QUANTUM COMPUTING AND ZEROES OF ZETA FUNCTIONS

WIM VAN DAM

ABSTRACT. A possible connection between quantum computing and Zeta functions of finite field equations is described. Inspired by the *spectral approach* to the Riemann conjecture, the assumption is that the zeroes of such Zeta functions correspond to the eigenvalues of finite dimensional unitary operators of natural quantum mechanical systems. The notion of universal, efficient quantum computation is used to model the desired quantum systems.

Using eigenvalue estimation, such quantum circuits would be able to approximately count the number of solutions of finite field equations with an accuracy that does not appear to be feasible with a classical computer. For certain equations (Fermat hypersurfaces) it is show that one can indeed model their Zeta functions with efficient quantum algorithms, which gives some evidence in favor of the proposal of this article.

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1. Introduction

The spectral approach to the Riemann conjecture tries to interpret the zeroes of the Riemann zeta function as the eigenvalues of a physical operator, like a Hamiltonian in quantum mechanics. This idea goes back, apparently, to Hilbert and Pólya, although neither have published about this. Evidence supporting the spectral interpretation comes from the observed spacing of the zeroes of Riemann's zeta function, which has a striking resemblance with the 'eigenvalue repulsion' of chaotic quantum mechanical systems[19]. An explicit construction of an operator whose spectrum coincides with the zeroes of ζ would show that these roots indeed lie on a line, thereby proving Riemann's conjecture. See [1] for an attempt to discover a Hamiltonian with such properties.

In this article we consider a different kind of Zeta functions: those that correspond to equations over finite fields. Such Zeta functions have a finite number of roots, and the corresponding 'Riemann hypothesis' of André Weil states that these roots lie on a circle in the complex plane. Unlike the case of the Riemann zeta function, this hypothesis has been proven. Here we propose to interpret the roots of such Zeta functions as eigenvalues of unitary transformations of finite dimensional quantum systems. More specifically we want these quantum systems to be natural, which we will define to mean that they correspond to efficient quantum algorithms. By the properties of these Zeta functions, such algorithms would enable us to estimate the number of solutions of the equations with an accuracy that does not seem to be possible classically.

Using earlier results on the quantum estimation of Gauss sums, we show that for certain diagonal equations, also known as Fermat hypersurfaces", we can indeed construct a corresponding quantum circuit. It is an open problem how to extend this result to general hypersurfaces and, ultimately, varieties.

Notation. Throughout the article we use the following conventions. The number of variables X in a polynomial f is n, such that in the affine \mathbb{A}^n setting $f \in \mathbb{F}[X_1,\ldots,X_n]$ and in the projective case \mathbb{P}^n we have the homogeneous polynomial $f \in \mathbb{F}[X_0,\ldots,X_n]$ of degree m. The finite field \mathbb{F} has $q=p^r$ elements and its extensions \mathbb{F}_{q^s} are indexed by s. The dimension of a variety V is denoted by s, such that for hypersurfaces defined by s we have s and s are ideal is defined by several polynomials s, then the number of polynomials will be s. The variable of a zeta function s is s, while those of Zeta functions s will be s.

2. Quantum Computing

In this Section we briefly describe the basic ingredients of the theory of quantum computation. See [15] or [18] for a thorough introduction to this field.

2.1. Quantum States and Quantum Transformations. A quantum state ψ of n quantum bits (qubits) is described by a 2^n dimensional complex valued vector, which represents a superposition over all possible n bit strings $\{0,1\}^n$. In the 'bra-ket' notation this is expressed as

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle,$$

with $\alpha_x \in \mathbb{C}$ and the normalization restriction $\sum_x |\alpha_x|^2 = 1$. When observing this state ψ in the computational basis $\{0,1\}^n$, the probability of observing a specific

string $x \in \{0,1\}^n$ is $|\alpha_x|^2$ (hence the normalization restriction). In general, the probability of observing the state ϕ when performing a measurement on ψ is the 'inner product squared' $|\langle \phi | \psi \rangle|^2$.

A coherent quantum mechanical transformation T can always be described by a linear transformation that preserves the normalization restriction of the state vectors. Thus the transformation T of an n qubit system can be represented as a $2^n \times 2^n$ dimensional unitary matrix: $T \in \mathsf{U}(2^n)$. Any such transformation is reversible as $T^\dagger \cdot T = I$. A k-qubit quantum gate G is a unitary transformation of k qubits, such that $G \in \mathsf{U}(2^k)$.

2.2. **Efficient Quantum Algorithms.** A quantum circuit $C \in U(2^n)$ on n qubits can be defined as sequence of 2 qubit gates (note that this includes single qubit gates and the two-qubit SWAP operation for the wiring of the circuit). The outcome of a circuit on a given input string $x \in \{0,1\}^n$ is the probability distribution of the output state $C|x\rangle$ over the computational basis states $\{0,1\}^n$. The depth of such a circuit equals its time complexity and we consider a family of circuits efficient if its depth as a function of n is upper bounded by $O(\operatorname{poly}(\log n))$.

Almost all quantum algorithms known to date rely on the properties and the efficiency of the quantum Fourier transform.

Definition 1 (Efficient Quantum Fourier Transformation). For a ring $\mathbb{Z}/m\mathbb{Z}$, the quantum Fourier transformation F is defined as the unitary mapping

$$F: |x\rangle \longmapsto \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} \omega_m^{xy} |y\rangle,$$

with $\omega_m := e^{2\pi i/m}$.

For a finite field \mathbb{F}_{p^r} the corresponding quantum Fourier transformation is defined by

$$F:|x\rangle \quad \longmapsto \quad \frac{1}{\sqrt{p^r}} \sum_{y \in \mathbb{F}_{p^r}} \omega_p^{\mathrm{Tr}(xy)} |y\rangle$$

for every $x \in \mathbb{F}_{p^r}$, where we used the trace function $\operatorname{Tr}: \mathbb{F}_{p^r} \to \mathbb{F}_p$ defined by $\operatorname{Tr}(x) := x + x^p + \dots + x^{p^{r-1}}$.

It is well-known that both quantum Fourier transforms can be implemented efficiently on a quantum computer (with circuit depth poly($\log n$) and poly($\log p^r$) respectively [5, 6, 2]). This fact is an important ingredient of the efficient quantum algorithms for factoring and the discrete logarithm problem[23].

2.3. Universality of Efficient Quantum Computation. An important aspect of the theory of efficient quantum computation is that it is independent of the computational model one uses. Just as with the classical theory of polynomial-time computation, it does not matter if one expresses it in terms of Turing machines, circuits, or in any other reasonable model. This universality is proven by, for example, the fact that a quantum Turing machine can efficiently simulate a quantum circuit and vice-versa. Furthermore we have the modern Church-Turing thesis, which says that the physical resources that are required to solve a computational problem are properly quantified by the space/time complexity of the problem in the abstract quantum Turing machine model. Again, see [15, 18] for more details.

3. Zeta Functions of Polynomial Equations in Finite Fields

3.1. Projective Spaces, Hypersurfaces and Varieties. Consider a homogeneous polynomial $f \in \mathbb{F}_q[X_0, \ldots, X_n]$ and its solutions obeying f(x) = 0. The preferred way of analyzing such a set of solutions is with the use of the projective space \mathbb{P}^n .

Definition 2 (Projective Spaces). The *projective space* $\mathbb{P}^n(\mathbb{F})$ is the set of distinct rays $x = \{(\lambda x_0, \lambda x_1, \dots, \lambda x_n) : \lambda \in \mathbb{F}^{\times}\}$, excluding the zero-point $(0, \dots, 0)$. If $|\mathbb{F}|$ is finite, each ray contains $|\mathbb{F}| - 1$ points from the affine space $\mathbb{A}^{n+1}(\mathbb{F})$. This affine space itself has $|\mathbb{F}|^{n+1} - 1$ non-zero points, hence $\mathbb{P}^n(\mathbb{F})$ has $(|\mathbb{F}|^{n+1} - 1)/(|\mathbb{F}| - 1) = |\mathbb{F}|^n + |\mathbb{F}|^{n-1} + \dots + 1$ elements.

In algebraic geometry, projective spaces are often preferred over affine ones because \mathbb{P}^n includes the 'points at infinity' such that it is a closed space.

By the homogeneity of the polynomial f it follows that if $f(x_0, \ldots, x_n) = 0$ then also $f(\lambda x_0, \ldots, \lambda x_n) = 0$ for all λ . We can therefore group the nontrivial roots of f as rays in \mathbb{P}^n and hence speak of the set of solutions of f(x) = 0 with $x \in \mathbb{P}^n(\mathbb{F})$.

The algebraic closure of \mathbb{F}_q is denoted by $\overline{\mathbb{F}}_q$, which is the infinite field that contains all extensions \mathbb{F}_{q^s} of \mathbb{F}_q . For $f \in \mathbb{F}_q[X_0, \ldots, X_n]$ we want to investigate the set of solutions $H = \{x : x \in \mathbb{P}^n(\overline{\mathbb{F}}_q) \text{ and } f(x) = 0\}$, which for obvious reasons we will call an (n-1)-dimensional hypersurface.

We can generalize the notion of a hypersurface by considering the joint solutions to several polynomial equations $\{f_i(x) = 0\}_i$. Such an algebraic set is called a variety in the context of algebraic geometry and is thus defined by $V = \{x : x \in \mathbb{P}^n(\bar{\mathbb{F}}_q), f_1(x) = 0, \dots, f_t(x) = 0\}$. A variety has a dimension, varieties of dimension 1 are called *curves*, and curves that are defined by a single polynomial f are planar curves.

3.2. **Zeta Function of a Variety.** Given a variety V of a set of polynomials $\{f_1, \ldots, f_t\}$, we can look at the solutions that V has in the finite extensions \mathbb{F}_{q^s} of the finite fields \mathbb{F}_q . That is, we consider the sizes of the intersections $|V \cap \mathbb{P}^n(\mathbb{F}_{q^s})|$ for $s = 1, 2, \ldots$ This brings us to the following important definition.

Definition 3 (Zeta Function of a Variety). Let V be a projective variety defined over $\mathbb{P}^n(\bar{\mathbb{F}}_q)$ and let $N_s := |V \cap \mathbb{P}^n(\mathbb{F}_{q^s})|$ for $s = \mathbb{Z}^+$. The Zeta function of the variety V is the power series $Z_V(T) \in \mathbb{Q}[T]$ defined by

$$Z_V(T) := \exp\left(\sum_{s=1}^{\infty} \frac{N_s T^s}{s}\right).$$

Why it is a good idea to define a function like this and why one can call it a Zeta function is not entirely trivial. The reader is referred to the appendices in this article and the many books and articles that deal with this topic, in order of increasing difficulty, [11, 22, 17, 25, 9]. For now, we will make do with a few small examples.

3.3. Three Examples of Zeta Functions of Hypersurfaces.

Example 1 (Zeta Function of a Straight Line). Consider the line $L \subset \mathbb{P}^2(\bar{\mathbb{F}}_q)$ defined by $f(X_0, X_1, X_2) = X_1 + X_2 = 0$. The solutions to this equation are the

rays $\{(\lambda, 0, 0) : \lambda \in \mathbb{F}^{\times}\}$ and $\{(\lambda z, \lambda, -\lambda) : \lambda \in \mathbb{F}^{\times}\}$ for every $z \in \mathbb{F}$. Hence L has $1 + |\mathbb{F}|$ solutions, and thus $N_s = 1 + q^s$. For the Zeta function \mathbf{Z}_L this means

$$Z_L(T) = \exp\left(\sum_{s=1}^{\infty} \frac{(1+q^s)T^s}{s}\right)$$
$$= \exp\left(\sum_{s=1}^{\infty} \frac{T^s}{s}\right) \exp\left(\sum_{s=1}^{\infty} \frac{(qT)^s}{s}\right)$$
$$= \frac{1}{(1-T)(1-qT)}.$$

Clearly every straight line with $N_s = 1 + q^s$ will have this Zeta function.

Example 2 (Zeta Function of a 2 Dimensional Hypersurface, Section 11.1 in [11]). Take the hypersurface H defined by $f(X_0, ..., X_3) = -X_0^2 + X_1^2 + X_2^2 + X_3^2 = 0$ over $\mathbb{P}^3(\overline{\mathbb{F}}_7)$. One can show that for this specific base field \mathbb{F}_7 we have $N_s = 1 + 7^s + (-7)^s + 49^s$, and thus

$$Z_H(T) = \exp\left(\sum_{s=1}^{\infty} \frac{(1+7^s+(-7)^s+49^s)T^s}{s}\right)$$
$$= \frac{1}{(1-T)(1-49T^2)(1-49T)}.$$

Example 3 (Zeta Function of a Quartic Diagonal Equation). Let $f(X_0, X_1, X_2) = X_0^4 + X_1^4 + X_2^4$ define a curve C in $\mathbb{P}^2(\bar{\mathbb{F}}_5)$. Its Zeta function equals

$$Z_C(T) = \frac{(1 - 2T + 5T^2)^3}{(1 - T)(1 - qT)}.$$

Note that the roots of the denominator are $T = \frac{1}{5} \pm \frac{2}{5}i$.

The kind of regularity of the above examples holds for all varieties and is expressed in the important Weil conjectures [7, 22, 27, 28], which have been proven during the period 1949–1973. As a result we now know that $Z_V(T)$ is always a rational function; but the most important part of the Weil conjectures concerns the roots and poles of the Zeta functions.

3.4. Weil's Riemann Hypothesis. For concreteness, we first describe part of Weil's conjecture regarding the Zeta function of curves.

Theorem 1 (Weil's Riemann Hypothesis for Curves [27]). Let C be a complete, nonsingular curve over \mathbb{F}_q . The corresponding Zeta function Z_C is a rational function $\in \mathbb{Z}(T)$ that can be decomposed as

$$Z_C(T) = \frac{P(T)}{(1-T)(1-qT)}$$
$$= \frac{\prod_{j=1}^{2g} (1-\alpha_j T)}{(1-T)(1-qT)},$$

where g is the genus of C. The polynomial P(T) has integer coefficients: $P(T) \in \mathbb{Z}[T]$, and the roots of P(T) can be grouped in pairs (α_j, α_{j+g}) such that $\alpha_j = \bar{\alpha}_{j+g}$. Most importantly, the magnitudes of the roots of P(T) obey $|\alpha_j| = \sqrt{q}$ for all j.

The genus of a curve reflects how different C is from a straight line (which has genus 0). This quantity has a close connection with the *geometric genus* of the curve f = 0 when viewed in the projective space $\mathbb{P}^2(\mathbb{C})$. Typically, the genus of a curve defined by f is approximately $\deg(f)^2$.

Example 3 is a typical instances of this theorem: The Zeta function of the curve $X_0^4 + X_1^4 + X_2^4 = 0$ in $\mathbb{P}^2(\bar{\mathbb{F}}_5)$ has 6 roots, all with norm $1/\sqrt{5}$, and when we view the equation in the complex projective space in $\mathbb{P}^2(\mathbb{C})$ we see that it has geometric genus 3, which corresponds with the fact that its Zeta functions has 2g = 6 roots.

The most general version of Weil's Riemann hypothesis applies to all 'proper' varieties.

Theorem 2 (Weil's Riemann Hypothesis for Varieties, [27]). Let V be a complete, nonsingular algebraic variety of dimension d over \mathbb{F}_q . The corresponding Zeta function \mathbb{Z}_V is a rational function that can be decomposed as

$$Z_{V}(T) = \frac{P_{1}(T) \cdots P_{2d-1}(T)}{P_{0}(T)P_{2}(T) \cdots P_{2d}(T)}$$

$$= \frac{\prod_{j=1}^{B_{1}} (1 - \alpha_{1,j}T) \cdots \prod_{j=1}^{B_{2d-1}} (1 - \alpha_{2d-1,j}T)}{\prod_{j=1}^{B_{0}} (1 - \alpha_{0,j}T) \prod_{j=1}^{B_{2}} (1 - \alpha_{2,j}T) \cdots \prod_{j=1}^{B_{2d}} (1 - \alpha_{2d,j}T)},$$

where B_i are the Betti numbers of V. All the polynomials $P_i(T)$ are elements of $\mathbb{Z}[T]$, the roots of the P_i polynomials can be paired such that $\alpha_{i,j} = \bar{\alpha}_{i,B_i/2+j}$. It always holds that $P_0(T) = 1 - T$ and $P_{2d}(T) = 1 - q^d T$, and in general the magnitudes of these roots obey $|\alpha_{i,j}| = q^{i/2}$ for all i, j.

As stated before, this theorem has been proven; see [7, 9, 17, 22, 25, 28].

For hypersurfaces, which are defined by a single polynomial, the Zeta functions is particularly simple. Let $f \in \mathbb{F}[X_0, \dots, X_n]$ be a homogeneous polynomial that defines a proper hypersurface H, then its Zeta function has the form

(1)
$$Z_H(T) = \frac{P(T)^{(-1)^n}}{(1-T)(1-qT)\cdots(1-q^{n-1}T)},$$

where each of the $[(\deg(f)-1)^{n+1}-(-1)^n(\deg(f)-1)]/\deg(f)\approx \deg(f)^n$ roots of P(T) have magnitude $q^{-(n-1)/2}$.

4. Zeroes of Zeta Functions as Eigenvalues of Quantum Circuits

We want to interpret the roots of Zeta functions of the previous section as the spectrum of a quantum mechanical process. This wish is a natural extension of the spectral approach to the Riemann conjecture with the added advantage that we already have more evidence in favor of it, especially in the case of curves.

First of all, we know that Weil's Riemann hypothesis is true, hence we know that the roots of the Zeta functions all lie on a circle in the complex plane. Second, it has also been proven by Katz and Sarnak [12, 13] that for curves the distribution of the zeroes of the Zeta functions obeys the kind 'eigenvalue repulsion' that one also sees in random quantum mechanical systems. Specifically, they showed that for 'generic' curves C in the double limit $g \to \infty$ and $q \to \infty$ the distribution of the zeroes of Z_C goes to the eigenvalue distribution of the circular unitary (symplectic) ensemble. (See also [14].)

Of course, given a curve C, one can always define a diagonal unitary matrix with as its diagonal entries the normalized roots of \mathbf{Z}_C and declare this matrix to

describe a quantum mechanical system. Clearly this is not a satisfactory answer because this definition does not give a natural description of a physical system. Hence, before we can proceed, we have to find a criterion for what does constitute a natural physical system.

Here we propose to use the class of efficient quantum algorithms to determine if a sequence of unitary operations is considered natural or not. Formally, a sequence U_1, U_2, \ldots of unitary matrices with $U_i \in \mathsf{U}(N_i)$ is natural if and only if there is a quantum algorithm that for given i efficiently implements the transformation U_i in time $O(\operatorname{poly}(\log N_i))$. More concretely, for a variety V we want to describe an efficient quantum circuit whose eigenvalues correspond to the phases of the zeroes of the Zeta function Z_V . The earlier described universality results in Section 2.3 show that this criterion is independent of the specific quantum computational model that we use, and the quantum Church-Turing thesis suggests that it captures exactly those quantum mechanical systems that can occur in Nature without some kind of exponential overhead.

The next section shows how for so called Fermat hypersurfaces H, which are defined by diagonal equations $c_0 X_0^m + c_1 X_1^m + \cdots + c_{d+1} X_{d+1}^m = 0$, we can indeed construct an efficient quantum algorithm that has the normalized roots of Z_H as its eigenvalues.

5. Quantum Algorithm for Zeta Functions of Some Hypersurfaces

For some hypersurfaces H we can express the roots of the corresponding Zeta functions \mathbf{Z}_H in terms of products of Gauss sums.[11, 16] Before we give the quantum algorithm whose eigenvalues are these roots, we will repeat some known results on the quantum computation of Gauss sums.

5.1. Multiplicative Characters and Quantum Computing Gauss Sums. A multiplicative character of a finite field is a function $\chi: \mathbb{F}_q \to \mathbb{C}$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x,y \in \mathbb{F}_q$. Let g be a primitive element of \mathbb{F}_q , i.e. the multiplicative group $\langle g \rangle$ generated by g equals $\mathbb{F}_q^{\times} := \mathbb{F}_q \backslash \{0\}$, and let $\omega_{q-1} := \mathrm{e}^{2\pi \mathrm{i}/(q-1)}$. For each $\alpha \in \{0,1,\ldots,q-2\}$, the function $\chi(g^j) := \omega_{q-1}^{\alpha j}$ (complemented with $\chi(0) := 0$) is a multiplicative character. Conversely, every multiplicative character can be written as such a function. The discrete logarithm with respect to g is defined for every $x = g^j \in \mathbb{F}_q^{\times}$ by $\log_g(x) := j \mod (q-1)$. Hence, every multiplicative character can be expressed as $\chi(x) := \omega_{q-1}^{\alpha \log_g(x)}$ for $x \neq 0$ and $\chi(0) := 0$. Fix a generator g and define the primitive multiplicative character χ by $\chi(x) := 0$.

Fix a generator g and define the primitive multiplicative character χ by $\chi(x) := \omega_{q-1}^{\log_g(x)}$. For every $\alpha \in \mathbb{Z}$ we have the α -th power of χ according to $\chi^{\alpha}(x) := (\chi(x))^{\alpha}$. Thus the *trivial multiplicative character* is denoted by χ^0 and is defined by $\chi^0(0) = 0$ and $\chi^0(x) = 1$ for all $x \neq 0$. Using the equality $\chi^{\alpha} \cdot \chi^{\beta} = \chi^{\alpha+\beta}$ we see that the set of characters $\{\chi^{\alpha} : \alpha \in \{0, \dots, q-2\}\}$ with pointwise multiplication defines a group isomorphic to the additive group $\mathbb{Z}/(q-1)\mathbb{Z}$. The inverse of χ obeys $\chi^{(-1)}(x) = \overline{\chi(x)}$ for all x, where \bar{z} denotes the complex conjugate of z.

Definition 4 (Gauss sums over Finite Fields). For a finite field \mathbb{F}_q with $q = p^r$ and a multiplicative character χ , we define the complex valued *Gauss sum g* by

$$g(\chi) \ := \ \sum_{x \in \mathbb{F}_q} \chi(x) \omega_p^{\mathrm{Tr}(x)},$$

where Tr is the standard trace function. Obviously, $g(\chi^0) = -1$ and for nontrivial characters χ we have that the norm of the Gauss sum obeys $|g(\chi)| = \sqrt{q}$.

To be a able to quantum compute we use the following states [4], which were also previously described by Watrous in [26].

Definition 5 (Chi States). Given a finite field \mathbb{F}_q and a generator $g \in \mathbb{F}_q^{\times}$ we define the *chi states* for every $\alpha \in \mathbb{Z}$ by

$$|\chi^{\alpha}\rangle \quad := \quad \frac{1}{\sqrt{q-1}} \sum_{x \in \mathbb{F}_q} \chi^{\alpha}(x) |x\rangle,$$

where χ^{α} refers to the multiplicative character defined by g in $\chi^{\alpha}(g^{j}) = \omega_{q-1}^{\alpha j}$. Note that $|\chi^{0}\rangle$ is the uniform superposition of the elements of \mathbb{F}_{q}^{\times} . Chi states can be produced in time $O(\operatorname{poly}(\log q))$ on a quantum computer.[4]

The assumption that multiplication and division in \mathbb{F}_q can be done efficiently implies that, using repeated powering $x\mapsto x^2\mapsto x^4\cdots$, we can efficiently calculate any power x^j in \mathbb{F}_q for -q< j< q. This is a useful operation in combination with χ states. It is straightforward to check that if we apply the reversible $|x,y\rangle\mapsto |x,y/x^\alpha\rangle$ mapping (for $x,y\in\mathbb{F}_q^\times$) to the superposition $|\chi^\beta\rangle|\chi^\gamma\rangle$, then we obtain $|\chi^{\beta+\alpha\gamma}\rangle|\chi^\gamma\rangle$. Hence, under the assumption that it is easy to create the uniform superposition $|\chi^0\rangle$, we can efficiently create arbitrary $|\chi^\alpha\rangle$ states from an initial state $|\chi\rangle$. Chi states are especially helpful if we want to induce a phase change $\mathrm{e}^{\mathrm{i}\theta}$ that is determined by a Gauss sum $g(\chi)=\mathrm{e}^{\mathrm{i}\theta}\sqrt{q}$ of a nontrivial character χ . For, if we create the state $|\chi\rangle$ and perform the Fourier transform over \mathbb{F}_q to it, we implement the evolution $|\chi\rangle\mapsto g(\chi)/\sqrt{q}\cdot|\chi^{-1}\rangle$. See [5, 6] and references therein for more detailed information about these topics.

5.2. Quantum Algorithm for Zeta Functions of Fermat Hypersurfaces. With the results of the previous subsection we have the following theorem.

Theorem 3. Let f be a homogeneous polynomial in $\mathbb{F}_q[X_0,\ldots,X_n]$:

$$f(X_0,\ldots,X_n) = c_0 X_0^m + c_1 X_1^m + \cdots + c_n X_n^m,$$

with $q = 1 \mod m$, and let H be the corresponding (n - 1)-dimensional projective hypersurface $H := \{x : x \in \mathbb{P}^n(\bar{\mathbb{F}}_q) \text{ and } f(x) = 0\}$. For every such H there exists an efficient quantum circuit whose eigenvalues are the roots of the Zeta function Z_H .

Proof. In Chapter 11 of [11] it is explained how the the corresponding Zeta function equals

$$Z_H(T) = \frac{P(T)^{(-1)^n}}{(1-T)(1-qT)\cdots(1-q^{n-1}T)},$$

where P (and hence \mathbf{Z}_H) has $(m-1)[(m-1)^n+(-1)^{n-1}]/m\approx m^n$ non-trivial zeroes, all on the circle $|T|=q^{-(n-1)/2}$.

Define $\tilde{\chi}$ to be the character $\tilde{\chi}(g^j) := \omega_{q-1}^{j(q-1)/m}$, with g a generator of \mathbb{F}_q^{\times} , such that $\tilde{\chi}^m = \chi^0$. Then, we also know that the roots $1/\alpha_j$ of $P = \prod_j (1 - \alpha_j T)$ are described by

(2)
$$\frac{1}{\alpha_j} = \frac{(-1)^{n-1}}{q^n} \cdot \frac{g(\chi_0) \cdots g(\chi_n)}{\chi_0(c_0) \cdots \chi_n(c_n)},$$

where the multiplicative characters are defined by $\chi_i := \tilde{\chi}^{b_i}$, with $b_i \in \{1, \dots, m-1\}$ for all i and $\sum_i b_i = 0 \mod m$.

We rewrite the roots as $1/\alpha_j = \mathrm{e}^{\mathrm{i}\theta_j}/\sqrt{q^{n-1}}$, such that we can focus on the unknown angles $\theta_j \in [0, 2\pi)$. Using the quantum algorithms for Gauss sum estimation and multiplicative character phase changing described in [5, 6], we can implement the evolution $|b_0,\ldots,b_n\rangle\mapsto \mathrm{e}^{\mathrm{i}\theta_j}|b_0,\ldots,b_n\rangle$ for every j and its corresponding sequence (b_0,\ldots,b_n) . Hence this evolution has $|b_0,\ldots,b_n\rangle$ as its eigenstates and the normalized roots $\sqrt{q^{n-1}}/\alpha_j = \mathrm{e}^{\mathrm{i}\theta_j}$ as the eigenvalues.

In order to make this theorem more explicit, we describe the quantum algorithm (note that the size of the input f is $\approx n \log q$ bits).

Algorithm 1. Given the polynomial f of Theorem 3, implement the following quantum evolution on a state $|b_0, \ldots, b_n\rangle$ that obeys $b_i \neq 0$ for all i and $\sum_i b_i = 0 \mod m$.

- (1) Attach the chi states $|\chi^0, \tilde{\chi}\rangle$, giving $|b_0, \dots, b_n\rangle \otimes |\chi^0, \tilde{\chi}\rangle$.
- (2) For every $0 \le i \le n$, perform the following steps:
 - (a) Apply $|x,y\rangle \mapsto |x,y/x^{b_i}\rangle$ to $|\chi^0, \tilde{\chi}\rangle$ such that we get $|\tilde{\chi}^{b_i}, \tilde{\chi}\rangle$.
 - (b) Apply a Fourier transformation over \mathbb{F}_q to the $\tilde{\chi}^{b_i}$ register, yielding the state $g(\tilde{\chi}^{b_i})/\sqrt{q}\cdot |\tilde{\chi}^{-b_i},\tilde{\chi}\rangle$.
 - (c) Again perform $|x,y\rangle\mapsto |x,y/x^{b_i}\rangle$ to the two chi registers such that the net effect of the subroutine is $|\chi^0,\tilde\chi\rangle\mapsto g(\tilde\chi^{b_i})/\sqrt q\cdot |\chi^0,\tilde\chi\rangle$.

We now have the state $g(\chi_0) \cdots g(\chi_n) / \sqrt{q^{n+1}} \cdot |b_0, \dots, b_n\rangle \otimes |\chi^0, \tilde{\chi}\rangle$.

- (3) For every $0 \le i \le n$, apply the reversible $|y\rangle \mapsto |y\cdot c_i^{b_i}\rangle$ to the $\tilde{\chi}$ register. This induces the phase change $|\tilde{\chi}\rangle \mapsto \chi^{-b_i}(c_i)|\tilde{\chi}\rangle = |\tilde{\chi}\rangle/\chi_i(c_i)$.
- (4) Finally, apply a general phase flip $(-1)^{n-1}$ to the state.

By multiplying the phase changes of the above steps, one sees that this algorithm establishes the overall evolution

$$|b_0,\ldots,b_n,\chi^0,\tilde{\chi}\rangle \longmapsto \frac{(-1)^{n-1}}{\sqrt{q^{n+1}}} \cdot \frac{g(\chi_0)\cdots g(\chi_n)}{\chi_0(c_0)\cdots \chi_n(c_n)} |b_0,\ldots,b_n,\chi^0,\tilde{\chi}\rangle,$$

which describes indeed the roots of Equation 2. The time and space complexity of the algorithm is $O(n \cdot \text{poly}(\log q))$.

6. Zeta Functions and Approximate Point Counting

In this section we look at the relevance of Zeta functions to the computational task of point counting. Directly from the definition $Z_V(T) := \exp(\sum_s N_s T^s/s)$ it is clear that knowledge about Z_V implies knowledge about the numbers of solutions N_s . Note for example that we have $Z_V(\delta) = 1 + N_1 \delta + O(\delta^2)$; hence the value of first derivative d/dT of $Z_V(T)$ at T = 0 answers the question " $N_1 = ?$ "

On the one hand this is good news because counting is a central problem in computational complexity theory about which we already have many results. On the other hand, we know that exact counting very quickly becomes #P-complete (where "very quickly" means determining N_1 for moderately complicated V). Because it is unlikely that quantum computers can solve #P-complete problems [8], this makes it at least as unlikely that we will be able to efficiently determine $\mathbf{Z}_V(T)$ exactly with a quantum algorithm. Fortunately, the proposal in this article concerns the design of algorithms that do not try to exactly count the values N_s , but rather only

try to approximate these quantities. Hence, the hardness of #P-complete problems does not directly contradict our spectral approach to Zeta functions.

6.1. Zeroes of Zeta Functions and Approximate Point Counting. By taking the sth derivative d^s/dT^s at T=0 of the logarithm of $Z_V(T)$, we see, using Theorem 2, that for the number of solutions N_s of V over the finite field extension \mathbb{F}_{q^s} we have

$$N_s = q^{ds} + 1 + \sum_{i=1}^{2d-1} (-1)^i \sum_{j=1}^{B_i} \alpha_{i,j}^s.$$

For a hypersurface H defined by a homogeneous polynomial $f \in \mathbb{F}[X_0, \dots, X_n]$, where most of the roots $\alpha_{i,j}$ are trivial, this equation becomes especially simple (see Equation 1):

$$N_s = \frac{q^{sn} - 1}{q^s - 1} - (-1)^n \sum_j \alpha_j^s,$$

where the summation goes over the $\approx \deg(f)^n$ non-trivial zeroes of \mathbf{Z}_H , each obeying $\alpha_j := \sqrt{q^{n-1}} \cdot \mathrm{e}^{\mathrm{i}\theta_j}$. The number of points of an (n-1)-dimensional plane in $\mathbb{P}^n(\mathbb{F}_{q^s})$ equals the first term $(q^{sn}-1)/(q^s-1)$, hence the α_j^s values express how much the number of points of H deviates from those of a straight plane. If we assume that we have a unitary transformation U_H whose spectrum consists of the $\mathrm{e}^{\mathrm{i}\theta_j}$ phases, then we can express this deviation as

$$N_s - \frac{q^{sn} - 1}{q^s - 1} = -(-1)^n \sqrt{q^{s(n-1)}} \cdot \text{Tr}(U_H^s).$$

Hence by estimating the trace of U_H we obtain a non-trivial estimation of the number N_1 of solutions to the equation f = 0 over \mathbb{F}_q .

6.2. Potential Quantum Algorithms for Approximate Counting. Assume that we can efficiently implement the unitary transformation U_H of the previous subsection. Here, "efficiently" means that the time/space complexity of the implementation is $\operatorname{poly}(\log q, \log(\dim(U_H))) = \operatorname{poly}(\log q, n\log(\deg(f)))$. Using standard phase estimation techniques and $1/\varepsilon$ repetitions, this enables us to estimate $-1 \leq \operatorname{Tr}(U_H)/\dim(U_H) \leq 1$ with precision ε . Overall, this gives an estimate \tilde{N}_1 with expected error

$$\left| N_1 - \tilde{N}_1 \right| \sim \sqrt{q^{n-1}} \deg(f)^n \cdot \varepsilon,$$

with time/space complexity poly $(1/\varepsilon, \log q, n \log(\deg(f)))$.

If one classically (and trivially) samples $1/\varepsilon$ times the space $\mathbb{P}^n(\mathbb{F})$ to estimate the number N_1 , then the estimated error will be $\sim q^n \cdot \varepsilon$, while recent results [10, 24] indicate that it might be possible to classically count N_1 exactly in time poly(log q, deg $(f)^n$).

This indicates that the conjectured quantum algorithm that uses U_H to approximately count N_1 would outperform classical computation in the case where $\deg(f)^n$ is exponentially big while $\deg(f)$ is smaller than \sqrt{q} . The algorithm that was described in Theorem 3 is an example of such a case if we fix both $\deg(f)$ and q with $\deg(f) < \sqrt{q}$.

7. Conclusion

The open problem that remains is obvious: For what other hypersurfaces H can we construct quantum circuits with eigenvalues corresponding to the roots of the Zeta function $\mathbf{Z}_H(T)$? We consider such circuits efficient if the space/time complexity is bounded by $\operatorname{poly}(\log q, n \log(\deg(f)))$, where $f \in \mathbb{F}_q[X_0, \ldots, X_n]$ is the homogeneous polynomial that defines H. In addition, if we want to know how useful such quantum circuits are in comparison with classical algorithms, we also need to know what the classical complexity is of approximating roots of Zeta functions.

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Appendix A. Varieties and Algebraic Geometry

In this section I will give a brief overview of the ingredients from commutative algebra and algebraic geometry that are necessary to appreciate Weil's Riemann conjectures. To make matters a bit easier the definitions are only done in the context of afine varieties V over finite fields \mathbb{F} . For the definition of the Zeta function we only deal with the case when the variety is a planar curve C. This restriction allows us to leave out various part of the general theory.

A.1. Ideals and Varieties. (See [3, 20, 21].) An *ideal* of the ring of polynomials $\mathbb{F}[X_1, \ldots, X_n]$ is a subset $\mathfrak{I} \subseteq \mathbb{F}[X_1, \ldots, X_n]$ such that:

- $0 \in \mathfrak{I}$
- If $f, g \in \Im$ then $f + g \in \Im$
- If $f \in \mathfrak{I}$ and $h \in \mathbb{F}[X_1, \dots, X_n]$ then $fh \in \mathfrak{I}$.

The ideal generated by a finite set of polynomials f_1, \ldots, f_t is the smallest possible ideal $\mathfrak{I} \subseteq \mathbb{F}[X_1, \ldots, X_n]$ such that $f_1, \ldots, f_t \in \mathfrak{I}$. This ideal equals

$$\langle f_1,\ldots,f_t\rangle := \{f_1h_1+\cdots+f_th_t:h_1,\ldots,h_t\in\mathbb{F}[X_1,\ldots,X_n]\}.$$

Hilbert's basis theorem says that each ideal $\subseteq \mathbb{F}[X_1,\ldots,X_n]$ can be finitely generated by a list of polynomials f_1,\ldots,f_t .

An affine variety $\subseteq \mathbb{A}^n(\bar{\mathbb{F}})$ is defined by the polynomials f_1, \ldots, f_t as the set of roots

$$\mathbf{V}(f_1, \dots, f_t) := \{x \in \mathbb{A}^n(\bar{\mathbb{F}}) : f_1(x) = 0, \dots, f_t(x) = 0\}.$$

For an affine variety V we define the corresponding ideal by

$$\mathbf{I}(V) := \{ f \in \mathbb{F}[X_1, \dots, X_n] : f(x) = 0 \text{ for all } x \in V \}.$$

The crux of the matter here is that for a given variety V there are many different sets of polynomials f_1, \ldots, f_t such that $V = \mathbf{V}(f_1, \ldots, f_t)$, but there is only one ideal $\mathbf{I}(V)$. Hence when studying V it is often more useful to look at $\mathbf{I}(V)$ rather than an arbitrary set of polynomials with $V = \mathbf{V}(f_1, \ldots, f_t)$.

We start with a few facts about multivariate polynomials over finite fields. Let $f \in \mathbb{F}[X_1,\ldots,X_n]$. It is important to realize that "f(X)=0 for all $X \in \mathbb{A}^n(\mathbb{F})$ " does not imply that f=0. Take for example $f(X_1)=X_1^2-X_1$ for $X_1\in\mathbb{F}_2$. If, however, "f(X)=0 for all $X\in\mathbb{A}^n(\bar{\mathbb{F}})$ " with $\bar{\mathbb{F}}$ algebraically closed, then indeed f=0. Similarly, if $f(X)\neq 0$ for all $X\in\mathbb{A}^n(\bar{\mathbb{F}})$, then $f=c\in\mathbb{F}^\times$. From this it follows that if the variety of a principal ideal $\langle f\rangle\subseteq\mathbb{F}[X_1,\ldots,X_n]$ is a subset of $\mathbb{A}^n(\bar{\mathbb{F}})$, then $\mathbf{V}(\langle f\rangle)=\mathbb{A}^n(\bar{\mathbb{F}})$ implies f=0, and $\mathbf{V}(\langle f\rangle)=\{\}$ implies $\langle f\rangle=\mathbb{F}[X_1,\ldots,X_n]$ (because $c\in\mathbb{F}^\times$ generates the whole ring $\mathbb{F}[X_1,\ldots,X_n]$).

For any variety V we have $\mathbf{V}(\mathbf{I}(V)) = V$. However, another important issue is the fact that although $\langle f_1, \ldots, f_t \rangle \subseteq \mathbf{I}(\mathbf{V}(f_1, \ldots, f_t))$, this inclusion is sometimes strict. Consider for example $\langle X^2 \rangle \subseteq \mathbf{I}(\mathbf{V}(X^2)) = \langle X \rangle \neq \langle X^2 \rangle$. Hence for some ideals \mathfrak{I} we have $\mathbf{I}(\mathbf{V}(\mathfrak{I})) \neq \mathfrak{I}$. The following is a necessary and sufficient condition on \mathfrak{I} such that $\mathbf{I}(\mathbf{V}(\mathfrak{I})) = \mathfrak{I}$. The radical ideal of an ideal \mathfrak{I} is the ideal

$$\sqrt{\mathfrak{I}} := \{f : f^m \in \mathfrak{I} \text{ for some } m \in \mathbb{Z}^+\}.$$

The Nullstellensatz says that for all ideals $\mathbf{I}(\mathbf{V}(\mathfrak{I})) = \sqrt{\mathfrak{I}}$. Or, in other words: $\mathbf{I}(\mathbf{V}(\mathfrak{I})) = \mathfrak{I}$ if and only if \mathfrak{I} is radical. A direct consequence of this "zeroes theorem" is that $\mathbf{V}(\mathfrak{I}) = \{\}$ implies $\mathfrak{I} = \mathbb{F}[X_1, \dots, X_n]$.

A variety V is *irreducible* if $V = V_1 \cup V_2$ with V_1 and V_2 varieties implies $V = V_1$ or $V = V_2$. An ideal $\mathfrak{P} \subseteq \mathbb{F}[X_1, \ldots, X_n]$ is *prime* if for all $fg \in \mathfrak{P}$ with $f, g \in \mathbb{F}[X_1, \ldots, X_n]$ we have $f \in \mathfrak{P}$ or $g \in \mathfrak{P}$. A variety V is irreducible if and only if the ideal $\mathbf{I}(V)$ is prime. Clearly, each prime ideal is radical.

For the algebraically closed field \mathbb{F} with varieties $V \subseteq \mathbb{A}^n(\mathbb{F})$ it also holds that for prime ideals \mathfrak{P} we have $\mathbf{I}(\mathbf{V}(\mathfrak{P})) = \mathfrak{P}$. Hence we see that \mathbf{I} and \mathbf{V} establish a bijection between the prime ideals $\mathfrak{P} \subseteq \mathbb{F}[X_1, \ldots, X_n]$ and the irreducible varieties $V \subseteq \mathbb{A}^n(\overline{\mathbb{F}})$. (That this is not true for varieties over finite fields is readily seen by $\mathfrak{P} = \langle X^2 + X + 1 \rangle \subseteq \mathbb{F}_2[X]$ with $\mathbf{V}(\mathfrak{P}) = \{\} \subseteq \mathbb{A}^1(\mathbb{F}_2)$ and hence $\mathbf{I}(\mathbf{V}(\mathfrak{P})) = \mathbb{F}_2[X] \neq \mathfrak{P}$. In the case of varieties $\subseteq \mathbb{A}^1(\overline{\mathbb{F}}_2)$ we have $\mathbf{V}(\mathfrak{P}) = \mathbb{F}_4 \setminus \mathbb{F}_2$, and hence indeed $\mathbf{I}(\mathbf{V}(\mathfrak{P})) = \mathfrak{P}$.) For each prime ideal \mathfrak{P} , the quotient ring $\mathbb{F}[X_1, \ldots, X_n]/\mathfrak{P}$ is an *integral domain*, which is a commutative ring with a "1" and without zero-divisors.

A.2. Addition and Multiplication of Ideals. One can add and multiply ideals $\mathfrak{I},\mathfrak{L}\subseteq R$ according to $\mathfrak{I}+\mathfrak{L}:=\{f+g:f\in\mathfrak{I},g\in\mathfrak{L}\}$ and $\mathfrak{I}\cdot\mathfrak{L}:=\{\sum_i f_ig_i:f_i\in\mathfrak{I},g_i\in\mathfrak{I},g_i\in\mathfrak{L}\}$, where \sum_i is a finite summation. Both constructions are again ideals with the inclusions $\mathfrak{I}\mathfrak{L}\subseteq\mathfrak{I},\mathfrak{L}\subseteq\mathfrak{I}+\mathfrak{L}$. The corresponding varieties obey the rules: $\mathbf{V}(\mathfrak{I}+\mathfrak{L})=\mathbf{V}(\mathfrak{I})\cap\mathbf{V}(\mathfrak{L})$ and $\mathbf{V}(\mathfrak{I}\cdot\mathfrak{L})=\mathbf{V}(\mathfrak{I})\cup\mathbf{V}(\mathfrak{L})$. The zero ideal is $\{0\}$, while the unit ideal is $\{1\}=\mathbb{F}[X_1,\ldots,X_n]$.

A.3. Coordinate Ring of a Planar Curve. Let $f \in \mathbb{F}[X,Y]$ define a planar curve such that $\langle f \rangle$ is a prime ideal and let $C := \mathbf{V}(\langle f \rangle) \subseteq \mathbb{A}^2(\overline{\mathbb{F}})$ be the corresponding irreducible variety. Let $\mathbb{F}[C] := \mathbb{F}[X,Y]/\langle f \rangle$ be the commutative coordinate ring of polynomials on the curve C. Because $\langle f \rangle$ is prime, $\mathbb{F}[C]$ is an integral domain, which means that gh = 0 implies g = 0 or h = 0 in $\mathbb{F}[C]$. In fact, the commutative ring $\mathbb{F}[C]$ is a Dedekind ring such that all ideals $\mathfrak{I} \subseteq \mathbb{F}[C]$ will have a unique factorization in terms of prime ideals $\mathfrak{I} \subseteq \mathbb{F}[C]$.

How do the ideals \mathfrak{I} of $\mathbb{F}[C]$ look like? Let $g \in \mathfrak{I}$ with a point $(a,b) \in \mathbb{A}^2(\bar{\mathbb{F}})$ on the curve C such that g(a,b) = 0. Define the *Frobenius automorphism* $\phi : \bar{\mathbb{F}} \to \bar{\mathbb{F}}$ by $\phi : z \mapsto z^q$, where g is the size $|\mathbb{F}|$ of the base field. It is straightforward to see that

 ϕ is indeed an automorphism with $\phi(x+y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \overline{\mathbb{F}}$. The rth power ϕ^r is defined by $\phi^r(z) := z^{(q^r)}$, such that ϕ^r acts as the identity on \mathbb{F}_{q^r} . Let $\phi(a,b)$ denote the point $(\phi(a),\phi(b))$. Because $g \in \mathbb{F}[X,Y]$ we see that g(a,b) = 0 implies $g(\phi^r(a,b)) = 0$ for $r = \mathbb{Z}^+$. Let d be the degree of the point (a,b), which means that d is the smallest integer such that $(a,b) \in \mathbb{A}^2(\mathbb{F}_{q^d})$. Then $\phi^d(a,b) = (a,b)$ and the orbit $O_{(a,b)} := \{(a,b),\phi(a,b),\phi^2(a,b),\ldots,\phi^{d-1}(a,b)\}$ consists of d different points in $\mathbb{A}^2(\overline{\mathbb{F}})$. Because $f \in \mathbb{F}[X,Y]$, all points $\phi^r(a,b)$ will also lie on the curve C. As a result, g will also be zero on C for all g points g po

Conversely, let $S \subseteq C$ be a set of points that is closed under the Frobenius automorphishm (that is: if $(a,b) \in S$, then $\phi(a,b) \in S$). It is straightforward to check that the set of polynomials $\{g \in \mathbb{F}[C] : g(a,b) = 0 \text{ for all } (a,b) \in S\}$ is an ideal in $\mathbb{F}[C]$.

The prime ideals of $\mathbb{F}[C]$ are in one-on-one correspondence with unique orbits in C. For each orbit $O_{(a,b)} \subset C$ the ideal $\mathbf{I}(O_{(a,b)})$ is prime, and for each prime ideal $\mathfrak{P} \subset \mathbb{F}[C]$ the variety $\mathbf{V}(\mathfrak{P}) \subset C$ is an orbit $O_{(a,b)}$.

A.4. Norms in Coordinate Rings. The polynomial $f \in \mathbb{F}[X,Y]$ is absolutely irreducible if and only if f is irreducible in $\overline{\mathbb{F}}[X,Y]$. A point (a,b) of the corresponding curve C is singular if and only if $\mathrm{d}f/\mathrm{d}X = \mathrm{d}f/\mathrm{d}Y = 0$ at (a,b). The curve C is nonsingular if and only if it has no singular points.

If f is absolutely irreducible and the corresponding curve C is nonsingular, then we can define a norm on the ideals of $\mathbb{F}[X,Y]/(f)$ as follows. Let \mathfrak{I} be a non-zero ideal, then its norm in the coordinate ring $\mathbb{F}[C]$ is defined by $\|\mathfrak{I}\|:=|\mathbb{F}[C]/\mathfrak{I}|$, which will always be a finite number. Recalling the definition of multiplication of ideals, one can show that $\|\mathfrak{I} \cdot \mathfrak{L}\| = \|\mathfrak{I}\| \cdot \|\mathfrak{L}\|$. For each prime ideal \mathfrak{P} , we have that $\|\mathfrak{P}\| = q^d$ where d is the size of the orbit corresponding to \mathfrak{P} .

APPENDIX B. WHY IT IS CALLED A ZETA FUNCTION

Here I will give a brief explanation of the connection between Riemann's zeta function and the Zeta function of finite field equations.

B.1. Riemann's Zeta Function. For $z \in \mathbb{C}$ with $\Re(z) > 1$, Riemann's zeta function is defined by $\zeta_R(z) := \sum_{n=1}^\infty n^{-z}$. For the other $z \in \mathbb{C}$ with $\Re(z) \le 1$, the zeta function $\zeta(z)$ is determined by the analytical continuation of the $\Re(z) > 1$ part of the function. This continuation gives explicitly $\zeta(z) = (\sum_{n=1}^\infty (-1)^{n-1}/n^z)/(1-2^{1-z})$ for $\Re(z) > 0$ and $\zeta(1-z) = 2(2\pi)^{-z}\cos(z\pi/2)\Gamma(z)\zeta(z)$ for $\Re(z) < 0$. (Compare this with the function $f(z) = \sum_{n=1}^\infty z^{-n}$, which is properly defined only for |z| > 1, but which can be continued to the whole $z \in \mathbb{C}$ plane by the function 1/(z-1).)

The Riemann zeta function diverges to $+\infty$ as $s\to 1$ and $\zeta(z)=0$ for $z=-2,-4,-6,\ldots$ (the *trivial zeroes*). The remaining roots of ζ lie in the strip $0<\Re(z)<1$. The *Riemann conjecture* states that all these non-trivial zeroes z are in fact on the line $\Re(z)=\frac{1}{2}$. Some well-known function values are: $\zeta(-2)=0$, $\zeta(-1)=-\frac{1}{12},\ \zeta(0)=-\frac{1}{2},\ \zeta(1)=+\infty,\ \zeta(2)=\pi^2/6,\ \zeta(3)=1.202056903\ldots$, and the first non-trivial root $\zeta(\frac{1}{2}\pm i\cdot 14.1347\ldots)=0$.

The connection between prime numbers and the zeta function is given by the important $Euler\ product$

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

$$= \prod_{p=2,3,5,\dots} \left(1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \dots \right)$$

$$= \prod_{p=2,3,5,\dots} \frac{1}{1 - p^{-z}}$$

where the product is over all primes p in \mathbb{Z} . For example, this shows how the value $\zeta(1) = +\infty$ implies the fact that there is an infinite number of primes in \mathbb{Z} . The Euler product relies crucially on the fact that each number $n = 1, 2, 3, \ldots$ has a unique factorization in terms of the primes $p = 2, 3, 5, \ldots$

The ideals of \mathbb{Z} are the subsets $\mathfrak{N} = n\mathbb{Z}$ with the norm $\|\mathfrak{N}\| := |\mathbb{Z}/n\mathbb{Z}| = n$. The zeta function is hence defined by $\zeta(\mathbb{Z},z) := \sum_{\mathfrak{N}} \|\mathfrak{N}\|^{-z} = \prod_{\mathfrak{P}} 1/(1 - \|\mathfrak{P}\|^{-z})$, which relies on the unique factorization of each ideal \mathfrak{N} into a sequence of prime ideals \mathfrak{P} :

$$\sum_{\mathfrak{N}\subset\mathbb{Z}}\frac{1}{\|\mathfrak{N}\|^z} = \prod_{\mathfrak{P}\subset\mathbb{Z}}\frac{1}{1-\|\mathfrak{P}\|^{-z}}.$$

B.2. **Dedekind Zeta Functions.** For number fields $K = \mathbb{Q}(\theta)$ other than \mathbb{Q} we can extend the notion of a zeta function to that of *Dedekind zeta functions*. We thus get a zeta function for the ring \mathbb{Z}_K (like the Gaussian integers $\mathbb{Z}[i]$) that are defined similar to the previous equation, except for that the summation (product) now ranges over the (prime) ideals $\mathfrak{N}, \mathfrak{P} \subseteq \mathbb{Z}_K$.

Example 4 (Dedekind Zeta Function of Gaussian Integers). Consider the elements in the ring $\mathbb{Z}[i] := \{a + bi : a, b \in \mathbb{Z}\}$, which is a *principal ideal domain* with units ± 1 and $\pm i$. Each ideal (a + bi) can be viewed as a lattice in $\mathbb{Z}[i]$ in that is spanned by the vectors a + bi and -b + ai, which shows that that we have the norm $\|(a + bi)\| := |\mathbb{Z}[i]/(a + bi)| = a^2 + b^2$ on the ideals of $\mathbb{Z}[i]$.

The prime ideals $\mathfrak{P} \subset \mathbb{Z}[i]$ are described as follows (see Section 9.7 in [11]):

- $\mathfrak{P} = (1 + i)$, with obviously $\|\mathfrak{P}\| = 2$.
- $\mathfrak{P} = (p)$ with $p = 3 \mod 4$ a prime in \mathbb{Z} , such that $\|\mathfrak{P}\| = p^2$.
- For each prime $p = 1 \mod 4$ in \mathbb{Z} there are two different prime ideals (a+bi) and (a-bi) (determined by $a^2 + b^2 = p$), such that for both ideals $\|\mathfrak{P}\| = p$.

Hence the Dedekind zeta function of $\mathbb{Z}[i]$ equals

$$\zeta(\mathbb{Z}[i], z) := \prod_{\mathfrak{P} \subset \mathbb{Z}[i]} \frac{1}{1 - \|\mathfrak{P}\|^{-z}} \\
= \frac{1}{1 - 2^{-z}} \prod_{p=3,7,11,\dots} \frac{1}{1 - p^{-2z}} \prod_{p=5,13,17,\dots} \frac{1}{(1 - p^{-z})^2}$$

The conjecture that the non-trivial zeroes of zeta functions ζ_K of all such K lie on the $\Re(z) = \frac{1}{2}$ line is known as the *extended Riemann hypothesis*.

B.3. Zeta Functions of Algebraic Varieties. Let $f \in \mathbb{F}[X,Y]$ be an absolutely irreducible function and let $C \subseteq \mathbb{A}^2(\bar{\mathbb{F}})$ be its nonsingular curve. In Section A.3 we mentioned that this implies that the ring $\mathbb{F}[C] := \mathbb{F}[X,Y]/\langle f \rangle$ is a Dedekind domain with finite quotients. This means that the nontrivial ideals $\mathfrak{I} \subseteq \mathbb{F}[C]$ have a unique factorization $\mathfrak{I} = \mathfrak{P}_1 \cdots \mathfrak{P}_s$ into the prime ideals \mathfrak{P}_i of $\mathbb{F}[C]$. Furthermore, for each nonzero ideal the norm $\|\mathfrak{I}\| := |\mathbb{F}[C]/\mathfrak{I}|$ will be a finite integer.

We can now define the zeta function of $\mathbb{F}[C]$ by

$$\zeta(\mathbb{F}[C], z) := \sum_{\gamma} \frac{1}{\|\mathfrak{I}\|^z},$$

where we sum over all non-zero ideals of $\mathbb{F}[C]$. Because of the unique factorization property of $\mathbb{F}[C]$, we can rewrite the zeta function according to the Euler product

$$\zeta(\mathbb{F}[C], z) = \prod_{\mathfrak{P}} \frac{1}{1 - \|\mathfrak{P}\|^{-z}},$$

with the product over all prime ideals \mathfrak{P} of $\mathbb{F}[C]$.

As mentioned in Section A.3, each prime ideal \mathfrak{P} corresponds to an orbit $O_{(a,b)} = \{(a,b),\phi(a,b),\ldots,\phi^{d-1}(a,b)\}$ of d points in C, where d is the degree of the point (a,b). The norm of such a prime ideal equals $\|\mathfrak{P}\| = q^d$. Let b_d be the number of ideals that have norm q^d and define $Z(\mathbb{F}[C],q^{-z}) := \zeta(\mathbb{F}[C],z)$ in combination with the substitution $T \leftarrow q^{-z}$, then

$$Z(\mathbb{F}[C], T) = \prod_{d=1}^{\infty} (1 - T^d)^{-b_d}.$$

Taking the natural logarithm of this infinite product gives the following summation

$$\log(\mathbf{Z}(\mathbb{F}[C], T)) = -\sum_{d=1}^{\infty} b_d \log(1 - T^d)$$
$$= \sum_{d,j=1}^{\infty} b_d \frac{T^{dj}}{j}.$$

By looking at fixed powers T^s , we can rewrite this summation as

$$\log(\mathbf{Z}(\mathbb{F}[C], T)) = \sum_{s=1}^{\infty} \sum_{d|s} db_d \frac{T^s}{s}.$$

Now, to count the number of points $N_s := C \cap \mathbb{A}^2(\mathbb{F}_{q^s})$, we add all orbits that have coefficients in \mathbb{F}_{q^s} . Because a degree d orbit lies in $\mathbb{A}^2(\mathbb{F}_{q^s})$ if and only if d|s (as $F_{q^d} \subseteq \mathbb{F}_{q^s}$ if and only if d|s), we see that $N_s = \sum_{d|s} db_d$. Going back to the Zeta function, we thus have

(3)
$$Z(\mathbb{F}[C], T) = \exp\left(\sum_{s=1}^{\infty} N_s \frac{T^s}{s}\right).$$

for $C \subseteq \mathbb{A}^2(\overline{\mathbb{F}})$ a nonsingular curve defined by a completely irreducible polynomial $f \in \mathbb{F}[X,Y]$. Note however that—with a leap of faith—Equation 3 can also be used to define a Zeta function $Z(\mathbb{F}[V],T)$ for general affine varieties $V \subseteq \mathbb{A}^n(\overline{\mathbb{F}})$ or projective varieties $V \subseteq \mathbb{P}^n(\overline{\mathbb{F}})$.

B.4. Connection Between Local and Global Zeta Functions. For each prime field $\mathbb{Z}/p\mathbb{Z}$ we say that $\zeta(\mathbb{Z}/p\mathbb{Z},z)=1/(1-p^{-z})$ for the following reason. Consider $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p[X]/(X)$ where X=0 has one solution $(N_s=1)$ for all fields \mathbb{F}_{p^s} . Using the definition of the Zeta function for varieties, we find that $\mathbb{Z}(\mathbb{F}_p,T)=1/(1-T)$ for all p, and hence indeed $\zeta(\mathbb{F}_p,z)=1/(1-p^{-z})$. Observe now that we can recover the Riemann zeta function (defined for \mathbb{Z}) by the product

$$\zeta(\mathbb{Z}, z) = \prod_{p=2,3,5,\dots} \zeta(\mathbb{Z}/p\mathbb{Z}, z),$$

over all primes p. We say that the global Riemann zeta function over \mathbb{Z} is composed by the local zeta functions over $\mathbb{Z}/p\mathbb{Z}$. The next example for the Gaussian integers $\mathbb{Z}[i]$ indicates that this is no coincidence.

Example 5 (Local and Global Zeta Functions of Gaussian Integers). Because $\mathbb{Z}[i] \cong \mathbb{Z}[X]/(X^2+1)$, we look at the zeta functions of the equation $X^2=-1$ for the various base fields \mathbb{F}_p :

- p=2: The equation $X^2=-1$ has one solution, hence $\zeta(\mathbb{F}_2[X]/(X^2+1),z)=1/(1-2^{-z})$.
- $p \neq 2$ and $p = 1 \mod 4$: The equation $X^2 = -1$ has two solutions in \mathbb{F}_{p^s} , hence $\zeta(\mathbb{F}_p[X]/(X^2+1), z) = 1/(1-p^{-z})^2$. • $p \neq 2$ and $p = 3 \mod 4$: The equation $X^2 = -1$ in \mathbb{F}_{p^s} has no solution if s
- $p \neq 2$ and $p = 3 \mod 4$: The equation $X^2 = -1$ in \mathbb{F}_{p^s} has no solution if s is odd, and it has two solutions if s is even. Hence (using $N_s = 1 + (-1)^s$) we have $\zeta(\mathbb{F}_p[X]/(X^2+1), z) = 1/(1-p^{-2z})$.

Altogether, this gives the product

$$\prod_{p=2,3,5,\dots} \zeta(\mathbb{F}_p[X]/(X^2+1),z) \quad = \quad \frac{1}{1-2^{-z}} \prod_{p=5,13,\dots} \frac{1}{(1-p^{-z})^2} \prod_{p=3,7,\dots} \frac{1}{1-p^{-2z}},$$

which corresponds exactly with the zeta function of $\mathbb{Z}[i] \cong \mathbb{Z}[X]/(X^2+1)$ in Example 4.

WIM VAN DAM
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CENTER FOR THEORETICAL PHYSICS
77 MASSACHUSETTS AVENUE
CAMBRIDGE, MA 02139-4307, USA
E-mail address: vandam@mit.edu