Preparing topological PEPS on a quantum computer

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Simulating of exotic phases of matter that are not amenable to classical techniques is one of the most important potential applications of quantum information processing. We present an efficient algorithm for preparing a large class of topological quantum states – the G-injective Projected Entangled Pair States (PEPS) – on a quantum computer. Important examples include the resonant valence bond (RVB) states, conjectured to be topological spin liquids. The runtime of the algorithm scales polynomially with the condition number of the PEPS projectors, and inverse-polynomially in the spectral gap of the PEPS parent Hamiltonian.

Creating and studying exotic phases of matter is one of the most challenging goals in contemporary physics. The increasingly sophisticated simulation abilities of systems such as cold atoms in optical lattices, trapped ions or superconducting qubits make this challenge accessible by means of Feynmann's original idea of using highly controllable quantum systems in order to simulate other quantum systems. Among those exotic phases, non-abelian topologically ordered states and topological spin liquids – such as resonating valence bond (RVB) states in frustrated lattices – are probably the holy grails of this area of quantum state engineering. Progress on the creation of such exotic phases in various experimental systems has accelerated rapidly in recent years, including cold atoms [1], ion traps [2], photonic devices [3] and superconducting devices [4].

Recently [5], a very general way of constructing quantum states on a quantum computer was proposed. The wide applicability of the method lies in the fact that there is a variational class of quantum states, called Projective Entangled Pair States (PEPS), which has a simple local description but is nonetheless complex enough to approximate the low-energy sector of local Hamiltonians. (A review of the analytical and numerical evidence for this can be found in [6] and the references therein.) However, a crucial technical assumption in the main result of [5], called 'injectivity', excludes any possibility of constructing quantum states with topological order.

The main aim of this article is to significantly extend the result of Ref. [5] to include exotic topological quantum phases, by proving:
Main result For any finite group G, a G-injective PEPS can be prepared on a quantum computer in polynomial time, when the inverse ground state gap of the associated parent Hamiltonian scales at most polynomially in the system size.

'G-injectivity', introduced only recently in [6] (and explained more fully below), is a substantially weaker requirement than injectivity, which explicitly allows for

topological order. A compelling example of the significance of this result is the very recently proven fact that the resonating valence bond (RVB) state in the Kagome lattice (conjectured to be a topological spin liquid), is a $\mathbb{Z}_2$-injective PEPS, with numerical evidence that the gap assumption is also verified [7]. Our result therefore gives one way in which the RVB state (and other topological states) can be prepared efficiently on a general quantum simulator. Engineering exotic quantum states by quantum simulation complements research aimed at finding materials that directly exhibit topological behaviour, and is already beginning to bear fruit experimentally [1–4].

In the following section, we summarise basic notions of PEPS required in this work, and introduce the class of G-injective PEPS which includes many of the important topological quantum states. We then briefly review the algorithm of Ref. [5] for preparing injective (non-topological) PEPS, before proceeding to show how this algorithm can be extended to the much larger class of G-injective PEPS, thereby allowing efficient preparation of many exotic topological quantum states. Finally, we close with some concluding remarks and open questions.

Projected Entangled Pair States For simplicity, we will focus in this letter on PEPS defined on a square lattice, but the results can be generalized to other lattices. An (unnormalized) PEPS can be described as follows. Place maximally entangled states of dimension D along all edges of the lattice. To each vertex ν, apply a linear map $A^\nu : (\mathbb{C}^D)^{\otimes 4} \to \mathbb{C}^d$ in the four D-dimensional systems labeled by l, t, r, b (for 'left', 'top', 'right' and 'bottom'), where $A^\nu = \sum_{i;l,t,r,b} A^\nu_{i;ltrb} |i\rangle\langle ltrb|$. The resulting vector in $(\mathbb{C}^d)^{\otimes N}$ is the unnormalized PEPS, N being the number of vertices in the lattice. For the purposes of this work, since local unitaries do not change the complexity of preparing a state, by taking the polar decomposition of A we can assume without loss of generality that A is positive-semidefinite. When A is invertible, we call the PEPS *injective* [6].

A particularly interesting class of PEPS is the class

of G-isometric PEPS, defined for any finite group G as follows. Take a semi-regular representation of G [6] – that is, a representation $U_g = \oplus_\alpha V_g^\alpha \otimes \mathbb{1}_{r_\alpha}$ having at least one copy of each irrep α. Note that the regular representation is exactly the one for which r_α is the dimension d_α of the irrep V_g^α for all α. We can define the re-weighting map

$$\Delta = \oplus_\alpha \left(\frac{d_\alpha}{r_\alpha}\right)^{\frac{1}{4}} \mathbb{1}_{d_\alpha} \otimes \mathbb{1}_{r_\alpha} \tag{1}$$

which is real, diagonal, commutes with U_g and satisfies $\mathrm{Tr}\,\Delta^4 U_g = |G|\delta_{g,e}$. (For the regular representation $\Delta = \mathbb{1}$.) The PEPS is then defined by taking, for all ν:

$$A^\nu = \frac{1}{|G|} \sum_{g \in G} \Delta \bar{U}_g \otimes \Delta \bar{U}_g \otimes \Delta U_g \otimes \Delta U_g . \tag{2}$$

G-isometric PEPS were originally defined in [6] only for the regular representation, and shown in that case to be exactly the quantum double models of Kitaev [8]. Here, we generalise the definition of G-isometric to any semi-regular representation [9]. If, on top of a G-isometric PEPS, we apply a further invertible (and w.l.o.g. positive-definite) linear map $A^\nu : \mathbb{C}^d \to \mathbb{C}^d$, we obtain a 'G-injective' PEPS [6]. (Here, d is the dimension of the symmetric subspace associated with the group.) The parallel with plain injective PEPS is clear. Both are defined by invertible maps on top of a G-isometric PEPS. In the case of injective PEPS, the group is the trivial one and the representation is simply $\mathbb{1}_d$ (d copies of the left-regular representation of the trivial group).

G-isometric PEPS have very nice properties, coming from their topological character, which are inherited by the more general G-injective PEPS. For instance, for each G-isometric PEPS $|\psi\rangle$ there exists a local frustration-free Hamiltonian (called the PEPS "parent Hamiltonian" [6]), consisting of commuting projectors and having as ground space the subspace (over-)spanned by $\{|\psi; K\rangle : K = (g,h), [g,h] = 0\}$. (Here, $|\psi; K\rangle$ is the PEPS obtained by the same maps A, except that we first apply an additional $U_g^{\otimes V}$ to exactly one vertical strip V and $U_h^{\otimes H}$ to exactly one horizontal strip H in the initial collection of maximally entangled states [6]). This generalises to G-injective PEPS, except that the local Hamiltonian terms are no longer necessarily commuting projectors.

We will denote by $|A^1 \cdots A^t\rangle$ the G-injective PEPS defined by applying the map A^j to vertex j for $j = 1, \ldots, t$ (and identity to the rest of the vertices) on top of the G-isometric PEPS, and define the states $|A^1 \cdots A^t; K\rangle$ analogously to above, which again (over-)span the ground space of a frustration-free local parent Hamiltonian H_t.

Preparing injective PEPS We first briefly review the algorithm of [5] for preparing injective PEPS on a quantum computer. Let H_t be the parent Hamiltonian of the partially constructed state $|A^1 \cdots A^t\rangle$. The algorithm starts at $t = 0$ with maximally entangled states between all pairs of adjacent sites in the lattice, and proceeds by successively projecting onto the ground states of H_t for $t = 1 \ldots N$ until the final state $|A^1 \cdots A^N\rangle$ is reached.

Since the ground state P_t of H_t is a complex, many-body quantum state, it is not immediately clear (i) how to efficiently perform the projective measurement $\{P_t, P_t^\perp\}$ onto the ground state. Furthermore, measurement in quantum mechanics is probabilistic, so even if this measurement can be performed, it is not at all clear (ii) how to guarantee the desired outcome P_t.

The answer to (i) is to run the coherent quantum phase estimation algorithm [10, 11] for the unitary generated by time-evolution under H_t. (Time-evolution under the local Hamiltonian H_t can be simulated efficiently by standard Hamiltonian simulation techniques [12].) If $\sum_k \alpha_k |\psi_k\rangle$ is the initial state expanded in the eigenbasis of H_t, then the phase estimation entangles this register with an output register containing an estimate of the corresponding eigenvalue: $\sum_k \alpha_k |\psi_k\rangle |E_k\rangle$. Performing a partial measurement on the output register to determine if its value is less than Δ_t (the spectral gap of H_t) completes the implementation of the measurement $\{P_t, P_t^\perp\}$. (See [5] for full details.)

The solution to (ii) is more subtle, and makes use of Camille Jordan's lemma of 1875 on the simultaneous block diagonalization of two projectors, which we first recall (see also the related CS decomposition [13]):

Lemma 1 (Jordan [14]) *Let R and Q be two projectors with rank $s_r = \mathrm{rank}\,R$ and $s_q = \mathrm{rank}\,Q$ respectively. Then both projectors can be decomposed simultaneously in the form*

$$R = \bigoplus_{k=1}^{s_r} R_k \quad Q = \bigoplus_{k=1}^{s_q} Q_k, \tag{3}$$

where R_k, Q_k denote rank-1 projectors acting on one- or two-dimensional subspaces. The eigenvectors $|r_k\rangle, |r_k^\perp\rangle$ and $|q_k\rangle, |q_k^\perp\rangle$ of the 2×2 projectors R_k and Q_k are related by

$$\begin{aligned}
|r_k\rangle &= \sqrt{d_k}|q_k\rangle + \sqrt{1-d_k}|q_k^\perp\rangle \\
|r_k^\perp\rangle &= -\sqrt{1-d_k}|q_k\rangle + \sqrt{d_k}|q_k^\perp\rangle \\
|q_k\rangle &= \sqrt{d_k}|r_k\rangle - \sqrt{1-d_k}|r_k^\perp\rangle \\
|q_k^\perp\rangle &= \sqrt{1-d_k}|r_k\rangle + \sqrt{d_k}|r_k^\perp\rangle .
\end{aligned}$$

Ref. [5] shows that if the current state is in the block containing the ground state of H_t, then the PEPS structure guarantees that the probability of a successful projection onto P_{t+1} is lower-bounded by $\kappa(A^{t+1})^{-2}$, where $\kappa(A^{t+1})$ is the condition number of the matrix A^{t+1}. Assume for induction that we have already successfully prepared the (unique) ground state of H_t. We first attempt to project from this state onto the unique ground state of H_{t+1} by measuring $\{P_{t+1}, P_{t+1}^\perp\}$. If this fails, we attempt to project back to the state we started from by measuring $\{P_t, P_t^\perp\}$, a technique introduced by Marriott and Watrous [15] in the

context of QMA-amplification. If this "rewind" measurement succeeds, then we're back to where we started and can try again. What if the "rewind" measurement fails? By Lemma 1, we can only be in the excited state from the same block, so we can still try to project "forwards" with the same lower bound on the success probability. Thus iterating forwards and backwards measurements until success generates a Markov chain with successful projection onto the ground state of H_{t+1} as the unique absorbing state. Moreover, since the success probability in each step is bounded away from zero, this converges rapidly to the desired state, allowing us to move from the ground state of H_t to the ground state of H_{t+1} in polynomial time.

Preparing G-injective PEPS Consider the algorithm of the preceding section from the perspective of G-injective PEPS. An injective PEPS can always be viewed as a G-injective PEPS for the representation $\mathbb{1}$ of the trivial group. The algorithm starts from the state consisting of maximally-entangled pairs between each site, and transforms this into the desired state by projecting onto the ground states of a sequence of injective parent Hamiltonians. But the initial state is none other than the G-*isometric* PEPS corresponding to the representation $\mathbb{1}$ of the trivial group. This hints at a generalisation of the algorithm to G-injective PEPS for arbitrary groups G: start by preparing the corresponding G-*isometric* PEPS, and successively transform this into the desired G-injective PEPS by projecting onto the ground states of the sequence of G-injective parent Hamiltonians (see Table I).

Input: G-injective A^v defined on an N-vertex lattice; $\epsilon > 0$.
Output: $|\psi\rangle \in \text{span}\,|A^1, \ldots, A^t; K\rangle$ with probability $\geq 1 - \epsilon$.

1. Prepare corresponding G-isometric PEPS
2. **For** $t = 1$ to N
   2.1. Measure $\{P_{t+1}, P_{t+1}^\perp\}$ on $|\psi\rangle$.
   2.2. **While** measurement outcome is $P_{t+1}^\perp$
      2.2.1. Measure $\{P_t, P_t^\perp\}$.
      2.2.2. Measure $\{P_{t+1}, P_{t+1}^\perp\}$.

TABLE I. Preparing a G-injective PEPS. H_t $(t = 1 \ldots N)$ is the parent Hamiltonian for the G-injective PEPS $|A^1 \ldots A^t\rangle$, P_t the projector onto its ground state subspace. Note that by specifying A^v, we are implicitly selecting a particular semi-regular representation of G.

There are, however, two obstacles to implementing this approach. (i) The initial G-isometric PEPS can be a substantially more complicated many-body quantum state than the trivial product of maximally-entangled pairs we must prepare in the injective case. (ii) Since G-injective parent Hamiltonians are topological, they have degenerate ground state subspaces. But the Marriott-Watrous "rewinding trick" [15] relies on the measurement projectors being 1-dimensional; it breaks down in general

for higher-dimensional projectors.

It turns out that there is a direct solution to (i). Ref. [6] proves that, for any group G, the parent Hamiltonian of the G-isometric PEPS for the regular representation corresponds precisely to a quantum-double model [8, 16]. But Ref. [17] shows that ground states of quantum-double models can be generated exactly by a polynomial-size quantum circuit. We can therefore use this circuit to efficiently prepare the G-isometric PEPS for the regular representation of G. What if the representation we require is semi-regular? In fact the G-isometric PEPS for any semi-regular representation is equivalent to the one for the regular representation (up to local isometries), by simply regrouping the tensors. (See the appendix for a proof.)

The second obstacle is more delicate. As described above, the Marriott-Watrous "rewinding trick" used in the injective case [5] works because, thanks to the Hamiltonians H_t and H_{t+1} at each step having unique ground states, there is only ever one 2×2 block involved in the back-and-forth measurements. However, in the G-injective case, the Hamiltonians no longer have unique ground states, and there are multiple 2×2 blocks corresponding to different ground states. Thus when we "rewind" a failed measurement, the backwards measurement could project us back into any superposition of states from any of the blocks corresponding to the ground state subspace. Now, A^{t+1} is only invertible on the G-symmetric subspace, so it necessarily has some zero eigenvalues. Hence $\kappa(A^{t+1}) = \infty$ and the lower bound $\kappa(A^{t+1})^{-2} = 0$ on the probability of a successful forward measurement is useless. Although there may still exist some ground state $|\psi_t^1\rangle$ of H_t which has positive probability of successful forward transition to a ground state of H_{t+1}, this does not rule out existence of another ground state $|\psi_t^2\rangle$ of H_t for which the probability of a successful forward transition is 0. In the worst case, if a forward measurement fails and we end up in a state $|\varphi_{t+1}^\perp\rangle$, the rewinding step could have probability 1 of transitioning back to $|\psi_t^2\rangle$, so that we remain stuck forever bouncing back and forth between $|\psi_t^2\rangle$ and $|\varphi_{t+1}^\perp\rangle$.

To overcome this, we must show that *if we start from the G-isometric state*, then the structure of G-injective PEPS ensures that this situation can never occur. To prove this, we need the following key lemma:

Lemma 2 *Let P_t and P_{t+1} denote two projectors on the ground state subspace of the partial PEPS parent Hamiltonains H_t and H_{t+1} for $|A^1 \ldots A^t\rangle$ and $|A^1 \ldots A^t, A^{t+1}\rangle$. The overlap d_k between P_t and P_{t+1} (cf. Lemma 1) is lower-bounded by $d_{\min} \geq \kappa(A^{t+1}|_{S_G})^{-2}$, where $\kappa(A^{t+1}|_{S_G}) := \sigma_{\max}(A_{t+1}|_{S_G})/\sigma_{\min}(A_{t+1}|_{S_G})$ is the condition number restricted to the G-symmetric subspace S_G.*

Proof The minimum overlap $d_{\min}$ between projectors P_t and P_{t+1} is given by

$$d_{\min} = \min_{|\psi_t\rangle} \max_{|\psi_{t+1}\rangle} |\langle\psi_t|\psi_{t+1}\rangle|^2, \tag{4}$$

where $|\psi_t\rangle$ and $|\psi_{t+1}\rangle$ are states in the respective ground state subspaces $\ker H_t$ and $\ker H_{t+1}$.

Now, $\ker H_t$ is spanned by the partially constructed PEPS $|A^1,\ldots,A^t;K\rangle$, with different boundary conditions K giving different ground states). Thus we can decompose any $|\psi_t\rangle \in \ker H_t$ as a linear combination of partial PEPS: $|\psi_t\rangle = \sum c_k |A^1,\ldots,A^t;K^k\rangle$.

H_{t+1} is obtained from H_t by replacing all the G-isometric local Hamiltonian terms at one vertex with the G-injective terms. So applying A_{t+1} to *any* $|A^1,\ldots,A^t;K\rangle$ takes us to the next ground state subspace. Therefore, the state

$$|\varphi_{t+1}\rangle = \frac{A_{t+1}|\psi_t\rangle}{\sqrt{\langle\psi_t|A_{t+1}^\dagger A_{t+1}|\psi_t\rangle}} \qquad (5)$$

is contained in $\ker H_{t+1}$. Choosing $|\psi_{t+1}\rangle = |\varphi_{t+1}\rangle$ in Eq. (4), we obtain the lower bound

$$d_{\min} \geq \min_{|\psi_t\rangle}|\langle\psi_t|\varphi_{t+1}\rangle|^2 \geq \min_{|\psi_t\rangle}\frac{|\langle\psi_t|A_{t+1}|\psi_t\rangle|^2}{\langle\psi_t|A_{t+1}^\dagger A_{t+1}|\psi_t\rangle}. \qquad (6)$$

It is immediate from the definition of G-injective PEPS that the ground states of H_t are symmetric, so that the projector P_t is supported on the symmetric subspace S_G. Thus the minimisation is over symmetric states and, recalling that w.l.o.g. A_t is positive-semidefinite, we obtain the claimed bound

$$d_{\min} \geq \min_{|\psi_t\rangle}\frac{\langle\psi_t|A_{t+1}|_{S_G}|\psi_t\rangle^2}{\langle\psi_t|A_{t+1}^2|_{S_G}|\psi_t\rangle} \geq \frac{\sigma_{\min}\left(A_{t+1}|_{S_G}\right)^2}{\sigma_{\max}\left(A_{t+1}|_{S_G}\right)^2}, \qquad (7)$$

by the variational characterisation of eigenvalues. □

Runtime We are now in a position to establish the runtime of the algorithm given in Table I. We start by bounding the failure probability of growing the partial PEPS by a single site.

Lemma 3 *The measurement sequence depicted in Fig. 1 with the two projective measurements $\{P_t, P_t^\perp\}$ and $\{P_{t+1}, P_{t+1}^\perp\}$ has a failure probability bounded by*

$$p_{fail}(m) < \frac{1}{2\, d_{\min}m} \qquad (8)$$

after m-subsequent measurement steps, where $d_{\min} = \min_k d_k$ is the minimal overlap between the eigenstates of P_t and P_{t+1}.

Proof Let $Q_1 = P_{t+1}$, $Q_0 = P_{t+1}^\perp$ and $R_1 = P_t$, $R_0 = P_t^\perp$, in accordance with the notation in Lemma 1. Hence Q_1 projects on to the new ground state subspace, whereas the R_1 is the projector on to the old ground state subspace. If we start in some state $|\psi\rangle = R_1|\psi\rangle$, the probability of failure of the measurement sequence depicted in Fig. 1 after m steps is $p_{\mathrm{fail}}(m) =$
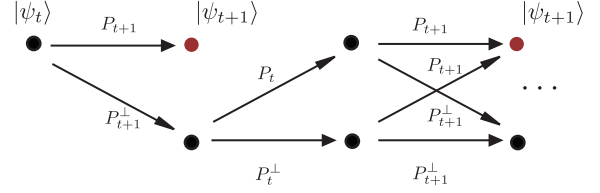

FIG. 1. The sequence of outcomes of the binary measurements $\{P_t, P_t^\perp\}$ and $\{P_{t+1}, P_{t+1}^\perp\}$. We are initially in an eigenstate $|\psi_t\rangle$ of the projector P_t, and want to transition to a state in the subspace P_{t+1}. With non-zero probability, the first $\{P_{t+1}, P_{t+1}^\perp\}$ succeeds with outcome P_{t+1}. If it fails, we have prepared a state in the $P_{t+1}^\perp$ subspace. We "unwind" the measurement by measuring $\{P_t, P_t^\perp\}$ again. Upon repeating the $\{P_{t+1}, P_{t+1}^\perp\}$ measurement, we again have a non-zero probability of successfully obtaining the P_{t+1} outcome. If we fail again, we repeat the procedure until success.

$\sum_{s_1,\ldots,s_m}\mathrm{Tr}(Q_0 R_{s_m}Q_0\ldots R_{s_1}Q_0|\psi\rangle\langle\psi|Q_0 R_{s_1}\ldots Q_0 R_{s_m}Q_0)$. Note that $[Q_0 R_s Q_0, Q_0 R_p Q_0] = 0$ for all s, p. We can therefore rearrange this to express $p_{\mathrm{fail}}(m)$ as the sum

$$\sum_{k=0}^m \binom{m}{k}\langle\psi|\left(Q_0 R_0 Q_0\right)^{2k}\left(Q_0 R_1 Q_0\right)^{2(m-k)}|\psi\rangle$$

$$= \langle\psi|\left(\left(Q_0 R_0 Q_0\right)^2 + \left(Q_0 R_1 Q_0\right)^2\right)^m|\psi\rangle.$$

If we work in the eigenbasis of Q_1, the individual 2×2 block matrices take the form

$$Q_1^k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R_1^k = \begin{pmatrix} d_k & \sqrt{d_k(1-d_k)} \\ \sqrt{d_k(1-d_k)} & 1-d_k \end{pmatrix}. \qquad (9)$$

Since $|\psi\rangle$ is left invariant by R_1, we have that $|\psi\rangle = \sum_k c_k|r_k\rangle$, where in this basis every $|r_k\rangle = \left(\sqrt{d_k}\quad\sqrt{1-d_k}\right)^T$ by Lemma 1. We are therefore left with

$$p_{\mathrm{fail}}(m) = \sum_k |c_k|^2\,(1-d_k)\,(1-2d_k(1-d_k))^m, \qquad (10)$$

with $d_k \in [0,1]$ and $\sum_k|c_k|^2 = 1$.

Since $(1-x) \leq e^{-x}$, we may bound $(1-d_k)(1-2d_k(1-d_k))^m \leq (1-d_k)e^{-2md_k(1-d_k)}$. Furthermore, we have that $(1-d_k)e^{-2md_k(1-d_k)} \leq 1/2md_k$ by Taylor expansion. If we now choose the largest factor $(2md_k)^{-1} \leq (2md_{\min})^{-1}$, we can bound the total failure probability by Eq. (8). □

We use this to bound the overall runtime.

Theorem 4 (Runtime) *Let A^v be G-symmetric tensors defining a PEPS on an N-vertex lattice. A state in the subspace spanned by the corresponding G-injective PEPS $|A^1\ldots A^N;K\rangle$ can be prepared on a quantum computer with probability $1 - \epsilon$ in time $O(N^4\kappa_G^2\Delta^{-1}\epsilon^{-1})$, with additional classical processing $O(Nd^6)$, where $\Delta = \min_t(\Delta_t)$ is the minimal spectral gap of the family of parent Hamiltonians H_t for $|A^1\ldots A^t\rangle$ $(t = 1\ldots N)$, and $\kappa_G = \max_t \kappa(A^t|_{S_G})$.*

Proof The algorithm in Table I first prepares the initial G-isometric PEPS, which can be done exactly in time $O(N \log N)$ [17], and then transforms this step by step into the G-injective PEPS, with one step for each of the N vertices of $\mathcal{G}$. Each step has a probability of failure $p_{\text{fail}}(m)$ if we repeat the back-and-forth measurement scheme m times. We need to ensure that the total success probability is lower-bounded by $(1 - p_{\text{fail}}(m))^N \geq 1 - \epsilon$. Since $(1 - x)^N \geq 1 - Nx$, we can use Lemma 3 to bound

$$(1 - p_{\text{fail}}(m))^N \geq 1 - \frac{N}{2md_{\min}}, \qquad (11)$$

so we want $N/2md_{\min} \leq \epsilon$. Since $d_{\min} \geq \kappa_G^{-2}$ by Lemma 2, we choose $m \geq N\kappa_G^2/2\epsilon$ at each step. We therefore need to perform $O(N^2\kappa_G^2\epsilon^{-1})$ quantum phase estimation procedures, each of which has runtime $\tilde{O}(N^2/\Delta^{-1})$ to ensure that we are able to resolve the energy gap of the parent Hamiltonian [18]. (Note that the notation $\tilde{O}(\cdot)$ suppresses more slowly growing terms such as $\exp(\sqrt{\ln N/\Delta})$.) The classical bookkeeping required to keep track of the Hamiltonians is the same as in [5]. Putting all this together, we arrive at the total runtime stated in the theorem. $\qquad\square$

Discussion We have shown how the Marriott-Watrous rewinding technique combined with the unique structure of G-injective PEPS can be used to transition from one state to the next (see Table I), successively building up the desired quantum state even when that state has topological order and the ground states are degenerate. There are a number of alternative techniques that could potentially be used to achieve the same thing. In each case, the key to proving an efficient runtime is our Lemma 2. In many cases the existing results in the literature assume non-degenerate ground states, so would need to be generalised before they would apply to the topologically degenerate ground states considered here.

Standard adiabatic state preparation has the disadvantage that it requires a polynomial energy gap along a continuous path joining the initial Hamiltonian with the final one. But the "jagged adiabatic lemma" of Ref. [19] shows that such a path connecting a discrete set of gapped Hamiltonians always exists if the ground states are unique and each ground state has sufficient overlap with the next. The latter is precisely what we prove in Lemma 2. For the 'injective' case of [5], this is sufficient to show that adiabatic state preparation is an efficient alternative to the "rewinding trick". Our results suggest that the jagged adiabatic lemma could be generalised to the case of non-unique ground states.

More general are the methods of [20], which subsume the jagged adiabatic lemma and the Marriott-Watrous technique. The results in [20] do not immediately apply to degenerate ground states, but if they can be generalised they could potentially improve the polynomial dependence on the required error probability to a logarithmic one. Another potential alternative is the recent quantum rejection sampling technique of [21], which gives quadratic improvement over Marriott-Watrous rewinding by a clever use of amplitude amplification. Finally, the spectral gap amplification technique of [22], which cites injective PEPS preparation [5] as a potential application, may also be applicable. In all cases, the techniques would first need to be generalised to handle degenerate ground states. If this can be done, our Lemma 2 would imply efficiency of the resulting algorithm.

The conditions required for efficient preparation in Theorem 4 (inverse-polynomial scaling with system size of the spectral gaps of the partial parent Hamiltonians and polynomial scaling of the condition numbers of the PEPS projectors) are very reminiscent of the conditions (local gap and local topological quantum order) required for stability of the spectral gap of local Hamiltonians [23]. It is also conjectured that the spectral gap of the parent Hamiltonian should be closely related to the condition number of the PEPS projectors. It would be interesting to understand better the relationships between these various conditions.

The technique we introduced, of constructing a complex many-body quantum state by starting from an easily-constructable state and successively transforming it into the desired state, is very general. Although we have applied it here to G-injective PEPS, as a class of states including many important topological quantum states such as the RVB state, our algorithm can be generalised to other classes of tensor network states, such as string-net models [24] and models constructed from Hopf algebras [25].

[1] I. Bloch, J. Dalibard, and S. Nascimbène, Nature Physics **8**, 267 (2012).

[2] R. Blatt and C. F. Roos, Nature Physics **8**, 277 (2012).

[3] A. Aspuru-Guzik and P. Walther, Nature Physics **8**, 285 (2012).

[4] A. A. Houck, H. E. Türeci, and J. Koch, Nature Physics **8**, 292 (2012).

[5] M. Schwarz, K. Temme, and F. Verstraete, Phys. Rev. Lett. **101**, 110502 (2012).

[6] N. Schuch, I. Cirac, and D. Peréz-García, Annals of

Physics **325**, 2153 (2010).

[7] D. Poilblanc, N. Schuch, D. Peréz-García, and J. I. Cirac, Physical Review B **86**, 014404 (2012).

[8] A. Kitaev, Annals of Physics **303**, 2 (2003).

[9] The generalisation of 'G-isometric to any semi-regular representation is justified by the fact that, for a given G, all G-isometric PEPS are equivalent (up to isometries) to the G-isometric PEPS for the regular representation by decomposing the tensor in a different way. A proof is given in the Appendix.

[10] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[11] E. Knill, G. Ortiz, and R. D. Somma, Physical Review A **75**, 012328 (2007).

[12] D. Berry, G. Ahokas, R. Cleve, and B. Sanders, Communications in Mathematical Physics **270**, 359 (2007), 10.1007/s00220-006-0150-x.

[13] G. H. Golub and C. F. van Loan, *Matrix Computations*, 3rd ed. (Johns Hopkins University Press, 1996).

[14] C. Jordan, Bull. Soc. Math. France **3**, 103 (1875).

[15] C. Marriott and J. Watrous, Computational Complexity **14**, 122 (2005).

[16] N. Schuch, D. Perez-Garcia, and I. Cirac, Phys. Rev. B **84**, 165139 (2011).

[17] M. Aguado and G. Vidal, Phys. Rev. Lett. **100**, 070404 (2008).

[18] A. Harrow, A. Hassidim, and S. Lloyd, Physical review letters **103**, 150502 (2009).

[19] D. Aharonov and A. Ta-Shma, SIAM Journal on Computing **37**, 47 (2007).

[20] S. Boixo, E. Knill, and R. Somma, (2010), arXiv:1005.3034.

[21] M. Ozols, M. Roetteler, and J. Roland, in *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*, ITCS '12 (ACM, New York, NY, USA, 2012) pp. 290–308.

[22] R. Somma and S. Boixo, (2011), arXiv:1110.2494.

[23] S. Michalakis, (2012), arXiv:1206.6900.

[24] N. Schuch, (private communication).

[25] O. Buerschaper, M. Mombelli, M. Christandl, and M. Aguado, (2010), arxiv:1007.5283[cond-mat].

Appendix: G-isometric PEPS

In this appendix, we use the argument described in [6] for the Toric Code and RVB states, generalised here to arbitrary G-isometric PEPS, to see that the G-isometric PEPS for any semi-regular representation is equivalent to the one for the regular representation. Let us start with a semi-regular representation U_g of a group G, and let

$$B = \frac{1}{|G|} \sum_g \Delta \bar{U}_g \otimes \Delta \bar{U}_g \otimes \Delta U_g \otimes \Delta U_g. \quad (12)$$

We will show how B can indeed be seen as the G-isometric PEPS corresponding to the regular representation – possibly composed with an isometry which embeds the initial Hilbert space into a sufficiently large one. As explained in the main text, the latter can be prepared efficiently on a quantum computer by other means.
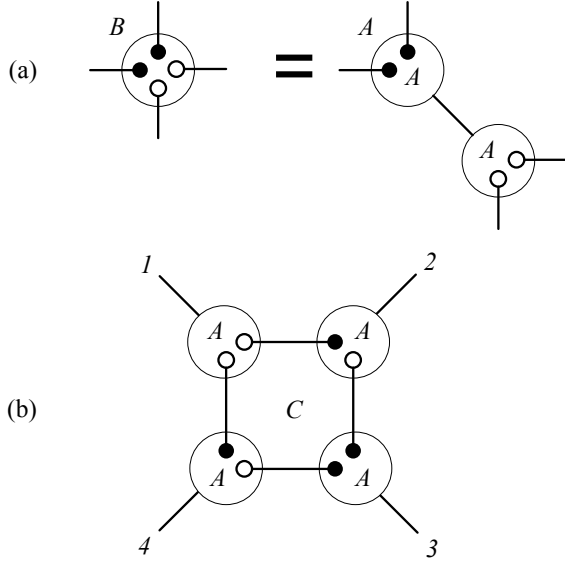
indices, is given by

$$C : |g_1 g_2 g_3 g_4\rangle \mapsto \frac{1}{|G|^2} \Delta^2 U_{g_1 g_2^{-1}} \otimes \Delta^2 U_{g_2 g_3^{-1}} \otimes \Delta^2 U_{g_4 g_3^{-1}} \otimes \Delta^2 U_{g_1 g_4^{-1}}. \quad (13)$$

By calling $g = g_1^{-1} g_1'$ and using Eq. (1) it is not difficult to see that

$$\langle g_1' g_2' g_3' g_4' | C^\dagger C | g_1 g_2 g_3 g_4 \rangle$$
$$= \frac{1}{|G|^4} \prod_{r=1}^{2} \mathrm{Tr}(\Delta^4 U_{g_r g_{r+1}^{-1} g_{r+1}' g_r'^{-1}}) \prod_{r=3}^{4} \mathrm{Tr}(\Delta^4 U_{g_{r+1} g_r^{-1} g_r' g_{r+1}'^{-1}}) \quad (14)$$

equals 1 if and only if there exist g such that $g_i g = g_i'$ for all i. Otherwise, the expression is identically zero.

Therefore $C^\dagger C = (|G|)^{-1} \sum_g R_g^{\otimes 4}$ for the regular representation R_g, hence the new PEPS C is the G-isometric PEPS corresponding to the regular representation.

FIG. 2. (a) illustrates the decomposition of the original tensor in the tensors A. We mark in white the bonds in which we have U_g and in black those in which we have $\bar{U}_g$. (b) illustrates the new way of grouping the tensors to get a G-isometric PEPS, called C. The bonds of this new tensor are numbered clockwise as in the figure.

To show this, we decompose the tensor B into two tensors of the form $A = (\sqrt{|G|})^{-1} \sum_g \Delta U_g \otimes \Delta U_g \otimes |g\rangle$ (where U_g and U_g are interchanged as needed, as shown in Fig. 2(a)). By regrouping these new tensors, we obtain a new PEPS decomposition of the same state, where now the bond dimension is $|G|$ (Fig. 2(b)). The resulting tensor C (Fig. 2(c)), as a map from the virtual to the physical