Quantum query complexity of some graph problems[*]

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Abstract

Quantum algorithms for graph problems are considered, both in the adjacency matrix model and in an adjacency list-like array model. We give almost tight lower and upper bounds for the bounded error quantum query complexity of CONNECTIVITY, STRONG CONNECTIVITY, MINIMUM SPANNING TREE, and SINGLE SOURCE SHORTEST PATHS. For example we show that the query complexity of MINIMUM SPANNING TREE is in $\Theta(n^{3/2})$ in the matrix model and in $\Theta(\sqrt{nm})$ in the array model, while the complexity of CONNECTIVITY is also in $\Theta(n^{3/2})$ in the matrix model, but in $\Theta(n)$ in the array model. The upper bounds utilize search procedures for finding minima of functions under various conditions.

Keywords: graph theory, quantum algorithm, lower bound, connectivity, minimum spanning tree, single source shortest paths

1 Introduction

A primary goal of the theory of quantum complexity is to determine when quantum computers may offer a computational speed-up over classical computers. Today there are only a few results which give a polynomial time quantum algorithm for some problem for which no classical polynomial time solution is known. We are interested in studying the potentialities for speed-up for problems for which there already are efficient classical algorithms. Basic graphs problems are interesting candidates.

We study the query complexity of these problems; meaning the minimal number of queries to the graph required for solving the problem. Throughout this paper, the symbol $[n]$ denotes the set $[0..n-1]$. We consider two query models for directed graphs:

The adjacency matrix model, where the graph is given as the adjacency matrix $M \in \{0,1\}^{n \times n}$, with $M_{ij} = 1$ if and only if $(v_i, v_j) \in E$.

The adjacency array model, where we are given the out-degrees of the vertices $d_1^+, \ldots, d_n^+$ and for every vertex u an array with its neighbors $f_i : [d_i^+] \to [n]$. So $f_i(j)$ returns the jth neighbor of vertex i, according to some arbitrary but fixed numbering of the outgoing edges of i. In this paper the upper bounds for this model are all at least n, so we assume henceforth that the degrees are given as part of

the input and we account only queries to the arrays f_i. In addition the arrays satisfy the *simple graph* promise

$$\forall i \in [n], j, j' \in [k], j \neq j' : f_i(j) \neq f_i(j')$$

ensuring the graph is not a multigraph, i.e. does not have multiple edges between any two vertices.

For undirected graphs we require an additional promise on the input, namely that M is symmetric in the matrix model, and for the array model that $\forall i, i' \in [n]$ if $\exists j \in [k] : f_i(j) = i'$ then $\exists j' \in [k] : f_{i'}(j') = i$. Note that in the matrix model this symmetry assumption does not make undirected graph problems, promise problems since we may assume that the input is upper triangular.

Weighted graphs are encoded by a weight matrix, where for convenience we set $M_{ij} = \infty$ if $(v_i, v_j) \notin E$. In the adjacency array model, the graph is encoded by a sequence of functions $f_i : [d_i^+] \to [n] \times \mathbb{N}$, such that if $f_i(j) = (i', w)$ then there is an edge $(v_i, v_{i'})$ and it has weight w.

We emphasize that the array model is different from the standard list model. In the latter, we have access to the neighbors of a given vertex only as a list, and thus querying the ith neighbor requires i accesses to the list. This is also true on a quantum computer, so its speedup is quite restricted.

Many other query models are of course possible, for example we could be given an array of edges $f : [m] \to [n] \times [n]$, or an ordered array (which is up to $O(n)$ preprocessing the same as the adjacency array model). For simplicity, we use the array model as presented above.

For the quantum query complexity of general monotone graph properties, a lower bound of $\Omega(\sqrt{n})$ is known in the matrix model, as shown by Buhrman, Cleve, de Wolf and Zalka [9].[1] We are not aware of any quantum nor classical lower bounds in the array model.

In this paper we show that the quantum query complexity of CONNECTIVITY is $\Theta(n^{3/2})$ in the matrix model and $\Theta(n)$ in the array model. The classical randomized query complexity of CONNECTIVITY in the matrix model is $\Omega(n^2)$ by a sensitivity argument: Distinguishing the graph consisting of two length $n/2$ paths from the graph consisting of those two paths, plus an additional edge connecting them, $\Omega(n^2)$ queries are required.

We study the complexity of three other problems. In STRONG CONNECTIVITY we are given a directed graph and have to decide if there is a directed path between any pair of vertices. In MINIMUM SPANNING TREE we are given a weighted graph and have to compute a spanning tree with minimal total edge weight. In SINGLE SOURCE SHORTEST PATHS we have to compute the shortest paths according to the total edge weight from a given source vertex to every other vertex. The quantum query complexity of these three problems is $\Omega(n^{3/2})$ in the matrix model and $\Omega(\sqrt{nm})$ in the array model. We give almost tight upper bounds.

problem	matrix model	array model
minimum spanning tree	$\Theta(n^{3/2})$	$\Theta(\sqrt{nm})$
connectivity	$\Theta(n^{3/2})$	$\Theta(n)$
strong connectivity	$\Theta(n^{3/2})$	$\Omega(\sqrt{nm})$, $O(\sqrt{nm}\log n)$
single src. short. paths	$\Omega(n^{3/2})$, $O(n^{3/2}\log^2 n)$	$\Omega(\sqrt{nm})$, $O(\sqrt{nm}\log^2 n)$

Table 1: Quantum query complexity of some graph problems

We note that for graphs with a large number of edges $(m = \Theta(n^2))$, the complexities are (almost) the same in the matrix and array model for all problems but CONNECTIVITY. However, the models still differ in that case. For example the test $(u, v) \in E$ costs a single query in the matrix model and $O(\sqrt{\min\{d_u^+, d_v^+\}})$ queries in the array model since we do not assume any order on the arrays f_u and f_v.

The time complexities of the algorithms are the same as the query complexities up to log-factors. The algorithms given for connectivity and strong connectivity can be altered to also output the (strongly) connected components without increasing the asymptotic complexity. The space requirement is $O(\log n)$ qubits

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[1]In a previous version of this paper, we said that Buhrman et al. conjectured $\Omega(n)$ for CONNECTIVITY, and since their conjecture concerns arbitrary monotone graph properties in general, we gave a false impression of improving their result. We apologize.

and $O(n \log n)$ classical bits. If we constraint the space (both classical and quantum) to $O(\log n)$ qubits, the problems may be solved by random walks. Quantum random walks has been the subject of several papers [1, 10, 15], in particular for the st-CONNECTIVITY problem [22].

Other work on the query complexity of graphs problems has been done independently of us. For example [4] shows that testing whether a given graph is bi-partite needs $\Omega(n^{3/2})$ queries in the matrix model. Note that a graph is bi-partite iff it does not contain an odd cycle. For the array model a lower bound can be constructed from our lower bound for connectivity, showing that $\Omega(\sqrt{nm})$ queries are necessary for any bounded error quantum algorithm which distinguishes a single even cycle from two disjoint odd cycles. Matching upper bounds follow from our connectivity algorithms: simply construct a spanning forest of the graph. Then color alternating black-white the nodes of each tree. And finally use the quantum search procedure to find an edge with endpoints of the same color. Such an edge creates an odd cycle, and exists iff the graph contains an odd cycle.

2   Tools used in this paper

We use two fundamental tools. The first is amplitude amplification [7, 8], which we use when proving the upper bounds, the second is Ambainis' lower bound technique [2].

Amplitude amplification is a generalization of Lov Grover's search algorithm [13]. Since it is the most important tool used in our algorithms, we restate the exact results we require. We are given a boolean function F defined on a domain of size n. The function is given as a black box so that the only way we can obtain information about F is via evaluating F on elements in the domain. The search problem considered by Grover is to find an element x for which $F(x) = 1$, provided one exists. We say that x is a *solution* to the search problem, and that x is *good*. We use three generalizations of the search algorithm—all of which we refer to as "the search algorithm".

- If there are t elements mapped to 1 under F, with $t > 0$, the search algorithm returns a solution after an expected number of at most $\frac{9}{10}\sqrt{n/t}$ queries to F. The output of the algorithm is chosen uniformly at random among the t solutions. The algorithm does not require prior knowledge of t [6].

- A second version uses $O(\sqrt{n})$ queries to F in the worst case and outputs a solution with probability at least a constant, provided there is one [6].

- A third version uses $O(\sqrt{n \log 1/\epsilon})$ queries to F and finds a solution with probability at least $1 - \epsilon$, provided there is one [9].

We note that for very sparse graphs given in the adjacency matrix model, it may for some applications be efficient to initially learn all entries of the matrix by reiterating the first version of the search algorithm, for instance as formalized in Fact 1.

Fact 1 *Let k be given. There is a quantum algorithm that takes as input an $n \times n$ boolean matrix M, uses $O(n\sqrt{k})$ queries to M, and outputs a set S of 1-entries of M so that with probability at least $\frac{1}{2}$, S is of cardinality at least k or contains all 1-entries of M in case M has less than k 1-entries.*

Our lower bounds uses a technique introduced by Andris Ambainis.

Theorem 2 ([2, theorem 6]) *Let $L \subseteq \{0,1\}^*$ be a decision problem. Let $X \subseteq L$ be a set of positive instances and $Y \subseteq \overline{L}$ a set of negative instances. Let $R \subseteq X \times Y$ be a relation between instances of same size. Let values $m, m', \ell_{x,i}$ and $\ell'_{y,i}$ for $x, y \in \{0,1\}^n$ and $x \in X$, $y \in Y$, $i \in [n]$ be so that*

- *for every $x \in X$ there are at least m different $y \in Y$ in relation with x,*

- *for every $y \in Y$ there are at least m' different $x \in X$ in relation with y,*

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- *for every $x \in X$ and $i \in [n]$ there are at most $\ell_{x,i}$ different $y \in Y$ in relation with x which differ from x at entry i,*

- *for every $y \in Y$ and $i \in [n]$ there are at most $\ell'_{y,i}$ different $x \in X$ in relation with y which differ from y at entry i.*

Then the quantum query complexity of L is $\Omega(\sqrt{mm'/\ell_{\max}})$ where $\ell_{\max}$ is the maximum of $\ell_{x,i}\ell'_{y,i}$ subject to xRy and $x_i \neq y_i$.

3   Minima finding

Many graph problems are optimization problems, as are finding a minimum spanning tree, single source shortest paths, and largest connected components. Most quantum algorithms for such optimization problems utilize the search algorithm discussed above. A very basic and abstract optimization problem is as follows. Suppose we are given a function f defined on a domain of size n, and we want to find an index i so that $f(i)$ is a minimum in the image of f. This minimization problem was considered in [11] which gives an optimal quantum algorithm that uses $O(\sqrt{n})$ queries to f and finds such an i with constant probability. It is very simple to analyze, and we present it now, since it will make the rest of the section easier to understand.

1. Initially let $j \in [N]$ be an index chosen uniformly at random.

2. Repeat forever

   (a) Find an index $i \in [N]$ such that $f(i) < f(j)$.
   (b) Set $j := i$.

Theorem 3 ([11]) *The expected number of queries to f, until j contains the index of a minimum in the image of f is $O(\sqrt{N})$.*

Proof: Without loss of generality assume that f is injective. Now every index $j \in [N]$ has a *rank*, which we define as the number of indexes i such that $f(i) \leq f(j)$.

Let p_r be the probability that at some moment of the algorithm, the index j will have rank r. We claim that $p_r = 1/r$. Consider the first moment when j will have rank less or equal r. This moment will happen with probability 1. At that moment, because of the uniform choice in step 1 and since step 2(a) uses the quantum search procedure, j will be uniformly chosen among all indexes with rank less or equal r, so $p_r = 1/r$.

If j has rank r, the search procedure of step 2(a) will require $c\sqrt{N/(r-1)}$ expected number of queries, for some constant c. Therefore the total number of queries until j contains the solution is

$$\sum_{r=2}^{N} p_r c\sqrt{N/(r-1)} < c\sqrt{N} \sum_{r=1}^{N-1} r^{-3/2} < c\sqrt{N}\left(1 + \int_{r=1}^{N-1} r^{-3/2}\mathrm{d}r\right) = O(\sqrt{N})$$

$\square$

Let $c'\sqrt{N}$ be the expected number of queries to f until j contains the index of a minima. Stopping the algorithm after $2c'$ queries gives a quantum algorithm with error probability upper bounded by $1/2$.

For the purposes of this paper, we require the following generalizations of the minimum finding problem, illustrated in Figure 1.

Problem 1 (Find d smallest values of a function) *Let $\mathbb{N}^*$ denote $\mathbb{N} \cup \{\infty\}$. Given function $f : [N] \to \mathbb{N}^*$ and an integer $d \in [N]$, we wish to find d distinct indexes mapping to smallest values, i.e. a subset $I \subseteq [N]$ of cardinality d such that for any $j \in [N] \setminus I$ we have that $f(i) \leq f(j)$ for all $i \in I$.*

In the rest of this section, we assume $d \leq N/2$. In the following problem we are given a different function $g : [N] \to \mathbb{N}$, such that $g(j)$ defines the *type* of j. Let $e = |\{g(j) : j \in [N]\}|$ be the number of different types.

Problem 2 (Find d elements of different type) *Given function g and an integer d' we wish to find integer $d = \min\{d', e\}$ and a subset $I \subseteq [N]$ of cardinality d such that $g(i) \neq g(i')$ for all distinct $i, i' \in I$.*

Now we present a generalization of both previous problems.

Problem 3 (Find d smallest values of different type) *Given two functions f, g and an integer d' we wish to find integer $d = \min\{d', e\}$ and a subset $I \subseteq [N]$ of cardinality d such that $g(i) \neq g(i')$ for all distinct $i, i' \in I$ and such that for all $j \in [N] \setminus I$ and $i \in I$, if $f(j) < f(i)$ then $f(i') \leq f(j)$ for some $i' \in I$ with $g(i') = g(j)$.*

4 points of min. value   4 points of diff. type   4 points of min. value
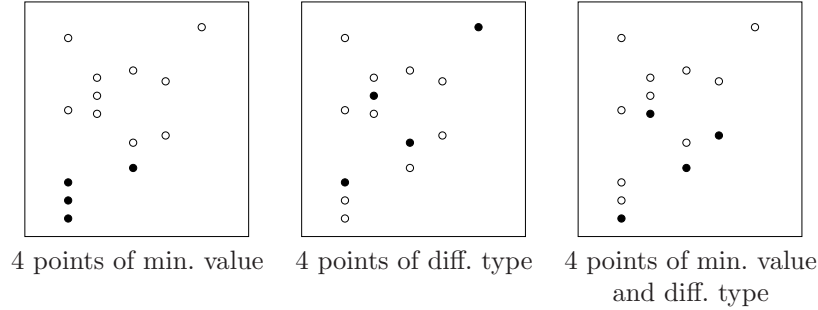   and diff. type

Figure 1: Illustration of the three problems. Each index i is illustrated by a point with horizontal coordinate $g(i)$ and vertical coordinate $f(i)$.

It is clear that Problems 1 and 2 are special cases of Problem 3. In this section, we give an upper bound of $O(\sqrt{dN})$ for Problem 3. In Section 8, we then show a lower bound of $\Omega(\sqrt{dN})$ for Problems 1 and 2, implying that all three problems are of complexity $\Theta(\sqrt{dN})$. We prove the upper bound by a simple greedy algorithm. Consider a subset $I \subseteq [N]$ of d indices of different types. We say an index $j \in [N]$ is *good for I* if

1. either $g(j) = g(i)$ and $f(j) < f(i)$ for some $i \in I$,

2. or $g(j) \notin g(I)$ and $f(j) < f(i)$ for some $i \in I$.

In the former case we say j is a good index of *known type*, in the latter that j is a good index of *unknown type*. In each iteration of the greedy algorithm, we find a good index j by the search algorithm and then improve I by replacing some index in I by j.

1. Initially, let $I = \{N + 1, \ldots, N + d'\}$ be a set of artificial indices of unique different types and unique maximal value.

2. Repeat forever

   (a) Let t denote the number of good elements for I. (Note: This step is not required, but only included for the purpose of simplifying the analysis of the algorithm.)

   (b) Use the first version of the search algorithm to find a good element $j \in [N]$ for I.

   (c) Set $I = \text{improve}(I, j)$ where we improve I by replacing with j the element in I that has the same type as j if j is of known type, and by replacing with j some element in I with largest f-value if j is of unknown type.

The next lemma shows we only need an expected number of $O(d)$ iterations of the main loop to eliminate a constant fraction of the remaining good elements.

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Lemma 4 *Let $I \subseteq [N]$ be any subset of d' indices of different types with $t > 0$ good elements of e types. After an expected number of $O(d)$ iterations of the main loop there are at most $\frac{3}{4}t$ good elements for I. Here $d = \min\{d', e\}$.*

Proof: For notational simplicity assume f is injective. Set $I_0 = I$ and let $T_0 = T$ be the set of good elements for I. Let T_j denote the set of good elements after j iterations of the main loop, for $j > 0$. Similarly, let I_j denote the selected index-set after j iterations, for $j > 0$. Set $t_k = |T_k|$. In particular $I_0 = I$ and $t_0 = t$. Let y_{mid} denote the $\lfloor t/2 \rfloor^{\mathrm{th}}$ smallest of the t elements according to f. For any subset $S \subseteq [N + d']$, let $\mathrm{low}(S)$ denote the number of elements in S that are no bigger than y_{mid} according to f.

Note that initially

- $\mathrm{low}(T_0) = \lfloor t/2 \rfloor$ and

- $\mathrm{low}(I_0) < d$.

By the nature of the greedy algorithm, $\mathrm{low}(T_{k+1}) \leq \mathrm{low}(T_k)$ and $\mathrm{low}(I_{k+1}) \geq \mathrm{low}(I_k)$ for any $k \geq 0$. Note that

- if $\mathrm{low}(T_k) < \frac{t}{4}$, then we have eliminated at least a fraction of $\frac{1}{4}$ of the initially t good elements for I, and similarly,

- if $\mathrm{low}(I_k) = d$, then we have eliminated at least a fraction of $\frac{1}{2}$ of the initially t good elements for I.

We claim that in each iteration of the main loop, as long as $\mathrm{low}(T_k) \geq \frac{t}{4}$, with probability at least $\frac{1}{32}$, at least one of the following two events happens

- $\mathrm{low}(T_{k+1}) \leq \mathrm{low}(T_k)\left(1 - \frac{1}{32d}\right)$,

- $\mathrm{low}(I_{k+1}) = \mathrm{low}(I_k) + 1$.

Assume $\mathrm{low}(T_k) \geq \frac{t}{4}$, since otherwise we are done. Consider the element j picked in Step 2b. First suppose the majority of the $\mathrm{low}(T_k)$ indices are of unknown type with respect to I_k. Then, with probability at least $\frac{1}{8}$, index j is among these, in which case $\mathrm{low}(I_{k+1}) = \mathrm{low}(I_k) + 1$.

Now suppose the majority of the $\mathrm{low}(T_k)$ indices are of known type with respect to I_k. Then, with probability at least $\frac{1}{8}$, index j is among these. Conditioned on this happens, with probability at least $\frac{1}{2}$, there are at least $\frac{\mathrm{low}(T_k)}{4d}$ good elements for I_k of the same type as j. With probability at least $\frac{1}{2}$, at least half of these are not good for I_{k+1}. Thus, with probability at least $\frac{1}{32}$, we have eliminated at least $\frac{t}{32d}$ of the remaining elements in T_j.

This proves the claim. It follows that after an expected number of $O(d)$ iterations of the main loop, we have eliminated at least a fraction of $\frac{1}{4}$ of the initially t good elements. $\qquad\square$

The above lemma implies that, for $t > 2d$, after an expected number of $O(d\sqrt{N/t}\,)$ applications of function f, the number of good elements is at most $\frac{t}{2}$. Hence, for any $t > 2d$, the expected number applications of function f required till we have that $t \leq 2d$ for the first time is in the order of

$$d\left(\sqrt{\frac{N}{d}} + \sqrt{\frac{N}{2d}} + \sqrt{\frac{N}{4d}} + \sqrt{\frac{N}{8d}} + \cdots\right) \in O(\sqrt{dN}).$$

Once $t \leq 2d$ for the first time, the expected number of applications of f required before $t = 0$ for the first time is in the order of $\sum_{j=1}^{2d} \sqrt{N/j}$ which is in $O(\sqrt{dN}\,)$.

Corollary 5 *In the greedy algorithm given above, after an expected number of $O(\sqrt{dN})$ applications of function f, there are no good elements for I, that is, $t = 0$.*

The next theorem follows immediately.

Theorem 6 *The problem* FIND d SMALLEST VALUES OF DIFFERENT TYPE *has bounded error quantum query complexity* $O(\sqrt{dN})$.

Nayak and Wu give in [19] a bounded error quantum algorithm that given a function $f : [N] \to \mathbb{N}$ and two integers d and Δ, outputs an index i such that the rank of $f(i)$ is between $d - \Delta$ and $d + \Delta$. The query complexity of their algorithm is $O(M \log M \log \log M)$ where $M = \sqrt{N/\Delta} + \sqrt{d(N-d)}/\Delta$. Setting $\Delta = \frac{1}{2}$ it would find the dth smallest element with $O(\sqrt{dN} \log N \log \log N)$ queries. Nayak [18] later improved this algorithm to $O(\sqrt{dN})$, matching the lower bound given in [19]. His method is different from ours.

Remark 1 *The algorithm above uses* $c\sqrt{dN}$ *queries for some constant c and outputs the solution with probability at least $1/2$. In order to reduce the error probability to $1/2^k$ one could run the algorithm k times and among the dk resulting indices, output the d smallest values of different type. However starting each run with randomly chosen points of different type regardless of the previous outcome, is a waste of information. So it is much more clever to run the algorithm only once and stop it after $kc\sqrt{dN}$ queries.*

4   Minimum Spanning Tree

In this section we consider undirected graphs with weighted edges. In MINIMUM SPANNING TREE we wish to compute a cycle-free edge set of maximal cardinality that has minimum total weight. To be precise if the graph is not connected this is actually a spanning forest.

Classically, there are a number of different approaches to finding minimum spanning trees efficiently, including the algorithms of Borůvka [5, 20], Kruskal [17], and Prim [21]. To construct an efficient quantum algorithm, we use Borůvka's algorithm since it is of a highly parallel nature. This allows us to use the minima finding algorithms given in Section 3.

Borůvka's algorithm consists of at most $\log n$ iterations. In brief, initially it starts with a collection of n spanning trees, each tree containing a single vertex. In each iteration, it finds a minimum weight edge out of each tree in the collection, adds the edges to the trees, and merges them into larger and fewer trees. After at most $\log n$ iterations, there is only one tree left, which is a minimum spanning tree. The correctness of Borůvka's algorithm rests on the following simple fact about spanning trees.

Fact 7 *Let $U \subset V$ be a set of vertices of a connected graph $G = (V, E)$ and let e be a minimum weight edge of $(U \times \overline{U}) \cap E$. Then there is a minimum spanning tree containing e.*

In our quantum version of Borůvka's algorithm, we make a few adjustments to keep the overall error probability small without sacrificing in the number of queries. We adjust it slightly so that the ℓth iteration errs with probability at most $\frac{1}{2^{\ell+2}}$, ensuring that the overall error is at most $\frac{1}{4}$. This increases the cost of the ℓth iteration by a factor of ℓ, but since the cost of the first few iterations dominates, this is asymptotically negligible. The details follow.

1. Let $T_1, T_2, \ldots, T_k$ be a spanning forest. Initially, $k = n$ and each tree T_j contains a single vertex.

2. Set $\ell = 0$.

3. Repeat until there is only a single spanning tree (i.e., $k = 1$).

    (a) Increment ℓ.

    (b) Find edges $e_1, e_2, \ldots, e_k$ satisfying that e_j is a minimum weight edge leaving T_j. Interrupt when the total number of queries is $(\ell + 2)c\sqrt{km}$ for some appropriate constant c.

    (c) Add the edges e'_j to the trees, merging them into larger trees.

4. Return the spanning tree T_1.

To find the minimum edges $e_1, \ldots, e_k$ in Step 3b, we use the following functions. In the array model, any edge (u, v) is coded twice, u appears as neighbor of v, but v also appears as neighbor of u. Enumerate the directed edges from 0 to $2m - 1$. Let $f : [2m] \to \mathbb{N}^*$ denote the function that maps every directed edge (u, v) to its weight if u and v belong to different trees of the current spanning forest and to ∞ otherwise. Let $g : [2m] \to [k]$ denote the function that maps every directed edge (u, v) to the index j of the tree T_j containing u. We then apply the algorithm for FINDING k SMALLEST VALUES OF DIFFERENT TYPE, interrupting it after $(\ell + 2)c\sqrt{km}$ queries to obtain an error probability at most $1/2^{\ell+2}$, see Remark 1.

Theorem 8 *Given an undirected graph with weighted edges, the algorithm above outputs a spanning tree that is minimum with probability at least $\frac{1}{4}$. The algorithm uses $O(\sqrt{nm})$ queries in the array model and $O(n^{3/2})$ queries in the matrix model.*

Proof: To simplify the proof, consider the matrix model an instance of the array model with $m = n(n-1)$ edges.

At the beginning of the ℓth iteration of the main loop, the number of trees k is at most $n/2^{\ell-1}$, and thus it uses at most $(\ell + 2)c\sqrt{nm/2^{\ell-1}}$ queries. Summing over all iterations, the total number of queries is at most $\sum_{\ell \geq 1}(\ell + 2)c\sqrt{nm/2^{\ell-1}}$, which is in $O(\sqrt{nm})$.

The ℓth iteration introduces an error with probability at most $\frac{1}{2^{\ell+2}}$. The overall error probability is thus upper bounded by $\sum_{\ell \geq 1} \frac{1}{2^{\ell+2}} \leq \frac{1}{4}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

For our algorithms, a $O(\log n)$ factor applies when considering the bit computational model, rather than the algebraic computational model. Apart from that, the time complexity is the same as the query complexity by using an appropriate data structure. Each vertex holds a pointer to another vertex in the same component, except for a unique vertex per component that holds a null pointer. This vertex is called the canonical representative of the component. To decide if two vertices are in the same component, we need only to determine the canonical representative of each vertex by pointer chasing. To merge two components, we change the pointer of one of the canonical representative to point to the other.

Using pointer chasing, the time complexity of the ℓth iteration is a factor of ℓ larger than its query complexity. However, as in the case of the error reduction, this is insignificant: $\sum_{\ell \geq 1} \ell(\ell + 2)c\sqrt{nm/2^{\ell-1}}$ is also in $O(\sqrt{nm})$. Thus, the time complexity is asymptotically the same as the query complexity.

5 Connectivity

A special case of MINIMUM SPANNING TREE when all edge weights are equal, is GRAPH CONNECTIVITY. The input is an *undirected* graph and the output is a spanning tree, provided the graph is connected.

For the matrix model, the algorithm for minimum spanning tree given in the previous section implies an $O(n^{3/2})$ upper bound for graph connectivity as well. Below, we give a somewhat simpler and arguably more natural quantum algorithm of query complexity $O(n^{3/2})$, which is optimal by the lower bound given in Section 8 below.

For the array model, we give a quantum algorithm that uses only $O(n)$ queries. Both algorithms start with a collection of n connected components, one for each vertex, and greedily construct a spanning tree by repeatedly picking an edge that connects two of the components.

Theorem 9 *Given the adjacency matrix M of an undirected graph G, the algorithm below outputs a spanning tree for G after an expected number of $O(n^{3/2})$ queries to M, provided G is connected, and otherwise runs forever.*

Proof: Consider the following algorithm.

1. Initially the edge set A is empty.

2. Repeat until A connects the graph.

(a) Search for a *good* edge, i.e., an edge that connects two different components in A, and add it to A. Use the version of the search algorithm that returns a solution in expected $O(n^2/t)$ queries if there are $t > 0$ good edges and otherwise runs forever.

3. Return the edge set A.

Suppose the graph is connected and consider the expected total number of queries used by the algorithm. There are exactly $n - 1$ iterations of the main loop. The number of good edges is at least $k - 1$ when A consists of k components, and thus the expected total number of queries is in the order of $\sum_{k=2}^{n} \sqrt{n^2/(k-1)}$, which is in $O(n^{3/2})$. □

When implementing the above algorithm, we maintain an appropriate data structure containing information about the connected components in the graph induced by A. This introduces an additional $O(n \log n)$ term in the running time of the algorithm which is negligible compared to $O(n^{3/2})$. We may choose to stop the algorithm after twice the expected total number of queries, giving an $O(n^{3/2})$ query algorithm with bounded one-sided error.

5.1 The array model

Lemma 10 *Given an undirected graph G in the array model, we can in $O(n)$ classical queries construct a set of connected components $\{C_1, \ldots, C_k\}$ for some integer k, so that for each component C, its total degree $m_C = \sum_{i \in C} d_i$ is no more than $|C|^2$.*

Proof: The algorithm is classical and is as follows.

1. Initially the edge set A is empty.

2. Let $S = V$ be the set of vertices not yet placed in some component.

3. Let $k = 0$ be the number of components constructed thus far.

4. While S is non-empty

   (a) Take the vertex v of highest degree in S and set $D = \{v\}$.

   (b) Go through v's list of neighbors one by one, each time adding the neighbor w to D and the edge (v, w) to A, until one of two events happens: (1) We reach the end of the list, or (2) we reach a neighbor w already assigned to some component C_j with $j \le k$.

   (c) In case (1), set $k = k + 1$, $C_k = D$, and remove D from S. In case (2), add D to C_j, and remove D from S.

5. Output k, A, and $C_1, C_2, \ldots, C_k$.

The algorithm uses $n - k$ queries in total, one query for each vertex but the first added to each component. Edge set A contains the union of spanning trees of the components C_1 through C_k.

To show correctness, let v be the vertex chosen in Step 4a and d its degree. Then $d \le |C_j|$ for each components constructed so far, since the size of a freshly created component is the degree of one of its vertices, which by choice in Step 4a must be no less than d, and components can only grow. To show that the total degree of every component C_j is no more than $|C_j|^2$, consider the two cases in Step 4b.

In case (1), D is the set of v and its neighbors, each neighbor having degree no larger than d, implying the total degree is at most $d(d + 1)$ which is strictly less than $(d + 1)^2 = |D|^2$. In case (2), let a be the size of the component C_j to which D is merged, and b the size of D. Then $b \le d \le a$. The total degree is no more than $a^2 + bd$ which is strictly less than $(a + b)^2$. □

Theorem 11 *Given an undirected graph G in the array model, the algorithm below outputs a spanning tree for G using an expected number of $O(n)$ queries, provided G is connected, and otherwise runs forever.*

Proof: Consider the following algorithm.

1. Construct the edge set A using the above lemma.

2. Repeat until A connects the graph.

   (a) Pick a connected component C in A with smallest total degree, i.e, a component minimizing $m_C = \sum_{i \in C} d_i$.

   (b) Search for an edge out of C, i.e., an edge that connects C to some other component in A, and add it to A. Use the version of the search algorithm that returns a solution in expected $O(\sqrt{m_C})$ queries if there is at least one such edge and otherwise runs forever.

3. Return the edge set A.

Suppose the graph is connected and consider the expected total number of queries used by the algorithm. We first construct k components, each component C having total degree m_C at most $|C|^2$. In each iteration of the main loop, we pick the component with smallest total degree and search for an edge out of C. The expected cost of finding such an edge is at most $\alpha\sqrt{m_C}$ for some constant α. We distribute this cost evenly among each of the m_C edge endpoints in C, each endpoint paying $\alpha/\sqrt{m_C}$.

Fix an arbitrary edge endpoint. Enumerate from 0 up to at most $\log m$ the successive components that were chosen by the algorithm for a search and that contain this fixed edge endpoint. Let m_i be the number of edge endpoint in the ith component. Then $m_{i+1} \geq 2m_i$. The total cost assigned to our fixed edge endpoint is upper bounded by

$$\sum_{i=0}^{\log m} \frac{\alpha}{\sqrt{m_i}} \leq \sum_{i=0}^{\log m} \frac{\alpha}{\sqrt{2^i m_0}} \leq \frac{4\alpha}{\sqrt{m_0}}.$$

Let C be any of the k components constructed in the first step. The total cost assigned over all edge endpoints in C is thus upper bounded by $4\alpha\sqrt{m_C}$, which is at most $4\alpha|C|$. Summing over all k components, the total cost assigned in the main loop is at most $4\alpha n$, which is linear in n. □

6 Strong Connectivity

We give two quantum algorithms for strong connectivity, first one for the matrix model and then one for the array model. The input is a *directed* graph and the output is a set of at most $2(n-1)$ edges that proves the graph is strongly connected, provided it is. It follows from the discussions below that such sets always exist.

Theorem 12 *Given the adjacency matrix M of a directed graph G and a vertex v_0, the algorithm below uses $O(\sqrt{nm\log n})$ queries to M and outputs a directed tree $A \subseteq E$ rooted at v_0. With probability at least $\frac{9}{10}$, A spans all of G, provided such a spanning tree exists.*

Proof: Consider the following simple algorithm.

1. Initially the edge set A is empty.

2. Let $S = \{v_0\}$ be a set of reachable vertices, and $T = \{v_0\}$ a stack of vertices to be processed.

3. While $T \neq \{\}$ do

   (a) Let u be the top most vertex of stack T.

   (b) Search for a neighbor v of u not in S. Use the version of the search algorithm that uses $O(\sqrt{d_u^+ \log n})$ queries and outputs a solution with probability at least $1 - \frac{1}{20n}$, provided one exists.

   (c) If succeed, add (u, v) to A, add v to S, and push v onto T.

(d) Otherwise, remove u from T.

4. Return edge set A.

For any vertex u let b_u^+ the out-degree in the tree A produced by the algorithm. Then the total number of queries spent in finding the b_u^+ neighbors of u is in the order of $\sum_{t=1}^{b_u^+} \sqrt{d_u^+/t}\sqrt{\log n}$ which is in $O(\sqrt{b_u^+ d_u^+ \log n})$. Summing over all vertices u this gives

$$\sum_{u \in V} \sqrt{b_u^+ d_u^+ \log n} \le \sqrt{\sum_u b_u^+}\sqrt{\sum_u d_u^+}\sqrt{\log n} = O(\sqrt{nm \log n}).$$

The first inequality follows from the general statement that the inner product of two vectors is upper bounded by the product of their ℓ_2-norms. The second inequality uses the fact that a tree has only $O(n)$ edges. The algorithm spends in addition $O(\sum_u \sqrt{d_u^+ \log n})$ queries for the unsuccessful neighbor searches, but this is dominated by the previous cost.

The overall error probability is upper bounded by $\frac{1}{10}$ since each of the at most $2n - 1$ searches has error at most $\frac{1}{20n}$. $\qquad\square$

Theorem 12 implies that STRONG CONNECTIVITY, too, can be solved using an expected number of queries in $O(n^{3/2}\sqrt{\log n})$ in the adjacency matrix model. In fact the previous algorithm can be used for the matrix model with $m = n(n-1)$. We first check that some fixed vertex v_0 can reach any other vertex u, producing some spanning tree rooted at v_0. We then check that all vertices can reach v_0 by repeating the previous step on the transposed adjacency matrix. The two stages produces a set at most $2(n-1)$ edges that proves the graph is strongly connected.

This is not possible in array model, since we store for any vertex u only the neighbors at which edges are pointing, and there is no easy access to vertices which are connected with a directed edge *to* u. The following theorem circumvents this obstacle.

Theorem 13 *Given a directed graph G in the array model and a vertex v_0, the algorithm below uses $O(\sqrt{nm \log n})$ queries and outputs an edge set $E' \subseteq E$ covering v_0. If G is strongly connected, then with probability at least $\frac{1}{4}$ E' is strongly connected.*

Proof: In a first stage we use the previous algorithm to construct a directed depth first spanning tree $A \subseteq E$ rooted in v_0. Assume vertexes to be named according to the order they are added to T.

Then in a second stage we search for every vertex $v_i \in V$, the neighbor v_j with smallest index. The result is a set of backward edges $B \subseteq E$. We claim that the graph $G(V, E)$ is strongly connected iff its subgraph $G'(V, A \cup B)$ is strongly connected.

Clearly if G' is strongly connected then so is G since $A \cup B \subseteq E$. Therefore to show the converse assume G strongly connected. For a proof by contradiction let v_i be the vertex with smallest index, which is not connected to v_0 in G'. However by assumption there is a path in G from v_i to v_0. Let $(v_l, v_{l'})$ be its first edge with $l \ge i$ and $l' < i$. We use the following property of depth first search.

> **Lemma 14** *Let v_l and $v_{l'}$ be two vertexes in the graph G with $l < l'$. If there is a path from v_l to $v_{l'}$ in $G(V, E)$ then $v_{l'}$ is in the subtree of $G''(V, A)$ with root v_l.*

Therefore we can replace in the original path the portion from v_i to v_l by a path only using edges from A. Let $v_{l''}$ be the neighbor of v_l with smallest index. Clearly $l'' \le l' < i$. By the choice of v_i, there exists a path from $v_{l''}$ to v_0 in G'. Together this gives a path from v_i to v_0 in G' contradicting the assumption and therefore concluding the correctness of the algorithm.

Now we analyze the complexity. During the first stage, set A is computed in time $O(\sqrt{n^{3/2} \log n})$. The second stage can be done with $O(\sqrt{nm})$ queries using the minima finding for the mapping from an edge number in $[1, m]$ to the source-target vertex pair. Both stages can be made succeed with probability at least $7/8$. $\qquad\square$

6.1 The matrix model

Here all we need is to construct an oriented tree, rooted in some vertex v_0, and it not need to be depth-first. We want this tree to cover all vertices reachable by v_0. There is a tricky method for constructing such a tree with bounded error, without a log-factor in the running time as the previous algorithm.

The idea is to classify vertices covered by the current tree, into sets $T_0, \ldots, T_q$ such that the *confidence* that vertices from T_i have no new neighbors is increasing with i. Whenever a search of an edge (u, v) with $u \in R$ and $v \notin T_0 \cup \ldots \cup T_q$ is successful, for some subset $R \subseteq T_i$, the vertices R and u will be moved into T_0, otherwise, R will be moved into T_{i+1}. We make it formal now.

1. Let S be the tree consisting of the single vertex v_0.
   Let be the partitioning of the vertex set covered by S into $T_0 = \{v_0\}$ and $T_1 = \ldots = T_q = \{\}$, for $q = \lfloor \log_2(n) \rfloor + 1$.

2. While there is a set T_i with $|T_i| \geq 2^i$ do

   (a) Let i be the smallest index such that $|T_i| \geq 2^i$.

   (b) If $|T_i| < 2^{i+1}$, $R = T_i$ otherwise R is an arbitrary subset of T_i with $|R| = 2^i$.

   (c) Remove R from T_i.

   (d) Search for an edge (u, v) with $u \in R$ and $v \notin S$ in search space of size $O(2^i n)$ with the version of the quantum search procedure which uses $O(2^{3i/4} \sqrt{n})$ queries and find a solution with probability $1 - 1/2^{\sqrt{2^{i+2}}}$ provided such an edge exists.

   (e) If the search was successful, add (u, v) to S and $R \cup \{v_0\}$ to T_0, otherwise add R to T_{i+1}.

3. Output S.

Now we show some properties of the algorithm. For convenience we define $t_j = |T_0| + |T_1| + \ldots + |T_j|$.

Lemma 15 *At the beginning and the end of an iteration we have the following invariant. Let k be the smallest index and ℓ be the largest index of a non-empty set T_j. Then*

$$|T_\ell| \geq 2^{\ell - 1} \tag{1}$$

$$\forall k \leq j < \ell : t_j \geq 2^j \tag{2}$$

Proof: by induction on the iterations of the algorithm. Initially, when T_0 is the unique non-empty set, the claim holds.

Assume the claim holds before an iteration.

First observe that by the induction assumption (2) if $k < \ell$ then $|T_k| \geq 2^k$, and so the index chosen by the algorithm is always $i = k$. Now if the search was successful, we have already $t_0 \geq 2^k$, and therefore $t_j \geq 2^k$ for all $0 \leq j \leq k$ and for all $j > k$, t_j increased by one, preserving condition (2). If the search was not successful, then by the choice of R, after the iteration either $t_k = 0$ or $t_k \geq 2^k$. For all values $j > k$, t_j is not modified, preserving condition (2).

Condition (1) is preserved because after decreasing every set T_j satisfies either $|T_j| = 0$ or $|T_j| \geq 2^j$, and because whenever a set T_j becomes non-empty, it contains at least 2^{j-1} elements. $\square$

As a consequence, when the algorithm stops, there is a unique non-empty set T_i and moreover $2^{i-1} \leq |T_i| < 2^i$. Also since at most $n-1$ searches can be successful, the algorithm stops after $O(n \log n)$ iterations.

Lemma 16 *When the algorithm stops, S covers all vertices reachable by v_0 with probability at least $2/3$.*

Proof: Suppose the algorithm failed so there is an edge (u, v) with $u \in S$ and $v \notin S$ which was not found for each call to the search procedure with a set T_j containing u. The probability of this event is the product of the failure probability over all calls. It is roughly upper bounded by the failure probability of the last

call, which is at most $p_i := 2^i/2^{\sqrt{2^{i+2}}}$, where $i = |S|$. Let q_i be the probability that the algorithm outputs a vertex set S with $|S| = i$. Then the failure probability is upper bounded by

$$\sum_{i=0}^{n} q_i p_i \leq \max_{i=0}^{n} p_i = p_1 \leq 1/3.$$

$\square$

Now we analyze the complexity of the algorithm.

Lemma 17 *The expected number of queries done by the algorithm is $O(|S|\sqrt{n})$.*

Proof: To analyze the total number of queries done by the search procedures, we group the calls of the search procedures into sequences of unsuccessful searches ending with a success, plus the last sequence of unsuccessful searches.

For the first case, let (u, v) be an arbitrary edge found by the algorithm. Then the probability that it was found when $u \in T_i$ is upper bounded by the probability that it was not found when $u \in T_{i-1}$, which is $1/2^{\sqrt{2^i}}$. The cost of this search and the $i - 1$ unsuccessful searches over sets R containing u is order of

$$\sum_{j=0}^{i} 2^{3i/4}\sqrt{n} = O(2^{3i/4}\sqrt{n}).$$

Therefore the expected cost of finding (u, v) is at most

$$\sum_{i=0}^{q} \frac{2^{3i/4}}{2^{\sqrt{2^i}}}\sqrt{n} = O(\sqrt{n}).$$

To complete the analysis we upper bound the total work of all the $O(\log n)$ unsuccessful searches which were made after the last successful search. Let i be such that $2^{i-1} \leq |S| < 2^i$. There where at most $2^i/2^j$ searches for sets R with $|R| = 2^j$. Therefore the total work is order of

$$\sum_{i=0}^{i} 2^{i-j} 2^{3j/4}\sqrt{n} = O(|S|\sqrt{n}).$$

This concludes the proof. $\square$

7 Single source shortest paths

Let G be a directed graph with non-negative edge weights and a fixed vertex v_0. We want to compute for every vertex v a shortest path from v_0 to v. It may happen that it is not unique. Using for example the lexicographical ordering on vertex sequences, we choose to compute a single canonical shortest path. From now on assume that different paths have different lengths. As a result, the union over all vertices v of the shortest paths from v_0 to v is a *shortest path tree*. Let $\nu(u, v)$ be the weight of edge (u, v) and $\delta(v_0, v)$ the shortest path length from v_0 to v.

Classically SINGLE SOURCE SHORTEST PATH may be solved by Dijkstra's algorithm. It maintains a subtree $\mathcal{T}$ with the "shortest path subtree" invariant: for any vertex $v \in \mathcal{T}$, the shortest path from v_0 to v uses only vertices from $\mathcal{T}$. An edge (u, v) is called a *border edge* (of $\mathcal{T}$) if $u \in \mathcal{T}$ and $v \notin \mathcal{T}$, and u is called the source vertex, v the target vertex. The *cost* of (u, v) is $\delta(v_0, u) + \nu(u, v)$. Dijkstra's algorithm starts with $\mathcal{T} = \{v_0\}$ and iteratively adds the cheapest border edge to it.

Our improvement lays in the selection of the cheapest border edge. We give the algorithm for the array model. Setting $m = n^2$ implies the required bound for the matrix model.

Theorem 18 *The bounded error query complexity of single source shortest path in the array model is* $O(\sqrt{nm}\log^{3/2}n)$.

Proof: As in Dijkstra's algorithm we construct iteratively a tree T, such that for every vertex $v \in T$, the shortest path from v_0 to v is in T. We also maintain a partition of the vertices covered by T, into a set sequence. Its length is denoted by l.

1. $T = \{v_0\}$, $l = 1$, $P_1 = \{v_0\}$

2. Repeat until T covers the graph

   (a) For P_l compute up to $|P_l|$ cheapest border edges with disjoint target vertices. For this purpose set $N = \sum_{v \in P_l} d(v)$, and number all edges with source in P_l from 1 to N. Define the functions $f : [N] \to \mathbb{N}^*$ and $g : [N] \to V$, where $g(i)$ is target vertex of the ith edge and $f(i)$ is its weight if $g(i) \notin T$ and ∞ otherwise. Apply the algorithm of section 3 on f and g with $d = |P_l|$ to find the d lowest cost edges with distinct target vertices. Let A_l be the resulting edge set.

   (b) Let (u, v) be the minimal weighted edge of $A_1 \cup \ldots \cup A_l$ with $v \notin P_1 \cup \ldots P_l$. Set $T = T \cup \{(u, v)\}$, $P_{l+1} = \{v\}$ and $l = l + 1$.

   (c) As long as $l \geq 2$ and $|P_{l-1}| = |P_l|$, merge P_l into P_{l-1}, and set $l = l - 1$.

All steps but 2(b) constructed a vertex set sequence $P_1, \ldots, P_l$, the cardinality of each being a power of 2, and of strictly decreasing sizes. Figure 2 shows an example of this partitioning of the vertices in T.
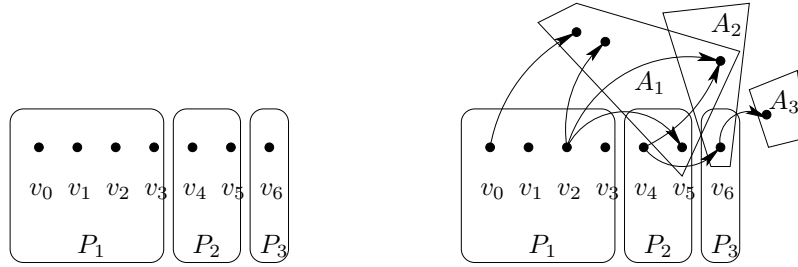

Figure 2: Left: example of the set decomposition for $|T| = 7$ into powers of 2. Right: example of corresponding edge sets. The closest border edge of T belongs to one of A_1, A_2, A_3.

Therefore each set P_i is strictly larger than the union of all the following sets, since $\sum_{i=0}^{k-1} 2^i = 2^k - 1$. If A_i contained $|P_i|$ edges, than at least one of them has its target vertex outside of $P_1 \ldots, P_l$. Let (u, v) be the cheapest border edge of T. Let P_i be the vertex set containing u. Then A_i must contain this edge, and step 2(b) selects it.

Only step 2(a) generates queries to the graph. What is the total number of queries related to sets P_i of some size s? There are at most n/s sets of this size s. Therefore total work is order of $\sum_{j=1}^{n/s} \sqrt{sm_j}$, where m_j is the number of edges with source in the jth vertex set. We have $\sum_{j=1}^{n/s} m_j = m$. This worst case is when $m_i = sm/n$. In that case the total work is $O(\sqrt{nm})$ for the fixed size s. There are $\log n$ different set sizes in the algorithm. Each of the $O(n \log n)$ queries to the minimum finding procedures should succeed with probability $1 - 1/2n \log n$ at least This introduces a $O(\log n)$ factor, see Remark 1, and we obtain the claimed complexity. □

8 The lower bounds

Theorem 19 *The problems* FIND d SMALLEST VALUES OF A FUNCTION, FIND d ELEMENTS OF DIFFERENT TYPE *and* FIND d SMALLEST VALUES OF DIFFERENT TYPE *require* $\Omega(\sqrt{dN})$ *queries.*

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Proof: For even k and odd d we consider $d \times k$ boolean matrices with a single 0 in every row. It is encoded by a function $f : [N] \to \{0, 1\}$ with $N = kd$, such that for every $i \in [d], j \in [k]$, $f(id + j)$ is the entry in row i and column j. Let function $g : [N] \to [d + 1]$ be such that $g(id + j)$ maps to i if $f(id + j) = 0$ and to d otherwise.

So the problems of FINDING $d+1$ SMALLEST VALUES OF A FUNCTION or $d+1$ ELEMENTS OF DIFFERENT TYPE or $d + 1$ SMALLEST VALUES OF DIFFERENT TYPE are all equivalent to finding the positions of the d zeroes in the matrix.

Let X be the set of matrices such that exactly $\lfloor d/2 \rfloor$ rows have their 0 in the first $k/2$ columns. And let Y be the set of matrices such that this number is exactly $\lceil d/2 \rceil$. We show a lower bound for distinguishing X and Y. We say that matrix $A \in X$ is in relation with $B \in Y$ iff both matrices differ at exactly two entries. It follows that there are indices $i \in [d], 0 \le j < k/2 \le j' < k$ with $A_{ij} = B_{ij'} = 1$ and $A_{ij'} = B_{ij} = 0$. The following example illustrates this definition.

$$A = \begin{pmatrix} \overbrace{011}^{k/2}\,\overbrace{111}^{k/2} \\ 111\ 011 \\ 111\ 101 \end{pmatrix} \quad B = \begin{pmatrix} \overbrace{011}^{k/2}\,\overbrace{111}^{k/2} \\ 110\ 111 \\ 111\ 101 \end{pmatrix}$$

Then the number of matrices which are in relation with a fixed matrix is at least $m = m' = \lceil d/2 \rceil k/2$. For fixed i, j the number of matrices in relation with M and differing at i, j is $k/2$ if $M_{ij} = 0$ and 1 if $M_{ij} = 1$. So $l_{\max} = k/2$ and $\Omega(\sqrt{mm'/l_{\max}})$ gives the required lower bound. □

Now, we give a simple lower bound for CONNECTIVITY (and STRONG CONNECTIVITY) in the list model, by a reduction from PARITY. As we recently found out, this reduction has first been used by Henzinger and Fredman for the on-line connectivity problem [14]. We show later how to improve this construction.

Lemma 20 STRONG CONNECTIVITY *requires* $\Omega(n)$ *queries in the array model.*

Proof: We use a straightforward reduction from PARITY.

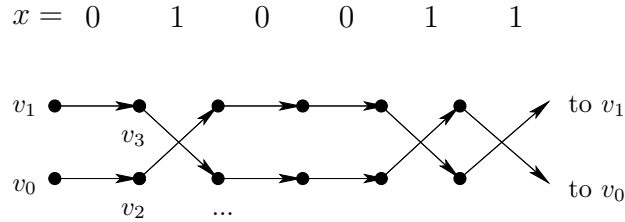$$x = \quad 0 \qquad 1 \qquad 0 \qquad 0 \qquad 1 \qquad 1$$

Figure 3: A standard reduction from parity

Let $x \in \{0, 1\}^p$ be an instance of the parity problem. We construct a permutation f on $V = \{v_0, \dots, v_{2p-1}\}$ which has exactly 1 or 2 cycles depending the parity of x. For any $i \in [p]$ and the bit $b = x_i$, we define $f(v_{2i}) = v_{2i+2+b}$ and $f(v_{2i+1}) = v_{2i+3-b}$, where addition is modulo $2p$. See figure 3. The graph defined by f has 2 levels and p columns, each corresponding to a bit of x. A directed walk starting at vertex v_0 will go from left to right, changing level whenever the corresponding bit in x is 1. So when x is even the walk returns to v_0 while having explored only half of the graph, otherwise it returns to v_1 connecting from there again to v_0 by p more steps. Since the query complexity of PARITY is $\Omega(n)$ — see for example [12] — this concludes the proof. □

The same technique can be extended to the undirected case.

Corollary 21 CONNECTIVITY *requires* $\Omega(n)$ *queries in the array model.*

We improve the lower bound in Lemma 20 by changing the construction slightly.

Theorem 22 STRONG CONNECTIVITY *requires* $\Omega(\sqrt{nm})$ *queries in the array model.*

Proof: Let m be such that $m = kn$ for some integer k. We construct the lower bound for graph with regular out-degree k.

    We use a similar construction as for Lemma 20, but now for every vertex the $k-1$ additional edges are redirected back to an origin. We would like to connect them back to a fixed vertex u_0, but this would generate multiple edges and we want the proof work for simple graphs. Therefore we connect them back to a k-clique which then is connected to u_0. See figure 4. Let vertex set $V = \{v_0, \ldots, v_{2p-1}, u_0, \ldots, u_{k-1}\}$ for some integer p. In the list model, the edges are defined by a function $f : V \times [k] \rightarrow V$. We consider only functions with the following restrictions:

    For every $i \in [k]$ we have $f(u_i, 0) = v_0$ and for $j \in \{1, \ldots, k-1\}$ $f(u_i, j) = u_{i+j}$, where addition is modulo k.

    For every $i \in [p]$ there exist $j_0, j_1 \in [k]$ and a bit b such that $f(v_{2i}, j_0) = v_{2i+2+b}$ and $f(v_{2i+1}, j_1) = v_{2i+3-b}$, where addition is modulo $2p$ this time. We call these edges the *forward edges*. The *backward edges* are for all $j \in [k]$ $f(v_{2i}, j) = u_j$ whenever $j \neq j_0$ and $f(v_{2i+1}, j) = u_j$ whenever $j \neq j_1$.
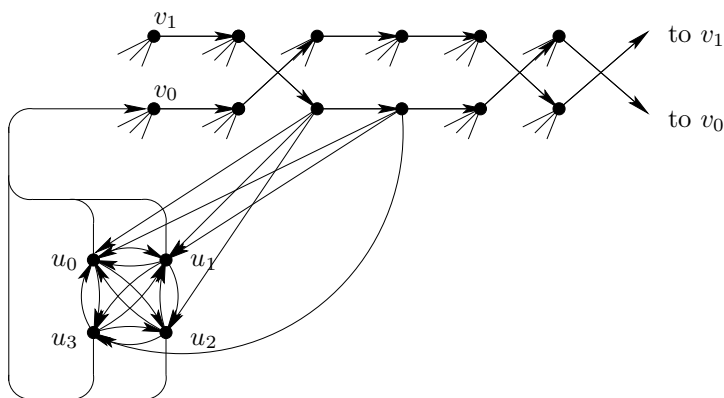

Figure 4: A strongly connected graph

    Now all the vertices are connected to the k-clique, the clique is connected to v_0, and the graph is strongly connected if and only if there is a path from v_0 to v_1.

    Let X be the set of functions which define a strongly connected graph, and Y the set of functions which do not. Function $f \in X$ is in relation with $g \in Y$ if there are numbers $i \in [p], j_0, j_1, h_0, h_1 \in [k]$ with $j_0 \neq h_0$, $j_1 \neq h_1$ such that the only places where f and g differ are

$$g(v_{2i}, h_0) = f(v_{2i+1}, j_1) \qquad g(v_{2i+1}, h_1) = f(v_{2i}, j_0)$$
$$g(v_{2i}, j_0) = u_{j_0} \qquad g(v_{2i+1}, j_1) = u_{j_1}$$
$$f(v_{2i}, h_0) = u_{h_0} \qquad f(v_{2i+1}, h_1) = u_{h_1}$$

Informally f and g are in relation if there is a level, where the forward edges are exchanged between a *parallel* and *crossing* configuration and in addition the edge labels are changed.

    Then $m = m' = O(nk^2)$, $p \in O(n)$ for the number of levels and $(k-1)^2$ for the number of possible forward edge labels. We also have $l_{f,(v,j)} = k-1$ if $f(v, j) \in \{u_0, \ldots, u_{k-1}\}$ and $l_{f,(v,j)} = (k-1)^2$ otherwise. The value $l'_{g,(v,j)}$ is the same. Since only one of $f(v, j), g(v, j)$ can be in $\{u_0, \ldots, u_{k-1}\}$ we have $l_{\max} = O(k^3)$ and the lower bound follows. $\qquad\square$

    For the matrix model, there is a much simpler lower bound which works even for undirected graphs.

Theorem 23 CONNECTIVITY *requires* $\Omega(n^{3/2})$ *queries in the matrix model.*

Proof: We use Ambainis' method for the following special problem. You are given a symmetric matrix $M \in \{0,1\}^{n \times n}$ with the promise that it is the adjacency matrix of a graph with exactly one or two cycles, and have to find out which is the case.

Let X be the set of all adjacency matrices of a unique cycle, and Y the set of all adjacency matrices with exactly two cycles each of length between $n/3$ and $2n/3$. We define the relation $R \subseteq X \times Y$ as $M \, R \, M'$ if there exist $a, b, c, d \in [n]$ such that the only difference between M and M' is that $(a,b), (c,d)$ are edges in M but not in M' and $(a,c), (b,d)$ are edges on M' but not in M. See figure 5. The definition of Y implies that in M the distance from a to c is between $n/3$ and $2n/3$.
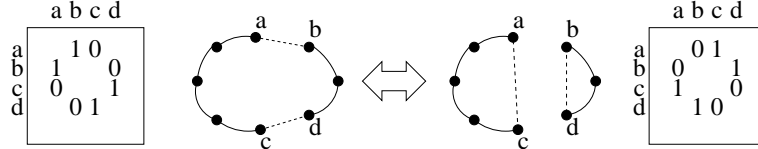

Figure 5: Illustration of the relation

Then $m = O(n^2)$ since there are $n-1$ choices for the first edge and $n/3$ choices for the second edge. Also $m' = O(n^2)$ since from each cycle one edge must be picked, and cycle length is at least $n/3$.

We have $l_{M,(i,j)} = 4$ if $M_{i,j} = 0$ since in M' we have the additional edge (i,j) and the endpoints of the second edge must be neighbors of i and j respectively. Moreover $l_{M,(i,j)} = O(n)$ if $M_{i,j} = 1$ since then (i,j) is one of the edges to be removed and there remains $n/3$ choices for the second edge.

The values $l'_{M',(i,j)}$ are similar, so in the product one factor will always be constant while the other is linear giving $l_{M,(i,j)} l'_{M',(i,j)} = O(n)$ and the theorem follows. $\qquad\square$

We give a lower bound for both minimum spanning tree and single-source shortest paths.

Theorem 24 *Finding minimum spanning tree and single source shortest paths require $\Omega(\sqrt{nm})$ queries.*

Proof: The proof is a reduction from minima finding. Let $m = k(n+1)$ for some integer k. Let M be a matrix with n rows and k columns and positive entries. The lower bound on minima finding, with $d = n, N = kn$, shows that $\Omega(\sqrt{kn^2})$ queries are required to find the minimum value in every row.

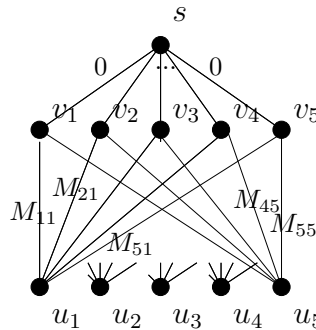
Figure 6: Reduction from finding minima to minimal spanning tree

We construct a weighted graph G from M, like this: The vertices are $V(G) = \{s, v_1, \ldots, v_k, u_1, \ldots, u_n\}$. The edges are all (s, v_i) with weight 0 and all (v_i, u_j) with weight M_{ji}. (See figure 6.) Then clearly a minimum spanning tree contains the 0-weight edges connecting s to all vertices v_i. And every vertex u_j will be connected to the rest of the graph only with the minimal weighted edge. $\qquad\square$

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