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UNIVERSITY OF SOUTHAMPTON

Competition between Demand-Side Intermediaries in Ad Exchanges

by

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degree of Doctor of Philosophy

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ABSTRACT

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS
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Online advertising constitutes one of the main sources of revenue for the majority of businesses on the web. Online advertising inventory was traditionally traded via bilateral contracts between publishers and advertisers, vastly through a number of intermediaries. However, what caused an explosion in the volume and, consequently, the revenue of online ads was the incorporation of auctions as the major mechanism for trading sponsored search ads in all major search engines. This reduced transaction costs and allowed for the advertisement of small websites which constitute the majority of Internet traffic. Auction-based markets were harder to establish in the display advertising industry due to the higher volume of inventory and the pre-existence of traditional intermediaries, often leading to inefficiencies and lack of transparency. Nevertheless, this has recently changed with the introduction of the *ad exchanges*, centralized marketplaces for the allocation of display advertising inventory that support auctions and real-time bidding. The appearance of ad exchanges has also altered the market structure of both demand-side and supply-side intermediaries which increasingly adopt auctions to perform their business operations. Hence, each time a user enters a publisher's website, the contracted ad exchange runs an auction among a number of demand-side intermediaries, each of which represents their interested advertisers and typically submits a bid by running a local auction among these advertisers.

Against this background, within this thesis, we look both at the auction design problem of the ad exchange and the demand-side intermediaries as well as at the strategies to be adopted by advertisers. Specifically, we study the revenue and efficiency effects of the introduction and competition of the demand-side intermediaries in a single-item auction setting with independent private valuations. The introduction of these intermediaries constitutes a major issue for ad exchanges since they hide some of the demand from the ad exchange and hence can make a profit by pocketing the difference between what they receive from their advertisers and what they pay at the exchange.

Ad exchanges were created to offer transparency to both sides of the market, so it is important to study the share of the revenue that intermediaries receive to justify their services offered given the competition they face by other such intermediaries. The existence of mediators is a well-known problem in other settings. For this reason, our formulation is general enough to encompass other areas where two levels of auctions arise, such as procurement auctions with subcontracting and auctions with colluding bidders.

In more detail, we study the effects of the demand-side intermediaries' choice of auction for three widely-used mechanisms, two variations of the second-price sealed-bid (known as Vickrey) auction, termed **PRE** and **POST**, and first-price sealed-bid (**FPSB**) auctions. We first look at a scenario with a finite number of intermediaries, each implementing the same mechanism, where we compare the profits attained for all stakeholders. We find that there cannot be a complete profit ranking of the three auctions: **FPSB** auctions yield higher expected profit for a small number of competing intermediaries, otherwise **PRE** auctions are better for the intermediaries. We also find that the ad exchange benefits from intermediaries implementing **POST** auctions.

We then let demand-side intermediaries set reserve (or floor) prices, that are known to increase an auctioneer's expected revenue. For issues of analytical tractability, we only consider scenarios with two intermediaries but we also compare the two Vickrey variations in heterogeneous settings where one intermediary implements the first whereas the other implements the second variation. We find that intermediaries, in general, follow mixed reserve-price-setting strategies whose distributions are difficult to derive analytically. For this reason, we use the fictitious play algorithm to calculate approximate equilibria and numerically compare the revenue and efficiency of the three mechanisms for specific instances. We find that **PRE** seems to perform best in terms of attained profit but is less efficient than **POST**. Hence, the latter might be a better option for intermediaries in the long term.

Finally, we extend the previous setting by letting advertisers strategically select one of the two intermediaries when the latter implement each of the two Vickrey variations. We analytically derive the advertisers' intermediary selection strategies in equilibrium. Given that, in some cases, these strategies are rather complex, we use again the fictitious play algorithm to numerically calculate the intermediaries' and the ad exchange's best responses for the same instances as before. We find that, when both intermediaries implement **POST** auctions, advertisers always select the low-reserve intermediary, otherwise they generally follow randomized strategies. Last, we find that the ad exchange benefits from intermediaries implementing the pre-award Vickrey variation compared to a setting with two heterogeneous Vickrey intermediary auctioneers, whereas the opposite is true for the intermediaries.

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Nomenclature

Chapter 2

Γ_N	Game in normal form
N	Set of players in a game
n	Number of players in a game
S_i	Set of pure strategies for player i
s_i	Pure strategy of player i
ξ_i	Mixed strategy of player i
$u_i(\cdot)$	Utility function of player i
Γ_E	Game in extensive form
Γ_{IE}	Imperfect-information game in extensive form
A	Set of actions in Γ_E
H	Set of non-terminal choice nodes in a game tree
Z	Set of terminal nodes in a game tree
$act(\cdot)$	Action function in Γ_E
$\rho(\cdot)$	Player function in Γ_E
$\sigma(\cdot)$	Successor function in Γ_E
I_i	Information set for player i
ξ_i	Mixed strategy for player i
Ξ_i	Set of mixed strategies for player i
Δ	Set of probability distributions
Γ_B	Bayesian game
Θ_i	Set of types for player i
θ_i	Type of player i
$\hat{\theta}_i$	Reported type of player i
$EU_i(\xi, \theta)$	Ex-post expected utility of player i with joint strategies and types, ξ, θ
$EU_i(\xi, \theta_i)$	Ex-interim expected utility of player i with type θ_i and joint strategies ξ
$EU_i(\xi)$	Ex-ante expected utility of player i with joint strategies ξ
ξ_i^*	Player i 's best response
$BR_i(\xi_{-i})$	The set of player i 's best responses in Γ_B
$scf(\cdot)$	Social choice function

\mathbb{O}	Set of outcomes
M	Mechanism
$g^*(\cdot)$	Outcome rule
L	Discrete choice set
x	Choice from set L
v_i	Valuation function of agent i
p	Vector of payments
p_i	Payment of agent i
r	Reserve price
r^*	Optimal reserve price
$\beta(\cdot)$	Symmetric equilibrium bidding strategy function in IPV FPSB auctions
$F(\cdot)$	Bidders' cumulative distribution function of private valuations
$f(\cdot)$	Bidders' probability density function of private valuations
$f_1^{(n)}(\cdot)$	Highest-order statistic among n samples i.i.d. drawn from F
v_0	Seller's private valuation
$\phi(\cdot)$	Virtual valuation function
E	Ad exchange
P	Publisher
adn_i	Ad network i
b_i	Bid price of ad network i
d_i	Ad to be shown by ad network i
c_i^*	Price to be paid to the ad exchange by ad network i
$ adn $	Number of ad networks
K	Total number of advertisers
ℓ	Level of a tree
w_ℓ	Bidder's revenue share at tree level ℓ
tlp	Take-it-or-leave-it price
di	Number of top bidders
T	Number of auctions
m	Number of bidders in a bidding ring
n_s	Number of competing sellers
n_b	Number of competing buyers
Φ	Potential function

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s_j	j^{th} intermediary
K	Total number of buyers
k_j	Number of buyers in intermediary s_j

n	Number of intermediaries
v_i	Buyer i 's private valuation
V	Support of private valuations
F, f	c.d.f. and p.d.f. of buyers' private valuations
ρ	Center's reserve price
r_j	Intermediary s_j 's reserve price
$\Pi_j(\cdot)$	A buyer's ex-interim expected surplus from intermediary s_j
$\alpha_j(\cdot)$	A buyer's probability of winning s_j 's local auction
p_j	A buyer's payment to intermediary s_j

Chapter 4

ρ_{OPT}^*	Optimal auctioneer's reserve price without intermediaries
ρ_ℓ^*	Optimal center's reserve price for $\ell = \{PRE, POST, FPSB\}$ intermediaries
ρ_{SINGLE}^*	Optimal center's take-it-or-leave-it price for a single intermediary
$G(\cdot), g(\cdot)$	c.d.f., p.d.f. of the second-highest-order statistic among k samples
	i.i.d. drawn from $F(\cdot)$
$G_2^{(n)}(\cdot), g_2^{(n)}(\cdot)$	c.d.f., p.d.f. of the second-highest-order statistic among n samples
	i.i.d. drawn from $G(\cdot)$
$H(\cdot), h(\cdot)$	c.d.f., p.d.f. of the second-highest-order statistic among k samples
	i.i.d. drawn from $F(\cdot)$
$H_1^{(n)}(\cdot), h_1^{(n)}(\cdot)$	c.d.f., p.d.f. of the second-highest-order statistic among n samples
	i.i.d. drawn from $H(\cdot)$
$\sigma_\ell(\cdot)$	FPSB equilibrium bidding function for buyers in one intermediary against $n - 1$ $\ell = \{PRE, POST\}$ opponents
$\beta(\cdot)$	FPSB equilibrium bidding function for buyers with n FPSB intermediaries
$F_\beta(\cdot)$	c.d.f. of bids in each FPSB intermediary with n homogeneous such intermediaries
$H_\beta(\cdot)$	c.d.f. of the highest-order statistic of k samples i.i.d. drawn from F_β

Chapter 5

r	Single intermediary's reserve price
r_{SINGLE}^*	Optimal single intermediary's reserve price
ρ_{SINGLE}^*	Optimal center's take-it-or-leave-it price for a single intermediary
r_ℓ^L, r_ℓ^H	Reserve prices for low- and high-reserve $\ell = \{PRE, POST, FPSB\}$ intermediaries

r_ℓ	Equal reserve prices for $\ell = \{PRE, POST, FPSB\}$ intermediaries
$\beta_\ell(\cdot)$	FPSB semi-separating equilibrium bidding strategy for $\ell = \{L, H\}$
$\beta_\ell^s(\cdot)$	FPSB separating equilibrium bidding strategy for $\ell = \{L, H\}$
$\beta(\cdot)$	FPSB equal reserve prices equilibrium bidding strategy
r_{min}, r_{max}	Minimum and maximum of support of MSNE reserve prices for PRE intermediaries with one buyer each
$\xi_r(\cdot)$	p.d.f. of MSNE reserve prices for PRE intermediaries with one buyer each

Chapter 6

$\theta(\cdot)$	Duopoly intermediary selection function
w_i	Cut-off point i
w	Low cut-off point in duopoly PRE MSNE intermediary selection function
a	High cut-off point in duopoly PRE MSNE intermediary selection function
$\theta_m(\cdot)$	Strictly mixed strategy part of duopoly PRE MSNE intermediary selection function
θ^*	Pure strategy part of duopoly PRE MSNE intermediary selection function

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To Christos, Maria, Evi, Vassilis and Foteini.

Declaration of Authorship

I, Lampros C. Stavrogiannis, declare that the thesis entitled *Competition between Demand-Side Intermediaries in Ad Exchanges* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as: (Stavrogiannis et al., 2013b) and (Stavrogiannis et al., 2014).

Signed:.....

Date:.....

Chapter 1

Introduction

Online advertising constitutes the main source of revenue for the majority of businesses on the web and is the reason why people can enjoy many online services for free. It is estimated that online advertising saves each Internet user approximately £140 a year¹. The first online advertisement (also called an ad) appeared on 27 October 1994, when HotWired, the first commercial web magazine, sold and displayed a banner clickable ad for AT&T (Kaye and Merdoff, 2001). According to some sources, Global Network Navigator (GNN) was the first company to serve an online advertisement on 1993² (Rubinfeld and Ratliff, 2010). However, it was not until 1999 that interest in online advertising increased, as the Internet bubble began attracting more and more advertisers. Since then, it has become a multi-billion business with an annual profit of \$ 42.8 billions only in the U.S. for 2013 (PwC, 2014), a 17% increase over 2012 (see Figure 1.1). In Europe, online advertising revenue for 2013 was approximately € 27.3 billions, 11.2% higher than 2012, with the U.K. having by far the highest total revenue of € 7.4 billions among the 26 countries included in the study (IHS, 2014).

The major reason for such a growth, besides the rapid expansion of the Internet, is the vast technological innovation that nowadays allows for the targeting of users browsing the web. Although traditional advertising has been around for several centuries, companies using it have no other option than showing their advertisements to an audience based on previous statistics on its expected demographics, using surveys with questionnaires. The main problem with this approach for advertisers working with traditional media, such as TV, radio and newspapers, was pointedly stated by John Wanamaker, the father of modern advertising, in 1875: "*Half the money I spend on advertising is wasted; the trouble is I don't know which half*". This problem is nowadays less apparent. Current

¹Source: <http://www.telegraph.co.uk/technology/news/11047801/Would-you-pay-140-a-year-for-an-ad-free-web.html>

²Probably Prodigy, a joint venture of IBM and Sears, was the first company to offer online but non-clickable advertisements.

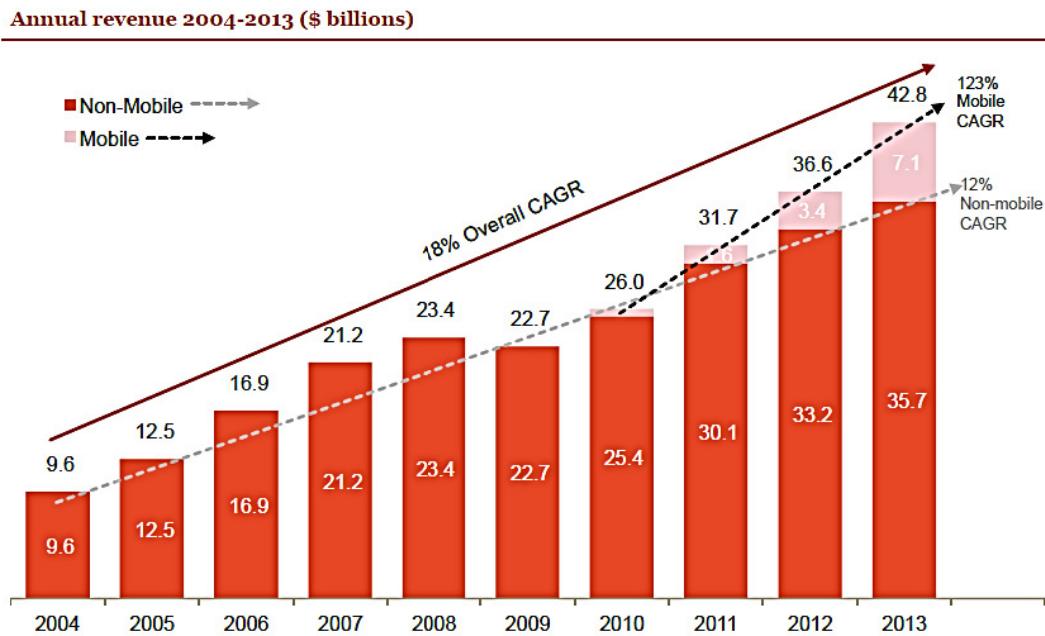


FIGURE 1.1: Online advertising annual revenue and compound annual growth rate
(Source: Interactive Advertising Bureau).

technology makes it possible in most cases³ to target each user separately, tracking their browsing behavior and transaction history via special files, known as “cookies”, that are installed on their browser. In this way, advertisers can now match each individual advertisement placement to a user.

The abundance of information about each specific user’s intentions and characteristics led to the development of new markets for the trading of online advertisements. More specifically, in 1998, GoTo.com (later renamed as Overture and now owned by Yahoo!) introduced the first sponsored search ads, i.e. advertisements on search engines, which since then have been traded using auctions based on query terms (keywords) that users enter. One of the novelties was also in the pricing rule, called cost-per-click (CPC), whereby payments are made only when a user clicks on an ad (Jansen and Mullen, 2008).

In contrast, marketeers in display (i.e. banner/video) advertising, although better established, followed the traditional cost-per-mille (CPM) (i.e. per thousand ad views, known as *impressions* in the context of online advertising) pricing model. Similar to the offline advertising process, in this market, known as *guaranteed delivery*, owners of web pages (called the *publishers*) would contact advertisers to trade advertising space in bulk on their websites via a negotiation process that led to bilateral contracts long before the

³This is not true for advertisements shown on social network platforms, such as Facebook, or other types of online advertising, such as viral marketing or mobile advertising. Moreover, one of the most prominent issues is how to perform cross-device targeting, i.e. how to recognize the same user on different devices (mobile, tablets and PCs) or browsers.

start of the advertising campaigns. These contracts specify an agreed upon volume and price for advertisements to be shown for specified dates to a future set of users that match the advertiser's desired demographics. For instance, an advertiser could agree with The New York Times to have 1 million advertisements displayed on its website during the following December and only to males, 25-35 years old, from California, with income \$100,000 - \$200,000 for \$5 CPM. This was a time-consuming process with high search costs for both sides (see Figure 1.2 for the steps involved in a typical display advertisement order). For this reason, the vast majority of publishers and advertisers started working with specialized intermediaries, known as *ad networks*. These intermediaries were responsible for matching the supply with the demand for ads, taking a percentage cut for their services. This, in turn, created a number of complications, the most important of which is that often complex, long chains of ad networks would form between publishers and advertisers, taking most of the surplus generated, and leading to inefficiencies as well as opaque trades on the two ends of this chain.

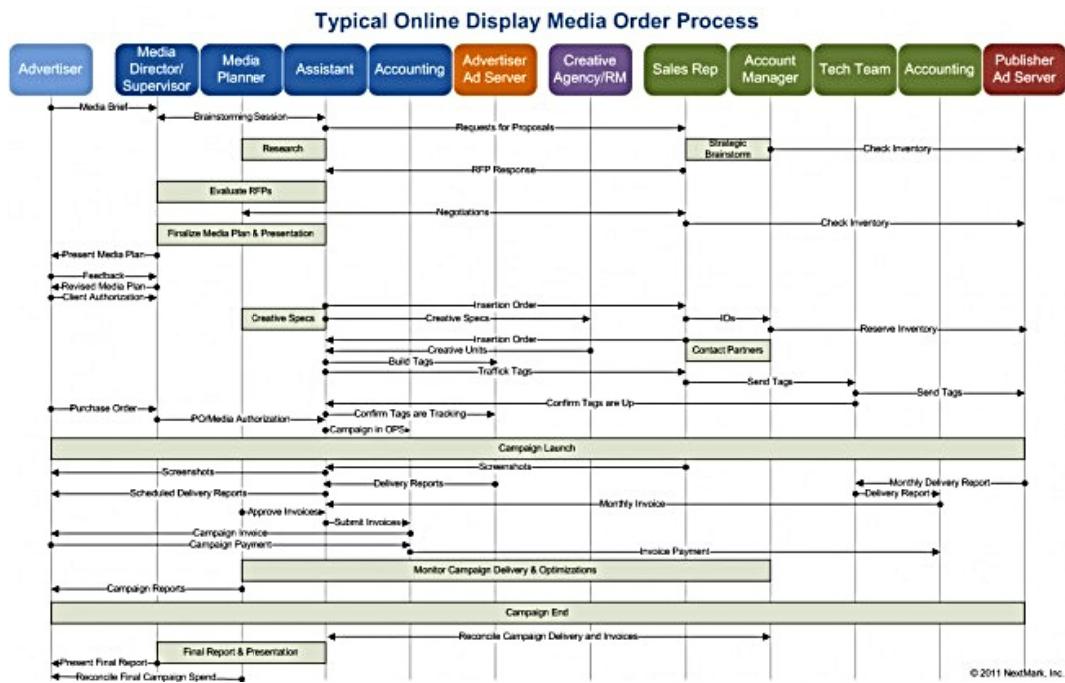


FIGURE 1.2: A typical online display advertisement order (Source: NextMark).

To alleviate the aforementioned problems, borrowing ideas from sponsored search, the use of auctions was adopted in 2005, when Right Media (now owned by Yahoo!) introduced the first auction-based marketplace for display advertisements, known as an *ad exchange*⁴ (Muthukrishnan, 2009). This allowed for the programmatic trading of advertising space, reducing the number of intermediaries, and increasing transparency. However, it was not until the early 2010 when ad exchanges gained substantial growth,

⁴There have been previous reported efforts on creating such exchanges in 2001 (MediaPort ad exchange, founded by the three largest ad networks of that time), but the market was not technologically mature enough to accept these institutions during that period, and the dot-com bubble was one of the main reasons for their collapse (Fiss and Kennedy, 2008).

with the introduction of *real-time bidding* (RTB), the ability to bid differently for each specific user visiting a specific website in almost real time⁵. In 2013, approximately \$4.5 billions have been traded worldwide on RTB, 66% higher than 2012 (IDC, 2013) and this number is expected to grow up to \$20.8 billions in 2017 (see Figure 1.3). Nowadays, two parallel markets are present: the guaranteed delivery market, trading mainly ads for brand recognition between large partners (known as *premium inventory*), and the ad exchange spot market, trading the remaining advertising space (known as the *remnant inventory*), mainly for performance-driven advertisers⁶ (Evans, 2008).

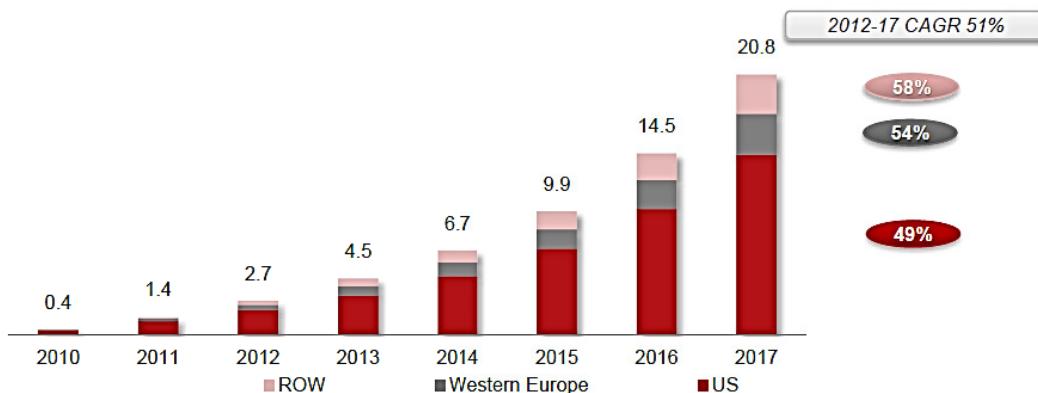


FIGURE 1.3: Real-time bidding historical and projected revenue (Source: International Data Corporation).

Ad exchanges are characterized by the extremely high speed of trades (each auction lasts approximately 100 ms) and immense number of advertising slots traded (billions of such auctions are conducted daily, (McAfee, 2011)). On top of this, there is an exponential number of attributes that advertisers can target for (Lahaie et al., 2008; Engel and Tennenholz, 2013). These facts make it impossible for humans to handle trading and so both the bidding and auctioning are performed by specialized, autonomous pieces of software, known as *intelligent agents* (Wooldridge, 2001). All these challenges make clear the importance of properly designing these auctions by careful analysis of the strategic interactions of all stakeholders.

Against this background, in the remainder of this chapter, we provide more details on the operation of ad exchanges and outline the requirements and contributions of our study. More specifically, Section 1.1 gives a general description of the online display advertising industry. Section 1.2 discusses the research challenges as well as the main motivation for this thesis. Then, Section 1.3 outlines our major contributions. Finally, in Section 1.4, we outline the content of the remaining chapters within this thesis.

⁵Source: <http://www.infectiousmedia.com/birth-real-time-bidding/>

⁶These are advertisers whose target is to get some immediate action from the user, such as clicking an ad, buying a product or filling in a form, an event known as *conversion*.

1.1 The Online Ad Exchange Landscape

As described before, ad exchanges are technology platforms that bring together buyers and sellers of advertising space in a centralized online auction-based marketplace, providing better liquidity, and thus increasing competition and efficiency. The main reason for their introduction was the opaque, bulk trading of impressions through a series of intermediaries. However, even today, the display advertising market is quite fragmented, as the landscape of Figure 1.4 illustrates. More specifically, the technological advancement requires specialization which the majority of both publishers and advertisers find difficult to acquire by themselves. For this reason, similar to financial exchanges, publishers and advertisers participate in the ad exchanges via the use of sell- and demand-side intermediaries, called *supply- or sell-side* (SSPs) and *demand-side platforms* (DSPs), respectively. These intermediaries provide the technical infrastructure, relevant tools, as well as a centralized point of access to the various ad exchanges, acting as brokers and executing orders on behalf of their customers. Another stakeholder that plays a crucial role in the trading of ads in real time are *data management platforms* (DMPs), also called *data exchanges* (O'Connell and Greene, 2011), which offer user profiling data to the intermediaries or their clients in order to increase the effectiveness of their targeting. The existence of these intermediaries creates a number of complications for designing the auction at the exchange, as will be shown in the remainder of this thesis. This is in contrast with sponsored search advertising where the publisher (search engine) contacts advertisers directly and which has been the focus of the majority of research (we refer the interested reader to (Maillé et al., 2010) for a survey). Other characteristics that need to be taken into account include:

- Goods are *extremely perishable*: as soon as an impression is generated, an appropriate advertisement must be shown, otherwise there is no value for any of the involved parties.
- Goods are *heterogeneous*: the expected profit of an advertiser for each ad is uniquely determined by the user that visits the web page and the context of that page.
- There are *information asymmetries*: different intermediaries (or advertisers) can have different information about the user visiting the web page based on the tracking cookies they have previously installed on his browser. This means that some intermediaries can target the user more effectively and have more precise information on how much he is worth to them (i.e. better estimation on the expected profit from showing him their selected ad).
- *Delivery* of ads must be *fast*: the time between the user visiting the web page and the display of impressions is infinitesimal, in the order of milliseconds, which means

that the rules of the auction must be simple and yet effective, while implementation must be robust to failures.

- The *volume* of impressions is extremely *high* whereas the *value* of the ads is *minuscule*: every day, billions of impressions are generated whereas pricing is performed on a per thousand scale, and the average CPM for an ad is usually around \$ 1-\$ 5 (which means that the cost for a single impression is small, typically from 0.1 to 0.5 cents)⁷.

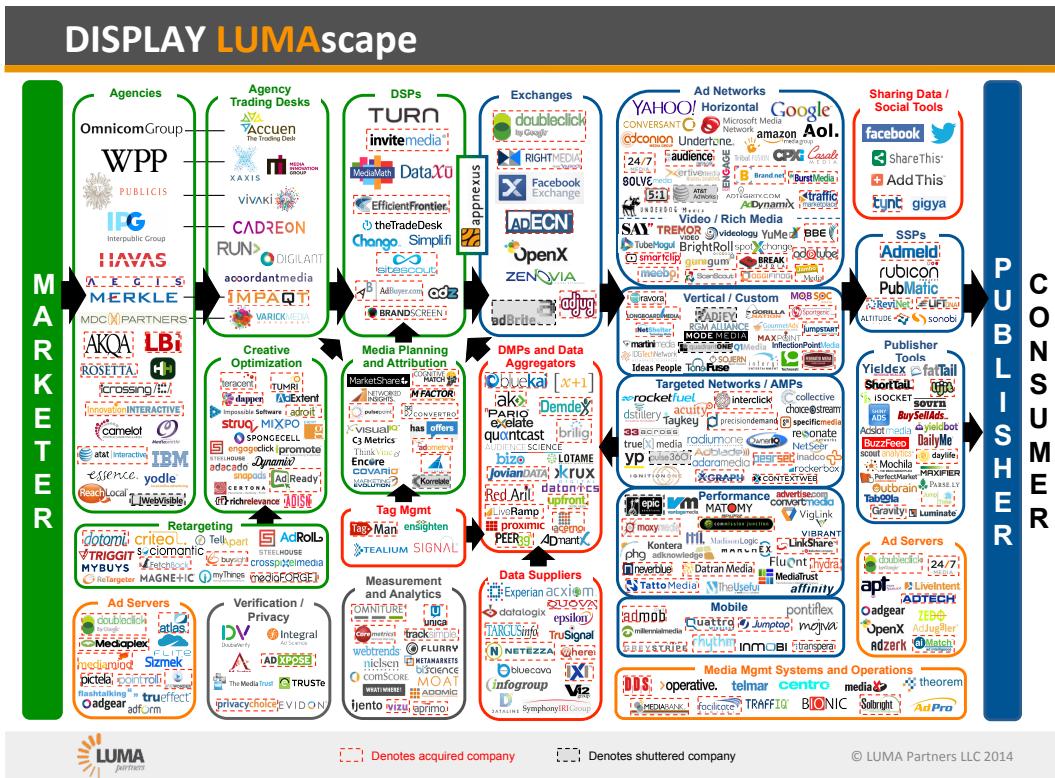


FIGURE 1.4: The online advertising landscape (Source: LUMA Partners).

The auction process for one advertising slot is shown in Figure 1.5. The timing is as follows. Whenever some user enters a web page, the owner of that web page (i.e. the publisher) contacts the ad exchange, either directly or indirectly through an SSP, sending relevant information about the user and the web page. The ad exchange/SSP then calls for bids from the DSPs each of which represents a number of advertisers. This process is known as a *call out* and lasts approximately 100 ms (Chakraborty et al., 2010). Then DSPs run some local mechanism of their choice to determine the bid(s) and the ad(s) to send on behalf of their advertisers. Then the ad exchange/SSP runs an auction with those bids, determining a winner and a payment, and the ad is finally displayed to the

⁷We refer the reader to <http://yourvalue.inrialpes.fr/> for an interesting online experiment on the value of impressions.

user. This whole process takes approximately 150 ms⁸. Formal models will be provided in sections 2.2.1 and 3.3.

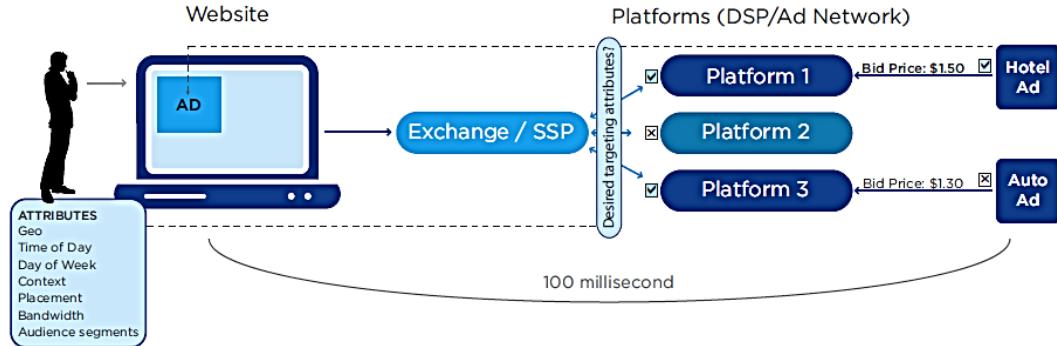


FIGURE 1.5: The online ad exchange auction (Netmining, 2011).

1.1.1 Research in Ad Exchanges

Given their recent appearance and complexity, research on ad exchanges has risen over the last years. Muthukrishnan (2009) offers a succinct introduction to the research challenges that arise in this area. More specifically, emphasis has been given on the publishers' problem of optimally allocating their inventory in the guaranteed and ad exchange markets so as to maximize revenue but also reduce their risk (Ghosh et al., 2009a; Yang et al., 2010; Chen, 2010; Balseiro et al., 2011). For the advertisers, most of the literature has focused on bidding strategies, usually employing optimization techniques (Ghosh et al., 2009b; Chen et al., 2011; Bartels et al., 2012; Amin et al., 2012; Zhang et al., 2014; Tran-Thanh et al., 2014). Notable exceptions are the works of Gummadi et al. (2012); Balseiro et al. (2013) who have also taken into account the strategic interactions between advertisers. Another problem that the stakeholders face is the asymmetry of information between publishers and advertisers but also among the advertisers themselves. This is due to the fact that some advertisers or DSPs have (free or paid) access to more data which increases the effectiveness of their targeting and leads to an effect called *cream skimming* or *cherry picking*, whereby informed advertisers obtain all good-quality inventory, leaving only low-quality inventory to the uninformed ones (McAfee, 2011). The publisher/ad exchange might also have information that can be shared and researchers have looked at the effect of such a revelation on its revenue (Levin and Milgrom, 2010; Abraham et al., 2011; Fu et al., 2012; Emek et al., 2012; Babaioff et al., 2012; Mahdian et al., 2012; Arnosti et al., 2014; Miltersen and Sheffet, 2012).

Probably one of the major challenges in this context is the design of the auction at the exchange. Some of the problems include, but are not limited to, the facts that the auctions are repeated, advertisers vary, have budgets and also have different incentives

⁸We refer the reader to <http://cmsummit.com/behindthebanner/> for a clear animated depiction.

(mainly brand promotion versus performance advertising), advertisements come in different sizes and multiple ad slots are usually available (see Section 2.2 for more details as well as the excellent articles of Muthukrishnan (2009); McAfee (2011)).

1.1.2 The Effect of DSPs

Probably the most crucial issue in auction design by ad exchanges is the introduction of the demand-side intermediaries, i.e. DSPs. These intermediaries typically⁹ submit a single bid at the ad exchange on behalf of their (multiple) advertisers, thus hiding some of the demand from the ad exchange. This can potentially reduce the exchange's revenue and can decrease its efficiency since the advertising space might not always be allocated to the advertiser that values it most (this will be more evident throughout the thesis, starting from the examples of Section 3.4). This situation is reminiscent of auctions with colluding bidders for the exchange (see Section 2.4 for an exposition). In the ad exchange setting, DSPs are seen as colluding groups by the ad exchange and hamper its successful operation. More specifically, the majority of DSPs run some local mechanism (i.e. decisions about which advertiser wins the advertising space and how much to get charged) and determine a typically single bid to send at the exchange.

There are currently two types of such intermediaries: *self-service* (also known as *self-serve*) and *managed service*. The latter type follow the classical ad network model, whereby the intermediary agrees with each advertiser the budget, pricing and number as well as type of delivered impressions, and the intermediary manages the campaign on behalf of each advertiser. On the other hand, self-service intermediaries, which appeared along with RTB, offer only the necessary tools and infrastructure and advertisers manage their campaigns by themselves¹⁰. For this reason, the predominant mechanism implemented by self-service intermediaries is the use of local auctions among their clients. Although managed-service intermediaries seem to be the prevalent type of such demand-side intermediaries, more and more such intermediaries are moving to the self-service type¹¹. The competition between the latter type of demand-side intermediaries where they pocket the difference between what they get paid by their advertisers and what they pay at the exchange is the problem studied within this thesis, as will be described in detail in the following section.

⁹Some DSPs might be contracted not to hide any of the demand from their advertisers (Mansour et al., 2012).

¹⁰There is a tendency in the advertising world to consider demand-side intermediaries true DSPs only if they belong to the self-service category (see <http://www.adexchanger.com/data-driven-thinking/not-every-demand-side-platform-dsp-is-created-equal-what-is-a-true-dsp/>).

¹¹See <http://www.adexchanger.com/data-driven-thinking/whats-behind-the-rise-in-self-serve-programmatic/>.

1.2 Research Objectives

The aim of the work in this thesis is to analyze the effects of the existence and competition between a small, finite number of demand-side intermediaries (i.e. imperfect competition) on the profits of each of the main stakeholders in these systems, namely the ad exchange/publisher, the DSPs and the advertisers.

More specifically, our research objectives within this thesis include:

- **Analysis of the competition between real-world demand-side intermediaries.** As has been mentioned before, DSPs implement local auctions that compete at the ad exchange for the advertising space. Hence, it becomes necessary to study the impact of the choice of auction and bid submission strategies at the exchange that are currently predominantly used by DSPs on the revenue generated as well as the efficiency of the ad exchange marketplace. This can help both the ad exchange and DSP auctioneers to efficiently design their markets as this will determine the future adoption of RTB in the world of online advertising.
- **Analysis of the profit maximization problem of the ad exchange and demand-side intermediaries when the latter compete.** Auction theory provides guidance on what auctioneers can do to increase their revenue, mainly by setting a *reserve* (also known as a *floor*) *price* that, if appropriately determined, has been shown to maximize the revenue of a monopolistic auctioneer. It is thus important to study the benefits and drawbacks, if any, of setting suitable reserve prices for both the ad exchange and the intermediaries.
- **Analysis of the advertisers' demand-side intermediary selection strategies.** The objective of advertisers in ad exchanges is to maximize their surplus, which is the difference between what they receive from displaying an advertisement and what they pay. This can be done by appropriately adjusting their bids based on the rules of the auction. However, in such markets, the advertisers must select the DSP to operate their trades. Advertisers are offered similar services from a multitude of demand-side intermediaries and must decide which is the best trading partner to work with for their campaigns. Doing so, they need to take into account the intermediary selections of the other advertisers, as this crucially determines their probability of winning and payment. Hence, our aim is to characterize stable outcomes of these strategic decisions of the advertisers that provide guidance and can serve as a benchmark against simpler, heuristic approaches that advertisers often take.
- **Analysis of the competition between demand-side intermediaries with strategic advertisers' allocation.** Given that advertisers have incentives to

strategically select a DSP for their trades, the latter should take advertisers' expected decisions into account along with the designs of their opponent intermediaries when designing their auctions. This is also true for the ad exchanges which need to consider the expected behavior of the intermediaries and the advertisers, when optimizing their auctions for revenue or any other objective.

Having explained our research objectives for this thesis, in the next section we detail the research contributions in this direction.

1.3 Research Contributions

Given the research objectives outlined in the previous section, the research reported in this thesis provides insights into the following aspects of ad exchanges:

1. **Competing real-world demand-side intermediaries (Chapter 4).** We first analyze the effect of competition between intermediaries that implement three widely-used auctions for their operations: two variations of the second-price sealed-bid auction and the first-price sealed-bid auction. Specifically, we consider a single advertising slot auctioned at an ad exchange (the publisher and the ad exchange will be considered a single entity throughout the rest of the thesis) that sets a reserve price. We focus on homogeneous intermediaries, i.e. intermediaries implementing the same auction. Moreover, we assume that all intermediaries have the same number of advertisers¹² and that advertisers have contractual agreements with their intermediaries and hence cannot move between them (i.e. are *captive*). We find that the reserve price of the ad exchange increases with the number of buyers and/or intermediaries even for a single-intermediary setting. We also show that there cannot be a complete profit ranking between the three auctions but first-price sealed-bid auctions seem to provide a good trade-off between profit and efficiency.
2. **Competing demand-side intermediaries with reserve prices (Chapter 5).** We extend the above-mentioned analysis by letting intermediaries set appropriate reserve prices. This problem has been first studied by Feldman et al. (2010) who have determined the optimal (i.e. profit maximizing) mechanism for the intermediaries, but only for the case of one advertiser per intermediary. They show that their results generalize to more than one advertiser per intermediary, but cannot analytically derive the reserve prices of the intermediaries in equilibrium. The authors show that intermediaries follow complex reserve-price-setting strategies that involve randomization of reserve prices from a defined interval. Given these

¹²Each intermediary has a different, finite set of advertisers whose cardinality is the same for all intermediaries.

issues of technical tractability, we limit our setting to two competing DSPs. In this duopoly scenario, we keep the symmetry and captivity assumptions for the intermediaries. We then numerically find the resulting approximate equilibria for the auction studied by Feldman et al. (2010) as well as the other two auctions. We show that our numerical technique provides a good approximation to the theoretical results, when those are available, and that, in general, intermediaries still follow randomized reserve-price-setting strategies. To the best of our knowledge, this is the first attempt to formally study the effects of reserve prices for different DSP mechanisms and the ad exchange. Our numerical examples depict that the mechanism studied by Feldman et al. (2010) yields the highest profit among the mechanisms studied but is less efficient than other mechanisms, i.e. the ad space is not always allocated to the advertiser that values it the most.

3. **Competing demand-side intermediaries with reserve prices and strategic intermediary selecting advertisers (Chapter 6).** We then remove the captivity assumption for the advertisers and let them strategically select their intermediary in a setting with two intermediaries, each implementing one of the second-price auction variations with a reserve price. We assume that advertisers *single-home*, i.e. select only one intermediary. Letting the advertisers select both intermediaries would give less insight since, in our model, that would mean that advertisers select all intermediaries, which cannot happen in reality where hundreds of DSPs operate. Moreover, there is an inherent cost of managing a campaign¹³. This is the first time that this problem has been addressed given the fact that little is known about the operation of ad exchanges but also due to the complexity increased by the presence of the intermediaries. We show that, in contrast to previous literature on competing auctions, in some settings, the advertisers' intermediary selection strategies involve non-uniform randomization between the intermediaries. Finally, we numerically derive the intermediaries' profit and the ad exchange's revenue in an approximate equilibrium using learning techniques. We find that the center and the ad exchange system as a whole benefits from intermediaries implementing the same auction from our restricted set of mechanisms, whereas the opposite is true for the intermediaries.

The work described in this thesis has led to the following peer-reviewed publications:

- L. C. Stavrogiannis, E. H. Gerding and M. Polukarov. Competing Intermediary Auctions. In *Proceedings of the 2013 international conference on Autonomous Agents and Multi-Agent Systems (AAMAS '13)*, pp. 667 - 674, St. Paul, MN, U.S.A., 2013.

¹³8-28% of each advertiser's budget is estimated to be spent on operational costs (Source: <http://www.admonsters.com/blog/get-ready-coming-operationally-driven-marketplace/>).

- L. C. Stavrogiannis, E. H. Gerdin and M. Polukarov. Auction mechanisms for demand-side intermediaries in online advertising exchanges. In *Proceedings of the 2014 international conference on Autonomous Agents and Multi-Agent Systems (AAMAS '14)*, pp. 1037 - 1044, Paris, France, 2014.

The author's work has also resulted in the following peer-reviewed publications that are not reported here as they do not fit perfectly into the context of this work:

- L. C. Stavrogiannis, E. H. Gerdin and M. Polukarov. Competing Intermediaries in Online Display Advertising. *Proceedings of AAMAS Joint Workshop on Trading Agent Design and Analysis and Agent-Mediated Electronic Commerce*, pp. 103 - 116, Valencia, Spain, 2012.

This work deals with the demand-side intermediary selection problem of the advertisers in a two-intermediary setting, where both intermediaries implement the same second-price sealed-bid auction, for a single item auctioned at an ad exchange when the latter sets no reserve price, i.e. minimum bid. The setting studied here is a complete-information one, where advertisers' valuations for the item are publicly known. We find that an infinite number of symmetric Nash equilibria exist for the advertisers' selection strategies, and that the reserve-price-setting problem of the intermediaries admits a symmetric subgame-perfect equilibrium, where both intermediaries set a reserve price equal to the second-highest valuation of the advertisers.

- L. Tran-Thanh, L. C. Stavrogiannis, V. Naroditskiy, V. Robu, N. R. Jennings, P. Key. Efficient regret bounds for online bid optimisation in budget-limited sponsored search auctions. *Proceedings of the 30th Conference on Uncertainty in Artificial Intelligence (UAI 2014)*, pp. 809 - 818, Quebec, Canada, 2014.

This work deals with the problem of an advertiser who needs to allocate her budget, B , across a number, T , of real-time second-price sealed-bid auctions. It is assumed that the highest opponent bid gets revealed at the end of each auction only if the advertiser wins the auction¹⁴, i.e. the data is right-censored, where it is assumed that this bid is independent and identically (i.i.d.) drawn from a fixed but unknown distribution. It is shown that two previously proposed algorithms achieve $O(\sqrt{T})$ regret with high probability compared to the optimal stochastic algorithm and another algorithm, ϵ -First, is proposed and is shown to achieve $O(T^{\frac{2}{3}})$ regret with high probability. The results are numerically verified using real sponsored search bidding data.

¹⁴This is commonly done in the vast majority of ad exchanges.

1.4 Thesis Outline

The remainder of this thesis is organized as follows:

- In Chapter 2, we provide a short introduction to the area of game theory that includes the tool set we use for our analysis. We then present the state-of-the-art in the ad exchanges literature from an auction theoretic perspective. We also summarize literature on competing mechanisms, auctions with intermediaries and auctions with bidding rings, which are closely related to the setting we study.
- Chapter 3 introduces a formal model of an ad exchange with competing demand-side intermediaries and then presents a number of motivating examples that shed some light in the issues related to the presence of these intermediaries.
- In Chapter 4, we present our analysis of the competing demand-side intermediaries with captive advertisers. We start with the special case of a single intermediary and then move to the more general setting with multiple homogeneous intermediaries where we compare, both theoretically and numerically, the three auction mechanisms studied. Finally, we conclude with the analysis of heterogeneous intermediary auctions.
- Chapter 5 deals with the competition of the same auction mechanisms but now including appropriate reserve prices. We start again with the motivating case of a single intermediary and then study the equilibrium reserve prices of the three intermediary mechanisms in a two-intermediary setting. We find that, in general, intermediaries should follow randomized reserve-price-setting strategies, so we conclude offering numerical results for the case where advertisers' private valuations are i.i.d. random variables following the uniform distribution.
- Chapter 6 then considers the intermediary selection problem of the advertisers in a duopoly setting with both homogeneous and heterogeneous second-price sealed-bid intermediary auctioneers. Based on these results, the reserve-price-setting problem is considered for the intermediaries and the ad exchange, where, given the complexity of their strategies, numerical results are depicted and comparisons are made against the previous settings with captive advertisers.
- Finally, Chapter 7 summarizes the contributions of this thesis and provides directions for future work that will increase the practical applicability of our work to real-world ad exchanges.

Chapter 2

Literature Review

In this chapter, we discuss related work on the field of online display advertising and auction theory. We begin by introducing the notions of game theory, emphasizing on two of its sub-areas, namely mechanism design and auction theory (Section 2.1) that are necessary to follow the remainder of this thesis. We then review related work in the field of online advertising exchanges, the main application area of this thesis, where we focus on the auction theoretic issues that arise (Section 2.2). Following this, we provide a short introduction to the areas of auctions with intermediaries and collusion in Sections 2.3 and 2.4 respectively, that share some similarities with the setting studied within this thesis. After that, Section 2.5 discusses previous work on competition between auctioneers. Finally, Section 2.6 summarizes.

2.1 Introduction to Game Theory, Mechanism Design and Auction Theory

In this section, we provide a short introduction to game theory, emphasizing on two of its sub-areas that are of immediate interest, namely mechanism design and auction theory. More specifically, we first formally define the notion of a game and the strategic interactions it encompasses, and then move to the description of the general task of designing such games for the allocation of resources, known as mechanism design. Following this, we review some of the most important results in the field of auction theory.

2.1.1 Game Theory

Game theory can be defined as the mathematical study of strategic interactions, such as conflict or cooperation, between intelligent, rational decision makers (Myerson, 1991;

Shoham and Leyton-Brown, 2008). A *game* is a formal representation of such interactions and comprises (i) a set of *players* (the decision makers), (ii) the *rules*, i.e. the order of moves and information available to each player as well as his available choices (called *actions*), (iii) the *outcomes*, which are the possible results of the game, given the actions by the players, (iv) the *payoffs* for each player related to each possible outcome (Mas-Colell et al., 1995). A player is *rational* if his decisions are made only with regard to his own objectives, and he is *intelligent* if he knows everything that is available to him and makes inferences about the situation based on his knowledge (Myerson, 1991). For the remainder of this thesis we will use the terms player and agent interchangeably, and assume that agents' rationality is common knowledge among them.

Games can be described using two main types of representation: the *strategic* or *normal form* and the *extensive form*. The former makes use of a matrix to specify player actions and their corresponding payoffs and is mostly used to represent simultaneous-move games, i.e. one-stage games where players perform actions at the same time. The extensive form is preferred in sequential-move games, where players take actions in turns. The game is represented by a tree (called the *game tree*) which specifies all possible states and actions until the end of the game, unraveling the tree at each action selection stage. Although there are many other forms to represent a game, each of them has an “*induced normal form*”, which is an equivalent normal-form representation that maintains game-theoretic properties (Shoham and Leyton-Brown, 2008). A formal definition of a normal-form game (NFG) follows.

Definition 2.1. A finite, n -person normal-form (or strategic-form) game, Γ_N , is a tuple (N, S, u) , where:

- N is a finite set of n players, indexed by i ;
- $S = \times_i S_i$, is the set of all possible strategy profiles, where S_i is a finite set of (pure) strategies available to player i . A player's strategy $s_i \in S_i$ is a complete contingency plan, i.e. a function mapping each state of the game to an action. A vector $s = (s_1, s_2, \dots, s_n)$ is called a strategy profile. The notation $s = (s_i, s_{-i})$ can be used instead, where $s_{-i} = (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$;
- $u = (u_1, u_2, \dots, u_n)$ where $u_i : S \rightarrow \mathbb{R}$ is a real-valued utility (or payoff) function for player i . It provides the von Neumann-Morgenstern utility levels associated with the outcome produced by strategies s .

An example of a normal-form game representation is shown in Figure 2.1(a), for a game known as the Battle of the Sexes. In this game, there are two players, namely the husband (H) and the wife (W), who must decide whether to go to a fight (F) or an opera (O) play. In the matrix shown, the row player is the husband and the column player is the wife. Each element of the matrix contains two numbers, the first corresponding to the utility of the row player and the second to that of the column player.

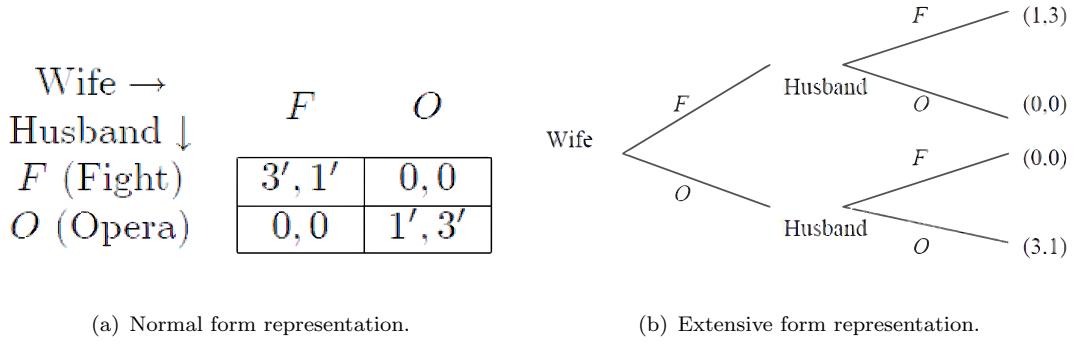


FIGURE 2.1: The Battle of the Sexes game.

Games can be separated in two broad categories based on the information available to each player on his opponents, namely *perfect-information* and *imperfect-information* games. Informally, in the former type of games, players can observe their opponents' previous moves (Mas-Colell et al., 1995). On the other hand, in the imperfect-information case, players might need to make decisions with limited or no knowledge of their opponents' past actions or they might even have limited record of their own actions (Shoham and Leyton-Brown, 2008). Let us define formally these two categories, making use of the extensive-form representation:

Definition 2.2. Perfect-information extensive-form game. A finite, perfect-information extensive-form game, Γ_E , is a tuple $(N, A, H, Z, act, \rho, \sigma, u)$, where:

- N is a finite set of n players, indexed by i ;
- A is a (single) set of actions;
- H is a set of non-terminal choice nodes;
- Z is a set of terminal nodes, disjoint from H ;
- $act : H \rightarrow 2^A$ is the action function, assigning a set of possible actions to each choice node;
- $\rho : H \rightarrow N$ is the player function, assigning to each non-terminal node a player i who selects an action at that node;
- $\sigma : H \times A \rightarrow H \cup Z$ is the successor function, mapping a choice node and action to a new choice or terminal node such that $\forall h_1, h_2 \in H$ and $\alpha_1, \alpha_2 \in A$, if $\sigma(h_1, \alpha_1) = \sigma(h_2, \alpha_2)$ then $h_1 = h_2$ and $\alpha_1 = \alpha_2$;
- $u = (u_1, u_2, \dots, u_n)$, where $u_i : Z \rightarrow \mathbb{R}$ is a real-valued utility (or payoff) function for player i on the terminal nodes Z .

The Battle of Sexes is a perfect-information game whose extensive form is illustrated in Figure 2.1(b).

Below we define a game with imperfect information:

Definition 2.3. Imperfect-information extensive-form game. A finite, imperfect-information extensive-form game, Γ_{IE} , is a tuple $(N, A, H, Z, act, \rho, \sigma, u, I)$, where:

- $(N, A, H, Z, act, \rho, \sigma, u)$ is a perfect-information game;
- $I = (I_1, I_2, \dots, I_n)$, where for each player i , $I_i = (I_{i,1}, I_{i,2}, \dots, I_{i,k_i})$ is a partition of $\{h \in H : \rho(h) = i\}$ with the property that $act(h) = act(h')$ whenever there exists a j for which $h \in I_{i,j}$ and $h' \in I_{i,j}$. That is, player i does not distinguish between nodes h and h' that belong to the same subset of partition I_i .

Until now we have assumed that a strategy of a player is deterministic, yielding a single action for each possible state of the game. These strategies are called *pure* strategies. However, in many cases, players may have to randomize over their possible choices, so as to be unpredictable in the eyes of their opponents. This leads to the concept of a *mixed* strategy (Mas-Colell et al., 1995):

Definition 2.4. Mixed strategy. Given player i 's (finite) pure strategy set, S_i , a mixed strategy for him, $\xi_i : S_i \rightarrow [0, 1]$, assigns to each pure strategy $s_i \in S_i$ a probability $\xi_i(s_i) \geq 0$ that s_i will be played, where $\sum_{s_i \in S_i} \xi_i(s_i) = 1$.

Given that players follow mixed strategies, a player's payoff is his expected utility where the expectation is taken with respect to the probabilities on the pure-strategy profiles induced by the incorporation of the former type of strategies.

Definition 2.5. Expected utility. Given a game Γ , the expected utility, $\mathbb{E}_\xi[u_i(s)]$, for player i of the mixed-strategy profile $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, where the expectation is taken with respect to the probabilities induced by ξ on pure strategy profiles $s = (s_1, s_2, \dots, s_n)$, is $\sum_{s \in S} u_i(s) \prod_{j=1}^n \xi_j(s_j)$, where $S = S_1 \times S_2 \cdots \times S_n$.

Moreover, if for any set X , $\Delta(X)$ denotes the set of all probability distributions over X :

$$\Delta(X) = \{q : X \rightarrow \mathbb{R} \mid \sum_{x \in X} q(x) = 1 \text{ and } q(x) \geq 0, \forall x \in X\} \quad (2.1)$$

then the set of mixed strategies for player i is $\Xi_i = \Delta(S_i)$.

In both types of games defined above, we have implicitly made the assumption that players have full knowledge of the parameters of the game played, such as the number of players, their possible actions and, most importantly, their corresponding payoffs.

This is not true in many real cases, where players must infer all this information and make decisions in this limited environment. This leads to the definition of an *incomplete-information game*, also known as a *Bayesian game*, introduced by Harsanyi (1967). This type of games can be modeled as an imperfect-information game with the incorporation of a special player, called Nature, that makes probabilistic choices in a way that is common knowledge to all agents and has no (or has constant) utility function. More specifically, we can imagine the user's private preferences (called his *type*) being determined by a random variable, whose prior probability distribution is common knowledge, and whose realization is performed by Nature¹. Formally (Shoham and Leyton-Brown, 2008):

Definition 2.6. Bayesian game. A Bayesian game, Γ_B , is a tuple (N, S, Θ, pr, u) where:

- N is a finite set of n players, indexed by i ;
- $S = S_1 \times S_2 \times \dots \times S_n$, where S_i is the set of strategies available to player i , known as the strategy space of i ;
- $\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_n$, where Θ_i is the space of (epistemic) types of player i ;
- $pr : \Theta \rightarrow [0, 1]$ is a common prior over types;
- $u = (u_1, u_2, \dots, u_n)$, where $u_i = S \times \Theta \rightarrow \mathbb{R}$ is the utility function of i .

In such a game, there is an additional source of uncertainty due to the introduction of players' types. Hence, we can have three different types of expected utilities for a player, defined as follows (Shoham and Leyton-Brown, 2008; Mas-Colell et al., 1995):

Definition 2.7. Ex-post expected utility. Given a Bayesian game, $\Gamma_B = (N, S, \Theta, pr, u)$, the ex-post expected utility, $EU_i(\xi, \theta)$, for player i of the mixed-strategy profile $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ on pure-strategy profiles $s = (s_1, s_2, \dots, s_n)$ when agent's types are given by $\theta = (\theta_1, \theta_2, \dots, \theta_n)$, is:

$$EU_i(\xi, \theta) = \sum_{s \in S} u_i(s, \theta) \prod_{j=1}^n \xi_j(s_j | \theta_j) \quad (2.2)$$

Definition 2.8. Ex-interim expected utility. Given a Bayesian game, $\Gamma_B = (N, S, \Theta, pr, u)$, the ex-interim expected utility, $EU_i(\xi, \theta_i)$, for player i of the mixed-strategy profile $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ on pure-strategy profiles $s = (s_1, s_2, \dots, s_n)$, when i 's type is θ_i , is:

$$EU_i(\xi, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} pr(\theta_{-i} | \theta_i) \sum_{s \in S} u_i(s, \theta_{-i}, \theta_i) \prod_{j=1}^n \xi_j(s_j | \theta_j) \quad (2.3)$$

¹We limit the scope of this definition to settings with common prior beliefs for all agents.

Definition 2.9. Ex-ante expected utility. Given a Bayesian game, $\Gamma_B = (N, S, \Theta, pr, u)$, the ex-ante expected utility, $EU_i(\xi)$, for player i of the mixed-strategy profile $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ on pure-strategy profiles $s = (s_1, s_2, \dots, s_n)$, is:

$$EU_i(\xi) = \sum_{\theta \in \Theta} pr(\theta) \sum_{s \in S} u_i(s, \theta) \prod_{j=1}^n \xi_j(s_j | \theta_j) \quad (2.4)$$

As mentioned above, the objective of game theory is to analyze the strategic behavior of players in a game so as to predict actual, stable where possible, outcomes and hence propose optimal actions to them at every state. We now define some of the involved so called solution concepts that help provide answers to these questions. Before this, it is useful to define the strategy that satisfies the single player's objective of utility maximization given the strategies of the others (Shoham and Leyton-Brown, 2008):

Definition 2.10. Best response. Player i 's best response to the strategy profile ξ_{-i} is a mixed strategy $\xi_i^* \in \Xi_i$ such that $u_i(\xi_i^*, \xi_{-i}) \geq u_i(\xi_i, \xi_{-i})$ for all strategies $\xi_i \in \Xi_i$.

However, in a Bayesian game, a player's set² of best responses is defined on her ex-ante expected utility (Shoham and Leyton-Brown, 2008):

Definition 2.11. Best response in a Bayesian game. The set of player i 's best responses to mixed-strategy profile ξ_{-i} are given by:

$$BR_i(\xi_{-i}) = \arg \max_{\xi'_i \in \Xi_i} EU_i(\xi'_i, \xi_{-i}) \quad (2.5)$$

The first solution concept to introduce is that of *dominance*. A strategy $s_i \in S_i$ for player i is (weakly) dominant if, no matter what other agents select, i will do at least as well as he would do if he would select any other strategy (Wooldridge, 2001). Formally:

Definition 2.12. Very weakly dominant strategy. A strategy $s_i \in S_i$ is a *very weakly dominant strategy* for player i in game $\Gamma_N = (N, S, u)$, if for all $s'_i \neq s_i$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}), \forall s_{-i} \in S_{-i}$.

Definition 2.13. Weakly dominant strategy. A strategy $s_i \in S_i$ is a *weakly dominant strategy* for player i in game $\Gamma_N = (N, S, u)$, if for all $s'_i \neq s_i$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}), \forall s_{-i} \in S_{-i}$, and for at least one $s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

Definition 2.14. Strictly dominant strategy. A strategy $s_i \in S_i$ is a *strictly dominant strategy* for player i in game $\Gamma_N = (N, S, u)$, if for all $s'_i \neq s_i$, $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}), \forall s_{-i} \in S_{-i}$.

All the above-mentioned notions of dominance generalize to mixed strategies in a straightforward manner.

²Since many strategies might lead to the same expected utility.

Dominance is quite a strong concept, as players may not always be in position to find a best action without taking into consideration the strategies of their opponents. A more widely-used solution concept is that of Nash equilibrium (NE) (Nash, 1951). Under this concept, when an agent has selected an action and knows the strategies of other agents, he cannot benefit from unilaterally deviating from the current strategy profile. This situation is formally expressed in the following definition for pure strategies (Mas-Colell et al., 1995):

Definition 2.15. Pure-strategy Nash equilibrium (PSNE). A pure strategy profile $s = (s_1, s_2, \dots, s_n)$ constitutes a (weak) *Nash equilibrium* of the game $\Gamma_N = (N, S, u)$, if for every $i = 1, 2, \dots, n$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}), \forall s'_i \in S_i$.

A stronger type of equilibrium is the *strict* Nash equilibrium:

Definition 2.16. Strict PSNE. A pure strategy profile $s = (s_1, s_2, \dots, s_n)$ constitutes a *strict Nash equilibrium* of the game $\Gamma_N = (N, S, u)$, if for every $i = 1, 2, \dots, n$, and $\forall s'_i \neq s_i, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

The concept of such an equilibrium naturally extends the case of mixed strategies:

Definition 2.17. Mixed-strategy Nash equilibrium (MSNE). A mixed strategy profile $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ constitutes a *Nash equilibrium* of the game $\Gamma_N = (N, \Delta(S), u)$ if for every $i = 1, 2, \dots, n$, $u_i(\xi_i, \xi_{-i}) \geq u_i(\xi'_i, \xi_{-i}), \forall \xi'_i \in \Delta(S_i)$.

Now, we can introduce the famous Nash's theorem, proving the existence of a NE in every finite strategic game (Nash, 1951):

Theorem 2.18. *Every game $\Gamma_N = (N, \Delta(S), u)$ in which the sets S_1, S_2, \dots, S_n have a finite number of elements has a mixed-strategy Nash equilibrium.*

It is clear that a strategy profile where every agent has a dominant strategy is also a Nash equilibrium, known as the *equilibrium in dominant strategies*, which will be unique for the case of strictly dominant strategies (Shoham and Leyton-Brown, 2008). Finally, another (weaker) solution concept is that of an ϵ -Nash equilibrium (Shoham and Leyton-Brown, 2008):

Definition 2.19. ϵ -Nash equilibrium (ϵ -NE). Fix (a small) $\epsilon > 0$. A strategy profile $s = (s_1, s_2, \dots, s_n)$ constitutes an ϵ -*Nash equilibrium* of the game $\Gamma_N = (N, S, u)$, if for every $i = 1, 2, \dots, n$, and $\forall s'_i \neq s_i, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) - \epsilon$.

This is an extremely useful concept for algorithms that try to find Nash equilibria, since the floating-point approximation of all computing devices means that the latter can only find such equilibria even though exact solutions may be claimed. It is also true

	<i>L</i>	<i>R</i>
<i>T</i>	1, 1	0, 0
<i>B</i>	$1 + \frac{\epsilon}{2}, 1$	20, 20

FIGURE 2.2: In this game (B, R) is the unique pure-strategy Nash equilibrium, but both (B, R) and (T, L) are ϵ -Nash equilibria.

that every NE is always surrounded by ϵ -NE, hence ϵ -NE always exist, although the opposite is not always true (cf. Figure 2.2).

Another useful solution concept that arises in perfect-information extensive-form games is that of a subgame-perfect equilibrium. To define this it is necessary to formally present what is known as a subgame of such games (Shoham and Leyton-Brown, 2008):

Definition 2.20. Subgame. Given a game, $\Gamma_E = (N, A, H, Z, act, \rho, \sigma, u)$, the subgame of Γ_E , rooted at node h , is the restriction of G to the descendants of h .

Following this, we can now define the solution concept of a subgame-perfect equilibrium, first introduced by Selten (1965), (Shoham and Leyton-Brown, 2008):

Definition 2.21. Subgame-perfect equilibrium (SPE). The subgame-perfect equilibria of a game, Γ_E , are all strategy profiles s such that for any subgame Γ'_E of Γ_E , the restriction of s to Γ'_E is a Nash equilibrium of Γ'_E .

Every SPE is also a NE, but the opposite does not always hold, and SPE always exist in every perfect-information extensive-form game.

We now move to the Bayesian game setting to obtain similar types of equilibria. In such a game, a strategy s_i of player i is a mapping from his type to an action, i.e. $s_i : \Theta_i \rightarrow A_i$, and similarly mixed strategies can be defined as probability distributions over the space of pure strategies. In this case, a player's expected payoff given the pure strategies of all agents can be written as $\mathbb{E}_\theta[u_i(s_1(\theta_1), s_2(\theta_2), \dots, s_n(\theta_n)), \theta_i]$. A Bayesian or Bayes-Nash equilibrium (BNE) can then be defined as follows (Mas-Colell et al., 1995):

Definition 2.22. Bayes-Nash equilibrium. A (pure-strategy) *Bayes-Nash* or *Bayesian equilibrium* for the game $\Gamma_B = (N, S, \Theta, pr, u)$ is a profile of strategies (s_1, s_2, \dots, s_n) that constitutes a Nash equilibrium of the game $\Gamma_N = (N, S, \mathbb{E}_\theta[u])$. That is, for every $i = 1, 2, \dots, n$, $\mathbb{E}_\theta[u_i(s_i, s_{-i})] \geq \mathbb{E}_\theta[u_i(s'_i, s_{-i})], \forall s'_i \in S_i$.

Until now, we have presented the basic notions of game theory and its fundamental concept, that of a Nash equilibrium. In what follows, we now describe one of the techniques that have been devised to numerically find such an equilibrium or an approximation of it (ϵ -NE), fictitious play.

2.1.2 The Fictitious Play Algorithm

Finding both pure- and mixed-strategy Nash equilibria is a topic of considerable research and a lot of effort has been taken in being able to do this using numerical techniques. There are currently two strands of literature on this area. First, researchers have devised optimization techniques to find exact or approximate Nash equilibria, starting with the celebrated Lemke-Howson algorithm by formulating the problem as a linear complementarity one (Von Stengel, 2002). However, finding Nash equilibria has been proved to be computationally hard, even in the simplest two-player case (Chen and Deng, 2006; Daskalakis et al., 2006).

Given the exponential time it sometimes takes to find a Nash equilibrium, the other approach taken is the development of learning algorithms that converge to exact or ϵ -NE. The main three categories of such learning algorithms are fictitious play, partial best response and replicator dynamics. However, in what follows, we just focus on the first, since this is the technique that we use in some of the settings to compute the equilibria where theoretical results are not available. We refer the reader to (Fudenberg and Levine, 1998) for a concise introduction to the theory of learning in games.

Fictitious play is the oldest and probably most well-studied learning algorithm (Brown, 1951; Robinson, 1951). In a two-player fictitious play, each player believes that the opponent is using an unknown but stationary mixed strategy. At each time-step, t , a player then keeps track of the sequence of the opponent's actions up to $t - 1$ and best responds to the observed opponent strategy distribution. This distribution is assumed to be uniform over the actions observed so far, i.e. the player best responds to the empirical frequency of opponent actions, called the *empirical distribution*. Hence, in this class of learning algorithms, players best respond myopically, disregarding the effect of their choices on the opponents' future play. In more detail, each player i has a function of initial weight, $\kappa_0^i : S^{-i} \mapsto \mathbb{R}^+$, exogenously given, which represents his initial beliefs about the opponent and which is updated every time the opponent plays as follows (Fudenberg and Levine, 1998):

$$\kappa_t^i(s^{-i}) = \kappa_{t-1}^i(s^{-i}) + \mathbb{1}_{\{s_{t-1}^{-i} = s^{-i}\}} \quad (2.6)$$

thus giving a probability of opponent play for each of her strategies:

$$\gamma_t^i(s^{-i}) = \frac{\kappa_t^i(s^{-i})}{\sum_{\tilde{s}^{-i} \in S^{-i}} \kappa_t^i(\tilde{s}^{-i})} \quad (2.7)$$

Given this, in fictitious play, a player plays a best response to these beliefs, $p_t^i(\gamma_t^i) \in BR^i(\gamma_t^i)$. This is a simple rule that assumes stationarity of strategies from the players' point of view, so myopic best response is consistent with the players' beliefs. This assumption is not in general realistic. However, it has been shown that, if a strategy

profile constitutes a strict Nash equilibrium and is played at some time, t , then it will be played in all subsequent steps (Fudenberg and Levine, 1998), and this strategy profile is called a steady or absorbing state (Shoham and Leyton-Brown, 2008). The relation of fictitious play and Nash equilibria is established in the following theorem (Shoham and Leyton-Brown, 2008):

Theorem 2.23. *If the empirical distribution of each player's strategies converges in fictitious play, then it converges to a Nash equilibrium.*

Fictitious play has been proved theoretically to always converge to a NE for two-player games that are zero-sum or solvable by iterated elimination of strictly dominated strategies, or when these are potential games, or $2 \times n$ and have generic payoffs³ (Shoham and Leyton-Brown, 2008; Fudenberg and Levine, 1998). In all other cases, fictitious play can converge but there are no theoretical guarantees for it. One famous example of non-convergence is that of Shapley (1964) who has shown cyclic behavior in a modification of the rock-scissors-paper game. Recently, Conitzer (2009) has shown that fictitious play is guaranteed to converge to an ϵ -NE with $\epsilon = \frac{t+1}{2t}$, t being the number of first time-steps for which both players uniformly randomize over their actions, for a two-player normal-form game where the players' utilities lie in $[0, 1]$.

Having introduced the basic notions of game theory and the fictitious play algorithm, we now continue by shortly presenting the foundations of mechanism design, the area of games with private information where the designer of the mechanism designs the structure of their payoffs.

2.1.3 Mechanism Design

Mechanism design is a sub-field of game theory that studies how social solutions with good system properties can be implemented when aggregating individual preferences that are privately known to each agent. Mechanism design deals with the problem of designing the rules of these social systems so that some desirable objectives are met. Some of these objectives can be expressed via the *social choice function*, which selects optimal outcomes given agents' private information (i.e. types) (Mas-Colell et al., 1995; Parkes, 2001):

Definition 2.24. Social choice function. A social choice function $scf : \Theta_1 \times \Theta_2 \times \dots \times \Theta_n \rightarrow \mathbb{O}$ chooses an outcome for each possible profile of the agent's types $\theta = (\theta_1, \theta_2, \dots, \theta_n)$. An element in the set of possible outcomes \mathbb{O} may define an allocation of items or a task assignment, a public good alternative, an elected committee or a candidate, or another social decision, depending on the problem in question.

³The definition of genericity in payoffs is out of the scope of this thesis.

⁴We consider an additive approximation.

Mechanism design is the art of engineering mechanisms. But what is a mechanism? Informally, a mechanism includes the possible strategies to players as well as the technique to select an outcome based on agents' actual strategies. More formally (Parkes, 2001):

Definition 2.25. Mechanism. A *mechanism* $M = (S_1, S_2, \dots, S_n, g^*(\cdot))$ includes the set of strategies S_i available to each agent i , and an outcome rule $g^* : S_1 \times S_2 \times \dots \times S_n \rightarrow \mathbb{O}$, such that $g^*(s)$ is the outcome implemented by the mechanism for the strategy profile $s = (s_1, s_2, \dots, s_n)$.

We say that a mechanism implements a social choice function $scf(\cdot)$ if the mechanism's game equilibrium outcome is a solution to $scf(\cdot)$ for each possible type profile of the agents $\theta = (\theta_1, \theta_2, \dots, \theta_n)$, as stated in the following definition (Mas-Colell et al., 1995):

Definition 2.26. Implementation. The mechanism $M = (S_1, S_2, \dots, S_n, g^*(\cdot))$ *implements* social choice function $scf(\cdot)$ if there is an equilibrium strategy profile $(s_1^*, s_2^*, \dots, s_n^*)$ of the game induced by M such that $g^*(s_1^*(\theta_1), s_2^*(\theta_2), \dots, s_n^*(\theta_n)) = scf(\theta_1, \theta_2, \dots, \theta_n)$, $\forall (\theta_1, \theta_2, \dots, \theta_n) \in \Theta_1 \times \Theta_2 \times \dots \times \Theta_n$.

In general, the equilibrium in the aforementioned definition can refer to any of the solution concepts stated before, such as Bayes-Nash or dominant-strategy equilibrium.

A common assumption in the mechanism design literature is that the utility functions of agents are *quasi-linear* (Shoham and Leyton-Brown, 2008):

Definition 2.27. Quasi-linear utility. A quasi-linear utility function for agent i with type θ_i in an n -player game, when the set of outcomes is $\mathbb{O} = L \times \mathbb{R}^n$ for a finite set L , has the form $u_i(o, \theta_i) = v_i(x, \theta_i) - f_i(p_i)$, where $o = (x, p) \in \mathbb{O}$ is an element of \mathbb{O} defining a choice $x \in L$ from a discrete choice set, $v_i : L \times \Theta_i \rightarrow \mathbb{R}$ is the valuation function, expressing his value for a choice $x \in L$, $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly monotonically increasing function, and p_i is the payment for the agent when p is the vector of all agent payments.

Quasi-linear utility functions allow for the separation of the outcome of a social choice function and an outcome rule into a choice $x \in L$ and a payment $p_i(\theta)$ to be made by each agent i .

As was previously stated, we are interested in designing mechanisms with desirable properties. We now focus on the most important and general of these properties that will be necessary for the following sections. The first of them is direct revelation. In this type of mechanisms, agent i 's strategy is to express his reported type $\hat{\theta}_i = s_i(\theta_i)$:

Definition 2.28. Direct-revelation mechanism. A direct-revelation mechanism $M = (\Theta_1, \Theta_2, \dots, \Theta_n, g^*(\cdot))$ restricts the strategy set S_i to $\Theta_i, \forall i$ and has outcome rule $g^* : \Theta_1 \times \Theta_2 \times \dots \times \Theta_n \rightarrow \mathbb{O}$, which selects an outcome $g^*(\hat{\theta})$ based on reported types $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_n)$.

Another significant property of some mechanisms is *incentive compatibility* (IC), according to which it is optimal for the agents to reveal their preferences truthfully in equilibrium, i.e. $s_i(\theta_i) = \theta_i \forall \theta_i \in \Theta_i$ for agent i . This type of strategies is called *truth-telling* or *truth-revealing*. This is crucial in many cases, as the designer of the mechanism can make agents' expected behavior to be according to his objectives and, moreover, knowing the valuations is many times the only way to evaluate the properties of the outcome. If we restrict our solution concept to dominant strategies⁵, then strategy-proofness can be defined as follows:

Definition 2.29. Strategy-proofness. A direct-revelation mechanism M is *strategy-proof* or *dominant-strategy incentive-compatible* if truth-revelation is a dominant strategy equilibrium.

Importantly, when designing a mechanism, we can restrict our attention to these truth-revealing direct-revelation mechanisms; this is due to a theorem called the *revelation principle* (Mas-Colell et al., 1995; Parkes, 2001), although details are out of the scope of this thesis. Finally, another important property of a mechanism is *individual rationality* (IR). A mechanism is individually rational if agents yield as much utility in expectation from participating in the mechanism than not taking part in it. Formally (Parkes, 2001):

Definition 2.30. Individual rationality. A mechanism M is (interim) individually rational if for all preferences θ_i it implements a social choice function $scf(\theta)$ with $\mathbb{E}_{\theta_{-i}}[u_i(scf(\theta_i, \theta_{-i}))] \geq \bar{u}_i(\theta_i)$, where $\bar{u}_i(\theta_i)$ is the expected utility for non-participation (usually zero).

From the perspective of the society, it is often desirable to achieve (*allocative*) *efficiency*, that is, to maximize total value over agents. First, let us define the corresponding social choice function:

Definition 2.31. Allocatively efficient social choice function. Social choice function $scf(\theta) = (x(\theta), p(\theta))$ is allocatively efficient if for all preferences $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ $\sum_{i=1}^n v_i(x(\theta), \theta_i) \geq \sum_{i=1}^n v_i(x'(\theta), \theta_i), \forall x' \in L$.

Consequently, an allocatively efficient mechanism is one that implements an allocatively efficiently social choice function $scf(\theta)$. Such a mechanism is said to *maximize the social welfare*, where the latter equals the sum of all agents' valuations. Formally:

Definition 2.32. Social welfare. The social welfare of a choice $x \in L$ is defined as the sum of all agents' valuations for x , i.e. $\sum_{i=1}^n v_i(x(\theta), \theta_i), \theta = (\theta_1, \theta_2, \dots, \theta_n)$.

A well-known family of direct mechanisms with all the previous properties (allocative efficiency, strategy-proofness, individual rationality) is the *Vickrey-Clarke-Groves* (VCG)

⁵Similar definitions exist for other solution concepts.

family (Mas-Colell et al., 1995; Parkes, 2001). For this type of mechanisms, we consider agents with quasi-linear utility functions. In a VCG mechanism, each agent i submits his (reported) type, $\hat{\theta}_i = s_i(\theta_i)$, and then the choice rule computes the corresponding optimal choice, x^* , based on agents' reported type profile $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_n)$, that maximizes the total reported value over the agents:

$$x^*(\hat{\theta}) = \arg \max_{x \in L} \sum_{i=1}^n v_i(x(\hat{\theta}), \hat{\theta}_i) \quad (2.8)$$

The payment, p_i , of agent i according to this class of mechanisms is then:

$$p_i(\hat{\theta}_{-i}) = h_i(\hat{\theta}_{-i}) - \sum_{j \neq i} v_j(x^*(\hat{\theta}), \hat{\theta}_j) \quad (2.9)$$

where $h_i : \Theta_{-i} \rightarrow \mathbb{R}$ is an arbitrary function.

A special case of the VCG family is the Pivotal mechanism (Clarke, 1971) where $h_i(\cdot)$ takes the form:

$$h_i(\hat{\theta}_{-i}) = \sum_{j \neq i} v_j(x_{-i}^*(\hat{\theta}_{-i}), \hat{\theta}_j) \quad (2.10)$$

where $x_{-i}^*(\hat{\theta}_{-i})$ is the optimal choice without the agent i in the mechanism: $x_{-i}^*(\hat{\theta}_{-i}) = \arg \max_{x \in L} \sum_{j \neq i} v_j(x, \hat{\theta}_j)$. In this way the mechanism achieves individual rationality and maximizes agent payments to the designer.

The most crucial property of the VCG family is that an agent's payment does not depend on his reported type but only on the others' reported types, thus providing incentives to the agent for truthfully expressing his private information, i.e. it is dominant-strategy incentive compatible (DSIC). After this (very) short introduction to the field of mechanism design, we now introduce the main application area of this work, auction theory, which is a special area of mechanism design where there is a seller.

2.1.4 Auction Theory

According to McAfee and McMillan (1987), an auction can be defined as “a market institution with an explicit set of rules determining resource allocation and prices on the basis of bids from the market participants” which naturally arises in settings with asymmetries of information between sellers and buyers of goods. From a game theoretic perspective, auctions are usually considered as games of incomplete information given the fact that valuations for the goods to be traded are, in general, private information to participants. There are two characteristics that distinguish auctions from other mechanisms, namely, that outcomes are reached based on information elicitation from the bidders, and the fact that these institutions are *anonymous*, meaning that all bidders are treated in the same way, so prices are only based on bids and not on their identities

(Krishna, 2010). Usually, it is assumed that valuations are independent between bidders (*independent private values* (IPV) setting), i.e. that each bidder receives a valuation which is independently drawn from a commonly-known distribution function. However, in some cases, valuations might be influenced by the information of opponent bidders (called *interdependent values* setting) or the valuations could be equal and unknown to all bidders (*common value* (CV) setting). A key concept in the latter setting is that of the “*winner’s curse*”; the winner always has the highest estimation among all participants and thus in many cases he will pay more than the actual value of the good. In this section, we focus on single-object, single-unit symmetric IPV auctions which is the setting studied throughout this thesis. For an in-depth introduction to auction theory, we refer the reader to the excellent textbooks of Krishna (2010); Menezes and Monteiro (2005); Milgrom (2004), and Cassady (1967) for a field study.

There are four widely-used formats of auctions, namely the *English*, the *second-price sealed-bid* (SPSB) (or *Vickrey* after the name of its inventor⁶ who is also the first researcher that has formally analyzed auctions (Vickrey, 1961)), the *Dutch* and the *first-price sealed-bid* (FPSB) auction (Krishna, 2010). In an English auction, an auctioneer initiates the auction by announcing a very low price that is steadily increased until there is only one bidder who is willing to buy the good⁷. English auctions are often used to sell pieces of art and are the most commonly-used form of auctions. On the other hand, in a Dutch auction, the auctioneer announces an artificially high price that is continuously lowered until a bidder declares his interest in buying the item at the current price. Applications of this auction include the selling of flowers in the Netherlands, as well as fishes in other countries such as Australia, Spain and France. An FPSB (or sealed-bid tender) auction requires bidders to submit bids in sealed envelopes to the auctioneer, and then the winner (i.e. the owner of the highest bid) pays the auctioneer the price of his bid. This is the most widely-used form of procurement auctions. However, probably the most theoretically important type of auctions is the Vickrey auction; in this auction, bidders also submit sealed bids, but the winner pays the second-highest bid instead of his own bid. The importance of this auction lies in the fact that it is a special case of the general VCG family of mechanisms, enjoying all its desirable properties (as described in Section 2.1.3). This is the main type of auction used to trade display ads. These four types of auctions are pairwise equivalent under some conditions. More specifically, an FPSB auction is strategically equivalent to the Dutch auction, and the English auction shares the same optimal strategies with Vickrey auction when valuations are independent (IPV) or when there are only two bidders (weak equivalence). More specifically, in a symmetric IPV FPSB or Dutch auction with n bidders whose private valuations are i.i.d. drawn from the strictly increasing cumulative distribution F with support in $[0, \omega]$, the equilibrium bidding strategy, $\beta(\cdot)$, of a bidder with private valuation v will

⁶It has been claimed that stamps have been traded via SPSB auctions long before Vickrey’s analysis though.

⁷Actually, this specific variation where there are no bid jumps is known as the Japanese auction.

be (Krishna, 2010):

$$\beta(v) = v - \frac{\int_0^v F^{n-1}(y)dy}{F^{n-1}(v)} \quad (2.11)$$

i.e. bidders shade their bids, bidding below their valuations. In contrast, bidders in IPV Vickrey or English auctions have a weakly dominant strategy of bidding their true valuations (Krishna, 2010).

Moreover, according to a well-known result, called the revenue equivalence principle, the expected revenue to the seller from any of these auctions is the same for independent private valuations. Formally (Riley and Samuelson, 1981; Myerson, 1981; Vickrey, 1961; Milgrom and Weber, 1982):

Theorem 2.33. *Revenue equivalence principle.* *If valuations are independent and identically distributed and all bidders are risk neutral (i.e. have quasi-linear utility functions), then any symmetric and increasing equilibrium of any standard auction⁸, such that the expected payment of a bidder with zero valuation is zero, yields the same expected revenue to the seller.*

It is important to note that this principle holds only under the assumptions of independence, risk neutrality, identical valuation distributions⁹ as well as absence of budget constraints for the bidders. If any of them is violated, then the principle is no longer valid. This is due to the allocation of the good; Myerson (1981) has first shown that all auctions that implement the same allocation rule should have the same expected payments.

As in any mechanism design problem, different objectives might be required by the designer. Nevertheless, probably the most commonly-found are (*allocative*) *efficiency* (or, equivalently, *social welfare*) and *optimality*. As already mentioned, an auction is efficient if the good is allocated to the bidder that values it most, which is indeed the case with the four aforementioned auction formats. On the other hand, optimal auctions maximize the auctioneer's revenue. Although not always, these two objectives are often in conflict (Krishna, 2010). Other common objectives include simplicity of the rules as well as prevention of collusion among bidders (see Section 2.4).

One of the tools that auctioneers usually implement to achieve optimality are reserve (or floor) prices. These are minimum acceptable prices attached to the good so as to guarantee a minimum desirable revenue for the seller. This of course requires that there is at least one bidder who is willing to bid no less than these prices. For a Vickrey

⁸An auction is called standard if it dictates that the good is awarded to the bidder with the highest bid.

⁹Myerson (1981) has shown that the revenue equivalence still holds for asymmetric bidders, i.e. whose private valuations are independently drawn from different distributions, but only if the probability of winning the auction is independent of the auction type for any realization of the valuations. This is not typically true in asymmetric auctions (Fibich et al., 2004).

auction with independent private valuations, where a reserve price is set at r , the ex-interim expected payment of a bidder whose valuation is $v \geq r$, is (Krishna, 2010):

$$p(v, r) = r \cdot F^{n-1}(r) + \int_r^v y(n-1)F^{n-2}(y)f(y)dy = vF^{n-1}(v) - \int_r^v F^{n-1}(y)dy \quad (2.12)$$

The ex-ante expected payment of such a bidder is then:

$$\begin{aligned} \mathbb{E}[p(v, r)] &= \int_r^\omega p(v, r)f(v)dv = \\ &= \frac{1}{n} \int_r^\omega vf_1^{(n)}(v)dv - \int_r^\omega f(v)\left(\int_r^v F^{n-1}(y)dy\right)dv \end{aligned} \quad (2.13)$$

where $f_1^{(n)}(x) = nF^{n-1}(x)f(x)$ is the probability density function (p.d.f.) of the highest-order statistic¹⁰ among n samples i.i.d. drawn from F . Then the ex-ante expected revenue of the auctioneer with a valuation of v_0 for the good in an auction with n such bidders will be equal to n times this expected payment plus v_0 if there is no sale (Riley and Samuelson, 1981):

$$revenue(r) = v_0F^n(r) + \int_r^\omega vf_1^{(n)}(v)dv - n \int_r^\omega f(v)\left(\int_r^v F^{n-1}(y)dy\right)dv \quad (2.14)$$

Integrating by parts yields:

$$\begin{aligned} \int_r^\omega f(v)\left(\int_r^v F^{n-1}(y)dy\right)dv &= \int_r^\omega F^{n-1}(v)dv - \int_r^\omega F^n(v)dv = \\ &= \frac{1}{n} \int_r^\omega f_1^{(n)}(v) \frac{1-F(v)}{f(v)} dv \end{aligned} \quad (2.15)$$

Hence the former equation yields:

$$revenue(r) = v_0F^n(r) + \int_r^\omega f_1^{(n)}(v)\left(v - \frac{1-F(v)}{f(v)}\right)dv \quad (2.16)$$

where the function $\phi(v) = v - \frac{1-F(v)}{f(v)}$ is known as the virtual valuation function, the difference between the valuation and the multiplicative inverse hazard rate, which can be translated as the auctioneer's marginal revenue from a bidder¹¹. Taking the first-order condition (FOC) on this equation yields the equation that the auctioneer's optimal reserve price¹², r^* , should solve (Riley and Samuelson, 1981):

$$r^* = v_0 + \frac{1-F(r^*)}{f(r^*)} \quad (2.17)$$

which is independent of the number of bidders. This solution is unique if the hazard rate of the distribution function, $\frac{f(v)}{1-F(v)}$, is strictly monotone increasing.

¹⁰Auction theorists call this the first-order statistic.

¹¹We refer the reader to Bulow and Roberts (1989) for an explanation of the connection of an auctioneer's revenue maximization and a monopolist's third-degree price discrimination.

¹²This is the reserve price that maximizes the auctioneer's revenue.

For an FPSB auction, the introduction of the reserve price naturally induces a different bid shading for bidders, who are bidding more aggressively compared to a setting with no reserve prices. Indeed, given that $\beta(r) = r$, the equilibrium bidding function for a bidder with valuation $v \geq r$ in this case will be:

$$\beta(v) = v - \frac{\int_r^v F^{n-1}(y)dy}{F^{n-1}(v)} \quad (2.18)$$

After our short introduction to the field of game theory and, more specifically, to the areas of mechanism design and auction theory, in the following section, we provide a survey of the literature on our main application area, advertising exchanges, emphasizing on its auction theoretic side, since this is the most relevant to the work within this thesis.

2.2 Online Advertising Exchanges

In this section, we provide a review of the relevant literature on ad exchanges, which is the main application area of this thesis. As stated in Chapter 1, the display advertising marketplace is a complex system, comprising two types of markets: a long-term market for guaranteed delivery where trading is performed via bilateral negotiations, and a short term one for the remnant inventory which is currently performed via ad exchanges, implementing some of the aforementioned auction types. The dominant form of pricing in the latter markets is cost per thousand impressions¹³ (CPM). In general, advertisers prefer paying per click (CPC pricing), given that they require visibility and good-quality inventory, whereas publishers prefer the current CPM model that reduces their risk. On the other hand, the latter pricing model induces high-quality advertisements (i.e. advertisements leading to clicks), whereas the former creates incentives for good traffic to the publisher (McAfee and Vassilvitskii, 2012). This is because, in a CPM model, advertisers pay irrespective of the users' interest on the ad, so are incentivized to provide relevant advertisements to increase their revenue. In contrast, in a CPC model, publishers take the responsibility of showing the advertisements to interested users since otherwise they receive zero revenue by getting no clicks. Advertisers can either buy advertisements in CPM or CPC (and less often cost-per-acquisition (CPA)), where the conversion is made in terms of expected click-through rate ($\mathbb{E}[\text{CTR}]$), the probability of a click given an impression, so the term effective CPM (eCPM) is often used instead ($e\text{CPM} = \mathbb{E}[\text{CTR}] \cdot \text{CPC} \cdot 1000$). This creates a number of complicating issues, such as the bias of the estimation (Edelman and Lee, 2008; Bax et al., 2012; Shanahan and Kurra, 2011). Hence, it is common for specialized intermediaries to undertake this risk of conversion, arbitraging between the two pricing formats (Cavallo et al., 2012). Another interesting pricing model was recently proposed by Goldstein et al. (2012), who consider time-based pricing of display advertisements and show that, under general assumptions, this will increase

¹³This is for historical reasons where ads have been traded in bulk. Ad exchanges allow for individual impression pricing, leading to an equivalent cost-per-impression (CPI) pricing model.

revenue for publishers and provide better targeting (i.e. increased value) for advertisers as well. However, it is still not clear which type of pricing will prevail. Hence, to avoid all aforementioned issues, CPM pricing is adopted throughout the remainder of this thesis. We now continue by presenting a generic model of ad exchanges.

2.2.1 The AdX Model

Although there is no standard format for the operation of ad exchanges (e.g. Yahoo!'s Right Media ad exchange organizes its market with the use of constrained path optimization algorithms¹⁴ (Lang et al., 2011)), Muthukrishnan (2009) has presented a general model to describe the operation of ad exchanges for a single advertising slot, which he calls the *AdX model*, illustrated in Figure 2.3. This is the basis of the model we have extended and used for our study (see Chapter 3 for details). The sequence of actions in AdX proceeds as follows:

1. User enters a web page belonging to publisher P .
2. P provides to the ad exchange E information about the web page, the user, as well as the publisher's minimum accepted price, r , for the ad to be placed.
3. E contacts ad networks $adn_1, adn_2, \dots, adn_{|adn|}$ with this information, which might mask some of the attributes of the web page or the user based on the contract with the publisher. Moreover, E will typically demand a higher reserve price than what P has asked.
4. Each adn_i returns a bid¹⁵ (b_i, d_i) to E on behalf of his more suitable (winning) advertiser, based on some local auction or other allocation mechanism, which comprises a price, b_i , and an ad, d_i , to be shown. However, ad networks have the possibility of not submitting a bid.
5. E determines a winner, i^* , and its price, c_i^* , such that $r \leq c_i^* \leq b_i^*$ via an auction and informs the winning ad network.
6. P shows the winning ad within the web page to the user for the current impression.

The AdX model is quite generic and many research questions remain to be answered. Some of those questions were pointedly stated in the same paper by Muthukrishnan (2009). However, since the focus of this work is on the auctions involved, in the following subsection we focus on research related to the design of auctions of for both the ad exchange and the ad networks.

¹⁴More recently, Right Media has implemented a mixture of first- and second-price auctions for its operation (Johnson, 2013).

¹⁵This model presupposes that each ad network submits a *single* bid.

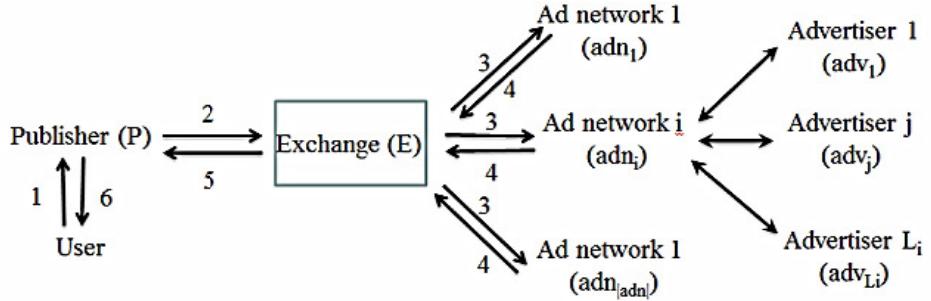


FIGURE 2.3: The AdX model.

2.2.2 Auction Design Issues

The vast majority of current ad exchanges implement variants of the SPSB auction with an appropriately set reserve price¹⁶. However, advertisers do not, in general, bid directly in the ad exchange but participate via a number of demand-side intermediaries, described as ad networks in the above-mentioned AdX model. Designing an appropriate auction for both the ad exchange and the demand-side intermediaries is a challenging task, especially given the volume and time constraints of the system (billions of impressions are traded every day, whereas each auction must be conducted in less than 100 milliseconds). Two main issues that should be taken into account when designing an auction for the exchange are as follows.

First, the ad exchange is in principle a matching platform and hence must optimize its operations so that these satisfy the short- and long-term objectives of both the publishers, the advertisers (and their representatives) in terms of relevance and revenue or profit, as well as maximizing its own long-term profit. Other desirable properties of an ad exchange include (McAfee and Vassilvitskii, 2012): (i) efficiency, i.e. maximizing the total value produced in the exchange, (ii) expressiveness, i.e. devising suitable bidding languages that describe the preferences of the participants in the best possible manner, (iii) neutrality, i.e. being fair towards all participants (being in accordance with the objectives of the exchange) and (iv) strategic simplicity, which makes it simple and less costly for the participants to enter the exchange.

Second, another crucial issue is the existence of the ad networks which extract some of the potential revenue from the ad exchange (an effect called double marginalization (Tirole, 1993)). In this vein, one of the open problems proposed by Muthukrishnan (2009) is how the ad exchange can extract a revenue close to the *book value*, i.e. the second among all advertisers' bids, given that the former does not have full information about the entire book of bids. This was recently studied in (Mansour et al., 2012), where the authors

¹⁶Variations include the OSP auction in Google's DoubleClick, explained later, or mixtures of first-price and second-price auctions using what is called hard and soft floor (reserve) prices (Yuan et al., 2013) or first-price/second-price discriminating auctions based on the type of bids (fixed or real-time) (Johnson, 2013).

describe the auction they have devised, and which is currently implemented in Google’s DoubleClick ad exchange. More specifically, the authors propose a variant of the Vickrey auction with a reserve price, called the *Optional Second Price* (OSP) auction. In this auction, ad networks submit a mandatory bid and an optional, second bid (lower than or equal to the first one). Then a regular Vickrey auction with a reserve price is run among these bids to determine the winner and the price to be paid. Although this might not be the case, the authors note that in many cases ad networks are contractually obliged to submit only truthful reports. OSP allows for a variety of auction formats for the ad networks, especially a second-price logic, as the one implemented by Google AdWords. The authors provide a simple analysis where K advertisers are uniformly assigned at random to one of $|adn|$ ad networks and show that the expected loss of the ad exchange from an ad network not truthfully reporting its second price is $\mathcal{O}(\frac{1}{K|adn|})$ of the book value, which is small as long as no network has a significantly high share of the market.

Feldman et al. (2010) also consider the problem of auction design at the ad exchange as well as the ad networks, when the latter compete for a single good in a Bayesian setting by running local auctions to determine a single bid to send at the exchange. This is the most closely-related work to this thesis, where it is assumed that advertisers are allocated to ad networks such that each ad network has exactly the same number of advertisers and advertisers cannot change their allocated ad network (i.e. remain *captive* in their ad network). The authors assume that both the ad exchange (called the *center* in their terminology) and the ad networks (called the *intermediaries*) implement Vickrey auctions with reserve prices, where the ad networks decide about the allocation and payment in their local mechanisms before the center’s auction. Hence the ad networks’ auctions are *contingent*, meaning that the ad network trades with its advertisers the obligation to deliver the ad at a specific price only if it wins at the central auction. The authors start their analysis by considering a single advertiser per ad network and prove that the only DSIC mechanism for them is to offer a take-it-or-leave-it price to their advertiser. They also prove that there is an MSNE for the ad networks where they offer such prices drawn randomly from an interval according to a derived probability distribution function. They show that the upper bound of this interval increases with the number of the ad networks and the mass of the probability distribution function gets closer to this upper bound under the same condition. On the other hand, the reserve price of the ad exchange decreases but remains strictly positive as the number of ad networks increases. Finally, the authors extend their results for the case of ad networks with more than one advertiser per network, where the number of advertisers is the same for each of the networks. They show that ad networks will use randomized reserve prices in equilibrium but do not manage to explicitly characterize the equilibrium distribution.

Ghosh et al. (2013) have considered a similar problem, albeit in a complete-information setting, where buyers compete for a good via a number of intermediaries in a series of levels on a tree, participating in a series of multiplicative revenue-sharing bargains, so

that the winning bidder at level ℓ gets a revenue share of w_ℓ and submits $(1 - w_\ell)$ of the bid to the next level until the root of the tree, i.e. the seller, is reached. These correspond to fee-based mechanisms that intermediaries implement sometimes. The authors prove the existence of a unique fixed point that can be efficiently (i.e. in polynomial time) calculated by reducing the problem to a path-bargaining one.

Related to this, Gomes and Mirrokni (2014) also study the design of the auction at the exchange when a publisher is present. However, they do not consider the presence of intermediaries. In more detail, the authors study the problem of designing an optimal revenue-sharing double auction for the ad exchange auctioneer, given that, in practice, ad exchanges often forward publishers' reserve prices and get a fixed share of the revenue. The authors find the optimal revenue-sharing mechanism. In doing so, they deal with the possibility of competition by other ad exchanges in their design by expressing the exchange's objective function as a weighted sum of its revenue and the publisher's profit. They then compare this mechanism with a revenue-sharing mechanism with fixed shares, as is common in practice, and find that such a scheme is optimal only when the publisher's distribution of opportunity cost has a power form, which can be translated to a constant elasticity of demand. Finally, they characterize optimal revenue-sharing mechanisms where the exchange's objective is the maximization of the seller's profit subject to a minimum revenue on its side, and find that such an optimization cannot be implemented by constant revenue shares. The use of a double auction for ad exchanges has also been proposed in (Deng et al., 2014).

For a similar setting, where ad networks and advertisers are treated in the same way (as bidders), Celis et al. (2011, 2012) consider a different type of mechanism for the exchange, called the Buy-It-Now or Take-A-Chance (BIN-TAC) mechanism, which is a hybrid of an auction and a take-it-or-leave-it price and which they claim to be truthful in expectation. Their argument is that, in reality, valuations are not drawn from a single distribution, but there are high-valued and low-valued advertisers based on the matching between their desired ad and the user or content available, i.e. there is a small probability of matching between an advertiser and a publisher-user pair, due to targeting. In their analysis, they assume two types of bidders, high-valued and low-valued ones, and that the virtual valuation function is increasing over two intervals (corresponding to low and high values). According to the BIN-TAC mechanism, the center first offers a (high) take-it-or-leave-it price to the bidders, tlp ; if more than one agree on this price, then a Vickrey auction with the former price as a reserve price is run among them. However, if no bidder agrees, then the top di bidders are considered and the slot is given to one of these bidders uniformly at random at the $(di + 1)$ -st price (a second reserve price, r , can be defined for this auction). The authors characterize the equilibrium of this mechanism: there is a unique pure-strategy BNE with a threshold so that bidders with valuations above it will go for the BIN option. Moreover, advertisers always bid their true valuations in the subsequent stage, if the good is not traded in this BIN stage. Although this

auction is not the optimal mechanism analyzed by Myerson (1981), it provides similar rules to the former in the setting under investigation. Finally, experiments on real data from Microsoft’s AdECN ad exchange show that, after determining the best di, r, tlp , BIN-TAC achieves 4.5% more revenue than the “optimal” Vickrey auction with reserve price, increasing at the same time consumer surplus by 11%.

Balseiro et al. (2013) have studied the auction design problem of the exchange in a repeated setting with no intermediaries but where bidders have budgets and stochastically enter and leave the exchange. The authors use a new type of equilibrium that they name the fluid mean-field equilibrium. This is a concept that can be used in games with very large populations, as is the case in ad exchanges. In such games, the agent is assumed to best respond to a mass of other agents instead of treating each of them individually, using a stochastic approximation¹⁷. Balseiro et al. also consider simplified bidding strategies for buyers that are only functions of their valuations, disregarding the history of play. They show that, in this unique fluid mean-field equilibrium, advertisers will shade their bids by a constant factor when their budgets are tight (otherwise they bid truthfully in equilibrium as in the Vickrey auction). They also study the publisher/ad exchange’s problem of optimizing the reserve price, the rate of impressions sent to the exchange and the information disclosed to the advertisers. They find that the reserve price is higher compared to a single-shot setting and that publishers should disclose all information when enforcing this optimal reserve price.

Another interesting design is proposed by Arnosti et al. (2014), who study the problem of auction design in a private ad exchange for a publisher with both brand and performance advertisers. In their setting, the former have contracts for guaranteed delivery, seeking brand recognition, whereas the latter are targeting for immediate interaction (i.e. clicks or conversions), getting the most valuable audience. The authors provide a list of properties that a mechanism should satisfy so as to be fair to both types of clients. More specifically, according to the authors, a qualifying mechanism should be deterministic, strategy-proof, false-name proof (i.e. advertisers should be incentivized not to submit multiple bids) and anonymous among performance advertisers, and only winning bidders should pay. Moreover, they propose a mechanism, called the *modified second-price auction*, that satisfies these properties for valuation distributions that are fat-tailed (such as the power law), in a setting with valuations that are the product of a common random variable (i.e. they are different for every impression) and an advertiser-specific variable. According to this mechanism, the highest performance advertiser’s bid wins and pays the second-highest performance bid, if their difference is above a pre-specified threshold. Otherwise the brand advertiser wins. Finally, they show that this mechanism is also adverse-selection free, meaning that the probability of the brand advertiser winning the impression is independent of the common factor of the performance advertisers’ valuations.

¹⁷For an introduction to this area, we refer the interested reader to (Lasry and Lions, 2007).

Game theorists have sometimes been criticized for setting unrealistic assumptions or information that is not available in real settings. Given this, a number of researchers have worked on the optimization of auctions for an ad exchange using learning-based methods. One such work is that of Cesa-Bianchi et al. (2013) who study the reserve price optimization problem of an ad exchange/publisher who runs a repeated Vickrey auction. In their setting, bidders are symmetric but their number is either unknown or stochastic and the second-highest bid is the only bid revealed given that this is higher than the reserve price (left-censored data). The authors propose a learning algorithm with regret, i.e. the difference between the optimal revenue and the revenue achieved, $\tilde{\mathcal{O}}(\sqrt{T})$, T being the number of auctions. This work has been extended by Mohri and Medina (2014) who also consider user features in their algorithm. A similar problem to that of Cesa-Bianchi et al. (2013) has been studied by Amin et al. (2013) who nevertheless consider a repeated single-bidder posted-price auction where the bidder is strategic. Kanoria and Nazerzadeh (2014) study a similar setting (with respect to the strategic aspects of bidders' strategies) with multiple bidders for repeated auctions where the good is of binary type which is unknown to the auctioneer (only probabilities of each type are available) but known to the bidders and their private valuations are i.i.d. drawn at the beginning and then remain fixed. The authors propose an approximately incentive compatible threshold mechanism where a low reserve price is set until there is a bid above a predefined threshold, at which point a higher reserve price is set. They show that, if the bidders' distribution of private valuations is regular¹⁸, then a Vickrey auction with an a priori set, fixed reserve price is no worse than setting a dynamic reserve price. However, if this is not the case, they show that their threshold mechanism yields higher revenue, albeit in a probably approximately correct (PAC) framework.

The problem of auction design for the ad exchanges has also been studied by Feige et al. (2013) who propose a mechanism to deal with the existence of the demand-side intermediaries in a complete-information single-item setting. The proposed mechanism is a randomized auction where one of the top di bidders (where di is selected by the auctioneer) obtains the good according to some predefined probability mass function and incurs VCG payments. The authors show that, when the advertisers are allowed to submit any number of bids both through intermediaries and directly at the exchange, they prefer bidding through the intermediaries and that the revenue generated is higher compared to the predominantly-used Vickrey auction.

2.3 Auctions Involving Intermediaries

As the previous section depicts, one of the main problems for ad exchanges is the existence of various types of intermediaries. This is not the only area where such intermediaries exist. For this reason, in this section we shortly survey literature on auctions where

¹⁸This means that the c.d.f. is strictly increasing and so is the virtual valuation function.

intermediaries are present. Besides the work of Feldman et al. (2010) on ad exchange auctions with intermediaries, there are two other relevant areas, namely procurement auctions¹⁹ with subcontracting and network resource allocation markets.

Our setting is similar to that of auctions with resale where bidders in an auction resell the won item either to their competitors or to other buyers. There is a significant literature on this topic (e.g. Bikhchandani and Huang (1989); Haile (2003)), however the majority focuses on the reselling to competitor bidders. Reselling to other (non-competitor) buyers seems more relevant to the setting with ad exchanges and demand-side intermediaries. However, in both cases, resale auctions typically take place after the primary auction, inducing different dynamics (Bose and Deltas, 2007).

Auctions with subcontracting are procurement auctions which are typically used by governments and other public organizations to assign public projects, such as highways. Given the size of such projects, often large companies, called the contractors, are unable to take on the whole project but instead assign parts of it to smaller companies, called the subcontractors. Although contractors can implement some other mechanism with the subcontractors, such as bargaining, they often organize local auctions before or after the allocation at the central auction to determine the ones with which they will share the project. Ad exchanges with intermediaries are a limiting case of such auctions where intermediaries, in contrast to contractors, have no valuation for the object. Traditional works in this area deal with the principal-agency relation of a contractor with its subcontractors, focusing on how and when (before or after the main auction) to optimally divide the project (e.g. Kawasaki and McMillan (1987); Maréchal and Morand (2003)). Nevertheless, there are two works which are closely related to the topic of this thesis.

A very relevant work is that of Wambach (2009) who studies the effects of auction design for subcontracting before the main auction, for a single contractor with exclusive, captive subcontractors. More specifically, the author compares FPSB and Vickrey subcontracting auctions with no reserve prices, taking the allocation function at the main auction as exogenous. Translated to our setting, the author shows that the contractor receives higher revenue with an FPSB than with a Vickrey auction, making a connection with auctions where bidders are risk-averse, and justifying this result to the fact that the subcontracting auction is contingent. Finally, the author takes a mechanism design approach to characterize the properties of the optimal contractor's mechanism. Specifically, the mechanism is efficient, the optimal contractor's bid depends only on the highest-valuation subcontractor and this bid is the same as with a bidder whose private valuation is the contractor's private valuation plus the virtual valuation of the highest subcontractor.

¹⁹Procurement auctions are reverse auctions where the lowest bidder wins, since the auctioneer is a buyer.

The second relevant work regarding auctions with subcontracting is that of Watanabe and Nakabayashi (2011). The authors have experimentally studied auctions with subcontracting in a single-object, two-contractor setting with two subcontractors each whose private valuations (costs) are i.i.d. drawn from the uniform distribution. The authors show theoretically and experimentally that Vickrey auctions yield higher profit to the contractors than FPSB auctions when all contractors use the same mechanism and that the latter auctions are more efficient than the former. They also emphasize on the risk aversion of bidders in their experimental setting. Nakabayashi (2010), in a subsequent work, studies the same problem in a more general symmetric setting, where he finds that FPSB auctions yield higher revenue for the contractors than Vickrey auctions, in agreement with the result of Wambach (2009). More specifically, he studies the equilibrium FPSB bidding function of the subcontractors for a standard main auction, showing that the aggressiveness level increases with the number of contractors and decreases with the auction's reserve price. Then, he proves that, if contractors' costs are fixed, the subcontractors' bidding function is non-concave and there is no reserve price, contractors increase their profit by implementing FPSB auctions compared to Vickrey. The author notes that the competition between contractors induces a downward shift of the distribution of bids at the central auction, making a connection to the work of Hansen (1988) on auctions with downstream markets where the quantity is induced by the winning bid. Finally, he also shows that the auctioneer's reserve price is a function of the number of bidders and that the FPSB auction is generally more efficient (this is in agreement with our results in Chapter 4).

Another relevant stream of literature is on the allocation of network (spectrum) resources, such as bandwidth, where there are multiple levels of markets, and the target is to achieve efficiency (Bitsaki et al., 2006). Tang and Jain (2012) have considered such a setting in a complete-information environment, where a seller auctions off a good to some intermediaries who then resell this to their exclusive lower-level bidders, who then do the same, in a tree-like structure, until the final buyers obtain the item. The authors compare FPSB and Vickrey auctions in a setting with a single unit, where they once again confirm the increased inefficiency of second-price auctions, and then continue with auctions involving multiple items where they show that VCG-type auctions, which they call the hierarchical network second-price auctions, can achieve efficiency and budget balance. Iosifidis (2012) considers a similar problem in a Bayesian setting where he proposes a novel mechanism for the case where the central auctioneer aims to maximize efficiency whereas intermediaries are purely profit maximizers. Other issues, such as the stability of tree structures with multiple intermediary levels, have been studied by Polanski and Cardona (2012) for an FPSB auction and uniformly distributed private valuations for buyers, where it was shown that only a single level of intermediaries is stable.

For the ad exchange auctioneers, demand-side intermediaries act as colluding bidders,

since only a single bid is submitted on behalf of their advertisers, thus suppressing competition at the exchange. For this reason, in the next section, we review related literature on collusion in auctions.

2.4 Bidding Rings

One of the most important issues that auction designers should take into account in selecting their mechanisms is the possibility of collusion between the bidders, which is known as *bid-rigging* in the context of auctions (Marshall and Marx, 2012; Hendricks et al., 2013; von Ungern-Sternberg, 1988). Often a subset of bidders form cartels, also called bidding rings (or sometimes “kippers” (Cassady, 1967)), in an effort to depress the closing price and, consequently, capture some of the revenue from the seller – called the collusive surplus, collusive gain or spoils (Hendricks et al., 2013; McAfee and McMillan, 1992). The latter surplus can be positive only when the cartel includes both the highest and second-highest valuation bidders, and is maximized when the former bidder is the one that bids in the main auction. In such a situation, the collusion is said to be *efficient* (Mailath and Zemsky, 1991).

Collusion between bidders can be either *explicit*, where side-payments between the ring members are permitted (strong cartels), or implicit, also known as *tacit collusion*, where bidders coordinate indirectly on their actions, using some bid rotation scheme (Comanor and Schankerman, 1976), so as to decrease the probability for the bidding ring to be detected by the auctioneer or other antitrust authorities (weak cartels). This section focuses on the former type of collusion in single-item single-shot IPV auction settings, since this is the main type of auctions implemented in advertising exchanges and there are no legal issues regarding the operation of the demand-side intermediaries²⁰.

The design of the bidding ring comprises three different decisions that need to be made: (i) the amount of the bids that will be submitted in the auction (including the potential winning bids, known as the serious bids, along with other, non-serious, bids that might be submitted so that the cartel cannot be detected or for enforcing the ring’s agreement); (ii) the allocation of the won items to the ring members; (iii) the way that the collusive surplus will be divided between the members of the ring for strong cartels. In line with the classical mechanism design approach, these decisions must take into account the incentive compatibility and individual rationality constraints for the ring members along with an additional “no-cheat” constraint; ring members should be incentivized to bid in the auction according to the ring’s directions (Marshall et al., 2012). Often the mechanism is assumed to be implemented by the ring’s “center”, a coordinating device that ensures that the rules of the ring are enforced as agreed, acting as a banker. One

²⁰For a general study on collusion, including tacit collusion in repeated auction settings, we refer the reader to (Marshall and Marx, 2012) and references therein.

of the most frequently-used mechanisms for the determination of winner and payments in single-item settings is for the ring to organize an internal auction before or after the main auction, known as pre- or post-auction knockout (PAKT) (Hendricks and Porter, 1989; McAfee and McMillan, 1992). For repeated settings, bidding rings often use some bid rotation scheme, such as bidding based on the phases of the moon, to determine the winner (Hendricks and Porter, 1989; Marshall and Marx, 2012; Hendricks et al., 2013).

In a simple complete-information setting, when there is no uncertainty about the valuations of all bidders, Marshall et al. (2012) illustrate that in both English, second-price and first-price sealed bid auctions with more than one ring, when no ring includes both the highest and second-highest bid, no surplus can be realized by collusion (the ring is “ineffective”). In the same setting, there is no incentive for the highest-valuation bidder to participate in the first place in such a ring since the side-payment that needs to be paid equals his extra surplus from participation. This effect becomes more apparent as the number of bidders increases, as the highest bidder’s surplus decreases compared to non-cooperative bidding. Also, in FPSB auctions, the collusion is not sustainable since the highest-valuation bidder’s necessary side-payments so that ring members refrain from cheating are too costly.

In his seminal work, Robinson (1984) provides a first attempt to formally study collusion in auctions where, among other results, he shows that English and SPSB auctions are more susceptible to bid rigging than FPSB or Dutch auctions in a complete-information case, both for IPV and CV settings, albeit when cartel membership is exogenously determined. Graham and Marshall (1987) study collusion in single-item SPSB and English auctions where there is a single bidding ring and both the auctioneer and non-ring members are unaware of this. They describe a mechanism whereby the ring members run an incentive-compatible pre-auction knockout auction to determine the bid to be submitted, which is the highest internal bid, and all ring members receive a fixed payment which is an equal share of the collusive surplus (i.e. the difference between the ring’s second-highest bid and the second-highest bid outside the auction, if it is positive, otherwise zero). They show that PAKT achieves budget balance in expectation and that revenue equivalence between the two auction formats still holds in such a collusive environment. Finally, they prove that the reserve price of the auctioneer increases with the number of collusive bidders and that bidders will form an all-inclusive ring in equilibrium. More specifically, if there are n total bidders, $m > 1$ of which form a bidding ring, whose private valuations are i.i.d. drawn from the same distribution, F , with support $[0, 1]$ and positive density $f > 0$, then the auctioneer’s ex-ante expected revenue, if a reserve price, r , is set, is:

$$\begin{aligned} \text{revenue}_{PAKT} = & r\{(n - m)[1 - F(r)]F^{n-1}(r) + [1 - F^m(r)]F^{n-m}(r)\} + \\ & + \int_r^1 y(n - m)[(n - 1)F^{n-2}(y) - nF^{n-1}(y) + F^{n-m-1}(y)]f(y)dy \end{aligned} \quad (2.19)$$

The first term corresponds to the probability of the second-highest bid being less than the reserve price times the reserve price, and the second term is the probability that the opposite happens times the conditional expected second-highest bid given that this is the case. Taking the first-order condition on the aforementioned revenue yields the following expression for the optimal reserve price, r^* :

$$(n - m)[1 - F(r^*)]F^{n-1}(r^*) + [1 - F^m(r^*)]F^{n-m}(r^*) = r^*nF^{n-1}(r^*)f(r^*) \quad (2.20)$$

Graham et al. (1990) extend the aforementioned analysis where nested cartels are also considered, i.e. cartels where a subset of the members form another cartel, participating as a single entity in the original cartel. They first establish an equivalence between a member's surplus and the Shapley value in a complete-information scenario with heterogeneous bidders. Then, they extend their analysis for the incomplete-information case where they assume that knockout auctions take place after the main auction and homogeneous bidders. The authors show that, in this case, members will overbid in equilibrium. Finally, their analysis includes the incomplete-information case of heterogeneous bidders, where they consider a single bidding ring and propose the same second-price PAKT as in (Graham and Marshall, 1987), where they again establish the correspondence between members' side-payments and their ex-ante Shapley value of the auction game. Heterogeneous bidder collusion in SPSB auctions is also studied in (Mailath and Zemsky, 1991), where, using a mechanism design approach, it is shown that an ex-post efficient collusive mechanism²¹ exists both for complete (i.e. all-inclusive) and partial rings, and, using concepts from cooperative game theory, that the payments can be designed in such a way so that once the grand coalition, i.e. an all-inclusive bidding ring, is formed, it will be stable. Finally, the authors confirm that the auctioneer's optimal reserve price increases with the ring size under conditions that guarantee that it is unique and is affected by the size of the ring.

McAfee and McMillan (1992) discuss collusion in an unrepeated single-item IPV FPSB setting, both for weak and strong all-inclusive cartels, i.e. when $m = n$. In the latter, more relevant case, they also describe an efficient PAKT where the highest bidder wins and pays each ring member (including himself) an equal share of the *expected* difference between his valuation and the reserve price, given that his valuation is higher than the reserve price, and also submits a bid equal to the latter price to the auctioneer. This mechanism can be implemented by using an FPSB PAKT or a Vickrey PAKT, and the authors show that no other mechanism can yield higher surplus for the ring. FPSB PAKT allows for ex-post budget balance but bidding strategies are in NE, whereas Vickrey PAKT allows for budget balance only in expectation but is DSIC. They then consider non-inclusive (partial) cartels, where, they show that both members and non-members will randomize their bids when their private valuations are i.i.d. drawn from

²¹A bidding ring is called ex-post efficient if for each vector of valuations, the outcome of the ring formation is Pareto undominated by any other allocation (Cramton and Palfrey, 1990).

a distribution with support $\{0, 1\}$ ²². Leyton-Brown et al. (2002) study the behavior of multiple bidding rings in single-item IPV FPSB auctions where they additionally assume that bidders have probabilistic estimates about the number of bidders and can voluntarily enter in a bidding ring, whereas non-ring members are not aware of the number of bids suppressed by a bidding ring, and it is required that the identity of the winner is announced after the end of the auction. They find that bidders truthfully reveal their valuations to the ring centers (called the coordinators) and both members and non-members benefit from the existence of the bidding rings, however they cannot characterize all resulting BNE of the game²³ and the ring centers are assumed to be incentiveless.

Another interesting work is that by Marshall and Marx (2007) who consider collusion in single-item FPSB and Vickrey auctions with heterogeneous bidders and a partial (less than all-inclusive) bidding ring where potential ring members are common knowledge and decide whether to enter the ring before learning their private valuations. The authors study the possibility of suppressing competition in the ring, given the competition from non-ring members, under two scenarios. In the first one, the ring can only provide bid and payment recommendations to its members but cannot enforce a single bid submission on behalf of the ring (BCM setting). According to the second scenario, the ring can additionally enforce submitting a single bid on behalf of the ring members (BSM setting). It is found that, in the former scenario, there cannot be a mechanism which is incentive compatible, individually rational, budget balanced and ex-post efficient that can enforce such a submission in FPSB auctions. This is because some members have an incentive to bid above their suggested amount and get the item, hence it is shown that at least two bids will be submitted that are very close to each other and higher than the single recommended bid. In contrast, for the BSM setting, rings can suppress within-ring competition for both types of auctions, under individual rationality constraints that need to be evaluated numerically for each private valuation distribution for the case of FPSB auctions. In line with the previous work, Lopomo et al. (2011) show that all-inclusive BSM bidding rings in FPSB auctions are not stable (i.e. they yield the same expected surplus to the colluders) in a simple 2-bidder setting with binary private valuations.

Finally, Laffont and Martimort (1997); Che and Kim (2006); Pavlov (2008) have studied the optimal auction design problem of an auctioneer in the presence of a group of colluders from a mechanism design perspective. They find that, in some environments, the auctioneer can design an optimal mechanism that yields the same revenue as a fully competitive optimal auction.

²²As the authors note, this setting is the same as asymmetric FPSB auctions where it is not, in general, known when equilibria of the bidding function exist.

²³Actually, a BNE is identified where all potential ring members decline the ring's invitation and bid non-cooperatively.

In this section, we shortly reviewed the literature in bidding rings for single-item auctions and strong cartels. As we have shown, most of the literature has focused on the ring's allocation of the bids and side-payments to its members, so that the ring is efficient, stable and budget balanced. However, the vast majority of the work in this area deals with a single bidding ring (with and without competition from non-ring members) and assumes that the ring center acts as a means of coordination. In contrast, in the advertising exchange marketplace, demand-side intermediaries correspond to such ring centers which are, in contrast, purely self-interested and profit-maximizing. Moreover, demand-side intermediaries compete both when bidding in the ad exchange but also in an effort to attract the qualified demand from the advertisers. For this reason, in the following section, we provide a summary of the closely related literature on competition between auctioneers.

2.5 Competing Mechanisms

The majority of studies in mechanism design and auction theory consider the case of a monopolist seller who is auctioning goods to a number of bidders so as to maximize revenue or allocative efficiency. In this case, the optimal reserve price is such that the marginal cost of increasing the reserve price (and hence excluding buyers with low valuations) equals the marginal profit due to extracting more surplus from high-valued buyers (Myerson, 1981). However, much less is known in settings where sellers compete to attract buyers by announcing appropriate mechanisms, a situation that often arises in real markets, such as search engines, financial markets, auction houses, or online auction sites. In contrast to the monopolistic setting, in these cases an increase in the reserve price might not yield enough surplus extraction from the buyers, as the latter now have the opportunity to select another mechanism.

There is relatively little research in the area of competing auctions. The main reasons for the scarcity of results in this area are the intractability of analysis and the fact that, until recently, auctions were standalone events and of limited use (Haruvy et al., 2008). Pai (2010) provides two major explanations for the former fact. First, there is no analog to the revelation principle; even when fixing the mechanisms of others, calculating the revenue of a seller's mechanism requires knowledge of the buyers' selection decisions in equilibrium (i.e. how many buyers will visit the seller and what is the distribution of their valuations, both of which are endogenously determined based on the selection of mechanism). This makes the seller's revenue maximization an optimization problem involving a fixed-point subroutine (see also (Epstein and Peters, 1999; Martimort and Stole, 1999; Peters, 2010; Attar et al., 2011a,b; Peters, 2012)). Second, the design of a mechanism will influence the response of other mechanisms, resulting in an infinite regress whose fixed point is difficult to find and utilize. For these reasons, the vast majority of research on competing auctions is either experimental (Anwar et al., 2006;

Haruvy et al., 2008) or focuses on the two ends of the competition, i.e. duopoly markets and markets with an infinite number of agents. Moreover, often the mechanisms considered are inevitably fixed, transforming the mechanism competition to that of price or quantity competition which are more amenable to analysis. Competition in prices and quantities between firms has a long history with the models of Bertrand and Cournot respectively (Mas-Colell et al., 1995), although the institutions considered are exogenously determined and information about demand is considered to be perfect.

Analysis of competing mechanisms starts with the seminal paper of McAfee (1993) in a multi-period setting with multiple sellers and buyers having single-unit supply and demand, where the latter *single-home* (i.e. can only select one seller at each period). This work is based on a “large market hypothesis”, meaning that the number of agents tends to infinity in the limit. Also, the ratio of buyers to sellers is kept constant and the work focuses on the steady-state result of the competition. The first hypothesis ensures that a change of one mechanism does not influence the available surplus of the buyers not participating in the mechanism and hence the response of the other mechanisms. McAfee shows that there is a unique symmetric pure-strategy equilibrium²⁴ where sellers hold Vickrey auctions with reserve prices that equal their production costs, whereas buyers select one of these auctions uniformly at random.

In a subsequent work, Peters (1997) validates the results of McAfee in a more general setting, relaxing McAfee’s assumptions about symmetric mechanisms and ignorance about a seller’s mechanism effect of deviation on the buyers and opponent sellers. In a follow-up paper, Peters and Severinov (1997) consider a similar setting for Vickrey auctions, where they study two different variations: cases where buyers select an auction before learning their valuation and cases where the selection is done after having acquired such information. The authors express the revenue of the sellers in a closed form for finite populations of agents and then get the limit as this number goes to infinity. They show that, in the former scenario, there exist symmetric equilibria for the sellers with reserve prices between their production costs and the monopolist optimal reserve price. In the latter case, they show that, in their formulation, there will also be a symmetric equilibrium where sellers set zero reserve prices, and buyers randomize equally between the auctions.

Hernando-Veciana (2005) extends the aforementioned results to a finite population of sellers with asymmetric production costs that implement Vickrey auctions²⁵ and can select among a finite set of reserve prices. He shows that there exists a unique MSNE for the buyers’ entry game, involving n_s cut-off points (where n_s is the number of sellers), so that buyers with valuations above each corresponding cut-off point select one of the eligible auctions (i.e. having a reserve price lower or equal than their valuations)

²⁴McAfee considers an equilibrium concept he terms the Competitive Subform Consistent Equilibrium (CSCE).

²⁵He mentions that the results can be generalized to FPSB auctions resulting from the revenue equivalence theorem.

uniformly at random. Moreover, he shows that, when the number of sellers is large enough, there is also a unique symmetric pure-strategy equilibrium in the sellers' game, where reserve prices equal production costs. A crucial assumption for this result is that production costs have the same support as the available reserve prices.

In a more recent study, Virág (2010) generalizes the previous results for finite markets with n_s Vickrey auctions with a continuum of feasible reserve prices, and n_b buyers where, in contrast to all previous works, he relaxes the requirement for the ratio n_s/n_b to be fixed. First, he confirms the result that the buyers' selection game admits a unique MSNE, involving n_s cut-off points, as before. Then, he also shows that there are mixed-strategy equilibria with reserve prices converging to the symmetric production cost, zero, in the limit, once the market is large enough. He argues that this happens because sellers in such a market cannot influence the surplus of buyers and hence should only decrease their reserve prices which will induce more buyer visits. Crucially, he is also able to provide necessary conditions for the existence of a symmetric pure-strategy equilibrium for the sellers that do not depend on the distribution of valuations, by assuming that the lowest valuation is higher than the production cost of the sellers (i.e. positive), in contrast to previous literature. Finally, he illustrates that these symmetric reserve prices quickly converge to zero as the market becomes large, as long as the ratio of buyers to sellers does not converge to infinity fast enough. Valverde (2012) extends these results to heterogeneous goods, showing that the buyers' entry game involves pure cut-off strategies, where buyers either select or do not enter a seller's auction based on their valuation for the item under consideration, and equilibrium reserve prices tend to production costs under the condition that (only) the buyers' population is large enough.

Damianov (2005) reaches a similar conclusion for a finite number of competing sellers (and buyers) implementing general mechanisms, where buyers select their auction before learning their valuation. He shows that the buyers' entry game admits a symmetric MSNE where they uniformly randomize between the sellers, and that the sellers' symmetric mechanism in equilibrium is a Vickrey auction with zero reserve price and an entrance fee which is independent of the number of buyers in the auction.

At the other end of the competing mechanisms spectrum, Burguet and Sákovics (1999) studied the duopoly competition of two Vickrey auctions with reserve prices. They show that the buyers' entry game admits a unique BNE involving a cut-off point, so that buyers with valuations below this point always select the low-reserve auction, whereas buyers with valuations higher than this cut-off point equally randomize between the two auctions. What's more, they show that the sellers' reserve-price-setting game cannot have a symmetric pure-strategy equilibrium, as is the case with large markets, but will involve a mixed-strategy equilibrium with reserve prices above their production costs (i.e. positive). However, they cannot fully characterize it. Finally, the authors extend their results for a larger class of mechanisms, called *quasi-efficient*. These are mechanisms that allocate the good to the buyer with the highest valuation given that buyers'

entry decisions are identical to the ones obtained under Vickrey auctions, hence inefficiencies arise only from the seller withholding the good. For this class of mechanisms, they show that the entry game of the buyers involves two cut-off points, so that buyers with valuations lower than the low cut-off point do not attend any seller's mechanism, buyers with valuations between the cut-off points all select the same seller, and buyers with valuations higher than the high cut-off point randomize equally between the mechanisms. Pai (2009) considers more general mechanisms, incorporating a “*hierarchical allocation rule*” (i.e. the allocation is based on a declared priority on the bidder types, each above a specified threshold), for a duopoly market. More specifically, he considers the best response of an auctioneer when his opponent implements a quasi-efficient mechanism. He shows that the equilibrium strategy of buyers' entry game is as before (Burguet and Sákovics, 1999) and then proves the existence of a symmetric weak-perfect BNE in the sellers' game where both implement a quasi-efficient mechanism when the probability distribution function of buyers' valuations satisfies the monotone hazard rate condition and, additionally, is a weakly decreasing function. As noticed, this means that auctioneers' mechanisms will involve a posted-price component.

Moreover, Gerding et al. (2007) consider the case where sellers offer Vickrey auctions with reserve prices in the presence of a mediator (such as eBay) and can make use of shill bids (i.e. bids submitted on behalf of the seller without buyers knowing it so that selling price is increased). The authors analytically find the equilibrium strategies of the sellers in the case of two auctions (by iteratively discretizing the space of reserve prices) and then implement evolutionary experiments to study the setting with more sellers. They show that sellers have incentives to adopt shill bids which can be deferred by the mediator by charging them appropriate auction fees. In another study, Ellison et al. (2004) study the competition of two mediators with multiple sellers and buyers, where both can select one of the mediators, and buyers learn their valuations after entering the corresponding market. Hence, in this model, participants face network effects, both from their opponents (i.e. higher competition leads to smaller expected profit) and their counterparts (i.e. large numbers of sellers/buyers induces higher expected profit for buyers/sellers respectively). The authors find that under some conditions, mediators with different sizes can co-exist.

In sponsored search auctions, Liu et al. (2008) have studied the competition between two search engines that differ in their ranking rule, which can either be according to bids or according to revenue (i.e. adjusted bids by quality). Moreover, advertisers are split in two categories, low-quality and high-quality ones. The authors find that if the search engines adopt different such rules, the equilibrium behavior of the advertisers depends on the ratio of high- to low-quality advertisers. If this ratio is greater than half, then low-quality bidders go to the bid-based ranking mechanism, and high-quality advertisers will go to each of them with a probability that is a function of the aforementioned ratio. Otherwise, low-quality advertisers prefer the revenue-based ranking mechanism with a

probability which is a function of their private valuation as well as the ratio of the low- and high-quality values selected. In this case, high-quality advertisers always select the search engine with the revenue-based ranking rule.

In line with the previous work, Ashlagi et al. (2011b) consider a setting with two auctions that differ in their click-through rate (i.e. popularity). One of their findings is that, when sellers offer VCG auctions with reserve prices, there is a unique equilibrium for the buyers' selection subgame, involving more complex strategies. More specifically, the equilibrium strategy is uniquely defined by two cut-off points, so that buyers with valuations in the interval defined by the cut-off points follow a strictly mixed strategy, whereas buyers with valuations outside of this interval follow pure strategies. However, none of the previous works considers the problem of competing intermediaries in a non-captive setting, whose presence fundamentally changes the nature of the problem. More specifically, the auctions are no longer independent, as they have to compete additionally as bidders at the central auction for the same good, and so the buyers' intermediary selection affects both the intermediaries' profit as well as the center's revenue. Finally, intermediary auctioneers face similar tradeoffs as the buyers: the higher their profit the smaller is their probability of obtaining the item at the center. In (Ashlagi et al., 2011a), the same authors extend their work allowing for buyer participation costs and capacity differences for generalized second-price search engines. They show that, under this model, large advertisers are likely to multi-home (where the buyers' entry game admits a PSNE with a single cut-off point), and that joining of the two search engines, such as the Microsoft-Yahoo! deal on 2008, may or may not benefit social welfare, based on the difference in capacities and popularities between the search engines.

Motivated by the online display advertising marketplace, Polevoy et al. (2014) extend the work of Emek et al. (2012) for multiple sellers that compete to attract buyers with binary valuations for a set of goods. The authors consider revenue maximizing sellers each of whom has a good (that corresponds to a flow of ads) and must decide what kind of attribute partitions should be revealed to the buyers. They show that the buyers' selection problem is a potential game²⁶ (hence admits a pure NE strategy) where the potential is the social welfare. Then, they study how competition affects social welfare, and they provide tight bounds, showing that competition might minimally increase social welfare but can also lead to significant loss compared to the setting with a monopolist.

Finally, another growing strand of literature deals with the competition between two-sided platforms, such as financial exchanges (Rochet and Tirole, 2006; Cantillon and Yin, 2011). However, the vast majority of this literature emphasizes on the network effects that influence the traders' decision about which platform to select as well as fee-based platforms where the focus is on the structure of the fees. Our model (see Section 3.3)

²⁶A game, $G = (N, S, u)$, is a *potential game* if there exists a function $\Phi : S \mapsto \mathbb{R}$ such that, for all $i \in N$, all $s_{-i} \in S_{-i}$ and $s_i, s'_i \in S_i$, $u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) = \Phi(s_i, s_{-i}) - \Phi(s'_i, s_{-i})$. $\Phi(\cdot)$ is known as the *potential function* (Monderer and Shapley, 1996).

departs from these settings, since intermediaries represent only one side of the market and are assumed to make a profit by pocketing the difference between what they get paid by their advertisers and what they pay at the exchange.

2.6 Summary

In this chapter, we reviewed related literature on auctions for ad exchanges and, more generally, auctions involving intermediaries. In particular, we began by introducing the key notions of game theory, the theory of strategic interaction among agents, and put an emphasis on auction theory, the main area of interest to this thesis.

We then presented related work in the area of online ad exchanges from an auction-theoretic perspective. More specifically, we described the AdX model along with the state-of-the-art literature in the auction design problems for the ad exchange and the publisher. As depicted, some of the problems that ad exchanges face include the existence of advertisers with different incentives and budgets, the fact that goods are heterogeneous and billions of auctions are conducted daily, so auctions need to be simple and fast. There are also important issues that need to be considered by all stakeholders, such as the limited information on opponents and bids that need to be learned over time. Finally, one of the most important issues is the existence of the intermediaries that hide some of the demand to the exchange and publisher.

This last issue is of immediate relevance to this thesis. For this reason, we continued our literature review on domains with auctions involving intermediaries where we showed the connection between ad exchanges, procurement auctions with subcontracting and auctions for network resource allocation. Specifically, we have seen that other researchers have compared first-price and second-price auctions in these settings. However, although some of the results share similarities with the work within this thesis, there are a number of issues that have not been addressed before to satisfy our research aims. Specifically, in these works, the intermediaries are assumed to be homogeneous and have no strategic tools to increase their profit, such as reserve prices. The only exception is the work by Feldman et al. (2010) who nevertheless derive results for intermediaries with a single buyer each. In addition, the related literature considers competing intermediary auctions where both the allocation and payments are determined before the central auction. Wambach (2009) has considered this timing issue but avoids explicitly modeling competition by other intermediaries. We aim to bridge this gap by considering both pre- and post-award intermediary auctioneers (see Section 3.4 for the mechanisms studied) and letting both the central auctioneer and the intermediaries set appropriate reserve prices (Chapter 5).

Intermediary auctions act as bidding rings in the eyes of the ad exchange auctioneer. For this reason, we followed with a summary of research in that area. We illustrated that

in explicit collusion, colluders often organize local auctions to determine the bid at the central auction, as well as the local payment and allocation if the ring wins. However, the emphasis in this stream of literature has been on a single bidding ring, in contrast to our competing intermediaries setting. Moreover, the main issues studied are related to the enforcement of the agreed bids, since bidders can directly bid at the central auction. Currently, this is not an issue in our ad exchange setting, since advertisers typically lack the expertise and infrastructure, or are not allowed to bid directly at the exchange. We complement some of the works in this area by considering different mechanisms for the rings, when multiple, competing such bidding rings are present in Chapter 4. More specifically, we find that the reserve price of the central auctioneer increases with the number of bidders and that it can also depend on the number of intermediaries for some types of PAKT. Finally, within the same chapter, our insights from competing, self-interested intermediary auctioneers, can also provide insights on which collusion mechanism colluders should choose in face of competition from other rings.

Finally, given that intermediary auctioneers are in direct competition, we offered a short survey of the growing literature on the problem of competition between auctioneers, which is the main broad area of our work. Specifically, we pointed out the difficulties in applying traditional mechanism design techniques, such as the revelation principle, in imperfect competition settings. This stream of literature considers the competition between auctioneers by predominantly fixing the mechanisms and converting the problem to that of price or quantity competition. Also, works in this literature consider either duopoly or perfect competition, mainly for tractability issues. In this thesis, we take the former approach, considering settings with two intermediaries that use predefined auction mechanisms and compete by setting appropriate reserve prices. This is the first time that this has been studied, since the vast majority of previous literature has focused on independent auctioneers, whereas in our ad exchange setting, intermediaries act as both auctioneers and bidders. In Chapter 6, we first study the intermediary selection problem faced by the advertisers, which is shown to be complex and remarkably different than that of previous works. Given this complexity, learning algorithms are used to calculate the intermediaries' equilibrium reserve-price-setting strategies.

In summary, we showed that the literature up to now has not sufficiently addressed the problem of competition between intermediary auctioneers. Against this background, in the next chapters, we aim to fill this gap, by analyzing the imperfect competition between demand-side intermediaries that satisfies our research objectives (Section 1.2). In particular, we study the imperfect (i.e. small, finite) competition between demand-side intermediaries with captive buyers when the latter do not impose any reserve prices (Chapter 4). We then look at the impact of such reserve prices in the revenue and efficiency of the ad exchange ecosystem, in Chapter 5. Finally, in Chapter 6, we remove the captivity limitation and instead let the advertisers select their favorite intermediary

in a simple two-intermediary setting. Before doing this, in the next chapter, we formally present our model and assumptions.

Chapter 3

The Problem of Competing Intermediary Auctioneers

In this chapter, we provide a general formalization of the problem of competing intermediaries that will be studied in the following chapters within this thesis. The aim of this discussion is to present a high-level model of the setting considered along with a number of necessary assumptions that will form the basis for the theoretical analysis presented in the remainder of this thesis.

To this end, first, the roles of each set of agents within our setting are outlined in Section 3.1. Then, Section 3.2 lays down the assumptions taken that were deemed necessary for the analytical tractability of the results. Following this, Section 3.3 presents a general formulation of the model that will be used in the next chapters. Next, Section 3.4 provides a description of the three mechanisms that are studied in the chapters that follow along with a number of motivating examples that show some of the implications of the presence of competing intermediaries and their strategic interaction. Finally, Section 3.5 concludes this chapter.

3.1 The Agents

As it has been discussed in Chapter 1, the online advertising industry is significantly complex, comprising a large variety of companies, each specializing in a different area. However, for issues of clarity and tractability of our model and since our focus is on the competition between demand-side platforms, in the setting studied, three different types of strategic participants are considered¹:

¹We borrow the terminology from Feldman et al. (2010).

- The **center** represents an ad exchange or a supply-side platform that calls out a number of interested advertisers (or, mainly, their representative intermediaries, as discussed below) every time a user visits a web page of a publisher whose supply is managed by the platform. The center takes the role of an auctioneer that forwards eligible information (if any) about the user and the web page to the bidders and conducts an auction in which the latter have to submit an advertisement tag along with a price offer (henceforth called a *bid*). The winning bidder's advertisement is then shown to the user (an event known as an *impression*) and the center shares the obtained revenue with the publisher based on a predetermined commission rate. In what follows, the publisher and third-party running the ad exchange or supply-side platform will be considered as a single entity.
- A number of demand-side **intermediaries**. These intermediaries provide advertisers with the technical infrastructure, expertise, relevant tools as well as a centralized point of access to the various ad exchanges. There are two broad categories of such intermediaries: self-serve and managed. The latter type follow the traditional ad network business model, whereby the intermediary agrees in advance with the advertiser on a budget to be spent on purchasing a specified number of qualified impressions. Furthermore, these intermediaries take full responsibility for managing the advertiser's campaign. In contrast, self-serve intermediaries, known as demand-side platforms, typically provide an application programming interface along with additional useful data to their advertisers who are now responsible for their bidding strategy and campaign management. Given that there is usually more than one interested advertiser for each impression in an intermediary's platform, the latter implements a mechanism, predominantly an auction, that decides which of the advertisers' bids to submit at the center and how much to ask for its services.
- A number of **buyers**. These are the advertisers that have some predefined demand for ad placement on web pages and for user types of their choice. The vast majority of buyers have a budget and have to decide how much to bid². In addition, they have to decide which advertisement to show for each available impression at the ad exchange, if called out. Buyers connect to the ad exchange via some intermediary, hence it is crucial for buyers to select an intermediary that maximizes their surplus.

In what follows, a buyer's utility will be called her *surplus*, an intermediary's utility will be called his *profit* and the center's utility will be called its *revenue*³.

Now that the main participant roles considered have been illustrated, in the following section, we present the assumptions that were taken in our model to make it amenable to theoretical analysis.

²Bid amounts are usually much smaller compared to a buyer's budget.

³For the remainder of this thesis, we follow the convention that buyers are female and intermediaries are male.

3.2 Assumptions

Before presenting our model formally, we first simplify the complexity of the real-world ad exchange area by making the following assumptions:

- All agents are self-interested.
- There is a single slot available for ad placement that is auctioned at the center⁴.
- There is a single center implementing a second-price sealed-bid auction with a reserve price and a fair tie-breaking rule. This mechanism is chosen since it is used in all major ad exchanges. Moreover, this mechanism is incentive compatible and is revenue-optimal when participants are symmetric (Myerson, 1981), which is a reasonable assumption for fair, open platforms like ad exchanges.
- The pricing model implemented is CPM, i.e. advertisements are traded on a per impression basis, as is the case in the vast majority of the current ad exchanges. Moreover, this pricing scheme alleviates problems related to the bias inherent in the estimation of the click-through rate of the advertisements (see Section 2.2).
- Buyers can only participate in the center's auction via demand-side intermediaries. This naturally arises in the complex online display advertising marketplace, as advertisers usually lack the required expertise and technical infrastructure, or are not allowed to bid directly in the ad exchange (e.g. Microsoft's advertising exchange).
- Buyers *single-home*, i.e. each buyer can select at most one intermediary. This is a first step towards understanding the interactions of buyers. Also, in practice advertisers typically select one demand-side intermediary for each type of campaigns; this is to avoid campaign management costs as well as the possibility of bidding against themselves⁵.

Having laid out the assumptions, we now continue by detailing the ad exchange model, presenting our setting along with the timing and actions available to the center, the intermediaries and the buyers.

⁴In practice, a page has often several ads. However, currently these ads are mostly sold independently, even though there are often interactions (so-called *externalities*) between ads (e.g. if the same ad is shown multiple times, or if competing brands are shown). Finding good mechanisms to deal with such interactions is still an open problem.

⁵For the PRE mechanism, bidding against one's self is not possible, however we can assume that there is a cost for entering in an intermediary's auction related to the cost of managing a campaign, identical for all intermediaries, that is normalized to zero.

3.3 Model

Our model extends that of Feldman et al. (2010) (see Section 2.2.2) by letting the buyers optionally strategically select an intermediary. Specifically, suppose that the center is auctioning an indivisible good to $K \in \mathbb{N}^+$ *ex-ante* symmetric, surplus-maximizing buyers via a number, $n \in \mathbb{N}^+$, of intermediary auctioneers s_j , $j = 1, \dots, n$. We assume that the center and the intermediaries have no valuation for the good⁶ and that the preferences of the buyers and auctioneers are described by von Neumann and Morgenstern utility functions. Buyers have independent valuations, v_i , $i = 1, \dots, K$, i.i.d. drawn from a commonly-known distribution F with a continuous, positive, differentiable density f , and a compact support $V = [0, 1]$. The center runs a second-price sealed-bid auction with a reserve price, $\rho \in V$, and a fair tie-breaking rule, and each intermediary is allowed to submit a single bid⁷. Hence, the center's revenue equals the maximum of the second-highest submitted bid and ρ , if there is at least a bid above ρ , and is otherwise zero. Each intermediary, s_j , $j = \{1, \dots, n\}$, runs a *contingent* auction among its set of $k_j \geq 1$ buyers (where $\sum_{j=1}^n k_j = K$). This auction determines the winning bid, the price to be paid by the winning buyer as well as the bidding amount to be submitted at the center. The intermediary's profit is the difference between the payment he receives from his winning buyer and the price he pays at the center, whereas the surplus of a buyer is the difference between her valuation and the price paid at the intermediary. Specifically, contingent on the intermediary winning at the central auction, the expected surplus for a buyer i with valuation v_i is $\Pi_j(v_i) = \alpha_j(v_i)(v_i - p_j)$, where $\alpha_j : V \mapsto [0, 1]$ is the probability of obtaining the item in intermediary s_j 's local auction, and $p_j \in [0, 1]$ the price to be paid to the intermediary. In more detail, the game proceeds as follows:

1. The center announces its reserve price, ρ , to the intermediaries.
2. Intermediaries announce their reserve prices, $r_j \geq \rho$, $j = \{1, \dots, n\}$, to the population of buyers.
3. Buyers learn their valuations for the good.
4. Buyers (optionally) simultaneously select their preferred (single) intermediary, s_j , and submit a bid to that intermediary.
5. Intermediaries run auctions among their buyers and submit their (single) bids (if any) to the center.
6. The center runs its auction with the intermediaries' bids, transfers the good to the winning intermediary (if any) and receives payment from that intermediary.
7. The winning intermediary (if any) transfers the good to his winning buyer and receives payment from that buyer.

⁶The center can have a valuation for the good that is normalized to zero.

⁷This is the predominant mechanism used in ad exchanges.

This formulation is general enough and so can be used in other settings, such as auctions for real estate or treasury bills, procurement auctions with subcontracting or resale, auctions with bidding rings, auctions for network resources allocation or any other auctions involving intermediaries.

Some of the implications of the introduction of intermediaries' auctions will be more clear in the examples of the following section where we present the intermediary auctions that we will study throughout the remainder of this thesis.

3.4 Auction Mechanisms for the Demand-Side Intermediaries

Due to the limitations inherent in the analysis of competing auctioneers, i.e. the lack of an analog to the revelation principle and the infinite regression resulting from auctioneers' best responses (see Section 2.5), in the following chapters, we fix the mechanisms of intermediaries. Specifically, we focus our analysis on three different widely-used auction mechanisms for the intermediaries, where they make a profit by pocketing the difference between the received payments from their buyers and the payment to the center, given that they win in the latter's auction. In more detail, we consider two variations of the Vickrey auction that we call PRE and POST based on the time of the determination of the intermediary's exact contingent payment, i.e. before or after the center's auction, as well as FPSB auctions. The three auction types are described in the following subsections.

3.4.1 Pre-Award Second-Price Sealed-Bid Auction (PRE)

In the first mechanism, named pre-award SPSB or PRE auction, the intermediary runs a local SPSB auction and determines the exact payment *before* the center's auction. Specifically, contingent on the intermediary winning the good at the central auction, the local highest bidder wins and pays the maximum of the local second-highest bid and the intermediary's announced reserve price given that her bid is higher than the latter price. Since this maximum is the intermediary's payment in case he wins at the central auction, it corresponds to his (private to other intermediaries) "valuation". As the center's auction is DSIC⁸ this is also the amount that the intermediary bids at the central auction (given that there is at least one bid above the intermediary's reserve price, otherwise the intermediary does not submit a bid). The PRE auction is the mechanism studied by Feldman et al. (2010) and is also DSIC for the buyers as the following proposition illustrates.

⁸For an exposition of all the bidding equilibria of the SPSB auction, we refer the reader to the work of Blume and Heidhues (2004).

Proposition 3.1. *Truthful bidding is a dominant-strategy equilibrium for the buyers in a PRE intermediary auction.*

Proof. Assume buyer i with private valuation v_i participating in intermediary j 's PRE auction submits a bid b_i . The highest opponent local bid in j 's auction is b^j and the highest opponent intermediary $q \neq j$'s bid submitted at the center is b^q . W.l.o.g. we assume that the intermediaries and the center do not impose any reserve prices. We proceed by case analysis.

- $v_i > b^j$. If $b^j \geq b^q$, then intermediary j wins (with probability $\frac{1}{2}$ if the two bids are equal), since he submits b^j , and hence buyer i best responds with a bid $b_i > b^j$ that yields positive surplus $v_i - b^j$. If $b^j < b^q$, then intermediary j loses and buyer i is indifferent across all bids. Hence, in both cases, $b_i = v_i$ is a weakly-dominant strategy.
- $v_i = b^j$. In this case, buyer i is indifferent across all bids since her surplus is always 0, so bidding v_i is a weakly-dominant strategy.
- $v_i < b^j$. If $b^j \geq b^q$, then intermediary j wins and buyer i best responds with bidding $b_i < b^j$, since otherwise she gets a negative surplus. If $b^j < b^q$, then buyer i is indifferent across all bids. Hence, in both cases, $b_i = v_i$ is a weakly-dominant strategy.

□

Having shown that the PRE auction is DSIC, in the following examples we aim to illustrate some of the effects of the competition between homogeneous intermediaries. More specifically, consider a setting with two intermediaries, s_1, s_2 , and a population of $K = 4$ buyers with discrete valuations $v_1 > v_2 > v_3 > v_4$ which are assumed to be common knowledge among all parties and who submit bids b_1, b_2, b_3, b_4 . This means that, if the intermediaries were not present, the center's revenue would be $\max\{\rho, v_2\}$.

In our first example, we assume that the center and the intermediaries do not set any reserve prices and that the allocation of the buyers to the intermediaries is exogenously determined.

Example 3.1. *Consider the following possible scenarios:*

- *Buyers 1 and 2 have been allocated to intermediary s_1 whereas the other buyers to intermediary s_2 . If both intermediaries implement PRE mechanisms, then $b_i = v_i$ for $i = \{1, 2, 3, 4\}$, hence intermediaries submit their local second-highest bids, v_2, v_4 , at the center. This means that the center's revenue is v_4 , intermediary s_1 wins at the central auction and obtains a profit of $v_2 - v_4$ and buyer 1 wins for a surplus of $v_1 - v_2$.*

- *Buyers 1 and 3 have been allocated to intermediary s_1 whereas the other buyers to intermediary s_2 . Then, PRE intermediaries submit v_3, v_4 at the center. Hence, the center again receives v_4 , intermediary s_1 wins at the central auction and obtains a profit of $v_3 - v_4$ and buyer 1 wins for a surplus of $v_1 - v_3$.*
- *Buyers 1 and 4 have been allocated to intermediary s_1 whereas the other buyers to intermediary s_2 . Then, PRE intermediaries submit v_4, v_3 at the center. Hence, the center again receives v_4 , intermediary s_2 in this case wins at the central auction and obtains a profit of $v_3 - v_4$ and buyer 2 wins for a surplus of $v_2 - v_3$.*

In this example, it can be seen that the outcome is not always efficient since, in the third case, buyer 2 who is not the buyer with the highest valuation obtains the item. Moreover, the center's revenue is invariant to the buyers' allocation to intermediaries when no reserve prices are present but it decreases compared to the case without intermediaries.

In the previous example, we assumed that buyers are exogenously allocated to intermediaries. In the following example, we remove this limitation (see (Stavrogiannis et al., 2013a) for a complete analysis).

Example 3.2. *Suppose that the center does not impose a reserve price but intermediaries s_1, s_2 set reserve prices r_1, r_2 respectively such that $r_1 \leq r_2$. If buyers strategically select one of the intermediaries based on the announced reserve prices, it is interesting to see what their decision will be in equilibrium, since this will determine the equilibrium reserve prices of the intermediaries. For PRE intermediary auctioneers, the intermediary selection is a 3-player game, that of the buyers with the three highest valuations, since the payment will be at least v_3 and both buyers 1 and 2 can obtain the good. More specifically, if $r_1 \leq r_2 < v_3$, buyer 1 tries to select the same intermediary as buyer 3, and different intermediary than buyer 2, since this will maximize her surplus. Also, there is a pressure for the intermediaries to increase their reserve prices, since these can be their payments contingent on winning at the center and, at the same time, will try not to set these too high so as to attract buyer 1. In equilibrium, both intermediaries set reserve prices $r_1 = r_2 = v_2$.*

This example shows the complexity of the buyers' intermediary selection problem for PRE intermediaries and the inherent pressure for increasing their reserve prices. In the next subsection, we present the other variation of the Vickrey auction for the intermediaries.

3.4.2 Post-Award Second-Price Sealed-Bid Auction (POST)

Similar to the previous mechanism, in the post-award SPSB or POST mechanism, the intermediary runs a local SPSB auction. However, he forwards the *local highest bid* to the center (given that this is higher than his reserve price), increasing his probability of

winning compared to the PRE mechanism. Moreover, his payment is determined *after* the central auction as the maximum of the local second-highest bid, the center's reserve price and the second-highest bid submitted at the center, contingent on the intermediary winning the good at the center. This mechanism is inspired by the operation of second-price PAKT found in the literature on bidding rings (see Section 2.4); the intermediary's profit is the collusive surplus, i.e. the difference between the local second-highest bid and the opponent intermediaries' highest submitted bid, only when this is positive, i.e. when the intermediary has both the highest and second-highest bids in total submitted in his local auction, otherwise the intermediary receives zero profit. Hence, compared to the PRE auction, this mechanism balances a higher probability of winning at the center (since the highest bid is submitted) with the possibility of receiving zero profit (when the second-highest bid among all buyers' population is submitted in another intermediary). The POST auction is also DSIC for the buyers as proved in the following proposition.

Proposition 3.2. *Truthful bidding is a dominant-strategy equilibrium for the buyers in a POST intermediary auction.*

Proof. Assume buyer i with private valuation v_i participating in intermediary j 's POST auction submits a bid b_i . The highest opponent local bid in j 's auction is b^j and the highest opponent intermediary $q \neq j$'s bid submitted at the center is b^q . W.l.o.g. we assume that the intermediaries and the center do not impose any reserve prices. We proceed by case analysis.

- $v_i > \max\{b^j, b^q\}$. Buyer i best responds by bidding above this maximum to obtain a positive surplus of $v_i - \max\{b^j, b^q\}$. Bidding anything lower than this leads to either losing the local auction (when $b_i < b^j$) or winning that auction but then losing at the central auction (when $b^j < b_i < b^q$). So, bidding $b_i = v_i$ is a weakly-dominant strategy.
- $v_i = \max\{b^j, b^q\}$. In this case, buyer i is indifferent across all bids since her surplus is always 0. More specifically, when $v_i = b^j > b^q$, then bidding (i) any amount above v_i guarantees the good but at a price of $b^j = v_i$, (ii) any amount lower than v_i leads to not winning at the local auction, and (iii) bidding exactly v_i gives her the good with probability $\frac{1}{2}$. In all these cases, her surplus is zero. Similarly, when $v_i = b^q > b^j$, bidding (i) any amount below b^q leads to either not winning at the local auction or winning the local auction but losing at the central auction, (ii) any amount above b^q leads to a negative surplus, $v_i - b^q$, and (iii) an amount equal to b^q gives i the item with probability $\frac{1}{2}$. In all cases, her surplus is also zero. Hence, bidding $b_i = v_i$ is a weakly-dominant strategy.
- $v_i < \max\{b^j, b^q\}$. Buyer i 's best response is to bid $b_i < \max\{b^j, b^q\}$, for zero surplus, irrespective of the bid, since otherwise she obtains a negative surplus. So, $b_i = v_i$ is a weakly-dominant strategy.

□

As we have shown, the POST auction is also DSIC for the buyers. We now illustrate some of the implications of the intermediaries using this mechanism compared to the case of PRE, using the same example as Example 3.1. More specifically, we consider a setting with two intermediaries, s_1, s_2 , now implementing POST auctions, and a population of $K = 4$ buyers with discrete valuations $v_1 > v_2 > v_3 > v_4$ and who submit bids b_1, b_2, b_3, b_4 . We also assume that both the center and the intermediaries do not any reserve prices and that buyers are exogenously allocated to the intermediaries.

Example 3.3. Consider the following possible scenarios:

- *Buyers 1 and 2 have been allocated to intermediary s_1 whereas the other buyers to intermediary s_2 . If both intermediaries implement POST mechanisms, then $b_i = v_i$ for $i = 1, 2, 3, 4$, hence intermediaries submit their local highest bids, v_1, v_3 , at the center. This means that the center's revenue is v_3 , intermediary s_1 wins at the central auction and obtains a profit of $v_2 - v_3$ and buyer 1 wins for a surplus of $v_1 - v_2$.*
- *Buyers 1 and 3 have been allocated to intermediary s_1 whereas the other buyers to intermediary s_2 . Then, POST intermediaries submit v_1, v_2 at the center. Hence, the center receives v_2 , intermediary s_1 wins at the central auction but obtains a profit of 0 and buyer 1 wins for a surplus of $v_1 - v_2$.*
- *Buyers 1 and 4 have been allocated to intermediary s_1 whereas the other buyers to intermediary s_2 . Then, POST intermediaries submit v_1, v_2 at the center. Hence, the center again receives v_2 , intermediary s_1 wins at the central auction but also obtains a profit of 0 and buyer 1 wins for a surplus of $v_1 - v_2$.*

In this example, we can see that the outcome is always efficient (i.e. the highest bidder obtains the item), the center's revenue is higher than that of Example 3.1 for PRE intermediaries, and is equal to that of a setting without intermediaries. Nevertheless, there are settings where the winning intermediary obtains zero profit even though he wins at the center (second and third case above).

We now move to present the third intermediary mechanism, FPSB.

3.4.3 First-Price Sealed-Bid Auction (FPSB)

In this FPSB mechanism, the intermediary uses an FPSB auction for his local buyers. Hence an intermediary's local winner obtains the good for a price equal to her bid only if her selected intermediary wins at the center. Since the payment of the intermediary is the aforementioned price, determined before the central auction, and the latter auction

is DSIC, intermediaries submit their local winning bids at the center (given that there is at least one bid above their reserve price, otherwise they do not submit a bid), as was the case with the PRE mechanisms. For this reason, the profit of a winning intermediary is the difference between his local winning bid and the maximum of the second-highest intermediaries' bid and the center's reserve price. As we will show, this mechanism is also more efficient than the PRE auction; however, buyers in this mechanism follow BNE bidding strategies.

We now illustrate some of the implications of the intermediaries using this mechanism compared to PRE and POST auctions, using the same example as examples 3.1 and 3.3. More specifically, we consider a setting with two intermediaries, s_1, s_2 , now implementing FPSB auctions, and a population of $K = 4$ buyers with discrete valuations $v_1 > v_2 > v_3 > v_4$ and who submit bids b_1, b_2, b_3, b_4 . Since FPSB auctions are not DSIC, $b_i < v_i$ in general for $i = \{1, 2, 3, 4\}$. We assume, as before, that both the center and the intermediaries do not have any reserve prices and that buyers are exogenously allocated to the intermediaries.

Example 3.4. Consider the following possible scenarios:

- *Buyers 1 and 2 have been allocated to intermediary s_1 whereas the other buyers to intermediary s_2 . If both intermediaries implement FPSB mechanisms, then intermediaries submit their local highest bids, b_1, b_3 , at the center. This means that the center's revenue is b_3 , intermediary s_1 wins at the central auction and obtains a profit of $b_1 - b_3$ and buyer 1 wins for a surplus of $v_1 - b_1$.*
- *Buyers 1 and 3 have been allocated to intermediary s_1 whereas the other buyers to intermediary s_2 . Then, FPSB intermediaries submit b_1, b_2 at the center. Hence, the center receives b_2 , intermediary s_1 wins at the central auction and obtains a profit of $b_1 - b_2$ and buyer 1 wins for a surplus of $v_1 - b_1$.*
- *Buyers 1 and 4 have been allocated to intermediary s_1 whereas the other buyers to intermediary s_2 . Then, FPSB intermediaries submit b_1, b_2 at the center. Hence, the center again receives b_2 , intermediary s_1 wins at the central auction and also obtains a profit of $b_1 - b_2$ and buyer 1 wins for a surplus of $v_1 - b_1$.*

In this example, and under the assumption about the ordering of the bids⁹, the outcome is always efficient as was the case with POST intermediaries. What's more, the winning intermediary always makes a positive profit, as was the case with PRE intermediaries. Hence, FPSB combine the benefits of both Vickrey variations but are not DSIC, so buyers must strategize about their bidding amounts. Finally, in all cases, the center's revenue is lower than that for POST intermediaries.

⁹This is actually true for this example. For more details, see Section 4.2.3

3.5 Summary

In this chapter, we introduced a generic model for the ad exchange ecosystem. Within this model, we made several simplifying assumptions that are necessary for tractability reasons. However, our model is a qualitatively reasonable abstraction of this complex marketplace. Having presented our assumptions, we then described the details of our model, including the timing and available actions of all the participants. In contrast to other models (AdX model of Muthukrishnan (2009) and Feldman et al. (2010)), we allow buyers to strategically select one of the two intermediaries, in the same manner that Burguet and Sákovics (1999) do for independent auctioneers. Following this, we presented three auction mechanisms for the intermediaries that form the basis of our analysis. Finally, we provided a number of examples that depict the resulting issues from the competition of intermediaries and their choice of mechanism for the center, the buyers and the intermediaries themselves.

In what follows, we analyze the competition between intermediaries in a Bayesian setting with captive buyers, where we consider intermediaries both without (Chapter 4) and with reserve prices (Chapter 5). We then remove the captivity assumption in Chapter 6 where we study the case with buyers strategically selecting one of the intermediaries, albeit in a simpler duopoly intermediary setting.

Chapter 4

Intermediaries with Captive Buyers: No Intermediary Reserve Prices

We study the model for ad exchanges presented in the previous chapter under an incomplete-information or Bayesian setting, where each buyer knows her private valuation, but has only probabilistic information about the private valuations of her opponents. More specifically, we consider a single-good setting with intermediaries whose buyers are captive, i.e. we ignore the issue of strategic selection of intermediaries from the buyers and assume that, after this exogenously determined allocation to the intermediaries, buyers cannot move between them. We assume that the center sets a reserve price but the intermediaries do not. Furthermore, we focus our analysis on the case where each intermediary has the same number of buyers with the same distribution of private valuations (i.e. the intermediary mechanisms are symmetric) and when all intermediaries implement the same auction (i.e. homogeneous population of intermediaries). Given this setting, we analyze the effect of the introduction of the intermediaries as well as their competition on the center's revenue, the intermediaries' profit and the buyers' surplus as well as on the social welfare (i.e. the sum of all agents' utilities).

In accordance to the model described in Section 3.3, in what follows, we assume that both the center and the intermediaries have selected their mechanism in advance and, in the setting studied within this chapter, that buyers are allocated to the intermediaries such that each intermediary has exactly the same number of buyers. Then, the center announces a reserve price for the good to be auctioned to the intermediaries who then forward this to their buyers. Buyers learn their private valuations for the good and submit a bid for the good to their allocated intermediary. Intermediaries then run local auctions with their buyers' bids to determine a winner, if any, and a payment contingent on winning the item at the central auction, and then submit a single bid (if there is some

qualified bid above the center's reserve price) to the center. The center then runs its auction with the intermediaries' bids, determines a winning intermediary, if any, and a payment and allocates the good to this intermediary who then allocates the good to his winning local buyer for the determined price.

To this end, we start our analysis in Section 4.1 with a simple case for a single intermediary and then move to the competing intermediaries case in Section 4.2. We then proceed to analyze the incentives of an intermediary to switch to a different mechanism in homogeneous intermediary settings for **PRE** and **POST** mechanisms in Section 4.3. Finally, Section 4.4 concludes.

4.1 Special Case: Single Intermediary

We start by showing that, even when only one intermediary is introduced, the center's optimal reserve price increases as the number of buyers increases and the social welfare¹ is smaller than that of a setting without intermediaries. Thus, these changes occur due to the very presence of the intermediaries, and not only due to their competition.

Feldman et al. (2010) have shown that, for single-buyer intermediaries with reserve prices, the center's optimal reserve price *decreases* with the number of intermediaries. Since each intermediary in their setting has exactly one buyer, this means that the optimal reserve price decreases with the number of buyers as well. However, it is not clear whether this is due to the number of intermediaries and/or number of buyers per intermediary. Regardless of this, as the authors notice, this is in contrast with the results by Myerson (1981) for a classical setting with no intermediaries, who has shown that the optimal reserve price, ρ_{OPT}^* , satisfies the equation:

$$\rho_{OPT}^* = \frac{1 - F(\rho_{OPT}^*)}{f(\rho_{OPT}^*)} \quad (4.1)$$

As can be seen, the optimal reserve price is independent of the number of buyers. In this case, the auctioneer's ex-ante expected revenue is:

$$revenue_{OPT}(\rho) = K \int_{\rho}^1 [xf(x) + F(x) - 1]F^{K-1}(x)dx \quad (4.2)$$

and a buyer's ex-interim expected surplus (with a valuation $v \geq \rho$) is:

$$\Pi_{OPT}(v, \rho) = \int_{\rho}^v F^{K-1}(y)dy \quad (4.3)$$

¹In accordance with Definition 2.32, the social welfare equals the sum of the center's expected revenue, intermediaries' expected profits and all buyers' expected surplus.

and her ex-ante expected surplus is:

$$\mathbb{E}[\Pi_{OPT}(\rho)] = \int_{\rho}^1 [1 - F(y)]F^{K-1}(y)dy \quad (4.4)$$

In our setting, we illustrate that, when there is no competition between intermediaries, the optimal reserve price *increases* as the number of buyers increases. This is in line with the results of the literature on bidding rings (cf. Lemma 1 of Graham and Marshall (1987) and McAfee and McMillan (1992)).

In more detail, since there is only one intermediary, the center's second-price sealed-bid auction with a reserve price is equivalent to the center offering a take-it-or-leave-it price, ρ , to the intermediary representing all the buyers' population (i.e. K buyers). Given the lack of competition between intermediaries, when allocated, the good is given to the highest bidder in all standard intermediary auctions, including the ones we study here. What's more, as equation (4.5) below expresses, the center in all three cases receives the take-it-or-leave-it price only if there is a single buyer in the intermediary's auction whose private valuation is above this price. This means that the center's optimal take-it-or-leave-it price is the same for the three mechanisms. Hence, as Riley and Samuelson (1981) have shown for auctions with the same minimum payment, all standard auctions yield the same expected center's expected revenue and intermediary's profit, and, since the allocation is identical, the same buyers' expected surplus. W.l.o.g., we assume that the intermediary runs a PRE auction. Then, the center's expected revenue equals ρ times the probability that there is at least one buyer that is willing to accept it:

$$revenue_{SINGLE}(\rho) = \rho[1 - F^K(\rho)] \quad (4.5)$$

which is maximized by setting an optimal ρ_{SINGLE}^* as:

$$\rho_{SINGLE}^* = \frac{1 - F^K(\rho_{SINGLE}^*)}{KF^{K-1}(\rho_{SINGLE}^*)f(\rho_{SINGLE}^*)} \quad (4.6)$$

This is essentially the same as equation (2.20) for $K = m$, i.e. the intermediary acts as an all-inclusive bidding ring with size K for the center. For this reason, we get the following theorem.

Theorem 4.1. (*Lemma 1 from (Graham and Marshall, 1987)*). *The reserve price, ρ_{SINGLE}^* , that would maximize the expected revenue of the center for a single intermediary with K buyers is an increasing function of K .*

The intermediary's ex-ante expected profit is the expected difference of the second-highest bid and the reserve price, ρ :

$$\text{profit}_{\text{SINGLE}}(\rho) = \int_{\rho}^1 (y - \rho) f_2^{(K)}(y) dy = 1 - \rho - \int_{\rho}^1 F_2^{(K)}(y) dy \quad (4.7)$$

where $f_2^{(K)}, F_2^{(K)}$ are the probability density function (p.d.f.) and cumulative distribution function (c.d.f.), respectively, of the second-highest-order statistic among K samples i.i.d. drawn from f, F . Finally, a buyer's ex-interim expected surplus for the same reserve price can be expressed in the same way as in equation (4.3).

Example 4.1. To illustrate these observations, we consider an example with buyers whose valuations are drawn from the uniform distribution $U(0, 1)$. Then, equation (4.6) yields:

$$\rho_{\text{SINGLE}}^* = \frac{1}{(K+1)^{\frac{1}{K}}} \quad (4.8)$$

which increases with the number of buyers². In this case, equations (4.1) - (4.3) and (4.7) yield:

$$\rho_{\text{OPT}}^* = \frac{1}{2} \quad (4.9)$$

$$\text{revenue}_{\text{OPT}}(\rho_{\text{OPT}}^*) = \frac{1}{K+1} \left[\left(\frac{1}{2} \right)^K + K - 1 \right] \quad (4.10)$$

$$\mathbb{E}[\Pi_{\text{OPT}}(\rho_{\text{OPT}}^*)] = \frac{2^{K+1} - K - 2}{K(K+1)2^{K+1}} \quad (4.11)$$

$$\text{revenue}_{\text{SINGLE}}(\rho_{\text{SINGLE}}^*) = \frac{K}{(K+1)^{\frac{K+1}{K}}} \quad (4.12)$$

$$\mathbb{E}[\Pi_{\text{SINGLE}}(\rho_{\text{SINGLE}}^*)] = \frac{1}{(K+1)^{\frac{2K+1}{K}}} \quad (4.13)$$

$$\text{profit}_{\text{SINGLE}}(\rho_{\text{SINGLE}}^*) = \frac{K}{K+1} - \frac{1}{(K+1)^{\frac{1}{K}}} \left[1 + \frac{K-1}{(K+1)^2} \right] \quad (4.14)$$

Given this, Figure 4.1 illustrates the center's ex-ante expected revenue and the social welfare with and without the intermediary, when the center sets the optimal reserve price, ρ^* , as the number of buyers increases. We can see that the social welfare decreases compared to the classical setting without the intermediary. This is due to the double

²It is easy to see that $\lim_{K \rightarrow \infty} (K+1)^{-\frac{1}{K}} = 1$.

marginalization effect from the presence of the intermediary. In more detail, the intermediary obtains some of the center's revenue, so, in response, the center increases its reserve price and that reduces the demand of the buyers (Tirole, 1993). Finally, it can be seen that the intermediary's expected profit decreases with the number of buyers, as ρ^* also increases.

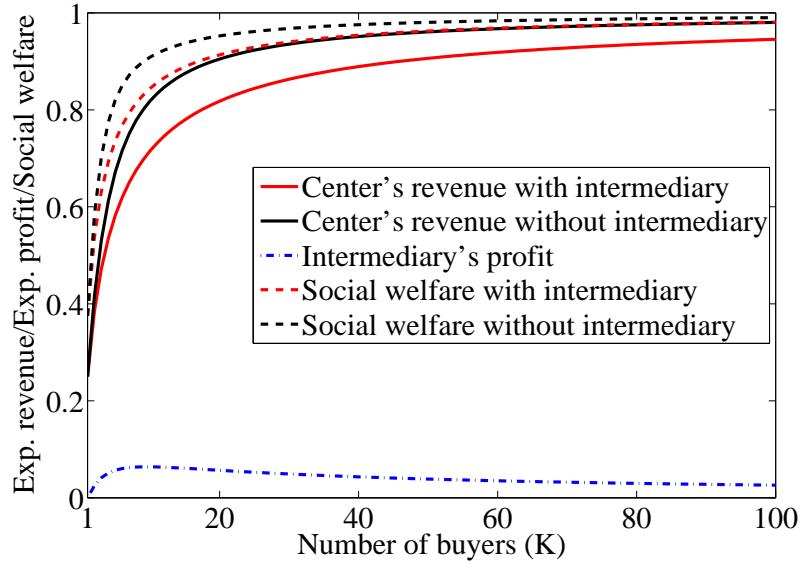


FIGURE 4.1: Center's ex-ante expected revenue and social welfare with and without a single intermediary and the latter's expected profit as a function of the number of buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

In the following section, we extend our analysis to the case where there is a homogeneous population of multiple intermediaries who compete in a central auction.

4.2 Homogeneous Symmetric Intermediaries

We now consider a scenario with a homogeneous population of $n > 1$ symmetric intermediaries³ (i.e. where all intermediaries implement the same mechanism and have exactly the same number of buyers each, with their private valuations drawn from the same distribution function). In line with Feldman et al. (2010), we assume that buyers are allocated to the intermediaries, such that each intermediary has exactly k buyers in his market, i.e. $k_j = k$ for all $j = 1, \dots, n$ and $K = nk$, and that buyers cannot move between intermediaries (i.e. they are captive) but, as before, intermediaries do not set any reserve prices.

We study three mechanisms for the intermediaries, as described in Section 3.4, where we show that they yield different intermediaries' expected profits and center's expected

³Typical numbers of bids submitted at an ad exchange vary considerably, ranging from 2 bids per auction for mobile advertising to a couple of dozens for the more well-established display advertising area.

revenue, although it is not possible to provide a complete ranking of the three. Finally, we show that the social welfare is the smallest for intermediaries using PRE auctions compared to the other two mechanisms; this is due to the inefficiency resulting from misallocations.

In what follows, we first characterize the expected ex-interim surplus, ex-ante profit and revenue for each buyer, intermediary and the center, respectively, under each mechanism for the more general case when $k \geq 2$. This will allow us to compare, both theoretically and numerically, the three mechanisms studied in Section 4.2.4.

4.2.1 Pre-Award Second-Price Sealed-Bid Auctions

We first study a setting with PRE intermediary auctions, i.e. auctions where the intermediaries run second-price sealed-bid auctions with their payments determined before the center's auction and hence submit the maximum of their second-highest bid and the center's reserve price. The center's ex-ante expected revenue in this case can be written as:

$$\begin{aligned} \text{revenue}_{\text{PRE}}(\rho) &= \rho[G^n(\rho) + n(1 - G(\rho))G^{n-1}(\rho) - F^{nk}(\rho)] + \int_{\rho}^1 yg_2^{(n)}(y)dy = \\ &= 1 - \rho F^{nk}(\rho) - \int_{\rho}^1 G_2^{(n)}(y)dy = 1 - \rho H^n(\rho) - \int_{\rho}^1 G_2^{(n)}(y)dy \end{aligned} \quad (4.15)$$

where $G = F_2^{(k)}$ is the c.d.f. of the second-highest-order statistic among k bids; $G_2^{(n)}, g_2^{(n)}$ are, respectively, the c.d.f. and the p.d.f. of the second-highest-order statistic among the n submitted bids of the intermediaries; $H_1^{(n)} = H^n$ is the c.d.f. of the highest-order statistic among n bids i.i.d. drawn from $H = F^k$ (i.e. the c.d.f. of the highest-order statistic among k bids). In more detail, the center expects to be paid its reserve price, ρ , with probability that either there is none or only one intermediary whose second-highest bid is greater than ρ (first and second terms), but when there is at least one bid higher than ρ (third term). In any other case, the center receives the expected second-highest among the intermediaries' submitted bids (fourth term). This means that, by taking the first-order condition on equation (4.15), the optimal center's reserve price will satisfy:

$$\rho_{\text{PRE}}^* = \frac{G_2^{(n)}(\rho_{\text{PRE}}^*) - H^n(\rho_{\text{PRE}}^*)}{h_1^{(n)}(\rho_{\text{PRE}}^*)} \quad (4.16)$$

where $h_1^{(n)}$ is the p.d.f. of the highest-order statistic among n bids i.i.d. drawn from $H = F^k$. Hence, we can see that the center's optimal reserve price not only depends on the number of buyers per intermediary but also on the number of participating

intermediaries. An intermediary's ex-ante expected profit in this case equals:

$$\text{profit}_{PRE}(\rho) = \int_{\rho}^1 f_2^{(k)}(y) \int_{\rho}^y (F_2^{(k)}(x))^{n-1} dx dy = \int_{\rho}^1 G^{n-1}(y)[1 - G(y)] dy \quad (4.17)$$

which is the expectation over the distribution of the second-highest-order statistic of the probability of winning against $n - 1$ bids. The c.d.f. of each such bid corresponds to that of the second-highest-order statistic over k samples i.i.d. drawn from F since intermediaries submit their local second-highest bids. Finally, a buyer with private valuation $v \in [\rho, 1]$ expects surplus:

$$\begin{aligned} \Pi_{PRE}(v) &= \int_{\rho}^v (v - y) f_1^{(k-1)}(y) G^{n-1}(y) dy + \\ &+ (v - \rho) F^{k-1}(\rho) \sum_{i=0}^{n-1} \frac{\binom{n-1}{i}}{i+1} F^{(n-1-i)k}(\rho) [k F^{k-1}(\rho) (1 - F(\rho))]^i \end{aligned} \quad (4.18)$$

Thus, a buyer expects positive surplus if her bid is the highest in the intermediary's auction and, at the same time, the second-highest bid is higher than the bids submitted at the center and the reserve price (first term). Finally, the buyer wins the good at the center's reserve price when all other intermediaries' submitted bids are less than ρ or when i other intermediaries also submit their reserve prices, winning with a probability of $\frac{1}{i+1}$ (second term)⁴.

This mechanism guarantees positive profit for the intermediary that wins at the center when there are at least two bids above the center's reserve price. However, besides the inefficiency due to the center's reserve price, this auction induces an additional (misallocation) inefficiency when more than one intermediary is present. To see this, consider a setting with two intermediaries, s_1, s_2 , a population of four buyers so that $v_1 > v_2 > v_3 > v_4$ where buyers 1 and 4 are bidding in intermediary s_1 's local auction and buyers 2 and 3 in s_2 's local auction. Given that s_1, s_2 submit bids equal to v_4, v_3 respectively, intermediary s_2 wins and the good is allocated to buyer 2, although buyer 1's valuation is higher. In the next subsection, we present an alternative mechanism for the intermediaries, which keeps the incentive compatibility property and does not suffer from this type of inefficiency.

4.2.2 Post-Award Second-Price Sealed-Bid Auctions

POST intermediary auctioneers run second-price sealed-bid auctions where the payment is determined after the center's auction and submit their highest local bid. Given this, it is straightforward to see that the highest overall bidder always wins in homogeneous settings. Hence, there are no misallocation inefficiencies. However, compared to the

⁴We assume a uniform tie-breaking rule at the center.

previous second-price sealed-bid auction, there is an apparent trade-off: intermediaries increase their probability of winning by submitting higher bids, but also decrease the number of times they make a positive profit (they make zero profit even if they win but their local second-highest bid is smaller than their payment at the center). In this case, an intermediary's expected payment to the center will be:

$$\begin{aligned} \text{payment}_{\text{POST}}(\rho) &= \int_{\rho}^1 f_1^{(k)}(y)[\rho H^{n-1}(\rho) + \int_{\rho}^y x h_1^{(n-1)}(x) dx] dy = \\ &= \rho H^{n-1}(\rho)[1 - H(\rho)] + \int_{\rho}^1 x h_1^{(n-1)}(x)[1 - H(x)] dx \end{aligned} \quad (4.19)$$

which is the expectation over the distribution of the intermediary's highest submitted bid of the payment for any submitted bid y . Then, the center's ex-ante expected revenue can be expressed as:

$$\text{revenue}_{\text{POST}}(\rho) = n \cdot \text{payment}_{\text{POST}}(\rho) = 1 - \rho H^n(\rho) - \int_{\rho}^1 H_2^{(n)}(y) dy \quad (4.20)$$

where $H_2^{(n)}, h_2^{(n)}$ are, respectively, the c.d.f. and the p.d.f. of the second-highest-order statistic among the n intermediaries' bids. Hence, the optimal center's reserve price will satisfy:

$$\rho_{\text{POST}}^* = \frac{1 - H(\rho_{\text{POST}}^*)}{h(\rho_{\text{POST}}^*)} \quad (4.21)$$

From this, we can see that the optimal reserve price for the center only depends on the number of buyers per intermediary and is independent of the number of intermediaries. Then, each intermediary's ex-ante expected profit can be written as:

$$\begin{aligned} \text{profit}_{\text{POST}}(\rho) &= F^{(n-1)k}(\rho) \int_{\rho}^1 (y - \rho) f_2^{(k)}(y) dy + \\ &+ \int_{\rho}^1 \int_{\rho}^{y_2} (y_2 - x_1) f_1^{(n-1)k}(x_1) f_2^{(k)}(y_2) dx_1 dy_2 = \\ &= \int_{\rho}^1 F^{(n-1)k}(y)[1 - F_2^{(k)}(y)] dy = \int_{\rho}^1 H^{n-1}(y)[1 - G(y)] dy \end{aligned} \quad (4.22)$$

That is, an intermediary expects to receive the difference between his local second-highest bid and the center's reserve price, ρ , only when there are at least two buyers with bids above ρ and all other opponent bids are less than ρ (first term). The second term is the expected profit in the other case where the highest opponent of $(n-1)k$ bids is lower than the second-highest among the winning intermediary's k bids. Finally, the ex-interim expected surplus of a buyer whose private valuation is $v \in [\rho, 1]$ is the same

as with a second-price sealed-bid auction with nk buyers and a reserve price of ρ :

$$\Pi_{POST}(v) = (v - \rho)F^{nk-1}(\rho) + \int_{\rho}^v (v - y)f_1^{(nk-1)}(y)dy = \int_{\rho}^v F^{nk-1}(y)dy \quad (4.23)$$

In the next subsection, we present the corresponding buyers' expected surplus, intermediaries' expected profit and center's expected revenue for the third mechanism.

4.2.3 First-Price Sealed-Bid Auctions

Intermediaries often employ an FPSB auction, usually for reasons of transparency. This mechanism also avoids the misallocation inefficiency of the first mechanism, but the strategies of buyers are no longer DSIC. Moreover, given that the allocation is the same as with the POST auction, the total revenue generated is the same, but, as will be shown (Theorem 4.8), the profit share of the intermediaries will be different. When intermediaries implement FPSB auctions, a buyer i with private valuation v wins only if his bid, b_i , is the highest submitted bid among *all* buyers' bids, i.e. if only $b_i \geq \max_{j=\{1, \dots, nk\}, j \neq i} b_j$. Hence, if buyers use the symmetric, increasing bidding strategy $\beta(\cdot) : [\rho, 1] \rightarrow [\underline{b}, \bar{b}]$, buyer i wins if $b_i \geq \beta(Y_1^{(nk-1)})$, where $Y_1^{(nk-1)}$ is the highest-order statistic among the other $nk - 1$ valuations. We assume that a buyer whose private valuation is less than ρ bids \underline{b} . Using standard equilibrium analysis (see Krishna (2010)), in the next theorem we show that buyers' symmetric BNE strategy is the same as in an FPSB auction without intermediaries, a reserve price ρ and $nk = K$ buyers.

Theorem 4.2. *The symmetric Bayes-Nash equilibrium strategy of $K = nk$ buyers participating in a homogeneous population of n intermediaries that implement FPSB auctions without reserve prices so that each intermediary has exactly k buyers when the center implements a SPSB auction with reserve price ρ is given by:*

$$\beta(v) = v - \frac{\int_{\rho}^v F^{nk-1}(x)dx}{F^{nk-1}(v)}, v \geq \rho \quad (4.24)$$

Proof. (Sketch) Without loss of generality, we take the perspective of buyer 1 in intermediary 1 whose private valuation is v . Then, let us assume that all other buyers use the symmetric, strictly increasing and differentiable bidding strategy $\beta(\cdot)$ with range $[\underline{b}, \bar{b}]$. Also, we assume that buyers $2, \dots, k$ with private valuations v_2, \dots, v_k , respectively, are in intermediary 1, buyers $k+1, \dots, 2k$ with private valuations v_{k+1}, \dots, v_{2k} are in intermediary 2, and so on. Also, let $y_1 = \max\{v_2, \dots, v_k\}$, $y_j = \max\{v_{(j-1)k+1}, \dots, v_{jk}\}$, $j = 2, \dots, n$. We also assume for the moment that the center's reserve price $\rho = 0$.

Then, if buyer 1 bids b_1 , she wins (for simplicity, we assume that the buyer loses in case of a draw) the intermediary's local auction only if $b_1 > \beta(y_1)$ and, since this is

the bid that will be sent at the center, conditional on this event, it should also be that $b_1 > \max\{\beta(y_2), \dots, \beta(y_n)\}$ for her to obtain the good. The buyer will never bid outside $[\underline{b}, \bar{b}]$, so there is a value $y \in [0, 1]$ such that $\beta(y) = b_1$ and that will be chosen by buyer 1 to maximize her expected surplus:

$$\begin{aligned}\Pi(v, y) &= [v - \beta(y)]Pr(\beta(y) > \beta(y_1))Pr(\beta(y) > \max\{\beta(y_2), \dots, \beta(y_n)\} | \beta(y) > \beta(y_1)) = \\ &= [v - \beta(y)]Pr(y > y_1)Pr(y > \max\{y_2, \dots, y_n\} | y > y_1) = \\ &= [v - \beta(y)]Pr(y > \max\{y_1, \dots, y_n\}) = \\ &= [v - \beta(y)]Pr(y > y_1^{(nk-1)}) = [v - \beta(y)]F^{nk-1}(y)\end{aligned}\tag{4.25}$$

where $y_1^{(nk-1)}$ is the highest-order statistic among all opponent buyers' private valuations and where we have used the fact that $\beta(\cdot)$ is strictly increasing. For the existence of a symmetric BNE, we take the first-order condition at $y = v$, i.e. $\Pi'(v, v) = 0$ that yields:

$$\begin{aligned}\Pi'(v, v) &= [v - \beta(v)](nk - 1)F(v)^{nk-2}f(v) - \beta'(v)F^{nk-1}(v) = 0 \implies \\ &[\beta(v)F^{nk-1}(v)]' = v(nk - 1)F^{nk-2}(v)f(v)\end{aligned}\tag{4.26}$$

Using the fundamental theorem of Calculus:

$$\beta(v)F^{nk-1}(v) = \int_0^v x(nk - 1)F^{nk-2}(x)f(x)dx + c\tag{4.27}$$

where $c = 0$ since $\beta(0) = 0$. Using integration by parts yields:

$$\beta(v) = v - \frac{\int_0^v F^{nk-1}(x)dx}{F^{nk-1}(v)}\tag{4.28}$$

When the center sets a reserve price $\rho > 0$, this does not affect the first-order condition of equation (4.26) but only the boundary condition which now becomes $\beta(\rho) = \rho$, that results in equation (4.24). The proof is standard and can be found e.g. in Menezes and Monteiro (2005). \square

Since the buyer with the highest overall private valuation, only if this is not less than ρ , obtains the good, FPSB homogeneous intermediary auctions have the same allocation as homogeneous POST mechanisms for the same center's reserve price.

Then, if $F_\beta(\cdot) = F(\beta^{-1}(\cdot))$ is the c.d.f. of the submitted bids in each intermediary, and $H_\beta = F_\beta^k$, the c.d.f. of the highest-order statistic of the k local bids, the ex-ante

expected payment of an intermediary to the center is:

$$\begin{aligned} payment_{FPSB}(\rho) &= \int_{\rho}^{\beta(1)} f_{\beta_1}^{(k)}(u)[\rho H_{\beta}^{n-1}(\rho) + \int_{\rho}^u y h_{\beta_1}^{(n-1)}(y) dy] du = \\ &= \rho H_{\beta}^{n-1}(\rho)[1 - H_{\beta}(\rho)] + \int_{\rho}^{\beta(1)} y h_{\beta_1}^{(n-1)}(y)[1 - H_{\beta}(y)] dy \end{aligned} \quad (4.29)$$

Hence the ex-ante expected revenue for the center is:

$$\begin{aligned} revenue_{FPSB}(\rho) &= n\rho H_{\beta}^{n-1}(\rho)[1 - H_{\beta}(\rho)] + \int_{\rho}^{\beta(1)} y h_{\beta_2}^{(n)}(y) dy = \\ &= 1 - \rho H^n(\rho) - \int_{\rho}^1 F^{nk-1}(x) dx - \int_{\rho}^1 H_2^{(n)}(x) \beta'(x) dx \end{aligned} \quad (4.30)$$

where, if $y = \beta(x) \implies dy = \beta'(x)dx$, and $H_{\beta}(\beta(x)) = F_{\beta}^k(\beta(x)) = F^k(x)$. Then, using the facts that:

$$\beta'(x) = \frac{(nk-1)f(x) \int_{\rho}^x F^{nk-1}(y) dy}{F^{nk}(x)} \quad (4.31)$$

and:

$$H_2^{(n)}(x) = nH^{n-1}(x) - (n-1)H^n(x) = nF^{(n-1)k}(x) - (n-1)F^{nk}(x) \quad (4.32)$$

equation (4.30) can be written as:

$$\begin{aligned} revenue_{FPSB}(\rho) &= 1 - \rho F^{nk}(\rho) - \int_{\rho}^1 F^{nk-1}(y) dy - \int_{\rho}^1 \frac{n(nk-1)f(x)}{F^k(x)} \left(\int_{\rho}^x F^{nk-1}(y) dy \right) dx + \\ &+ \int_{\rho}^1 (n-1)(nk-1)f(x) \left(\int_{\rho}^x F^{nk-1}(y) dy \right) dx = 1 - \rho F^{nk}(\rho) - \int_{\rho}^1 F^{nk-1}(y) dy - \\ &- n(nk-1) \int_{\rho}^1 F^{nk-1}(y) \left(\int_y^1 \frac{f(x)}{F^k(x)} dx \right) dy + (n-1)(nk-1) \int_{\rho}^1 F^{nk-1}(y)(1 - F(y)) dy \end{aligned} \quad (4.33)$$

Then, for $k > 1$, letting $u = F(x) \implies du = f(x)dx$, then $\int_y^1 \frac{f(x)}{F^k(x)} dx = \int_{F(y)}^1 \frac{du}{u^k} = \frac{1-F^{k-1}(y)}{(k-1)F^{k-1}(y)}$ and the above equation becomes:

$$\begin{aligned} revenue_{FPSB}(\rho) &= 1 - \rho F^{nk}(\rho) - \int_{\rho}^1 F^{nk-1}(y) dy - \\ &- \frac{n(nk-1)}{k-1} \int_{\rho}^1 F^{(n-1)k}(y)(1 - F^{k-1}(y)) dy + (n-1)(nk-1) \int_{\rho}^1 F^{nk-1}(y)(1 - F(y)) dy \end{aligned} \quad (4.34)$$

Taking first-order condition w.r.t. ρ for the center's optimal reserve price, ρ_{FPSB}^* , for $k > 1$ yields:

$$\rho_{FPSB}^* = \frac{(nk - 1)[1 - F^{k-1}(\rho_{FPSB}^*)] - [(n-1)k - 1](k-1)F^{k-1}(\rho_{FPSB}^*)[1 - F(\rho_{FPSB}^*)]}{k(k-1)F^{k-1}(\rho_{FPSB}^*)f(\rho_{FPSB}^*)} \quad (4.35)$$

When $k = 1$, $\int_y^1 \frac{f(x)}{F^k(x)} dx = \int_{F(y)}^1 \frac{du}{u} = -\ln F(y)$, so equation (4.33) becomes:

$$\begin{aligned} revenue_{FPSB}(\rho) &= 1 - \rho F^n(\rho) - \int_{\rho}^1 F^{n-1}(y) dy + \\ &+ n(n-1) \int_{\rho}^1 F^{n-1}(y) \ln F(y) dy + (n-1)^2 \int_{\rho}^1 F^{n-1}(y)(1 - F(y)) dy \end{aligned} \quad (4.36)$$

Taking first-order condition in this case, i.e. $k = 1$, yields the following equation for the center's optimal reserve price, ρ_{FPSB}^* :

$$\rho_{FPSB}^* = -\frac{(n-1) \ln F(\rho_{FPSB}^*) + (n-2)[1 - F(\rho_{FPSB}^*)]}{f(\rho_{FPSB}^*)} \quad (4.37)$$

The ex-ante expected profit of an intermediary is:

$$\begin{aligned} profit_{FPSB}(\rho) &= \int_{\rho}^{\beta(1)} f_{\beta_1}^{(k)}(y) \int_{\rho}^y H_{\beta}^{n-1}(u) du dy = \\ &= \int_{\rho}^{\beta(1)} H_{\beta}^{n-1}(u)[1 - H_{\beta}(u)] du = \int_{\rho}^1 H^{n-1}(y)[1 - H(y)]\beta'(y) dy \end{aligned} \quad (4.38)$$

Finally, a buyer expects the same surplus as with a POST auction for the same center's reserve price, given that the allocation in both mechanisms is the same for that reserve price, i.e. $\Pi_{FPSB}(v) = \Pi_{POST}(v)$ for all $v \in [0, 1]$ when $\rho_{FPSB} = \rho_{POST}$.

In what follows, we provide a comparison of the aforementioned intermediary mechanisms, combining our theoretical insights with numerical results.

4.2.4 Comparison of the Three Intermediary Mechanisms

Having expressed the expected utilities for all scenarios, in this subsection, we compare, both theoretically and numerically, the resulting intermediaries' expected profits, center's expected revenue and social welfare under the three mechanisms for homogeneous populations of intermediaries.

We start with Lemma 4.3 below, comparing the optimal reserve prices of the center under the three mechanisms.

Lemma 4.3. *When all intermediaries implement PRE auctions, the center's optimal reserve price, ρ_{PRE}^* , is always not less than the optimal reserve price when all intermediaries implement POST auctions, ρ_{POST}^* .*

Proof. Let ρ_{PRE}^* , ρ_{POST}^* be the optimal reserve prices for equations (4.16) and (4.21), respectively. Taking the first-order derivative with respect to ρ in equation (4.15) yields:

$$\frac{d\text{revenue}_{PRE}(\rho)}{d\rho} = -H^n(\rho) - n\rho H^{n-1}(\rho)h(\rho) + G_2^{(n)}(\rho) \quad (4.39)$$

Now, applying the condition of (4.21) in the equation above, we get:

$$\begin{aligned} \frac{d\text{revenue}_{PRE}(\rho)}{d\rho}|_{\rho=\rho_{POST}^*} &= -H^n(\rho_{POST}^*) - n \frac{1 - H(\rho_{POST}^*)}{h(\rho_{POST}^*)} H^{n-1}(\rho_{POST}^*)h(\rho_{POST}^*) + \\ &+ G_2^{(n)}(\rho_{POST}^*) = (n-1)H^n(\rho_{POST}^*) - nH^{n-1}(\rho_{POST}^*) + G_2^{(n)}(\rho_{POST}^*) = \\ &= G_2^{(n)}(\rho_{POST}^*) - H_2^{(n)}(\rho_{POST}^*) = \\ &= nG^{n-1}(\rho_{POST}^*) - (n-1)G^n(\rho_{POST}^*) - [nH^{n-1}(\rho_{POST}^*) - (n-1)H^n(\rho_{POST}^*)] \geq 0 \end{aligned} \quad (4.40)$$

since $G(y) - H(y) = kF^{k-1}(y) - (k-1)F^k(y) - F^k(y) = (k-1)F^{k-1}(y)[1 - F(y)] \geq 0 \implies G(y) \geq H(y)$ for any $y \in [0, 1]$, and the function $nx^{n-1} - (n-1)x^n$ is an increasing function of x . The above inequality is strict for $k > 1$ and $\rho_{POST}^* \in (0, 1)$. Hence, since for the existence of an optimal reserve price, the function $\text{revenue}_{PRE}(\rho)$ should be concave, the above equation means that $\rho_{POST}^* \leq \rho_{PRE}^*$. \square

This result allows us to compare the center's expected revenue for the two SPSB auctions as follows:

Theorem 4.4. *When all intermediaries implement POST auctions, the center's optimal expected revenue is at least the expected revenue when all intermediaries implement PRE auctions.*

Proof. Taking the difference of (4.15) and (4.20), for the same reserve price, ρ , we obtain that:

$$\text{revenue}_{POST}(\rho) - \text{revenue}_{PRE}(\rho) = \int_{\rho}^1 [G_2^{(n)}(y) - H_2^{(n)}(y)]dy \geq 0 \quad (4.41)$$

where $G_2^{(n)} = nG^{n-1} - (n-1)G^n$, $H_2^{(n)} = nH^{n-1} - (n-1)H^n$. This is since $G(y) \geq H(y)$ and the function $nx^{n-1} - (n-1)x^n$ is a strictly increasing function of x . We should also notice that the inequality is strict for any reasonable reserve, $\rho \in [0, 1)$.

If ρ_{PRE}^* , ρ_{POST}^* are the optimal reserve prices for (4.16) and (4.21), from the previous result, we have that $revenue_{POST}(\rho_{POST}^*) \geq revenue_{POST}(\rho_{PRE}^*) \geq revenue_{PRE}(\rho_{PRE}^*)$. This, combined with the result of Lemma 4.3, concludes the proof. \square

Proposition 4.5 below compares the intermediaries' expected profits for the same SPSB variations.

Proposition 4.5. *For any reserve price, ρ , of the center, the expected profits of PRE intermediary auctions, $profit_{PRE}(\rho)$, are always not less than the corresponding profits of POST intermediary auctions, $profit_{POST}(\rho)$, when all intermediaries implement the same mechanism.*

Proof. This happens if $G^{n-1}(y) \geq H^{n-1}(y)$ from equations (4.17) and (4.22), which is true (see proof of Lemma 4.3). \square

We now compare the corresponding values for POST and FSPSB intermediaries. We start with our result for the center's optimal reserve price for homogeneous intermediaries of these two kinds.

Lemma 4.6. *When all intermediaries implement FSPSB auctions, the center's optimal reserve price, ρ_{FSPSB}^* , is always not less than the optimal reserve price when all intermediaries implement POST auctions, ρ_{POST}^* .*

Proof. Let ρ_{FSPSB}^* , ρ_{POST}^* be the optimal reserve prices for the center when all intermediaries implement FSPSB and POST auctions respectively.

We start with the case of $k > 1$ buyers per intermediary. Taking the first-order derivative with respect to ρ in equation (4.34) yields:

$$\begin{aligned} \frac{d revenue_{FSPSB}(\rho)}{d\rho} &= nF^{(n-1)k}(\rho) \left\{ \frac{nk-1}{k-1} [1 - F^{k-1}(\rho)] - \right. \\ &\quad \left. - [(n-1)k-1]F^{k-1}(\rho)[1 - F(\rho)] - kF^{k-1}(\rho)f(\rho)\rho \right\} \end{aligned} \quad (4.42)$$

Then, replacing $\rho = \rho_{POST}^*$ from equation (4.21), gives:

$$\begin{aligned} \frac{d revenue_{FSPSB}(\rho)}{d\rho} \Big|_{\rho=\rho_{POST}^*} &= nF^{(n-1)k}(\rho_{POST}^*) \left\{ \frac{nk-1}{k-1} [1 - F^{k-1}(\rho_{POST}^*)] - \right. \\ &\quad \left. - [(n-1)k-1]F^{k-1}(\rho_{POST}^*)[1 - F(\rho_{POST}^*)] - [1 - F^k(\rho_{POST}^*)] \right\} = \\ &= nF^{(n-1)k}(\rho_{POST}^*) \left\{ \frac{nk-1-(k-1)}{k-1} - \frac{nk-1-(k-1)}{k-1} F^{k-1}(\rho_{POST}^*) - \right. \\ &\quad \left. - (n-1)k[F^{k-1}(\rho_{POST}^*) - F^k(\rho_{POST}^*)] \right\} = \end{aligned}$$

$$\begin{aligned}
 &= nF^{(n-1)k}(\rho_{POST}^*) \left\{ \frac{(n-1)k}{k-1} [1 - F^{k-1}(\rho_{POST}^*)] - (n-1)kF^{k-1}(\rho_{POST}^*)[1 - F(\rho_{POST}^*)] \right\} = \\
 &= \frac{n(n-1)k}{k-1} F^{(n-1)k}(\rho_{POST}^*) \left\{ 1 - F^{k-1}(\rho_{POST}^*) - (k-1)F^{k-1}(\rho_{POST}^*)[1 - F(\rho_{POST}^*)] \right\} = \\
 &= \frac{n(n-1)k}{k-1} F^{(n-1)k}(\rho_{POST}^*)[1 - G(\rho_{POST}^*)] \geq 0
 \end{aligned} \tag{4.43}$$

since $G(\rho_{POST}^*) = kF^{k-1}(\rho_{POST}^*) - (k-1)F^k(\rho_{POST}^*) \leq 1$. Hence, since for the existence of an optimal reserve price, the function $\text{revenue}_{FPSB}(\rho)$ should be concave, the above equation means that $\rho_{POST}^* \leq \rho_{FPSB}^*$.

Similarly, for $k = 1$, taking the first-order derivative of equation (4.36) w.r.t. ρ yields:

$$\frac{d \text{revenue}_{FPSB}(\rho)}{d\rho} = -nF^{n-1}(\rho) \left\{ (n-2)[1 - F(\rho)] + \rho f(\rho) + (n-1)\ln F(\rho) \right\} \tag{4.44}$$

Hence, taking this derivative at ρ_{POST}^* from equation (4.21) gives:

$$\begin{aligned}
 \frac{d \text{revenue}_{FPSB}(\rho)}{d\rho} \Big|_{\rho=\rho_{POST}^*} &= -nF^{n-1}(\rho_{POST}^*) \left\{ (n-2)[1 - F(\rho_{POST}^*)] + \right. \\
 &\quad \left. + 1 - F(\rho_{POST}^*) + (n-1)\ln F(\rho_{POST}^*) \right\} = \\
 &= -n(n-1)F^{n-1}(\rho_{POST}^*)[1 - F(\rho_{POST}^*) + \ln F(\rho_{POST}^*)] \geq 0
 \end{aligned} \tag{4.45}$$

since the function $1 - x + \ln x \leq 0$ for $x \in [0, 1]$. For the same reason, this means that $\rho_{POST}^* \leq \rho_{FPSB}^*$. \square

Given this last result, we are now able to compare the center's ex-ante expected revenue for these two types of intermediaries.

Theorem 4.7. *The expected revenue of the center when intermediaries implement POST auctions, revenue_{POST} , is always at least the expected revenue when the latter implement FPSB auctions, revenue_{FPSB} .*

Proof. Let us for the moment assume that the buyers' private valuations, v_i , are known, that $v_i > v_j$ when $i < j$, and that $\rho = 0$. If all intermediaries implement POST auctions, then the center receives v_2 when buyers 1 and 2 are in different intermediaries. It receives v_3 when buyers 1 and 2 are in the same intermediary but buyer 3 is not, v_4 when buyers 1, 2 and 3 are in the same intermediary and buyer 4 is not and so on, until the case where buyers $1, \dots, k$ are in the same intermediary and buyer $k+1$ is not when the center obtains v_{k+1} . For the same allocation of buyers to FPSB intermediaries, the center receives $\beta(v_2), \beta(v_3), \beta(v_4), \dots, \beta(v_{k+1})$ respectively. When the private valuations are not known, in BNE (equation (4.24)), $\beta(v) < v$ for all $v \in (\rho, 1]$, hence $\text{revenue}_{FPSB} < \text{revenue}_{POST}$. Also, since the allocation of the good is the

same for both types of intermediary auctions and the same center's reserve price ρ , then, for $\rho > 0$, $revenue_{FPSB}(\rho) = \mathbb{E}[\max\{\beta((Y_1^{(k)})_2^{(n)}), \rho\}] \leq \mathbb{E}[\max\{(Y_1^{(k)})_2^{(n)}, \rho\}] = revenue_{POST}(\rho)$, where $(Y_1^{(k)})_2^{(n)}$ is a random variable that corresponds to the second-highest-order statistic among n samples, each of which is the highest-order statistic among k samples. So, $revenue_{FPSB}(\rho_{FPSB}^*) \leq revenue_{POST}(\rho_{FPSB}^*) < revenue_{POST}(\rho_{POST}^*)$. \square

The above theorem along with the fact that the allocation is the same for both FPSB and POST intermediaries when the center's reserve price is the same lead to the opposite direction for the intermediaries' ex-ante expected profits, as shown in the next corollary.

Corollary 4.8. *The expected profits of intermediaries implementing FPSB auctions, $profit_{FPSB}(\rho)$, are always at least the corresponding profits for POST auctions, $profit_{POST}(\rho)$, for the same center's reserve price, ρ .*

Proof. The total revenue obtained by the center and the intermediaries for POST auctioneers (equations (4.20) and (4.22) respectively) can be written as follows:

$$\begin{aligned}
revenue_{POST}(\rho) + n \cdot profit_{POST}(\rho) &= 1 - \rho H^n(\rho) - \int_{\rho}^1 H_2^{(n)}(y) dy + \\
&+ n \int_{\rho}^1 H^{n-1}(y)[1 - G(y)] dy = 1 - \rho H^n(\rho) - n \int_{\rho}^1 H^{n-1}(y) dy + \\
&+ (n-1) \int_{\rho}^1 H^n(y) dy + n \int_{\rho}^1 H^{n-1}(y) dy - n \int_{\rho}^1 H^{n-1}(y) G(y) dy = \\
&= 1 - \rho F^{nk}(\rho) + (n-1) \int_{\rho}^1 F^{nk}(y) dy - n \int_{\rho}^1 F^{(n-1)k}(y) [kF^{k-1}(y) - (k-1)F^k(y)] dy = \\
&= 1 - \rho F^{nk}(\rho) + (nk-1) \int_{\rho}^1 F^{nk}(y) dy - nk \int_{\rho}^1 F^{nk-1}(y) dy = \\
&= 1 - \rho F^K(\rho) - \int_{\rho}^1 F_2^{(K)}(y) dy
\end{aligned} \tag{4.46}$$

Similarly, using equations (4.30), (4.38) and (4.31), the total revenue obtained by the center and the FPSB intermediaries can be written as:

$$\begin{aligned}
revenue_{FPSB}(\rho) + n \cdot profit_{FPSB}(\rho) &= 1 - \rho H^n(\rho) - \int_{\rho}^1 F^{nk-1}(x) dx - \\
&- \int_{\rho}^1 H_2^{(n)}(x) \beta'(x) dx + n \int_{\rho}^1 H^{n-1}(x)[1 - H(x)] \beta'(x) dx =
\end{aligned}$$

$$\begin{aligned}
&= 1 - \rho H^n(\rho) - \int_{\rho}^1 F^{nk-1}(x)dx - n \int_{\rho}^1 H^{n-1}(x)\beta'(x)dx + \\
&+ (n-1) \int_{\rho}^1 H^n(x)\beta'(x)dx + n \int_{\rho}^1 H^{n-1}(x)\beta'(x)dx - n \int_{\rho}^1 H^n(x)\beta'(x)dx = \\
&= 1 - \rho F^{nk}(\rho) - \int_{\rho}^1 F^{nk-1}(x)dx - \int_{\rho}^1 F^{nk}(x)\beta'(x)dx = \\
&= 1 - \rho F^{nk}(\rho) - \int_{\rho}^1 F^{nk-1}(x)dx - \int_{\rho}^1 F^{nk}(x)(nk-1) \frac{f(x)}{F^{nk}(x)} \int_{\rho}^x F^{nk-1}(y)dy dx = \\
&= 1 - \rho F^{nk}(\rho) - \int_{\rho}^1 F^{nk-1}(x)dx - (nk-1) \int_{\rho}^1 F^{nk-1}(y)[1-F(y)]dy = \\
&= 1 - \rho F^{nk}(\rho) - nk \int_{\rho}^1 F^{nk-1}(x)dx + (nk-1) \int_{\rho}^1 F^{nk}(x)dx = \\
&= 1 - \rho F^K(\rho) - \int_{\rho}^1 F_2^{(K)}(x)dx
\end{aligned} \tag{4.47}$$

So $revenue_{FPSB}(\rho) + n \cdot profit_{FPSB}(\rho) = revenue_{POST}(\rho) + n \cdot profit_{POST}(\rho)$. From Theorem 4.7, $revenue_{FPSB}(\rho) \leq revenue_{POST}(\rho) \implies profit_{FPSB}(\rho) \geq profit_{POST}(\rho)$.

□

Our theoretical analysis above shows that the center prefers POST auctions for the intermediaries. However, no ranking between FPSB and PRE auctions has been provided. Similarly, it is not clear which mechanism is better for the intermediaries. Corollary 4.8 is valid for the same center's reserve price, however $\rho_{FPSB}^* \geq \rho_{POST}^*$ from Lemma 4.6, so the equilibrium profits are not generally comparable for all distributions. Moreover, when each intermediary has a single buyer, we can see that the FPSB auction is the only mechanism that yields positive profit, but for $k > 1$ buyers per intermediary, it is not possible to obtain a general ranking of the expected profits for the intermediaries. We show this and other results using a numerical evaluation.

In more detail, we consider a setting where buyers have private valuations that are i.i.d. drawn from the uniform distribution $U(0, 1)$. Furthermore, we first consider a population of $K = 100$ buyers and vary the number of intermediaries ($n = 2, 4, 5, 10, 20, 25, 50$), keeping the total number of buyers fixed. Figure 4.2 shows the resulting expected profits for the intermediaries. It is clear that intermediaries' expected profits are very small, in the order of 10^{-3} , ranging from approximately 0.1% up to 0.5% of the center's revenue in total, significantly decreasing as the number of intermediaries increases. As shown, in this example, PRE auctions yield higher expected profits for a small number of intermediaries, whereas FPSB auctions yield higher expected profits in the remaining cases. POST auctions seem to perform worse in terms of profit than their PRE counterpart, but one can verify that for $n = 2, k = 2$, the opposite happens⁵, so a general ranking of

⁵For $n = 2, k = 2$, $\rho_{POST}^* = \frac{\sqrt{3}}{3}$, $\rho_{PRE}^* = \frac{2}{3}$ and $profit_{POST}(\rho_{POST}^*) = \frac{4}{45} - \frac{2}{15\sqrt{3}} \approx 0.0119 > profit_{PRE}(\rho_{PRE}^*) = \frac{14}{1215} \approx 0.0115$

the two is not possible. Nevertheless, POST auctioneers receive a smaller expected profit in all cases shown than FPSB intermediaries.

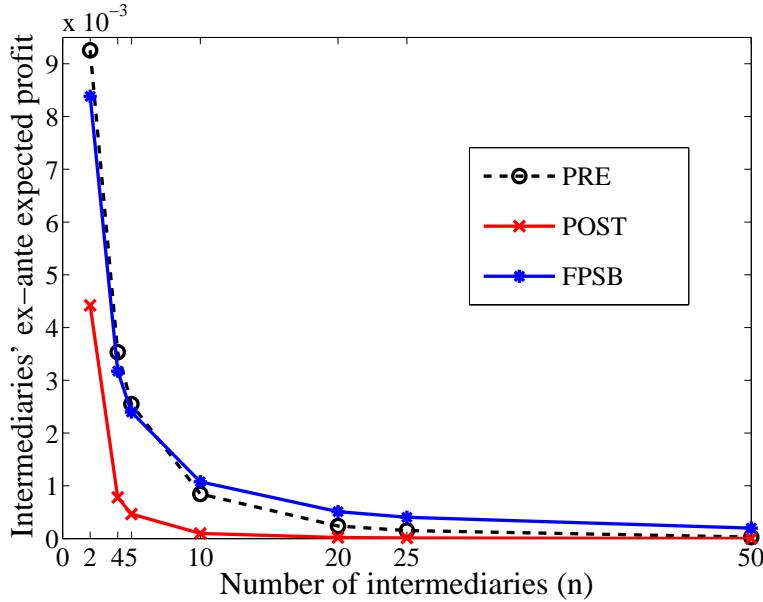


FIGURE 4.2: Intermediaries' ex-ante expected profits for the three different mechanisms with increasing number of opponents for a fixed number of $K = 100$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

Next, Figure 4.3 shows the buyers' ex-ante expected surplus for the three mechanisms where it is clear that a population of PRE intermediaries yields the highest surplus to the buyers among all three mechanisms, and the expected surplus for POST intermediaries is marginally higher than that for FPSB auctioneers. Figure 4.4 illustrates the center's expected revenue and corresponding social welfare for the mechanisms for the same example. As can be seen, the social welfare for the more efficient FPSB and POST auctions slightly increases with the number of intermediaries (getting very close to the social welfare of 0.9901 for the setting without intermediaries), whereas the opposite effect happens for the PRE mechanism. The latter is due to the fact that, as the number of buyers per intermediary decreases, the misallocation inefficiency increases, thus further decreasing the corresponding social welfare. Also POST intermediaries are the most efficient among the three mechanisms given that the center's optimal reserve price is lower than that for FPSB intermediary auctioneers. As for the revenue of the center, this slightly increases with the number of intermediaries for the more efficient FPSB and POST mechanisms, whereas the opposite effect happens for the setting of PRE mechanisms. However, in all cases, the center's expected revenue is higher for FPSB auctioneers compared to PRE intermediaries. Finally, Figure 4.5 depicts the optimal reserve price of the center for the three mechanisms. As shown, for POST and FPSB intermediaries, it decreases with the number of intermediaries. For POST auctioneers this is due to the fact that the corresponding number of buyers per intermediary decreases, and the optimal

reserve price in these cases is only a function of the latter number. The corresponding decrease for FPSB mechanisms is less apparent. In contrast, the center's optimal reserve price for PRE intermediaries remains almost constant and is always higher than the corresponding price for POST and FPSB intermediaries.

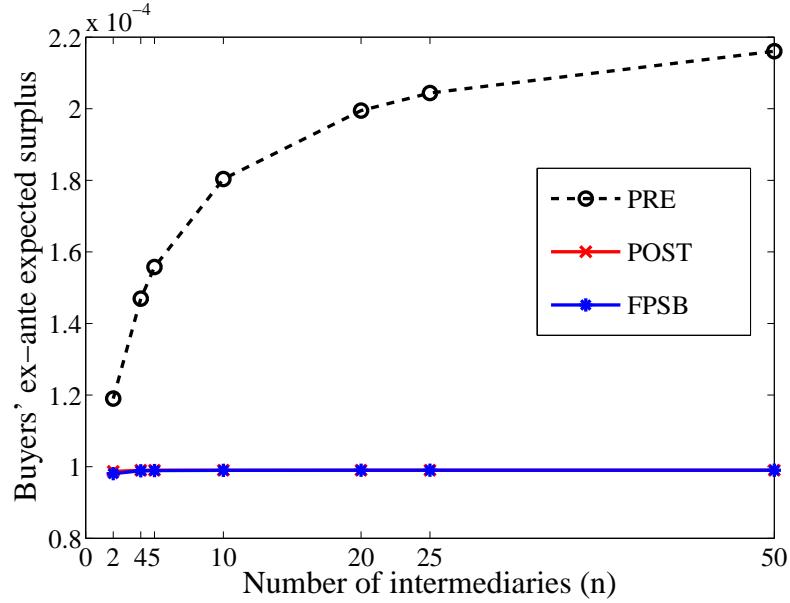


FIGURE 4.3: Buyers' ex-ante expected surplus for the three different mechanisms with increasing number of intermediaries for a fixed number of $K = 100$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

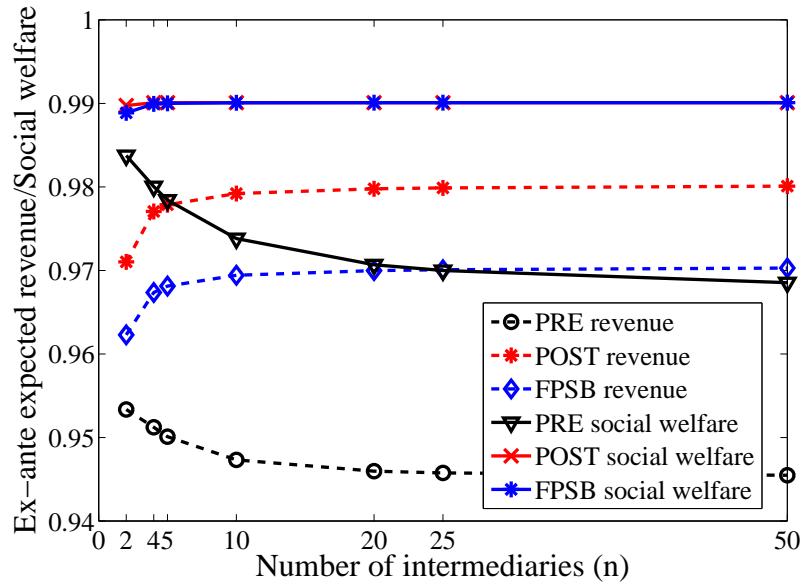


FIGURE 4.4: Center's ex-ante expected revenue and social welfare for the three different mechanisms with an increasing population of intermediaries and a fixed number of $K = 100$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

As was previously shown, the generated revenue, profit and surplus for the center, the intermediaries and the buyers depends, in general, both on the number of intermediaries

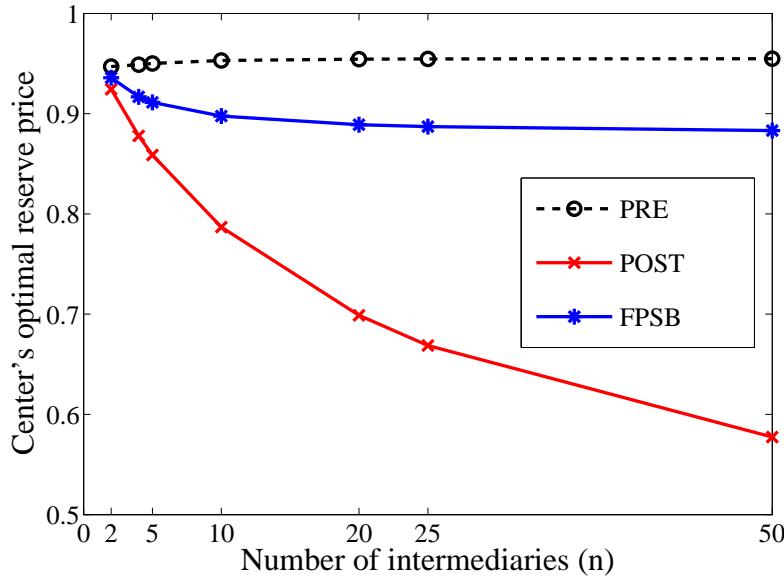


FIGURE 4.5: Center's optimal reserve price for the three different mechanisms with an increasing population of intermediaries and a fixed number of $K = 100$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

and the size of their local markets (i.e. number of buyers per intermediary). We complete our numerical evaluation by removing the limitation of fixed K , considering the full set of $n = 2, \dots, 50$ intermediaries, each with a number of $k = 2, \dots, 50$ local buyers. Figures 4.6, 4.7, 4.8, 4.9 and 4.10 illustrate the intermediaries' expected profits, buyers' expected surplus, the center's expected revenue, the social welfare and the center's optimal reserve price, respectively, for the three mechanisms. As can be seen, FPSB intermediary auctions yield higher profit for a large number of intermediaries with a small number of buyers each, whereas PRE intermediaries are more profitable in the opposite case, with fewer intermediaries having more buyers each. Also, the ordering of the mechanisms for the center seems consistent, with the center having higher expected revenue for POST, followed by FPSB and PRE mechanisms, and as the number of intermediaries and buyers increase, this revenue approaches the expected revenue without intermediaries (OPT). Buyers' expected surplus is more difficult to compare due to the very small absolute differences, however PRE mechanisms seem to be better in sum, followed by POST and FPSB auctions. What's more, the social welfare is higher for FPSB intermediaries, followed by their POST and then PRE counterparts. Finally, the center's optimal reserve price increases both with the number of buyers and intermediaries for PRE and FPSB intermediaries and is always higher for the former type of intermediaries compared to the latter. Both reserve prices are also higher than the corresponding reserve price for POST intermediary mechanisms.

In the next section, we continue our analysis for heterogeneous intermediaries where we look at the incentives of intermediaries to switch to another mechanism from a homogeneous population of other intermediaries.

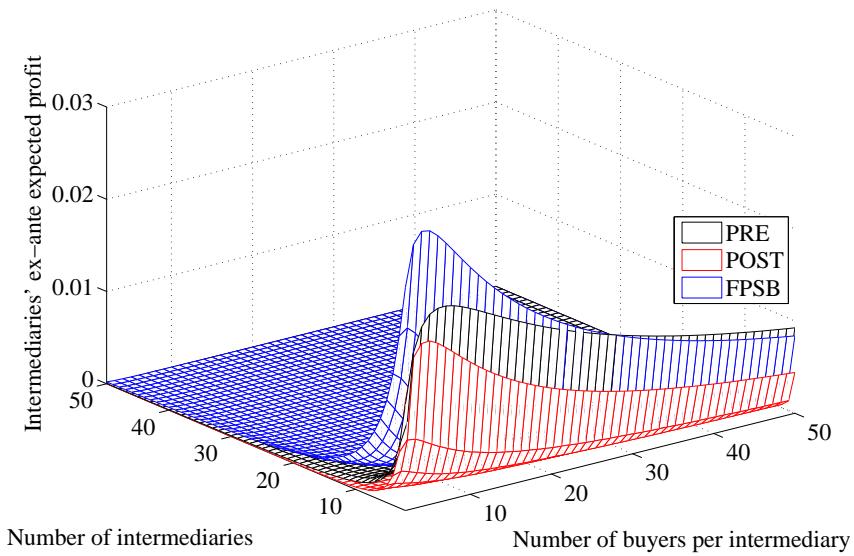


FIGURE 4.6: Intermediaries' ex-ante expected profits for the three different mechanisms with increasing number of opponents and buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

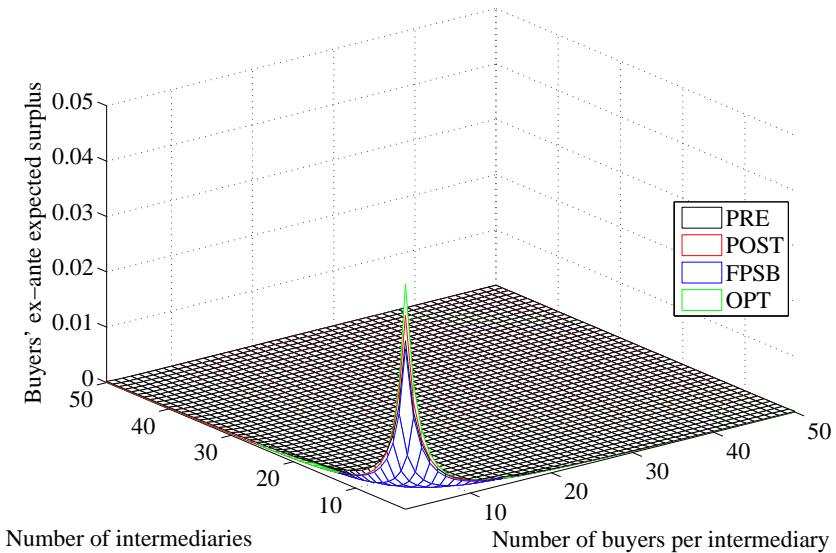


FIGURE 4.7: Buyers' ex-ante expected surplus for the three different mechanisms with increasing number of intermediaries and buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

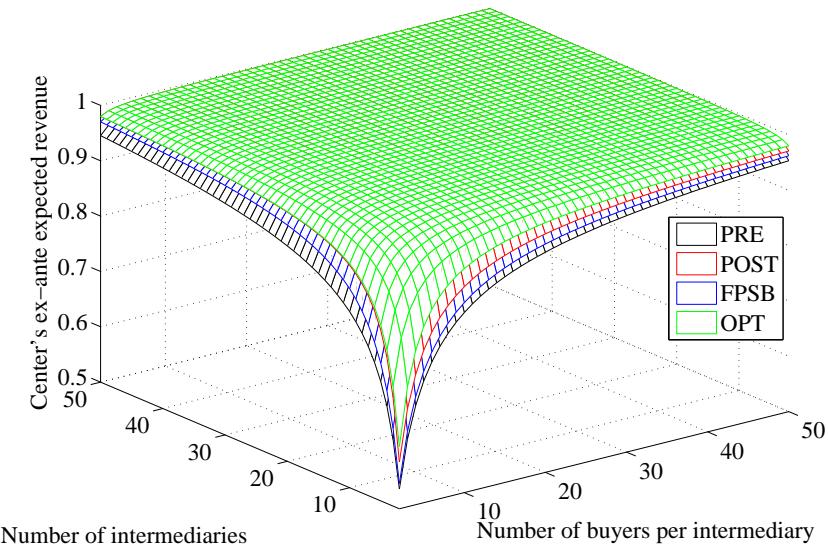


FIGURE 4.8: Center's ex-ante expected revenue for the three different mechanisms with an increasing population of intermediaries and buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

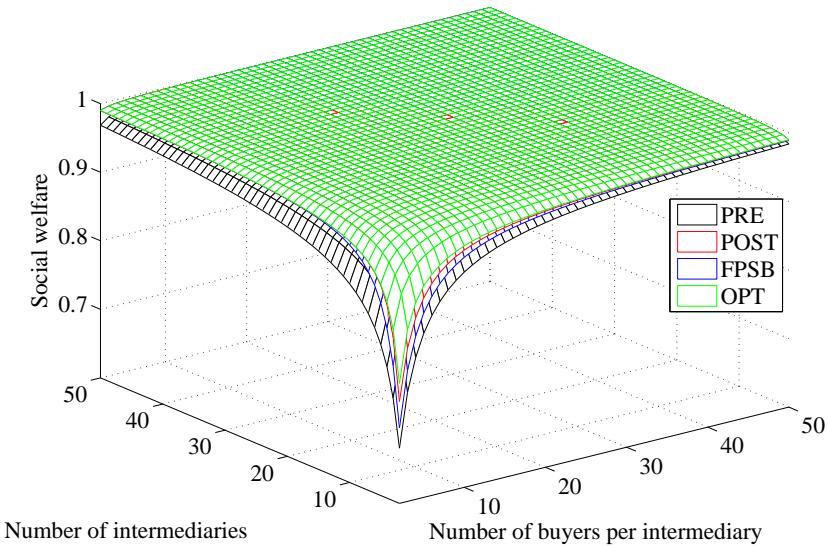


FIGURE 4.9: Social welfare for the three different mechanisms with an increasing population of intermediaries and buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

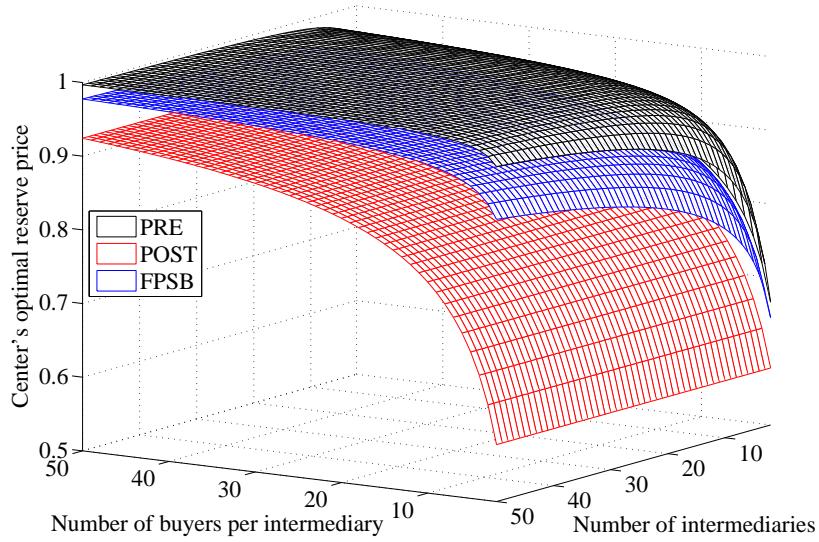


FIGURE 4.10: Center's optimal reserve price for the three different mechanisms with increasing number of intermediaries and buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

4.3 Heterogeneous Symmetric Intermediaries

In the previous section, we have assumed intermediaries are homogeneous, i.e. that all intermediaries implement the same mechanism. In this section, we remove this limitation, considering the pairwise competition between the three auction mechanisms. Specifically, for tractability reasons, we consider a homogeneous population of $n - 1$ intermediaries implementing one mechanism and one intermediary switching to a different mechanism. As will be seen, the complexity of the equilibrium bidding strategies for FPSB auctions makes it difficult to draw conclusions. Hence, our results are only for the competition in the two Vickrey variations. In more detail, for the competition between PRE and POST mechanisms, in the next subsection, we show that intermediaries do not have a unilateral incentive to deviate from the majority mechanism to the other, albeit only when keeping the center's reserve price fixed.

4.3.1 Pre- versus Post-Award Second-Price Sealed-Bid Auctions

Assume that there are $n - 1$ intermediaries that implement PRE (POST) auctions and one intermediary that switches to a POST (PRE) auction, and that each intermediary, s_i , has, as before, exactly k buyers in his local market. Then, *keeping the reserve price of the center fixed*⁶, we show that no intermediary has a strict incentive to deviate from homogeneous PRE (POST) to POST (PRE) auctions.

⁶In equilibrium this might not happen since the center's optimal reserve price with asymmetric intermediaries might be different for the two cases.

Proposition 4.9. *For any reserve price, ρ , of the center, an intermediary has no strict incentive to switch from a PRE (POST) to a POST (PRE) auction when all other intermediaries implement PRE (POST) auctions.*

Proof. First, assume that $n - 1$ intermediaries implement PRE auctions, and one intermediary switches to a POST auction. Then the deviator's expected profit will be:

$$\begin{aligned} \text{profit}_{\text{dev.POST}}(\rho) &= G(\rho)^{n-1} \int_{\rho}^1 (y - \rho) g(y) dy + \\ &+ \int_{\rho}^1 \int_{\rho}^{y_2} (y_2 - x_1) g_1^{(n-1)}(x_1) g_2(y_2) dx_1 dy_2 = \int_{\rho}^1 G^{n-1}(x)(1 - G(x)) dx \end{aligned} \quad (4.48)$$

which is the same as when implementing pre-award payments. In contrast, if $n - 1$ intermediaries use POST auctions, then a deviating intermediary's expected profit when implementing a PRE auction will be:

$$\text{profit}_{\text{dev.PRE}}(\rho) = \int_{\rho}^1 g(y) \int_{\rho}^y H^{n-1}(x) dx dy = \int_{\rho}^1 H^{n-1}(x)(1 - G(x)) dx \quad (4.49)$$

which is again the same as with POST. \square

We now continue with a preliminary discussion on the competition between first-price sealed-bid and our two Vickrey variations.

4.3.2 Pre- and Post-Award Second-Price Sealed-Bid versus First-Price Sealed-Bid Auctions

In this subsection, we characterize the condition for the FPSB equilibrium bidding function when an intermediary implements an FPSB auction against $n - 1$ PRE or POST mechanisms respectively. Ignoring for the moment the center's reserve price, let us assume that buyers in this intermediary follow a symmetric, increasing bidding strategy $\sigma_{\ell}(\cdot)$, for $\ell = \{PRE, POST\}$ when bidding against $n - 1$ PRE or POST intermediaries respectively. Then, a buyer with private valuation v who bids s expects surplus that can be written as:

$$\Pi_{FPSB-\ell}(v, s) = (v - s) F^{k-1}(\sigma_{\ell}^{-1}(s)) M^{n-1}(s) \quad (4.50)$$

where $M(s) = F^k(s)$ for POST and $M(s) = F_2^{(k)}(s)$ for PRE opponent intermediaries respectively. More specifically, the winning FPSB local bid must be higher than the bids of the two Vickrey variations, which in this case will also equal their private valuations.

Taking the first-order condition and assuming that in equilibrium $s = \sigma_\ell(v)$, the above equation yields:

$$\begin{aligned} \frac{\partial \Pi_{FPSB-\ell}(v, s)}{\partial s} &= 0 \implies \frac{d}{dv}[F^{k-1}(v)M^{n-1}(\sigma_\ell(v))\sigma_\ell(v)] = \\ &= v \frac{d}{dv}[F^{k-1}(v)M^{n-1}(\sigma_\ell(v))] \end{aligned} \quad (4.51)$$

Solving this, and taking the condition that $\sigma_\ell(\rho_\ell) = \rho_\ell$ yields:

$$\sigma_\ell(v) = v - \frac{\int_{\rho_\ell}^v F^{k-1}(y)M^{n-1}(\sigma_\ell(y))dy}{F^{k-1}(v)M^{n-1}(\sigma_\ell(v))} \quad (4.52)$$

Given that this is a non-linear differential equation, a generic closed form solution seems implausible. This is also true for the uniform distribution when reserve prices are present⁷. This concludes our analysis for competing intermediaries with no reserve prices and with captive buyers.

4.4 Summary

In this chapter, we analyzed the competition between three intermediary mechanisms, focusing on the case of symmetric homogeneous intermediaries. Our analysis suggests that FPSB performs well both in terms of profit and efficiency. The advantages of FPSB auctions are also verified in practice by the fact that this is the dominant intermediary mechanism implemented, both in ad exchanges (Elmeleegy et al., 2013) and auctions with subcontracting (Nakabayashi, 2010) that our model encompasses. This is true not only from an economic point of view but also from a business perspective since FPSB offer greater transparency to the final buyers. However, the latter need to employ BNE bidding strategies that might be difficult to derive and coordinate on. Moreover, FPSB auctions are known for suffering from stability issues in repeated settings, such as the ones we observe in ad exchanges (Edelman and Ostrovsky, 2007). From the experiments with uniform distribution, we see that, when buyers are captive, POST auctions generally yield lower expected profit than their counterpart, and so are less likely to be adopted in this scenario. Interestingly, next, we show that, when buyers strategically select their intermediary, the opposite in general holds. Before we do this, in the following chapter, we keep the captivity assumption but let intermediaries set reserve prices. As will be shown, the problem becomes technically challenging and so we make use of learning techniques to get an approximation of the resulting profits and social welfare.

⁷When the reserve price is zero, and there are $n-1$ other POST intermediaries, the equilibrium bidding function for the uniform distribution $U(0, 1)$ is $\sigma_\ell(v) = \frac{nk-1}{nk}v$, i.e. the same as in an FPSB auction with nk bidders.

Chapter 5

Intermediaries with Captive Buyers: The Effects of Reserve Prices

So far, we have assumed that intermediaries do not impose reserve prices. In this chapter, we relax this restriction, as a reserve price is known to increase an auctioneer's revenue. However, given that the reserve-price-setting problem for competing auctioneers is technically challenging, we restrict our analysis to a duopoly intermediary setting.

Specifically, in accordance to the model described in Section 3.3, in what follows, we assume that both the center and the intermediaries have selected their mechanism in advance and, in the setting studied within this chapter, that buyers are allocated to the intermediaries such that each intermediary has exactly the same number of buyers. Then, the center announces a reserve price for the good to be auctioned to the intermediaries who then, based on this information, strategically select and announce their local reserve prices to their buyers. Buyers learn their private valuations for the good and submit a bid for the good to their allocated intermediary. Intermediaries then run local auctions with their buyers' bids subject to the constraint imposed by the reserve price to determine a winner, if any, and a payment contingent on winning the good at the central auction, and then submit a single bid (if there was some qualified bid) to the center. The center then runs its auction with intermediaries' bids, determines a winning intermediary, if any, and payment and allocates the good to this intermediary, if there is a winner, who then allocates the good to his winning local buyer for the pre-determined price.

To this end, we start with the motivating scenario of a single intermediary in Section 5.1. We then extend our analysis for the case of competing intermediaries in Section 5.2 where we characterize all players' expected utilities and present our theoretical and numerical results for homogeneous intermediaries and heterogeneous PRE versus POST

intermediary mechanisms. Then, in Section 5.3 we compare the profits and efficiency of the previous settings in equilibrium and contrast with the results of the previous chapter. Finally, Section 5.4 summarizes our findings.

5.1 Special Case: Single Intermediary

Let us again consider the case where the center offers a take-it-or-leave-it price, ρ , to one intermediary representing all the buyers' population (i.e. K buyers). Given the lack of intermediary competition, no misallocation inefficiencies arise in this setting. For this reason, since in addition the center's ex-ante expected revenue is the same for the three mechanisms, as in the case of no intermediary's reserve price (Section 4.1), the center's and intermediary's expected revenue and profit are the same for all standard auctions (Riley and Samuelson, 1981), so we can assume, without loss of generality, that the intermediary runs a PRE auction. Hence, the center proposes ρ to the intermediary, who runs a sealed-bid second-price auction with reserve price, $r \geq \rho$. We will now derive the intermediary's and center's ex-ante expected profit and revenue respectively and then use these to calculate the optimal reserve prices for both. In this case, the distribution of the second price in the intermediary's auction, y , is given by:

- $y = 0$ with probability $F^K(r)$.
- $y = r$ with probability $K(1 - F(r))F^{K-1}(r)$ (i.e. the probability that $K - 1$ bids are less than r and one above r).
- $y > r$ with density $K(K - 1)(1 - F(y))f(y)F^{K-2}(y)$ (i.e. the density of the second-highest-order statistic, $Y_2^{(K)}$).

Hence his expected profit is:

$$\begin{aligned} profit_{SINGLE}(r, \rho) &= \int_r^1 f_2^{(K)}(y) \int_0^y \alpha(b) db dy + \\ &\quad + K(1 - F(r))F^{K-1}(r) \int_0^r \alpha(b) db \end{aligned} \tag{5.1}$$

Since the center offers the unique intermediary a take-it-or-leave-it price, the allocation probability of the item, $\alpha(y)$, is: $\alpha(y) = 1$ if $y \geq \rho$ and $\alpha(y) = 0$ otherwise. So, we can write the previous equation as:

$$\begin{aligned} profit_{SINGLE}(r, \rho) &= \int_r^1 (y - \rho) f_2^{(K)}(y) dy + (r - \rho)[F_2^{(K)}(r) - F_1^{(K)}(r)] = \\ &= 1 - \rho - \int_r^1 F_2^{(K)}(y) dy - (r - \rho)F_1^{(K)}(r) \end{aligned} \tag{5.2}$$

Taking its first-order derivative to find the optimal profit yields:

$$\begin{aligned} \text{profit}'(r) &= F_2^{(K)}(r) - F_1^{(K)}(r) - (r - \rho)f_1^{(K)}(r) = \\ &= KF^{K-1}(r)[1 - F(r)] - K(r - \rho)F^{K-1}(r)f(r) = \\ &= KF^{K-1}(r)[1 - F(r) - f(r)(r - \rho)] \end{aligned} \quad (5.3)$$

The first-order condition yields $F(r_{\text{SINGLE}}^*) = 0$ or $r_{\text{SINGLE}}^* = \rho + \frac{1 - F(r_{\text{SINGLE}}^*)}{f(r_{\text{SINGLE}}^*)}$ for the intermediary's optimal reserve price, r_{SINGLE}^* .

As before, the center wants to maximize its revenue, which is ρ times the probability that the second-highest-order statistic is higher or equal to the intermediary's reserve price:

$$\text{revenue}_{\text{SINGLE}}(\rho, r) = \rho[1 - F^K(r)] \quad (5.4)$$

Hence, for the optimal intermediary's reserve, r_{SINGLE}^* , the center's ex-ante expected revenue given the optimal response of the intermediary is:

$$\begin{aligned} \text{revenue}_{\text{SINGLE}}(\rho, r_{\text{SINGLE}}^*) &= \rho[1 - F^K(r_{\text{SINGLE}}^*)] = \\ &= [r_{\text{SINGLE}}^* - \frac{1 - F(r_{\text{SINGLE}}^*)}{f(r_{\text{SINGLE}}^*)}][1 - F^K(r_{\text{SINGLE}}^*)] \end{aligned} \quad (5.5)$$

This is maximized by setting $\frac{d\text{revenue}_{\text{SINGLE}}}{dr_{\text{SINGLE}}^*} = 0$:

$$\begin{aligned} \frac{d\text{revenue}_{\text{SINGLE}}}{dr_{\text{SINGLE}}^*} &= [2 + \frac{(1 - F(r_{\text{SINGLE}}^*))f'(r_{\text{SINGLE}}^*)}{f^2(r_{\text{SINGLE}}^*)}][1 - F^K(r_{\text{SINGLE}}^*)] - \\ &\quad - [r_{\text{SINGLE}}^* - \frac{1 - F(r_{\text{SINGLE}}^*)}{f(r_{\text{SINGLE}}^*)}]Kf(r_{\text{SINGLE}}^*)F^{K-1}(r_{\text{SINGLE}}^*) = 0 \end{aligned} \quad (5.6)$$

Also, the ex-ante expected surplus for a buyer is:

$$\mathbb{E}[\Pi_{\text{SINGLE}}(r)] = \int_r^1 \int_r^x F^{K-1}(y)dyf(x)dx = \int_r^1 F^{K-1}(y)(1 - F(y))dy \quad (5.7)$$

For illustration, we continue with the following example.

Example 5.1. Let us again consider the example with buyers whose valuations are drawn from a uniform distribution $U(0, 1)$. Then, equations (5.2), (5.4) and (5.7) become:

$$\text{profit}_{\text{SINGLE}}(r, \rho) = -\frac{2K}{K+1}r^{K+1} + (1 + \rho)r^K + \frac{K-1}{K+1} - \rho \quad (5.8)$$

$$\text{revenue}(\rho, r) = \rho(1 - r^K) \quad (5.9)$$

$$\mathbb{E}[\Pi_{SINGLE}(r)] = \frac{1 - r^K(K + 1 - Kr)}{K(K + 1)} \quad (5.10)$$

The first-order condition for the intermediary's expected profit yields $r_{SINGLE}^* = 0$ or $r_{SINGLE}^* = \frac{1+\rho}{2}$. However, $\frac{d^2\text{profit}}{dr^2} = Kr^{K-2}[(K-1)(1+\rho) - 2Kr]$. The function is increasing for $r \in [0, \frac{1+\rho}{2}]$, decreasing otherwise, and is convex for $r \in [0, \frac{K-1}{K}\frac{1+\rho}{2}]$, concave otherwise. Since $\frac{K-1}{K}\frac{1+\rho}{2} < \frac{1+\rho}{2}$, the maximum occurs at $r_{SINGLE}^* = \frac{1+\rho}{2}$. Also, taking the first-order condition for the revenue gives $\frac{d\text{revenue}_{SINGLE}(r_{SINGLE}^*)}{dr_{SINGLE}^*} = -2(K+1)r_{SINGLE}^{K-1} + Kr_{SINGLE}^{K-2} + 2 = 0$. The second-order derivative of the revenue yields $\frac{d^2\text{revenue}_{SINGLE}(r_{SINGLE}^*)}{d(r_{SINGLE}^*)^2} = Kr_{SINGLE}^{K-2}[K - 1 - 2(K+1)r_{SINGLE}^*]$, so the revenue function is concave as long as¹ $r_{SINGLE}^* \geq \frac{K-1}{2(K+1)}$. Given this, we can write the conditions for the intermediary's and center's optimal reserve prices, $\rho_{SINGLE}^*, r_{SINGLE}^*$:

$$-2(K+1)r_{SINGLE}^{K-1} + Kr_{SINGLE}^{K-2} + 2 = 0 \quad (5.11)$$

$$\rho_{SINGLE}^* = 2r_{SINGLE}^* - 1 \quad (5.12)$$

Then, the intermediary's ex-ante expected profit can be expressed as:

$$\text{profit}_{SINGLE}(r_{SINGLE}^*) = \frac{2}{K+1}r_{SINGLE}^{K+1} - 2r_{SINGLE}^* + \frac{2K}{K+1} \quad (5.13)$$

The center's ex-ante expected revenue will be:

$$\text{revenue}_{SINGLE}(r_{SINGLE}^*) = (2r_{SINGLE}^* - 1)(1 - r_{SINGLE}^{K-1}) \quad (5.14)$$

Finally, a buyer's ex-ante expected surplus will be:

$$\mathbb{E}[\Pi_{SINGLE}(r)] = \frac{1}{K(K+1)}[1 - (K+1 - Kr_{SINGLE}^*)(r_{SINGLE}^*)^K] \quad (5.15)$$

Figures 5.1 - 5.4 illustrate the intermediary's ex-ante expected profit, buyers' ex-ante expected surplus, center's ex-ante expected revenue and the social welfare with and without the intermediary, who may or may not set an optimal reserve price, r_{SINGLE}^* , when the center sets its optimal reserve price, ρ_{SINGLE}^* , as the number of buyers increases. Again, we can see that the social welfare is lower when the intermediary is present and that the latter's expected profit decreases as the number of buyers increases. However, compared to the setting without an intermediary's reserve price in Figure 4.1, we can see that the incorporation of the intermediary's reserve price increases his expected profit and further decreases social welfare. The latter effect is due to the inefficiency caused by the increase in the reserve price for the buyers, as Figure 5.5 shows. As can be seen,

¹ $\frac{K-1}{2(K+1)} < \frac{1}{2}$, so this should hold for sure as long as the intermediary's optimal price in equilibrium is greater than or equal to half.

the center's optimal reserve price increases both with the intermediary setting a reserve price and not, however the intermediary's reserve price is always higher than the center's reserve price in the latter scenario. Also, reserve prices are always higher than the center's optimal reserve price when the intermediary is not present.

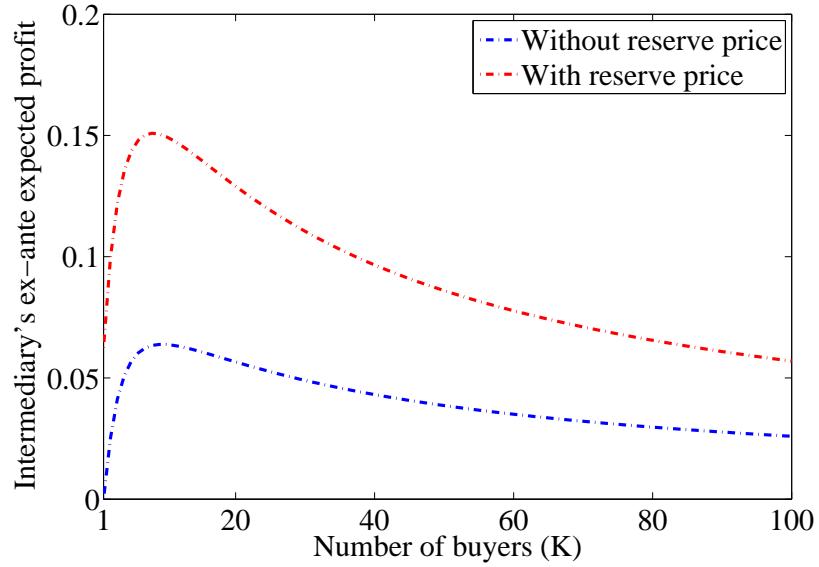


FIGURE 5.1: Intermediary's ex-ante expected profit with and without a reserve price as a function of the number of buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

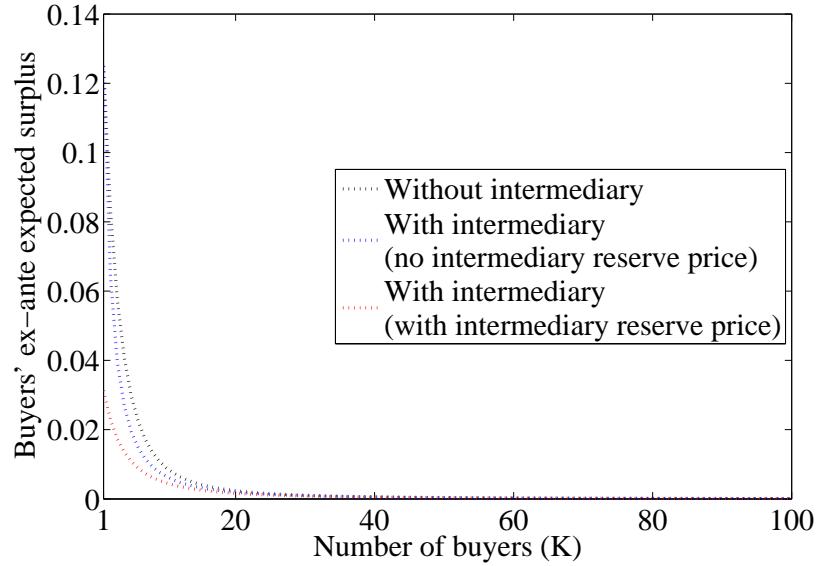


FIGURE 5.2: Buyers' ex-ante expected surplus with and without a single intermediary who sets or does not set a reserve price as a function of the number of buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

Although this example is only valid for one distribution, some of the results generalize to other distributions. Specifically, the center's optimal reserve price is expected to

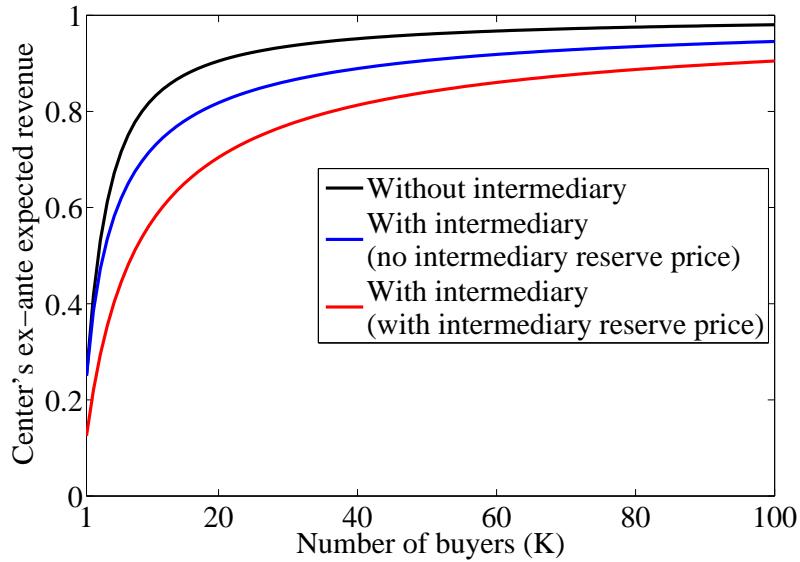


FIGURE 5.3: Center's ex-ante expected revenue with and without a single intermediary who sets or does not set a reserve price as a function of the number of buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

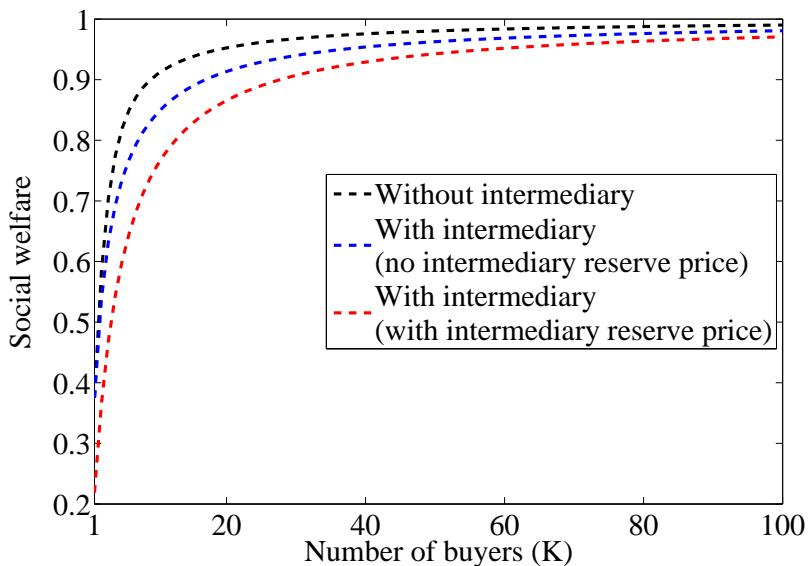


FIGURE 5.4: Social welfare with and without a single intermediary who sets or does not set a reserve price as a function of the number of buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

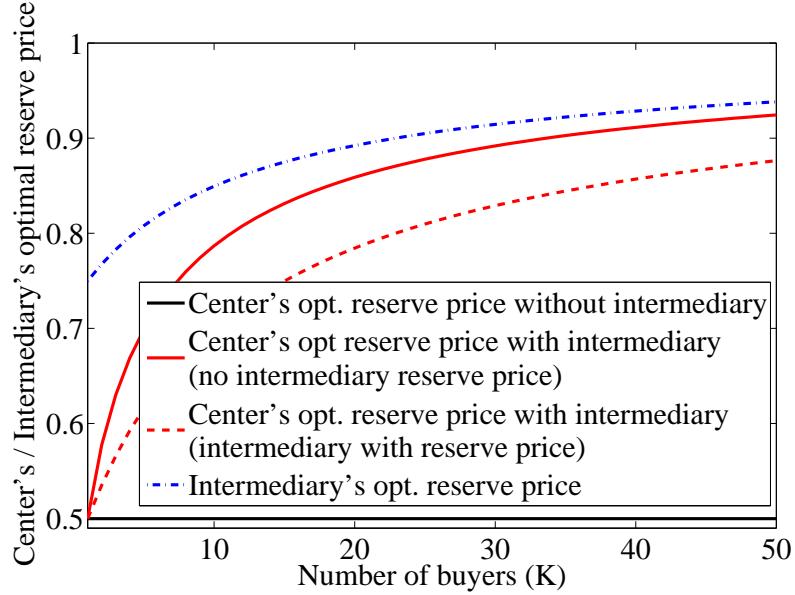


FIGURE 5.5: Center's optimal reserve price with and without an intermediary along with the intermediary's optimal reserve price as a function of the number of buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

increase with the number of buyers, as was the case without an intermediary's reserve price in Section 4.1, since the latter price is a constant markup on the center's reserve price; so the results from the bidding rings literature (cf. Theorem 4.1) will also hold in this setting. For the same reason, this will also be true for the intermediary's optimal reserve price. The center's optimal reserve price will be lower (higher) for positively (negatively respectively) skewed distributions compared to symmetric distributions of private valuations such as a truncated Gaussian or symmetric Beta distribution and the same ordering will hold for the center's expected revenue. Finally, regarding the intermediary's expected profit, this should also exhibit a similar pattern to that of Figure 5.1, increasing up to a critical point where the number of buyers is large enough for his reserve price to have a smaller impact and the difference between their expected highest valuation and the center's (high) reserve price gradually decreasing with an increasing population of buyers.

5.2 Multiple Intermediaries

Having analyzed the impact of a reserve price for the intermediary in a monopoly setting, we now move to the more interesting case of competition between intermediaries. It has been previously observed (Feldman et al., 2010) that finding equilibrium reserve-price setting strategies in this setting is nontrivial. Hence, in what follows, we limit our analysis to a duopoly homogeneous intermediary setting, i.e. an environment with two intermediaries. Specifically, we assume that each intermediary imposes the same mechanism with a reserve price, keeping the assumption of symmetry for the number

and distribution of buyers' private valuations in each intermediary's market. We also analyze a duopoly heterogeneous setting with one PRE and one POST intermediary, but do not compare FPSB auctions with their other counterparts, given that one needs to derive the resulting equilibrium bidding functions which is technically challenging. As it will be shown, even in this simple scenario, intermediaries, in general, follow mixed-equilibrium reserve-price-setting strategies. The intuition behind this is that intermediaries have an incentive to increase their reserve price, since this increases their chance of obtaining the good, but at the same time decreases the probability of having a buyer that is able to pay that high. Hence, in contrast to classical models of competition, where prices are driven downwards, in this setting, the opposite happens: reserve prices being driven upwards up to a critical point, where competition from other intermediaries and buyers' strategies drive reserve prices to the other direction, leading to cycles which in turn lead to mixed-equilibrium strategies. For this reason, we employ numerical techniques to find the resulting reserve-price-setting Nash equilibria in specific instances. This will shed some light in the reserve-price-setting problem of the intermediaries and its impact in their profit, the center's revenue and the buyers' surplus. In more detail, we run the fictitious play algorithm for $k = 1, 2$ and 5 buyers per intermediary whose private valuations are i.i.d. drawn from the uniform distribution $U(0, 1)$. We begin our discussion by describing the fictitious play setup we have used for our experiments.

5.2.1 Fictitious Play Setup

In this subsection, we describe the setup of the fictitious play (see Section 2.1.2) experiments that were used to calculate ϵ -NE for the intermediaries' reserve-price-setting strategies. Other techniques can be used to solve for Nash equilibria (McKelvey and McLennan, 1996), such as the Lemke-Howson algorithm, which might take exponential time to converge. We have tried the latter algorithm using the well-known game theory software called Gambit², but it failed to find NE. Another reason that such techniques might not work is that in some cases, such as the case with competing FPSB auctioneers (Section 5.2.4) or non-captive buyers (Chapter 6), the payoff matrix consists of sample averages that are only approximations of the actual expected utilities, so only what are known as empirical game-theoretic techniques can be used in such cases (Jordan et al., 2008). However, it is required for consistency reasons to use the same technique for all experiments and fictitious play is a natural technique to use in two-player games.

More specifically, in all cases, we have constructed, for each (discretized) center's reserve price, ρ , an $|r| \times |r|$ payoff matrix with the intermediaries' expected profits (or average profits when these cannot be expressed in a closed form) for each combination of (discretized) intermediary reserve prices (of size $|r|^2$) in $[\rho, 1]$. This is a natural range since profit-maximizing intermediaries are expected to set a reserve price that is at least ρ . For

²<http://gambit.sourceforge.net/>

FPSB intermediaries or settings with non-captive buyers (next chapter), where, as will be seen, closed-form formulas are not available for the intermediaries' expected profits, the numerical averages of their profits for 500,000 repetitions and for each combination of intermediary reserve prices at each ρ were used instead.

In the fictitious play experiments, we have discretized the intermediaries' reserve prices using a step of 0.01 and the center's reserve price using a step of 0.1 at $[0, 1]$. Using a higher discretization step (i.e. 0.01) for the center led to variations in the obtained expected revenues that made the results more prone to numerical errors. Also, a higher discretization step for the intermediaries would make the number of simulations and corresponding time needed to obtain good estimates of the average profits prohibitive.

We ran one fictitious play experiment for each center's reserve price ρ , each combination of intermediary mechanisms and for each of three populations of $k = 1, 2, 5$ buyers per intermediary respectively whose private valuations are i.i.d. drawn from the uniform distribution $U(0, 1)$. These values for k allow for utility comparisons of the three mechanisms as the number of buyers increases and are close to actual numbers of buyers per intermediary that might be interested at a specific impression. Moreover, these were deemed appropriate for comparison with the results of Chapter 6 taking into account issues of computational tractability that arise there. Each fictitious play experiment was conducted once for 500,000 rounds in total where we have used a random tie-breaking rule and we have also assumed uniform initial beliefs over $[\rho, 1]$.

In accordance with Definition 2.19, we consider an additive approximation for ϵ -NE where the value for ϵ at each round is calculated as the difference between the utility produced by each player's best response (pure strategy) and the utility of the current mixed strategy produced by the fictitious play algorithm given the player's current beliefs. Figures 5.6(a) - 5.6(c) illustrate the values for ϵ per round for all the cases considered with varying number of buyers at the optimal center's reserve price, as these will be illustrated in the following subsections. As can be seen, in all cases the algorithm converges to very small values for ϵ . For this reason, Figures 5.6(a) - 5.6(c) depict these values only for the first 50,000 rounds.

As will be shown in the next section, we compared our fictitious play results with the only available theoretical results of Feldman et al. (2010) for two PRE intermediaries and $k = 1$ buyer each, where we have shown the convergence of the intermediaries' fictitious beliefs to the expected distributions of actions. This is a good evidence of the effectiveness of the technique and its configuration used.

In what follows, we start with the competition between two PRE intermediaries where we validate fictitious play results for $k = 1$ and then extend these for the cases of $k = 2$ and $k = 5$ buyers per intermediary.

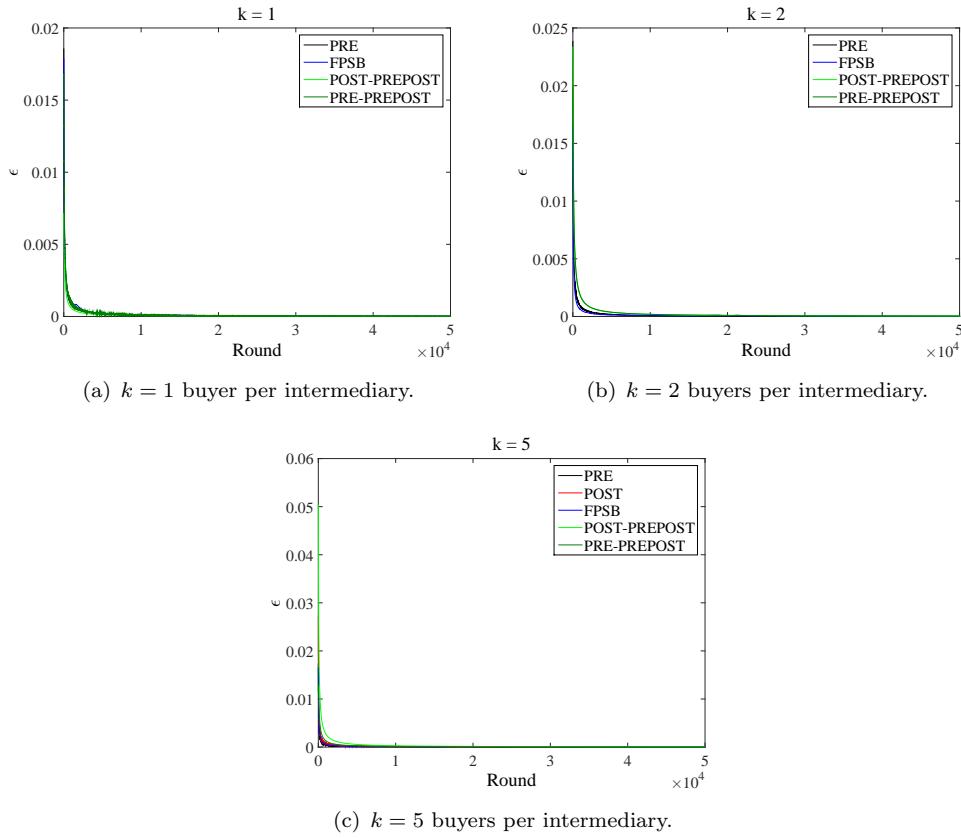


FIGURE 5.6: ϵ values of the fictitious play experiments for duopoly PRE, POST, FPSB and PRE-POST settings with intermediary reserve prices, where each intermediary has $k = 1$ (left), $k = 2$ (right) or $k = 5$ (bottom) buyers whose private valuations are i.i.d. drawn from $U(0, 1)$ and the center impose its optimal reserve price.

5.2.2 Pre-Award Second-Price Sealed-Bid Auctions

We begin our analysis with a duopoly setting with PRE intermediaries that differ in their reserve prices. We will call the intermediary with the higher (lower respectively) reserve price, the high- (low- respectively) intermediary. Following this, we consider the same setting for POST and FPSB intermediary auctioneers, and then study a setting with a PRE and a POST intermediary auctioneer. We first characterize the expected utilities for all the above-mentioned cases and then present our theoretical analysis along with our numerical results for these four settings.

For the remainder of this thesis, we will denote the high- and low-reserve intermediaries as s_ℓ^L, s_ℓ^H and their reserve prices as $r_\ell^L \leq r_\ell^H$ respectively, where $\ell = \{PRE, POST, FPSB\}$.

Assume two intermediaries, s_{PRE}^L, s_{PRE}^H , each implementing a PRE mechanism with reserve price r_{PRE}^L, r_{PRE}^H respectively and with an equal number of buyers, $k = \frac{K}{2}$, in their market. Feldman et al. (2010) have analyzed this problem and have analytically derived the (mixed-) equilibrium reserve-price-setting strategies of the intermediaries in a more general setting with n intermediaries but assuming that each intermediary has

only a single buyer in his market. They have shown that a similar equilibrium will arise in the general case where each intermediary has $k > 1$ buyers, providing the conditions that should hold. However, they have not managed to explicitly characterize such an equilibrium for issues of analytical tractability.

In this section, we characterize the center's ex-ante expected revenue along with the intermediaries' ex-ante expected profits and the buyers' expected surplus. We then look at the intermediaries' best responses in three examples with buyers whose private valuations are i.i.d. drawn from the uniform distribution $U(0, 1)$. Finally, we employ the fictitious play algorithm to find the resulting ϵ -NE reserve prices, first validating its good approximation for the case of $k = 1$ where the exact Nash equilibria are known. We begin with the derivation of intermediaries' expected profits.

The ex-ante expected profits of the low- and the high-reserve intermediary respectively can be expressed as:

$$\begin{aligned} profit_{PRE}^L(r_{PRE}^L) &= F^k(r_{PRE}^H)[kF^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))(r_{PRE}^L - \rho) + \\ &+ \int_{r_{PRE}^L}^1 (y - \rho)f_2^{(k)}(y)dy] + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^H}^1 (y - r_{PRE}^H)f_2^{(k)}(y)dy + \\ &+ \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^H}^y (y - x)f_2^{(k)}(x)dxdy \end{aligned} \quad (5.16)$$

and

$$\begin{aligned} profit_{PRE}^H(r_{PRE}^H) &= F^k(r_{PRE}^L)[kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))(r_{PRE}^H - \rho) + \int_{r_{PRE}^H}^1 (y - \rho)f_2^{(k)}(y)dy] + \\ &+ kF^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))[kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))(r_{PRE}^H - r_{PRE}^L) + \\ &+ \int_{r_{PRE}^H}^1 (y - r_{PRE}^L)f_2^{(k)}(y)dy] + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^L}^{r_{PRE}^H} (r_{PRE}^H - y)f_2^{(k)}(y)dy + \\ &+ \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y (y - x)f_2^{(k)}(x)dxdy \end{aligned} \quad (5.17)$$

That is, the low-reserve intermediary either receives the difference between the maximum of his reserve price and his buyers' second-highest bid, and the center's reserve price, if all opponent bids are above the high reserve (first term in (5.16)), or he gets the difference between his local second-highest bid and the high reserve price if only one opponent local bid is above the latter (second term in (5.16)). Otherwise, he receives the difference between his buyers' second-highest bid and the opponent intermediary buyers' second-highest bid (third term in (5.16)). Similarly, the high-reserve intermediary receives the difference between the maximum of his reserve price and his buyers' second-highest bid, and the center's reserve price, if all opponent bids are below his opponent's low reserve price (first term in (5.17)), or the difference between his reserve price and the opponent intermediary's reserve price, if there is only one valid bid submitted in the latter and, at

the same time, there is at least one bid above his reserve price (second term in (5.17)). He also receives the difference between his reserve price and the local second-highest opponent bid, if the latter is below his reserve price (third term in (5.17)), as well as the difference between his buyers' second-highest bid and the opponent intermediary buyers' second-highest bid (fourth term in (5.17)). After some manipulation (see Section A.1 in the Appendix), these equations simplify to the following:

$$\begin{aligned} profit_{PRE}^L(r_{PRE}^L) &= F^k(r_{PRE}^H)[r_{PRE}^H - \rho - F^k(r_{PRE}^L)(r_{PRE}^L - \rho) - \int_{r_{PRE}^L}^{r_{PRE}^H} F_2^{(k)}(y)dy] + \\ &+ \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^H}^y F_2^{(k)}(x)dx dy \end{aligned} \quad (5.18)$$

$$\begin{aligned} profit_{PRE}^H(r_{PRE}^H) &= F^k(r_{PRE}^L)(1 - F^k(r_{PRE}^H))(r_{PRE}^L - \rho) + \\ &+ kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^L}^{r_{PRE}^H} F_2^{(k)}(y)dy + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y F_2^{(k)}(x)dx dy \end{aligned} \quad (5.19)$$

The center's ex-ante expected revenue can be expressed as:

$$\begin{aligned} revenue_{PRE}(\rho) &= \rho[F^k(r_{PRE}^L)(1 - F^k(r_{PRE}^H)) + F^k(r_{PRE}^H)(1 - F^k(r_{PRE}^L))] + \\ &+ r_{PRE}^L k F^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))(1 - F^k(r_{PRE}^H)) + \\ &+ r_{PRE}^H k F^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))(1 - F_2^{(k)}(r_{PRE}^H)) + \\ &+ \int_{r_{PRE}^H}^1 2y(1 - F_2^{(k)}(y))f_2^{(k)}(y)dy + (1 - F^k(r_{PRE}^H)) \int_{r_{PRE}^L}^{r_{PRE}^H} y f_2^{(k)}(y)dy \end{aligned} \quad (5.20)$$

That is, the center receives its reserve price if there is at least one bid above one intermediary's reserve price but no eligible bids in the other intermediary's market (first term in (5.20)). It also receives the low reserve price if there is only a single eligible bid submitted in the low-reserve intermediary, and at least one eligible bid submitted in the high-reserve intermediary (second term in (5.20)). The center similarly receives the high reserve price if there is only a single bid above the high reserve price submitted in the high-reserve intermediary auction and, at the same time, the second-highest local bid in the low-reserve intermediary is above the high reserve price (third term in (5.20)). If none of the above holds, then the center receives the second-highest submitted bid (fourth and fifth terms in (5.20)).

Finally, a buyer with valuation v expects the following surplus from the low- and high-reserve intermediaries, when $r_{PRE}^L < r_{PRE}^H$:

$$\Pi_{PRE}^L(v) = \begin{cases} 0 & \text{if } v \in [0, r_{PRE}^L] \\ F^k(r_{PRE}^H) \int_{r_{PRE}^L}^v F^{k-1}(y) dy & \text{if } v \in [r_{PRE}^L, r_{PRE}^H) \\ F^k(r_{PRE}^H)[(v - r_{PRE}^L)F^{k-1}(r_{PRE}^L) + \int_{r_{PRE}^L}^{r_{PRE}^H} (v - y)f_1^{(k-1)}(y)dy] + \\ + \int_{r_{PRE}^H}^v (v - y)f_1^{(k-1)}(y)F_2^{(k)}(y)dy & \text{if } v \in [r_{PRE}^H, 1] \end{cases} \quad (5.21)$$

$$\Pi_{PRE}^H(v) = \begin{cases} 0 & \text{if } v \in [0, r_{PRE}^H] \\ (v - r_{PRE}^H)F^{k-1}(r_{PRE}^H)F_2^{(k)}(r_{PRE}^H) + \\ + \int_{r_{PRE}^H}^v (v - y)f_1^{(k-1)}(y)F_2^{(k)}(y)dy & \text{if } v \in [r_{PRE}^H, 1] \end{cases} \quad (5.22)$$

In more detail, a buyer with valuation $v \in [r_{PRE}^H, 1]$ in the low-reserve intermediary wins and pays him his reserve price, r_{PRE}^L , or the second-highest local bid if there is no eligible bid submitted in the other intermediary (first and second terms in (5.21)). Otherwise, the buyer pays the local second-highest bid, if this is above the high reserve price and, at the same time, the local second-highest bid in the high-reserve intermediary is below it (third term in (5.21)). Similarly, such a buyer in the high-reserve intermediary wins and pays the high reserve price, r_{PRE}^H , if all other bids in this intermediary are below this reserve price and, at the same time, the local second-highest bid in the low-reserve intermediary is below r_{PRE}^H (first term in (5.22)). Finally, she wins and pays the local second-highest bid in the high-reserve intermediary if this is lower than her bid and, simultaneously, the opponent intermediary submitted bid at the center is also below her bid (second term in (5.22)).

For the special case where both intermediaries set the same reserve price, i.e. $r_{PRE}^L = r_{PRE}^H = r_{PRE}$, a buyer's ex-interim expected surplus will be:

$$\Pi_{PRE}^{eq}(v) = \begin{cases} 0 & \text{if } v \in [0, r_{PRE}] \\ (v - r_{PRE})F^{k-1}(r_{PRE})[F^k(r_{PRE}) + \frac{1}{2}kF^{k-1}(r_{PRE})(1 - F(r_{PRE}))] + \\ + \int_{r_{PRE}}^v (v - y)f_1^{(k-1)}(y)F_2^{(k)}(y)dy & \text{if } v \in [r_{PRE}, 1] \end{cases} \quad (5.23)$$

The only essential difference compared to the non-equal reserve prices situation is that, when both intermediaries submit their reserve prices, a random tie-breaking rule yields a probability of $\frac{1}{2}$ of winning (second term in (5.23)).

In what follows, we depict the resulting equilibrium for our duopoly setting with $k = 1$ buyer for each intermediary, assuming a uniform distribution, $U(0, 1)$, for buyers' private valuations and compare this with a numerical approximation that we derive by using

the fictitious play algorithm. We then repeat the same technique for the two examples of $k = 2$ and $k = 5$ (i.e. greater than one) buyers.

To see why intermediaries in this case ($k = 1$) follow mixed-equilibrium strategies, let us illustrate the best response function for our duopoly setting. In this case, the low-reserve intermediary's expected profit is the difference between his reserve price, r_{PRE}^L and the center's reserve price, ρ , given that his buyer's private valuation is higher than his reserve price and, at the same time, the other buyer's private valuation is lower than the high-reserve intermediary's reserve price, r_{PRE}^H , i.e.

$$\text{profit}_{PRE}^L(r_{PRE}^L) = (1 - F(r_{PRE}^L))F(r_{PRE}^H)(r_{PRE}^L - \rho) \quad (5.24)$$

whereas the high-reserve intermediary additionally expects positive profit even if his opponent submits his reserve price:

$$\text{profit}_{PRE}^H(r_{PRE}^H) = (1 - F(r_{PRE}^H))[F(r_{PRE}^L)(r_{PRE}^H - \rho) + (1 - F(r_{PRE}^L))(r_{PRE}^H - r_{PRE}^L)] \quad (5.25)$$

The best-response function for $U(0, 1)$ and $\rho = 0$ is shown in Figure 5.7. As can be seen, this function does not cross the 45° line, meaning that it is unlikely that there is a symmetric pure-strategy equilibrium³, which would naturally arise since both intermediaries are ex-ante identical. What's more, for low reserve prices, the opponent intermediary best responds by setting a higher reserve price up to a point where it is best to set a lower reserve price, creating a vicious cycle, leading to a mixed-equilibrium behavior for the intermediaries' reserve-price-setting strategies.

As Feldman et al. have shown, the resulting mixed-equilibrium reserve-price-setting strategies of n intermediaries with $k = 1$ buyer each, involve each intermediary offering a random take-it-or-leave-it price, r , in an interval $[r_{min}, r_{max}]$ with density $\xi_r(r)$, where $r_{min}, r_{max}, \xi_r(\cdot)$ are found by solving the following system of equations (Feldman et al., 2010):

$$r_{min} = \rho + \frac{1 - F(r_{min})}{f(r_{min})} \quad (5.26)$$

$$\int_{r_{min}}^{r_{max}} \frac{\zeta'(v)}{(n-1)\zeta^{\frac{n-2}{n-1}}(v)(1-F(v))} dv = (\zeta(r_{max}))^{\frac{1}{n-1}} \quad (5.27)$$

$$\xi_r(r) = \left(\frac{1}{\zeta(r_{max})}\right)^{\frac{1}{n-1}} \frac{\zeta'(r)}{(n-1)\zeta^{\frac{n-2}{n-1}}(r)(1-F(r))} \quad (5.28)$$

³This happens regardless of the reserve price of the center and the granularity of the discretization.

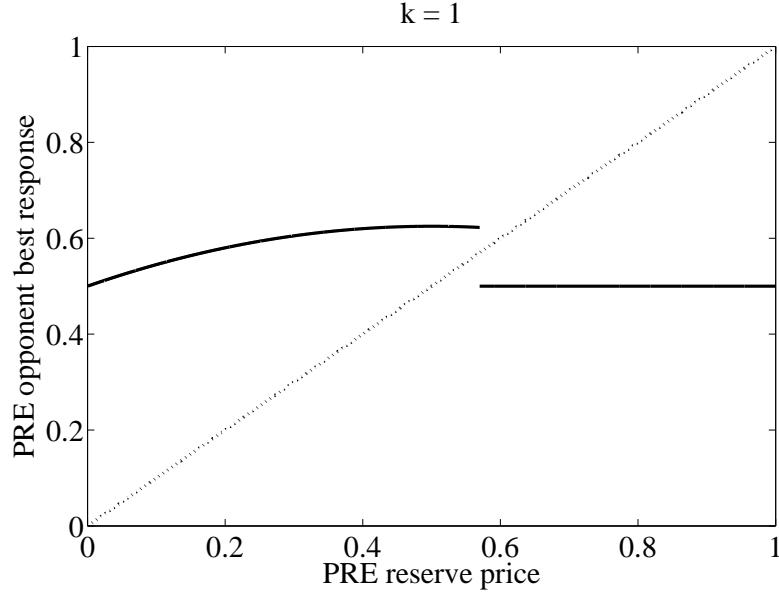


FIGURE 5.7: Best-response reserve price function for a duopoly PRE intermediary setting with intermediary reserve prices, where each intermediary has exactly $k = 1$ buyer whose private valuation is i.i.d. drawn from $U(0, 1)$ and the center does not impose a reserve price.

where $\zeta(r) = \frac{f(r)}{(1-F(r))^2}$. Then the center's reserve-price optimization problem is given by (Feldman et al., 2010):

$$\rho_{PRE}^* = \arg \max_{\rho} \left\{ r_{max} - \frac{1 - F(r_{max})}{f(r_{max})} - \rho \left(\frac{\zeta(r_{min})}{\zeta(r_{max})} \right)^{\frac{n-1}{n}} - \frac{n-1}{\zeta(r_{max})} \right\} \quad (5.29)$$

subject to equations (5.26) - (5.28). For $n = 2$ intermediaries and buyers whose private valuations are i.i.d. drawn from the uniform distribution $U(0, 1)$, equations (5.26) - (5.29) become:

$$r_{min} = \frac{1 + \rho}{2} \quad (5.30)$$

$$\frac{1}{(1 - r_{max})^3} - \frac{8}{(1 - \rho)^3} = \frac{3}{2(1 - r_{max})^2} \quad (5.31)$$

$$\xi_r(r) = \frac{2(1 - r_{max})^2}{(1 - r)^4} \quad (5.32)$$

$$\rho_{PRE}^* = \arg \max_{\rho} \left\{ 4r_{max} - (r_{max})^2 - 16\rho \left(\frac{1 - r_{max}}{1 - \rho} \right)^4 - 2 \right\} \quad (5.33)$$

For the uniform distribution, $U(0, 1)$, the equations above yield $\rho_{PRE}^* = 0.5$, $r_{min} = 0.75$, $r_{max} = 0.78$.

We have used fictitious play in this setting to compare our results with the only available theoretical results of Feldman et al.. Given the technical difficulties in characterizing the exact equilibrium distribution for settings with more than one buyer per intermediary, we use these theoretical results as a benchmark for fictitious play. Specifically,

the convergence of the intermediaries' utilities to the theoretically derived ones in our fictitious play experiments is good evidence of the effectiveness of fictitious play in the remaining cases studied within this thesis, where there are no theoretical guarantees.

Figure 5.8 illustrates the intermediaries' utilities for each round. The acquired utilities are very close to the theoretical expected profits of 0.0479 in equilibrium that are achieved according to the results of Feldman et al.. Moreover, Figure 5.9 illustrates the resulting c.d.f. (black line) of the intermediaries' reserve-price-setting strategies in our approximate equilibrium compared to the theoretical c.d.f. (gray line). As can be seen, the two functions are quite close, illustrating the effectiveness of our experiments in this setting.

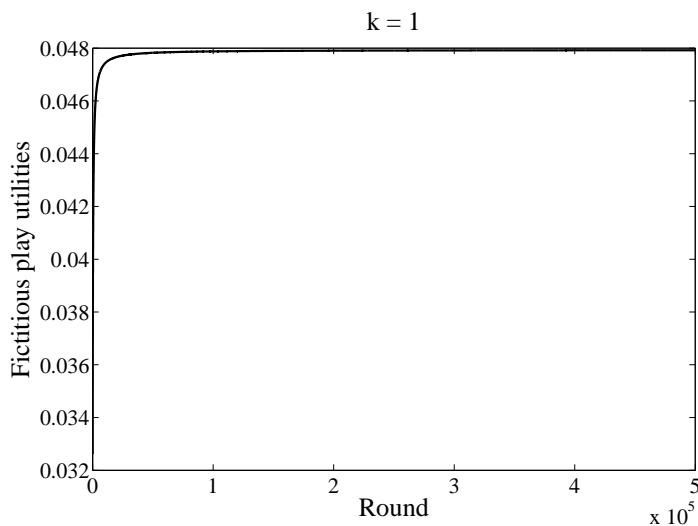


FIGURE 5.8: Intermediary fictitious play utilities per round for a duopoly PRE intermediary setting, where each intermediary has $k = 1$ buyer whose private valuation is i.i.d. drawn from $U(0, 1)$ and the center imposes its optimal reserve price (0.5).

Having shown the effectiveness of fictitious play in the $k = 1$ intermediary duopoly setting, we now move to the more general case of $k > 1$ buyers per intermediary in the same scenario with i.i.d. uniform $U(0, 1)$ buyers' private valuations. Feldman et al. have shown that intermediaries in this case also implement mixed-strategy equilibrium reserve prices but did not manage to analytically characterize the resulting equilibrium. The best-response functions for two examples with $k = 2, 5$ buyers for each intermediary and $\rho = 0$ in Figure 5.10 imply the same. As can be seen, the low best-response reserve price remains fixed whereas the high best-response reserve price increases with the number of buyers per intermediary.

We now depict the results of the fictitious play experiments for $k = 2$ and 5 buyers per intermediary, where we vary the center's reserve price. Figure 5.11 illustrates the center's ex-ante expected revenue for $k = 2$ (left) and $k = 5$ (right). As can be seen from these examples, the center's optimal reserve price increases with the number of buyers per intermediary. Feldman et al. have shown that, for a single buyer per intermediary, the

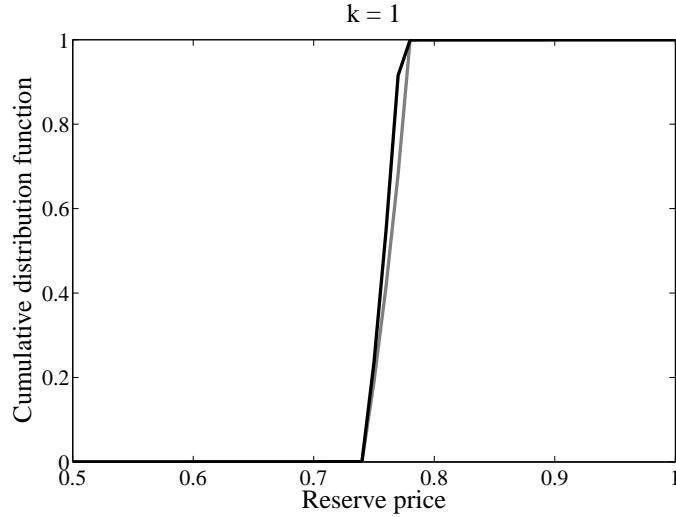


FIGURE 5.9: C.d.f. of the intermediaries' equilibrium reserve-price-setting strategies (gray) along with the corresponding empirical c.d.f. of the fictitious play run (black) for a duopoly PRE intermediary setting, where each intermediary has $k = 1$ buyer whose private valuation is i.i.d. drawn from $U(0, 1)$ and the center imposes its optimal reserve price (0.5).

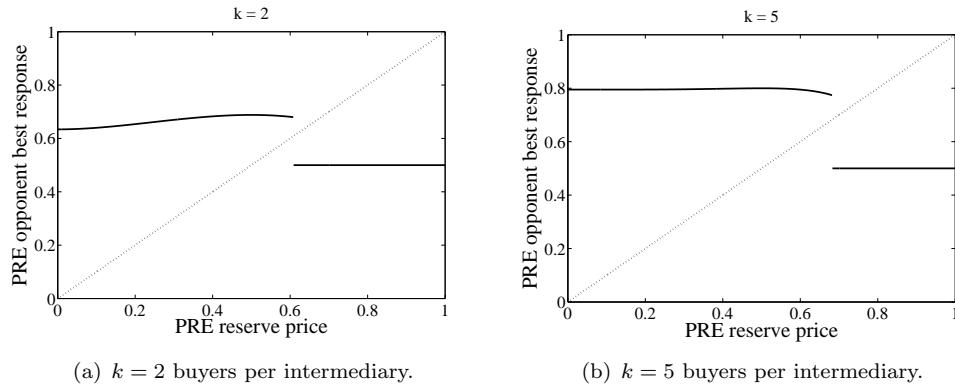


FIGURE 5.10: Best-response reserve price functions for a duopoly PRE intermediary setting with intermediary reserve prices, where each intermediary has $k = 2$ (left) or $k = 5$ (right) buyers whose private valuations are i.i.d. drawn from $U(0, 1)$ and the center does not impose a reserve price.

latter optimal reserve price decreases with the number of intermediaries. Our example depicts that their result is due to the increased competition of the intermediaries. Hence, these two factors (number of intermediaries, number of buyers per intermediary) drive the optimal reserve price for the center in opposite directions.

Figure 5.12 illustrates the c.d.f. of the resulting ϵ -NE reserve-price-setting strategies from our experiments when the center implements its optimal reserve price (0.5 for $k = 2$, 0.6 for $k = 5$). As shown, ϵ -NE reserve prices tend to increase with increasing number of buyers and the support of this equilibrium increases as well.

We now move to the setting with two intermediaries both implementing POST mechanisms.

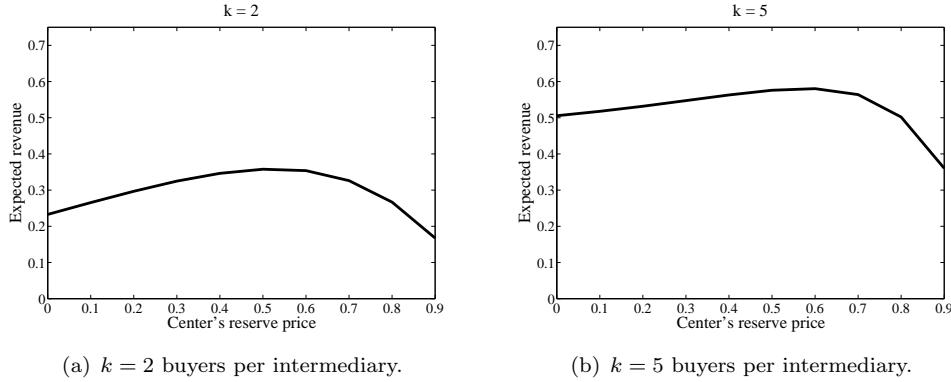


FIGURE 5.11: Center's ex-ante expected revenue when intermediaries follow the fictitious play ϵ -NE reserve-price-setting strategies for a duopoly PRE intermediary setting, where each intermediary has $k = 2$ (left) or $k = 5$ (right) buyers whose private valuations are i.i.d. drawn from $U(0, 1)$ and the center does not impose a reserve price.

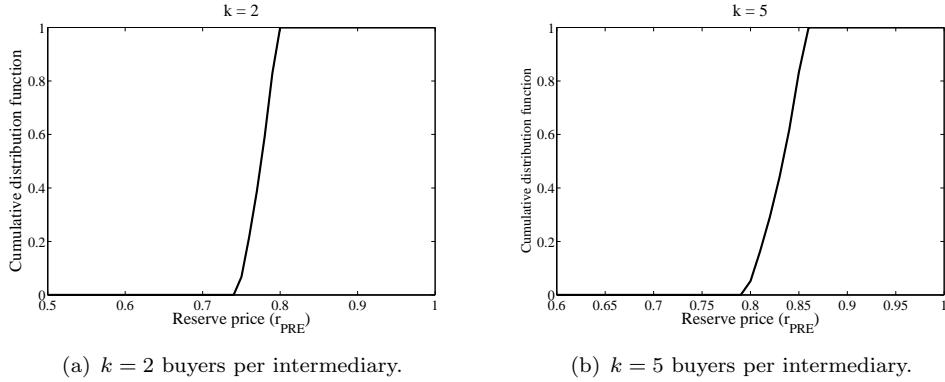


FIGURE 5.12: Empirical c.d.f. of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly PRE intermediary setting, where each intermediary has $k = 2$ (left) and $k = 5$ (right) buyers whose private valuations are i.i.d. drawn from $U(0, 1)$ and the center imposes its optimal reserve price.

5.2.3 Post-Award Second-Price Sealed-Bid Auctions

Let us consider the duopoly competition between two intermediaries implementing POST mechanisms where both set a reserve price to increase their profit, so that a buyer wins and pays the maximum of the local second-highest bid, the center's second-highest bid or its reserve price, and the intermediary's reserve price. In this section, as before, we first characterize the expected utilities for all agents and then analyze the resulting equilibrium reserve-price-setting behavior of the intermediaries. In this setting, we show the existence of a symmetric pure-strategy equilibrium in the intermediaries' strategies under some conditions. As will be seen, these conditions are met for our example with the uniform distribution $U(0, 1)$ for $k = 1$ and 2 buyers per intermediary. For this reason, we will present our fictitious play results only for the case of $k = 5$ buyers per intermediary.

We assume that intermediaries set reserve prices r_{POST}^L, r_{POST}^H such that $r_{POST}^L \leq r_{POST}^H$. In this case, the low- and high-reserve intermediary expected profits can be expressed as follows:

$$\begin{aligned} profit_{POST}^L(r_{POST}^L) &= F^k(r_{POST}^H)[kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))(r_{POST}^L - \rho) + \\ &+ \int_{r_{POST}^L}^1 (y - \rho)f_2^{(k)}(y)dy] + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^H}^y (y - x)f_1^{(k)}(x)dxdy \end{aligned} \quad (5.34)$$

and

$$\begin{aligned} profit_{POST}^H(r_{POST}^H) &= F^k(r_{POST}^L)[kF^{k-1}(r_{POST}^L)(1 - F(r_{POST}^L))(r_{POST}^H - \rho) + \\ &+ \int_{r_{POST}^H}^1 (y - \rho)f_2^{(k)}(y)dy] + kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H)) \int_{r_{POST}^L}^{r_{POST}^H} (r_{POST}^H - y)f_1^{(k)}(y)dy + \\ &+ \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^L}^y (y - x)f_1^{(k)}(x)dxdy \end{aligned} \quad (5.35)$$

In more detail, the low-reserve intermediary receives the difference between the maximum of his buyers' local second-highest bid and the center's reserve price, if there is at least one bid above his reserve price and, at the same time, all opponent bids are below the high reserve price (first term in (5.34)). Otherwise, he receives the difference between his local second-highest bid and the center's second-highest bid if the former is higher than the latter (second term in (5.34)). The high-reserve intermediary correspondingly receives the difference between the maximum of his local second-highest bid and his reserve price, and the center's reserve price or the difference between his local second-highest bid and the center's second-highest bid in the same cases (first and third terms in (5.35)), and additionally obtains the difference between his reserve price and the opponent highest bid, if the latter is higher than the low reserve price but lower than the high reserve price (second term in (5.35)). The above equations simplify to the following:

$$\begin{aligned} profit_{POST}^L(r_{POST}^L) &= F^k(r_{POST}^H)[r_{POST}^H - \rho - F^k(r_{POST}^L)(r_{POST}^L - \rho) - \\ &- \int_{r_{POST}^L}^{r_{POST}^H} F_2^{(k)}(y)dy] + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^H}^y F^k(x)dxdy \end{aligned} \quad (5.36)$$

$$\begin{aligned} profit_{POST}^H(r_{POST}^H) &= F^k(r_{POST}^L)(1 - F^k(r_{POST}^H))(r_{POST}^L - \rho) + \\ &+ kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H)) \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^H}^y F^k(x)dxdy \end{aligned} \quad (5.37)$$

Having expressed the intermediaries' expected profits, we now characterize the center's ex-ante expected revenue; the center receives its reserve price, ρ , whenever there is at

least one bid above this reserve in one intermediary but no such eligible bid in the other intermediary's local auction. Additionally, it receives the second-highest intermediaries' bid if both intermediaries' bids are above the high reserve price, r_{POST}^H , or when the high-reserve intermediary's bid is above r_{POST}^H and the low-reserve intermediary's bid is higher than r_{POST}^L but lower than r_{POST}^H . Hence, the center's expected revenue can be expressed as follows:

$$\begin{aligned} \text{revenue}_{POST}(\rho) &= [F^k(r_{POST}^H)(1 - F^k(r_{POST}^L)) + F^k(r_{POST}^L)(1 - F^k(r_{POST}^H))] \rho + \\ &+ \int_{r_{POST}^H}^1 2y(1 - F^k(y))f_1^{(k)}(y)dy + (1 - F^k(r_{POST}^H)) \int_{r_{POST}^L}^{r_{POST}^H} yf_1^{(k)}(y)dy \end{aligned} \quad (5.38)$$

Finally, a buyer with valuation v expects surplus from the low- and high-reserve intermediaries that can be expressed as follows:

$$\Pi_{POST}^L(v) = \begin{cases} 0 & \text{if } v \in [0, r_{POST}^L) \\ F^k(r_{POST}^H) \int_{r_{POST}^L}^v F^{k-1}(y)dy & \text{if } v \in [r_{POST}^L, r_{POST}^H) \\ F^k(r_{POST}^H)[(v - r_{POST}^L)F^{k-1}(r_{POST}^L) + \int_{r_{POST}^L}^{r_{POST}^H} (v - y)f_1^{(k-1)}(y)dy] + \\ + \int_{r_{POST}^H}^v (v - y)f_1^{(2k-1)}(y)dy & \text{if } v \in [r_{POST}^H, 1] \end{cases} \quad (5.39)$$

and

$$\Pi_{POST}^H(v) = \begin{cases} 0 & \text{if } v \in [0, r_{POST}^H) \\ (v - r_{POST}^H)F^{2k-1}(r_{POST}^H) + \int_{r_{POST}^H}^v (v - y)f_1^{(2k-1)}(y)dy & \text{if } v \in [r_{POST}^H, 1] \end{cases} \quad (5.40)$$

Thus, a buyer with valuation $v \geq r_{POST}^H$ in the low-reserve intermediary wins and pays the low reserve price, r_{POST}^L , or the highest opponent local bid in this intermediary's auction if no other eligible bid is submitted in the high-reserve intermediary (first and second terms in (5.39)). Otherwise, she pays the second-highest opponent bid overall (third term in (5.39)). On the other hand, if the buyer is in the high-reserve intermediary, she pays the high reserve price, r_{POST}^H , if all other bids are less than this reserve price (first term in (5.40)). Otherwise, she wins and pays the highest opponent bid overall, given that the latter is higher than r_{POST}^H but lower than her bid (second term in (5.40)).

Again, we start with the motivating scenario where each intermediary has only one buyer participating in his auction. In this case, the expected profits of the two intermediaries can be expressed as:

$$\text{profit}_{POST}^L(r_{POST}^L) = (1 - F(r_{POST}^L))F(r_{POST}^H)(r_{POST}^L - \rho) \quad (5.41)$$

$$\begin{aligned} profit_{POST}^H(r_{POST}^H) &= (1 - F(r_{POST}^H))[(r_{POST}^H - \rho)F(r_{POST}^L) + \int_{r_{POST}^L}^{r_{POST}^H} (r_{POST}^H - y)f(y)dy] = \\ &= (1 - F(r_{POST}^H))[\int_{r_{POST}^L}^{r_{POST}^H} F(x)dx + (r_{POST}^L - \rho)F(r_{POST}^L)] \end{aligned} \quad (5.42)$$

Looking at the intermediaries' reserve-price-setting strategies in this setting, surprisingly, in this case we find that, under some conditions on the distribution of buyers' private valuations, there is a symmetric pure-strategy Nash equilibrium where both intermediaries set the same reserve price as in the monopoly intermediary setting.

Specifically, the low-reserve intermediary's optimal reserve price solves the first-order condition of (5.41):

$$\frac{\partial profit_{POST}^L(r_{POST}^L)}{\partial r_{POST}^L} = -f(r_{POST}^L)(r_{POST}^L - \rho) + 1 - F(r_{POST}^L) = 0 \quad (5.43)$$

that gives:

$$r_{POST}^* = \rho + \frac{1 - F(r_{POST}^*)}{f(r_{POST}^*)} \quad (5.44)$$

Then, taking first-order condition for the high-reserve intermediary yields:

$$\begin{aligned} \frac{\partial profit_{POST}^H(r_{POST}^H)}{\partial r_{POST}^H} &= -f(r_{POST}^H)[\int_{r_{POST}^L}^{r_{POST}^H} F(y)dy + F(r_{POST}^L)(r_{POST}^H - \rho)] + \\ &\quad + (1 - F(r_{POST}^H))F(r_{POST}^H) = 0 \end{aligned} \quad (5.45)$$

which is satisfied for $r_{POST}^L = r_{POST}^H = r_{POST}^*$. For this to be a pure-strategy Nash equilibrium, the second-order derivative at r_{POST}^* has to be negative. Taking the second-order derivative yields:

$$\begin{aligned} \frac{\partial^2 profit_{POST}^H(r_{POST}^H)}{\partial(r_{POST}^H)^2} &= -f'(r_{POST}^H)[\int_{r_{POST}^L}^{r_{POST}^H} F(y)dy + F(r_{POST}^L)(r_{POST}^H - \rho)] + \\ &\quad + f(r_{POST}^H)[1 - 3F(r_{POST}^H)] \end{aligned} \quad (5.46)$$

which at $r_{POST}^L = r_{POST}^H = r_{POST}^*$ yields:

$$\begin{aligned} \frac{\partial^2 profit_{POST}^H(r_{POST}^H)}{\partial(r_{POST}^H)^2}|_{r_{POST}^*} &= -f'(r_{POST}^*) \frac{F(r_{POST}^*)(1 - F(r_{POST}^*))}{f(r_{POST}^*)} + \\ &\quad + f(r_{POST}^*)[1 - 3F(r_{POST}^*)] \end{aligned} \quad (5.47)$$

Setting $\frac{\partial^2 profit_{POST}^H(r_{POST}^H)}{\partial(r_{POST}^H)^2}|_{r_{POST}^*} < 0$ yields the necessary condition for the existence of a pure-strategy Nash equilibrium.

It is easy to see that such a symmetric pure-strategy Nash equilibrium will always exist for the example of the uniform distribution $U(0, 1)$, with each intermediary setting a reserve price equal to $\frac{1+\rho}{2}$.

Given this equilibrium behavior of the intermediaries, the center's ex-ante expected revenue will be:

$$\text{revenue}_{POST}(\rho) = \int_{r_{POST}^*}^1 y f_2^{(2)}(y) dy + 2F(r_{POST}^*)(1 - F(r_{POST}^*))\rho \quad (5.48)$$

Substituting the intermediaries' equilibrium reserve prices in the equation above yields:

$$\text{revenue}_{POST}(r_{POST}^*) = \int_{r_{POST}^*}^1 y f_2^{(2)}(y) dy + 2F(r_{POST}^*)(1 - F(r_{POST}^*))[r_{POST}^* - \frac{1 - F(r_{POST}^*)}{f(r_{POST}^*)}] \quad (5.49)$$

Taking the first-order derivative of this w.r.t. r_{POST}^* yields:

$$2\{F(r_{POST}^*)[3(1 - F(r_{POST}^*)) - r_{POST}^* f(r_{POST}^*)] - (1 - F(r_{POST}^*))^2[1 - F(r_{POST}^*)]\frac{f'(r_{POST}^*)}{f^2(r_{POST}^*)}\} \quad (5.50)$$

Setting this equal to zero then yields the optimal reserve price of the center. For the case of the uniform distribution, we obtain $r_{POST}^* = 0.7236$ and the optimal center's reserve then will be $\rho_{POST}^* = 2r_{POST}^* - 1 = 0.4472$.

We now show that the above-mentioned result for the existence of a symmetric pure-strategy Nash equilibrium in the duopoly POST intermediary reserve-price-setting problem generalizes to settings with more than one buyer per intermediary.

Theorem 5.1. *There exists a symmetric pure-strategy Nash equilibrium in the duopoly POST - POST intermediary reserve-price-setting game with intermediaries having $k \in \mathbb{N}^+$ captive buyers each, where each intermediary sets a reserve price, r_{POST}^* , that solves:*

$$r_{POST}^* = \rho + \frac{1 - F(r_{POST}^*)}{f(r_{POST}^*)} \quad (5.51)$$

if

$$f'(r_{POST}^*) > \frac{[k - (k+2)F(r_{POST}^*)]f^2(r_{POST}^*)}{kF(r_{POST}^*)(1 - F(r_{POST}^*))} \quad (5.52)$$

Proof. Taking the first- and second-order derivatives of the high-reserve intermediary expected profit (equation (5.37)) w.r.t. r_{POST}^H yields:

$$\begin{aligned} \frac{\partial \text{profit}_{POST}^H(r_{POST}^H)}{\partial r_{POST}^H} &= -kF^{k-1}(r_{POST}^H)f(r_{POST}^H)[F^k(r_{POST}^L)(r_{POST}^L - \rho) + \\ &+ \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy] + kF^{2k-1}(r_{POST}^H)(1 - F(r_{POST}^H)) \end{aligned} \quad (5.53)$$

$$\begin{aligned}
\frac{\partial^2 \text{profit}_{POST}^H(r_{POST}^H)}{\partial(r_{POST}^H)^2} &= k(2k-1)F^{2k-2}(r_{POST}^H)(1-F(r_{POST}^H))f(r_{POST}^H) - \\
&- 2kF^{2k-1}(r_{POST}^H)f(r_{POST}^H) - [k(k-1)F^{k-2}(r_{POST}^H)f^2(r_{POST}^H) + \\
&+ kF^{k-1}(r_{POST}^H)f'(r_{POST}^H)][F^k(r_{POST}^L)(r_{POST}^L - \rho) + \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy] - \\
&- kF^{k-1}(r_{POST}^H)f(r_{POST}^H)F^k(r_{POST}^H)
\end{aligned} \tag{5.54}$$

It is easy to see that setting r_{POST}^H as in (5.51) satisfies the first-order condition of (5.53). However, it still remains to be shown whether this maximizes the expected profit of the high-reserve intermediary. For this to happen, the second-order derivative of this profit expression should be negative:

$$\begin{aligned}
\frac{\partial^2 \text{profit}_{POST}^H(r_{POST}^H)}{\partial(r_{POST}^H)^2}|_{r_L^H=r_{POST}^H=r_{POST}^*} &= kF^{2k-2}(r_{POST}^*)[k(1-F(r_{POST}^*))f(r_{POST}^*) - \\
&- 2F(r_{POST}^*)f(r_{POST}^*) - kF(r_{POST}^*)f'(r_{POST}^*)\frac{1-F(r_{POST}^*)}{f(r_{POST}^*)}]
\end{aligned} \tag{5.55}$$

yielding the condition of (5.52). \square

For the uniform distribution $U(0, 1)$ the condition for the existence of such a pure-strategy Nash equilibrium becomes $r_{POST}^* > \frac{k}{k+1}$ or, equivalently for the center's reserve price, $\rho, \rho > \frac{k-2}{k+1}$, since $r_{POST}^* = \frac{1+\rho}{2}$. This will be always satisfied for $k \leq 2$ when $\rho > 0$, however for the remaining cases we need to derive the center's optimal reserve price under this equilibrium and study the feasibility of such an optimal reserve price.

If both intermediaries set the same reserve price, r_{POST}^* , the center's expected revenue (equation (5.38)) becomes:

$$\begin{aligned}
\text{revenue}(r_{POST}^*) &= 2F^k(r_{POST}^*)(1-F^k(r_{POST}^*))[r_{POST}^* - \frac{1-F(r_{POST}^*)}{f(r_{POST}^*)}] + \\
&+ \int_{r_{POST}^*}^1 2y(1-F^k(y))kF^{k-1}(y)f(y)dy
\end{aligned} \tag{5.56}$$

Taking the first-order derivative w.r.t. r_{POST}^* then yields:

$$\begin{aligned}
\frac{\partial \text{revenue}(r_{POST}^*)}{\partial r_{POST}^*} &= 2F^{k-1}(r_{POST}^*)\{kF^k(r_{POST}^*)[1-F(r_{POST}^*) - r_{POST}^*f(r_{POST}^*)] - \\
&- (1-F^k(r_{POST}^*))[k - (k+2)F(r_{POST}^*) - F(r_{POST}^*)(1-F(r_{POST}^*))\frac{f'(r_{POST}^*)}{f^2(r_{POST}^*)}]\}
\end{aligned} \tag{5.57}$$

The optimal reserve price for the intermediaries, ρ_{POST}^* , solves the first-order condition of this equation subject to the constraint of (5.52). For the case of the uniform distribution $U(0, 1)$, such an equilibrium exists only for the case of $k = 2$, where $r_{POST}^* = 0.7071$ and hence $\rho_{POST}^* = 0.4142$.

For $k > 2$ buyers per intermediary, intermediaries are likely to follow mixed equilibrium reserve-price setting strategies, as the example for the best response function of Figure 5.13 illustrates for $k = 5$ and $\rho = 0$. For this reason, we run the fictitious play algorithm for this setting, varying as before the center's reserve price from 0 to 1 with a step of 0.1.

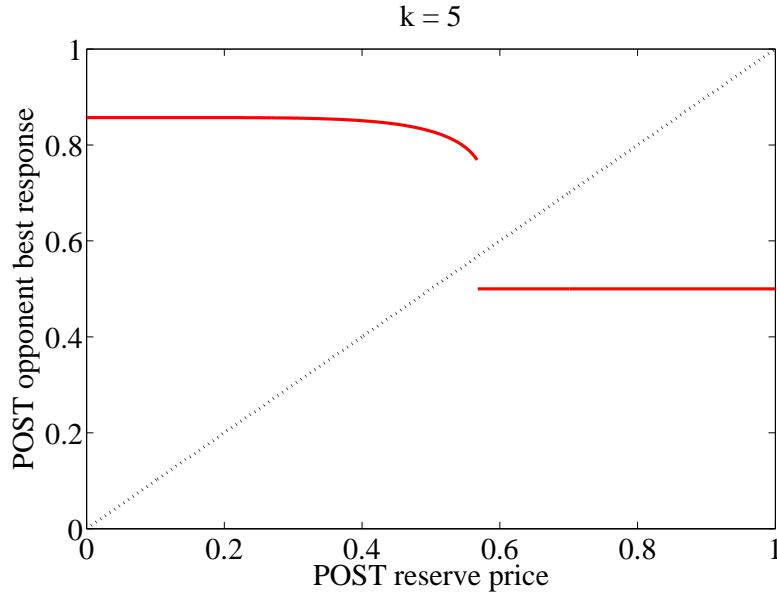


FIGURE 5.13: Best-response reserve price function for a duopoly POST intermediary setting with intermediary reserve prices, where each intermediary has $k = 5$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$ and the center does not impose a reserve price.

Figure 5.14 (left) illustrates the center's expected revenue in the ϵ -NE of the fictitious play experiments, as a function of the reserve price. As can be seen, the optimal reserve price is 0.3, further decreasing compared to the cases of $k = 1$, and $k = 2$ analyzed above. This is due to the fact that, since intermediaries submit their highest-local bid, as the number of buyers increases, the impact of a higher reserve price is offset by the probability of setting these too high and hence missing a trade. This drives the center's optimal reserve price downwards. Figure 5.14 (right) shows the resulting ϵ -NE reserve-price-setting strategy. As can be seen, equilibrium reserve prices are concentrated on $[0.65, 0.71]$, hence are generally smaller compared to the cases of $k = 1$ and $k = 2$.

Next, we analyze an FPSB duopoly intermediary setting. As will be seen, the BNE bidding functions are complex, so the derivation of the expected utilities for all stakeholders becomes technically challenging.

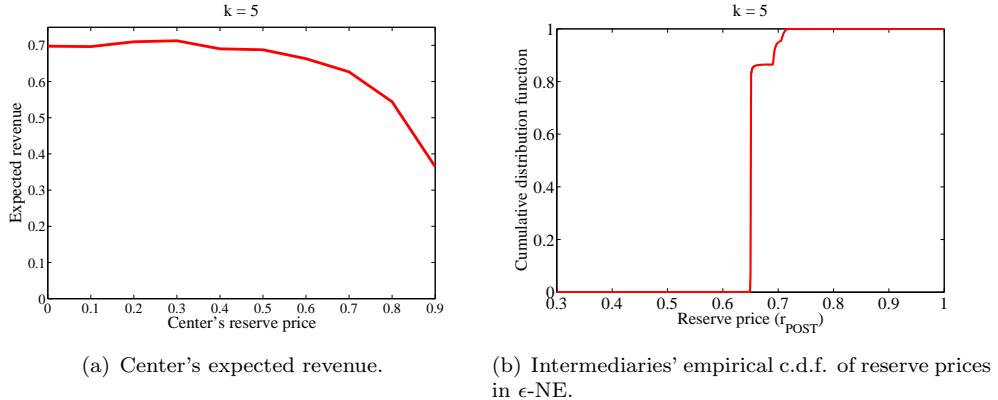


FIGURE 5.14: Center's ex-ante expected revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly POST intermediary setting, where each intermediary has $k = 5$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

5.2.4 First-Price Sealed-Bid Auctions

When intermediaries implement FPSB auctions, the analytical derivations of the expected utilities for all parties become more cumbersome, since one has to derive the equilibrium bidding functions of the buyers. Kotowski (2014) has considered such equilibrium functions for a different problem that perfectly fits our setting. This is the problem of an auctioneer who separates bidders in two groups and discriminates in favor of one of the groups by setting different reserve prices for each group. Among others, the author shows that, when the auctioneer uses a first-price sealed-bid auction with different reserve prices for each group, the bidders' equilibrium bidding functions are nontrivial to derive and might depict discontinuities. We make use of the author's results to numerically calculate the center's average revenue and intermediaries' average profits so as to compare those with the other two auction mechanisms studied. We now illustrate the author's results for the equilibrium bidding functions of the bidders.

Kotowski shows that, when the reserve prices are not very different, bidders' equilibrium strategies are identical for some interval in the support of the valuations, and the bids' profile is called *semi-separating*. Otherwise, the bids of the two groups are disjoint sets and their profile is called *separating*. In the limiting case where both reserve prices are the same, the bids' profile is called *pooling* and coincides with the equilibrium bidding function of buyers when intermediaries do not impose any reserve prices (equation (4.24)). In our terminology, for the existence of a corresponding semi-separating equilibrium, the reserve prices of the two groups (i.e. intermediaries), $r_{FPSB}^L \leq r_{FPSB}^H$, should satisfy the following condition:

$$F^k(r_{FPSB}^H) \int_{r_{FPSB}^L}^1 F^{k-1}(y) dy \leq \int_{r_{FPSB}^H}^1 F^{k-1}(y) dy \quad (5.58)$$

In this case, there is a cut-off point, \hat{v} , in the support of the valuations where the equilibrium bidding function changes. This cut-off point is the unique solution of:

$$F^k(r_{FPSB}^H) \int_{r_{FPSB}^L}^{\hat{v}} F^{k-1}(y) dy = F^k(\hat{v}) \int_{r_{FPSB}^H}^{\hat{v}} F^{k-1}(y) dy \quad (5.59)$$

Then, if:

$$b_\ell(v) = v - \frac{\int_{r_{FPSB}^\ell}^v F^{k-1}(y) dy}{F^{k-1}(v)} \quad (5.60)$$

$$\hat{b}(v) = v - \frac{F^k(\hat{v}) \int_{r_{FPSB}^H}^{\hat{v}} F^{k-1}(y) dy}{F^{2k-1}(v)} - \frac{\int_{\hat{v}}^v F^{2k-1}(y) dy}{F^{2k-1}(v)} \quad (5.61)$$

for $\ell = \{L, H\}$, the semi-separating equilibrium bidding strategies of the buyers in the low- and high-reserve intermediaries, $\beta_L(\cdot), \beta_H(\cdot)$ respectively will be:

$$\beta_\ell(v) = \begin{cases} b_\ell(v) & \text{if } v \in [r_{FPSB}^\ell, \hat{v}] \\ \hat{b}(v) & \text{if } v \in (\hat{v}, 1] \end{cases} \quad (5.62)$$

On the other hand, when the reserve prices are very different, i.e. when

$$F^k(r_{FPSB}^H) \int_{r_{FPSB}^L}^1 F^{k-1}(y) dy > \int_{r_{FPSB}^H}^1 F^{k-1}(y) dy \quad (5.63)$$

then buyers follow the following separating equilibrium bidding strategies:

$$\beta_\ell^s(v) = v - \frac{\int_{r_{FPSB}^\ell}^v F^{k-1}(y) dy}{F^{k-1}(v)} \quad (5.64)$$

Finally, if both intermediaries set the same reserve price, r_{FPSB} , then all buyers follow the same equilibrium bidding strategy:

$$\beta_L(v) = \beta_H(v) = \beta(v) = v - \frac{\int_{r_{FPSB}}^v F^{2k-1}(y) dy}{F^{2k-1}(v)} \quad (5.65)$$

An example of the equilibrium bidding functions when the reserve prices are different are shown in Figure 5.15 where each intermediary has 2 buyers whose private valuations are i.i.d. drawn from the uniform distribution $U(0, 1)$. Given the complexity of the resulting equilibrium function, we do not provide any closed form expressions for the expected revenue and profits for the center and the intermediary.

We now analyze the equilibrium reserve-price-setting strategies of the two FPSB intermediaries for the examples of $k = 1, 2$ and 5 buyers per intermediary whose private valuations are i.i.d. drawn from $U(0, 1)$. Our fictitious play results are shown in Figures

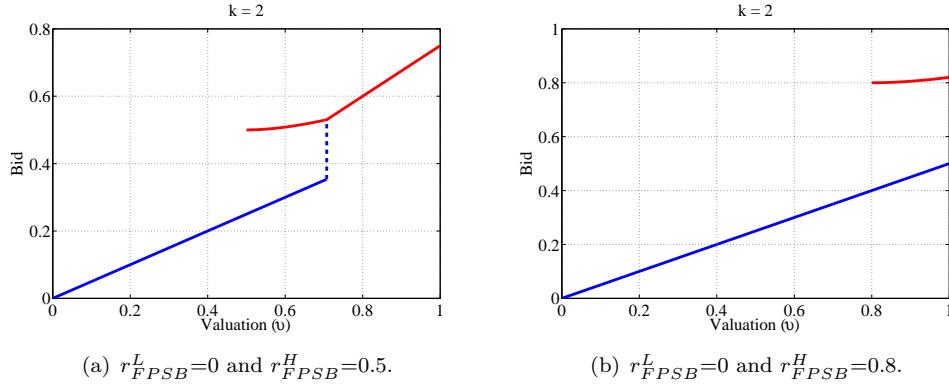


FIGURE 5.15: Two examples of semi-separating (left) and separating (right) equilibrium first-price sealed-bid bidding strategies when there are $k = 2$ buyers in each intermediary whose valuations are i.i.d. drawn from $U(0, 1)$. The bidding functions of buyers in the low-and high-reserve intermediaries are shown with red and blue colour respectively.

5.16 - 5.18. As can be seen, the center's optimal reserve price increases with the number of buyers per intermediary and that also leads to an increase of the reserve prices of the intermediaries. Moreover, this reserve price is also higher compared to all previous scenarios. In all cases, intermediaries in our approximate equilibria follow strictly mixed strategies.

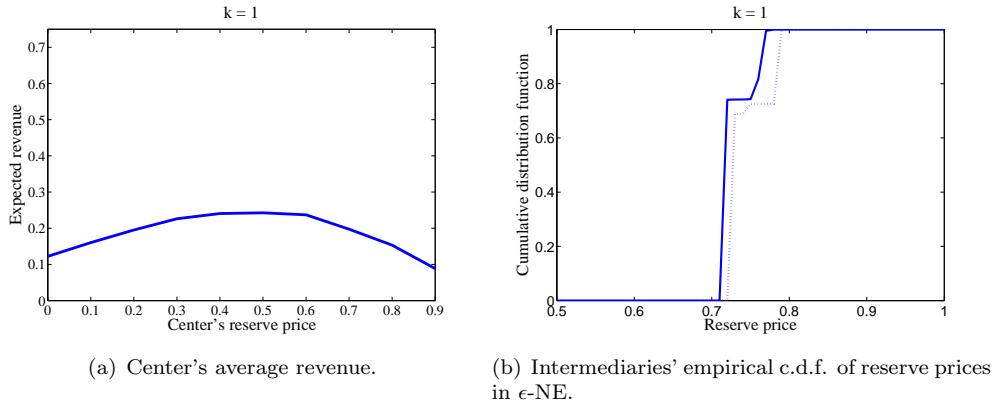


FIGURE 5.16: Center's average revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly FPSB intermediary setting, where each intermediary has $k = 1$ buyer whose private valuation is i.i.d. drawn from $U(0, 1)$.

In what follows, we study a heterogeneous Vickrey setting, where one intermediary implements a PRE and the other a POST mechanism.

5.2.5 Pre- versus Post-Award Second-Price Sealed-Bid Auctions

Let us now assume that intermediaries select different Vickrey mechanisms. We start with the case where the low-reserve intermediary implements a POST mechanism with a

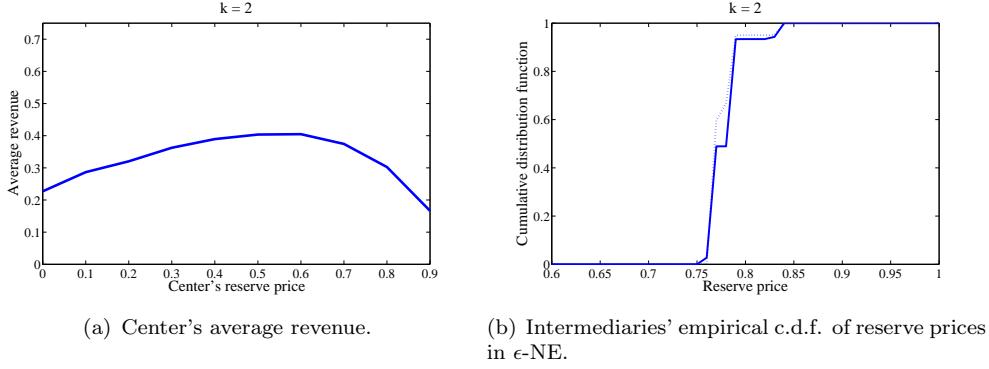


FIGURE 5.17: Center's average revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly FSB intermediary setting, where each intermediary has $k = 2$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

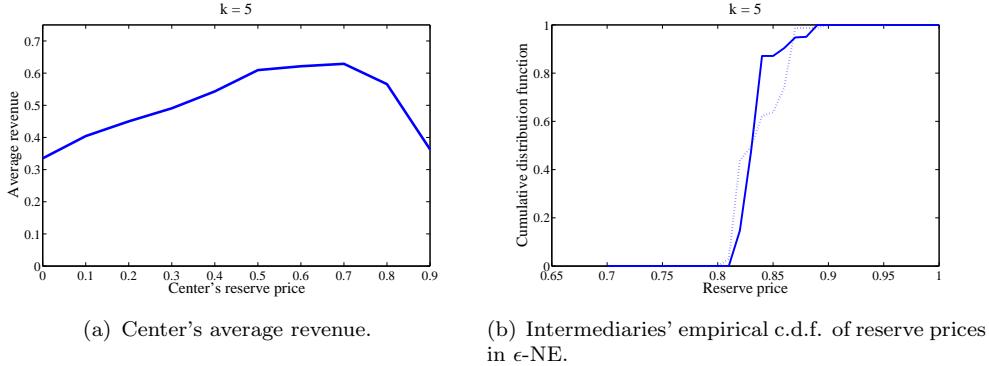


FIGURE 5.18: Center's average revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly FSB intermediary setting, where each intermediary has $k = 5$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

reserve price r_{POST}^L whereas the high-reserve intermediary implements a PRE mechanism with a reserve price $r_{PRE}^H \geq r_{POST}^L$. Then the intermediaries' ex-ante expected profits can be expressed as:

$$\begin{aligned} profit_{POST}^L(r_{POST}^L) &= F^k(r_{PRE}^H)[kF^{k-1}(r_{POST}^L)(1 - F(r_{POST}^L))(r_{POST}^L - \rho) + \\ &+ \int_{r_{POST}^L}^1 (y - \rho)f_2^{(k)}(y)dy] + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^H}^1 (y - r_{PRE}^H)f_2^{(k)}(y)dy + \\ &+ \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^H}^y (y - x)f_2^{(k)}(x)dxdy \end{aligned} \quad (5.66)$$

$$\begin{aligned}
profit_{PRE}^H(r_{PRE}^H) &= F^k(r_{POST}^L)[kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))(r_{PRE}^H - \rho) + \\
&+ \int_{r_{PRE}^H}^1 (y - \rho)f_2^{(k)}(y)dy] + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{POST}^L}^{r_{PRE}^H} (r_{PRE}^H - y)f_1^{(k)}(y)dy + \\
&+ \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^L}^y (y - x)f_1^{(k)}(x)dxdy
\end{aligned} \tag{5.67}$$

In more detail, the low-reserve POST intermediary receives the difference between his reserve price or his local second-highest bid and the center's reserve price, if the other intermediary has no eligible bid and, at the same time, there is at least one bid above his reserve price, r_{POST}^L (first and second term in (5.66)). He also receives the difference between his local second-highest bid and r_{PRE}^H , if there is only a single eligible bid submitted in the other intermediary's auction and, simultaneously, his local second-highest bid is above his reserve price (third term in (5.66)). Finally, he obtains the difference between his local second-highest bid and the other intermediary's submitted bid, if the latter is below the former and both are above the high reserve price (fourth term in (5.66)).

Similarly, the high-reserve PRE intermediary obtains the difference between his reserve price or his local second-highest bid and the center's reserve price when there is at least one bid above r_{PRE}^H and, at the same time, all bids in the other intermediary are below the low reserve price (first and second terms in (5.67) respectively). He also obtains the difference between his reserve price and the opponent's highest local bid, if the latter is above r_{POST}^L but below r_{PRE}^H (third term in (5.67)). Finally, if the highest local bid in the high-reserve intermediary is below his second-highest bid but above the low reserve price, and his second-highest local bid is above r_{PRE}^H , then the intermediary receives the difference between these two bids (fourth term in (5.67)).

The above equations simplify to the following:

$$\begin{aligned}
profit_{POST}^L(r_{POST}^L) &= F^k(r_{PRE}^H)[r_{PRE}^H - \rho - F^k(r_{POST}^L)(r_{POST}^L - \rho) - \\
&- \int_{r_{POST}^L}^{r_{PRE}^H} F_2^{(k)}(y)dy] + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^H}^y F_2^{(k)}(x)dxdy
\end{aligned} \tag{5.68}$$

$$\begin{aligned}
profit_{PRE}^H(r_{PRE}^H) &= F^k(r_{POST}^L)(1 - F^k(r_{PRE}^H))(r_{POST}^L - \rho) + \\
&+ kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{POST}^L}^{r_{PRE}^H} F^k(y)dy + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^L}^y F^k(x)dxdy
\end{aligned} \tag{5.69}$$

Then the center's ex-ante expected revenue will be:

$$\begin{aligned} \text{revenue}_{\text{POST-PRE}}(\rho) = & \rho[F^k(r_{\text{POST}}^L)(1 - F^k(r_{\text{PRE}}^H) + \\ & + F^k(r_{\text{PRE}}^H)(1 - F^k(r_{\text{POST}}^L)] + kr_{\text{PRE}}^H F^{k-1}(r_{\text{PRE}}^H)(1 - F(r_{\text{PRE}}^H))(1 - F^k(r_{\text{PRE}}^H) + \\ & + \int_{r_{\text{PRE}}^H}^1 f_1^{(k)}(y) \int_{r_{\text{PRE}}^H}^y x f_2^{(k)}(x) dx dy + (1 - F^k(r_{\text{PRE}}^H)) \int_{r_{\text{POST}}^L}^{r_{\text{PRE}}^H} y f_1^{(k)}(y) dy + \\ & + \int_{r_{\text{PRE}}^H}^1 f_2^{(k)}(y) \int_{r_{\text{PRE}}^H}^y x f_1^{(k)}(x) dx dy \end{aligned} \quad (5.70)$$

Finally, a buyer with valuation v expects surplus from the low- and high-reserve intermediaries that can be expressed as follows:

$$\Pi_{\text{POST}}^L(v) = \begin{cases} 0 & \text{if } v \in [0, r_{\text{POST}}^L) \\ F^k(r_{\text{PRE}}^H) \int_{r_{\text{POST}}^L}^v F^{k-1}(y) dy & \text{if } v \in [r_{\text{POST}}^L, r_{\text{PRE}}^H) \\ F^k(r_{\text{PRE}}^H)[(v - r_{\text{POST}}^L)F^{k-1}(r_{\text{POST}}^L) + \int_{r_{\text{POST}}^L}^{r_{\text{PRE}}^H} (v - y)f_1^{(k-1)}(y)dy] + \\ + F^{k-1}(r_{\text{PRE}}^H)kF^{k-1}(r_{\text{PRE}}^H)(1 - F(r_{\text{PRE}}^H))(v - r_{\text{PRE}}^H) + \\ + \int_{r_{\text{PRE}}^H}^v (v - y)f_1^{(k-1)}(y)F_2^{(k)}(y)dy + \\ + \int_{r_{\text{PRE}}^H}^v (v - y)f_2^{(k)}(y)F^{k-1}(y)dy & \text{if } v \in [r_{\text{PRE}}^H, 1] \end{cases} \quad (5.71)$$

$$\Pi_{\text{PRE}}^H(v) = \begin{cases} 0 & \text{if } v \in [0, r_{\text{PRE}}^H) \\ (v - r_{\text{PRE}}^H)F^{2k-1}(r_{\text{PRE}}^H) + \int_{r_{\text{PRE}}^H}^v (v - y)f_1^{(k-1)}(y)F^k(y)dy & \text{if } v \in [r_{\text{PRE}}^H, 1] \end{cases} \quad (5.72)$$

Specifically, a buyer with private valuation $v \geq r_{\text{PRE}}^H$ in the low-reserve POST intermediary pays the low reserve price or the highest local bid in this intermediary if all bids in the other intermediary are less than his reserve price and, at the same time, no eligible bid is submitted in her intermediary or the highest local bid in her intermediary is also below his reserve price (first and second terms in (5.71)). She also pays the high reserve price if all bids in the same intermediary are less than his reserve price and there is only a single eligible bid in the opponent intermediary (third term in (5.71)). She also pays the highest local bid in her selected intermediary if this is above r_{PRE}^H , below her valuation, and, at the same time, the highest opponent bid is below this bid (fourth term in (5.71)). Otherwise, she pays the second-highest opponent intermediary's local bid if this is higher than r_{PRE}^H , below her bid, and, at the same time, the highest local opponent bid in her intermediary's auction is less than the former bid (fifth term in (5.71)).

A buyer with private valuation $v \geq r_{\text{PRE}}^H$ in the high-reserve PRE intermediary pays the high reserve price if there is no other eligible bid in both intermediary auctions (first term in (5.72)), otherwise she pays the highest opponent local bid in her intermediary's

auction if this is higher than r_{PRE}^H , lower than her valuation, and, at the same time, the highest bid of the other intermediary is less than the former bid (second term in (5.72)).

On the other hand, when the low-reserve intermediary implements a PRE mechanism with reserve price r_{PRE}^L and the high-reserve intermediary implements a POST mechanism with reserve price $r_{POST}^H \geq r_{PRE}^L$, the intermediaries' ex-ante expected profits will be:

$$\begin{aligned} profit_{PRE}^L(r_{PRE}^L) &= F^k(r_{POST}^H)[kF^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))(r_{PRE}^L - \rho) + \\ &+ \int_{r_{PRE}^L}^1 (y - \rho)f_2^{(k)}(y)dy] + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^H}^y (y - x)f_1^{(k)}(x)dxdy \end{aligned} \quad (5.73)$$

$$\begin{aligned} profit_{POST}^H(r_{POST}^H) &= F^k(r_{PRE}^L)[kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))(r_{POST}^H - \rho) + \\ &+ \int_{r_{POST}^H}^1 (y - \rho)f_2^{(k)}(y)dy] + kF^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))[kF^{k-1}(r_{POST}^H) \\ &(1 - F(r_{POST}^H))(r_{POST}^H - r_{PRE}^L) + \int_{r_{POST}^H}^1 (y - r_{PRE}^L)f_2^{(k)}(y)dy] + \\ &+ kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H)) \int_{r_{PRE}^L}^{r_{POST}^H} (r_{POST}^H - y)f_2^{(k)}(y)dy + \\ &+ \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y (y - x)f_2^{(k)}(x)dxdy \end{aligned} \quad (5.74)$$

That is, the low-reserve PRE intermediary receives the difference between his reserve price or his second-highest local bid and ρ when all opponent intermediary's submitted bids are below r_{POST}^H and there is at least one bid in his auction that is higher than r_{PRE}^L (first and second terms in (5.73)). Otherwise, this intermediary obtains the difference between his local second-highest bid and the opponent intermediary's highest submitted bid if both are above the high reserve price, r_{POST}^H (third term in (5.73)).

Similarly, the high-reserve POST intermediary receives the difference between his local second-highest bid or his reserve price and ρ , if there is no bid submitted by the other intermediary and, simultaneously, there is at least one eligible bid in his auction (first and second terms in (5.74)). He also receives the difference between his local second-highest bid or his reserve price and the low reserve price, r_{PRE}^L , if there is only a single eligible bid in the opponent intermediary and, at the same time, there is at least one eligible bid in his local auction (third and fourth terms in (5.74)). The intermediary receives the difference between his reserve price and the opponent local highest bid when the latter is in $[r_{PRE}^L, r_{POST}^H]$ and there is only a single eligible bid in his local auction (fifth term in (5.74)). Finally, the intermediary receives the difference between his local second-highest bid and the opponent second-highest bid locally if the former is above r_{POST}^H whereas the latter is above r_{PRE}^L but below the former (last term in (5.74)).

The equations above simplify to the following:

$$\begin{aligned} profit_{PRE}^L(r_{PRE}^L) &= F^k(r_{POST}^H)[r_{POST}^H - \rho - F^k(r_{PRE}^L)(r_{PRE}^L - \rho) - \\ &- \int_{r_{PRE}^L}^{r_{POST}^H} F_2^{(k)}(y)dy] + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^H}^y F^k(x)dxdy \end{aligned} \quad (5.75)$$

$$\begin{aligned} profit_{POST}^H(r_{POST}^H) &= F^k(r_{PRE}^L)(1 - F^k(r_{POST}^H))(r_{PRE}^L - \rho) + \\ &+ kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H)) \int_{r_{PRE}^L}^{r_{POST}^H} F_2^{(k)}(y)dy + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y F_2^{(k)}(x)dxdy \end{aligned} \quad (5.76)$$

In this case, the center's ex-ante expected revenue will then be:

$$\begin{aligned} revenue_{PRE-POST}(\rho) &= \rho[F^k(r_{PRE}^L)(1 - F^k(r_{POST}^H)) + \\ &+ F^k(r_{POST}^H)(1 - F^k(r_{PRE}^L))] + kr_{PRE}^L F^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))(1 - F^k(r_{POST}^H) + \\ &+ \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^H}^y xf_1^{(k)}(x)dxdy + \int_{r_{POST}^H}^1 f_1^{(k)}(y) \int_{r_{PRE}^L}^y xf_2^{(k)}(x)dxdy \end{aligned} \quad (5.77)$$

Finally, a buyer with valuation v expects surplus from the low- and high-reserve intermediaries that can be expressed as follows:

$$\Pi_{PRE}^L(v) = \begin{cases} 0 & \text{if } v \in [0, r_{PRE}^L) \\ F^k(r_{POST}^H) \int_{r_{PRE}^L}^v F^{k-1}(y)dy & \text{if } v \in [r_{PRE}^L, r_{POST}^H) \\ F^k(r_{POST}^H)[(v - r_{PRE}^L)F^{k-1}(r_{PRE}^L) + \int_{r_{PRE}^L}^{r_{POST}^H} (v - y)f_1^{(k-1)}(y)dy] + \\ + \int_{r_{POST}^H}^v (v - y)f_1^{(k-1)}(y)F^k(y)dy & \text{if } v \in [r_{POST}^H, 1] \end{cases} \quad (5.78)$$

$$\Pi_{POST}^H(v) = \begin{cases} 0 & \text{if } v \in [0, r_{POST}^H) \\ (v - r_{POST}^H)F^{k-1}(r_{POST}^H)F_2^{(k)}(r_{POST}^H) + \int_{r_{POST}^H}^v (v - y)f_1^{(k-1)}(y)F_2^{(k)}(y)dy + \\ + \int_{r_{POST}^H}^v (v - y)f_2^{(k)}(y)F^{k-1}(y)dy & \text{if } v \in [r_{POST}^H, 1] \end{cases} \quad (5.79)$$

Specifically, a buyer with private valuation $v \geq r_{POST}^H$ in the low-reserve PRE intermediary pays the low reserve price or the highest local bid in this intermediary if all bids in the other intermediary are less than his reserve price and, at the same time, no eligible bid is submitted in her intermediary or the highest local bid in her intermediary is also below his reserve price (first and second terms in (5.78)). Otherwise, she pays the highest local bid in her intermediary if this is above r_{POST}^H , below her valuation, and, at the same time, the highest local bid in the other intermediary is less than this former bid (third term in (5.78)).

A buyer with private valuation $v \geq r_{POST}^H$ in the high-reserve POST intermediary pays the high reserve price if there is no other eligible bid in her local auction and the second-highest local bid in the high-reserve intermediary is also less than the high reserve price (first term in (5.79)). Otherwise she pays the highest opponent local bid in her intermediary's auction if this is higher than r_{POST}^H , lower than her valuation, and, at the same time, the second-highest bid of the other intermediary is less than the former bid (second term in (5.79)). Finally, she pays the second-highest opponent local bid if this is higher than r_{POST}^H , lower than her valuation, and, at the same time, the highest opponent intermediary's local bid is lower than the former bid (third term in (5.79)).

We now present our fictitious play results. In accordance with the previous cases, we consider intermediaries with $k = 1, 2$ and 5 buyers each whose private valuations are i.i.d. drawn from the uniform distribution $U(0, 1)$. Our results for the center's expected revenue and the resulting intermediary ϵ -NE reserve-price-setting strategies when the former sets its optimal reserve price are shown in Figures 5.19 - 5.21. With $k = 1$ buyer per intermediary, fictitious play strategies converge to a single reserve price. However, that does not necessarily mean that intermediaries should follow pure strategies if we increase the level of discretization. The support of the mixed strategies followed by the intermediaries also evidently increases with increasing number of buyers per intermediary.

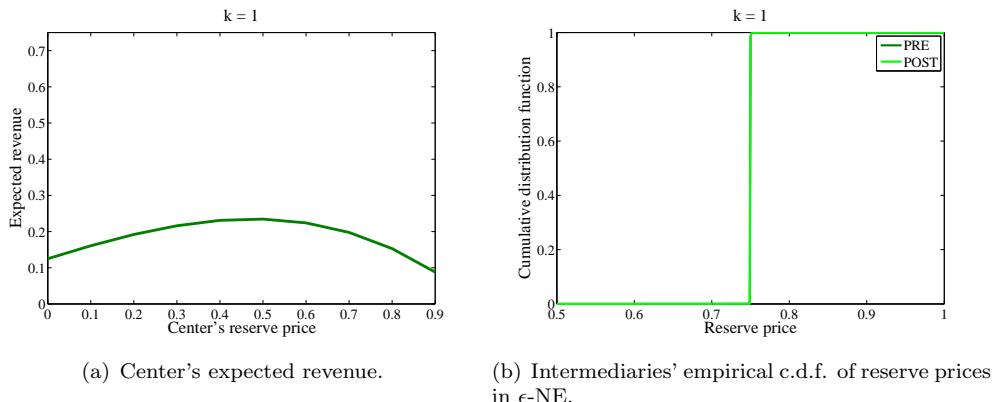


FIGURE 5.19: Center's ex-ante expected revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly PRE-POST intermediary setting, where each intermediary has $k = 1$ buyer whose private valuation is i.i.d. drawn from $U(0, 1)$.

In the following section, we compare the efficiency and revenue attained in all the aforementioned settings for the numerical examples provided.

5.3 Comparison of the Three Intermediary Mechanisms

Having analyzed the (approximate) equilibrium behavior of the three mechanisms along with the PRE - POST heterogeneous duopoly competition, in this section, we compare their

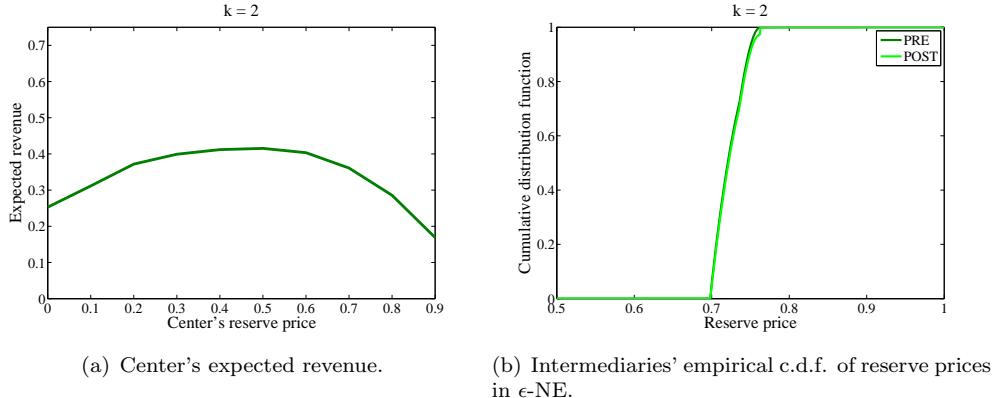


FIGURE 5.20: Center's ex-ante expected revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly PRE-POST intermediary setting, where each intermediary has $k = 2$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

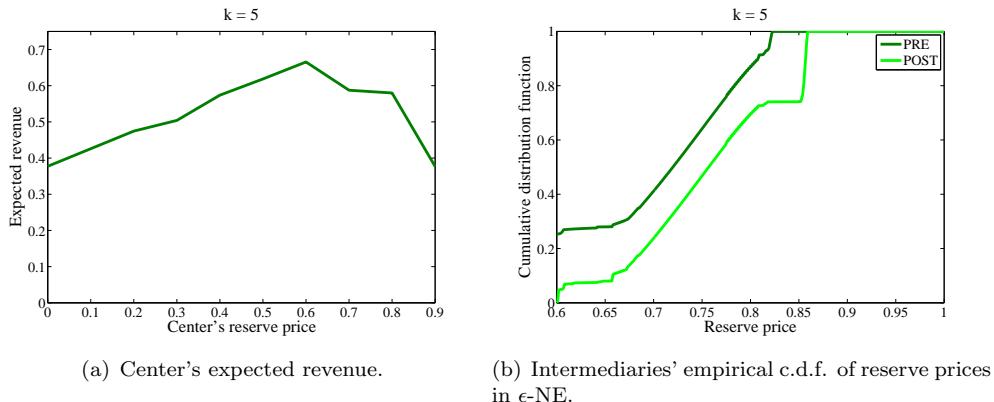


FIGURE 5.21: Center's ex-ante expected revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly PRE-POST intermediary setting, where each intermediary has $k = 5$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

revenue and efficiency. Specifically, we combine the theoretical and numerical results for the uniform distribution $U(0, 1)$ and compare the center's average revenue, intermediary profits and the social welfare for these mechanisms as well as against the corresponding results for the setting of Chapter 4 with no reserve prices.

Figure 5.22 (left) illustrates intermediaries' profits for $k = 1, 2$ and 5 buyers per intermediary. As can be seen, for $k = 1$ buyer per intermediary, all intermediaries obtain similar profit on average, although FPSB and PRE auctions seem to perform slightly better. For more buyers per intermediary, PRE auctioneers perform best with the notable exception of the POST auction that seems to yield higher expected profit in homogeneous settings with 2 buyers per intermediary. We contrast these with the results of Chapter 4 (Figure 5.22 (right)). As can be seen, reserve prices significantly benefit intermediaries in all cases. It is important to note that the reserve prices change the ranking of the mechanisms in terms of profits: when reserve prices are absent, FPSB seem to perform

better, whereas, for the same examples, PRE intermediaries are superior when imposing reserve prices. Finally, we see that the heterogeneous competition between the Vickrey variations benefits the POST mechanism against its PRE counterpart.

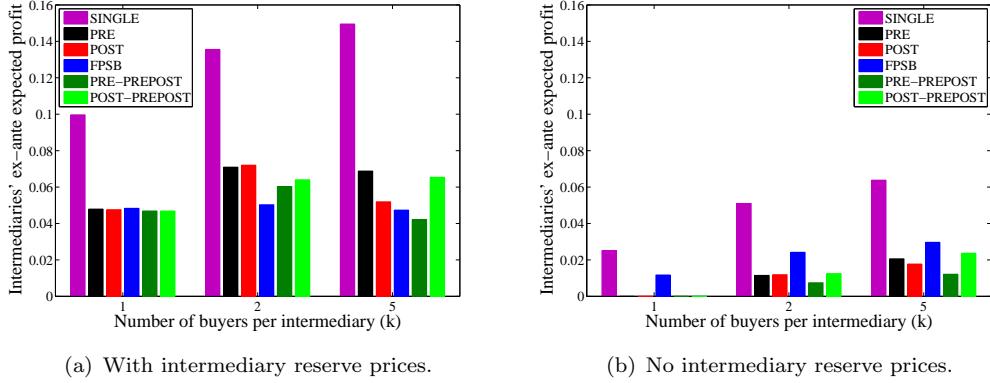


FIGURE 5.22: Intermediaries' ex-ante expected/average profits with (left) and without (right) intermediary reserve prices for the three intermediary mechanisms for varying number of buyers per intermediary whose private valuations are i.i.d. drawn from $U(0, 1)$.

Regarding the buyers' expected surplus, as Figure 5.23 depicts, this is higher for POST intermediaries (both in homogeneous settings and, especially, against a PRE auctioneer), followed by that for homogeneous FPSB mechanisms. In all cases, intermediaries' reserve prices significantly decrease the surplus of buyers as expected.

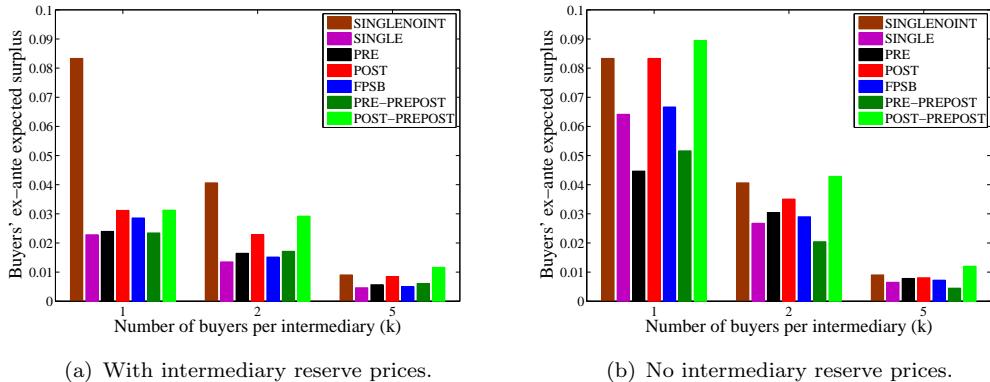


FIGURE 5.23: Buyers' ex-ante expected/average surplus with (left) and without (right) intermediary reserve prices for the three intermediary mechanisms for varying number of buyers per intermediary whose private valuations are i.i.d. drawn from $U(0, 1)$.

Figure 5.24 illustrates the revenue effects of the intermediaries for the center. As can be seen, the center, similar to the results of Chapter 4, benefits from intermediaries adopting the POST mechanism. As illustrated, the center's revenue is significantly smaller compared to the setting with intermediaries that do not impose reserve prices. This is more apparent as the number of buyers decreases.

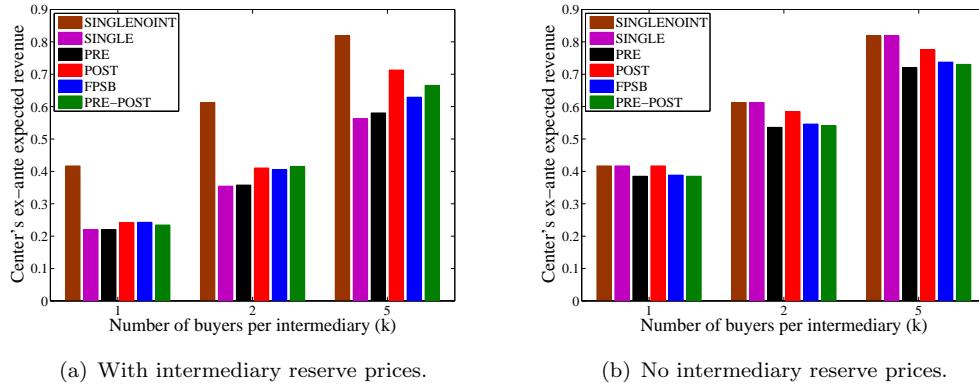


FIGURE 5.24: Center's ex-ante expected/average revenue with intermediaries imposing reserve prices (left) or not (right) for the three intermediary mechanisms for varying number of buyers per intermediary whose private valuations are i.i.d. drawn from $U(0, 1)$.

Finally, Figure 5.25 shows our results for the social welfare. Homogeneous POST mechanisms are not only beneficent to the center but to the system as a whole since setting symmetric reserve prices as a fixed markup on the center's reserve price lead to a smaller center's reserve price compared to the other settings. At this point we should note that numerical and discretization errors in fictitious play against the theoretically derived results for POST auctioneers with $k = 1, 2$ buyers might increase the observed differences. However, the higher social welfare is also apparent in the PRE - POST duopoly setting where the introduction of the POST auction increases the social welfare compared to a homogeneous PRE duopoly. On the other hand, in contrast to the setting with no reserve prices, FPSB appear to be less efficient. This is because of the higher reserve prices that the center imposes that increases the number of lost trades. Also, the social welfare of FPSB is now comparable to that of PRE auctioneers for $k = 2, 5$. This is probably due to the fact that the lower center's reserve prices for PRE intermediaries partially compensates for the increased inefficiency due to misallocation against their FPSB counterpart.

5.4 Summary

In this chapter, we studied the effects of the intermediary reserve prices for the center's revenue, the intermediaries' profits and the buyers' surplus. We have limited our analysis to settings with two intermediaries given the technical challenges that arise in the imperfect competition between auctioneers (Section 2.5).

Specifically, we started with the case of a single intermediary where we depicted the benefits of setting a reserve price for the intermediary. We then studied the duopoly competition between homogeneous PRE, POST and FPSB intermediaries and also looked

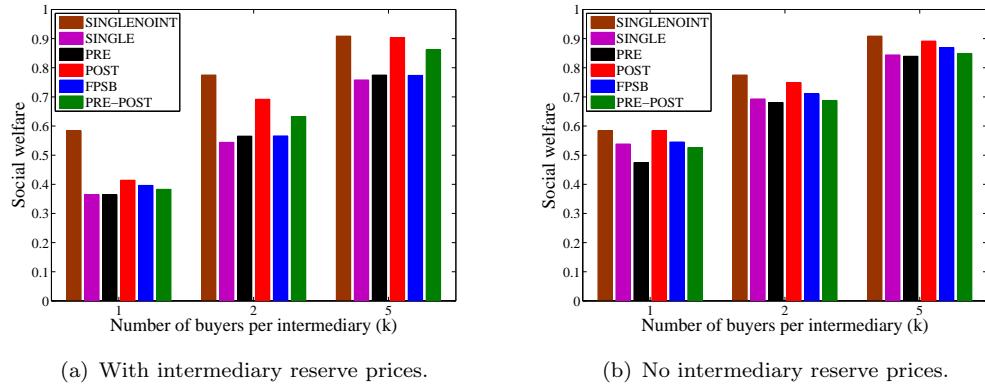


FIGURE 5.25: Social welfare with intermediaries imposing reserve prices (left) or not (right) for the three intermediary mechanisms for varying number of buyers per intermediary whose private valuations are i.i.d. drawn from $U(0, 1)$.

at the heterogeneous competition between the two Vickrey variations. Following this, we characterized the expected utilities of all agents where this was possible. We also proved the existence of a symmetric pure-strategy Nash equilibrium in the reserve-price-setting problem of two **POST** intermediaries where they both set the monopolistic intermediary reserve price, under some conditions on the distribution of private valuations and the number of buyers.

Nevertheless, as the best-response functions for the example of the uniform distribution $U(0, 1)$ depict, intermediaries, in general follow mixed-strategy equilibrium reserve prices. The latter strategies are difficult to derive. For this reason, we turned our attention to learning techniques to obtain ϵ -NE of these strategies. In more detail, we used the fictitious play algorithm to derive approximate equilibria for three examples of $k = 1, 2$ and 5 buyers per intermediary in all the remaining cases. To do this, we first compared our numerical solution to the theoretical one, as was derived by Feldman et al. (2010) for the case of $k = 1$ buyer per intermediary and **PRE** intermediaries.

Our numerical results show that **PRE** auctioneers are indeed profit-superior to the other counterparts, at least in the examples studied. Nevertheless, they are less efficient than their **POST** opponents, in agreement with the results of the previous chapter. Surprisingly, **FPSB** intermediary auctions are also less efficient compared to the **POST** mechanisms as well as to the setting without reserve prices. This is probably due to the higher optimal reserve price for the center that increases the probability of non-allocation.

Up to now, we have assumed that buyers are non-strategic in their selection of intermediary, being exogenously allocated to the available intermediaries. In the following chapter, we remove this captivity assumption and let the intermediaries compete to attract them by setting appropriate reserve prices.

Chapter 6

Intermediaries with Non-Captive Buyers

In the previous chapter, we studied the revenue and efficiency effects of the competition between intermediaries with captive buyers. In this chapter, we remove this limitation and let the buyers simultaneously and strategically select one of the intermediaries, albeit in a restrictive duopoly setting. Given the issues related to competition between auctioneers, described in Section 2.5, as well as the analytical tractability problems of calculating BNE bidding strategies in FPSB auctions (Kotowski, 2014), we only consider PRE and POST intermediaries.

Specifically, in accordance to the model described in Section 3.3, in what follows, we assume that both the center and the intermediaries have selected their mechanism in advance and, in the setting studied within this chapter, that buyers strategically select one of the intermediaries. First, the center announces a reserve price for the good to be auctioned to the intermediaries who then, based on this information, strategically select and announce their reserve prices to the population of buyers. Buyers then learn their private valuations for the good and strategically choose one of the intermediaries to submit a bid for the good. Intermediaries then run local auctions with their selected buyers' bids subject to the constraint imposed by their reserve price to determine a winner, if any, and a payment contingent on winning the good at the central auction, and then submit a single bid (if there was some qualified bid) to the center. The center then runs its auction with intermediaries' bids, determines a winning intermediary, if any, and payment and allocates the good to this intermediary, if there is a winner, who then allocates the good to his winning local buyer for the pre-determined price.

To this end, in Section 6.1, we study the intermediary selection problem that the buyers face in a duopoly setting with homogeneous PRE and POST mechanisms and then extend our analysis for a heterogeneous setting with one PRE and one POST mechanism. Then, in Section 6.2, we study the intermediaries' and center's best responses given the buyers'

selection of intermediary. However, for the last problem, we are only able to provide numerical results for the case of buyers with private valuations i.i.d. drawn from the uniform distribution $U(0, 1)$. Finally, Section 6.3 concludes.

6.1 Buyers’ Duopoly Intermediary Selection Problem with Vickrey Auctioneers

We begin our analysis with the buyers’ problem of selecting one of $n = 2$ intermediaries. We assume that buyers single-home, i.e. can only select one intermediary. This is for a variety of reasons. First, in the case of POST mechanisms, bidding in both auctions means that the winner is likely to pay her bid, creating a number of complications for the bidding strategies of the buyers. Second, in practice, advertisers tend to select one intermediary for each type of campaign, since there is an underlying cost of managing each campaign that we normalize to be zero here. Third, our aim is to study the competition between intermediaries in finite markets where more than one intermediaries are present and, advertisers cannot fully multi-home (i.e. select all intermediaries).

We begin our analysis for homogeneous intermediaries, starting with the case of two PRE intermediaries and then moving to POST intermediaries. We then extend our analysis to heterogeneous mechanisms, comparing PRE versus POST competition.

6.1.1 Pre-Award Second-Price Sealed-Bid Intermediary Auctions

In accordance with our model description in Chapter 3, we consider a setting with a unique indivisible good and assume a population of $K > n$, buyers that compete for this good, but are allowed to participate only via two qualified *intermediaries*, s_{PRE}^L, s_{PRE}^H , that both implement PRE mechanisms with reserve prices $r_{PRE}^L \leq r_{PRE}^H$. In this subsection, we characterize the resulting Bayes-Nash equilibria of the intermediary selection problem that the buyers face.

Since we are in a probabilistic environment, our equilibrium concept is *symmetric Bayes-Nash*, assuming that buyers cannot coordinate, and act anonymously. Keeping the notation of Chapter 3, we denote by $\theta : V \mapsto [0, 1]$ the selection strategy of the buyers¹, which is a mapping from a buyer’s private valuation to the probability of selecting the *low-reserve* intermediary, s_{PRE}^L . Thus, $1 - \theta(v)$ is the probability that a buyer with private valuation v selects intermediary s_{PRE}^H . In what follows, we start by providing a closed-form expression for the surplus from each intermediary that the buyers expect.

¹Since we consider symmetric selection strategies, we drop the index from the selection function for notational convenience, i.e. $\theta_{(i)}(\cdot) = \theta(\cdot)$ for all $i \in \{1, \dots, K\}$.

The expected ex-interim surplus for a buyer with private valuation v from selecting the low- and high-reserve intermediary, $\Pi_{PRE}^L(v), \Pi_{PRE}^H(v)$ respectively, when $r_{PRE}^L < r_{PRE}^H$, can be written as:

$$\Pi_{PRE}^L(v) = \begin{cases} 0 & \text{if } v \in [0, r_{PRE}^L) \\ \int_{r_{PRE}^L}^v F_1^{(K-1)}(y) dy & \text{if } v \in [r_{PRE}^L, r_{PRE}^H) \\ (v - r_{PRE}^L)F_1^{(K-1)}(r_{PRE}^L) + \int_{r_{PRE}^L}^{r_{PRE}^H} (v - y)f_1^{(K-1)}(y)dy + \\ + \int_{r_{PRE}^H}^v (v - y)\theta(y)f_1^{(K-1)}(y)dy + \\ + \int_{y_2=r_{PRE}^H}^v \int_{y_1=y_2}^1 (v - y_2)(1 - \theta(y_1))\theta(y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 & \text{if } v \in [r_{PRE}^H, 1] \end{cases} \quad (6.1)$$

$$\Pi_{PRE}^H(v) = \begin{cases} 0 & \text{if } v \in [0, r_{PRE}^H) \\ (v - r_{PRE}^H)F_1^{(K-1)}(r_{PRE}^H) + \int_{r_{PRE}^H}^v (v - y)(1 - \theta(y))f_1^{(K-1)}(y)dy + \\ + (v - r_{PRE}^H) \int_{y_2=0}^{r_{PRE}^H} \int_{y_1=r_{PRE}^H}^1 \theta(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\ + \int_{y_2=r_{PRE}^H}^v \int_{y_1=y_2}^1 (v - y_2)\theta(y_1)(1 - \theta(y_2))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 & \text{if } v \in [r_{PRE}^H, 1] \end{cases} \quad (6.2)$$

where $F_1^{(K-1)}(y) = F^{K-1}(y)$, $f_1^{(K-1)}(y) = (K-1)F^{K-2}(y)f(y)$ are the cumulative distribution and density functions of the first-order statistic, and $f_{1,2}^{(K-1)}(y_1, y_2) = (K-1)(K-2)f(y_1)f(y_2)F^{K-3}(y_2)$ is the joint density of the first- and second-order statistics among $K-1$ bids.

In more detail, a buyer with valuation in $[r_{PRE}^H, 1]$ expects positive surplus from the low-reserve intermediary auction, s_{PRE}^L , when all opponent bids are less than or equal to r_{PRE}^L (first term in (6.1)), or when the expected highest opponent bid over the population of buyers is higher than r_{PRE}^L , lower than her valuation, and is submitted in the same auction (second and third terms in (6.1)), as this bid will always win at the center. Finally, she expects positive surplus from s_{PRE}^L when the expected second highest opponent bid over the population of buyers is higher than r_{PRE}^H , lower than her valuation, and is submitted in the same auction, and, at the same time, the expected highest opponent bid is submitted in the high-reserve intermediary, s_{PRE}^H (fourth term in (6.1)). This is because the local second-highest bids compete at the center, and hence her local second-highest bid (which will be the third-highest global bid) is guaranteed to win against the local second-highest bid in the other auction (which will be at most the fourth-highest global bid or r_{PRE}^H).

Similarly, a buyer with valuation in $[r_{PRE}^H, 1]$ expects positive surplus from the high-reserve intermediary auction, s_{PRE}^H , when all opponent bids are less than or equal to r_{PRE}^H (first term in (6.2)), or when the expected highest opponent bid over the population of buyers is higher than r_{PRE}^H , lower than her valuation, and is submitted in the same

auction (second term in (6.2)). She also expects positive surplus from s_{PRE}^H when the expected second-highest opponent bid over the population of buyers is higher than r_{PRE}^H , lower than her valuation, and is submitted in the same auction, and, at the same time, the expected highest opponent bid is submitted in the low-reserve intermediary auction, s_{PRE}^L (fourth term in (6.2)). Finally, the third term in (6.2) corresponds to the case where the expected highest opponent bid is higher than r_{PRE}^H and submitted in s_{PRE}^L , and, at the same time, the expected second-highest opponent bid is less than r_{PRE}^H . Then, the buyer's expected payment is r_{PRE}^H , as the forwarded bid by s_{PRE}^L (which will be at most the third-highest global bid or r_{PRE}^L) will always be less than r_{PRE}^H in this case.

In the special case where $r_{PRE}^L = r_{PRE}^H = r_{PRE}$, assuming a fair tie-breaking rule by the center, the expected ex-interim surplus for a buyer with valuation v from the low- and high-reserve intermediary, $\Pi_{PRE}^{Leq}(v), \Pi_{PRE}^{Heq}(v)$ respectively, can be written as:

$$\Pi_{PRE}^{Leq}(v) = \begin{cases} 0 & \text{if } v \in [0, r_{PRE}) \\ (v - r_{PRE})F_1^{(K-1)}(r_{PRE}) + \int_{r_{PRE}}^v (v - y)\theta(y)f_1^{(K-1)}(y)dy + \\ + \frac{1}{2}(v - r_{PRE}) \int_0^{r_{PRE}} \int_{r_{PRE}}^1 (1 - \theta(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\ + \int_{y_2=r_{PRE}}^v \int_{y_1=y_2}^1 (v - y_2)(1 - \theta(y_1))\theta(y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 & \text{if } v \in [r_{PRE}, 1] \end{cases} \quad (6.3)$$

$$\Pi_{PRE}^{Heq}(v) = \begin{cases} 0 & \text{if } v \in [0, r_{PRE}) \\ (v - r_{PRE})F_1^{(K-1)}(r_{PRE}) + \int_{r_{PRE}}^v (v - y)(1 - \theta(y))f_1^{(K-1)}(y)dy + \\ + \frac{1}{2}(v - r_{PRE}) \int_{y_2=0}^{r_{PRE}} \int_{y_1=r_{PRE}}^1 \theta(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\ + \int_{y_2=r_{PRE}}^v \int_{y_1=y_2}^1 (v - y_2)\theta(y_1)(1 - \theta(y_2))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 & \text{if } v \in [r_{PRE}, 1] \end{cases} \quad (6.4)$$

The third term in both equations above represents the expected surplus of a buyer when the highest opponent bid is submitted in the other intermediary, whereas all remaining bids are less than the reserve price set by the intermediaries. In this case, both intermediaries will submit r_{PRE} at the center, where a fair tie breaking rule yields the same probability of winning the auction.

In the next theorem, we show that buyers in this case select each intermediary with equal probability in equilibrium unless both reserve prices are 0 or 1. In the former, more interesting case, if a buyer selects a different intermediary than all other opponent buyers, he obtains zero surplus. This is because the intermediary submits zero at the center that can never win against the other intermediary. Formally:

Theorem 6.1. *Whenever $r_{PRE}^L = r_{PRE}^H = r_{PRE}$, randomizing with equal probability is a mixed-strategy Bayes-Nash equilibrium for the buyers in the buyer PRE - PRE duopoly*

intermediary selection problem. Moreover, whenever the reserve prices are such that $F(r_{PRE}) = 0$ or $F(r_{PRE}) = 1$, there exists a pure-strategy Bayes-Nash equilibrium where all buyers select either the low-reserve intermediary or the high-reserve intermediary auction.

Proof. It is easy to see that the only mixed equilibrium strategy $\theta(v) \in (0, 1)$ equals $\frac{1}{2}$ for all $v \in [r_{PRE}, 1]$, due to the symmetry of (6.3) and (6.4). For the pure-strategy BNE, suppose without loss of generality that all (other) buyers select intermediary s_{PRE}^L , i.e. $\theta(v) = 1$ for all $v \in [r_{PRE}, 1]$. Then, the surplus difference that a buyer with valuation $v \in [r_{PRE}, 1]$ expects will be:

$$\begin{aligned} \Pi_{PRE}^{Leq}(v) - \Pi_{PRE}^{Heq}(v) &= \int_{r_{PRE}}^v F_1^{(K-1)}(y)dy - (v - r_{PRE})[\frac{K-1}{2}F^{K-2}(r_{PRE}) - \\ &\quad - \frac{K-3}{2}F^{K-1}(r_{PRE})] \end{aligned} \quad (6.5)$$

The second partial derivative of this function w.r.t. v is:

$$\frac{\partial^2}{\partial v^2}(\Pi_1^{eq}(v) - \Pi_2^{eq}(v)) = (K-1)F^{K-2}(v)f(v) \geq 0 \quad (6.6)$$

This means that the function is convex, so its global minimum at a valuation we denote v_c will satisfy the FOC:

$$F^{K-1}(v_c) = \frac{K-1}{2}F^{K-2}(r_{PRE}) - \frac{K-3}{2}F^{K-1}(r_{PRE}) \quad (6.7)$$

For the existence of a pure-strategy BNE, we need to show that $\Pi_1^{eq}(v_c) - \Pi_2^{eq}(v_c) \geq 0$. Using (6.7), this means that $\int_{r_{PRE}}^{v_c} F^{K-1}(y)dy \geq F^{K-1}(v_c)(v_c - r_{PRE})$. However, from the first mean value theorem for integration, $\int_{r_{PRE}}^{v_c} F^{K-1}(y)dy = F^{K-1}(\omega)(v_c - r_{PRE})$, where $r_{PRE} < \omega < v_c$. So, we would have that $F^{K-1}(\omega)(v_c - r_{PRE}) \geq F^{K-1}(v_c)(v_c - r_{PRE})$, which can only happen for $v_c = r_{PRE}$, since $f > 0 \Rightarrow F(\omega) < F(v_c)$. Using this last fact in (6.7) yields:

$$\begin{aligned} F^{K-1}(r_{PRE}) &= \frac{K-1}{2}F^{K-2}(r_{PRE}) - \frac{K-3}{2}F^{K-1}(r_{PRE}) \implies \\ &\implies F(r_{PRE}) = 0 \text{ or } F(r_{PRE}) = 1 \end{aligned} \quad (6.8)$$

Similarly, when all (other) buyers select intermediary s_{PRE}^H , i.e. $\theta(v) = 0$ for all $v \in [r_{PRE}, 1]$, we reach at the same conclusion due to the symmetry in (6.3) and (6.4). \square

In what follows, we will consider the most interesting cases where r_{PRE}^L is strictly lower than r_{PRE}^H . In more detail, in the next section we will show that, when the reserve prices are sufficiently different, a unique pure-strategy BNE arises where all buyers select the low-reserve intermediary.

6.1.1.1 Pure-Strategy Bayes-Nash Equilibria

We start by proving the existence of a pure-strategy BNE in the PRE - PRE intermediary selection problem where all buyers select s_{PRE}^L :

Theorem 6.2. *There exists a pure-strategy Bayes-Nash equilibrium in the buyer PRE - PRE duopoly intermediary selection problem where all buyers select the low-reserve auction if the reserve price of the low-reserve intermediary, r_{PRE}^L , is lower or equal than a critical value r_c ($r_{PRE}^L \leq r_c < r_{PRE}^H$) satisfying:*

$$\int_{r_c}^{v_c} F_1^{(K-1)}(y)dy = F_1^{(K-1)}(v_c)(v_c - r_{PRE}^H) \quad (6.9)$$

where v_c is such that:

$$F_1^{(K-1)}(v_c) = F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] \quad (6.10)$$

Proof. Consider the case that all (other) buyers select the low-reserve intermediary auction, that is $\theta(v) = 1$ for all $v \in [r_{PRE}^L, 1]$. Then, using equations (6.1) and (6.2), we can write the difference in surplus that a buyer with valuation $v \in [r_{PRE}^H, 1]$ expects as:

$$\begin{aligned} \Pi_{PRE}^L(v) - \Pi_{PRE}^H(v) &= \int_{r_{PRE}^L}^v F_1^{(K-1)}(y)dy - \\ &[F_1^{(K-1)}(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))F_1^{(K-2)}(r_{PRE}^H)](v - r_{PRE}^H) \end{aligned} \quad (6.11)$$

We can derive the first- and second-order partial derivatives of this function with respect to v :

$$\begin{aligned} \frac{\partial}{\partial v}(\Pi_{PRE}^L(v) - \Pi_{PRE}^H(v)) &= F_1^{(K-1)}(v) - [F_1^{(K-1)}(r_{PRE}^H) + \\ &+ (K-1)(1 - F(r_{PRE}^H))F_1^{(K-2)}(r_{PRE}^H)] \end{aligned} \quad (6.12)$$

$$\frac{\partial^2}{\partial v^2}(\Pi_{PRE}^L(v) - \Pi_{PRE}^H(v)) = f_1^{(K-1)}(v) \geq 0 \quad (6.13)$$

This means that the function is convex, so its global minimum at a valuation we denote v_c will satisfy the condition:

$$\begin{aligned} \frac{\partial}{\partial v}(\Pi_{PRE}^L(v_c) - \Pi_{PRE}^H(v_c)) &= 0 \implies \\ \implies F_1^{(K-1)}(v_c) &= F_1^{(K-1)}(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))F_1^{(K-2)}(r_{PRE}^H) \end{aligned} \quad (6.14)$$

For this to be a pure-strategy BNE, we require that $\Pi_{PRE}^L(v_c) - \Pi_{PRE}^H(v_c) \geq 0$. The equality $\Pi_{PRE}^L(v_c) - \Pi_{PRE}^H(v_c) = 0$ gives us an upper bound for r_{PRE}^L , which we call the critical reserve price, r_c . As can be seen from (6.14), v_c is only dependent on

r_{PRE}^H . Additionally, $\Pi_{PRE}^L - \Pi_{PRE}^H$ is a decreasing function of r_{PRE}^L (see (6.11)). This means that, for a given r_{PRE}^H , setting $r_{PRE}^L = r_c + \epsilon$, where $\epsilon > 0$ is a strictly positive quantity, $\Pi_{PRE}^L(v_c) - \Pi_{PRE}^H(v_c) \leq 0$. This bound is strict if $F(\cdot)$ is strictly increasing. On the other hand, setting $r_{PRE}^L = r_c - \epsilon$ gives a non-negative surplus difference, i.e. $\Pi_{PRE}^L(v_c) - \Pi_{PRE}^H(v_c) \geq 0$. So, a pure-strategy BNE exists for any $r_{PRE}^L \leq r_c$. \square

To illustrate the above theorem, Figure 6.1 depicts the reserve-price combinations for which the pure-strategy BNE exists for a uniform distribution $F = U(0, 1)$, and for varying numbers of buyers, where equations (6.9) - (6.10) give us the following condition²:

$$(1 - K)[r_{PRE}^{H-K-1} + (K - 1)(1 - r_{PRE}^H)r_{PRE}^{H-K-2}]^{\frac{K}{K-1}} + Kr_{PRE}^H[r_{PRE}^{H-K-1} + (K - 1)(1 - r_{PRE}^H)r_{PRE}^{H-K-2}] - r_c^K = 0 \quad (6.15)$$

Here, the top frontier of each region corresponds to the critical reserve price, r_c . As can be seen, for the pure-strategy BNE where all buyers go to the low-reserve intermediary to exist, the required difference between the reserve prices has to be quite large when r_{PRE}^H is relatively low, but the required minimum difference rapidly becomes smaller as r_{PRE}^H increases. Moreover, as the number of buyers increases, the minimum difference between the reserve prices significantly increases, and the region where the pure-strategy BNE exists shrinks. In what follows, we prove, through a number of steps, that the equilibria of Theorem 6.2 are the only pure-strategy BNE of the intermediary selection problem. First, we show that it is not a pure-strategy BNE for the buyers to always select the high-reserve intermediary.

Theorem 6.3. *There is no pure-strategy Bayes-Nash equilibrium in the buyer PRE - PRE duopoly intermediary selection problem where all buyers always select the high-reserve intermediary.*

Proof. Similarly to the proof of Theorem 6.2, when other buyers select the high-reserve intermediary auction, the difference in expected surplus for a buyer with valuation $v \in [r_{PRE}^H, 1]$ will be:

$$\Pi_{PRE}^H(v) - \Pi_{PRE}^L(v) = \int_{r_{PRE}^H}^v F_1^{(K-1)}(y)dy - \int_{r_{PRE}^L}^{r_{PRE}^H} F_1^{(K-1)}(y)dy - (v - r_{PRE}^H)F_1^{(K-1)}(r_{PRE}^H) \quad (6.16)$$

By taking the first- and second-order derivative, we get:

$$\frac{\partial}{\partial v}(\Pi_{PRE}^H(v) - \Pi_{PRE}^L(v)) = F_1^{(K-1)}(v) - F_1^{(K-1)}(r_{PRE}^H) \quad (6.17)$$

²Solving (6.10) for v_c yields $v_c = [r_{PRE}^{H-K-1} + (K - 1)(1 - r_{PRE}^H)r_{PRE}^{H-K-2}]^{\frac{1}{K-1}}$. Then $K(\Pi_{PRE}^L(v_c) - \Pi_{PRE}^H(v_c)) = v_c^K - r_c^K - K[r_{PRE}^{H-K-1} + (K - 1)(1 - r_{PRE}^H)r_{PRE}^{H-K-2}]v_c + Kr_{PRE}^H[r_{PRE}^{H-K-1} + (K - 1)(1 - r_{PRE}^H)r_{PRE}^{H-K-2}] = 0$.

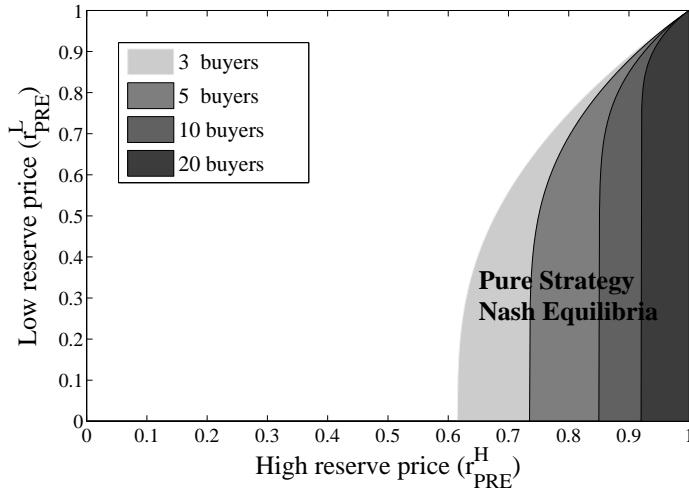


FIGURE 6.1: Figure showing the existence of pure-strategy BNE for the buyers' PRE - PRE intermediary selection problem, for varying numbers of buyers whose private valuations are i.i.d. drawn from $U(0, 1)$. Each gray area indicates, for a given number of buyers, the set of reserve price pairs for which all buyers select the low-reserve intermediary in equilibrium. This is the unique equilibrium, and no pure-strategy BNE exists outside of this set.

$$\frac{\partial^2}{\partial v^2}(\Pi_{PRE}^H(v) - \Pi_{PRE}^L(v)) = f_1^{(K-1)}(v) \geq 0 \quad (6.18)$$

This means that the function is convex, so there is a global minimum at v_c where $F_1^{(K-1)}(v_c) = F_1^{(K-1)}(r_{PRE}^H)$. But $\Pi_{PRE}^H(v_c) - \Pi_{PRE}^L(v_c) \leq 0$ (in general $\Pi_{PRE}^H(v_c) - \Pi_{PRE}^L(v_c) < 0$, unless there are no bidders with valuations in $[r_{PRE}^L, r_{PRE}^H]$), so this can never be a symmetric pure-strategy BNE. \square

The only remaining case to consider is a selection strategy $\theta(v)$ consisting of a number of intervals involving pure strategies, defined by cut-off points, so that the pure strategies change between two successive intervals. Such cut-off strategies are commonly found in the literature on competing auctions (see, for example, Burguet and Sákovics (1999); Hernando-Veciana (2005); Virág (2010)). Suppose that there are $m \geq 1$ points, $w_i, i = 1, \dots, m$, in V , which we call *cut-off points*, so that $\theta(v) = \theta_1$ for $v \in [r_{PRE}^H, w_1]$, $\theta(v) = \theta_2$ for $v \in [w_1, w_2]$ and so on. Moreover, it has to hold that $\Pi_{PRE}^L(w_i) = \Pi_{PRE}^H(w_i) \forall i = 1, \dots, m$, i.e. a buyer with a valuation equal to a cut-off point has to be indifferent between choosing either intermediary. The next theorem shows that, if $\theta_i \in \{0, 1\}$, i.e. buyers follow pure strategies in each interval, no pure-strategy BNE with such cut-off points exists.

Theorem 6.4. *Let $w_1, w_2, \dots, w_k \in (r_{PRE}^H, 1]$, $m \in \mathbb{N}^*$ denote cut-off points and let $\theta : V \mapsto [0, 1]$ be a strategy profile where $\theta(v_i) = \theta_1$ if $r_{PRE}^H \leq v_i < w_1$, $\theta(v_i) = \theta_2$ if $w_1 \leq v_i < w_2$ and so on, $\theta(v_i) = \theta_{m+1}$ if $w_k \leq v_i \leq 1$, for each buyer i , $i = 1, \dots, K$. Then, $\theta(\cdot)$ is not a pure Bayes-Nash equilibrium strategy profile of the buyer PRE - PRE duopoly intermediary selection problem.*

Proof. We prove this statement by contradiction. Specifically, if such equilibrium strategies existed, they should have the form $\theta(v) = \theta_{\lambda+1} \in \{0, 1\}$, for $v \in [w_\lambda, w_{\lambda+1})$, $\lambda = 1, \dots, m+1$ such that $\Pi_{PRE}^L(w_\lambda) = \Pi_{PRE}^H(w_\lambda)$ and $\theta_\lambda \neq \theta_{\lambda+1}$, so the w_λ are points where buyers change their selection strategies.

A buyer whose private valuation equals r_{PRE}^H expects positive surplus from the low-reserve intermediary and zero surplus from the high-reserve intermediary. This means that $\theta(r_{PRE}^H) = \theta_1 = 1$. Given that Π_{PRE}^L, Π_{PRE}^H are continuous functions of v , it will be true that $\theta_1 = 1$ for all $v \in [r_{PRE}^H, w_1]$. This means that $\theta_{\lambda+1} = 1$ for even λ and $\theta_{\lambda+1} = 0$ for odd λ . For a pure-strategy BNE to exist, we require that $\Pi_{PRE}^L(v) - \Pi_{PRE}^H(v) \geq 0$ when $\theta_{\lambda+1} = 1$, and $\Pi_{PRE}^L(v) - \Pi_{PRE}^H(v) \leq 0$ otherwise. We have to consider two cases, when $m = 1$ and $m \geq 2$. This is because, as will be seen, for such equilibrium strategies to exist, there should be discontinuities within the intervals defined by two successive cut-off points. This cannot happen with only two cut-off points.

- *Single cut-off point ($m = 1$).*

Let us start with the case that $m = 1$, i.e. there exists a single cut-off point. Since we are interested in pure-strategy BNE ($\theta_\lambda \in \{0, 1\}$, $\lambda = 1, 2$), this means that $\theta_2 = 0$. For this to be a pure-strategy BNE, we need that $(\Pi_{PRE}^L - \Pi_{PRE}^H)(r_{PRE}^H \leq v < w) \geq 0$ and $(\Pi_{PRE}^H - \Pi_{PRE}^L)(w \leq v \leq 1) \geq 0$. Then the difference in expected surplus for $v \in [r_{PRE}^H, w)$ and $v \in [w, 1]$ can be written as:

$$\begin{aligned} D_l = (\Pi_{PRE}^L - \Pi_{PRE}^H)(r_{PRE}^H \leq v < w) &= \int_{r_{PRE}^L}^v F_1^{(K-1)}(y) dy - (v - r_{PRE}^H) F_1^{(K-1)}(r_{PRE}^H) - \\ &- (K-1)(v - r_{PRE}^H)(1 - F(r_{PRE}^H)) F_1^{(K-2)}(r_{PRE}^H) + (K-1)(1 - F(w)) \int_{r_{PRE}^H}^v F_1^{(K-2)}(y) dy \end{aligned} \quad (6.19)$$

$$\begin{aligned} D_h = (\Pi_{PRE}^H - \Pi_{PRE}^L)(v \geq w) &= - \int_{r_{PRE}^L}^w F_1^{(K-1)}(y) dy + \int_w^v F_1^{(K-1)}(y) dy + \\ &+ (v - r_{PRE}^H) F_1^{(K-1)}(r_{PRE}^H) - 2(v - w) F_1^{(K-1)}(w) + \\ &+ (K-1)(v - r_{PRE}^H)(F(w) - F(r_{PRE}^H)) F_1^{(K-2)}(r_{PRE}^H) - \\ &- (K-1)(1 - F(w))[(v - w) F_1^{(K-2)}(w) - (v - r_{PRE}^H) F_1^{(K-2)}(r_{PRE}^H) + \int_{r_{PRE}^H}^w F_1^{(K-2)}(y) dy] \end{aligned} \quad (6.20)$$

Let us take the first- and second-order derivatives of equations (6.19) and (6.20) (the second-order derivatives can only be defined for $[r_2, w) \cup (w, 1]$):

$$\begin{aligned} \frac{\partial D_l}{\partial v} &= F_1^{(K-1)}(v) - F_1^{(K-1)}(r_{PRE}^H) - (K-1)(1 - F(r_{PRE}^H)) F_1^{(K-2)}(r_{PRE}^H) + \\ &+ (K-1)(1 - F(w)) F_1^{(K-2)}(v) \end{aligned} \quad (6.21)$$

$$\frac{\partial^2 D_l}{\partial v^2} = f_1^{(K-1)}(v) + (K-1)(1-F(w))f_1^{(K-2)}(v) \geq 0 \quad (6.22)$$

$$\begin{aligned} \frac{\partial D_h}{\partial v} &= F_1^{(K-1)}(v) + F_1^{(K-1)}(r_{PRE}^H) - 2F_1^{(K-1)}(w) + (K-1)(F(w) - F(r_{PRE}^H))F_1^{(K-2)}(r_{PRE}^H) - \\ &- (K-1)(1-F(w))[F_1^{(K-2)}(w) - F_1^{(K-2)}(r_{PRE}^H)] \end{aligned} \quad (6.23)$$

$$\frac{\partial^2 D_h}{\partial v^2} = f_1^{(K-1)}(v) \geq 0 \quad (6.24)$$

Given that both D_l, D_h are convex, for the existence of pure-strategy BNE, it will be that $\frac{\partial D_l}{\partial v} \leq 0, \forall v \in [r_{PRE}^H, w]$ and $\frac{\partial D_h}{\partial v} \geq 0, \forall v \in [w, 1]$. Moreover, given that $\frac{\partial D_l}{\partial v}|_{v=w} = \frac{\partial D_h}{\partial v}|_{v=w}$, they should both be equal to zero. Since the surplus difference at w should be zero, w must solve the following system of equations;

$$\begin{aligned} \frac{\partial D_h}{\partial v}|_{v=w} = 0 &\implies F_1^{(K-1)}(w) - F_1^{(K-1)}(r_{PRE}^H) - (K-1)(1-F(r_{PRE}^H))F_1^{(K-2)}(r_{PRE}^H) + \\ &+ (K-1)(1-F(w))F_1^{(K-2)}(w) = 0 \end{aligned} \quad (6.25)$$

$$\begin{aligned} (\Pi_{PRE}^L - \Pi_{PRE}^H)(w) = 0 &\implies \int_{r_{PRE}^L}^w F_1^{(K-1)}(y)dy - (w - r_{PRE}^H)F_1^{(K-1)}(r_{PRE}^H) - \\ &- (K-1)(w - r_{PRE}^H)(1 - F(r_{PRE}^H))F_1^{(K-2)}(r_{PRE}^H) + \\ &+ (K-1)(1 - F(w)) \int_{r_{PRE}^H}^w F_1^{(K-2)}(y)dy = 0 \end{aligned} \quad (6.26)$$

Equation (6.25) gives the following condition (w is a global minimum):

$$\begin{aligned} F_1^{(K-1)}(w) + (K-1)(1-F(w))F_1^{(K-2)}(w) &= F_1^{(K-1)}(r_{PRE}^H) + \\ &+ (K-1)(1-F(r_{PRE}^H))F_1^{(K-2)}(r_{PRE}^H) \end{aligned} \quad (6.27)$$

However, the function $x^{K-1} + (K-1)(1-x)x^{K-2}$ is strictly increasing for $0 < x < 1$, so the only valid case is when $F(w) = F(r_{PRE}^H)$, which means that $w = r_{PRE}^H$ where one can easily show that the difference is strictly positive for $r_{PRE}^L < r_{PRE}^H$. Hence, this cannot constitute an equilibrium.

- *Multiple cut-off points ($m \geq 2$).*

We now continue with the case of $m \geq 2$ cut-off points. We can write again the difference in expected surplus $\Pi_{PRE}^L - \Pi_{PRE}^H$ for valuations $v \in [r_{PRE}^H, w_1], v \in [w_\lambda, w_{\lambda+1}]$, for $\lambda = 1, \dots, m-1$, and $v \in [w_k, 1]$, along with their first- and second-order derivatives³ (see Section B.1.1 in Appendix B for the derivation). Then, the second-order derivative

³For all double integrals, the outer part refers to y_2 and the inner part to y_1 .

of the expected surplus difference in every interval is:

$$\begin{aligned} \frac{\partial^2(\Pi_{PRE}^L - \Pi_{PRE}^H)(r_{PRE}^H \leq v < w)}{\partial v^2} &= f_1^{(K-1)}(v) + \\ &+ (K-1)\left[\sum_{i=1}^{m-1}\left\{(1-\theta_{i+1})(F(w_{i+1}) - F(w_i))\right\} + (1-\theta_{m+1})(1-F(w_k))\right]f_1^{(K-2)}(v) \geq 0 \end{aligned} \quad (6.28)$$

$$\begin{aligned} \frac{\partial^2(\Pi_{PRE}^L - \Pi_{PRE}^H)(w_\lambda \leq v < w_{\lambda+1})}{\partial v^2} &= (2\theta_{\lambda+1} - 1)f_1^{(K-1)}(v) + \\ &+ (K-1)\left[\sum_{i=\lambda+1}^{m-1}\left\{(\theta_{\lambda+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i))\right\} + \right. \\ &\left. + (\theta_{\lambda+1} - \theta_{m+1})(1-F(w_k))\right]f_1^{(K-2)}(v) \end{aligned} \quad (6.29)$$

$$\frac{\partial^2(\Pi_{PRE}^L - \Pi_{PRE}^H)(w_k \leq v \leq 1)}{\partial v^2} = (2\theta_{m+1} - 1)f_1^{(K-1)}(v) \quad (6.30)$$

From equations (6.28), (6.29), (6.30), we can see that, when $\theta_{\lambda+1} = 1$, the difference in expected surplus is a convex function, whereas when $\theta_{\lambda+1} = 0$ a concave function of v . For this to be an equilibrium, we should have $\Pi_{PRE}^L - \Pi_{PRE}^H > 0$ in the decreasing convex interval $[r_{PRE}^H, w_1]$ followed by the concave interval $[w_1, w_2]$ where $\Pi_{PRE}^L - \Pi_{PRE}^H < 0$, then by the convex interval $[w_2, w_3]$ where $\Pi_{PRE}^L - \Pi_{PRE}^H > 0$ and so on, as shown in a sketch of the profit difference of Figure 6.2. For this to happen (i.e. concave negative values followed by convex positive ones), given that we start from a positive convex interval, the function should have discontinuities in the local optima in every intermediate interval $[w_\lambda, w_{\lambda+1}]$, $\lambda = 1, \dots, m-1$. However, one can easily see that the function is differentiable everywhere inside each interval, contradicting the initial statement. This means that there cannot be a pure-strategy BNE with $m \geq 2$ cut-off points, which ends our proof. \square

From theorems 6.3 and 6.4, we derive the following corollary.

Corollary 6.5. *The equilibrium of Theorem 6.2 is the unique pure-strategy Bayes-Nash equilibrium of the buyer PRE - PRE duopoly intermediary selection problem subject to the conditions of equations (6.9) and (6.10) that the intermediaries' reserve prices must satisfy.*

Proof. The nonexistence results of theorems 6.3 - 6.4 show the uniqueness of the pure-strategy BNE attained by Theorem 6.2, where, additionally equations (6.9) and (6.10) provide the necessary conditions for its existence. \square

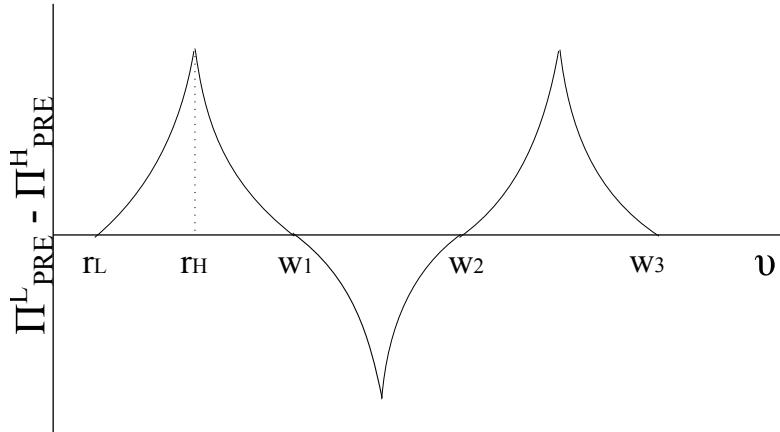


FIGURE 6.2: Figure showing the form of the difference of expected surplus, $\Pi_{PRE}^L - \Pi_{PRE}^H$, under pure-strategy BNE involving multiple (here $m = 3$) cut-off points. As shown, in this sketch of the surplus difference function, there should be discontinuities at the local optima of this function.

This fact has serious implications for the buyers, who, in contrast to the complete-information scenario (Stavrogiannis et al., 2013a), have an incentive to select the low-reserve intermediary, given that the difference between the reserve prices is large enough. However, as Figure 6.1 shows, the pure-strategy BNE of Theorem 6.2 are not the only BNE of the intermediary selection problem. Since no pure-strategy equilibrium exists when the condition of Theorem 6.2 does not hold, buyers should follow mixed strategies in equilibrium. Given this observation, in the next section we will identify the mixed-strategy BNE of the problem in question.

6.1.1.2 Mixed-Strategy Bayes-Nash Equilibria

As has been discussed in the previous section, when the reserve prices do not satisfy the conditions of Theorem 6.2, buyers will follow a mixed strategy $\theta_m(v) \in (0, 1)$ at an appropriate interval in equilibrium. Buyers whose valuations are slightly higher than r_{PRE}^H will always select the low-reserve intermediary where they expect strictly positive surplus, in contrast to the high-reserve intermediary auction where their expected surplus is arbitrarily close to zero. This means that there will be at least a single cut-off point, $w \in (r_{PRE}^H, 1]$, so that buyers with valuations in $[r_{PRE}^H, w)$ always select s_{PRE}^L . We will now show that their strategy will include a second, higher cut-off point $a \in (w, 1]$ so that buyers with valuations $v \in [w, a]$ randomize with a certain probability $\theta_m(v)$ between the intermediaries, whereas buyers with valuations $v \in (a, 1]$ follow pure strategies $\theta^*(v) \in \{0, 1\}$. The following lemma provides the conditions that the mixed strategy, $\theta_m(\cdot)$, should satisfy.

Lemma 6.6. *Let $\theta : V \mapsto [0, 1]$ be a mixed-strategy Bayes-Nash equilibrium profile involving an interval $[w, a] \subseteq (r_{PRE}^H, 1]$, where $\theta(v_i) = \theta_m(v_i) \in (0, 1)$ for each buyer i ,*

$i = 1, \dots, K$. Then $\theta_m(\cdot)$ satisfies the condition:

$$[2F(v_i) + (K-2)(1-F(v_i))] \theta_m(v_i) = (K-2) \left[\int_{v_i}^a \theta_m(y) f(y) dy + \int_a^1 \theta^*(y) dy \right] + F(v_i) \quad (6.31)$$

Proof. Suppose that buyers follow a pure strategy $\theta(v) = \theta_p(v)$ for all $v \in [r_{PRE}^H, w]$, then follow a mixed strategy $\theta(v) = \theta_m(v) \in (0, 1)$ for all $v \in [w, a]$ and then follow again a pure strategy $\theta(v) = \theta^*(v)$ for all $v \in (a, 1]$, i.e. the selection strategy involves an interval $[w, a]$ where buyers randomize between the two intermediary auctions. Then, for the existence of a mixed-strategy BNE, $\Pi_{PRE}^L(v) - \Pi_{PRE}^H(v)$ as well as all of its higher-order derivatives should be zero⁴ for all $v \in [w, a]$. Under this assumption, the second-order derivative of the surplus difference for a buyer with valuation v in $[w, a]$ can be written as (see Section B.1.2 in Appendix B):

$$\frac{\partial^2(\Pi_{PRE}^L(v) - \Pi_{PRE}^H(v))}{\partial v^2} = (K-1)F^{K-3}(v)f(v) \left\{ [2F(v) + (K-2)(1-F(v))] \theta_m(v) - \right. \\ \left. - F(v) - (K-2) \left[\int_v^a \theta_m(y) f(y) dy + \int_a^1 \theta^*(y) dy \right] \right\} \quad (6.32)$$

where setting $\frac{\partial^2(\Pi_{PRE}^L(v) - \Pi_{PRE}^H(v))}{\partial v^2} = 0$ gives the condition of (6.31). We should note that the form of (6.32) is independent of our assumptions on the form of the pure strategies in $[r_{PRE}^H, w]$ and $(a, 1]$, i.e. as long as there is an interval where buyers will randomize, (6.32) will always hold in this interval. \square

Equation (6.31) is a Volterra integral equation of the second kind (Corduneanu, 1991). However, solving it requires, in general, knowledge of the distribution function. Hence, the form of the selection function will depend on our assumption about the valuations, parametrized by the values of a (and K). Nevertheless, when $a = 1$, (6.31) has a solution $\theta(v) = \frac{1}{2}$ for all $v \geq w$. This strategy is identical to the one proposed by Burguet and Sákovics for two independent auctions (Burguet and Sákovics, 1999) (see Section 2.5). Substituting the proposed $\theta(\cdot)$, $\Pi_{PRE}^L(v) - \Pi_{PRE}^H(v)$, when $v \geq w$, will have the following form:

$$\begin{aligned} \Pi_{PRE}^L(v) - \Pi_{PRE}^H(v) &= \int_{r_{PRE}^L}^w F^{K-1}(y) dy + \frac{K-1}{2}(1-F(w)) \int_{r_{PRE}^H}^w F^{K-2}(y) dy + \\ &+ r_{PRE}^H [(K-1)F^{K-2}(r_{PRE}^H) - (K-2)F^{K-1}(r_{PRE}^H)] + w \left[\frac{K-3}{2} F^{K-1}(w) - \right. \\ &\left. - \frac{K-1}{2} F^{K-2}(w) \right] + v [(K-2)F^{K-1}(r_{PRE}^H) - (K-1)F^{K-2}(r_{PRE}^H) - \\ &- \frac{K-3}{2} F^{K-1}(w) + \frac{K-1}{2} F^{K-2}(w)] \end{aligned} \quad (6.33)$$

⁴Setting all derivatives of the surplus difference equal to zero yields necessary but not sufficient conditions for the existence of the equilibrium.

For this to be a BNE for all $v \geq w$, w must make both the first-order and zero-order coefficients of this polynomial zero. However, this cannot be true but for a single pair of reserve prices at most: in the zero-order coefficient, w is uniquely defined by both r_{PRE}^L, r_{PRE}^H , whereas in the first-order it only depends on r_{PRE}^H . So, given that the system of equations is under-defined, w cannot be the solution of both equations for all valid pairs of r_{PRE}^L, r_{PRE}^H . This means that there should be at least one more cut-off point, $a \leq 1$.

We have shown that there should be at least two cut-off points, w, a , so that buyers with valuations in $[r_{PRE}^H, w)$ and $(a, 1]$ follow pure strategies. We continue by showing that there can only be a single cut-off point, $w \in (r_{PRE}^H, 1]$, before and at most a single cut-off point, $a \in (w, 1]$, after randomizing, where w, a are such that $\Pi_{PRE}^L(w) = \Pi_{PRE}^H(w)$, $\Pi_{PRE}^L(a) = \Pi_{PRE}^H(a)$, and $\theta(r_{PRE}^H \leq v < w) = 1$, $\theta(w \leq v \leq a) = \theta_m(v) \in (0, 1)$, $\theta(a < v \leq 1) \in \{0, 1\}$.

Lemma 6.7. *The PRE - PRE duopoly intermediary selection strategy of a buyer in a mixed-strategy Bayes-Nash equilibrium involves at most three intervals in the support of the buyers' private valuations defined by two cut-off points, $w \in (r_{PRE}^H, 1]$, $a \in (w, 1]$: buyers with private valuations in $[r_{PRE}^H, w)$ and $(a, 1]$ follow strictly pure Bayes-Nash equilibrium strategies whereas buyers with private valuations in $[w, a]$ follow strictly mixed Bayes-Nash equilibrium strategies.*

Proof. As has been mentioned at the beginning of this section, there will be at least a single cut-off point, $w \in (r_{PRE}^H, 1]$, so that buyers with valuations in $[r_{PRE}^H, w)$ will always select the low-reserve intermediary, expecting surplus arbitrarily close to zero from the high-reserve intermediary. This means that always $\theta_1 = 1$. Using a similar reasoning as in Theorem 6.4, we can first show that there can only be one cut-off point, w , before randomizing by taking the second-order derivative of the expected surplus difference, $\Pi_{PRE}^L(v) - \Pi_{PRE}^H(v)$, from the two intermediaries at any interval $[w_\lambda, w_{\lambda+1})$, $\lambda = 0, \dots, \sigma' - 1$, where, for notational convenience, we denote $r_{PRE}^H = w_0$, and showing that the selection function $\theta_{\lambda+1}$ controls the convexity of this difference. More specifically, if we do not assume anything about the pure strategy after randomizing, i.e. $\theta(v) = \theta^*(v)$ for all $v \in (a, 1]$, the second-order derivative of this difference is (see Section B.1.3 in Appendix B for the derivation):

$$\begin{aligned} \frac{\partial^2(\Pi_{PRE}^L - \Pi_{PRE}^H)(w_\lambda \leq v < w_{\lambda+1})}{\partial v^2} &= (K-1)F^{K-3}(v)f(v)\{(2\theta_{\lambda+1} - 1)F(v) + \\ &+ (K-2)\left[\sum_{i=\lambda+1}^{\sigma'-1} \left\{(\theta_{\lambda+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i))\right\}\right] + \theta_{\lambda+1}(1 - F(w_{\sigma'})) - \\ &- \int_{w_{\sigma'}}^a \theta_m(y)f(y)dy - \int_a^1 \theta^*(y)f(y)dy\} \end{aligned} \quad (6.34)$$

This means that when $\theta_{\lambda+1} = 1$, the corresponding surplus difference is convex, whereas when $\theta_{\lambda+1} = 0$, it is concave. This means that for the existence of a mixed-strategy BNE with multiple ($\sigma' \geq 2$) cut-off points before randomizing, the surplus difference for $v \in [r_{PRE}^H, w_\sigma']$ will consist of non-negative convex intervals followed by non-positive concave intervals, which cannot happen unless there are discontinuities at the local optima (Figure 6.2), a fact which is not supported by the well-defined first-order derivative, so there can only be a single cut-off point w before randomizing.

Similarly, if there are $m' \geq 2$ cut-off points, a_i , after randomizing so that $\theta(v) = \theta_\lambda^* \in \{0, 1\}$ for valuations $v \in [a_\lambda, a_{\lambda+1})$, $\lambda = 1, \dots, m'$ (with $a_{m+1} = 1$), and $\theta_i^* \neq \theta_j^*$ for $|i - j| = 1, i, j = 1, \dots, m'$, then the second-order derivative of the difference in expected surplus, $\frac{\partial^2(\Pi_{PRE}^L - \Pi_{PRE}^H)}{\partial v^2}$, for $v \in [a_\lambda, a_{\lambda+1})$ will be (see Section B.1.4 in Appendix B for the derivation):

$$\begin{aligned} \frac{\partial^2(\Pi_{PRE}^L - \Pi_{PRE}^H)(a_\lambda \leq v < a_{\lambda+1})}{\partial v^2} &= (K-1)F^{K-3}(v)f(v)[(2\theta_\lambda^* - 1)F(v) + \\ &(K-2)\sum_{i=\lambda+1}^{m'}(\theta_\lambda^* - \theta_i^*)(F(a_{i+1}) - F(a_i))] \end{aligned} \quad (6.35)$$

Again, the selection strategy controls the convexity of the surplus difference. More specifically, if $\theta_\lambda^* = 1$ ($\theta_\lambda^* = 0$ respectively), the function in the corresponding interval is convex (concave respectively) and we would then have a series of non-negative convex surplus difference intervals followed by non-positive concave alternating intervals if $\theta_1^* = 1$, or the opposite when $\theta_1^* = 0$. This means that there should be discontinuities at the local optima of the corresponding intervals, which is in contrast with the well defined first-order derivative of $\Pi_{PRE}^L - \Pi_{PRE}^H$, and hence there can be at most a single cut-off point a in $(w, 1]$. \square

Lemma 6.7 thus implies that the mixed-strategy equilibrium selection of the buyers will involve three intervals defined by two cut-off points: buyers with valuations in the first interval always select the low-reserve intermediary, buyers with valuations in the middle interval will randomize between the intermediary with a probability that is given by the solution to (6.31), and buyers whose valuations lie in the third interval will also follow a pure strategy. We formalize this finding in the following theorem where we also give the conditions that w and a should satisfy.

Theorem 6.8. *Let $\theta : V \mapsto [0, 1]$ be a strategy profile where $\theta(v_i) = 1$ if $r_{PRE}^H \leq v_i < w$, $\theta(v_i) = \theta_m(v_i)$ if $w \leq v_i \leq a$, and $\theta(v_i) = \theta^* \in \{0, 1\}$ if $a < v_i \leq 1$, for each buyer i , $i = 1, \dots, K$, where $\theta_m(\cdot)$ satisfies the condition:*

$$[2F(v_i) + (K-2)(1-F(v_i))]\theta_m(v_i) = (K-2)\left[\int_{v_i}^a \theta_m(y)f(y)dy + \theta^*(1-F(a))\right] + F(v_i) \quad (6.36)$$

and w, a are given by:

$$\begin{aligned} F^{K-2}(w) \int_{r_{PRE}^L}^w F^{K-1}(y) dy - F^{K-1}(w) \int_{r_{PRE}^H}^w F^{K-2}(y) dy &= [(w - r_{PRE}^H)F^{K-2}(w) - \\ &- \int_{r_{PRE}^H}^w F^{K-2}(y) dy]F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] \end{aligned} \quad (6.37)$$

$$\begin{aligned} F^{K-2}(w) \left\{ F(w) + (K-1)[1 - F(w) - \int_w^a \theta_m(y) f(y) dy - \right. \\ \left. - \theta^*(1 - F(a))] \right\} &= F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] \end{aligned} \quad (6.38)$$

Then, $\theta(\cdot)$ is a unique mixed-strategy Bayes-Nash equilibrium profile of the buyer PRE - PRE duopoly intermediary selection problem.

Proof. Equation (6.36) can be directly derived from Lemmas 6.6 and 6.7, where we have used the fact that there can only be a single w and a . Given that this makes the surplus difference a linear function of the valuation, for the existence of a mixed-strategy BNE, both the surplus difference and its first-order derivative should be zero for all $v \in [w, a]$. We will continue by writing the difference in expected surplus, $\Pi_{PRE}^L(v) - \Pi_{PRE}^H(v)$, as well as its first-order condition ($\frac{\partial(\Pi_{PRE}^L - \Pi_{PRE}^H)(v)}{\partial v} = 0$), for a buyer with valuation $v \in [w, a]$ (for the derivation see Section B.1.5 in Appendix B), and then use the fact that $\Pi_{PRE}^L(w) - \Pi_{PRE}^H(w) = 0$ and $\frac{\partial(\Pi_{PRE}^L - \Pi_{PRE}^H)(v)}{\partial v}|_{v=w} = 0$ to get the conditions for w and a :

$$\begin{aligned} (\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a) &= \int_{r_{PRE}^L}^w F^{K-1}(y) dy - \int_w^v F^{K-1}(y) dy - \\ &- (v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + \\ &+ (K-1)(1 - F(r_{PRE}^H))] + (v - w)F^{K-2}(w)[2F(w) + (K-1)(1 - F(w) - \int_w^a \theta_m(y) f(y) dy)] + \\ &+ (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y) dy[1 - F(w) - \int_w^a \theta_m(y) f(y) dy - \theta^*(1 - F(a))] - \\ &- (K-1)\theta^*(1 - F(a)) \int_w^v F^{K-2}(y) dy + (K-1)(1 - F(a)) \int_w^v (v - y)\theta_m(y) f_1^{(K-2)}(y) dy + \\ &+ 2 \int_w^v (v - y)\theta_m(y) f_1^{(K-1)}(y) dy + \int_w^v \int_{y_2}^a (v - y_2)(\theta_m(y_2) - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \end{aligned} \quad (6.39)$$

$$\begin{aligned}
\frac{\partial(\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a)}{\partial v} = 0 &\implies F^{K-2}(v)[F(v) + \theta^*(K-1)(1-F(a))] = \\
&= F^{K-2}(w)[2F(w) + (K-1)(1-F(w) - \int_w^a \theta_m(y)f(y)dy)] - \\
&- F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1-F(r_{PRE}^H))] + \\
&+ 2 \int_w^v \theta_m(y)f_1^{(K-1)}(y)dy + (K-1)(1-F(a)) \int_w^v \theta_m(y)f_1^{(K-2)}(y)dy + \\
&+ \int_w^v \int_{y_2}^a (\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2
\end{aligned} \tag{6.40}$$

This should also be true at w , where the last equation above yields:

$$\begin{aligned}
F^{K-2}(w)[F(w) + \theta^*(K-1)(1-F(a))] &= F^{K-2}(w)[2F(w) + (K-1)(1-F(w) - \\
&- \int_w^a \theta_m(y)f(y)dy)] - F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1-F(r_{PRE}^H))]
\end{aligned} \tag{6.41}$$

Given this last condition, (6.39) becomes:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a) &= \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy[1-F(w) - \\
&- \int_w^a \theta_m(y)f(y)dy - \theta^*(1-F(a))] + r_{PRE}^H F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1-F(r_{PRE}^H))] - \\
&- w F^{K-2}(w)[2F(w) + (K-1)(1-F(w) - \int_w^a \theta_m(y)f(y)dy)] - \int_w^v F^{K-1}(y)dy - \\
&- (K-1)\theta^*(1-F(a)) \int_w^v F^{K-2}(y)dy - 2 \int_w^v y \theta_m(y)f_1^{(K-1)}(y)dy - \\
&- (K-1)(1-F(a)) \int_w^v y \theta_m(y)f_1^{(K-2)}(y)dy - \\
&- \int_w^v \int_{y_2}^a y_2(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
&+ v F^{K-2}(v)[F(v) + (K-1)\theta^*(1-F(a))]
\end{aligned} \tag{6.42}$$

Substituting for $v = w$ yields:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(w) = 0 &\implies \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy[1-F(w) - \\
&- \int_w^a \theta_m(y)f(y)dy - \theta^*(1-F(a))] = \\
&= (w - r_{PRE}^H) F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1-F(r_{PRE}^H))]
\end{aligned} \tag{6.43}$$

The system of equations (6.41) and (6.43) provides the conditions that w and a should jointly satisfy. However, we can eliminate a from these equations and get a solution for

w that is independent of the former:

$$\begin{aligned} F^{K-2}(w) \int_{r_{PRE}^L}^w F^{K-1}(y) dy - F^{K-1}(w) \int_{r_{PRE}^H}^w F^{K-2}(y) dy &= [(w - r_{PRE}^H)F^{K-2}(w) - \\ &- \int_{r_{PRE}^H}^w F^{K-2}(y) dy]F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] \end{aligned} \quad (6.44)$$

and then find a by substituting the w found in any of the equations (6.41) or (6.43).

We have reasoned about the existence of these cut-off points (due to Theorem 6.2 and equation (6.33) respectively). To show that the w and a found are unique, we rearrange (6.37) and (6.38):

$$\begin{aligned} H_w &= F^{K-2}(w) \int_{r_{PRE}^L}^w F^{K-1}(y) dy - F^{K-1}(w) \int_{r_{PRE}^H}^w F^{K-2}(y) dy - [(w - r_{PRE}^H)F^{K-2}(w) - \\ &- \int_{r_{PRE}^H}^w F^{K-2}(y) dy]F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] \end{aligned} \quad (6.45)$$

$$\begin{aligned} H_a &= F^{K-2}(w)[F(w) + (K-1)(1 - F(w))] - F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + \\ &+ (K-1)(1 - F(r_{PRE}^H))] - (K-1)F^{K-2}(w)[\int_w^a \theta_m(y)f(y)dy + \theta^*(1 - F(a))] \end{aligned} \quad (6.46)$$

Taking the first-order derivative of (6.45) with respect to w yields:

$$\begin{aligned} \frac{\partial H_w}{\partial w} &= (K-2)F^{K-3}(w)f(w)\left\{ \int_{r_{PRE}^L}^w F^{K-1}(y) dy - (w - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + \right. \\ &\left. + (K-1)(1 - F(r_{PRE}^H))] \right\} - (K-1)F^{K-2}(w)f(w)\int_{r_{PRE}^H}^w F^{K-2}(y) dy \end{aligned} \quad (6.47)$$

which is strictly negative, given that $\int_{r_{PRE}^L}^w F^{K-1}(y) dy - (w - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] < 0$ as can be directly derived from (6.43), which means that the solution for w is unique.

Similarly, taking the first-order derivative of (6.46) with respect to a yields:

$$\frac{\partial H_a}{\partial a} = -(K-1)F^{K-2}(w)f(a)[\theta_m(a) - \theta^*] \quad (6.48)$$

which is either strictly positive when $\theta^* = 1$ or strictly negative when $\theta^* = 0$. Finally, uniqueness of solution to (6.36) is guaranteed because this integral equation can be transformed to a first-order linear differential equation with continuous coefficients (existence and uniqueness theorem⁵). \square

⁵According to this theorem, “if the functions h and g are continuous on an open interval, I , containing the initial value point $x = x_0$, then there exists a unique function $y = \phi(x)$ that satisfies the differential equation $y' + h(x)y = g(x)$ for each $x \in I$, and that also satisfies the initial condition $y(x_0) = y_0$, where y_0 is an arbitrary prescribed initial value” (Theorem 2.4.1 in (Boyce and DiPrima, 2009)).

6.1.1.3 Numerical Examples

To gain intuition, two examples of the equilibrium selection strategy of the buyers when $\theta^* = 0$ and $\theta^* = 1$ for the uniform distribution $U(0, 1)$ and $K = 5$ buyers are given in Figures 6.3 and 6.4 respectively. In this case, the selection strategy $\theta_m(\cdot)$ will have the form (see Section B.1.6 in Appendix B for the derivation):

$$\theta_m(v) = \begin{cases} \frac{1}{2} - \frac{(1-2\theta^*)(1-a)}{2 \exp(-a)} \exp(-v) & \text{if } K=4 \\ \frac{1}{2} + (K-2) \frac{(1-2\theta^*)(1-a)}{2[(K-4)a-(K-2)]^{\frac{K-2}{K-4}}} [(K-4)v - (K-2)]^{\frac{2}{K-4}} & \text{otherwise} \end{cases} \quad (6.49)$$

and w, a are given by the following equations respectively:

$$[w^{K-1} - r_{PRE}^H]^{K-1} - (K-1)(w - r_{PRE}^H)w^{K-2}]r_{PRE}^H]^{K-2}[r_{PRE}^H + (K-1)(1 - r_{PRE}^H)] + \\ + \frac{K-1}{K}w^{K-2}(w^K - r_{PRE}^L)^K - (w^{K-1} - r_{PRE}^H)^{K-1})w^{K-1} = 0 \quad (6.50)$$

$$\left\{ \begin{array}{l} w^2 \{ w + \frac{3}{2} [1 - w + (1 - 2\theta^*)(1 - a) \exp(a - w)] \} = r_{PRE}^H {}^2 [r_{PRE}^H + 3(1 - r_{PRE}^H)] \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{if } K=4 \\ w^{K-2} \{ w + \frac{K-1}{2} [1 - w + (1 - 2\theta^*)(1 - a) (\frac{(K-4)w-(K-2)}{(K-4)a-(K-2)})^{\frac{K-2}{K-4}}] \} = \\ = r_{PRE}^H {}^{K-2} [r_{PRE}^H + (K - 1)(1 - r_{PRE}^H)] \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{otherwise} \end{array} \right. \quad (6.51)$$

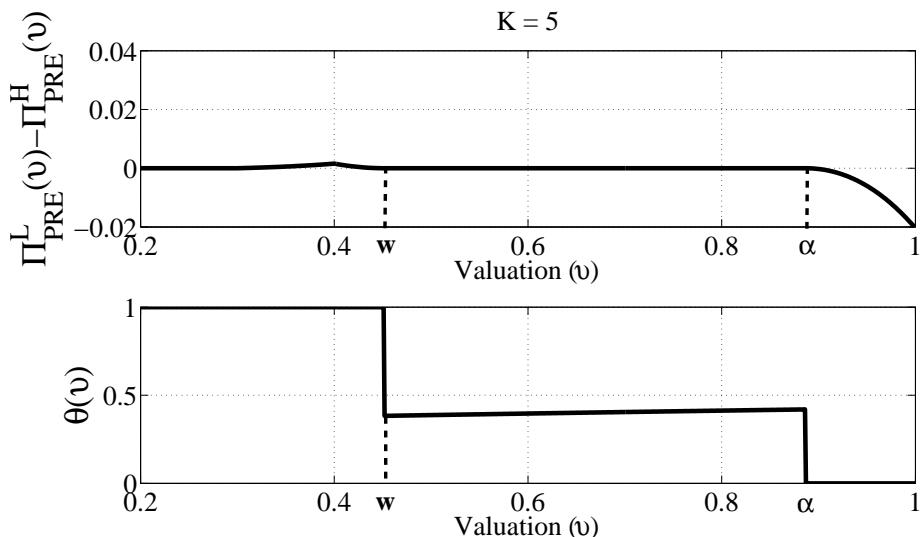


FIGURE 6.3: Figure showing the equilibrium strategy, θ , (bottom) and the corresponding surplus difference, $\Pi_{PRE}^L - \Pi_{PRE}^H$, (top) for the buyers' PRE - PRE intermediary selection problem when there are $K = 5$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$ and reserve prices are $r_{PRE}^L = 0.2$, $r_{PRE}^H = 0.4$. In this case, buyers with high valuations select the high-reserve intermediary ($\theta^* = 0$).

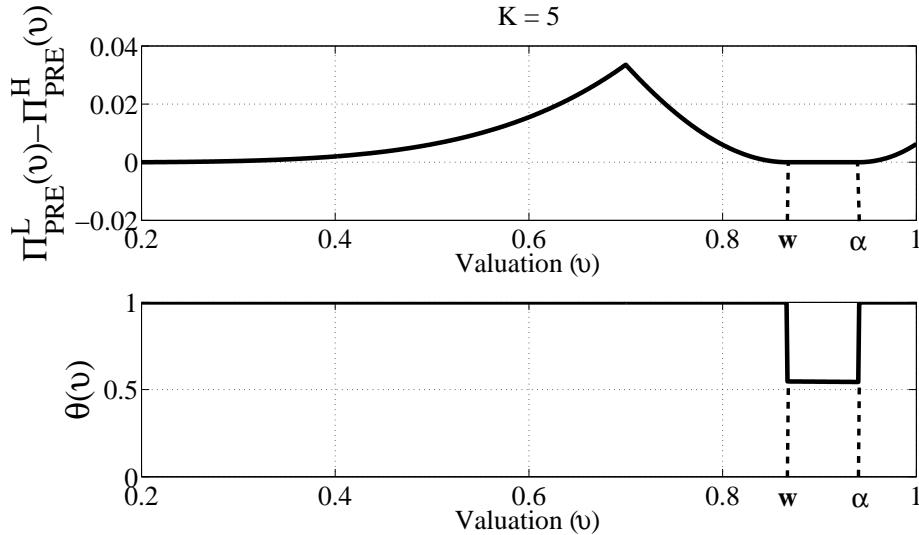


FIGURE 6.4: Figure showing the equilibrium strategy, θ , (bottom) and the corresponding surplus difference, $\Pi^L_{PRE} - \Pi^H_{PRE}$, (top) for the buyers' PRE - PRE intermediary selection problem when there are $K = 5$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$ and reserve prices are $r^L_{PRE} = 0.2$, $r^H_{PRE} = 0.7$. In this case, buyers with high valuations select the low-reserve intermediary ($\theta^* = 1$).

Figures 6.5 and 6.6 illustrate two examples for the reserve price combinations for which BNE exist when valuations are i.i.d. drawn from a uniform distribution $F = U(0, 1)$ when $K = 5$ or $K = 10$ buyers are present respectively. As in Figure 6.1, there is a region of pure-strategy BNEs where all buyers select s^L_{PRE} , followed by a region of mixed-strategy BNEs where buyers with valuations $v \in (a, 1]$ always select s^L_{PRE} but buyers with valuations in $[w, a]$ follow mixed strategies. Finally, there is a region where buyers with high valuations $v \in [w, a]$ randomize but buyers with higher valuations ($v > a$) in contrast always select the high-reserve intermediary. This means that intermediaries have incentives to increase their reserve prices up to a point, in contrast with the classical setting without intermediaries. Moreover, the regions where buyers select the low-reserve intermediaries shrinks as the number of participating buyers increases.

6.1.2 Post-Award Second-Price Sealed-Bid Intermediary Auctions

Having studied the duopoly competition between two PRE intermediaries, we now study the competition between two POST intermediary auctioneers. More specifically, we consider a population of K buyers that can participate at the center's auction only via two intermediaries, s^L_{POST}, s^H_{POST} , that implement POST auctions with reserve prices $r^L_{POST} \leq r^H_{POST}$ respectively. As we show in this section, the buyers' intermediary selection problem in this setting is much simpler than that of selecting between two PRE intermediaries: all buyers select the intermediary with the lowest reserve price, thus driving the reserve prices down to the center's reserve price, ρ .

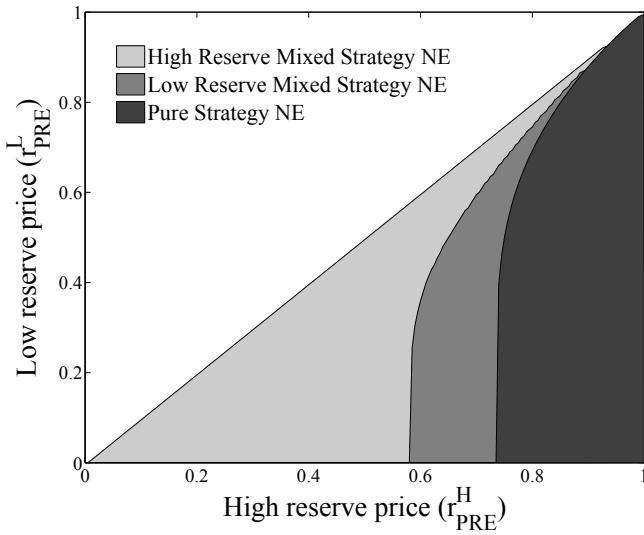


FIGURE 6.5: Figure showing BNE for the buyers' PRE - PRE intermediary selection problem when there are $K = 5$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$. There are three distinct regions for the reserve prices: (i) pure-strategy BNE: buyers always select the low-reserve intermediary (right), (ii) mixed-strategy BNE: buyers with valuations in $[r_{PRE}^H, w)$ select the low-reserve intermediary, buyers with valuations in $[w, a]$ randomize between the intermediaries, and buyers with valuations in $(a, 1]$ either select the low-reserve intermediary (center) or select the high-reserve intermediary (left).

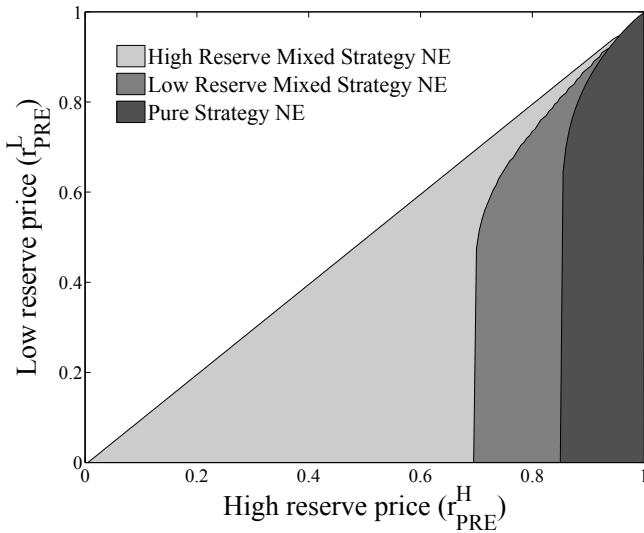


FIGURE 6.6: Figure showing BNE for the buyers' PRE - PRE intermediary selection problem when there are $K = 10$ buyers whose valuations are i.i.d. drawn from $U(0, 1)$. There are three distinct regions for the reserve prices: (i) pure-strategy BNE: buyers always select the low-reserve intermediary (right), (ii) mixed-strategy BNE: buyers with valuations in $[r_{PRE}^H, w)$ select the low-reserve intermediary, buyers with valuations in $[w, a]$ randomize between the intermediaries, and buyers with valuations in $(a, 1]$ either select the low-reserve intermediary (center) or select the high-reserve intermediary (left).

As before, let $\theta : V \mapsto [0, 1]$ denote the selection strategy of the buyers, which is a mapping from a buyer's private valuation to the probability of selecting the *low-reserve* intermediary, s_{POST}^L , so $1 - \theta(v)$ is the probability that the buyer selects intermediary s_{POST}^H . Then, the expected ex-interim surplus for a buyer with private valuation v from selecting the low- and high-reserve intermediary, $\Pi_{POST}^L(v), \Pi_{POST}^H(v)$ respectively, when $r_{POST}^L \leq r_{POST}^H$, can be written as:

$$\Pi_{POST}^L(v) = \begin{cases} 0 & \text{if } v \in [0, r_{POST}^L) \\ \int_{r_{POST}^L}^v F_1^{(K-1)}(y) dy & \text{if } v \in [r_{POST}^L, 1] \end{cases} \quad (6.52)$$

$$\Pi_{PRE}^H(v) = \begin{cases} 0 & \text{if } v \in [0, r_{POST}^H) \\ \int_{r_{POST}^H}^v F_1^{(K-1)}(y) dy & \text{if } v \in [r_{POST}^H, 1] \end{cases} \quad (6.53)$$

More specifically, a buyer in each intermediary's auction expects to pay the intermediary's reserve price, if all buyers have valuations below the latter reserve price, or pays the highest opponent bid if this is not the case, irrespective of where this opponent bid is placed. Hence, whenever $r_{POST}^L < r_{POST}^H$, the following proposition holds.

Proposition 6.9. *Whenever $r_{POST}^L < r_{POST}^H$, it is a weakly dominant strategy for a buyer to select the low-reserve intermediary in the buyer POST - POST duopoly intermediary selection problem.*

In fact, the aforementioned result generalizes to any number of intermediaries:

Proposition 6.10. *Whenever $r_{POST}^1 < r_{POST}^2 < \dots < r_{POST}^n$, it is a weakly dominant strategy for a buyer to select the lowest-reserve intermediary in the n -POST intermediary selection problem, $n \in \mathbb{N}^+, n \geq 2$.*

Given that the expected surplus of a buyer is independent of the decisions of her opponent buyers, when $r_{POST}^L = r_{POST}^H = r_{POST}$, her ex-interim expected surplus from the two intermediaries is the same, i.e. $\Pi_{POST}^{Leg}(v) = \Pi_{POST}^{Heg}(v) = \int_r^v F_1^{(K-1)}(y) dy$. This means that our model's prediction of the buyers' selection becomes limited.

Proposition 6.11. *Whenever $r_{POST}^L = r_{POST}^H$, there are an infinite number of equilibria in weakly dominant strategies of the buyer POST - POST duopoly intermediary selection problem.*

Similar to the previous results, Proposition 6.11 generalizes to the case of n POST intermediaries.

Finally, we turn our attention to the intermediary selection problem of buyers when one intermediary implements a PRE whereas the other implements a POST intermediary mechanism.

6.1.3 Pre-Award versus Post-Award Intermediary Auctions

We now consider the duopoly competition between two intermediaries with reserve prices, one implementing a PRE and the other a POST mechanism. As before, we study the more interesting case where a population of $K > n$ buyers select one of two intermediaries s_{PRE} , s_{POST} that implement a PRE and POST mechanism respectively.

We first consider the special case where both intermediaries set the same reserve price r . Then a buyer with private valuation $v \geq r$ expects surplus from the PRE intermediary that can be expressed as:

$$\Pi_{PRE}^{eq}(v) = (v - r)F^{K-1}(r) + \int_r^v (v - y)\theta(y)f_1^{(K-1)}(y)dy \quad (6.54)$$

That is, the buyer expects positive surplus when all other buyers' valuations are below r (first term) or when the highest opponent bid is less than the buyer's bid and is submitted in the same intermediary auction (second term). On the other hand, the expected surplus from the POST intermediary is:

$$\begin{aligned} \Pi_{POST}^{eq}(v) = & (v - r)F^{K-1}(r) + (v - r) \int_0^r \int_r^1 \theta(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\ & + \int_r^v (v - y)(1 - \theta(y))f_1^{(K-1)}(y)dy + \\ & + \int_r^v \int_{y_2}^1 (v - y_2)\theta(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \end{aligned} \quad (6.55)$$

That is, the buyer expects positive surplus when all other buyers' valuations are below r (first term) or when there is only one buyer with bid above r submitted in the other intermediary auction (second term), paying the center's reserve price, r . Moreover, the buyer pays the highest opponent bid when it is below v and submitted in the same auction (third term) as well as the second-highest opponent bid, wherever this is submitted, as long as it is above r , below v , and, at the same time, the highest opponent bid is submitted in the opponent intermediary. This leads to the following theorem.

Theorem 6.12. *There exists a unique equilibrium in weakly dominant strategies in the buyer PRE - POST duopoly intermediary selection problem where all buyers select the intermediary implementing a POST auction, when the other intermediary implements a PRE auction if both intermediaries set the same reserve price.*

Proof. Taking the difference of the expected surplus from both intermediaries yields:

$$\begin{aligned} \Pi_{POST}^{eq}(v) - \Pi_{PRE}^{eq}(v) &= (K-1)(v-r)F^{K-2}(r) \int_r^1 \theta(y)f(y)dy + \\ &+ \int_r^v (v-y)(1-2\theta(y))f_1^{(K-1)}(y)dy + \\ &+ \int_r^v \int_{y_2}^1 (v-y_2)\theta(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \end{aligned} \quad (6.56)$$

The partial derivative of this difference w.r.t. v is:

$$\begin{aligned} \frac{\partial \Pi_{POST}^{eq}(v) - \Pi_{PRE}^{eq}(v)}{\partial v} &= (K-1)F^{K-2}(r) \int_r^1 \theta(y)f(y)dy + \\ &+ \int_r^v (1-2\theta(y))f_1^{(K-1)}(y)dy + \int_r^v \int_{y_2}^1 \theta(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \end{aligned} \quad (6.57)$$

However, we can write:

$$\begin{aligned} \int_r^v \int_{y_2}^1 \theta(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 &= \int_r^v \int_{y_2}^v \theta(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\ &+ \int_r^v \int_v^1 \theta(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 = \\ &= (K-1)[F^{K-2}(v) \int_v^1 \theta(y)f(y)dy - F^{K-2}(r) \int_r^1 \theta(y)f(y)dy] + \\ &+ \int_r^v \theta(y)f_1^{(K-1)}(y)dy \end{aligned} \quad (6.58)$$

Hence, equation (6.57) can be written as:

$$\begin{aligned} \frac{\partial \Pi_{POST}^{eq}(v) - \Pi_{PRE}^{eq}(v)}{\partial v} &= (K-1)F^{K-2}(v) \int_v^1 \theta(y)f(y)dy + \\ &+ \int_r^v (1-\theta(y))f_1^{(K-1)}(y)dy \geq 0 \end{aligned} \quad (6.59)$$

Hence, given that $\Pi_{POST}^{eq}(r) = \Pi_{PRE}^{eq}(r)$ and $\Pi_{POST}^{eq}(\cdot)$ grows faster than $\Pi_{PRE}^{eq}(\cdot)$ for every $v > r$, it should always be $\Pi_{POST}^{eq}(v) > \Pi_{PRE}^{eq}(v)$, so the only equilibrium strategy is $\theta(y) = 0$. \square

Having derived the equilibrium selection strategies of the buyers for equal intermediary reserve prices, we now move to the more general case when one intermediary sets a lower reserve price than the other. Let us start with the first case where the POST intermediary sets a reserve price $r_{POST}^L < r_{PRE}^H$, where r_{PRE}^H is the PRE intermediary's reserve price. In this case, the ex-interim expected surplus of a buyer with valuation v from the POST

and PRE intermediary, $\Pi_{POST}^L, \Pi_{PRE}^H$ respectively, can be expressed as:

$$\Pi_{POST}^L(v) = \begin{cases} 0 & \text{if } v \in [0, r_{POST}^L) \\ \int_{r_{POST}^L}^v F_1^{(K-1)}(y) dy & \text{if } v \in [r_{POST}^L, r_{PRE}^H) \\ (v - r_{POST}^L)F_1^{(K-1)}(r_{POST}^L) + \\ +(v - r_{PRE}^H) \int_{y_2=0}^{r_{PRE}^H} \int_{y_1=r_{PRE}^H}^1 (1 - \theta(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\ + \int_{r_{POST}^L}^{r_{PRE}^H} (v - y) f_1^{(K-1)}(y) dy + \int_{r_{PRE}^H}^v (v - y) \theta(y) f_1^{(K-1)}(y) dy + \\ + \int_{y_2=r_{PRE}^H}^v \int_{y_1=y_2}^1 (v - y_2)(1 - \theta(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 & \text{if } v \in [r_{PRE}^H, 1] \end{cases} \quad (6.60)$$

$$\Pi_{PRE}^H(v) = \begin{cases} 0 & \text{if } v \in [0, r_{PRE}^H) \\ (v - r_{PRE}^H)F_1^{(K-1)}(r_{PRE}^H) + \int_{r_{PRE}^H}^v (v - y)(1 - \theta(y)) f_1^{(K-1)}(y) dy & \text{if } v \in [r_{PRE}^H, 1] \end{cases} \quad (6.61)$$

That is, a buyer with valuation $v \geq r_{PRE}^H$ that selects the low-reserve POST mechanism pays the latter his reserve price, r_{POST}^L , when all other bids are below this reserve price (first term in (6.60)), while she pays the high reserve price, r_{PRE}^H , when the highest opponent bid is higher than r_{PRE}^H and is submitted in the PRE intermediary and, at the same time, the second-highest opponent bid, lower than r_{PRE}^H , is submitted in her selected intermediary (second term in (6.60)). She also pays the highest opponent bid if it is submitted in the POST mechanism (third and fourth terms in (6.60)) and the second-highest opponent bid if it is higher than r_{PRE}^H , lower than her bid, and, at the same time, the highest opponent bid is submitted in the PRE intermediary auction.

On the other hand, if she selects the high-reserve PRE mechanism, she pays the latter's reserve price if all opponent bids are below this reserve price (first term in (6.61)), and pays the highest opponent bid, if it is higher than r_{PRE}^H , lower than her bid, and is submitted in the same intermediary (second term in (6.61)).

Having expressed the expected ex-interim surplus of a buyer from the two intermediaries, we can now derive the resulting equilibria of the intermediary selection problem in this setting, namely that there is a unique equilibrium in weakly dominant strategies where all buyers select the low-reserve POST intermediary:

Theorem 6.13. *There exists a unique equilibrium in weakly dominant strategies in the buyer PRE - POST duopoly intermediary selection problem where all buyers select the intermediary implementing a POST auction with reserve price r_{POST}^L , when the other intermediary implements a PRE auction with reserve price r_{PRE}^H and $r_{POST}^L < r_{PRE}^H$.*

Proof. Taking the difference of the expected surplus from both intermediaries yields:

$$\begin{aligned} \Pi_{POST}^L(v) - \Pi_{PRE}^H(v) &= (K-1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H) \int_{r_{PRE}^H}^1 (1 - \theta(y))f(y)dy + \\ &+ \int_{r_{PRE}^H}^v (v - y)(2\theta(y) - 1)f_1^{(K-1)}(y)dy + \\ &+ \int_{r_{PRE}^H}^v \int_{y_2}^1 (v - y_2)(1 - \theta(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \int_{r_{POST}^L}^{r_{PRE}^H} F^{K-1}(y)dy \end{aligned} \quad (6.62)$$

The partial derivative of this difference w.r.t. v is:

$$\begin{aligned} \frac{\partial \Pi_{POST}^L(v) - \Pi_{PRE}^H(v)}{\partial v} &= (K-1)F^{K-2}(r_{PRE}^H) \int_{r_{PRE}^H}^1 (1 - \theta(y))f(y)dy + \\ &+ \int_{r_{PRE}^H}^v (2\theta(y) - 1)f_1^{(K-1)}(y)dy + \int_{r_{PRE}^H}^v \int_{y_2}^1 (1 - \theta(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \end{aligned} \quad (6.63)$$

Similar to equation (6.58), we can write:

$$\begin{aligned} &\int_{r_{PRE}^H}^v \int_{y_2}^1 (1 - \theta(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 = \\ &= (K-1)[F^{K-2}(v) \int_v^1 (1 - \theta(y))f(y)dy - F^{K-2}(r_{PRE}^H) \int_{r_{PRE}^H}^1 (1 - \theta(y))f(y)dy] + \\ &+ \int_{r_{PRE}^H}^v (1 - \theta(y))f_1^{(K-1)}(y)dy \end{aligned} \quad (6.64)$$

Hence, equation (6.63) can be written as:

$$\begin{aligned} \frac{\partial \Pi_{POST}^L(v) - \Pi_{PRE}^H(v)}{\partial v} &= (K-1)F^{K-2}(v) \int_v^1 (1 - \theta(y))f(y)dy + \\ &+ \int_{r_{PRE}^H}^v \theta(y)f_1^{(K-1)}(y)dy \geq 0 \end{aligned} \quad (6.65)$$

Hence, given that $\Pi_{POST}^L(r_{PRE}^H) > \Pi_{PRE}^H(r_{PRE}^H)$ and $\Pi_{POST}^L(\cdot)$ grows faster than $\Pi_{PRE}^H(\cdot)$ for every $v > r_{PRE}^H$, it should always be $\Pi_{POST}^L(v) > \Pi_{PRE}^H(v)$, so the only equilibrium strategy is $\theta(y) = 1$. \square

Until now, we have shown that buyers always select the POST intermediary against a PRE competing intermediary given that the former's reserve price is lower or equal to the latter's. We complete our analysis with the scenario where the PRE intermediary sets a reserve price, r_{PRE}^L , strictly lower than the POST mechanism's, r_{POST}^H . In this last case, the ex-interim expected surplus of a buyer with private valuation v from the two

mechanisms will be:

$$\Pi_{PRE}^L(v) = \begin{cases} 0 & \text{if } v \in [0, r_{PRE}^L) \\ \int_{r_{PRE}^L}^v F_1^{(K-1)}(y) dy & \text{if } v \in [r_{PRE}^L, r_{POST}^H) \\ (v - r_{PRE}^L) F_1^{(K-1)}(r_{PRE}^L) + \int_{r_{PRE}^L}^{r_{POST}^H} (v - y) f_1^{(K-1)}(y) dy + \\ + \int_{r_{POST}^H}^v (v - y) \theta(y) f_1^{(K-1)}(y) dy & \text{if } v \in [r_{POST}^H, 1] \end{cases} \quad (6.66)$$

$$\Pi_{POST}^H(v) = \begin{cases} 0 & \text{if } v \in [0, r_{POST}^H) \\ (v - r_{POST}^H) \int_{y_2=0}^{r_{POST}^H} \int_{y_1=y_2}^1 \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\ + \int_{r_{POST}^H}^v (v - y)(1 - \theta(y)) f_1^{(K-1)}(y) dy + \\ + \int_{y_2=r_{POST}^H}^v \int_{y_1=y_2}^1 (v - y_2) \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 & \text{if } v \in [r_{POST}^H, 1] \end{cases} \quad (6.67)$$

More specifically, a buyer with valuation $v \geq r_{POST}^H$ that selects the low-reserve PRE mechanism pays the latter his reserve price when all opponent bids are less than this reserve price (first term in (6.66)), and pays the highest opponent bid if this is lower than her bid and is submitted in the same intermediary (second and third terms in (6.66)).

On the other hand, if she selects the high-reserve POST intermediary, she pays his reserve price, r_{POST}^H , when the second-highest opponent bid is less than this reserve price and, at the same time, the highest opponent bid is submitted in the other (PRE) intermediary (first term in (6.67)). She also pays the highest opponent bid if it is higher than r_{POST}^H , less than her bid, and is submitted in the same intermediary (second term in (6.67)). Finally, she pays the second-highest opponent bid if it is also less than her bid, higher than r_{POST}^H , and, at the same time, the highest opponent bid is submitted in the other (PRE) intermediary (third term in (6.67)).

As we will immediately show, there is a pure-strategy BNE where all buyers select the PRE intermediary given that the intermediaries' reserve prices satisfy a condition, similar in concept to Section 6.1.1. Formally:

Theorem 6.14. *There exists a pure-strategy Bayes-Nash equilibrium in the buyer PRE - POST duopoly intermediary selection problem where all buyers select the low-reserve intermediary implementing a PRE auction with reserve price r_{PRE}^L , when the other intermediary implements a POST auction with reserve price r_{POST}^H and $r_{PRE}^L < r_{POST}^H$, if the intermediary reserve prices satisfy the condition:*

$$\int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y) dy \geq (K-1) \int_{r_{POST}^H}^1 [1 - F(y)] F^{K-2}(y) dy \quad (6.68)$$

Proof. Equation (6.67) for valuations $v \geq r_{POST}^H$ can be written as:

$$\Pi_{PRE}^L(v) = \int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy + (v - r_{POST}^H)F^{K-1}(r_{POST}^H) + \int_{r_{POST}^H}^v (v - y)\theta(y)f_1^{(K-1)}(y)dy \quad (6.69)$$

Using equation (B.92), a buyer's ex-interim expected surplus from the high-reserve, POST intermediary, when her valuation is $v \geq r_{POST}^H$ will be:

$$\begin{aligned} \Pi_{POST}^H(v) &= (v - r_{POST}^H)F^{K-1}(r_{POST}^H) + \int_{r_{POST}^H}^v (v - y)f_1^{(K-1)}(y)dy + \\ &+ (K-1) \int_{r_{POST}^H}^v \int_{y_2}^1 \theta(y_1)f(y_1)F^{K-2}(y_2)dy_1 dy_2 \end{aligned} \quad (6.70)$$

Taking their difference yields:

$$\begin{aligned} \Pi_{PRE}^L(v) - \Pi_{POST}^H(v) &= \int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy - \int_{r_{POST}^H}^v (v - y)(1 - \theta(y))f_1^{(K-1)}(y)dy - \\ &- (K-1) \int_{r_{POST}^H}^v \int_{y_2}^1 \theta(y_1)f(y_1)F^{K-2}(y_2)dy_1 dy_2 \end{aligned} \quad (6.71)$$

Consider the case that all (other) buyers select the low-reserve intermediary auction, that is $\theta(v) = 1$ for all $v \in [r_{POST}^H, 1]$. Then equation (6.71) yields:

$$\Pi_{PRE}^L(v) - \Pi_{POST}^H(v) = \int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy - (K-1) \int_{r_{POST}^H}^v \int_{y_2}^1 f(y_1)F^{K-2}(y_2)dy_1 dy_2 \quad (6.72)$$

Taking the first-order derivative of this difference w.r.t. v yields:

$$\frac{\partial}{\partial v}(\Pi_{PRE}^L(v) - \Pi_{POST}^H(v)) = -(K-1)F^{K-2}(v)[1 - F(v)] \leq 0 \quad (6.73)$$

i.e. the ex-interim expected surplus difference is decreasing and, since $\Pi_{PRE}^L(r_{POST}^H) - \Pi_{POST}^H(r_{POST}^H) > 0$, the only condition for $\theta(\cdot) = 1$ to be a pure-strategy BNE, is for this difference to be non-negative for the maximum valuation, $v = 1$:

$$\Pi_{PRE}^L(1) - \Pi_{POST}^H(1) = \int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy - (K-1) \int_{r_{POST}^H}^1 [1 - F(y)]F^{K-2}(y)dy \geq 0 \quad (6.74)$$

thus yielding equation (6.68). \square

Similarly, it is easy to see that buyers will never all select the high-reserve POST intermediary.

Proposition 6.15. *There is no pure-strategy Bayes-Nash equilibrium in the buyer PRE - POST duopoly intermediary selection problem where all buyers always select the high-reserve intermediary implementing a POST auction.*

This is because $\Pi_{PRE}^L(r_{POST}^H) - \Pi_{POST}^H(r_{POST}^H) = \int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy > 0$ irrespective of the selection strategies of the other buyers.

As in Section 6.1.1, it still remains to show what are the resulting equilibrium intermediary selection strategies of the buyers when the condition of equation (6.68) does not hold. Since $\Pi_{PRE}^L(r_{POST}^H) - \Pi_{POST}^H(r_{POST}^H) > 0$ and the first-order derivative of the expected surplus difference for any strategy $\theta(\cdot)$ is:

$$\begin{aligned} \frac{\partial}{\partial v}(\Pi_{PRE}^L(v) - \Pi_{POST}^H(v)) &= - \int_{r_{POST}^H}^v (1 - \theta(y))f_1^{(K-1)}(y)dy - \\ &\quad - (K-1)F^{K-2}(v) \int_v^1 \theta(y)f(y)dy < 0 \end{aligned} \quad (6.75)$$

this means that there should be at least one cut-off point $w \in (r_{POST}^H, 1)$ so that $\Pi_{PRE}^L(w) = \Pi_{POST}^H(w)$ and where the intermediary selection strategy changes. Theorem 6.16 shows that this is indeed the case.

Theorem 6.16. *Let $\theta : V \mapsto [0, 1]$ be a strategy profile where $\theta(v_i) = 1$ if $r_{POST}^H \leq v_i < w$, and $\theta(v_i) = 0$ if $w \leq v_i \leq 1$, for each buyer i , $i = 1, \dots, K$, where $w \in (r_{POST}^H, 1]$, a cut-off value given by:*

$$\int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy = (K-1)[F(w) \int_{r_{POST}^H}^w F^{K-2}(y)dy - \int_{r_{POST}^H}^w F^{K-1}(y)dy] \quad (6.76)$$

Then, when the condition of Theorem 6.14 does not hold, $\theta(\cdot)$ is a pure-strategy BNE profile of the buyer PRE - POST duopoly intermediary selection problem where one intermediary implements a PRE auction with a reserve price r_{PRE}^L and the other implements a POST auction with a reserve price $r_{POST}^H > r_{PRE}^L$.

Proof. $\theta(v) = 1$ when $r_{POST}^H \leq v < w$, since a buyer expects a positive surplus from the low-reserve PRE intermediary if her valuation is close to r_{POST}^H from above, but arbitrarily close to zero surplus from the POST intermediary. If $\theta(\cdot)$ is the intermediary selection strategy of the other buyers with valuations $v \geq w$, then the difference in

expected surplus of a buyer with valuation $v \in [r_{POST}^H, w)$ will be:

$$\begin{aligned}\Pi_{PRE}^L(v) - \Pi_{POST}^H(v) &= \int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy - (K-1)[\int_{r_{POST}^H}^v \int_{y_2}^w f(y_1)F^{K-2}(y_2)dy_1dy_2 + \\ &+ \int_{r_{POST}^H}^v \int_w^1 \theta(y_1)f(y_1)F^{K-2}(y_2)dy_1dy_2] = \\ &= \int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy - (K-1)[\int_{r_{POST}^H}^v [F(w) - F(y)]F^{K-2}(y)dy + \\ &+ \int_{r_{POST}^H}^v \int_w^1 \theta(y_1)f(y_1)F^{K-2}(y_2)dy_1dy_2]\end{aligned}\tag{6.77}$$

At $v = w$, the difference in expected surplus should be zero, i.e.:

$$\begin{aligned}\int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy &= (K-1)[\int_{r_{POST}^H}^w [F(w) - F(y)]F^{K-2}(y)dy + \\ &+ \int_{r_{POST}^H}^w \int_w^1 \theta(y_1)f(y_1)F^{K-2}(y_2)dy_1dy_2]\end{aligned}\tag{6.78}$$

Then, the corresponding difference in expected surplus for a buyer with private valuation $v \in [w, 1]$ is:

$$\begin{aligned}\Pi_{PRE}^L(v) - \Pi_{POST}^H(v) &= \int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy - \int_w^v (v-y)(1-\theta(y))f_1^{(K-1)}(y)dy - \\ &- (K-1)[\int_{r_{POST}^H}^w \int_{y_2}^1 \theta(y_1)f(y_1)F^{K-2}(y_2)dy_1dy_2 + \int_w^v \int_{y_2}^1 f(y_1)F^{K-2}(y_2)dy_1dy_2] = \\ &= \int_{r_{PRE}^L}^{r_{POST}^H} F^{K-1}(y)dy - \int_w^v (v-y)(1-\theta(y))f_1^{(K-1)}(y)dy - \\ &- (K-1)[\int_{r_{POST}^H}^w \int_{y_2}^w f(y_1)F^{K-2}(y_2)dy_1dy_2 + \int_{r_{POST}^H}^w \int_w^1 \theta(y_1)f(y_1)F^{K-2}(y_2)dy_1dy_2 + \\ &+ \int_w^v \int_{y_2}^1 \theta(y_1)f(y_1)F^{K-2}(y_2)dy_1dy_2]\end{aligned}\tag{6.79}$$

Using equation (6.78), this can be written as:

$$\begin{aligned}\Pi_{PRE}^L(v) - \Pi_{POST}^H(v) &= - \int_w^v (v-y)(1-\theta(y))f_1^{(K-1)}(y)dy - \\ &- (K-1) \int_w^v \int_{y_2}^1 \theta(y_1)f(y_1)F^{K-2}(y_2)dy_1dy_2\end{aligned}\tag{6.80}$$

which is negative for all $\theta(v) \in [0, 1]$. This means that a buyer with valuation $v \in [w, 1]$ will always select the high-reserve, POST intermediary irrespective of the selection

strategy of the other buyers. Hence, $\theta(v) = 0$ for $v \in [w, 1]$ and then equation (6.78) yields the condition of equation (6.76). \square

This concludes our analysis on the intermediary selection strategy of the buyers. In what follows, we consider the intermediaries' and center's best responses to this selection.

6.2 Intermediaries' and Center's Best Responses

Having characterized the equilibrium intermediary selection strategy of the buyers, in this section, we look at the equilibrium reserve-price-setting problem of the intermediaries and the center which take into account the selection strategies of the buyers (and the expected behavior of the intermediaries for the center) when announcing their reserve prices.

Given the non-trivial equilibrium selection strategies of the buyers, in the following subsections, we numerically find ϵ -NE for the intermediaries' reserve prices. As before, we run the fictitious play algorithm for the example with buyers whose private valuations are i.i.d. drawn from $U(0, 1)$ for a population of $K = 4$ and $K = 10$ buyers. These numbers allow us to compare the results of this chapter with those in Chapter 5 for captive buyers. We start with the homogeneous PRE and POST intermediary duopoly competitions in Sections 6.2.1 and 6.2.2, and then move to the heterogeneous PRE-POST competition in Section 6.2.3.

6.2.1 Pre-Award Second-Price Sealed-Bid Intermediary Auctions

As has been shown, in this setting, the buyers' intermediary selection strategy involves three intervals in the support of their private valuations, unless the intermediary reserve prices are sufficiently different. In the former unique equilibrium, buyers whose valuations lie in the low-valuation interval always choose the low-reserve intermediary. Buyers whose valuations lie in the middle interval follow a strictly mixed strategy. Finally, the strategy of the buyers with valuations in the high-valuation interval is for all of them to go to either the high-reserve intermediary, or the low-reserve one, but not both. Given the complexity of this strategy, it seems unlikely that it is possible to analytically derive the Nash equilibrium reserve-price-setting strategies of the intermediaries.

For this reason, in accordance with the methodology of Chapter 5, we run the fictitious play algorithm in this setting to shed some light on the effects of the buyers' non-captivity to the generated profits and social welfare. Our results for $K = 4$ and $K = 10$ buyers with uniformly i.i.d. drawn private valuations in $[0, 1]$ are shown in Figure 6.7 and Figure 6.8 respectively. As can be seen, the center best responds by setting a higher reserve price compared to the setting with captive buyers. Furthermore, the

center benefits from the competition of the intermediaries to attract buyers since its average revenue is also higher compared to the captive buyers setting (cf. Figure 5.11). Additionally, it can be seen that the support of the mixing of reserve prices shrinks as the number of buyers increases.

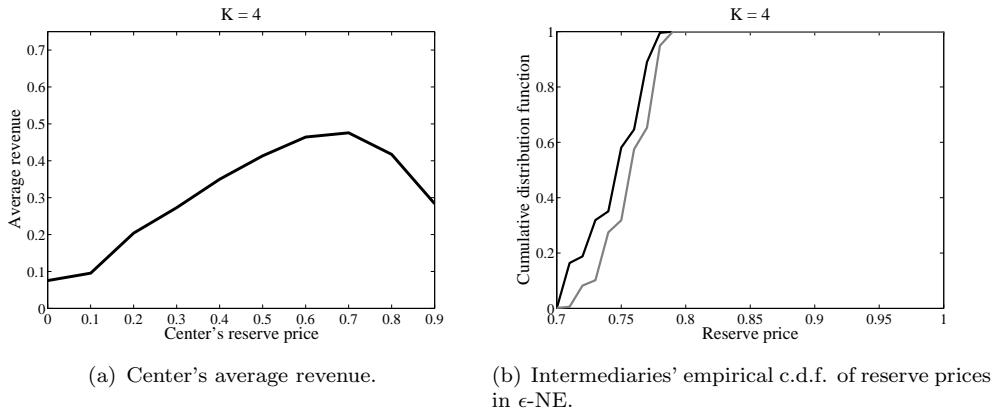


FIGURE 6.7: Center's average revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly PRE intermediary setting with non-captive $K = 4$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

This concludes our exposition of the intermediaries equilibrium reserve prices in the PRE duopoly non-captive intermediary setting. We continue with the case of two POST intermediaries where it will be trivially shown that their reserve prices are driven towards the center's reserve price.

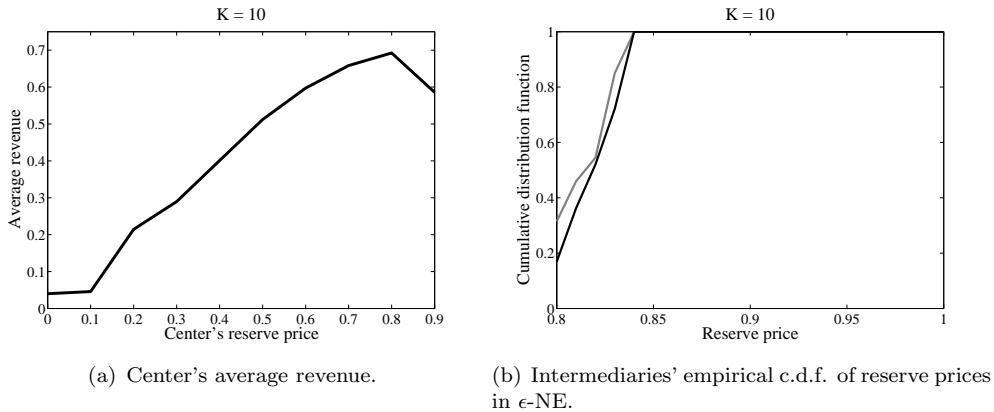


FIGURE 6.8: Center's average revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly PRE intermediary setting with non-captive $K = 10$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

6.2.2 Post-Award Second-Price Sealed-Bid Intermediary Auctions

In Section 6.1.2, we have seen that buyers have a dominant strategy of selecting the lowest-reserve intermediary. This creates zero demand for the remaining intermediaries and naturally suppresses their reserve prices to the minimum level, i.e. the center's reserve price, ρ , yielding a positive profit only when the buyers with the highest and the second-highest valuations select him or when the buyer with the highest valuation is in his market but all opponent intermediary bids are less than ρ . We formalize this observation in the following proposition.

Proposition 6.17. *It is a pure-strategy equilibrium in the n -POST intermediary reserve-price-setting problem, $n \in \mathbb{N}^+, n \geq 2$, for all intermediaries to set a reserve price $r_{POST}^i = \rho$, $i = 1, \dots, n$.*

Since the intermediary selection problem admits an infinite number of equilibria, there is no sensible way for the center to optimize its reserve price unless it has knowledge of the exact selection strategies of the buyers. This concludes our analysis of the competition between POST intermediaries. In what follows, we remove the limitation of homogeneity between intermediaries and let one intermediary implement a PRE auction and the other a POST auction.

6.2.3 Pre- versus Post-Award Second-Price Sealed-Bid Intermediary Auctions

As we have shown in Section 6.1.3, buyers always choose the POST intermediary against a PRE one, as long as the latter sets a reserve price that is not less than that of the POST intermediary. If this is not the case, then low-valuation buyers select the low-reserve PRE intermediary, whereas high-valuation buyers select the high-reserve POST intermediary. It is hence not clear whether the POST mechanism is better off by undercutting his PRE opponent, obtaining all the available market-share and driving his reserve price downwards, or he prefers setting a high-enough reserve price that increases his profit.

To shed some light on this, we conducted a number of numerical experiments, using the fictitious play algorithm, for the examples of $K = 4$ and 10 buyers whose private valuations are i.i.d. drawn from $U(0, 1)$. Our results for these two cases are shown in Figures 6.9 and 6.10 respectively. As can be seen, the supports of both intermediaries' equilibrium strategies are very small. Moreover, it can be seen that the PRE intermediary best responds by setting very high reserve prices in an effort to increase his profit against the more efficient POST mechanism. Also, it can be observed that all reserve prices are higher compared to the case with captive buyers.

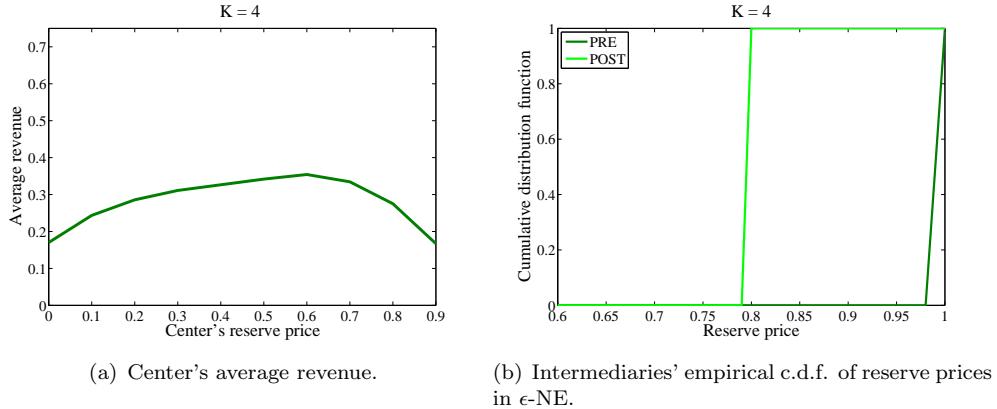


FIGURE 6.9: Center's average revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly PRE-POST intermediary setting with non-captive $K = 4$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

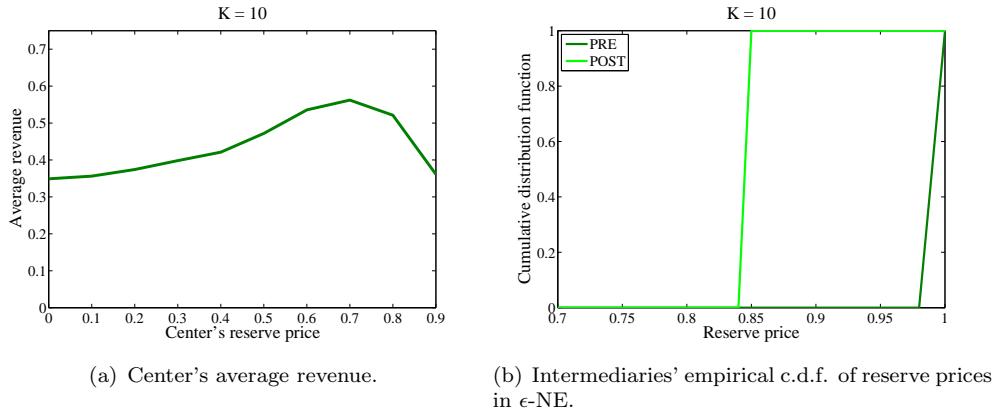


FIGURE 6.10: Center's average revenue (left) and empirical c.d.f. at the center's optimal reserve price (right) of the fictitious play intermediaries' ϵ -NE reserve-price-setting strategies for a duopoly PRE-POST intermediary setting with non-captive $K = 10$ buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

Having illustrated the approximate equilibrium behavior of the intermediaries and the center, in the following subsection, we compare the obtained profits and social welfare for the two examples above.

6.2.4 Comparison of the Two Intermediary Mechanisms

In this subsection, we provide a comparison of the homogeneous PRE and heterogeneous PRE - POST duopoly competitions. In more detail, Figures 6.11 - 6.14 depict the intermediaries' average profits, buyers' average surplus, the center's average revenue as well as the social welfare respectively under the two settings.

In these two examples, intermediaries seem to benefit from their heterogeneity since they are both better off compared to the case of two PRE mechanisms. One of the reasons for

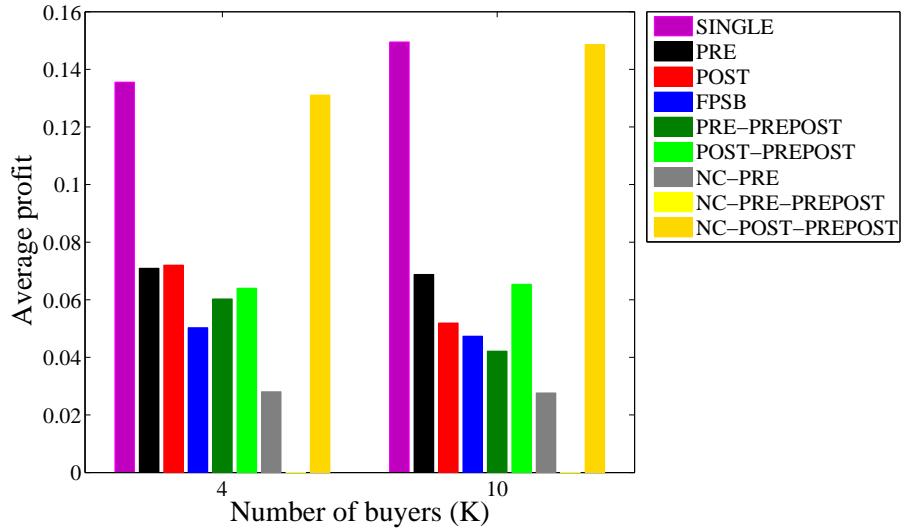


FIGURE 6.11: Intermediaries' average profits with reserve prices for the three intermediary mechanisms for $K = 4$ and $K = 10$ captive and non-captive buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

this might be the fact that the center's optimal reserve price in the heterogeneous setting is smaller, thus increasing their probability of having a buyer whose bid is above their reserve price. As can also be seen, the POST intermediary yields higher average profit than his PRE opponent. This is in agreement with the results of Chapter 5 when buyers are captive. In general, intermediaries' average profits decrease as the number of buyers increases for the two examples, with the notable exception of POST intermediaries that benefit from competition against PRE auctioneers. This decrease can be attributed to the disproportionate increase of the center's reserve price (as Figure 6.15 shows) compared to the profit that should increase because of the higher number of buyers. This decrease is in accordance with the results for a single intermediary (cf. Figure 5.1), although, in this latter, case this decrease is apparent for a higher number of buyers. What's more, numerical errors due to the discretization used might make this effect even more apparent.

Regarding the buyers' average surplus, this also decreases with the total number of their population, as both the center's reserve price and intermediaries' support of randomization for their reserve-price-setting strategies increase. As Figure 6.12 reveals, buyers benefit more from the competition between two PRE intermediaries compared to a heterogeneous PRE - POST competition. Also, the strategic selection of an intermediary increases buyers' average surplus for the duopoly PRE case, but not for the heterogeneous PRE - POST scenario. However, these results are subject to numerical errors due to the fact that the experiments for captive buyers correspond to expected surplus whereas the experiments for non-captive buyers are averages over a number of simulations.

The center's average revenue is also higher for homogeneous PRE intermediaries as well as

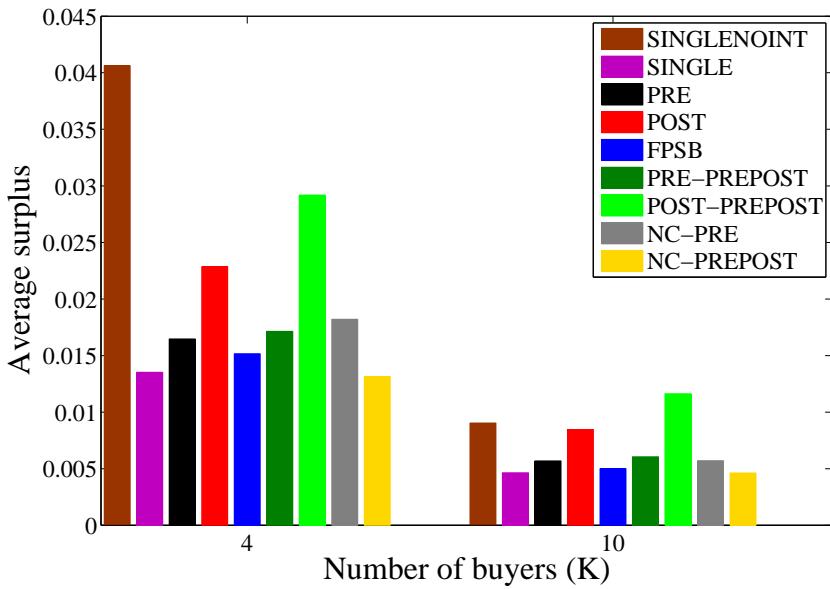


FIGURE 6.12: Buyers' average surplus with intermediaries imposing reserve prices for the three intermediary mechanisms for $K = 4$ and $K = 10$ captive and non-captive buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

compared to the case where buyers are captive. In contrast, the center's attained revenue is smaller for the PRE - POST competition compared to the scenario with captive buyers. Similarly, for non-captive buyers, the social welfare is higher for PRE intermediaries, again due to the higher optimal center's reserve price, compared to the PRE - POST competition. Also the ad exchange system seems to be worse off as a whole for PRE versus

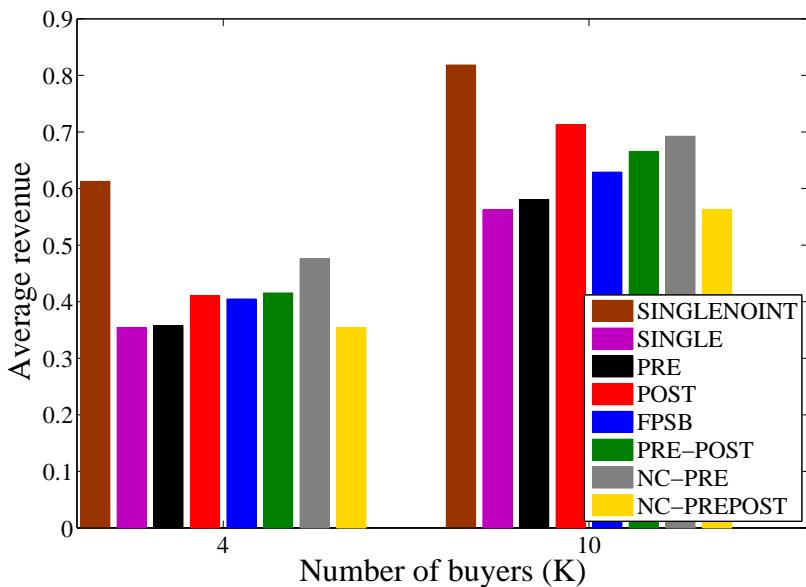


FIGURE 6.13: Center's average revenue with intermediaries imposing reserve prices for the three intermediary mechanisms for $K = 4$ and $K = 10$ captive and non-captive buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

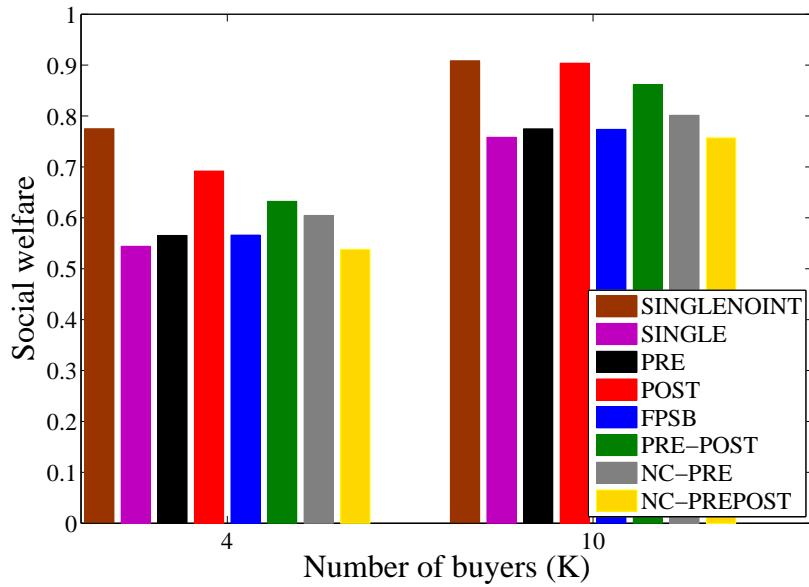


FIGURE 6.14: Social welfare with intermediaries imposing reserve prices for the three intermediary mechanisms for $K = 4$ and $K = 10$ captive and non-captive buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

POST intermediaries whose buyers strategically select one of the two mechanisms. This is expected since both the center's average revenue and the buyers' average surplus is smaller in the latter case. This ends our analysis of the duopoly intermediary competition with non-captive buyers.

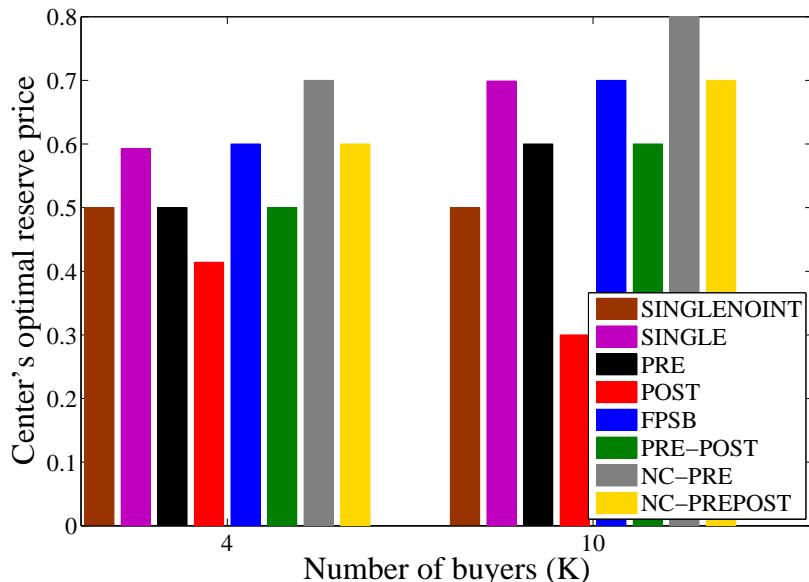


FIGURE 6.15: Center's optimal reserve price with intermediaries imposing reserve prices for the three intermediary mechanisms for $K = 4$ and $K = 10$ captive and non-captive buyers whose private valuations are i.i.d. drawn from $U(0, 1)$.

6.3 Summary

In contrast to the previous chapters, where the focus was on the competition of intermediaries with captive buyers that are non-strategically allocated to the intermediaries, within this chapter we studied the imperfect intermediaries' competition with non-captive buyers. Since the intermediary selection problem of the buyers constitutes a key research challenge of this thesis, we focused on deriving such Bayes-Nash equilibrium intermediary selection strategies, albeit in a simple duopoly setting.

To this end, we first studied the intermediary selection problem of buyers in a setting with two PRE intermediaries. We proved the existence of a unique pure-strategy Bayes-Nash equilibrium under some defined conditions on the difference between the intermediaries' reserve prices. We then showed that, when these conditions are not met, there is a unique mixed-strategy Nash equilibrium involving three intervals in the support of the buyers valuations: buyers with low valuations always select the low-reserve intermediary, buyers whose valuations lie in the rightmost interval deterministically select one of the intermediaries, whereas buyers whose valuations lie in the middle interval follow a strictly mixed strategy whose form will be different for different distributions of private valuations and numbers of buyers. Our results for the duopoly PRE competition are of interest to the general literature on competing auctions where, in the majority of cases, bidders have been found to equally randomize between auctioneers in equilibrium (see Section 2.5 for details). We then repeated our analysis for two POST intermediaries where we showed that buyers always select the low-reserve intermediary but there are an infinite number of Nash equilibria when the intermediaries' reserve prices are equal. Following this, we analyzed the heterogeneous competition between a PRE and a POST intermediary, where we proved that buyers always choose the more efficient POST mechanism if his reserve price is not higher than that of the PRE. If, in contrast, this does not hold, then buyers with private valuations above a cut-off point still select the POST intermediary but buyers with lower valuations select the low-reserve PRE intermediary.

Given the intermediary selection strategies of the buyers, we looked at the equilibrium reserve-price-setting problem of the intermediaries. Letting the buyers select an intermediary makes the analysis too technical, since the derivations are involved even when buyers are captive and symmetrically allocated to the intermediaries. For this reason, we limited our analysis to numerical simulations for two examples with buyers whose private valuations are i.i.d. drawn from the uniform distribution $U(0, 1)$. These examples showed that both the center's average revenue and the social welfare targets are aligned, being higher for homogeneous PRE intermediaries, however intermediaries are better off implementing heterogeneous mechanisms. In this last case, in accordance with the results of Chapter 5, the POST intermediary obtains higher average profit.

Chapter 7

Conclusions and Future Work

In this final chapter, we conclude by reviewing the contributions of this work towards the research objective of studying the impact of the auction design under competition for demand-side intermediaries in online advertising exchanges. To this end, in Section 7.1, we summarize the main results within each chapter of this thesis and discuss their implications for real-world ad exchanges. Thereafter, in Section 7.2, we shortly provide a number of suggestions to the designers of DSPs based on our analysis in the previous chapters. Finally, in Section 7.3, we identify promising lines of future work that could be pursued continuing the research of this work.

7.1 Summary of Results

Advertising exchanges are becoming the de facto means of trading ads online. Real-time bidding allows advertisers to achieve what has never been possible in the past, that is to target their ads only to potentially interested users, reducing their cost and simultaneously the number of annoying ads that people see on the web.

Two of the most important parties of ad exchanges are the demand- and supply-side intermediaries that take the role of brokers on behalf of their customers. These intermediaries are vital to the successful adoption of ad exchanges, since they provide all the tools and infrastructure that allows medium and smaller advertisers to participate in these exchanges. Our focus within this thesis has been the competition between the demand-side intermediaries along with their effect on the exchange and the advertisers. Modern such intermediaries, known as DSPs, typically run their own local auctions before the exchange's central auction and submit (usually a single) bid at the exchange on behalf of their (multiple) advertisers. Hence, demand-side intermediaries hide some of the demand from the exchange and behave as bidding rings whose centers are profit maximizing. This creates a number of complications for the revenue of the exchange

and the ad exchange ecosystem as a whole. Hence, a careful study of some of the currently-used mechanisms can provide guidance on the proper design and optimization of all the auctions involved which will determine the prosperous operation of this new marketplace.

We have seen that previous literature does not satisfy our research aims of studying the imperfect competition between such intermediaries. Specifically, the literature on bidding rings focuses on cases with a single ring and does not consider profit maximizing ring centers, as is the case in the ad exchange setting. Furthermore, the literature on competing auctioneers focuses on the more general case of independent competing auctioneers which do not subsequently compete at another auction. Finally, the more relevant work of Feldman et al. (2010) is only for the limiting case of one buyer per intermediary and the paper by Mansour et al. (2012) does not fully take into account the strategic actions of the intermediaries (such as setting appropriate reserve prices) and the advertisers (such as selecting an intermediary).

For this reason, within this thesis, we looked at the intermediaries' competition in a simplified single-item IPV setting. Specifically, in Chapter 4 we studied three widely-used mechanisms, two variations of the Vickrey auction, called the **PRE** and **POST** mechanisms, and **FPSB** auctions, in a simple symmetric setting with intermediaries implementing the same mechanism (i.e. homogeneous). We assumed that they do not impose any reserve prices but instead make a profit by pocketing the difference between their local bids and their payment at the exchange. We have shown that none of the three mechanisms yields the highest profit in all settings but homogeneous **POST** auctions are the most efficient. Moreover, our numerical results showed that **FPSB** intermediary auctions provide a good trade-off between profit and efficiency, especially for small numbers of intermediaries and buyers per intermediary, as is usually the case nowadays.

We then continued our analysis in Chapter 5 where we considered the duopoly competition between homogeneous mechanisms along with the competition between **PRE** and **POST** intermediary auctioneers. This duopoly simplification was necessary for reasons of analytical tractability, but even in this simple setting, a number of insights about the effects of intermediary competition can be obtained. In this setting, we proved the existence of a symmetric pure-strategy Nash equilibrium for the intermediaries' reserve prices in a **POST** duopoly competition subject to some constraints on the distribution of private valuations and the number of buyers. In this equilibrium, both intermediaries implement the optimal monopolistic intermediary reserve price, i.e. a fixed markup on top of the center's reserve price. For the remaining cases, we have reasoned why intermediaries are likely to follow mixed strategies due to their competition and the double marginalization effect, as Feldman et al. (2010) have shown for the **PRE** auctions. Given the inherent difficulty in analytically expressing these strategies, after characterizing the expected utilities of the stakeholders, wherever this was possible, we conducted a number of numerical experiments to shed some light on the revenue and efficiency effects

of the intermediaries. In more detail, we used the fictitious play algorithm to obtain approximate Nash equilibria of the intermediaries' reserve-price-setting game, and compared the resulting approximate equilibrium utilities of all the stakeholders as well as the social welfare for the above-mentioned mechanisms. Our results depict the profit superiority of the PRE mechanism for the intermediaries in this setting in accordance with some of the results of Feldman et al. (2010). However, POST mechanisms are by far better in terms of the efficiency attained, that might be more desirable in the long-term. What's more, buyers in total benefit from intermediaries using different mechanisms, as our results for the setting with one PRE and one POST intermediary. In all cases, the introduction of the reserve prices benefit only the intermediaries, since the utilities of the other players as well as the total social welfare decrease compared to the setting of Chapter 4.

Following this, in Chapter 6, we let buyers strategically select one of the intermediaries again in a duopoly intermediary setting. The derivation of FPSB equilibrium bidding strategies when taking into account both the participation decision of the other advertisers becomes cumbersome. For this reason, we analyzed the homogeneous and heterogeneous competition of the two Vickrey variations. Specifically, we first looked at the intermediary selection problem of the buyers and derived their corresponding equilibrium strategies. We found that, in the case of two PRE intermediaries, the resulting equilibrium intermediary selection strategy of the buyers is rather complex; it is in mixed strategies unless the reserve prices are very different, where, in the latter case, all buyers select the low-reserve intermediary. We then continued our analysis with a duopoly POST setting and have found in a straightforward manner that buyers always select the low-reserve intermediary, thus driving the intermediaries' reserve prices downwards, towards the center's reserve price. The intermediary selection strategies in a heterogeneous Vickrey setting are also simpler to derive; buyers always select the POST auctioneer as long as his reserve price is not above the one for the PRE mechanism, otherwise high-valuation buyers still select the former mechanism but low-valuation buyers switch to the PRE auctioneer. Finally, in most cases, given the mixed Bayes-Nash equilibrium selection strategies of the buyers, the intermediaries' equilibrium reserve prices become too technical to derive. Hence, we repeated fictitious play in this setting in two examples with advertisers whose private valuations are i.i.d. drawn from the uniform distribution $U(0, 1)$. We found that the center's average revenue, the buyers' average surplus and the social welfare obtained are higher for the case of two PRE mechanisms but, in contrast, intermediaries' profit increases if they implement different Vickrey mechanisms, where the POST intermediary benefits from this heterogeneity.

Our formulation is general enough to include other settings with intermediaries (as described in Section 2.3) and is of relevance to the growing literature on bidding rings. In the next section, we use our insights gained from the analysis within this thesis to

make a suggestion on the intermediary mechanism that seems more suitable in each of the studied environments.

7.2 Demand-Side Intermediary Policy Recommendations

The work within this thesis has combined theoretical insights and numerical simulations to compare three natural choices for demand-side intermediary auctioneers that make a profit by getting the difference of what they get paid by their advertisers and what they pay at the exchange. Given the complexity of the ad exchange system, our results are bounded by the limitations of such simulations, such as the distributions of private valuations considered that might not always be representative of the actual such distributions as well as the number of intermediaries considered in some scenarios. Nevertheless, these results can qualitatively shed some light in the proper design of demand-side intermediary auctions and give intuition on the issues of their competition.

Given that the vast majority of DSPs at this time do not impose reserve prices, our results from Chapter 4 show that current DSPs should benefit from using FPSB local auctions; currently, less than a handful of intermediaries participate at each individual auction due to the vast number of available impressions, and each has a small number of interested advertisers. Hence, as Figure 4.6 illustrates, the FPSB auction yields the highest profit among the three mechanisms studied. What's more, FPSB auctions are more efficient than their closest-in-profit PRE competitors and efficiency is vastly important for the future adoption of ad exchanges and real-time bidding. Another benefit of FPSB auctions is their transparency, since there is no need to reveal the whole book of bids. Nevertheless, advertisers will need to continuously adapt their bids and BNE bids are less predictable from the auctioneer's point of view. For this reason, PRE auctions that are DSIC might also be a good option in this case due to their strategic simplicity, although their smaller social welfare might be detrimental for the future adoption of RTB.

As we have shown, a reserve price can increase an intermediary's attained profit, so we expect more and more DSPs to enforce such prices. Our analysis in this case suggests that, when advertisers strategically select their DSP, they are more likely to select a POST intermediary compared to a PRE intermediary. This means that all intermediaries are likely to adopt the former mechanism. However, given that such a strategic selection on behalf of the advertisers is likely to take place less often, the analysis of Chapter 5 becomes useful. Specifically, the results show that both POST intermediaries' and the ad exchange's reserve prices are driven downwards, increasing the efficiency and advertisers' surplus at the small expense of the intermediaries' attained profit, compared to e.g. PRE intermediaries. What's more, POST auctions are more transparent and their operation is more easy to explain to advertisers compared to their PRE counterpart.

Finally, regarding the ad exchange's (or publisher's) reserve price optimization, we have shown that this will generally depend both on the number of buyers and intermediaries, in accordance with the results of Feldman et al. (2010), and that this reserve price will be higher than a monopolist's optimal reserve price without intermediaries, with the notable exception of POST intermediaries setting reserve prices, where the exchange's reserve price is driven to zero as the number of buyers increases. Hence, knowledge of the type of intermediary mechanisms participating in an ad exchange is helpful for the exchange's auction designer to effectively optimize its reserve price.

This concludes the analysis of the work done to date. A number of open challenges remain to be tackled, as outlined in the following section.

7.3 Future Work

As has been summarized in Section 7.1, the contributions within this thesis have addressed, to the extent that this was possible, our initial research aims. Despite these achievements, a number of open problems yet remain to be solved in this complex marketplace. Indeed, our results show that different auction formats are not easily comparable and that one needs to perform a market analysis before selecting one mechanism over the other. Moreover, our analysis constitutes only a first step in the analysis of competition between demand-side intermediaries. For the sake of analytical tractability, we had to make a number of simplifying assumptions at various places throughout the thesis such as symmetry, homogeneity and duopoly markets for the intermediaries. Some of these assumptions seem to be binding for an analysis using the tool set provided by traditional auction theory. Hence, new solution concepts and methodologies need to be derived. Some potential lines of investigation to extend the scope of this work include:

- **Heterogeneous and asymmetric intermediary auctioneers.** The first immediate extension of our results is for the case of symmetric heterogeneous intermediaries, when first-price sealed-bid auctioneers compete against second-price sealed-bid ones. Kotowski (2014) has derived the equilibrium bidding functions for two first-price sealed-bid auctions in auctions with discrimination, so deriving similar functions for our setting would nicely supplement the existing results in this direction. The other direct extension is for intermediaries with different number of buyers each, a setting which is more likely to arise in practice; a study by Nicholls et al. (2013) has shown that in the U.S., the top 3 DSPs accounted for 50-60% of the market share in 2013. Nevertheless, this creates asymmetries at the central auction, which, along with the problems of competition between auctions (see Section 2.5), would also limit the analytical tractability of the results.

- **Managed- versus self-service demand-side intermediaries.** The work described in the previous chapters deals with the competition of self-service intermediaries which run local auctions, given the intermediaries' lack of knowledge of their advertisers' valuations. Since, even today, most of the demand-side intermediaries still follow the managed-service model, it would be interesting to see what is the cost or benefit of such intermediaries with full knowledge of advertisers' valuations against the other type, taking also into account the cost of managing a campaign that intermediaries of managed-service type undertake. This will shed light in the future of online advertising and the future of traditional ad networks.
- **Repeated setting.** Throughout this thesis, we have considered the auction of a single good that corresponds to an advertising slot. In reality, billions of such auctions take place every day for different goods and with different participants, each of which might enter and leave at any point and also has some budget. Hence, it is important to study this more realistic setting. Given the insights gained from our analysis, this seems to be technically challenging, hence new methodologies and concepts need to be developed. We consider mean-field approximation methods that have recently gained momentum and have also been used in the context of advertising auctions (Balseiro et al., 2013; Gummadi et al., 2012; Iyer et al., 2011; Athey and Nekipelov, 2010) a promising direction for research in this area. Specifically, a study on the competition of DSPs which do not have full information about their opponents and advertisers in such a context would fully complement the analysis within this thesis.

We believe that these extensions could further increase the applicability and insights of the analysis within this thesis, offering insightful guidance and contributing to the successful operation of this new, complex marketplace.

Appendix A

Supplement for Chapter 5

A.1 Derivations for Duopoly PRE - PRE Intermediaries with Reserve Prices

When both intermediaries set reserve prices $r_{PRE}^L \leq r_{PRE}^H$, each having exactly $k > 1$ buyers, the low-reserve intermediary's expected profit can be written as:

$$\begin{aligned}
profit_{PRE}^L(r_{PRE}^L) &= F^k(r_{PRE}^H)[kF^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))(r_{PRE}^L - \rho) + \\
&+ \int_{r_{PRE}^L}^1 (y - \rho)f_2^{(k)}(y)dy] + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^H}^1 (y - r_{PRE}^H)f_2^{(k)}(y)dy + \\
&+ \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^H}^y (y - x)f_2^{(k)}(x)dxdy = \\
&= F^k(r_{PRE}^H)[kF^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))(r_{PRE}^L - \rho) + \int_{r_{PRE}^L}^1 (y - \rho)f_2^{(k)}(y)dy] + \\
&+ kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^H}^1 (y - r_{PRE}^H)f_2^{(k)}(y)dy + \\
&+ \int_{r_{PRE}^H}^1 f_2^{(k)}(y)[- (y - r_{PRE}^H)F_2^{(k)}(r_{PRE}^H) + \int_{r_{PRE}^H}^y F_2^{(k)}(x)dx]dy = \\
&= F^k(r_{PRE}^H)[kF^{k-1}(r_{PRE}^L)(r_{PRE}^L - \rho) - kF^k(r_{PRE}^L)(r_{PRE}^L - \rho) + 1 - \rho - \\
&- (r_{PRE}^L - \rho)F_2^{(k)}(r_{PRE}^L) - \int_{r_{PRE}^L}^1 F_2^{(k)}(y)dy] + [kF^{k-1}(r_{PRE}^H) - kF^k(r_{PRE}^H) - kF^{k-1}(r_{PRE}^H) + \\
&+ (k - 1)F^k(r_{PRE}^H)] \int_{r_{PRE}^H}^1 (y - r_{PRE}^H)f_2^{(k)}(y)dy + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^H}^y F_2^{(k)}(x)dx dy =
\end{aligned}$$

$$\begin{aligned}
&= F^k(r_{PRE}^H)[-F^k(r_{PRE}^L)(r_{PRE}^L - \rho) + 1 - \rho - \int_{r_{PRE}^L}^1 F_2^{(k)}(y)dy] - \\
&\quad - F^k(r_{PRE}^H) \int_{r_{PRE}^H}^1 (y - r_{PRE}^H) f_2^{(k)}(y) dy + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^H}^y F_2^{(k)}(x) dx dy = \\
&= F^k(r_{PRE}^H)[-F^k(r_{PRE}^L)(r_{PRE}^L - \rho) + 1 - \rho - \int_{r_{PRE}^L}^1 F_2^{(k)}(y)dy] - \\
&\quad - (1 - r_{PRE}^H - \int_{r_{PRE}^H}^1 F_2^{(k)}(y)dy)] + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^H}^y F_2^{(k)}(x) dx dy = \\
&= F^k(r_{PRE}^H)[r_{PRE}^H - \rho - F^k(r_{PRE}^L)(r_{PRE}^L - \rho) - \int_{r_{PRE}^L}^{r_{PRE}^H} F_2^{(k)}(y)dy] + \\
&\quad + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^H}^y F_2^{(k)}(x) dx dy
\end{aligned} \tag{A.1}$$

The high-reserve intermediary's ex-ante expected profit can be written as:

$$\begin{aligned}
profit_{PRE}^H(r_{PRE}^H) &= F^k(r_{PRE}^L)[kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))(r_{PRE}^H - \rho) + \int_{r_{PRE}^H}^1 (y - \rho) f_2^{(k)}(y) dy] + \\
&\quad + kF^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))[kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))(r_{PRE}^H - r_{PRE}^L) + \int_{r_{PRE}^H}^1 (y - r_{PRE}^L) f_2^{(k)}(y) dy] + \\
&\quad + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^L}^{r_{PRE}^H} (r_{PRE}^H - y) f_2^{(k)}(y) dy + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y (y - x) f_2^{(k)}(x) dx dy = \\
&= F^k(r_{PRE}^L)[(kF^{k-1}(r_{PRE}^H) - kF^k(r_{PRE}^H))(r_{PRE}^H - \rho) + 1 - \rho - \\
&\quad - (r_{PRE}^H - \rho)(kF^{k-1}(r_{PRE}^H) - (k-1)F^k(r_{PRE}^H) - \int_{r_{PRE}^H}^1 F_2^{(k)}(y)dy] + \\
&\quad + kF^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))(r_{PRE}^H - r_{PRE}^L) + \\
&\quad + [kF^{k-1}(r_{PRE}^L) - kF^k(r_{PRE}^L) - kF^{k-1}(r_{PRE}^L) + (k-1)F^k(r_{PRE}^L)] \int_{r_{PRE}^H}^1 (y - r_{PRE}^L) f_2^{(k)}(y) dy + \\
&\quad + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^L}^{r_{PRE}^H} (r_{PRE}^H - y) f_2^{(k)}(y) dy + \\
&\quad + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y F_2^{(k)}(x) dx dy =
\end{aligned}$$

$$\begin{aligned}
&= F^k(r_{PRE}^L)[1 - \rho - F^k(r_{PRE}^H)(r_{PRE}^H - \rho) - \int_{r_{PRE}^H}^1 F_2^{(k)}(y)dy] + \\
&\quad + kF^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))(r_{PRE}^H - r_{PRE}^L) - \\
&\quad - F^k(r_{PRE}^L) \int_{r_{PRE}^L}^1 (y - r_{PRE}^L)f_2^{(k)}(y)dy + \\
&\quad + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^L}^{r_{PRE}^H} (r_{PRE}^H - y)f_2^{(k)}(y)dy + \\
&\quad + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y F_2^{(k)}(x)dxdy = \\
&= F^k(r_{PRE}^L)[1 - \rho - F^k(r_{PRE}^H)(r_{PRE}^H - \rho) - \int_{r_{PRE}^H}^1 F_2^{(k)}(y)dy - \\
&\quad - (1 - r_{PRE}^L - (r_{PRE}^H - r_{PRE}^L)F_2^{(k)}(r_{PRE}^H) - \int_{r_{PRE}^H}^1 F_2^{(k)}(y)dy)] + \\
&\quad + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))[kF^{k-1}(r_{PRE}^L)(1 - F(r_{PRE}^L))(r_{PRE}^H - r_{PRE}^L) - \\
&\quad - (r_{PRE}^H - r_{PRE}^L)F_2^{(k)}(r_{PRE}^L) + \int_{r_{PRE}^L}^{r_{PRE}^H} F_2^{(k)}(y)dy] + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y F_2^{(k)}(x)dxdy = \\
&= F^k(r_{PRE}^L)[r_{PRE}^L - \rho - F^k(r_{PRE}^H)(r_{PRE}^H - \rho) + F_2^k(r_{PRE}^H)(r_{PRE}^H - r_{PRE}^L)] + \\
&\quad + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))[(kF^{k-1}(r_{PRE}^L) - kF^k(r_{PRE}^L) - kF^{k-1}(r_{PRE}^L) + \\
&\quad + (k-1)F^{k-1}(r_{PRE}^L))(r_{PRE}^H - r_{PRE}^L) + \int_{r_{PRE}^L}^{r_{PRE}^H} F_2^{(k)}(y)dy] + \\
&\quad + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y F_2^{(k)}(x)dxdy = \\
&= F^k(r_{PRE}^L)[r_{PRE}^L - \rho - F^k(r_{PRE}^H)(r_{PRE}^H - \rho) + F_2^k(r_{PRE}^H)(r_{PRE}^H - r_{PRE}^L)] + \\
&\quad + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H))[\int_{r_{PRE}^L}^{r_{PRE}^H} F_2^{(k)}(y)dy - F^k(r_{PRE}^L)(r_{PRE}^H - r_{PRE}^L)] + \\
&\quad + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y F_2^{(k)}(x)dxdy = \\
&= F^k(r_{PRE}^L)[r_{PRE}^L - \rho - F^k(r_{PRE}^H)(r_{PRE}^H - \rho) + F^k(r_{PRE}^H)(r_{PRE}^H - r_{PRE}^L)] + \\
&\quad + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^L}^{r_{PRE}^H} F_2^{(k)}(y)dy + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y F_2^{(k)}(x)dxdy = \\
&= F^k(r_{PRE}^L)(1 - F^k(r_{PRE}^H))(r_{PRE}^L - \rho) + kF^{k-1}(r_{PRE}^H)(1 - F(r_{PRE}^H)) \int_{r_{PRE}^L}^{r_{PRE}^H} F_2^{(k)}(y)dy + \\
&\quad + \int_{r_{PRE}^H}^1 f_2^{(k)}(y) \int_{r_{PRE}^L}^y F_2^{(k)}(x)dxdy \tag{A.2}
\end{aligned}$$

A.2 Derivations for Duopoly POST - POST Intermediaries with Reserve Prices

When both intermediaries implement POST mechanisms with equilibrium reserve prices $r_{POST}^L = r_{POST}^H = r_{POST}^* = \rho + \frac{1-F(r_{POST}^*)}{f(r_{POST}^*)}$, the center's ex-ante expected revenue is:

$$\text{revenue}_{POST}(r_{POST}^*) = \int_{r_{POST}^*}^1 y f_2^{(2)}(y) dy + 2F(r_{POST}^*)(1-F(r_{POST}^*))[r_{POST}^* - \frac{1-F(r_{POST}^*)}{f(r_{POST}^*)}] \quad (\text{A.3})$$

Taking the first-order derivative of this w.r.t. r_{POST}^* yields

$$\begin{aligned} \frac{\partial \text{revenue}_{POST}(r_{POST}^*)}{\partial r_{POST}^*} &= -2r_{POST}^*(1-F(r_{POST}^*))f(r_{POST}^*) + 2[f(r_{POST}^*)(1-F(r_{POST}^*)) - \\ &- f(r_{POST}^*)F(r_{POST}^*)][r_{POST}^* - \frac{1-F(r_{POST}^*)}{f(r_{POST}^*)}] \\ &+ 2F(r_{POST}^*)(1-F(r_{POST}^*))[1 - \frac{-f^2(r_{POST}^*) - (1-F(r_{POST}^*))f'(r_{POST}^*)}{f^2(r_{POST}^*)}] = \\ &= 2\{-r_{POST}^*f(r_{POST}^*)F(r_{POST}^*) - (1-F(r_{POST}^*))(1-2F(r_{POST}^*)) + \\ &+ F(r_{POST}^*)(1-F(r_{POST}^*))(2 + \frac{(1-F(r_{POST}^*))f'(r_{POST}^*)}{f^2(r_{POST}^*)})\} \\ &= 2\{F(r_{POST}^*)[3(1-F(r_{POST}^*)) - r_{POST}^*f(r_{POST}^*)] - (1-F(r_{POST}^*))^2[1-F(r_{POST}^*)\frac{f'(r_{POST}^*)}{f^2(r_{POST}^*)}]\} \end{aligned} \quad (\text{A.4})$$

When both intermediaries implement POST mechanisms with reserve prices $r_{POST}^L \leq r_{POST}^H$ with $k > 1$ buyers each, then their ex-ante expected profits will be:

$$\begin{aligned}
profit_{POST}^L(r_{POST}^L) &= F^k(r_{POST}^H)[kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))(r_{POST}^L - \rho) + \\
&+ \int_{r_{POST}^L}^1 (y - \rho)f_2^{(k)}(y)dy] + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^H}^y (y - x)f_1^{(k)}(x)dxdy = \\
&= F^k(r_{POST}^H)[kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))(r_{POST}^L - \rho) + \int_{r_{POST}^L}^1 (y - \rho)f_2^{(k)}(y)dy - \\
&- \int_{r_{POST}^H}^1 (y - r_{POST}^H)f_2^{(k)}(y)dy] + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^H}^y F^k(x)dxdy = \\
&= F^k(r_{POST}^H)[kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))(r_{POST}^L - \rho) + 1 - \rho - F_2^{(k)}(r_{POST}^L)(r_{POST}^L - \rho) - \\
&- \int_{r_{POST}^L}^1 F_2^{(k)}(y)dy - (1 - r_{POST}^H - \int_{r_{POST}^H}^1 F_2^{(k)}(y)dy)] + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^H}^y F^k(x)dxdy = \\
&= F^k(r_{POST}^H)[r_{POST}^H - \rho - F^k(r_{POST}^L)(r_{POST}^L - \rho) - \int_{r_{POST}^L}^{r_{POST}^H} F_2^{(k)}(y)dy] + \\
&+ \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^H}^y F^k(x)dxdy
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
profit_{POST}^H(r_{POST}^H) &= F^k(r_{POST}^L)[kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))(r_{POST}^H - \rho) + \\
&+ \int_{r_{POST}^H}^1 (y - \rho)f_2^{(k)}(y)dy] + kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H)) \int_{r_{POST}^L}^{r_{POST}^H} (r_{POST}^H - y)f_1^{(k)}(y)dy + \\
&+ \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^L}^y (y - x)f_1^{(k)}(x)dxdy = \\
&= F^k(r_{POST}^L)[kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))(r_{POST}^H - \rho) + \int_{r_{POST}^H}^1 (y - \rho)f_2^{(k)}(y)dy - \\
&- \int_{r_{POST}^H}^1 (y - r_{POST}^L)f_2^{(k)}(y)dy] + kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))[\int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy - \\
&- F^k(r_{POST}^L)(r_{POST}^H - r_{POST}^L)] + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^L}^y F^k(x)dxdy =
\end{aligned}$$

$$\begin{aligned}
&= F^k(r_{POST}^L)[kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))(r_{POST}^H - \rho) + 1 - \rho - F_2^{(k)}(r_{POST}^H)(r_{POST}^H - \rho) - \\
&\quad - \int_{r_{POST}^H}^1 F_2^{(k)}(y)dy - (1 - r_{POST}^L - F_2^{(k)}(r_{POST}^H)(r_{POST}^H - r_{POST}^L) - \int_{r_{POST}^H}^1 F_2^{(k)}(y)dy) - \\
&\quad - kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))(r_{POST}^H - r_{POST}^L)] + \\
&\quad + kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H)) \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^L}^y F^k(x)dxdy = \\
&= F^k(r_{POST}^L)[r_{POST}^L - \rho - F^k(r_{POST}^H)(r_{POST}^H - \rho) + F^k(r_{POST}^H)(r_{POST}^H - r_{POST}^L)] + \\
&\quad + kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H)) \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^L}^y F^k(x)dxdy = \\
&= F^k(r_{POST}^L)(1 - F^k(r_{POST}^H))(r_{POST}^L - \rho) + kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H)) \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy + \\
&\quad + \int_{r_{POST}^H}^1 f_2^{(k)}(y) \int_{r_{POST}^L}^y F^k(x)dxdy \tag{A.6}
\end{aligned}$$

Taking the first-order derivatives w.r.t. the reserve prices of the above equations yields:

$$\begin{aligned}
\frac{\partial profit_{POST}^L(r_{POST}^L)}{\partial r_{POST}^L} &= F^k(r_{POST}^H)[-kF^{k-1}(r_{POST}^L)f(r_{POST}^L)(r_{POST}^L - \rho) - \\
&\quad - F^k(r_{POST}^L) + F_2^{(k)}(r_{POST}^L)] = -F^k(r_{POST}^H)[kF^{k-1}(r_{POST}^L)f(r_{POST}^L)(r_{POST}^L - \rho) + \\
&\quad + F^k(r_{POST}^L) - kF^{k-1}(r_{POST}^L) + (k-1)F^k(r_{POST}^L)] = \\
&= -kF^{k-1}(r_{POST}^L)F^k(r_{POST}^H)[f(r_{POST}^L)(r_{POST}^L - \rho) + F(r_{POST}^L) - 1] \tag{A.7}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial profit_{POST}^H(r_{POST}^H)}{\partial r_{POST}^H} &= -kF^{k-1}(r_{POST}^H)f(r_{POST}^H)F^k(r_{POST}^L)(r_{POST}^L - \rho) + \\
&\quad + [k(k-1)F^{k-2}(r_{POST}^H)f(r_{POST}^H)(1 - F(r_{POST}^H)) - kF^{k-1}(r_{POST}^H)f(r_{POST}^H)] \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy + \\
&\quad + kF^{k-1}(r_{POST}^H)(1 - F(r_{POST}^H))F^k(r_{POST}^H) - f_2^{(k)}(r_{POST}^H) \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy = \\
&= -kF^{k-1}(r_{POST}^H)f(r_{POST}^H)[F^k(r_{POST}^L)(r_{POST}^L - \rho) + \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy] + \\
&\quad + kF^{2k-1}(r_{POST}^H)(1 - F(r_{POST}^H)) \tag{A.8}
\end{aligned}$$

Taking the first-order derivative of the last equation w.r.t. r_{POST}^H above yields:

$$\begin{aligned}
\frac{\partial^2 profit_{POST}^H(r_{POST}^H)}{\partial(r_{POST}^H)^2} &= k(2k-1)F^{2k-2}(r_{POST}^H)f(r_{POST}^H)(1-F(r_{POST}^H)) - \\
&- kF^{2k-1}(r_{POST}^H)f(r_{POST}^H) - [k(k-1)F^{k-2}(r_{POST}^H)f^2(r_{POST}^H) + \\
&+ kF^{k-1}(r_{POST}^H)f'(r_{POST}^H)][F^k(r_{POST}^L)(r_{POST}^L - \rho) + \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy] - \\
&- kF^{k-1}(r_{POST}^H)f(r_{POST}^H)F^k(r_{POST}^H) = \\
&= k(2k-1)F^{2k-2}(r_{POST}^H)(1-F(r_{POST}^H))f(r_{POST}^H) - 2kF^{2k-1}(r_{POST}^H)f(r_{POST}^H) - \\
&- [k(k-1)F^{k-2}(r_{POST}^H)f^2(r_{POST}^H) + kF^{k-1}(r_{POST}^H)f'(r_{POST}^H)][F^k(r_{POST}^L)(r_{POST}^L - \rho) + \\
&+ \int_{r_{POST}^L}^{r_{POST}^H} F^k(y)dy] - kF^{k-1}(r_{POST}^H)f(r_{POST}^H)F^k(r_{POST}^H)
\end{aligned} \tag{A.9}$$

For $r_L^{POST} = r_{POST}^H = r_{POST}^* = \rho + \frac{1-F(r_{POST}^*)}{f(r_{POST}^*)}$ the last equation yields:

$$\begin{aligned}
\frac{\partial^2 profit_{POST}^H(r_{POST}^H)}{\partial(r_{POST}^H)^2}|_{r_L^{POST}=r_{POST}^H=r_{POST}^*} &= k(2k-1)F^{2k-2}(r_{POST}^*)(1-F(r_{POST}^*))f(r_{POST}^*) - \\
&- 2kF^{2k-1}(r_{POST}^*)f(r_{POST}^*) - k(k-1)F^{2k-2}(r_{POST}^*)f^2(r_{POST}^*)(r_{POST}^* - \rho) - \\
&- kF^{2k-1}(r_{POST}^*)f'(r_{POST}^*)(r_{POST}^* - \rho) = \\
&= kF^{2k-2}(r_{POST}^*)[(2k-1)(1-F(r_{POST}^*))f(r_{POST}^*) - 2F(r_{POST}^*)f(r_{POST}^*) - \\
&- (k-1)(1-F(r_{POST}^*))f(r_{POST}^*) - kF(r_{POST}^*)f'(r_{POST}^*)\frac{1-F(r_{POST}^*)}{f(r_{POST}^*)}] = \\
&= kF^{2k-2}(r_{POST}^*)[k(1-F(r_{POST}^*))f(r_{POST}^*) - 2F(r_{POST}^*)f(r_{POST}^*) - \\
&- kF(r_{POST}^*)f'(r_{POST}^*)\frac{1-F(r_{POST}^*)}{f(r_{POST}^*)}]
\end{aligned} \tag{A.10}$$

When both intermediaries set this reserve price, the center's ex-ante expected revenue is:

$$\begin{aligned}
revenue(r_{POST}^*) &= 2F^k(r_{POST}^*)(1-F^k(r_{POST}^*))[r_{POST}^* - \frac{1-F(r_{POST}^*)}{f(r_{POST}^*)}] + \\
&+ \int_{r_{POST}^*}^1 2y(1-F^k(y))kF^{k-1}(y)f(y)dy
\end{aligned} \tag{A.11}$$

Taking the first-order derivative w.r.t. r_{POST}^* yields:

$$\begin{aligned}
\frac{\partial \text{revenue}(r_{POST}^*)}{\partial r_{POST}^*} &= 2\{[kF^{k-1}(r_{POST}^*)(1 - F^k(r_{POST}^*)) - kF^{k-1}(r_{POST}^*)F^k(r_{POST}^*)]f(r_{POST}^*)r_{POST}^* - \\
&- k(1 - 2F^k(r_{POST}^*))(1 - F(r_{POST}^*)) - k(1 - F^k(r_{POST}^*))F^{k-1}(r_{POST}^*)f(r_{POST}^*) + \\
&+ F^k(r_{POST}^*)(1 - F^k(r_{POST}^*))[2 + \frac{(1 - F(r_{POST}^*))f'(r_{POST}^*)}{f^2(r_{POST}^*)}]\} = \\
&= 2F^{k-1}(r_{POST}^*)\{-kF^k(r_{POST}^*)f(r_{POST}^*)r_{POST}^* - k(1 - F(r_{POST}^*))(1 - 2F^k(r_{POST}^*)) + \\
&+ F(r_{POST}^*)(1 - F^k(r_{POST}^*))[2 + \frac{(1 - F(r_{POST}^*))f'(r_{POST}^*)}{f^2(r_{POST}^*)}]\} = \\
&= 2F^{k-1}(r_{POST}^*)\{kF^k(r_{POST}^*)[1 - F(r_{POST}^*) - r_{POST}^*f(r_{POST}^*)] + (1 - F^k(r_{POST}^*))[2F(r_{POST}^*) - \\
&- k(1 - F(r_{POST}^*)) + F(r_{POST}^*)(1 - F(r_{POST}^*))\frac{f'(r_{POST}^*)}{f^2(r_{POST}^*)}\}\} = \\
&= 2F^{k-1}(r_{POST}^*)\{kF^k(r_{POST}^*)[1 - F(r_{POST}^*) - r_{POST}^*f(r_{POST}^*)] - (1 - F^k(r_{POST}^*))[k - \\
&- (k + 2)F(r_{POST}^*) - F(r_{POST}^*)(1 - F(r_{POST}^*))\frac{f'(r_{POST}^*)}{f^2(r_{POST}^*)}\} \tag{A.12}
\end{aligned}$$

Appendix B

Supplement for Chapter 6

B.1 Derivations for Buyer PRE - PRE Duopoly Intermediary Selection

In this section, we will show all the derivations used in Section 6.1.1. Wherever needed, we will use the relationships between the highest- and second-highest-order statistics¹ among $K - 1$ draws from a distribution F with density f , $F_1^{(K-1)}(\cdot)$ and $F_2^{(K-1)}(\cdot)$ respectively, as well as their joint density, $f_{1,2}^{(K-1)}(\cdot)$ (Krishna, 2010):

$$F_1^{(K-1)}(y) = F^{K-1}(y) \quad (\text{B.1})$$

$$f_1^{(K-1)}(y) = (K-1)F^{K-2}(y)f(y) \quad (\text{B.2})$$

$$F_2^{(K-1)}(y) = (K-1)F^{K-2}(y) - (K-2)F^{K-1}(y) \quad (\text{B.3})$$

$$f_2^{(K-1)}(y) = (K-1)(K-2)(1-F(y))F^{K-3}(y)f(y) \quad (\text{B.4})$$

$$f_{1,2}^{(K-1)}(y_1, y_2) = (K-1)(K-2)f(y_1)f(y_2)F^{K-3}(y_2) = (K-1)f(y_1)f_1^{(K-2)}(y_2) \quad (\text{B.5})$$

B.1.1 Pure-Strategy Bayes-Nash Equilibria with Multiple Cut-Off Points

We consider m cut-off points $w_1, w_2, \dots, w_k, m \geq 1$, where $\Pi_{PRE}^L(w_i) = \Pi_{PRE}^H(w_i)$ so that buyers switch their strategies, $\theta_1, \theta_2, \dots, \theta_{m+1} \in \{0, 1\}$ in the sub-intervals $[r_{PRE}^H, w_1]$, $[w_1, w_2], \dots, [w_k, 1]$. Given that at r_{PRE}^H , the expected utility from the high-reserve intermediary is zero, whereas for the low-reserve is, in general, positive, $\theta_1 = 1$ always. In what follows, we provide a closed form representation of the expected utility from both intermediaries and then take their difference $\Pi_{PRE}^L - \Pi_{PRE}^H$, which we denote D .

¹In classical statistics, these are the K - and $(K - 1)$ -order statistics, however economists tend to denote these as the first- and second-order statistics respectively (we refer the reader, for example, to Appendix C of Krishna (2010)).

So, there are three cases: (i) valuations in the interval $[r_{PRE}^H, w_1]$, (ii) valuations in a random interval $[w_\lambda, w_{\lambda+1}]$, $\lambda \in \{1, \dots, m-1\}$, and (iii) valuations greater or equal to the last cut-off point up to 1. Then, we can write the expected utilities as ²:

$$\begin{aligned} \Pi_{PRE}^L(r_{PRE}^H \leq v < w_1) = & \int_{r_{PRE}^L}^v F_1^{(K-1)}(y)dy + (K-1) \sum_{i=1}^{m-1} \left\{ (1-\theta_{i+1})(F(w_{i+1}) - \right. \\ & \left. - F(w_i)) \int_{r_{PRE}^H}^v (v-y)f_1^{(K-2)}(y)dy \right\} + (K-1)(1-\theta_{m+1})(1-F(w_k)) \int_{r_{PRE}^H}^v (v-y)f_1^{(K-2)}(y)dy \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} \Pi_{PRE}^H(r_{PRE}^H \leq v < w_1) = & (v - r_{PRE}^H)F_1^{(K-1)}(r_{PRE}^H) + (K-1)(v - r_{PRE}^H)F_1^{(K-2)}(r_{PRE}^H)[F(w) - \\ & - F(r_{PRE}^H) + \sum_{i=1}^{m-1} \left\{ \theta_{i+1}(F(w_{i+1}) - F(w_i)) \right\} + \theta_{m+1}(1 - F(w_k))] \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} D_0 = (\Pi_{PRE}^L - \Pi_{PRE}^H)(r_{PRE}^H \leq v < w_1) = & \int_{r_{PRE}^L}^v F_1^{(K-1)}(y)dy - (v - r_{PRE}^H)F_1^{(K-1)}(r_{PRE}^H) - \\ & - (K-1)(v - r_{PRE}^H)F_1^{(K-2)}(r_{PRE}^H)[1 - F(r_{PRE}^H)] + (K-1) \left[\sum_{i=1}^{m-1} \left\{ (1-\theta_{i+1})(F(w_{i+1}) - F(w_i)) \right\} + \right. \\ & \left. + (1-\theta_{m+1})(1 - F(w_k)) \right] \int_{r_{PRE}^H}^v F_1^{(K-2)}(y)dy \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} \frac{\partial D_0}{\partial v} = & F_1^{(K-1)}(v) - F_1^{(K-1)}(r_{PRE}^H) - (K-1)F_1^{(K-2)}(r_{PRE}^H)(1 - F(r_{PRE}^H)) + \\ & + (K-1) \left[\sum_{i=1}^{m-1} \left\{ (1-\theta_{i+1})(F(w_{i+1}) - F(w_i)) \right\} + (1-\theta_{m+1})(1 - F(w_k)) \right] F_1^{(K-2)}(v) \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} \frac{\partial^2 D_0}{\partial v^2} = & f_1^{(K-1)}(v) + (K-1) \left[\sum_{i=1}^{m-1} \left\{ (1-\theta_{i+1})(F(w_{i+1}) - F(w_i)) \right\} + \right. \\ & \left. + (1-\theta_{m+1})(1 - F(w_k)) \right] f_1^{(K-2)}(v) \geq 0 \end{aligned} \quad (\text{B.10})$$

²For all double integrals, the outer part refers to y_2 (second highest valuation) and the inner part to y_1 (highest valuation).

$$\begin{aligned}
\Pi_{PRE}^L(w_\lambda \leq v < w_{\lambda+1}) = & \int_{r_{PRE}^L}^{w_1} F_1^{(K-1)}(y) dy + (v - w_1) F_1^{(K-1)}(w_1) + \\
& + \sum_{i=1}^{\lambda-1} \left\{ \theta_{i+1}(v - w_{i+1}) F_1^{(K-1)}(w_{i+1}) \right\} - \sum_{i=1}^{\lambda-1} \left\{ \theta_{i+1}(v - w_i) F_1^{(K-1)}(w_i) \right\} - \\
& - \theta_{\lambda+1}(v - w_\lambda) F_1^{(K-1)}(w_\lambda) + \\
& + \sum_{i=1}^{\lambda-1} \left\{ \theta_{i+1} \int_{w_i}^{w_{i+1}} F_1^{(K-1)}(y) dy \right\} + \theta_{\lambda+1} \int_{w_\lambda}^v F_1^{(K-1)}(y) dy + \\
& + (K-1) \sum_{i=1}^{m-1} \left\{ (1 - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{r_{PRE}^H}^{w_1} (v - y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1)(1 - \theta_{m+1})(1 - F(w_k)) \int_{r_{PRE}^H}^{w_1} (v - y) f_1^{(K-2)}(y) dy + \\
& + (K-1) \sum_{j=1}^{\lambda-1} \left\{ \theta_{j+1}(1 - \theta_{j+1}) \int_{w_j}^{w_{j+1}} (v - y)(F(w_{j+1}) - F(y)) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1) \sum_{j=1}^{\lambda-1} \sum_{i=j+1}^{m-1} \left\{ \theta_{j+1}(1 - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{w_j}^{w_{j+1}} (v - y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1) \sum_{j=1}^{\lambda-1} \left\{ \theta_{j+1}(1 - \theta_{m+1})(1 - F(w_k)) \int_{w_j}^{w_{j+1}} (v - y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1)\theta_{\lambda+1}(1 - \theta_{\lambda+1}) \int_{w_\lambda}^v (v - y)(F(w_{\lambda+1}) - F(y)) f_1^{(K-2)}(y) dy + \\
& + (K-1) \sum_{i=\lambda+1}^{m-1} \left\{ \theta_{\lambda+1}(1 - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{w_\lambda}^v (v - y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1)\theta_{\lambda+1}(1 - \theta_{m+1})(1 - F(w_k)) \int_{w_\lambda}^v (v - y) f_1^{(K-2)}(y) dy
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
\Pi_{PRE}^H(w_\lambda \leq v < w_{\lambda+1}) = & (v - r_{PRE}^H) F_1^{(K-1)}(r_{PRE}^H) + \sum_{i=1}^{\lambda-1} (1 - \theta_{i+1})(v - w_{i+1}) F_1^{(K-1)}(w_{i+1}) - \\
& - \sum_{i=1}^{\lambda-1} (1 - \theta_{i+1})(v - w_i) F_1^{(K-1)}(w_i) - (1 - \theta_{\lambda+1})(v - w_\lambda) F_1^{(K-1)}(w_\lambda) + \\
& + (K-1)(v - r_{PRE}^H) F_1^{(K-2)}(r_{PRE}^H) [F(w_1) - F(r_{PRE}^H) + \sum_{i=1}^{m-1} \theta_{i+1}(F(w_{i+1}) - F(w_i)) + \\
& + \theta_{m+1}(1 - F(w_k))] + \sum_{i=1}^{\lambda-1} (1 - \theta_{i+1}) \int_{w_i}^{w_{i+1}} F_1^{(K-1)}(y) dy + (1 - \theta_{\lambda+1}) \int_{w_\lambda}^v F_1^{(K-1)}(y) dy +
\end{aligned}$$

$$\begin{aligned}
& + (K-1) \sum_{j=1}^{\lambda-1} \left\{ (1-\theta_{j+1})\theta_{j+1} \int_{w_j}^{w_{j+1}} (v-y)(F(w_{j+1}) - F(y)) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1) \sum_{j=1}^{\lambda-1} \sum_{i=j+1}^{m-1} \left\{ (1-\theta_{j+1})\theta_{i+1}(F(w_{i+1}) - F(w_i)) \int_{w_j}^{w_{j+1}} (v-y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1) \sum_{j=1}^{\lambda-1} \left\{ (1-\theta_{j+1})\theta_{m+1}(1-F(w_k)) \int_{w_j}^{w_{j+1}} (v-y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1)(1-\theta_{\lambda+1})\theta_{\lambda+1} \int_{w_\lambda}^v (v-y)(F(w_{\lambda+1}) - F(y)) f_1^{(K-2)}(y) dy + \\
& + (K-1) \sum_{i=\lambda+1}^{m-1} \left\{ (1-\theta_{\lambda+1})\theta_{i+1}(F(w_{i+1}) - F(w_i)) \int_{w_\lambda}^v (v-y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1)(1-\theta_{\lambda+1})\theta_{m+1}(1-F(w_k)) \int_{w_\lambda}^v (v-y) f_1^{(K-2)}(y) dy
\end{aligned} \tag{B.12}$$

$$\begin{aligned}
D_\lambda = (\Pi_{PRE}^L - \Pi_{PRE}^H)(w_\lambda < v \leq w_{\lambda+1}) &= \int_{r_{PRE}^L}^{w_1} F_1^{(K-1)}(y) dy - (v - r_{PRE}^H) F_1^{(K-1)}(r_{PRE}^H) + \\
& + (v - w_1) F_1^{(K-1)}(w_1) + \sum_{i=1}^{\lambda-1} \left\{ (2\theta_{i+1} - 1)[(v - w_{i+1}) F_1^{(K-1)}(w_{i+1}) - (v - w_i) F_1^{(K-1)}(w_i)] \right\} - \\
& - (2\theta_{\lambda+1} - 1)(v - w_\lambda) F_1^{(K-1)}(w_\lambda) - (K-1)(v - r_{PRE}^H) F_1^{(K-2)}(r_{PRE}^H) [1 - F(r_{PRE}^H)] + \\
& + \sum_{i=1}^{\lambda-1} \left\{ (2\theta_{i+1} - 1) \int_{w_i}^{w_{i+1}} F_1^{(K-1)}(y) dy \right\} + (2\theta_{\lambda+1} - 1) \int_{w_\lambda}^v F_1^{(K-1)}(y) dy + \\
& + (K-1)[(v - w_1) F_1^{(K-2)}(w_1) + \int_{r_{PRE}^H}^{w_1} F_1^{(K-2)}(y) dy] \sum_{i=1}^{m-1} \left\{ (1 - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \right\} + \\
& + (1 - \theta_{m+1})(1 - F(w_k)) + \\
& + (K-1) \sum_{j=1}^{\lambda-1} \sum_{i=j+1}^{m-1} \left\{ (\theta_{j+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{w_j}^{w_{j+1}} (v-y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1) \sum_{j=1}^{\lambda-1} \left\{ (\theta_{j+1} - \theta_{m+1})(1 - F(w_k)) \int_{w_j}^{w_{j+1}} (v-y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1) \sum_{i=\lambda+1}^{m-1} \left\{ (\theta_{\lambda+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{w_\lambda}^v (v-y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1)(\theta_{\lambda+1} - \theta_{m+1})(1 - F(w_k)) \int_{w_\lambda}^v (v-y) f_1^{(K-2)}(y) dy
\end{aligned} \tag{B.13}$$

$$\begin{aligned}
\frac{\partial D_\lambda}{\partial v} = & -F_1^{(K-1)}(r_{PRE}^H) + F_1^{(K-1)}(w_1) + \sum_{i=1}^{\lambda-1} \left\{ (2\theta_{i+1} - 1)[F_1^{(K-1)}(w_{i+1}) - F_1^{(K-1)}(w_i)] \right\} - \\
& -(2\theta_{\lambda+1} - 1)F_1^{(K-1)}(w_\lambda) - (K-1)F_1^{(K-2)}(r_{PRE}^H)(1 - F(r_{PRE}^H)) + (2\theta_{\lambda+1} - 1)F_1^{(K-1)}(v) + \\
& +(K-1)F_1^{(K-2)}(w_1) \left[\sum_{i=1}^{m-1} \left\{ (1 - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \right\} + (1 - \theta_{m+1})(1 - F(w_k)) \right] + \\
& +(K-1) \sum_{j=1}^{\lambda-1} \sum_{i=j+1}^{m-1} \left\{ (\theta_{j+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{w_j}^{w_{j+1}} f_1^{(K-2)}(y) dy \right\} + \\
& +(K-1) \sum_{j=1}^{\lambda-1} \left\{ (\theta_{j+1} - \theta_{m+1})(1 - F(w_k)) \int_{w_j}^{w_{j+1}} f_1^{(K-2)}(y) dy \right\} + \\
& +(K-1) \sum_{i=\lambda+1}^{m-1} \left\{ (\theta_{\lambda+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{w_\lambda}^v f_1^{(K-2)}(y) dy \right\} + \\
& +(K-1)(\theta_{\lambda+1} - \theta_{m+1})(1 - F(w_k)) \int_{w_\lambda}^v f_1^{(K-2)}(y) dy
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
\frac{\partial^2 D_\lambda}{\partial v^2} = & (2\theta_{\lambda+1} - 1)f_1^{(K-1)}(v) + (K-1) \left[\sum_{i=\lambda+1}^{m-1} \left\{ (\theta_{\lambda+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \right\} + \right. \\
& \left. + (\theta_{\lambda+1} - \theta_{m+1})(1 - F(w_k)) \right] f_1^{(K-2)}(v)
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
\Pi_{PRE}^L(v \geq w_k) = & \int_{r_{PRE}^L}^{w_1} F_1^{(K-1)}(y) dy + (v - w_1)F_1^{(K-1)}(w_1) + \\
& + \sum_{i=1}^{m-1} \left\{ \theta_{i+1}(v - w_{i+1})F_1^{(K-1)}(w_{i+1}) \right\} - \sum_{i=1}^{m-1} \left\{ \theta_{i+1}(v - w_i)F_1^{(K-1)}(w_i) \right\} - \\
& - \theta_{m+1}(v - w_k)F_1^{(K-1)}(w_k) + \sum_{i=1}^{m-1} \left\{ \theta_{i+1} \int_{w_i}^{w_{i+1}} F_1^{(K-1)}(y) dy \right\} + \theta_{m+1} \int_{w_k}^v F_1^{(K-1)}(y) dy + \\
& +(K-1) \sum_{i=1}^{m-1} \left\{ (1 - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{r_{PRE}^H}^{w_1} (v - y) f_1^{(K-2)}(y) dy \right\} + \\
& +(K-1)(1 - \theta_{m+1})(1 - F(w_k)) \int_{r_{PRE}^H}^{w_1} (v - y) f_1^{(K-2)}(y) dy + \\
& +(K-1) \sum_{j=1}^{m-1} \left\{ \theta_{j+1}(1 - \theta_{j+1}) \int_{w_j}^{w_{j+1}} (v - y)(F(w_{j+1}) - F(y)) f_1^{(K-2)}(y) dy \right\} + \\
& +(K-1) \sum_{j=1}^{m-1} \sum_{i=j+1}^{m-1} \left\{ \theta_{j+1}(1 - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{w_j}^{w_{j+1}} (v - y) f_1^{(K-2)}(y) dy \right\} +
\end{aligned}$$

$$\begin{aligned}
& + (K-1) \sum_{j=1}^{m-1} \left\{ \theta_{j+1}(1-\theta_{m+1})(1-F(w_k)) \int_{w_j}^{w_{j+1}} (v-y) f_1^{(K-2)}(y) dy \right\} \\
& + (K-1)\theta_{m+1}(1-\theta_{m+1}) \int_{w_k}^v (v-y)(1-F(y)) f_1^{(K-2)}(y) dy
\end{aligned} \tag{B.16}$$

$$\begin{aligned}
\Pi_{PRE}^H(v \geq w_k) & = (v - r_{PRE}^H) F_1^{(K-1)}(r_{PRE}^H) + \sum_{i=1}^{m-1} (1-\theta_{i+1})(v-w_{i+1}) F_1^{(K-1)}(w_{i+1}) - \\
& - \sum_{i=1}^{m-1} (1-\theta_{i+1})(v-w_i) F_1^{(K-1)}(w_i) - (1-\theta_{m+1})(v-w_k) F_1^{(K-1)}(w_k) + \\
& + (K-1)(v - r_{PRE}^H) F_1^{(K-2)}(r_{PRE}^H) [F(w_1) - F(r_{PRE}^H) + \sum_{i=1}^{m-1} \theta_{i+1}(F(w_{i+1}) - F(w_i)) + \\
& + \theta_{m+1}(1-F(w_k))] + \sum_{i=1}^{m-1} (1-\theta_{i+1}) \int_{w_i}^{w_{i+1}} F_1^{(K-1)}(y) dy + (1-\theta_{m+1}) \int_{w_k}^v F_1^{(K-1)}(y) dy + \\
& + (K-1) \sum_{j=1}^{m-1} \left\{ (1-\theta_{j+1}) \theta_{j+1} \int_{w_j}^{w_{j+1}} (v-y)(F(w_{j+1}) - F(y)) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1) \sum_{j=1}^{m-1} \sum_{i=j+1}^{m-1} \left\{ (1-\theta_{j+1}) \theta_{i+1} (F(w_{i+1}) - F(w_i)) \int_{w_j}^{w_{j+1}} (v-y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1) \sum_{j=1}^{m-1} \left\{ (1-\theta_{j+1}) \theta_{m+1} (1-F(w_k)) \int_{w_j}^{w_{j+1}} (v-y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1)(1-\theta_{m+1}) \theta_{m+1} \int_{w_k}^v (v-y)(1-F(y)) f_1^{(K-2)}(y) dy
\end{aligned} \tag{B.17}$$

$$\begin{aligned}
D_k & = (\Pi_{PRE}^L - \Pi_{PRE}^H)(v > w_k) = \int_{r_{PRE}^L}^{w_1} F_1^{(K-1)}(y) dy - (v - r_{PRE}^H) F_1^{(K-1)}(r_{PRE}^H) + \\
& + (v-w_1) F_1^{(K-1)}(w_1) + \sum_{i=1}^{m-1} \left\{ (2\theta_{i+1}-1)[(v-w_{i+1}) F_1^{(K-1)}(w_{i+1}) - (v-w_i) F_1^{(K-1)}(w_i)] \right\} - \\
& - (2\theta_{m+1}-1)(v-w_k) F_1^{(K-1)}(w_k) - (K-1)(v - r_{PRE}^H) F_1^{(K-2)}(r_{PRE}^H) [1 - F(r_{PRE}^H)] + \\
& + \sum_{i=1}^{m-1} \left\{ (2\theta_{i+1}-1) \int_{w_i}^{w_{i+1}} F_1^{(K-1)}(y) dy \right\} + (2\theta_{m+1}-1) \int_{w_k}^v F_1^{(K-1)}(y) dy + \\
& + (K-1)[(v-w_1) F_1^{(K-2)}(w_1) + \int_{r_{PRE}^H}^{w_1} F_1^{(K-2)}(y) dy] \sum_{i=1}^{m-1} \left\{ (1-\theta_{i+1})(F(w_{i+1}) - F(w_i)) \right\} + \\
& + (1-\theta_{m+1})(1-F(w_k))
\end{aligned}$$

$$\begin{aligned}
& + (K-1) \sum_{j=1}^{m-1} \sum_{i=j+1}^{m-1} \left\{ (\theta_{j+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{w_j}^{w_{j+1}} (v-y) f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1) \sum_{j=1}^{m-1} \left\{ (\theta_{j+1} - \theta_{m+1})(1 - F(w_k)) \int_{w_j}^{w_{j+1}} (v-y) f_1^{(K-2)}(y) dy \right\}
\end{aligned} \tag{B.18}$$

$$\begin{aligned}
\frac{\partial D_k}{\partial v} = & -F_1^{(K-1)}(r_{PRE}^H) + F_1^{(K-1)}(w_1) + \sum_{i=1}^{m-1} \left\{ (2\theta_{i+1} - 1)[F_1^{(K-1)}(w_{i+1}) - F_1^{(K-1)}(w_i)] \right\} - \\
& - (2\theta_{m+1} - 1)F_1^{(K-1)}(w_k) - (K-1)F_1^{(K-2)}(r_{PRE}^H)(1 - F(r_{PRE}^H)) + (2\theta_{m+1} - 1)F_1^{(K-1)}(v) + \\
& + (K-1)F_1^{(K-2)}(w_1) \left[\sum_{i=1}^{m-1} \left\{ (1 - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \right\} + (1 - \theta_{m+1})(1 - F(w_k)) \right] + \\
& + (K-1) \sum_{j=1}^{m-1} \sum_{i=j+1}^{m-1} \left\{ (\theta_{j+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \int_{w_j}^{w_{j+1}} f_1^{(K-2)}(y) dy \right\} + \\
& + (K-1) \sum_{j=1}^{m-1} \left\{ (\theta_{j+1} - \theta_{m+1})(1 - F(w_k)) \int_{w_j}^{w_{j+1}} f_1^{(K-2)}(y) dy \right\}
\end{aligned} \tag{B.19}$$

$$\frac{\partial^2 D_k}{\partial v^2} = (2\theta_{m+1} - 1)f_1^{(K-1)}(v) \tag{B.20}$$

B.1.2 Mixed-Strategy Bayes-Nash Equilibria with Arbitrary Pure Strategies Before and After Randomizing

Suppose that buyers follow a pure strategy $\theta(v) = \theta_p(v)$ for valuations $v \in [r_{PRE}^H, w]$, a mixed strategy $\theta(v) = \theta_m(v) \in (0, 1)$ for valuations $v \in [w, a]$, and a pure strategy $\theta(v) = \theta^*(v)$ for all $v \in (a, 1]$. Then, the expected utility of a buyer with valuation $v \in [w, a]$ from the low- and high-reserve intermediary, Π_{PRE}^L and Π_{PRE}^H respectively, will be:

$$\begin{aligned}
\Pi_{PRE}^L(w \leq v \leq a) &= (v - r_{PRE}^L) F^{K-1}(r_{PRE}^L) + \int_{r_{PRE}^L}^{r_{PRE}^H} (v - y) f_1^{(K-1)}(y) dy + \\
&+ \int_{r_{PRE}^H}^w (v - y) \theta_p(y) f_1^{(K-1)}(y) dy + \int_w^v (v - y) \theta_m(y) f_1^{(K-1)}(y) dy + \\
&+ \int_{r_{PRE}^H}^w \int_{y_2}^w (v - y_2)(1 - \theta_p(y_1)) \theta_p(y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_{r_{PRE}^H}^w \int_w^a (v - y_2)(1 - \theta_m(y_1)) \theta_p(y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_{r_{PRE}^H}^w \int_a^1 (v - y_2)(1 - \theta^*(y_1)) \theta_p(y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_w^v \int_{y_2}^a (v - y_2)(1 - \theta_m(y_1)) \theta_m(y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_w^v \int_a^1 (v - y_2)(1 - \theta^*(y_1)) \theta_m(y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 = \\
&= \int_{r_{PRE}^L}^{r_{PRE}^H} F^{K-1}(y) dy + (v - r_{PRE}^H) F^{K-1}(r_{PRE}^H) + \\
&+ \int_{r_{PRE}^H}^w (v - y) \theta_p(y) f_1^{(K-1)}(y) dy + \int_w^v (v - y) \theta_m(y) f_1^{(K-1)}(y) dy + \\
&+ \int_{r_{PRE}^H}^w \int_{y_2}^w (v - y_2)(1 - \theta_p(y_1)) \theta_p(y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_{r_{PRE}^H}^w \int_w^a (v - y_2)(1 - \theta_m(y_1)) \theta_p(y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_{r_{PRE}^H}^w \int_a^1 (v - y_2)(1 - \theta^*(y_1)) \theta_p(y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_w^v \int_{y_2}^a (v - y_2)(1 - \theta_m(y_1)) \theta_m(y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_w^v \int_a^1 (v - y_2)(1 - \theta^*(y_1)) \theta_m(y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2
\end{aligned} \tag{B.21}$$

$$\begin{aligned}
\Pi_{PRE}^H(w \leq v \leq a) &= (v - r_{PRE}^H) F^{K-1}(r_{PRE}^H) + \\
&+ \int_{r_{PRE}^H}^w (v - y)(1 - \theta_p(y)) f_1^{(K-1)}(y) dy + \int_w^v (v - y)(1 - \theta_m(y)) f_1^{(K-1)}(y) dy + \\
&+ (K-1)(v - r_{PRE}^H) F^{K-2}(r_{PRE}^H) [\int_{r_{PRE}^H}^w \theta_p(y) f(y) dy + \int_w^a \theta_m(y) f(y) dy + \int_a^1 \theta^*(y) f(y) dy] + \\
&+ \int_{r_{PRE}^H}^w \int_{y_2}^w (v - y_2)\theta_p(y_1)(1 - \theta_p(y_2)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_{r_{PRE}^H}^w \int_w^a (v - y_2)\theta_m(y_1)(1 - \theta_p(y_2)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_{r_{PRE}^H}^w \int_a^1 (v - y_2)\theta^*(y_1)(1 - \theta_p(y_2)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_w^v \int_{y_2}^a (v - y_2)\theta_m(y_1)(1 - \theta_m(y_2)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_w^v \int_a^1 (v - y_2)\theta^*(y_1)(1 - \theta_m(y_2)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2
\end{aligned} \tag{B.22}$$

Hence, their difference will be:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a) &= \int_{r_{PRE}^L}^{r_{PRE}^H} F^{K-1}(y) dy + \\
&+ \int_{r_{PRE}^H}^w (v - y)(2\theta_p(y) - 1) f_1^{(K-1)}(y) dy + \int_w^v (v - y)(2\theta_m(y) - 1) f_1^{(K-1)}(y) dy - \\
&- (K-1)(v - r_{PRE}^H) F^{K-2}(r_{PRE}^H) [\int_{r_{PRE}^H}^w \theta_p(y) f(y) dy + \int_w^a \theta_m(y) f(y) dy + \int_a^1 \theta^*(y) f(y) dy] + \\
&+ \int_{r_{PRE}^H}^w \int_{y_2}^w (v - y_2)(\theta_p(y_2) - \theta_p(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_{r_{PRE}^H}^w \int_w^a (v - y_2)(\theta_p(y_2) - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_{r_{PRE}^H}^w \int_a^1 (v - y_2)(\theta_p(y_2) - \theta^*(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_w^v \int_{y_2}^a (v - y_2)(\theta_m(y_2) - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_w^v \int_a^1 (v - y_2)(\theta_m(y_2) - \theta^*(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2
\end{aligned} \tag{B.23}$$

Taking the first and second derivatives of the utility difference $\Pi_{PRE}^L - \Pi_{PRE}^H$ yields the following equations:

$$\begin{aligned} \frac{\partial(\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a)}{\partial v} &= \int_{r_{PRE}^H}^w (2\theta_p(y) - 1)f_1^{(K-1)}(y)dy + \int_w^v (2\theta_m(y) - 1)f_1^{(K-1)}(y)dy - \\ &- (K-1)F^{K-2}(r_{PRE}^H)[\int_{r_{PRE}^H}^w \theta_p(y)f(y)dy + \int_w^a \theta_m(y)f(y)dy + \int_a^1 \theta^*(y)f(y)dy] + \\ &+ \int_{r_{PRE}^H}^w \int_{y_2}^w (\theta_p(y_2) - \theta_p(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\ &+ \int_{r_{PRE}^H}^w \int_w^a (\theta_p(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\ &+ \int_{r_{PRE}^H}^w \int_a^1 (\theta_p(y_2) - \theta^*(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\ &+ \int_w^v \int_{y_2}^a (\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\ &+ \int_w^v \int_a^1 (\theta_m(y_2) - \theta^*(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \end{aligned} \tag{B.24}$$

$$\begin{aligned} \frac{\partial^2(\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a)}{\partial v^2} &= (2\theta_m(v) - 1)f_1^{(K-1)}(v) + \\ &+ \int_v^a (\theta_m(v) - \theta_m(y))f_{1,2}^{(K-1)}(y, v)dy + \int_a^1 (\theta_m(v) - \theta^*(y))f_{1,2}^{(K-1)}(y, v)dy = \\ &= (2\theta_m(v) - 1)f_1^{(K-1)}(v) + \theta_m(v) \int_v^1 f_{1,2}^{(K-1)}(y, v)dy - \\ &- \int_v^a \theta_m(y)f_{1,2}^{(K-1)}(y, v)dy - \int_a^1 \theta^*(y)f_{1,2}^{(K-1)}(y, v)dy = \\ &= (K-1)(2\theta_m(v) - 1)F^{K-2}(v)f(v) + (K-1)(K-2)\theta_m(v)f(v)F^{K-3}(v)(1 - F(v)) - \\ &- (K-1)(K-2)f(v)F^{K-3}(v)[\int_v^a \theta_m(y)f(y)dy + \int_a^1 \theta^*(y)f(y)dy] = \\ &= (K-1)f(v)F^{K-3}(v)\{\theta_m(v)[2F(v) + (K-2)(1 - F(v))] - F(v) - \\ &- (K-2)[\int_v^a \theta_m(y)f(y)dy + \int_a^1 \theta^*(y)f(y)dy]\} \end{aligned} \tag{B.25}$$

B.1.3 Mixed-Strategy Bayes-Nash Equilibria with Multiple Cut-Off Points Before Randomizing

In what follows, we will analytically derive the closed-form expression of the expected utilities from the low-reserve and high-reserve intermediary, Π_{PRE}^L and Π_{PRE}^H respectively, when buyers follow a pure strategy $\theta(v) = \theta_1 = 1$ for all $v \in [r_{PRE}^H, w_1]$, $\theta(v) = \theta_2 = 0$ for all $v \in [w_1, w_2]$ and so on, $\theta(v) = \theta_{\sigma'} \in \{0, 1\}$ for all $v \in [w_{\sigma'-1}, w_{\sigma'}]$, and then a mixed strategy $\theta(v) = \theta_m(v) \in (0, 1)$ for all $v \in [w, a]$, and $\theta(v) = \theta^*(v) \in \{0, 1\}$ if $v \in (a, 1]$.

For a buyer whose valuation $v \in [w_\lambda, w_{\lambda+1})$ for some $\lambda = 1, \dots, \sigma' - 1$, the expected utilities from the low- and high-reserve intermediaries will be:

$$\begin{aligned}
 \Pi_{PRE}^L(w_\lambda \leq v < w_{\lambda+1}) = & (v - r_{PRE}^L) F^{K-1}(r_{PRE}^L) + \int_{r_{PRE}^L}^{w_1} (v - y) f_1^{(K-1)}(y) dy + \\
 & + \sum_{i=1}^{\lambda-1} \theta_{i+1} \int_{w_i}^{w_{i+1}} (v - y) f_1^{(K-1)}(y) dy + \theta_{\lambda+1} \int_{w_\lambda}^v (v - y) f_1^{(K-1)}(y) dy + \\
 & + \int_{r_{PRE}^H}^{w_1} \int_{y_2}^{w_1} (v - y_2)(1 - 1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
 & + \sum_{i=1}^{\sigma'-1} \left\{ (1 - \theta_{i+1}) \int_{r_{PRE}^H}^{w_1} \int_{w_i}^{w_{i+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \\
 & + \int_{r_{PRE}^H}^{w_1} \int_{w_{\sigma'}}^a (v - y_2)(1 - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
 & + \int_{r_{PRE}^H}^{w_1} \int_a^1 (v - y_2)(1 - \theta^*(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
 & + \sum_{j=1}^{\lambda-1} \left\{ (1 - \theta_{j+1}) \theta_{j+1} \int_{w_j}^{w_{j+1}} \int_{y_2}^{w_{j+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \right. \\
 & \left. + \sum_{i=j+1}^{\sigma'-1} \left\{ (1 - \theta_{i+1}) \theta_{j+1} \int_{w_j}^{w_{j+1}} \int_{w_i}^{w_{i+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \right. \\
 & \left. + \theta_{j+1} \int_{w_j}^{w_{j+1}} \int_{w_{\sigma'}}^a (v - y_2)(1 - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \right. \\
 & \left. + \theta_{j+1} \int_{w_j}^{w_{j+1}} \int_a^1 (v - y_2)(1 - \theta^*(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \\
 & + (1 - \theta_{\lambda+1}) \theta_{\lambda+1} \int_{w_\lambda}^v \int_{y_2}^{w_{\lambda+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
 & + \sum_{i=\lambda+1}^{\sigma'-1} \left\{ (1 - \theta_{i+1}) \theta_{\lambda+1} \int_{w_\lambda}^v \int_{w_i}^{w_{i+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \\
 & + \theta_{\lambda+1} \int_{w_\lambda}^v \int_{w_{\sigma'}}^a (v - y_2)(1 - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
 & + \theta_{\lambda+1} \int_{w_\lambda}^v \int_a^1 (v - y_2)(1 - \theta^*(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2
 \end{aligned} \tag{B.26}$$

$$\begin{aligned}
\Pi_{PRE}^H(w_\lambda \leq v < w_{\lambda+1}) &= (v - r_{PRE}^H) F^{K-1}(r_{PRE}^H) + \sum_{i=1}^{\lambda-1} \left\{ (1 - \theta_{i+1}) \int_{w_i}^{w_{i+1}} (v - y) f_1^{(K-1)}(y) dy \right\} + \\
&+ (1 - \theta_{\lambda+1}) \int_{w_\lambda}^v (v - y) f_1^{(K-1)}(y) dy + (K-1)(v - r_{PRE}^H) F^{K-2}(r_{PRE}^H) [F(w_1) - F(r_{PRE}^H)] + \\
&+ \sum_{i=1}^{\sigma'-1} \left\{ \theta_{i+1} (F(w_{i+1}) - F(w_i)) \right\} + \int_{w'_\sigma}^a \theta_m(y) f(y) dy + \int_a^1 \theta^*(y) f(y) dy + \\
&+ \int_{r_{PRE}^H}^{w_1} \int_{y_2}^{w_1} (v - y_2) (1 - 1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \sum_{i=1}^{\sigma'-1} \left\{ \theta_{i+1} \int_{r_{PRE}^H}^{w_1} \int_{w_i}^{w_{i+1}} (v - y_2) (1 - 1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \\
&+ \int_{r_{PRE}^H}^{w_1} \int_{w'_{\sigma'}}^a (v - y_2) \theta_m(y_1) (1 - 1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \int_{r_{PRE}^H}^{w_1} \int_a^1 (v - y_2) \theta^*(y_1) (1 - 1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \sum_{j=1}^{\lambda-1} \left\{ \theta_{j+1} (1 - \theta_{j+1}) \int_{w_j}^{w_{j+1}} \int_{y_2}^{w_{j+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \right. \\
&+ \sum_{i=j+1}^{\sigma'-1} \left\{ \theta_{i+1} (1 - \theta_{j+1}) \int_{w_j}^{w_{j+1}} \int_{w_i}^{w_{i+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \\
&+ (1 - \theta_{j+1}) \int_{w_j}^{w_{j+1}} \int_{w'_{\sigma'}}^a (v - y_2) \theta_m(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ (1 - \theta_{j+1}) \int_{w_j}^{w_{j+1}} \int_a^1 (v - y_2) \theta^*(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \Big\} + \\
&+ \theta_{\lambda+1} (1 - \theta_{\lambda+1}) \int_{w_\lambda}^v \int_{y_2}^{w_{\lambda+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ \sum_{i=\lambda+1}^{\sigma'-1} \left\{ \theta_{i+1} (1 - \theta_{\lambda+1}) \int_{w_\lambda}^v \int_{w_i}^{w_{i+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \\
&+ (1 - \theta_{\lambda+1}) \int_{w_\lambda}^v \int_{w'_{\sigma'}}^a (v - y_2) \theta_m(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&+ (1 - \theta_{\lambda+1}) \int_{w_\lambda}^v \int_a^1 (v - y_2) \theta^*(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2
\end{aligned} \tag{B.27}$$

So, their difference will be:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(w_\lambda \leq v < w_{\lambda+1}) &= \int_{r_{PRE}^L}^{w_1} F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) + \\
&+ (v - w_1)F^{K-1}(w_1) - (K-1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w_1) - F(r_{PRE}^H)] + \\
&+ \sum_{i=1}^{\sigma'-1} \left\{ \theta_{i+1}(F(w_{i+1}) - F(w_i)) \right\} + \int_{w'_\sigma}^a \theta_m(y)f(y)dy + \int_a^1 \theta^*(y)f(y)dy] + \\
&+ \sum_{i=1}^{\lambda-1} (2\theta_{i+1} - 1) \int_{w_i}^{w_{i+1}} (v - y)f_1^{(K-1)}(y)dy + (2\theta_{\lambda+1} - 1) \int_{w_\lambda}^v (v - y)f_1^{(K-1)}(y)dy + \\
&+ \sum_{i=1}^{\sigma'-1} \left\{ (1 - \theta_{i+1}) \int_{r_{PRE}^H}^{w_1} \int_{w_i}^{w_{i+1}} (v - y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
&+ \int_{r_{PRE}^H}^{w_1} \int_{w'_{\sigma'}}^a (v - y_2)(1 - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
&+ \int_{r_{PRE}^H}^{w_1} \int_a^1 (v - y_2)(1 - \theta^*(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
&+ \sum_{j=1}^{\lambda-1} \left\{ \sum_{i=j+1}^{\sigma'-1} \left\{ (\theta_{j+1} - \theta_{i+1}) \int_{w_j}^{w_{j+1}} \int_{w_i}^{w_{i+1}} (v - y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \right. \\
&+ \int_{w_j}^{w_{j+1}} \int_{w'_{\sigma'}}^a (v - y_2)(\theta_{j+1} - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
&\left. + \int_{w_j}^{w_{j+1}} \int_a^1 (v - y_2)(\theta_{j+1} - \theta^*(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
&+ \sum_{i=\lambda+1}^{\sigma'-1} \left\{ (\theta_{\lambda+1} - \theta_{i+1}) \int_{w_\lambda}^v \int_{w_i}^{w_{i+1}} (v - y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
&+ \int_{w_\lambda}^v \int_{w'_{\sigma'}}^a (v - y_2)(\theta_{\lambda+1} - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
&+ \int_{w_\lambda}^v \int_a^1 (v - y_2)(\theta_{\lambda+1} - \theta^*(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2
\end{aligned} \tag{B.28}$$

Taking the first- and second-order derivatives of this utility difference then yields:

$$\begin{aligned}
& \frac{\partial(\Pi_{PRE}^L - \Pi_{PRE}^H)(w_\lambda \leq v < w_{\lambda+1})}{\partial v} = -F^{K-1}(r_{PRE}^H) + F^{K-1}(w_1) - (K-1)F^{K-2}(r_{PRE}^H)[F(w_1) - F(r_{PRE}^H)] + \\
& + \sum_{i=1}^{\sigma'-1} \left\{ \theta_{i+1}(F(w_{i+1}) - F(w_i)) \right\} + \int_{w'_\sigma}^a \theta_m(y)f(y)dy + \int_a^1 \theta^*(y)f(y)dy] + \\
& + \sum_{i=1}^{\lambda-1} \left\{ (2\theta_{i+1} - 1) \int_{w_i}^{w_{i+1}} f_1^{(K-1)}(y)dy \right\} + (2\theta_{\lambda+1} - 1) \int_{w_\lambda}^v f_1^{(K-1)}(y)dy + \\
& + \sum_{i=1}^{\sigma'-1} \left\{ (1 - \theta_{i+1}) \int_{r_{PRE}^H}^{w_1} \int_{w_i}^{w_{i+1}} f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
& + \int_{r_{PRE}^H}^{w_1} \int_{w'_{\sigma'}}^a (1 - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
& + \int_{r_{PRE}^H}^{w_1} \int_a^1 (1 - \theta^*(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
& + \sum_{j=1}^{\lambda-1} \left\{ \sum_{i=j+1}^{\sigma'-1} \left\{ (\theta_{j+1} - \theta_{i+1}) \int_{w_j}^{w_{j+1}} \int_{w_i}^{w_{i+1}} f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} \right\} + \\
& + \int_{w_j}^{w_{j+1}} \int_{w'_{\sigma'}}^a (\theta_{j+1} - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
& + \int_{w_j}^{w_{j+1}} \int_a^1 (\theta_{j+1} - \theta^*(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
& + \sum_{i=\lambda+1}^{\sigma'-1} \left\{ (\theta_{\lambda+1} - \theta_{i+1}) \int_{w_\lambda}^v \int_{w_i}^{w_{i+1}} f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
& + \int_{w_\lambda}^v \int_{w'_{\sigma'}}^a (\theta_{\lambda+1} - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
& + \int_{w_\lambda}^v \int_a^1 (\theta_{\lambda+1} - \theta^*(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \quad (B.29)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2(\Pi_{PRE}^L - \Pi_{PRE}^H)(w_\lambda \leq v < w_{\lambda+1})}{\partial v^2} = (2\theta_{\lambda+1} - 1)f_1^{(K-1)}(v) + \\
& + \sum_{i=\lambda+1}^{\sigma'-1} \left\{ (\theta_{\lambda+1} - \theta_{i+1}) \int_{w_i}^{w_{i+1}} f_{1,2}^{(K-1)}(y, v)dy \right\} + \\
& + \int_{w'_{\sigma'}}^a (\theta_{\lambda+1} - \theta_m(y))f_{1,2}^{(K-1)}(y, v)dy + \int_a^1 (\theta_{\lambda+1} - \theta^*(y))f_{1,2}^{(K-1)}(y, v)dy = \\
& = (K-1)F^{K-2}(v)f(v)(2\theta_{\lambda+1} - 1) + \\
& + (K-1)(K-2)F^{K-3}(v)f(v) \sum_{i=\lambda+1}^{\sigma'-1} \left\{ (\theta_{\lambda+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i)) \right\} + \\
& + (K-1)(K-2)F^{K-3}(v)f(v)[\theta_{\lambda+1} \int_{w'_{\sigma'}}^1 f(y)dy - \int_{w'_{\sigma'}}^a \theta_m(y)f(y)dy - \int_a^1 \theta^*(y)f(y)dy] =
\end{aligned}$$

$$\begin{aligned}
&= (K - 1)F^{K-3}(v)f(v)\{(2\theta_{\lambda+1} - 1)F(v) + \\
&\quad + (K - 2)\left[\sum_{i=\lambda+1}^{\sigma'-1}\left\{(\theta_{\lambda+1} - \theta_{i+1})(F(w_{i+1}) - F(w_i))\right\} + \theta_{\lambda+1}(1 - F(w_{\sigma'})) - \right. \\
&\quad \left.- \int_{w_{\sigma'}}^a \theta_m(y)f(y)dy - \int_a^1 \theta^*(y)f(y)dy]\}\}
\end{aligned} \tag{B.30}$$

This means that when $\theta_{\lambda+1} = 1$, the corresponding utility difference is convex, whereas when $\theta_{\lambda+1} = 0$, it is concave.

Similarly, for the special case where $v \in [r_{PRE}^H, w_1)$, the corresponding utility difference and its second order derivative (given that $\theta_1 = 1$ always) will be:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(r_{PRE}^H \leq v < w_1) &= \int_{r_{PRE}^L}^v F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) - \\
&\quad - (K - 1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w_1) - F(r_{PRE}^H) + \sum_{i=1}^{\sigma'-1}\left\{\theta_{i+1}(F(w_{i+1}) - F(w_i))\right\} + \\
&\quad + \int_{w_{\sigma'}}^a \theta_m(y)f(y)dy + \int_a^1 \theta^*(y)f(y)dy] + \\
&\quad + \sum_{i=1}^{\sigma'-1}\left\{(1 - \theta_{i+1})\int_{r_{PRE}^H}^v \int_{w_i}^{w_{i+1}} (v - y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2\right\} + \\
&\quad + \int_{r_{PRE}^H}^v \int_{w_{\sigma'}}^1 (v - y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 - \\
&\quad - \int_{r_{PRE}^H}^v \int_{w_{\sigma'}}^a (v - y_2)\theta_m(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 - \\
&\quad - \int_{r_{PRE}^H}^v \int_a^1 (v - y_2)\theta^*(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2
\end{aligned} \tag{B.31}$$

$$\begin{aligned}
\frac{\partial^2(\Pi_{PRE}^L - \Pi_{PRE}^H)(r_{PRE}^H \leq v < w_1)}{\partial v^2} &= (K - 1)F^{K-3}(v)f(v)\{F(v) + \\
&\quad + (K - 2)\left[\sum_{i=1}^{\sigma'-1}\left\{(1 - \theta_{i+1})(F(w_{i+1}) - F(w_i))\right\} + 1 - F(w_{\sigma'}) - \right. \\
&\quad \left.- \int_{w_{\sigma'}}^a \theta_m(y)f(y)dy - \int_a^1 \theta^*(y)f(y)dy]\}\} \geq 0
\end{aligned} \tag{B.32}$$

B.1.4 Mixed-Strategy Bayes-Nash Equilibria with Multiple Cut-Off Points After Randomizing

In what follows, we will analytically derive the closed-form expression of the expected utilities from the low-reserve and high-reserve intermediary, Π_{PRE}^L and Π_{PRE}^H respectively, when buyers follow a pure strategy $\theta(v) = 1$ for all $v \in [r_{PRE}^H, w)$, a mixed

strategy $\theta(v) = \theta_m(v) \in (0, 1)$ for all $v \in [w, a_1]$, and, in the general case, then follow pure strategies $\theta(v) = \theta^*(v)$ for all $v \in (a_1, 1]$, so that $\theta^*(v) = \theta_1^*$ if $v \in (a_1, a_2)$, $\theta^*(v) = \theta_2^*$ if $v \in [a_2, a_3]$ and so on, $\theta^*(v) = \theta_{m'}^*$ if $v \in [a_{m'}, 1]$, where $\theta_i^* \in \{0, 1\}$ and $\theta_i^* \neq \theta_{i+1}^*, \forall i \in \{1, \dots, m'\}$.

For a buyer whose valuation $v \in [a_\lambda, a_{\lambda+1})$ for some $\lambda = 1, \dots, m'$, the expected utilities from the low- and high-reserve intermediaries will be (we use the convention that $a_{m'+1} = 1$ for notational convenience):

$$\begin{aligned}
\Pi_{PRE}^L(a_\lambda \leq v < a_{\lambda+1}) &= \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (v-w)F^{K-1}(w) + \\
&+ \int_w^{a_1} (v-y)\theta_m(y)f_1^{(K-1)}(y)dy + \sum_{i=1}^{\lambda-1} \left\{ \theta_i^* \int_{a_i}^{a_{i+1}} (v-y)f_1^{(K-1)}(y)dy \right\} + \\
&+ \theta_\lambda^* \int_{a_\lambda}^v (v-y)f_1^{(K-1)}(y)dy + \int_{r_{PRE}^H}^w \int_w^{a_1} (v-y_2)(1-\theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
&+ \sum_{i=1}^{m'} \left\{ (1-\theta_i^*) \int_{r_{PRE}^H}^w \int_{a_i}^{a_{i+1}} (v-y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
&+ \int_w^{a_1} \int_{y_2}^{a_1} (v-y_2)(1-\theta_m(y_1))\theta_m(y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
&+ \sum_{i=1}^{m'} \left\{ (1-\theta_i^*) \int_w^{a_1} \int_{a_i}^{a_{i+1}} (v-y_2)\theta_m(y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
&+ \sum_{j=1}^{\lambda-1} \left\{ (1-\theta_j^*)\theta_j^* \int_{a_j}^{a_{j+1}} \int_{y_2}^{a_{j+1}} (v-y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \right. \\
&\quad \left. + \theta_j^* \sum_{i=j+1}^{m'} \left\{ (1-\theta_i^*) \int_{a_j}^{a_{j+1}} \int_{a_i}^{a_{i+1}} (v-y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} \right\} + \\
&+ (1-\theta_\lambda^*)\theta_\lambda^* \int_{a_\lambda}^v \int_{y_2}^{a_{\lambda+1}} (v-y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
&+ \theta_\lambda^* \sum_{i=\lambda+1}^{m'} \left\{ (1-\theta_i^*) \int_{a_\lambda}^v \int_{a_i}^{a_{i+1}} (v-y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} \tag{B.33}
\end{aligned}$$

$$\begin{aligned}
\Pi_{PRE}^H(a_\lambda \leq v < a_{\lambda+1}) &= (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) + (K-1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - F(r_{PRE}^H) + \\
&+ \int_w^{a_1} \theta_m(y)f(y)dy + \sum_{i=1}^{m'} \left\{ \theta_i^*(F(a_{i+1}) - F(a_i)) \right\}] + \int_w^{a_1} (v-y)(1-\theta_m(y))f_1^{(K-1)}(y)dy + \\
&+ \sum_{i=1}^{\lambda-1} \left\{ (1-\theta_i^*) \int_{a_i}^{a_{i+1}} (v-y)f_1^{(K-1)}(y)dy \right\} + (1-\theta_\lambda^*) \int_{a_\lambda}^v (v-y)f_1^{(K-1)}(y)dy + \\
&+ \int_w^{a_1} \int_{y_2}^{a_1} (v-y_2)\theta_m(y_1)(1-\theta_m(y_2))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 +
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{m'} \left\{ \theta_i^* \int_w^{a_1} \int_{a_i}^{a_{i+1}} (v - y_2) (1 - \theta_m(y_2)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \\
& + \sum_{j=1}^{\lambda-1} \left\{ \theta_j^* (1 - \theta_j^*) \int_{a_j}^{a_{j+1}} \int_{y_2}^{a_{j+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \right. \\
& \quad \left. + (1 - \theta_j^*) \sum_{i=j+1}^{m'} \left\{ \theta_i^* \int_{a_j}^{a_{j+1}} \int_{a_i}^{a_{i+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} \right\} + \\
& + \theta_\lambda^* (1 - \theta_\lambda^*) \int_{a_\lambda}^v \int_{y_2}^{a_{\lambda+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
& + (1 - \theta_\lambda^*) \sum_{i=\lambda+1}^{m'} \left\{ \theta_i^* \int_{a_\lambda}^v \int_{a_i}^{a_{i+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} \tag{B.34}
\end{aligned}$$

So, their difference will be:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(a_\lambda \leq v < a_{\lambda+1}) & = \int_{r_{PRE}^L}^w F^{K-1}(y) dy + (v - w) F^{K-1}(w) - (v - r_{PRE}^H) F^{K-1}(r_{PRE}^H) - \\
& - (K-1)(v - r_{PRE}^H) F^{K-2}(r_{PRE}^H) [F(w) - F(r_{PRE}^H) + \int_w^{a_1} \theta_m(y) f(y) dy + \\
& + \sum_{i=1}^{m'} \left\{ \theta_i^* (F(a_{i+1}) - F(a_i)) \right\}] + \int_w^{a_1} (v - y) (2\theta_m(y) - 1) f_1^{(K-1)}(y) dy + \\
& + \sum_{i=1}^{\lambda-1} \left\{ (2\theta_i^* - 1) \int_{a_i}^{a_{i+1}} (v - y) f_1^{(K-1)}(y) dy \right\} + (2\theta_\lambda^* - 1) \int_{a_\lambda}^v (v - y) f_1^{(K-1)}(y) dy + \\
& + \int_{r_{PRE}^H}^w \int_w^{a_1} (v - y_2) (1 - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
& + \sum_{i=1}^{m'} \left\{ (1 - \theta_i^*) \int_{r_{PRE}^H}^w \int_{a_i}^{a_{i+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \\
& + \int_w^{a_1} \int_{y_2}^{a_1} (v - y_2) (\theta_m(y_2) - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
& + \sum_{i=1}^{m'} \left\{ \int_w^{a_1} \int_{a_i}^{a_{i+1}} (v - y_2) (\theta_m(y_2) - \theta_i^*) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \\
& + \sum_{j=1}^{\lambda-1} \sum_{i=j+1}^{m'} \left\{ (\theta_j^* - \theta_i^*) \int_{a_j}^{a_{j+1}} \int_{a_i}^{a_{i+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} + \\
& + \sum_{i=\lambda+1}^{m'} \left\{ (\theta_\lambda^* - \theta_i^*) \int_{a_\lambda}^v \int_{a_i}^{a_{i+1}} (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \right\} \tag{B.35}
\end{aligned}$$

and its first and second order derivatives will have the following form:

$$\begin{aligned}
\frac{\partial(\Pi_{PRE}^L - \Pi_{PRE}^H)(a_\lambda \leq v < a_{\lambda+1})}{\partial v} = & F^{K-1}(w) - F^{K-1}(r_{PRE}^H) - \\
& -(K-1)F^{K-2}(r_{PRE}^H)[F(w) - F(r_{PRE}^H) + \int_w^{a_1} \theta_m(y)f(y)dy + \int_{a_1}^1 \theta^*(y)f(y)dy] + \\
& + \int_w^{a_1} (2\theta_m(y) - 1)f_1^{(K-1)}(y)dy + \sum_{i=1}^{\lambda-1} \left\{ (2\theta_i^* - 1) \int_{a_i}^{a_{i+1}} f_1^{(K-1)}(y)dy \right\} + \\
& + (2\theta_\lambda^* - 1) \int_{a_\lambda}^v f_1^{(K-1)}(y)dy + \\
& + \int_{r_{PRE}^H}^w \int_w^{a_1} (1 - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
& + \sum_{i=1}^{m'} \left\{ (1 - \theta_i^*) \int_{r_{PRE}^H}^w \int_{a_i}^{a_{i+1}} f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
& + \int_w^{a_1} \int_{y_2}^{a_1} (\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
& + \sum_{i=1}^{m'} \left\{ \int_w^{a_1} \int_{a_i}^{a_{i+1}} (\theta_m(y_2) - \theta_i^*)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
& + \sum_{j=1}^{\lambda-1} \sum_{i=j+1}^{m'} \left\{ (\theta_j^* - \theta_i^*) \int_{a_j}^{a_{j+1}} \int_{a_i}^{a_{i+1}} f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} + \\
& + \sum_{i=\lambda+1}^{m'} \left\{ (\theta_\lambda^* - \theta_i^*) \int_{a_\lambda}^v \int_{a_i}^{a_{i+1}} f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} \tag{B.36}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2(\Pi_{PRE}^L - \Pi_{PRE}^H)(a_\lambda \leq v < a_{\lambda+1})}{\partial v^2} = & (2\theta_\lambda^* - 1)f_1^{(K-1)}(v) + \\
& +(K-1)(K-2) \sum_{i=\lambda+1}^{m'} (\theta_\lambda^* - \theta_i^*) \int_{a_i}^{a_{i+1}} f(y)f(v)F^{K-3}(v)dy = \\
& =(K-1)F^{K-3}(v)f(v)[(2\theta_\lambda^* - 1)F(v) + (K-2) \sum_{i=\lambda+1}^{m'} (\theta_\lambda^* - \theta_i^*)(F(a_{i+1}) - F(a_i))] \tag{B.37}
\end{aligned}$$

B.1.5 Mixed-Strategy Bayes-Nash Equilibria with Two Cut-Off Points

We will now derive the closed-form expression of the expected utilities from the low-reserve and high-reserve intermediary, Π_{PRE}^L and Π_{PRE}^H respectively, when buyers follow a pure strategy $\theta(v) = 1$ for all $v \in [r_{PRE}^H, w]$, a mixed strategy $\theta(v) = \theta_m(v) \in (0, 1)$ for all $v \in [w, a]$, and then follow pure strategies $\theta(v) = \theta^* \in \{0, 1\}$ for all $v \in (a, 1]$.

If $v \in [0, r_{PRE}^L]$, the buyer can not participate in any of the two auctions, so in this case $\Pi_{PRE}^L(v) = \Pi_{PRE}^H(v) = 0$.

If $v \in [r_{PRE}^L, r_{PRE}^H)$, then a buyer expects positive utility only from the low-reserve intermediary, as this is the only possible auction to participate in. So, in this case:

$$\Pi_{PRE}^L(r_{PRE}^L \leq v < r_{PRE}^H) = \int_{r_{PRE}^L}^v F^{K-1}(y) dy \quad (\text{B.38})$$

and $\Pi_{PRE}^H(r_{PRE}^L \leq v < r_{PRE}^H) = 0$.

More specifically, when $v \in [r_{PRE}^H, w)$, then we can write the expected utility from the two intermediaries as:

$$\begin{aligned} \Pi_{PRE}^L(r_{PRE}^H \leq v < w) &= \int_{r_{PRE}^L}^v F^{K-1}(y) dy + \int_{r_{PRE}^H}^v \int_w^a (v - y_2)(1 - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\ &+ \int_{r_{PRE}^H}^v \int_a^1 (v - y_2)(1 - \theta^*) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 \end{aligned} \quad (\text{B.39})$$

$$\begin{aligned} \Pi_{PRE}^H(r_{PRE}^H \leq v < w) &= (v - r_{PRE}^H) F^{K-1}(r_{PRE}^H) + (K-1)(v - r_{PRE}^H) F^{K-2}(r_{PRE}^H) [F(w) - \\ &- F(r_{PRE}^H) + \int_w^a \theta_m(y) f(y) dy + \int_a^1 \theta^* f(y) dy] \end{aligned} \quad (\text{B.40})$$

So, their difference will be:

$$\begin{aligned} (\Pi_{PRE}^L - \Pi_{PRE}^H)(r_{PRE}^H \leq v < w) &= \int_{r_{PRE}^L}^v F^{K-1}(y) dy - (v - r_{PRE}^H) F^{K-1}(r_{PRE}^H) - \\ &- (K-1)(v - r_{PRE}^H) F^{K-2}(r_{PRE}^H) [F(w) - F(r_{PRE}^H) + \int_w^a \theta_m(y) f(y) dy + \int_a^1 \theta^* f(y) dy] + \\ &+ \int_{r_{PRE}^H}^v \int_w^a (v - y_2)(1 - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\ &+ \int_{r_{PRE}^H}^v \int_a^1 (v - y_2)(1 - \theta^*) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 = \\ &= \int_{r_{PRE}^L}^v F^{K-1}(y) dy - (v - r_{PRE}^H) F^{K-1}(r_{PRE}^H) - (K-1)(v - r_{PRE}^H) F^{K-2}(r_{PRE}^H) [F(w) - \\ &- F(r_{PRE}^H) + \int_w^a \theta_m(y) f(y) dy + \int_a^1 \theta^* f(y) dy] + \int_{r_{PRE}^H}^v \int_w^1 (v - y_2) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 - \\ &- \int_{r_{PRE}^H}^v \int_w^a (v - y_2) \theta_m(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 - \int_{r_{PRE}^H}^v \int_a^1 (v - y_2) \theta^* f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 = \end{aligned}$$

$$\begin{aligned}
&= \int_{r_{PRE}^L}^v F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) - (K-1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - \\
&\quad - F(r_{PRE}^H) + \int_w^a \theta_m(y)f(y)dy + \int_a^1 \theta^*f(y)dy] + (K-1)[1 - F(w)] \int_{r_{PRE}^H}^v (v - y)f_1^{(K-2)}(y)dy - \\
&\quad - (K-1) \int_{r_{PRE}^H}^v (v - y_2)f_1^{(K-2)}(y_2)dy_2 [\int_w^a \theta_m(y_1)f(y_1)dy_1 + \int_a^1 \theta^*f(y_1)dy_1] = \\
&= \int_{r_{PRE}^L}^v F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) - (K-1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - \\
&\quad - F(r_{PRE}^H) + \int_w^a \theta_m(y)f(y)dy + \int_a^1 \theta^*f(y)dy] + (K-1)[1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \\
&\quad - \int_a^1 \theta^*f(y)dy] [-(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H) + \int_{r_{PRE}^H}^v F^{K-2}(y)dy] = \\
&= \int_{r_{PRE}^L}^v F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] + \\
&\quad + (K-1) \int_{r_{PRE}^H}^v F^{K-2}(y)dy [1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \theta^*(1 - F(a))] \tag{B.41}
\end{aligned}$$

Whenever $v \in [w, a]$, the expected utility from the low- and high-reserve intermediaries will be:

$$\begin{aligned}
\Pi_{PRE}^L(w \leq v \leq a) &= (v - r_{PRE}^L)F^{K-1}(r_{PRE}^L) + \int_{r_{PRE}^L}^w (v - y)f_1^{(K-1)}(y)dy + \\
&\quad + \int_w^v (v - y)\theta_m(y)f_1^{(K-1)}(y)dy + \int_{r_{PRE}^H}^w \int_w^a (v - y_2)(1 - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_{r_{PRE}^H}^w \int_a^1 (v - y_2)(1 - \theta^*)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^v \int_{y_2}^a (v - y_2)(1 - \theta_m(y_1))\theta_m(y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^v \int_a^1 (v - y_2)(1 - \theta^*)\theta_m(y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 = \\
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (v - w)F^{K-1}(w) + \int_w^v (v - y)\theta_m(y)f_1^{(K-1)}(y)dy + \\
&\quad + \int_{r_{PRE}^H}^w \int_w^a (v - y_2)(1 - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_{r_{PRE}^H}^w \int_a^1 (v - y_2)(1 - \theta^*)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^v \int_{y_2}^a (v - y_2)(1 - \theta_m(y_1))\theta_m(y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^v \int_a^1 (v - y_2)(1 - \theta^*)\theta_m(y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 \tag{B.42}
\end{aligned}$$

$$\begin{aligned}
\Pi_{PRE}^H(w \leq v \leq a) &= (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) + (K-1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - \\
&\quad - F(r_{PRE}^H) + \int_w^a \theta_m(y)f(y)dy + \int_a^1 \theta^*f(y)dy] + \int_w^v (v-y)(1-\theta_m(y))f_1^{(K-1)}(y)dy + \\
&\quad - F(r_{PRE}^H) + \int_w^a \theta_m(y)f(y)dy + \int_a^1 \theta^*f(y)dy] + \int_w^v (v-y)(1-\theta_m(y))f_1^{(K-1)}(y)dy + \\
&\quad + \int_w^v \int_{y_2}^a (v-y_2)\theta_m(y_1)(1-\theta_m(y_2))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^v \int_a^1 (v-y_2)\theta^*(1-\theta_m(y_2))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2
\end{aligned} \tag{B.43}$$

Hence, their difference is:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a) &= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) + \\
&\quad + (v-w)F^{K-1}(w) + \int_w^v (v-y)(2\theta_m(y)-1)f_1^{(K-1)}(y)dy - \\
&\quad - (K-1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - F(r_{PRE}^H) + \int_w^a \theta_m(y)f(y)dy + \int_a^1 \theta^*f(y)dy] + \\
&\quad + \int_{r_{PRE}^H}^w \int_w^a (v-y_2)(1-\theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_{r_{PRE}^H}^w \int_a^1 (v-y_2)(1-\theta^*)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^v \int_{y_2}^a (v-y_2)(\theta_m(y_2)-\theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^v \int_a^1 (v-y_2)(\theta_m(y_2)-\theta^*)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 = \\
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) + (v-w)F^{K-1}(w) + \\
&\quad + 2 \int_w^v (v-y)\theta_m(y)f_1^{(K-1)}(y)dy - \int_w^v (v-y)f_1^{(K-1)}(y)dy - \\
&\quad - (K-1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - F(r_{PRE}^H) + \int_w^a \theta_m(y)f(y)dy + \theta^*(1-F(a))] + \\
&\quad + \int_{r_{PRE}^H}^w \int_w^a (v-y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 - \int_{r_{PRE}^H}^w \int_w^a (v-y_2)\theta_m(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_{r_{PRE}^H}^w \int_a^1 (v-y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 - \theta^* \int_{r_{PRE}^H}^w \int_a^1 (v-y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^v \int_{y_2}^a (v-y_2)(\theta_m(y_2)-\theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^v \int_a^1 (v-y_2)\theta_m(y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 - \theta^* \int_w^v \int_a^1 (v-y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 =
\end{aligned}$$

$$\begin{aligned}
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) + (v - w)F^{K-1}(w) + \\
&+ 2 \int_w^v (v - y)\theta_m(y)f_1^{(K-1)}(y)dy + (v - w)F^{K-1}(w) - \int_w^v F^{K-1}(y)dy - \\
&- (K - 1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - F(r_{PRE}^H) + \int_w^a \theta_m(y)f(y)dy + \theta^*(1 - F(a))] + \\
&+ \int_{r_{PRE}^H}^w \int_w^1 (v - y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 - \theta^* \int_{r_{PRE}^H}^v \int_a^1 (v - y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 - \\
&- \int_{r_{PRE}^H}^w \int_w^a (v - y_2)\theta_m(y_1)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \int_w^v \int_a^1 (v - y_2)\theta_m(y_2)f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
&+ \int_w^v \int_{y_2}^a (v - y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 = \\
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - \int_w^v F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) + 2(v - w)F^{K-1}(w) + \\
&+ 2 \int_w^v (v - y)\theta_m(y)f_1^{(K-1)}(y)dy - (K - 1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - F(r_{PRE}^H) + \\
&+ \int_w^a \theta_m(y)f(y)dy + \theta^*(1 - F(a))] + (K - 1)(1 - F(w)) \int_{r_{PRE}^H}^w (v - y)f_1^{(K-2)}(y)dy - \\
&- \theta^*(K - 1)(1 - F(a)) \int_{r_{PRE}^H}^v (v - y)f_1^{(K-2)}(y)dy - \\
&- (K - 1) \int_{r_{PRE}^H}^w (v - y_2)f_1^{(K-2)}(y_2)dy_2 \int_w^a \theta_m(y_1)f(y_1)dy_1 + \\
&+ (K - 1)(1 - F(a)) \int_w^v (v - y)\theta_m(y)f_1^{(K-2)}(y)dy + \\
&+ \int_w^v \int_{y_2}^a (v - y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 = \\
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - \int_w^v F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) + 2(v - w)F^{K-1}(w) + \\
&+ 2 \int_w^v (v - y)\theta_m(y)f_1^{(K-1)}(y)dy - (K - 1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - F(r_{PRE}^H) + \\
&+ \int_w^a \theta_m(y)f(y)dy + \theta^*(1 - F(a))] + (K - 1)(1 - F(w))[(v - w)F^{K-2}(w) - \\
&- (v - r_{PRE}^H)F^{K-2}(r_{PRE}^H) + \int_{r_{PRE}^H}^w F^{K-2}(y)dy] - \theta^*(K - 1)(1 - F(a))[-(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H) + \\
&+ \int_{r_{PRE}^H}^w F^{K-2}(y)dy + \int_w^v F^{K-2}(y)dy] - (K - 1) \int_w^a \theta_m(y)f(y)dy[(v - w)F^{K-2}(w) - \\
&- (v - r_{PRE}^H)F^{K-2}(r_{PRE}^H) + \int_{r_{PRE}^H}^w F^{K-2}(y)dy] + (K - 1)(1 - F(a)) \int_w^v (v - y)\theta_m(y)f_1^{(K-2)}(y)dy + \\
&+ \int_w^v \int_{y_2}^a (v - y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 =
\end{aligned}$$

$$\begin{aligned}
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - \int_w^v F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-1}(r_{PRE}^H) + 2(v - w)F^{K-1}(w) + \\
&+ 2 \int_w^v (v - y)\theta_m(y)f_1^{(K-1)}(y)dy - (K - 1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - F(r_{PRE}^H) + \\
&+ \int_w^a \theta_m(y)f(y)dy + \theta^*(1 - F(a))] + (K - 1)(1 - F(w))(v - w)F^{K-2}(w) - \\
&- (K - 1)(1 - F(w))(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H) + (K - 1)(1 - F(w)) \int_{r_{PRE}^H}^w F^{K-2}(y)dy + \\
&+ \theta^*(K - 1)(1 - F(a))(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H) - \theta^*(K - 1)(1 - F(a)) \int_{r_{PRE}^H}^w F^{K-2}(y)dy - \\
&- \theta^*(K - 1)(1 - F(a)) \int_w^v F^{K-2}(y)dy - (K - 1)(v - w)F^{K-2}(w) \int_w^a \theta_m(y)f(y)dy + \\
&+ (K - 1)(v - r_{PRE}^H)F^{K-2}(r_{PRE}^H) \int_w^a \theta_m(y)f(y)dy - (K - 1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy \int_w^a \theta_m(y)f(y)dy + \\
&+ (K - 1)(1 - F(a)) \int_w^v (v - y)\theta_m(y)f_1^{(K-2)}(y)dy + \\
&+ \int_w^v \int_{y_2}^a (v - y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 = \\
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - \int_w^v F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + \\
&+ (K - 1)(1 - F(r_{PRE}^H))] + 2(v - w)F^{K-1}(w) + 2 \int_w^v (v - y)\theta_m(y)f_1^{(K-1)}(y)dy + \\
&+ (K - 1)(1 - F(a)) \int_w^v (v - y)\theta_m(y)f_1^{(K-2)}(y)dy + (K - 1)(v - w)F^{K-2}(w)[1 - F(w) - \\
&- \int_w^a \theta_m(y)f(y)dy] + (K - 1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy[1 - F(w) - \theta^*(1 - F(a)) - \\
&- \int_w^a \theta_m(y)f(y)dy] - (K - 1)\theta^*(1 - F(a)) \int_w^v F^{K-2}(y)dy + \\
&+ \int_w^v \int_{y_2}^a (v - y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 = \\
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - \int_w^v F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + \\
&+ (K - 1)(1 - F(r_{PRE}^H))] + (v - w)F^{K-2}(w)[2F(w) + (K - 1)(1 - F(w) - \int_w^a \theta_m(y)f(y)dy)] + \\
&+ (K - 1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy[1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \theta^*(1 - F(a))] - \\
&- (K - 1)\theta^*(1 - F(a)) \int_w^v F^{K-2}(y)dy + (K - 1)(1 - F(a)) \int_w^v (v - y)\theta_m(y)f_1^{(K-2)}(y)dy + \\
&+ 2 \int_w^v (v - y)\theta_m(y)f_1^{(K-1)}(y)dy + \int_w^v \int_{y_2}^a (v - y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2
\end{aligned} \tag{B.44}$$

Finally, using (B.35) for $m' = 1$ and $\lambda = 1$, we get the equivalent utility difference for buyers with valuations in $(a, 1]$:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(a < v \leq 1) &= \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (v-w)F^{K-1}(w) - (v-r_{PRE}^H)F^{K-1}(r_{PRE}^H) - \\
&\quad - (K-1)(v-r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - F(r_{PRE}^H) + \int_w^a \theta_m(y)f(y)dy + \int_a^1 \theta^*f(y)dy] + \\
&\quad + 2[\int_w^a (v-y)\theta_m(y)f_1^{(K-1)}(y)dy + \int_a^v (v-y)\theta^*f_1^{(K-1)}(y)dy] - \\
&\quad - \int_w^v (v-y)f_1^{(K-1)}(y)dy + (K-1)\int_{r_{PRE}^H}^w \int_w^1 (v-y_2)f(y_1)f_1^{(K-2)}(y_2)dy_1dy_2 - \\
&\quad - (K-1)\int_{r_{PRE}^H}^w \int_w^a (v-y_2)f_1^{(K-2)}(y_2)\theta_m(y_1)f(y_1)dy_1dy_2 - \\
&\quad - (K-1)\int_{r_{PRE}^H}^w \int_a^1 (v-y_2)f_1^{(K-2)}(y_2)\theta^*f(y_1)dy_1dy_2 + \\
&\quad + \int_w^a \int_{y_2}^a (v-y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^a \int_a^1 (v-y_2)(\theta_m(y_2) - \theta^*)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_a^v \int_{y_2}^1 (v-y_2)(\theta^* - \theta^*)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 = \\
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - \int_w^v F^{K-1}(y)dy - (v-r_{PRE}^H)F^{K-1}(r_{PRE}^H) + 2(v-w)F^{K-1}(w) - \\
&\quad - (K-1)(v-r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(w) - F(r_{PRE}^H) + \int_w^a \theta_m(y)f(y)dy + \int_a^1 \theta^*f(y)dy] + \\
&\quad + 2[\int_w^a (v-y)\theta_m(y)f_1^{(K-1)}(y)dy + \int_a^v (v-y)\theta^*f_1^{(K-1)}(y)dy] + \\
&\quad + (K-1)[1 - F(w)][(v-w)F^{K-2}(w) - (v-r_{PRE}^H)F^{K-2}(r_{PRE}^H) + \int_{r_{PRE}^H}^w F^{K-2}(y)dy] - \\
&\quad - (K-1)[(v-w)F^{K-2}(w) - (v-r_{PRE}^H)F^{K-2}(r_{PRE}^H) + \int_{r_{PRE}^H}^w F^{K-2}(y)dy] \\
&[\int_w^a \theta_m(y)f(y)dy + \int_a^1 \theta^*f(y)dy] + \int_w^a \int_{y_2}^a (v-y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^a \int_a^1 (v-y_2)(\theta_m(y_2) - \theta^*)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 = \\
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - \int_w^v F^{K-1}(y)dy - (v-r_{PRE}^H)F^{K-1}(r_{PRE}^H) + 2(v-w)F^{K-1}(w) - \\
&\quad - (K-1)(v-r_{PRE}^H)F^{K-2}(r_{PRE}^H)[1 - F(r_{PRE}^H)] + 2[\int_w^a (v-y)\theta_m(y)f_1^{(K-1)}(y)dy + \\
&\quad + \int_a^v (v-y)\theta^*f_1^{(K-1)}(y)dy] + (K-1)[(v-w)F^{K-2}(w) + \\
&\quad + \int_{r_{PRE}^H}^w F^{K-2}(y)dy][1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \int_a^1 \theta^*f(y)dy] + \\
&\quad + \int_w^a \int_{y_2}^a (v-y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + \int_w^a \int_a^1 (v-y_2)(\theta_m(y_2) - \theta^*)f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 =
\end{aligned}$$

$$\begin{aligned}
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - \int_w^v F^{K-1}(y)dy - (v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - \\
&\quad - F(r_{PRE}^H))] + 2(v-w)F^{K-1}(w) + 2 \int_w^a (v-y)\theta_m(y)f_1^{(K-1)}(y)dy - 2\theta^*(v-a)F^{K-1}(a) + \\
&\quad + 2\theta^* \int_a^v F^{K-1}(y)dy + (K-1)(v-w)F^{K-2}(w)[1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \\
&\quad - \theta^*(1 - F(a))] + (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy[1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \\
&\quad - \theta^*(1 - F(a))] + \int_w^a \int_{y_2}^a (v-y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 + \\
&\quad + (K-1)(1 - F(a)) \int_w^a (v-y)\theta_m(y)f_1^{(K-2)}(y)dy - \\
&\quad - (K-1)\theta^*(1 - F(a))[(v-a)F^{K-2}(a) - (v-w)F^{K-2}(w) + \int_w^a F^{K-2}(y)dy] = \\
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy - \int_w^a F^{K-1}(y)dy - (1-2\theta^*) \int_a^v F^{K-1}(y)dy - \\
&\quad - (v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] + (v-w)F^{K-2}(w)[2F(w) + \\
&\quad + (K-1)(1 - F(w) - \int_w^a \theta_m(y)f(y)dy)] - \theta^*(v-a)F^{K-2}(a)[F(a) + \\
&\quad + (K-1)(1 - F(a))] + (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy[1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \\
&\quad - \theta^*(1 - F(a))] - (K-1)\theta^*(1 - F(a)) \int_w^a F^{K-2}(y)dy + \\
&\quad + 2 \int_w^a (v-y)\theta_m(y)f_1^{(K-1)}(y)dy + (K-1)(1 - F(a)) \int_w^a (v-y)\theta_m(y)f_1^{(K-2)}(y)dy + \\
&\quad + \int_w^a \int_{y_2}^a (v-y_2)(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2
\end{aligned} \tag{B.45}$$

As has been mentioned, it should also be that $\frac{\partial(\Pi_{PRE}^L - \Pi_{PRE}^H)(v)}{\partial v} = 0$ for all $v \in [w, a]$. Hence, using (B.44), the first-order derivative of the expected utility difference from the two intermediaries in $[w, a]$ will be:

$$\begin{aligned}
&\frac{\partial(\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a)}{\partial v} = -F^{K-2}(v)[F(v) + \theta^*(K-1)(1 - F(a))] - F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + \\
&\quad + (K-1)(1 - F(r_{PRE}^H))] + F^{K-2}(w)[2F(w) + (K-1)(1 - F(w) - \int_w^a \theta_m(y)f(y)dy)] + \\
&\quad + 2 \int_w^v \theta_m(y)f_1^{(K-1)}(y)dy + (K-1)(1 - F(a)) \int_w^v \theta_m(y)f_1^{(K-2)}(y)dy + \\
&\quad + \int_w^v \int_{y_2}^a (\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1dy_2 = 0 \implies
\end{aligned}$$

$$\begin{aligned}
&\implies F^{K-2}(v)[F(v) + \theta^*(K-1)(1-F(a))] = F^{K-2}(w)[2F(w) + (K-1)(1-F(w)) - \\
&\quad - \int_w^a \theta_m(y)f(y)dy] - F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1-F(r_{PRE}^H))] + \\
&\quad + 2 \int_w^v \theta_m(y)f_1^{(K-1)}(y)dy + (K-1)(1-F(a)) \int_w^v \theta_m(y)f_1^{(K-2)}(y)dy + \\
&\quad + \int_w^v \int_{y_2}^a (\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2
\end{aligned} \tag{B.46}$$

This should also be true at w , where the equation above yields:

$$\begin{aligned}
&F^{K-2}(w)[F(w) + \theta^*(K-1)(1-F(a))] = F^{K-2}(w)[2F(w) + (K-1)(1-F(w)) - \\
&\quad - \int_w^a \theta_m(y)f(y)dy] - F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1-F(r_{PRE}^H))]
\end{aligned} \tag{B.47}$$

After rearranging and using the condition of (B.46), the expected utility difference of (B.44) gives the following equation:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a) &= \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy[1 - F(w) - \\
&\quad - \int_w^a \theta_m(y)f(y)dy - \theta^*(1-F(a))] + r_{PRE}^H F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1-F(r_{PRE}^H))] - \\
&\quad - w F^{K-2}(w)[2F(w) + (K-1)(1-F(w)) - \int_w^a \theta_m(y)f(y)dy] - \int_w^v F^{K-1}(y)dy - \\
&\quad - (K-1)\theta^*(1-F(a)) \int_w^v F^{K-2}(y)dy - 2 \int_w^v y \theta_m(y)f_1^{(K-1)}(y)dy - \\
&\quad - (K-1)(1-F(a)) \int_w^v y \theta_m(y)f_1^{(K-2)}(y)dy - \\
&\quad - \int_w^v \int_{y_2}^a y_2 (\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 + \\
&\quad + v \left\{ F^{K-2}(w)[2F(w) + (K-1)(1-F(w)) - \int_w^a \theta_m(y)f(y)dy] - \right. \\
&\quad - F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1-F(r_{PRE}^H))] + 2 \int_w^v \theta_m(y)f_1^{(K-1)}(y)dy + \\
&\quad + (K-1)(1-F(a)) \int_w^v \theta_m(y)f_1^{(K-2)}(y)dy + \\
&\quad \left. + \int_w^v \int_{y_2}^a (\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 \right\} =
\end{aligned}$$

$$\begin{aligned}
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy [1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \\
&\quad - \theta^*(1 - F(a))] + r_{PRE}^H F^{K-2}(r_{PRE}^H) [F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] - w F^{K-2}(w) [2F(w) + \\
&\quad + (K-1)(1 - F(w) - \int_w^a \theta_m(y)f(y)dy)] - \int_w^v F^{K-1}(y)dy - \\
&\quad - (K-1)\theta^*(1 - F(a)) \int_w^v F^{K-2}(y)dy - 2 \int_w^v y \theta_m(y) f_1^{(K-1)}(y) dy - \\
&\quad - (K-1)(1 - F(a)) \int_w^v y \theta_m(y) f_1^{(K-2)}(y) dy - \\
&\quad - \int_w^v \int_{y_2}^a y_2 (\theta_m(y_2) - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&\quad + v F^{K-2}(v) [F(v) + (K-1)\theta^*(1 - F(a))]
\end{aligned} \tag{B.48}$$

Substituting for $v = w$ in the equation above and using (B.47) yields:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(w) &= \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy [1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \\
&\quad - \theta^*(1 - F(a))] + r_{PRE}^H F^{K-2}(r_{PRE}^H) [F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] - w F^{K-2}(w) [2F(w) + \\
&\quad + (K-1)(1 - F(w) - \int_w^a \theta_m(y)f(y)dy)] + w F^{K-2}(w) [F(w) + (K-1)\theta^*(1 - F(a))] = \\
&= \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy [1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \\
&\quad - \theta^*(1 - F(a))] + r_{PRE}^H F^{K-2}(r_{PRE}^H) [F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] - \\
&\quad - w [F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] = 0 \implies \\
&\implies \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy [1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \\
&\quad - \theta^*(1 - F(a))] = (w - r_{PRE}^H) F^{K-2}(r_{PRE}^H) [F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))]
\end{aligned} \tag{B.49}$$

By using this last equation along with (B.47), equation (B.48) becomes:

$$\begin{aligned}
(\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a) &= - \int_w^v F^{K-1}(y)dy + \theta^*(K-1)(1 - F(a)) \int_w^v F^{K-2}(y)dy - \\
&\quad - 2 \int_w^v y \theta_m(y) f_1^{(K-1)}(y) dy - (K-1)(1 - F(a)) \int_w^v y \theta_m(y) f_1^{(K-2)}(y) dy - \\
&\quad - \int_w^v \int_{y_2}^a y_2 (\theta_m(y_2) - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \\
&\quad + v F^{K-2}(v) [F(v) + (K-1)\theta^*(1 - F(a))] - w F^{K-2}(w) [F(w) + (K-1)\theta^*(1 - F(a))] = 0
\end{aligned} \tag{B.50}$$

The system of equations (B.47) and (B.49) gives a solution for the two cut-off points, w and a . However, we can eliminate a and find w directly. Then, we can use this

solution to find the appropriate value for a using any of these two last equations. More specifically, if we set $x_a = 1 - F(w) - \int_w^a \theta_m(y)f(y)dy - \theta^*(1 - F(a))$, (B.47) becomes:

$$\begin{aligned} F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] &= F^{K-2}(w)[F(w) + (K-1)x_a] \implies \\ \implies (K-1)F^{K-2}(w)x_a &= F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] - F^{K-1}(w) \end{aligned} \quad (\text{B.51})$$

Similarly, (B.49) becomes:

$$\begin{aligned} x_a(K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy &= (w - r_{PRE}^H)F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + \\ &+ (K-1)(1 - F(r_{PRE}^H))] - \int_{r_{PRE}^L}^w F^{K-1}(y)dy \end{aligned} \quad (\text{B.52})$$

Eliminating x_a from these two last equations yields:

$$\begin{aligned} F^{K-2}(w) \int_{r_{PRE}^L}^w F^{K-1}(y)dy - F^{K-1}(w) \int_{r_{PRE}^H}^w F^{K-2}(y)dy &= \\ = [(w - r_{PRE}^H)F^{K-2}(w) - \int_{r_{PRE}^H}^w F^{K-2}(y)dy]F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] & \end{aligned} \quad (\text{B.53})$$

and then a can be found from (B.47) for the provided w .

Using equations (B.46), (B.47), (B.49) and B.50, (B.45) can be significantly simplified, giving:

$$\begin{aligned} (\Pi_{PRE}^L - \Pi_{PRE}^H)(a < v \leq 1) &= \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (K-1) \int_{r_{PRE}^H}^w F^{K-2}(y)dy[1 - F(w) - \\ &- \int_w^a \theta_m(y)f(y)dy - \theta^*(1 - F(a))] + r_{PRE}^H F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] - \\ &- wF^{K-2}(w)[2F(w) + (K-1)(1 - F(w) - \int_w^a \theta_m(y)f(y)dy)] - \int_w^a F^{K-1}(y)dy - \\ &- \theta^*(K-1)(1 - F(a)) \int_w^a F^{K-2}(y)dy - 2 \int_w^a y\theta_m(y)f_1^{(K-1)}(y)dy - \\ &- (K-1)(1 - F(a)) \int_w^a y\theta_m(y)f_1^{(K-2)}(y)dy - \\ &- \int_w^a \int_{y_2}^a y_2(\theta_m(y_2) - \theta_m(y_1))f_{1,2}^{(K-1)}(y_1, y_2)dy_1 dy_2 - \\ &- (1 - 2\theta^*) \int_a^v F^{K-1}(y)dy + \theta^* a F^{K-2}(a)[2F(a) + (K-1)(1 - F(a))] + \\ &+ v \left\{ -F^{K-2}(r_{PRE}^H)[F(r_{PRE}^H) + (K-1)(1 - F(r_{PRE}^H))] + F^{K-2}(w)[2F(w) + \right. \\ &\left. + (K-1)(1 - F(w) - \int_w^a \theta_m(y)f(y)dy)] + 2 \int_w^a \theta_m(y)f_1^{(K-1)}(y)dy + \right. \end{aligned}$$

$$\begin{aligned}
& + (K-1)(1-F(a)) \int_w^a \theta_m(y) f_1^{(K-2)}(y) dy + \\
& + \int_w^a \int_{y_2}^a (\theta_m(y_2) - \theta_m(y_1)) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 - \\
& - \theta^* F^{K-2}(a)[2F(a) + (K-1)(1-F(a))] \Big\} = \\
& = (w - r_{PRE}^H) F^{K-2}(r_{PRE}^H) [F(r_{PRE}^H) + (K-1)(1-F(r_{PRE}^H))] + r_2 F^{K-2}(r_{PRE}^H) [F(r_{PRE}^H) + \\
& + (K-1)(1-F(r_{PRE}^H))] - w F^{K-2}(w) [2F(w) + (K-1)(1-F(w) - \int_w^a \theta_m(y) f(y) dy)] + \\
& + w F^{K-2}(w) [F(w) + (K-1)\theta^*(1-F(a))] - a F^{K-2}(a) [F(a) + (K-1)\theta^*(1-F(a))] - \\
& - (1-2\theta^*) \int_a^v F^{K-1}(y) dy + \theta^* a F^{K-2}(a) [2F(a) + (K-1)(1-F(a))] + \\
& + v \left\{ F^{K-2}(a) [F(a) + (K-1)\theta^*(1-F(a))] - \theta^* F^{K-2}(a) [2F(a) + (K-1)(1-F(a))] \right\} = \\
& = -w F^{K-2}(w) [F(w) + (K-1)\theta^*(1-F(a))] + w F^{K-2}(w) [F(w) + (K-1)\theta^*(1-F(a))] - \\
& - a F^{K-2}(a) [F(a) + (K-1)\theta^*(1-F(a))] - (1-2\theta^*) \int_a^v F^{K-1}(y) dy + \\
& + \theta^* a F^{K-2}(a) [2F(a) + (K-1)(1-F(a))] + v \left\{ F^{K-2}(a) [F(a) + (K-1)\theta^*(1-F(a))] - \right. \\
& \left. - \theta^* F^{K-2}(a) [2F(a) + (K-1)(1-F(a))] \right\} = \\
& = -(1-2\theta^*) \int_a^v F^{K-1}(y) dy + (v-a) F^{K-2}(a) [F(a) + (K-1)\theta^*(1-F(a))] - \\
& - \theta^* (v-a) F^{K-2}(a) [2F(a) + (K-1)(1-F(a))] = \\
& = -(1-2\theta^*) \left[\int_a^v F^{K-1}(y) dy - (v-a) F^{K-1}(a) \right] \tag{B.54}
\end{aligned}$$

After all derivations, the final expressions for the expected surplus of a buyer from the two intermediaries in equilibrium will be:

$$\Pi_{PRE}^L(r_{PRE}^L \leq v < r_{PRE}^H) = \int_{r_{PRE}^L}^v F^{K-1}(y) dy \tag{B.55}$$

$$\begin{aligned}
\Pi_{PRE}^L(r_{PRE}^H \leq v < w) &= \int_{r_{PRE}^L}^v F^{K-1}(y) dy + (K-1) \left[\int_{r_{PRE}^H}^v F^{K-2}(y) dy - \right. \\
&\quad \left. - (v - r_{PRE}^H) F^{K-2}(r_{PRE}^H) \right] [1 - F(w) - \int_w^a \theta_m(y) f(y) dy - \theta^*(1-F(a))] \tag{B.56}
\end{aligned}$$

$$\begin{aligned}
\Pi_{PRE}^H(r_{PRE}^H \leq v < w) &= (v - r_{PRE}^H) F^{K-2}(r_{PRE}^H) \{F(r_{PRE}^H) + (K-1)[F(w) - F(r_{PRE}^H) + \\
& + \int_w^a \theta_m(y) f(y) dy + \theta^*(1-F(a))] \} \tag{B.57}
\end{aligned}$$

$$\begin{aligned}
\Pi_{PRE}^L(w \leq v \leq a) = & \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (v-w)F^{K-1}(w) + (K-1)\left[\int_{r_{PRE}^H}^w F^{K-2}(y)dy + \right. \\
& + (v-w)F^{K-2}(w) - (v-r_{PRE}^H)F^{K-2}(r_{PRE}^H)][1-F(w) - \int_w^a \theta_m(y)f(y)dy - \theta^*(1-F(a))] + \\
& - (K-1)(K-3) \int_w^v (v-y)\theta_m(y)F^{K-2}(y)f(y)dy + \\
& + (K-1)(K-2)[1-\theta^*(1-F(a))] \int_w^v (v-y)\theta_m(y)F^{K-3}(y)f(y)dy - \\
& - (K-1)(K-2) \int_w^v (v-y_2)\theta_m(y_2)F^{K-3}(y_2)f(y_2)dy_2 \int_{y_2}^a \theta_m(y_1)f(y_1)dy_1
\end{aligned} \tag{B.58}$$

$$\begin{aligned}
\Pi_{PRE}^H(w \leq v \leq a) = & (v-r_{PRE}^H)F^{K-2}(r_{PRE}^H)\{F(r_{PRE}^H) + (K-1)[F(w) - F(r_{PRE}^H) + \\
& + \int_w^a \theta_m(y)f(y)dy + \theta^*(1-F(a))]\} + \int_w^v F^{K-1}(y)dy - (v-w)F^{K-1}(w) + \\
& + (K-1)\theta^*[1-F(a)][\int_w^v F^{K-2}(y)dy - (v-w)F^{K-2}(w)] - \\
& - (K-1) \int_w^v (v-y)\theta_m(y)F^{K-2}(y)f(y)dy - \\
& + (K-1)(K-2)[- \theta^*(1-F(a)) \int_w^v (v-y)\theta_m(y)F^{K-3}(y)f(y)dy + \\
& + \int_w^v (v-y_2)F^{K-3}(y_2)f(y_2)dy_2 \int_{y_2}^a \theta_m(y_1)f(y_1)dy_1 - \\
& - \int_w^v (v-y_2)\theta_m(y_2)F^{K-3}(y_2)f(y_2)dy_2 \int_{y_2}^a \theta_m(y_1)f(y_1)dy_1]
\end{aligned} \tag{B.59}$$

$$\begin{aligned}
\Pi_{PRE}^L(a < v \leq 1) = & \int_{r_{PRE}^L}^w F^{K-1}(y)dy + (v-w)F^{K-1}(w) + \\
& + (K-1)\left[\int_{r_{PRE}^H}^w F^{K-2}(y)dy + (v-w)F^{K-2}(w) - (v-r_{PRE}^H)F^{K-2}(r_{PRE}^H)\right] \\
& [1-F(w) - \int_w^a \theta_m(y)f(y)dy - \theta^*(1-F(a))] + \theta^*\left[\int_a^v F^{K-1}(y)dy - (v-a)F^{K-1}(a)\right] + \\
& + (K-1)\theta^*(1-\theta^*)\left[\int_a^v F^{K-2}(y)dy - (v-a)F^{K-2}(a)\right] - \\
& - (K-2)\theta^*(1-\theta^*)\left[\int_a^v F^{K-1}(y)dy - (v-a)F^{K-1}(a)\right] + \\
& + (K-1)(K-3) \int_w^a (v-y)\theta_m(y)F^{K-2}(y)f(y)dy +
\end{aligned}$$

$$\begin{aligned}
& + (K-1)(K-2)\{[1-\theta^*(1-F(a))]\int_w^a(v-y)\theta_m(y)F^{K-3}(y)f(y)dy - \\
& - \int_w^a(v-y_2)\theta_m(y_2)F^{K-3}(y_2)f(y_2)dy_2 \int_{y_2}^a\theta_m(y_1)f(y_1)dy_1\}
\end{aligned} \tag{B.60}$$

$$\begin{aligned}
\Pi_{PRE}^H(a < v \leq 1) = & (v - r_{PRE}^H)F^{K-2}(r_{PRE}^H)\{F(r_{PRE}^H) + (K-1)[F(w) - F(r_{PRE}^H) + \\
& + \int_w^a\theta_m(y)f(y)dy + \theta^*(1-F(a))] + \int_w^vF^{K-1}(y)dy - (v-w)F^{K-1}(w) + \\
& + (K-1)\theta^*(1-F(a))[\int_w^aF^{K-2}(y)dy + (v-a)F^{K-2}(a) - (v-w)F^{K-2}(w)] - \\
& - \theta^*[\int_a^vF^{K-1}(y)dy - (v-a)F^{K-1}(a)] + \\
& + (K-1)\theta^*(1-\theta^*)[\int_a^vF^{K-2}(y)dy - (v-a)F^{K-2}(a)] - \\
& - (K-2)\theta^*(1-\theta^*)[\int_a^vF^{K-1}(y)dy - (v-a)F^{K-1}(a)] - \\
& - (K-1)\int_w^a(v-y)\theta_m(y)F^{K-2}(y)f(y)dy + \\
& + (K-1)(K-2)[- \theta^*(1-F(a))\int_w^a(v-y)\theta_m(y)F^{K-3}(y)f(y)dy + \\
& + \int_w^a(v-y_2)F^{K-3}(y_2)f(y_2)dy_2 \int_{y_2}^a\theta_m(y_1)f(y_1)dy_1 - \\
& - \int_w^a(v-y_2)\theta_m(y_2)F^{K-3}(y_2)f(y_2)dy_2 \int_{y_2}^a\theta_m(y_1)f(y_1)dy_1]
\end{aligned} \tag{B.61}$$

B.1.6 Mixed-Strategy Bayes-Nash Equilibrium Example: The Uniform Distribution $U(0, 1)$

In this case, $F(v) = v$, $f(v) = 1$, $\forall v \in [w, a]$. Then the second-order derivative condition becomes:

$$[2v + (K-2)(1-v)]\theta_m(v) = (K-2)\int_v^a\theta_m(y)dy + (K-2)\theta^*(1-a) + v \tag{B.62}$$

where $\theta_m(a) = \frac{(K-2)\theta^*(1-a)+a}{2a+(K-2)(1-a)}$. Equation (B.62) can be written as:

$$[2v + (K-2)(1-v)]\theta_m(v) + (K-2)\int_a^v\theta_m(y)dy = (K-2)\theta^*(1-a) + v \tag{B.63}$$

We set:

$$\begin{aligned}\Theta_m(v) &= \int_a^v \theta_m(y) dy = - \int_v^a \theta_m(y) dy = -[\int_0^a \theta_m(y) dy - \int_0^v \theta_m(y) dy] = \\ &= \int_0^v \theta_m(y) dy - \int_0^a \theta_m(y) dy\end{aligned}\quad (\text{B.64})$$

So, from the first fundamental theorem of calculus $\Theta'_m(v) = \theta_m(v)$. Equation (B.63) can now be written in terms of $\Theta_m(\cdot)$ as:

$$[2v + (K-2)(1-v)]\Theta'_m(v) + (K-2)\Theta_m(v) = (K-2)\theta^*(1-a) + v \quad (\text{B.65})$$

The form of the differential equation changes for $K = 4$ so we have to consider two cases: i) $K \neq 4$ and ii) $K = 4$.

We start with the latter case ($K = 4$). Dividing by 2 and multiplying both sides with $\exp(v)$ yields:

$$\exp(v)\Theta'_m(v) + \exp(v)\Theta_m(v) = -\exp(v)(-\theta^*(1-a) - \frac{v}{2}) \quad (\text{B.66})$$

Then, we substitute with $(\exp(v))' = \exp(v)$ in the left-hand side gives:

$$\exp(v)\Theta'_m(v) + (\exp(v))'\Theta_m(v) = -\exp(v)(-\theta^*(1-a) - \frac{v}{2}) \quad (\text{B.67})$$

Applying the reverse product rule ($f \frac{dg}{dx} + \frac{df}{dx}g = \frac{d}{dx}(fg)$) to the left-hand side yields:

$$(\exp(v)\Theta_m(v))' = -\exp(v)(-\theta^*(1-a) - \frac{v}{2}) \quad (\text{B.68})$$

where integrating both sides with respect to v gives:

$$\exp(v)\Theta_m(v) = \frac{1}{2}\exp(v)[2\theta^*(1-a) + v - 1] + k_4 \quad (\text{B.69})$$

where k_4 is an arbitrary constant. Then, dividing both sides with $\exp(v)$ yields:

$$\Theta_m(v) = k_4 \exp(-v) + \theta^*(1-a) + \frac{v-1}{2} \quad (\text{B.70})$$

So,

$$\theta_m(v) = \Theta'_m(v) = \frac{1}{2} - k_4 \exp(-v) \quad (\text{B.71})$$

where the initial condition $\theta_m(a) = \frac{2\theta^*(1-a)+a}{2}$ gives $k_4 = \frac{(1-2\theta^*)(1-a)}{2 \exp(-a)}$, hence:

$$\theta_m(v) = \frac{1}{2} - \frac{(1-2\theta^*)(1-a)}{2 \exp(-a)} \exp(-v) \quad (\text{B.72})$$

Similarly, we will solve (B.65) for the general case ($K \neq 4$). Dividing by $-[2v + (K - 2)(1 - v)] = (K - 4)v - (K - 2)$ and rearranging gives:

$$\Theta'_m(v) - \frac{K-2}{(K-4)v-(K-2)}\Theta_m(v) = -\frac{(K-2)\theta^*(1-a)+v}{(K-4)v-(K-2)} \quad (\text{B.73})$$

Multiplying both sides by $\exp(\int \frac{-(K-2)}{(K-4)v-(K-2)} dv) = [(K-4)v-(K-2)]^{-\frac{K-2}{K-4}}$ gives:

$$\begin{aligned} & [(K-4)v-(K-2)]^{-\frac{K-2}{K-4}}\Theta'_m(v) - (K-2)[(K-4)v-(K-2)]^{-\frac{K-2}{K-4}-1}\Theta_m(v) = \\ & = -[(K-2)\theta^*(1-a)+v][(K-4)v-(K-2)]^{-\frac{K-2}{K-4}-1} \end{aligned} \quad (\text{B.74})$$

We then substitute $-(K-2)[(K-4)v-(K-2)]^{-\frac{K-2}{K-4}-1} = \frac{d}{dv}\{[(K-4)v-(K-2)]^{-\frac{K-2}{K-4}}\}$:

$$\begin{aligned} & [(K-4)v-(K-2)]^{-\frac{K-2}{K-4}}\Theta'_m(v) + \frac{d}{dv}\{[(K-4)v-(K-2)]^{-\frac{K-2}{K-4}}\}\Theta_m(v) = \\ & = -[(K-2)\theta^*(1-a)+v][(K-4)v-(K-2)]^{-\frac{K-2}{K-4}-1} \end{aligned} \quad (\text{B.75})$$

Applying the reverse product rule to the left-hand side yields:

$$\frac{d}{dv}\{[(K-4)v-(K-2)]^{-\frac{K-2}{K-4}}\Theta_m(v)\} = -[(K-2)\theta^*(1-a)+v][(K-4)v-(K-2)]^{-\frac{K-2}{K-4}-1} \quad (\text{B.76})$$

and afterwards integrating both sides with respect to v gives:

$$[(K-4)v-(K-2)]^{-\frac{K-2}{K-4}}\Theta_m(v) = \frac{1}{2}[2\theta^*(1-a)+v-1][(K-4)v-(K-2)]^{-\frac{K-2}{K-4}}+k_n \quad (\text{B.77})$$

where k_n is an arbitrary constant. Finally, dividing both sides by $[(K-4)v-(K-2)]^{-\frac{K-2}{K-4}}$ produces the following solution for $\Theta_m(v)$:

$$\Theta_m(v) = \theta^*(1-a) + \frac{v-1}{2} + k_n[(K-4)v-(K-2)]^{\frac{2}{K-4}} \quad (\text{B.78})$$

and hence:

$$\theta_m(v) = \Theta'_m(v) = \frac{1}{2} + k_n(K-2)[(K-4)v-(K-2)]^{\frac{2}{K-4}} \quad (\text{B.79})$$

Finally, using the fact that $\theta_m(a) = \frac{(K-2)\theta^*(1-a)+a}{2a+(K-2)(1-a)}$, we get that $k_n = \frac{(1-2\theta^*)(1-a)}{2[(K-4)a-(K-2)]^{\frac{K-2}{K-4}}}$, so the final form of $\theta_m(v)$ will be:

$$\theta_m(v) = \frac{1}{2} + \frac{(K-2)(1-2\theta^*)(1-a)}{2[(K-4)a-(K-2)]^{\frac{K-2}{K-4}}}[(K-4)v-(K-2)]^{\frac{2}{K-4}} \quad (\text{B.80})$$

Then, equations (B.38), (B.41), (B.50) and (B.54) yield:

$$(\Pi_{PRE}^L - \Pi_{PRE}^H)(r_{PRE}^L \leq v < r_{PRE}^H) = \frac{v^K - r_{PRE}^L^K}{K} \quad (\text{B.81})$$

For $K \neq 4$:

$$\begin{aligned} (\Pi_{PRE}^L - \Pi_{PRE}^H)(r_{PRE}^H \leq v < w) &= \frac{v^K - r_{PRE}^{L^K}}{K} - (v - r_{PRE}^H)r_{PRE}^{H^{K-2}}[r_{PRE}^H + (K-1)(1 - r_{PRE}^H)] + \\ &+ (v^{K-1} - r_{PRE}^{H^{K-1}})[\frac{1-w}{2} + \frac{(1-2\theta^*)(1-a)}{2[(K-4)a-(K-2)]^{\frac{K-2}{K-4}}}[(K-4)w-(K-2)]^{\frac{K-2}{K-4}}] \end{aligned} \quad (\text{B.82})$$

For $K = 4$:

$$\begin{aligned} (\Pi_{PRE}^L - \Pi_{PRE}^H)(r_{PRE}^H \leq v < w) &= \frac{v^4 - r_{PRE}^{L^4}}{4} - (v - r_{PRE}^H)r_{PRE}^{H^2}[r_{PRE}^H + 3(1 - r_{PRE}^H)] + \\ &+ (v^3 - r_{PRE}^{H^3})[\frac{1-w}{2} + \frac{(1-2\theta^*)(1-a)}{2 \exp(-a)} \exp(-w)] \end{aligned} \quad (\text{B.83})$$

$$(\Pi_{PRE}^L - \Pi_{PRE}^H)(w \leq v \leq a) = 0 \quad (\text{B.84})$$

$$(\Pi_{PRE}^L - \Pi_{PRE}^H)(a < v \leq 1) = -(1-2\theta^*)[\frac{v^K - a^K}{K} - (v-a)a^{K-1}] \quad (\text{B.85})$$

Moreover, the conditions of equations (B.53), (B.47) yield:

$$\begin{aligned} [w^{K-1} - r_{PRE}^{H^{K-1}} - (K-1)(w - r_{PRE}^H)w^{K-2}]r_{PRE}^{H^{K-2}}[r_{PRE}^H + (K-1)(1 - r_{PRE}^H)] + \\ + \frac{K-1}{K}w^{K-2}(w^K - r_{PRE}^{L^K}) - w^{K-1}(w^{K-1} - r_{PRE}^{H^{K-1}}) = 0 \end{aligned} \quad (\text{B.86})$$

and for a , when $K \neq 4$:

$$\begin{aligned} w^{K-2}\{w + (K-1)[\frac{1-w}{2} + \frac{(1-2\theta^*)(1-a)}{2[(K-4)a-(K-2)]^{\frac{K-2}{K-4}}}[(K-4)w-(K-2)]^{\frac{K-2}{K-4}}]\} - \\ - r_{PRE}^{H^{K-2}}[r_{PRE}^H + (K-1)(1 - r_{PRE}^H)] = 0 \end{aligned} \quad (\text{B.87})$$

whereas when $K = 4$:

$$\begin{aligned} w^2\{w + 3[\frac{1-w}{2} + \frac{(1-2\theta^*)(1-a)}{2 \exp(-a)} \exp(-w)]\} - \\ - r_{PRE}^{H^2}[r_{PRE}^H + 3(1 - r_{PRE}^H)] = 0 \end{aligned} \quad (\text{B.88})$$

B.2 Derivations for Buyer PRE - POST Duopoly Intermediary Selection

We can write:

$$\begin{aligned}
& \int_{y_2=0}^{r_{POST}^H} \int_{y_1=y_2}^1 \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 = \\
&= \int_0^{r_{POST}^H} \int_{y_2}^{r_{POST}^H} \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \int_0^{r_{POST}^H} \int_{r_{POST}^H}^1 \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 = \\
&= \int_0^{r_{POST}^H} \int_0^{y_1} \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_2 dy_1 + (K-1) F^{K-2}(r_{POST}^H) \int_{r_{POST}^H}^1 \theta(y) f(y) dy = \\
&= (n-1) \int_0^{r_{POST}^H} \theta(y) f(y) F^{K-2}(y) dy + (K-1) F^{K-2}(r_{POST}^H) \int_{r_{POST}^H}^1 \theta(y) f(y) dy = \\
&\quad (B.89)
\end{aligned}$$

$$= \int_0^{r_{POST}^H} \theta(y) f_1^{(K-1)}(y) dy + (K-1) F^{K-2}(r_{POST}^H) \int_{r_{POST}^H}^1 \theta(y) f(y) dy \quad (B.90)$$

We can also write:

$$\begin{aligned}
& \int_{y_2=r_{POST}^H}^v \int_{y_1=y_2}^1 (v-y_2) \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 = \\
&= \int_{r_{POST}^H}^v \int_{y_2}^v (v-y_2) \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 + \int_{r_{POST}^H}^v \int_v^1 (v-y_2) \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 = \\
&= \int_{r_{POST}^H}^v \int_{r_{POST}^H}^{y_1} (v-y_2) \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_2 dy_1 + \int_{r_{POST}^H}^v \int_v^1 (v-y_2) \theta(y_1) f_{1,2}^{(K-1)}(y_1, y_2) dy_1 dy_2 = \\
&= (K-1) \int_{r_{POST}^H}^v \int_{r_{POST}^H}^{y_1} (v-y_2) \theta(y_1) f(y_1) f_1^{(K-2)}(y_2) dy_2 dy_1 + \\
&+ (K-1) \int_{r_{POST}^H}^v \int_v^1 (v-y_2) \theta(y_1) f(y_1) f_1^{(K-2)}(y_2) dy_1 dy_2 = \\
&= (K-1) \int_{r_{POST}^H}^v \theta(y_1) f(y_1) [(v-y_1) F^{K-2}(y_1) - (v-r_{POST}^H) F^{K-2}(r_{POST}^H) + \int_{r_{POST}^H}^{y_1} F^{K-2}(y_2) dy_2] dy_1 \\
&+ (K-1) [-(v-r_{POST}^H) F^{K-2}(r_{POST}^H) + \int_{r_{POST}^H}^v F^{K-2}(y_2) dy_2] \int_v^1 \theta(y_1) f(y_1) dy_1 = \\
&= \int_{r_{POST}^H}^v (v-y) \theta(y) f_1^{(K-1)}(y) dy - (K-1)(v-r_{POST}^H) F^{K-2}(r_{POST}^H) \int_{r_{POST}^H}^1 \theta(y) f(y) dy + \\
&+ (K-1) \int_{r_{POST}^H}^v \int_{r_{POST}^H}^{y_1} \theta(y_1) f(y_1) F^{K-2}(y_2) dy_2 dy_1 + \\
&+ (K-1) \int_{r_{POST}^H}^v \int_v^1 \theta(y_1) f(y_1) F^{K-2}(y_2) dy_1 dy_2 = \\
&= \int_{r_{POST}^H}^v (v-y) \theta(y) f_1^{(K-1)}(y) dy - (K-1)(v-r_{POST}^H) F^{K-2}(r_{POST}^H) \int_{r_{POST}^H}^1 \theta(y) f(y) dy +
\end{aligned}$$

$$\begin{aligned}
& + (K-1) \int_{r_{POST}^H}^v \int_{y_2}^v \theta(y_1) f(y_1) F^{K-2}(y_2) dy_1 dy_2 + (K-1) \int_{r_{POST}^H}^v \int_v^1 \theta(y_1) f(y_1) F^{K-2}(y_2) dy_1 dy_2 = \\
& = \int_{r_{POST}^H}^v (v-y) \theta(y) f_1^{(K-1)}(y) dy - (K-1)(v-r_{POST}^H) F^{K-2}(r_{POST}^H) \int_{r_{POST}^H}^1 \theta(y) f(y) dy + \\
& + (K-1) \int_{r_{POST}^H}^v \int_{y_2}^1 \theta(y_1) f(y_1) F^{K-2}(y_2) dy_1 dy_2
\end{aligned} \tag{B.91}$$

This means that equation (6.67) can be written as:

$$\begin{aligned}
\Pi_{POST}^H(v) &= (v-r_{POST}^H) \left[\int_0^{r_{POST}^H} \theta(y) f_1^{(K-1)}(y) dy + (K-1) F^{K-2}(r_{POST}^H) \int_{r_{POST}^H}^1 \theta(y) f(y) dy \right] + \\
& + \int_{r_{POST}^H}^v (v-y)(1-\theta(y)) f_1^{(K-1)}(y) dy + \int_{r_{POST}^H}^v (v-y) \theta(y) f_1^{(K-1)}(y) dy - \\
& - (K-1)(v-r_{POST}^H) F^{K-2}(r_{POST}^H) \int_{r_{POST}^H}^1 \theta(y) f(y) dy + \\
& + (K-1) \int_{r_{POST}^H}^v \int_{y_2}^1 \theta(y_1) f(y_1) F^{K-2}(y_2) dy_1 dy_2 = \\
& = (v-r_{POST}^H) \int_0^{r_{POST}^H} \theta(y) f_1^{(K-1)}(y) dy + \int_{r_{POST}^H}^v (v-y) f_1^{(K-1)}(y) dy + \\
& + (K-1) \int_{r_{POST}^H}^v \int_{y_2}^1 \theta(y_1) f(y_1) F^{K-2}(y_2) dy_1 dy_2
\end{aligned} \tag{B.92}$$

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