Single quantum querying of a database

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We present a class of fast quantum algorithms, based on Bernstein and Vazirani's parity problem, that retrieve the entire contents of a quantum database Y in a single query. The class includes binary search problems and coin-weighing problems. Our methods far exceed the efficiency of classical algorithms which are bounded by the classical information-theoretic bound. We show the connection between classical algorithms based on several compression codes and our quantum-mechanical method.

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I. INTRODUCTION

Quantum computers have been shown recently to be able to solve certain problems faster than any known algorithm running on a classical computer [1–3]. These problems include factoring, which can be performed in polynomial time on a quantum computer [2], but is widely believed to be exponentially difficult on a classical computer, and database lookup, which is provably faster on a quantum computer [3]. Understanding the power of quantum algorithms and developing new algorithms is of major interest as the building of a quantum computer will require a huge investment.

In this paper we present quantum algorithms for binary search and coin-weighing problems in which the information in a quantum database is retrieved with a single query. These are applications of Bernstein and Vazirani parity problem [5] and provide a strong illustration of the power of quantum computation and point out the limitations of classical information-theoretic bounds applied to quantum computers.

Information theory is a useful tool for analyzing the efficiency of classical algorithms. Problems involving information retrieval from a database are particularly amenable to such analysis. Consider this database search problem: we have a database Y that contains n items, of which a single one is marked. This database is represented as a bit string y of length n with Hamming weight one (y has exactly one "1"). One would like to locate the

marked item in as few queries to the database as possible. The queries are bit strings x of length n such that the database returns the answer

$$a(x,y) = x \cdot y \equiv \left(\sum_{i=1}^{n} x_i y_i\right) \mod 2$$
 (1.1)

where x_i and y_i are the i^{th} bits of x and y. A simple version of this problem is the case in which the allowed queries x have Hamming weight 1. The information retrieved by a single query $x_j = \delta_{ij}$ is small—it adds or eliminates item i from the set of possible marked items. It thus takes n-1 queries to locate the marked item in the worst case. A surprising result of Grover [3] is that a quantum mechanical algorithm can be faster than this and find the marked item with high probability in $O(\sqrt{n})$ quantum queries, contrary to one's "classical" intuition. Grover's algorithm does not, however, violate the information theoretic lower bound on the minimal number of queries M.

The information-theoretic lower bound [4] on M is given by the amount of information in the database divided by the maximal amount of information retrieved by a query which has A possible answers, i.e.

$$M \ge \frac{H(Y)}{\log_2 A},\tag{1.2}$$

where $H(Y) = -\sum_y p_y \log_2 p_y$, p_y is the probability for Y to contain y and $\sum_y p_y = 1$.

A quantum algorithm employs a database which responds to superpositions of queries with superpositions of answers. The quantum database acts on two input registers: register X containing the query state $|x\rangle$ and register B, an output register of dimension A initially containing state $|b\rangle$. We define the operation of querying the database as

$$R_y: |x,b\rangle \to |x,[b+a(x,y)] \mod A$$
 (1.3)

where R_y is a classical reversible transformation which maps basis states to basis states (that is, a permutation matrix) depending on the contents of the database, and a(x,y) is the answer to query x, given database state y. In a classical query only query basis states $|x\rangle$ are used and the output register B is initially set to $|0\rangle$. However,

a quantum database is not restricted to working only on basis states but can handle arbitrary superpositions of inputs [7]. Because of this the information that is retrieved by a single quantum query is not bounded by $\log_2 A$. The relevant quantity in the quantum setting is the accessible information in the registers X and B (together called XB) and the internal state of the quantum computer Φ about the database Y. Together these are always in a one of a set of pure states $\{|\psi_y\rangle, p_y\}_{y\in Y}$ and the accessible information on y is bounded by the Kholevo bound [8]

$$I_{\rm acc}(\Phi XB) \le S(\Phi XB),$$
 (1.4)

where $S(\Phi XB) = -\text{Tr}\rho_{\Phi XB}\log_2\rho_{\Phi XB}$ is the Von Neumann entropy of ΦXB and $\rho_{\Phi XB} = \sum_y p_y |\psi_y\rangle\langle\psi_y|$. In the case of a classical query, the Von Neumann entropy $S(\Phi XB)$ is strictly less than $\log_2 \dim(B) = \log_2 A$ which gives rise to the classical bound (1.2). The quantum algorithms that will be presented in this paper "violate" the classical information-theoretic bound by extracting extra information in the phases of the query register X. It is notable that Grover's Hamming weight one problem [3] has been proven optimal [9,10]; no quantum algorithm for this problem can violate the classical information-theoretic bound (1.2).

The quantum algorithms presented here are a major improvement on the classical algorithms in terms of computation time if the computation performed by the database is costly. All our algorithms make use of an interaction with the database of the form $a(x,y) = x \cdot y$. A direct implementation of such a database takes $O(\log n)$ time using n Toffoli and n XOR gates in parallel. The database can however be more general than this. Any function f(x) with the property that it can be written as $f(x) = x \cdot y$ can function as a database. The circuit that computes f(x) for any input $x \in \{0,1\}^n$ runs in some time T(n). We will compare the running time of the quantum algorithms including this circuit to the classical running time.

II. THE PARITY PROBLEM AND COIN WEIGHING

Bernstein and Vazirani [5,6] have given the first problem in which a single quantum query to the database is sufficient and a strong violation of the classical information theoretic bound comes about. In their parity problem they consider a database Y which contains an arbitrary n-bit string y. The answer to queries represented by n-bit strings x to the database is the parity of the bits common to x and y given by $a(x,y) = x \cdot y$. Note the problem is to determine y in its entirety, not to merely determine the parity of y. Bernstein and Vazirani show that y can be determined in a single query to the database. Here we apply this quantum algorithm to the coin weighing problem.

Coin weighing problems are a group of problems in which a set of defective coins is to be identified in a total set of coins. Assume there are two types of coins, good and bad ones, and we can weigh arbitrary sets of coins with a spring-scale (which gives the weight of the set of coins directly, as opposed to a balance which compares two sets of coins). All sets of coins are equiprobable. A set of n coins is represented as a bit string y of length n where $y_i = 1$ indicates that coin i is defective. A weighing can be represented by a query string x, where x_i specifies whether coin i is included in the set to be weighed. The result of a classical weighing is the Hamming weight of the bitwise product of x and y, $w_H(x \land y)$. For this problem the information theoretic bound (1.2) gives

$$M \ge \frac{n}{\log_2(n+1)}. (2.1)$$

This is close to what the best predetermined algorithm which perfectly identifies the set of coins can achieve [4]

$$\lim_{n \to \infty} M_{\text{pre}}(n) = \frac{2n}{\log_2(n)}.$$
 (2.2)

If one has a spring scale capable of performing weighings in superposition, then, one can use the Bernstein-Vazirani algorithm to identify the defective coins perfectly with a single weighing.

Define $n' \equiv 2\lceil \frac{n+1}{2} \rceil - 1$. We construct a query state

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x} |x\rangle \otimes \frac{1}{\sqrt{n'+1}} \sum_{b=0}^{n'} (-1)^b |b\rangle. \tag{2.3}$$

The special preparation of register B will make the result of a quantum query end up in the phases of X while leaving register B itself unchanged (cf. [9]). After the query we have (using $(-1)^{w_h(x \wedge y)} = (-1)^{x \cdot y}$)

$$|\psi_y\rangle = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{x \cdot y} |x\rangle \otimes \frac{1}{\sqrt{n'+1}} \sum_{b=0}^{n'} (-1)^b |b\rangle.$$
 (2.4)

Thereafter we perform a Hadamard transform H on the query register

$$H: |x\rangle \to \frac{1}{\sqrt{2^n}} \sum_{z} (-1)^{x \cdot z} |z\rangle.$$
 (2.5)

This results in the final state

$$|y\rangle \otimes \frac{1}{\sqrt{n'+1}} \sum_{b=0}^{n'} (-1)^b |b\rangle, \tag{2.6}$$

thus retrieving y within a single query.

Note that this coin-weighing algorithm uses only the parity of the Hamming weight of the answer whereas the full Hamming weight is available from the database and is fruitfully used in the classical algorithm.

We can compare the total running time of this quantum algorithm with that of the classical algorithm. The preprocessing and postprocessing of the register X of the query state and the preprocessing of the B register all consist of the Hadamard transforms on individual bits

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1\\ 1 & -1 \end{pmatrix}. \tag{2.7}$$

which can be done in parallel. The total running time is then simply 2 + T(n). In the classical algorithm the database circuit is used at least (see Eq. 2.1) $n/\log_2(n+1)$ times resulting in a total running time of at least $nT(n)/\log_2(n+1)$.

III. COMPRESSIVE ALGORITHMS

In this section we will consider modifications of this problem in which the information in the database H(Y) is less than n bits. In these cases retrieving the data from the source can be viewed as a problem of data compression of a source Y. We will restrict ourselves here to the compression of the data from a single use of the source. In the classical case, each query to the database retrieves a single digit of the word into which the bit string y will be encoded. The minimal set of predetermined classical queries will serve to construct the single quantum query algorithm. We will use coding schemes that minimize the amount of pre/postprocessing in the single query quantum algorithm. A classically optimal encoding scheme (cf. [12]) has

$$H(Y) \le \sum_{i} p_i l_i \le H(Y) + 1,$$
 (3.1)

where l_i the length of the compacted y_i and p_i the probability that the database contains bit string y_i . It is not guaranteed however that such an optimal encoding scheme can be implemented by the type of database interaction that one is given to use, namely (1.3).

In the following section we present single query quantum algorithms of which the construction is based on optimal classical encoding schemes, namely Huffman coding. In the section thereafter we consider more general type of databases and use a random coding scheme. Each of these schemes will require precomputation of a set of queries based on the encoding schemes. The time for this computation will not be counted in the total running time as the queries can be precomputed once and reused on subsequent problems.

A. Binary search problem

Binary search problems are defined as problems in which the database responds with two answers to the query. Here we look at such a search problem in which the queries have Hamming weight n/2. The database contains a bit string y with Hamming weight 1. Let us first look at the problem in which all these bit strings are equiprobable. We assume that n is an integer power of two. For other n one simply extends the database size to the next higher power of two.

Classically it is well known that the marked item can be found in $\log_2 n$ queries, which achieves the classical information-theoretic bound (1.2). The k^{th} query, g_k , is a string of 2^{k-1} zeros alternating with a string of 2^{k-1} ones, where $k = 1 \dots \log_2 n$, i.e.

$$g_1 = 01010101...$$

 $g_2 = 00110011...$
 $g_3 = 00001111...$
etc (3.2)

The result of query g_k is

$$z_k \equiv g_k \cdot y = a(g_k, y), \tag{3.3}$$

where z_k is the $k^{\rm th}$ bit of the encoding z of y. Each y will have a different encoding z and thus z uniquely determines y. The g_k s are the generators of the group F of Walsh functions whose group multiplication rule is addition modulo 2. We can represent a Walsh function f_s as

$$f_s = \sum_{i=1}^{\log_2 n} g_i s_i \mod 2, \tag{3.4}$$

where s is an arbitrary bit string of length $\log_2 n$. The quantum-mechanical algorithm makes use of superpositions of all the Walsh functions. We construct the query state

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{s} |s, f_s\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$
 (3.5)

After one query the state becomes

$$|\psi_y\rangle = \frac{1}{\sqrt{n}} \sum_s (-1)^{a(f_s,y)} |s, f_s\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$
 (3.6)

It can be shown that

$$\langle \psi_y | \psi_{y'} \rangle = \frac{1}{n} \sum_s (-1)^{a(f_s, y) + a(f_s, y')} = \delta_{yy'}.$$
 (3.7)

We can write

$$a(f_s, y) = \sum_{k=1}^{\log_2 n} s_k a(g_k, y) \mod 2.$$
 (3.8)

Using (3.3) it follows that $a(f_s, y) = s \cdot z$, and with this we can rewrite (3.7) as

$$\langle \psi_y | \psi_{y'} \rangle = \frac{1}{n} \sum_s (-1)^{s \cdot z + s \cdot z'} = \delta_{zz'} = \delta_{yy'}. \tag{3.9}$$

As all states $|\psi_y\rangle$ and $|\psi_{y'}\rangle$ are orthogonal, they can be distinguished by a measurement and no further queries to the database are required.

What are the transformations that are required for preand postprocessing of the query? The preparation of the queries takes $1 + n/2 \log_2 n$ steps. Register s is prepared in superposition using parallel one-bit Hadamard transforms and used as input to the circuit shown in Figure 1. The circuit in Figure 1 uses multi-bit XORs which we have counted as being in series. The same sequence in reverse is used as the postprocessing. The total time is thus $2 + n \log_2 n + T(n)$. In the classical case the queries are also prepared using the circuit in Figure 1, but the multi-bit XORs can be done in parallel, and the total time is $(\log_2 n)/2 + T(n)\log_2 n$. Note that in the Cirac-Zoller ion-trap model [13] of quantum computation a multi-bit XOR gate can be done in parallel by using the "bus phonon" modes. Such quantum computers run our algorithm in time $2 + \log_2 n + T(n)$.

Note that we could have used all possible queries as in section II to retrieve y and subsequently compacted y to z. This would have taken 2+T(n) for the algorithm plus $n\log_2 n$ steps for the compression, which is the same as this direct compression.

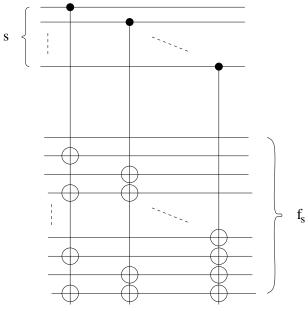


FIG. 1. "Walsh" Circuit

If we generalize this problem to databases that have unequal probabilities assigned to different y's, a scheme based on Huffman coding [12] is sometimes more efficient in terms of pre/post processing. We assume that the probability distribution, or H(Y) is known beforehand.

Huffman coding is a fixed-to-variable-length encoding of a source, in our case the database, which is optimal in the sense that it minimizes the average codeword length $\sum_i p_i l_i \leq H(Y) + 1$ with H(Y) the entropy of the source. The encoding prescribes a set of queries that play the role

of the Walsh generators in the equal probabilities case. This is illustrated with an example in Figure 2. (In fact, the Huffman construction results in Walsh queries in the equal-probability-case.)

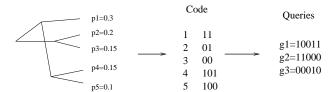


FIG. 2. Example of a Huffman code

Classically, instead of querying with the Walsh generators, one can use these Huffman queries, until the marked item has been found. The optimality of the Huffman code assures that the expected number of queries is minimized. Choose the set of queries that will take the place of the Walsh generators in the following way. Select a set of m queries, the first m queries that are used in the classical case, such that the probability of not finding the marked item after these m queries is very small. The value of m will depend on the probability distribution. If

$$m \le \lceil \log_2 n \rceil,\tag{3.10}$$

this Huffman scheme can be more efficient than a Walsh scheme.

Note that this requirement is not necessarily satisfied for all probability distributions. For example for the distribution $p_1 = 1/10, p_i = \frac{9}{10n}, i = 1 \dots n$, the length of n-1 encoded words will be about $\lceil \log_2 n \rceil$. If we choose $\lceil \log_2 n \rceil - s$ queries, the probability of error will be go to 9/10 exponentially as 2^{-s} .

This set of m queries will take the place of the Walsh generators in our quantum algorithm. The circuit which implements the Huffman queries will be as in Fig. 1 but with a different pattern of XORs corresponding the Huffman queries. All the database states that gave rise to distinct codewords after these m classical queries will give rise to distinct $|\psi_y\rangle$ in our quantum algorithm.

If we query only once, the total running time will be 2+mn+T(n) if we are willing to accept a small chance of error. A classical algorithm that uses the same Huffman queries and has the same probability of error takes m/2+mT(n) time. Thus for some probability distributions, m can be significantly smaller than $\log_2 n$ and the algorithm is faster than a straightforward search with the Walsh queries.

B. Random coding

The binary search and the coin weighing problem are special cases of a more general problem in which we have a database Y that contains k arbitrary base A strings of length n. Here we restrict ourselves to databases that contain k equally probable strings.

The queries x are all possible base A strings of length n, the elements of $(\mathbb{Z}_A)^n$. The database returns the answer

$$a(x,y) = \sum_{i=1}^{n} x_{i} y_{i} \mod A \equiv x \cdot y . \tag{3.11}$$

The information H(Y) is equal to $\log_A k$. A classical predetermined algorithm to determine y with high probability makes use of $m = \log_A k + l$ random strings, where l is a small integer. Pick m linearly independent random base A strings of length n; these are the queries g_i , $i = 1 \dots m$. Similarly to (3.3) we define the encoding as

$$z_k = g_k \cdot y \tag{3.12}$$

where z_k is the k^{th} digit of z. The g_i s are used to compress the string y of length n to the codewords z of length m. What is the probability that the codeword z determines y uniquely? The probability that two base A strings of length m are mapped onto the same codeword is equal to $\left(\frac{1}{A}\right)^m$. Thus, the probability of a collision with y is

$$p_{\text{col}} = 1 - \left(1 - \left(\frac{1}{A}\right)^m\right)^{k-1}.$$
 (3.13)

For small 2^{-l} we approximate

$$p_{\text{col}} = 1 - \left(1 - \frac{2^{-l}}{k}\right)^{k-1} \sim 2^{-l}(1 - 1/k) + O(2^{-2l})$$
 (3.14)

This probability can be made arbitrarily small for only a relatively small l. Thus $O(\log_A k)$ random g_i s are sufficient to retrieve the information with arbitrarily low probability of error. It is clear that for negative l the length of the codewords is not sufficiently large to avoid collisions. A codeword length of $O(\log_A k)$ is thus necessary as well as sufficient. If the contents of the database are to be determined with certainty, the codeword length must be made larger. A code with no collisions and with codeword length $O(2\log_A k)$ always exists (cf. the discussion of the birthday problem [15]).

Our quantum algorithm to determine the contents of the database in a single query with high probability makes use of this classical random coding construction. The random strings g_i , $i = 1 \dots m$ are the generators of a group C_A . The multiplication rule for this group is a digit-wise addition modulo A and the identity element is the string 0. Members of C_A can be written as

$$c(s) \in C_A \Rightarrow c(s)_k = \sum_i (g_i)_k s_i \mod A$$
 (3.15)

with $c(s)_k$ the k^{th} digit of a group element c(s) and s is a base A string of length m. Due to the linear independence of the generators, C_A is a subgroup of $(\mathbb{Z}_A)^n$ with A^m elements.

For $c(s) \in C_A$ there is a one-to-one map between c(s) and its encoding z defined in (3.12). This is true as

$$\forall_i \ c \in C_A, \ c \cdot g_i = 0 \Leftrightarrow c = 0 \tag{3.16}$$

which follows from the linear independence of the generators.

In the quantum algorithm we construct a state

$$|\psi\rangle = \frac{1}{\sqrt{m}} \sum_{s|c(s) \in C_A} |s, c(s)\rangle \otimes \frac{1}{\sqrt{A}} \sum_{b=0}^{A-1} \omega_A^b |b\rangle, \quad (3.17)$$

with $\omega_A = e^{\frac{i2\pi}{A}}$. The query results in the state

$$|\psi_y\rangle = \frac{1}{\sqrt{m}} \sum_{s|c(s)\in C_A} \omega_A^{-a(c(s),y)} |s,c(s)\rangle \otimes \frac{1}{\sqrt{A}} \sum_{b=0}^{A-1} \omega_A^b |b\rangle.$$
(3.18)

We can write, using the encoding z of y defined in (3.12)

$$a(c(s), y) = s \cdot z. \tag{3.19}$$

Thus we have

$$\langle \psi_y | \psi_{y'} \rangle = \frac{1}{m} \sum_s \omega_A^{s \cdot (z - z')} = \delta_{zz'}. \tag{3.20}$$

If two different strings y and y' are mapped onto a different codeword, they are thus distinguishable by a measurement. The probability that this occurs (3.14) can be made arbitrary small just as in the classical case since the encoding is the same. In order to measure, we reverse the preparation steps and then we perform a Fourier transform over $(\mathbb{Z}_A)^n$

$$H_A: |s\rangle \to \frac{1}{\sqrt{m}} \sum_{z} \omega_A^{s,z} |z\rangle.$$
 (3.21)

A measurement in the query basis determines z and, with high probability, y.

The circuit used to implement the random coding is similar to that in Figure 1 but the XORS are replaced by summation base A operators,

$$XOR_A(a,b) = (a+b) \bmod A \tag{3.22}$$

and their locations are according to the random queries. The total quantum running time is 2 + mn + T(n). Here the basic unit of time is an operation on an Adimensional Hilbert space. The classical time using the same random codewords is m/2 + mT(n). Since m is less than n, this algorithm is better than the direct coin-

weighing algorithm provided we are willing to tolerate a small chance of error bounded by $p_{\rm col}$.

IV. DISCUSSION

We have discussed the complexity of our quantum algorithms compared to a classical setup and shown that the quantum algorithms are faster in situations in which T(n) > O(n). In problems where querying the database would occur repeatedly, a bigger (real) separation between the classical computation time and the quantum computation time could be achieved (see [6] for an instance of such a problem).

It is noteworthy that in the binary search problem in the classical case only the generators of the Walsh functions are required, while the quantum algorithm needs all the Walsh functions to achieve this speedup. It would be interesting to find out whether any speedup is possible if the database only responds to queries which are the generators.

We have chosen the quantum database R_y as defined in (1.3) to make a fair comparison with the classical setting. A unitary U_y could easily become more powerful as was pointed out in [14]. At its most general, a quantum database could be defined by an arbitrary unitary transformation acting on an input register and a hidden quantum state (the database). This has no good classical analogue and might be be worthwhile to explore.

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- [1] D. Simon, "On the power of quantum computation," Proceedings of the 35th Annual Symposium on the Foundations of Computer Science (IEEE Computer Society Press, Los Alamitos, CA 1994), p. 116.
- [2] P.W. Shor, "Algorithms for quantum computation: discrete log and factoring," Proceedings of the 35th Annual Symposium on the Foundations of Computer Science (IEEE Computer Society Press, Los Alamitos, CA 1994), p. 124.
- [3] L.K. Grover, "A fast quantum mechanical algorithm for database search," Proceedings of the 28th Annual ACM Symposium on Theory of Computing, 1996, pp. 212-219.
- [4] cf. Martin Aigner, Combinatorial Search, John Wiley & Sons, 1988.
- [5] E.Bernstein, U.Vazirani, "Quantum complexity theory", Proceedings of the 25th Annual ACM Symposium on Theory of Computing, 1993, pp.11-20.
- [6] E.Bernstein, U.Vazirani, "Quantum complexity theory", final version of [5]. To appear in SIAM Journal on Computing.
- [7] One could have a nonunitary quantum database where

- the X register is always reset to $|0\rangle$ to ensure it gives no information on Y. This would not matter to a classical query where all the information is in B, but severely cripples the quantum algorithm. It is not known if such a database will allow any improvement over classical algorithms.
- [8] A.S. Kholevo, Problemy Peredachi Informatsii; 9, 3 (1973). This paper has appeared in English translation in Problems of Information Transmission 9, 177 (1973).
- [9] C.H. Bennett, E. Bernstein, G. Brassard and U. Vazirani, to appear in SIAM Journal on Computing, also Report-No. quant-ph/9701001.
- [10] M. Boyer, G. Brassard, P. Hoyer, and A. Tapp, to appear in the Proceedings of *Physics of Computation '96*, also Report-No. quant-ph/9605034.
- [11] C.E. Shannon, Bell Syst. Tech. J. 27, 379,623 (1948).
- [12] T.M.Cover, J.A.Thomas, Elements of information theory, Wiley Series in Telecommunications, 1991.
- J.I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
 cf S. Braunstein and J.A. Smolin, Phys. Rev. A. **55**, 945 (1997) for a discussion.
- [14] J.Machta, "Phase information in Quantum Oracle Computing", Physics Department, University of Massachusetts at Amherst, manuscript, May 1996.
- [15] W.Feller An introduction to probability theory and its applications, Vol. I, John Wiley & Sons, 1957, p.33.