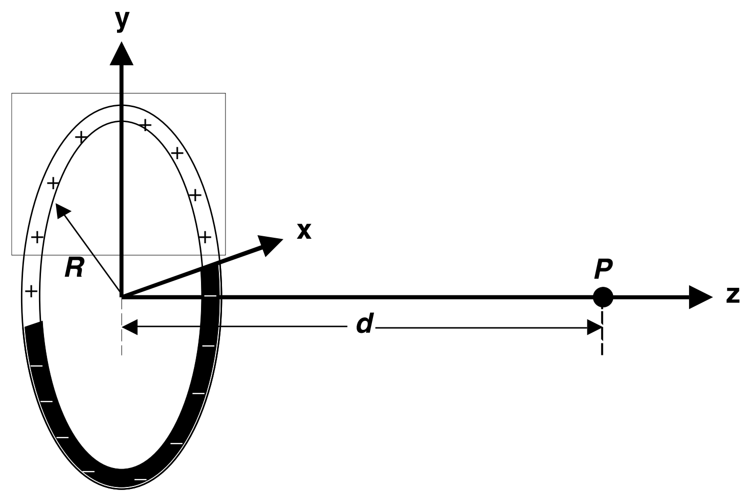
b) What minimum path-radius can this apparatus accommodate.

c) If $B = 2$ Tesla, what is the minimum measurable mass for a ion with charge q = +1 e (where e is the magnitude of charge on an electron), accelerated to an energy 25 eV? What is the maximum mass? Do you think there may be any difficulty in trying to detect extremely heavy ions? If so, why? Note: an *electron volt* (eV) is another unit for energy. *[Brian]*

HINTS: $\bar{F} = q\bar{v} \times \bar{B}$ is the force experienced by a charged particle moving through a magnetic field. In this equation, q is the charge of the particle, v is the velocity of the particle and B is the magnetic field. The right-hand rule for cross products may be useful in determining which *way* the particle curves.

5) A ringing charge

A thin ring of radius R positioned in the x-y plane has a positive *linear charge density* $+\rho$ [C/m] on the top half and an equal-magnitude negative linear charge density $-\rho$ [C/m] on the bottom half. Derive an expression for the *magnitude* and *direction* of the electric field at point P, a distance d away from the center of the ring.



HINT: what is contribution to the electric field at point P from any small segment of the ring? *[Brian]*

6) Scoping out electrical charge

Two identical spheres of mass m are identically charged, then each suspended from the same point by two non-conducting threads. Take each thread to have zero mass and fixed length L. The electrostatic forces acting on each object will act to push them apart.

THE THEORY

a) Derive an expression for the charge q on the spheres, found from their mass, the length of strings, and the angle θ between the threads.

THE EXPERIMENT

You will need the following:

1) 2 pieces of light string or sewing thread, and sewing needle

2) Styrofoam™ (or similar) packing material — packing 'peanuts' or blocks

3) utility knife or paring knife or razor blade

4) A protractor or ruler

5) A bit of tape, or push-pin

Procedure:

From the Styrofoam block cut out two cubes and try to cut the corners off to make something nearly spherical. Try fairly small cubes — about a centimeter or less. Using the sewing needle and thread, thread equal lengths of thread through the centers of the spheres. Using the tape or push-pin, attach the free ends of the thread together and suspend the whole thing freely — I tied the two ends together and then taped the end to the edge of my bookshelf.

Charge the spheres up — I rubbed mine against my hair, but you could rub a balloon against a sweater and then touch it for a few seconds to the styrofoam. The suspended charged balls will now hang with some space between them.

b) Now use the results of your measurements, and part (a) above, to determine the charge on the Styrofoam spheres. [*Brian*]

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2000-2001 Physics Olympiad Preparation Program

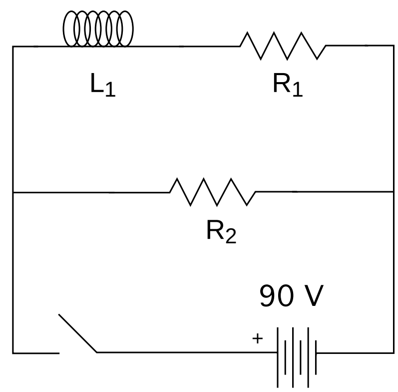
— University of Toronto —

Problem Set 6: AC Circuits and Electronics

Due *April 6, 2001 (revised date)*

1) Inducting Henry

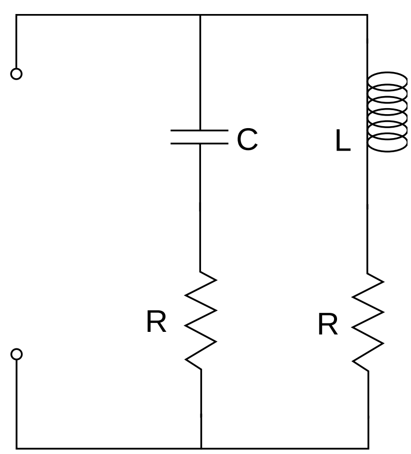
A circuit contains an inductor L_1 = 70 mH in series with a resistor R_1 = 50 Ω, both of which are in parallel with a R_2 = 350 Ω resistor and a 90 V battery. The switch is closed for 110 μs and then opened.



What is the value of the current in both resistors at the moment the switch is opened at 110 μs? *[Sal]*

2) Inducing Resistance

a) Graph the current that would flow from a power supply in the circuit at right (across terminals marked with circles) if the following voltage were applied:

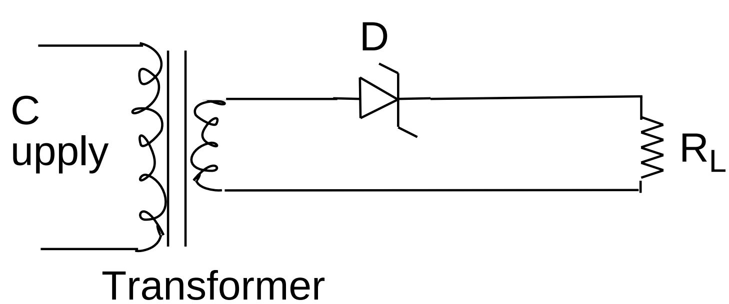


$$V(t) = \begin{cases} 0 & t < 0 \\ V_o & t \geq 0 \end{cases}$$

b) What condition on L, R, and C will make the impedance seen by the power supply one which is purely *resistive*? (You may want to see the POPTOR Primer™ at the back of this set). *[Robin & Cambridge Tripos]*

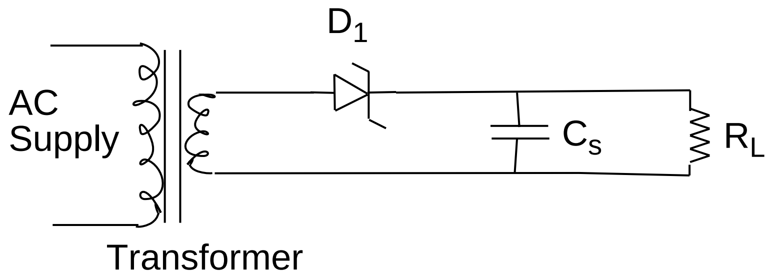
3) Rectifying a situation

In its simplest description, a *diode* is an electrical component that permits current to flow in only one direction. It can be used as a *rectifier* to change alternating current (AC) to direct current (DC).



a) In the circuit shown at right, a step-down transformer converts 110V AC to 6.3V AC. *D* is a diode and the load resistor R_L is 100 Ω. If the frequency of the AC power supply is 60 Hz, plot the voltage across R_L versus time.

b) In order to get a smoother output voltage, capacitors can be used as shown in the circuit at right. If C_s is 330 μF, plot the voltage across R_L for three values of R_L, so that



$R_L C_s$ infinitely large

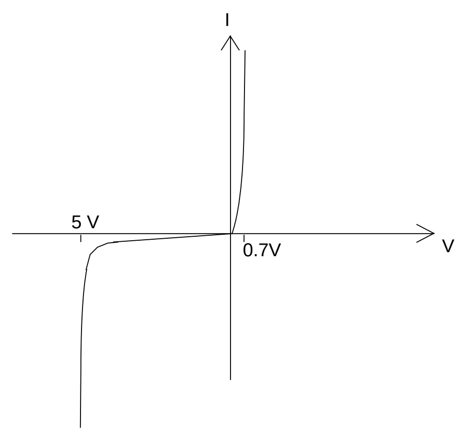
$R_L C_s = 1/6$ sec

$R_L C_s = 1/60$ s

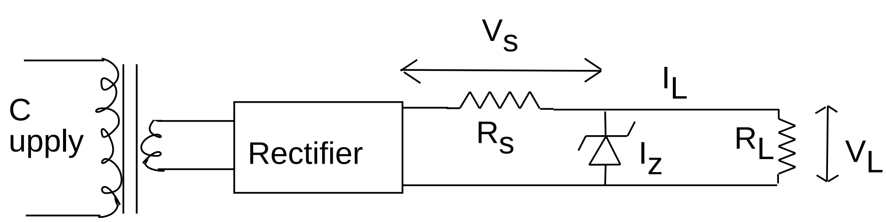
[Yaser]

4) Zeners and the art of electronics maintenance

A *Zener diode* maintains a more or less constant voltage across it in the reverse direction independent of current passing through it for a wide range of currents. This property may be used to stabilize a power supply against variations in input supply voltage and in load current drawn.



Assume that there is a Zener diode, which the I-V curve is as below. This diode is placed in the circuit, which is shown. Transformer gives an output voltage reduced from the AC supply by a factor of 11. Rectifier is used to change the AC voltage to DC.



a) If R_s is 100 Ω, for R_L = 100 Ω, what is the voltage across it if the voltage of AC supply is varied from 85 volts to 130 volts?

b) Assume that the voltage of AC supply is 110 volts. What is the voltage across R_L if we vary its resistance from 50 Ω to 150 Ω? *[Yaser]*

5) Active courage

a) By replacing an op-amp with its ideal model (see the POPTOR Primer™ below), derive an expression for V_{out} as a function of V_{in} and use this expression to calculate the ratio $\frac{V_{out}}{V_{in}}$.

b) What is the limit of the equation $\frac{V_{out}}{V_{in}}$ you derived in part (a) as the gain A approaches infinity? What effect does the op-amp have on the output signal with respect to the input signal?

c) Is the equation in part (b) a good approximation to the equation in part (a) if the gain A is equal to 10,000? (less than 1% is good)

d) Why doesn't the op-amp amplify the input voltage by the op-amp gain A?

e) BONUS (Worth an *insane* number of marks): When the op-amp gain A is very large, what does the voltage on the negative terminal V_1 become? What happens to the voltage V_1 when V_2 is non-zero (not grounded)?

[The result of this bonus question illustrates the real usefulness of op-amps: op-amps in negative feedback mode allow one to arbitrarily set the voltage of any point in a circuit *without sinking current (i.e., with infinite effective resistance).* This is the basis for circuits such as the Miller Integrator.] *[Brian]*

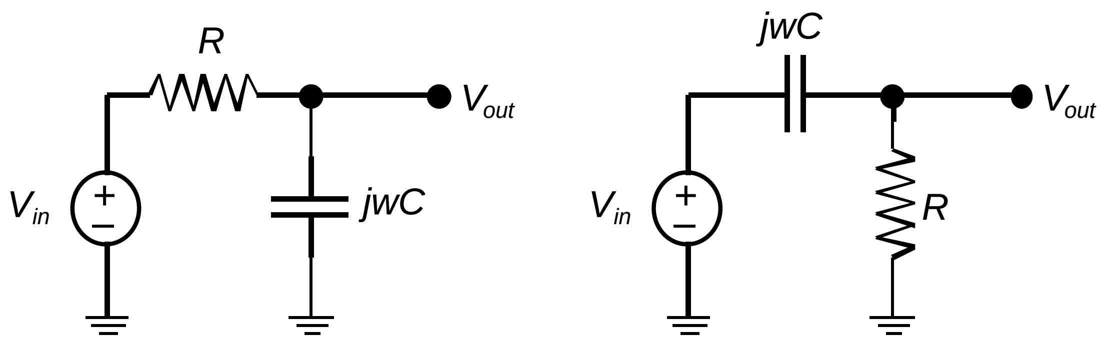
6) Freaky Filters for "Phreaky" Physicists

Various types of frequency filter appear in many devices. Here are some examples:

Electrical – radio antenna
Mechanical – automotive shock absorbers
Optical – sunglasses
Acoustical – earplugs
Data – Net-Nanny

This problem will concentrate on understanding how simple first-order electrical filters work. But before getting into solving the actual problem, you'll need to understand the useful concept of *impedance*: See the POPTOR Primer™ below, for a briefing, if you don't know about impedances very well.

For the two filters on the next page, answer the following questions.



Filter A *Filter B*

a) Derive expressions for $\frac{V_{out}}{V_{in}}$ (these should complex functions of ω).

b) From part (a) produce expressions for the magnitude and phase of $\frac{V_{out}}{V_{in}}$ as a function of frequency.

c) What is the magnitude and phase of $\frac{V_{out}}{V_{in}}$ when $\omega = 0$ and as ω approaches infinity (HINT: you may want to use l'Hôpital's Rule)?

d) Two-way speakers usually have a *tweeter* for high-frequency sounds and a *woofer* for low to mid-frequency sounds. The reason why two-way speakers are used is because a single speaker cannot efficiently produce both high and low frequency sounds (FYI: speakers act like filters too!). The tweeter and woofer each have a separate filter that filters the incoming signal from the amplifier. If you were building your own two-way speaker, which filter (A or B) would you use on the tweeter and which filter would you use on the woofer? Since the magnitude of the filter responses do not cut off sharply, the tweeter and woofer will have some overlap in the sound frequencies they produce. Use the equations for magnitude developed in part (b) to calculate the *crossover frequency* for the two filters if $R = 10{,}000\ \Omega$ and $C = 0.02\ \mu\text{F}$.

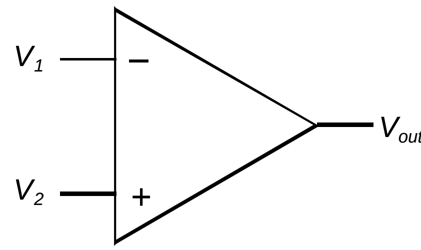
BONUS (Worth a reasonable number of marks): What effect will the phase have on the music played through our two-way speaker? *[Brian]*

Remember to check the POPTOR web-page for hints and any necessary corrections!

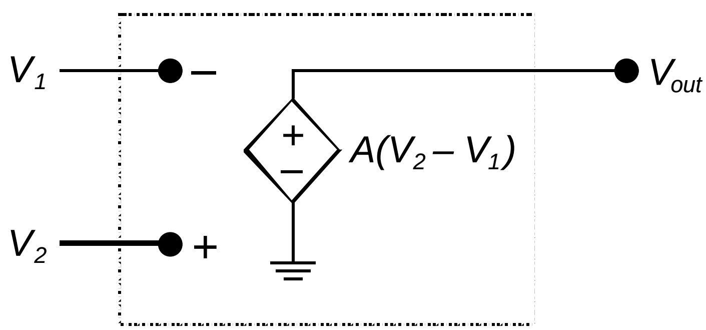
www.physics.utoronto.ca/~poptor

Active Circuits

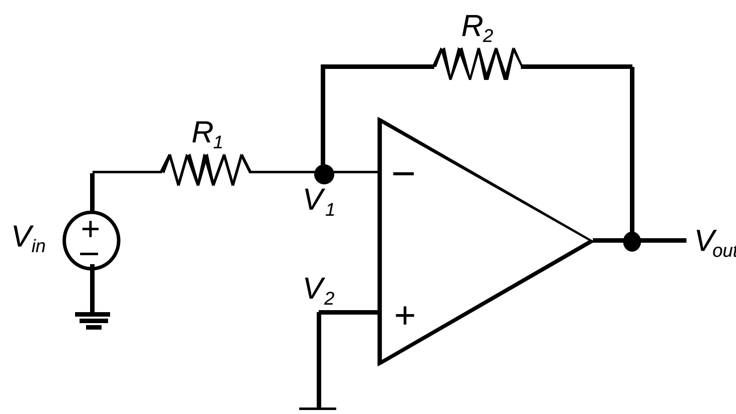
For many of you this may be your first encounter with an active circuit element. Unlike resistors, capacitors, inductors and diodes which are passive circuit elements, active components like op-amps and transistors have the ability to put energy into a circuit in a controlled manner based on electrical input from the circuit itself!



Op-amp stands for "operational amplifier" and is one of the simplest of all the active elements. The figure at right shows the circuit symbol for an op-amp. An op-amp is a three-terminal device that amplifies the difference in voltages between the "+" and "–" input terminals by a factor A and outputs the result ("A" is called the *gain* or *amplification factor*). The gain for a typical op-amp is usually between 10,000 and 1,000,000. Electrical characteristics of a "real-life" op-amp are very complex. However, for most applications it is more than reasonable to represent the op-amp by a simple model.



The circuit at left shows a model of an ideal op-amp – that is, if we could build a perfect op-amp, this model would completely describe its electrical response. Circuits containing op-amps can be analyzed using standard circuit techniques (such as Kirchoff's Voltage and Current Laws and Ohm's Law) by replacing all the op-amp symbols with its model.



The circuit on the right shows an op-amp in a *negative feedback* configuration. All negative feedback means is the output is fed back into the "–" input terminal; in this case, the negative feedback occurs through resistor R_2.

If you go on to take electronics courses at university or college, you will learn more about negative feedback. In the meantime, for a motivating explanation, negative feedback is used to eliminate non-linearity in circuit response and increase stability – but you do not need to know this in order to solve this question!

AC Impedance

Ohm's Law $V = IR$ relates the voltage V across a resistor R to the current I through the resistor. But circuits can include capacitors and coils in addition to resistors. How then can we analyze such circuits? Is there an Ohm's Law for capacitors & coils?

The answer is "yes" – it is the *impedance* version of Ohm's Law, $\boldsymbol{V} = \boldsymbol{IZ}$, where $\boldsymbol{Z}$ is called the impedance (boldface denotes a complex number, here). The impedance $\boldsymbol{Z}$ can be thought of as a complex-valued "resistance" having the form $a + jb$ where a is the "real" part and b is the "imaginary" part [1]. Don't get scared of complex-valued impedances. Complex numbers are only used to simplify the bookkeeping when determining things like magnitude and phase as you will soon find out [2]. The great thing about impedances is that the familiar techniques used to analyze DC circuits carry over!

When we want to extract a physically meaningful number from the impedance version of Ohm's Law, we just have to take the magnitude and phase of the complex expression. There are a few things about impedances you need to know about: The phase of a positive imaginary impedance is 90° and the phase of negative imaginary impedance is –90°. Similarly, the phase of a positive real impedance is 0° and the phase of a negative real impedance is 180°. The phase of an impedance with real *and* complex parts can be anywhere between 0° and 360°.

Experiments show that when a sinusoidal (sine-wave) voltage is applied across a capacitor, the current through the capacitor is 90° out of phase with respect to the applied voltage and *increases proportionally* with frequency. Similar experiments show that when a sinusoidal voltage is applied across an inductor (coil), the current through the coil is –90° out of phase with respect the applied voltage and *decreases inversely* with frequency. (N.B.: the frequency of the applied voltage and current are the same for both the capacitor and coil). Without proof (or you can take this to be an experimental result [3]), the impedance of a capacitor is $j\varpi C$ and the impedance of a coil is $\frac{-j}{\varpi L}$ where C is the capacitance, L is the inductance and ω is the frequency.

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1. In circuit analysis the engineer's "j" is used to denote √(–1), instead of using the mathematician's "i" which can be confused with current.

2. In principle, we can solve these problems without complex numbers. However, doing so would induce a really bad case of trigonometric diarrhea.

3. The impedances of a capacitor and coil can be derived from Maxwell's equations. However, Maxwell's equations themselves cannot be derived, but are based upon empirical results.

2001-2002 Physics Olympiad Preparation Program

– University of Toronto –

Problem Set 1: General

Due 9 November 2001

Since this is the first problem set, a few comments before starting:

- If you want to know what kind of physics you might need to solve these problems, look at the Physics Olympiad syllabus at http://www.jyu.fi/ipho/syllabus.html. You don't need to know the syllabus by heart to do the problems, but you do need to be able to recognize what you need to know so you can look it up.
- Don't forget to look at the information given in the POPBits™ section at the end of the problem set, it sometimes has information helpful or necessary for particular problems.
- Pay attention to words like "estimate" or "about". They indicate that the expected answer is not exact because either the input data is not precisely known or because approximations or simplifying assumptions are necessary. Much of real physics is learning how to turn insoluble exact problems into soluble approximations.
- "*Nothing ventured, nothing gained*": Whether you finish a problem or not, please make sure your reasoning and analysis are clear. If you write down nothing, it is easy for us to mark – we just give you zero – but pretty boring. Your basic ideas may be right even if you make a mistake or get stuck.
- Now, on with the show!

1) Let the sun shine in!

Hold a small, flat, square mirror toward the sun and send the reflected light onto a wall. Describe how the shape of the bright spot on wall changes as you move the mirror closer to and farther from the wall. What information about the sun can you obtain with this experiment? Write us your measured value and its error.

(*Hints*: I made my mirror by masking a small round mirror with masking tape and paper so that only a 0.5 cm square was exposed; this way you can also easily experiment with different size or shape mirrors by just changing the tape. The mirror does not have to be an exact square, but it does have to be flat, so don't use a mirror which magnifies or reduces, as do many round makeup or hand mirrors. If it is more convenient, you can leave the mirror fixed and look at the reflection on something like a piece of flat cardboard which you move closer to and farther from the mirror.)

[Yaser]

2) Ping-Pong!

A ball will bounce back from a wall with the same speed it hits, if the collision is elastic. What happens if the ball (which is moving with a speed v) hits a "wall" (e.g. a table tennis paddle) that is moving (with a speed u) toward the ball?

(a) What is the velocity of the ball after it bounces back from the wall?
(*Hint*: Think of a frame in which the speed is the same before and after the bounce.)

(b) What is the change in the kinetic energy of the ball if its mass is m?

(c) Try to show that your answer to part (b) is equal to the work done by the wall on the ball.
(*Hint*: Assuming the duration of the collision is ΔT, find the force exerted on the ball and its displacement.)

[Yaser]

3) Don't break the window!

You have a ball of mass m attached to a massless string. The maximum tension the string can handle before breaking is T. (What we are actually talking about is a yo-yo that has been left outside until its string starts to rot, but we have translated this into "textbook physics".)

(a) If you hold one end of the string at a height h above the ground, what is the maximum length the string can have so that you can still swing the ball in circles around you without the ball touching the ground or the string breaking?

(b) If you do swing it around in circles slightly too hard and the string breaks, for what length string will the ball fly the longest horizontal distance before hitting the ground?

[Alex]

4) Cowabunga!

First a well known joke: Farmer Smith was not satisfied with the yield of his milk cows, so he decided to called in an animal psychologist, an engineer and a physicist to try to improve matters. All three inspected the farm and the cows and made their recommendations. The animal psychologist went first, "If you paint the milking shed green the cows will be happier and happy cows will give more milk." Then came the turn of the engineer. "If you narrow the milking stalls by 10 centimeters you will be able to add an extra stall and thus be able to milk an extra cow in the same time." Farmer Smith was very happy so far, now it came to the turn of the physicist. She got out a black board, drew a circle and said: "First, we assume a spherical cow."

(a) Estimate the capacitance of a cow.
(*Hint*: To help any city-slickers, I'll point out that a typical run-of-the-mill Holstein[1] dairy cow weighs about 700kg and is more or less neutrally buoyant in water.)

(b) Estimate the resistance of a cow.
(*Hints*: People and cows have similar resistivity.
Sometimes different approximations are necessary for different parts of a problem.)

(c) When a cow walks across a carpet in winter, it can easily charge itself up to 10KV, and it can get a nasty shock if its nose touches a doorknob. Estimate the peak power in such a shock.
(*Hint*: The total resistance of a mammal in an electrical circuit is normally dominated by

[1] http://www.holstein.ca/

skin/contact resistance which can be quite variable, but cows tend to lick their noses which keeps the skin conductivity up, so let's ignore the skin/contact resistance.)

[David]

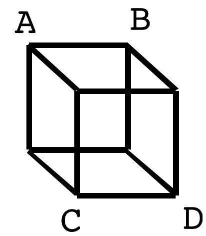
5) Symmetry Cubed!

Here's is a great project for your art class: an electrified hanging mobile made from wire models of the 5 platonic solids.

Consider a wire tetrahedron where each edge has a resistance R.

(a) What is the resistance between any two corners?

What if we replace the tetrahedron with a cube?



(b) What is the resistance between two opposite corners (*e.g.* AD)?

(c) What is the resistance between two diagonally opposite corners on the same side (e.g. AC)?

(d) What is the resistance between two adjacent corners (*e.g.* AB)?

We'll leave calculating the resistance of the various vertices of the octahedron, the dodecahedron, and the icosahedron for your own amusement.

[Alex]

6) Is it hot enough for you?

The Mars Pathfinder mission's Sojourner rover[1] carried 3 small radioactive plutonium heaters to keep its electronics warm. How warm would your radioactivity keep you?

The two largest sources of natural radioactivity in your body are the radioactive isotopes Carbon-14 and Potassium-40. (Carbon-14 is continually produced in the earth's atmosphere by cosmic rays, while the Potassium-40 is part of the primordial composition of the solar system.) About 10^{-12} of the carbon in your body is Carbon-14, and 0.0117% by weight of all potassium is Potassium-40. The decay of a Carbon-14 atom releases an energy of 0.16×10^{6}eV, while the decay of a Potassium-40 atom releases 1.3×10^{6}eV. Both these decays are beta decays, so on average about half the energy released in a decay is carried out of your body by a neutrino, but the other half of the energy is deposited in your body as ionizing radiation. A typical human is about 19% carbon by weight, and about 0.3% potassium,

(a) What is the total activity (i.e. decays per second) of the Carbon and Potassium in your body?

(b) What is the total power (in Watts) deposited in your body by these decays?

(c) If you were put into suspended animation and fired into interstellar space, what would your equilibrium surface temperature be if your internal Carbon-14 and Potassium-40 were the only source of heat keeping your body warm?

[David]

[1] http://mpfwww.jpl.nasa.gov/MPF/index1.html

POPBits™ – Possibly useful bits of information

Constants and units[1,2]

astronomical unit (mean earth-sun distance)	au	149 597 870 660±20 m
atomic mass unit: (mass ^{12}C atom)/12	u	$(1.66053873\pm0.00000013)\times10^{-27}$ kg
elementary (*i.e.* electron) charge	e	$(1.602176462\pm0.000000063)\times10^{-19}$ C
electron volt	eV	$(1.602176462\pm0.000000063)\times10^{-19}$ J
Newtonian gravitational constant	G_N	$(6.673\pm0.010)\times10^{-11}$ m^3/kg/s^2
solar luminosity	$L_{\odot}$	$(3.846 \pm 0.008)\times10^{26}$W
speed of light in vacuum	c	299 792 458 m/s
standard acceleration of gravity at the earth's surface	g	9.80665 m/s^2
Stephan-Boltzmann radiation constant	σ	$(5.670400\pm0.000040)\times10^{-8}$ W/m^2/K^4
tropical year (2001)	yr	31556925.2 s

Radioactive half-lives of some isotopes[3]

^{14}C	(Carbon-14)	5730±40 years
^{40}K	(Potassium-40)	$1.277\pm0.001 \times 10^9$ years
^{238}Pu	(Plutonium-238)	87.7±0.1 years

Resistivity of human tissue[4]

blood	1.7 Ω·m
bones	160 Ω·m
fat	27 Ω·m
muscle	7.0 Ω·m
nerve	2.5 Ω·m

Great excuse for a party

Birthday of Marie Curie[5] (1867) and Lise Meitner[6] (1878) November 7

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1 http://physics.nist.gov/cuu/Constants/index.html
2 http://pdg.lbl.gov/2000/contents_sports.html
3 http://ie.lbl.gov/education/isotopes.htm
4 http://www.acusd.edu/~mmorse/BMES2000.shtml
5 http://www.mariecurie.org/mariecurie
6 http://www.users.bigpond.com/Sinclair/fission/LiseMeitner.html

2001-2002 Physics Olympiad Preparation Program

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Solution Set 1: General

1) Let the sun shine in!

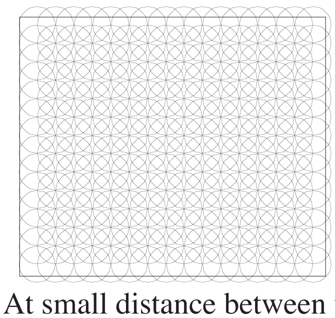
Let's say the mirror is 0.5*0.5 cm^2 square. It is important that your mirror shouldn't be round to observe the effect. When the wall is close to the mirror the bright spot has the same shape as the mirror, in this case square. When you go farther and farther it changes gradually to the circle. This effect comes from the fact that the rays coming from the sun are not exactly parallel due to the size of the sun.

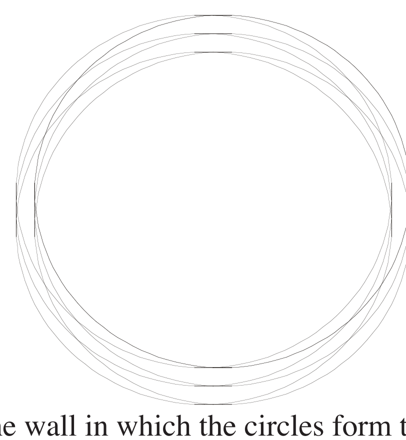
We can solve this question by considering that the square shape mirror consists of infinitely large number of very tiny mirrors. Each tiny mirror makes a circular image of the sun on the wall. If the distance between wall and the mirror is small all these small circular images will form together the shape of the mirror. At large distances, these circular images are big and they all overlap with each other form a big circle. By geometry, the ratio between the radius of the image to the distance between the wall and the mirror is equal to the ratio between the radius of the sun and the distance of sun to earth. For 0.5*0.5 cm^2 mirror, up to about 50 cm the spot on the wall looks square. As we go farther the spot changes to circle and at about 1.5 meter away, the circle has the radius of about 0.75 cm. Therefore the ratio between diameter of the sun and its distance to the earth is about 0.01.

The distance x where the image shape starts changing from square to the circle, is about where you get the smallest image and by geometry

$$\frac{x}{Diameter-of-the-mirror} = \frac{Distance-between-sun-and-earth}{Diameter-of-the-sun}$$

which is about half a meter in this case.





(Right) At small distance between the mirror and the wall in which the circles form the square .
(Left) At large distance between the mirror and the wall in which the circles overlap and the whole image looks like a circle.

2) **Ping-Pong!**

(a) A table tennis paddle is much heavier than the ping-pong ball. Therefore we can assume that it's mass is infinity compared to the ball. Therefore, in the wall's frame (or that of a table tennis paddle!), the velocity of the ball is the same before and after it hits the wall, but in the opposite direction. The velocity of the ball in this frame is $v+u$ and so it bounces back with $v+u$ relative to the wall. Therefore its velocity relative to the ground is $v+2u$.

(b) The change in the kinetic energy is

$$\Delta K = K_f - K_i = \frac{1}{2}m(v+2u)^2 - \frac{1}{2}mv^2 = 2mu(u+v)$$

where m is mass of the ball and K is the kinetic energy.

(c) The change in the linear momentum of the ball is

$$\Delta P = P_f - P_i = m(v+2u) - (-mv) = 2m(v+u)$$

Therefore if the collision time is ΔT, then the force is

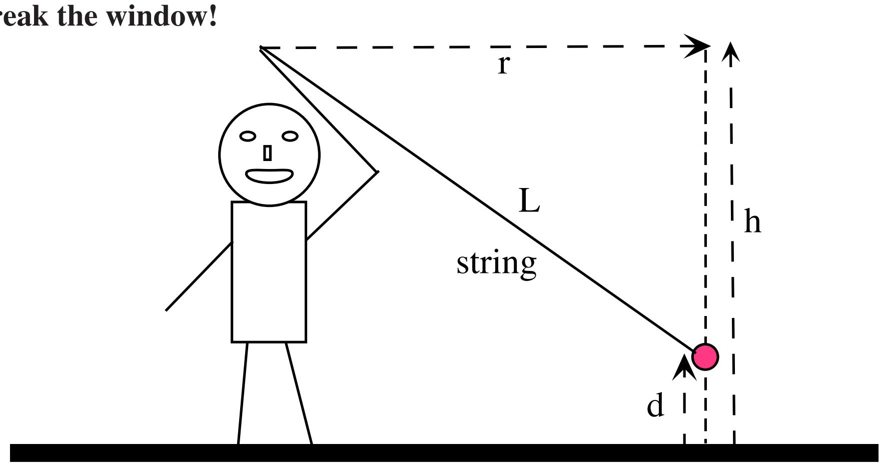
$$F = \frac{\Delta P}{\Delta T} = \frac{2m(u+v)}{\Delta T}$$

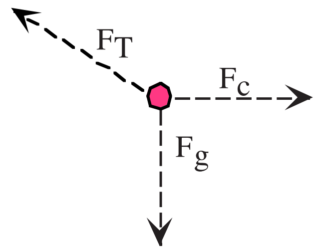
The distance that the ball moves during the collision is equal to the distance that the wall moves which is $u\Delta T$. Therefore the work done by the wall on the ball is

$$W = Fu\Delta T = 2mu(u+v)$$

which is equal to the change in the kinetic energy of the ball as we expect.

3) **Don't break the window!**





The faster you spin the ball around you, the higher it will go, but the more stress we put on the string. Looking at the above diagrams, we see that

$$\frac{h-d}{r} = \frac{F_g}{F_c} = \frac{mg}{m\omega^2 r} \quad \Rightarrow \quad d = h - \frac{g}{\omega^2}$$

This is independent of the length of the string, and as we intuitively feel, $d \to h$ as $\omega \to \infty$. The tension in the string is, however, proportional to the length of the string:

$$\frac{L}{h-d} = \frac{F_T}{F_g} = \frac{F_T}{mg} \quad \Rightarrow \quad F_T = \frac{mgL}{h-d} = m\omega^2 L$$

(a) Rearranging the above equation, the length of the string

$$L = \frac{F_T(h-d)}{mg}$$

So L is a maximum when F_T is a maximum and d is a minimum, *i.e.* when F_T=T and d=0.

$$\therefore L_{\max} = \frac{Th}{mg}$$

(b) The distance the ball flies is equal to its horizontal velocity, v=$r\omega$, multiplied by the time it takes to fall to the ground, t. If the starts to fall at a height, d, then the time it takes to hit the ground is

$$d = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2d}{g}}$$

What we need to do is express the distance of travel, s=vt, in terms of L, and find the value of L which maximizes s.

$$s = vt = r\omega\sqrt{\frac{2d}{g}} = \sqrt{r^2\omega^2\frac{2d}{g}}$$

The maximum distance will occur for the maximum tension, F_T=T, and by geometry

$$r^2 = L^2 - (h-d)^2$$

and using previous results

$$\omega^2 = \frac{F_T}{mL} = \frac{T}{mL} \quad \text{and} \quad d = h - \frac{g}{\omega^2} = h - \frac{g}{\left(\frac{F_T}{mL}\right)} = h - \frac{mgL}{T}$$

we have

$$s=\sqrt{\left(L^2-\left(h-\left(h-\frac{mgL}{T}\right)\right)^2\right)\left(\frac{T}{mL}\right)\frac{2(h-mgL/T)}{g}}$$

$$=\sqrt{2L\left(1-\left(\frac{mg}{T}\right)^2\right)\left(\frac{hT}{mg}-L\right)}$$

Looking at this we easily see that the maximum distance will be when we maximize

$$L\left(\frac{hT}{mg}-L\right)$$

This is just a parabola in L with an obvious maximum at

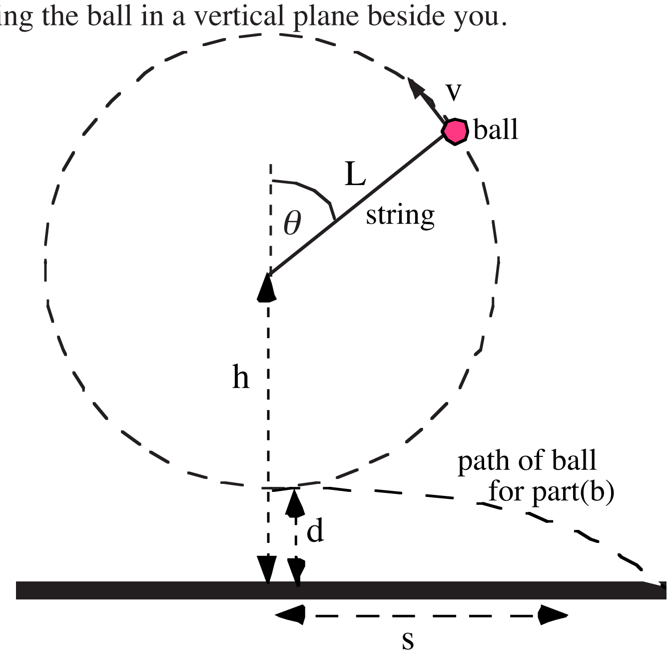
$$L_{\max}=\frac{hT}{2mg}$$

For our interest, we can now calculate the maximum distance

$$s_{\max}=\frac{h}{\sqrt{2}}\sqrt{\frac{T^2}{m^2g^2}-1}$$

Note

Some people assumed the motion of the ball was vertical instead of horizontal. You can't actually "swing the ball in circles around you" in a vertical plane, but semantics aside, let's see what happens if you swing the ball in a vertical plane beside you.



- The ball must orbit fast enough so the centrifugal force can overcome gravity at the top of its orbit, so the velocity of the ball at the top of the orbit must be such that $v^2/L>g$.
- If the string breaks, it will break at the bottom of the orbit since the tension must be a maximum since gravity and the centrifugal force add together at that point.

To figure this out in detail, let's assume you have got the ball going and it is now orbiting with no further energy input from you. The constant total energy of the ball is

$$
\begin{aligned}
E_{tot} &= E_{potential} + E_{kinetic} \\
&= mgL\cos\theta + \frac{1}{2}mv^2
\end{aligned}
$$

where we have set the gravitational potential energy zero at the height h. The velocity of the ball thus depends on where in the orbit it is

$$
v^2 = 2\frac{E_{tot} - mgL\cos\theta}{m}
$$

Not surprisingly, the maximum velocity is at the bottom of the orbit (θ=180°).

The tension in the string must balance the sum of the centrifugal force plus the component of the gravitational force along the string:

$$
\begin{aligned}
F_T &= m\frac{v^2}{L} - mg\cos\theta \\
&= \frac{2\left(E_{tot} - mgL\cos\theta\right)}{L} - mg\cos\theta \\
&= \frac{2E_{tot}}{L} - 3mg\cos\theta
\end{aligned}
$$

As expected, this shows the tension is a maximum at the bottom of the orbit and a minimum at the top; to minimize the tension we want E_{tot} to also be a minimum. The tension at the top (θ=0°) cannot be less than zero or the string will not be taut and the ball will fall. So we want

$$
\begin{aligned}
E_{tot} &= \frac{3}{2}mgL \\
\therefore F_T &= 3mg - 3mg\cos\theta = 3mg(1 - \cos\theta)
\end{aligned}
$$

The tension at the bottom must be less than the breaking tension, T, so we must have

$$
T > 6mg
$$

This is independent of the length of the string, so as long as the string is strong enough to support 6mg, the maximum length of the string is just set by the height you hold the end of the string above the ground, *i.e.*

$$
L_{\max} = h
$$

If we take the same string and swing it horizontally, we have $L_{max}=(T/mg)h=(6mg/mg)h=6h$, so you can swing a much longer string horizontally instead of vertically. Even the weakest string that can support the weight of the ball, *i.e.* T>mg, has $L_{max}>h$ if swung horizontally.

Finally, let's check how far the ball will fly if the string breaks when swinging vertically. Since the tension is maximum at the bottom, the ball will fly horizontally from the bottom of the orbit and the time to hit the ground is

$$
t = \sqrt{\frac{2d}{g}} = \sqrt{\frac{2\left(h - L\right)}{g}}
$$

Again we want to maximize $s=vt$,

$$
\begin{aligned}
s &= vt \\
&= \sqrt{2\frac{E_{tot} - mgL\cos(180^\circ)}{m}}\sqrt{\frac{2(h-L)}{g}} \\
&= \sqrt{4\frac{\frac{3}{2}mgL + mgL}{m}\left(\frac{h-L}{g}\right)} \\
&= \sqrt{10L(h-L)}
\end{aligned}
$$

So the maximum value is $L=h/2$, and the maximum distance the ball will fly in the vertical orbit case is

$$s_{\max} = \sqrt{\frac{5}{2}}h$$

Comparing this with the horizontal orbit, we see that the minimum strength string strong enough to be swung[1] vertically (*i.e.* $T>6mg$) will fly exactly the same distance if swung horizontally or vertically. In the vertical case, however, the maximum distance is independent of the string strength, but in the horizontal case the maximum distance increases with the strength of the string without limit (except for mundane factors such as air resistance and the size of the earth).

4) Cowabunga!

(a) For this question, it is convenient to assume that the cow is a sphere, since that is a shape we know how to deal with and we only want an estimate. The capacitance of an conducting object is the ratio of its electric charge to its electric potential or "voltage", *i.e.*

$$C = \frac{Q}{\phi}$$

By Gauss's Law and symmetry, the potential at the surface of a conducting sphere with charge Q and radius r is the same as the potential a distance r from of a point charge Q:

$$\phi = \frac{q}{4\pi\varepsilon_0 r}$$

(Where, by convention, V=0 at infinity defines the zero potential.) The capacitance of a sphere of radius r is therefore:

$$C = \frac{Q}{\phi} = 4\pi\varepsilon_0 r$$

The radius of a sphere with the same volume, V, as a Holstein[2] cow (mass M=700kg, density ρ=1000kg/m^3) is

$$r = \left(\frac{V}{\frac{4}{3}\pi}\right)^{1/3} = \left(\frac{M/\rho}{\frac{4}{3}\pi}\right)^{1/3} = \left(\frac{700kg/1000kg/m^3}{\frac{4}{3}\pi}\right)^{1/3} = 0.55m$$

The cowpacitance is thus

[1] Try saying "the minimum strength string strong enough to be swung" 5 times fast.

[2] http://www.holstein.ca/

$$C = 4\pi\varepsilon_0 r = 4\pi\left(8.854187817\times10^{-12}\,F/m\right)0.55m = 60\,pF$$

(While writing this problem, I discovered that the standard capacitance of an isolated cow – I am not making this up – is 0.1nF, in good agreement with our estimate. If the cow is standing in a field, the earth acts as a conducting plane and this increases the capacitance to about 0.2nF, but we don't care about a factor of 2 in this problem.)
Note that you had to provide the value of ε_0; we do not always give the values of basic constants available in any appropriate physics textbook.

(b) A sphere is not a useful approximation for resistance, since it is hard to calculate the resistance of a sphere. We'll assume a cube, since the resistance of a cube of side length L and resistivity ρ is very simple:

$$R = \rho\frac{L}{A} = \rho\frac{L}{L^2} = \frac{\rho}{L}$$

For cow-size cube

$$L = \left(\frac{M_{cow}}{\rho_{cow}}\right)^{1/3} = 0.88m$$

The capacitance of a cow just depends on the cow being a conductor of a certain shape, but resistivity depends on the internal structure of a cow. My rough estimate of the average resistivity is based on the values given for people's insides in the POPBits™ is ρ_{cow}~5Ω·m. The resistance is dominated by the low resistance pathways (e.g. the blood vessels), so the high resistivity of bones and fat are largely irrelevant. So my estimate of the resistance of the cow is just

$$R = \frac{5\Omega/m}{0.88m} = 6\Omega \sim 10\Omega$$

This is much cruder than our capacitance estimate. This is because parallel capacitances add, so the size of the object pretty much determines the total capacitance; parallel resistances add inversely, so any small scale structure (e.g. blood vessels, noses) can have a big effect. The resistance is also sensitive to how the cow is connected in a circuit. In this case the resistance of the cow is probably dominated by the cow's nose and is probably an order of magnitude higher than our simple estimate since the cow's nose is an order of magnitude smaller than the whole cow. (Note: This seems consistent with the 150Ω I get when I hold one probe in each hand and measure my resistance with a multimeter, I have not yet tried to measure the resistance of my nose.)

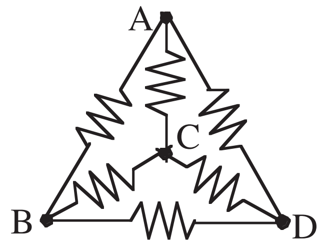
(c) The peak power is just the initial power, since the cow's voltage drops as it discharges:

$$P_{peak} = I_{peak}V_{peak} = \frac{V_{peak}^2}{R} = \frac{V_{initial}^2}{R} = \frac{(10kV)^2}{10\Omega} \sim 10MW$$

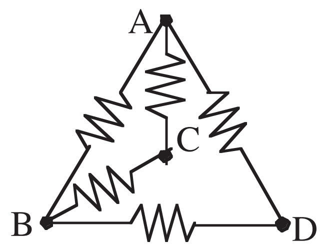
No wonder static hurts!
Note: By mistake, a draft pdf version of this problem set was originally posted with 600kg cows at 20KV, instead of 700kg cows at 10KV as in the print version. This is an approximate problem so either set of numbers are equally good.

5) Symmetry Cubed!

(a) By symmetry, the resistance between any two vertices must be the same, so let's choose vertices to measure the resistance between A and B.



By symmetry, C and D will have the same potential so no current will flow between them, so that resistor can be ignored and the tetrahedron is equivalent to

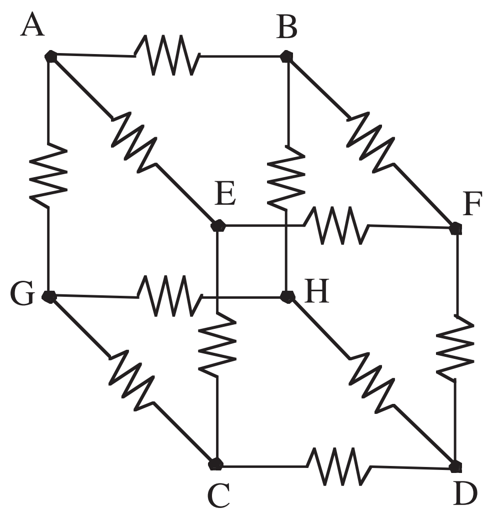


So the total resistant between A and B can be calculated by adding the 3 paths in parallel, i.e.

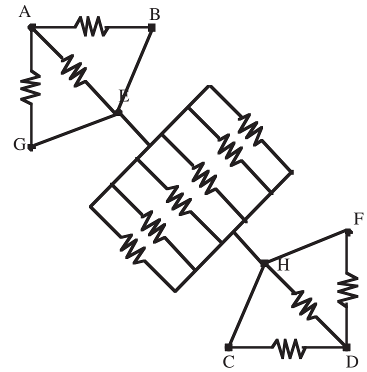
$$\frac{1}{R_{AB}} = \frac{1}{R} + \frac{1}{2R} + \frac{1}{R} = \frac{2}{R}$$

$$\therefore R_{AB} = \frac{R}{2}$$

(b) First we calculate the resistance between A and D.



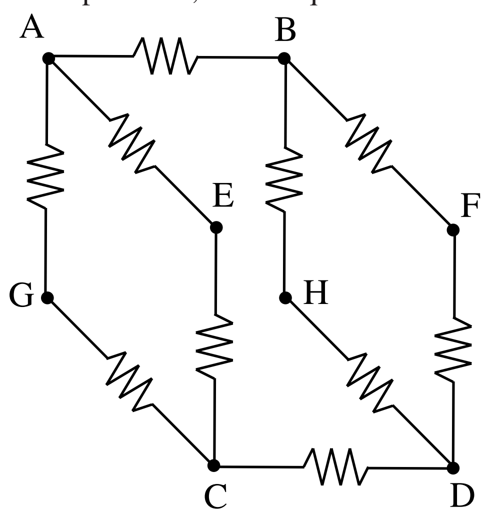
There is lots of symmetry, so it easy to see that G, E, & B will have the same potential and can treated as the same point, and similarly C, H, & F are at the same potential and can be treated as the same point, so the circuit is equivalent to



So we see that

$$R_{AD} = \frac{R}{3} + \frac{R}{6} + \frac{R}{3} = \frac{5R}{6}$$

(c) Now we want the resistance between A and C. By symmetry, E, F, G, & H are all at the same potential. As with part (a), the circuit is not affected by ignoring the resistors connecting points at the same potential, so our equivalent circuit is:



So

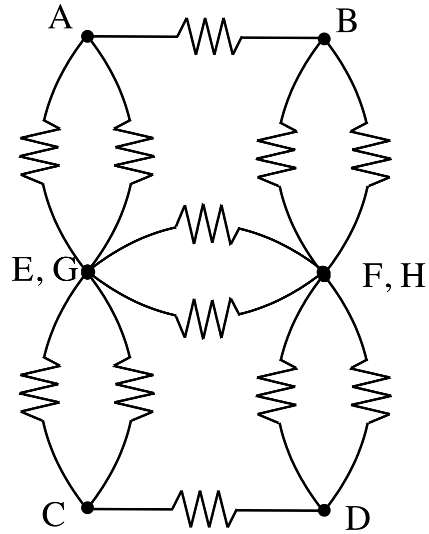
$$\frac{1}{R_{AC}} = \frac{1}{R_{AEC}} + \frac{1}{R_{AGC}} + \frac{1}{\left(R_{AB} + \left(\frac{1}{R_{BFD}} + \frac{1}{R_{BHD}}\right)^{-1} + R_{CD}\right)}$$

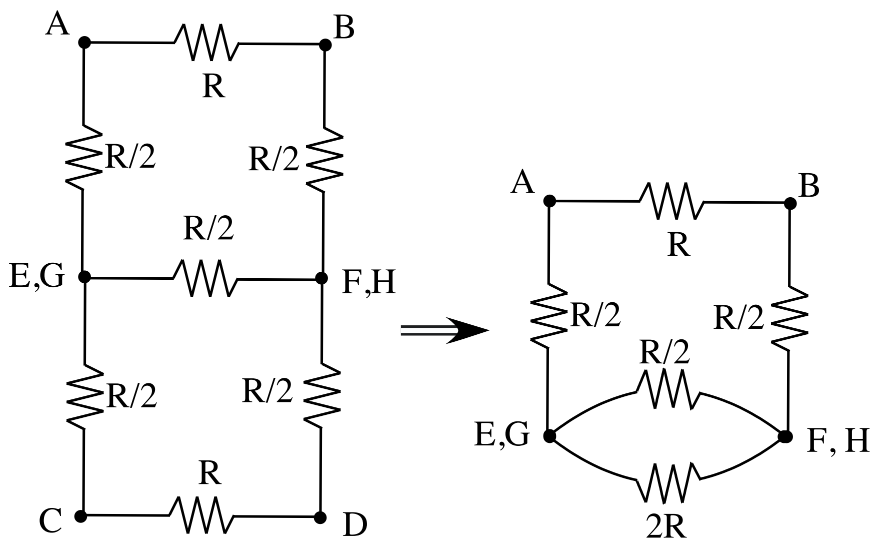
$$= \frac{1}{2R} + \frac{1}{2R} + \frac{1}{\left(R + \left(\frac{1}{2R} + \frac{1}{2R}\right)^{-1} + R\right)}$$

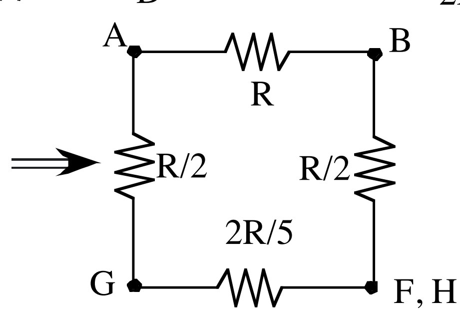
$$= \frac{1}{R} + \frac{1}{(R + R + R)}$$

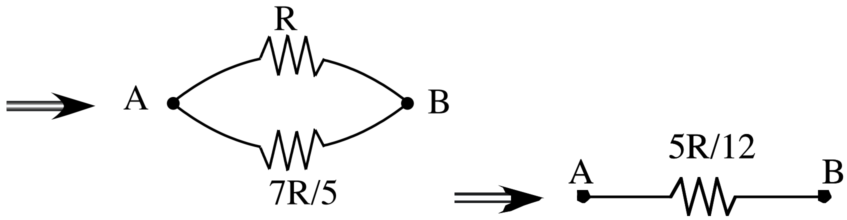
$$\therefore R_{AC} = \frac{3}{4}R$$

(d) Now we want the resistance between A and B. This is the toughest one yet, but there is still some symmetry, and we see that vertices E & G are at the same potential and can be connected without changing the circuit, and F & H are at the same potential and be connected. Our new equivalent circuit is









$$\therefore R_{AB} = \frac{7}{12}R$$

Note: You can, of course, solve all these systems using Kirchoff's rules and Ohm's law to write down systems of linear equations which can be solved by substitution or matrix algebra. This is fine, but we thought using symmetry was more fun.

6) Is it hot enough for you?

(a) The activity A is

$$A = \frac{N}{\tau_{1/2}/\ln 2}$$

where N is the number of atoms and $\tau_{1/2}$ is the half-life of the radioactive isotope. The number of atoms for each isotope is simply its mass divide by its atomic weight: 15 atomic mass units (amu) for Carbon-14 and 40 amu for Potassium-40. For a 70kg person the masses of each isotope are:

$$m_{C14} \sim 70kg \times 19\% \times 10^{-12} \sim 1.33 \times 10^{-11}\,kg = 13\text{ng}$$
$$m_{K40} \sim 70kg \times 0.3\% \times 0.0117\% \sim 2.46 \times 10^{-5}\,kg$$

Therefore the activities are

$$A_{C14} \sim \frac{1.33 \times 10^{-11}\,kg}{(5730\,yr/\ln 2)} \frac{1}{(14\,amu)\left(1.66 \times 10^{-27}\,kg/amu\right)} = 2193/s$$

$$A_{K40} \sim \frac{2.46 \times 10^{-5}\,kg}{(1.277 \times 10^{9}\,yr/\ln 2)} \frac{1}{(40\,amu)\left(1.66 \times 10^{-27}\,kg/amu\right)} = 6363/s$$

So the total activity is A_{total}=8600 decays per second.

(b) About half the energy is deposited in the person's body, therefore the power deposited by each isotope is:

$$P_{C14} = \frac{1}{2}A_{C14}\left(0.16 \times 10^{6}\,eV\right)\left(1.60 \times 10^{-19}\,J/eV\right) = 2.8 \times 10^{-11}\,W$$
$$P_{K40} = \frac{1}{2}A_{K40}\left(1.3 \times 10^{6}\,eV\right)\left(1.60 \times 10^{-19}\,J/eV\right) = 6.63 \times 10^{-10}\,W$$

So the total activity is about P_{total}=0.7nW.

(c) In thermal equilibrium you will radiate as much heat, P, as you produce, so your temperature, T, will be

$$T = \left(\frac{P}{\varepsilon A \sigma}\right)^{1/4}$$

where A is your surface area (about 2m^2), and σ is the Stefan-Boltzmann constant. Your emissivity, ε, is less than the black body value of 1, but it doesn't matter too much since the temperature only varies as the 4th root, so we'll assume it is 1. So

$$T \sim \left(\frac{0.7nW}{2m^2\left(5.670400 \times 10^{-8}\,\text{W/m}^2/\text{K}^4\right)} \right)^{1/4} = 0.3K$$

The cosmic microwave background radiation (http://background.uchicago.edu/~whu/beginners/introduction.html) is 2.7K so it will keep you warmer than than your internal radiation.

University of Toronto

2001-2002 Physics Olympiad Preparation Program

Problem Set 2: Mechanics

Due Monday 17 December 2001

(Note small extension to due date!)

Welcome back.

- This and the next 4 problem sets each concentrate on different areas of physics, but this does not mean that you do not need to use a broad range of physics. For example, every question on this problem set requires knowledge of mechanics[1], but knowledge of mechanics alone may not be enough to solve every problem.
- Each problem set has an experimental question. They typically require a few common items, but you should look at them early in case you have to find something you don't immediately have. Your Physics teacher may be able to help.
- If you can't do all of a question, do the parts you can do.
- Unless otherwise specified, assume everything takes place on the earth's surface and gravity points down in all our diagrams.
- We often provide links to websites, but the problems do not require any information from these websites. Often the links are just for fun.

1) Salty log!

Salt water is denser than fresh water, and in the ocean you will sometimes find a sharp vertical discontinuity in salinity (known as a "halocline") between fresher water on top and saltier water underneath. This often happens near coasts where fresh water runs into the sea or where glaciers or sea ice are melting. Variations in seawater salinity and temperature drive the circulation of deep ocean waters[2] and have a major impact on climate.

Consider a log washed down a river and out to sea. Eventually the log becomes saturated with water and it starts to sink, but if it reaches a halocline it may float on the boundary. If the log has a uniform density ρ_L, and the (assumed) uniform densities of the surface and deep water are ρ_S and ρ_D, what fraction of the volume, V, of the log will be above the halocline in the fresher water.

[Isamu]

[1] At least "mechanics" as defined by us.

[2] http://www.oceansonline.com/watermasses.htm

2) Don't leave me!

(a) Show that the Sun's gravitational force on the Moon is more than twice the force of the Earth on the Moon. (The distance from the moon to the earth is about 400000 km.)

(b) So why doesn't the Sun pull the Moon away from the Earth?

[David]

3) Singing in the rain!

If you are suddenly caught out in the rain, is it better to run or walk to the nearest shelter?

(a) First consider the case where there is no wind and you have no umbrella. Assume it is a distance L to the shelter, the raindrops are falling with a constant velocity u, and the rain is falling at a constant rate of n mm/hour. Calculate how wet will you get (*i.e.* how mach rain will hit you) if you walk with a constant speed v to the shelter. What speed, v, should you walk or run to minimize how wet you get? (Your maximum speed is v_{max}.)

(b) What is your answer if you have an umbrella?

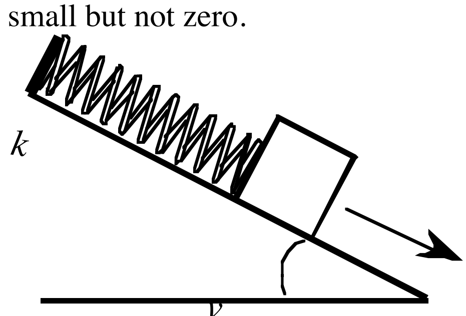
(c) What is your answer if you have an umbrella and the rain is failing at an angle θ in to the direction you are walking? *i.e.* θ=0° is the rain falling vertically, θ=90° is the rain blowing horizontally into your face, θ=–90° is the rain blowing horizontally into your back. (Assume the vertical component of the velocity of the rain is still u.)

Hints: Simplify a person as a rectangular cube. An umbrella prevents water from hitting the top of the cube, but not the front, back, or sides. By "rate", I mean that after one hour an open container left out in the rain will have a layer of water n mm deep.

[Yaser]

4) Sproing!

A block of mass m sliding on a ramp is connected to a spring that has a negligible mass and a force constant of k, as shown in the figure. The spring is unstretched when the system is as shown in the figure, the block has an initial velocity v down the ramp, and there is friction between the incline and the block. The static and kinetic coefficients of friction are μ_s and μ_k; you can assume the friction is small but not zero.



(a) Estimate how much smaller the kinetic energy of the block will be the next time it passes through the initial point. α

(b) About how often is the block's velocity zero? (*i.e.* What is the frequency that v=0?)

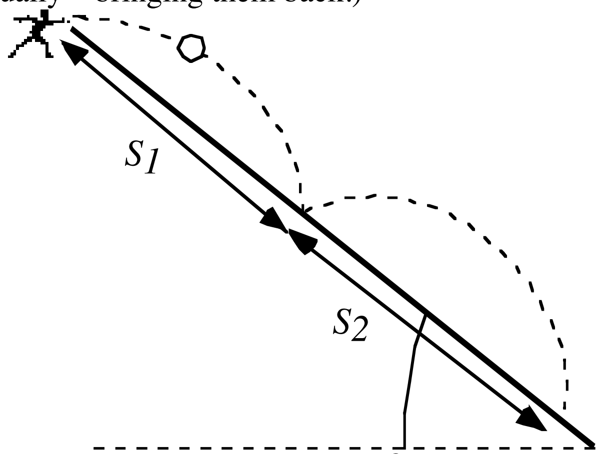
(c) Plot qualitatively the position of the block as a function of time until it permanently stops.

(d) Where is it more likely for the block to permanently stop: at a point higher or at a point lower than the initial point (where the spring is unstretched)? Explain your answer.

[Isamu]

5) Fun before television!

Professor Bailey lived on a steep hill when he was a kid, and he and his friends used to throw balls down the hill to try to get them into the woods at the bottom with only one bounce. (Fortunately they had some helpful dogs who thought it was great fun to do the hard part of finding the balls and – usually – bringing them back.)



If you are standing on top of a smooth linear hill with a slope of $\tan\theta$ and throw a ball out horizontally with an initial velocity v,

(a) how far down the hill, S_1, will the ball hit on its first bounce?

(b) What is the ratio, S_2/S_1, of the distance (S_2) travelled by the ball on its second bounce over the distance (S_1) travelled on the first bounce?

Hints: Assume the bounces are perfectly elastic. Ignore air resistance.

[Alex]

6) Soup's on! Roll over Galileo!

Dr. Zed loves science, but even he sometimes gets misquoted in the press. As reported in the Summer 1996 issue of the popular Canadian children's magazine "Chickadee"[3], Dr. Zed likes to race empty and full soup cans against each other down a ramp. The magazine said that full cans always won because they were heavier.

(a) Is it true that full cans roll faster than empty cans? Was the magazine's explanation correct?

Note: The empty and full cans in each race are identical except – Duh! – the empty cans are empty and the full cans are full, and both the top and bottom of the empty cans are removed by a traditional can opener that leaves the rims on the can. The ramp is not too steep, so the cans roll on the rims without slipping. Dr. Zed used condensed cream soup so the contents of the full cans are jelly-like and act more or less like a solid as long as they are not subject to too high a stress. Dr. Zed likes soup a lot, so he has a big supply of empty and full soup cans. He also has cans full of all sorts of other stuff – beans, tomato paste, pumpkin pie filling – if he wants some variety for his can races or for his dinner.

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[3] http://www.owlkids.com/

(b) What is the relative acceleration, a_{full}/a_{empty}, of the two cans?
   i) Measure it experimentally. (Explain your methods and present your data.)
   ii) Calculate it theoretically.
   iii) Discuss any differences and give possible qualitative explanations for any discrepancies (larger than your uncertainties) between your experimental data and your theoretical calculation.

(c) What happens if you race a can of condensed Cream of Mushroom (or any other soup which is a non-fluid jelly in the can) against the same size can of chicken or vegetable broth (or consommé or any other soup with a water-like consistency in the can)?
   i) Which can is first? What is the experimentally observed relative acceleration, a_{cream}/a_{broth}, of the two cans?
   ii) Give a qualitative theoretical explanation for your results.

Hints: Measure the relative accelerations either by timing each can separately or by racing them and seeing where the second can is when the first can reaches the bottom – whatever works best for you. Many cheap digital wristwatches have stopwatch functions. The second method works best if you have two people. For a ramp I just used a shelf propped on some books. Don't forget to tell us any possibly relevant experimental parameters, *e.g.* the size of the cans you used and the slope and length of your ramp.

[David]

POPBits™ – Possibly useful bits of information

Constants and units[4,5]

astronomical unit (mean earth-sun distance)	au	149 597 870 660±20 m
Earth mass	$M_\oplus$	$(5.974\pm0.009)\times10^{24}$ kg
Newtonian gravitational constant	G_N	$(6.673\pm0.010)\times10^{-11}$ m^3/kg/s^2
Solar mass	$M_\odot$	$(1.98892 \pm 0.00025)\times10^{30}$kg
standard acceleration of gravity at the earth's surface	g	9.80665 m/s^2
tropical year (2001)	yr	31556925.2 s

David's Maxim of the Month

"A measurement without an uncertainty[6] is just a number;
a measurement with an uncertainty is data."

Great excuse for a party

Nobel Prize awards[7] (Eric Cornell, Wolfgang Ketterle, Carl Wieman) December 10

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4 http://physics.nist.gov/cuu/Constants/index.html
5 http://pdg.lbl.gov/2000/contents_sports.html
6 Also known as an "error bar".
7 http://www.nobel.se/physics/laureates/2001/index.html

2001-2002 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 2: Mechanics

1) Salty log!

If the log has a density greater than the fresher surface water but less than the deeper saltier water, it will float at the halocline such that the net sinking force on the fraction, *f*, of the volume of the log above the halocline will be equal and opposite to the upward buoyancy force on the fraction, 1-*f*, below the halocline, *i.e.*

$$fV(\rho_L - \rho_S) = (1-f)V(\rho_D - \rho_L)$$
$$\therefore\ f\rho_L - f\rho_S = \rho_D - \rho_L - f\rho_D + f\rho_L$$
$$\therefore -f\rho_S = \rho_D - \rho_L - f\rho_D$$
$$\therefore\ f = \frac{\rho_D - \rho_L}{\rho_D - \rho_S}$$

As a quick check of our calculation, we can consider the case before the log begins to sink and it is less dense than the fresh water. In this case it will float on the surface with a fraction in the water

$$f = 1 - \frac{\rho_S - \rho_L}{\rho_S - \rho_{Air}} = \frac{\rho_L}{\rho_S}$$

which is exactly what we expect.

2) Don't leave me!

(a) The ratio of the forces of the earth and the sun on the moon are

$$\frac{F_{Sun}}{F_{Earth}} = \frac{G_N m_{Moon} m_{Sun} / r_{Sun}^2}{G_N m_{Moon} m_{Earth} / r_{Earth}^2}$$

$$= \frac{m_{Sun} / r_{Sun}^2}{m_{Earth} / r_{Earth}^2} = \frac{2.0\times10^{30}\,\text{kg} / \left(1.5\times10^{11}\,m\right)^2}{6.0\times10^{24}\,\text{kg} / \left(400000\,km\right)^2} = 2.4$$

(b) This question is not trivial, since the three-body problem is not exactly soluble for a inverse-square law force. We cannot write down the exact general equations of motion for an arbitrary system of three (or more) bodies interacting by gravity, so we cannot prove that the earth and the moon will never part company, even if we assume they are ideal point masses. But if they do part company, it certainly won't be for a very, very, very, very long time. This is because the acceleration of the moon due to the sun is the same, on average, as the acceleration of the earth due to the sun, so once they are moving together around the sun (as they do) they will continue to move together. The maximum difference in acceleration between the earth and the moon towards the sun is much smaller than the acceleration of the moon due to the earth.

$$\frac{a_{Earth-Sun}-a_{Moon-Sun}}{a_{Moon-Earth}} = \frac{{G_N m_{Sun}}\big/{r_{Sun}^2} - {G_N m_{Sun}}\big/{\left(r_{Sun}-r_{Earth}\right)^2}}{{G_N m_{Earth}}\big/{r_{Earth}^2}}$$

$$= \frac{m_{Sun}\left({1}\big/{r_{Sun}^2} - {1}\big/{\left(r_{Sun}-r_{Earth}\right)^2} \right)}{{m_{Earth}}\big/{r_{Earth}^2}}$$

$$\approx \frac{2m_{Sun}r_{Earth}^3}{m_{Earth}r_{Sun}^3}$$

$$= \frac{2\times 2.0\times 10^{30}\,\mathrm{kg}\left(400000\,km\right)^3}{6.0\times 10^{24}\,\mathrm{kg}\left(1.5\times 10^{11}\,m\right)^3} = 0.013$$

For the moon to separate from the earth, it would have to gain or loose angular momentum from the sun or the earth and there is no simple mechanism to do so.

3) Singing in the rain!

(a) We make the approximation that we are vertical rectangular cube with a top surface area *T* and a front (or back) surface area *F*. The time it takes us to reach the shelter is $t=L/v$, so our "top wetness" (*i.e.* the amount of rain landing on our top) is

$$W_T = nTt = nTL/v$$

and the rain we sweep out of the air onto our front is

$$W_F = \rho FL = (n/u)FL$$

where, ρ, is the volume fraction of the air taken up by rain drops.

Our total wetness is

$$W = W_T + W_F = nTL/v + (n/u)FL = nL(T/v + F/u)$$

So our minimum wetness is when we run as fast as we can, *i.e.*

$$v = v_{max}$$

(b) If we have an umbrella, then $W_T=0$ and our total wetness will just be

$$W = W_F = (n/u)FL$$

So wetness is independent of our speed, v.

(c) If we have an umbrella, and the rain is falling at an angle (along our Front/Back plane) then the amount of rain reaching our front or back is not just the amount we sweep out as we walk, but also includes a contribution due to the horizontal movement of the rain,

$$W = W_F = (n/u)FL + (n\tan\theta)Ft = (n/u)FL + (n\tan\theta)FL/v$$

$$= nFL\,(1/u + \tan\theta/v)$$

where $n\ tan\theta$ is the rate at which the rain is falling horizontally. If $\theta = 0$ then we just recover our answer to part (b) as expected.

If $\theta > 0$, then the rain is blowing in our face, $(1/u + tan\theta/v) > 0$, so the best we can do is run as fast as we can, *i.e.* $v = v_{max}$, as for part (a), and our total wetness will be

$$W = nFL\,(1/u + tan\theta/v_{max})$$

If $\theta < 0$, then the rain is blowing from behind, $(1/u + tan\theta/v) > 0$, so we can keep dry if we just walk at the horizontal speed of the rain so that it is falling vertically in our reference frame, *i.e.* our optimum speed is $v = -u\ tan\theta = |u\ tan\theta|$ and our total wetness is W=0.

(Note: We want to get to the shelter so we assume v is positive. If our goal is just not to get wet, an alternate for $\theta > 0$ is just to walk away from the shelter with speed $v = -u\ tan\theta = -|u\ tan\theta|$)

4) Sproing!

The total force on the block along the ramp is the sum of the spring, gravitational, and frictional forces:

$$F = m\frac{d^2x}{dt^2} = kx + mg\ sin\alpha - \mu_k\ mg\ cos\alpha$$

where x is the displacement of the block along the ramp from the position where the spring force is zero, and $\hat{v} = \frac{dx/dt}{\left|dx/dt\right|}$ is the unit direction of velocity of the block along the ramp. *i.e.* The frictional force always opposes the direction of motion, and we have made the usual (but not always accurate) textbook assumption that sliding friction is independent of speed.

Without the frictional damping term, it is just an harmonic oscillator

$$m\frac{d^2x}{dt^2} = kx + mg\sin\alpha$$

which you are expected to recognize after some rearrangement

$$\frac{d^2\left(x + \frac{mg\sin\alpha}{k}\right)}{dt^2} = \frac{k}{m}\left(x + \frac{mg\sin\alpha}{k}\right)$$

and know the solution to be

$$\left(x + \frac{mg\sin\alpha}{k}\right) = x_0\sin(\omega t), \qquad \omega = \sqrt{k/m}$$

$$\therefore x = x_0\sin(\omega t) - \frac{mg\sin\alpha}{k}$$

where x_0 is a (as yet unknown) constant. The block's velocity (in the zero friction limit) is thus

$$\frac{dx}{dt} = \omega x_0\cos(\omega t)$$

We know the initial velocity at t=0 is v (I probably should have said v_0), so $x_0 = \frac{v}{\omega}$. The minimum (*i.e.* lowest) and maximum positions of the block (in the zero friction limit) are

$$x_{\min} = -x_0 - \frac{mg\sin\alpha}{k}, \qquad x_{\max} = x_0 - \frac{mg\sin\alpha}{k}$$

which will occur at times

$$\omega t_{\min} = \frac{\pi}{2} + 2n\pi, \qquad \omega t_{\min} = \frac{3\pi}{2} + 2n\pi, \qquad n = 0,1,2,\ldots$$

The total kinetic (T) and potential (U) energy of the block are:

$$\mathrm{T} = \frac{1}{2}\,\mathrm{m}\left(\frac{dx}{dt}\right)^2$$

$$\mathrm{U} = \frac{1}{2}\,\mathrm{k}x^2 + \mathrm{mg}\; \sin\alpha\; x$$

where we have chosen the (arbitrary) zero for the gravitation potential energy to be the initial point x=0 so that the total potential energy is zero at the initial point.. The initial kinetic energy is

$$\mathrm{T}_0 = \frac{1}{2}\,\mathrm{mv}^2$$

and we can find the maximum extension points where the block's motion reverses itself since the velocity and hence kinetic energy at these points is zero so all the energy must in the form of potential energy, *i.e.*

$$\frac{1}{2}mv^2 = \frac{1}{2}kx_{\min}^2 + mg\sin\alpha\; x_{\min}$$

$$\therefore x_{\min} = \frac{-mg\sin\alpha \pm \sqrt{(mg\sin\alpha)^2 - 4\left(\frac{1}{2}k\right)\left(-\frac{1}{2}mv^2\right)}}{2\left(\frac{1}{2}k\right)}$$

$$= \frac{-mg\sin\alpha \pm \sqrt{(mg\sin\alpha)^2 + kmv^2}}{k}$$

By definition, x_{min} must be less than zero, so only one of the two solutions is possible, *i.e.*

$$x_{\min} = \frac{-mg\sin\alpha - \sqrt{(mg\sin\alpha)^2 + kmv^2}}{k}$$

(a) The frictional force is small, so the motion of the block is damped only very slowly. The kinetic energy lost is just the integrated frictional energy loss. On the way down to its minimum position the frictional energy loss is

$$W = \int_0^{x_{\min}} \mu_k mg\cos\alpha dx = \mu_k mg\cos\alpha x_{\min}$$

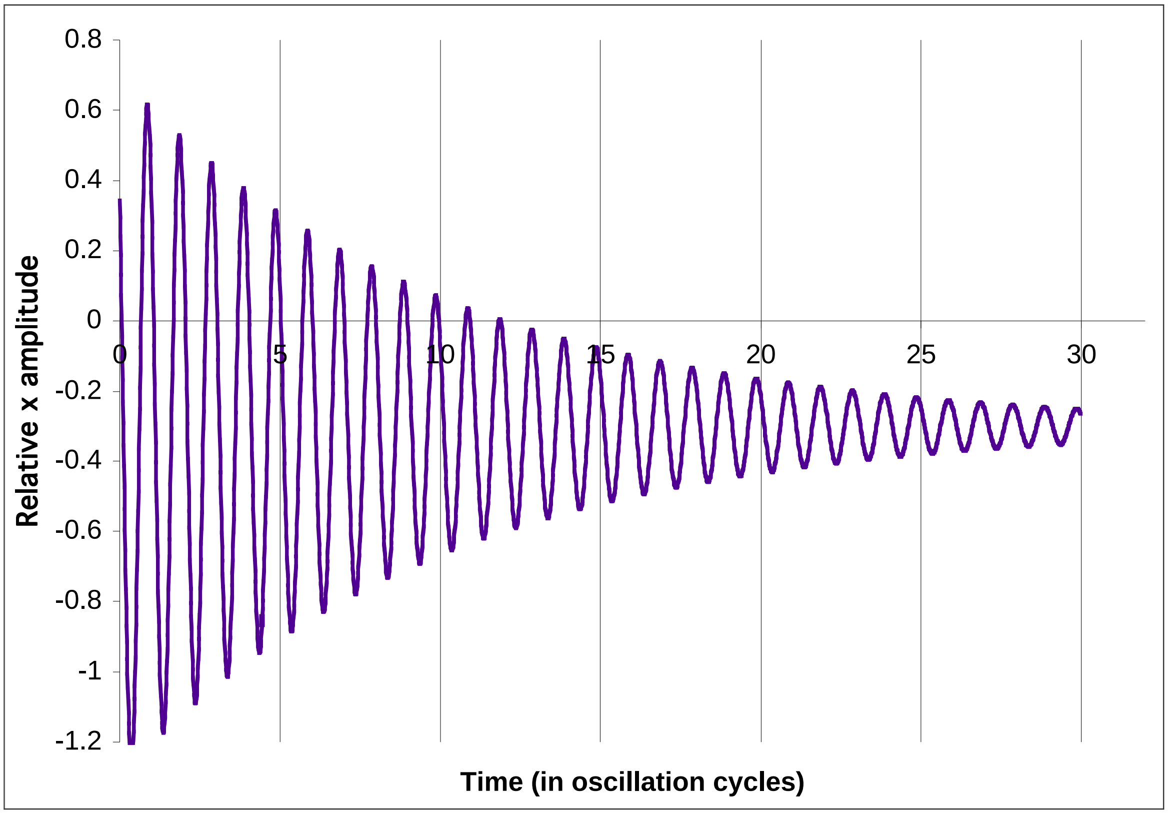
$$= \mu_k mg\cos\alpha\left(\frac{-mg\sin\alpha - \sqrt{(mg\sin\alpha)^2 + kmv^2}}{k}\right)$$

On the way back up to the initial point the block will lose the same amount of energy since it travels the same distance, so the kinetic energy will be smaller by an amount

$$\Delta T = 2W = 2\mu_k mg\cos\alpha\left(\frac{-mg\sin\alpha - \sqrt{(mg\sin\alpha)^2 + kmv^2}}{k}\right)$$

(b) The block's velocity is zero twice every oscillation cycle, *i.e.* at a frequency $2\omega/(2\pi)=\omega/\pi=\frac{1}{\pi}\sqrt{\frac{k}{m}}$.

(c)



(d) Because of gravity, the minimum potential energy position of the block is below the initial position (which is the position where the spring force and potential energy are zero). The springs will oscillate about a position below the initial position (*i.e.* from our solutions about the middle point of the oscillation is $x_{middle} = -\frac{mg\sin\alpha}{k}$), so the block as it slows down will be more likely to stop below its initial position.

5) Fun before television!

(a) Lets choose an x-y coordinate system in which the positive x direction points vertically down and the positive y direction points horizontally away from the hill. Before the first bounce has a constant horizontal velocity and constant vertical acceleration and the path of the ball as a function of time is

$$y = v\,t\ , \qquad x = \frac{1}{2}\,\mathrm{g}\,t^2$$

The parameterization for the slope of the hill is

$$x/y = \tan\theta$$

and the ball will bounce when its coordinates intercept those of the hill, *i.e.* when

$$x_1 = \frac{1}{2}g\left(\frac{y_1}{v}\right)^2 = \frac{1}{2}g\left(\frac{x_1}{v\tan\theta}\right)^2$$

$$\therefore \frac{1}{2}g{x_1}^2 = x_1 v^2 \tan^2\theta$$

$$\therefore x_1 = 2\frac{v^2\tan^2\theta}{g}$$

The distance down the hill for the first bounce is

$$S_1 = \sqrt{{x_1}^2 + {y_1}^2} = \sqrt{{x_1}^2 + \frac{{x_1}^2}{\tan^2\theta}} = \sqrt{1+\frac{1}{\tan^2\theta}}\,2\frac{v^2\tan^2\theta}{g} = 2\frac{v^2\tan^2\theta}{g\sin\theta}$$

$$= 2\frac{v^2\tan\theta}{g\cos\theta}$$

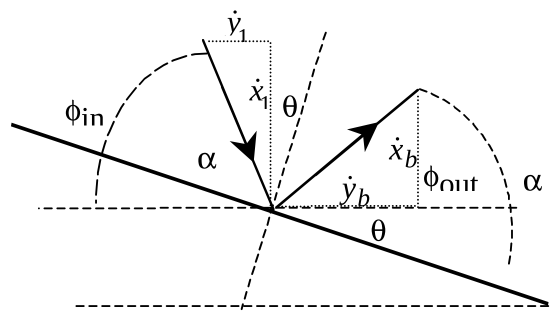
(b) This is pretty much the same as part (a), except a bit messier. First we need the ball's vector velocity $(\dot{x},\dot{y})$ before the bounce.

$$\dot{y}_1 = v\ , \qquad \dot{x}_1 = gt$$

To get the x component we need the time of the bounce:

$$t_1 = \sqrt{\frac{2x_1}{g}} = \sqrt{\frac{2}{g}2\frac{v^2\tan^2\theta}{g}} = \frac{2v\tan\theta}{g}$$

$$\therefore \dot{x}_1 = gt_1 = 2v\tan\theta$$



By energy conservation in an elastic collision the speed (v_1) of the ball must be the same before and after the bounce, so looking at the angles in the bounce, we can see that the velocity components before $(\dot{x},\dot{y})$ and after $(\dot{x}_b,\dot{y}_b)$ the bounce are related by

$$\dot{y}_1 = v_1\cos\phi_{in},\ \ \dot{x}_1 = v_1\sin\phi_{in},\ \ \phi_{in} = \theta + \alpha$$

$$\dot{y}_b = v_1\cos\phi_{out},\ \ \dot{x}_b = v_1\sin\phi_{out},\ \ \phi_{out} = -\theta + \alpha$$

These can be solved in various ways, some of them involving messy trigonometry, but I think the easiest is to think about how each component of $(\dot{x},\dot{y})$ "bounces" separately and then adding them back together, *i.e.*

$$\dot{y}_b = \dot{x}_1\sin(2\theta) + \dot{y}_1\cos(2\theta),\ \ \dot{x}_b = -\dot{x}_1\cos(2\theta) + \dot{y}_1\sin(2\theta)$$

So the path of the ball after the collision is given by

$$y = \dot{y}_b t + y_1, \quad x = \frac{1}{2}gt^2 + \dot{x}_b t + x_1$$

We can make things easier for ourselves by choosing to redefine our coordinate system such that $x_1 = y_1 = 0$, since the distance bounced can only depend on the initial vector velocity , the acceleration due to gravity, and the angle of the slope. So

$$y = \dot{y}_b t \qquad \Rightarrow \qquad t = \frac{y}{\dot{y}_b}$$

$$x = \frac{1}{2}gt^2 + \dot{x}_b t = \frac{1}{2}g\left(\frac{y}{\dot{y}_b}\right)^2 + \dot{x}_b \frac{y}{\dot{y}_b}$$

Once again the bounce will occur when the ball's coordinates intersect that of the hill (using our redefined coordinate system)

$$\therefore y\tan\theta = \frac{1}{2}g\left(\frac{y}{\dot{y}_b}\right)^2 + \dot{x}_b \frac{y}{\dot{y}_b}$$

$$\therefore y = \frac{2\dot{y}_b{}^2}{g}\left(\tan\theta - \frac{\dot{x}_b}{\dot{y}_b}\right)$$

$$= \frac{2(\dot{x}_1\sin(2\theta) + \dot{y}_1\cos(2\theta))^2}{g}\left(\tan\theta - \frac{-\dot{x}_1\cos(2\theta) + \dot{y}_1\sin(2\theta)}{\dot{x}_1\sin(2\theta) + \dot{y}_1\cos(2\theta)}\right)$$

$$= \frac{2(2v\tan\theta\sin(2\theta) + v\cos(2\theta))^2}{g}\left(\tan\theta - \frac{-2v\tan\theta\cos(2\theta) + v\sin(2\theta)}{2v\tan\theta\sin(2\theta) + v\cos(2\theta)}\right)$$

$$= \frac{2v^2}{g}\tan\theta\left(1 + 2\sin^2\theta\right)$$

So distance travelled along the slope on the second bounce is

$$S2 = \sqrt{x^2 + y^2} = \sqrt{\tan^2\theta + 1}y = \frac{y}{\cos\theta}$$

and the ratio compared to the first bounce is

$$\frac{S2}{S1} = \frac{y}{S1\cos\theta} = \frac{g}{2v^2\tan\theta}y = \frac{g}{2v^2\tan\theta}\frac{2v^2}{g}\tan\theta\left(1 + 2\sin^2\theta\right)$$

$$= \left(1 + 2\sin^2\theta\right)$$

This does not depend on *v* or g; it is 1 when the slope is near horizontal and goes to infinity when the slope is near vertical, both of which make sense since in the former case the ball picks up very little gravitational energy so every bounce should be the same length, and in the latter case the ball picks up a lot of gravitational energy before bouncing.

6) Soup's on! Roll over Galileo!

(a) I practically choked on my breakfast cereal when I read the magazine's explanation in 1996. One of the most famous experiment's in all of science is Galileo dropping balls of different weights from the Leaning Tower of Pisa to show that gravitational acceleration is independent of mass. (For a virtual version of this experiment, see http://www.pbs.org/wgbh/nova/pisa/galileo.html. For a more modern version, check out the video of David Scott dropping a hammer and a feather on the moon at http://cass.jsc.nasa.gov/expmoon/Apollo15/apo15g.avi.) Whether Galileo actually dropped balls from the Tower is somewhat controversial, but we know for sure that Galileo studied acceleration of gravity using balls rolling down inclined planes. From Newton's Laws (of Gravity and Motion) I knew heavier cans should not roll faster, except for effects such a friction and air resistance.

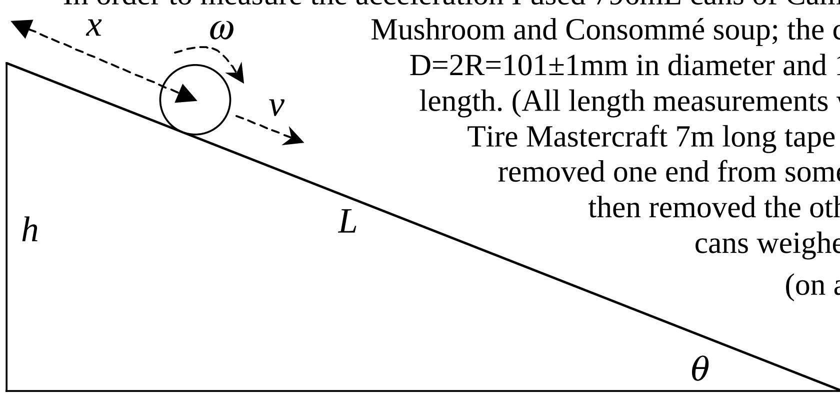
Jumping up from the table, I immediately took a bookshelf (825±10 mm long) and put one end on some books. I took some full cans and empty cans (from our recycling bin) and quickly established that full cans do roll faster than empty cans, but it has nothing to do with how heavy they are. A full 284ml can rolls just as fast as a full 796ml can; similarly a small empty can rolls at about the same speed as a large empty can. The difference must have to do with how the mass is distributed, not how much their is.

I must admit I find it hard to believe that Aristotle actually thought acceleration was proportional to mass. Yes, a flat piece of paper and a rock do fall at very different speeds, but all you need to do is drop a small pebble and a large rock to see that acceleration isn't even close to being proportional to mass. (Even a crumpled up piece of paper hits the ground at almost the same time as a rock if dropped from chest height.) Aristotle was not an idiot (e.g. see http://www.batesville.k12.in.us/Physics/PhyNet/AboutScience/WasAristotle.html), but in this case he seems to have created a classic example of how "common sense" can produce spectacularly wrong but long lasting dogma.

Note: The answers I give below is far longer and more detailed than we expect from you, and even so I have left out much of the details of what I did.

(b) For this problem set, I redid my 5 year old experiment more carefully.

i) In order to measure the acceleration I used 796mL cans of Campbell's Cream of Mushroom and Consommé soup; the cans were D=2R=101±1mm in diameter and 118±1mm in length. (All length measurements with a Canadian Tire Mastercraft 7m long tape measure.) I removed one end from some cans, made soup, then removed the other end. The empty cans weighed m_{empty}=70±5g (on a kitchen scale) and the full cans weighed



m_{full}=0.9±0.1kg. (The kitchen scale only went to 250g, so I had to empty the can and measure the contents in several portions which is why the error is large. I roughly checked the calibration of the scale using some water, but the quoted uncertainties are based on how reproducible my measurements were.)

After playing around with various cans and ramps ("playing around" is an essential part of good experimental technique – it is how you get a feel for how to do things), I decided to use a ramp made from a L=825±10mm long Ikea wooden bookshelf resting on VHS video cassettes. I used the cassettes instead of books since they were all the same size (each about 30mm thick). I used this ramp because it was straight, wide, not as slippery like laminate surfaces (so the cans always rolled instead of starting to slide), and it was short enough so I could release the can with one hand and catch with the other. (This last point saved me a lot of time since I didn't have to get up to fetch the can after each run.)

When I raced two cans, the full Cream of Mushroom can beat the empty can by 10±2cm, 8±3cm, 13±3cm for ramp heights of h = 120±5mm, 60±5mm, and 182±5mm respectively. I found I could not release the cans simultaneously and see their separation accurately at the bottom (I was alone sine my kids lost interest after about 30minutes), so I decided it was better for me to time the cans one at a time with the stop watch function on my $30 Casio AQ-140 wrist-watch. After more playing around the best method seemed for me to hold the watch in my right hand which also blocked the can at the top of the ramp. I would start the watch with my thumb while raising my hand which released the can. I would stop the watch when the can hit my hand at the bottom of the ramp. I hoped that if I was consistent my reaction times would mostly cancel out.

In order to have a consistent set of data (e.g. what would happen if my reaction times changed) did runs alternating an empty can, a full Cream of Mushroom can, and a full Consommé can so I would have the data for part (c) as well. I rolled each can about 10 times for a given ramp height. I did this for 3 ramp heights (h=120±5, 60±5, and 182±5mm), before realizing that my timings depended on where I was looking when I released and caught the cans, and also that by lifting my hand up to release the can, I might sometimes give the can a little push. I decided to continuously look at my catching hand from release to catch, and I pulled my hand away from the can and up to avoid pushing the can (although there is still the possibility of air movements "pulling" the empty can after my hand).

I started again and did a consistent set of measurements ("Can Runs") which are given in a table at the end of this solution set. The times (in seconds) are given for 5 runs for each type of can (Empty, Cream of Mushroom, Full Consommé). The data are presented in the order I took them, but notice that I did not just steadily increase (or decrease) the height of the ramp, to avoid the possibility of a correlated time bias (e.g. maybe I got tired and this affected how I measured the time). I was actually quite pleasantly surprised by how reproducible my measurements were; I didn't realize my reflexes are consistent to 0.1s.

If constant forces (e.g. gravity) on the cans are dominant, then the mean acceleration of the cans is just given by

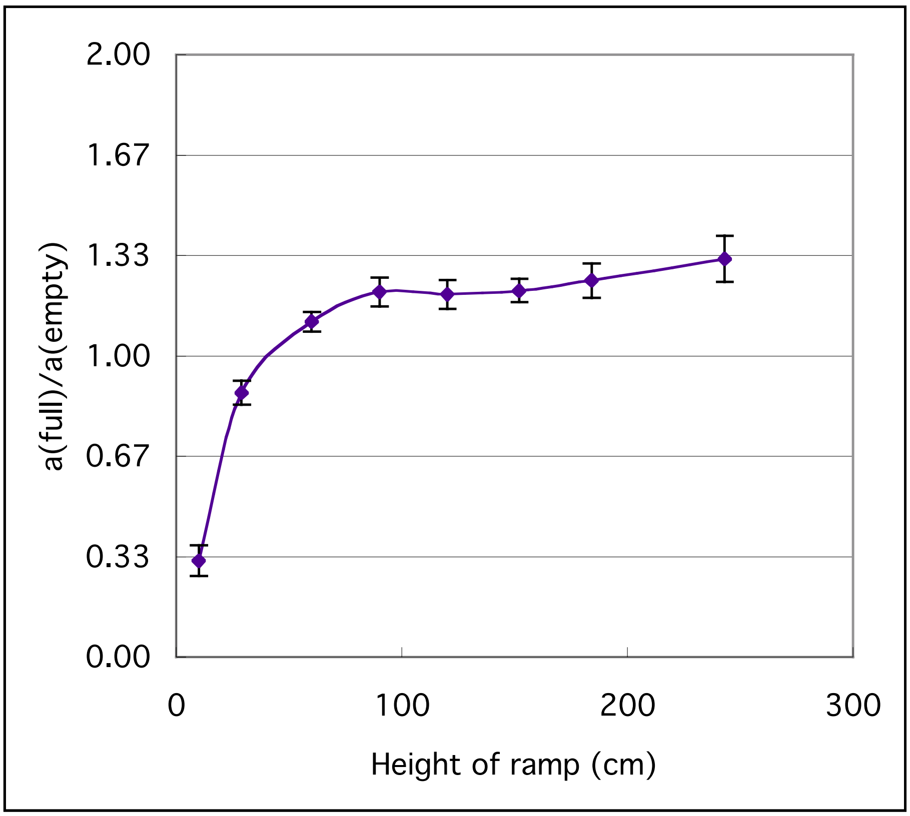
$$a = \frac{2L}{t^2}$$

So since L is a fixed distance,

$$\frac{a_{full}}{a_{empty}} = \left(\frac{t_{empty}}{t_{full}} \right)^2$$

The date for the full and empty cream of mushroom soup is summarized in the table and chart below:

h (in videos	h (in mm)	σ(h)	t_{empty}	σ(t_{empty})	t_{full}	σ(t_{full})	$\frac{a_{full}}{a_{empty}}$	$\sigma(\frac{a_{full}}{a_{empty}})$	Correct ed
0	10	2	4.72	0.07	8.35	1.33	0.32	0.05	0.30
1	29	2	3.02	0.05	3.22	0.13	0.88	0.04	0.87
2	60	2	2.17	0.03	2.05	0.05	1.11	0.03	1.13
3	90	3	1.83	0.06	1.66	0.03	1.21	0.05	1.26
4	120	3	1.57	0.03	1.43	0.05	1.20	0.05	1.25
5	152	4	1.43	0.04	1.30	0.03	1.22	0.04	1.28
6	184	4	1.31	0.04	1.17	0.04	1.25	0.06	1.33
8	243	5	1.20	0.03	1.04	0.05	1.32	0.08	1.44
11	337	5	1.02	0.04	0.94	0.03	1.17	0.06	1.25



For shallow ramp heights the full can is slower than the empty can, but for steeper ramps the relative acceleration seems to flatten out approaching about 4/3. If I average the last 5 points, I get

$$\frac{a_{full}}{a_{empty}} = 1.23 \pm 0.05$$

ii) The empty can is a hollow cylindrical (radius *R*) with a moment of inertia:

$$I_{empty} = m_{empty} R^2$$

If the cream soup contents rotate uniformly with the can, then the contents of the can are a solid cylinder with a moment of inertia

$$I_{contents} = \frac{1}{2} m_{contents} R^2$$

The total moment of inertial of the full cream of soup can is the sum of the moments of the metal shell and the contents, *i.e.*

$$I_{full} = \frac{1}{2} m_{contents} R^2 + m_{empty} R^2 = \frac{1}{2}\left(m_{contents} + 2m_{empty}\right) R^2 = \frac{1}{2}\left(m_{full} + m_{empty}\right) R^2$$

where $m_{full} = m_{contens} + m_{empty}$.

There are various ways of calculating the acceleration, but I like to follow the energy. When the can rolls down the ramp (height *h*, length *L*, angle with the floor θ), the can gains both translational and rotational kinetic energy, *i.e.*

$$T = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$$

Since the can rolls without slipping, the angular velocity is ω=*v/R*, so

$$T = \frac{1}{2} mv^2 + \frac{1}{2} I \frac{v^2}{R^2} = \frac{v^2}{2}\left(m + \frac{I}{R^2}\right)$$

The can's kinetic energy after it has rolled a distance *x* down the ramp must equal the gravitational potential energy it has lost, *i.e.*

$$\frac{v^2}{2}\left(m + \frac{I}{R^2}\right) = mgx \sin\theta$$

$$\therefore x = \frac{v^2}{2mg\sin\theta}\left(m + \frac{I}{R^2}\right)$$

$$\therefore \frac{dx}{dt} = \frac{v}{mg\sin\theta} \frac{d^2x}{dt^2}\left(m + \frac{I}{R^2}\right)$$

but $v \equiv \frac{dx}{dt}$, so

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{g\sin\theta}{\left(1 + \frac{I}{mR^2}\right)}$$

$$\frac{a_{full}}{a_{empty}} = \frac{\left(1+\frac{I_{empty}}{m_{empty}R^2}\right)}{\left(1+\frac{I_{full}}{m_{full}R^2}\right)} = \frac{\left(1+\frac{m_{empty}R^2}{m_{empty}R^2}\right)}{\left(1+\frac{\frac{1}{2}\left(m_{full}+m_{empty}\right)R^2}{m_{full}R^2}\right)} = \frac{4}{\left(3+\frac{m_{empty}}{m_{full}}\right)}$$

So in the limit where we ignore the weight of the empty can compared to the full can, we have

$$\frac{a_{full}}{a_{empty}} \xrightarrow[\frac{m_{empty}}{m_{full}}\to 0]{} \frac{4}{3} = 1.333$$

or using my measurements for my cans

$$\frac{a_{full}}{a_{empty}} = \frac{4}{\left(3+\frac{70\pm 5g}{0.9\pm 0.1kg}\right)} = 1.300 \pm 0.004$$

(Note: If you solve the torques with respect to the point of contact and you use the wrong moment of inertia - the one with respect to the center of the can- you get a fraction for the two accelerations that is really close to my experimental result: 1.23. This is a coincidence.)

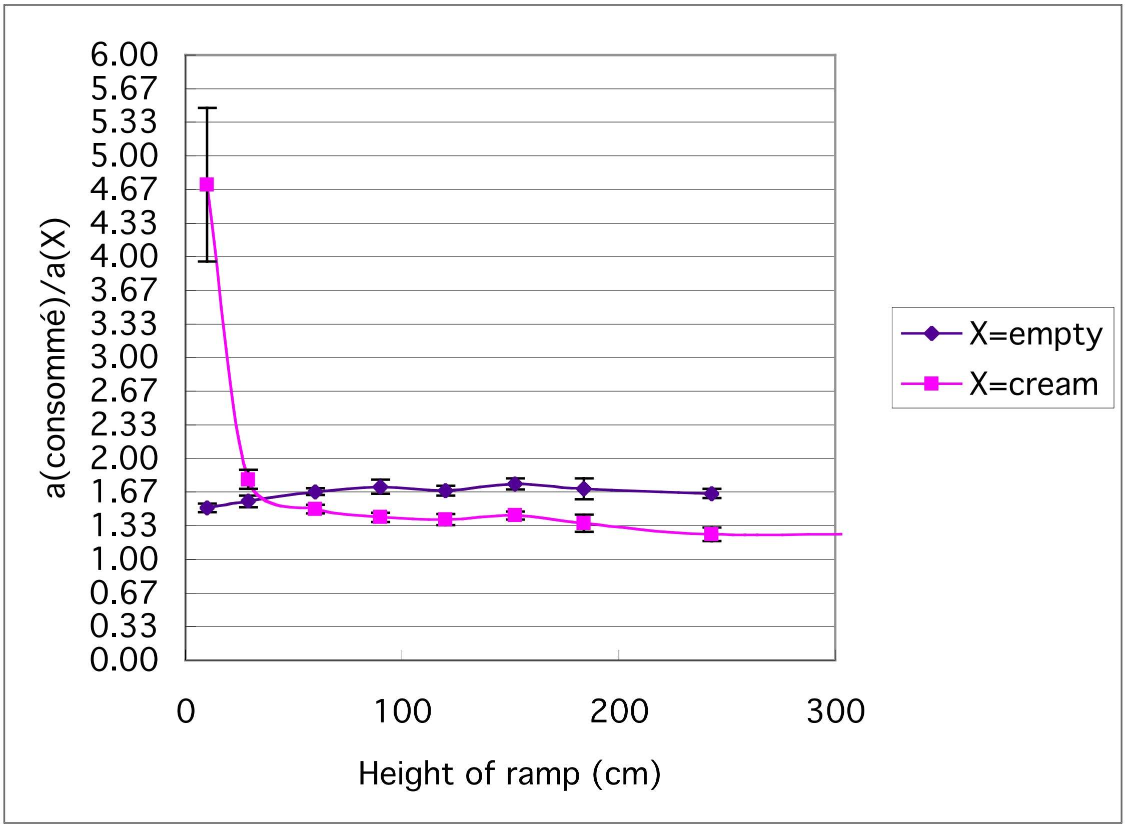
iii) Our theoretical calculation is very close to the experimentally observed value for larger slopes, but several factors could be responsible for the difference and the odd results for small slopes: friction, a constant offset in my timings, air resistance.

Air resistance would have a larger affect at higher speeds, so we would expect it to affect the higher ramp data more, which is not what we see.

I investigated the vary large variations or the shallowest slope (h=10mm, which had no videos and only a thin book propping up the ramp), by trying another can. This can rolled slowly in jerks and almost always stopped and did not reach the bottom of the slope. A little playing around showed that this can preferred to rest in one orientation, apparently because its centre of gravity did not lie on its axis. Since mushroom soup has chunks, this is easy to understand if the mushroom chunks were resting on one side inside the can. The original can probably had a smaller imbalance which was enough to affect its results.

(c) The data at the end of the solution set also includes data for a consommé can.

i) The consommé was clearly the fastest. The plot below shows the data for the relative accelerations of the consommé compared to both the Cream of Mushroom and the empty can.



If we again only consider the last 5 data points, then the average measured ratios are

$$\frac{a_{consommé}}{a_{empty}} = 1.66 \pm 0.09$$

$$\frac{a_{consommé}}{a_{cream\ \ of\ \ mushroom}} = 1.35 \pm 0.08$$

ii) The consommé flows freely inside the can. If friction and viscosity were zero, the can would roll down the slope but the consommé would not rotate within the can. This means that almost all the gravitational potential energy gained by the can would go into translational, not rotational, motion so it would roll with maximum possible acceleration.

$$a_{consommé} = \frac{g\sin\theta}{\left(1+\frac{I_{consommé}}{m_{consommé}R^2}\right)} = \frac{g\sin\theta}{\left(1+\frac{m_{empty}R^2}{m_{consommé}R^2}\right)} = \frac{g\sin\theta}{\left(1+\frac{m_{empty}}{m_{consommé}}\right)}$$

$$\frac{a_{consommé}}{a_{empty}} = \frac{2}{\left(1 + \frac{m_{empty}}{m_{consommé}}\right)} \xrightarrow[\frac{m_{empty}}{m_{consommé}}=0]{} 2$$

$$\frac{a_{consommé}}{a_{cream}} = \frac{\left(1 + \frac{\frac{1}{2}\left(m_{full} + m_{empty}\right)}{m_{full}}\right)}{\left(1 + \frac{m_{empty}}{m_{consommé}}\right)} = \frac{\left(\frac{3}{2} + \frac{m_{empty}}{m_{full}}\right)}{\left(1 + \frac{m_{empty}}{m_{consommé}}\right)} \xrightarrow[\frac{m_{empty}}{m_{consommé}}=0]{} \frac{3}{2}$$

So data indicate the consommé is not quite as fast as our model, which is reasonable since the consommé undoubtedly does rotate somewhat..

Data from Can Runs for Problem 6 (19 December 2001)

height (in videos)	Empty (s)	Cream (s)	Consommé (s)
5	1.45	1.31	1.11
height in mm	1.4	1.33	1.09
152	1.43	1.28	1.06
±	1.48	1.26	1.07
4	1.39	1.3	1.08
3	1.9	1.67	1.39
height in mm	1.82	1.68	1.45
90	1.76	1.63	1.4
±	1.78	1.62	1.36
3	1.88	1.7	1.37
1	3.04	3.39	2.4
height in mm	3	3.15	2.5
29	3.02	3.34	2.45
±	2.95	3.09	2.4
2	3.1	3.15	2.29
6	1.36	1.13	0.95
height in mm	1.32	1.21	0.97
184	1.27	1.19	0.97
±	1.31	1.12	1.05
4	1.27	1.19	1.07
2	2.15	2.05	1.68
height in mm	2.13	2.03	1.67
60	2.18	2.09	1.64
±	2.21	1.98	1.69
2	2.16	2.11	1.7
4	1.57	1.43	1.19
height in mm	1.53	1.41	1.23
120	1.62	1.37	1.22
±	1.59	1.49	1.19
3	1.56	1.47	1.24
8	1.25	1.01	0.93
height in mm	1.17	1.05	0.94
243	1.18	0.99	0.92
±	1.19	1.03	0.94
5	1.2	1.13	0.93
11	0.99	0.91	0.79
height in mm	1.03	0.95	0.82
337	1.06	0.91	0.78
±	1.01	0.91	0.84
5	0.96	0.99	0.9
	1.05	0.96	0.88
0	4.8	8.4	3.77
height in mm	4.78	7.99	3.84
10	4.66	6.85	3.93
±	4.71	10.49	3.95
2	4.66	8.01	3.73

University of Toronto

2001-2002 Physics Olympiad Preparation Program

Problem Set 3: Thermodynamics

Due Monday 14 January 2002

1) How much heat could a heat pump pump if a heat pump could pump heat?

The heat in my house (and many other old houses in Toronto) is provided by an ancient coal boiler converted to burn oil. It is starting to leak water so we are considering what to replace it with, and one possibility is an air heat pump. (Geothermal heat pumps[1] work better, but our back yard is too small to fit the pipes.)

As its name implies, an air heat pump pumps heat by absorbing some heat (Q_{out}) from the outside air (temperature T_{out}) and supplying some heat (Q_{in}) to the air inside the house (temperature T_{in}). The heatpump works cyclically and for each cycle the heat pump motor does an amount of work (W). Assume the heat pump has the maxiumum possible efficiency.

(a) What is the relationship between Q_{in}, Q_{out}, and W?

(b) How much heat will be pumped into the house for every joule of work done by the motor if the inside temperature is 20°C and the outside temperature is –20°C?

(c) If the outside temperature was absolute zero, the heat pump would still pump heat into the house. How is this possible and where does the heat come from?

[David]

2) Relatively heavy

When I have a fever, I sometimes feel lethargic and heavy. Estimate the fractional amount I am heavier if my temperature increases from 37°C to 39°C?

Hints: In addition to writing down the most famous equation in all of science, did you know that Einstein had over 40 patents on refrigerators?[2]

All experimental evidence is consistent with General Relativity's principle that inertial and gravitational mass are equivalent.

Assume I neither gain nor lose any molecules as I heat up, and you may make the crude approximation that I am a bag of ideal gas water molecules. (My kids sometimes think I am full of gas, but not ideal.)

[David]

[1] http://www.earthenergy.ca/tech.html

[2] http://gtalumni.org/news/magazine/sum98/einsrefr.html

3) Jet cooled!

The New Zealand physicist Ernest Rutherford[1], who discovered the atomic nucleus in 1911, is often described as the greatest experimental physicist of the 20^{th} century. His great experimental tradition still lives on in New Zealand and manifests itself in such recent accomplishments as the jet powered beverage cooler[2]. This cooler consists of a tank of liquid petroleum gas (LPG) in a container of water. When gas is released from the tank, the water is cooled very quickly. To avoid accidental and explosive accidental ignition of the very flammable gas, the released LPG is burned in a jet engine with a very satisfying 125dB roar.

The specifications of the cooler are a bit sparse, but it appears to be able to cool about 10 litres of water from 14°C to 2°C in about 5 minutes.

(a) What is the average rate (in Watts) that the cooler removes heat from the water?

I don't have any LPG in my lab (and I have enough sense not to risk explosion, fire, and deafness just to cool a drink), but I do have a 50 litre bottle of helium gas (atomic weight 4.003) at a pressure of 170 Atmospheres and a temperature of 21°C.

(b) If I release gas from my bottle and allow it to expand adiabatically to atmospheric pressure, estimate how cold (in °C) will it get?

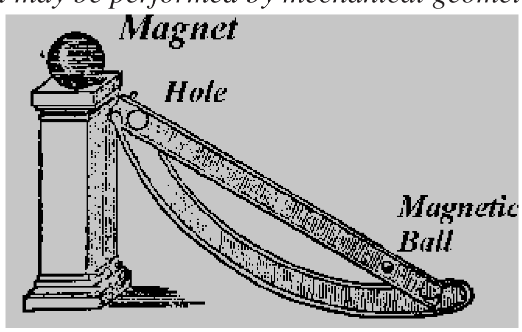
(c) If I then use this cold gas to cool water, how much gas (in litres measured at 170 Atmospheres and a temperature of 21°C) do I need to release in order to cool 10 litres of water from 21°C to 9°C?

Hint: Assume the water and gas are brought into thermal contact and left to adiabatically reach equilibrium.

[David]

4) Perpetual motion?

Of the many fascinating perpetual machines proposed through history, that proposed Johannes Taisnierus in 1570 and discussed by Bishop John Wilkins of Chester[3] in his book "*Mathematical Magick, or the wonders that may be performed by mechanical geometry*" is one of the simplest.



The magnet fixed on the top of the column was to draw the iron ball up the ramp until the ball reached the hole in the ramp. The ball would then fall and be returned to the ramp's base, and the procedure would begin again.[4]

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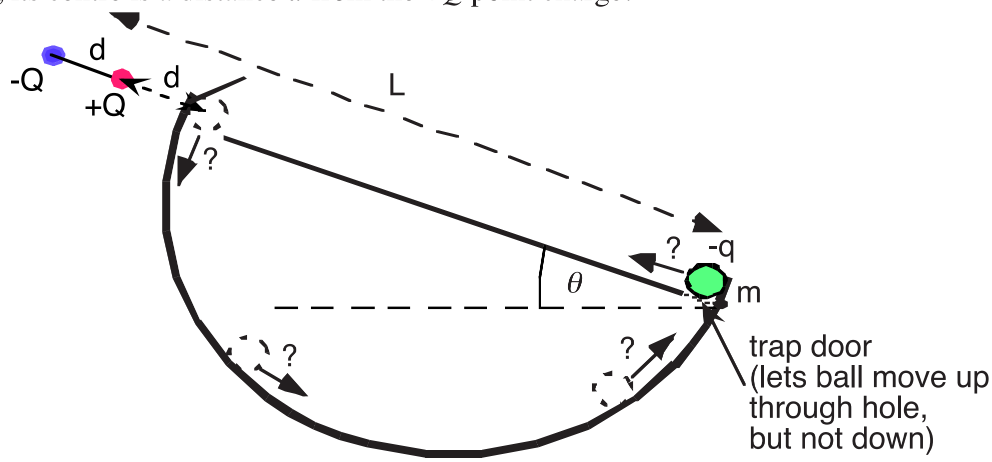
[1] http://www.nobel.se/chemistry/laureates/1908/
[2] http://www.asciimation.co.nz/beer/
[3] http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Wilkins.html
[4] A modern "perpetual motion machine" involving rolling balls and magnets can be found at http://www.theverylastpageoftheinternet.com/magneticDev/finsrud/finsrud.htm.

Let's consider an modern electrostatic version of this device, as shown in the diagram below. The electric dipole (point charges $\pm Q$, separated by a distance d) at the upper left is aligned with the path of the ball (charge $-q$, mass m) rolling up the ramp. The length L ($>>d$) is the distance from the centre of the charged ball to the centre of the dipole. When the ball is over the hole, its centre is a distance d from the $+Q$ point charge.



If you <u>really</u> made such a device, describe and discuss what would happen when the ball is released (from the initial position shown in the figure) until it either stops or it finishes one complete cycle of an infinitely repeated loop. Does your answer depend on the ramp angle, θ, or any other parameters (possibly including parameters not listed here)? Assume that the device is as well made as is humanly possible, and feel free to make any realistic minor design changes that might improve the performance of the device.

[Isamu]

5) A slow day on the Bruce Trail![1]

If all else fails, a pleasant walk is better than moping in bed, and random walks are particularly important in physics. A one-dimensional random walk is when you walk along a line, and for each step the direction – forwards or backwards – is determined randomly. If the probabilities for going forwards or backwards are equal, then after N steps you will find yourself on (root-mean-square) average a distance of $\sqrt{N}$ steps away from where you started. This is only recommended for hikers who aren't in a hurry to get anywhere.

A classic example of a random walk process is diffusion, and the classic elementary school demonstration of diffusion is dye[2] dropped into water. I think these demonstrations are great, but it is not obvious to me that diffusion is the dominant process distributing the dye in most such demonstrations. One website[3] I looked at said it took 7 minutes for dye to diffuse throughout a glass of cold water, but only 4 minutes for a same-sized glass of hot water.

(a) Show that such a result cannot be due to random walk diffusion. Suggest possible explanations for the reported result (or convince me that I am wrong in saying that the result cannot be due to random walk diffusion).

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[1] http://www.brucetrail.org/

[2] The "dye" is usually food colouring, but you can use ink, juice, pop, ….

[3] http://hastings.ci.lexington.ma.us/staff/SLee/science/molecules.html

Hints: Calculate the ratio of temperatures, T_{hot}/T_{cold}, of the water in the two glasses if it the results are due to random walk diffusion. The mean speed of the molecules depends on the molecular mass and on the temperature, and the mean step size is roughly given by the mean spacing between water molecules. Diffusion in 3-dimensions can be treated as 3 independent one-dimensional random walks.

(b) Do the experiment yourself by dropping some dye into hot and cold glasses of water. Do your results agree with the results I report from the elementary school website?

(c) Design, do, and describe an improved experiment to study the diffusion of dye in room temperature water. Is the movement of the colour in your container in your new experiment consistent with random walk diffusion?

Hints: Here are some of the "improvements" I tried: I found a spot where my water containers were be safe from being moved or shaken or subject to sudden temperature changes. (I discovered the top of my fridge or a south facing window are NOT good places.) I waited for the water to be motionless and in thermal equilibrium before adding the dye. I tried adding the dye to the bottom, not the top. Chunks of leftover Hallowe'en lollipops gently dropped into the containers initially seemed to work well (and were used to bribe young children not to mess with my experiments). The candy dissolved in the water and released its colouring, but a dense bottom layer of sugar water was also created which seemed to affect the diffusion.[1] Next I tried injecting food colouring into the bottom of a container with an eyedropper[2], but often the dye leaked out before I reached the bottom. I also tried putting a few drops of food colouring into a tiny paper pocket which I weighted with a nickel and taped the edges shut before dropping into the container.

[David]

[1] Note: A good experimenter always reports any interesting effects observed, even if they don't match initial expectations.
[2] Actually a plastic straw with the top taped shut since I didn't have an eyedropper.

POPBits™ – Possibly useful bits of information

Constants and units[1,2]

astronomical unit (mean earth-sun distance)	au	149 597 870 660±20 m
atomic mass unit: (mass ^{12}C atom)/12	u	$(1.66053873 \pm 0.00000013) \times 10^{-27}$ kg
Boltzmann constant	k	$(1.3806503 \pm 0.0000024) \times 10^{-23}$ J/K
elementary (*i.e.* electron) charge	e	$(1.602176462 \pm 0.000000063) \times 10^{-19}$ C
electron volt	eV	$(1.602176462 \pm 0.000000063) \times 10^{-19}$ J
Newtonian gravitational constant	G_N	$(6.673 \pm 0.010) \times 10^{-11}$ $m^3/kg/s^2$
permittivity of free space	ε_0	$8.854187817 \times 10^{-12}$ F/m
Planck constant	h	$(6.6260755 \pm 0.000004) \times 10^{-34}$ J·s
solar luminosity	$L_\odot$	$(3.846 \pm 0.008) \times 10^{26}$W
speed of light in vacuum	c	299 792 458 m/s
standard acceleration of gravity at the earth's surface	g	9.80665 m/s^2
standard atmospheric pressure at the earth's surface	atm	101325 Pa
Stephan-Boltzmann radiation constant	σ	$(5.670400 \pm 0.000040) \times 10^{-8}$ $W/m^2/K^4$
tropical year (2001)	yr	31556925.2 s

Interesting facts about water

Specific heat = 4186J/kg/K
Freezing temperature at standard atmospheric pressure = 273.15K
Elephants can smell water more than 5 km away.

Great excuses for a party

Birthday of the quantum[3] (1900)	December 14
Isaac Newton's birthday (1642)	December 25

[1] http://physics.nist.gov/cuu/Constants/index.html
[2] http://pdg.lbl.gov/2000/contents_sports.html
[3] http://www.nobel.se/physics/educational/tools/quantum/energy-1.html

2001-2002 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 3: Thermodynamics

1) How much heat could a heat pump pump if a heat pump could pump heat?

(a) By energy conservation, the work must provide the difference between the heat pumped from the outside and the heat delivered to the inside, *i.e.* $|W|=|Q_{in}|-|Q_{out}|$.

(b) The maximum efficiency possible is for a Carnot engine,

$$\therefore \left|\frac{Q_{hot}}{Q_{cold}}\right| = \frac{T_{hot}}{T_{cold}} \quad \Rightarrow \quad \left|\frac{Q_{in}}{Q_{out}}\right| = \frac{T_{in}}{T_{out}}$$

To convert from °C to absolute temperature in °K, we just add 273°. The heat pumped per joule of work is

$$\frac{Q_{in}}{W} = \frac{Q_{in}}{Q_{in}-Q_{out}} = \left(1-\frac{Q_{out}}{Q_{in}}\right)^{-1} = \left(1-\frac{T_{out}}{T_{in}}\right)^{-1} = \left(1-\frac{273°-20°}{273°+20°}\right)^{-1} = 7.3$$

(c) The motor's work is converted to heat, so the heat pump always provides at least *W* heat. Rereading the question, one could argue that pump does not "pump heat into the house" (especially if the motor is inside the house), so maybe "provide heat to the house" would be better wording.

2) Relatively heavy

Another easy one, as long as you remember $E=mc^2$. Some students (equally correctly) used the heat capacity of an ideal gas or water, but this is how I did it. The total energy of a molecule is the sum of its rest mass and its thermal energy. (We can define its potential energy to be zero.) An ideal gas molecule has no internal or rotational degrees of freedom, so its only thermal energy is its kinetic energy and the total energy is

$$E = mc^2 = m_0c^2 + \frac{3}{2}kT$$

$$\therefore m = m_0 + \frac{3kT}{2c^2}$$

Where m_0 is the rest mass of the molecule at T=0. In this approximation, my mass is just proportional to my molecules masses, so the fractional increase in my mass going from T_1=37°C to T_2=39°C is

$$\frac{m(T_2)-m(T_1)}{m(T_1)} = \frac{\left(m_0+\frac{3kT_2}{2c^2}\right)-\left(m_0+\frac{3kT_1}{2c^2}\right)}{m_0+\frac{3kT_1}{2c^2}} = \frac{\frac{3k}{2c^2}(T_2-T_1)}{m_0+\frac{3kT_1}{2c^2}} = \frac{T_2-T_1}{\frac{2m_0c^2}{3k}+T_1}$$

As is well known (*i.e.* you are supposed to know it) water is H_2O and hence has an atomic mass of 18, so m_0=18u, where u is the atomic mass unit. Thus

$$\frac{m(39°C)-m(37°C)}{m(37°C)}=\frac{39°C-37°C}{\dfrac{2\left(18\times1.661\times10^{-27}kg\right)\left(299792458m/s\right)^2}{3\left(1.3807\times10^{-23}J/K\right)}+\left(37°C+273°K\right)}$$

$$=1.5\times10^{-14}$$

So the fractional mass increase is about 10^{-14}. (You get the same order of magnitude no matter which way you do it.)

3) Jet cooled!

(a) The average cooling rate is just

$$\frac{dQ}{dt}=\frac{(10l\ \times\ 1kg/l)(14°-2°)K}{(5\min\times\ 60s/\min)}\left(4186J/kg/K\right)=1674\frac{J}{s}=1.7kW$$

where we have used the well known density of water (1kg/litre).

(b) The reason I chose helium[1] for this problem is that it is a noble gas, and noble gases[2] act almost like an ideal gas at normal temperatures. For an ideal gas undergoing adiabatic expansion, $P^{(1-\gamma)}T^{\gamma}$ is a constant, where P is the pressure, T is the absolute temperature, and γ=5/3 is a constant.

$$\therefore T_f=T_i\left(\frac{P_i}{P_f}\right)^{(1-\gamma)/\gamma}=\left(273°+21°\right)K\left(\frac{170atm}{1atm}\right)^{\left(\frac{3}{5}-1\right)}=38K=-235°C$$

(c) To cool 10 litres of water by 12°, we need

$$\Delta Q=\left(10l\ \times\ 1kg/l\right)\left(21°-9°\right)K\left(4186J/kg/K\right)=502320J$$

of heat removed. The thermal energy of an ideal gas molecule is just 3kT/2, so using the ideal gas law, our cold expanded helium gas has a thermal energy

$$Q=N\frac{3}{2}kT=\frac{2}{3}PV\ .$$

To provide the required ΔQ, we must have

$$N=\frac{2\Delta Q}{3k\Delta T}.$$

where ΔT =9°C-(-235°C)=244K is the temperature increase of the gas as it reaches equilibrium with the water. Using the ideal gas law the volume of gas (at 170 Atmospheres and 21°C) required is

$$V=\frac{NkT}{P}=\frac{2T\Delta Q}{3P\Delta T}=\frac{2(21K+273K)(502320J)}{3(170atm\times101325\text{Pa/atm})244K}=23\times10^{-3}m^3=23l.$$

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[1] http://www.webelements.com/webelements/elements/text/He/key.html

[2] http://www.xrefer.com/entry.jsp?xrefid=490629&secid=.-&hh=1

That corresponds to about 4 cubic metres at atmospheric temperature - enough helium to inflate about 300 11" balloons.

Once again, many students used (equally correctly) the heat capacity of an ideal gas. My answer corresponds to using the heat capacity at constant volume, but you get 14*l* if you use the heat capacity at constant pressure. Rereading how I worded the question, assuming constant pressure probably does make more sense (since the gas is being released and expands in air at atmospheric pressure), but we accepted both answers.

4) Perpetual motion?

When the ball is released, it will not move unless the attractive force of the dipole can overcome gravity and static friction. We will assume that we can make it such that the static friction is very small, so we must have

$$\frac{Qq}{4\pi\varepsilon_0\left(L-d/2\right)^2}-\frac{Qq}{4\pi\varepsilon_0\left(L+d/2\right)^2}>mg\sin\theta$$

$$\therefore \frac{Qq}{4\pi\varepsilon_0}\left(\frac{2Ld}{\left(L^2-d^2/4\right)^2}\right)>mg\sin\theta$$

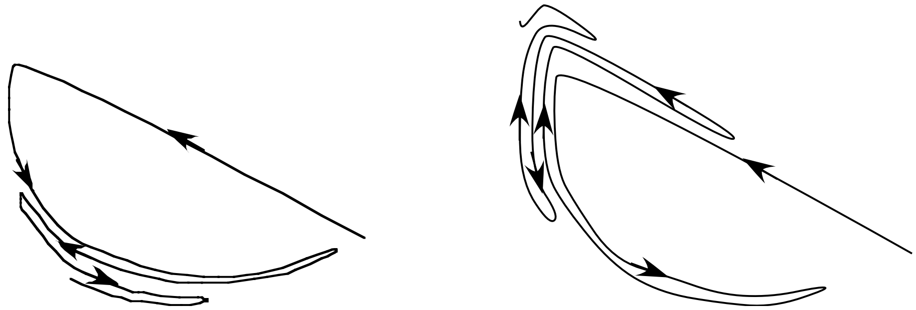
Since L>>d, this requirement becomes

$$\frac{Qq}{2\pi\varepsilon_0}\frac{d}{L^3}>mg\sin\theta$$

This is possible for appropriate charges and masses, although it may not be easy. (For example, everything would have to be constructed from insulators and the experiment done in a vacuum to prevent charged particles in the air being attracted to our charges.)

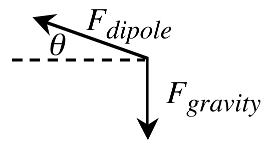
Once the ball starts to move, it will roll up the ramp until it hits the barrier where it will be deflected down. (If there is just an wall perpendicular to the direction of the ball, the ball will bounce backwards with a strength proportional to the elasticity of the collision, so I added a deflector to make sure the ball goes down in the right direction.) The ball will roll (if the deflector is just right, otherwise it will bounce) around the circular return path. Because of the 2nd Law of Thermodynamics (e.g. friction, inelasticity, …) the ball will not quite get back to its initial position. (For example, even if friction is essentially zero, the ball will not have quite enough energy to raise the flap since in the initial position the flap was down, and the flap has some mass which must be raised against gravity.)

Since the ball can't quite get back to its starting point, it will roll back down the circular return path and end up oscillating back and forth with small and smaller amplitude until it finally stops.



Note that the oscillations may include bouncing back down the ramp, but because of the energy losses, the ball never can get back to its start. The exact path depends on the amount of friction and the inelasticity of any bounces.

The equilibrium position depends on the strength of the dipole. If it is strong enough to overcome gravity, the ball will stick at the top of the ramp. (It will stick the first time it reaches the top of the ramp if the collision with the wall is inelastic enough, but we have assumed we are doing a good job in designing this device.) If we want the ball end up with the ball finally stopping on the circular path, we must build the dipole such that



$$F_{dipole} \sin\theta < F_{gravity}$$

$$i.e.\ \frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{d^2} - \frac{1}{(2d)^2}\right)\sin\theta < mg$$

$$\therefore \frac{3Qq}{16\pi\varepsilon_0 d^2}\sin\theta < mg$$

This requirement combined with our earlier requirement to get the ball started gives

$$\frac{Qq}{2\pi\varepsilon_0}\frac{d}{L^3} > mg\sin\theta > \frac{3Qq}{16\pi\varepsilon_0 d^2}\sin^2\theta$$

$$\therefore d > \left(\tfrac{3}{8}\sin^2\theta\right)^{1/3} L$$

Since we want $L>>d$, this means the angle θ must be pretty small. When the ball stops on the circular path, it will stop slightly to the left of the bottom of the ramp.

5) A slow day on the Bruce Trail!

A couple of comments about experimental questions on POPTOR. First, the important thing is make measurements and then to interpret them to the best of your ability. Second, in many ways, one never "completes" a real experiment, since one can always improve the experiment in some way.[3] Third, it is traditional in schools and universities – and this is also the case for POPTOR – that experiments do not usually receive a weight in marking proportional to the effort required to do them well.[4]

(a) The density of liquid water doesn't change very much between freezing and boiling temperatures, so the number of steps to diffuse over the whole glass is pretty much

[3] An interesting book discussing these kinds of issues is "How Experiments End", see http://www.fas.harvard.edu/~hsdept/faculty/galison/how_experiments_end.html.

[4] For POPTOR, we "correct" for this by keeping a separate record of how people do on experiments.

constant independent of temperature. *i.e*. The mean free path that a diffusing dye molecule travels before colliding with a water molecule is about the same for both temperatures. The change in diffusion times is dominated by the change in mean molecular velocities at different temperatures. The diffusion time, τ, is inversely proportional to the mean velocity, *i.e*.

$$\tau \propto \frac{1}{\langle v \rangle}$$

Using the relationship between kinetic energy and velocity, and temperature and kinetic energy for an ideal gas we have

$$\begin{aligned} \tau &\propto \frac{1}{\sqrt{\langle E_{kinetic} \rangle}} && \Leftarrow \quad E_{kinetic} = \tfrac{1}{2} m v^2 \\ &\propto \frac{1}{\sqrt{T}} && \Leftarrow \quad \sqrt{\langle E_{kinetic} \rangle} = \tfrac{3}{2} kT \end{aligned}$$

So the diffusion time is inversely proportional to the square-root of the absolute temperature. The maximum temperature range of liquid water is from freezing (273K) to boiling (373K), so the maximum ratio of diffusion times is

$$\max\left(\frac{\tau_{cold}}{\tau_{hot}}\right) = \max\left(\frac{\sqrt{T_{hot}}}{\sqrt{T_{cold}}}\right) = \sqrt{\frac{373}{273}} = 1.17$$

So the maximum ratio of diffusion times in water is about 1.2, which is much less than the reported ratio of 7min/4min=1.75.

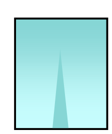
One obvious possible reason for the reported results is that hot water was moving more which mixes the dye faster. One might expect convection in the hot water driven by temperature differences at the water surfaces. Another possibility is that the measurement errors were very large, *i.e*. people aren't very good at judging when the dye has completely diffused, especially if they have prior expectations about what should happen. The whole point of the demo was to show that diffusion is faster in hot water, and a 17% (or less) difference is not subjectively "large enough".

(b) I reproduced the experiment using identical cylindrical glass glasses 10±0.2cm tall and 6.3±0.2cm in diameter (measured with a plastic ruler). I used water from my taps and the hot water was 50±1°C and the cold water was 15±1°C, where the uncertainties are just my reading errors since I didn't calibrate my rather cheap outdoor thermometer.

7:35pm — Food colouring added to each glass. The hot water clearly had more swirling. In the cold glass the dye when mostly to the bottom and then swirled slowly. Clearly the residual motion from adding the water to the glasses had not stopped.

7:38pm — Movement is clearly visible in the hot water, consistent with convection. The is only slow movement in the cold water, and the dye is in thin strands..

7:40 — The dye is almost uniformly distributed in the hot glass, but it is clearly slightly darker at the top than the bottom. More clear is a dark vertical swirl along the axis of the glass, which is wider and darker at the bottom. The cold glass is still has clearly visible strands of colour.



7:51	Can still see dark central region in hot glass, and strands in cold, but mixing is steadily progressing.
8:02	Dark central region in hot glass is fading, but still just visible. The strands in the cold water are still visible, but fading.
8:12	Both hot and cold are still fading and mixing, but otherwise the same
8:31	Both hot and cold look uniform to me.

It took about an hour before both hot and cold glasses looked uniform to me, and there was clearly non-diffusive mixing at the beginning. There was no obvious difference in the diffusion.

(c) I used the same glasses as in part (a), and as mentioned in the question, I let the water reach equilibrium before adding the dye.

The candy dropped to the bottom was interesting in that after dissolving the dye slowly diffused up a couple of centimetres, darker at the bottom than the top, but then all movement stopped until I threw it out after a week. There was a constant 2 cm layer of coloured water at the bottom, and no sign of movement. (Within the 2cm layer it was darker near the bottom.) When I swirled the glass before throwing out the water I could clearly see the sugar (because of the change in index of refraction). My guess (and it is only a guess) is that so the candy formed a network of molecules which stopped the diffusion.

The thing that worked best was a better eyedropper than the one I mentioned in the question. This one had a narrow tip and very little food colouring (my preferred dye) leaked out while I gently inserted it, and I could deposit a nice layer of dye at the bottom. The first time I did this, I put it in a window and I noticed a beautiful (but very slow) convection cycle starting with water on the warm (room) side moving up about 1.5cm after an hour, but no movement on the cold (window) side.

Once put in a uniform temperature position, I observed the following:

11:30pm	Food colouring (red and blue) added to two glasses. It formed a thin layer on the bottom of each glass, but initially not all the bottom was covered.
11:39pm	Dye has almost covered the bottom of each glass.
11:49pm	Dye covers bottom of both glasses uniformly with a layer 1-2mm thick. (The range is because it is hard to determine the top of the layer.)
12:20am	Dye layers now 2-3mm thick
12:43am	4-5mm thick.
1:15am	5-6mm thick
2:03am	6-8mm thick
5:03am	There is a light green-blue colour throughout the blue glass, but also a dark 5-10mm thick layer at the bottom. The red glass has a uniform pink colour througout, with a similar darker red layer on bottom.
7:15am	The bulk colour is a bit darker maybe, but the dark 5-10mm thick layer at the bottom is still there.
8:00pm	The liquid is uniform blue in the "blue" glass, but the red still has a darker layer on the bottom.
8:00am	One glass is a unifrom blue, the other a uniform red.

It is very hard to measure the progress, to see if it follows a square-root time spread. This is because diffusion does not have a sharp edge. Nevertheless the time scale for diffusion of the food colouring throughout my 10cm tall glasses is about a day, with the red taking a bit longer than the blue.

We can't confirm the square root time dependence, but we can estimate the diffusion time to see if it is consistent. The molecular weight of water is 18, so the density of water molecules is

$$n = \frac{\rho_{H_2O}}{m_{molecule(H_2O)}} = \frac{1kg/m^3}{18(1.66053873\times10^{-27}\text{kg})} = 3.35\times10^{25}m^{-3}$$

and the mean free path between collisions should be about

$$\lambda = \frac{1}{\sqrt[3]{n}} = \sqrt[3]{\frac{m_{molecule(H_2O)}}{\rho_{H_2O}}} = \left(3.35\times10^{25}m^{-3}\right)^{-1/3} = 3.1\times10^{-9}m = 3.1nm$$

The mean velocity of the water molecules is calculated from the kinetic thermal energy, *i.e.*

$$\left\langle \tfrac{1}{2} m_{molecule(dye)} v^2 \right\rangle = \tfrac{3}{2}kT$$

$$\therefore \langle v \rangle = \sqrt{\frac{3kT}{m_{molecule(dye)}}}$$

For water molecules diffusing at room temperature

$$\therefore \langle v_{H_20} \rangle = \sqrt{\frac{3kT}{m_{molecule(H_20)}}} = \sqrt{\frac{3\left(1.3807\times10^{-23}J/K\right)(273+20K)}{18(1.66053873\times10^{-27}\text{kg})}} = 637m/s$$

and the mean time between collisions is

$$\tau = \frac{\lambda}{\langle v \rangle}$$

The mean number of steps to diffuse a distance L is about

$$\langle steps \rangle = \left(\frac{L}{\lambda}\right)^2$$

and the mean time to diffuse a distance L is about

$$\langle t \rangle = \left(\frac{L}{\lambda}\right)^2 \tau = \frac{L^2}{\lambda\langle v \rangle} = L^2 \sqrt[3]{\frac{\rho_{H_2O}}{m_{molecule(H_2O)}}} \sqrt{\frac{m_{molecule(dye)}}{3kT}}$$

For water this is

$$\langle t \rangle = \left(\frac{L}{\lambda}\right)^2 \tau = \frac{L^2}{\lambda\langle v \rangle} = \frac{L^2}{3.1nm(637m/s)} = 5.1\times10^5 \frac{s}{m^2} L^2$$

For our 10cm tall glass we expect

$$\langle t \rangle = 5\times10^5 \frac{s}{m^2}(0.1m)^2 = 5\times10^3 s$$

So the water molecules take well over an hour to diffuse through the glass, and the heavier dye molecules should take even longer. A quick search on the web and we can find that most common food colourings have molecular weights[5] of 400-1000, with most around 500, so scaling by the square root of the molecular mass (from the above formulae) we expect their diffusion times to be about

$$\langle t_{dye} \rangle \approx 5 \times 10^3 s \sqrt{\frac{500}{18}} \approx 7hr$$

which is quite consistent with what I saw. (Note that the diffusion time tells us the average time for a molecule to travel the 10cm, but some molecules will be slower and some faster, so it will take longer than the diffusion time to get a perfectly uniform colour.)

[5] http://www.kiriya-chem.co.jp/eng/syokuyou/chem-2.html

University of Toronto

2001-2002 Physics Olympiad Preparation Program

Problem Set 4: Optics and Waves

Due Monday 11 February 2002

1) Yes, Officer!

A car driven by a physicist is stopped by a police officer for running a red light. The physicist explains that the light appeared yellow because of the Doppler shift. (The wavelengths of red and yellow light are about 690 nm and 600 nm respectively.) Does the physicist deserve the ticket?

[David]

2) You push, I'll pull.

Radiation pressure is a very important factor for interstellar dust clouds[1], comet tails[2], and stellar and planetary system formation[3]. Once a cloud of gas and dust collapses and a star ignites, its radiation blows away smaller particles.

How big must a spherical dust particle be to be currently attracted by the sun instead of being repelled. *i.e*. What is its minimum radius, R?

Hints: Assume the dust has a typical mineral density of about density 3 g/cm^3 and that the dust totally absorbs the sunlight.

[David]

3) 500 channels - but nothing to watch!

For years I have been waiting for my favourite TV show, Fraggle Rock[4], to be available again in Toronto. (That part isn't made up, but from here on I may wander just a teeny bit from reality.) I recently learned that the show is playing on a small station broadcasting on Channel 2 (57 Mhz).which is not on Toronto cable. The station is 50 km away, but fortunately my house is on a hill and with a good antenna I can pick up the station by direct line of sight.

One day in the middle of the show I notice that the signal is fading in and out, going from maximum to minimum and back again 8 times per minute. I then learn from the radio that one of my atmospheric physics colleagues has lost control[5] of a humongous balloon[6] and it is currently just halfway between me and the TV station. The balloon is 20 km up and slowly rising at a

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[1] http://www.seds.org/messier/more/m045_i349.html
[2] http://antwrp.gsfc.nasa.gov/apod/ap001227.html
[3] http://www.seds.org/messier/more/m042_hst2.html
[4] http://www.henson.com/television/series/fraggle_home.html, http://home.no.net/fraggel/fragglerock.htm
[5] http://www.atmosp.physics.utoronto.ca/MANTRA/press.html
[6] http://www.msc-smc.ec.gc.ca/events/balloon/launch/launch17_e.cfm

constant speed, and the signal I am receiving is the sum of the direct line of sight signal plus the reflected signal from the balloon. How fast is the balloon rising?

[David]

4) An intense experience!

Nonlinear optical materials have an index of refraction that is dependent on the intensity of the incident beam. For a linear non-linear optical material, $n = n_1 + n_2 I$, where I is the intensity of the beam, n is the index of refraction, and n_1 & n_2 are positive, real constants.

Most laser beams have a Gaussian distribution of intensity in the xy plane normal to the direction of the beam, z. If for simplicity we only consider one axis we have

$$I(x) = I_0 e^{-\frac{x^2}{w^2}}$$

(a) Plot the intensity versus x. What parameter determines the size of the beam in this formula?

(b) If a Gaussian beam hits a nonlinear material, how does it travel inside it? Show qualitatively that the beam has to focus down. How does the beam continue after it focuses?

(c) Estimate the distance from the surface where the beam focuses. Use the fact that different rays which meet each other at the focal point have to travel the same optical distance (Optical distance=Index of refraction * distance). If it helps simplify your answer, you may assume $n_1 >> n_2 I_0$.

(d) How do your answers change if n_2 is negative?

[Yaser]

5) All the world in a drop of water

A microscope can be made using a drop of water as a lens.[1]

(a) What is the focal length of roughly 5mm diameter drop of water supported by a horizontal piece of transparent plastic?

(b) If you put your eye <u>very close</u> to the water drop and look through it you can see quite impressive magnification of objects. (I thought the ridges on my fingers were fascinating, but I am known to be easily entertained. By "very close" I mean your eyelashes may be touching the plastic.) Estimate the magnification of your water drop microscope.

If you can calculate the answer to these questions I'll be very impressed, since the only way I know to get the answer is to do the experiment. (To calculate the answer you'd have to calculate the shape of the drop which depends on its mass, the acceleration due to gravity, the surface tension of water, the molecular forces at the water/plastic boundary, …)

(c) Play around with different drops. Very roughly, discuss how the optical parameters (*e.g.* the focal length, magnification, and distortion) depend on the physical parameters (*e.g.* diameter) of your droplets?

Make sure you describe what you did to get your answers.

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[1] http://www.microscopy-uk.org.uk/mag/art98/watermic.html

Hints: The drop should be more or less circular as seen from the top. I have used shiny transparent tape, transparent food wrap, transparent CD or audio cassette cases…. (Plastic seems to work better than glass.) You want the drop to be as rounded as possible, not smeared out, e.g. like the drop on the left, not like the one on the right.

I use a drinking straw to gently deposit my drops on the plastic. Your droplet lens is fairly sensitive to vibration, so it is better to support your piece of plastic so you don't have to hold it and you can put things underneath your lens without touching it, but this isn't necessary. Make sure you have good lighting; I used either bright daylight or a flashlight. With a little practice, I could put samples (*e.g.* salt crystals) on the bottom of an audio cassette case and consistently put the right sized drop on the top which gave the right focal length.

The magnification is conventionally defined as the apparent size compared to the size seen at the naked eye at a distance of 25cm from the eye. I found pieces of linear graph paper very useful for studying magnification and distortion; I darkened some of the lines to make them more visible if looked at out of focus while looking through a droplet at another piece of graph paper in focus. Overhead lights provided a convenient light source for the focal length measurement.

[David]

<u>**POPBits™** – Possibly useful bits of information</u>

Constants and units[1,2]

astronomical unit (mean earth-sun distance)	au	149 597 870 660±20 m
atomic mass unit: (mass ^{12}C atom)/12	u	$(1.66053873 \pm 0.00000013)\times 10^{-27}$ kg
Boltzmann constant	k	$(1.3806503 \pm 0.0000024)\times 10^{-23}$ J/K
elementary (*i.e.* electron) charge	e	$(1.602176462 \pm 0.000000063)\times 10^{-19}$ C
electron volt	eV	$(1.602176462 \pm 0.000000063)\times 10^{-19}$ J
index of refraction of water	n_{H20}	1.33
Newtonian gravitational constant	G_N	$(6.673 \pm 0.010)\times 10^{-11}$ $m^3/kg/s^2$
solar luminosity (i.e. light output)	$L_\odot$	$(3.846 \pm 0.008)\times 10^{26}$W
solar mass	$M_\odot$	$(1.98892 \pm 0.00025)\times 10^{30}$kg
speed of light in vacuum	c	299 792 458 m/s
standard acceleration of gravity at the earth's surface	g	9.80665 m/s^2
Stephan-Boltzmann radiation constant	σ	$(5.670400 \pm 0.000040)\times 10^{-8}$ $W/m^2/K^4$
tropical year (2001)	yr	31556925.2 s

Great excuses for a party

High energy Nobel birthdays:

Hideki Yukawa[3] (1907)	January 23
Samuel Chao Chung Ting[4] (1931)	January 27
Abdus Salam[5] (1926)	January 29

[1] http://physics.nist.gov/cuu/Constants/index.html
[2] http://pdg.lbl.gov/2000/contents_sports.html
[3] http://www.nobel.se/physics/laureates/1949/press.html
[4] http://www.nobel.se/physics/laureates/1976/press.html
[5] http://www.nobel.se/physics/laureates/1979/salam-bio.html

2001-2002 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 4: Optics

1) Yes, Officer!

If the driver is relativistic, the police officer clearly should issue a ticket, so we can just use the non-relativistic Doppler formula for frequency (f) or wavelength (λ):

$$\frac{f'}{f} = \left(1 + \frac{v}{c}\right) \qquad or \qquad \frac{\lambda}{\lambda'} = \left(1 + \frac{v}{c}\right)$$

$$\therefore \qquad \frac{690nm}{600nm} - 1 = \frac{v}{c}$$

$$\therefore \qquad v = 0.15c = 4.5 \times 10^7 \frac{m}{s} = 1.6 \times 10^8 \frac{km}{hour}$$

So the driver is clearly speeding and deserves a ticket. The exact relativistic Doppler shift differs by only a factor of $\sqrt{1 - (v/c)^2} = 0.99$.

2) You push, I'll pull.

The momentum and energy of the photons are

$$p_\gamma = \frac{h}{\lambda} \qquad and \qquad E_\gamma = \frac{hc}{\lambda}$$

The number of photons produced per second by the sun are

$$N_\gamma = \frac{L_\odot}{E_\gamma} = \frac{L_\odot \lambda}{hc}$$

A dust particle absorbs the momentum of any photon hitting it, so the pressure on a dust particle at a distance r from the sun is the total momentum of the photons per unit area per unit time, *i.e.*

$$P_\odot = \frac{N_\gamma p_\gamma}{4\pi r^2} = \frac{L_\odot h\lambda}{hc 4\pi r^2 \lambda} = \frac{L_\odot}{4\pi c r^2}$$

The total light force on a spherical dust grain of radius R is

$$F_L = P_\odot \times Area = P_\odot \pi R^2 = \frac{L_\odot \pi R^2}{4\pi c r^2} = \frac{L_\odot R^2}{4cr^2}$$

The mass of a spherical dust grain of radius R and density ρ is

$$m = \tfrac{4}{3}\pi R^3 \rho$$

The dust grains will be attracted to the sun when the light pressure force is weaker than the gravitational attraction of the sun, *i.e.* when

$$\therefore F_L < F_G$$

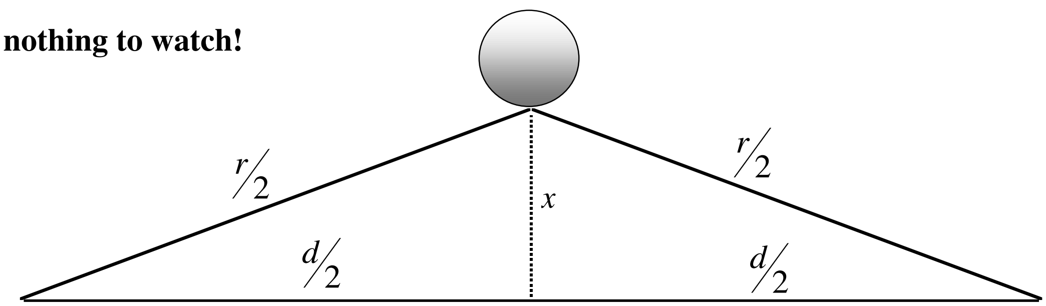
$$\therefore \frac{L_\odot R^2}{4cr^2} < G_N \frac{mM_\odot}{r^2}$$

$$\therefore \frac{L_\odot R^2}{4cr^2} < G_N \frac{\frac{4}{3}\pi R^3 \rho M_\odot}{r^2}$$

$$\therefore R > \frac{3L_\odot}{16\pi c G_N \rho M_\odot} = 1.9\times 10^{-7}\, m$$

So the dust grains must be larger than 0.2μm to fall into the sun. This is why the clouds of gas and small dust are blown away when a star ignites. Another nice example of the effect of radiation pressure on interstellar dust is the Trifid Nebula (http://antwrp.gsfc.nasa.gov/apod/ap990608.html).

3) 500 channels - but nothing to watch!



The distance between me and the TV transmitter are fixed, so the interference indicates that the signal path length is increasing by 8 wavelengths per minute. The path length of the reflected signal is

$$r = 2\sqrt{x^2 + (d/2)^2} = \sqrt{4x^2 + d^2} = \sqrt{4(vt + x_0)^2 + d^2}$$

where t=0 is defined to be when the balloon has a height $x_0 = 20km$. The balloon rises with a constant velocity, $v=dx/dt$, so

$$\frac{dr}{dt} = \frac{d}{dt}\sqrt{4(vt + x_0)^2 + d^2} = \frac{4(vt + x_0)v}{\sqrt{4(vt + x_0)^2 + d^2}}$$

We are interested in the time t=0, when the rate of change of pathlength is

$$\left.\frac{dr}{dt}\right|_{t=0} = \frac{2v}{\sqrt{1 + \left(d/2x_0\right)^2}}$$

The wavelength of the signal is

$$\lambda = \frac{c}{f} = \frac{3\times 10^8 m/s}{57\times 10^6 /s} = \frac{100}{19} m$$

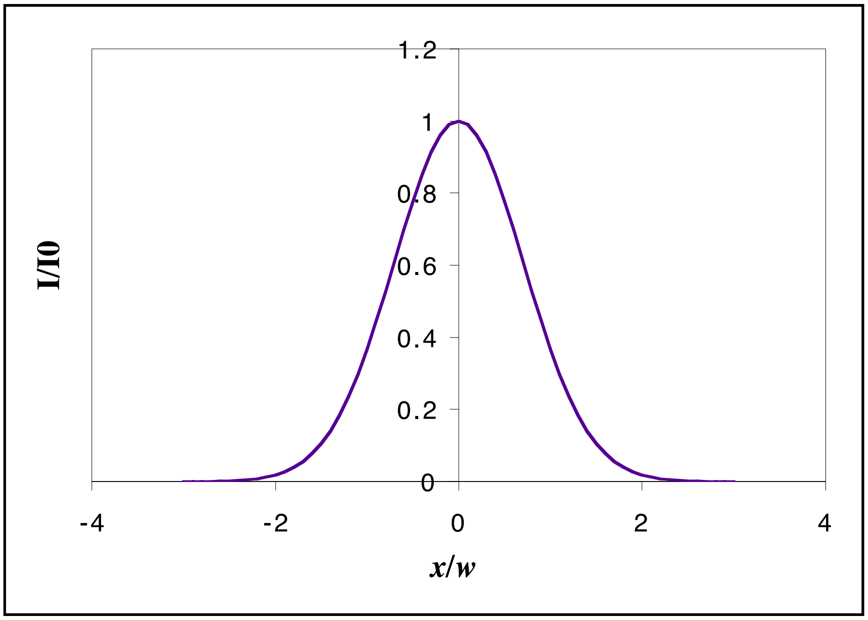
So the balloon is rising with a velocity of

$$v = \frac{1}{2}\left.\frac{dr}{dt}\right|_{t=0}\sqrt{1 + \left(d/2x_0\right)^2} = \frac{1}{2}\frac{8\lambda}{60s}\sqrt{1 + \left(50km/(2\times 20km)\right)^2} = \frac{1}{15s}\left(\frac{100}{19}m\right)\sqrt{1 + \frac{25}{16}} = 0.56 m/s$$

Note: Some people used differences instead of derivatives to get almost the same answer. This works fine in this case since the change in distance is so small compared to the initial distance.

4) An intense experience!

(a)



The width is determined by the parameter w.

(b) The beam focuses down because the index of refraction is proportional to the intensity and so is higher nearer to the axis. When passing through varying indices of refraction, light always bends in the direction of higher index of refraction. This follows from Snell's Law or the law of optical distance given as a hint in part (c). In this case there is positive feedback (*i.e.* the light bends towards the region of high index of refraction which increases the intensity there which increases the index of refraction, ...), so the light will form a thin beam. After it focuses, it diverges and again focuses and the focusing and diverging will be repeated in the medium, but the net result is a beam. (See, for example http://focus.aps.org/v8/st26.html.)

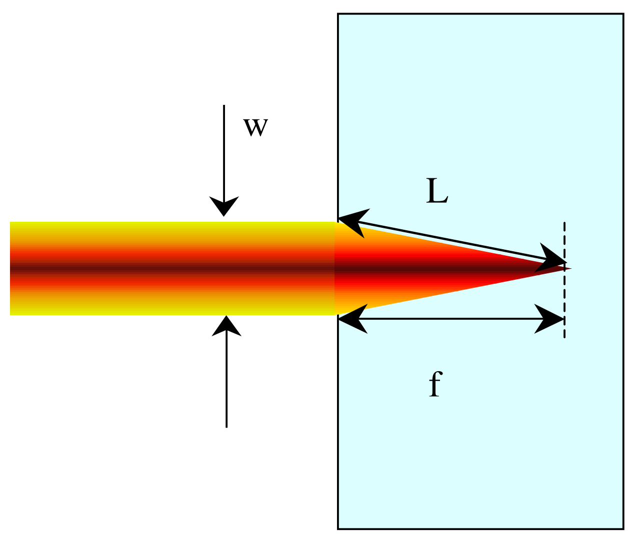
(c) The variation in n drops over a distance w, so we can make the approximation of triangular focusing

$$L^2 = w^2 + f^2$$

At the edge the index of refraction is n_1, while at the centre it is $n_1+n_2I_0$, so using the hint that all the light meeting at a focus will have travelled the same optical distance, we have

$$Ln_1 \sim f\left(n_1 + \frac{n_2}{2}I_0\right)$$

where we use the average index of refraction between the centre and the edge.



$$\therefore \sqrt{w^2 + f^2}\, n_1 \sim f\left(n_1 + \frac{n_2}{2} I_0\right)$$

$$\therefore \left(w^2 + f^2\right) n_1^2 \sim f^2\left(n_1 + \frac{n_2}{2} I_0\right)^2 = f^2\left(n_1^2 + n_1 n_2 I_0 + \frac{n_2^2}{4} I_0^2\right)$$

$$\therefore f \sim \frac{w n_1}{\sqrt{\left(n_1 + \frac{n_2}{4} I_0\right) n_2 I_0}} \xrightarrow[n_1 >> n_2 I_0]{} \sqrt{\frac{n_1}{n_2 I_0}}\, w$$

(d) If $n_2<0$, then the beam will blow up instead of focusing, since the index of refraction will now be lower in the high intensity region.

5) All the world in a drop of water

I used the top part of a CD case to support my drops. The overhead lights in my office provided a nice light source. I had some linear graph paper which had 1mm squares; this made it easy to measure the diameter of the droplets by simply resting the plastic on the graph paper and counting the squares under the drops.

I measured the focal length by seeing how how high above my desk I had to hold the drop to get a sharp image. I initially used a ruler, but then I realized that since I had 2 long parallel fluorescent lights separated by 128±1cm at a distance 203±1cm above my desk, I could get the focal length by measuring the separation of the images on the graph paper and multiplying by 203/128. The drops were rather thick (1.5 mm for the smallest drops, almost 3mm for the biggest diameters), so the thin lens approximation was not very accurate, but probably good enough.

Measuring the magnification was a bit tricky, but first you need to make your drop work as a microscope. Since it is a microscope you have to put your eye very close to the droplet (as the question says quite emphatically). If you don’t put your eye close, all you have is a poor magnifying glass. When you do put your eye close, you have to move what your looking at (e.g. your finger nail) nearer and farther until it is in focus. (When it does come into focus the typical reaction, at least for Grade 5 students in my kids’ school, is to say “Wow!”.)

To measure the magnification for a drop, I first measured the diameter and focal length of a drop using my graph paper. I then rested the CD case on a box of facial tissue which was resting on its end. The box was 23cm long, which is very close to the 25cm standard reference distance, so the drop was held at (approximately) the right distance above the table. I then put a piece of graph paper on the table underneath the drop, and looked at another small piece of graph paper held close underneath the drop. With my eyes very close to the drop I would adjust the small graph paper and move my head until the small graph paper was in focus (often with my eyelashes touching the CD case). I could then see both the graph paper on the table and the magnified small graph paper, and could easily (if not terribly accurately) measure the magnification by comparing the magnified and non-magnified squares. This was hard to do with the smallest drops. It also required a steady hand holding the small piece of graph paper.

Diameter (mm)	Focal Length (mm)	Magnification
3.1±0.3	4.4±0.6	40±5
3.9±0.3	4.8±0.6	32±4
5±1	6.0±0.8	30±3
6±0.4	8.2±0.8	22±3
7.7±0.5	13.5±1	17±2
12.5±0.5	29±3	9±1
18±1	54±6	4±1
5±1	6.0±0.8	30±3
3±0.3	3.5±0.5	40±3

(a) As you see from the above table, I found a 5 mm drop to have a focal length of 6±1mm.

(b) A 5mm drop has a magnification of 30±3.

(c) In general, the larger diameters gave larger focal lengths, larger field of view, and smaller magnification. There are several different kinds of distortion. When looking at graph paper, the lines were more curved near the edges for small drops, but for large drops it was less likely all the field of view would be in focus and the drop was less likely to be circular which lead to more distortion.

One thing I noticed was that difference support plastics made different height drops. CD cases made very high (*i.e.* thick) drops, while the drops were lower on other plastics (*e.g.* overhead transparencies, name tag holders, …). I assume this depended on how hydrophobic the plastic is. I could also make the drops lower by waiting for them to evaporate or poking them with a paper clip. Shorter drops had longer focal lengths and less magnification.

You can see a nice example of a water droplet microscope in the journal Physics Education 36 (March 2001) 97-101 (http://www.iop.org/EJ/S/1/NTO803278/abstract/0031-9120/36/2/301). If you cannot access the IOP website, it is also available (although the beautiful colour photos are only in black and white) from the Applied Spectroscopy Laboratory at the University of Indonesia at http://www.geocities.com/spectrochemical/int_papers/ip34.pdf.

University of Toronto

2001-2002 Physics Olympiad Preparation Program

Problem Set 5: Electricity and Magnetism

Due Monday 11 March 2002

1) Purple Haze!

My old friend Steve studies top quarks using the CDF[1] experiment which has a very large superconducting solenoidal magnet with a uniform horizontal field of 1.5 Tesla. In addition to physics, Steve also loves guitars.[2] Is it safe for Steve to play his guitar inside the CDF magnet? In particular, is he likely to shock himself or burn his fingers on the E_4 string if it is vibrating with a peak-to-peak amplitude of 1mm?

Hints: The E_4 string (330 Hz) on Steve's guitar is 66 cm long, 0.23mm in diameter, and has an electrical resistivity of 12 $\mu\Omega$ cm, a thermal conductivity of 80 W/m/K, a linear mass density of 0.3g/m, and a specific heat of 500 J/kg/K. (You may not need all this data, but we didn't want to leave out anything potentially useful.) The string vibrates sinusoidally in its lowest mode, and both ends of the string are connected to pins at constant temperature (21°C). Assume that any heat produced in the string can only escape through the ends of the string or through Steve's fingers, *i.e.* thermal radiation and convection are negligible. The time scale for heat flow out of the wire is much longer than the oscillation period of the wire. Thermal conductivity and electrical conductivity are completely analogous and the equations have identical form, *i.e.* electrical conductivity relates charge flow and voltage differences, thermal conductivity relates heat flow and temperature differences.

[David]

2) Relatively electric!

Near the surface of the earth there is typically a vertical electric field of about -120 V/m. (The field can reach 20 kV/m underneath a thundercloud - this is why your hair may stand on end just before you get hit by lightning - but we'll ignore this interesting but irrelevant safety tip for the rest of the problem.)

(a) Why aren't people electrocuted by the typical 200 V potential difference between the air at their feet and the air at their head?

(b) What is the typical surface electric charge density on the earth's surface? Give your answer in units of electron charges per metre squared.

(c) What is the typical surface magnetic field (magnitude and direction) at the equator produced by this surface electric charge density, as measured by a magnetic field meter just above the surface of the earth

[1] http://www-cdf.fnal.gov/cdf.html

[2] http://webug.physics.uiuc.edu/courses/phys398/

i) if it is attached to the earth and rotates with it?

ii) just above the surface of the earth, letting the earth rotate under it?

(d) Are either of your answers to the two parts of (c) consistent with the observed magnitude (typically 30μT) or observed direction of the surface magnetic field at the earth's equator?

Hints: My coordinates are such that an electric field pointing up is positive, pointing down is negative. The surface of the earth, cows, and people, all have similar (non-infinite) resistivities. Don't worry about the fact that the meter sitting above the surface of the earth would be moving supersonically, *e.g.* assume it is in a jet flying in an eternal sunrise, or sunset, or noon. The north pole of a magnet is the one which is attracted to the north pole of the earth.

[David]

3) Spheres within spheres

An insulating sphere of radius R with uniform volume charge density of r has an electric field inside it equal to $\vec{E} = \frac{\rho\ \vec{r}}{3\varepsilon_0}$ where r is the distance from the center of the sphere ($r<R$). Note that this is a vector formula.

(a) Consider two overlapping spheres of radii R_1 and R_2 with uniform charge densities ρ and $-\rho$ and at a distance d from each other ($d<R_1+R_2$). Show that the electric field in the region of overlap is uniform and calculate its magnitude and direction.

(b) If the two spheres have the same radii R and the distance between them is much smaller than the radius ($d<<R$), you can consider the whole thing as one sphere with a surface charge density σ.

i) Find σ on the surface in terms of the angle that it makes with the line connecting the two centres. What is the maximum surface charge density σ_0? What is the electric field inside the sphere in terms of σ_0?

ii) What is the electric potential, V, on the surface of the sphere in terms of the angle that it makes with the line connecting the two centres? (The potential is defined to be zero at infinite distance.)

[Yaser]

4) Does it (anti)matter?

When I was a kid watching Star Trek (during its original run), I sometimes wondered if antimatter falls up. According to General Relativity it should fall down at the same rate as ordinary matter, but some exotic theories predict it will fall with a different acceleration. Experiments trying to observe the gravitational force on antimatter usually use conductors to shield the antiparticles from external electromagnetic forces. Imagine a positron inside a solid piece of copper on the surface of the earth. What is the vertical acceleration of the positron?

Hints: Assume General Relativity is correct. For the purposes of this question, a positron is just like an electron, but with opposite charge. You can ignore the fact that the positron will quickly annihilate with an electron, and assume it can move freely through the copper just like conduction electrons. Before thinking about positrons, you may first want to think about the net force on the free conduction electrons inside the copper.

[David]

5) Refrigerator Art

Experimentally and quantitatively determine the arrangement of north and south poles on a flat, flexible refrigerator magnet. Describe what you did and what your conclusions are.

Hints: We are talking about the flexible magnetic cards which usually have advertising on one side and a dark smooth surface on the back. We don't care which poles are north and which poles are south, but what is the arrangement of the poles. I figured it out just by playing around with two similar fridge magnets and making a measurement or two with a ruler, but there may be other methods.

[David]

POPBits™ – Possibly useful bits of information

Constants and units[3,4]

Boltzmann constant	k	$(1.3806503 \pm 0.0000024) \times 10^{-23}$ J/K
Earth equatorial radius	$R_\oplus$	6.378140×10^{6} m
elementary (*i.e.* electron) charge	e	$(1.602176462 \pm 0.000000063) \times 10^{-19}$ C
electron (or positron) inertial mass	m_e	(510998.902 ± 0.021) eV/c^2
electron volt	eV	$(1.602176462 \pm 0.000000063) \times 10^{-19}$ J
Newtonian gravitational constant	G_N	$(6.673 \pm 0.010) \times 10^{-11}$ m^3/kg/s^2
permittivity of free space	ε_0	$8.854187817 \times 10^{-12}$ F/m
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
speed of light in vacuum	c	299 792 458 m/s
standard acceleration of gravity at the earth's surface	g	9.80665 m/s^2
Stephan-Boltzmann radiation constant	σ	$(5.670400 \pm 0.000040) \times 10^{-8}$ W/m^2/K^4

Interesting (?) Integrals (a is a constant)

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) \qquad \int \cos(ax)dx = \frac{1}{a}\sin(ax)$$

$$\int \sin^2(ax)dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

Great excuses for a party

Galileo's birthday (1564)	February 14
Discovery of radioactivity (1896)	March 1

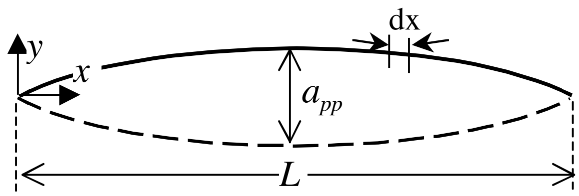
[3] http://physics.nist.gov/cuu/Constants/index.html
[4] http://pdg.lbl.gov/2000/contents_sports.html

2001-2002 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 5: Electricity and Magnetism

1) Purple Haze!



This can be done using the Lorentz force law, but we'll use Faraday's Law. Assuming the string is oscillating sinusoidally, the induced electromotive force produced by the string sweeping out an area A in the magnetic field B is

$$\varphi = B\frac{dA}{dt} = B\frac{d}{dt}\int_0^L \frac{a_{pp}}{2}\sin(\omega t)\sin\left(\frac{\pi x}{L}\right)dx$$

$$= B\frac{a_{pp}}{2}\omega\cos(\omega t)\frac{L}{\pi}\left[\cos\left(\frac{\pi x}{L}\right)\right]_0^L = -2BLa_{pp}\nu\cos(\omega t)$$

where ω=2πν is the radial frequency and ν=330Hz is the frequency of oscillation. The magnitude of the peak e.m.f. is then

$$\varphi_p = \left|-2BLa_{pp}\nu\right| = 2(1.5T)(0.66m)(0.001m)330s^{-1} = 0.7V$$

This is not enough voltage to shock Steve, but is the string temperature safe?

By Joule's and Ohm's Laws, the power produced in a segment, dx, of the string with voltage drop dV across its resistance dR is

$$dP = \frac{(dV)^2}{dR} = \frac{\left(B\frac{a_{pp}}{2}\omega\cos(\omega t)\sin\left(\frac{\pi x}{L}\right)dx\right)^2}{\frac{\rho}{A}dx} = \frac{AB^2a_{pp}^2\omega^2}{4\rho}\cos^2(\omega t)\sin^2\left(\frac{\pi x}{L}\right)dx$$

where A is now the cross-sectional area of the string and ρ is it's resistivity. The average power produced over one complete oscillation of the string is

$$\langle dP\rangle = \frac{AB^2a_{pp}^2\omega^2}{4\rho}\sin^2\left(\frac{\pi x}{L}\right)dx\left\langle\cos^2(\omega t)\right\rangle = \frac{AB^2a_{pp}^2(2\pi\nu)^2}{4\rho}\sin^2\left(\frac{\pi x}{L}\right)dx\frac{\int_0^{2\pi}\cos^2(\omega t)dt}{\int_0^{2\pi}dt}$$

$$= \frac{AB^2a_{pp}^2\pi^2\nu^2}{\rho}\sin^2\left(\frac{\pi x}{L}\right)dx\frac{\left[\frac{1}{2}-\frac{1}{4\omega}\sin(2\omega t)\right]_0^{2\pi}}{2\pi} = \frac{AB^2a_{pp}^2\pi^2\nu^2}{2\rho}\sin^2\left(\frac{\pi x}{L}\right)dx$$

Since heat is being produced, the string will heat up. By symmetry, the middle of the string must be the hottest point, and the net heat flow in the right half of the string (in the figure) is to the right, and the flow in the left half is too the left. The string will heat up until it reaches a steady state where the heat flowing through each segment just matches the joule heating of the part of the string whose heat must flow out through the segment (*i.e.* integrated from the midpoint to the segment):

$$\frac{dQ}{dt} = -kA\frac{dT}{dx} = \int_{L/2}^{x} \langle dP \rangle = \int_{L/2}^{x} \frac{AB^2 a_{pp}^2 \pi^2 \nu^2}{2\rho} \sin^2\left(\frac{\pi x'}{L}\right) dx'$$

$$= \frac{AB^2 a_{pp}^2 \pi^2 \nu^2}{2\rho}\left[\frac{x'}{2} - \frac{L}{4\pi}\sin\left(\frac{2\pi x'}{L}\right)\right]_{L/2}^{x} = \frac{AB^2 a_{pp}^2 \pi^2 \nu^2}{2\rho}\left[\frac{x}{2} - \frac{L}{4} - \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\right]$$

(k is the thermal conductivity of the string.)

$$\therefore \frac{dT}{dx} = \frac{B^2 a_{pp}^2 \pi^2 \nu^2}{8k\rho}\left[L - 2x + \frac{L}{\pi}\sin\left(\frac{2\pi x}{L}\right)\right]$$

The temperature profile of the string is thus

$$T(x) - T(0) = \int_0^x \frac{dT}{dx'}dx' = \int_0^x \frac{B^2 a_{pp}^2 \pi^2 \nu^2}{8k\rho}\left[L - 2x' + \frac{L}{\pi}\sin\left(\frac{2\pi x'}{L}\right)\right]dx'$$

$$= \frac{B^2 a_{pp}^2 \pi^2 \nu^2}{8k\rho}\left[Lx - x^2 - \frac{L^2}{2\pi^2}\left(\cos\left(\frac{2\pi x}{L}\right) - 1\right)\right]$$

The peak temperature of the string is for x=L/2

$$T(x) = \frac{B^2 a_{pp}^2 \nu^2 L^2}{8k\rho}\left[\frac{\pi^2}{4} + 1\right] + T(0)$$

$$= \frac{(1.5T)^2 (0.001m)^2 (330Hz)^2 (0.66m)^2}{8\left(12 \times 10^{-6}\Omega \cdot 10^{-2}m\right)(80W/m/K)}\left[\frac{\pi^2}{4} + 1\right] + (21K + 273K)$$

$$= 5100K$$

This means that Steve will burn his fingers, and if he tries to keep on playing with an insulated pick the string will melt (the element with the highest melting point, 3800K, is carbon, but a carbon fibre or diamond string would burst into flames first; the metal with the highest melting point, 3695K, is Tungsten, see http://www.webelements.com). I think Steve should keep his guitar out of the CDF magnet.

Alternate shorter approach from Alex:

The area swept out by the string is

$$A = \int_0^{660mm} (1mm)\sin\left(\frac{\pi x}{660mm}\right)dx = 420mm^2$$

and this area is swept out twice per cycle, so the average e.m.f. is

$$\langle \varphi \rangle \cong B\left\langle \frac{dA}{dt} \right\rangle = 1.5T \cdot 2 \cdot 330Hz \cdot 420mm^2 = 0.42V$$

The resistance is

$$\langle \varphi \rangle \cong BR = \frac{\rho L}{A} = \frac{\left(12 \times 10^{-6}\,\Omega \cdot cm\right) 66cm}{\pi\left(0.023cm\right)} = 1.9\Omega$$

So the total power is

$$\langle \varphi \rangle \cong P = \frac{V^2}{R} = 93mW$$

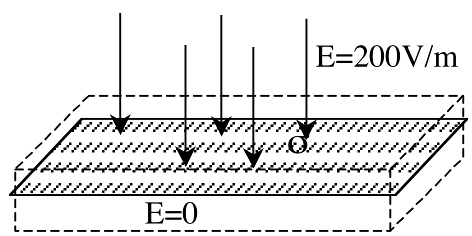
Half the power is lost through the end of the string (with most power being produced in the middle). Thus

$$\frac{P}{2} = \frac{\sigma \pi d^2}{4L/2}\Delta T$$
$$\therefore \Delta T = \frac{LP}{\sigma \pi d^2} = 4600K$$
$$\therefore T_{middle} = T_{end} + \Delta T \approx 4900K$$

2) Relatively electric!

(a) People are much better conductors than air, so they short out the field so their whole body is at a constant potential. The energy density of the atmospheric "battery" is so low that only an infinitesimal surface current flows through the body.

(b) The earth is a conductor, so it will have zero field inside it and a charge density on its surface which can be calculated from the electric field at the surface using Gauss's Law. Near the surface of the earth it looks like a plane, so we can just use a Gaussian box with top (and bottom) area A, *i.e.*



$$\frac{q_{insidebox}}{\varepsilon_0} = \vec{E}_{top} \cdot \vec{A}_{top} + \vec{E}_{bottom} \cdot \vec{A}_{bottom} + \vec{E}_{sides} \cdot \vec{A}_{sides}$$
$$\therefore \frac{\sigma A}{\varepsilon_0} = EA + 0 \cdot A + E_{sides} A_{sides} \cos\tfrac{\pi}{2}$$
$$\therefore \frac{\sigma}{\varepsilon_0} = E$$

So the surface charge density is

$$\sigma = \varepsilon_0 E = \frac{\left(8.854 \times 10^{-12} F/m\right)\left(-120V/m\right)}{1.6 \times 10^{-19} C/e} = -6.6 \times 10^{6} \left|electron\ \ charges\right|/m^2$$

Note that the surface charge density is negative.

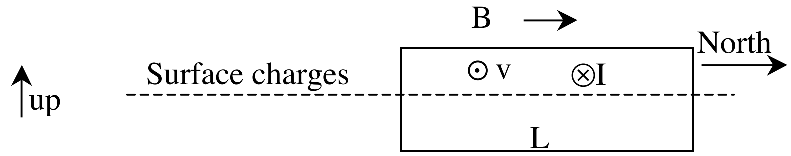
(c)

i) In the rest frame of the earth's surface, the magnetic field generated by the surface charge density is zero, since it isn't moving in this reference frame.

ii) In this frame, the charge is moving with a velocity

$$v = \frac{2\pi R_{\oplus}}{1 day} = \frac{2\pi 6.378\times 10^{6} m}{24 hours \times 3600 s/hour} = 464\frac{m}{s}$$

underneath the observer. This corresponds to a current sheet with a current density σv. By symmetry and the Biot-Savart Law, the magnetic field must run at right angles to the velocity and parallel t to the surface. The direction of the magnetic field can be determined to be from south to north using the right hand rule. so we can use Ampere's Law to calculate the magnetic field:



$$\oint_{loop} \vec{B}\cdot d\vec{s} = \tfrac{1}{2}\mu_0 I_{enclosed}$$

$$\therefore BL = \tfrac{1}{2}\mu_0 L\sigma v$$

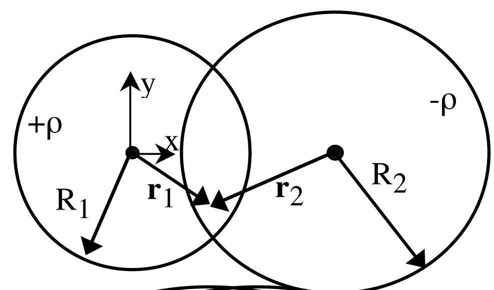
$$\therefore B = \tfrac{1}{2}\mu_0\sigma v = \tfrac{1}{2}\mu_0\varepsilon_0 E v = \frac{Ev}{2c^2} = \tfrac{1}{2}\beta\frac{E}{c} = \tfrac{1}{2}\frac{(120V/m)(464 m/s)}{\left(3\times 10^{8} m/s\right)^2} = 0.3pT$$

(d) So the magnetic field is the right direction, but much too weak.

3) Spheres within spheres

(a) The electric field in the region of overlap is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\rho\mathbf{r}_1}{3\varepsilon_0} + \frac{-\rho\mathbf{r}_2}{3\varepsilon_0} = \frac{\rho}{3\varepsilon_0}(\mathbf{r}_1 - \mathbf{r}_2)$$

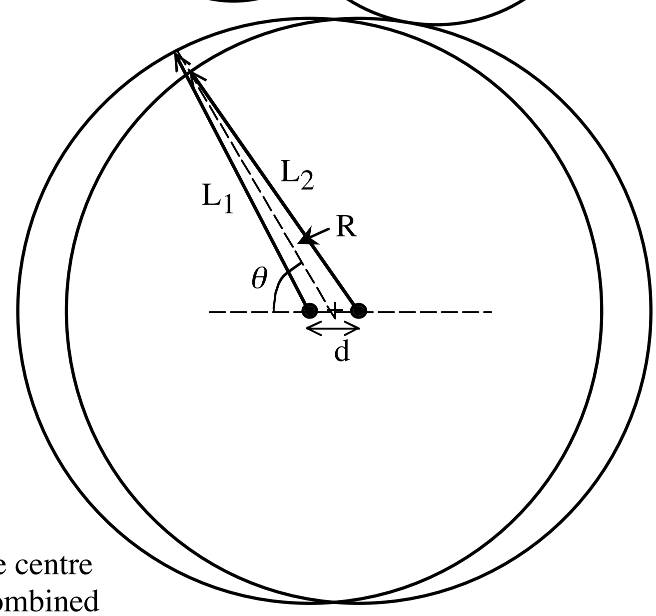


Choosing sphere 1 as the origin, and defining the x axis to be the direction from the centre of sphere 1 to the centre of sphere 2, then

$$\mathbf{E} = \frac{\rho}{3\varepsilon_0}(x_1 - x_2, 0, 0)$$

$$\therefore \mathbf{E} = \frac{\rho}{3\varepsilon_0} d\ \hat{\mathbf{x}}$$

(b)



i) The charges cancel where the spheres overlap, so the surface charge density is just $\sigma(\theta) = \rho\, t(\theta)$, where $t(\theta)$ is the thickness of the uncancelled charge layer at the surface. The maximum charge thicness is obviously at $t(0) = d$, $\sigma_0 = \rho d$.

Using the cosine rule and the fact that the centre of each sphere is ofset by ±d/2 from their combined centre, we can calculate the thickness in the limit where d<<R, *i.e.*

$$
\begin{aligned}
t(\theta) &= L_2(\theta) - L_1(\theta) \\
&= \sqrt{\left(L_2(\theta)\right)^2} - \sqrt{\left(L_1(\theta)\right)^2} \\
&= \sqrt{R^2 + \left(\frac{d}{2}\right)^2 - \frac{2Rd}{2}\cos\theta} - \sqrt{R^2 + \left(\frac{d}{2}\right)^2 - \frac{2Rd}{2}\cos(\pi - \theta)} \\
&\cong R\left(1 + \frac{d}{R}\cos\theta\right) - R\left(1 - \frac{d}{R}\cos\theta\right) \\
&= d\cos\theta
\end{aligned}
$$

where we have used the approximation that $\sqrt{1+\varepsilon} \cong 1 + \frac{1}{2}\varepsilon$ when ε<<1. So the surface charge density is

$$\sigma = \rho \;\; \mathrm{d}\cos\theta \;\; = \;\; \sigma_0 \cos\theta$$

and the electric field strength inside the (overlapped) sphere is

$$\therefore \mathbf{E} = \frac{\sigma_0}{3\varepsilon_0} \;\; \mathbf{x}$$

ii) Outside the spheres, this charge configuration is just an electric dipole with dipole moment

$$\mathrm{p} = Q_{sphere}\mathrm{d} = \tfrac{4}{3}\pi R^3 \rho \mathrm{d} = \tfrac{4}{3}\pi R^3 \sigma_0$$

The electric potential outside the spheres is just the dipole potential

$$V(r) = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3} = \frac{\mathrm{p}}{4\pi\varepsilon_0 r^2}\cos\theta$$

so at the surface of the sphere

$$V(r) = \frac{\frac{4}{3}\pi R^3 \sigma_0}{4\pi\varepsilon_0 R^2}\cos\theta = \frac{R\sigma_0}{3\varepsilon_0}\cos\theta = \frac{R\rho \mathrm{d}}{3\varepsilon_0}\cos\theta = \frac{R\sigma}{3\varepsilon_0}$$

4) Does it (anti)matter?

Conduction electrons are free to move, so if they are at rest they must feel no net force. In a piece of copper on the surface of the earth, the conduction electrons feel the force of gravity, but they don't all fall out the bottom of the copper, so there must be a force balancing gravity. This force is an equal and opposite electrostatic force generated by the electrons themselves sagging under the force of gravity. *i.e.* They start to fall but this very quickly causes the bottom of the copper to become slightly negatively charged relative to the top, and the electrons will stop moving when the charge imbalance is such that it exactly balances gravity everywhere inside the conductor. The electrons natural arrange themselves for this to happen, and this is just the same explanation why the electric field inside a conductor is always zero (except they never mention that the effect of gravity is to create a tiny non-zero electrostatic field).

For a positron, the electrostatic force is the equal and opposite to that on a electron, but the gravitational force is equal and in the same direction (according to General Relativity), so if the net force on a conduction electron is zero, the net force on a positron must be

$a_{positron} = 2g = 19.6 m/s/s$ straight down.

(g is the standard acceleration of gravity at the earth's surface.)

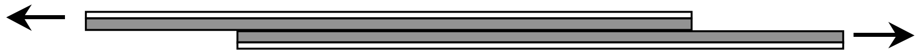
5) Refrigerator Art

Playing around with two fridge magnets, I noticed that if you stick them together like this (where the white area is the

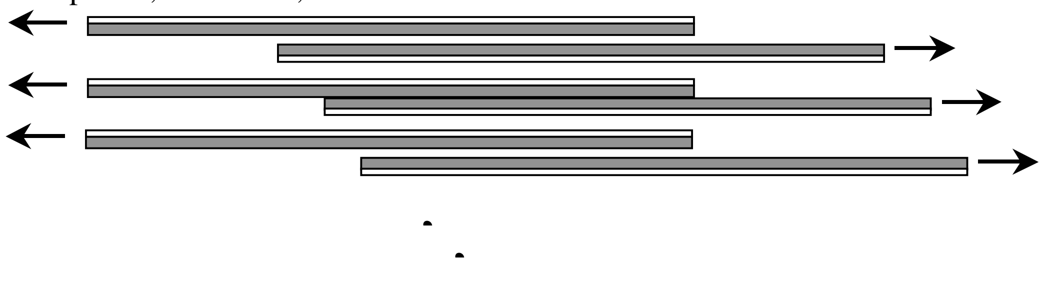
printed face and the dark area is the magnetic part), and then gently pull them like this, they

make a "clack, clack, clack" sound. Studying this effect more closely it is clear that what is happening is that they start to repel each other as I pull them

and then they are attracted,

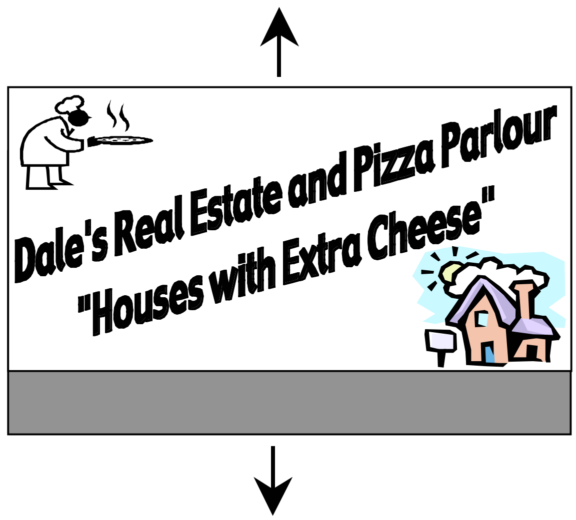


and then repelled, and so on,

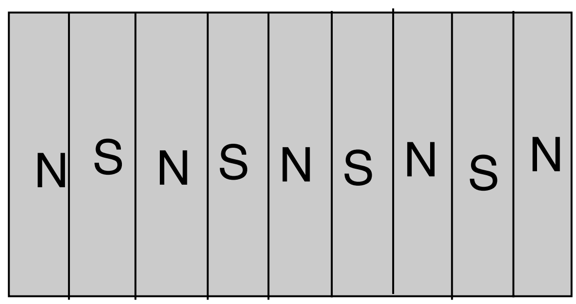


and each time they are attracted together they make a nice "clack!". This only happens when I pull them along their long axis, *i.e.* like this

not when I pull them like this

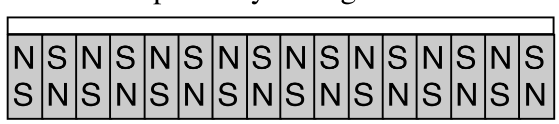


This must mean that fridge magnet must consist of many magnetic domains (essentially little dipole magnets) whose North and South poles must be arranged like

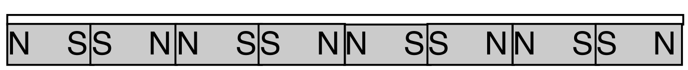


(I can't tell which is north and which is south, but they must be alternating.). The positions where the fridge magnets were attracted together were at 5, 9, 13, and 17 mm, so the width of the domains is about 4 $^{1}/_{2}$ mm. (Your magnets may differ, although all the ones I tried seemed similar.)

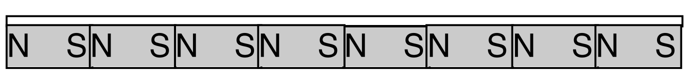
Looking from the side, the domains are probably arranged like this

(A) 

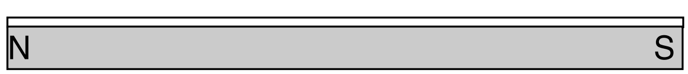
or less probably like this

(B) 

It cannot be like this

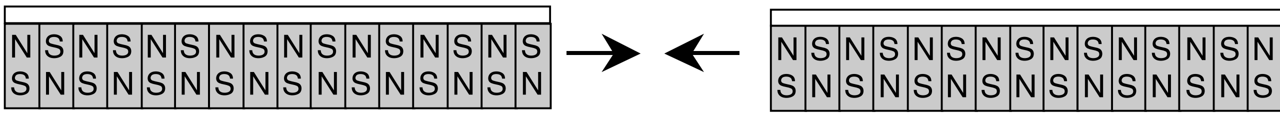
(C) 

since that us equivalent to this

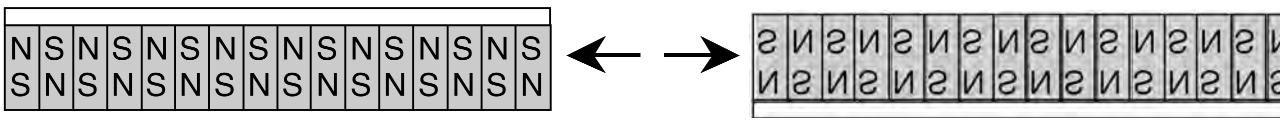


(assuming all domains are equal strength magnets) which would not give the correct “clack, clack behaviour.

To check that arrangement (A) is correct, I held the ends of the fridge magnets close to each other. They either attracted or repelled, *e.g.*



When I flipped them over, they did the opposite *e.g.*



(This took some care to see since the forces are weak and the main observable affect is not the repulsion but the fact that the magnets tend to push to one side to try to align North to South poles.) These observations are not consistent with arrangement (B), in which case the force should not change direction.

University of Toronto

2001-2002 Physics Olympiad Preparation Program

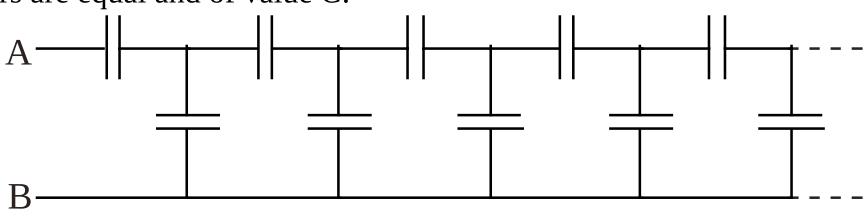
Problem Set 6: Circuits and (a bit of) Modern Physics

Due Monday 8 April 2002

- You need a multimeter or ohmmeter for the last problem. If you don't have one, ask your Physics Teacher or a friend.
- Please be prompt with this problem set; we will be selecting students for our exciting POPTOR training weekend right after the due date.

1) Everyone has the capacity to learn the quadratic formula!

Calculate the equivalent capacitance between points A and B of the following infinite circuit, if all the capacitors are equal and of value *C*.



Hint: If you got rid of the two left-most capacitors (one horizontal and one vertical), what would the capacitance be of the remaining infinite chain?

[Isamu]

2) I always resist getting a shock

You may have noticed warnings on television sets or computer monitors not to mess about inside even if the power is unplugged. This is because there are some large capacitors inside which are charged up to high voltages – a potentially lethal combination. Large inductors can be equally lethal but in a different way. If you have a circuit with a current flowing through a large inductor, if you try to turn it off by opening a switch there can be spectacular sparks generated by huge voltages across the switch. I work on the ATLAS[1] experiment at the CERN[2] laboratory and we have a very large inductor. It is a superconducting solenoidal magnet 5.3m long and 1.2m in radius. The solenoid is wrapped uniformly with about 1200 turns of superconducting wire which carry a current of 8000A.

(a) Estimate the magnetic field at the centre of the solenoid.

[1] http://pdg.lbl.gov/atlas/atlas.html
[2] http://public.web.cern.ch/Public/, "Where the Web was born!"

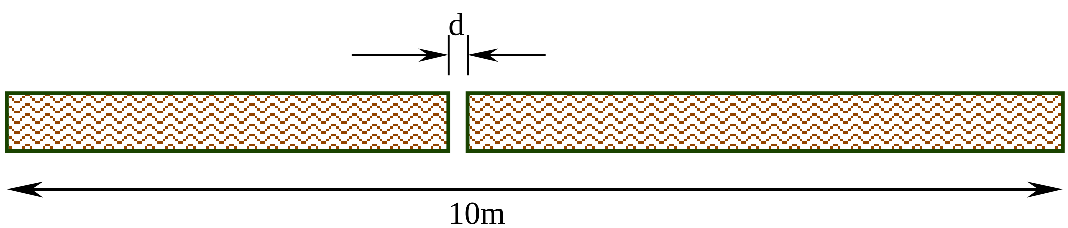
(b) When we talk about "how big" a resistor is, we need to specify both its resistance (in Ohms) and its power rating (in Watts). (The power rating tells us how much power the resistor can dissipate continuously without melting, blowing up, bursting into flames, … whatever.) How big a resistor would you need to put in series with the solenoid so that if you had to switch the magnet off suddenly, the voltage across the switch would not exceed 80V?

(c) With the resistor in parallel, how long will it take for the voltage across the open switch to drop to a safe value of 8V?

[David]

3) Cracked!

After buying a new TV, I connect it to an satellite dish with a 10m long rubber coated copper wire. I then discover that I have trouble picking up any channel below Channel 19 (500MHz). (I switched to satellite after dealing with that pesky balloon in Problem Set 4, but I seem to be doomed to never manage to get Fraggle Rock.) When I measure the resistance of the wire with a DC ohmmeter I discover it has infinite resistance so I decide it must have a hairline crack of width *d* (see figure, not to scale). The reason the low frequency channels can't get through is that



the wire has an RC time constant of about $1/(500\text{MHz})=2\times10^{-9}$s. What is the width of the crack, d?

Hints: Assume the sides of the crack are parallel and straight, and that the crack is filled with rubber. The resistivity of copper is $1.7\times10^{-8}\,\Omega\cdot\text{m}$; the dielectric constant of rubber is $\kappa=7$. Assume all resistances and capacitances other than the wire's are negligible.

[David]

4) Gravity isn't Bohring

Some cosmologies allow the possibility that other universes exist in which the forces and constituents differ from our universe. Imagine a universe in which there is no long range electromagnetic force, so the only long range force is gravity. Assume the proton, neutron and the electron exist with exactly the same masses as in our universe, but there are no electromagnetic forces between them. A hydrogen atom would still consist of an electron bound to a proton, but it would only be bound by gravity.

(a) What would be the energy difference (in electron volts) between the ground state and the first excited state of a hydrogen atom in the imaginary universe without electromagnetism? Assume the value of Planck's and Newton's constants (and all the other laws of physics except for electromagnetism) are the same in the imaginary universe as in our universe.

(b) Assume the current size of the imaginary universe is about the same as ours (radius ~ 10^{10} light years). Can gravitationally bound hydrogen atoms exist in the imaginary universe?

[David]

5) Resisting the light

When a light bulb heats up, does its electrical resistance go up or down?

Find an unbroken 120V incandescent light bulb; any typical size (15W, 25W, 40W, 60W, 75W, 100W) will do. (If you – temporarily – take a light bulb from a lamp, make sure that the lamp is turned off, unplugged, the bulb has cooled down before removing it, and you have permission from the owner of the lamp.)

(a) Measure the resistance of the light bulb when it is at room temperature. (Just use an ohmmeter or multimeter; ask your physics teacher for help if you don't have one at home.)

(b) <u>Calculate</u> the resistance of the light bulb when it is turned on and hot under normal operating conditions. (*Do not try to measure the resistance while the light bulb is plugged into a circuit!* Just calculate the resistance from the bulb's voltage – 120V - and power rating, *e.g*. 60W.)

(c) Assuming your measurement and calculation are accurate, answer our original question: When a light bulb heats up, does its electrical resistance go up or down?

[David]

POPBits™ – Possibly useful bits of information

Constants and units[3,4]

astronomical unit (mean earth-sun distance)	au	149597870660±20 m
atomic mass unit: (mass ^{12}C atom)/12	u	$(1.66053873 \pm 0.00000013)\times10^{-27}$ kg
Boltzmann constant	k	$(1.3806503 \pm 0.0000024)\times10^{-23}$ J/K
elementary (*i.e.* electron) charge	e	$(1.602176462 \pm 0.000000063)\times10^{-19}$ C
electron (or positron) mass	m_e	(510998.902 ± 0.021) eV/c^2
		$(9.10938188 \pm 0.00000072)\times10^{-31}$ kg
electron volt (eV)	eV	$(1.602176462 \pm 0.000000063)\times10^{-19}$ J
Newtonian gravitational constant	G_N	$(6.673\pm0.010)\times10^{-11}$ m^3/kg/s^2
permittivity of free space	ε_0	$8.854187817\times10^{-12}$ F/m
permeability of free space	μ_0	$4\pi\times10^{-7}$ N/A^2
Planck constant	h	$(6.6260755 \pm 0.000004)\times10^{-34}$ J·s
		$(4.13566727 \pm 0.00000016)\times10^{-15}$ eV·s
proton mass	m_e	(938271998 ± 38) eV/c^2
		$(1.67262158 \pm 0.00000013)\times10^{-27}$ kg
solar luminosity	$L_\odot$	$(3.846 \pm 0.008)\times10^{26}$W
speed of light in vacuum	c	299 792 458 m/s
standard acceleration of gravity at the earth's surface	g	9.80665 m/s^2
Stephan-Boltzmann radiation constant	σ	$(5.670400 \pm 0.000040)\times10^{-8}$ W/m^2/K^4
tropical year (2001)	yr	31556925.2 s

Great excuses for a party

Most famous physics class demonstration ever — April 1820
(Oersted discovers that electric currents generate magnetic fields)

[3] http://physics.nist.gov/cuu/Constants/index.html
[4] http://pdg.lbl.gov/2000/contents_sports.html

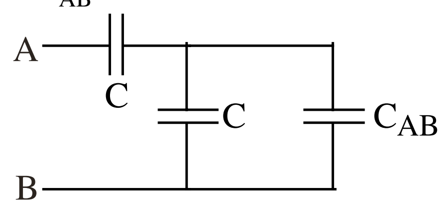
2001-2002 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 6: Electronics and Modern Physics

1) Everyone has the capacity to learn the quadratic formula!

Since this is an infinite chain, removing the two leftmost capacitors still leaves the same chain with the same total capacitance C_{AB}.



From this we see that we must have

$$\frac{1}{C_{AB}} = \frac{1}{C} + \frac{1}{C + C_{AB}}$$

$$\therefore C\left(C + C_{AB}\right) = C_{AB}\left(C + C_{AB} + C\right)$$

$$\therefore C_{AB}^2 + C_{AB}C - C^2 = 0$$

$$\therefore C_{AB} = \frac{-C \pm \sqrt{C^2 + 4C^2}}{2} = C\frac{-1 \pm \sqrt{5}}{2}$$

We can't have a negative capacitance, so

$$C_{AB} = \frac{\sqrt{5} - 1}{2}C$$

2) I always resist getting a shock

(a) We can estimate the magnetic field using the formula for an infinitely long solenoid:

$$B = \mu_0 nI$$

$$\therefore B \approx \mu_0 \frac{N}{L} I = 4\pi \times 10^{-7} \frac{\mathrm{N}}{\mathrm{A}^2} \frac{1200}{5.3m} 8000A = 2.3T$$

So the magnetic field will be about 2 Tesla.

(b) Since the resistance is in parallel, no current flows in it as long as current can flow throught the superconducting (and hence zero resistance) solenoid, but when the solenoid is switched off, all the current flows through the resistor and the voltage across the resistor is the voltage across the switch. The initial current is just the current originally flowing through the solenoid, and the maximum voltage is just the initial voltage:

$$V_{\max} = I_0 R$$

and the required resistance is

$$R = \frac{V_{\max}}{I_0} = \frac{80V}{8000A} = 0.01\Omega$$

The peak power dissipated is

$$P = I_0 V_{\max} = 8000A \times 80V = 0.64\,MW$$

To be safe we should round up the size, so what we need is a 0.01Ω, 1MW resistor. This is not something you find at your local Radio Shack[1].

(c) The resistor and the solenoid form an LR circuit with an exponential time constant, τ=L/R. The inductance per unit length for an infinite solenoid is

$$\frac{L}{l} = \mu_0 n^2 \times Area$$

so the inductance of the solenoid is approximately

$$L = \mu_0 \frac{N^2}{l^2}\left(\pi r^2\right) l = \pi\mu_0 \frac{N^2 r^2}{l}$$

and the time constant is

$$\begin{aligned}\tau &= \frac{L}{R} = \pi\mu_0 \frac{N^2 r^2}{Rl} \\ &= 4\pi^2 \times 10^{-7} \frac{kg \cdot m}{C^2} \frac{1200^2 (1.2m)^2}{(0.01\Omega)5.3m} \\ &= 154s\end{aligned}$$

The current decay is exponential, so to drop to one tenth of the original voltage will take

$$\tau_{1/10} = \tau \ln 10 = (154s)2.3 = 356s$$

i.e. about 6 minutes.

3) Cracked!

The broken wire is equivalent to two resistances and the crack is a dielectric filled parallel plate capacitor, all in series. Since they are part of the same wire, their resistivity, ρ, and cross-sectional area, *A*, are the same. The resistances of the two pieces of wire are

$$R_1 = \rho\frac{L_1}{A} \qquad\qquad R_2 = \rho\frac{L_2}{A}$$

And the total resistance is

$$R = R_1 + R_2 = \rho\frac{L_1 + L_2}{A} = \rho\frac{L}{A}$$

where L=10m is the total length of the wire.

The capacitance of the crack is

$$C = \kappa\varepsilon_0 \frac{A}{d}$$

where d is the width of the crack and κ is the dielectric constant.

The RC time constant is

[1] http://www.radioshack.ca/en/

$$\tau = RC = \rho \frac{L}{A} \kappa \varepsilon_0 \frac{A}{d} = \kappa \varepsilon_0 \rho \frac{L}{d}$$

So the crack width is

$$\begin{aligned} d &= \kappa \varepsilon_0 \rho \frac{L}{\tau} \\ &= 7 \cdot 8.854 \times 10^{-8} F/\mathrm{m} \cdot 1.7 \times 10^{-8} \Omega \cdot \mathrm{m} \frac{10m}{2 \times 10^{-9}\mathrm{s}} \\ &= 5 \times 10^{-9} m \\ &= 5nm \end{aligned}$$

4) Gravity isn't Bohring

(a) The energy levels of an atom are determined by the quantization condition that orbital angular momentum is quantized in units of Planck's constant over 2π, *i.e.*

$$L_n = m_e v_n r_n = n \frac{h}{2\pi} = n\hbar \qquad n = 1,2,3,\ldots$$

Using a simple Bohr atomic model for our gravitational hydrogen atom, the attractive gravitational force must balance the centripetal force, *i.e.*

$$G_N \frac{m_e m_p}{r_n^2} = \frac{m_e v_n^2}{r_n}$$

Using our quantization condition and solving this gives

$$r_n = \frac{n^2 \hbar^2}{G_N m_e^2 m_p}$$

The total (potential+kinetic) energy of the electron is thus

$$\begin{aligned} E_n &= \tfrac{1}{2} m_e v_n^2 - G_N \frac{m_e m_p}{r_n} \\ &= \tfrac{1}{2} G_N \frac{m_e m_p}{r_n} - G_N \frac{m_e m_p}{r_n} \\ &= -\tfrac{1}{2} G_N^2 \frac{m_p^2 m_e^3}{n^2 \hbar^2} \\ &= \frac{-0.26 \times 10^{-77} eV}{n^2} \end{aligned}$$

The energy difference between the ground (n=1) and first excited state (n=2) is

$$\Delta E = -2.6 \times 10^{-79} eV \left(\frac{1}{2^2} - \frac{1}{1^2} \right) = 2 \times 10^{-78} eV$$

This is 79 orders of magnitude smaller than the corresponding energy in our universe (10.2eV).

(b) The smallest radius of a gravitationally bound hydrogen atom is that of its n=1 ground state

$$r_1 = \frac{\hbar^2}{G_N m_e^2 m_p} = 1.2 \times 10^{29} m = 1.2 \times 10^{13} \ \ light \ \ years$$

This is 3 orders of magnitude large than the universe, so a gravitationally bound hydrogen atom cannot exist in the imaginary universe.

5) Resisting the light

(a) I used a 15W bulb, the measured resistance was about 85±1 ohms.

(b) The resistance of the bulb when operating should be $R=V^2/Power=(120V)^2/W=960$ ohms.

(c) The operating resistance is an order of magnitude higher than the measured room temperature resistance, so the the resistance of the filament must increase with temperature.

2002-2003 Physics Olympiad Preparation Program

– University of Toronto –

Problem Set 1: General

Due 28 October 2002

Here are a few comments before starting your first problem set.

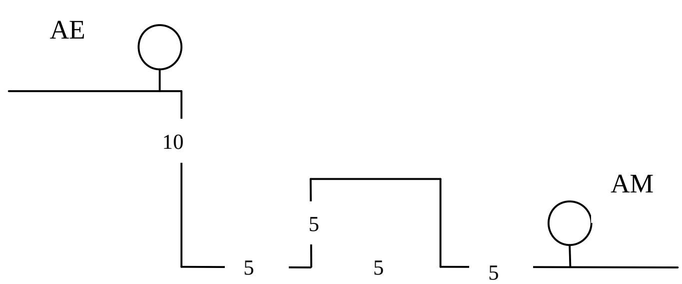
- If you want to know what kind of physics you might need to solve these problems, look at the Physics Olympiad syllabus at http://www.jyu.fi/ipho/syllabus.html. You do not have to know the syllabus by heart to do the problems, but you should recognize what you need to know so you can look it up.
- Don't forget to look at the information given in the POPBits™ section at the end of the problem set. It sometimes has information helpful or necessary for particular problems.
- Pay attention to words like "estimate" or "about". They indicate that the expected answer is not exact because either the input data is not precisely known or approximations or simplifying assumptions are necessary. Much of real physics is learning how to turn insoluble exact problems into soluble approximations.
- "*Nothing ventured, nothing gained.*" Whether you finish a problem or not, please make sure your reasoning and analysis are clear. If you write down nothing, we can only give you zero. Your basic ideas may be right even if you make a mistake or get stuck.

1. Two Alberts

AE must throw a ball so that it reaches AM after hitting the ground one time. AM remains fixed at his position. The distances shown in the diagram are in metres.

(a) What should AE do to achieve the above? Ignore air resistance.

(b) Is there more than one way for AE to throw the ball so that it hits the ground only once before arriving at AM?



2. Two violins

How much louder are two violins compared to one violin? Suppose that each violin produces a single pitch (frequency) wave $f(t)=A\sin(2\pi\nu t+\phi)$, where A is the constant amplitude, ν is the frequency, and ϕ is a phase factor that might vary randomly in time for each instrument. You will need to consider several cases.

3. Temperature of the planet Venus

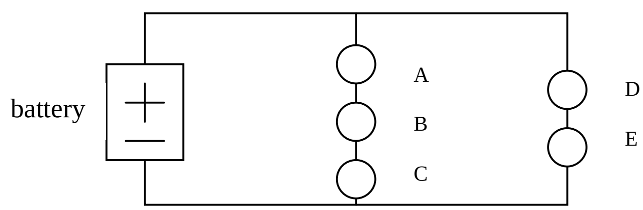
All objects emit radiation. The energy flux density J of this radiation is the energy emitted per unit time and per unit area. The Stefan-Boltzmann law is $J=e\sigma T^4$, where e=1 for a perfect radiator, σ is a constant, and T is the temperature of the object. What is the equilibrium temperature of the planet Venus? Assume that the only radiation incident on Venus is from the Sun. Assume that Venus and the Sun are perfect radiators. If your calculated T is different from the measured temperature of Venus of about 700 Kelvin, speculate on why this is the case.

4. Can the can roll?

Rest an empty aluminum pop can on its side and on a table. Bring a charged object close to the can, but do not touch the can. Can you get the can to roll? If it does start to roll, why does it do so? Can you halt the can's motion by using the charged object, but without hitting the can? Describe the charged object you used in this experiment.

5. Circuits

When the following circuit is connected, how bright are the bulbs A, B, C, D, and E relative to one another? The bulbs are identical. If bulb A is removed from the circuit, what happens to the brightness of B, C, D, and E? If bulb A is replaced by two new bulbs, what happens to the brightness of the remaining bulbs? You can try this question experimentally and/or theoretically.



<u>**POPBits™** – Possibly useful bits of information</u>

Constants and units[1,2]

Stefan-Boltzmann radiation constant	σ	$(5.670400\pm0.000040)\times10^{-8}$ W/m^2/K^4
Mean Sun-Venus distance	R_{sv}	1.08×10^{8} km
Temperature of the Sun	T_S	5800 K
Radius of the Sun	R_S	6.96×10^{5} km

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[1] http://physics.nist.gov/cuu/Constants/index.html
[2] http://pdg.lbl.gov/2000/contents_sports.html

2002-2003 Physics Olympiad Preparation Program - University of Toronto - Solution Set 1: General

1. The equations of kinematics in one dimension are

$$x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2 , \quad (1)$$
$$v_x^2 = v_{x0}^2 + 2a_x(x - x_0) , \quad (2)$$

where x_0 is the initial position, t is the time, a_x is the acceleration, and v_{x0} is the initial velocity. The ball starts at $(x_0, y_0) = (0, 10)$ m. It hits the ground at (x_1, y_1) and arrives at $(x_2, y_2) = (15, 0)$ m, the position of AM. At (x_1, y_1), we have

$$x_1 = x_0 + v_{x0}t_1 + \frac{1}{2}a_x t_1^2 , \quad (3)$$
$$y_1 = y_0 + v_{y0}t_1 + \frac{1}{2}a_y t_1^2 . \quad (4)$$

The acceleration in the x-direction (y-direction) is $a_x = 0$ ($a_y = -g \approx -10$ m/s^2). Rearranging Eq. 3 yields $t_1 = \frac{x_1 - x_0}{v_{x0}}$. This can be substituted into Eq. 4 to give

$$(y_0 - y_1)v_{x0}^2 + (x_1 - x_0)v_{y0}v_{x0} - \frac{1}{2}(x_1 - x_0)^2 g = 0 . \quad (5)$$

There is a similar equation involving (x_1, y_1), (x_2, y_2), v_{x1}, and v_{y1}. Since $a_x = 0$, $v_{x1} = v_{x0}$. To find v_{y1}, solve $v_{y1}^2 = v_{y0}^2 - 2g(y_1 - y_0)$. Choose a particular x_1, and y_1, and then compute v_{x0} and v_{y0}. For example, if AE (Albert Einstein) throws the ball with an initial velocity $(v_{x0}, v_{y0}) \approx (3.29, 6.39)$ m/s, it reaches AM (Albert Michelson) after one bounce at $(x_1, y_1) = (6, 5)$ m. You should check that the ball clears the raised ledge following its impact at (x_1, y_1).

2. Superpose the waves from the violins by adding them

$$F(t) = f_1(t) + f_2(t) = Asin(2\pi\nu t + \phi_1) + Asin(2\pi\nu t + \phi_2) \quad (6)$$

The loudness or intensity of a wave is proportional to the square of its wave function. The intensity for one violin is

$$\text{Intensity} = C(f_1(t))^2 = CA^2 sin^2(2\pi\nu t + \phi_1) , \quad (7)$$

where C is a constant and for two violins

$$\begin{aligned} \text{Intensity} &= C(f_1(t) + f_2(t))^2 = CA^2[sin^2(2\pi\nu t + \phi_1) + sin^2(2\pi\nu t + \phi_2) \\ &+ 2sin(2\pi\nu t + \phi_1)sin(2\pi\nu t + \phi_2)] \end{aligned} \quad (8)$$

Consider two cases. If the phase factors do not vary randomly, and in particular $\phi_1 = \phi_2$, then at certain times the intensity reaches $I = 4CA^2$. This is four times as loud as one violin. If the phase factors vary in a completely random way, then the last term in Eq. 8 is on average 0. The loudness is on average $I = 2CA^2$. This is twice as loud as one violin.

3. The energy flux density emitted by the Sun is $J_S = \sigma T_S^4$. This gives the energy per unit time flowing through unit area. The total energy per unit time going through the surface of the Sun is the energy flux density times the surface area of the Sun, namely, $(4\pi R_S^2)J_S$. If there is no energy lost as this radiation propagates between the Sun and Venus, then the same amount of energy per unit time must reach the orbit of Venus. In equilibrium, the radiation that Venus intercepts equals the amount it emits.

$$(4\pi R_S^2)\sigma T_S^4 \frac{\pi R_V^2}{4\pi R_{SV}^2} = (4\pi R_V^2)\sigma T_V^4 \ , \tag{9}$$

$$T_V^4 = \frac{R_S^2}{4R_{SV}^2} T_S^4 \ , \tag{10}$$

$$T_V = \sqrt{\frac{R_S}{2R_{SV}}} T_S \ . \tag{11}$$

Substituting for R_S, R_{SV}, and T_S gives $T_V \approx 330$ Kelvin. This is lower than the measured temperature of Venus of about 700 Kelvin. The atmosphere of Venus contains a large amount of carbon dioxide. This is a "greenhouse" gas. It traps the radiation re-emitted by the planet and leads to a higher surface temperature. Could this effect happen on Earth?

4. You can charge an object such as a comb or balloon by brushing or rubbing it through your hair. Bringing a charged object O close to the can redistributes the charges in the can. If object O is negatively charged, then it repels the negatively charged electrons in the can. There is an excess of positive charge on the surface of the can close to where O is located. There is an attractive electric force between the can and O. Friction between the can and the table produces a turning force or torque about the can's central axis (centre of mass or CM). Depending on the placement of O, the electric force could also produce a torque about the CM. The torque results in the can's rolling motion. The can's motion can be halted by bringing O to the side of the can that is opposite to its direction of motion.

5. Each bulb acts as a resistor with resistance r. The electromotive force of the battery is V. When all of the bulbs are in place, there is a current $i_1 = \frac{V}{3r}$ ($i_2 = \frac{V}{2r}$) through the branch with ABC (DE). The brightness of A, B, and C are equal. They will be less bright than that of D and E, since less current is going through the ABC branch. If bulb A is removed, then no current flows through the ABC branch of the circuit. The brightness of B and C are zero. The brightness of D and E are the same and unchanged from the situation in which all of the bulbs were connected. If bulb A is replaced by two new bulbs, then bulbs A1, A2, B, and C will be equally bright. The current flowing through the ABC branch is $i_1 = \frac{V}{4r}$. The bulbs A1, A2, B, and C will be dimmer than before when ABCDE were connected. Bulbs D and E will have equal brightness and maintain the same level in all of the above situations.

2002-2003 Physics Olympiad Preparation Program
- University of Toronto -
Problem Set 2: Mechanics
Due: 9 December 2002

1. A cylinder of radius R and mass M rolls down an incline as shown in Figure 1. It starts from rest. What is the cylinder's acceleration a? What is the minimum coefficient of static friction μ_s that is required for the cylinder to roll down the incline without slipping?

2. Four planets, each of mass M, are at the corners of a square, the sides of which are length L. At what speed v must the planets move so that they maintain a circular orbit that circumscribes the square?

3. An object consists of two identical masses m joined by a weightless rod of length L. It collides elastically with a mass $2m$. See Figure 2. How does the system evolve?

4. A gas of molecules exerts a pressure P on the walls of a closed box. A small hole is created in one side of the box. Estimate the speed v with which the molecules exit the box.

5. Blow up a balloon B1. Insert one end of a plastic drinking straw into the mouth of B1. Tie down the mouth of B1 to the straw. Do not allow any air to escape through the open end of the straw. Blow up a second balloon B2 with a smaller volume than B1. Insert the open end of the straw into the mouth of B2. Tie down the mouth of B2 to the straw, but do not allow any air from B2 into the straw. Now allow the system to equilibrate. What happens to the balloons? Explain your observations.

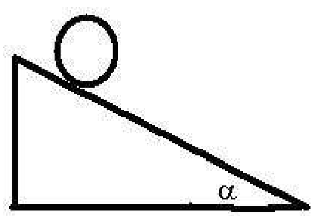


Figure 1

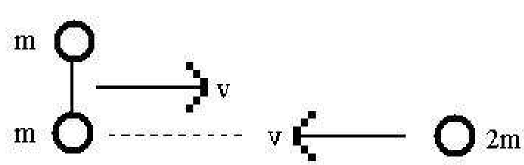


Figure 2

2002-2003 Physics Olympiad Preparation Program - University of Toronto - Solution Set 2: Mechanics

1. The equation of motion for the cylinder along the surface of the incline is

$$Ma \quad = \quad Mgsin\alpha - f \ , \tag{1}$$

where a is the linear acceleration and f is the force due to friction (Figure 1). The equation for the angular motion about the cylinder's axis is

$$IA \quad = \quad fR \, , \tag{2}$$

where $I = \frac{1}{2}MR^2$ is the moment of inertia of the cylinder and $A = \frac{a}{R}$ is the angular acceleration. Solve Eq. 2 for f. Substitute $f = \frac{IA}{R} = \frac{Ia}{R^2}$ into Eq. 1 and solve for a.

$$Ma \quad = \quad Mgsin\alpha - \frac{Ia}{R^2} \to a = \frac{Mgsin\alpha}{M + \frac{I}{R^2}} = \frac{2}{3}gsin\alpha \ . \tag{3}$$

The coefficient of static friction is specified by

$$\mu_s \geq \frac{f}{N} = \frac{\frac{1}{3}Mgsin\alpha}{Mgcos\alpha} = \frac{1}{3}tan\alpha \ , \tag{4}$$

where N is the normal force. The minimum coefficient that is required is $\frac{1}{3}tan\alpha$.

2. As they travel in a circular orbit of radius r and speed v, each planet feels a centripetal force $F_c = \frac{Mv^2}{r}$ (Figure 2). The gravitational force on each planet is

$$F_G \quad = \quad F_{12} + F_{14} + F_{13} = 2cos(\frac{\pi}{4})\frac{GMM}{L^2} + \frac{GMM}{(2r)^2} = \frac{GM^2}{L^2}(\sqrt{2} + \frac{1}{2}) \ , \tag{5}$$

where $(2r)^2 = L^2 + L^2$ implies that $r = \frac{L}{\sqrt{2}}$. Equate the centripetal and gravitational forces to find v

$$\frac{Mv^2}{\frac{L}{\sqrt{2}}} \quad = \quad \frac{GM^2}{L^2}(\sqrt{2} + \frac{1}{2}) \to v^2 = \frac{GM}{L}(1 + \frac{\sqrt{2}}{4}) \ . \tag{6}$$

The speed at which the planets must move is $v = \sqrt{\frac{GM}{L}(1 + \frac{\sqrt{2}}{4})}$.

3. Due to the conservation of linear momentum, the initial and final linear centre of mass (CM) momenta of the objects are zero. Conservation of energy tells us that

$$2 \times \frac{1}{2}(2m)v^2 \quad = \quad 2 \times \frac{1}{2}(2m)v'^2 + \frac{1}{2}I\omega^2 \to v^2 = v'^2 + \frac{1}{8}L^2\omega^2 \ , \tag{7}$$

where v (v') is the speed before (after) the collision, $I = m(\frac{1}{2}L)^2 + m(\frac{1}{2}L)^2 = \frac{1}{2}mL^2$ is the moment of inertia of the left object (LO), and ω is the angular velocity of LO about its CM. Conservation of angular momentum requires that

$$(2m)(\frac{L}{2})v \quad = \quad (2m)(\frac{L}{2})v' + I\omega = mLv' + \frac{1}{2}mL^2\omega \ , \tag{8}$$

$$v' \quad = \quad v - \frac{1}{2}L\omega \ , \tag{9}$$

where the first two terms of Eq. 8 are angular momenta calculated with respect to the collision point of the objects and the third term of Eq. 8 is the angular momentum about the CM of LO. Substitute Eq. 9 into Eq. 7 to establish

$$v^2 \;=\; (v - \frac{1}{2}L\omega)^2 + \frac{1}{8}L^2\omega^2 \rightarrow -vL\omega + \frac{3}{8}L^2\omega^2 = 0\,. \qquad (10)$$

One solution is $\omega = 0$ and $v' = v$. This implies that the objects pass through one another with no collision. Another solution is $\omega = \frac{8v}{3L}$ and $v' = -\frac{1}{3}v$. Following the collision, the objects rebound from each other. Their speeds are reduced and LO rotates about its CM.

4. Suppose the hole is rectangular in shape and has a cross-sectional area A and length z. The energy of each molecule leaving the box is $\frac{1}{2}mv^2$. If there are N molecules, then the total energy per unit volume is $\frac{N\frac{1}{2}mv^2}{Az} = \frac{\frac{1}{2}(mN)v^2}{Az} = \frac{1}{2}\rho v^2$ where $\rho = \frac{Nm}{Az}$ is the mass density of the molecules. Pressure and energy per unit volume have the same dimensions. To obtain an estimate of the speed of the molecules, equate the pressure P to the above total energy per unit volume to find $P = \frac{1}{2}\rho v^2$ and $v = \frac{2P}{\rho}$.

5. Each part of the wall of the smaller balloon B2 is more curved and under greater tension than that of the larger balloon B1. In order to support the high tension in the wall of B2, the air must exert a bigger pressure P2 than that in B1. Once the balloons are allowed to equilibrate, the larger P2 forces air from B2 into B1. So the smaller balloon decreases in size and the larger balloon becomes even bigger.

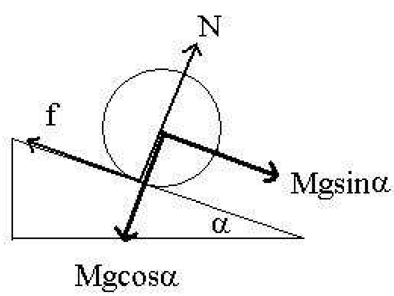


Figure 1

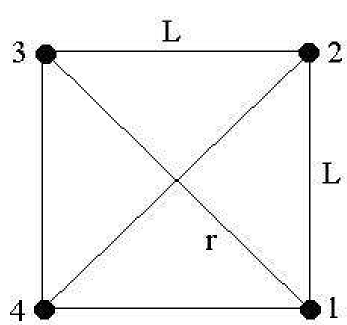


Figure 2

2002-2003 Physics Olympiad Preparation Program
- University of Toronto -
Problem Set 3: Thermodynamics
Due: 13 January 2003

1. Why does the Moon not have an atmosphere?

2. The rate of conduction of heat H between a hot reservoir at temperature T_h and cooler reservoir at temperature T_c ($T_h > T_c$) through a conducting medium is

$$H = kA\frac{T_h - T_c}{L}\ . \tag{1}$$

The thermal conductivity, cross-sectional area, and length of the conducting medium are k, A, and L. You can wear a parka filled with goose down or two wool sweaters. Which is going to keep you warmer? The cross-sectional area of the parka and sweaters is $A = 2 \times 20\text{cm} \times 20\text{cm}$. Your body is at 20°C and the cool winter air is at 0°C. The goose down parka has $k = 0.0250$ J/(s·m·°C) and $L = 0.5$ cm. A wool sweater has $k = 0.0400$ J/(s·m·°C) and $L = 0.2$ cm. There is an air layer between the sweaters which has $k = 0.0256$ J/(s·m·°C) and $L = 0.1$ cm.

3. A fridge takes heat Q_c from a low temperature reservoir at T_c and delivers heat Q_h to a high temperature reservoir at T_h. Work W is done on the system.
a. Explain why $K = Q_c/W$ is a measure of the performance of the fridge.
b. Is it possible to have a fridge do the following? If yes, then what is K?
(i) Extract 200 J from a cold reservoir and discharge 200 J to a hot reservoir.
(ii) Extract heat from a cold reservoir at $T_c = 5$°C and discharge heat to a hot reservoir at $T_h = 20$°C.

4. An ideal gas composed of relativistic particles is in a box of dimensions $L \times L \times L$. The pressure, volume, and total energy of the gas are P, V, and U. Each particle has energy $E = pc$, where p is the momentum and c is the speed of light. Verify that $PV = \frac{1}{3}U$.

5. Make some ice cubes of equal volume. Bring them out of the freezer and sprinkle the upper surface of some of the cubes with equal amounts of either salt or sand. Compared to clean ice cubes, how quickly do the latter melt at room temperature? Why does one type of ice cube melt faster than another? What should we use to melt the ice on Ontario roads during winter?

2002-2003 Physics Olympiad Preparation Program - University of Toronto - Solution Set 3: Thermodynamics

1. Suppose the Moon has an atmosphere made of molecules each of mass m_o. To determine whether the Moon can hold onto such molecules, determine the escape speed v_e of an object on the lunar surface via

$$\frac{1}{2}m_o v_e^2 = \frac{G m_o M_M}{R_M} \rightarrow v_e = \sqrt{\frac{2GM_M}{R_M}} \ , \tag{1}$$

where M_M (R_M) is the mass (radius) of the Moon. This equates the kinetic energy of the object to the work needed to move the object from the Moon's surface to infinity. Using $M_M = 7.36 \times 10^{22}$ kg, $R_M = 1.74 \times 10^6$ m, and $G = 6.67 \times 10^{-11}$ m^3/kg·s^2, gives $v_e \approx 2.4$ km/s. From kinetic theory, the average speed of a molecule in an ideal gas is $v = \sqrt{\frac{3k_BT}{m_o}}$, where $k_B = 1.381 \times 10^{-23}$ J/K is the Boltzmann constant and T is the temperature of the gas. If the lunar temperature and the mass of an oxygen molecule are $T = 100°$C=373 K and $m_o = 5.32 \times 10^{-26}$ kg, then the mean speed of an oxygen molecule is $v \approx 0.54$ km/s. Although the latter is less than v_e, there are many oxygen molecules with speeds above the average v. These molecules will eventually depart from the lunar surface and no atmosphere will remain.

2. The rate of heat conduction through the parka is

$$H_{parka} = k_{goose} A \frac{T_h - T_c}{L_{parka}} = (0.025)(2)(0.2)^2(20-0)/0.005 = 8 \text{ J/s} \ , \tag{2}$$

where T_h (T_c) is the temperature of your body (outside air). For the wool sweaters, there are three layers through which heat is conducted.

body | wool (1) | air (2) | wool (3) | outside air

The rates of heat conduction through the three layers are

$$\begin{aligned} H_1 &= k_{wool} A \frac{T_h - T_{12}}{L_{wool}} \ , \\ H_2 &= k_{air} A \frac{T_{12} - T_{23}}{L_{air}} \ , \\ H_3 &= k_{wool} A \frac{T_{23} - T_c}{L_{wool}} \ , \end{aligned}$$

where T_{12} (T_{23}) is the temperature at the boundary between layers 1 and 2 (2 and 3). In steady-state, $H_1 = H_2 = H_3 \equiv H_{sweaters}$. The above equations can be solved to find,

$$\begin{aligned} H_{sweaters} &= \frac{A(T_h - T_c)}{L_{wool}/k_{wool} + L_{air}/k_{air} + L_{wool}/k_{wool}} \\ &= (2)(0.2)^2(20-0)/(2 \times 0.002/0.04 + 0.001/0.0256) = 11.5 \text{ J/s} \ . \end{aligned} \tag{3}$$

The parka will keep you warmer since the rate of heat conducted through the goose down is less than that through the wool sweaters.

3(a) The ratio $K = Q_c/W$ compares the amount of heat Q_c removed from the low temperature reservoir to the amount of work $W = Q_h - Q_c$ needed to achieve this extraction. Since we want Q_c to be large and W to be small, K is a measure of the efficiency of the fridge. For example, K is small when Q_c is small and W is large.
3(b)(i) Since $W = Q_h - Q_c = 0$, no work is required to move heat from a cool to hot reservoir. This cannot happen since heat naturally flows from hotter to colder objects.
3(b)(ii) For ideal fridges and engines, $\frac{Q_h}{Q_c} = \frac{T_h}{T_c}$. The factor K becomes

$$\begin{aligned} K &= Q_c/W = Q_c/(Q_h - Q_c) = 1/(Q_h/Q_c - 1) = 1/(T_h/T_c - 1) \\ &= T_c/(T_h - T_c) = 278\ K/(293\ K - 278\ K) \approx 18.5\ . \end{aligned}$$

4. During a short time interval Δt, if a particle is within a distance Δx of a wall, it will undergo a collision.

$$\begin{array}{cc|} & \\ -p_x & \rightarrow \\ +p_x & \leftarrow \\ & \end{array}$$

There is a momentum change of $\Delta p = 2p_x$. The number of collisions during Δt is $\frac{1}{2}n(A\Delta x)$, where A is the area of the wall and n is the number of particles N divided by volume V. The pressure applied by the particles is given by the momentum change of a particle per time interval, times the number of collision during the time interval, and divided by the area A. This equals the total force exerted by the particles per unit area.

$$\begin{aligned} P &= \frac{(\frac{1}{2}nA\Delta x)(\frac{\Delta p}{\Delta t})}{A} = (\frac{1}{2}n\Delta x)(\frac{2p_x}{\Delta t}) \\ &= nv_x p_x = nv_x(E_x/c) = nc(E_x/c) = nE_x = n(\frac{1}{3}E) = \frac{NE}{3V} = \frac{U}{3V}\ , \end{aligned} \tag{4}$$

where $v_x \approx c$ and $U = NE$ is the total energy of the gas. The energy E_x refers to the energy of the particles in the x-direction and on average it is one third of the total energy E. Equation 4 implies that $PV = \frac{1}{3}U$.

5. Use equal amounts of salt and sand that are at the same temperature. Salt will melt the ice cubes more quickly. Salt dissolves in water and reduces the melting temperature of the ice. Sand is not soluble in water. It just sits on the surface of the ice. We can use salt on our roads in winter in order to melt ice more quickly. But the salt ends up in our water supply, which is not a good thing. Sand can be used on icy roads. It provides some traction between people and vehicles and the ice.

2002-2003 Physics Olympiad Preparation Program
- University of Toronto -
Problem Set 4: Waves and Optics
Due: 10 February 2003

1. A cube of ice C, the sides of which are length $L = 1$ metre, is placed in water. When the water and C come to rest, 0.5 metre of C is above the surface of the water. The cube is raised vertically out of the water by a small amount and then released. Prove that C exhibits simple harmonic oscillation. Ignore damping (friction). What is the frequency of the water waves produced by the oscillating ice cube? The density of water (ice) is $\rho_{water} = 1000$ kg/m^3 ($\rho_{ice} = 900$ kg/m^3).

2. A point source of sound waves S moves with constant speed v. Its power output is P. What is the wave intensity measured at a distance r in front of S? The speed of sound is v_s.

3. A light beam travels from air through a medium of thickness L and index of refraction $n > 1.2$ (Figure 1). Upon entering the medium, the beam makes an angle θ with the surface normal. Prove that the beam exits the medium shifted by a distance x with respect to the direction of entry. Express the index of refraction of the medium in terms of θ, L, and x.

4. Two point sources are a distance d apart. They emit spherical waves of wavelength λ. The intensity of the waves is detected along the z-axis as indicated in Figure 2. Where are the points of maximum intensity along the z-axis? Is there an infinite number of maxima on the z-axis? Explain your answer. What is the intensity at the minimum on the z-axis closest to the sources?

5. Place an extended object O at the bottom of a container filled with water (Figure 3). View O from various angles and distances. Try looking at O with your head oriented at different angles. Is the horizontal distance to O that you see altered by the water? Explain your observations.

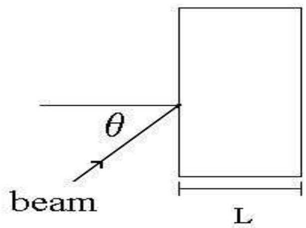


Figure 1

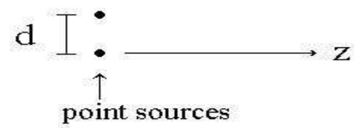


Figure 2

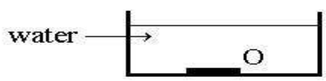


Figure 3

2002-2003 Physics Olympiad Preparation Program
- University of Toronto - Solution Set 4: Waves and Optics

1. In equilibrium, the net force on the ice cube is

$$F_{net} = -W_i + B = 0 \rightarrow W_i = B \rightarrow \rho_{ice} V_{ice} g = \rho_{water} V_{water} g \ , \tag{1}$$

where W_i (B) is the gravitational (buoyancy) force on the cube, $V_{ice} = L^3$ (V_{water}) is the volume of the ice (water displaced by the cube), and g is the acceleration due to gravity. When the ice cube is raised by a small amount x, the net force is

$$F_{net} = -W_i + B' \rightarrow Ma = -\rho_{ice} V_{ice} g + \rho_{water}(V_{water} - L^2 x)g = -\rho_{water} L^2 g x \ , \tag{2}$$

where $M = \rho_{ice} V_{ice}$ and $a = \frac{d^2x}{dt^2}$ are the mass and acceleration of the cube. Eq. (1) was used to simplify Eq. (2). The latter reduces to

$$\frac{d^2x}{dt^2} + \frac{\rho_{water} L^2 g}{\rho_{ice} L^3} x = \frac{d^2x}{dt^2} + \omega^2 x = 0 \ , \tag{3}$$

which is the equation for simple harmonic oscillation. The angular frequency is $\omega = \sqrt{\frac{\rho_{water} g}{\rho_{ice} L}} = \sqrt{\frac{1000 \cdot 9.8}{900 \cdot 1}} \approx 3.3 \text{ s}^{-1}$ and the frequency is $\nu = \frac{\omega}{2\pi} \approx 0.53 \text{ s}^{-1}$.

2. At $t = 0$, the source S emits a spherical wave front with a radius that increases as $R = v_s t$. We are interested in the intensity of the sound waves reaching a detector D at a distance $r = R - vt$ in front of S. The energy flowing through area A during time T is $E_A = \frac{PT}{A}$, where T is the period of the sound wave. Since S is moving towards D, the period of the waves T' is Doppler shifted such that $\frac{1}{T'} = \frac{1}{T}\frac{v_s}{v_s - v}$. The above equations reveal that

$$r = R - vt = v_s t - vt \rightarrow t = \frac{r}{v_s - v} \rightarrow R = \frac{v_s}{v_s - v} r \ , \tag{4}$$

$$E_A = \frac{PT}{4\pi R^2} = \frac{PT(v_s - v)^2}{4\pi v_s^2 r^2} \ . \tag{5}$$

The intensity, or energy flowing through unit area per unit time, at D is

$$I = \frac{1}{T'} E_A = \frac{P}{4\pi r^2} \frac{v_s - v}{v_s} \ . \tag{6}$$

3. Light travelling from air to glass, and then from glass to air, refracts according to $n_1 sin\theta_1 = n_2 sin\theta_2$ and $n_2 sin\theta_2 = n_1 sin\theta_3$, where $\theta_1 \equiv \theta$ and $n_2 = n$. (See Figure 1.) These equations indicate that $\theta_1 = \theta_3$. The beam enters and exits from the glass at the same angle to the surface normal. From the geometry shown in Figure 1, $cos\theta_2 = \frac{L}{D}$ and $sin(\theta - \theta_2) = \frac{x}{D}$. The latter implies that $x = D sin(\theta - \theta_2)$. The beam will be shifted by a distance x with respect to the direction of entry as long as $\theta = \theta_2 \neq 0$.

$$x = D sin(\theta - \theta_2) = \frac{L}{cos\theta_2}[sin\theta cos\theta_2 - cos\theta sin\theta_2] \ , \tag{7}$$

$$\frac{x}{L} = sin\theta - tan\theta_2 cos\theta \rightarrow tan\theta_2 = \frac{sin\theta - x/L}{cos\theta} \tag{8}$$

The index of refraction of the medium is

$$n = n_1 \frac{sin\theta}{sin\theta_2} = n_1 \frac{sin\theta}{sin[tan^{-1}(tan\theta - \frac{x}{Lcos\theta})]} \ . \tag{9}$$

4. To find the maxima on the z-axis, set the path difference of the waves originating from the sources equal to an integral number of wavelengths. (See Figure 2.)

$$r_2 - r_1 = m\lambda \to \sqrt{d^2 + z^2} - z = m\lambda \quad \to z = \frac{d^2 - m^2\lambda^2}{2m\lambda}\ , \tag{10}$$

where $m = 1, 2, \ldots$. The above must be restricted so that $z > 0$. This implies that

$$d^2 - m^2\lambda^2 \quad = \quad (d - m\lambda)(d + m\lambda) > 0 \to m < \frac{d}{\lambda}\ . \tag{11}$$

There are a finite number of maxima on the z-axis corresponding to $m = 1, 2, \ldots, int(\frac{d}{\lambda})$, where $int(x)$ is the largest integer less than x. Consider the spacing of the maxima along a line $z = D$ that is perpendicular to the z-axis. The maxima are located at $d sin\theta = n\lambda$, where n is an integer. For large D, $d sin\theta = d\frac{y}{\sqrt{y^2+D^2}} \approx \frac{dy}{D} = n\lambda \to y = \frac{n\lambda D}{d}$. The spacing of the maxima in the direction orthogonal to $\hat{z}$ increases with D. For large enough D, no maxima will be found along the z-axis. The minima appear at

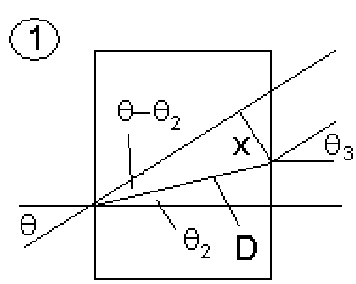
$$r_2 - r_1 \quad = \quad (m + \frac{1}{2})\lambda \to z = \frac{d^2 - (m + \frac{1}{2})^2\lambda^2}{2(m + \frac{1}{2})\lambda}\ . \tag{12}$$

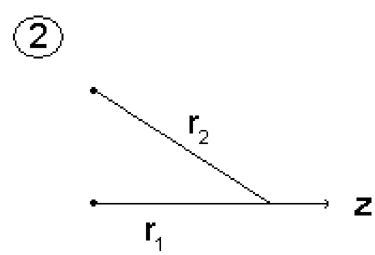
The restriction $z > 0$ yields minima corresponding to $m = 0, 1, 2, \ldots, int(\frac{d}{\lambda} - \frac{1}{2})$. The minimum closest to the sources is z_{min} such that $m = int(\frac{d}{\lambda} - \frac{1}{2})$. The intensity at $z = z_{min}$ is

$$\begin{aligned} I(0, 0, z_{min}, t) &= \frac{1}{2}\epsilon_0 c|E|^2 = \frac{1}{2}\epsilon_0 c|\frac{E_0}{r_1}e^{i(kr_1 - \omega t)} + \frac{E_0}{r_2}e^{i(kr_2 - \omega t)}|^2 && (13)\\ &= \frac{1}{2}\epsilon_0 c E_0^2(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{2}{r_1 r_2}cos[k(r_1 - r_2)]) && (14)\\ &= \frac{1}{2}\epsilon_0 c E_0^2(\frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{2}{r_1 r_2})\ , && (15) \end{aligned}$$

where $r_1 = z_{min}$ and $r_2 = \sqrt{z_{min}^2 + d^2}$.

5. Light that originates from the submerged object O, refracts towards the surface of the water when it enters the air. The object appears higher than it actually is. If your eyes are horizontal with respect to the water's surface, then there is no alteration of the horizontal distance to O. If your eyes are vertical, then there is a difference between the light reaching your left and right eye. This light has been refracted by different amounts. The object looks closer and there is a change in its apparent horizontal distance.





2002-2003 Physics Olympiad Preparation Program
- University of Toronto -
Problem Set 5: Electricity and Magnetism
Due: 10 March 2003

1. A current of one ampere flows through a wire made of copper and aluminum. See Figure 1. They have the same cross section A. The resistivity of copper (aluminum) is $\rho_{Cu} = 1.7 \times 10^{-8}\ \Omega\cdot\text{m}$ ($\rho_{Al} = 2.8 \times 10^{-8}\ \Omega\cdot\text{m}$). The permittivity of free space is $\epsilon_0 = 8.85 \times 10^{-12}$ F/m. Does any charge accumulate at the boundary between the metals? If yes, how much charge? Discuss your result.

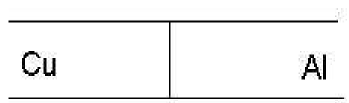
2. A capacitor consists of two parallel metal plates of area A separated by a distance d. A dielectric slab of area A, thickness b, and dielectric constant κ is placed inside the capacitor. What is the capacitance of the empty capacitor (C) and the capacitor with the dielectric (C_κ)? How must κ and b be restricted so that $C_\kappa = 2C$?

3. A toroid is a solenoid wrapped into a doughnut. See Figure 2. There N turns of wire wound around the surface of the toroid. The wire carries a current I. (a) What is the magnetic field $\mathbf{B}$ inside the toroid? (b) At $t = 0$, an electron inside the toroid has a velocity $\mathbf{v}_0 = v_0\hat{z}$. What is the path of the electron for $t > 0$?

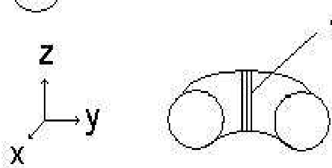
4. A large circular loop (L) of radius a is centred at (0,0,0) in the xy-plane. See Figure 3. A current I flows counterclockwise through L. A small circular loop (S) of radius b ($b \ll a$) falls along the z-axis starting from $(0, 0, z_0)$ such that $z_0 \gg a$. The mass (resistance) of S is m (R). Find the current induced in S and describe the motion of S.

5. Design an electric motor using a battery, tape, and some pieces of conducting wire. Draw a diagram showing how your motor is assembled and functions. Explain the physics behind the operation of your device.

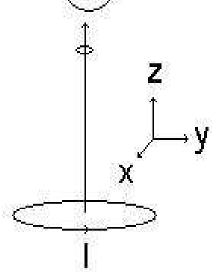
F1



F2



F3



2002-2003 Physics Olympiad Preparation Program - University of Toronto - Solution Set 5: Electricity and Magnetism

1. Gauss' law is $q = \epsilon_0 \int \mathbf{E} \cdot d\mathbf{S}$, where $\mathbf{E}$ is the electric field. The integral is over a surface, such as a cylinder, enclosing an amount of charge q. Over the sides of the cylinder, $\mathbf{E} \cdot d\mathbf{S} = 0$. The contributions from the ends of the cylinder give

$$\begin{aligned} q &= \epsilon_0(-E_{Cu}A + E_{Al}A) = \epsilon_0 A(E_{Al} - E_{Cu}) = \epsilon_0 A(\frac{\rho_{Al}I}{A} - \frac{\rho_{Cu}I}{A}) \qquad (1) \\ &= \epsilon_0 I(\rho_{Al} - \rho_{Cu}) = (8.85 \times 10^{-12})(1)(2.8 - 1.7) \times 10^{-8} \qquad (2) \\ &\approx 9.735 \times 10^{-20} \text{ C} \approx 0.608 \text{ e} , \qquad (3) \end{aligned}$$

where A is the area of the cylinder's ends, I is the current flowing through the wire, and E_{Cu} (E_{Al}) is the magnitude of the electric field in the copper (aluminum). The electric field in a wire with a cross sectional area A and resistivity ρ is $E = \frac{\rho I}{A}$. The amount of charge that accumulates at the boundary between the metals is less than e, the charge of an electron. Something is incorrect about the above calculation since under ordinary conditions, fractional charges are not observed in nature.

2. The electric potential between the two parallel metal plates is

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = xE + bE_\kappa + (d - b - x)E = (\frac{b}{\kappa} + d - b)E , \qquad (4)$$

where the integral is evaluated along a line between the plates, E is the magnitude of the electric field between the plates, and $E_\kappa = \frac{E}{\kappa}$ is the field inside the dielectric slab when it is within the capacitor. According to Gauss' law, $q = \epsilon_0 AE$ and $E = \frac{q}{\epsilon_0 A}$, where q is the charge on the plates. The capacitance with the dielectric slab present is

$$C_\kappa = \frac{q}{V} = \frac{q}{(\frac{b}{\kappa} + d - b)E} = \frac{\epsilon_0 A}{\frac{b}{\kappa} + d - b} . \qquad (5)$$

Set $b = 0$ to find $C = \frac{\epsilon_0 A}{d}$, the capacitance of the empty capacitor. If $C_\kappa = 2C$, then $\frac{\epsilon_0 A}{\frac{b}{\kappa} + d - b} = 2\frac{\epsilon_0 A}{d}$ implies $\kappa = \frac{2b}{2b - d}$. We must also have that $\kappa > 0$ and $b \leq d$. The restrictions on κ and b are $\kappa = \frac{2b}{2b-d}$ and $\frac{d}{2} < b \leq d$.

3(a) Ampere's Law is $\int \mathbf{B} \cdot d\mathbf{l} = \mu_0 IN$, where $\mathbf{B}$ is the magnetic field, μ_0 is the permeability, and IN is the total current contained in the N turns of wire. By symmetry, the field inside the toroid is $\mathbf{B} = B(r)\hat{\theta}$, where $B(r)$ is given by

$$\int \mathbf{B} \cdot d\mathbf{l} = \int B dl = B \int dl = B 2\pi r = \mu_0 IN \rightarrow B(r) = \frac{\mu_0 IN}{2\pi r} . \qquad (6)$$

The integral is performed along a circular path of radius r.
(b) The force on the electron is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \rightarrow F = qvB = \frac{mv^2}{R} \rightarrow RB = \frac{mv}{q} , \qquad (7)$$

where R is the radius of curvature of the electron's path and $B = B(r)$ from part (a). Since $\mathbf{F}$ and the direction of the electron's motion are always orthogonal, the work done on the electron by $\mathbf{B}$ is $\mathbf{F} \cdot \mathbf{x} = 0$. The speed of the electron v_0 does not change and $RB(r) = R\frac{\mu_0 IN}{2\pi r}$ remains constant. The radii of curvature at two different distances from the centre of the toroid are related via $\frac{R_1}{r_1} = \frac{R_2}{r_2} \rightarrow R_2 = \frac{r_2}{r_1}R_1$. This shows that if $r_2 < r_1$, then $R_2 < R_1$. The radius of curvature becomes smaller as r decreases. The electron executes a curved path as indicated in Figure 3.

4. Find the magnetic field due to the current in L via the Biot-Savart law

$$dB = \frac{\mu_0}{4\pi}\frac{I sin\theta dl}{r^2} . \tag{8}$$

By symmetry, only the z component of $\mathbf{B}$ is nonzero at points on the z-axis.

$$dB_z = dB cos\alpha = \frac{\mu_0}{4\pi}\frac{I sin(\frac{\pi}{2}) dl \frac{a}{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{I a dl}{r^3} \tag{9}$$

$$B_z = \int dB_z = \frac{\mu_0}{4\pi}\frac{Ia}{r^3}\int dl = \frac{\mu_0}{4\pi}\frac{Ia(2\pi a)}{(z^2+a^2)^{3/2}} = \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}} . \tag{10}$$

Assume that the above magnetic field is constant across the area of S. The magnetic flux Φ through S is

$$\Phi = \mathbf{B} \cdot \mathbf{A} = B\pi b^2 = \frac{\mu_0 \pi I a^2 b^2}{2(z^2+a^2)^{3/2}} . \tag{11}$$

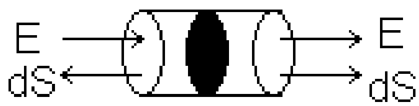
The electromotive force (voltage) in S is

$$\varepsilon(t) = -\frac{\partial \Phi}{\partial t} = -\frac{-3\mu_0 \pi I a^2 b^2 z}{2(z^2+a^2)^{5/2}}\frac{dz}{dt} = \frac{3\mu_0 \pi I a^2 b^2 z(t) v_z(t)}{2(z(t)^2+a^2)^{5/2}} . \tag{12}$$

Following its release, the small loop accelerates towards the large loop. The former's position and velocity are $z = z(t) = z_0 - \frac{1}{2}gt^2$ and $\mathbf{v} = v_z(t)\hat{z} = -gt\hat{z}$. The net force on the current in S due to L's magnetic field is zero. The time dependent current in S is $I_S(t) = \frac{\varepsilon(t)}{R}$ and it flows clockwise to produce a magnetic field that opposes the field of L.

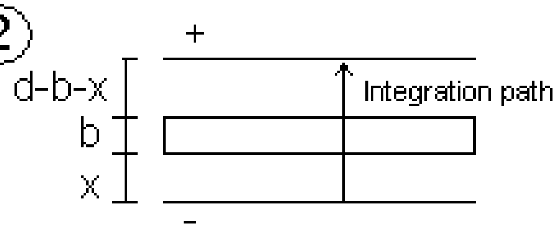
5. A motor converts electrical energy into mechanical energy. The design in Figure 5 consists of a wire loop L connected to a battery. The magnetic field due to the magnet M exerts a force on the current in L. If the current always flows in the same direction, L will rotate once or twice until it is horizontal with respect to the field. The net force on L is then zero. The loop may also twitch in a random way and some peculiar motions are possible. If the current flows one direction during a half rotation of L and changes direction during the next half rotation, then it is possible for a continuous net force to be applied to the current carrying wire. This depends on the nature of the contact between L and the wire holder. The loop will rotate in one direction until the battery is exhausted.

1



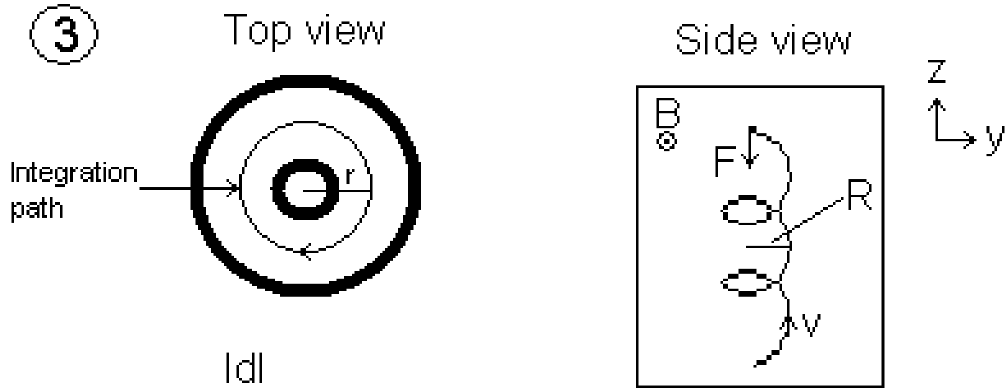
E
dS
E
dS

2



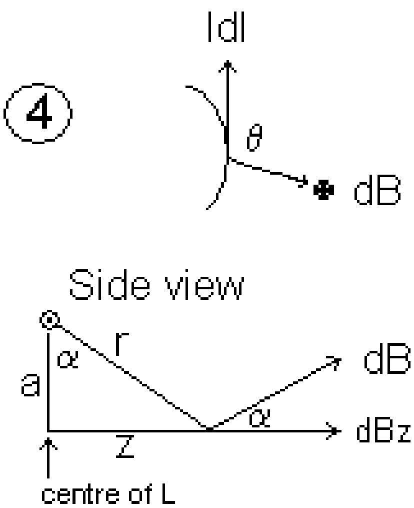
+
d-b-x
b
x
-
Integration path

3



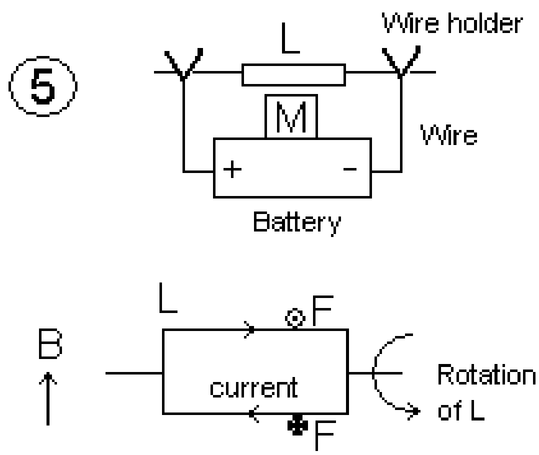
Top view
Integration path
r
Side view
B
F
R
v
z
y

4



Idl
θ
dB
Side view
a
α
r
dB
α
dBz
z
centre of L

5



L
Wire holder
M
Wire
+
-
Battery
L
F
B
current
F
Rotation of L

2002-2003 Physics Olympiad Preparation Program
- University of Toronto -
Problem Set 6: Circuits and Modern Physics
Due: 7 April 2003

1. A circuit consists of a battery of voltage V, resistor of resistance R, and a capacitor of capacitance C in series.
(a) What is the voltage across the capacitor as a function of time when the circuit is complete? A detailed mathematical solution is not required.
(b) If the capacitor is not ideal, then a spark can be produced when the voltage between the plates reaches V'. Assume that after the spike, the capacitor does not break and begins recharging. Draw a graph of the voltage across the capacitor as a function of time for $V' < V$. Show that it would result in a sawtooth voltage waveform.
(c) How can the peak voltage or the period of this sawtooth voltage waveform be changed?

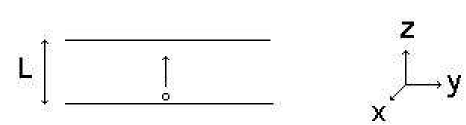
2. An electromagnetic wave in a vacuum has an electric field $\mathbf{E} = E_0 cos(\omega t - kz)\hat{x} + E_0 sin(\omega t - kz)\hat{y}$. What is direction of propagation and the momentum of the wave?

3. A light pulse travels with speed c along $\hat{z}$ between two parallel mirrors that are separated by a distance L. See Figure 1. How long does it take for the pulse to make a round trip according to an observer who is (a) at rest with respect to the mirrors and (b) travelling along $\hat{y}$ with a speed V relative to the mirrors? Is there a situation in which the pulse is seen to never return to its starting point?

4. The energy needed to break a system into its components is called its binding energy E_b. Calculate E_b for a hydrogen atom by (a) assuming that the electron is a classical particle orbiting a proton at a radius $r_0 = 5.29 \times 10^{-11}$ m and (b) using the data $m_e = 5.48579903 \times 10^{-4}$ u, $m_p = 1.007276470$ u, and $m_H = 1.007825035$ u (u=931.49432 MeV=1.66053873 $\times 10^{-27}$ kg). Are your computations in agreement?

5. Make a circuit consisting of a battery (V), some conducting wire and material, and a small light bulb (B). See Figure 2. Alter the resistance R of the circuit by introducing different lengths of conducting material between the points X and Y. How does R vary with the length of the path between X and Y? Determine the resistance according to the brightness of the bulb.

F1



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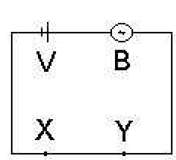

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1(a) At $t = 0$, the capacitor is not charged. The voltage across the capacitor, $V(t)$, is zero. For $t > 0$, charges build up on the capacitor's plates. One plate becomes positively and the other negatively charged; $V(t)$ increases, but it cannot do so indefinitely. Only a finite number of charges can be added or removed from the plates. The voltage smoothly approaches V, the voltage of the battery.
1(b) As in part (a), $V(t)$ starts at zero and increases with time. However, when $V(t) = V' < V$, $V(t)$ sharply drops to zero. This corresponds to the spark between the plates. The capacitor then begins recharging and $V(t)$ increases as before until $V(t) = V'$, after which $V(t)$ falls to zero again. The graph has the appearance of a sawtooth waveform.
1(c) The peak voltage or period can be changed by altering the resistance and/or capacitance. For example, R determines how quickly $V(t)$ approaches V'. If it takes a long time for $V(t)$ to reach V', the period of the sawtooth waveform will be longer. The same is true if the capacitor is nearly an ideal device with $V' \approx V$. If the capacitor is very poor, then $V' \ll V$ and the sparking happens more often. The period of the waveform will be smaller.

2(i) The periodic functions $cos(\omega t - kz)$ and $sin(\omega t - kz)$ advance in the positive z-direction. The electromagnetic wave propagates in the z-direction with a phase velocity $v_{phase} = \frac{\omega}{k} = c$, where c is the speed of light. (ii) Suppose the wave consists of quantized energy packets known as photons. According to quantum theory, each photon has an energy $\epsilon = pc$ and a momentum $p = \frac{\epsilon}{c}$. The electromagnetic wave therefore has a linear momentum $P_z = \frac{E}{c}$ since it propagates along $\hat{z}$. (The total energy of the wave is $E = IAT$, where I is the wave intensity, A is an area, and T is a time.) (iii) If the wave interacts with a particle that has a charge $q > 0$, mass m, and is confined to the xy plane ($z = 0$), then the particle's motion is described by $\mathbf{F} = m\mathbf{a} = q\mathbf{E}$ or

$$ma_x = qE_x \rightarrow m\frac{d^2x}{dt^2} = qE_0 cos(\omega t) \ , \quad (1)$$
$$ma_y = qE_y \rightarrow m\frac{d^2y}{dt^2} = qE_0 sin(\omega t) \ . \quad (2)$$

Verify by differentiation with respect to t that the following

$$x(t) = -\frac{qE_0}{m\omega^2}cos(\omega t) \ , \quad v_x(t) = \frac{qE_0}{m\omega}sin(\omega t) \ , \quad (3)$$
$$y(t) = -\frac{qE_0}{m\omega^2}sin(\omega t) \ , \quad v_y(t) = -\frac{qE_0}{m\omega}cos(\omega t) \ , \quad (4)$$

satisfy the above equations. The particle acquires an angular momentum

$$\mathbf{L} = \mathbf{r}(t) \times \mathbf{p}(t) = \frac{q^2E_0^2}{m\omega^3}\hat{z} \ , \quad (5)$$

about the z-axis, where $\mathbf{r}(t) = x(t)\hat{x} + y(t)\hat{y}$ and $\mathbf{p}(t) = m(v_x(t)\hat{x} + v_y(t)\hat{y})$ are the particle's position and momentum. The electromagnetic wave therefore has an angular momentum about the z-axis.

3(a) An observer who is at rest with respect to the mirrors will "see" the light pulse travel straight up and down. The time required for a round trip is $t_0 = \frac{L}{c} + \frac{L}{c} = \frac{2L}{c}$, where c is the speed of light.

3(b) An observer who is travelling along $\hat{y}$ at speed V relative to the mirrors will "see" the clock travel along $-\hat{y}$ and the light pulse move at an angle to the vertical. The speed of light c is the same for all observers. The time t required for a round trip is

$$2\sqrt{\left(\frac{Vt}{2}\right)^2 + L^2} = ct \rightarrow t = \frac{2L}{\sqrt{c^2 - V^2}} = \frac{\frac{2L}{c}}{\sqrt{1 - V^2/c^2}} = \frac{t_0}{\sqrt{1 - V^2/c^2}} \ . \tag{6}$$

As the observer's speed V approaches c, the time it takes for the light pulse to return to its origin approaches infinity. Since material objects such as an observer cannot move at the speed of light, the light pulse is always seen to return to its starting point, although the time t could be very large.

4(a) An electron (e) orbiting a proton (p) at a radius r_0 experiences electrostatic and centripetal forces. To find the kinetic energy K of the electron set

$$F_E = F_c \rightarrow \frac{1}{4\pi\epsilon_0}\frac{|q_e q_p|}{r_0^2} = m_e\frac{v^2}{r_0} \rightarrow K = \frac{1}{2}m_e v^2 = \frac{1}{8\pi\epsilon_0}\frac{|q_e q_p|}{r_0} \ . \tag{7}$$

The potential energy U and total energy E_{tot} of the electron are

$$U = -\int_{\infty}^{r_0} F_E dr = -\int_{\infty}^{r_0} \frac{1}{4\pi\epsilon_0}\frac{q_e q_p}{r^2}dr = -\frac{1}{4\pi\epsilon_0}q_e q_p\left(-\frac{1}{r}\right)_{\infty}^{r_0} = \frac{1}{4\pi\epsilon_0}\frac{q_e q_p}{r_0} \ , \tag{8}$$

$$E_{tot} = K + U = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2r_0} \ , \tag{9}$$

where $q_e = -e$ and $q_p = +e$. The binding energy is $E_b = -E_{tot} \approx 13.6166972$ eV where $e = 1.60217733 \times 10^{-19}$ C and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m.

4(b) The mass of the hydrogen atom is less than the sum of the electron and proton masses when they are measured separately. This mass difference is equivalent to the atom's binding energy according to Einstein's mass-energy relation,

$$E_b = -\Delta E = -\Delta mc^2 = -(m_H - m_e - m_p)c^2 = 13.87926537\text{eV} \ , \tag{10}$$

where $c = 2.99 \times 10^8$ m/s. The two calculations are in fairly good agreement. This is surprising since the electron in a hydrogen atom cannot be viewed as a classical particle orbiting the proton. The measurements of the masses and fundamental constants have to be very accurate in order for this comparison to be reasonable.

5. Try several measurements using different lengths of wire L between X and Y. Use the same type of wire in each case in order to keep the resistivity ρ and the cross section A of the material between X and Y constant. The light bulb shines less brightly as L is increased. Less current flows through the circuit for a given battery voltage V. The resistance R of the wire increases with L. If a voltage V is applied to a conducting object, then a current I and a current density $j = \frac{I}{A}$ are generated. The resistivity of the object is defined to be

$$\rho \equiv \frac{E}{j} = \frac{V/L}{I/A} = \frac{V/I}{L/A} = \frac{R}{L/A} \rightarrow R = \frac{\rho L}{A} \ , \tag{11}$$

where $E = \frac{V}{L}$ is the applied electric field and R is the resistance of the object. This demonstrates the relationship between R and L, if the currents and potentials are not too large.

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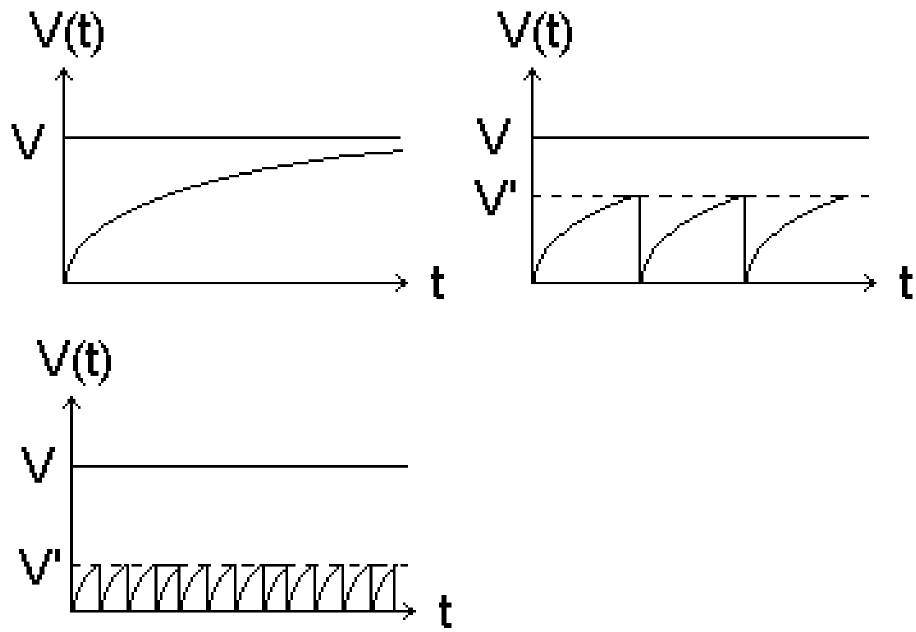

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