THE ACTUAL CONTENT OF QUANTUM THEORETICAL KINEMATICS AND MECHANICS

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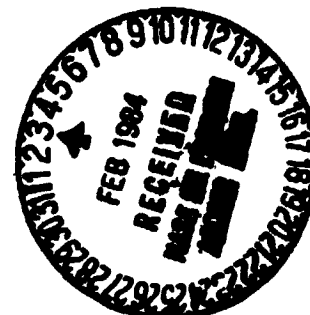
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16. Abstract

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anics; then we shall show that canonically conjugated variables
can be determined simultaneously only with a characteristic
uncertainty. This uncertainty is the intrinsic reason for the
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THE ACTUAL CONTENT OF QUANTUM THEORETICAL KINEMATICS AND MECHANICS

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SUMMARY   First, exact definitions are supplied in this paper for the terms: position, velocity, energy, etc. (of the electron, for instance), such that they are valid also in quantum mechanics; then we shall show that canonically conjugated variables can be determined simultaneously only with a characteristic uncertainty (§1). This uncertainty is the intrinsic reason for the occurrence of statistical relations in quantum mechanics. Their mathematical formulation is made possible by the Dirac-Jordan theory (§2). Beginning from the basic principles thus obtained, we shall show how macroscopic processes can be understood from the viewpoint of quantum mechanics (§3). Several imaginary experiments are discussed to elucidate the theory (§4).

We believe to understand a theory intuitively, if in all simple cases we can qualitatively imagine the theory's experimental consequences and if we have simultaneously realized that the application of the theory excludes internal contradictions. For instance: we believe to understand Einstein's concept of a finite three-dimensional space intuitively, because we can imagine the experimental consequences of this concept without contradictions. Of course, these consequences contradict our customary intuitive space-time beliefs. But we can convince ourselves that the possibility of applying this customary view of space and time can not be deduced either from our laws of thinking, or from experience. The intuitive

* Numbers in the margin indicate foreign pagination

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interpretation of quantum mechanics is still full of internal
contradictions, which become apparent in the battle of opin-
ions on the theory of continuums and discontinuums, corpuscles
and waves. This alone tempts us to believe that an interpre-
tation of quantum mechanics is not going to be possible in the
customary terms of kinematic and mechanical concepts. Quantum
theory, after, derives from the attempt to break with these
customary concepts of kinematics and replace them with rela-
tions between concrete, experimentally derived values. Since
this appears to have succeeded, the mathematical structure of
quantum mechanics won't require revision, on the other hand.
By the same token, a revision of the space-time geometry for
small spaces and times will also not be necessary, since by a
choice of arbitrarily heavy masses the laws of quantum mechan-
ics can be made to approach the classic laws as closely as   /173
desired, no matter how small the spaces and times. The fact
that a revision of the kinematic and mechanic concepts is re-
quired seems to follow immediately from the basic equations
of quantum mechanics. Given a mass $\underline{m}$, it is readily understand-
able, in our customary understanding, to speak of the position
and of the velocity of the center of gravity of that mass $\underline{m}$.
But in quantum mechanics, a relation $pq - qp = \frac{h}{2\pi i}$ exists
between mass, position and velocity. We thus have good reasons
to suspect the uncritical application of the terms "position"
and "velocity". If we admit that for very small spaces and
times discontinuities are somehow typical, then the failure
of the concepts precisely of "position" and "velocity" become
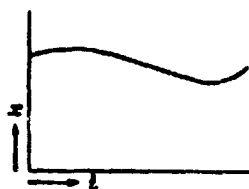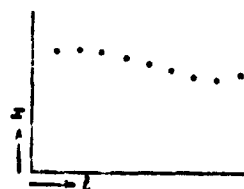immediately plausible: if, for instance, we imagine the uni-

Fig. 1.   Fig. 2.

dimensional motion of a mass point, then in a continuum theory

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it will be possible to trace the trajectory curve x(t) for the particle's trajectory (or rather, that of its center of mass) (see Fig. 1, above), with the tangent to the curve indicating the velocity, in each case. In a discontinuum theory, in contrast, instead of the curve we shall have a series of points at finite distances (see Gig. 2, above). In this case it is obviously pointless to talk of the velocity at a certain position, since the velocity can be defined only by means of two positions and consequently and inversely, two different velocities corresponded to each point.

The question thus arises whether it might not be possible, by means of a more precise analysis of those kinematic and mechanical concepts, to clear up the contradictions currently existing in an intuitive interpretation of quantum mechanics, to thus achieve an intuitive understanding of the relations of quantum mechanics.*

§ 1   The concepts: position, path, velocity, energy   /174

In order to be able to follow the quantum-mechanical behavior of any object, it is necessary to know the object's mass and and the interactive forces with any fields or other objects. Only then is it possible to set up the hamiltonian function for the quantum-mechanical system. [The considerations below

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shall in general refer to non-relativistic quantum mechanics, since the laws of quantum-theory electrodynamics are not completely known yet.* No further statements regarding the object's "gestalt" are necessary: the totality of those interactive forces is best designated by the term "gestalt".

If we want to clearly understand what is meant by the word "position of the object" - for instance, an electron - (relative co a given reference system), then we must indicate the definite experiments by means of which we intend to determine the "position of the electron." Otherwise the word is meaningless. In principle, there is no shortage of experiments that permit a determination of the "position of the electron", to any desired precision, even. For instance: illuminate the electron and look at it under the microscope. The highest precision attainable here in the determination of the position is substantially determined by the wavelength of the light used. But let us build in principle, a Γ-ray microscope and by means of it determine the position as precisely as desired. But in this determination a secondary circumstance becomes essential: the Compton effect. Any observation of the scattered light coming from the electron (into the eye, onto a photographic plate, into a photocell) presupposes a photoelectric effect, that is, it can also be interpreted as a light quantum striking the electron, there being reflected or diffracted to then
- deflected once again by the microscope's lense - finally
triggering the photoelectric effect. At the instant of the determination of its position - i.e., the instant at which the light quantum is diffracted by the electron - the electron discontinuously changes its impulse. That change will be more pronounced, the smaller the wavelength of the light used, i.e. the more precise the position determination is to be. In the

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* However, significant progress was made very recently through the work of P. Dirac [Proc. Roy. Soc. (A), 114, 243 (1927) and subsequent studies.]

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instant at which the electron's position is known, therefore, its impulse can become known only to the order of magnitude corresponding to that discontinuous change. That is, the more precisely the position is determined, the more imprecisely will the impulse be known, and vice-versa. This provides us with a direct, intuitive clarification of the relation $pq - qp = \frac{h}{2\pi i}$. Let q_1 be the precision to which the value of $\underline{q}$ is known (q_1 is approximately the average error of $\underline{q}$), or here, the wavelength of the light; p_1 is the precision to which the value of $\underline{p}$ can be determined, or in this case, the discontinuous change in $\underline{p}$ during the Compton effect. According to the basic equations of the Compton effect, the relation between p_1 and q_1 is then

$$p_1 q_1 \sim h. \tag{1}$$

That relation (1) above stands in a direct mathematical connection with the commutation relation $pq - qp = \frac{h}{2\pi i}$ shall be shown below. Here we shall point out that equation (1) is the precise expression for the fact that we once sought to describe by dividing the phase space into cells of size $\underline{h}$.

Other experiments can also be performed to determine the electron's position, such as impact tests. A very precise determination of the position requires impacts with very fast particles, since for slow electrons the diffraction phenomena - which according to Einstein are a consequence of the de Broglie waves (see for instance the Ramsay effect) - preclude a precise determination of the position. Thus, once again for a precise position measurement the electron's impulse changes disontinuously and a simple estimate of the precision with the equations of the de Broglie waves once again leads to equation (1).

This discussion seems to define the concept "position of the electron" clearly enough and we only need to add a word about

the "size" of the electron. If two very fast particles strike the electron sequentially in the very brief time interval Δt, then the two positions of the electron defined by these two particles lie very close together, separated by a distance Δl. From the laws observed for α-particles we conclude that Δl can be reduced to a magnitude of the order of 10^{-12} cm, provided Δt is sufficiently small and the particles selected are suf- ficiently fast. That is the meaning, when we say that the electron is a particle whose radius is not greater than 10^{-12} cm.

Let us move on to the concept of the "path of the electron." By path or trajectory we mean a series of points in space (in a given reference system) that the electron adopts as sucessive "positions." Since we already know what "position at a certain time" means, there are no new difficulties, here. It is still readily understood that the often used expression, for instance, "the 1-S orbit of the electron in the hydrogen atom" makes no sense, from out point of view. Because in order to measure this 1S orbit, we would have to illuminate the atom with light such that its wavelength is considerably shorter than 10^{-8} cm. But one light quantum of this kind of light would be sufficient to completely throw the electron out of its "orbit" (for which reason never more than a single point of this "path" could be defined, in space) and hence the word "path" is not very sensible or meaningful, here. This can be easily derived from the experimental possibilities, even without any knowledge of the new theories.

In contrast, the imaginary position measurements can be performed for many atoms in a 1S state. (Atoms in a given "stationary" state, for instance, can in principle be isolated by the Stern-Gerlach experiment.) Thus, for a given state, for instance 1S, of an atom, a probability function must exist for the electron's positions, such that it corresponds, on the average, to the classical trajectory over all phases, and that can be

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established by measurements to any desired precision. According to Born* this function is given by $\psi_{1S}(q)\overline{\psi}_{1S}(q)$, if $\psi_{1S}(q)$ is the Schroedinger wave function corresponding to the state 1S. I want to join Dirac* and Jordan*, in view of subsequent generalizations, in saying: the probability is given by $S(1S,q)\overline{S}(1S,q)$, where $S(1S,q)$ is that column of the transformation matrix $S(E,q)$ from E to $\underline{q}$, which corresponds to $E = E_{1S}$ (E = energies).

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In the fact that in quantum theory for a given state - for instance 1S - only the probability function for the electron position can be given, we may see a characteristic statistical feature of quantum theory, as do Born and Jordan, quite in contrast to the classical theory. On the other hand, if we want to we can say with Dirac that the statistics came in via our experiments. Because also in classical theory only the probability of a certain electron position could be given, if and as long as we do not know the atom's phases. Rather, the difference between classical and quantum mechanics consists in this: classically, we can always assume the phases to have been determined in a previous experiment. But in reality this is impossible, because every experiment to determine the phase would either destroy or modify the atom. In a definite stationary "state" of the atom, the phases are indetermined in

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* The statistical meaning of the de Broglie waves was first formulated by A. Einstein [Sitzungsber.d.preuss.Akad.d. Wiss. 1925, p.3). This statistical element then plays a significant role for M. Born, W. Heisenberg and P. Jordan, "Quantum mechanics II." [Z.f.Phys. 35, 557 (1926)], especially chapter 4, §3, and P. Jordan [Z.f.Phys. 37, 376 (1926)]; it is analyzed mathematically in a fundamental paper by M. Born [Z.f.Phys. 38, 803 (1926)] and used for the interpretation of the collision phenomena. The foundation for using the probability theorem from the transformation theory for matrices can be found in: W. Heisenberg [Z. f.Phys. 40, 501 (1926)], P. Jordan [ibid. 40, 661 (1926)], W. Pauli [Anm. in Z.f.Phys. 41, 81 (1927)], P. Dirac [Proc. Roy.Soc.(A) 113, 621 (1926)], P. Jordan [Z.f.Phys. 40, 809 (1926)]. The statistical side of quantum mechanics in general is discussed by P. Jordan (Naturwiss. 15, 105 (1927)] and M. Born [Naturwiss. 15, 238 (1927)].

principle, which we may consider a direct clarification of the
known equations

$$Et - tE = \frac{h}{2\pi i} \quad \text{or} \quad Jw - wJ = \frac{h}{2\pi i}$$

(J = action variable, w = angular variable).

The word "velocity" of an object is easily defined by measure-
ment, if it is a force-free motion. For instance, the object
can be illuminated with red light and then the particle's ve-
locity can be determined by the Doppler effect of the scatter-
ed light. The determination of the velocity will be the more
precise, the longer the wavelength of the light used is, since
then the particle's velocity change per light quantum due to
Compton effect will be the smaller. The position determination
becomes correspondingly uncertain, as required by equation(1).
If the velocity of the electron in an atom is to be measured
at a certain instant, we should have to make the nuclear charge
and the forces due to the other electrons disappear, at that
instant, so that the motion may proceed force free, after that
instant, to then perform the determination described above. As
was the case earlier, we once again can convince ourselves that
a function p(t) for a certain state of the atom - say, 1S - can
not be defined. In contrast, there again will be a function for /178
the probability of $\underline{p}$ in this state, which according to Dirac
and Jordan will have the value $S(1S,p)\overline{S}(1S,p)$. Again, $S(1S,p)$
means the column of the transformation matrix $S(E,p)$ of E into
p that corresponds to $E = E_{1S}$.

Finally, let us point out the experiments that allow the meas-
urement of the energy or the value of the action variables J.
Such experiments are particularly important since only with
their aid will we be able to define what we mean, when we talk
about the discontinuous change of the energy or of J. The

Franck-Hertz collision experiments permit the tracing back of the energy measurements on atoms to the energy measurements of electrons moving in a straight line, because of the validity of the energy theorem in the quantum theory. In principle, this measurement can be made as precise as desired, if only we forego the simultaneous determination of the electron position, i.e., of the phase (see above, the determination of $\underline{p}$), corresponding to the relation $Et - tE = \frac{h}{2\pi i}$. The Stern-Gerlach experiment permits the determination of the magnetic or an average electric moment of the atom, i.e., the measurement of magnitudes that depend only the action variables J. The phases remain undetermined in principle. If it is not sensible to talk of the frequency of a light wave at a given instant, it is not possible either to speak of the energy of an atom at a particular instant. In the Stern-Gerlach experiment this corresponds to the situation that the precision of the energy measurement will be the smaller, the shorter the time interval during which the atom is under the influence of the deflecting force*. Because an upper limit for the deflecting force is given by the fact that the potential energy of that deflecting force inside the beam of rays can vary only by quantities that are considerably smaller than the energy differences of the stationary states, if a determination of the stationary states' energy is to be possible. If E_1 is the quantity of energy that satisfies that condition (E_1 at the same time is a measure of the precision of that energy measurement), then E_1/d is the maximum value for the deflecting force, if $\underline{d}$ is the width of the ray beam (measurable by means of the width of the slit used. The angular deflection of the atom beam is then $E_1 t_1/dp$, where t_1 is the period of time during which the atoms are under the effect of the deflecting force, $\underline{p}$ the impulse of the atoms /179 in the direction of the beam. This deflection must be at least of the same order of magnitude as the natural beam broadening caused by diffraction in the slit, in order for a measurement

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* Cf. also W. Pauli, l.c. p.61

to be possible. The angular deflection due to diffraction is approximately λ/d, where λ is the de Broglie wavelength, i.e.,

$$\frac{\lambda}{d} \sim \frac{E_1 t_1}{dp} \quad \text{cr since} \quad \lambda = \frac{h}{p}.$$

$$E_1 t_1 \sim h. \tag{2}$$

This equation corresponds to equation (1) and it shows that a precise energy determination can be attained only through a corresponding uncertainty in the time.

§ 2   The Dirac-Jordan theory

We would like to summarize the results of the previous section and generalize them in this statement: All concepts used in classical theory to describe a mechanical system can also be defined exactly for atomic processes, in analogy to the classic concepts. But purely from experimentation, the experiments that serve for such definitions carry an inherent uncertainty, if we expect from them the simultaneous determination of two canonically conjugated variables. The degree of this uncertainty is given by equation (1), widened to include any canonically conjugated variables. It is reasonable to here compare the quantum theory with the special theory of relativity. According to the theory of relativity, the term "simultaneous" can only be defined by experiments in which the propagation velocity of light plays an essential role. If there were a "sharper" definition of simultaneity - for instance, signals that propagate infinitely rapidly - then the theory of relativity would be impossible. But since such signals do not exist - because the velocity of light already appears in the definition of simultaneity - room is available for the postulate of a constant velocity of light and therefore this postulate is not contradicted by the appropriate use of the terms, "position, velocity, time". The situation is similar in regard to the definition of the

concepts "electron position and velocity", in quantum theory. All the experiments we could use to define these terms necessarily contain the uncertainty expressed by equation (1), even though they permit an exact definition of the individual concepts p and q. If experiments existed that allowed a "more precise" definition of p and q than that corresponding to equation (1), then the quantum theory would be impossible. This uncertainty - which is fixed by equation (1) - now provides the space for the relations that find their terse expression in the commutation relations of quantum mechanics,

$$pq - qp = \frac{h}{2\pi i} \quad .$$

This equation becomes possible without having to change the physical meaning of the variables p and q.

For those physical phenomena for which a quantum theory formulation is still unknown (for instance, electrodynamics), equation (1) represents a demand that may be helpful in finding the new laws. For quantum mechanics, equation (1) can be derived from the Dirac-Jordan formulation, by means of a minor generalization. If for a certain value η of an arbitrary parameter we can determine the position q of the electron at q' with a precision q_1, then we can express this fact by means of a probability amplitude $S(\eta, q)$ that will be noticeably different from zero only in an area of approximate dimension q_1 around q'. We can thus say, more specifically

$$S(\eta, q) \text{ prop } e^{-\frac{(q-q')^2}{2q_1^2} - \frac{i}{\hbar} p'(q-q')}, \text{ i.e., } S\bar{S} \text{ prop } e^{-\frac{(q-q')^2}{q_1^2}}. \quad (3)$$

We thus have for the probability amplitude corresponding to p:

$$S(\eta, p) = \int S(\eta, q) S(q, p) dq. \quad (4)$$

In agreement with Jordan, we can say for $S(q,p)$ that

$$S(q, p) = e^{\frac{2\pi i pq}{h}}. \quad (5)$$

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In that case, according to (4), $S(\eta,p)$ will be noticeably different from zero only for values of $\underline{p}$ for which $2\pi(p-p')q_1/h$ is not substantially larger than 1. More especially, in the case of (3) we shall have:

$$S(\eta \cdot p) \text{ prop} \int e^{\frac{2\pi i(p-p')q}{h} - \frac{(q'-q)^2}{2\eta_1^2}} dq,$$

i.e.,

$$S(\eta, p) \text{ prop } e^{-\frac{(p-p')^2}{2p_1^2} + \frac{2\pi i}{h} q'(p-p')}, \text{ that is } S\bar{S} \text{ prop } e^{-\frac{(p-p')^2}{p_1^2}}.$$

where

$$p_1 q_1 = \frac{h}{2\pi}. \qquad (6)$$

Thus, assumption (3) for $S(\eta,q)$ corresponds to the experimental fact that the value p' of $\underline{p}$ and the value q' of $\underline{q}$ were measured [with the precision restriction (6)].

The purely mathematical characteristic of the Dirac-Jordan formulation of quantum mechanics is that the relations between p, q, E , etc., can be written as equations between very general matrices, such that any variable indicated by quantum theory appears as the diagonal matrix. The feasibility of such a notation seem reasonable if we visualize the matrices as tensors (for instance, moments of inertia) in multidimensional spaces, among which mathematical relations exist. The axes of the coordinate system in which these mathematical relations are expressed can always be placed along the main axis of one of these tensors. It is after all always possible to characterize the mathematical relation between two tensors A and B by means of transformation formulae that will convert a system of coordinates oriented along the main axis of A, into one oriented along the main axis of B. The latter formulation corresponds to Schroedinger's theory. In contrast, Dirac's notation of the q-numbers must be considered the truly "invariant"

formulation of quantum mechanics, independent of all coordinate systems. If we wanted to derive physical results from that mathematical model, then we must assign numerical values to the quantum mechanics variables, i.e., the matrices (or "tensors" in multidimensional space). This is to be understood as meaning that in that multidimensional space a certain direction is arbitrarily chosen (that is, established by the kind of experiment performed), and then the "value" of the matrix is asked for (for instance, the value of the moment of inertia, in that picture), in the direction chosen. This question has unequivocal meaning only if the direction chosen coincides with one of the matrix' main axes: in that case there will be an exact answer to the question. If the direction chosen deviates but little from one of the matrix' main directions, we can still talk with a certain imprecision, given by the relative inclination, with a certain probable error, of the "value" of the matrix in the direction chosen. We can thus state: it is possible to assign a number to every quantum theory variable, or matrix, which provides its "value", with a certain probable error. The probable error depends on the system of coordinates. For each quantum mechanics variable there /182 exists one system of coordinates for which the probable error vanishes, for that variable. Thus, a given experiment can never provide precise information on all quantum mechanics variables: rather, it divides the physical variables into "known" and "unknown" (or: more or less precisely known variables), in a manner characteristic for that experiment. The results of two experiments can be derived precisely from each other only when the two experiments divide the physical variables in the same manner into "known" and "unknown" (i.e., if the tensors in that multidimensional space already used for visualization are "viewed" from the same direction, in both experiments.) If two experiments cause two different distributions into "known" and "unknown" variables, then the relation of the results of those experiments can be given appropriately only statistically.

Let us perform an imaginary experiment, to more precisely discuss these statistical relations. We shall start by sending a Stern-Gerlach beam of atoms through a field F_1 that is so inhomogeneous in the beam direction, that it causes noticeably numerous transitions due to a "shaking effect". The atom beam is then allowed to run unimpeded, but then a second field shall begin, F_2, as inhomogeneous as F_1. We shall assume that it is possible to measure the number of atoms in the different stationary states, between F_1 and F_2 and also beyond F_2, by means of an eventually applied magnetic field. Let us assume the atoms' radiative forces to be zero. If we know that an atom was in the energy state E_n before passing through F_1, then we can express this experimental fact by assigning a wave function to the atom - for instance, in p-space - with a certain energy E_p and the indetermined phase β_n

$$S(E_n \cdot p) = \psi(E_n, p) e^{-\frac{2\pi i E_n(\alpha + \beta_n)}{h}} \quad .$$

After passing through field F_1, the function will have become*

$$S(E_n; p) \xrightarrow{F_1} \sum_m c_{nm} \psi(E_m, p) e^{-\frac{2\pi i E_m(\alpha + \beta_m)}{h}} . \tag{7}$$

Let us assume that here the β_m are arbitrarily fixed, such /183 that the c_{nm} is unequivocally determined by F_1. The matrix c_{nm} transforms the energy value before passing through F_1 to that after passing through F_1. If behind F_1 we perform a determination of the stationary states - for instance, by means of an inhomogeneous magnetic field - then we shall find, with a probability of $c_{nm}\bar{c}_{nm}$ that the atom has passed from the state n to the state m. If we determine experimentally that the atom has actually acquired the state m, then in the subsequent calculations we shall have to assign it the function

* See P. Dirac, Proc.Roy.Soc. (A)112, 661 (1926) and M. Born, Z. f. Phys. 40, 167 (1926).

S_m with an indeterminate phase, instead of the function $\sum_m c_{nm} S_m$. Through the experimental determination "state m" we select, from among the different possibilities (c_{nm}), a certain m and simultaneously destroy, as we shall explain below, whatever remained of phase relations in the variables c_{nm}. When the beam passes through F_2, we repeat the same procedure used for F_1. Let d_{nm} be the coefficients of the transformation matrix that converts the energies before F_2 to those after F_2. If no determination of the state is performed between F_1 and F_2, then the eigen-function is transformed according to the following pattern:

$$S(E_n, p) \xrightarrow{F_1} \sum_m c_{nm} S(E_m, p) \xrightarrow{F_2} \sum_m \sum_l c_{nm} d_{ml} S(E_l, p). \quad (8)$$

Let $\sum_m c_{nm} d_{ml} = e_{nl}$. If the stationary state of the atom is determined, after F_2, we shall find the state l with a probability of $e_{nl} \bar{e}_{nl}$. If, in contrast, we determined "state m" between F_1 and F_2, then the probability for l behind F_2 is given by $d_{ml} \bar{d}_{ml}$. Repeating the entire experiment several times (determining the state, each time, between F_1 and F_2) we shall then observe the state l, behind F_2, with the relative frequency $Z_{nl} = \sum_m c_{nm} \bar{c}_{nm} d_{ml} \bar{d}_{ml}$. This expression does not agree with $e_{nl} \bar{e}_{nl}$. For this reason Jordan (l.c.) mentions an "interference of the probabilities". I, for one, would not agree with this. Because the two experiments leading to $e_{nl} \bar{e}_{nl}$ or Z_{nl}, respectively, are really physically different. In one case the atom suffers no disturbance between F_1 and F_2; in the other it is disturbed by the equipment that makes the determination of the stationary states possible. The consequence of this equipment is that the "phase" of the atom changes by quantities that are uncontrollable in principle, just as the impulse was changed /184 in the determination of the electron's position (cf. § 1). The magnetic field for the determination of the state between F_1 and F_2 will change the eigen-values E and during the observation of the atom beam (I am thinking of something like a Wilson track) the atoms will be slowed down in different degrees,

statistically, and in an uncontrollable manner. As a consequence, the final transformation matrix e_{nl} (from the energy values before F_1 to those after leaving F_2) is no longer given by $\sum_m c_{nm} d_{ml}$, and instead each term of the sum will have, in addition, an unknown phase factor. Hence, all we can expect is for the average value of $e_{nl}\bar{e}_{nl}$, over all eventual phase changes, to be equal to Z_{nl}. A simple calculation shows this to be the case.

Thus, following certain statistical rules, we can draw conclusions, based on one experiment, regarding the results possible for another. The other experiment selects, by itself and from among all the possibilities, one particular one, thus limiting the possibilities for all subsequent experiments. This interpretation of the equation for the transformation matrix S, or Schroedinger's wave equation, is possible only because the sum of all solutions is also a solution. Here we can see the deeper meaning of the linearity of Schroedinger's equations and hence they can be understood only as waves in the phase space; for this same reason we would consider any attempt to replace these equations - for instance, in the relativistic case (for several electrons) - by non-linear equations as doomed to fail.

§ 3   The transition from micro to macromechanics

I believe the analyses performed in the preceding sections of the terms "electron position", "velocity", "energy", etc., have sufficiently clarified the concepts of quantum theory kinematics and mechanics, so that an intuitive understanding of the macroscopic processes must also be possible, from the point of view of quantum mechanics. The transition from micro to macro mechanics has already been dealt with by Schroedinger[*], but I

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[*] E. Schroedinger, Naturwiss. 14, 664 (1926)

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do not believe that Schroedinger's considerations address the essence of the problem, for the following reasons: according to Schroedinger, in highly excited states a sum of the eigen-vibrations will yield a not overly large wave packet, that in its turn, under periodic changes of its size, performs the periodic motions of the classical "electron". The following objections can be raised here: If the wave packet had such properties as described here, then the radiation emitted by the atom could be developed into a Fourier series in which the frequencies of the harmonic vibrations are integer multiples of the fundamental frequency. Instead, the frequencies of the spectral lines emitted by the atom are never integer multiples of a fundamental frequency, according to quantum mechanics - with the exception of the special case of the harmonic oscil-lator. Thus Schroedinger's consideration is applicable only to the harmonic oscillator considered by him, while in all other cases in the course of time the wave packet spreads over all space surrounding the atom. The higher the atom's excitation state, the slower will be the scattering of the wave packet. But it will occur, if one waits long enough. The argument used above for the radiation emitted by an atom can be used, for the time being, against all attempts of a direct transition from quantum to classical mechanics, for high quantum numbers. For this reason, it used to be attempted to circumvent that argu-ment by pointing to the natural beam width of the stationary states; certainly improperly, since in the first place this way out is already blocked for the hydrogen atom, because of insufficient radiation at higher states; in the second place, the transition from quantum to classical mechanics must be un-derstandable without borrowing from electrodynamics. Bohr* has repeatedly pointed out these known difficulties, in the past, that make a direct connection between quantum and classical theory difficult. If we explained them here again in such

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* N. Bohr, Basic Postulates of Quantum Theory, l.c.

detail, it is because apparently they have been forgotten.

I believe the genesis of the classical "orbit" can be precisely formulated thus: the "orbit" only comes into being by our observing it. Let us assume an atom in its thousandth excitation state. The dimensions of the orbit are relatively large here, already, so that it is sufficient, in the sense of § 1, to determine the electron's position with a light of relatively long wavelength. If the determination of the electron's position is not to be too uncertain, then one consequence of Compton recoil will be that after the collision, the atom will be in some state between, say, the 950th and the 1050th. At the same time, the electron's impulse can be derived - to a precision given by equation (1) - from the Doppler effect. The experimental fact so obtained can be characterized by means of /186 a wave packet - or better, probability packet - in q-space, by a variable given by the wavelength of the light used, essentially composed of eigenfunctions between the 950th and the 1050th eigen-function, and through the corresponding packet in p-space. After a certain time, a new position determination is performed, to the same precision. According to § 2, its result can be expressed only statistically; possible positions are all those within the now already spread wave packet, with a calculable probability. This would in no way be different in classical theory, since in classical theory the result of the second position could also be given only statistically, due to the uncertainty in the first determination; in addition, the system's orbits would also spread in classical theory similarly to the wave packet. However, the laws of statistics themselves are different, in quantum mechanics and classical theory. The second position determination selected a q from among all those possible, thus limiting the possibilities for all subsequent determinations. After the second position determination, the results for later measurements can be calculated only by again assigning to the electron a "smaller" wave packet of dimension

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λ (wavelength of the light used for the observation). Thus, each position determination reduces the wave packet again to its original dimension λ. The "values" of the variables p and q are known to a certain precision, during all experiments. Since within these limits of precision the values of p and q follow the classical equations of motion, we can conclude, directly from the laws of quantum mechanics,

$$\dot{p} = -\frac{\partial H}{\partial q}; \qquad \dot{q} = \frac{\partial H}{\partial p} \qquad (9) .$$

But as we mentioned, the orbit can only be calculated statistically from the initial conditions, which we may consider a consequence uncertainty existing in principle, in the initial conditions. The laws of statistics are different for quantum mechanics and classical theory. Under certain conditions, this can lead to gross macroscopic differences between classical and quantum theory. Before discussing an example of this, I want to show by means of a simple mechanical system – the force-free motion of a mass point – how the transition to the classical theory discussed above is to be formulated mathematically. The /187 equations of motion are (for unidimensional motion)

$$H = \frac{1}{2m}p^2; \quad \dot{q} = \frac{1}{m}p; \quad \dot{p} = 0. \qquad (10)$$

Since time can be treated as a parameter (as a "c-number") if there are no external, time-dependent forces, then the solution to this equation is:

$$q = \frac{1}{m}p_0 t + q_0; \qquad p = p_0. \qquad (11)$$

where p_0 and q_0 represent impulse and position at time t=0. At time t=0 [see equations (3) to (6)], let $q_0 = q'$ be measured with precision q_1, $p_0 = p'$ with precision p_1. If from the values" of p_0 and q_0 we are to derive the "value" of q at time $\underline{t}$, then according to Dirac and Jordan we must find that transformation function, that transforms all matrices

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in which q_0 appears as a diagonal matrix, into matrices in which q appears as the diagonal matrix. In the matrix pattern in which q_0 appears as the diagonal matrix, p_0 can be replaced by the operator $\dfrac{h}{2\pi i}\dfrac{\partial}{\partial q_0}$. According to Dirac [l.c. equation (11)] we then have for the transformation amplitude sought, $S(q_0,q)$, the differential equation

$$\left\{ \frac{t}{m}\frac{h}{2\pi i}\frac{\partial}{\partial q_0} + q_0 \right\} S(q_0, q) = q\, S(q_0, q) \qquad (12)$$

$$\frac{t}{m}\frac{h}{2\pi i}\frac{\partial S}{\partial q_0} = \left(q_0 - q\right) S(q_0, q)$$

$$S(q_0, q) = const.\, e^{\dfrac{2\pi i m \int \left(q - q_0\right) dq_0}{h \cdot t}} \qquad (13) \quad .$$

Thus $S\bar{S}$ is independent of q_0, i.e., if at time $t = 0$, q_0 is known exactly, then at any time $t > 0$ all values of q are equally likely, i.e., the probability that q lies within a finite range, is generally zero. This is quite clear, intuitively. Because the exact determination of q_0 leads to an infinitely large Compton recoil. The same would of course be true of any mechanical system. However, if at time $t = 0$, q_0 is known only to a precision q_1 and p_0 to precision p_1, then [cf. equation (3)]

$$S(\eta, q_0) = const.\, e^{-\dfrac{(q_0 - q')^2}{2q_1^2} - \dfrac{2\pi i}{h} p'(q_0 - q')},$$

and the probability function for q will have be calculated from the equation

$$S(\eta, q) = \int S(\eta, q_0)\, S(q_0, q)\, dq_0 \quad .$$

We obtain

$$S(\eta, q) = const.\int e^{\dfrac{2\pi i m}{th}\left[q_0\left(q - \frac{t}{m}p'\right) - \frac{q_0^2}{2}\right] - \dfrac{(q' - q_0)^2}{2q_1^2}} dq_0. \qquad (14)$$

If we introduce the abbreviation

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$$\beta = \frac{th}{2\pi m q_1^2} \qquad (15)$$

then the exponent in (14) becomes

$$-\frac{1}{2 q_1^2}\left\{q_0^2\left(1+\frac{i}{\beta}\right) - 2q_0\left(q' + \frac{i}{\beta}\left(q - \frac{t}{m}p'\right)\right) + q'^2\right\}.$$

The term in q'^2 can be included in the constant factor (independent of q); by integration we obtain

$$S(\eta, q) = \text{const}.\, e^{\frac{1}{2 q_1^2}\frac{\left[q'+\frac{i}{\beta}\left(q-\frac{t}{m}p'\right)\right]^2}{1+\frac{i}{\beta}}} , \qquad (16$$

$$= \text{const}.\, e^{-\frac{\left(q-\frac{t}{m}p'-i\beta q'\right)^2\left(1-\frac{i}{\beta}\right)}{2 q_1^2(1+\beta^2)}} .$$

From which follows

$$S(\eta, q)\, S(\eta, \overline{q}) = \text{const}.\, e^{-\frac{\left(q-\frac{t}{m}p'-q'\right)^2}{q_1^2(1+\beta^2)}} . \qquad (17)$$

Thus, at time $\underline{t}$ the electron is at position $(t/m)p' + q'$, to a precision $q_1\sqrt{1+\beta^2}$. The "wave packet" or better, the "probability packet" has become larger by a factor of $\sqrt{1+\beta^2}$. According to (15), β is proportional to the time $\underline{t}$, inversely proportional to the mass - this is immediately plausible - and inversely proportional to q_1^2. Too great a precision in q_0 has a greater uncertainty in p_0 as a consequence and hence also leads to an increased uncertainty in q. The parameter η, which we introduced above for formal reasons, could be eliminated in all equations, here, since it does not enter in the calculations.

As an example that the difference between the classical laws of statistics and those from quantum theory can lead to gross macroscopic differences in the results from both theories, under certain conditions, shall be briefly discussed for the reflection of an electron flow by a grating. If the lattice

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constant is of the order of magnitude of the de Broglie wave-
length of the electron, then the reflection will occur in
certain discrete directions in space, as does the light at a
grating. Here, classical theory yields macroscopically some-
thing grossly different. And yet, we can not find a contradic-
tion against classical theory in the orbit of a single electron.
We could do it, if somehow we could direct the electron to a
certain location on a grating line and there establish that the
reflection did not occur classically. But if we want to deter-
mine the electron's position so precisely that we could say at
which location on a grating line it would impact, then the elec-
tron would acquire such a velocity, due to this determination,
that the de Broglie wavelength of the electron would be reduced
to the point that in this approximation, the electron would be
actually reflected in the direction prescribed by classical
theory, without contradicting the laws of quantum theory.

§ 4   Discussion of some special, imaginary experiments

According to the intuitive interpretation of quantum theory at-
tempted here, the points in time at which transitions - the
"quantum jumps" - occur should be experimentally determinable
in a concrete manner, such as energies of stationary states,
for instance. The precision to which such a point in time can
be determined is given by equation (2) as $h/\Delta E$*, if ΔE is the
change in energy accompanying the transition. We are thinking
of an experiment such as the following: Let an atom, in state
2 at time t=0, return to its normal state 1 by emitting radia-
tion. We could then assign to the atom, in analogy to equation
(7), the eigenfunction

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* See W. Pauli, l.c., p.12

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$$S(t,p) = e^{-\alpha t}\psi(E_2,p)e^{-\frac{2\pi i E_2 t}{h}} + \sqrt{1 - e^{-2\alpha t}}\,\psi(E_1,p)e^{-\frac{2\pi i E_1 t}{h}} \quad (18)$$

if we assume that the radiation damping will express itself in
the eigen-function by means of a factor of the form $e^{-\alpha t}$(the
true dependence may not be that simple). Let us send this atom
through an inhomogeneous magnetic field, to measure its energy,
as is customary in the Stern-Gerlach experiment, except that
the inhomogeneous field shall follow the atom beam for a good
portion of the path. The corresponding acceleration could be
measured by dividing the entire path followed by the atom beam
in the magnetic field, into small partial paths, at the end of
each of which we measure the beam's deflection. Depending on   /190
the atom beam's velocity, the division into partial paths will
correspond, for the atom, also to division into partial time
intervals Δt. According to § 1, equation (2), to the interval
Δt corresponds a precision in the energy of $h/\Delta t$. The probabil-
ity of measuring a certain energy can be directly derived from
$S(p,E)$ and is hence calculated in the interval from $n\Delta t$ to
$(n+1)\Delta t$ by means of

$$S(p,E) = \int_{n\Delta t}^{(n+1)\Delta t} S(p,t)e^{\frac{2\pi i E t}{h}}\,dt.$$

If at time $(n+1)\Delta t$ we make the determination, "state 2", then
for all subsequent events we may no longer assign to the atom
the eigen-function (18), but one derived from (18) if we re-
place t with $t-(n+1)\Delta t$. If, in contrast, we determine "state
1", then from then on we must assign to the atom the eigen-
function

$$\psi(E_1,p)e^{-\frac{2\pi i E_1 t}{h}} \; .$$

Thus, in a series of intervals Δt we would first observe "state
2", then continuously "state 1." To make a differentiation of
the two states possible, Δt must not fall below $h/\Delta E$. Thus, the

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transition-point in time can be determined with that precision. We conceive of the experiment above entirely in the sense of the old interpretation of quantum theory, as explained by Planck, Einstein and Bohr, when we speak of a discontinuous change of energy. Since such an experiment can be performed, in principle, agreement as to its results must be possible.

In Bohr's basic postulate of the quantum theory, the energy of an atom, as well as the values of the action variables J, has the privilege over other items to be determined (such as the position of the electron, etc.) that its numerical value can always be given. This privileged position held by energy over other quantum mechanics magnitudes is owed strictly to the circumstance that in a closed system, it represents an integral of the equation of motion (for the energy matrix we have E = const.). In contrast, in open systems the energy has no preference over other quantum mechanics variables. In particular, it will be possible to conceive of experiments, in which the atom's phases w are precisely measurable and for which then the energy will remain, in principle, indetermined, corresponding to a relation $Jw - wJ = \frac{h}{2\pi i}$, or $J_1 w_1 \approx h$. Such an experiment is provided by resonance fluorescence, for instance. If an atom is irradiated with an eigen-frequency of say, $v_{12} = (E_2 - E_1)/h$, then the atom will vibrate in phase with the external radiation, in which case in principle it is senseless to ask, in which state - E_1 or E_2 - the atom is vibrating. The phase relation between atom and external radiation can be determined, for instance, by means of the phase relations among many atoms (Woods experiment). If one does not want to use experiments involving radiation, the phase relation can also be measured by performing precise position measurements in the sense of § 1 for the electron, at different times, relative to the phase of the light used for illumination (for many atoms). To each atom we could then assign a "wave function" such as

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$$S(y, t) = c_2 \psi_2 (E_2, y) e^{-\frac{2\pi i (E_2 t + \beta)}{\hbar}} + \sqrt{1 - c_2^2} \, \psi_1 (E_1, y) e^{-\frac{2\pi i E_1 t}{\hbar}} \quad (19) .$$

Here c_2 depends on the intensity and β on the phase of the illuminating light. Thus, the probability q of a certain position is

$$S(q, t) \overline{S(q, t)} = c_2^2 \psi_2 \bar{\psi}_2 + (1 - c_2^2) \psi_1 \bar{\psi}_1$$
$$+ c_2 \sqrt{1 - c_2^2} \left(\psi_2 \bar{\psi}_1 \, e^{-\frac{2\pi i}{\hbar} [(E_2 - E_1) t + \beta]} + \bar{\psi}_2 \psi_1 \, e^{+\frac{2\pi i}{\hbar} [(E_2 - E_1) t + \beta]} \right) \quad (20) .$$

The periodic term in (20) can be experimentally separated from the non-periodical, since the position determination can be performed at different phases of the illuminating light.

In a known imaginary experiment proposed by Bohr, the atoms of a Stern-Gerlach atom beam are initially excited to resonance fluorescence, at a certain location, by means of light irradiation. After a certain length, the atoms pass through an inhomogeneous magnetic field; the radiation emitted by the atoms can be observed over the entire length of their path, before and behind the magnetic field. Before the atoms enter the magnetic field, they exhibit normal resonance fluorescence, i.e., in analogy to the dispersion theory, we must assume that all atoms emit in phase with the incident , spherical light waves. At first, this latter interpretation stands in conflict with what a rough application of the light quanta theory or the basic /192 rules of quantum theory indicate: from it one would conclude that that only a few atoms would be raised to an "upper state" by the absorption of a light quantum and hence, that all of the resonance radiation would come from intensively radiating excited centers. Thus, it used to be tempting to say: the concept of light quanta can be called upon here only for the energy impulse balance; "in reality" all atoms radiate in lower states as a weak and coherent spherical wave. Once the atoms have passed through the magnetic field, there can hardly be

any doubt left that the atom beam has split into two beams
of which one corresponds to atoms in the higher state and the
other, to atoms in the lower state. If the atoms in the lower
state were radiating, this would be a gross infringement of
the energy theorem, because all of the excitation energy is
contained in the fraction with the higher state. Rather, there
can be no doubt that behind the magnetic field, only the atom
beam with the upper states is emitting light - and non-coherent
light, at that - from the few intensively radiating atoms in
the upper state. As Bohr showed, this imaginary experiment makes
particularly clear how careful we must be with the application
of the concept "stationary state". From the conception of the
quantum theory developed here, it is easy to discuss Bohr's ex-
periment without any difficulty. In the outer radiation field
the phases of the atoms are determined and hence there is no
sense in talking of the energy of the atom. Even after the atom
has left the radiation field we can not say that it is in a
certain stationary state, if we are asking for coherence charac-
teristics of the radiation. But experiments can be performed to
test in which state the atom is; the result of this experiment
can only be given statistically. Such an experiment is actual-
ly performed by the inhomogeneous magnetic field. Behind the
magnetic field, the energies of the atoms are determined and
hence their phases are undetermined. The radiation is incoher-
ent and emitted only by atoms in the upper state. The magnetic
field determined the energies and hence destroys the phase re-
lations. Bohr's imaginary experiment provides a beautiful
clarification of the fact that the energy of the atom is also,
"in reality, not a number, but a matrix." The law of conserva-
tion applies to the matrix energy and hence also to the value
of the energy, as precisely as it is measured, in each case.
Analytically, the cancellation of the phase relations can be   /193
followed approximately thus: let Q be the coordinates of the
atom's center of mass; we can then assign to the atom (instead
of (19)) the eigen-function

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$$S(Q, t) S(q, t) = S(Q, q, t) \qquad (21)$$

where $S(Q,t)$ is a function that [as $S(\eta,q)$ in (16)] is different from zero in only a small area around a point in Q-space, and propagates with the velocity of the atoms in the direction of the beam. The probability of a relative amplitude q for some values Q is given by the integral of

$$S(Q,q,t)S(Q,q,t) \text{ over } Q, \text{ i.e., via } (20).$$

The eigen-function (21), however, will change in the magnetic field in a calculable manner, and because of the differing deflection of the atoms in the upper and the lower state, will have become, behind the magnetic field,

$$S(Q, q, t) = c_2 S_2(Q, t) \psi_2(E_2, q) e^{\frac{2\pi i (E_2 t + \beta)}{h}}$$
$$+ \sqrt{1 - c_2^2} S_1(Q, t) \psi_1(E_1, q) e^{\frac{2\pi i E_1 t}{h}}. \qquad (22)$$

$S_1(Q,q,t)$ and $S_2(Q,t)$ will be functions in Q-space differing from zero only in a small area surrounding the point. But this point is different for S_1 and for S_2. Hence $S_1 S_2$ is zero everywhere. Hence, the probability of a relative amplitude q and a definite value Q is

$$S(Q, q, t) \bar{S}(Q, q, t) = c_2^2 S_2 \bar{S}_2 \psi_2 \bar{\psi}_2 + (1 - c_2^2) S_1 \bar{S}_1 \psi_1 \bar{\psi}_1. \qquad (23)$$

The periodic term in (20) has disappeared and with it, the possibility of measuring a phase relation. The result of the statistical position determination will always be the same, regardless of the phase of the incident light for which it was determined. We may assume that experiments with radiation whose theory has not yet been fully elaborated will yield the same results regarding the phase relations of atoms to the incident light.

Finally, let us examine the relation between equation (2), $E_1 t_1 \simeq h$, and a problem complex discussed by Ehrenfest* and two other researchers by means of Bohr's correspondence principle,

in two important papers[**]. Ehrenfest and Tolman speak of "weak quantization" when a quantified periodic motion is subdivided, by quantum jumps or other disturbances, into time intervals /194 that can not be considered long in relation to the system's period. Supposedly, in this case there are not only the exact energy values from quantum theory, but also - with a lower a priori probability that can be qualitatively indicated - energy values that do not differ too much from the quantum theory-based values. In quantum mechanics, such a behavior is to be inter- pretated as follows: since the energy is really changed, due to other disturbances or to quantum jumps, each energy measurement has to be performed in the interval between two disturbances, if it is to be unequivocal. This provides an upper limit to t_1 in the sense of § 1. Thus the energy value E_0 of a quantified state is also measured only with a precision $E_1 \simeq t/t_1$. Here, the question whether the system "really" adopts energy values E that differ from E_0 - with the correspondingly smaller statis- tical weight - or whether their experimental determination is due only to the uncertainty of the measurement, is pointless, in principle. If t_1 is smaller than the system's period, then there is no longer any sense in talking of discrete stationary states or discrete energy values.

In a similar context, Ehrenfest and Breit (l.c.) point out the following paradox: let us imagine a rotator - for instance, in the shape of a gear wheel - fitted with a mechanism that after f revolutions just reverses the direction of rotation. Let us further assume that the gear wheel acts on a rack that can be linearly displaced between two blocks. After the specified num- ber of revolutions, the blocks force the rack, and hence the wheel, to reverse direction. The true period T of the system is

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[*] P. Ehrenfest and G. Breit, ZS.f.Phys. 9, 207 (1922) and P. Ehrenfest and R.C. Tolman, Phys.Rev. 24, 287 (1924); see also the discussion in N. Bohr, Basic postulates of quantum theory, l.c.
[**] Mr. W. Pauli pointed this relation out to me.

long in relation to the period t of the wheel; the discrete
energy steps are correspondingly dense, and denser, the greater
T is. Since from the point of view of a consistent quantum theo-
ry all stationary states have the same statistical weight, for
a sufficiently large T practically all energy values will occur
with the same frequency - in contrast to what we would expect
for the rotator. Initially, this paradox becomes even sharper
when we consider our points of view. Because in order to es-
tablish whether the system will adopt the discrete energy val-
ues corresponding to a pure rotator singly or with special
frequency, or whether it will adopt all possible values (i.e.,
values corresponding to the small energy steps h/T) with the
same probability, a time t_1 is sufficient, which is small in
relation to T (but $\gg t$). That is, although the large period $/195$
for such measurements never becomes effective, it apparently
manifests itself in that all possible energy values can occur.
We believe that such experiments for the determination of the
system's total energy would actually yield all possible energy
values with the same probability; and this is not due to the
large period T, but to the linearly displaceable rack. Even if
the system should find itself in a state whose energy corres-
ponds to the rotator quantification, by means of external
forces acting on the rack it can be easily taken to states,
that do not correspond to the rotator quantification*. The
coupled system rotator-rack simply has periodicity character-
istics that are different from those of the rotator. The solu-
tion of the paradox rather lies in the following: if we wanted
to measure the energy of the rotator alone, then we shall first
have to dissolve the coupling between rotator and rack. In
classical theory, for a sufficiently small mass of the rack the
dissolution of the coupling could occur without energy changes
and therefore there the energy of the total system could be
equated to that of the rotator (for a small rack mass). In

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* According to Ehrenfest and Breit, this can occur not at all,
or only rarely, due to forces acting on the wheel.

quantum mechanics, the interaction energy between rack and wheel is at least of the same order of magnitude, as one of the rotator's energy steps (even for a small rack mass, a high zero-point energy remains for the elastic interaction between wheel and rack!) Once the coupling is dissolved, the rack and the wheel individually adopt their quantum theory energy values. Thus, to the extent that we can measure the energy values of the rotator alone, we will always find the values prescribed by quantum theory, with the precision allowed by the experiment. Even for a vanishingly small rack mass will the energy of the coupled system be different from that of the rotator. The energy of the coupled system can adopt all possible values (those allowed by T-quantification) with the same probability.

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Quantum theory kinematics and mechanics are vastly different from classical. But the applicability of classical kinematic and mechanical concepts can not be deduced either from the laws that govern our thinking, or from experience. We are entitled to this conclusion by the relation (1) $p_1 q_1 \simeq h$. Since /196 the impulse, position, energy, etc., of an electron are precisely defined concepts, we need not be discouraged by the fact that the fundamental equation (1) contains only a qualitative statement. Since, in addition, we can qualitatively conceive of the theory's experimental consequences, in all simple cases, we shall no longer have to view quantum mechanics as not intuitive or abstract*. If we admit this, then we would of course

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* Schroedinger described quantum mechanics as a formal theory, of frightening, even repulsive un-intuitiveness and abstraction. The value of the mathematical (and to that extent, intuitive) penetration of the laws of quantum mechanics accomplished by Schroedinger can certainly not be praised highly enough. However, in terms of the principled, physical questions, I believe the popular intuitiveness of wave mechanics has deflected it from the straight path that had been marked

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also like to be able to derive the quantitative laws of quantum mechanics directly from the intuitive foundations, i.e., essentially, from relation (1). For this reason Jordan attempted to interpret the equation

$$S(q\,q'') = \int S(q\,q')\, S(q'\,q'')\, dq'$$

as a probability relation. We can not agree, however, with that interpretation (§ 2). Rather, we believe that the quantitative laws can be understood, to begin with, according to the principle of the greatest possible simplicity, starting from the intuitive foundations. If, for instance, the X coordinate of the electron no longer is a "number" - as can be concluded experimentally, from equation (1) - then the simplest imaginary assumption [that does not contradict (1)] is that this X coordinate is a diagonal term of a matrix whose non-diagonal terms are expressed in an uncertainty, or respectively, by other kinds of transformations (cf. for instance § 4). Perhaps the statement that the velocity in the X-direction "in reality" is not a number, but a diagonal term in a matrix is no more unintuitive and abstract than the determination, that the electric field intensity "in reality" is the time portion of an antisymmetrical tensor of the space-time world. The expression "in reality" is just as much or as little justified here as it is for any other description of natural phenomena in mathematical terms. As soon as we admit that all quantum theory variables "in reality" are matrices, the quantitative laws follow without difficulty.

If one assumes that the interpretation of quantum mechanics at- tempted here is valid at least in its essential points, then we may be allowed to discuss its main consequences, in a few words. We have not assumed that quantum theory - in contrast to classical theory - is essentially a statistical theory, in the sense that starting from exact data we can only draw statistical

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by the works of Einstein and de Broglie on the one hand, and by quantum mechanics, on the other.

conclusions. Among others, the known experiments by Geiger and
Bothe speak against such an assumption. Rather, in all cases
in which relations exist between variables, in classical theo-
ry, that can really be measured precisely, the corresponding
exact relations exist also in quantum theory (impulse and en-
ergy theorems). But in the rigorous formulation of the law of
causality - "If we know the present precisely, we can calculate
the future" - it is not the conclusion that is faulty, but the
premise. We simply can not know the present in principle in all
its parameters. Therefore all perception is a selection from a
totality of possibilities and a limitation of what is possible
in the future. Since the statistical nature of quantum theory
is so closely to the uncertainty in all observations or percep-
tions, one could be tempted to conclude that behind the ob-
served, statistical world a "real" world is hidden, in which
the law of causality is applicable. We want to state explicit-
ly that we believe such speculations to be both fruitless and
pointless. The only task of physics is to describe the relation
between observations. The true situation could rather be des-
cribed better by the following: Because all experiments are
subject to the laws of quantum mechanics and hence to equation
(1), it follows that quantum mechanics once and for all estab-
lishes the invalidity of the law of causality.

Addendum at the time of correction. After closing this paper,
new investigations by Bohr have led to viewpoints that allow a
considerable broadening and refining of the analysis of quantum
mechanics relations attempted here. In this context, Bohr cal-
led my attention to the fact that I had overlooked some essen-
tial points in some discussions of this work. Above all, the
uncertainty in the observation is not due exclusively to the
existence of discontinuities, but is directly related to the
requirement of doing justice simultaneously to the different
experiences expressed by corpuscular theory on the one hand,

and by wave theory on the other. For instance, in the use of
an imaginary Γ-ray microscope, the divergence of the ray beam
must be taking into account. The first consequence of this is
that in the observation of the electron's position, the direc-
tion of the Comptom recoil will only be known with some uncer-
tainty, which will then lead to relation (1). It is further-
more not sufficiently stressed that rigorously, the simple
theory of the Compton effect can be applied only to free elec-
trons. As professor Bohr made very clear, the care necessary in
the application of the uncertainty relationship is essential
above all in a general discussion of the transition from micro
to macro-mechanics. Finally, the considerations on resonance
fluorescence are not entirely correct, because the relation
between the phase of the light and that of the motion of the
electrons is not as simple as assumed here. I am greatly in-
debted to professor Bohr for being permitted to know and discuss
during their gestation those new investigations by Bohr, men-
tioned above, dealing with the conceptual structure of quantum
theory, and to be published soon.