5. Transpose of a Determinant:

The transpose of a determinant is a determinant obtained after interchanging the rows & columns.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6. Symmetric, Skew-Symmetric, Asymmetric Determinants:

(i) A determinant is symmetric if it is identical to its transpose. Its i^{th} row is identical to its i^{th} column i.e. $a_{ij} = a_{ji}$ for all values of 'i' and 'j'

(ii) A determinant is skew-symmetric if it is identical to its transpose having sign of each element inverted i.e. $a_{ij} = -a_{ji}$ for all values of 'i' and 'j'. A skew-symmetric determinant has all elements zero in its principal diagonal.

(iii) A determinant is asymmetric if it is neither symmetric nor skew-symmetric.

7. Properties of Determinants:

(i) $D = D'$

(ii) If any two rows (or columns) of a determinant be interchanged, then $D' = -D$.

(iii) If a determinant has all the elements zero in any row or column then $D = 0$.

(iv) If a determinant has any two rows (or columns) identical, then $D = 0$.

(v) If all the elements of any row (or column) be multiplied by the same number k, then $D' = kD$

(vi) If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants,

i.e. $$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(vii) The value of a determinant is not altered by adding to the elements of any row (or column) a constant multiple of the corresponding elements of any other row (or column),

i.e. $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}. \text{ Then } D' = D.$$

8. Multiplication Of Two Determinants:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1\ell_1+b_1\ell_2 & a_1m_1+b_1m_2 \\ a_2\ell_1+b_2\ell_2 & a_2m_1+b_2m_2 \end{vmatrix}$$

9. Summation of Determinants

$$\text{Let } \Delta(r) = \begin{vmatrix} f(r) & g(r) & h(r) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \text{ then } \sum_{r=1}^{n} \Delta(r) = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are constants indepedent of r

10. Integration of a determinant

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$, then $\int_a^b \Delta(x)\,dx = \begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx & \int_a^b h(x)\,dx \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants independent of x.

11. Differentiation of Determinant:

Let $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$

then $\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$

12. Cramer's Rule: System of Linear Equations

(i) Two Variables

(a) Consistent Equations: Definite & unique solution. [intersecting lines]

(b) Inconsistent Equation: No solution. [Parallel line]

(c) Dependent equation: Infinite solutions. [Identical lines]

Let $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ then:

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent &

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are dependent

(ii) Three Variables

Let, $a_1x + b_1y + c_1z = d_1$ .... (I)

$a_2x + b_2y + c_2z = d_2$ .... (II)

$a_3x + b_3y + c_3z = d_3$ .... (III)

Then, $x = \frac{D_1}{D}$, $Y = \frac{D_2}{D}$, $Z = \frac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$.

(iii) Consistency of a system of Equations

(a) If $D \neq 0$ and alteast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.

(b) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$ then the given system of equations are consistent and have trivial solution only.

(c) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations have either infinite solutions or no solution.

- Cofactors of D
  - All zeros
    - All the cofactors of D_1, D_2 and D_3 are zero, then system will have infinite solutions
    - Cofactors of atleast one of D_1, D_2 and D_3 is non zero, then system will have no solution.
  - Atleast one non zero, then system will have infinite solutions

(Refer **Example & Self Practice Problem** with*)

(d) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

(e) If a given system of linear equations have Only Zero Solution for all its variables then the given equations are said to have TRIVIAL SOLUTION.

(iv) Three equation in two variables :

If x and y are not zero, then condition for $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ &

$a_3x + b_3y + c_3 = 0$ to be consistent in x and y is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

13. Application of Determinants:

Following examples of short hand writing large expressions are:

(i) Area of a triangle whose vertices are (x_r, y_r); $r = 1, 2, 3$ is:

$$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If D = 0 then the three points are collinear.

VECTORS

1. Vectors & Their Representation:

Vector quantities are specified by definite magnitude and definite directions. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the **initial point** & B is called the **terminal point**. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.

Zero Vector : A vector of zero magnitude is a zero vector.

Unit Vector : A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically, $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

Equal Vectors: Two vectors are said to be equal if they have the same magnitude, direction & represent the same physical quantity.

Collinear Vectors: Two vectors are said to be collinear if their directed line segments are parallel irrespective of their directions. If $\vec{a}$ and $\vec{b}$ are collinear, then $\vec{a} = k\vec{b}$, where $k \in R, k \neq 0$

Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Coplanar Vectors:

A given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note that "**Two Vectors Are Always Coplanar**".

2. **Angle Between two Vectors**

It is the smaller angle formed when the initial points or the terminal points of the two vectors are brought together. It should be noted that $0^\circ \leq \theta \leq 180^\circ$.

3. **Addition Of Vectors:**

- If two vectors $\vec{a}$ & $\vec{b}$ are represented by $\overrightarrow{OA}$ & $\overrightarrow{OB}$, then their sum $\vec{a} + \vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.
- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative)
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (associativity)
- $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$
- $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$
- $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$
- $|\vec{a} - \vec{b}| \geq ||\vec{a}| - |\vec{b}||$
- $\left|\vec{a} \pm \vec{b}\right| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta}$ where θ is the angle between the vectors
- A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\lambda\left(\hat{a} + \hat{b}\right)$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ & $\vec{b}$ is $\lambda\left(\hat{a} - \hat{b}\right)$, $\lambda \in R^+$.

4. **Multiplication Of A Vector By A Scalar:**

If $\vec{a}$ is a vector & m is a scalar, then m $\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called **Scalar Multiplication**. If $\vec{a}$ and $\vec{b}$ are vectors & m, n are scalars, then:

$m(\vec{a}) = (\vec{a})m = m\vec{a}$

$m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

$(m+n)\vec{a} = m\vec{a} + n\vec{a}$

$m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

5. Position Vector Of A Point:

let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then, $\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B – pv of A.

DISTANCE FORMULA : Distance between the two points $A(\vec{a})$ and $B(\vec{b})$ is $AB = \left|\vec{a} - \vec{b}\right|$

SECTION FORMULA : $\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{m+n}$. Mid point of AB $= \dfrac{\vec{a}+\vec{b}}{2}$.

6. Scalar Product Of Two Vectors:

$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$, where $|\vec{a}|, |\vec{b}|$ are magnitude of $\vec{a}$ and $\vec{b}$ respectively and θ is angle between $\vec{a}$ and $\vec{b}$.

(i) $i.i = j.j = k.k = 1$; $\quad i.j = j.k = k.i = 0$ ☞ projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

(ii) if $\vec{a} = a_1 i + a_2 j + a_3 k$ & $\vec{b} = b_1 i + b_2 j + b_3 k$ then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $\quad |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

(iii) the angle ϕ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}\vec{b}}{|\vec{a}|\,|\vec{b}|}$, $0 \le \phi \le \pi$

(iv) $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta \ (0 \le \theta \le \pi)$, note that if θ is acute, then $\vec{a}.\vec{b} > 0$ & if θ is obtuse, then $\vec{a}.\vec{b} < 0$

(v) $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2$, $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative) ☞ $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)

(vi) $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $\quad (\vec{a} \ne 0\ \vec{b} \ne 0)$

(vii) $(m\vec{a}).\vec{b} = \vec{a}.(m\vec{b}) = m(\vec{a}.\vec{b})$ (associative) where m is scalar.

Note:

(a) Maximum value of $\vec{a}.\vec{b}$ is $|\vec{a}|\,|\vec{b}|$

(b) Minimum value of $\vec{a}.\vec{b}$ is $-|\vec{a}|\,|\vec{b}|$

(c) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a}.\hat{i})\,\hat{i} + (\vec{a}.\hat{j})\,\hat{j} + (\vec{a}.\hat{k})\,\hat{k}$.

7. Vector Product Of Two Vectors:

(i) If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them then $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ forms a right handed screw system.

(ii) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.

(iii) $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$; $\hat{i}\times\hat{j}=\hat{k}, \hat{j}\times\hat{k}=\hat{i}, \hat{k}\times\hat{i}=\hat{j}$

(iv) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

(v) $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ (not commutative)

(vi) $(m\vec{a})\times\vec{b}=\vec{a}\times(m\vec{b})=m(\vec{a}\times\vec{b})$ (associative) where m is a scalar.

(vii) $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$ (distributive)

(viii) $\vec{a}\times\vec{b}=\vec{0}\Leftrightarrow\vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a}\neq 0\ ,\ \vec{b}\neq 0)$ i.e. $\vec{a}=K\vec{b}$, where K is a scalar.

(ix) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n}=\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

☞ If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

☞ If $\vec{a},\vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle ABC = $\frac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right]$. The points A, B & C are collinear if $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}$

☞ Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$

☞ Lagrange's Identity: for any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}.\vec{b})^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b}\\ \vec{a}.\vec{b} & \vec{b}.\vec{b}\end{vmatrix}$

8. Scalar Triple Product:

☞ The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ is defined as: $\vec{a}\times\vec{b}.\vec{c}=|\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & φ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also written as $[\vec{a}\ \vec{b}\ \vec{c}]$ and spelled as box product.

☞ Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a},\vec{b}$ & $\vec{c}$ i.e. $V=[\vec{a}\ \vec{b}\ \vec{c}]$

☞ In a scalar triple product the position of dot & cross can be interchanged i.e.

$\vec{a}.(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b}).\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$

☞ $\vec{a}.(\vec{b}\times\vec{c})=-\vec{a}.(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}]=-[\vec{a}\ \vec{c}\ \vec{b}]$

☞ If $\vec{a} = a_1 i + a_2 j + a_3 k$; $\vec{b} = b_1 i + b_2 j + b_3 k$ & $\vec{c} = c_1 i + c_2 j + c_3 k$ then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

In general, if $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$ & $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \left[\vec{l}\,\vec{m}\,\vec{n}\right]$; where $\vec{\ell}$, $\vec{m}$ & $\vec{n}$ are non coplanar vectors.

☞ If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\;\vec{b}\;\vec{c}] = 0$.

☞ $[i\;j\;k] = 1$ ☞ $[K\vec{a}\;\vec{b}\;\vec{c}] = K[\vec{a}\;\vec{b}\;\vec{c}]$ ☞ $[(\vec{a}+\vec{b})\;\vec{c}\;\vec{d}] = [\vec{a}\;\vec{c}\;\vec{d}] + [\vec{b}\;\vec{c}\;\vec{d}]$

☞ Volume of tetrahedron OABC with O as origin & A($\vec{a}$), B($\vec{b}$) and C($\vec{c}$) be the vertices = $\left|\frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]\right|$

☞ The positon vector of the centroid of a tetrahedron if the pv's of its vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}[\vec{a}+\vec{b}+\vec{c}+\vec{d}]$.

9. Vector Triple Product: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$, $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$

☞ $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$, in general

10. Reciprocal System Of Vectors:

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$ then the two systems are called Reciprocal System of vectors, where $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

11. Linear Combinations:

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}$,...... then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots\ldots..$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}$,...... for any x, y, z..... $\in$ R. We have the following results:

(i) If $\vec{a}, \vec{b}$ are non zero, non–collinear vectors then $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'$; $y = y'$

(ii) **FundamentalTheorem:** Let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. Then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$

i.e. There exist some uniquely x, y $\in$ R such that $x\vec{a} + y\vec{b} = \vec{r}$.

(iii) If $\vec{a},\vec{b},\vec{c}$ are non–zero, non–coplanar vectors then:

$$x\vec{a}+y\vec{b}+z\vec{c}=x'\vec{a}+y'\vec{b}+z'\vec{c} \Rightarrow x=x' , y=y' , z=z'$$

(iv) **Fundamental Theorem In Space:** Let $\vec{a},\vec{b},\vec{c}$ be non–zero, non–coplanar vectors in space. Then any vector $\vec{r}$, can be uniqly expressed as a linear combination of $\vec{a},\vec{b},\vec{c}$ i.e. There exist some unique x,y ∈ R such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

(v) If $\vec{x}_1,\vec{x}_2,......\vec{x}_n$ are n non zero vectors, & k_1, k_2,.....k_n are n scalars & if the linear combination $k_1\vec{x}_1+k_2\vec{x}_2+........k_n\vec{x}_n=0 \Rightarrow k_1=0, k_2=0.....k_n=0$ then we say that vectors $\vec{x}_1,\vec{x}_2,......\vec{x}_n$ are **Linearly Independent Vectors**.

(vi) If $\vec{x}_1,\vec{x}_2,......\vec{x}_n$ are not **Linearly Independent** then they are said to be **Linearly Dependent** vectors. i.e. if $k_1\vec{x}_1 + k_2\vec{x}_2 +........+ k_n\vec{x}_n = 0$ & if there exists at least one $k_r \neq 0$ then $\vec{x}_1,\vec{x}_2,......\vec{x}_n$ are said to be **Linearly Dependent**.

Note 1: If $k_1\vec{x}_1+k_2\vec{x}_2+k_3\vec{x}_3+....+k_n\vec{x}_n=\vec{o}$, for some $k_r \neq 0$, then $\vec{x}_1, \vec{x}_2, \vec{x}_3, ..., \vec{x}_n$ form a linearly dependent set of vectors.

- $\hat{i},\hat{j},\hat{k}$ are **Linearly Independent** set of vectors. For $K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = 0 \Rightarrow K_1 = K_2 = K_3 = 0$
- Two vectors $\vec{a}$ & $\vec{b}$ are linearly dependent $\Rightarrow \vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a} \times \vec{b} = 0 \Rightarrow$ linear dependence of $\vec{a}$ & $\vec{b}$. Conversely if $\vec{a}\times\vec{b}=\vec{o}$, then $\vec{a}$ and $\vec{b}$ are linearly independent.
- If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}, \vec{b}, \vec{c}] = 0$, conversely, if $[\vec{a}, \vec{b}, \vec{c}] \neq 0$, then the vectors are linearly independent.

Note: **Test Of Collinearity:**

Three points A,B,C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalars x· y· z not all zero simultaneously such that; $x\vec{a}+y\vec{b}+z\vec{c} = 0$, where x + y + z = 0.

Note: Test Of Coplanarity:

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d} = 0$ where, x + y + z + w = 0.

THREE DIMENSIONAL GEOMETRY

1. **Vector representation of a point :** Position vector of point P (x, y, z) is $x\hat{i} + y\hat{j} + z\hat{k}$.

2. **Distance formula :** $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$, AB = $|\overrightarrow{OB} - \overrightarrow{OA}|$

3. **Distance of P from coordinate axes :** PA = $\sqrt{y^2+z^2}$, PB = $\sqrt{z^2+x^2}$, PC = $\sqrt{x^2+y^2}$

4. **Section Formula :** $x = \dfrac{mx_2+nx_1}{m+n}$, $y = \dfrac{my_2+ny_1}{m+n}$, $z = \dfrac{mz_2+nz_1}{m+n}$

Mid point : $x = \dfrac{x_1+x_2}{2}$, $y = \dfrac{y_1+y_2}{2}$, $z = \dfrac{z_1+z_2}{2}$

5. **Centroid of a triangle :** $G \equiv \left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}, \dfrac{z_1+z_2+z_3}{3}\right)$

6. **Incentre of triangle ABC :** $\left(\dfrac{ax_1+bx_2+cx_3}{a+b+c}, \dfrac{ay_1+by_2+cy_3}{a+b+c}, \dfrac{az_1+bz_2+cz_3}{a+b+c}\right)$

7. **Centroid of a tetrahedron :** $\left(\dfrac{\sum x_i}{4}, \dfrac{\sum y_i}{4}, \dfrac{\sum z_i}{4}\right)$

8. **Direction Cosines And Direction Ratios**

(i) Direction cosines: Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the line. The direction cosines are usually denoted by (ℓ, m, n). Thus $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) Direction ratios: Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

If a, b, c, are the direction ratios of any line L then $a\hat{i}+b\hat{j}+c\hat{k}$ will be a vector parallel to the line L.

If ℓ, m, n are direction cosines of line L then $\ell\hat{i} + m\hat{j} + n\hat{k}$ is a unit vector parallel to tne line L.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\ell = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

or $$\ell = \frac{-a}{\sqrt{a^2+b^2+c^2}}, m = \frac{-b}{\sqrt{a^2+b^2+c^2}}, n = \frac{-c}{\sqrt{a^2+b^2+c^2}}$$

(v) If OP = r, the direction cosines of OP are ℓ, m, n then the coordinates of P are $(\ell r, mr, nr)$.
If direction cosines of the line AB are ℓ, m,n, |AB| = r, and the coordinates of A is (x_1, y_1, z_1) then the coordinates of B is given as $(x_1 + r\ell, y_1 + rm, z_1 + rn)$

(vi) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}$, $m = \frac{y_2 - y_1}{|PQ|}$ and $n = \frac{z_2 - z_1}{|PQ|}$

(vii) Direction cosines of axes:

Direction cosines of x–axis, y -axis and z-axis are (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively

9. Angle Between Two Line Segments:

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|.$$

The line will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$, parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

10. Projection of a line segment on a line

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then the projection of PQ on a line having direction cosines ℓ, m, n is

$$|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$$

A PLANE

If line joining any two points on a surface lies completely on it then the surface is called a plane.

11. Equation Of A Plane : General form: $ax + by + cz + d = 0$, where a, b, c are not all zero, a, b, c, d $\in$ R.

(i) Normal form : $\ell x + my + nz = p$

(ii) Plane through the point (x_1, y_1, z_1) : $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

(iii) Intercept Form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

(iv) Vector form: $(\vec{r} - \vec{a}).\vec{n} = 0$ or $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

(v) **Planes parallel to the axes:**

(a) plane parallel to the x–axis is $by + cz + d = 0$.

(b) planes parallel to y and z–axis are $ax + cz + d = 0$ and $ax + by + d = 0$ respectively.

(vi) **Plane through origin:** Equation of plane passing through origin is $ax + by + cz = 0$.

(vii) **Transformation of the equation of a plane to the normal form:** $ax + by + cz - d = 0$

in normal form is $\frac{ax}{\pm\sqrt{a^2 + b^2 + c^2}} + \frac{by}{\pm\sqrt{a^2 + b^2 + c^2}} + \frac{cz}{\pm\sqrt{a^2 + b^2 + c^2}} = \frac{d}{\pm\sqrt{a^2 + b^2 + c^2}}$

(viii) Any plane parallel to the given plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.

Distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $= \dfrac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

(ix) **Equation of a plane passing through a given point & parallel to the given vectors:**

$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.

or $\vec{r}.(\vec{b}\times\vec{c}) = \vec{a}.(\vec{b}\times\vec{c})$ (non parametric form)

(x) A plane $ax + by + cz + d = 0$ divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2). in the ratio $\left(-\dfrac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$

(xi) Coplanarity of four points : The points $A(x_1\ y_1\ z_1)$, $B(x_2\ y_2\ z_2)$ $C(x_3\ y_3\ z_3)$ and $D(x_4\ y_4\ z_4)$ are coplaner then $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix} = 0$

12. A Plane & A Point

(i) Distance of the point (x', y', z') from the plane $ax + by + cz + d = 0$ is given by $\dfrac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}$.

(ii) Length of the perpendicular from a point $(\vec{a})$ to plane $\vec{r}.\vec{n} = d$ is given by $p = \dfrac{|\vec{a}.\vec{n}-d|}{|\vec{n}|}$.

(iii) Foot (x', y', z') of perpendicular drawn from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is given by $\dfrac{x'-x_1}{a} = \dfrac{y'-y_1}{b} = \dfrac{z'-z_1}{c} = -\dfrac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(iv) **To find image of a point w.r.t. a plane:**
Let P (x_1, y_1, z_1) is a given point and $ax + by + cz + d = 0$ is given plane Let (x', y', z') is the image point. then $\dfrac{x'-x_1}{a} = \dfrac{y'-y_1}{b} = \dfrac{z'-z_1}{c} = -2\dfrac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(v) The distance between two parallel planes $ax + by + cz + d = 0$ and $ax + by + cz + d' = 0$ is given by $\dfrac{|d-d'|}{\sqrt{a^2+b^2+c^2}}$

13. Angle Between Two Planes:

$$\cos\theta = \left|\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}\right|$$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

The angle θ between the planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by, $\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1|.|\vec{n}_2|}$

Planes are perpendicular if $\vec{n}_1.\vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$, λ is a scalar

14. Angle Bisectors

(i) The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(ii) Equation of bisector of the angle containing origin: First make both the constant terms positive.

Then the positive sign in $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$ gives the bisector of the angle which contains the origin.

(iii) Bisector of acute/obtuse angle: First make both the constant terms positive. Then

$a_1a_2 + b_1b_2 + c_1c_2 > 0 \Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 < 0 \Rightarrow$ origin lies in acute angle

15. Family of Planes

(i) Any plane through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is $a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ & $\vec{r}.\vec{n}_2 = d_2$ is $\vec{r}.(n_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

16. Area of a triangle:

Let A (x_1, y_1, z_1), B (x_2, y_2, z_2), C (x_3, y_3, z_3) be the vertices of a triangle, then $\Delta = \sqrt{(\Delta_x^2 + \Delta_y^2 + \Delta_z^2)}$

where $\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$, $\Delta_y = \frac{1}{2}\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Vector Method – From two vector $\vec{AB}$ and $\vec{AC}$. Then area is given by $\frac{1}{2}|\vec{AB} \times \vec{AC}|$

17. Volume Of A Tetrahedron:

Volume of a tetrahedron with vertices A (x_1, y_1, z_1), B(x_2, y_2, z_2), C (x_3, y_3, z_3) and D (x_4, y_4, z_4) is given by $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

A LINE

18. Equation Of A Line

(i) A straight line is intersection of two planes.
it is reprsented by two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + +c_2z + d_2 = 0$.

(ii) Symmetric form : $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r$.

(iii) Vector equation: $\vec{r} = \vec{a} + \lambda\vec{b}$

(vi) Reduction of cartesion form of equation of a line to vector form & vice versa

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \quad \Leftrightarrow \quad \vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}).$$

19. To find image of a point w. r. t a line

Let $L \equiv \frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c}$ is a given line

Let (x', y', z') is the image of the point $P(x_1, y_1, z_1)$ with respect to the line L. Then

(i) $a(x_1 - x') + b(y_1 - y') + c(z_1 - z') = 0$, (ii) $\dfrac{\frac{x_1+x'}{2} - x_2}{a} = \dfrac{\frac{y_1+y'}{2} - y_2}{b} = \dfrac{\frac{z_1+z'}{2} - z_2}{c} = \lambda$

from (ii) get the value of x', y', z' in terms of λ as

$x' = 2a\lambda + 2x_2 - x_1$, $y' = 2b\lambda - 2y_2 - y_1$, $z' = 2c\lambda + 2z_2 - z_1$

now put the values of x', y', z' in (i) get λ and resubtitute the value of λ to to get $(x'\ y'\ z')$.

20. Angle Between A Plane And A Line:

(i) If θ is the angle between line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$,

then $\sin\theta = \left|\dfrac{a\ell + bm + cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{\ell^2+m^2+n^2}}\right|$.

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\cdot\vec{n} = d$ then $\sin\theta = \left[\dfrac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right]$.

(iii) Condition for perpendicularity $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$, $\vec{b} \times \vec{n} = 0$

(iv) Condition for parallel $a\ell + bm + cn = 0$ $\vec{b}\cdot\vec{n} = 0$

21. Condition For A Line To Lie In A Plane

(i) Cartesian form: Line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ would lie in a plane

$ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) Vector form: Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r}\cdot\vec{n} = d$ if $\vec{b}\cdot\vec{n} = 0$ & $\vec{a}\cdot\vec{n} = d$

22. **Skew Lines:**

(i) The straight lines which are not parallel and non–coplanar i.e. non–intersecting are called skew lines. If $\Delta = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, then lines are skew.

(ii) Shortest distance: Suppose the equation of the lines are

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \text{ and } \frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$$

$$\text{S.D.} = \frac{(\alpha-\alpha')(mn'-m'n)+(\beta-\beta')(n\ell-n'\ell)+(\gamma-\gamma')(\ell m'-\ell' m)}{\sqrt{\sum (mn'-m'n)^2}}$$

$$= \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \div \sqrt{\sum (mn'-m'n)^2}$$

(iii) Vector Form: For lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ to be skew $(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1) \neq 0$

(iv) Shortest distance between lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}$ & $\vec{r} = \vec{a}_2 + \mu \vec{b}$ is $d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \times \vec{b}}{|\vec{b}|} \right|$.

23. **Sphere**

General equation of a sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. $(-u, -v, -w)$ is the centre and $\sqrt{u^2 + v^2 + w^2 - d}$ is the radius of the sphere.

FOR JEE(MAIN)

SETS & RELATIONS

1. **SET**

A set is a collection of well defined objects which are distinct from each other. Set are generally denoted by capital letters A, B, C, ... etc. and the elements of the set by a, b, c .... etc.

2. **Methods to write a set :**

(i) Roster Method : In this method a set is described by listing elements, separated by commas and enclose then by curly brackets.

(ii) Set builder from : In this we write down a property or rule which gives us all the element of the set. A = {x " P(x)} where P(x) is the property by which $x \in A$.

3. **Types of sets:**

(i) Null set or empty set : A set having no element in it is called an empty set or a null set or void set, it is denoted by ϕ or { }

(ii) Singleton : A set consisting of a single element is called a singleton set.

(iii) Finite set : A set which has only finite number of elements is called a finite set.

(iv) Infinite set : A set which has an infinite number of elements is called an infinite set.

(v) Equal sets : Two sets A and B are said to be equal if every element of A is member of B, and every element of B is a member of A.

(vi) Equivalent sets : Two finite sets A and B are equivalent if their number of elements are same i.e. n(A) = n(B)

(vii) **Subset :** Let A and B be two sets if every element of A is an element B, then A is called a subset of B if A is a subset of B. We write $A \subseteq B$

(viii) **Proper subset :** If A is a subset of B s.t. $A \neq B$ then A is a proper subset of B. and we write $A \subset B$

(ix) **Universal set :** A set consisting of all possible elements which occur in the discussion is called a universal set and is denoted by U

(x) **Power set :** Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A)

4. Some Operation on sets :

(i) **Union of two sets :** $A \cup B = \{x : x \in A \text{ or } x \in B\}$

(ii) **Intersection of two sets :** $A \cap B = \{x : x \in A \text{ and } x \in B\}$

(iii) **Difference of two sets :** $A - B = \{x : x \in A \text{ and } x \notin B\}$

(iv) **Complement of a set :** $A' = \{x : x \notin A \text{ but } x \in U\} = U - A$

(v) **De-morgan laws :** $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$

(vi) **Distributive laws :** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(vii) **Commutative laws :** $(A \cup B) = B \cup A$; $A \cap B = B \cap A$

(viii) **Associative laws :** $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$

(ix) $A \cap \phi = \phi$; $A \cap U = A$ $\quad$ $A \cup \phi = A$; $A \cup U = U$

(x) $A \cap B \subseteq A$; $A \cap B \subseteq B$ $\quad$ (xi) $A \subseteq A \cup B$; $B \subseteq A \cup B$ $\quad$ (xii) $A \subseteq B \Rightarrow A \cap B = A$

(xiii) $A \subseteq B \Rightarrow A \cup B = B$

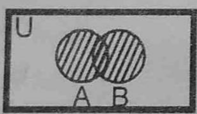
5. Disjoint sets :

If $A \cap B = \phi$, then A, B are disjoint

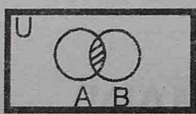
6. Symmetric difference of sets :

$A \Delta B = (A - B) \cup (B - A)$ $\quad$ $(A')' = A$ $\quad$ $A \subseteq B \Rightarrow B' \subseteq A'$

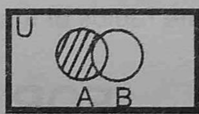
7. Venn diagrame



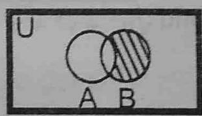
$A \cup B$



$A \cap B$

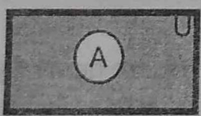


$A - B$

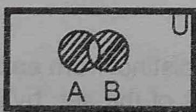


$B - A$

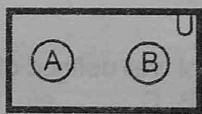
Clearly $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$



A'



$(A \Delta B) = (A - B) \cup (B - A)$



Disjoint

8. Some important results on number of elements in sets :

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B)$ $\quad \Rightarrow \quad$ A, B are disjoint non-void sets.

(iii) $n(A - B) = n(A) - n(A \cap B)$ $\quad$ i.e. $\quad$ $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(v) Number of elements in exactly two of the sets A, B, C

$$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$$

(vi) Number of elements in exactly one of the sets A, B, C

$$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$$

(vii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

RELATION

Introduction :

1. **Cartesian Product :** The set of all possible ordered pairs (a, b), where $a \in A$ and $b \in B$ i.e. $\{(a, b) ; a \in A \text{ and } b \in B\}$ is called the cartesian product of A to B and is denoted by $A \times B$, usually $A \times B \neq B \times A$.

Relation : Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$. Thus, R is a relation from A to B $\Rightarrow R \subseteq A \times B$.

Total number of relations : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}.

Domain and Range of a relation : Let R be a relation from a set A to a set B. Then the set of all first components of coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components of coordinates of the ordered pairs in R is called the range of R.

Thus, Dom (R) = $\{a : (a, b) \in R\}$ and, Range (R) = $\{b : (a, b) \in R$

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

Inverse relation : Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

2. **Types of relations :**

Void relation : Let A be a set. Then $\phi \subset A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

Universal relation : Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

Identity relation : Let A be a set. Then the relation $I_A = \{(a, a) : a \in A$ on A is called the identity relation on A.

Reflexive relation : A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus, R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Transitive relation : Let A be any set. A relation R on A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c, \in A$ i.e. a R b and b R c $\Rightarrow$ a R c for all $a, b, c \in A$

Antisymmetric relation : Let A be any set. A relation R on set A is said to be an antisymmetric relation iff $(a, b) \in R$ and $(b, a) \in R$ $\Rightarrow a = b$ all $a, b \in A$

Equivalence relation : A relation R on a set A is said to be an equivalence relation of A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b, \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, \in A$

Partial order relation :

A relation R on set A is said to be an partial order relation on A if

(i) R is reflexive i.e. $(a, a) \in R, \forall a \in A$

(ii) R is antisymmetric i.e. $(a, b) \in R$ and $(b, a) \in R$ only possible when $a = b \ \forall a, b \in A$

(iii) R is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R \ \forall a, b, c \in R$

STATISTICS

MEASURES OF CENTRAL TENDENCY :

It is that single value which may be taken as the most suitable representative of the data. This single value is known as the average. Averages are generally, the central part of the distribution and therefore, they are also called the measures of **Central Tendency**.

It can be divided into two groups :

(a) **MATHEMATICAL AVERAGE :**

I. Arithmetic mean or mean

II. Geometric mean

III. Harmonic mean

(b) **POSITIONAL AVERAGE :**

I. Median

II. Mode or positional average

ARITHMETIC MEAN :

Individual observation or unclassified data :

If $x_1, x_2, ..., x_n$ be n observations, then their arithmetic mean is given by

$$\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n} \text{ or } \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Arithmetic mean of discrete frequency distribution :

Let $x_1, x_2, ..., x_n$ be n observation and let $f_1, f_2, ..., f_n$ be their corresponding frequencies, then their

$$\text{mean } \bar{x} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} \text{ or } \bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

Short cut method :

If the values of x or (and) f are large, the calculation of arithmetic mean by the previous formula used, is quite tedious and time consuming. In such case we take the deviation from an arbitrary point A.

$$\bar{x} = A + \frac{\Sigma f_i \ d_i}{\Sigma f_i}$$

where A = Assumed mean

$d_i = x_i - A$ = deviation for each term

Step deviation method :

Sometimes during the application of shortcut method of finding the mean, the deviation d_i are divisible by a common number h(say). In such case the arithmetic is reduced to a great extent taken by

$$u_i = \frac{x_i - A}{h}, \ i = 1, 2, ..., n$$

$$\therefore \quad \text{mean } \bar{x} = A + h\left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right)$$

Weighted arithmetic mean : If $w_1, w_2, w_3, ..., w_n$ are the weight assigned to the value $x_1, x_2, x_3, ..., x_n$ respectively, then the weighted average is defined as -

$$\text{Weighted A.M.} = \frac{w_1x_1 + w_2w_2 + ... + w_nx_n}{w_1 + w_2 + ... + w_n}$$

$$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

Combined mean : If $\bar{x}_1, \bar{x}_2, ..., \bar{x}_k$ are the mean of k series of sizes $n_1, n_2, ..., n_k$ respectively then the mean $\bar{x}$ of the composite series is given by $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2 + ... + n_k\bar{x}_k}{n_1 + n_2 + ... + n_k}$

Properties of Arithmetic Mean :

(i) If $\bar{x}$ is the mean of $x_1, x_2, ..., x_n$. The mean of $ax_1 + b, \ ax_2 + b, ... ax_n + b$ is $a\bar{x} + b$ where a, b are any real numbers ($a \neq 0$).

(ii) Arithmetic mean is dependent on change of origin & scale.

Merits of Arithmetic Mean :

(i) It is rigidly defined.
(ii) It is based on all the observation taken.
(iii) It is calculated with reasonable ease and repidity.
(iv) It is least affected by fluctuations in sampling.
(v) It is based on each observation and so it is a better representative of the data.
(vi) It is relatively reliable
(vii) Mathematical analysis of mean is possible.

Demerits of Arithmetic Mean :

(i) It is severely affected by the extreme values.
(ii) It cannot be represented in the actual data since the mean does not coincide with any of the observed value.
(iii) It cannot be computed unless all the items are known.
(iv) It cannot be calculated for qualitative data incapable of numerical measurements.
(v) It cannot be used in the study of ratios, rates etc.

MEDIAN :

Median is the middle most or the central values of the variate in a set of observations, when the observations are arranged either in ascending or in descending order of their magnitudes. It divides the arranged series in two equal parts.

Median of an individual series : Let n be the number of observations-

(i) arrange the data in ascending or descending order.
(ii) (a) If n is odd then -

$$\text{Median (M)} = \text{Value of } \left(\frac{n+1}{2}\right)^{th} \text{ observation}$$

(b) If n is even then

$$\text{Median (M)} = \text{mean of } \left(\frac{n}{2}\right)^{th} \text{ and } \left(\frac{n}{2}+1\right)^{th} \text{ observation}$$

i.e. $$M = \frac{\left(\frac{n}{2}\right)^{th} \text{observation} + \left(\frac{n}{2}+1\right)^{th} \text{observation}}{2}$$

Median of the discrete frequency distribution : Algorithm to find the median :

Step - I : Find the cumulative frequency (C.F.)

Step - II : Find $\frac{N}{2}$, where $N = \sum_{i=1}^{n} f_i$

Step - III : See the cumulative frequency (C.F.) just greater than $\frac{N}{2}$ and determine the corresponding value of the variable.

Step - IV : The value obtained in step III is the median.

Median of grouped data or continuous series : Let the number of observation be n

(i) Prepare the cumulative frequency table
(ii) Find the median class i.e. the class in which the $\left(\frac{N}{2}\right)^{th}$ observation lies
(iii) The median value is given by the formulae

$$\text{Median (M)} = \ell + \left[\frac{\left(\frac{N}{2}\right) - F}{f}\right] \times h$$

N = total frequency = Σf_i
l = lower limit of median class
f = frequency of the median class
F = cumulative frequency of the class preceding the median class
h = class interval (width) of the median class

MODE :

Mode is that value in a series which occurs most frequently. In a frequency distrinuction, mode is that variate which has the maximum frequency.

Computation of Mode :

Mode for individual series :

In the case of individual series, the value which is repeated maximum number of times is the mode of the series.

Mode for grouped data (discreate frequency distribution series) :

In the case of discrete frequency distribution, mode is the value of the variate corresponding to the maximum frequency.

Mode for continuous frequency distribution :

(i) First find the model class i.e. the class which has maximum frequency. The model class can be determined either by inspecting or with the help of grouping data.

(ii) The mode is given by the formula

$$\text{Mode} = \ell + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} h$$

where ℓ → lower limit of the model class
h → width of the model class
f_m → frequency of the model class
f_{m-1} → frequency of the class preceding model class
f_{m+1} → frequency of the class succeeding model class

Relationship between mean, mode and median

(i) In symmetrical distribution
Mean = Mode = Median

(ii) In skew (moderately symmetrical) distribution, mode = 3 median – 2 mean

MEASURES OF DISPERSION :

Dispersion is the measure of the variations. The degree to which numerical data tend to spread about an average value is classed the dispersion of the data. The measures of dispersion commonly used are:

(i) Range
(ii) Mean deviation
(iii) Standard Deviation

RANGE :

It is the difference between maximum and minimum values of variate i.e. range is = L – S where L is the largest value and S is the smallest value.

MEAN DEVIATION :

Mean deviation is defined as the arithmetic mean of the absolute deviations of all the values taken about any central value.

I. **Mean deviation of individual observations :** If $x_1, x_2, \ldots, x_n$ are n values of a variable x, then the mean deviation from an average A (median or AM) is given by

$$\text{M.D.} = \frac{1}{4}\sum_{i=1}^{n} |x_i - A| = \frac{1}{4}\Sigma|d_i|, \text{ where } d_i = x_i - A$$

II. **Mean deviation of a discrete frequency distribution :** If $x_1, x_2, \ldots, x_n$ are n observation with frequencies $f_1, f_2, \ldots, f_n$, then mean deviation from an average A is given by -

$$\text{Mean Deviation} = \frac{1}{N} \Sigma f_i |x_i - A|$$

$$\text{where } N = \sum_{i=1}^{n} f_i$$

III. **Mean deviation of a grouped or continuous frequency distribution :** For calculating mean deviation of a continuous frequency distribution the procedure is same as for a discrete frequency distribution. The only difference is that the here we have to obtain the midpoint of the various classes and take the deviations of these mid point from the given central value (median or mean).

VARIANCE AND STANDARD DEVIATION :

The variance of a variate x is the arithmetic mean of the squares of all deviations of x from the arithmetic mean of the observations and is denoted by var(x) or σ^2.

The positive square root of the variance of a variate x is known as standard deviation

i.e. standard deviation = $+\sqrt{\text{var}(x)}$

(i) Variance of Individual observations : If $x_1, x_2, \ldots\ldots, x_n$ are n values of a variable x, then by definition

$$\text{var}(x) = \frac{1}{4}\left[\sum_{i=1}^{n} (x_i - \bar{x})^2\right] = \sigma^2 \quad \ldots\ldots\ldots(i)$$

or $$\text{var}(x) = \frac{1}{4}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 \quad \ldots\ldots\ldots(ii)$$

If the value of variable x are large, the calculation of variance from the above formulae is quite tedious and time consuming. In that case, we take deviation from an arbitrary point A (say) then

$$\text{var}(x) = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2 \quad \ldots\ldots\ldots(iii)$$

(ii) **Variance of a discrete frequency distribution :** If $x_1, x_2, \ldots, x_n$ are n observation with frequencies $f_1, f_2, \ldots, f_n$, then $\text{var}(x) = \frac{1}{N}\left\{\sum_{i=1}^{n} f_i(x_i - \bar{x})^2\right\}$ .....(i)

or $$\text{var}(x) = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \bar{x}^2 \quad \ldots\ldots(ii)$$

If the value of x or f are large, we take the deviations of the values of variable x from an arbitrary point A. (say)

$\therefore$ $d_i = x_i - A$; i = 1, 2, ..., n

$$\therefore \quad \text{Var}(x) = \frac{1}{N}\left(\sum_{i=1}^{n} f_i d_i^2\right) - \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i^2\right) \quad \ldots\ldots(iii)$$

$\therefore$ where $N = \sum_{i=1}^{n} f_i$

Sometime $d_i = x_i - A$ are divisible by a common number h(say)

then $u_i = \frac{x_i - A}{h} = \frac{d_i}{h}$, i = 1, 2, ..., n

then $\text{var}(x) = h^2 \left[\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right]$ .....(iv)

(iii) **Variance of a grouped or continuous frequency distribution :** In a grouped or continuous frequency distribution any of the formulae discussed in discrete frequency distribution can be used.

COEFFICIENT OF VARIATION :

The mean deviation and standard deviation have the same units in which the data is given. Whenever we want to compare the variability of two series with data expressed in different units, we require measure of dispersion which is independent of the units. This measure is coefficient of variation (C.V.)

$$C.V. = \frac{S.D}{Mean} \times 100 = \frac{\sigma}{\bar{x}} \times 100, \quad \bar{x} \neq 0$$

The series having greater C.V. is said to be more variable and less consistent than the other.

MATHEMATICAL REASONING

STATEMENTS :

In reasoning we communicate our ideas or thoughts with the help of sentences in a particular language. "A sentence is called a mathematically acceptable statement if it either true or false but not both". A statement is assumed to be either true or false. A true statement is known as a valid statement and a false statement is known as an invalid statement.

Note : A statement can not be both true and false at the same time.

Illustration 1 :

consider the following sentences :

(i) Three plus two equals five.

(ii) The sum of two negative number is negative

(iii) Every square is a rectangle.

Each of these sentences is a true sentence, therefore they are statements.

Illustration 2: Consider the following sentences :

(i) Three plus four equals six.

(ii) All prime numbers are odd.

(iii) Every relation is a function.

Each of these sentences is a false sentence, therefore they are statements.

Illustration 3:

Consider the following sentences :

(i) The sum of x and y is greater than 0.

(ii) The square of a number is even.

Here, we are not in a position to determine whether it is true of false unless we know what the numbers are. Therefore these sentences are not a statement.

Note : (i) Imperative (expresses a request or command), exclamatory sentences (expresses some strong feeling), Interrogative sentences (asks some questions) do not considered as a statement in mathemetical language.

(ii) Sentences involving variable time such as "today", "tomorrow" or "yesterday" are not statements.

Illustration 4: Consider the sentences :

(i) Give me a glass of water.
(ii) Is very set finite ?
(iii) How beautiful ?
(iv) Tomorrow is Monday.

These all the sentences are not a statement.

TRUTH TABLE :

Truth table is that which gives truth values of compound statements.
It has a number of rows and columns. The number of rows depend upon the number of simple statements.
Note truth table for simple statement :
Note that for n statements, there are 2^n rows,

(i) Truth table for single statement p:
Number of rows = $2^1 = 2$

p
T
F

(ii) Truth table for two statements p and q :
Number of rows = $2^2 = 4$

p	q
T	T
T	F
F	T
F	F

(iii) Truth table for three statements p, q are r.
Number of rows = $2^3 = 8$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

NEGATION OF A STATEMENT :

The denial of a statement p is called its negation and is written as ~ p, and read as 'not p'. Negation of any statement p is formed by writting "It is not the case that ...."
or "It is false that ...."
or inserting the word "not" in p.
For example consider the statement:

COMPOUND STATEMENTS :

If a statement is combination of two or more statements, then it is said to be a compound statement. And each statement which form a compound statement are known as its sub-statements or component statements.

BASIC CONNECTIVES :

In the compound statement, we have learnt that the words 'or' and 'and' connect two or more statements. These are called connectives. When we use these compound statements, it is necessary to understand the role of these words.

THE WORD "AND" :

Any two statements can be connected by the word "and" to form a compound statement. The compound statement with word "and" is true if all its component statements are true. The compound statement with word "and" is false if any or all of its component statements are false. The compound statement "p and q" is denoted by "$p \wedge q$".

Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

THE WORD "OR" :

Any two statements can be connected by the word "OR" to form a compound statement. For example, consider the statement. The compound statement with word "or" is true if any or all of its component statements are true. The compound statement with word "or" is false if all its component statement are false.

The compound statement "p or q" is denoted by "$p \vee q$."

Truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

NEGATION OF "AND, OR" :

~(p or q) is ~ p and ~ q $\boxed{\sim(p \vee q) = \sim p \wedge \sim q}$

~(p and q) is ~ p or ~ q $\boxed{\sim(p \wedge q) = \sim p \vee \sim q}$

ALGEBRA OF STATEMENTS :

Statements satisfy many laws some of which are given below -

(1) Idempotent Laws : If p is any statement then
(i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$

(2) Associative Laws : If p, q, r are any three statements, then
(i) $p \vee (q \vee r) = (p \vee q) \vee r$
(ii) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

(3) Commutative Laws : If p, q are any two statements, then
(i) $p \vee q = q \vee p$ (ii) $p \wedge q = q \wedge p$

(4) Distributive Laws : If p,q,r are any three statements, then
(i) $p \vee (q \vee r) = (p \wedge r) \vee (p \wedge r)$ (ii) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(5) Identity Laws : If p is any statement-1, t is tautalogy and c is a couthdiction, then
(i) $p \vee t = t$ (ii) $p \wedge t = p$ (iii) $p \vee c = p$ (iv) $p \wedge c = c$

(6) Complement Laws : If t is a tautology, c is a contradiction and p is any statement, then
(i) $p \vee (\sim p) = t$ (ii) $p \wedge (\sim p) = c$ (iii) $\sim t = c$ (iv) $\sim c = t$

(7) Involution law : If p is any statement, then $\sim(\sim p) = p$

(8) De morgan's law : if p and q are two statements, then
(i) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

(ii) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

CONDITIONAL STATEMENTS OR IMPLICATIONS :

If p and q are any two statement then the compound statement in the form "If p then q" is called a conditional statement or an implication. The statement "If p then q" is denoted by $\boxed{p \to q}$ or $\boxed{p \Rightarrow q}$ (to be read as p implies q) In the implication "$p \to q$", p is called the antecedent (or the hypothesis) and q the consequent (or the conclusion)

For examples : Consider the statements -

(i) If $x = 4$, then $x^2 = 16$
(ii) If ABCD is a parallelogram, then AB = CD
(iii) If Mumbai is in England, then 2 + 2 = 5
(iv) If Shikha works hard, then it will rain today.

Remark :

In the first two statements given above we observe that the hypothesis and conclusion have related subject matters where as in the last two statements do not have related subject matters. In mathematical logic such type of statements are also accepted as a conditional statements.

Truth table for a conditional statement :

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Note: $p \to q$ is false only when p is true and q is false.

NEGATION AND CONTRAPOSITIVE OF A CONDITIONAL STATEMENT :

(1) **Negation :** If p and q are two statements then negation of $p \to q$ is $\sim(p \to q)$ which is equivalent to $p \wedge \sim q$

(2) **Contrapositive :** If p and q are two statements, then the contrapositive of $p \to q$ is $(\sim q) \to (\sim p)$

BICONDITIONAL STATEMENTS :

If p and q are any two statements then the compound statement in the form of "p if and only if q" is called a biconditional statements and is written in symbolic form $\boxed{p \leftrightarrow q}$ or $\boxed{p \Leftrightarrow q}$

Ex. The following statements are biconditional statements
(i) A number is divisible by 3 if and only if the sum of the digits forming the number is divisible by 3.
(ii) One is less than seven if and only if two is less than eight.
(iii) A triangle is equilateral if and only if it is equiangular.

Truth table for a biconditional statement :

Rule : $p \leftrightarrow q$ is true only when both p and q have the same value.

p	q	$p \to q$	$q \leftrightarrow p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Negation of biconditional statement :

$\sim(p \leftrightarrow q) = (p \wedge \sim q) \vee (\sim p \wedge q)$

TAUTOLOGY AND FALLACY (CONTRADICTIONS) :

(a) Tautology : This is a statement which is true for all truth values of its components.

Ex. Consider p v ~ p
Truth table

p	~p	p v ~p
T	F	T
F	T	T

we observe that last columns is always true
Hence p v ~ p is a tautology.

(b) **Fallacy (contradiction) :** This is statement which is false for all truth values of its components.

Ex. Consider p ∧ ~ p

p	~p	p∧~p
T	F	F
F	T	F

We observe that last columns is always false.
Hence p ∧ ~ p is a fallacy (contradiction).

DUALITY :

The compound statements s_1 and s_2 are said to be dauls of each other if one can be obtained from the other by replacing ∧ by v and by v by ∧.The connectives ∧ and v are also called duals of each other. Let s(p, q) = p ∧ q be a compound statement then s * (p, q) = p v q where s * (p, q) is the dual statement of s(p,q).

ARGUMENTS AND THEIR VALIDITY :

An argument is an assertion that a given set of statements $s_1, s_2, \ldots\ldots s_n$ implies other statement. In other word, an argument is an assertion that the statement s follows from statements $S_1, S_2, \ldots, S_n$ are called hypotheses (or premises) and the statement s is called the conclusion.
We denote the argument containing hypotheses
$s_1, s_2, \ldots\ldots s_n$ and conclusion s by

$s_1, s_2, \ldots\ldots s_n$; s
or
$s_1, s_2 \ldots\ldots s_n$ /- s
or
$(s_1 \wedge s_2 \wedge \ldots\ldots \wedge s_n) \rightarrow s$
or
s_1
s_2
s_3
...
...
s_n
______
so s

The symbol "/-" is read as turnstile.

VALID ARGUMENT :

An argument is said to be a valid argument if the conclusion s is true whenever all the hypotheses $s_1, s_2, \ldots$ s_n are true or equivalently argument is valid when it is a tautology, otherwise the argument is called an invalid argument.

Method of testing the validity of arguments :

Step I - Construct the truth table for conditional statements $s_1 \wedge s_2 \wedge s_3 \wedge \ldots \wedge s_n \rightarrow s$.

Step II - Check the last column of truth table. If the last column contains T only, then the given argument is valid. otherwise, it is an invalid argument.