$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}{r\,!}\,x^r$$

Expansions to be remembered (|x| < 1)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots. + (-1)^r x^r + \ldots.$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^5 + \ldots. + x^r + \ldots..$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots. + (-1)^r (r + 1) x^r + \ldots.$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots. + (r + 1)x^r + \ldots.$

PERMUTATION & COMBINNATION

Permutations are arrangements and combinations are selections.

1. **Fundamental Principle of Counting :**

(i) Principle of Multiplication:

If an event can occur in 'm' different ways, following which another event can occur in 'n', different ways, then total number of different ways of simultaneous occurrence of both the events in a definite order is m × n.

(ii) Principle of Addition:

If an event can occur in 'm' different ways, and another event can occur in 'n' different ways, then exactly one of the events can happen in m + n ways.

2. **Arrangement :** number of permutations of n different things taken r at a time =

$${}^nP_r = n\,(n-1)\,(n-2)\ldots\,(n-r+1) = \frac{n\,!}{(n-r)!}$$

3. **Circular Permutation :**

The number of circular permutations of n different things taken all at a time is; (n – 1)!.

4. **Selection :** Number of combinations of n different things taken r at a time = ${}^nC_r = \frac{n\,!}{r!\,(n-r)!} = \frac{{}^nP_r}{r!}$

5. The number of permutations of 'n' things, taken all at a time, when 'p' of them are similar & of one type, q of them are similar & of another type, 'r' of them are similar & of a third type & the remaining n – (p + q + r) are all different is $\frac{n\,!}{p\,!\,q\,!\,r\,!}$.

6. **Formation of Groups :**

Number of ways in which (m + n + p) different things can be divided into three different groups containing m, n & p things respectively is $\frac{(m+n+p)!}{m\,!\,n\,!\,p\,!}$,

If m = n = p and the groups have identical qualitative characteristic then the number of groups $= \frac{(3n)!}{n!\,n!\,n!\,3!}$.

However, if 3n things are to be divided equally among three people then the number of ways $= \frac{(3n)!}{(n!)^3}$.

7. **Selection of one or more objects**

(a) Number of ways in which atleast one object be selected out of 'n' distinct objects is
$${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$$

(b) Number of ways in which atleast one object may be selected out of 'p' alike objects of one type 'q' alike objects of second type and 'r' alike of third type is
$$(p + 1)(q + 1)(r + 1) - 1$$

(c) Number of ways in which atleast one object may be selected from 'n' objects where 'p' alike of one type 'q' alike of second type and 'r' alike of third type and rest
$n - (p + q + r)$ are different, is
$$(p + 1)(q + 1)(r + 1)\,2^{n-(p+q+r)} - 1$$

8. **Multinomial Theorem:**

Coefficient of x^r in expansion of $(1 - x)^{-n} = {}^{n+r-1}C_r\ (n \in N)$

Number of ways in which it is possible to make a selection from $m + n + p = N$ things, where p are alike of one kind, m alike of second kind & n alike of third kind taken r at a time is given by coefficient of x^r in the expansion of
$$(1 + x + x^2 + \ldots + x^p)(1 + x + x^2 + \ldots + x^m)(1 + x + x^2 + \ldots + x^n).$$

(i) For example the number of ways in which a selection of four letters can be made from the letters of the word **PROPORTION** is given by coefficient of x^4 in
$$(1 + x + x^2 + x^3)(1 + x + x^2)(1 + x + x^2)(1 + x)(1 + x)(1 + x).$$

(ii) **Method of fictious partition :**

Number of ways in which n identical things may be distributed among p persons if each person may receive none, one or more things is; ${}^{n+p-1}C_n$.

9. Let $N = p^a\, q^b\, r^c \ldots$ where p, q, r...... are distinct primes & a, b, c..... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is $= (a + 1)(b + 1)(c + 1)\ldots\ldots$

(b) The sum of these divisors is =
$$(p^0 + p^1 + p^2 + \ldots + p^a)(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)\ldots\ldots$$

(c) Number of ways in which N can be resolved as a product of two factors is
$$= \begin{cases} \frac{1}{2}(a+1)(b+1)(c+1)\ldots & \text{if N is not a perfect square} \\ \frac{1}{2}\left[(a+1)(b+1)(c+1)\ldots+1\right] & \text{if N is a perfect square} \end{cases}$$

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

10. Let there be 'n' types of objects, with each type containing atleast r objects. Then the number of ways of arranging r objects in a row is n^r.

11. **Dearrangement :**

Number of ways in which 'n' letters can be put in 'n' corresponding envelopes such that no letter goes to correct envelope is $n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \ldots\ldots\ldots + (-1)^n \frac{1}{n!}\right)$

PROBABILITY

1. (i) ***Random Experiment*** : An experiment which on repeated performances gives different results.
   (ii) ***Sample Space :*** Collection of all possible outcomes of a random experiment is the sample
   (iii) ***Event :*** It is subset of sample space.
   (iv) ***Complement of event :*** An event is called complement of the event A if it occurs when ever A does not occur and vice-versa. It is denoted by A', $\bar{A}$ or A^C.
   (v) ***Compound Event :*** $A \cap B$, $A \cup B$, $A - B$ etc. are compound events.
   (vi) ***Equally likely Events :*** If events have same chance of occurrence, then they are said to be equally likely.
   (vii) ***Mutually Exclusive / Disjoint / Incompatible Events :*** $A \cap B = \Phi$
   (viii) ***Exhaustive Events :*** $E_1, E_2, E_3, \ldots\ldots E_n$ are exhaustive if $E_1 \cup E_2 \cup E_3 \ldots\ldots E_n = S$.

2. **Classical (A priori) Definition of Probability :**
   If an experiment results in a total of $(m + n)$ outcomes which are equally likely and mutually exclusive with one another and if 'm' outcomes are favorable to an event 'A' while 'n' are unfavorable, then the probability of occurrence of the event 'A' $= P(A) = \dfrac{m}{m+n} = \dfrac{n(A)}{n(S)}$.

   We say that odds in favour of 'A' are $m : n$, while odds against 'A' are $n : m$.

   $$P(\bar{A}) = \frac{n}{m+n} = 1 - P(A)$$

3. **Addition theorem of probability :** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
   De Morgan's Laws : (a) $(A \cup B)^c = A^c \cap B^c$ (b) $(A \cap B)^c = A^c \cup B^c$
   Distributive Laws : (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (i) $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
   (ii) $P(\text{at least two of } A, B, C \text{ occur}) = P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$
   (iii) $P(\text{exactly two of } A, B, C \text{ occur}) = P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$
   (iv) $P(\text{exactly one of } A, B, C \text{ occur}) =$
   $P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

4. **Conditional Probability :** $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$.

5. **Independent and dependent events :** $P(A \cap B) = P(A)\,P(B)$, then A and B are independent.
   (i) If A and B are independent, then (a) A' and B' are independent, (b) A and B' are independent and (c) A' and B are independent.
   (ii) If A and B are independent, then $P(A / B) = P(A)$.

Independency of three or more events

Three events A, B & C are independent if & only if all the following conditions hold :

$P(A \cap B) = P(A) \,.\, P(B)$; $P(B \cap C) = P(B) \,.\, P(C)$

$P(C \cap A) = P(C) \,.\, P(A)$; $P(A \cap B \cap C) = P(A) \,.\, P(B) \,.\, P(C)$

6. **Binomial Probability Theorem**
   If an experiment is such that the probability of success or failure does not change with trials, then the probability of getting exactly r success in n trials of an experiment is ${}^nC_r\, p^r q^{n-r}$, where 'p' is the probability of a success and q is the probability of a failure. Note that $p + q = 1$.

7. **Expectation :**

If a value M_i is associated with a probability of p_i, then the expectation is given by $\Sigma p_i M_i$.

8. **Total Probability Theorem :** $P(A) = \sum_{i=1}^{n} P(B_i).P(A/B_i)$

9. **Bayes' Theorem :**

If an event A can occur with one of the n mutually exclusive and exhaustive events $B_1, B_2, \ldots, B_n$ and the probabilities $P(A/B_1), P(A/B_2) \ldots P(A/B_n)$ are known, then

$$P(B_i/A) = \frac{P(B_i).P(A/B_i)}{\sum_{i=1}^{n} P(B_i).P(A/B_i)}$$

$B_1, B_2, B_3, \ldots, B_n$

$$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots\ldots \cup (A \cap B_n)$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots\ldots + P(A \cap B_n) = \sum_{i=1}^{n} P(A \cap B_i)$$

10. **Binomial Probability Distribution :**

(i) Mean of any probability distribution of a random variable is given by : $\mu = \frac{\Sigma p_i x_i}{\Sigma p_i} = \Sigma p_i x_i$

(ii) Variance of a random variable is given by, $\sigma^2 = \Sigma (x_i - \mu)^2 . p_i = \Sigma p_i x_i^2 - \mu^2$

COMPLEX NUMBER

1. **The complex number system**

A number of the form a + bi where a and b are real number and $i^2 = -1$ is called a complex number. It is denoted by z i.e. z = a + ib. a is called as real part of z which is denoted by (Re z) and b is called imaginary part of z which is denoted by (Im z).

(i) Purely real, if b = 0 (ii) Purely imaginary, if a = 0 (iii) Imaginary, if $b \neq 0$.

z = a + ib, then a – ib is called congugate of z and is denoted by $\bar{z}$.

2. **Algebraic Operations:**

Fundamental operations with complex numbers

1. $(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$
2. $(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$
3. $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$
4. $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-bi}{c-di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2i^2} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$

3. **Equality In Complex Number:** $z_1 = z_2 \Rightarrow Re(z_1) = Re(z_2)$ and $I_m(z_1) = I_m(z_2)$.

4. **Representation Of A Complex Number:**

(a) **Cartesian Form (Geometric Representation) :**

Every complex number z = x + i y can be represented by the point P(x, y).

Length OP is called modulus of z and is denoted by $|z|$. Angle $\angle XOP = \theta$ is called argument or amplitude of z.

$$|z| = \sqrt{x^2 + y^2} \ \& \ \tan\theta = \frac{y}{x}$$

(i) If θ is the argument of z, then $2n\pi + \theta$; $n \in I$ is also argument of z. Any two arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument.

(iii) For the complex number 0 + 0 i the argument is not defined.

(b) **Trignometric/Polar Representation :**

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

$\cos\theta + i\sin\theta$ is also written as CiS θ or $e^{i\theta}$.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2}$ are known as Euler's identities.

(c) **Euler's Representation :**

$z = re^{i\theta}$; $|z| = r$; $\arg z = \theta$; $\bar{z} = re^{-i\theta}$

5. Properties of modulus

(i) $|z_1 z_2| = |z_1|\,|z_2|$ (ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (provide $z_2 \neq 0$)

(iii) $|z_1 + z_2| \le |z_1| + |z_2|$ (iv) $|z_1 - z_2| \ge ||z_1| - |z_2||$

(Equality in (iii) and (iv) holds if and only if origin, z_1 and z_2 are collinear with z_1 and z_2 on the same side of origin).

6. Properties of arguments

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2m\pi$ for some integer m.

(ii) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) + 2m\pi$ for some integer m.

(iii) $\arg(z^2) = 2\arg(z) + 2m\pi$ for some integer m.

(iv) $\arg(z) = 0 \quad \Leftrightarrow \quad$ z is a positive real number

(v) $\arg(z) = \pm\pi/2 \quad \Leftrightarrow \quad$ z is purely imaginary and $z \neq 0$

7. Properties of conjugate

(i) $|z| = |\bar{z}|$ (ii) $z\bar{z} = |z|^2$ (iii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iv) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (v) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ (vi) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ $(z_2 \neq 0)$

(vii) $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

(viii) $\overline{(\bar{z}_1)} = z$ (ix) If $w = f(z)$, then $\bar{w} = f(\bar{z})$

(x) $\arg(z) + \arg(\bar{z})$

8. Rotation theorem

(i) If $P(z_1)$ and $Q(z_z)$ are two complex numbers such that $|z_1| = |z_2|$, then $z_2 = z_1 e^{i\theta}$ where $\theta = \angle POQ$

(ii) If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then $\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$

(iii) If $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are complex numbers and $\angle STQ = \theta$, then $\frac{z_3 - z_4}{z_1 - z_2} = \left|\frac{z_3 - z_4}{z_1 - z_2}\right| e^{i\theta}$

9. **Demoivre's Theorem:**

Case I : If n is any integer then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots$
$.(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots\ \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots + \theta_n)$

Case II : If $p, q \in Z$ and $q \neq 0$ then $(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$

where $k = 0, 1, 2, 3, \ldots\ldots, q - 1$

10. **Cube Root Of Unity :**

(i) The cube roots of unity are $1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$.

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(iii) In polar form the cube roots of unity are : $\cos 0 + i\sin 0$; $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

(iv) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

(v) The following factorisation should be remembered : ($a, b, c \in R$ & ω is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$; $a^2 + ab + b^2 = (a - bw)(a - bw^2)$

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

11. **n^{th} Roots of Unity :**

If $1, \alpha_1, \alpha_2, \alpha_3 \ldots\ \alpha_{n-1}$ are the n, n^{th} root of unity then :

(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(ii) $1^p + \alpha_1^p + \alpha_2^p + \ldots + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n
$= n$ if p is an integral multiple of n

(iii) $(1 - \alpha_1)(1 - \alpha_2)\ldots\ldots(1 - \alpha_{n-1}) = n$ &
$(1 + \alpha_1)(1 + \alpha_2)\ldots\ldots(1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(iv) $1.\alpha_1.\alpha_2.\alpha_3\ldots\ldots\alpha_{n-1} = 1$ or -1 according as n is odd or even.

12. **The Sum Of The Following Series Should Be Remembered :**

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos\left(\frac{n+1}{2}\right)\theta.$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\dfrac{n+1}{2}\right)\theta$.

13. Logarithm Of A Complex Quantity :

(i) $\mathrm{Log}_e(\alpha + i\beta) = \dfrac{1}{2}\mathrm{Log}_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\dfrac{\beta}{\alpha}\right)$ where $n \in I$.

(ii) i^i represents a set of positive real numbers given by $e^{-\left(2n\pi + \frac{\pi}{2}\right)}$, $n \in I$.

14. Geometrical Properties:

Distance formula : $|z_1 - z_2|$.

Section formula : $z = \dfrac{mz_2 + nz_1}{m+n}$ (internal division), $z = \dfrac{mz_2 - nz_1}{m-n}$ (external division)

If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(1) If the vertices A, B, C of a Δ represent the complex nos. z_1, z_2, z_3 respectively and a, b, c are the length of sides then,

(i) Centroid of the Δ ABC $= \dfrac{z_1 + z_2 + z_3}{3}$:

(ii) Orthocentre of the Δ ABC =

$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \text{ or } \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

(iii) Incentre of the Δ ABC $= (az_1 + bz_2 + cz_3) \div (a + b + c)$.

(iv) Circumcentre of the Δ ABC = :
$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

(2) $\mathrm{amp}(z) = \theta$ is a ray emanating from the origin inclined at an angle θ to the x– axis.

(3) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(4) The equation of a line joining z_1 & z_2 is given by, $z = z_1 + t(z_1 - z_2)$ where t is a real parameter.

(5) $z = z_1(1 + it)$ where t is a real parameter is a line through the point z_1 & perpendicular to the line joining z_1 to the origin.

(6) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$

This is also the condition for three complex numbers to be collinear. The above equation on manipulating, takes the form $\bar{\alpha}z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.

NOTE : If we replace z by $ze^{i\theta}$ and $\bar{z}$ by $\bar{z}e^{-i\theta}$ then we get equation of a straight line which. Passes through the foot of the perpendicular from origin to given straight line and makes an angle θ with the given straight line.

(7) The equation of circle having centre z_0 & radius ρ is :
$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form
$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + k = 0$, k is real. Centre is $-\alpha$ & radius $= \sqrt{\alpha\bar{\alpha} - k}$.
Circle will be real if $\alpha\bar{\alpha} - k \geq 0$..

(8) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is
$\arg\dfrac{z - z_2}{z - z_1} = \pm\dfrac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.

(9) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is the number
$\dfrac{z_3 - z_1}{z_3 - z_2} \cdot \dfrac{z_4 - z_2}{z_4 - z_1}$ should be real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real

$$\Rightarrow \quad \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \frac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}.$$

(10) $\operatorname{Arg}\left(\dfrac{z - z_1}{z - z_2}\right) = \theta$ represent (i) a line segment if $\theta = \pi$
(ii) Pair of rays if $\theta = 0$ (iii) a part of circle, if $0 < \theta < \pi$.

(11) Area of triangle formed by the points z_1, z_2 & z_3 is $\left|\dfrac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|$

(12) If $|z_1 - z_1| + |z - z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose focii are z_1 & z_2

(13) If $\left|\dfrac{z - z_1}{z - z_2}\right| = k \neq 1, 0$, then locus of z is circle.

(14) If $\left||z - z_1| - |z - z_2|\right| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose focii are z_1 & z_2.

TRIGONOMETRY RATIOS AND IDENTITIES

1. Basic Trigonometric Identities:

(a) $\sin^2\theta + \cos^2\theta = 1; \ -1 \leq \sin\theta \leq 1; \ -1 \leq \cos\theta \leq 1 \ \forall \ \theta \in R$

(b) $\sec^2\theta - \tan^2\theta = 1 \ ; \ |\sec\theta| \geq 1 \ \forall \ \theta \in R - \left\{(2n+1)\dfrac{\pi}{2}, n \in I\right\}$

(c) $\operatorname{cosec}^2\theta - \cot^2\theta = 1 \ ; \ |\operatorname{cosec}\theta| \geq 1 \ \forall \ \theta \in R - \{n\pi, \ n \in I\}$

2. Trigonometric Functions Of Allied Angles:

If θ is any angle, then $-\theta$, $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ etc. are called **ALLIED ANGLES**.

(a) $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$
(b) $\sin(90° - \theta) = \cos\theta$; $\cos(90° - \theta) = \sin\theta$
(c) $\sin(90° + \theta) = \cos\theta$; $\cos(90° + \theta) = -\sin\theta$
(d) $\sin(180° - \theta) = \sin\theta$; $\cos(180° - \theta) = -\cos\theta$

(e) $\sin(180° + \theta) = -\sin\theta$; $\cos(180° + \theta) = -\cos\theta$

(f) $\sin(270° - \theta) = -\cos\theta$; $\cos(270° - \theta) = -\sin\theta$

(g) $\sin(270° + \theta) = -\cos\theta$; $\cos(270° + \theta) = \sin\theta$

(h) $\tan(90° - \theta) = \cot\theta$; $\cot(90° - \theta) = \tan\theta$

3. Trigonometric Functions of Sum or Difference of Two Angles:

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B).\sin(A-B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)$

(e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ (f) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(g) $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$.

4. Factorisation of the Sum or Difference of Two Sines or Cosines:

(a) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (b) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (d) $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

5. Transformation of Products into Sum or Difference of Sines & Cosines:

(a) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ (b) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ (d) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

6. Multiple and Sub-multiple Angles :

(a) $\sin 2A = 2\sin A\cos A$; $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$; $2\cos^2\dfrac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\dfrac{\theta}{2} = 1 - \cos\theta$.

(c) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$; $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$ (d) $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$, $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(e) $\sin 3A = 3\sin A - 4\sin^3 A$ (f) $\cos 3A = 4\cos^3 A - 3\cos A$

(g) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

7. Important Trigonometric Ratios:

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)$; $\tan n\pi = 0$, where $n \in I$

(b) $\sin 15°$ or $\sin\dfrac{\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\dfrac{5\pi}{12}$;

$\cos 15°$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\frac{5\pi}{12}$;

$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$; $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

(c) $\sin\frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4}$ & $\cos 36°$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

8. **Conditional Identities:**

If $A + B + C = \pi$ then :

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(vii) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} . \cot\frac{B}{2} . \cot\frac{C}{2}$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(ix) $A + B + C = \frac{\pi}{2}$ then $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

9. **Sine and Cosine Series:**

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\left(\alpha + \overline{n-1}\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos\left(\alpha + \overline{n-1}\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

TRIGONOMETRIC EQUATIONS

1. **Principal Solutions:** Solutions which lie in the interval $[0, 2\pi)$ are called **Principal solutions.**

2. **General Solution :**

(i) $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

(ii) $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

(iii) $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

(iv) $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$, $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

3. **Types Of Trigonometric Equations:**

(i) which can be factorised.

(ii) Reducible to quadratic equations.

(iii) $a \sin x + b \cos x = \sqrt{a^2 + b^2}$

(iv) $a \sin x + b \cos x = k$, where $|k| \le \sqrt{a^2 + b^2}$

(v) Involving trigonometric identities

(vi) Involving $\cos x \pm \sin x$

(vii) Involving Boundary conditions

(vii) Involving sum of squares = 0

SOLUTION OF TRIANGLE

1. Sine Rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

2. Cosine Formula: (i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ (ii) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ (iii) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

3. Projection Formula: (i) $a = b \cos C + c \cos B$ (ii) $b = c \cos A + a \cos C$ (iii) $c = a \cos B + b \cos A$

4. Napier's Analogy - tangent rule:

(i) $\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}$ (ii) $\tan \dfrac{C-A}{2} = \dfrac{c-a}{c+a} \cot \dfrac{B}{2}$ (iii) $\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$

5. Trigonometric Functions of Half Angles:

(i) $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$; $\sin \dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$; $\sin \dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

(ii) $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$; $\cos \dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ca}}$; $\cos \dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$

(iii) $\tan \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \dfrac{\Delta}{s(s-a)}$ where $s = \dfrac{a+b+c}{2}$ is semi perimetre of triangle.

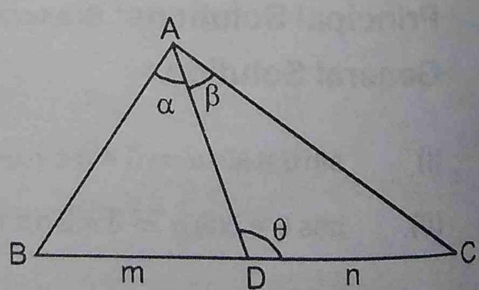
(iv) $\sin A = \dfrac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} = \dfrac{2\Delta}{bc}$

6. Area of Triangle (Δ) : $\Delta = \dfrac{1}{2} ab \sin C = \dfrac{1}{2} bc \sin A = \dfrac{1}{2} ca \sin B = \sqrt{s(s-a)(s-b)(s-c)}$

7. m - n Rule:

If BD : DC = m : n, then

$$(m+n)\cot\theta = m \cot\alpha - n \cot\beta$$
$$= n \cot B - m \cot C$$



8. Radius of Circumcirlce :

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

9. Radius of The Incircle :

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(iii) $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on

(iv) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

10. Radius of The Ex-Circles :

(i) $r_1 = \frac{\Delta}{s-a}$; $r_2 = \frac{\Delta}{s-b}$; $r_3 = \frac{\Delta}{s-c}$

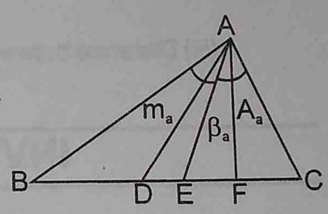
(ii) $r_1 = s\tan\frac{A}{2}$; $r_2 = s\tan\frac{B}{2}$; $r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ & so on

(iv) $r_1 = 4R\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$

11. Length of Angle Bisectors, Medians & Altitudes :

(i) Length of an angle bisector from the angle $A = \beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$;



(ii) Length of median from the angle $A = m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

& (iii) Length of altitude from the angle A $= A_a = \frac{2\Delta}{a}$

12. The Distances of The Special Points from Vertices and Sides of Triangle:

(i) Circumcentre (O) : OA = R & O_a = R cos A (ii) Incentre (I) : IA = r cosec $\frac{A}{2}$ & $I_a = r$

(iii) Excentre (I_1) : $I_1A = r_1$ cosec $\frac{A}{2}$ (iv) Orthocentre : HA = 2R cos A & H_a = 2R cos B cos C

(v) Centroid (G) : $GA = \frac{1}{3}\sqrt{2b^2+2c^2-a^2}$ & $G_a = \frac{2\Delta}{3a}$

13. Orthocentre and Pedal Triangle:

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.

(i) Its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

(ii) Its sides are a cosA = R sin 2A,

b cosB = R sin 2B and

c cosC = R sin 2C

(iii) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

14. Excentral Triangle:

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

(i) ΔABC is the pedal triangle of the $\Delta I_1 I_2 I_3$.

(ii) Its angles are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ & $\frac{\pi}{2}-\frac{C}{2}$.

(iii) Its sides are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ & $4R\cos\frac{C}{2}$.

(iv) $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

(v) Incentre I of ΔABC is the orthocentre of the excentral $\Delta I_1 I_2 I_3$.

15. Distance Between Special Points :

(i) Distance between circumcentre and orthocentre $OH^2 = R^2(1 - 8\cos A\cos B\cos C)$

(ii) Distance between circumcentre and incentre $OI^2 = R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = R^2 - 2Rr$

(iii) Distance between circumcentre and centroid $OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$

INVERSE TRIGONOMETRIC FUNCTIONS

1. Principal Values & Domains of Inverse Trigonometric/Circular Functions:

	Function		Domain	Range
(i)	$y = \sin^{-1}x$	where	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(ii)	$y = \cos^{-1}x$	where	$-1 \le x \le 1$	$0 \le y \le \pi$
(iii)	$y = \tan^{-1}x$	where	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv)	$y = \text{cosec}^{-1}x$	where	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$
(v)	$y = \sec^{-1}x$	where	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi;\ y \ne \frac{\pi}{2}$
(vi)	$y = \cot^{-1}x$	where	$x \in R$	$0 < y < \pi$

2. Properties of Inverse Trigonometric Functions:

P - 1 (i) $\sin(\sin^{-1}x) = x, \quad -1 \le x \le 1$ (ii) $\cos(\cos^{-1}x) = x, \quad -1 \le x \le 1$

(iii) $\tan(\tan^{-1}x) = x, \quad x \in R$ (iv) $\cot(\cot^{-1}x) = x, \quad x \in R$

(v) $\sec(\sec^{-1}x) = x, \quad x \le -1, x \ge 1$ (vi) $\text{cosec}(\text{cosec}^{-1}x) = x, \quad x \le -1, x \ge 1$

P - 2 (i) $\sin^{-1}(\sin x) = x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(ii) $\cos^{-1}(\cos x) = x; \quad 0 \le x \le \pi$

(iii) $\tan^{-1}(\tan x) = x; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $\cot^{-1}(\cot x) = x; \quad 0 < x < \pi$

(v) $\sec^{-1}(\sec x) = x; \quad 0 \le x \le \pi, x \ne \frac{\pi}{2}$

(vi) $\text{cosec}^{-1}(\text{cosec } x) = x; \quad x \ne 0, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

P - 3 (i) $\sin^{-1}(-x) = -\sin^{-1}x, \quad -1 \le x \le 1$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x, \quad x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x, \quad -1 \le x \le 1$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x, \quad x \in R$

P - 4 (i) $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}; x \le -1, x \ge 1$

(ii) $\sec^{-1}x = \cos^{-1}\frac{1}{x}; x \le -1, x \ge 1$

(iii) $\cot^{-1}x = \begin{cases} \tan^{-1}\frac{1}{x} & ; x > 0 \\ \pi + \tan^{-1}\frac{1}{x} & ; x < 0 \end{cases}$

P - 5 (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, -1 \le x \le 1$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad x \in R$

(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, \quad |x| \ge 1$

P - 6 (i) $\sin(\cos^{-1}x) = \cos(\sin^{-1}x) = \sqrt{1 - x^2}, -1 \le x \le 1$

(ii) $\tan(\cot^{-1}x) = \cot(\tan^{-1}x) = \frac{1}{x}, x \in R, x \ne 0$

(iii) $\text{cosec}(\sec^{-1}x) = \sec(\text{cosec}^{-1}x) = \frac{|x|}{\sqrt{x^2 - 1}}, \quad |x| > 1$

3. Identities of Addition and Substraction:

I - 1 (i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right], x \ge 0, y \ge 0 \ \& \ (x^2 + y^2) \le 1$

$= \pi - \sin^{-1}\left[x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right], x \ge 0, y \ge 0 \ \& \ x^2 + y^2 > 1$

Note that: $x^2 + y^2 \le 1 \Rightarrow 0 \le \sin^{-1}x + \sin^{-1}y \le \frac{\pi}{2}$

$x^2 + y^2 > 1 \Rightarrow \frac{\pi}{2} < \sin^{-1}x + \sin^{-1}y < \pi$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1 - x^2}\sqrt{1 - y^2}\right], \quad x \ge 0, y \ge 0$

(iii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x + y}{1 - xy}, x > 0, y > 0 \ \& \ xy < 1$

$= \pi + \tan^{-1}\frac{x + y}{1 - xy}, x > 0, y > 0 \ \& \ xy > 1 \quad = \frac{\pi}{2}, x > 0, y > 0 \ \& \ xy = 1$

I - 2 (i) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1 - y^2} - y\sqrt{1 - x^2}\right], x \ge 0, y \ge 0$

(ii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1 - x^2}\sqrt{1 - y^2}\right], \quad x \ge 0, y \ge 0, x \le y$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x - y}{1 + xy}, x \ge 0, y \ge 0$

I - 3 (i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{bmatrix} 2\sin^{-1}x & \text{if } |x| \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if } x > \frac{1}{\sqrt{2}} \\ -\left(\pi + 2\sin^{-1}x\right) & \text{if } x < -\frac{1}{\sqrt{2}} \end{bmatrix}$

(ii) $\cos^{-1}(2x^2 - 1) = \begin{bmatrix} 2\cos^{-1}x & \text{if } 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & \text{if } -1 \le x < 0 \end{bmatrix}$

(iii) $\tan^{-1}\frac{2x}{1-x^2} = \begin{bmatrix} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -\left(\pi - 2\tan^{-1}x\right) & \text{if } x > 1 \end{bmatrix}$

(iv) $\sin^{-1}\frac{2x}{1+x^2} = \begin{bmatrix} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -\left(\pi + 2\tan^{-1}x\right) & \text{if } x < -1 \end{bmatrix}$

(v) $\cos^{-1}\frac{1-x^2}{1+x^2} = \begin{bmatrix} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{bmatrix}$

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if, $x > 0$, $y > 0$, $z > 0$ & $(xy + yz + zx) < 1$

NOTE:

(i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$ then $xy + yz + zx = 1$

(iii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ (iv) $\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{2}$

FUNCTION

1. Definition :

Function is a special case of relation, from a non empty set A to a non empty set B, that associates each member of A to a unique member of B. Symbolically, we write f: A → B. We read it as "f is a function from A to B".

Set 'A' is called **domain** of f and set 'B' is called **co-domain** of f.

2. Image of a point and Range of a Function :

Let f: A → B, then the set A is known as the domain of f & the set B is known as co–domain of f. If a member 'a' of A is associated to the member 'b' of B, then 'b' is called **the f-image** of 'a' and we write b = f (a). Further 'a' is called **a pre-image** of 'b'. The set {f(a): ∀ a ∈A} is called **the range** of f and is denoted by f(A). Clearly f(A) ⊆ B.

3. Classification of Functions :

One – One Function (Injective Mapping) : $x_1 \ne x_2 \Leftrightarrow f(x_1) \ne f(x_2)$.

Many – One function : $f(x_1) = f(x_2)$ for some $x_1 \ne x_2$.

Onto function (Surjective mapping) : Range = codomain

Into function : range ≠ codomain

4. Various Types of Functions :

(i) Polynomial Function : $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$ is called a polynomial function of degree n.

There are two polynomial functions, satisfying the relation; $f(x).f(1/x) = f(x) + f(1/x)$, which are $f(x) = 1 \pm x^n$

(ii) Exponential Function : A function $f(x) = a^x = e^{x \ln a}$ $(a > 0, a \neq 1, x \in R)$

(iii) Logarithmic Function : $f(x) = \log_a x$ where $a > 0$ and $a \neq 1$ and $x > 0$.

(iv) Absolute Value Function / Modulus Function : $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

(v) Signum Function : $\text{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases} = \begin{cases} \dfrac{|x|}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$

(vi) Greatest Integer Function : $f(x) = [x]$ is greatest integer $\leq x$.

Properties of greatest integer function :

$x - 1 < [x] \leq x$, $\quad [x \pm m] = [x] \pm m$ if $m \in I$. $\quad [x] + [-x] = \begin{cases} 0; & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$

(vii) Fractional Part Function: $f(x) = \{x\} = x - [x]$.

(viii) Identity function : $f : A \to B$, $f(x) = x$, $\forall x \in A$ is called identity function on A.

(ix) Constant function : $f : A \to B$; $f(x) = c$, $\forall x \in A$, $c \in B$ is a constant function.

5. Odd & Even Functions :

(i) If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.

(ii) If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.

If an odd function is defined at $x = 0$, then $f(0) = 0$

6. Periodic Function : If $T \neq 0$, $x - T$ and $x + T \in$ domain of f, $f(x) = f(x + T)$ $\forall x \in$ domain of f, then f is called periodic and T is its period.

The **least positive period is called principal or fundamental period** of f or simply the period of f.

Properties of Periodic Function

(a) If f(x) has a period T, then $\dfrac{1}{f(x)}$ and $\sqrt{f(x)}$ also have a period T.

(b) If f(x) has a period T then $f(ax + b)$ has a period $\dfrac{T}{|a|}$.

(c) If f(x) has a period T_1 & g(x) has a period T_2, then period of $f(x) \pm g(x)$ or $f(x) . g(x)$ or $\dfrac{f(x)}{g(x)}$ is L.C.M. of T_1 & T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists $f(x) \pm g(x)$ or $f(x) . g(x)$ or $\dfrac{f(x)}{g(x)}$ is aperiodic.

7. Composite Function :

Let $f: X \to Y_1$ and $g: Y_2 \to Z$ be two functions and the set $D = \{x \in X: f(x) \in Y_2\}$. If $D \neq \phi$, then the function h defined on D by $h(x) = g\{f(x)\}$ is called composite function of g and f and is denoted by gof. It is also called function of a function. $D \subseteq X$

Properties of Composite Functions :

If f and g both are one-one, then gof if defined is also be one-one.

If f and g both are onto, then gof may or may not be onto.

The composite of two bijections is a bijection iff f & g are two bijections such that gof is defined, then gof is also a bijection only when co-domain of f is equal to the domain of g.

If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods. Further if g is one-one, then T is the period of gof ; if g is also periodic with T′ as the period and the range of f is a sub-set of [0, T′], then T is the period of gof

8. Inverse of a Function :

Let $f : A \to B$ be a function. Then f is invertible iff there is a function $g : B \to A$ such that gof is an identity function on A and fog is an identity function on B. Then g is called inverse of f and is denoted by f^{-1}.

For a function to be invertible it must be bijective

9. Equal or Identical Function :

Two functions f & g are said to be identical (or equal) iff :The domain of f $\equiv$ the domain of g,

The range of f $\equiv$ the range of g and f(x) = g(x), for every x belonging to their common domain.

LIMITS

1. Limit of a function f(x) is said to exist as $x \to a$ when,

$\underset{h\to 0^+}{\text{Limit}}\ f(a-h) \quad = \quad \underset{h\to 0^+}{\text{Limit}}\ f(a+h)$ = some finite value M.

(Left hand limit) (Right hand limit)

Note that we are not interested in knowing about what happens at x = a. Also note that if L.H.L. & R.H.L. are both tending towards '∞' or '$-\infty$' then it is said to be infinite limit.

Remember, $\underset{x\to a}{\text{Limit}} \Rightarrow x \neq a$

2. Indeterminant Forms:

$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, \infty^0, 0^0$, and 1^∞.

3. Method of Removing Indeterminancy : Factorisation, Rationalisation or double rationalisation, Substitution, Using standard limits, Expansions of functions.

4. Fundamental Theorems on Limits:

Let $\underset{x\to a}{\text{Limit}}\ f(x) = \ell$ & $\underset{x\to a}{\text{Limit}}\ g(x) = m$. If ℓ & m exist then:

(i) $\underset{x\to a}{\text{Limit}}\ \{f(x) \pm g(x)\} = \ell \pm m$ (ii) $\underset{x\to a}{\text{Limit}}\ \{f(x).\ g(x)\} = \ell\ .\ m$

(iii) $\underset{x\to a}{\text{Limit}}\ \frac{f(x)}{g(x)} = \frac{\ell}{m}$, provided $m \neq 0$ (iv) $\underset{x\to a}{\text{Limit}}\ kf(x) = k\underset{x\to a}{\text{Limit}}\ f(x)$; where k is a constant.

(v) $\underset{x\to a}{\text{Limit}}\, f[g(x)] = f\left(\underset{x\to a}{\text{Limit}}\, g(x)\right) = f(m)$; provided f is continuous at $g(x) = m$.

5. **Standard Limits:**

$$\underset{x\to 0}{\text{Limit}}\frac{\sin x}{x} = \underset{x\to 0}{\text{Limit}}\frac{\tan x}{x} = \underset{x\to 0}{\text{Limit}}\frac{\tan^{-1}x}{x} = \underset{x\to 0}{\text{Limit}}\frac{\sin^{-1}x}{x} = \underset{x\to 0}{\text{Limit}}\frac{e^x-1}{x} = \underset{x\to 0}{\text{Limit}}\frac{\ell n(1+x)}{x} = 1$$

$$\underset{x\to 0}{\text{Limit}}\,(1+x)^{1/x} = \underset{x\to\infty}{\text{Limit}}\left(1+\frac{1}{x}\right)^x = e,\quad \underset{x\to 0}{\text{Limit}}\frac{a^x-1}{x} = \log_e a,\ a>0,\quad \underset{x\to a}{\text{Limit}}\frac{x^n-a^n}{x-a} = na^{n-1}.$$

6. **Limits Using Expansion**

(i) $a^x = 1 + \frac{x\ln a}{1!} + \frac{x^2\ln^2 a}{2!} + \frac{x^3\ln^3 a}{3!} + \ldots\ldots\ldots a>0$ (ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$ for $-1 < x \le 1$ (iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$ (vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$

(vii) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$ (viii) $\sin^{-1}x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots\ldots$

(ix) $\sec^{-1}x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots\ldots$

(x) for $|x| < 1$, $n \in R$ $(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots\ldots\ldots\ldots\infty$

7. **Limits of form $1^\infty, 0^0, \infty^0$**

All these forms can be convered into $\frac{0}{0}$ form in the following ways

(i) If $x \to 1$, $y \to \infty$, then $z = (x)^y$ $\Rightarrow$ $\ell n\, z = y\,\ell n\, x$ $\Rightarrow \ell n\, z = \frac{\ell nx}{(1/y)}$; $\frac{0}{0}$ form

Since $y \to \infty$ hence $\frac{1}{y} \to 0$ and $x \to 1$ hence $\ell nx \to 0$

(ii) If $x \to 0$, $y \to 0$, then $z = x^y$ $\Rightarrow$ $\ell n\, z = y\,\ell n\, x$ $\Rightarrow$ $\ell n\, z = \frac{y}{1/\ell ny}$; $\frac{0}{0}$ form

(iii) If $x \to \infty$, $y \to 0$, then $z = x^y$ $\Rightarrow$ $\ell n\, z = y\,\ell n\, x$ $\Rightarrow$ $\ell n\, z = \frac{y}{1/\ell nx}$; $\frac{0}{0}$ form

also for $(1)^\infty$ type of problems we can use following rules.

$\underset{x\to 0}{\lim}\,(1+x)^{1/x} = e$, $\underset{x\to a}{\lim}\,[f(x)]^{g(x)}$, where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$ = $\underset{x\to a}{\lim} = e^{\underset{x\to a}{\lim}[f(x)-1]\,g(x)}$

8. Sandwich Theorem or Squeeze Play Theorem:

If $f(x) \le g(x) \le h(x)\ \forall\ x$ & $\underset{x\to a}{\text{Limit}}\ f(x) = \ell = \underset{x\to a}{\text{Limit}}\ h(x)$ then $\underset{x\to a}{\text{Limit}}\ g(x) = \ell$.

9. Some Important Notes :

(i) $\lim\limits_{x\to\infty} \dfrac{\ell nx}{x} = 0$ (ii) $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$

As $x \to \infty$, $\ell n\, x$ increnes much slower than any (+ve) power of x where e^x increases much faster than (+ve) power of x

(iii) $\underset{n\to\infty}{\text{Limit}}\ (1-h)^n = 0$ & $\underset{n\to\infty}{\text{Limit}}\ (1+h)^n \to \infty$, where $h > 0$.

(iv) If $\underset{x\to a}{\text{Limit}}\ f(x) = A > 0$ & $\underset{x\to a}{\text{Limit}}\ \phi(x) = B$ (a finite quantity) then;

$\underset{x\to a}{\text{Limit}}\ [f(x)]^{\phi(x)} = e^z$ where $z = \underset{x\to a}{\text{Limit}}\ \phi(x).\ \ln[f(x)] = e^{B\ln A} = A^B$

CONTINUITY & DERIVABILITY

CONTINUITY

1. A function f(x) is said to be continuous at x = c, if $\underset{x\to c}{\text{Limit}}\ f(x) = f(c)$.

2. Types of Discontinuity :

(a) **Removable Discontinuity :** $\underset{x\to c}{\text{Limit}}\ f(x)$ exists but is not equal to f(c)

Removable type of discontinuity can be further classified as :

(i) **Missing Point Discontinuity :** Where $\underset{x\to a}{\text{Limit}}\ f(x)$ exists finitely but f(a) is not defined.

(ii) **Isolated Point Discontinuity:** Where $\underset{x\to a}{\text{Limit}}\ f(x)$ exists but $\underset{x\to a}{\text{Limit}}\ f(x) \ne f(a)$.

(b) **Irremovable Discontinuity:** $\underset{x\to c}{\text{Limit}}\ f(x)$ does not exist. However if both the limits (i.e. L.H.L. & R.H.L.) are finite, then discontinuity is said to be of first kind otherwise it is non–removable discontinuity of second kind.

Irremovable type of discontinuity can be further classified as:

(i) **Finite discontinuity** e.g. $f(x) = x - [x]$ at all integral x.

(ii) **Infinite discontinuity** e.g. $f(x) = \dfrac{1}{x-4}$ or $g(x) = \dfrac{1}{(x-4)^2}$ at x = 4.

(iii) **Oscillatory discontinuity** e.g. $f(x) = \sin\dfrac{1}{x}$ at x = 0.

In all these cases the value of f (a) of the function at x = a (point of discontinuity) may or may not exist but $\underset{x\to a}{\text{Limit}}$ does not exist.

(c) **Discontinuity of I[st] kind**

If L.H.L. and R.H.L both exist finitely then discontinuity is said to be of I[st] kind

(d) **Discontinuity of II^{nd} kind**

If either L.H.L. or R.H.L does not exist then discontinuity is said to be of II^{nd} kind

(e) Point functions defined at single point only are to be treated as discontinuous.

eg. $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1.

3. **Jump of discontinuity :** $\left| \underset{x \to a^+}{\text{Limit}} f(x) - \underset{x \to a^-}{\text{Limit}} f(x) \right|$

4. **Continuity on an Interval :**

(a) A function f is said to be continuous on (a, b) if f is continuous at each & every point $\in (a, b)$.

(b) A function f is said to be continuous on a closed interval [a, b] if:

(i) f is continuous in the open interval (a, b) &

(ii) f is right continuous at a and left continuous at b

(c) All Polynomials, Trigonometrical functions, Exponential and Logarithmic functions are continuous at every point in their domains.

(d) {f(x)} and [f(x)] may not be continuous where f(x) becomes integer.

(e) sgn (f(x)) may not be continuous where f(x) = 0

5. If f & g are two functions which are continuous at x = c then the functions defined by:

$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, K any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x = c.

Further, if g (c) is not zero, then $F_4(x) = \dfrac{f(x)}{g(x)}$ is also continuous at x = c.

6. **Continuity of Composite Function :** If f is continuous at x = c & g is continuous at x = f(c) then the composite g[f(x)] is continuous at x = c.

7. **Intermediate Value Theorem :**

A function f which is continuous in $[a, b]$ possesses the following properties:

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b).

(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open inetrval (a, b).

DERIVABILITY

1. **Differentiability of a function at a point:**

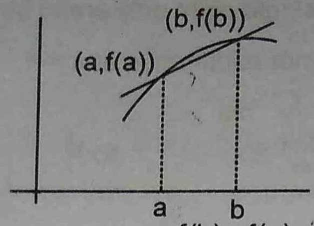
$f'(a^+) = \underset{h \to 0^+}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$ is the right hand derivative at x = a, provided the limit exists.

$f'(a^-) = \underset{h \to 0^+}{\text{Limit}} \dfrac{f(a-h)-f(a)}{-h}$ is the left hand derivative at x = a, provided the limit exists.

A function f(x) is said to be differentiable(finitely) at x = a if $f'(a^+) = f'(a^-)$ = finite

2. Concept of Tangent and its Association with Derivability:

Tangent :- The tangent is defined as the limiting case of a chord or a secant



slope of chord joining (a, f(a)) and (b, (f(b)) = $\dfrac{f(b)-f(a)}{b-a}$

slope of the line joining a and (a,f(a)) and (a + h, f(a + h)) = $\dfrac{f(a+h)-f(a)}{h}$

R.H.D. = $f'(a^+) = \lim\limits_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h}$, L.H.D. = $f'(a^-) = \lim\limits_{h\to 0^+} \dfrac{f(a-h)-f(a)}{-h}$

A function will have a tangent at point x = a if $f'(a^+) = f'(a^-)$ (may or may be finite) and equation of tangent at (a, f(a)) is given by $y - f(a) = f'(a)(x - a)$

3. Relation between Differentiability & Continuity:

If f(x) is differentiable at a point of its domain, then it is continuous at that point but the converse is not true.

4. Differentiability Over an Interval:

f (x) is said to be differentiable over an open interval if it is differentiable at every point of the interval and f(x) is said to be differentiable over a closed interval [a, b] if f is differentiable on (a, b) and

$f'(a^+)$ = a finite number and $f'(b^-)$ = a finite number

Important : -

All polynomial, exponential, logrithimic and trigonometric (inverse trigonomettric not included) are differentiable at every point of their domain.

5. Differentiabilty of sum, product & composition of functions

At x = a, if f(x) & g(x) are diffrerentiable, then $f(x) \pm g(x)$, $f(x).\,g(x)$ are also differentiable. further if $g(a) \neq 0$ then the function f(x)/g(x) is also be differentiable.

If f(x) is not differentiable at x = a & g(x) is differentiable at x = a, then f + g is not differentiable, $f(x)\cdot g(x)$ may or may not be differentiable at x = a

If f(x) & g(x) both are not differentiable at x = a then each of $f(x)\cdot g(x)$ and f + g may or may not be differentiable.

If f is differentiable at x = a, then $\lim\limits_{h\to 0} \dfrac{f(a+g(h))-f(a+p(h))}{g(h)-p(h)} = f'(a)$, where , $\lim\limits_{h\to 0} P(h) = \lim\limits_{h\to 0} g(x) = 0$

METHOD OF DIFFERENTIATION

A. First Principle Of Differentiation

1. The derivative of a given function f at a point x = a on its domain is defined as:

$\text{Limit}_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ or $\text{Limit}_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ provided the limit exists & is denoted by $f'(a)$.

2. If x and x + h belong to the domain of a function f defined by y = f(x), then

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ if it exists, is called the Derivative of f at x & is denoted by $f'(x)$ or $\frac{dy}{dx}$. i.e.,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

This method of differentiation is also called ab-initio method or first principle.

3. Differentiation of some elementary functions.

1. $\frac{d}{dx}(x^n) = nx^{n-1}$ 2. $\frac{d}{dx}(a^x) = a^x \ln a$ 3. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ 4. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

5. $\frac{d}{dx}(\sin x) = \cos x$ 6. $\frac{d}{dx}(\cos x) = -\sin x$ 7. $\frac{d}{dx}(\sec x) = \sec x \tan x$

8. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ 9. $\frac{d}{dx}(\tan x) = \sec^2 x$ 10. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

4. Basic Theorems

1. $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$ 2. $\frac{d}{dx}(k\, f(x)) = k \frac{d}{dx} f(x)$ 3. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)\, g'(x) + g(x)\, f'(x)$

4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\, f'(x) - f(x) g'(x)}{g^2(x)}$ 5. $\frac{d}{dx}(f(g(x))) = f'(g(x))\, g'(x)$

6. $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$ 7. $\frac{dy}{dx} = \frac{1}{dx/dy}$

another way of expressing the same concept is by considering y = f(x) and x = g(y) as inverse functions of each other.

$$\frac{dy}{dx} = f'(x) \quad \text{and} \quad \frac{dx}{dy} = g'(y) \quad \Rightarrow \quad g'(y) = \frac{1}{f'(x)}$$

B. Derivative Of Inverse Trigonometric Functions.

$$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d\cos^{-1}x}{dx} = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1.$$

$$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}, \quad \frac{d\cot^{-1}x}{dx} = -\frac{1}{1+x^2} \quad (x \in R)$$

$$\frac{d\sec^{-1}x}{dx} = \frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d\operatorname{cosec}^{-1}x}{dx} = -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } x \in (-\infty, -1) \cup (1, \infty)$$

1. Implicit differentiation

If $f(x, y) = 0$, is an implicit function then in order to find dy/dx, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx.

2. Differentiation using substitution

Following substitutions are normally used to sumplify these expression.

(i) $\sqrt{x^2+a^2}$ by substituting $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(ii) $\sqrt{a^2-x^2}$ by substituting $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(iii) $\sqrt{x^2-a^2}$ by substituting $x = a\sec\theta$, where $\theta \in [0, \pi]$, $\theta \ne \frac{\pi}{2}$

(iv) $\sqrt{\frac{x+a}{a-x}}$ by substituting $x = a\cos\theta$, where $\theta \in (0, \pi]$.

3. Parametric Differentiation

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

4. Derivative of one function with respect to another

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

5. If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

APPLICATION OF DERIVATIVES

1. Equation of tangent and normal

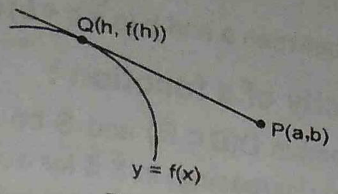
Tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

And normal at (x_1, y_1) is $(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

2. Tangent from an external point

Given a point P(a, b) which does not lie on the curve $y = f(x)$, then the equation of possible tangents to the curve $y = f(x)$, passing through (a, b) can be found by solving for the point of contact Q.

$$f'(h) = \frac{f(h)-b}{h-a}$$



Figure

And equation of tangent is $y - b = \frac{f(h) - b}{h - a}(x - a)$

3. **Length of tangent, normal, subtangent, subnormal**

(i) $PT = |k|\sqrt{1+\frac{1}{m^2}}$ = Length of Tangent

(ii) $PN = |k|\sqrt{1+m^2}$ = Length of Normal

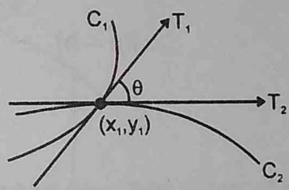
(iii) $TM = \left|\frac{k}{m}\right|$ = Length of subtangent

(iv) $MN = |km|$ = Length of subnormal.

4. **Angle between the curves**

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves (as shown in figure).

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$



where m_1 & m_2 are the slopes of tangents at the intersection point (x_1, y_1).

Notes : (i) The angle is defined between two curves if the curves are intersecting. This can be ensured by finding their point of intersection or graphically.

(ii) If the curves intersect at more than one point then angle between curves is found out with respect to the point of intersection.

(iii) Two curves are said to be orthogonal if angle between them at **each** point of intersection is right angle. i.e. $m_1 m_2 = -1$.

5. **Shortest distance between two curves**

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

6. **Rolle's Theorem :**

If a function f defined on [a, b] is

(i) continuous on [a, b]

(ii) derivable on (a, b) and

(iii) f(a) = f(b),

then there exists at least one real number c between a and b (a < c < b) such that f′(c) = 0

7. **Lagrange's Mean Value Theorem (LMVT) :**

If a function f defined on [a, b] is

(i) continuous on [a, b] and

(ii) derivable on (a, b)

then there exists at least one real numbers between a and b (a < c < b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

B. Monotonicity of a function :

Let f be a real valued function having domain $D(D \subset R)$ and S be a subset of D. f is said to be monotonically increasing (non decreasing) (increasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$. f is said to be monotonically decreasing (non increasing) (decreasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

f is said to be strictly increasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Similarly, f is said to be strictly decreasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

1. Application of differentiation for detecting monotonicity :

Let I be an interval (open or closed or semi open and semi closed)

(i) If $f'(x) > 0 \ \forall \ x \in I$, then f is strictly increasing in I

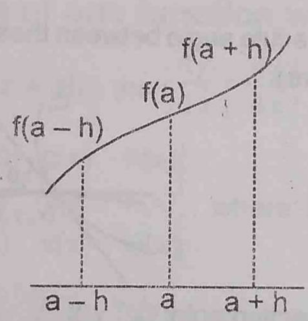
(ii) If $f'(x) < 0 \ \forall \ x \in I$, then f is strictly decreasing in I

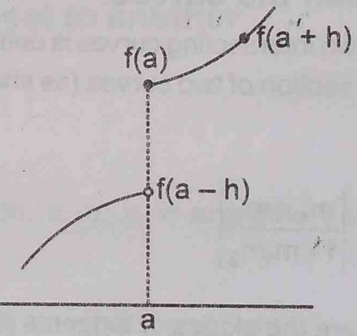
Note : Let I be an interval (or ray) which is a subset of domain of f. If $f'(x) > 0$, $\forall \ x \in I$, except for countably many points where $f'(x) = 0$, then f(x) is strictly increasing in I.

{$f'(x) = 0$ at countably many points $\Rightarrow f'(x) = 0$ does not occur on an interval which is a subset of I}

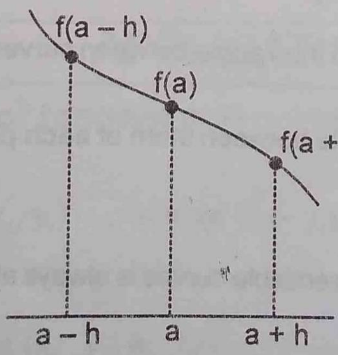
2. Monotonicity of function about a point :

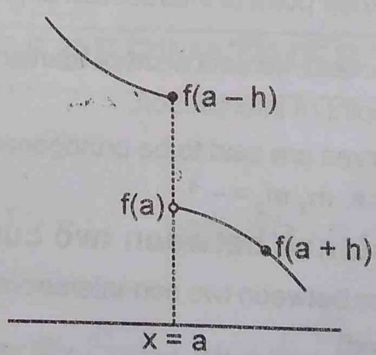
(i) A function f(x) is called as a strictly increasing function about a point (or at a point) $a \in D_f$ if it is strictly increasing in an open interval containing a (as shown in figure).



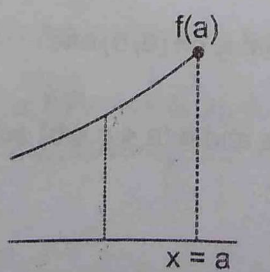


(ii) A function f(x) is called a strictly decreasing function about a point x = a, if it is strictly decreasing in an open interval containing a (as shown in figure).

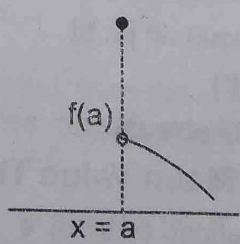




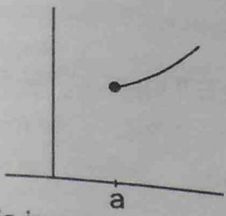
Note : If x = a is a boundary point then use the appropriate one sided inequality to test monotonicity of f(x).



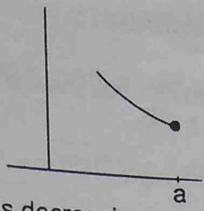
f is increasing at x = a



f is decreasing at x = a



f is increasing at $x = a$



f is decreasing at $x = a$

3. Test for increasing and decreasing functions about a point

Let $f(x)$ be differentiable.

(i) If $f'(a) > 0$ then $f(x)$ is increasing at $x = a$.

(ii) If $f'(a) < 0$ then $f(x)$ is decreasing at $x = a$.

(iii) If $f'(a) = 0$ then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a.

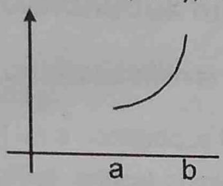
(a) If $f'(x)$ is positive on both the neighbourhoods, then f is increasing at $x = a$.

(b) If $f'(x)$ is negative on both the neighbourhoods, then f is decreasing at $x = a$.

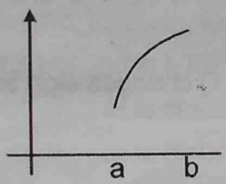
(c) If $f'(x)$ have opposite signs on these neighbourhoods, then f is non-monotonic at $x = a$.

Results :

1. If $f''(x) > 0 \ \forall \ x \in (a, b)$, then the curve $y = f(x)$ is concave in (a, b)



2. If $f''(x) < 0 \ \forall \ x \in (a, b)$ then the curve $y = f(x)$ is convex in (a, b)



3. If f is continuous at $x = c$ and $f''(x)$ has opposite signs on either sides of c, then the point $(c, f(c))$ is a point of inflection of the curve

4. If $f''(c) = 0$ and $f'''(c) \neq 0$, then the point $(c, f(c))$ is a point of inflection

4. Proving Inequalities using curvature :

Generally these inequalities involve comparison between values of two functions at some particular points.

C. Maxima and Minima

1. (i) Global Maximum :

A function $f(x)$ is said to have global maximum on a set E if there exists at least one $c \in E$ such that $f(x) \leq f(c)$ for all $x \in E$.

We say global maximum occurs at $x = c$ and global maximum (or global maximum value) is $f(c)$.

1. (ii) Local Maxima :

A function $f(x)$ is said to have a local maximum at $x = c$ if $f(c)$ is the greatest value of the function in a small neighbourhood $(c - h, c + h)$, $h > 0$ of c.

i.e. for all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \leq f(c)$.

i.e. $f(c - \delta) \leq f(c) \geq f(c + \delta)$, $0 < \delta < h$

Note : If $x = c$ is a boundary point then consider $(c - h, c)$ or $(c, c + h)$ $(h > 0)$ appropriately.

1. **(iii) Global Minimum :**

A function f(x) is said to have a global minimum on a set E if there exists at least one $c \in E$ such that $f(x) \geq f(c)$ for all $x \in E$.

1. **(iv) Local Minima :**

A function f(x) is said to have a local minimum at x = c if f(c) is the least value of the function in a small neighbourhood $(c - h, c + h)$, $h > 0$ of c.

i.e. for all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \geq f(c)$.

i.e. $f(c - \delta) \geq f(c) \leq f(c + \delta)$, $0 < \delta < h$

2. **Extrema :**

A maxima or a minima is called an extrema.

3. **Maxima, Minima for differentiable functions :**

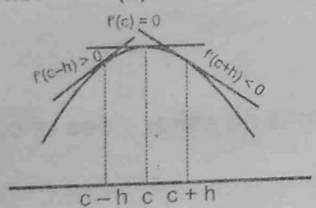
A necessary condition for f(c) to be an extremum of f(x) is that $f'(c) = 0$.

i.e. f(c) is extremum $\Rightarrow$ $f'(c) = 0$

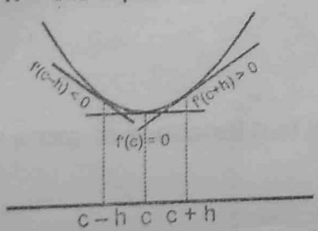
4. **Sufficient condition for an extrema :**

A sufficient condition for f(c) to be an extremum of f(x) is that $f'(x)$ changes sign as x passes through c.

i.e. f(c) is an extrema (see figure) $\Leftrightarrow$ $f'(x)$ changes sign as x passes through c.



x = c is a point of maxima. $f'(x)$ changes sign from positive to negative.



x = c is a point of local minima (see figure), $f'(x)$ changes sign from negative to positive.

5. **Stationary points :**

The points on graph of function f(x) where $f'(x) = 0$ are called stationary points.

Rate of change of f(x) is zero at a stationary point.

6. **First Derivative Test :**

Let f(x) be continuous and differentiable function.

Step - I. Find $f'(x)$

Step - II. Solve $f'(x) = 0$, let x = c be a solution. (i.e. Find stationary points)

Step - III. Observe change of sign

(i) If $f'(x)$ changes sign from negative to positive as x crosses c from left to right then x = c is a point of local minima

(ii) If $f'(x)$ changes sign from positive to negative as x crosses c from left to right then x = c is a point of local maxima.

(iii) If $f'(x)$ does not changes sign as x crosses c then x = c is neither a point of maxima nor minima.

Note : In case of continuous functions points of maxima and minima are alternate.

7. **Critical points :**

The points where $f'(x) = 0$ or $f(x)$ is not differentiable are called critical points.

Stationary points $\subseteq$ Critical points.

Important Note :

For f(x) defined on a subset of R, points of extrema (if exists) occur at critical points

Note : Critical points are always interior points of an interval.

8. **Global extrema for continuous functions :**

(i) Function defined on closed interval

Let $f(x)$, $x \in [a, b]$ be a continuous function

Step - I : Find critical points. Let it be $c_1, c_2, \ldots\ldots, c_n$

Step - II : Find $f(a), f(c_1)\ldots\ldots\ldots, f(c_n), f(b)$

Let $M = \max \cdot \{f(a), f(c_1), \ldots\ldots\ldots, f(c_n), f(b)\}$

$m = \min \cdot \{f(a), f(c_1), \ldots\ldots f(c_n), f(b)\}$

Step - III : M is global maximum.

m is global minimum.

(ii) Function defined on open interval.

Let $f(x)$, $x \in (a, b)$ be continuous function.

Step - I Find critical points . Let it be $c_1, c_2, \ldots\ldots c_n$

Step - II Find $f(c_1), f(c_2), \ldots\ldots\ldots, f(c_n)$

Let $M = \max \cdot \{f(c_1), \ldots\ldots f(c_n)\}$;

$m = \min \cdot \{f(c_1), \ldots\ldots\ldots\ldots, f(c_n)\}$

Step - III $\lim\limits_{x\to a^+} f(x) = \ell_1$ (say), $\lim\limits_{x\to b^-} f(x) = \ell_2$ (say).

Let $\ell = \min. \{\ell_1, \ell_2\}$, $L = \max. \{\ell_1, \ell_2\}$

Step - IV

(i) If $m \le \ell$ then m is global minimum

(ii) If $m > \ell$ then f(x) has no global minimum

(iii) If $M \ge L$ then M is global maximum

(iv) If $M < L$, then f(x) has no global maximum

9. **Maxima, Minima by higher order derivatives :**

Second derivative test :

Let f(x) have derivatives up to second order

Step - I. Find $f'(x)$

Step - II. Solve $f'(x) = 0$. Let $x = c$ be a solution

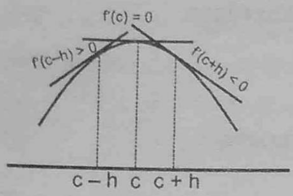
Step - III. Find $f''(c)$

Step - IV.

(i) If $f''(c) = 0$ then further investigation is required

(ii) If $f''(c) > 0$ then $x = c$ is a point of minima.

(iii) If $f''(c) < 0$ then $x = c$ is a point of maxima.



For maxima $f'(x)$ changes from positive to negative (as shown in figure).

$\Rightarrow$ $f'(x)$ is decreasing hence $f''(c) < 0$

10. n^{th} Derivative test :

Let $f(x)$ have derivatives up to nth order

If $f'(c) = f''(c) = \ldots\ldots\ldots = f^{n-1}(c) = 0$ and

$f^n(c) \neq 0$ then we have following possibilities

(i) n is even, $f^{(n)}(c) < 0 \Rightarrow x = c$ is point of maxima

(ii) n is even, $f^{(n)}(c) > 0 \Rightarrow x = c$ is point of minima.

(iii) n is odd, $f^{(n)}(c) < 0 \Rightarrow f(x)$ is decreasing about $x = c$

(iv) n is odd, $f^{(n)} > 0 \Rightarrow f(x)$ is increasing about $x = c$.

11. Application of Maxima, Minima :

Useful Formulae of Mensuration to Remember :

1. Volume of a cuboid = ℓbh.
2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.
3. Volume of cube = a^3
4. Surface area of cube = $6a^2$
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.
6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height)
7. Curved surface area of a cylinder = $2\pi rh$.
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
9. Volume of a sphere = $\frac{4}{3}\pi r^3$.
10. Surface area of a sphere = $4\pi r^2$.
11. Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.
12. Volume of a prism = (area of the base) × (height).
13. Lateral surface area of a prism = (perimeter of the base) × (height).

14. Total surface area of a prism = (lateral surface area) + 2 (area of the base)
(Note that lateral surfaces of a prism are all rectangle).

15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).

16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).
(Note that slant surfaces of a pyramid are triangles).

INDEFINITE INTEGRATION

1. If f & g are functions of x such that $g'(x) = f(x)$ then,

$\int f(x)\,dx = g(x) + c \Leftrightarrow \frac{d}{dx}\{g(x)+c\} = f(x)$, where c is called the **constant of integration**.

2. **Standard Formula:**

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n\,(ax+b) + c$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

(iv) $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ell n\,a} + c;\ a > 0$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\ell n\,\sec(ax+b) + c$

(viii) $\int \cot(ax+b)\,dx = \frac{1}{a}\ell n\,\sin(ax+b) + c$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \mathrm{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \sec(ax+b).\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$

(xii) $\int \mathrm{cosec}(ax+b).\cot(ax+b)\,dx = -\frac{1}{a}\mathrm{cosec}(ax+b) + c$

(xiii) $\int \sec x\,dx = \ell n\,(\sec x + \tan x) + c$ **OR** $\ell n\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(xiv) $\int \mathrm{cosec}\,x\,dx = \ell n\,(\mathrm{cosec}\,x - \cot x) + c$ **OR** $\ell n\tan\frac{x}{2} + c$ **OR** $-\ell n\,(\mathrm{cosec}\,x + \cot x) + c$

(xv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

(xvi) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xvii) $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x + \sqrt{x^2+a^2}\right] + c$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x + \sqrt{x^2-a^2}\right] + c$

(xx) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right| + c$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right| + c$

(xxii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxiii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ell n\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) + c$

(xxiv) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ell n\left(\frac{x+\sqrt{x^2-a^2}}{a}\right) + c$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

3. Theorems on integration

(i) $\int c\, f(x).dx = c\int f(x).dx$

(ii) $\int (f(x) \pm g(x))dx = \int f(x)dx \pm g(x)dx$

(iii) $\int f(x)dx = g(x) + c \Rightarrow \int f(ax+b)dx = \frac{g(ax+b)}{a} + c$

4. Integration by Subsitutions

If we subsitute $f(x) = t$, then $f'(x)\,dx = dt$

5. Integration by Part :

$$\int (f(x)\,g(x))\,dx = f(x)\int (g(x))dx - \int\left(\frac{d}{dx}(f(x))\int (g(x))\,dx\right)dx$$

(i) when you find integral $\int g(x)dx$ then it will not contain arbitarary constant.

(ii) $\int g(x)dx$ should be taken as same both terms.

(iii) the choice of $f(x)$ and $g(x)$ is decided by ILATE rule.

6. **Integration of type** $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\, dx$

Make the substitution $x + \frac{b}{2a} = t$

7. **Integration of type**

$$\int \frac{px+q}{ax^2+bx+c}\, dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}\, dx, \int (px+q)\sqrt{ax^2+bx+c}\, dx$$

Make the substitution $x + \frac{b}{2a} = t$, then split the integral as some of two integrals one containing the linear term and the other containing constant term.

8. **Integration of trigonometric functions**

(i) $\int \frac{dx}{a + b\sin^2 x}$ OR $\int \frac{dx}{a + b\cos^2 x}$ OR $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$ **put tan x = t.**

(ii) $\int \frac{dx}{a + b\sin x}$ OR $\int \frac{dx}{a + b\cos x}$ OR $\int \frac{dx}{a + b\sin x + c\cos x}$ **put** $\tan\frac{x}{2} = t$

(iii) $\int \frac{a.\cos x + b.\sin x + c}{\ell.\cos x + m.\sin x + n}\, dx$. **Express** $Nr \equiv A(Dr) + B\frac{d}{dx}(Dr) + c$ **& proceed.**

9. **Integration of type** $\int \sin^m x . \cos^n x\, dx$

Case - I
If m and n are even natural number then converts higher power into higher angles.

Case - II
If at least m or n is odd natural number then if m is odd put cosx = t and vice-versa.

Case - III
When m + n is a negative even integer then put tan x = t.

10. **Integration of type**

$$\int \frac{x^2 \pm 1}{x^4 + Kx^2 + 1}\, dx$$ **where K is any constant.**

Divide Nr & Dr by x^2 & put $x \mp \frac{1}{x} = t$.

11. **Integration of type**

$$\int \frac{dx}{(ax+b)\sqrt{px+q}} \text{ OR } \int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$$ **; put** $px + q = t^2$.

12. **Integration of type**

$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, **put** $ax + b = \frac{1}{t}$; $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$, **put** $x = \frac{1}{t}$

13. **Integration of type**

$$\int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx \text{ or } \int \sqrt{(x-\alpha)(\beta-x)}\,; \quad \text{put } x = \alpha\cos^2\theta + \beta\sin^2\theta$$

$$\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx \text{ or } \int \sqrt{(x-\alpha)(x-\beta)}\,; \quad \text{put } x = \alpha\sec^2\theta - \beta\tan^2\theta$$

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}\,; \quad \text{put } x-\alpha = t^2 \text{ or } x-\beta = t^2.$$

14. **Reduction formula of** $\int \tan^n x\,dx$, $\int \cot^n x\,dx$, $\int \sec^n x\,dx$, $\int \text{cosec}^n x\,dx$

(i) If $I_n = \int \tan^n x\,dx$, then $I_n = \dfrac{\tan^{n-1}x}{n-1} - I_{n-2}$

2. If $I_n = \int \cot^n x\,dx$, then $I_n = -\dfrac{\cot^{n-1}x}{n-1} - I_{n-2}$

(ii) If $I_n = \int \sec^n x\,dx$, then $I_n = \dfrac{\tan x \sec^{n-2}x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

(iii) If $I_n = \int \text{cosec}^n x\,dx$, then $I_n = \dfrac{\cot x\,\text{cosec}^{n-2}x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

(iv) If $I_{m,n} = \int (\sin x)^m (\cos x)^n\,dx$ then $I_{m,n} = \dfrac{(\sin x)^{m+1}(\cos x)^{n-1}}{m+n} + \dfrac{n-1}{m+n} \cdot I_{m,n-2}$

DEFINITE INTEGRATION

Properties of definite integral

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4. $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x)+f(-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(-x) = f(x) \\ 0, & f(-x) = -f(x) \end{cases}$

5. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

6. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

7. $\int_0^{2a} f(x)\,dx = \int_0^a (f(x)+f(2a-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$

8. If f(x) is a periodic function with period T, then

$$\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in z, \qquad \int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in z,\ a \in R$$

$$\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx,\ m, n \in z, \qquad \int_{nT}^{a+nT} f(x)\,dx = \int_0^a f(x)\,dx,\ n \in z,\ a \in R$$

$$\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx,\ n \in z,\ a, b \in R$$

9. If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then $\int_a^b \psi(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

10. If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

11. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

12. If $f(x) \ge 0$ on $[a, b]$ then $\int_a^b f(x)\,dx \ge 0$

Leibnitz Theorem : If $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$, then $\frac{dF(x)}{dx} = h'(x)\,f(h(x)) - g'(x)\,f(g(x))$

PART - I

Definite Integral as a Limit of Sum.

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} h\,f(a+rh) = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a+\frac{(b-a)r}{n}\right)$$

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx \quad , m, n \in N \text{ then,}$$

$$I_{m,n} = \begin{cases} \dfrac{(m-1)(m-3)(m-5)\ldots\ldots(n-1)(n-3)(n-5)\ldots\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots}\,\dfrac{\pi}{2} & \text{when both m, n are even} \\[2ex] \dfrac{(m-1)(m-3)(m-5)\ldots\ldots(n-1)(n-3)(n-5)\ldots\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots} & \text{otherwise} \end{cases}$$

AREA UNDER CURVE

1. **Curve Tracing :**

To find the approximate shape of a curve, the following procedure is adopted in order:

(i) **Symmetry about x – axis,**
(ii) **Symmetry about y – axis:**
(iii) **Symmetry about the line y = x,**
(v) **Symmetry in opposite quadrants:**

(v) Find the points where the curve crosses the x–axis and also the y–axis.

(vi) Find $\frac{dy}{dx}$ and equate it to zero to find the points on the curve where you have horizontal tangents.

(vii) Examine if possible the intervals when f(x) is increasing or decreasing.

(viii) Examine what happens to 'y' when $x \to \infty$ or $x \to -\infty$.

(ix) **Asymptotes :**

(a) If $\displaystyle\mathop{Lt}_{x \to a} f(x) = \infty$ or $\displaystyle\mathop{Lt}_{x \to a} f(x) = -\infty$, then $x = a$ is asymptote of $y = f(x)$

(b) If $\displaystyle\mathop{Lt}_{x \to +\infty} f(x) = k$ or $\displaystyle\mathop{Lt}_{x \to -\infty} f(x) = k$, then $y = k$ is asymptote of $y = f(x)$

(c) If $\displaystyle\mathop{Lt}_{x \to \infty} \frac{f(x)}{x} = m_1$, $\displaystyle\mathop{Lt}_{x \to \infty} (f(x) - m_1x) = c$, then $y = m_1x + c_1$ is an asymptote.

2. Quadrature :

(i) If $f(x) \geq 0$ for $x \in [a, b]$, then area bounded by curve $y = f(x)$, x-axis, $x = a$ and $x = b$ is $\displaystyle\int_a^b f(x)\,dx$

(ii) If $f(x) \leq 0$ for $x \in [a, b]$, then area bounded by curve $y = f(x)$, x-axis, $x = a$ and $x = b$ is $-\displaystyle\int_a^b f(x)\,dx$

(iii) If $f(x) \geq 0$ for $x \in [a,c]$ and $f(x) \leq 0$ for $x \in [c,b]$ $(a < c < b)$ then area bounded by curve $y = f(x)$ and x–axis between $x = a$ and $x = b$ is $\displaystyle\int_a^c f(x)dx - \int_c^b f(x)dx$.

(iv) If $f(x) \geq g(x)$ for $x \in [a,b]$ then area bounded by curves $y = f(x)$ and $y = g(x)$ between ordinates $x = a$ and $x = b$ is $\displaystyle\int_a^b (f(x) - g(x))dx$.

(v) If $g(y) \geq 0$ for $y \in [c,d]$ then area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $\displaystyle\int_{y=c}^{d} g(y)dy$

DIFFERENTIAL EQUATIONS

1. Order and Degree of a Differential Equation:

2. Order : Order is the highest differential appearing in a differential equation.

3. Degree :

It is determined by the degree of the highest order derivative present in it after the differential equation is cleared of radicals and fractions so far as the derivatives are concerned.

$$f_1(x, y)\left[\frac{d^my}{dx^m}\right]^{n_1} + f_2(x, y)\left[\frac{d^{m-1}y}{dx^{m-1}}\right]^{n_2} + \ldots\ldots\ f_k(x, y)\left[\frac{dy}{dx}\right]^{n_k} = 0$$

The above differential equation has the order m and degree n_1.

4. **Differential Equation of First Order and First Degree :** $\frac{dy}{dx} + f(x, y) = 0$

5. **Elementary Types of First Order and First Degree Differential Equations :**

Variables separable : $\int f(x)\,dx = \int \phi(y)\,dy + c$

Homogeneous Differential Equations :

A differential equation of the from $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are homogeneous function of x and y, and of the same degree, is called homogeneous differential equaiton and can be solved easily by putting $y = vx$.

Exact Differential Equation :

The differential equation $M + N\frac{dy}{dx} = 0$ ....(1)

Where M and N are functions of x and y is said to be exact if it can be derived by direct differentiation (without any subsequent multiplication, elimination etc.) of an equation of the form $f(x, y) = c$

$xdy + y\,dx = d(xy)$, $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$, $2(x\,dx + y\,dy) = d\,(x^2 + y^2)$, $\frac{xdy - ydx}{xy} = d\left(\ln\frac{y}{x}\right)$

$\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$, $\frac{xdy + ydx}{xy} = d(\ln xy)$, $\frac{xdy + ydx}{x^2 y^2} = d\left(-\frac{1}{xy}\right)$

6. **Linear Differential Equation :**

Linear differential equations of first order :

The differential equation $\frac{dy}{dx} + Py = Q$, is linear in y, where P and Q are functions of x.

I.F for linear differential equation $= e^{\int Pdx}$

The solution is given by $y.e^{\int Pdx} = \int Q.e^{\int Pdx} + C$.

Bernoulli's equation : Equations of the form $\frac{dy}{dx} + Py = Q.y^n$, $n \neq 0$ and $n \neq 1$

where P and Q are functions of x, is called Bernoulli's equation and can be made linear in v by dividing by y^n and putting $y^{-n+1} = v$. Now its solution can be obtained as in (v).

7. **Clairaut's Equation :** $y = px + f(p)$, where $p = \frac{dy}{dx}$ is known as Clairout's Equation.

To solve , differentiate it w.r.t. x, which gives

either $\frac{dp}{dx} = 0 \Rightarrow p = c$ or $x + f'(p) = 0$ p is eliminated

8 **Orthogonal Trajectory :**

An orthogonal trajectory of a given system of curves is defined to be a curve which cuts every member of a given family of curve at right angle.

Steps to find orthogonal trajectory :

(i) Let f (x, y, c) = 0 be the equation of the given family of curves, where 'c' is an arbitrary constant.

(ii) Differentate the given equation w.r.t. x and then eliminate c.

(iii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the equation obtained in (ii).

(iv) Solve the differential equation obtained in (iii).

Hence solution obtained in (iv) is the required orthogonal trajectory.

MATRICES & DETERMINANTS

MATRICES : Any rectangular arrangement of numbers in m rows and n columns is called a **matrix of order** m × n.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1j} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2j} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{i1} & a_{i2} & a_{i3} & \ldots\ldots & a_{ij} & \ldots\ldots & a_{in} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mj} & \ldots\ldots & a_{mn} \end{bmatrix}$$

where a_{ij} denote the element of i^{th} row & j^{th} column. The above matrix is usually denoted as $[a_{ij}]_{m\times n}$.

The elements $a_{11}, a_{22}, a_{33}, \ldots\ldots$ are called as **diagonal elements**. Their sum is called as **trace** of A denoted as $T_r(A)$

1. Basic Definitions

(i) Row matrix : A matrix having only one row is called as row matrix

(ii) Column matrix : A matrix having only one column is called as column matrix.

(iii) Square matrix : A matrix of order m × n is called square matrix if m = n.

(iv) Zero matrix : $A = [a_{ij}]_{m\times n}$ is called a zero matrix, if $a_{ij} = 0\ \forall$ i & j.

(v) Upper triangular matrix : $A = [a_{ij}]_{m\times n}$ is said to be upper triangular, if $a_{ij} = 0$ for i > j

(vi) Lower triangular matrix : $A = [a_{ij}]_{m\times n}$ is said to be a lower triangular, if $a_{ij} = 0$ for i < j.

(vii) Diagonal matrix : A square matrix $[a_{ij}]_n$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.
Diagonal matrix of order n is denoted as Diag $(a_{11}, a_{22}, \ldots\ldots a_{nn})$.

(viii) Scalar matrix : A diagonal matrix $A = [a_{ij}]_n$ is a scalar matrix, if $a_{ij} = k$ for i = j.

(ix) Unit matrix (Identity matrix) : A diagonal matrix $A = [a_{ij}]_n$ is a unit matrix, if $a_{ij} = 1$ for i = j.

(x) Comparable matrices : Two matrices A & B are said to be comparable, if they have the same order.

(xi) **Equality of matrices :** Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are said to be equal if $m = p$, $n = q$ and $a_{ij} = b_{ij}$ $\forall$ i & j.

(xii) **Multiplication of matrix by scalar :**
Let λ be a scalar, then $\lambda A = [b_{ij}]_{m \times n}$ where $b_{ij} = \lambda a_{ij}$ $\forall$ i & j.

(xiii) **Addition of matrices :** Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. be two matrices, then $A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$. $= [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$ $\forall$ i & j.

(xiv) **Substraction of matrices :** Then $A - B = A + (-B)$, where $-B$ is (-1) B.

(xv) **Properties of addition & scalar multiplication :**
(i) $\lambda(A + B) = \lambda A + \lambda B$ (ii) $\lambda A = A\lambda$ (iii) $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$

(xvi) **Multiplication of matrices :** Let $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$, then

$$AB = [c_{ij}]_{m \times n} \text{ where } c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj},$$

(xvii) **Properties of matrix multiplication :** We have the following when ever defined
(i) In general $AB \neq BA$ (ii) $(AB)C = A(BC)$ (iii) $AI_n = A = I_n A$
(iv) For every non singular square matrix A (i.e., $|A| \neq 0$) there exist a unique matrix B so that $AB = I_n = BA$. In this case we say that A & B are multiplicative inverses of one another. In notations, we write $B = A^{-1}$ or $A = B^{-1}$.

2. Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then A'(or A^T) the transpose of A is defined as $A' = [b_{ji}]_{n \times m}$ where $b_{ji} = a_{ij}$ $\forall$ i & j.

(i) $(A')' = A$ (ii) $(\lambda A)' = \lambda A'$ (iii) $(A + B)' = A' + B'$ & $(A - B)' = A' - B'$
(iv) $(AB)' = B'A'$
(v) **Symmetric & skew symmetric matrix :** A square matrix A is said to be symmetric if $A' = A$
A square matrix A is said to be skew symmetric if $A' = -A$

3. Submatrix of a Matrix

Submatrix : Let A be a given matrix. The matrix obtained by deleting some rows or columns of A is called as submatrix of A.

(i) **Some properties of determinant**
(a) $|A| = |A'|$ for any square matrix A.
(b) If two rows are identical (or two columns are identical) then $|A| = 0$.
(c) $|\lambda A| = \lambda^n |A|$, when $A = [a_{ij}]_n$.
(d) If A and B are two square matrices of same order, then $|AB| = |A|\,|B|$.

(ii) **Singular & non singular matrix :** A square matrix A is said to be singular or non singular according as |A| is zero or non zero respectively.

(iii) **Cofactor & adjoint matrix :** Let $A = [a_{ij}]_n$ be a square matrix. The matrix obtained by replacing each element of A by corresponding cofactor is called as matrix of cofactor of A. The transpose of matrix of cofactor of A is called as adjoint of A, denoted as adj A.

(iv) **Properties of adj A:**

(a) $A \cdot \text{adj } A = |A|\, I_n = (\text{adj } A)\, A$ where $A = [a_{ij}]_n$.

(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A.

(c) If A is a symmetric matrix, then adj A is also symmetric matrix.

(d) If A is singular, then adj A is also singular.

(v) **Inverse of a matrix (reciprocal matrix) :** Let A be a non singular matrix. Then the matrix $\frac{1}{|A|}$ adj A is the multiplicative inverse of A and is denoted by A^{-1}.

Remarks :

1. *A^{-1} is always non singular.*
2. *If $A = \text{dia}\,(a_{11}, a_{12}, \ldots, a_{nn})$ where $a_{ii} \neq 0\ \forall\, i$, then $A^{-1} = \text{diag}\,(a_{11}^{-1}, a_{22}^{-1}, \ldots, a_{nn}^{-1})$.*
3. *$(A^{-1})' = (A')^{-1}$ for any non singular matrix A. Also $\text{adj}\,(A') = (\text{adj } A)'$.*
4. *$(A^{-1})^{-1} = A$ if A is non singular.*
5. *Let k be a non zero scalar & A be a non singular matrix. Then $(kA)^{-1} = \frac{1}{k} A^{-1}$.*
6. *$|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$.*
7. *Let A be a nonsingular matrix. Then $AB = AC \Rightarrow B = C$ & $BA = CA \Rightarrow B = C$.*
8. *A is non-singular and symmetric $\Rightarrow A^{-1}$ is symmetric.*
9. *In general AB = 0 does not imply A = 0 or B = 0. But if A is non singular and AB = 0, then B = 0. Similarly B is non singular and $AB = 0 \Rightarrow A = 0$. Therefore, $AB = 0 \Rightarrow$ either both are singular or one of them is 0.*

4. System of Linear Equations & Matrices

System of linear equations AX = B is said to be consistent if it has atleast one solution.

(i) **System of linear equations and matrix inverse:**

If A is nonsingular, solution is given by $X = A^{-1}B$.

If A is singular, (adj A) B = 0 and no two columns of A are proportional, then the system has infinite many solution.

If A is singular and (adj A) $B \neq 0$, then the system has no solution

(ii) **Homogeneous system and matrix inverse:**

If the above system is homogeneous, n equations in n unknowns, then in the matrix form it is AX = O. ($\because$ in this case $b_1 = b_2 = \ldots\ldots\ b_n = 0$), where A is a square matrix.

If A is nonsingular, the system has only the trivial solution (zero solution) X = 0

If A is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non trivial solutions.

(iii) **Elementary row transformation of matrix :**

The following operations on a matrix are called as elementary row transformations.

(a) Interchanging two rows.

(b) Multiplications of all the elements of row by a nonzero scalar.

(c) Addition of constant multiple of a row to another row.

Remark :

Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write $A \approx B$.

5. More on Matrices

(i) **Characteristic polynomial & Characteristic equation :**

Let A be a square matrix. Then the polynomial $|A - xI|$ is called as characteristic polynomial of A & the equation $|A - xI| = 0$ is called as characteristic equation of A.

Remark :

Every square matrix A satisfy its characteristic equation (Cayley - Hamilton Theorem).

i.e. $a_0 x^n + a_1 x^{n-1} + \dots\dots + a_{n-1}x + a_n = 0$ is the characteristic equation of A, then
$a_0A^n + a_1A^{n-1} + \dots\dots + a_{n-1}A + a_n I = 0$

(ii) **More Definitions on Matrices :**

(a) **Nilpotent matrix:** A square matrix A is called nilpotent if $A^p = O$ for some positive integer
p. If p is smallest such positive integer then p is called its nilpotency

(b) **Idempotent matrix:** A square matrix A is said to be idempotent if, $A^2 = A$.

(c) **Involutory matrix:** A square matrix A is said to be involutory if $A^2 = I$.

(d) **Orthogonal matrix:** A square matrix A is said to be orthogonal, if $A'A = I = A'A$.

(e) **Unitary matrix:** A square matrix A is said to be unitary if $A(\overline{A})' = I$, where $\overline{A}$ is the complex conjugate of A.

DETERMINANTS

1. Definition:

We write the expression $a_1b_2 - a_2b_1$ as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a determinant of order 2.

2. Expansion of Determinant:

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order three.

Its value can be found as:

$$D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \dots \text{ \& so on.}$$

In this manner we can expand a determinant in 6 ways using elements of ; R_1, R_2, R_3 or C_1, C_2, C_3.

3. Minors:

The minor of a_{ij} is obtained deleting i^{th} row & j^{th} column from the determinant. It is denoted by M_{ij}.

4. Cofactor:

Cofactor of the element a_{ij} is $C_{ij} = (-1)^{i+j} \cdot M_{ij}$;

$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$