Notes on the hidden subgroup problem on some semi-direct product groups

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We consider the hidden subgroup problem on the semi-direct product of cyclic groups $\mathbb{Z}_N \rtimes \mathbb{Z}_p$ with some restriction on N and p. By using the homomorphic properties, we present a class of semi-direct product groups in which the structures of subgroups can be easily classified. Furthermore, we show that there exists an efficient quantum algorithm for the hidden subgroup problem on the class.

I. INTRODUCTION

Most of exponentially fast quantum algorithms, such as the Simon algorithm [1] and the period-finding algorithm [2], can be regarded as one for a special problem called the hidden subgroup problem (HSP). The HSP is to find a subgroup H of a given group G with an oracle function f defined on G such that $f(a) = f(b)$ if and only if $aH = bH$ for all a, b in G. It is well known that if the group G is abelian then the HSP on G can be efficiently solved by a quantum algorithm of running time polynomial in $\log |G|$, while no solution is known for the general case of nonabelian groups. In particular, since the graph isomorphism problem and certain lattice problem can be reduced to the HSP on the symmetric group and the HSP on the dihedral group, respectively [3, 4], it was naturally asked whether there exists an efficient quantum algorithm for the HSP on nonabelian groups, and has been actively studied [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

One way to construct a quantum algorithm for the HSP is first to explicitly investigate the structures of all subgroups of a given group, and then to find a quantum algorithm applicable to each subgroup structure. Recently, Inui and Le Gall [13] presented an efficient quantum algorithm for the HSP on the groups $\mathbb{Z}_{p^r} \rtimes \mathbb{Z}_p$ for odd prime p by classifying all the possible subgroups. Employing such a method, we can also show that there exists an efficient quantum algorithm for the HSP on $\mathbb{Z}_{2p^r} \rtimes \mathbb{Z}_p$ for odd prime p as in Appendix.

Since multiplicative groups $\mathbb{Z}_{p^r}^*$ and $\mathbb{Z}_{2p^r}^*$ are cyclic, it can be verified that $\mathbb{Z}_{p^r} \rtimes \mathbb{Z}_p$ and $\mathbb{Z}_{2p^r} \rtimes \mathbb{Z}_p$ have the same form of subgroup structures, and hence we can obtain the same result for the HSP. However, if one exploits the above method to solve the HSP on general semi-direct product groups, then it is hard to find a quantum algorithm for the HSP on the groups since it is difficult, even mathematically, to classify the subgroup structures.

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In this work, we consider the HSP on the semi-direct products of cyclic groups $\mathbb{Z}_N \rtimes \mathbb{Z}_p$, where N is factorized as $N = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$, and a prime p does not divide each $p_j - 1$. By using the homomorphic properties, we show that there exists an efficient quantum algorithm for the HSP on the groups.

This paper is organized as follows. In Sec. II we briefly introduce the semi-direct product groups, and explain some homomorphic properties of the semi-direct product groups. In Sec. III we show that there exists an efficient quantum algorithm for the HSP on the groups. Finally, in Sec. IV we summarize our result.

II. SEMI-DIRECT PRODUCT GROUPS

For any positive integer N and p, and any group homomorphism ϕ from the group $\mathbb{Z}_p$ into the group of automorphisms of $\mathbb{Z}_N$, the semi-direct product group $\mathbb{Z}_N \rtimes_\phi \mathbb{Z}_p$ is the set $\{(a, b) : a \in \mathbb{Z}_N, b \in \mathbb{Z}_p\}$ with the group operation $(a_1, b_1)(a_2, b_2) = (a_1 + \phi(b_1)(a_2), b_1 + b_2)$.

We note that the group $\mathbb{Z}_N \rtimes_\phi \mathbb{Z}_p$ is obviously generated by the two elements $x = (1, 0)$ and $y = (0, 1)$, and that ϕ is completely determined by the function value $\phi(1)(1)$, since $\phi : \mathbb{Z}_p \to \mathrm{Aut}(\mathbb{Z}_N)$ is a homomorphism and $\phi(a)$ is an automorphism for every $a \in \mathbb{Z}_p$. Using the fact that $\phi(b)(a) = a\phi(1)(1)^b$, we obtain the relation

$$y^b x^a = x^{a\phi(1)(1)^b} y^b. \tag{1}$$

Due to the facts that $\phi(1)(1)$ is relatively prime to N and that $\phi(1)(1)^p = 1 \pmod{N}$, $\mathbb{Z}_N \rtimes_\phi \mathbb{Z}_p$ is completely determined by $\phi(1)(1)$. For example, the case $\phi(1)(1) = 1$ leads to the direct product $\mathbb{Z}_N \times \mathbb{Z}_p$, and if $\phi(1)(1) \neq 1$ and p is prime then p is the smallest positive integer satisfying

$$\phi(1)(1)^p = 1 \pmod{N}, \tag{2}$$

that is, $\phi(1)(1)$ is one of elements of $\mathbb{Z}_N^*$ with order p, and p is a divisor of $\varphi(N)$, where φ is the Euler phi-function.

We now consider the semi-direct product group $\mathbb{Z}_{q^s p^r} \rtimes_\phi \mathbb{Z}_p$, where p and q are distinct primes, and s

and r are positive integers. Since $\mathbb{Z}_{q^s p^r} \rtimes_\phi \mathbb{Z}_p$ is isomorphic to $(\mathbb{Z}_{q^s} \times \mathbb{Z}_{p^r}) \rtimes_\phi \mathbb{Z}_p$, for each $\alpha \in \mathbb{Z}_p$, $\phi(\alpha)$ can be regarded as an automorphism on $\mathbb{Z}_{q^s} \times \mathbb{Z}_{p^r}$ such that $\phi(1)(1)$ is an element in $\mathbb{Z}_{q^s}^* \times \mathbb{Z}_{p^r}^*$ of order p. Then we can obtain the following lemma:

Lemma 1. *For each $\alpha \in \mathbb{Z}_p$, $a \in \mathbb{Z}_{q^s}$ and $b \in \mathbb{Z}_{p^r}$, $\phi(\alpha)(a, 0) = (a', 0)$ and $\phi(\alpha)(0, b) = (0, b')$ for some $a' \in \mathbb{Z}_{q^s}$ and $b' \in \mathbb{Z}_{p^r}$.*

Proof. Suppose $\phi(\alpha)(1, 0) = (c, d)$ for some $c \in \mathbb{Z}_{q^s}$ and $d \in \mathbb{Z}_{p^r}$. Then since $\phi(\alpha)$ is an homomorphism, $\phi(\alpha)(0, 0) = (0, 0)$ and

$$
\begin{aligned}
\phi(\alpha)(0, 0) &= \phi(\alpha)(q^s, 0) \\
&= q^s \phi(\alpha)(1, 0) \\
&= (q^s c, q^s d). \quad (3)
\end{aligned}
$$

Since $\gcd(p, q) = 1$ and $q^s d = 0 \pmod{p^r}$, $d = 0 \pmod{p^r}$ and $\phi(\alpha)(1, 0) = (c, 0)$. It follows that for any $a \in \mathbb{Z}_{q^s}$ $\phi(\alpha)(a, 0) = (a', 0)$ for some $a' \in \mathbb{Z}_{q^s}$. Similarly, we can also obtain that for any $b \in \mathbb{Z}_{p^r}$, $\phi(\alpha)(0, b) = (0, b')$ for some $b' \in \mathbb{Z}_{p^r}$. □

Lemma 2. *Let p and q be distinct primes satisfying $p \nmid q - 1$. Then*

$$
(\mathbb{Z}_{q^s} \times \mathbb{Z}_{p^r}) \rtimes_\phi \mathbb{Z}_p = \mathbb{Z}_{q^s} \times (\mathbb{Z}_{p^r} \rtimes_\psi \mathbb{Z}_p) \quad (4)
$$

for some homomorphism ψ from $\mathbb{Z}_p$ to $\mathrm{Aut}(\mathbb{Z}_{p^r})$.

Proof. Since $\phi(p) = \phi(0)$ is the identity map $\mathcal{I}$ on $\mathbb{Z}_{q^s} \times \mathbb{Z}_{p^r}$, by Lemma 1, we can see

$$
(1, 0) = \mathcal{I}(1, 0) = \phi(p)(1, 0) = \phi(1)^p(1, 0) = (a^p, 0), \quad (5)
$$

where $(a, 0) = \phi(1)(1, 0)$ and $a^p = 1 \pmod{q^s}$. Since the order of $\mathbb{Z}_{q^s}^*$ is $q^{s-1}(q-1)$ and $p \nmid q-1$, we obtain that a must be 1, that is, ϕ trivially acts on $\mathbb{Z}_{q^s}$. Thus, for each $\alpha \in \mathbb{Z}_p$, $\phi(\alpha) = \mathcal{I}_0 \times \psi(\alpha)$, where $\mathcal{I}_0$ is the identity map on $\mathbb{Z}_{q^s}$ and ψ is a homomorphism from $\mathbb{Z}_p$ to $\mathrm{Aut}(\mathbb{Z}_{p^r})$.

Therefore, the operation of the semi-direct product group is as follows:

$$
\begin{aligned}
((a, b), c)((a', b'), c') &= ((a, b) + \phi(c)(a', b'), c + c') \\
&= (a + a', b + \psi(c)(b'), c + c'),
\end{aligned}
\quad (6)
$$

and this implies that Eq. (4) holds. □

Lemma 3. *Let G_0 and G_1 be finite groups such that the order of G_0 is relatively prime to the order of G_1, and let H be a subgroup of $G_0 \times G_1$. Then $H = H_0 \times H_1$ for some subgroups H_0 of G_0 and H_1 of G_1.*

Proof. Let H be a subgroup of $G_0 \times G_1$ of order rs, where r and s divide the order of G_0 and the order of G_1, respectively. Then it is trivial that r and s are relatively prime to each other. Now, for each $j = 0, 1$, let π_j be the natural projection from $G_0 \times G_1$ onto G_j, and let $H_j = \pi_j(H) \subset G_j$. Then we can readily show that $H = H_0 \times H_1$. □

III. QUANTUM ALGORITHM

In this section, we present an efficient quantum algorithm that can solve the HSP over the semi-direct product of cyclic groups $\mathbb{Z}_N \rtimes_\phi \mathbb{Z}_p$, which is not a direct product of $\mathbb{Z}_N$ and $\mathbb{Z}_p$, where N is factorized as $N = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$, and a prime p does not divide each $p_j - 1$.

Since the p_j's are all distinct primes and p is a divisor of $\varphi(N) = p_1^{r_1 - 1} p_2^{r_2 - 1} \cdots p_n^{r_n - 1}(p_1 - 1)(p_2 - 1) \cdots (p_n - 1)$, $p = p_k$ and $r_k \geq 2$ for some $k \in \{1, 2, \ldots, n\}$. Thus, N is a multiple of p^2.

Due to the factorization of N, $\mathbb{Z}_N$ is isomorphic to the direct product of cyclic groups $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$, and we have

$$
\mathbb{Z}_N \rtimes_\phi \mathbb{Z}_p \cong (\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}) \rtimes_\phi \mathbb{Z}_p. \quad (7)
$$

By Lemma 1, for each $\alpha \in \mathbb{Z}_p$, the automorphism $\phi(\alpha)$ on $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$ acts trivially on each component of $\mathbb{Z}_{p_j^{r_j}}$ such that p differs p_j. Let $\mathcal{I}_1$ be the identity on $\mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_{k-1}^{r_{k-1}}}$ and $\mathcal{I}_2$ be the identity on $\mathbb{Z}_{p_{k+1}^{r_{k+1}}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$. Then, by Lemma 2, we obtain

$$
\mathbb{Z}_N \rtimes_\phi \mathbb{Z}_p \cong \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \widehat{\mathbb{Z}_{p_k^{r_k}}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}} \times (\mathbb{Z}_{p_k^{r_k}} \rtimes_\psi \mathbb{Z}_p), \quad (8)
$$

where $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \widehat{\mathbb{Z}_{p_k^{r_k}}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$ is the direct product of $\mathbb{Z}_{p_j^{r_j}}$'s except $\mathbb{Z}_{p_k^{r_k}}$ and ψ is a homomorphism from $\mathbb{Z}_p$ to $\mathrm{Aut}(\mathbb{Z}_{p_k^{r_k}})$ such that $\phi(\alpha) = \mathcal{I}_1 \times \psi(\alpha) \times \mathcal{I}_2$ for each $\alpha \in \mathbb{Z}_p$. Moreover, since $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \widehat{\mathbb{Z}_{p_k^{r_k}}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$ is a cyclic group of order $N/p_k^{r_k}$,

$$
\mathbb{Z}_N \rtimes_\phi \mathbb{Z}_p \cong \mathbb{Z}_{N/p_k^{r_k}} \times (\mathbb{Z}_{p_k^{r_k}} \rtimes_\psi \mathbb{Z}_p), \quad (9)
$$

by Eq. (8). Hence, the HSP on $\mathbb{Z}_N \rtimes_\phi \mathbb{Z}_p$ is essentially equivalent to the HSP on $\mathbb{Z}_{N/p_k^{r_k}} \times (\mathbb{Z}_{p_k^{r_k}} \rtimes_\psi \mathbb{Z}_p)$.

Furthermore, since the order of $\mathbb{Z}_{N/p_k^{r_k}}$ is relatively prime to the order of the group $\mathbb{Z}_{p^{r_k}} \rtimes_\psi \mathbb{Z}_p$, by Lemma 3, any subgroup H of $\mathbb{Z}_{N/p_k^{r_k}} \times (\mathbb{Z}_{p_k^{r_k}} \rtimes_\psi \mathbb{Z}_p)$ is of the form $H_1 \times H_2$, where H_1 and H_2 are subgroups of $\mathbb{Z}_{N/p_k^{r_k}}$ and $\mathbb{Z}_{p^{r_k}} \rtimes_\psi \mathbb{Z}_p$, respectively. We note that the HSP on a cyclic group and the HSP on a group of the form $\mathbb{Z}_{p^{r_k}} \rtimes_\psi \mathbb{Z}_p$ can be efficiently solved by quantum algorithms [2, 13]. Therefore, we can obtain the following result:

Theorem 1. *There exists an efficient quantum algorithm for the HSP on $\mathbb{Z}_N \rtimes_\phi \mathbb{Z}_p$, where N is factorized as $N = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$, and a prime p does not divide each $p_j - 1$.*

IV. SUMMARY

We have considered the hidden subgroup problem on the semi-direct product of cyclic groups $\mathbb{Z}_N \rtimes \mathbb{Z}_p$ with

some restriction on N and p. By using the homomorphic properties, we have presented a class of semi-direct product groups in which the structures of subgroups can be easily classified. Furthermore, we have shown that there exists an efficient quantum algorithm for the HSP on the class.

APPENDIX

In [13], Inui and Le Gall showed that there exists an efficient quantum algorithm solving the HSP on the group $\mathbb{Z}_{p^r} \rtimes \mathbb{Z}_p$. In this appendix, we show that Inui-Le Gall's method can be clearly applied to the case that the group is $\mathbb{Z}_{2p^r} \rtimes \mathbb{Z}_p$, where p is an odd prime.

We first note that the multiplicative group $\mathbb{Z}_{2p^r}^*$ is cyclic and its order is $p^{r-1}(p-1)$, and that $\mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$ is completely determined by the elements of order p in $\mathbb{Z}_{2p^r}^*$.

Proposition 1. *Let p be an odd prime. Then $\alpha \in \mathbb{Z}_{2p^r}^*$ and $|\alpha| = p$ if and only if*

$$\alpha = (2p^{r-1}+1)^i = 2ip^{r-1}+1 \pmod{2p^r}, \quad (A.1)$$

for $i \in \{1, 2, \ldots, p-1\}$.

Proof. It is clear that $(2p^{r-1}+1)^{ip} = 1 \pmod{2p^r}$ and $(2ip^{r-1}+1)^k = 2ikp^{r-1}+1 \neq 1 \pmod{2p^r}$ for any $k \in \{1, 2, \cdots, p-1\}$ since p cannot divide ik.

Furthermore, since $\mathbb{Z}_{2p^r}^*$ is a cyclic group of order $p^{r-1}(p-1)$,

$$A = \{2ip^{r-1}+1 : i = 0, 1, 2, \cdots, p-1\} \quad (A.2)$$

is a cyclic subgroup of $\mathbb{Z}_{2p^r}^*$. Let m be the minimal positive integer such that $w^m \in A$ where w is a generator of $\mathbb{Z}_{2p^r}^*$. Suppose there exists $w^b \in \mathbb{Z}_{2p^r}^*$ such that $|w^b| = p$. Then since $b = sm + r$ for some $s, r \in \mathbb{Z}$ with $0 \leq r < m$, $(w^r)^p = (w^b)^p = 1 \pmod{2p^r}$. Hence, by the minimality of m, we have $r = 0$. Therefore, $w^b \in A$, that is, every element of order p in $\mathbb{Z}_{2p^r}^*$ is in A. $\square$

1. Structure of the semi-direct product group $\mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$

Proposition 1 implies that $\phi(1)(1) \in A$ if and only if $\mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$ is well-defined. It follows that there are $p-1$ semi-direct product groups of the form $\mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$ other than $\mathbb{Z}_{2p^r} \times \mathbb{Z}_p$.

Proposition 2. *For each $i = 0, 1, 2, \ldots, p-1$, let ϕ_i be a homomorphism from $\mathbb{Z}_p$ to $\mathrm{Aut}(\mathbb{Z}_{2p^r})$ defined as $\phi_i(1)(1) = (2p^{r-1}+1)^i$. Then all $\mathbb{Z}_{2p^r} \rtimes_{\phi_i} \mathbb{Z}_p$'s are isomorphic to each other.*

Proof. For each $i = 1, 2, \ldots, p-1$, let Ψ_i be a map from $\mathbb{Z}_{2p^r} \rtimes_{\phi_1} \mathbb{Z}_p$ to $\mathbb{Z}_{2p^r} \rtimes_{\phi_i} \mathbb{Z}_p$ defined as $\Psi_i(x^a y^b) = x^a y^{bi^{-1}}$, where i^{-1} is the inverse of i in $\mathbb{Z}_p^*$. Then it can be easily checked that Ψ_i is a group isomorphism. $\square$

By Proposition 2, it suffices to consider $\mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$ with $\phi(1)(1) = 2p^{r-1}+1$. Now, we formalize the group presentation and the properties of the semi-direct product group $\mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$, and we then classify all of its subgroups in terms of the group presentation.

The group presentation of $\mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$ is

$$\left\langle x, y \mid x^{2p^r} = y^p = e, \ yx = x^{\phi(1)(1)}y = x^{2p^{r-1}+1}y \right\rangle. \quad (A.3)$$

Then we can have

Proposition 3.

$$\begin{aligned} y^b x^a &= x^{\phi(b)(a)} y^b = x^{a(2bp^{r-1}+1)} y^b, \\ (x^a y^b)^k &= x^{ak((k-1)bp^{r-1}+1)} y^{bk}, \\ (x^a y^b)^p &= x^{ap} \end{aligned} \quad (A.4)$$

and

$$|x^a y^b| = \begin{cases} \dfrac{2p^{r+1}}{\gcd(a, 2p^r)}, & \text{if } p^r | a \text{ and } b \neq 0, \\ \dfrac{2p^r}{\gcd(a, 2p^r)}, & \text{otherwise.} \end{cases} \quad (A.5)$$

for any $0 \leq a \leq 2p^r - 1$ and $0 \leq b \leq p-1$.

Proof. It follows from straightforward calculations that the first three properties are true, and for the last one, we divide it into several cases.

Case 1. $p^r | a$ and $b \neq 0$. If $p^r | a$, then $a = 0$ or $a = p^r$.
(a) If $a = 0$, then $x^a y^b = y^b$. Since $b \neq 0$,

$$|x^a y^b| = |y^b| = p = \frac{2p^{r+1}}{\gcd(a, 2p^r)}. \quad (A.6)$$

(b) If $a = p^r$, then $x^a y^b = x^{p^r} y^b$. Let $|x^{p^r} y^b| = d$. Then, by (A.4), we have

$$(x^{p^r} y^b)^d = x^{p^r d((d-1)bp^{r-1}+1)} y^{db} = e. \quad (A.7)$$

Since $b \neq 0$, $d = kp$ for some $k \in \mathbb{Z}$, and hence

$$(x^{p^r} y^b)^d = (x^{p^{r+1}})^k = e \quad (A.8)$$

which implies $2|k$. Thus, $2p|d$ and $(x^{p^r} y^b)^{2p} = e$, and hence

$$d = 2p = \frac{2p^{r+1}}{\gcd(a, 2p^r)}. \quad (A.9)$$

Case 2. $p^r \nmid a$ or $b = 0$.
(a) If $b = 0$ then it is clear that

$$|x^a y^b| = |x^a| = \frac{2p^r}{\gcd(a, 2p^r)}. \qquad (A.10)$$

(b) If $b \neq 0$ then $p^r \nmid a$. Let $|x^a y^b| = d$. Then since

$$(x^a y^b)^d = x^{ad((d-1)bp^{r-1}+1)} y^{db} = e, \qquad (A.11)$$

we have $p | d$ and so $(x^a y^b)^d = x^{ad} = (x^a)^d = e$. Thus, we have

$$\frac{2p^r}{\gcd(a, 2p^r)} \Big| d. \qquad (A.12)$$

Since $p^r \nmid a$, the left-hand side of (A.12) is a multiple of p, and hence

$$(x^a y^b)^{2p^r/\gcd(a,2p^r)} = x^{a2p^r/\gcd(a,2p^r)} = e. \qquad (A.13)$$

Therefore, by the minimality of d, we obtain

$$d = \frac{2p^r}{\gcd(a, 2p^r)}. \qquad (A.14)$$

$\square$

Furthermore, for cyclic subgroups generated by the element $x^a y^b$, we can have the following property.

Proposition 4. *Let $0 \leq a \leq 2p^r - 1$ and $0 \leq b \leq p - 1$. If $p^r | a$ and $b \neq 0$ then*

$$\langle x^a y^b \rangle = \left\{ x^{ai} y^{bi} : 0 \leq i \leq \frac{2p^{r+1}}{\gcd(a, 2p^r)} - 1 \right\}. \quad (A.15)$$

Otherwise,

$$\langle x^a y^b \rangle = \left\{ x^{ai} y^{bi} : 0 \leq i \leq \frac{2p^r}{\gcd(a, 2p^r)} - 1 \right\}. \quad (A.16)$$

Proof. Let i be an arbitrary integer. Then, by (A.4),

$$\begin{aligned}
(x^a y^b)^i &= x^{ai(i-1)bp^{r-1}} x^{ai} y^{bi} \\
&= ((x^a y^b)^p)^{i(i-1)bp^{r-2}} x^{ai} y^{bi}. \quad (A.17)
\end{aligned}$$

Thus

$$\left(((x^a y^b)^p)^{i(i-1)bp^{r-2}} \right)^{-1} (x^a y^b)^i = x^{ai} y^{bi}, \quad (A.18)$$

and hence

$$\{ x^{ai} y^{bi} : i \in \mathbb{Z} \} \subseteq \langle x^a y^b \rangle. \qquad (A.19)$$

Furthermore, $x^{ai} y^{bi} \neq x^{aj} y^{bj}$, for any i, j with $1 \leq i < j \leq \frac{2p^{r+1}}{\gcd(a,2p^r)} - 1$ if $p^r | a$ and $b \neq 0$, and with $1 \leq i < j \leq \frac{2p^r}{\gcd(a,2p^r)} - 1$ otherwise. Hence, we obtain (A.15) and (A.16). $\square$

Now, we are ready to classify all possible subgroups of $\mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$ in terms of group presentation. Let $H' = H \cap \langle x \rangle$. Then $H' = \left\langle x^{2^t p^s} \right\rangle$ for some integers t and s with $0 \leq t \leq 1$ and $0 \leq s \leq r$.

We assume that $H \neq \left\langle x^{2^t p^s} \right\rangle$. Then it is clear that $x^{a_0} y^{b_0} \in H$ for some integers a_0 and b_0 with $0 \leq a_0 \leq 2p^r - 1$ and $1 \leq b_0 \leq p - 1$. By Proposition 4, $x^{a_0 b_0^{-1}} y \in \left\langle x^{a_0} y^{b_0} \right\rangle \subseteq H$, where b_0^{-1} is the multiplicative inverse of b_0 in $\mathbb{Z}_p^*$. Furthermore, since $(x^{a_0 b_0^{-1}} y)^p = x^{a_0 b_0^{-1} p} \in H$, $2^t p^{s-1} | a_0 b_0^{-1}$. Thus, we have $x^{h 2^t p^{s-1}} y \in H$ where h is an integer such that $h 2^t p^{s-1} = a_0 b_0^{-1} \pmod{2^t p^s}$ and $0 \leq h \leq p - 1$. Hence we obtain $\left\langle x^{2^t p^s}, x^{h 2^t p^{s-1}} y \right\rangle \subseteq H$.

Let $x^a y^b \in H$ with $0 \leq a \leq 2p^r - 1$ and $0 \leq b \leq p - 1$. Then since $x^{h 2^t p^{s-1} b} y^b \in \left\langle x^{h 2^t p^{s-1}} y \right\rangle$ by Proposition 4, its inverse $(x^{h 2^t p^{s-1} b} y^b)^{-1} = y^{-b} x^{-h 2^t p^{s-1} b}$ is also in $\left\langle x^{h 2^t p^{s-1}} y \right\rangle \subseteq H$. Thus, $(x^a y^b)(y^{-b} x^{-h 2^t p^{s-1} b}) = x^{a - h 2^t p^{s-1} b}$ is contained in H, and so it is in H'. Hence we obtain $2^t p^s | a - h 2^t p^{s-1} b$, that is, $a = 2^t p^s m + h 2^t p^{s-1} b$ for some integer m. Thus,

$$x^a y^b = \left(x^{2^t p^s} \right)^m \left(x^{h 2^t p^{s-1} b} y^b \right) \in \left\langle x^{2^t p^s}, x^{h 2^t p^{s-1}} y \right\rangle, \qquad (A.20)$$

that is, $H = \left\langle x^{2^t p^s}, x^{h 2^t p^{s-1}} y \right\rangle$.

Therefore we obtain the following proposition.

Proposition 5. *For any subgroup $H \in \mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$, H is one of $\left\langle x^{2^t p^s} \right\rangle$, or $\left\langle x^{2^t p^s}, x^{h 2^t p^{s-1}} y \right\rangle$ for some integers t, s, and h with $0 \leq t \leq 1$, $0 \leq s \leq r$, and $0 \leq h \leq p - 1$.*

Proposition 6. *Let t, s, and h be integers with $0 \leq t \leq 1$, $0 \leq s \leq r$, and $0 \leq h \leq p - 1$. Then*

$$\left\{ x^{(bh \bmod p) 2^t p^{s-1}} y^b \right\}_{0 \leq b \leq p-1} \subseteq \left\langle x^{2^t p^s}, x^{h 2^t p^{s-1}} y \right\rangle.$$

Proof. For any $b = 0, \cdots, p - 1$, and $h = 0, \cdots, p - 1$,

$$x^{bh 2^t p^{s-1}} y^b = x^{l 2^t p^s} x^{(bh \bmod p) 2^t p^{s-1}} y^b, \qquad (A.21)$$

where l is an integer such that $bh = lp + (bh \bmod p)$. Since $x^{l 2^t p^s} \in \left\langle x^{2^t p^s} \right\rangle$ and $x^{bh 2^t p^{s-1}} y^b \in \left\langle x^{h 2^t p^{s-1}} y \right\rangle$ by Proposition 4,

$$x^{(bh \bmod p) 2^t p^{s-1}} y^b = (x^{l 2^t p^s})^{-1} x^{bh 2^t p^{s-1}} y^b \qquad (A.22)$$

is in $\left\langle x^{2^t p^s}, x^{h 2^t p^{s-1}} y \right\rangle$.

$\square$

2. Quantum algorithm

Here, we present a quantum algorithm solving the HSP over $\mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$, which is the same as that in [13]. The

hidden subgroup of $\mathbb{Z}_{2p^r} \rtimes_\phi \mathbb{Z}_p$ with respect to the oracle function f is denoted by H. Then the procedure is as follows. First, employing the abelian HSP algorithm on a cyclic group $\langle x \rangle$, we find $H' = H \cap \langle x \rangle = \left\langle x^{2^t p^s} \right\rangle$, and then we determine whether $H = \left\langle x^{2^t p^s} \right\rangle$ or $H = \left\langle x^{2^t p^s}, x^{h2^t p^{s-1}} y \right\rangle$, and find h if $H = \left\langle x^{2^t p^s}, x^{h2^t p^{s-1}} y \right\rangle$, by means of the following quantum algorithm:

1. Prepare the state

$$\frac{1}{p} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} |a\rangle |b\rangle \left| f(x^{a2^t p^{s-1}} y^b) \right\rangle. \qquad (A.23)$$

2. Measure the third register.

3. Apply $\mathcal{F}_p \otimes \mathcal{F}_p$ to the first two registers, where $\mathcal{F}_p$ is the quantum Fourier transform over $\mathbb{Z}_p$, that is, $\iota = \sqrt{-1}$ and

$$\mathcal{F}_p |l\rangle = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} e^{2\pi\iota kl/p} |k\rangle \ \text{ for all } \ 0 \le l \le p-1. \ (A.24)$$

4. Measure the first and the second register: we get two values $\tilde{c}$ and $\tilde{d}$. If $\tilde{c} = 0$, then we regard the procedure as failed. Otherwise, compute $\tilde{h} = -\tilde{c}^{p-2}\tilde{d} \pmod{p}$ and output $\tilde{h}$.

We repeat this procedure $k = \Theta(1)$ times, where the details are in the next section. If we obtain the same $\tilde{h}$ at every repetition, we conclude that $H = \left\langle x^{2^t p^s}, x^{h2^t p^{s-1}} y \right\rangle$ with $h = \tilde{h}$. If we obtain at least two different values for $\tilde{h}$ during the k repetitions, we conclude that $H = H'$. Furthermore, we can readily show that the total time complexity is $O((r \log p)^2)$.

3. Analysis of the algorithm

In this section, we show that we can find H with sufficiently high probability by the above algorithm.

a. The case when $H = \left\langle x^{2^t p^s}, \ x^{h2^t p^{s-1}} y \right\rangle$

For the initial state

$$\frac{1}{p} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} |a\rangle |b\rangle \left| f(x^{a2^t p^{s-1}} y^b) \right\rangle, \qquad (A.25)$$

take $a_0 = a - bh \bmod p \in \mathbb{Z}_p$. Then the state in (A.25) can be rewritten by use of the summation on a_0 instead of the summation on a as follows.

$$\frac{1}{p} \sum_{a_0=0}^{p-1} \sum_{b=0}^{p-1} |a_0 + bh \bmod p\rangle |b\rangle$$
$$\left| f(x^{a_0 2^t p^{s-1}} x^{(bh \bmod p)2^t p^{s-1}} y^b) \right\rangle. \qquad (A.26)$$

By Proposition 6, for each integer b with $0 \le b \le p-1$, the element $x^{(bh \bmod p)2^t p^{s-1}} y^b$ is in the hidden subgroup H, and thus all the elements $x^{a_0 2^t p^{s-1}} x^{(bh \bmod p)2^t p^{s-1}} y^b$ are mapped to the same value, that is,

$$f(x^{a_0 2^t p^{s-1}} x^{(bh \bmod p)2^t p^{s-1}} y^b) = f(x^{a_0 2^t p^{s-1}}), \quad (A.27)$$

for any $0 \le a_0, b \le p-1$. Thus, discarding the third register after the second step, the resulting state becomes

$$|\psi\rangle = \frac{1}{\sqrt{p}} \sum_{b=0}^{p-1} |a_0 + bh \bmod p\rangle |b\rangle, \qquad (A.28)$$

where a_0 is randomly determined by the measurement.

By applying the quantum Fourier transform to $|\psi\rangle$, the state becomes

$$\begin{aligned} \mathcal{F}_p \otimes \mathcal{F}_p |\psi\rangle &= \frac{1}{p\sqrt{p}} \sum_{c,d=0}^{p-1} e^{2\pi\iota a_0 c/p} \sum_{b=0}^{p-1} e^{2\pi\iota b(ch+d)/p} |c\rangle |d\rangle \\ &= \frac{1}{\sqrt{p}} \sum_{p | ch+d} e^{2\pi\iota a_0 c/p} |c\rangle |d\rangle. \qquad (A.29) \end{aligned}$$

Since, after the fourth step, the measured values $\tilde{c}$ and $\tilde{d}$ satisfy $p | \tilde{c}h + \tilde{d}$, if $\tilde{c} \ne 0$ then we obtain $\tilde{h}$ by computing $-\tilde{c}^{p-2}\tilde{d} \bmod p$. Hence, with probability $1 - 1/p$, we obtain the value of h.

b. The case when $H = \left\langle x^{2^t p^s} \right\rangle$

In this case, we note that if $x^a y^b H = x^a y^{b'} H$ then $b = b'$, and that if $x^{a2^t p^{s-1}} y^b H = x^{a'2^t p^{s-1}} y^b H$ then since $x^{a2^t p^{s-1}} y^b (x^{a'2^t p^{s-1}} y^b)^{-1} = x^{(a-a')2^t p^{s-1}} \in H$, we have $p \mid a - a'$, which implies $a = a'$. Thus, the oracle function f is an injective map on the set $\left\{ x^{a2^t p^{s-1}} y^b : 0 \le a, b \le p-1 \right\}$.

After measuring and discarding the third register, the state is $|\psi'\rangle = |a_0\rangle |b_0\rangle$, where a_0 and b_0 are determined by the measurement outcome of the third register $\left| f(x^{a_0 2^t p^{s-1}} y^{b_0}) \right\rangle$.

By applying quantum Fourier transform, we can have

$$\mathcal{F}_p \otimes \mathcal{F}_p |\psi'\rangle = \frac{1}{p} \sum_{c,d=0}^{p-1} e^{2\pi\iota(a_0 c + b_0 d)/p} |c\rangle |d\rangle, \qquad (A.30)$$

which is a uniform superposition of the values c and d in $\mathbb{Z}_p$.

If the measurement outcome of the first register is zero, that is, $\tilde{c} = 0$, then we disregard the result. If $\tilde{c}$ is not zero, then $\{\tilde{h} = -\tilde{c}^{p-2}\tilde{d} \bmod p\}$ forms a uniform distribution over $\mathbb{Z}_p$. Therefore, with high probability, we obtain at least two different values $\tilde{h}$'s.

c. Success probability

We now consider the success probability when repeating the procedure in the previous section k times, after obtaining t and s by the abelian HSP algorithm.

If $H = \left\langle x^{2^t p^s}, x^{h2^t p^{s-1}}y \right\rangle$, then, with probability $1/p^k$, it fails to output the correct h.

If $H = \left\langle x^{2^t p^s} \right\rangle$, then the probability that $\tilde{c} = 0$ in the fourth step for every repetition is $1/p^k$. In addition, although all values of $\tilde{c}$'s are not zero, the probability of incorrect output, that is, the probability of deciding that H is $H = \left\langle x^{2^t p^s},\ x^{h2^t p^{s-1}}y \right\rangle$ for a value $0 \le h \le p-1$, is $(2^k - 1)/p^{k-1}$, which can be easily shown.

Therefore, the total success probability is at least $1 - (2^k p - p + 1)/p^k$, and hence taking some proper constant k, we can obtain the correct h with high probability by k repetitions.

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