Quantum Algorithms for Weighing Matrices and Quadratic Residues

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Abstract

In this article we investigate how we can employ the structure of combinatorial objects like Hadamard matrices and weighing matrices to device new quantum algorithms. We show how the properties of a weighing matrix can be used to construct a problem for which the quantum query complexity is significantly lower than the classical one. It is pointed out that this scheme captures both Bernstein & Vazirani's inner-product protocol, as well as Grover's search algorithm.

In the second part of the article we consider Paley's construction of Hadamard matrices, which relies on the properties of quadratic characters over finite fields. We design a query problem that uses the Legendre symbol χ (which indicates if an element of a finite field \mathbb{F}_q is a quadratic residue or not). It is shown how for a shifted Legendre function $f_s(i) = \chi(i+s)$, the unknown $s \in \mathbb{F}_q$ can be obtained exactly with only two quantum calls to f_s . This is in sharp contrast with the observation that any classical, probabilistic procedure requires more than $\log q + \log(\frac{1-\varepsilon}{2})$ queries to solve the same problem.

1 Introduction

The theory of quantum computation investigates how we can use quantum mechanical effects to solve computational problems more efficiently than we can by classical means. So far, the strongest evidence that there is indeed a real and significant difference between quantum and classical computation is provided by Peter Shor's polynomial-time factoring algorithm[31]. Most other quantum complexity results are expressed in the black-box, or oracle, setting of computation with various degrees of separation between the two models. The algorithms of—for example—Deutsch[13], Deutsch & Jozsa[14], Berthiaume & Brassard[6], Bernstein & Vazirani[4], Simon[26], Grover[15], and Buhrman & van Dam[8]

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define problems for which we have a quantum reduction in the query complexity, whereas the lower bounds of Jozsa[20], Bennett et al.[2], and Beals et al.[1] show that there are limits to the advantage that quantum computation can give us. The general picture that has emerged from these results is that we can only expect a superpolynomial difference between classical and quantum computation if we can use the specific structure of the problem that we try to solve. The promise on the function of Simon's problem is a typical example of such a structure that establishes an exponential quantum improvement over the classical complexity.[26] It was this same improvement that inspired Shor for his result.

In this article we introduce a general family of structured problems for which we prove a better-than-classical query complexity. These results rely heavily on the properties of weighing matrices (as defined in combinatorics) and encompass the earlier query protocols by Grover[15] and Bernstein & Vazirani[4]. Following Paley's construction of Hadamard matrices, we also define a more specific problem that concerns the determination of a shifted Legendre sequence over finite fields.

In the next section we start with a brief overview of the essential ingredients of quantum computation and some of the relevant complexity results for the black-box model. Section 3 then explains how the theory of weighing matrices can be used as a source for non-trivial, yet structured, unitary operations. In the last part of the article (Section 5) our attention will focus on Raymond Paley's construction of Hadamard matrices and the theory of quadratic residues for finite fields that it uses. This will lead to the definition of a query problem which is akin to the inner-product problem of Bernstein & Vazirani[4].

2 Quantum Computation

We assume the reader to be familiar with the theory of quantum computation. (Otherwise, see the standard references by Berthiaume[5], Nielsen and Chuang[23], or Preskill[25].) Here we will mainly fix the terminology and notation for the rest of the article.

2.1 Quantum Information Processing

A system ψ of n quantum bits (qubits) is a superposition of all possible n-bit strings. It can therefore be represented as a normalized vector (or "ket") $|\psi\rangle$ in a 2^n -dimensional Hilbert space:

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle,$$

with $\alpha_x \in \mathbb{C}$ and the normalization restriction $\sum_x |\alpha_x|^2 = 1$. The probability of observing the outcome "x" when measuring the state ψ equals $|\alpha_x|^2$. More general, when we try to determine if ψ equals the measurement vector $|m\rangle =$

 $\sum_{x} \beta_{x} |x\rangle$, we will get an affirmative answer with probability

$$\operatorname{Prob}(m|\psi) := |\langle m|\psi\rangle|^2 = \left|\sum_{x\in\{0,1\}^n} \bar{\beta}_x \alpha_x\right|^2$$

(with $\bar{\beta}$ the complex conjugate of β). An orthogonal measurement basis for an N-dimensional Hilbert space \mathcal{H}_N is a set $\{m_1, m_2, \ldots, m_N\}$ of mutually orthogonal state vectors $|m_i\rangle$. For such a basis it holds that $\sum_{i=1}^N \operatorname{Prob}(m_i|\psi) = 1$, for every state $|\psi\rangle \in \mathcal{H}_N$, and that if $\psi = m_s$ for a certain s, then $\operatorname{Prob}(m_i|\psi) = 1$ if i = s, and $\operatorname{Prob}(m_i|\psi) = 0$ otherwise.

The quantum mechanical time evolution of a system ψ is a linear transformation that preserves the normalization restriction. Hence, for a finite-dimensional state space \mathcal{H}_N , such a transformation can be represented by a unitary matrix $M \in \mathsf{U}(N)$, for which we can write $M|\psi\rangle = \sum_{x=1}^N \alpha_x M|x\rangle$. An example of a one-qubit transformation is the 'Hadamard transform', which is represented by the unitary matrix

$$\mathbf{H} \ := \ \frac{1}{\sqrt{2}} \left(\begin{array}{cc} +1 & +1 \\ +1 & -1 \end{array} \right).$$

On the standard zero/one basis for a bit this transformation has the following effect: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

2.2 Quantum versus Classical Query Complexity

Consider a problem that is defined in terms of n (unknown) values $f(1), \ldots, f(n)$. The (probabilistic) query complexity of such a problem is the minimum number of times that an algorithm has to 'consult' the string $f(1), \ldots, f(n)$ to solve the problem (with high probability). A typical example of this setting is the calculation of the \mathbb{OR} of n bit values: the question whether there is an index i with f(i) = 1. The classical probabilistic, query complexity of this task is $\Omega(n)$, whereas in the quantum setting we only need $O(\sqrt{n})$ calls to f to solve the problem with high probability. We therefore say that we have a 'quadratic' separation between the classical and the quantum query complexity of the \mathbb{OR} function. The question which tasks allow a quantum reduction in the query complexity (and if so, how much) is a central one in quantum complexity research.

2.3 Some Earlier Results in Quantum Computing

In this article we are especially concerned with the query complexity of procedures that prepare a state that depends on the values of a black-box function. For example, how often do we have to read out the bit values of a function $f: \mathbb{N} \to \mathbb{C}$ if we want to create the state $\sum_i f(i)|i\rangle$? The following lemma shows us that if the range of the function is limited to $\{-1, +1\}$, this can be done with a single query.

Fact 1 (Phase-kick-back trick) Given a function $f: \mathbb{N} \to \{-1, +1\}$, the phase changing transition $\sum_i \alpha_i |i\rangle \to \sum_i f(i)\alpha_i |i\rangle$ can be established with only one call to the unknown binary values of f.

Proof: (See [9] for the original proof.) First, attach to the superposition of $\sum_{i} \alpha_{i} |i\rangle$ the (two qubit state)

$$|\varphi\rangle \quad := \quad \tfrac{1}{2}(|0\rangle + \sqrt{-1}|1\rangle - |2\rangle - \sqrt{-1}|3\rangle).$$

Then, in superposition, add modulo 4 the values f(i) to this register (step a). Finally, apply a general phase change $|\Psi\rangle \to \sqrt{-1}|\Psi\rangle$ (step b). It is straightforward to see that this yields the desired phase change according to the equation:

$$|i\rangle \otimes |\varphi\rangle \longrightarrow_a \begin{cases} -\sqrt{-1}|i\rangle \otimes |\varphi\rangle & \text{if } f(i) = +1\\ \sqrt{-1}|i\rangle \otimes |\varphi\rangle & \text{if } f(i) = -1\\ \longrightarrow_b f(i)|i\rangle \otimes |\varphi\rangle, \end{cases}$$

for every i in the superposition.

The usefulness of such a phase-changing operation is made clear by the following result, which is mentioned because the Theorems 1 and 2 of this article are of a similar fashion. In 1993 Bernstein & Vazirani gave the following example of a family of functions g_1, g_2, \ldots that are more easily distinguished with quantum queries to g than with classical ones.

Fact 2 (Inner-Product Problem) Let the black-box function $g_s: \{0,1\}^n \to \{0,1\}$ be defined by $g_s(x) = (x,s) := \sum_{i=1}^n s_i x_i \mod 2$, where $s = (s_1, \ldots, s_n) \in \{0,1\}^n$ is an unknown n-bit mask. A quantum computer can determine the value s exactly with one call to the function g_s , whereas any probabilistic, classical algorithm needs at least $n + \log(1 - \varepsilon)$ queries to g_s to perform the same task with an error rate of at most ε .

Proof: See [4] for the original proof by Bernstein & Vazirani, and [9] for the single query version of it. \Box

The above result uses the unitarity of $\mathbb{H}^{\otimes n}$ and its connection with the inner-product function. In Section 5 of this article we will do a similar thing for a different family of unitary matrices and the Legendre function that it uses.

Another key result in quantum computation is the square-root speed-up that one can obtain when querying a database for a specific element.

Fact 3 (Grover's search algorithm) Let the function values $f(1), \ldots, f(n)$ form a string of n-k zeros and k ones. Knowing k, but not the specific entries s for which f(s) = 1, the amplitude changing evolution

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle \longrightarrow \frac{1}{\sqrt{k}} \sum_{i=1}^{n} f(i)|i\rangle$$

can be established exactly with $\left\lceil \frac{\pi}{4} \sqrt{\frac{n}{k}} \right\rceil$ quantum queries to the function f.

Proof: See the original article by Lov Grover[15], or better yet, the excellent analysis of it by Boyer $et\ al.$ [7]

3 Hadamard Matrices and Weighing Matrices in Combinatorics

The matrix H that we mentioned in the previous section is—in the context of quantum computation—called the 'Hadamard matrix'. This terminology is a bit unfortunate because the same term has already been used in combinatorics to cover a much broader concept. (See the 1893 article by Jacques Hadamard[16] for the origin of this term.)

Definition 1 (Hadamard matrix in combinatorics) In combinatorics, a matrix $M \in \{-1, +1\}^{n \times n}$ is called a Hadamard matrix if and only if $M \cdot M^T = n \cdot I_n$, where "T" denotes the transpose of a matrix.

Obviously, when M is a Hadamard matrix in the above sense, then $\frac{1}{\sqrt{n}}M$ is a unitary matrix $\in U(n)$. Also, if M_1 and M_2 are Hadamard matrices, then their tensor product $M_1 \otimes M_2$ is a Hadamard matrix as well. It is a famous open problem if there exists a Hadamard matrix for every dimension 4k.

The $\mathbb{H}^{\otimes n}$ matrices that we encountered in the section on quantum computation form only a small subset of all the Hadamard matrices that we know in combinatorics. Instead, the matrices $\sqrt{2^n} \cdot \mathbb{H}^{\otimes n}$ should perhaps be called "Hadamard matrices of the Sylvester kind" after the author who first discussed this specific family of matrices.[32]

The properties of Hadamard matrices (especially the above mentioned 4k-conjecture) is an intensively studied topic in combinatorics, and its complexity is impressive given the simple definition.[11, 17, 27, 28, 29] In 1933, Raymond Paley proved the existence of two families of Hadamard matrices that are very different from Sylvester's 2^n -construction.

Fact 4 (Paley construction I and II) Construction I: For every prime p with $p = 3 \mod 4$ and every integer k, there exists a Hadamard matrix of dimension $(p^k+1)\times(p^k+1)$. Construction II: For every prime p with $p=1 \mod 4$ and every integer k, there exists a Hadamard matrix of dimension $(2p^k+2)\times(2p^k+2)$.

Proof: See the original article [24], or any other standard text on combinatorial objects[11, 22, 29].

For here it sufficient to say that Paley's construction uses the theory of quadratic residues of finite fields \mathbb{F}_{p^k} . Its properties that are relevant for this article are discussed in the appendix.

We can extend the notion of Hadamard matrices by allowing three possible matrix entries $\{-1, +1, 0\}$, while still requiring the $M \cdot M^T \propto I_n$ restriction. We thus reach the following definition.

Definition 2 (Weighing matrix [11, 28]) In combinatorics, a matrix $M \in \{-1, 0, +1\}^{n \times n}$ is called a weighing matrix if and only if $M \cdot M^T = k \cdot I_n$ for some $0 \le k \le n$. We will denote the set of such matrices by W(n, k).

Every column and row of a W(n, k) weighing matrix has n-k zeros, and k entries "+1" or "-1". Clearly, W(n, n) are the Hadamard matrices again, whereas W(n, n-1) are also called *conference matrices*. The identity matrix I_n is an example of a W(n, 1) matrix. If $M_1 \in W(n_1, k_1)$ and $M_2 \in W(n_2, k_2)$, then their tensor product $M_1 \otimes M_2$ is an element of $W(n_1 n_2, k_1 k_2)$. This implies that for every weighing matrix $M \in W(n, k)$ we have in fact a whole family of matrices $M^{\otimes t} \in W(n^t, k^t)$, indexed by $t \in \mathbb{N}$.

Example 1

$$\begin{pmatrix} +1 & +1 & +1 & 0 \\ +1 & -1 & 0 & +1 \\ +1 & 0 & -1 & -1 \\ 0 & +1 & -1 & +1 \end{pmatrix}^{\otimes t}$$
 is a W(4^t, 3^t) weighing matrix for every $t \in \mathbb{N}$.

The observation that for every $M \in W(n,k)$ the matrix $\frac{1}{\sqrt{k}} \cdot M \in U(n)$ is a unitary matrix makes the connection between combinatorics and quantum computation that we explore in this article. In the next section we will see how the mutually orthogonal basis of such a matrix can be used for a query efficient quantum algorithm. The classical lower bound for the same problem is proven using standard, decision tree arguments.

4 Quantum Algorithms for Weighing Matrices

In this section we will describe a general weighing-matrix-problem and its quantum solution. But before doing so, we first mention the following state construction lemma which follows directly from earlier results on Grover's search algorithm.

Lemma 1 (State construction lemma) Let $f : \{1, ..., n\} \rightarrow \{-1, 0, +1\}$ be a black-box function. If we know that k of the function values are "+1" or "-1", and the remaining n - k entries are "0", then the preparation of the state

$$|f\rangle := \frac{1}{\sqrt{k}} \sum_{i=1}^{n} f(i)|i\rangle,$$

requires no more than $\lceil \frac{\pi}{4} \sqrt{\frac{n}{k}} \rceil + 1$ quantum evaluations of the black-box function f. When k = n, a single query is sufficient.

Proof: First, we use the amplitude amplification process of Grover's search algorithm[15] described in Fact 3 to create exactly the state $\frac{1}{\sqrt{k}} \sum_{i=1,\dots,n}^{f(i)\neq 0} |i\rangle$ with no more than $\lceil \frac{\pi}{4} \sqrt{\frac{n}{k}} \rceil$ queries to f. (See the article by Boyer $et\ al.\ [7]$ for a derivation of this upper bound. Obviously, no queries are required if k=n.) After that, following Fact 1, one additional f-call is sufficient to insert the proper amplitudes, yielding the desired state $|f\rangle$.

4.1 Weighing Matrix Problem and Its Quantum Solution

We will now define the central problem of this article, which assumes the existence of a weighing matrix.

Definition 3 (Weighing Matrix Problem) Let M be a W(n,k) weighing matrix. Define a set of n functions $f_s^M: \{1,\ldots,n\} \to \{-1,0,+1\}$ for every $s \in \{1,\ldots,n\}$ by $f_s^M(i) := M_{si}$. Given a function f_s^M in the form of a blackbox, we want to determine the parameter s. The (probabilistic) query complexity of the weighing matrix problem is the minimum number of calls to the function f that is necessary to determine the value s (with error probability at most $1-\varepsilon$).

With the quantum protocol of Lemma 1 we can solve this problem in a straightforward way.

Theorem 1 (Quantum procedure for the Weighing Matrix Problem) For every weighing matrix $M \in W(n,k)$ with the corresponding Weighing Matrix Problem of Definition 3, there exists a quantum algorithm that determines s exactly with at most $\lceil \frac{\pi}{4} \sqrt{\frac{n}{k}} \rceil + 1$ queries to f_s^M . When n = k, the problem can be solved with one query to the function.

Proof: First, prepare the state $|f_s^M\rangle = \frac{1}{\sqrt{k}}\sum_{i=1}^n f_s^M(i)|i\rangle$ with $\left\lceil \frac{\pi}{4}\sqrt{\frac{n}{k}}\right\rceil + 1$ queries to the function f (Lemma 1). After that, measure the state in the basis spanned by the vectors $|f_1^M\rangle,\ldots,|f_n^M\rangle$. Because M is a weighing matrix, this basis is orthogonal and hence the outcome of the measurement gives us the value s (via the outcome $|f_s^M\rangle$) without error.

4.2 Classical Bounds for Weighing Matrix Problems

For every possible weighing matrix, the above result establishes a separation between the quantum and the classical query complexity of the problem, as is shown by the following classical lower bound.

Lemma 2 (Classical lower bounds for the Weighing Matrix Problem) Consider the Weighing Matrix Problem of Definition 3 for a matrix $M \in W(n, k)$. Let d be the number of queries used by a classical algorithm that recovers s with an error probability of at most ε . This query complexity d is bounded from below by the following three inequalities:

$$d \geq \log_3 n + \log_3 (1 - \varepsilon),$$

$$d \geq (1 - \varepsilon) \frac{n}{k} - \frac{1}{k},$$

$$d \geq \log(\frac{n}{n-k+1}) + \log(1 - \varepsilon).$$

For the case where k = n, this last lower bound equals $d \ge \log n + \log(1 - \varepsilon)$.

Proof: We will prove these bounds by considering the decision trees that describe the possible classical protocols. The procedure starts at the root of the tree and this node contains the first index i that the protocol queries to

the function f. Depending on the outcome $f(i) \in \{-1,0,+1\}$, the protocol follows one of the three outgoing edges to a new node v, which contains the next query index i_v . This routine is repeated until the procedure reaches one of the leaves of the tree. At that point, the protocol guesses which function it has been querying. With this representation, the depth of such a tree reflects the number of queries that the protocol uses, while the number of leaves (nodes without outgoing edges) indicates how many different functions the procedure can distinguish.

For a probabilistic algorithm with error probability ε , we need to use decision trees with at least $(1-\varepsilon)n$ leaves. Because the number of outgoing edges cannot be bigger than 3, a tree with depth d has maximally 3^d leaves. This proves the first lower bound via $3^d > (1-\varepsilon)n$.

For the second and third bound we have to analyze the maximum size of the decision tree as it depends on the values k and n. We know that for every index i_v , there are only k different functions with $f(i_v) \neq 0$. This implies that at every node v the joint number of leaves of the two subtrees associated with the outcomes $f(i_v) = -1$ and +1 cannot be bigger than k. Hence, by considering the path (starting from the root) along the edges that correspond to the answers $f(i_v) = 0$, we see that a decision tree with d queries, can distinguish no more than dk + 1 functions. The second bound is thus obtained by the resulting inequality $dk + 1 \geq (1 - \varepsilon)n$. (The case k = 1 is the strongest example of this bound.)

In a similar fashion, we can use the observation that there are exactly n-k functions with $f(i_v)=0$ for every node v. Now we should consider the binary subtree that is spanned by the edges that correspond to the answers $f(i_v)=+1$ and $f(i_v)=-1$. With depth d, this subtree has at most 2^d leaves and 2^d-1 internal nodes. For the complete tree, each such internal node v gives at most n-k additional leaves, which are the functions with $f(i_v)=0$. In sum, this tells us that the total tree (with depth d) has a maximumber of leaves of $2^d+(2^d-1)(n-k)$, leading to the third result: $d \geq \log(\frac{n}{n-k+1}) + \log(1-\varepsilon)$. \square

4.3 Additional Remarks

The above bounds simplify significantly when we express them as functions of big enough n, giving us the following table:

k	quantum upper bound	classical lower bound
o(n)	$\frac{\pi}{4}\sqrt{\frac{n}{k}}+2$	$(1-\varepsilon)\frac{n}{k} - O(1)$
$\Theta(n)$	O(1)	$\log_3 n + \log_3 (1 - \varepsilon)$
\overline{n}	1	$\log n + \log(1 - \varepsilon)$

Note that the n-dimensional identity matrix is a W(n,1) weighing matrix, and that for this I_n the previous theorem and lemma are just a rephrasing (with k=1) of the results on Grover's search algorithm for exactly one matching entry. The algorithm of Bernstein & Vazirani is also captured by the above as the case where k has the maximum value k=n (with the weighing matrices

 $(\sqrt{2} \cdot \mathbb{H})^{\otimes t} \in W(2^t, 2^t)$). Hence we can think of those two algorithms as the extreme instances of the more general weighing matrix problem.

As we phrased it, a weighing matrix $M \in W(n,k)$ gives only one specific problem for which there is a classical/quantum separation, but not a problem that is defined for every input size N, as is more customary. We know, however, that for every such matrix M, the tensor products $M^{\otimes t}$ are also $W(n^t, k^t)$ weighing matrices (for all $t \in \mathbb{N}$). We therefore have the following direct consequence of our results.

Corollary 1 Every weighing matrix $M \in W(n,k)$ defines an infinite family of W(N,K) weighing matrix problems, with parameters $N=n^t$ and $K=k^t=N^{\log_n k}$ for every $t \in \mathbb{N}$. By defining $\gamma=1-\log_n k$ we have, for every suitable N, a quantum algorithm with query complexity $\frac{\pi}{4}\sqrt{N^{\gamma}}$ for which there is a classical, probabilistic lower bound of $(1-\varepsilon) \cdot N^{\gamma} + o(1)$.

Example 2 Using the W(4^t, 3^t) weighing matrices of Example 1, we have $\gamma = 1 - \frac{1}{2} \log 3 \approx 0.21$, and hence a quantum algorithm with query complexity $\frac{\pi}{4} N^{0.10...}$. The corresponding classical probabilistic, lower bound of this problem is $(1 - \varepsilon) \cdot N^{0.21...} + o(1)$.

A legitimate objection against the Weighing Matrix Problem is that it does not seem to be very useful for solving real-life problems. In order to obtain more natural problems one can try to look into the specific structure that constitutes the weighing matrix or matrices. An example of such an approach will be given in the next section via Paley's construction of Hadamard matrices. We will see how this leads to the definition of a problem dealing with quadratic residues of finite fields that has a quantum solution that is more efficient than any classical protocol.

5 The Shifted Legendre Sequence Problem

The query task that we will define in this section relies on some standard properties of finite fields. See the appendix of this article for a short but sufficient overview of this theory. Especially important is the following function which generalizes the Legendre symbol $(\frac{i}{p})$ over $\mathbb{Z}/(p\mathbb{Z})$ to all finite fields \mathbb{F}_q . (From now on, p will alway denote an odd prime, and $q = p^k$ a power of such a prime.)

Definition 4 (Legendre symbol over a Finite Field) For every finite field \mathbb{F}_q , with $q = p^k$ an odd prime power, the Legendre symbol function $\chi : \mathbb{F}_q \to \{-1, 0, +1\}$ indicates if a number is a quadratic residue or not:

$$\chi(i) := \begin{cases} 0 & \text{if } i = 0 \\ +1 & \text{if } \exists j \neq 0 : j^2 = i \\ -1 & \text{if } \forall j : j^2 \neq i. \end{cases}$$

This function is a quadratic, multiplicative character over \mathbb{F}_q , which implies the following result.

Fact 5 (Near Orthogonality of Shifted Legendre Sequences) For the 'inner product' between two Legendre sequences that are shifted by s and $r \in \mathbb{F}_q$ it holds that

$$\sum_{i \in \mathbb{F}_a} \chi(i+r) \chi(i+s) \quad = \quad \left\{ \begin{array}{cc} q-1 & \text{if } s=r, \\ -1 & \text{if } s \neq r. \end{array} \right.$$

Proof: See the proof of Fact 6 in the appendix.

Raymond Paley used this near orthogonality property for the construction of his Hadamard matrices.[24] Here we will use the same property to describe a problem that, much the like the weighing matrix problem of the previous section, has a clear gap between its quantum and classical query complexity. In light of Theorem 1 and Lemma 2 the results of this section are probably not very surprising. Rather, we wish to give an example of how we can borrow the ideas behind the construction of combinatorial objects for the design of new quantum algorithms. In this case this is done by stating a problem that uses the Legendre symbol over finite fields.

Definition 5 (Shifted Legendre Sequence/SLS Problem) Assume that we have a black-box for a shifted Legendre function $f_s : \mathbb{F}_q \to \{-1,0,+1\}$ that obeys $f_s(i) := \chi(i+s)$, with the—for us unknown—shift parameter $s \in \mathbb{F}_q$. The task is to determine (with probability $1 - \varepsilon$) the value s with a minimum number of calls to the function f.

5.1 Classical Query Complexity of the SLS Problem

Before describing the quantum algorithm for the Shifted Legendre Sequence Problem, we will first determine its classical query complexity. The following lower bound is established in a way similar to the proof of Lemma 2.

Lemma 3 (Classical lower bound SLS Problem) Assume a classical algorithm that tries to solve the shifted Legendre sequence problem over a finite field \mathbb{F}_q . To determine the requested value s with a maximum error rate ε , requires more than $\log q + \log(\frac{1-\varepsilon}{2})$ queries to the function f_s .

Proof: Consider, like in the proof of Lemma 2, a decision tree with nodes v and corresponding query indices i_v . For every index i_v there is exactly one function with $f(i_v) = 0$. For the tree this implies that every node v can only have two proper subtrees (corresponding to the answers f(i) = +1 and -1) and one deciding leaf (the case $f_{(-i)}(i) = 0$). Hence, a decision tree of depth d can distinguish no more than $2^{d+1} - 1$ different functions. In order to be able to differentiate between $(1 - \varepsilon)q$ functions, we thus need a depth d of at least $\log((1 - \varepsilon)q + 1) - 1$.

This lower bound of $\log q$ queries is also optimal as is shown by the following result.

Lemma 4 (Classical upper bound SLS Problem) There exists a deterministic, classical protocol that solves the Shifted Legendre Sequence Problem with $O(\log q)$ queries to the black-box function f_s .

Proof: Let $S \subseteq \mathbb{F}_q$ be the set of possible values of s at a given moment during the execution of the protocol. (Thus, initially we have $S = \mathbb{F}_q$, and we want to end with the unique answer determined by |S| = 1.) Below we will show that if S has at least 4 elements, then there always exists an index i such that all three possible answers to the query "f(i)?" lead to a reduced the set of options S' with $|S'| < \frac{3}{4}|S|$. By repeating this procedure no more than $\log q/\log(\frac{4}{3})$ times, we can reduce the initial set $S = \mathbb{F}_q$ to four possibilities, which can then be checked with three additional queries.

What follows is the existence proof of such an index for every possible subset $S \subseteq \mathbb{F}_q$. Given a set S and an index i, we have a partition of S in three subsets according to the answer $\chi(s+i)$ to the query "f(i)?"

$$\begin{array}{lll} S_i^+ &:= & \{j|j \in S \text{ and } \chi(s+j) = +1\}, \\ S_i^- &:= & \{j|j \in S \text{ and } \chi(s+j) = -1\}, \\ S_i^0 &:= & \{j|j \in S \text{ and } \chi(s+j) = 0\}. \end{array}$$

Note that, depending on whether -s is an element of S or not, S_i^0 is either $\{-s\}$ or the empty set. Clearly, one of these three sets will be the reduced set S' mentioned in the first part of the proof.

Define the Legendre matrix $L \in \{-1,0,+1\}^{q \times q}$ by $L_{ij} := \chi(i+j)$, and let z_S be the characteristic vector $\in \{0,1\}^q$ of the subset $S \subseteq \mathbb{F}_q$. The product of L and z_S yields a new vector w_s with the following property for its i-th entry:

$$(w_S)_i = \sum_{j \in \mathbb{F}_q} \chi(i+j)(z_S)_j = \sum_{j \in S} \chi(i+j) = |S_i^+| - |S_i^-|.$$

By the near orthogonality property of the Legendre sequence (Fact 5) we know that for the matrix L we have $L^T \cdot L = q \mathbf{I}_q - \mathbf{J}_q$, where \mathbf{J}_q is the 'all ones' matrix of dimension $q \times q$. The inner product $z_S^T z_S$ is the Hamming weight of z_S and hence equals the size |S| of the set S. This implies for the inner product of Lz_S with itself: $(z_S^T L^T)(Lz_S) = z_S^T (q \mathbf{I}_q - \mathbf{J}_q) z_S = q |S| - |S|^2$. By the previous equation for w_S , we thus see that

$$q|S| - |S|^2 = w_S^T w_S = \sum_{i \in \mathbb{F}_q} (|S_i^+| - |S_i^-|)^2.$$

This proves that there exist at least one index i for which $(|S_i^+| - |S_i^-|)^2 \le \frac{1}{q}(q|S|-|S|^2)$, and hence $-\sqrt{|S|} < |S_i^+|-|S_i^-| < \sqrt{|S|}$. In combination with the general bound $|S_i^+|+|S_i^-| \le |S|$, this gives that for this i both $|S_i^+|$ and $|S_i^-|$ are less than $\frac{1}{2}|S|+\frac{1}{2}\sqrt{|S|}$. For $|S| \ge 4$ this proves that indeed all three S_i^+ , S_i^- and S_i^0 have size less than $\frac{3}{4}|S|$.

5.2 Quantum Query Complexity of the SLS Problem

The above classical upper bound for the Shifted Legendre Sequence Problem relied on the near orthogonality property of the Legendre sequence. The same is done by the quantum protocol for the SLS Problem, but in a more efficient way: only 2 quantum queries are required.

Theorem 2 (Two query quantum protocol for the SLS Problem) For any finite field \mathbb{F}_q , the Shifted Legendre Sequence Problem of Definition 5 can be solved exactly with two quantum queries to the black-box function f_s .

Proof: We exhibit the quantum algorithm in detail. We start with the superposition

$$\frac{1}{\sqrt{q+1}} \left(\sum_{i \in \mathbb{F}_q} |i\rangle |0\rangle + |\operatorname{dummy}\rangle |1\rangle \right).$$

(The reason for the "dummy" part of state that we use will be clear later in the analysis.) The first oracle call is used to calculate the different $f_s(i) = \chi(i+s)$ values for the non-dummy states, giving a superposition of states $|i,\chi(i+s)\rangle$. At this point, we measure the rightmost register to see if it contains the value "zero". If this is indeed the case (probability $\frac{1}{q+1}$), the state has collapsed to $|-s\rangle|0\rangle$ which directly gives us the desired answer s. Otherwise, we continue with the now reduced state

$$\frac{1}{\sqrt{q}} \left(\sum_{i \in \mathbb{F}_q \setminus \{-s\}} |i\rangle |\chi(i+s)\rangle + |\mathrm{dummy}\rangle |1\rangle \right),$$

on which we apply a conditional phase change (depending on the χ values in the rightmost register). We finish the computation by 'erasing' this rightmost register with a second call to f_s . (For the dummy part, we just reset the value to "zero".) This gives us the final state ψ , depending on s, of the form

$$|\psi_s\rangle|0\rangle := \frac{1}{\sqrt{q}} \left(\sum_{i\in\mathbb{F}_q} \chi(i+s)|i\rangle + |\text{dummy}\rangle \right) |0\rangle.$$

What is left to show is that $\{|\psi_s\rangle|s\in\mathbb{F}_q\}$ forms a set of orthogonal vectors. Fact 5 tells us that for the inner product between two states ψ_s and ψ_r it holds that $\langle \psi_r | \psi_s \rangle = 1$ if s = r, and $\langle \psi_r | \psi_s \rangle = 0$ if $s \neq r$. In other words, the states ψ_s for $s \in \mathbb{F}_q$ are mutually orthogonal. Hence, by measuring the final state in the ψ -basis, we can determine without error the shift factor $s \in \mathbb{F}_q$ after only two oracle calls to the function f_s .

More recently, Peter Høyer has shown the existence of a one query protocol for the same problem [private communication].

5.3 Query versus Time Complexity Issues

The above algorithm only reduces the query complexity to f_s . The time complexity of the protocol is another matter, as we did not explain how to perform the final measurement along the ψ axes in a time-efficient way. This question, whether there exists a tractable implementation of the unitary mapping

$$|s\rangle \quad \longleftrightarrow \quad \frac{1}{\sqrt{q}} \left(\sum_{x \in \mathbb{F}_q} \chi(x+s) |x\rangle \quad + \quad |\text{dummy}\rangle \right),$$

is discussed and solved in an independent article[12].

6 Conclusion

We have established a connection between the construction of weighing matrices in combinatorics, and the design of new quantum algorithms. It was shown how every weighing matrix leads to a query problem that has a more efficient quantum solution than is possible classically. The earlier known results of Bernstein & Vazirani[4] and Grover[15] were shown to be specific instances of this more general problem.

Starting from Paley's construction of Hadamard matrices[16], we used the structure of quadratic residues over finite fields to give a more explicit example of a Weighing Matrix Problem. This led to the definition of the Shifted Legendre Sequence Problem, which has a constant quantum query complexity, compared to a logarithmic classical query complexity.

Although the results in this article only concern the query complexity of black-box problems, it should be viewed as the first step towards the construction of quantum protocols that are also time-efficient and that deal with more realistic problems. Constructions of Hadamard matrices that are especially interesting in this context are, for example, the complex Hadamard matrices of Turyn[33] and the Hadamard matrices of the dihedral group type[21, 30]. Also, the time-efficiency of the quantum solution of (a generalization of) the Shifted Legendre Sequence Problem was proven more recently in [12].

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A Quadratic Residues of Finite Fields

This appendix describes some standard results about quadratic residues and Legendre symbols over finite fields. For more background information one can look up references like [10] or [18].

A.1 Finite Field Factoids

Let $q=p^k$ denotes a power of an odd prime p. There always exists a generator ζ for the multiplicative group $\mathbb{F}_q^{\star}=\mathbb{F}_q\setminus\{0\}$. This means that the sequence $\zeta,\zeta^2,\zeta^3,\ldots$ will generate all non-zero elements of \mathbb{F}_q . As this is a set of size q-1, it follows that $\zeta^q=\zeta$, and hence $\zeta^{(q-1)}=1$. Hence we have the equivalence relation

$$\zeta^i = \zeta^j$$
 if and only if $i = j \mod (q-1)$

for every integer i and j.

We now turn our attention to the definition of the generalized Legendre symbol $\chi : \mathbb{F}_q \to \{-1, 0, +1\}$ which is defined by:

$$\chi(i) := \begin{cases} 0 & \text{if } i = 0 \\ +1 & \text{if } \exists j \neq 0 : j^2 = i \\ -1 & \text{if } \forall j : j^2 \neq i. \end{cases}$$

By the above mentioned equivalence relation, the quadratic expression $(\zeta^j)^2 = \zeta^{2j} = \zeta^i$ is correct if and only if $2j = i \mod q - 1$. As p is odd, q-1 will be even, and hence there can only exists a j with $(\zeta^j)^2 = \zeta^i$ when i is even. Obviously, if i is even, then ζ^j with $j = \frac{i}{2}$ gives a solution to our quadratic equation. This proves that 50% of the elements of \mathbb{F}_q^* are a quadratic residue with $\chi(x) = +1$, while the other half has $\chi(x) = -1$. In short: $\chi(\zeta^i) = (-1)^i$, and hence for the total sum of the function values: $\sum_x \chi(x) = 0$.

A.2 Multiplicative Characters over Finite Fields

The rule $\chi(\zeta^i) \cdot \chi(\zeta^j) = \chi(\zeta^{i+j})$, in combination with $\chi(0) = 0$, shows that the Legendre symbol χ is a multiplicative character with $\chi(x) \cdot \chi(y) = \chi(xy)$ for all $x, y \in \mathbb{F}_q$.

Definition 6 (Multiplicative characters over finite fields) The function χ : $\mathbb{F}_q \to \mathbb{C}$ is a multiplicative character if and only if $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in \mathbb{F}_q$. The constant function $\chi(x) = 1$ is called the trivial character. (We do not consider the other trivial function $\chi(x) = 0$.)

See [10, 18] for the usage of multiplicative characters in number theory. Some of the properties that we will use in this article are: $\chi(1) = 1$, if χ is nontrivial, then $\chi(0) = 0$, the inverse of nonzero x obeys $\chi(x^{-1}) = \chi(x)^{-1} = \overline{\chi(x)}$, and $\sum_{x} \chi(x) = 0$ for nontrivial χ .

With these properties we can prove the following fact that we use in the article.

Fact 6 (Near orthogonality of shifted characters) Consider a nontrivial character $\chi : \mathbb{F}_q \to \mathbb{C}$. For the 'complex inner product' between two χ -sequences that are shifted by s and $r \in \mathbb{F}_q$ it holds that

$$\sum_{x \in \mathbb{F}_q} \overline{\chi(x+r)} \chi(x+s) = \begin{cases} q-1 & \text{if } s=r, \\ -1 & \text{if } s \neq r. \end{cases}$$

Proof: (For the quadratic character, these are simple instances of so-called *Jacobsthal sums*[19]; see for example Section 6.1 in [3].) Rewrite

$$\sum_{x \in \mathbb{F}_q} \overline{\chi(x+r)} \chi(x+s) = \sum_{x \in \mathbb{F}_q} \overline{\chi(x)} \chi(x+\Delta)$$

with $\underline{\Delta} := s - r$. If s = r this sum equals q - 1. Otherwise, we can use the fact that $\overline{\chi(x)}\chi(x + \Delta) = \chi(1 + x^{-1}\Delta) = \chi(\Delta)\chi(\Delta^{-1} + x^{-1})$ (for $x \neq 0$) to reach

$$\sum_{x \in \mathbb{F}_q} \overline{\chi(x)} \chi(x + \Delta) = \chi(\Delta) \sum_{x \in \mathbb{F}_q^*} (\Delta^{-1} + x^{-1}).$$

Earlier we noticed that $\sum_x \chi(x) = 0$, and therefore in the above summation (where the value x = 0 is omitted) we have $\sum_x \chi(x^{-1} + \Delta^{-1}) = -\chi(\Delta^{-1})$.

This confirms that indeed

$$\chi(\Delta) \sum_{x \in \mathbb{F}_q^{\star}} \chi(x^{-1} + \Delta^{-1}) = -1,$$

which finishes the proof.

Note that the above property should not be confused with the orthogonality relation

$$\sum_{x\in\mathbb{F}_q}\overline{\chi(x)}\chi'(x) \quad = \quad \left\{ \begin{array}{cc} q-1 & \text{if } \chi=\chi' \\ 0 & \text{if } \chi\neq\chi' \end{array} \right.$$

between two (possibly different) characters χ and χ' .