A Constructive Quantum Lovász Local Lemma for Commuting Projectors

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Abstract

The Quantum Satisfiability problem generalizes the Boolean satisfiability problem to the quantum setting by replacing classical clauses with local projectors. The Quantum Lovász Local Lemma gives a sufficient condition for a Quantum Satisfiability problem to be satisfiable [AKS12], by generalizing the classical Lovász Local Lemma.

The next natural question that arises is: can a satisfying quantum state be *efficiently* found, when these conditions hold? In this work we present such an algorithm, with the additional requirement that all the projectors commute. The proof follows the information theoretic proof given by Moser's breakthrough result in the classical setting [Mos09].

Similar results were independently published in [CS11, CSV13].

1 Introduction and main results

Given a set of independent events, $B_1, \ldots, B_m$ such that $\Pr(B_i) < 1$, the probability that none of the events happen is strictly positive:

$$\Pr(\bigwedge_{i=1}^{m} \overline{B_i}) = \prod_{i=1}^{m}(1 - \Pr(B_i)) > 0.$$

What if the events are not independent? The Lovász Local Lemma (LLL) provides a definition of "weakly dependent" events, and is an important tool to guarantee that the probability that none of the events happens is strictly positive.

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Theorem 1.1 ([EL75], see also Ref. [AS04, Chapter 5]) *Let $B_1, B_2, \ldots, B_m$ be events with $Pr(B_i) \leq p$ and such that each event is mutually independent of all but $g - 1$ events. If $p \cdot e \cdot g \leq 1$ then $Pr(\bigwedge_{i=1}^{m} \overline{B_i}) > 0$.*

One of the many uses of the LLL is providing a sufficient condition for the satisfiability of a k-SAT formula.

Corollary 1.2 ([KST93]) *Let Φ be a k-SAT formula in CNF-form in which each clause shares variables with at most $g - 1$ other clauses. Then if $g \leq \frac{2^k}{e}$, Φ is satisfiable.*

The corollary follows from Thm. 1.1 by letting B_i be the event that the i-th clause is not satisfied for a random assignment, which happens with probability $p = 2^{-k}$, and noting that by definition, every such event is independent of all but at most $g - 1$ other events.

In Ref. [AKS12], a quantum version of Theorem 1.1 was proved, known as the *Quantum Lovász Local Lemma (QLLL)*. In the quantum settings, one replaces the probability space by the quantum Hilbert space, events by subspaces and their probabilities by the "relative dimension", which is the ratio of the subspace dimension to the total dimension of the Hilbert space. The formal statement of this theorem, as well as the exact definitions, are not relevant for this work, and therefore will be omitted here; these details can be found in Ref. [AKS12].

The QLLL also has a simple corollary, which is equivalent to Corollary 1.2. To state it, we first need to define the The *quantum satisfiability problem*, k-QSAT, which is the quantum analog of the classical k-SAT problem. Introduced by Bravyi in Ref. [Bra06], k-QSAT is the following promise problem. We are given a set of projectors $\{\Pi_1, \ldots, \Pi_m\}$ that are defined on the Hilbert space of n qubits. Each projector Π_i acts non-trivially on at most k qubits, which means that it can be written as $\Pi_i = \hat{\Pi}_i \otimes \mathbb{1}_{n-k}$, where $\hat{\Pi}_i$ is a projection defined in the Hilbert space of k qubits, and $\mathbb{1}_{n-k}$ is the identity operator on the rest of the qubits. In addition, we are given a parameter $\epsilon = \Omega(1/\text{poly}(n))$, as well as a promise that either there exists a quantum state $|\psi\rangle$ in the intersection of the null spaces of all the projectors (in which case we say that the instance is *satisfiable*), or that every state $|\psi\rangle$ satisfies $\sum_{i=1}^{m} \langle \psi | \Pi_i | \psi \rangle \geq \epsilon$. Our goal is to decide which of these possibilities holds. We note that just as in the classical case, where k-SAT is NP-complete for $k \geq 3$, k-QSAT is *quantum* NP-complete for $k \geq 3$ [GN13].[1]

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[1] More precisely, it is complete for the class QMA_1, which is the usual quantum NP class QMA, but with a one-sided error (see Ref. [Bra06] for details).

Next, we say that the *neighborhood* of a projector Π_i, denoted $\Gamma^+(\Pi_i)$, is the set of projectors that act nontrivially on at least one of the qubits on which Π_i acts nontrivially (note that $\Pi_i \in \Gamma^+(\Pi_i)$). We say that the problem is a (k-local, rank-r, g-neighborhood)-QSAT if it is a k-QSAT problem with the additional properties that each projector is of rank at most r, and $|\Gamma^+(\Pi_i)| \leq g$ for every projector Π_i. In terms of these definitions, we have the following corollary of the QLLL:

Corollary 1.3 ([AKS12]) *Let* $\{\Pi_1, \ldots, \Pi_m\}$ *be a (k-local, rank-r, g-neighborhood)-* QSAT *instance. If* $g \leq \frac{2^k}{re}$, *the instance is satisfiable.*

A rank-1 k-QSAT instance is similar to a k-SAT instance in the following sense: in a k-CNF formula, each clause excludes one out of 2^k configurations of the relevant variables; in a rank-1 k-QSAT, each projector excludes one dimension out of the 2^k relevant dimensions. The parameters of Corollary 1.3 and Corollary 1.2 coincide in the case of rank-1 instances. We keep the generality of rank-r instances for reasons that are related to commuting instances, to be clarified shortly.

Corollary 1.2 guarantees the satisfiability of a SAT instance under a certain condition. How difficult is it to find a satisfying assignment under this condition, in terms of computational resources? Efficient constructive versions of the LLL started with the work of Beck [Bec91], which provided an algorithm that worked under stronger conditions than the LLL. The results were improved by others, and culminated in the work by Moser and Tardos [MT10](see the references therein for the complete line of research), which provided an efficient algorithm under the same conditions as in the LLL.

Our main result is Algorithm 1, an efficient quantum algorithm for a *commuting* k-QSAT instance, which satisfies the conditions of the QLLL, i.e., Corollary 1.3. The COMMUTING QSAT problem adds the following requirement to QSAT: $[\Pi_i, \Pi_j] = 0$ for all i, j. The COMMUTING QSAT (and more generally, the COMMUTING LOCAL HAMILTONIAN problem) is an intermediate problem between the classical and quantum regime. On one hand, all the terms can be diagonalized simultaneously (like the classical setting); on the other hand, this basis may be highly entangled, and possesses rich non-classical phenomena (for example, Kitaev's toric code [KSV02]). Various results concerning the complexity of this problem appeared in [BV05, AE11, Sch11]. Our proof for the correctness of the algorithm is based on an information theoretic argument, which is similar to the one used in Moser's information theoretic proof (see [For09]).

Theorem 1.4 (Main result) *Let* $\{\Pi_1, \ldots, \Pi_m\}$ *be a (k-local, rank r, g-neighborhood)-*QSAT *instance with* $g < \frac{2^k}{re}$. *Then for every* $\delta > 0$, *Algorithm 1 returns "Success" with probability* $\geq 1 - \delta$ *and a running time of* $m \cdot \tilde{\mathcal{O}}(\eta)$, *where* $\eta \overset{\text{def}}{=} \frac{1}{\delta[k - \log(ger)]}$.
  If $\forall i, j \leq m \ [\Pi_i, \Pi_j] = 0$ *then the output is a satisfying state.*

Algorithm 1 Commuting QLLL solver

Initialize:

 1: For $\eta \overset{\text{def}}{=} \frac{1}{\delta[k - \log(ger)]}$, fixed integer

$$T \overset{\text{def}}{=} \lceil 4m\eta \cdot \log(\eta + 2) \rceil \ . \tag{1}$$

 2: System register: n qubits, prepared in a fully mixed state.
 3: Stock register: $N = Tk$ qubits in fully mixed state.

Algorithm:

 1: $t \leftarrow 0$
 2: **for** $i \leftarrow 1$ to m **do**
 3:     Fix(Π_i)
 4: **end for**
 5: return "Success"

 6: **procedure** Fix(Π)
 7:     measure $\{\Pi, \mathbb{1} - \Pi\}$; in the system register
 8:     **if** result $= \Pi$ (projector violated, energy $= 1$) **then**
 9:         $t \leftarrow t + 1$
10:         **if** $t = T$ **then**
11:             abort and return "Failure"
12:         **end if**
13:         Replace the k measured qubits with k maximally mixed qubits from the stock register
14:             **for all** $\Pi_j \in \Gamma^+(\Pi)$ **do**
15:                 Fix(Π_j)
16:             **end for**
17:     **end if**
18: **end procedure**

Very similar variants of the main result were discovered independently. A talk that described the different approaches was given in Ref. [CSV$^+$12]. The main open question emerging from these works is how to find a constructive version for the QLLL in the non-commuting case. We hope that at least one of these approaches turns out to be useful for proving the non-commuting case. We now compare the differences between these approaches. We believe that the main (and only) advantage of our approach is its simplicity. The result given in Ref. [CS11] holds in a more general setting which is called the *asymmetric* QLLL, whereas our version only holds in the so-called *symmetric* QLLL. The asymmetric LLL (see, e.g. Ref. [AS04, Lemma 5.1.1]) is useful when there are differences between the upper bounds on the probabilities of the events. Furthermore, in the non-commuting case, the termination of the algorithm in Ref. [CS11] implies that the state has low energy (so, the remaining task to prove the non-commuting version is to prove a fast termination). On the other hand, our algorithm terminates also in the non-commuting case, and the missing part is proving that the state has low-energy. These results are complementary in that sense. The approach in Ref. [CSV13] has a much better trade-off between the running time and the probability of success.

Before proving Theorem 1.4, let us briefly explain why we need to work in the general setting with rank-r projectors, and not only with rank-1. It is easy to verify that a (k-local, rank r)-QSAT is equivalent to a (k-local, rank 1)-QSAT by replacing each rank-r projector with r different rank-1 projectors. This is because a rank-r projector can be written as the sum of r rank-1 projectors. Unfortunately, this transformation breaks the commutativity property for COMMUTING QSAT; for example, it is possible that $[A, B] = 0$ for some projectors A and B, but for the decomposition to rank-1 projectors $A = \sum_i A_i$, $B = \sum_j B_j$, $[A_i, B_j] \neq 0$ for some i and j.

2   Analyzing the algorithm – proof of Theorem 1.4

In the commuting case, if the algorithm succeeds (i.e., returns "Success"), it produces a satisfying state ρ: the set of satisfied projectors monotonically increases when a FIX call returns, and furthermore, after FIX(Π_i) returns, $\mathrm{Tr}(\Pi_i \rho) = 0$. Because FIX is called for every projector (line 3), if the algorithm succeeds, the resulting state on the system register satisfies all the projectors.

Therefore, in order to prove Theorem 1.4, we need to show that the success probability is at least $1 - \delta$. We analyze the success probability of

the algorithm by deriving an inequality that relates the initial entropy of the system to that of the different possible branches in the running history of the algorithm.

Initially, the two registers are completely mixed, so the system is in the state $\rho_{init} = 2^{-(n+N)}\mathbb{1}_{n+N}$, and

$$S(\rho_{init}) = n + N \ .$$

To analyze the final state of the system, we note that all possible sequences of measurements form a history tree. We characterize each history branch by a binary string that records the result of the measurements in the branch. 0 means a "success" — a projection into the zero energy subspace of the projector, and 1 means a "failure". We denote such string by $\bar{s} = (s_1, s_2, s_3, \ldots)$, where s_i denotes the outcome of the ith measurement, and let $|\bar{s}|$ denote the Hamming weight of $\bar{s}$, which is exactly the number of failures in a particular branch. Denote by $\rho_{\bar{s}}$ the normalized resulting state of such a series of projective measurements, and $p_{\bar{s}}$ the probability of that branch to occur.

The heart of the analysis is the following simple claim, that upper-bounds the entropy of the initial state by an expression involving $\{p_{\bar{s}}\}$ and $\{\rho_{\bar{s}}\}$.

Claim 2.1 *Consider an (adaptive) quantum algorithm that applies a series of projective measurements to an initial state ρ_{init}, and let O denote the set of all possible measurements outcomes. For each $s \in O$, let p_s denote the probability with which outcome s occurs, and ρ_s the resultant final state of the system. Then*

$$S(\rho_{init}) \leq H(\{p_s\}) + \sum_{s \in O} p_s S(\rho_s) \ ,$$

where $H(\cdot)$ is the Shanon entropy of classical probability distributions, and $S(\cdot)$ is the von Neumann entropy.

Proof: By adding a sufficient amount of ancilla qubits to the system, initialized in the pure state $|0\rangle$, we can use standard techniques from the theory of quantum computation, and move all intermediate measurements to the end of the algorithm, where they will be performed on the ancilla qubits. Specifically, if L is the number of ancilla qubits we add, then we can assume without loss of generality that we start with the state

$$\rho'_{init} \stackrel{\text{def}}{=} \rho_{init} \otimes |0^{\otimes L}\rangle\langle 0^{\otimes L}| \ , \tag{2}$$

apply to it some unitary circuit U, obtain the state

$$\rho_1' \stackrel{\text{def}}{=} U^\dagger \rho_{init}' U \;, \tag{3}$$

and in the end perform a projective measurement on the ancilla qubits. The measurement is defined by the set of orthogonal projectors $\{P_s\}_{s \in O}$, such that $\sum_{s \in O} P_s = \mathbb{1}$, and we are guaranteed that

$$p_s = \text{Tr}\big(P_s \rho_1'\big) \;,$$
$$\rho_s = \frac{1}{p_s} \text{Tr}_{ancilla}\big(P_s \rho_1' P_s\big) \;.$$

Let us now define the state $\rho_{final}' \stackrel{\text{def}}{=} \sum_{s \in O} P_s \rho_1' P_s$, which is the state of the system after we measured the ancillary qubits, and let us define $\rho_s' \stackrel{\text{def}}{=} \frac{1}{p_s} P_s \rho_1' P_s$. Clearly, $\rho_{final}' = \sum_s p_s \rho_s'$, and ρ_s' have orthogonal supports. We now apply the following two elementary results from the theory of quantum information:

1. A projective measurement can only increase the von Neumann entropy (see Nielsen & Chuang Theorem 11.9):

$$S(\rho_1') \leq S(\rho_{final}') \;.$$

2. The entropy of a sum of mixed states with orthogonal support (see Nielsen & Chuang Theorem 11.8(4)):

$$S(\sum_s p_s \rho_s') = H(\{p_s\}) + \sum_s p_s S(\rho_s') \;.$$

Combining these two results, together with the easy observation that $S(\rho_{init}) = S(\rho_{init}') = S(\rho_1')$, we conclude that $S(\rho_{init}) \leq H(\{p_s\}) + \sum_s p_s S(\rho_s')$. Finally, $\rho_s' = \rho_s \otimes |s\rangle\langle s|$. Therefore $S(\rho_s') = S(\rho_s)$, and this complete the proof.

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We now return to the analysis of the algorithm. Applying Claim 2.1, we get

$$n + N = S(\rho_{init}) \leq H\big(\{p_{\bar{s}}\}\big) + \sum_{\bar{s}} p_{\bar{s}} S(\rho_{\bar{s}}) \;. \tag{4}$$

We define p_t as the probability that Algorithm 1 ended with exactly t failures:

$$p_t \stackrel{\text{def}}{=} \sum_{|\bar{s}|=t} p_{\bar{s}}.$$

Let us upper bound the RHS of Eq. (4) in terms of the probabilities $\{p_t\}$.

1. $H(\{p_{\bar{s}}\})$:

By definition, $H(\{p_{\bar{s}}\}) = H(\{p_t\}) + \sum_{t=0}^{T} p_t H\big(\{p_{\bar{s}}\}\big|\,|\bar{s}| = t\big)$. Since t can take values between 0 and T then trivially $H(\{p_t\}) \leq \log T$.

To upper-bound $H\big(\{p_{\bar{s}}\}\big|\,|\bar{s}| = t\big)$, we count the number of strings with exactly t "1". Every such string corresponds to a branch with exactly t failures. The total length of the string is the total number of measurements, which is the total number of calls to FIX. This is at most $m + gt$ because we had at most m external calls (we had exactly m such calls when $t < T$, but for $t = T$ we could have had less) and every failure triggers at most g recursive calls. Therefore,

$$H\big(\{\hat{p}_{\bar{s}}\}\big|\,|\bar{s}| = t\big) \leq \log \binom{m + gt}{t}$$

$$\leq m + \log \binom{gt}{t}$$

$$\leq m + \log \left(\frac{egt}{t}\right)^t = m + t\log(ge) \ .$$

The second inequality is valid for $g \geq 2, t \geq 1$ and $m \geq 0$, and is proved in Appendix A.1. The second inequality follows from the standard bound $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$. All together,

$$H(\{p_{\bar{s}}\}) \leq \log T + \sum_{t=0}^{T} p_t \cdot \big(m + t\log(ge)\big) \ .$$

$\sum_{\bar{s}} p_{\bar{s}} S(\rho_{\bar{s}})$:

We write, $\sum_{\bar{s}} p_{\bar{s}} S(\rho_{\bar{s}}) = \sum_{t=0}^{T} \sum_{\bar{s}, |\bar{s}|=t} p_{\bar{s}} S(\rho_{\bar{s}})$. Each branch in the internal sum has exactly t failures, and each such failure is a measurement which puts k qubits in an r-dimensional subspace (because the Π_i are rank-r projectors). The entropy of these qubits is therefore at most $k - \log r$, and so for such branch, $S(\rho_{\bar{s}}) \leq N + n - t(k - \log r)$.

Therefore,

$$\sum_{\bar{s}} p_{\bar{s}} S(\rho_{\bar{s}}) \leq \sum_{t=0}^{T} \sum_{\bar{s}, |\bar{s}|=t} p_{\bar{s}} \big(N + n - t(k - \log r)\big)$$

$$= N + n - (k - \log r) \sum_{t=0}^{T} p_t \cdot t \ .$$

Plugging these bounds into Eq. (4), we get

$$n + N \leq \log T + N + n + m - [k - \log(ger)] \sum_{t=0}^{T} p_t \cdot t \ ,$$

which implies $[k - \log(ger)] \sum_{t=0}^{T} p_t \cdot t \leq \log T + m$. By assumption, $k - \log(ger) > 0$, and therefore,

$$\Pr(\text{failure}) = p_T \leq \frac{\log T + m}{T[k - \log(ger)]} \ . \tag{5}$$

Recalling that $T \stackrel{\text{def}}{=} \lceil 4m\eta \cdot \log(\eta + 2) \rceil$, it is now a simple algebra to show that when $m \geq 2$, the RHS of the above equation is upper bounded by δ. See Appendix A.2 for a full proof. This finishes the proof of Theorem 1.4.

Acknowledgments

A Proofs

A.1 An upper bound on $\binom{m+gt}{t}$

Let $g \geq 2, t \geq 1$ be some integers. We would like to show that for every $m \geq 0$,

$$\binom{m+gt}{t} \leq 2^m \binom{gt}{t} . \tag{6}$$

The proof is by induction. For $m = 0$, the inequality in (6) is trivially satisfied. Assume then that it is true for $m > 0$, and let us prove its validity for $m + 1$. Writing $\binom{m+gt}{t} = \frac{(M+gt)!}{(m+gt-t)!t!}$, one can easily verify that

$$\binom{m+gt}{t} = \frac{m+gt}{m+gt-t} \binom{m-1+gt}{t} .$$

Therefore, by the induction assumption, $\binom{m+gt}{t} \leq \frac{m+gt}{m+gt-t} 2^{m-1}$. Finally, using the fact that $g \geq 2, t \geq 1$ and $m \geq 0$, it is easy to see that $\frac{m+gt}{m+gt-t} \leq 2$, and therefore $\binom{m+gt}{t} \leq 2^m$.

A.2 Bounding the RHS of Eq. (5)

To show that the RHS of Eq. (5) is upperbounded by δ, we need to show that

$$\frac{\log T + m}{T} \leq \delta[k - \log(ger)] \stackrel{\text{def}}{=} \frac{1}{\eta} . \tag{7}$$

Since the LHS of the above inequality is a decreasing function of T for every $T \geq 1$ and $m \geq 2$, we can safely drop the $\lceil \cdot \rceil$ from the definition of T, and prove the inequality for $T = 4m\eta \cdot \log(\eta + 2)$. Substituting this in Eq. (7), we obtain the following equivalent inequality

$$\frac{\log(4m) + \log \eta + \log \log(\eta + 2) + m}{m \log(\eta + 2)} \leq 4 .$$

It is now straight forward to verify that since $\eta > 0$ then as long as $m \geq 2$,

$$\frac{\log(4m)}{m} \leq 2 , \qquad \frac{\log \eta + \log \log(\eta + 2)}{m \log(\eta + 2)} \leq 1 , \qquad \frac{m}{m \log(\eta + 2)} \leq 1 ,$$

and this finishes the proof.

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