Definition :

Function is a rule (or correspondence), from a non empty set A to a non empty set B, that associates each member of A to a unique member of B. Symbolically, we write f: A $\to$ B. We read it as "f is a function from A to B".

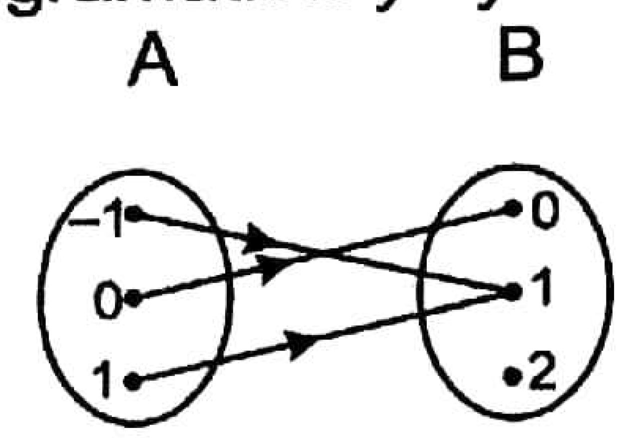
For example, let A $\equiv$ {–1, 0, 1} and B $\equiv$ {0, 1, 2}.

Then A $\times$ B $\equiv$ {(–1, 0), (–1, 1), (–1, 2), (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)}

Now, " f : A $\to$ B defined by f(x) = x^2 " is the function such that

f $\equiv$ {(–1, 1), (0, 0), (1, 1)}

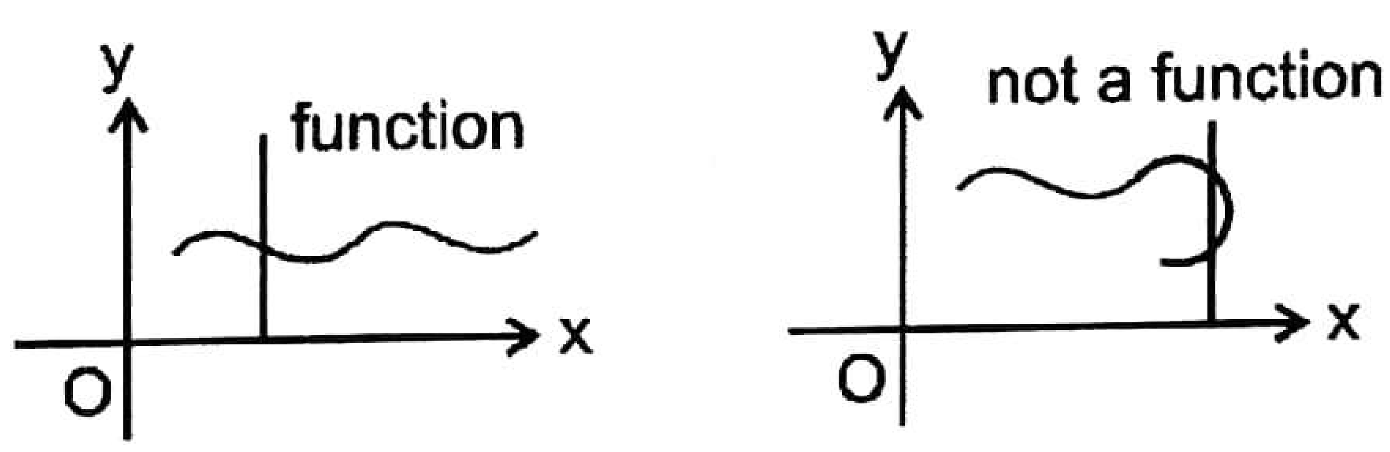
f can also be shown diagramatically by following mapping.



Note : Every function say y = f(x) : A $\to$ B. Here x is independent variable which takes its values from A while 'y' takes its value from B. A relation will be a function if and only if

(i) x must be able to take each and every value of A and

(ii) one value of x must be related to one and only one value of y in set B.



(a) (b)

Graphically : If any vertical line cuts the graph at more than one point, then the graph does not represent a function.

Example # 1 : (i) Which of the following correspondences can be called a function ?

(A) $f(x) = x^3$; {–1, 0, 1} $\to$ {0, 1, 2, 3}

(B) $f(x) = \pm\sqrt{x}$; {0, 1, 4} $\to$ {–2, –1, 0, 1, 2}

(C) $f(x) = \sqrt{x}$; {0, 1, 4} $\to$ {–2, –1, 0, 1, 2}

(D) $f(x) = -\sqrt{x}$; {0, 1, 4} $\to$ {–2, –1, 0, 1, 2}

(ii) Which of the following pictorial diagrams represent the function

Solution :

(i) f(x) in (C) and (D) are functions as definition of function is satisfied. while in case of (A) the given relation is not a function, as f(–1) $\notin$ 2nd set. Hence definition of function is not satisfied. While in case of (B), the given relation is not a function, as f(1) = $\pm$ 1 and f(4) = $\pm$ 2 i.e. element 1 as well as 4 in 1st set is related with two elements of 2nd set.Hence definition of function is not satisfied.

(ii) B and D. In (A) one element of domain has no image, while in (C) one element of 1st set has two images in 2nd set

Self practice problem :

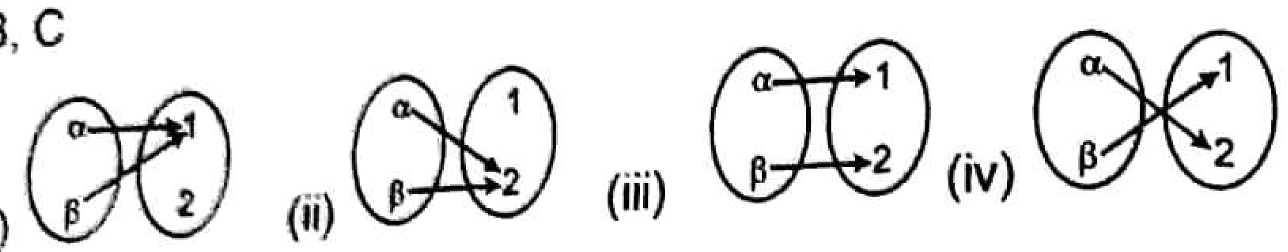
(1) Let g(x) be a function defined on [–1, 1]. If the area of the equilateral triangle with two of its vertices at (0,0) and (x,g(x)) is $\sqrt{3}/4$ sq. unit, then the function g(x) may be.

(A) $g(x) = \pm\sqrt{(1-x^2)}$ (B) $g(x) = \sqrt{(1-x^2)}$ (C) $g(x) = -\sqrt{(1-x^2)}$ (D) $g(x) = \sqrt{(1+x^2)}$

(2) Represent all possible functions defined from {α, β} to {1, 2}.

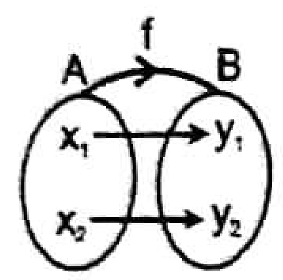
Answers : (1) B, C

(2) (i) (ii) (iii) (iv)



Domain, Co-domain and Range of a Function :

Let y = f(x) : A → B, then the set A is known as the domain of f and the set B is known as co-domain of f.



If x_1 is mapped to y_1, then y_1 is called as image of x_1 under f. Further x_1 is a pre-image of y_1 under f.
If only expression of f (x) is given (domain and co-domain are not mentioned), then domain is **complete** set of those values of x for which f (x) is real, while codomain is considered to be (– ∞, ∞) (except in inverse trigonometric functions).
Range is the complete set of values that y takes. Clearly range is a subset of Co-domain.
A function whose domain and range are both subsets of real numbers is called **a real function.**

Example # 2 : Find the domain of following functions :

(i) $f(x) = \sqrt{x^2-5}$ (ii) $\sin(x^3 - x)$

Solution : (i) $f(x) = \sqrt{x^2-5}$ is real iff $x^2 - 5 \geq 0$

$\Rightarrow$ $|x| \geq \sqrt{5}$ $\Rightarrow$ $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$

$\therefore$ the domain of f is $(-\infty, -\sqrt{5}\,] \cup [\sqrt{5}, \infty)$

(ii) $x^3 - x \in R$ $\therefore$ domain is $x \in R$

Algebraic Operations on Functions :

If f and g are real valued functions of x with domain set A and B respectively, then both f and g are defined in A ∩ B. Now we define f + g, f – g, (f . g) and (f /g) as follows:

(i) $(f \pm g)(x) = f(x) \pm g(x)$
(ii) $(f.g)(x) = f(x).\ g(x)$
— domain in each case is A ∩ B

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is {x | x ∈ A ∩ B such that g(x) ≠ 0}.

For domain of $\phi(x) = \{f(x)\}^{g(x)}$, conventionally, the conditions are f(x) > 0 and g(x) must be real.
For domain of $\phi(x) = {}^{f(x)}C_{g(x)}$ or $\phi(x) = {}^{f(x)}P_{g(x)}$ conventional conditions of domain are $f(x) \geq g(x)$ and $f(x) \in N$ and $g(x) \in W$.

Example # 3 : Find the domain of function $f(x) = \frac{3}{\sqrt{4-x^2}}\log(x^3 - x)$

Solution : Domain of $\sqrt{4-x^2}$ is [–2, 2] but $\sqrt{4-x^2} = 0$ for x = ± 2 ⇒ x ∈ (–2, 2)
$\log(x^3 - x)$ is defined for $x^3 - x > 0$ i.e. $x(x-1)(x+1) > 0$.
$\therefore$ domain of $\log(x^3 - x)$ is (–1, 0) ∪ (1, ∞).
Hence the domain of the given function is {(–1, 0) ∪ (1, ∞)} ∩ (–2, 2) = (–1, 0) ∪ (1, 2).

Self practice problems :

(3) Find the domain of following functions.

(i) $f(x) = \frac{1}{\log(2-x)} + \sqrt{x+1}$ (ii) $f(x) = \sqrt{1-x} - \sin\frac{2x-1}{3}$

Answers : (i) [–1, 1) ∪ (1, 2) (ii) [–1, 1]

Methods of determining range :

(i) **Representing x in terms of y**
If y = f(x), try to express as x = g(y), then domain of g(y) represents possible values of y, which is range of f(x).

(ii) **Graphical Method :**
The set of y– coordinates of the graph of a function is the range.

Example # 4 : Find the range of $f(x) = \frac{x^2+x+1}{x^2+x-1}$

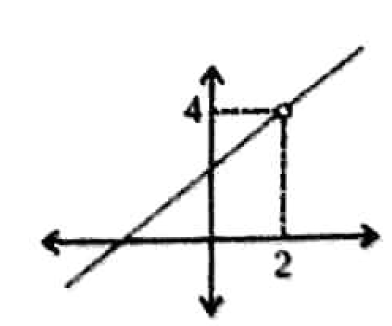
Solution : $f(x) = \frac{x^2+x+1}{x^2+x-1}$ {$x^2 + x + 1$ and $x^2 + x - 1$ have no common factor}

$y = \frac{x^2+x+1}{x^2+x-1}$

$\Rightarrow$ $yx^2 + yx - y = x^2 + x + 1$
$\Rightarrow$ $(y - 1)x^2 + (y - 1)x - y - 1 = 0$
If y = 1, then the above equation reduces to –2 = 0. Which is not true.
Further if y ≠ 1, then $(y - 1)x^2 + (y - 1)x - y - 1 = 0$ is a quadratic and has real roots if
$(y - 1)^2 - 4(y - 1)(-y - 1) \geq 0$
i.e. if y ≤ –3/5 or y ≥ 1 but y ≠ 1
Thus the range is (–∞, –3/5] ∪ (1, ∞)

Example # 5 : Find the range of $f(x) = \frac{x^2-4}{x-2}$

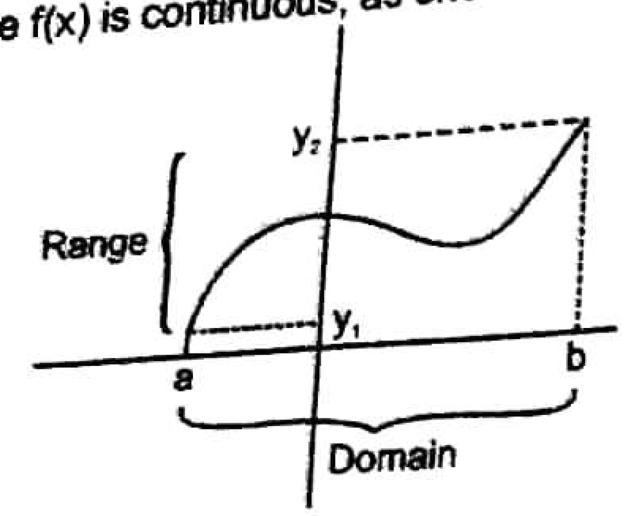
Solution :



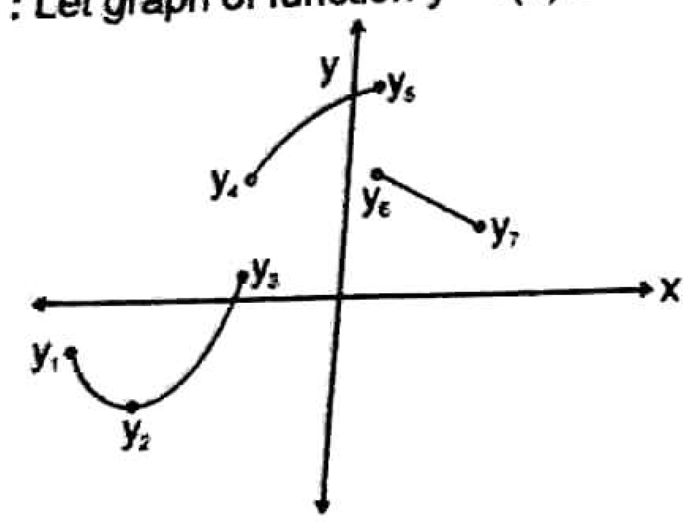
$f(x) = \frac{x^2-4}{x-2} = x + 2;\ x \neq 2$

$\therefore$ graph of f(x) would be
Thus the range of f(x) is R – {4}

Further if f(x) happens to be continuous in its domain then range of f(x) is [min f(x), max. f(x)]. However for sectionally continuous functions, range will be union of [min f(x), max. f(x)] over all those intervals where f(x) is continuous, as shown by following example.



Example # 6 : Let graph of function y = f(x) is



Then range of above sectionally continuous function is $[y_2, y_3] \cup [y_7, y_6) \cup (y_4, y_5]$

(iii) **Using monotonocity :** Many of the functions are monotonic increasing or monotonic decreasing. In case of monotonic continuous functions the minimum and maximum values lie at end points of domain. Some of the common function which are increasing or decreasing in the interval where they are continuous is as under.

Monotonic increasing	Monotonic decreasing
$\log_a x$, a > 1	$\log_a x$, 0 < a < 1
e^x	e^{-x}
$\sin^{-1} x$	$\cos^{-1} x$
$\tan^{-1} x$	$\cot^{-1} x$
$\sec^{-1} x$	$\text{cosec}^{-1} x$

For monotonic increasing functions in [a, b]
(i) $f'(x) \geq 0$ (ii) range is [f(a), f(b)]
for monotonic decreasing functions in [a, b]
(i) $f'(x) \leq 0$ (ii) range is [f(b), f(a)]

Example # 7 : Find the range of function $y = \ell n\,(2x - x^2)$

Solution : (i) **Step – 1**
We have $2x - x^2 \in (-\infty, 1]$
Step – 2 Let $t = 2x - x^2$
For $\ell n t$ to be defined accepted values are (0, 1]
Now, using monotonocity of ℓn t,
$\ell n\,(2x - x^2) \in (-\infty, 0]$
$\therefore$ range is $(-\infty, 0]$ **Ans.**

Self practice problems :

(4) Find domain and range of following functions.

(i) $y = x^3$ (ii) $y = \dfrac{x^2 - 2x + 5}{x^2 + 2x + 5}$ (iii) $y = \dfrac{1}{\sqrt{x^2 - x}}$

Answers : (i) domain R; range R (ii) domain R ; range $\left[\dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}\right]$
(iii) domain R – [0, 1] ; range $(0, \infty)$

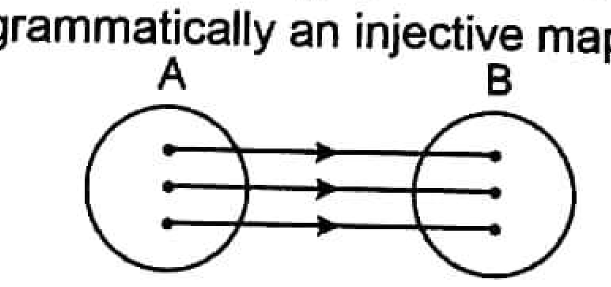
Classification of Functions :

Functions can be classified as "One – One Function (Injective Mapping)" and "Many – One Function" :

One-One Function :

A function f : A → B is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus for $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.
Diagrammatically an injective mapping can be shown as



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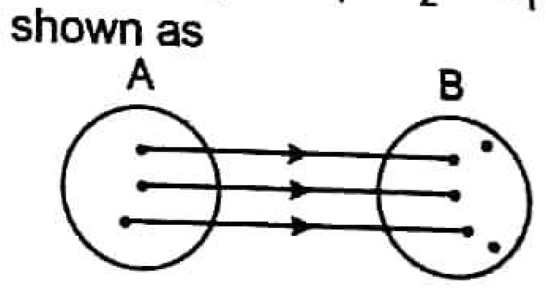
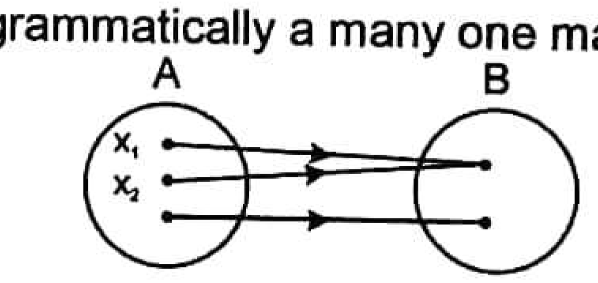


Many-One function :

A function f : A → B is said to be a many one function if there exist at least two or more elements of A having the same f image in B.

Thus f : A → B is many one iff there exist atleast two elements $x_1, x_2 \in A$, such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.
Diagrammatically a many one mapping can be shown as



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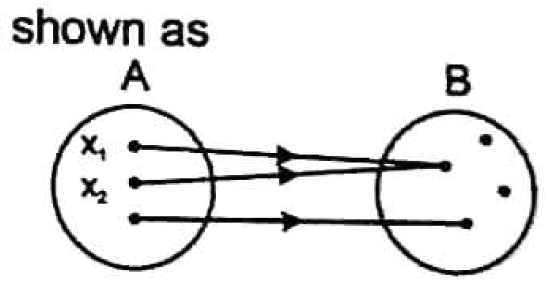


Note : ☞ If a function is one–one, it cannot be many–one and vice versa.

Methods of determining whether a given function is ONE-ONE or MANY-ONE :

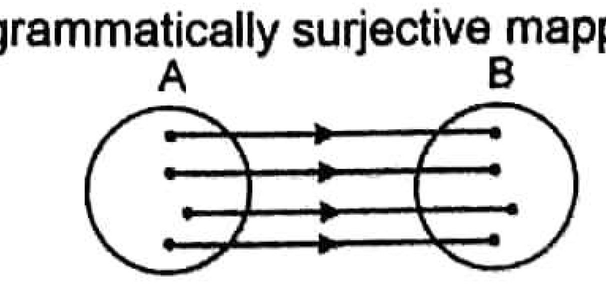
(a) If $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$, equate $f(x_1)$ and $f(x_2)$ and if it implies that $x_1 = x_2$, then and only then function is ONE-ONE otherwise MANY-ONE.
(b) If there exists a straight line parallel to x-axis, which cuts the graph of the function atleast at two points, then the function is MANY-ONE, otherwise ONE- ONE.
(c) If either $f'(x) \geq 0$, $\forall x \in$ domain or $f'(x) \leq 0$ $\forall x \in$ domain, where equality can hold at discrete point(s) only i.e. strictly monotonic, then function is ONE-ONE, otherwise MANY-ONE.

Note : If f and g both are one-one, then gof and fog would also be one-one (if they exist). Functions can also be classified as "Onto function (Surjective mapping)" and "Into function":

Onto function :

If the function f : A → B is such that each element in B (co–domain) must have atleast one pre–image in A, then we say that f is a function of A 'onto' B. Thus f : A → B is surjective iff $\forall$ b $\in$ B, there exists some a $\in$ A such that f (a) = b.

Diagrammatically surjective mapping can be shown as



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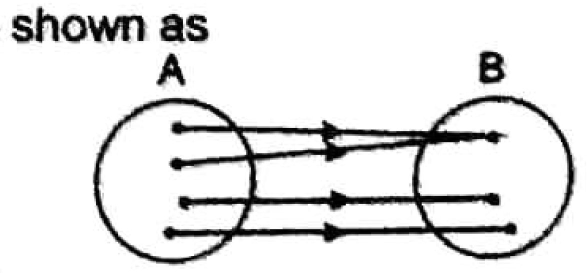
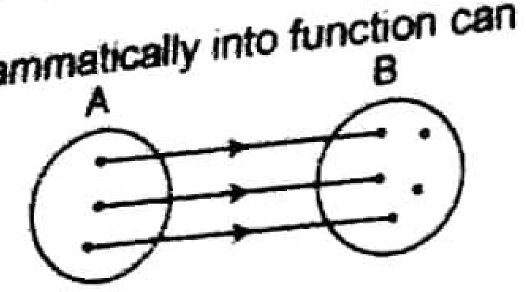


Into function :

If $f : A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then f(x) is into.

Diagrammatically into function can be shown as



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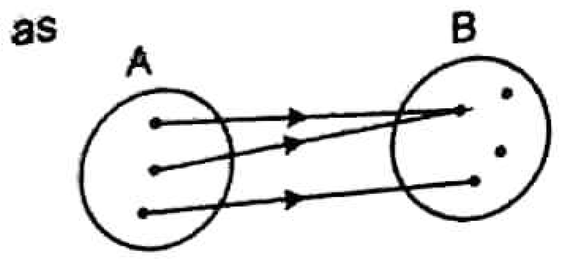
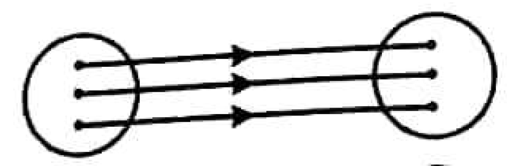


Note : (i) If range ≡ co–domain, then f(x) is onto, otherwise into

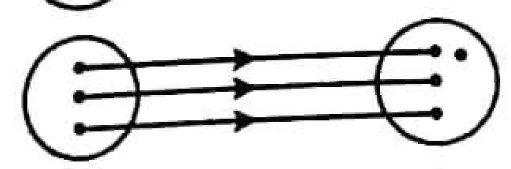
(ii) If a function is onto, it cannot be into and vice versa.

A function can be one of these four types:

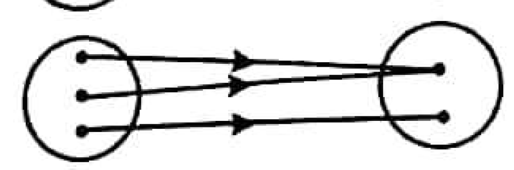
(a) one–one onto (injective and surjective)



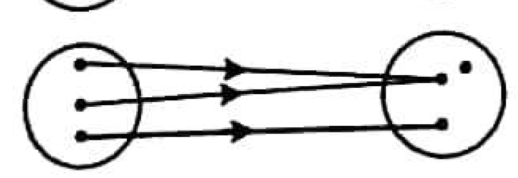
(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Note : (i) If f is both injective and surjective, then it is called a **bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains 'n' distinct elements, then the number of different functions defined from $A \to A$ is n^n and out of which n! are one one.

(iii) If f and g both are onto, then gof or fog may or may not be onto.

(iv) The composite of two bijections is a bijection iff f and g are two bijections such that gof is defined, then gof is also a bijection only **when co-domain of f is equal to the domain of g.**

Example # 8 : (i) Find whether $f(x) = x + \cos x$ is one-one.

(ii) Identify whether the function $f(x) = -x^3 + 3x^2 - 2x + 4$ for $f : R \to R$ is ONTO or INTO

(iii) $f(x) = x^2 - 2x$; $[0, 3] \to A$. Find whether f(x) is injective or not. Also find the set A, if f(x) is surjective.

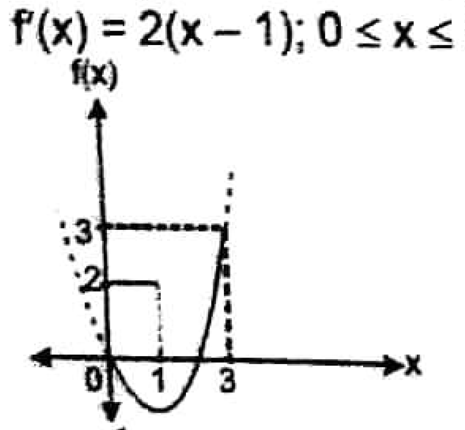
Solution : (i) The domain of f(x) is R. $f'(x) = 1 - \sin x$.

∴ $f'(x) \geq 0 \ \forall \ x \in$ complete domain and equality holds at discrete points only

∴ f(x) is strictly increasing on R. Hence f(x) is one-one.

(ii) As range ≡ codomain, therefore given function is ONTO

(iii) $f'(x) = 2(x - 1)$; $0 \leq x \leq 3$



∴ $f'(x) = \begin{cases} -ve & ; \ 0 \leq x < 1 \\ +ve & ; \ 1 < x < 3 \end{cases}$

∴ f(x) is non monotonic. Hence it is not injective.

For f(x) to be surjective, A should be equal to its range. By graph range is [–1, 3]

∴ A ≡ [–1, 3]

Self practice problems :

(5) For each of the following functions find whether it is one-one or many-one and also into or onto

(i) $f(x) = 2 \tan x$; $(\pi/2, 3\pi/2) \to R$ (ii) $f(x) = \dfrac{1}{1+x^2}$; $(-\infty, 0) \to R$

(iii) $f(x) = x^2 + \ell n\, x$

Answers : (i) one-one onto (ii) one-one into (iii) one-one onto

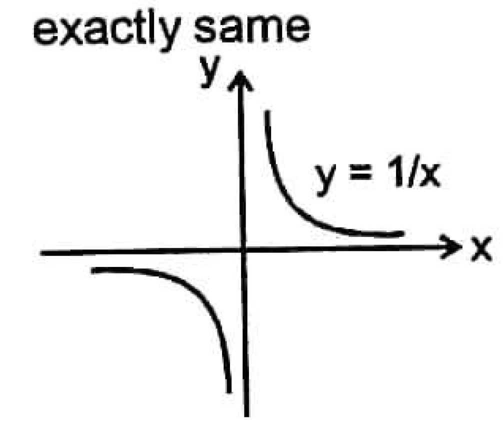
Equal or Identical Functions :

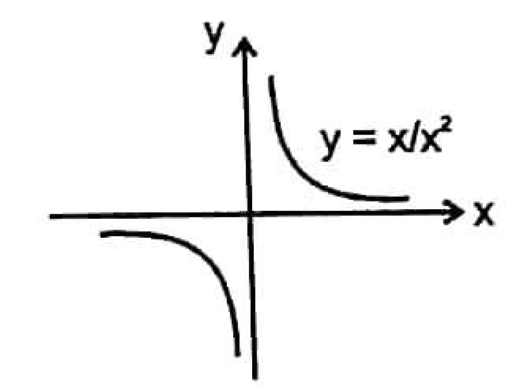
Two functions f and g are said to be identical (or equal) iff :

(i) The domain of f ≡ the domain of g.

(ii) f(x) = g(x), for every x belonging to their common domain.

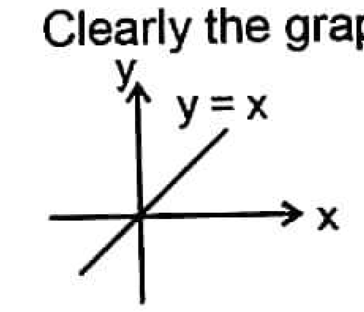
e.g. $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x}{x^2}$ are identical functions. Clearly the graphs of f(x) and g(x) are exactly same

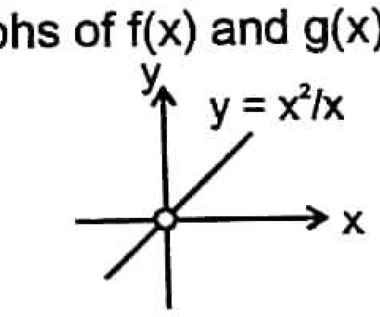




But $f(x) = x$ and $g(x) = \dfrac{x^2}{x}$ are not identical functions.

Clearly the graphs of f(x) and g(x) are different at x = 0.





Example # 9 : Examine whether following pair of functions are identical or not ?

(i) $f(x) = \dfrac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$

(ii) $f(x) = \sin^2 x + \cos^2 x$ and $g(x) = \sec^2 x - \tan^2 x$

Solution : (i) No, as domain of f(x) is R – {1} while domain of g(x) is R

(ii) No, as domain are not same. Domain of f(x) is R while that of g(x) is $R - \left\{(2n+1)\dfrac{\pi}{2};\ n \in I\right\}$

Self practice problems

(6) Examine whether the following pair of functions are identical or not :

(i) $f(x) = \text{sgn}(x)$ and $g(x) = \begin{cases} \dfrac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(ii) $f(x) = \text{cosec}^2 x - \cot^2 x$ and $g(x) = 1$

Answers : (i) Yes (ii) No

Composite Function :

Let $f: X \to Y_1$ and $g: Y_2 \to Z$ be two functions and D is the set of values of x such that if $x \in X$, then $f(x) \in Y_2$. If $D \neq \phi$, then the function h defined on D by $h(x) = g\{f(x)\}$ is called composite function of g and f and is denoted by gof. It is also called function of a function.

Note : Domain of gof is D which is a subset of X (the domain of f). Range of gof is a subset of the range of g. If D = X, then $f(X) \subseteq Y_2$.

Pictorially gof(x) can be viewed as under $\longrightarrow \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$

Note that gof(x) exists only for those x when range of f(x) is a subset of domain of g(x).

Properties of Composite Functions :

(a) In general gof $\neq$ fog (i.e. not commutative)

(b) The composition of functions are associative i.e. if three functions f, g, h are such that fo (goh) and (fog) oh are defined, then fo (goh) = (fog) oh.

Example # 10 : Describe fog and gof wherever is possible for the following functions

(i) $f(x) = \sqrt{x+3}$, $g(x) = 1 + x^2$ (ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Solution : (i) Domain of f is $[-3, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[1, \infty)$.

<u>**For gof(x)**</u>

Since range of f is a subset of domain of g,

$\therefore$ domain of gof is $[-3, \infty)$ {equal to the domain of f }

gof (x) = g{f(x)} = g ($\sqrt{x+3}$) = 1 + (x+3) = x + 4. Range of gof is $[1, \infty)$.

<u>**For fog(x)**</u>

since range of g is a subset of domain of f,

$\therefore$ domain of fog is R {equal to the domain of g}

fog (x) = f{g(x)}= f(1+ x²) = $\sqrt{x^2+4}$ Range of fog is $[2, \infty)$.

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Domain of f is $[0, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[-1, \infty)$.

For gof(x)

Since range of f is a subset of the domain of g,

$\therefore$ domain of gof is $[0, \infty)$ and $g\{f(x)\} = g(\sqrt{x}) = x - 1$. Range of gof is $[-1, \infty)$

For fog(x)

Since range of g is not a subset of the domain of f

i.e. $[-1, \infty) \not\subset [0, \infty)$

$\therefore$ fog is not defined on whole of the domain of g.

Domain of fog is $\{x \in R$, the domain of g : $g(x) \in [0, \infty)$, the domain of f$\}$.

Thus the domain of fog is $D = \{x \in R: 0 \le g(x) < \infty\}$

i.e. $D = \{ x \in R: 0 \le x^2 - 1\} = \{ x \in R: x \le -1 \text{ or } x \ge 1 \} = (-\infty, -1] \cup [1, \infty)$

$fog (x) = f\{g(x)\} = f(x^2-1) = \sqrt{x^2-1}$ Its range is $[0, \infty)$.

Example #11 : Let $f(x) = e^x$; $R^+ \to R$ and $g(x) = \sin x$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$. Find domain and range of fog(x)

Solution : Domain of f(x) : $(0, \infty)$ Range of g(x) : $[-1, 1]$

values in range of g(x) which are accepted by f(x) are $\left(0, \frac{\pi}{2}\right]$

$\Rightarrow$ $0 < g(x) \le 1$ $\Rightarrow$ $0 < \sin x \le 1$ $\Rightarrow$ $0 < x \le \frac{\pi}{2}$

Hence domain of fog(x) is $x \in (0, \frac{\pi}{2}]$

Therefore Domain : $(0, \frac{\pi}{2}]$

Range : $(1, e]$

Example #12 : If $f(x) = -1 + |x-2|, 0 \le x \le 4$

$g(x) = 2 - |x|, -1 \le x \le 3$

Then find fog (x) and gof (x). Also draw their rough sketch.

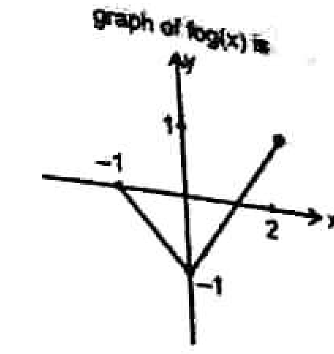
Solution : fog (x) = $\{-1 + |g(x) - 2|, 0 \le g(x) \le 4, -1 \le x \le 3$

$= \{-1 + |2 - |x| - 2|, 0 \le 2 - |x| \le 4, -1 \le x \le 3$

$= \{-1 + |x|, -2 \le x \le 2, -1 \le x \le 3$

$$= \begin{cases} -(1+x) , & -1 \le x \le 0 \\ x - 1 , & 0 < x \le 2 \end{cases} \quad ;$$

graph of fog(x) is



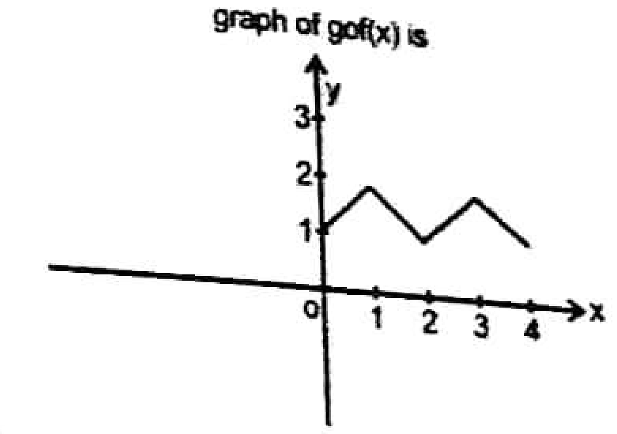
gof(x) $= \{2 - |f(x)|, -1 \le f(x) \le 3, 0 \le x \le 4$

$= \{2 - |-1 + |x-2||, -1 \le -1 + |x-2| \le 3, 0 \le x \le 4$

$= \{2 - |-1 + |x-2||, -2 \le x \le 6, 0 \le x \le 4$

$$= \begin{cases} x+1 , & 0 \le x < 1 \\ 3 - x , & 1 \le x \le 2 \\ x - 1 , & 2 < x \le 3 \\ 5 - x , & 3 < x \le 4 \end{cases} ;$$

graph of gof(x) is



Self practice problems

(7) Define fog(x) and gof(x). Also find their domain and range.

(i) f(x) = [x], g(x) = sin x

(ii) $f(x) = \tan x$, $x \in (-\pi/2, \pi/2)$; $g(x) = \sqrt{1-x^2}$

(8) Let $f(x) = e^x : R^+ \to R$ and $g(x) = x^2 - x : R \to R$. Find domain and range of fog (x) and gof (x)

Answers :

(7) (i) gof = sin [x] domain : R range { sin a : a ∈ I}

fog = [sin x] domain : R range : {–1, 0, 1}

(ii) gof $\equiv \sqrt{1-\tan^2 x}$, domain : $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ range : [0, 1]

fog $\equiv \tan \sqrt{1-x^2}$ domain : [–1, 1] range [0, tan 1]

(8) fog (x)

Domain : $(-\infty, 0) \cup (1, \infty)$

Range : $(1, \infty)$

gof (x)

Domain : $(0, \infty)$

Range : $(0, \infty)$

Odd and Even Functions :

(i) If f (–x) = f (x) for all x in the domain of 'f', then f is said to be an even function. e.g. f (x) = cos x; g (x) = x² + 3.

(ii) If f (–x) = –f (x) for all x in the domain of 'f', then f is said to be an odd function. e.g. f (x) = sin x; g (x) = x³ + x.

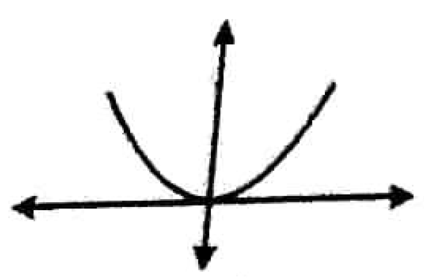
Note : (i) A function may neither be odd nor even. (e.g. $f(x) = e^x$, $\cos^{-1}x$)

(ii) If an odd function is defined at x = 0, then f(0) = 0

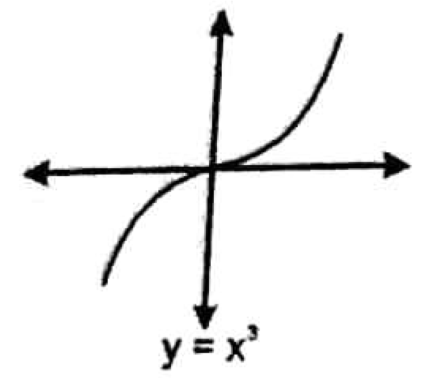
...rties of Even/Odd Function

(a) The graph of every even function is symmetric about the y–axis and that of every odd function is symmetric about the origin.

For example graph of $y = x^2$ is symmetric about y-axis, while graph of $y = x^3$ is symmetric about origin



$y = x^2$



$y = x^3$

(b) All functions (whose domain is symmetrical about origin) can be expressed as the sum of an even and an odd function, as follows

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$$

(c) The only function which is defined on the entire number line and is even and odd at the same time is $f(x) = 0$.

(d) If f and g both are even or both are odd, then the function f.g will be even but if any one of them is odd and the other even then f.g will be odd.

(e) If f(x) is even then $f'(x)$ is odd while derivative of odd function is even. Note that same cannot be said for integral of functions.

Example#13 : Show that $a^x + a^{-x}$ is an even function.

Solution : Let $f(x) = a^x + a^{-x}$

Then $f(-x) = a^{-x} + a^{-(-x)} = a^{-x} + a^x = f(x)$. Hence f(x) is an even function

Example#14 : Prove that $f(x) = x\left(\dfrac{x}{e^x-1}+\dfrac{x}{2}\right)$ is odd function

Solution : Let $g(x) = \left(\dfrac{x}{e^x-1}+\dfrac{x}{2}\right)$ then $g(-x) = \left(\dfrac{-x}{e^{-x}-1}+\dfrac{-x}{2}\right) = \left(\dfrac{x}{e^x-1}+\dfrac{x}{2}\right)$

$\Rightarrow$ g(x) is even

hence $f(x) = x.g(x) = x\left(\dfrac{x}{e^x-1}+\dfrac{x}{2}\right)$ is odd function.

Self practice problems

(9) Determine whether the following functions are even / odd / neither even nor odd?

(i) $f(x) = \dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

(ii) $f : [-2, 3] \to [0, 9]$, $f(x) = x^2$

(iii) $f(x) = x \log\left(x+\sqrt{x^2+1}\right)$

Answers (i) Odd (ii) neither even nor odd (iii) Even

...riodic Functions :

A function f(x) is called periodic with a period T if there exists a real number T > 0 such that for each x in the domain of f the numbers x – T and x + T are also in the domain of f and f(x) = f(x + T) for all x in the domain of f(x). Graph of a periodic function with period T is repeated after every interval of 'T'.

e.g. The function sin x and cos x both are periodic over 2π and tan x is periodic over π.

The least positive period is called the principal or fundamental period of f(x) or simply the period of the function.

...e of a periodic function does not exist.

Properties of Periodic Functions :

(a) If f(x) has a period T, then $\dfrac{1}{f(x)}$ and $\sqrt{f(x)}$ also have a period T.

(b) If f(x) has a period T, then f (ax + b) has a period $\dfrac{T}{|a|}$.

(c) Every constant function defined for all real x, is always periodic, with no fundamental period.

(d) If f (x) has a period T_1 and g (x) also has a period T_2 then period of f(x) ± g(x) or f(x) . g(x) or $\dfrac{f(x)}{g(x)}$ is L.C.M. of T_1 and T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists then f(x) ± g(x) or f(x) . g(x) or $\dfrac{f(x)}{g(x)}$ is nonperiodic.

L.C.M. of $\left(\dfrac{a}{b}, \dfrac{p}{q}, \dfrac{\ell}{m}\right) = \dfrac{\text{L.C.M.}(a, p, \ell)}{\text{H.C.F.}(b, q, m)}$

e.g. |sinx| has the period π, | cosx | also has the period π

$\therefore$ |sinx| + |cosx| also has a period π. But the fundamental period of |sinx| + |cosx| is $\dfrac{\pi}{2}$.

(e) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods.

Example #15 : Find period of the following functions

(i) $f(x) = \sin\dfrac{x}{2} + \cos\dfrac{x}{3}$

(ii) f(x) = {x} + sin x, where {.}denotes fractional part function

(iii) f(x) = 4 cos x . cos 3x + 2 (iv) $f(x) = \sin\dfrac{3x}{2} - \cos\dfrac{x}{3} - \tan\dfrac{2x}{3}$

Solution : (i) Period of $\sin\dfrac{x}{2}$ is 4π while period of $\cos\dfrac{x}{3}$ is 6π. Hence period of $\sin\dfrac{x}{2} + \cos\dfrac{x}{3}$ is 12π {L.C.M. of 4 and 6 is 12}

(ii) Period of sin x = 2π

Period of {x} = 1

but L.C.M. of 2π and 1 is not possible as their ratio is irrational number

$\therefore$ it is aperiodic

(iii) f(x) = 4 cos x . cos 3x + 2

period of f(x) is L.C.M. of $\left(2\pi, \dfrac{2\pi}{3}\right) = 2\pi$

but 2π may or may not be fundamental periodic, but fundamental period $= \dfrac{2\pi}{n}$, where $n \in N$. Hence cross-checking for n = 1, 2, 3, ...we find π to be fundamental period $f(\pi + x) = 4(-\cos x)(-\cos 3x) + 2 = f(x)$

(iv) Period of f(x) is L.C.M. of $\dfrac{2\pi}{3/2}, \dfrac{2\pi}{1/3}, \dfrac{\pi}{2/3}$ = L.C.M. of $\dfrac{4\pi}{3}, 6\pi, \dfrac{3\pi}{2} = 12\pi$

Inverse of a Function :

Let y = f(x) : A → B be a one-one and onto function. i.e. bijection, then there will always exist bijective function x = g(y) : B → A such that if (p, q) is an element of f, (q, p) will be an element of g and the functions f(x) and g(x) are said to be inverse of each other. g(x) is also denoted by $f^{-1}(x)$ and f(x) is denoted by $g^{-1}(x)$

Note : (i) The inverse of a bijection is unique.

(ii) Inverse of an even function is not defined.

Self practice problems :

(10) Determine $f^{-1}(x)$, if given function is invertible
$f : (-\infty, -1) \to (-\infty, -2)$ defined by $f(x) = -(x + 1)^2 - 2$

Answers : $-1 - \sqrt{-x-2}$

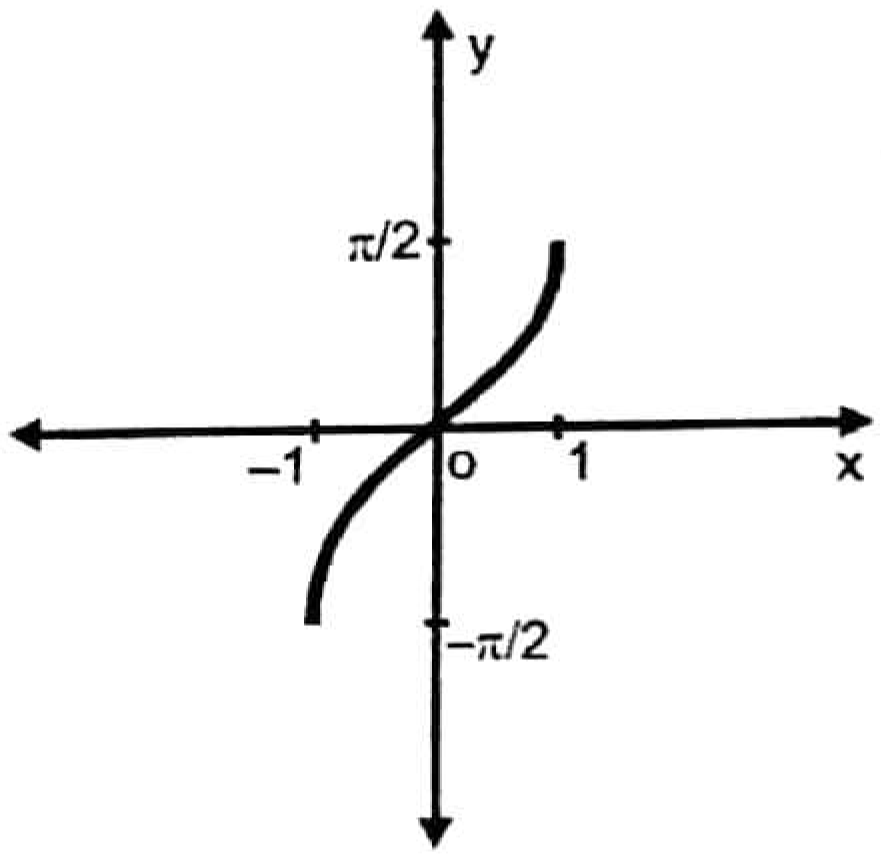
Inverse Trigonometry Functions

Introduction : The student may be familiar about trigonometric functions viz sin x, cos x, tan x, cosec x, sec x, cot x with respective domains R, R, R – {(2n + 1) π/2}, R – {nπ}, R – {(2n + 1) π/2}, R – {nπ} and respective ranges [–1, 1], [–1, 1], R, R – (–1, 1), R – (–1, 1), R.

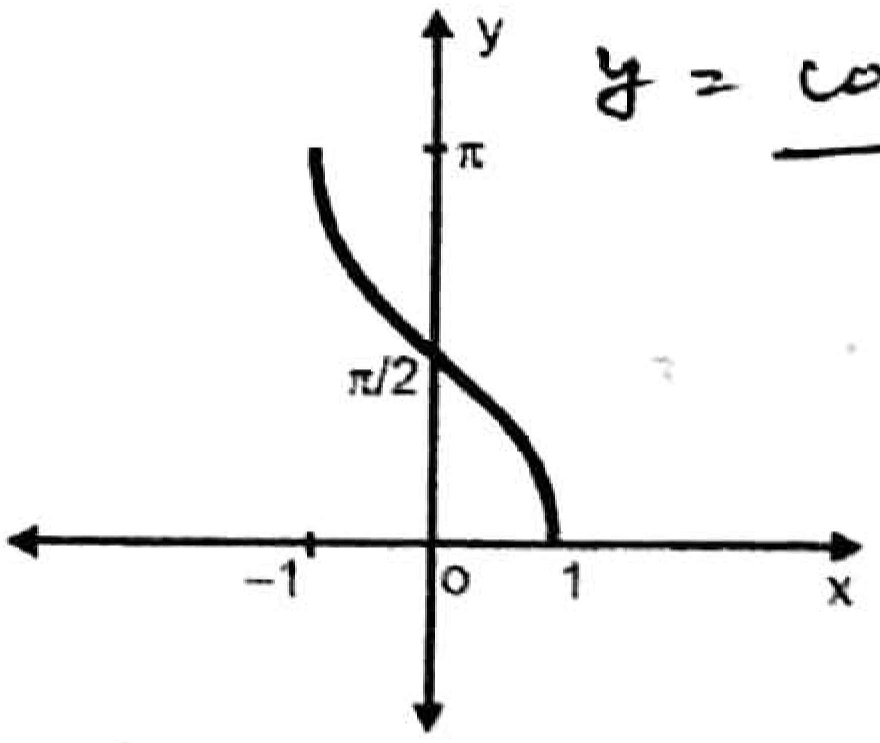
Correspondingly, six inverse trigonometric functions (also called inverse circular functions) are defined.

$\sin^{-1}x$: The symbol $\sin^{-1}x$ or arcsinx denotes the angle θ so that sin θ = x. As a direct meaning, $\sin^{-1}x$ is not a function, as it does not satisfy the requirements for a rule to become a function. But by a suitable choice [–1, 1] as its domain and standardized set [–π/2, π/2] as its range, then rule $\sin^{-1}x$ is a single valued function.

Thus $\sin^{-1}x$ is considered as a function with domain [–1, 1] and range [–π/2, π/2].
The graph of $y = \sin^{-1}x$ is as shown below, which is obtained by taking the mirror image, of the portion of the graph of y = sin x, from x = –π/2 to x = π/2, on the line y = x.

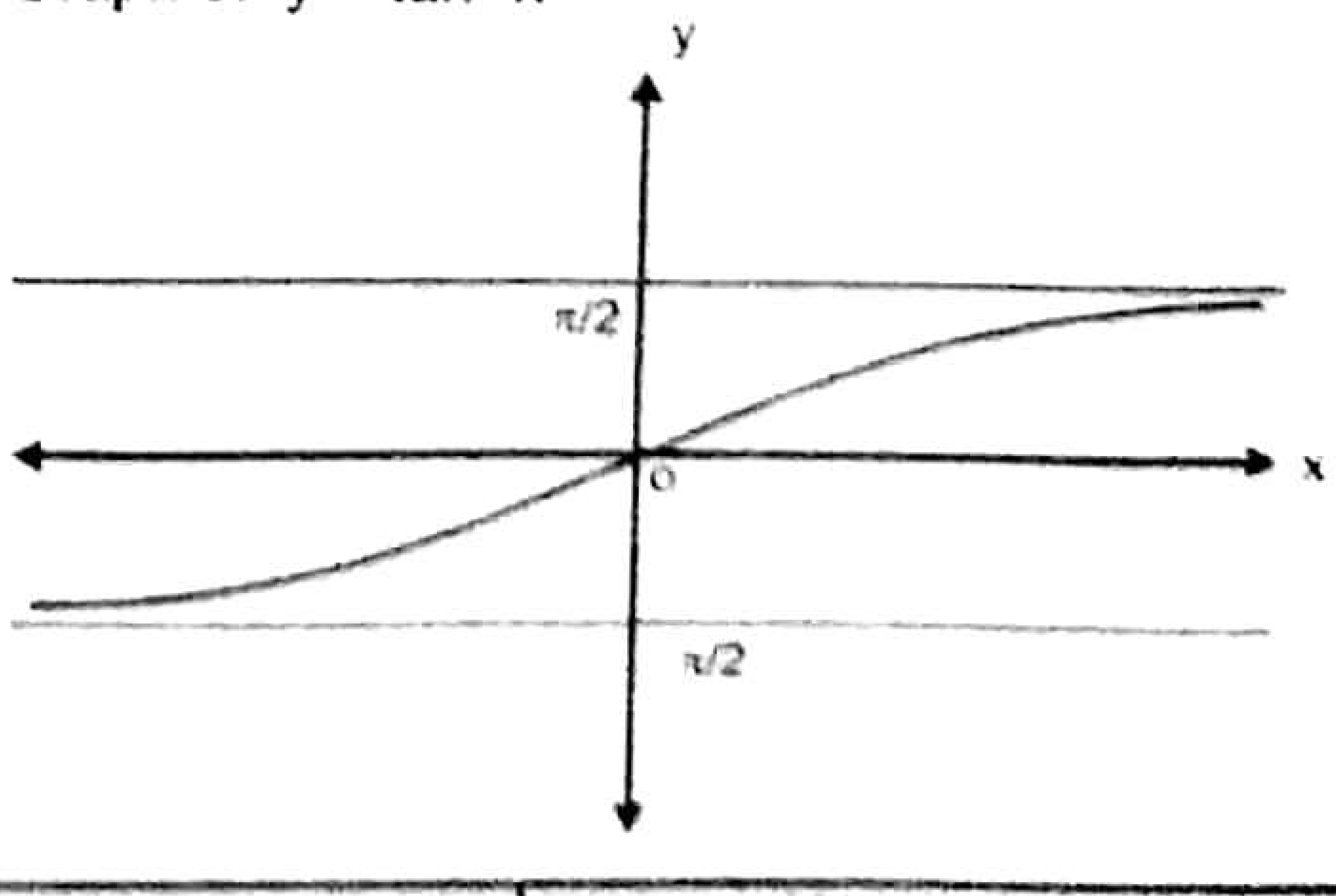


$\cos^{-1}x$: By following the discussions, similar to above, we have $\cos^{-1}x$ or arccos x as a function with domain [–1, 1] and range [0, π].
The graph of $y = \cos^{-1}x$ is similarly obtained as the mirror image of the portion of the graph y = cos x from x = 0 to x = π.



$y = \cos^{-1}x$

$\tan^{-1}x$: We get $\tan^{-1}x$ or arctanx as a function with domain R and range (–π/2, π/2).
Graph of $y = \tan^{-1}x$



Self practice problems :

(11) Find the value of $\cos\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

(12) Find the value of cosec $[\sec^{-1}(\sqrt{2}) + \cot^{-1}(1)]$

(13) Find the domain of $y = \sec^{-1}(x^2 + 3x + 1)$

(14) Find the domain of $y = \sin^{-1}\left(\frac{x^2}{1+x^2}\right)$

(15) Find the domain of $y = \cot^{-1}(\sqrt{x^2 - 1})$

Answers : (11) 0 (12) 1 (13) $(-\infty, -3] \cup [-2, -1] \cup [0, \infty)$
(14) R (15) $(-\infty, -1] \cup [1, \infty)$

Property 1 : "–x"

The graphs of $\sin^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$ are symmetric about origin.

Hence we get $\sin^{-1}(-x) = -\sin^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}x$

$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$.

Also the graphs of $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ are symmetric about the point $(0, \pi/2)$. From this, we get

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$.

Property 2 : $T(T^{-1})$

(i) $\sin(\sin^{-1}x) = x$, $\quad -1 \le x \le 1$

Proof : Let $\theta = \sin^{-1}x$. Then $x \in [-1, 1]$ & $\theta \in [-\pi/2, \pi/2]$.

$\Rightarrow$ $\sin\theta = x$, by meaning of the symbol

$\Rightarrow$ $\sin(\sin^{-1}x) = x$

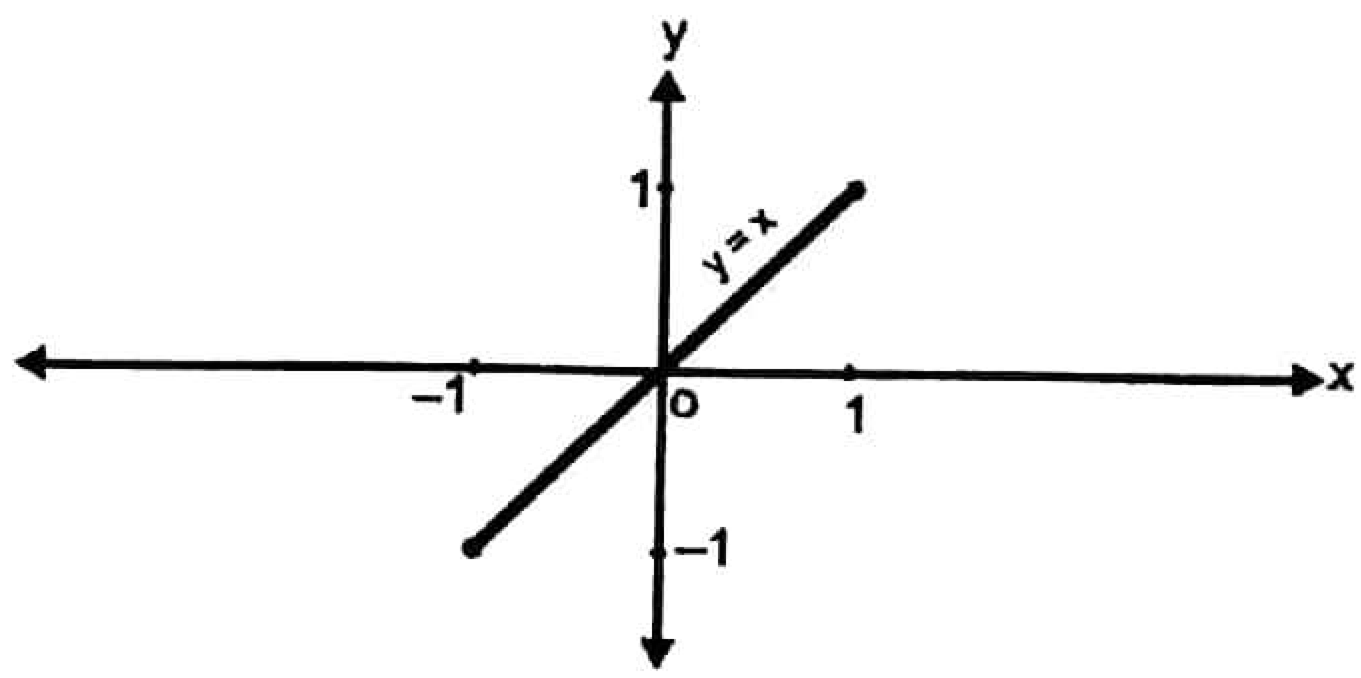
Similar proofs can be carried out to obtain

(ii) $\cos(\cos^{-1}x) = x$, $\quad -1 \le x \le 1$ (iii) $\tan(\tan^{-1}x) = x$, $x \in R$

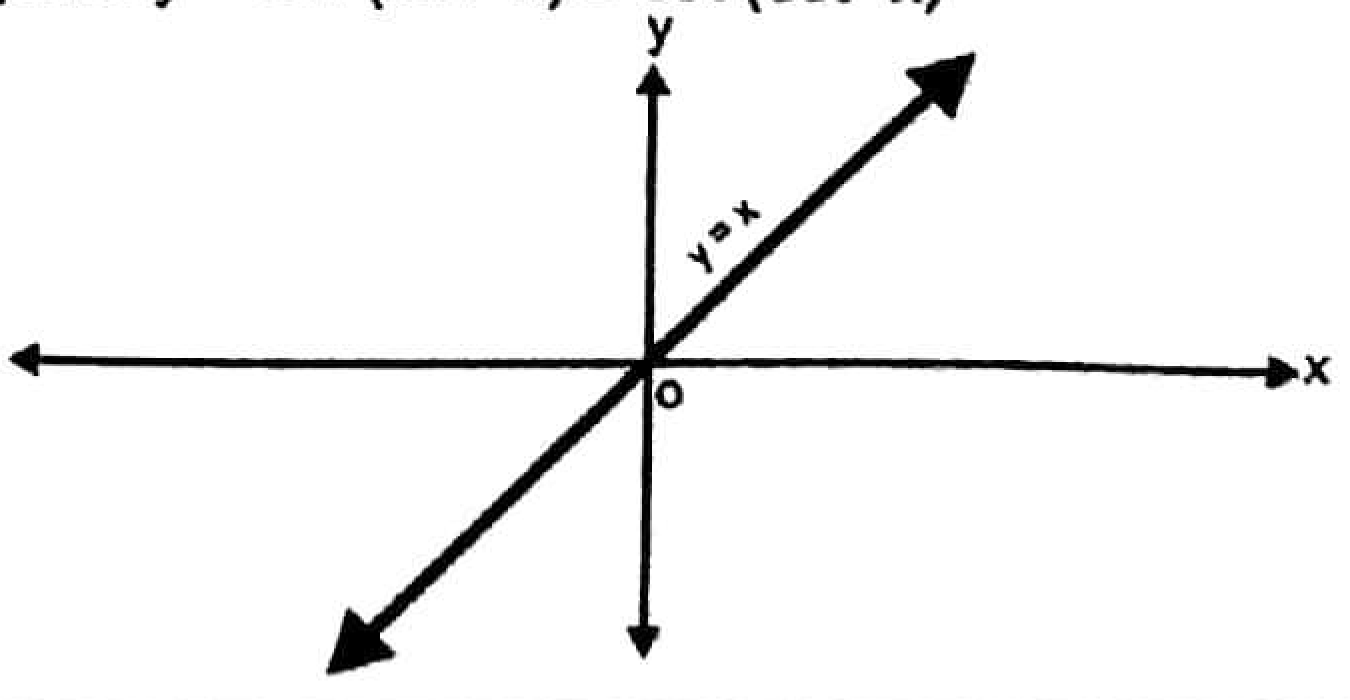
(iv) $\cot(\cot^{-1}x) = x$, $x \in R$ (v) $\sec(\sec^{-1}x) = x$, $\quad x \le -1, x \ge 1$

(vi) $\text{cosec}(\text{cosec}^{-1}x) = x$, $\quad |x| \ge 1$

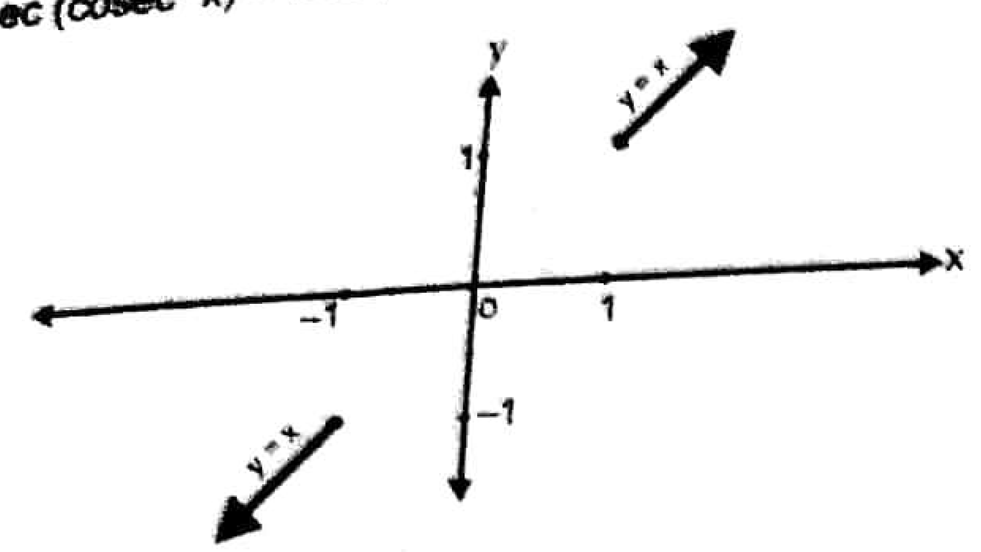
The graph of $y = \sin(\sin^{-1}x) \equiv \cos(\cos^{-1}x)$



The graph of $y = \tan(\tan^{-1}x) \equiv \cot(\cot^{-1}x)$



... graph of $y = \text{cosec}(\text{cosec}^{-1}x) = \sec(\sec^{-1}x)$



Property 3 : $T^{-1}(T)$

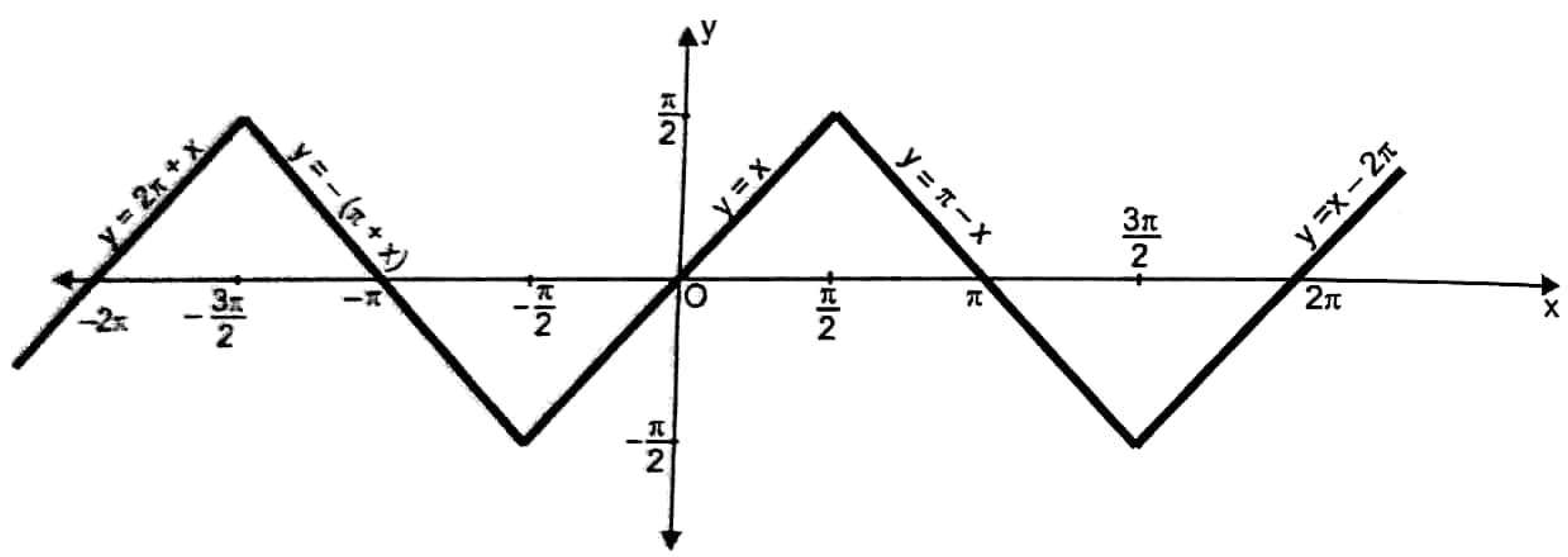
(i) $\sin^{-1}(\sin x) = \begin{cases} -2n\pi + x, & x \in [2n\pi - \pi/2,\ 2n\pi + \pi/2] \\ (2n+1)\pi - x, & x \in [(2n+1)\pi - \pi/2,\ (2n+1)\pi + \pi/2],\ n \in Z \end{cases}$

Proof : If $x \in [2n\pi - \pi/2,\ 2n\pi + \pi/2]$, then $-2n\pi + x \in [-\pi/2, \pi/2]$ and $\sin(-2n\pi + x) = \sin x$.

Hence $\sin^{-1}(\sin x) = -2n\pi + x$ for $x \in [2n\pi - \pi/2,\ 2n\pi + \pi/2]$.

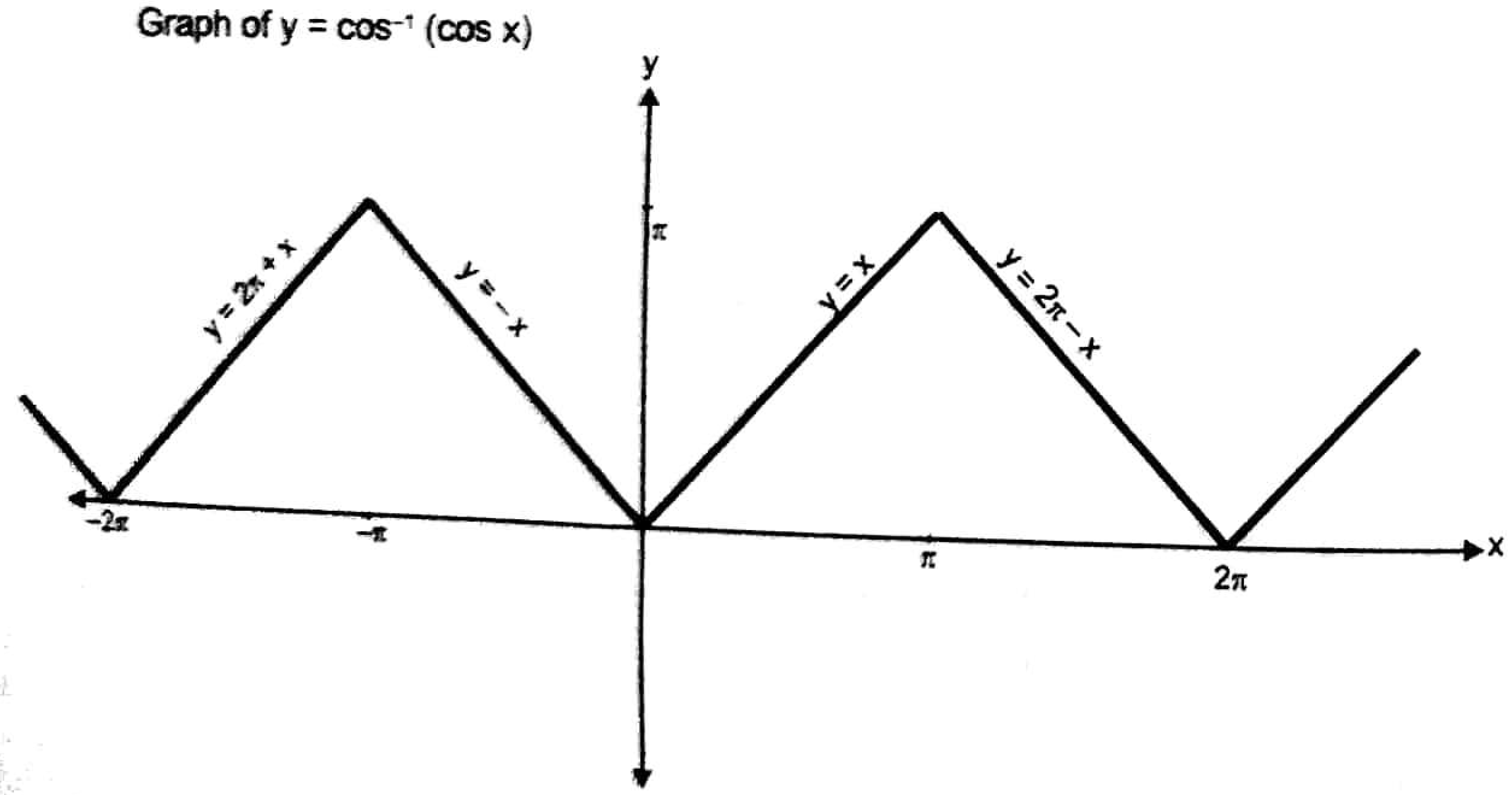
Proof of 2nd part is left for the students.

Graph of $y = \sin^{-1}(\sin x)$



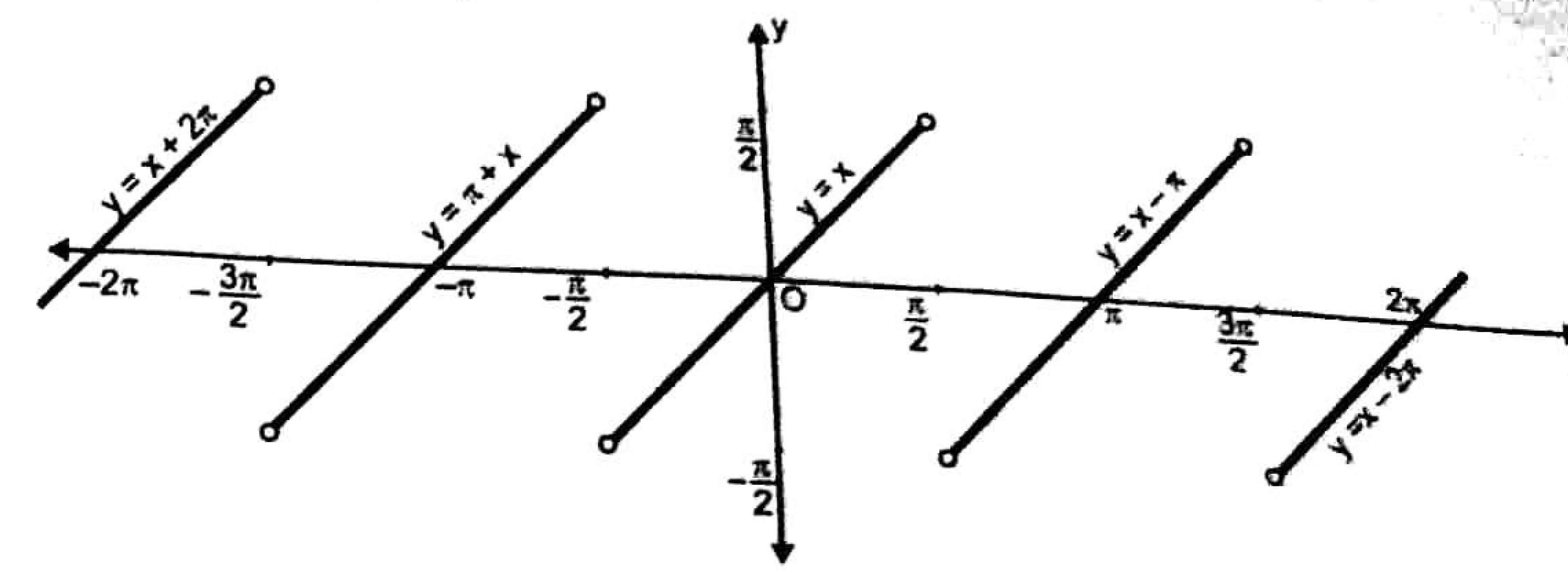
(ii) $\cos^{-1}(\cos x) = \begin{cases} -2n\pi + x, & x \in [2n\pi,\ (2n+1)\pi] \\ 2n\pi - x, & x \in [(2n-1)\pi,\ 2n\pi],\ n \in I \end{cases}$

Graph of $y = \cos^{-1}(\cos x)$



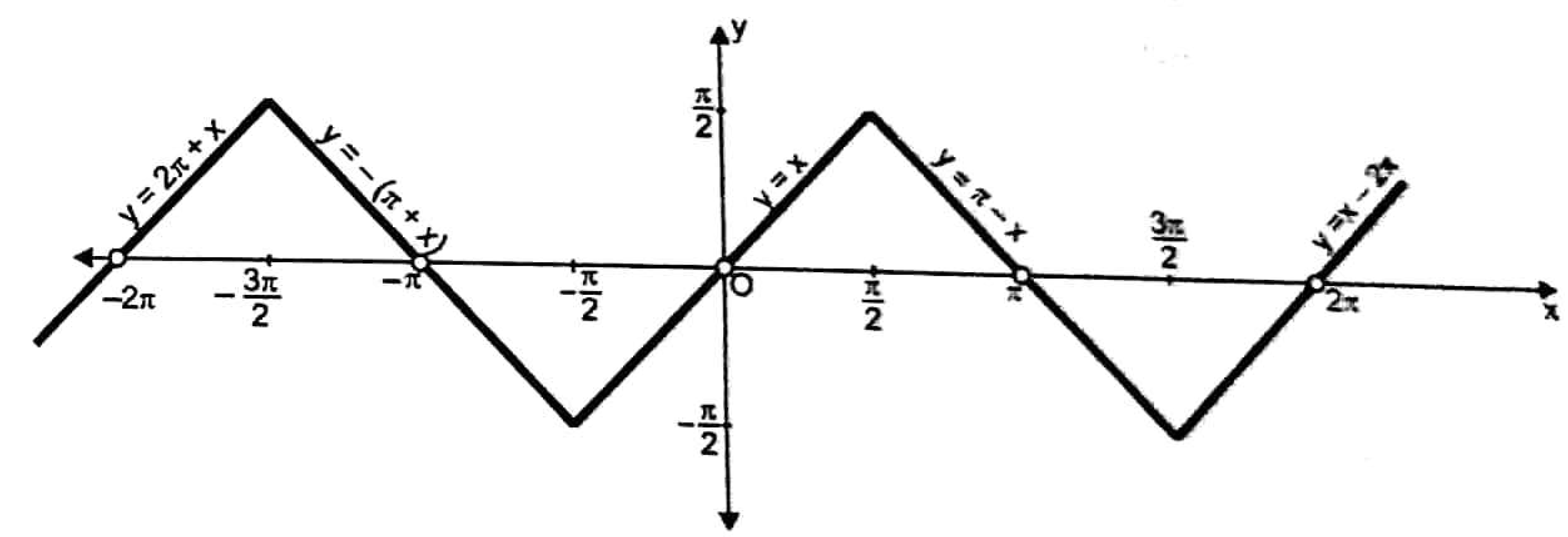
(iii) $\tan^{-1}(\tan x) = -n\pi + x,\ n\pi - \pi/2 < x < n\pi + \pi/2,\ n \in z$

Graph of $y = \tan^{-1}(\tan x)$



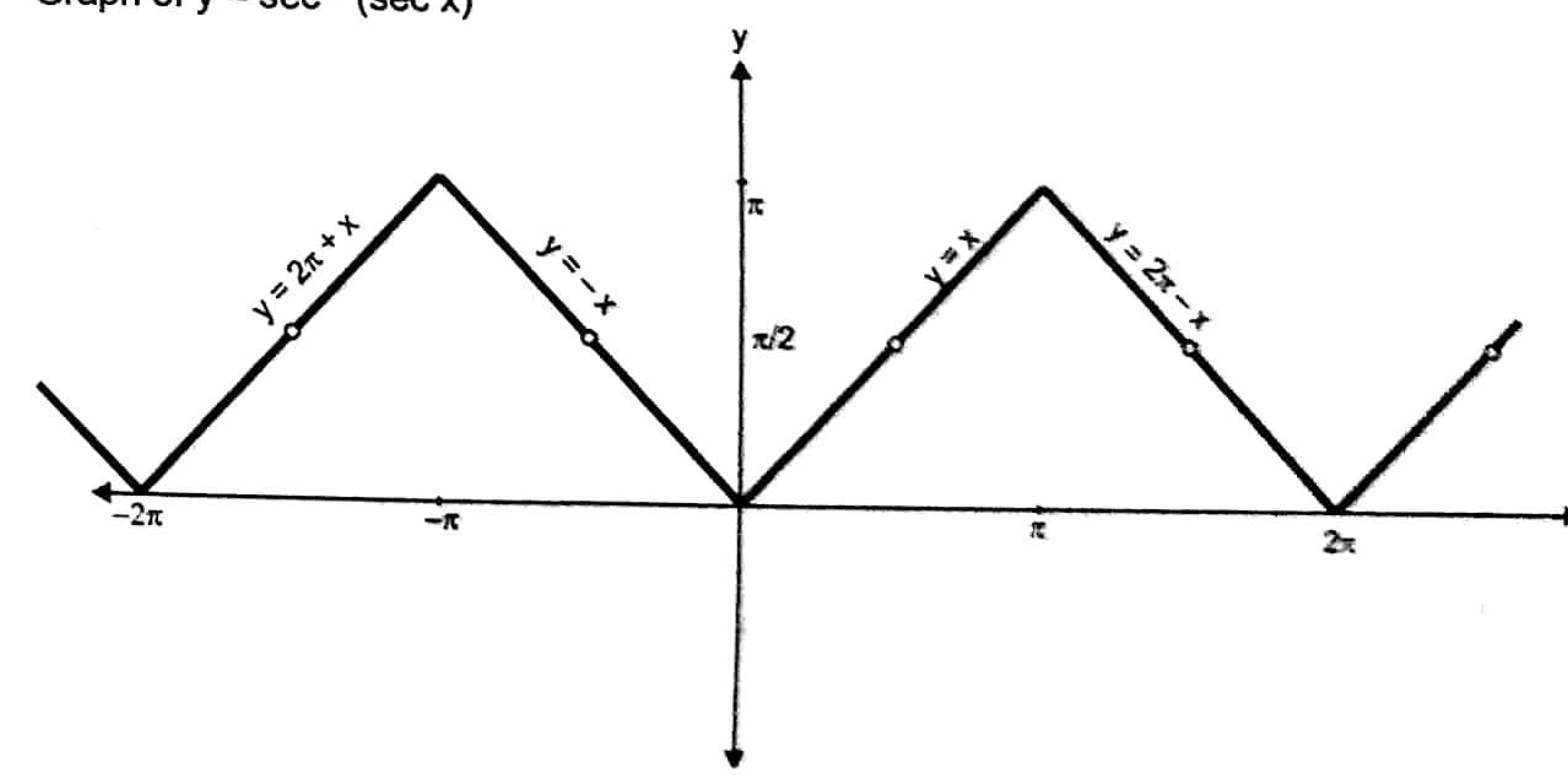
(iv) $\text{cosec}^{-1}(\text{cosec}\,x)$ is similar to $\sin^{-1}(\sin x)$

Graph of $y = \text{cosec}^{-1}(\text{cosec}\,x)$



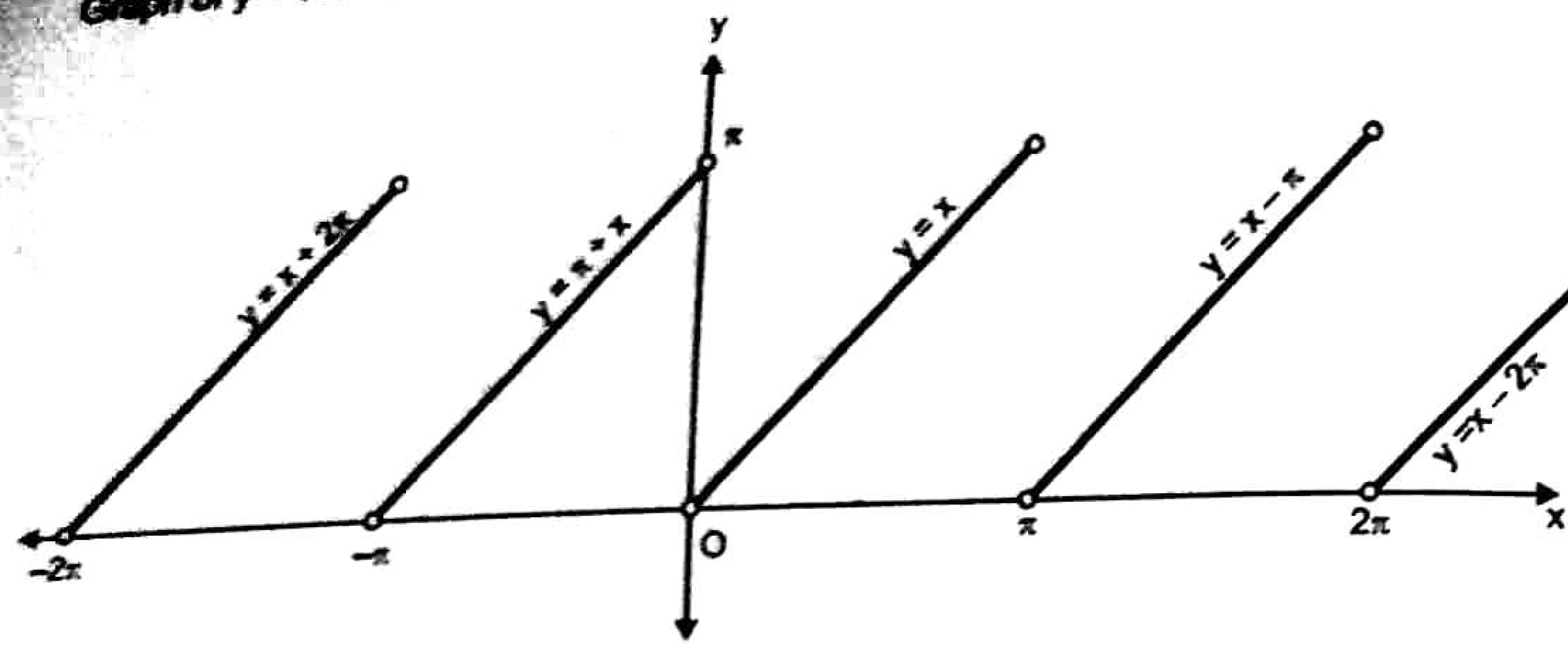
(v) $\sec^{-1}(\sec x)$ is similar to $\cos^{-1}(\cos x)$

Graph of $y = \sec^{-1}(\sec x)$



(vi) $\cot^{-1}(\cot x) = -n\pi + x,\ x \in (n\pi, (n+1)\pi),\ n \in \mathbb{Z}$

Graph of $y = \cot^{-1}(\cot x)$



Remark : $\sin(\sin^{-1}x), \cos(\cos^{-1}x), \ldots \cot(\cot^{-1}x)$ are aperiodic (non periodic) functions where as $\sin^{-1}(\sin x), \ldots, \cot^{-1}(\cot x)$ are periodic functions.

Property 4 : "1/x"

(i) $\text{cosec}^{-1}(x) = \sin^{-1}(1/x),\quad |x| \ge 1$

Proof : Let $\text{cosec}^{-1} x = \theta$

$\Rightarrow$ $1/x = \sin\theta$

$\Rightarrow$ $\sin^{-1}(1/x) = \sin^{-1}(\sin\theta)$

$= \theta$ (as $\theta \in [-\pi/2, \pi/2] - \{0\}$)

$= \text{cosec}^{-1}x$

(ii) $\sec^{-1} x = \cos^{-1}(1/x),\quad |x| \ge 1$

(iii) $\cot^{-1}x = \begin{cases} \tan^{-1}(1/x), & x > 0 \\ \pi + \tan^{-1}(1/x), & x < 0 \end{cases}$

Property 5 : "π/2"

(i) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2},\ -1 \le x \le 1$

Proof : Let $A = \sin^{-1}x$ and $B = \cos^{-1}x$ $\Rightarrow$ $\sin A = x$ and $\cos B = x$

$\Rightarrow$ $\sin A = \cos B$ $\Rightarrow$ $\sin A = \sin(\pi/2 - B)$

$\Rightarrow$ $A = \pi/2 - B$, because A and $\pi/2 - B \in [-\pi/2, \pi/2]$

$\Rightarrow$ $A + B = \pi/2$.

Similarly, we can prove

(ii) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2},\ x \in R$ (iii) $\text{cosec}^{-1}x + \sec^{-1}x = \dfrac{\pi}{2},\ |x| \ge 1$

Example # 19 : Find the value of $\text{cosec}\left\{\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right)\right\}$.

Solution :

$\because$ $\cot(\cot^{-1} x) = x,\ \forall x \in R$

$\therefore$ $\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right) = \dfrac{3\pi}{4}$

$\text{cosec}\left\{\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right)\right\} = \text{cosec}\left(\dfrac{3\pi}{4}\right) = \sqrt{2}$

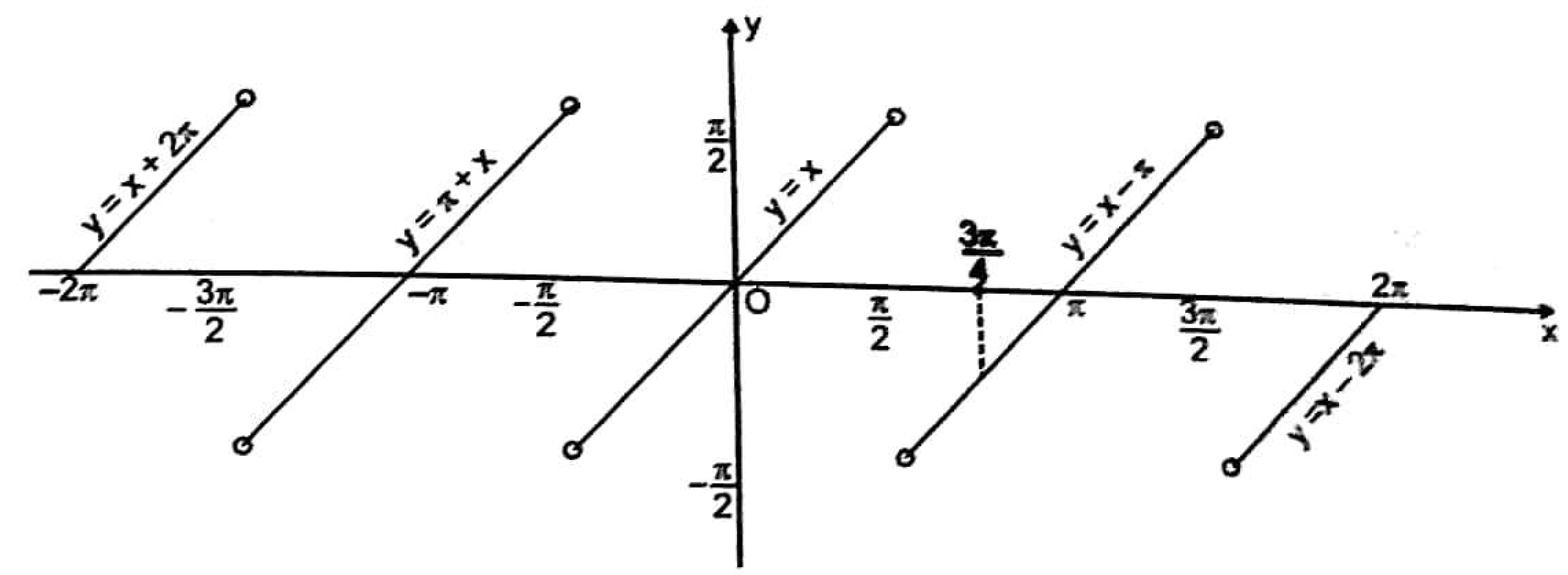
Example # 20 Find the value of $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$.

Solution : $\because$ $\tan^{-1}(\tan x) = x$ if $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

As $\dfrac{3\pi}{4} \notin \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ $\therefore$ $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right) \ne \dfrac{3\pi}{4}$

$\therefore$ $\dfrac{3\pi}{4} \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

graph of $y = \tan^{-1}(\tan x)$ is as :



$\because$ from the graph we can see that if $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$,

then $\tan^{-1}(\tan x) = x - \pi$

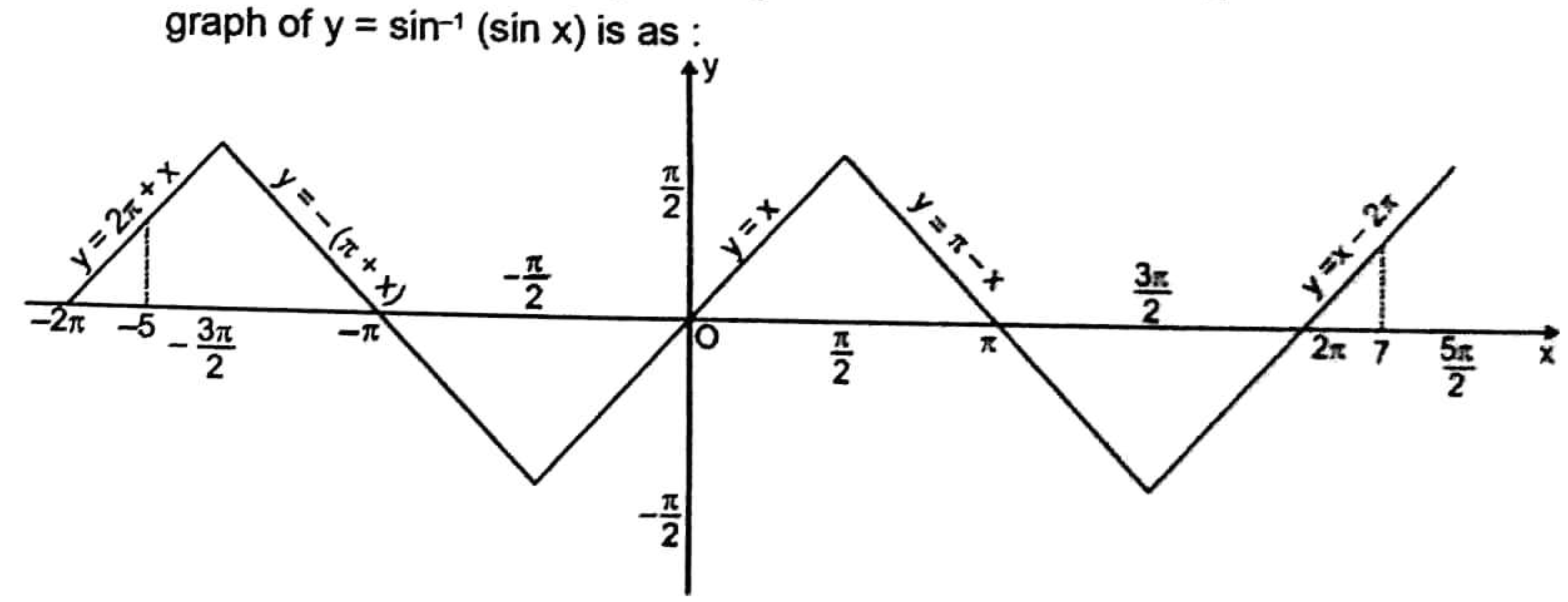
$\therefore$ $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right) = \dfrac{3\pi}{4} - \pi = -\dfrac{\pi}{4}$

Example # 21 : Find the value of $\sin^{-1}(\sin 7)$ and $\sin^{-1}(\sin(-5))$.

Solution. Let $y = \sin^{-1}(\sin 7)$

$\sin^{-1}(\sin 7) \ne 7$ as $7 \notin \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ $\because$ $2\pi < 7 < \dfrac{5\pi}{2}$

graph of $y = \sin^{-1}(\sin x)$ is as :



From the graph we can see that if $2\pi \le x \le \dfrac{5\pi}{2}$, then

$y = \sin^{-1}(\sin x)$ can be written as :

$y = x - 2\pi$

$\therefore$ $\sin^{-1}(\sin 7) = 7 - 2\pi$

Similarly if we have to find $\sin^{-1}(\sin(-5))$ then

$\because$ $-2\pi < -5 < -\dfrac{3\pi}{2}$

$\therefore$ from the graph of $\sin^{-1}(\sin x)$, we can say that $\sin^{-1}(\sin(-5)) = 2\pi + (-5) = 2\pi - 5$

Example # 22 : Find the value of $\sin\left(\pi \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}\right)$

Solution : Let $y = \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ ....(i)

$\because$ $\cot^{-1}(-x) = \pi - \cot^{-1}x,\ x \in R$

(i) can be written as

$$y = \tan\left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$$

$$y = -\tan\left(\cot^{-1}\frac{2}{3}\right)$$

$\because$ $\cot^{-1} x = \tan^{-1}\frac{1}{x}$ if $x > 0$

$\therefore$ $y = -\tan\left(\tan^{-1}\frac{3}{2}\right) \Rightarrow y = -\frac{3}{2}$ so $\sin\left(\pi \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}\right) = \sin\left(-\frac{3\pi}{2}\right) = 1$

Example # 23 : Find the value of $\sin\left(2\tan^{-1}\frac{1}{2}\right)$

Solution : $\sin\left(2\tan^{-1}\frac{1}{2}\right) = 2\sin\left(\tan^{-1}\frac{1}{2}\right)\cos\left(\tan^{-1}\frac{1}{2}\right) = 2\sin\left(\sin^{-1}\frac{1}{\sqrt{5}}\right) \times \cos\left(\cos^{-1}\frac{2}{\sqrt{5}}\right)$

$$= 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}} = \frac{4}{5}$$

Example # 24 : Find the value of $\tan\left(\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\frac{\sqrt{5}}{3}\right)$

Solution : $\tan\left(\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\frac{\sqrt{5}}{3}\right) = \tan\left(\frac{1}{2}\left(\frac{\pi}{2} - \sin^{-1}\frac{\sqrt{5}}{3}\right)\right) = \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

Let $y = \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$ ....(i)

Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$ and $\cos\theta = \frac{\sqrt{5}}{3}$

$\therefore$ (i) becomes $y = \tan\left(\frac{\theta}{2}\right)$ ....(ii)

$\because$ $\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}} = \frac{3-\sqrt{5}}{3+\sqrt{5}} = \frac{(3-\sqrt{5})^2}{4}$

$\tan\frac{\theta}{2} = \pm\left(\frac{3-\sqrt{5}}{2}\right)$ ....(iii)

$\frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right) \Rightarrow \tan\frac{\theta}{2} > 0$

$\therefore$ from (iii), we get $y = \tan\frac{\theta}{2} = \left(\frac{3-\sqrt{5}}{2}\right)$

Example # 25 : Find the value of $\cos(2\cos^{-1}x + \sin^{-1}x)$ when $x = \frac{1}{5}$

Solution : $\cos\left(2\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5}\right) = \cos\left(\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5} + \cos^{-1}\frac{1}{5}\right)$

$= \cos\left(\frac{\pi}{2} + \cos^{-1}\frac{1}{5}\right) = -\sin\left(\cos^{-1}\left(\frac{1}{5}\right)\right)$ ....(i)

$= -\sqrt{1-\left(\frac{1}{5}\right)^2} = -\frac{2\sqrt{6}}{5}$.

***Aliter* :** Let $\cos^{-1}\frac{1}{5} = \theta \Rightarrow \cos\theta = \frac{1}{5}$ and $\theta \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ $\sin\theta = \frac{\sqrt{24}}{5}$

$\therefore$ $\sin^{-1}(\sin\theta) = \sin^{-1}\left(\frac{\sqrt{24}}{5}\right)$ ....(ii)

$\because$ $\theta \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin^{-1}(\sin\theta) = \theta$

$\therefore$ equation (ii) can be written as

$\theta = \sin^{-1}\left(\frac{\sqrt{24}}{5}\right)$ $\because$ $\theta = \cos^{-1}\left(\frac{1}{5}\right)$

$\Rightarrow$ $\cos^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{\sqrt{24}}{5}\right)$

Now equation (i) can be written as $y = -\sin\left\{\sin^{-1}\left(\frac{\sqrt{24}}{5}\right)\right\}$ ....(iii)

$\because$ $\frac{\sqrt{24}}{5} \in [-1, 1]$ $\therefore$ $\sin\left\{\sin^{-1}\left(\frac{\sqrt{24}}{5}\right)\right\} = \frac{\sqrt{24}}{5}$

$\therefore$ from equation (iii), we get $y = -\frac{\sqrt{24}}{5}$

Example # 26 : Solve $\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

Solution : $\sin^{-1}(f(x)) + \cos^{-1}(g(x)) = \frac{\pi}{2} \Leftrightarrow f(x) = g(x)$ and $-1 \le f(x), g(x) \le 1$

$x^2 - 2x + 1 = x^2 - x \Leftrightarrow x = 1$, accepted as a solution

Self practice problems :

(16) Find the value of $\cos\left\{\sin\left(\sin^{-1}\frac{\pi}{6}\right)\right\}$

(17) Find the value of $\sin\left\{\cos\left(\cos^{-1}\frac{3\pi}{4}\right)\right\}$

(18) Find the value of $\cos^{-1}(\cos 13)$

(19) Find $\sin^{-1}(\sin\theta)$, $\cos^{-1}(\cos\theta)$, $\tan^{-1}(\tan\theta)$, $\cot^{-1}(\cot\theta)$ for $\theta \in \left(\frac{5\pi}{2}, 3\pi\right)$

(20) Find the value of $\cos^{-1}(-\cos 4)$ (21) Find the value of $\tan^{-1}\left\{\tan\left(-\frac{7\pi}{8}\right)\right\}$

(22) Find the value of $\tan^{-1}\left\{\cot\left(-\frac{1}{4}\right)\right\}$ (23) Find the value of $\sec\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$

(24) Find the value of $\text{cosec}\left(\sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$

(25) Find the value of $\sin(2\cos^{-1}x + \sin^{-1}x)$ when $x = \frac{1}{5}$

(26) Solve the following equations (i) $5\tan^{-1}x + 3\cot^{-1}x = 2\pi$ (ii) $4\sin^{-1}x = \pi - \cos^{-1}x$

(27) Evaluate $\tan\left(\text{cosec}^{-1}\frac{\sqrt{41}}{4}\right)$ (28) Evaluate $\sec\left(\cot^{-1}\frac{16}{63}\right)$

(29) Evaluate $\sin\left\{\frac{1}{2}\cot^{-1}\left(\frac{-3}{4}\right)\right\}$ (30) Evaluate $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$

(31) Solve $\sin^{-1}(x^2 - 2x + 3) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

Answers :

(16)	$\frac{\sqrt{3}}{2}$	(17)	not defined		
(18)	$13 - 4\pi$	(19)	$3\pi - \theta$, $\theta - 2\pi$, $\theta - 3\pi$, $\theta - 2\pi$		
(20)	$4 - \pi$	(21)	$\frac{\pi}{8}$	(22)	$\left(\frac{1}{4} - \frac{\pi}{2}\right)$
(23)	$\frac{3}{2}$	(24)	$-\sqrt{3}$	(25)	$\frac{1}{5}$
(26). (i)	$x = 1$	(ii)	$x = \frac{1}{2}$		
(27) $\frac{4}{5}$	(28) $\frac{65}{16}$	(29)	$\frac{2\sqrt{5}}{5}$	(30)	$\frac{-7}{17}$
(31)	No solution				

Property 6 : Identities on addition and subtraction:

(i) $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), & x \ge 0,\ y \ge 0 \ \&\ (x^2+y^2) \le 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), & x \ge 0,\ y \ge 0 \ \&\ x^2+y^2 \ge 1 \end{cases}$

Proof : Let $A = \sin^{-1}x$ and $B = \sin^{-1}y$ where $x, y \in [0, 1]$.

$\sin(A + B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \sin^{-1}\sin(A + B) = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

$\Rightarrow \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

$= \begin{cases} A+B & \text{for } 0 \le A+B \le \pi/2 \\ \pi-(A+B) & \text{for } \pi/2 \le A+B \le \pi \end{cases} = \begin{cases} \sin^{-1}x + \sin^{-1}y, & x^2+y^2 \le 1 \\ \pi - (\sin^{-1}x + \sin^{-1}y), & x^2+y^2 \ge 1 \end{cases}$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$; $x, y \in [0, 1]$

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$; $x, y \in [0, 1]$

(iv) $\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); & 0 \le x < y \le 1 \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); & 0 \le y < x \le 1 \end{cases}$

(v) $\tan^{-1}x + \tan^{-1}y = \begin{cases} \pi/2 & \text{if } x, y > 0 \ \&\ xy = 1 \\ -\pi/2 & \text{if } x, y < 0 \ \&\ xy = 1 \\ \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x, y \ge 0 \ \&\ xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x, y \ge 0 \ \&\ xy > 1 \end{cases}$

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, $x \ge 0, y \ge 0$

Notes : (i) $x^2 + y^2 \le 1$ & $x, y \ge 0$ $\Rightarrow$ $0 \le \sin^{-1}x + \sin^{-1}y \le \frac{\pi}{2}$

and $x^2 + y^2 \ge 1$ & $x, y \ge 0$ $\Rightarrow$ $\frac{\pi}{2} \le \sin^{-1}x + \sin^{-1}y \le \pi$

(ii) $xy < 1$ and $x, y \ge 0 \Rightarrow 0 \le \tan^{-1}x + \tan^{-1}y < \frac{\pi}{2}$; $xy > 1$ and $x, y \ge 0 \Rightarrow \frac{\pi}{2} < \tan^{-1}x + \tan^{-1}y < \pi$

(iii) For $x < 0$ or $y < 0$ these identities can be used with the help of property "$-x$" i.e. change x or y to $-x$ or $-y$ which are positive .

Example # 27 : Show that $\cos^{-1}\frac{4}{5} + \sin^{-1}\frac{15}{17} = +\cos^{-1}\frac{84}{85}$

Solution : $\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{3}{5}$

$\because$ $\frac{3}{5} > 0$, $\frac{15}{17} > 0$ and $\left(\frac{3}{5}\right)^2 + \left(\frac{15}{17}\right)^2 = \frac{8226}{7225} > 1$

$\therefore$ $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{225}{289}} + \frac{15}{17}\sqrt{1-\frac{9}{25}}\right)$

$= \pi - \sin^{-1}\left(\frac{3}{5} \cdot \frac{8}{17} + \frac{15}{17} \cdot \frac{4}{5}\right) = \pi - \sin^{-1}\left(\frac{84}{85}\right) = \pi - \frac{\pi}{2} + \cos^{-1}\frac{84}{85} = \frac{\pi}{2} + \cos^{-1}\frac{84}{85}$

Example # 28 : Evaluate $\cot^{-1}\frac{1}{9} + \cot^{-1}\frac{4}{5} + \cot^{-1}1$

Solution : $\cot^{-1}\frac{1}{9} + \cot^{-1}\frac{4}{5} + \cot^{-1}1 = \tan^{-1}9 + \tan^{-1}\frac{5}{4} + \cot^{-1}1$

$\because$ $9 > 0$, $\frac{5}{4} > 0$ and $\left(9 \times \frac{5}{4}\right) > 1$

$\therefore$ $\tan^{-1}9 + \tan^{-1}\frac{5}{4} + \cot^{-1}1 = \pi + \tan^{-1}\left(\frac{9 + \frac{5}{4}}{1 - 9 \cdot \frac{5}{4}}\right) + \cot^{-1}1 = \pi + \tan^{-1}(-1) + \cot^{-1}1$

$= \pi - \frac{\pi}{4} + \cot^{-1}1 = \pi$.

Example # 29 : Define $y = \cos^{-1}(4x^3 - 3x)$ in terms of $\cos^{-1}x$ and also draw its graph.

Solution : Part - 1: Let $y = \cos^{-1}(4x^3 - 3x)$

$\because$ Domain : $[-1, 1]$ and range : $[0, \pi]$

Let $\cos^{-1}x = \theta \Rightarrow \theta \in [0, \pi]$ and $x = \cos\theta$

$\therefore$ $y = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$y = \cos^{-1}(\cos 3\theta)$ ....(i)

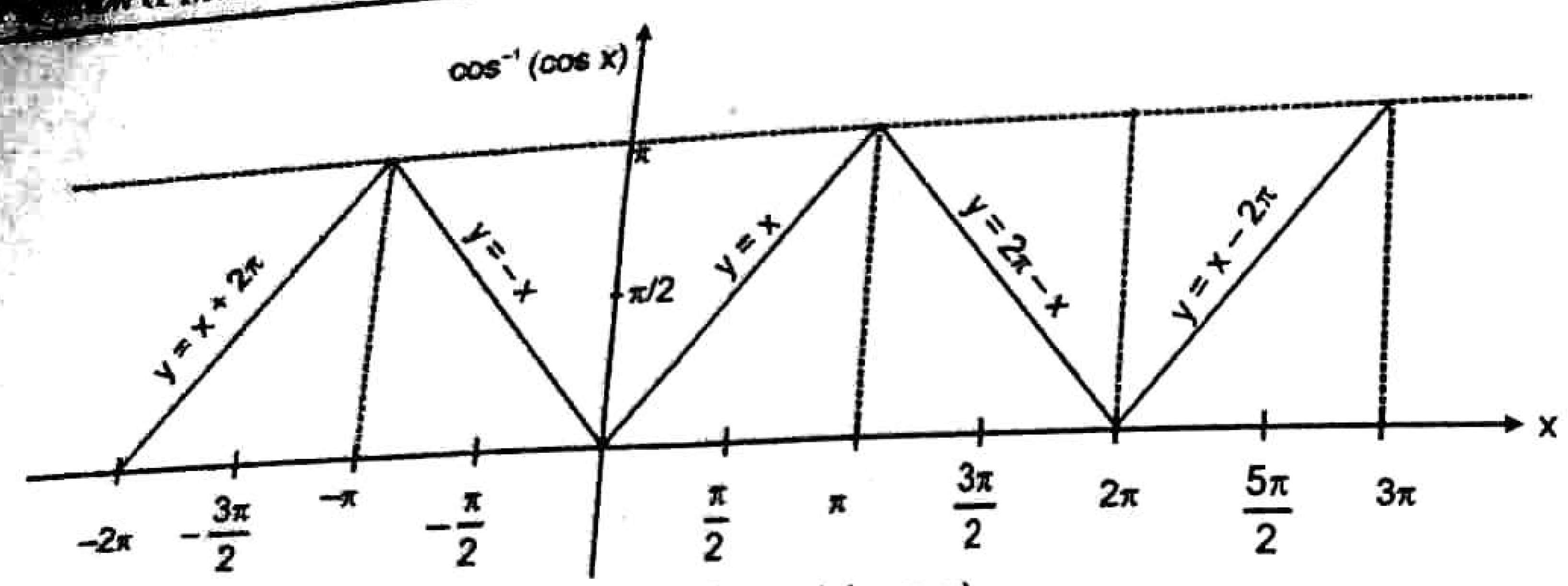


Fig.: Graph of $\cos^{-1}(\cos x)$

$\because$ $\theta \in [0, \pi]$

$\therefore$ $3\theta \in [0, 3\pi]$

$\therefore$ to define $y = \cos^{-1}(\cos 3\theta)$, we consider the graph of $\cos^{-1}(\cos x)$ in the interval $[0, 3\pi]$. Now, from the above graph we can see that

(i) if $0 \le 3\theta \le \pi$ $\Rightarrow$ $\cos^{-1}(\cos 3\theta) = 3\theta$

$\therefore$ from equation (i), we get

$y = 3\theta$ if $\theta \le 3\theta \le \pi$

$\Rightarrow$ $y = 3\theta$ if $0 \le \theta \le \dfrac{\pi}{3}$

$\Rightarrow$ $y = 3\cos^{-1}x$ if $\dfrac{1}{2} \le x \le 1$

(ii) if $\pi < 3\theta \le 2\pi \Rightarrow$ $\cos^{-1}(\cos 3\theta) = 2\pi - 3\theta$

$\therefore$ from equation (i), we get

$y = 2\pi - 3\theta$ if $\pi < 3\theta \le 2\pi$

$\Rightarrow$ $y = 2\pi - 3\theta$ if $\dfrac{\pi}{3} < \theta \le \dfrac{2\pi}{3}$

$y = 2\pi - 3\cos^{-1}x$ if $-\dfrac{1}{2} \le x < \dfrac{1}{2}$

(iii) $2\pi < 3\theta \le 3\pi$ $\Rightarrow$ $\cos^{-1}(\cos 3\theta) = -2\pi + 3\theta$

$\therefore$ from equation (i), we get

$\Rightarrow$ $y = -2\pi + 3\theta$ if $2\pi < 3\theta \le 3\pi$

$\Rightarrow$ $y = -2\pi + 3\theta$ if $\dfrac{2\pi}{3} < \theta \le \pi$

$\Rightarrow$ $y = -2\pi + 3\cos^{-1}x$ if $-1 \le x < -\dfrac{1}{2}$

$\therefore$ from (i), (ii) & (iii), we get

$$y = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x & ; \quad \dfrac{1}{2} \le x \le 1 \\ 2\pi - 3\cos^{-1}x & ; \quad -\dfrac{1}{2} \le x < \dfrac{1}{2} \\ -2\pi + 3\cos^{-1}x & ; \quad -1 \le x < -\dfrac{1}{2} \end{cases}$$

Part-2 : For $y = \cos^{-1}(4x^3 - 3x)$

domain : $[-1, 1]$

range : $[0, \pi]$

(i) if $\dfrac{1}{2} \le x \le 1$, $y = 3\cos^{-1}x$.

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{-3}{\sqrt{1-x^2}} = -3(1-x^2)^{-1/2}$ ....(i)

$\Rightarrow$ $\dfrac{dy}{dx} < 0$ if $x \in \left[\dfrac{1}{2}, 1\right)$ $\Rightarrow$ decreasing if $x \in \left[\dfrac{1}{2}, 1\right)$

again if we differentiate equation (i) w.r.t. 'x', we get

$\dfrac{dy}{dx} = -\dfrac{3x}{(1-x^2)^{3/2}}$ $\Rightarrow$ $\dfrac{d^2y}{dx^2} < 0$ if $x \in \left[\dfrac{1}{2}, 1\right) \Rightarrow$ concavity downwards if $x \in \left[\dfrac{1}{2}, 1\right)$

(ii) if $-\dfrac{1}{2} \le x < \dfrac{1}{2}$, $y = 2\pi - 3\cos^{-1}x$.

$\therefore$ $\dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-x^2}}$ $\Rightarrow$ $\dfrac{dy}{dx} > 0$ if $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right)$

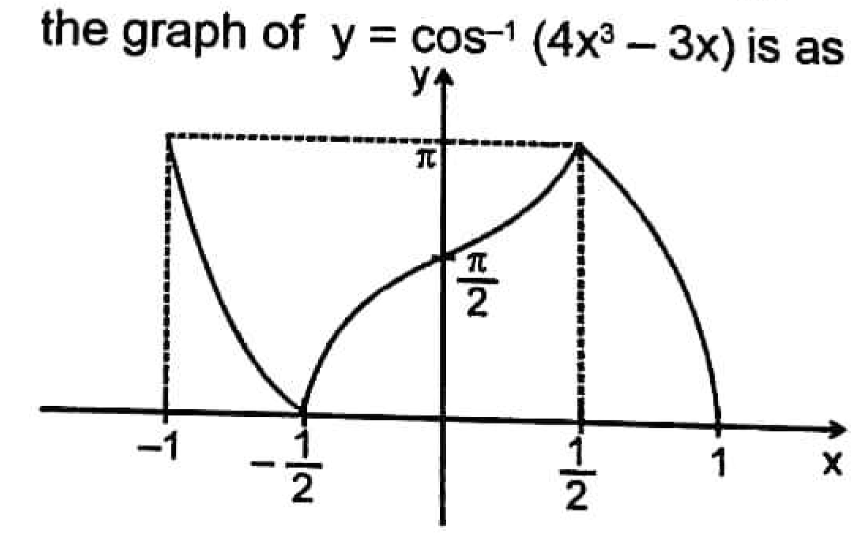
$\Rightarrow$ increasing if $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right)$ and $\dfrac{d^2y}{dx^2} = \dfrac{3x}{(1-x^2)^{3/2}}$

(a) if $x \in \left[-\dfrac{1}{2}, 0\right)$ then $\dfrac{d^2y}{dx^2} < 0$ $\Rightarrow$ concavity downwards if $x \in \left[-\dfrac{1}{2}, 0\right)$

(b) if $x \in \left(0, \dfrac{1}{2}\right)$ then $\dfrac{d^2y}{dx^2} > 0$ $\Rightarrow$ concavity upwards if $x \in \left(0, \dfrac{1}{2}\right)$

(iii) Similarly if $-1 \le x < -\dfrac{1}{2}$ then $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} > 0$.

$\therefore$ the graph of $y = \cos^{-1}(4x^3 - 3x)$ is as



Self practice problems:

(32) Evaluate $\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{5}{13} + \sin^{-1}\dfrac{16}{65}$

(33) If $\tan^{-1}4 + \tan^{-1}5 = \cot^{-1}\lambda$, then find 'λ'

(34) Prove that $2\cos^{-1}\dfrac{3}{\sqrt{13}} + \cot^{-1}\dfrac{16}{63} + \dfrac{1}{2}\cos^{-1}\dfrac{7}{25} = \pi$

(35) Solve the equation $\sin^{-1}x + \sin^{-1}2x = \dfrac{2\pi}{3}$

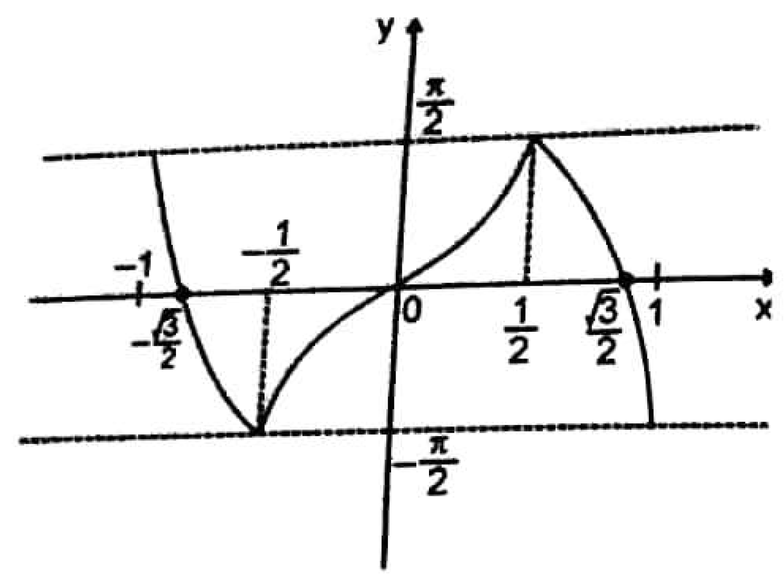
(36) Define $y = \sin^{-1}(3x - 4x^3)$ in terms of $\sin^{-1}x$ and also draw its graph.

(37) Define $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$ in terms of $\tan^{-1}x$ and also draw its graph.

Answers. (32) $\dfrac{\pi}{2}$ (33) $\lambda = -\dfrac{19}{9}$ (35) $x = \dfrac{1}{2}$

(36) $$y = \sin^{-1}(3x - 4x^3) = \begin{cases} 3\sin^{-1}x \; ; & -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x \; ; & \dfrac{1}{2} < x \le 1 \\ -\pi - 3\sin^{-1}x \; ; & -1 \le x < -\dfrac{1}{2} \end{cases}$$

$\therefore$ graph of $y = \sin^{-1}(3x - 4x^3)$



(37) $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \begin{cases} 3\tan^{-1}x\ ; & -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x\ ; & -\infty < x < -\frac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x\ ; & \frac{1}{\sqrt{3}} < x < \infty \end{cases}$

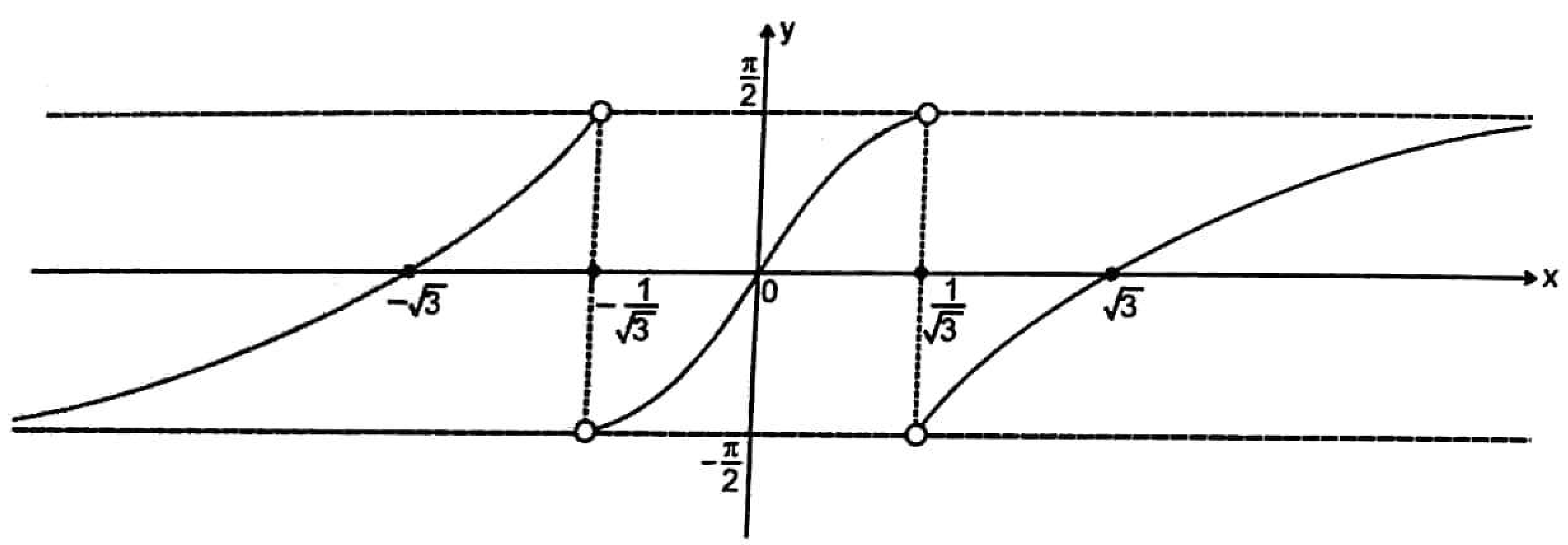
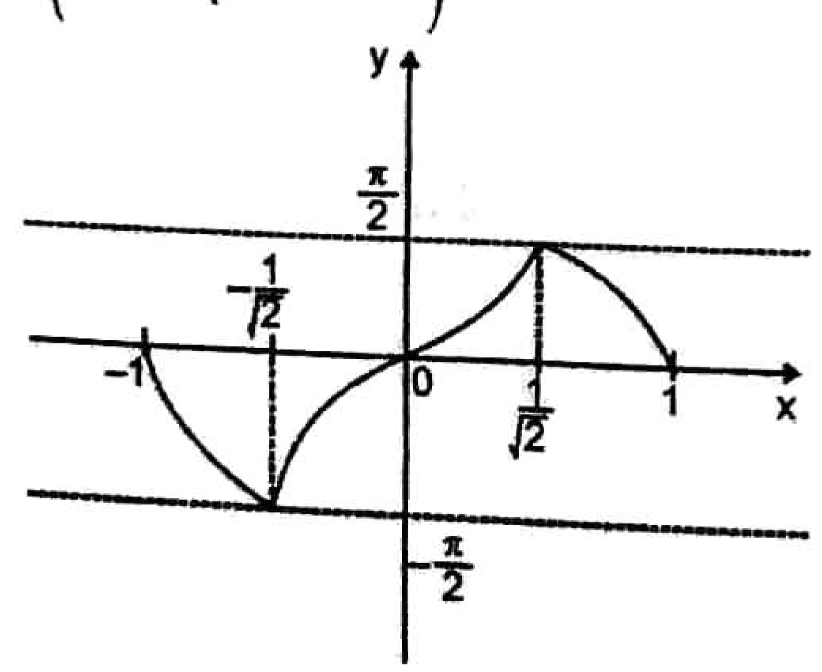


Fig.: Graph of $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Property 7 : Miscellaneous Identities

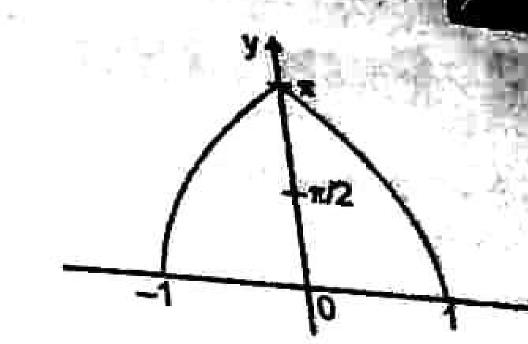
(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} 2\sin^{-1}x & \text{if } |x| \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if } \frac{1}{\sqrt{2}} < x \le 1 \\ -\left(\pi + 2\sin^{-1}x\right) & \text{if } -1 \le x < -\frac{1}{\sqrt{2}} \end{cases}$

graph of $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$



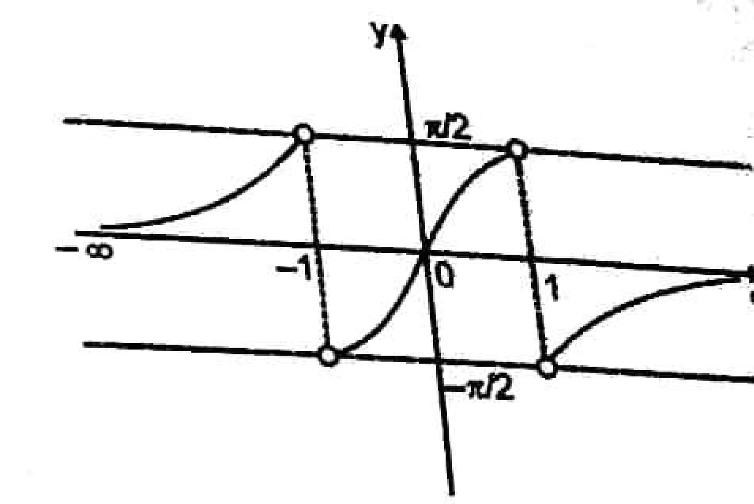
(ii) $\cos^{-1}(2x^2-1) = \begin{cases} 2\cos^{-1}x & \text{if } 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & \text{if } -1 \le x < 0 \end{cases}$

graph of $y = \cos^{-1}(2x^2-1)$



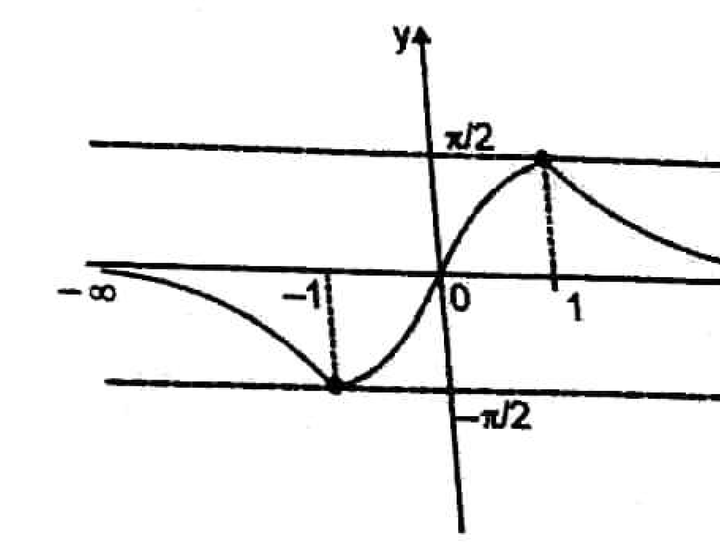
(iii) $\tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -\left(\pi - 2\tan^{-1}x\right) & \text{if } x > 1 \end{cases}$

graph of $y = \tan^{-1}\frac{2x}{1-x^2}$

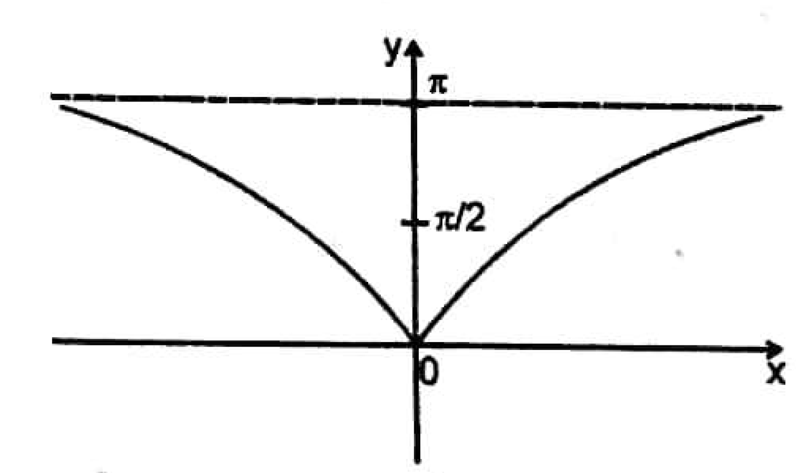


(iv) $\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -\left(\pi + 2\tan^{-1}x\right) & \text{if } x < -1 \end{cases}$

graph of $y = \sin^{-1}\frac{2x}{1+x^2}$



(v) $\cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$



graph of $y = \cos^{-1}\frac{1-x^2}{1+x^2}$

(vi) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $x + y + z = xyz$

(vii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, then $xy + yz + zx = 1$

(viii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(ix) $\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{2}$

EFGH KUNAL $x(x^2-12)$

Exercise-1

Marked Questions may have for Revision Questions.

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Definition of function, Domain and Range, Classification of Functions

A-1. Check whether the followings represent function or not

(i) $x^2 + y^2 = 36, \quad y \in [0, 6]$ (ii) $x^2 + y^2 = 36, \quad x \in [0, 1]$

(iii) $x^2 + y^2 = 36, \quad x \in [-6, 6]$ (iv) $x^2 + y^2 = 36$

A-2. Find the domain of each of the following functions :

(i) $f(x) = \dfrac{x^3 - 5x + 3}{x^2 - 1}$ (ii) $f(x) = \sqrt{\sin(\cos x)}$

(iii) $f(x) = \dfrac{1}{\sqrt{x + |x|}}$ (iv) $f(x) = e^{x+\sin x}$

(v) $f(x) = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x+2}$ (vi) $f(x) = \sqrt{\dfrac{\log_2(x-2)}{\log_{1/2}(3x-1)}}$

(vii) $f(x) = \ell n\,[x^2 + x + 1]$, where [.] GIF. (viii) $f(x) = \dfrac{\sqrt{\cos x - \frac{1}{2}}}{\sqrt{6 + 35x - 6x^2}}$

A-3. Find the domain of definitions of the following functions :

(i) $f(x) = \sqrt{3 - 2^x - 2^{1-x}}$ let $2^x = t$ (ii) $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$

(iii) $f(x) = (x^2 + x + 1)^{-3/2}$ (iv) $f(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}}$

(v) $f(x) = \sqrt{\tan x - \tan^2 x}$ (vi) $f(x) = \dfrac{1}{\sqrt{1 - \cos x}}$

(vii) $f(x) = \sqrt{\ell og_{1/4}\left(\dfrac{5x - x^2}{4}\right)}$ (viii) $f(x) = \ell og_{10}\,(1 - \ell og_{10}(x^2 - 5x + 16))$

A-4. Find the range of each of the following functions :

(i) $f(x) = |x - 3|$ (ii) $f(x) = \dfrac{x}{1+x^2}$ (iii) $f(x) = \sqrt{16 - x^2}$ (iv) $f(x) = \dfrac{|x-4|}{x-4}$

A-5. Find the domain and the range of each of the following functions :

(i) $f(x) = \dfrac{1}{\sqrt{4 + 3\sin x}}$ (ii) $f(x) = x\,!$ (iii) $f(x) = \dfrac{x^2 - 9}{x - 3}$ (iv) $f(x) = \sin^2(x^3) + \cos^2(x^3)$

A-6. Find the range of each of the following functions : (where {.} and [.] represent fractional part and greatest integer part functions respectively)

(i) $f(x) = 5 + 3\sin x + 4\cos x$ (ii) $f(x) = \dfrac{1}{1 + \sqrt{x}}$

(iii) $f(x) = 2 - 3x - 5x^2$ $\left(-\infty, \frac{-D}{4a}\right]$ (iv) $f(x) = 3|\sin x| - 4|\cos x|$

(v) $f(x) = \dfrac{\sin x}{\sqrt{1 + \tan^2 x}} + \dfrac{\cos x}{\sqrt{1 + \cot^2 x}}$ (vi) $f(x) = \ell n\left(\dfrac{\sqrt{8 - x^2}}{x - 2}\right)$

(vii) $f(x) = \left[\dfrac{1}{\sin\{x\}}\right]$

A-7. Find the range of the following functions : (where {.} and [.] represent fractional part and greatest integer part functions respectively)

(i) $f(x) = 1 - |x - 2|$ (ii) $f(x) = \dfrac{1}{\sqrt{16 - 4^{x^2 - x}}}$

(iii) $f(x) = \dfrac{1}{2 - \cos 3x}$ (iv) $f(x) = \dfrac{x+2}{x^2 - 8x - 4}$

(v) $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$ (vi) $f(x) = 3\sin\sqrt{\dfrac{\pi^2}{16} - x^2}$

(vii) $f(x) = x^4 - 2x^2 + 5$ (viii) $f(x) = x^3 - 12x$, where $x \in [-3, 1]$

(ix) $f(x) = \sin^2 x + \cos^4 x$

(x) $f(x) = [\sin x + [\cos x + [\tan x + [\sec x]]]]$ Here $x \in (0, \pi/4)$

(xi) $f(x) = \sec^2 x - \tan^2 x + \sin(\sin x + \cos x)$

A-8. Find whether the following functions are one-one or many-one & into or onto if $f : D \to R$ where D is its domain.

(i) $f(x) = |x^2 + 5x + 6|$ many, into (ii) $f(x) = |\ell n x|$ many, into,

(iii) $f(x) = \sin 4x : \left(-\dfrac{\pi}{8}, \dfrac{\pi}{8}\right) \to (-1, 1)$ many (iv) $f(x) = x + \dfrac{1}{x}, x \in (0, \infty)$ many into

(v) $f(x) = \sqrt{1 - e^{\left(\frac{1}{x} - 1\right)}}$ into, one (vi) $f(x) = \dfrac{3x^2}{4\pi} - \cos \pi x$ into, many-one

(vii) $f(x) = \dfrac{1 + x^6}{x^3}$ many, into (viii) $f(x) = x\cos x$ many, onto (ix) $f(x) = \dfrac{1}{\sin\sqrt{|x|}}$ into, $(x \neq 0)$ many

A-9. Classify the following functions f(x) defined in $R \to R$ as injective, surjective, both or none.

(i) $f(x) = x|x|$ (ii) $f(x) = \dfrac{x^2}{1+x^2}$ (iii) $f(x) = x^3 - 6x^2 + 11x - 6$

A-10. Check whether the following functions is/are many-one or one-one & into or onto

(i) $f(x) = \tan(2\sin x)$ (ii) $f(x) = \tan(\sin x)$

A-11. Let $f : A \to A$ where $A = \{x : -1 \le x \le 1\}$. Find whether the following functions are bijective.

(i) $x - \sin x$ (ii) $x|x|$ (iii) $\tan\dfrac{\pi x}{4}$ (iv) x^4

A-12. Let A be a set of n distinct elements. Then find the total number of distinct functions from A to A ? How many of them are onto functions ?

Section (B) : Identical functions, Composite functions

$\cot x = 0 \Rightarrow x = \frac{\pi}{2}$

B-1. Check whether following pairs of functions are identical or not ?

(i) $f(x) = \sqrt{x^2}$ and $g(x) = \left(\sqrt{x}\right)^2$ (ii) $f(x) = \tan x$ and $g(x) = \dfrac{1}{\cot x}$

(iii) $f(x) = \sqrt{\dfrac{1 + \cos 2x}{2}}$ and $g(x) = \cos x$ (iv) $f(x) = x$ and $g(x) = e^{\ell n x}$

B-2. Find for what values of x, the following functions would be identical.

$f(x) = \log(x-1) - \log(x-2)$ and $g(x) = \log\left(\dfrac{x-1}{x-2}\right)$ $x \in (2, \infty)$

B-3. Let $f(x) = x^2 + x + 1$ and $g(x) = \sin x$. Show that fog $\neq$ gof

B-4. Let $f(x) = x^2$, $g(x) = \sin x$, $h(x) = \sqrt{x}$, then verify that [fo (goh)] (x) and [(fog) oh] (x) are equal.

B-5. Find fog and gof, if
(i) $f(x) = e^x$; $g(x) = \ell n\, x$ (ii) $f(x) = |x|$; $g(x) = \sin x$
(iii) $f(x) = \sin x$; $g(x) = x^2$ (iv) $f(x) = x^2 + 2$; $g(x) = 1 - \frac{1}{1-x}$, $x \neq 1$

B-6. If $f(x) = \ell n(x^2 - x + 2)$; $R^+ \to R$ and
$g(x) = \{x\} + 1$; $[1, 2] \to [1, 2]$, where {x} denotes fractional part of x.
Find the domain and range of f(g(x)) when defined.

B-7. If $f(x) = \begin{cases} 1+x^2; & x \le 1 \\ x+1; & 1 < x \le 2 \end{cases}$ and $g(x) = 1 - x$; $-2 \le x \le 1$, then define the function fog(x).

B-8. If $f(x) = \frac{x+2}{x+1}$ and $g(x) = \frac{x-2}{x}$, then find the domain of
(i) fog(x) (ii) gof(x) (iii) fof(x) (iv) fogof(x)

B-9. If $f(x) = \begin{cases} \sqrt{2}x & x \in Q - \{0\} \\ 3x & x \in Q^c \end{cases}$, then define fof(x) and hence define fofof.....f(x) where f is 'n' times.

B-10. Let $f(x) \begin{cases} x+1 & x \le 4 \\ 2x+1 & 4 < x \le 9 \\ -x+7 & x > 9 \end{cases}$ and $g(x) = \begin{cases} x^2 & -1 \le x < 3 \\ x+2 & 3 \le x \le 5 \end{cases}$ then, find f(g(x)).

Section (C) : Even/Odd Functions & Periodic Functions

C-1. Determine whether the following functions are even or odd or neither even nor odd :
(i) $\sin(x^2 + 1)$ (ii) $x + x^2$ (iii) $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$ even
(iv) $f(x) = \sin x + \cos x$ (v) $f(x) = (x^2 - 1)\,|x|$ even
(vi) $f(x) = \begin{cases} |\ell n e^x| & ; \; x \le -1 \\ [2+x] + [2-x] & ; \; -1 < x < 1 \text{, where [.] is GIF.} \\ e^{\ell n x} & ; \; x \ge 1 \end{cases}$

$\frac{\frac{1-a^x}{a^x}}{\frac{1+a^x}{a^x}} = \frac{1-a^x}{1+a^x}$

C-2. Examine whether the following functions are even or odd or neither even nor odd, where [ ] denotes greatest integer function.
(i) $f(x) = \frac{(1 + 2^x)^7}{2^x}$ (ii) $f(x) = \frac{\sec x + x^2 - 9}{x \sin x}$
(iii) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$ (iv) $f(x) = \begin{cases} x\,|x|, & x \le -1 \\ [1+x] + [1-x], & -1 < x < 1 \\ -x\,|x|, & x \ge 1 \end{cases}$

C-3. Prove that the following functions are not periodic :
(i) $f(x) = \sin\sqrt{x}$ (ii) $f(x) = x + \sin x$

C-4. Find the fundamental period of the following functions : (where [.] denotes greatest integer fu...
(i) $f(x) = 2 + 3\cos(x - 2)$ (ii) $f(x) = \sin 3x + \cos^2 x + |\tan x|$
(iii) $f(x) = \sin\frac{\pi x}{4} + \sin\frac{\pi x}{3}$ (iv) $f(x) = \cos\frac{3}{5}x - \sin\frac{2}{7}x$.
(v) $f(x) = [\sin 3x] + |\cos 6x|$ (vi) $f(x) = \frac{1}{1-\cos x}$
(vii) $f(x) = \frac{\sin 12x}{1+\cos^2 6x}$ (viii) $f(x) = \sec^3 x + \text{cosec}^3 x$

Section (D) : Inverse of a function

D-1. Let $f : D \to R$, where D is the domain of f. Find the inverse of f, if it exists
(i) $f(x) = 1 - 2^{-x}$ (ii) $f(x) = \left(4 - (x-7)^3\right)^{1/5}$
(iii) $f(x) = \ell n\,(x + \sqrt{1+x^2})$
(iv) Let $f : [0, 3] \to [1, 13]$ is defined by $f(x) = x^2 + x + 1$, then find $f^{-1}(x)$.

D-2. Let $f : R \to R$ be defined by $f(x) = \frac{e^{2x} - e^{-2x}}{2}$. Is f(x) invertible ? If yes, then find its inverse.

D-3. **(a)** If $f(x) = -x|x|$, then find $f^{-1}(x)$ and hence find the number of solutions of $f(x) = f^{-1}(x)$.
(b) Solve $2x^2 - 5x + 2 = \frac{5 - \sqrt{9+8x}}{4}$, where $x < \frac{5}{4}$

D-4. If g is inverse of $f(x) = x^3 + x + \cos x$, then find the value of $g'(1)$.

D-5. If $f(x) = \begin{cases} (\alpha - 1)x & x \in Q^c \\ -\alpha^2 x + \alpha + 3x - 1 & x \in Q \end{cases}$ and $g(x) = \begin{cases} x & x \in Q^c \\ 1 - x & x \in Q \end{cases}$ are inverse to each other then find possible values of α.

Section (E) : Definition, graphs and fundamentals & Inverse Trigonometry

E-1. Find the domain of each of the following functions :
(i) $f(x) = \frac{\sin^{-1} x}{x}$ $[-1, 0) \cup (0, 1]$ (ii) $f(x) = \sqrt{1-2x} + 3\sin^{-1}\left(\frac{3x-1}{2}\right)$ (iii) $f(x) = 2^{\sin^{-1} x} + \ldots$

E-2. Find the range of each of the following functions :
(i) $f(x) = \ell n\,(\sin^{-1} x)$ (ii) $f(x) = \sin^{-1}\left(\frac{\sqrt{3x^2+1}}{5x^2+1}\right)$
(iii) $f(x) = \cos^{-1}\left(\frac{(x-1)(x+5)}{x(x-2)(x-3)}\right)$

E-3. **Find the simplified value of the following expressions :**
(i) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ (ii) $\tan\left[\cos^{-1}\frac{1}{2} + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$
(iii) $\sin^{-1}\left[\cos\left\{\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right]$

E-4. (i) If $\sum_{i=1}^{n} \cos^{-1} \alpha_i = 0$, then find the value of $\sum_{i=1}^{n} i \cdot \alpha_i$

(ii) If $\sum_{i=1}^{2n} \sin^{-1} x_i = n\pi$, then show that $\sum_{i=1}^{2n} x_i = 2n$

E-5. Solve the following inequalities:

(i) $\cos^{-1} x > \cos^{-1} x^2$ (ii) $\tan^{-1} x > \cot^{-1} x$.

(iii) $\text{arccot}^2 x - 5 \,\text{arccot}\, x + 6 > 0$

Section (F) : Trig. (trig^{-1}x), trig^{-1} (trig x) trig (–x) and Properties

F-1. Evaluate the following inverse trigonometric expressions :

(i) $\sin^{-1}\left(\sin \frac{7\pi}{6}\right)$ (ii) $\tan^{-1}\left(\tan \frac{2\pi}{3}\right)$

(iii) $\cos^{-1}\left(\cos \frac{5\pi}{4}\right)$ (iv) $\sec^{-1}\left(\sec \frac{7\pi}{4}\right)$

F-2. Find the value of the following inverse trigonometric expressions :

(i) $\sin^{-1} (\sin 4)$ (ii) $\cos^{-1} (\cos 10)$

(iii) $\tan^{-1} (\tan (-6))$ (iv) $\cot^{-1} (\cot (-10))$

(v) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\left(\cos \frac{9\pi}{10} - \sin \frac{9\pi}{10}\right)\right)$

$\cos \frac{\pi}{4} = \sin \frac{\pi}{4}$

F-3. Find the value of following expressions :

(i) $\cot (\tan^{-1} a + \cot^{-1} a)$ → $\cot \frac{\pi}{2} = 0$ (ii) $\sin (\sin^{-1}x + \cos^{-1}x)$, $|x| \le 1$ → $\sin \frac{\pi}{2} = 1$

(iii) $\tan\left[\cos^{-1}\left(\frac{3}{4}\right) + \sin^{-1}\left(\frac{3}{4}\right) - \sec^{-1}3\right]$

→ $\frac{\pi}{2}$

Section (G) : Interconversion

G-1. Evaluate the following expressions :

(i) $\sin\left(\cos^{-1}\frac{3}{5}\right)$ (ii) $\tan\left(\cos^{-1}\frac{1}{3}\right)$

(iii) $\text{cosec}\left(\sec^{-1}\frac{\sqrt{41}}{5}\right)$ (iv) $\tan\left(\text{cosec}^{-1}\frac{65}{63}\right)$

(v) $\sin\left(\frac{\pi}{6} + \cos^{-1}\frac{1}{4}\right)$ (vi) $\cos\left(\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{2}{3}\right)$

(vii) $\sec\left(\tan\left\{\tan^{-1}\left(-\frac{\pi}{3}\right)\right\}\right)$ (viii) $\cos \tan^{-1} \sin \cot^{-1}\left(\frac{1}{2}\right)$

G-2. Find the value of $\sin^{-1} (\cos(\sin^{-1}x)) + \cos^{-1} (\sin (\cos^{-1}x))$

G-3. If $\tan^{-1}x + \cot^{-1}\frac{1}{y} + 2\tan^{-1}z = \pi$, then prove that $x + y + 2z = xz^2 + yz^2 + 2xyz$

G-4. If $\cos^{-1}x + 2\sin^{-1}x + 3\cot^{-1}y + 4\tan^{-1}y = 4\sec^{-1}z + 5\text{cosec}^{-1}z$, then prove that $\sqrt{z^2 - 1} = \frac{\sqrt{1-x^2} - xy}{x + y\sqrt{1-x^2}}$

C-2. The function $f(x) = [x] + \frac{1}{2}$, $x \notin I$ is a/an (where [.] denotes greatest integer function)
(A) Even (B) odd
(C) neither even nor odd (D) Even as well as odd

C-3. The graph of the function y = f(x) is symmetrical about the line x = 2, then :
(A) f(x + 2) = f(x – 2) (B) f(2 + x) = f(2 – x) (C) f(x) = f(–x) (D) f(x) = –f(–x)

C-4. Fundamental period of f(x) = sec (sin x) is
(A) $\frac{\pi}{2}$ (B) 2π (C) π (D) aperiodic

C-5. If $f(x) = \sin\left(\sqrt{[a]}\, x\right)$ (where [.] denotes the greatest integer function) has π as its fundamental period, then
(A) a = 1 (B) a = 9 (C) $a \in [1, 2)$ (D) $a \in [4, 5)$

C-6. Find the area below the curve $y = \left[\sqrt{2+2\cos 2x}\right]$ but above the x-axis in $[-3\pi, 6\pi]$ is (where [.] denotes the greatest integer function) :
(A) 2π square units (B) π square units (C) 6π square units (D) 8π square units

Section (D) : Inverse of a function

D-1. The inverse of the function $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ is
(A) $\frac{1}{2}\ln\frac{1+x}{1-x}$ (B) $\frac{1}{2}\ln\frac{2+x}{2-x}$ (C) $\frac{1}{2}\ln\frac{1-x}{1+x}$ (D) $2\ln(1 + x)$

D-2. If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals:
(A) $\frac{x+\sqrt{x^2-4}}{2}$ (B) $\frac{x}{1+x^2}$ (C) $\frac{x-\sqrt{x^2-4}}{2}$ (D) $1-\sqrt{x^2-4}$

D-3. If $f : R \to R$ is an invertible function such that f(x) and $f^{-1}(x)$ are also mirror image to each other about the line y = –x, then
(A) f(x) is odd
(B) f(x) and $f^{-1}(x)$ may not be mirror image about the line y = x
(C) f(x) may not be odd
(D) f(x) is even

D-4. If $f(x) = \frac{ax+b}{cx+d}$, then (fof) (x) = x, provided that
(A) d + a = 0 (B) d – a = 0 (C) a = b = c = d = 1 (D) a = b = 1

D-5. Let $f(x) = \begin{cases} x & -1 \le x \le 1 \\ x^2 & 1 < x \le 2 \end{cases}$ the range of $h^{-1}(x)$, where h(x) = fof(x) is
(A) $\left[-1, \sqrt{2}\right]$ (B) [–1, 2] (C) [–1, 4] (D) [–2, 2]

D-6. Statement – 1 All points of intersection of y = f (x) and $y = f^{-1}(x)$ lies on y = x only.
Statement – 2 If point P (α, β) lies on y = f (x), then Q (β, α) lies on $y = f^{-1}(x)$.
Statement – 3 Inverse of invertible function is unique and its range is equal to the function domain.
Which of the following option is correct for above statements in order
(A) T T F (B) F T T (C) T T T (D) T F T

Section (E) : Definition, graphs and fundamentals of Inverse Trigonometric functions

E-1. The domain of definition of $f(x) = \sin^{-1}(|x - 1| - 2)$ is:
(A) $[-2, 0] \cup [2, 4]$ (B) $(-2, 0) \cup (2, 4)$ (C) $[-2, 0] \cup [1, 3]$ (D) $(-2, 0) \cup (1, 3)$

E-2. The function $f(x) = \cot^{-1}\sqrt{(x+3)x} + \cos^{-1}\sqrt{x^2+3x+1}$ is defined on the set S, where S is equal to:
(A) {0, 3} (B) (0, 3) (C) {0, – 3} (D) [– 3, 0]

E-3. Domain of $f(x) = \cos^{-1}x + \cot^{-1}x + \operatorname{cosec}^{-1}x$ is
(A) [– 1, 1] (B) R (C) $(-\infty, -1] \cup [1, \infty)$ (D) {– 1, 1}

E-4. Range of $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$ is
(A) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (B) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (C) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ (D) $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

E-5. $\operatorname{cosec}^{-1}(\cos x)$ is real if
(A) $x \in [-1, 1]$ (B) $x \in R$
(C) x is an odd multiple of $\frac{\pi}{2}$ (D) x is a multiple of π

E-6. Domain of definition of the function $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ for real valued 'x' is:
(A) $\left[-\frac{1}{4}, \frac{1}{2}\right]$ (B) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (C) $\left(-\frac{1}{2}, \frac{1}{9}\right)$ (D) $\left[-\frac{1}{4}, \frac{1}{4}\right]$

Section (F) : Trig. (trig⁻¹x), trig⁻¹ (trig x) trig (–x) and Properties

F-1. If $\pi \le x \le 2\pi$, then $\cos^{-1}(\cos x)$ is equal to
(A) x (B) $\pi - x$ (C) $2\pi + x$ (D) $2\pi - x$

F-2. If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1}y$ is equal to
(A) $\frac{2\pi}{3}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) π

F-3. If $x \ge 0$ and $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x$, then
(A) $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$ (B) $0 \le \theta \le \frac{\pi}{4}$ (C) $0 \le \theta < \frac{\pi}{2}$ (D) $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

F-4. If x < 0 then value of $\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$ is equal to
(A) $\frac{\pi}{2}$ (B) $-\frac{\pi}{2}$ (C) 0 (D) $-\pi$

F-5. If $\sin^{-1}x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to
(A) 0 (B) $\frac{1}{\sqrt{5}}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{3}}{2}$

Section (G) : Interconversion

G-1. The numerical value of $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$ is
(A) $\frac{-7}{17}$ (B) $\frac{7}{17}$ (C) $\frac{17}{7}$ (D) $-\frac{2}{3}$

G-2. The numerical value of $\cot\left(2\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{3}{5}\right)$ is

(A) $\frac{-4}{3}$ (B) $\frac{-3}{4}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$

G-3. STATEMENT-1 : $\tan^2(\sec^{-1}2)+\cot^2(\csc^{-1}3)=11$.
STATEMENT-2 : $\tan^2\theta+\sec^2\theta=1=\cot^2\theta+\csc^2\theta$.
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

G-4. If α is a real root of the equation $x^3+3x-\tan 2=0$ then $\cot^{-1}\alpha+\cot^{-1}\frac{1}{\alpha}-\frac{\pi}{2}$ can be equal to

(A) 0 (B) $\frac{\pi}{2}$ (C) π (D) $\frac{3\pi}{2}$

G-5. If $\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)+\sin^{-1}\left(\sqrt{1-\frac{x}{4}}\right)+\tan^{-1}y=\frac{2\pi}{3}$, then :

(A) maximum value of x^2+y^2 is $\frac{49}{3}$ (B) maximum value of x^2+y^2 is 4

(C) minimum value of x^2+y^2 is $\frac{1}{2}$ (D) minimum value of x^2+y^2 is 3

Section (H) : Additional Subtraction Rule and Equations

H-1. If $f(x)=\tan^{-1}\left(\frac{\sqrt{3}x-3x}{3\sqrt{3}+x^2}\right)+\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$, $0\le x\le 3$, then range of f(x) is

(A) $\left[0,\frac{\pi}{2}\right]$ (B) $\left[0,\frac{\pi}{4}\right]$ (C) $\left[\frac{\pi}{6},\frac{\pi}{3}\right]$ (D) $\left[0,\frac{\pi}{3}\right]$

H-2. The solution of the equation $\sin^{-1}\left(\tan\frac{\pi}{4}\right)-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)-\frac{\pi}{6}=0$ is

(A) x = 2 (B) x = – 4 (C) x = 4 (D) x = 3

H-3. Number of solutions of the equation $\cot^{-1}\sqrt{4-x^2}+\cos^{-1}(x^2-5)=\frac{3\pi}{2}$ is :

(A) 2 (B) 4 (C) 6 (D) 8

H-4. Number of solutions of equation $\tan^{-1}(e^{-x})+\cot^{-1}(|\ell nx|)=\pi/2$ is :

(A) 0 (B) 1 (C) 3 (D) 2

H-5. **STATEMENT-1 :** If $a>0$, $b>0$, $\tan^{-1}\left(\frac{a}{x}\right)+\tan^{-1}\left(\frac{b}{x}\right)=\frac{\pi}{2}\Rightarrow x=\sqrt{ab}$.

STATEMENT-2 : If $m, n\in N$, $n\ge m$, then $\tan^{-1}\left(\frac{m}{n}\right)+\tan^{-1}\left(\frac{n-m}{n+m}\right)=\frac{\pi}{4}$.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

H-6. If $\cos^{-1}x-\cos^{-1}\frac{y}{2}=\alpha$, then $4x^2-4xy\cos\alpha+y^2$ is equal to-

(A) $2\sin 2\alpha$ (B) 4 (C) $4\sin^2\alpha$ (D) $-4\sin^2\alpha$

PART - III : MATCH THE COLUMN

1.

Column – I		Column – II	
(A)	The period of the function $y=\sin(2\pi t+\pi/3)+2\sin(3\pi t+\pi/4)+3\sin 5\pi t$ is	(p)	1/2
(B)	$y=\{\sin(\pi x)\}$ is a many one function for $x\in(0,a)$, where {x} denotes fractional part of x, then a may be	(q)	8
(C)	The fundamental period of the function $y=\frac{1}{2}\left(\frac{\lvert\sin(\pi/4)x\rvert}{\cos(\pi/4)x}+\frac{\sin(\pi/4)x}{\lvert\cos(\pi/4)x\rvert}\right)$ is	(r)	2
(D)	If $f:[0,2]\to[0,2]$ is bijective function defined by $f(x)=ax^2+bx+c$, where a, b, c are non-zero real numbers, then f(2) is equal to	(s)	0

2. Let $f(x)=\sin^{-1}x$, $g(x)=\cos^{-1}x$ and $h(x)=\tan^{-1}x$. For what complete interval of variation of x the following are true.

Column – I		Column – II	
(A)	$f(\sqrt{x})+g(\sqrt{x})=\pi/2$	(p)	$[0,\infty)$
(B)	$f(x)+g\left(\sqrt{1-x^2}\right)=0$	(q)	[0, 1]
(C)	$g\left(\frac{1-x^2}{1+x^2}\right)=2h(x)$	(r)	$(-\infty, 1)$
(D)	$h(x)+h(1)=h\left(\frac{1+x}{1-x}\right)$	(s)	[– 1, 0]

3.

Column – I		Column – II	
(A)	If S be set of all triangles and $f:S\to R^+$, $f(\Delta)=$ Area of Δ, then f is	(p)	one-one B
(B)	$f:R\to\left[\frac{3\pi}{4},\pi\right)$ and $f(x)=\cot^{-1}(2x-x^2-2)$, then f(x) is	(q)	many one A C D
(C)	If $f:R\to R$ such that $f(x)=\frac{2x^2-x+1}{7x^2-4x+4}$, then f(x) is	(r)	onto function A B D
(D)	$f:R\to R$ and $f(x)=e^{px}\sin qx$ where $p, q\in R^+$, then f(x) is	(s)	into function C

4. Match The column

Column – I		Column – II	
(A)	If f(x) is even & g(x) is odd	(p)	then fog must be odd
(B)	If g(x) is periodic	(q)	then fog must be manyone
(C)	If f(x) & g(x) are bijective	(r)	then fog is periodic
(D)	If f(x) is into	(s)	then fog is injective
		(t)	then fog is into

5. Match the column

Column - I		Column - II	
(A)	Let a, b, c be three positive real numbers $\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$ then θ equal	(p)	π
(B)	The value of the expression $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$ for $0 < A < (\pi/4)$	(q)	$-\frac{\pi}{2}$
(C)	If $x < 0$, then $\frac{1}{2}\{\cos^{-1}(2x^2 - 1) + 2\cos^{-1} x\}$	(r)	$-\pi$
(D)	The value of $\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{16}{65}\right)$	(s)	$\frac{\pi}{2}$

Exercise-2

Marked Questions may have for Revision Questions.

PART - I : ONLY ONE OPTION CORRECT TYPE

1. The domain of the function $f(x) = \log_{1/2}\left(-\log_2\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$ is:

(A) $0 < x < 1$ (B) $0 < x \le 1$ (C) $x \ge 1$ (D) null set

2. If $q^2 - 4pr = 0$, $p > 0$, then the domain of the function $f(x) = \log(px^3 + (p + q)x^2 + (q + r)x + r)$ is:

(A) $R - \left\{-\frac{q}{2p}\right\}$ (B) $R - \left[(-\infty, -1] \cup \left\{-\frac{q}{2p}\right\}\right]$

(C) $R - \left[(-\infty, -1) \cap \left\{-\frac{q}{2p}\right\}\right]$ (D) R

3. Let $f(x) = \frac{x - [x]}{1 + x - [x]}$, $R \to A$ is onto then find set A. (where {.} and [.] represent fractional part and greatest integer part functions respectively)

(A) $\left(0, \frac{1}{2}\right]$ (B) $\left[0, \frac{1}{2}\right]$ (C) $\left[0, \frac{1}{2}\right)$ (D) $\left(0, \frac{1}{2}\right)$

4. Let f be a real valued function defined by $f(x) = \frac{e^x - e^{-|x|}}{e^x + e^{|x|}}$, then the range of f(x) is :

(A) R (B) [0, 1] (C) [0, 1) (D) $\left[0, \frac{1}{2}\right)$

5. The range of the function $f(x) = \log_{\sqrt{2}}\left(2 - \log_2\left(16\sin^2 x + 1\right)\right)$ is

(A) $(-\infty, 1)$ (B) $(-\infty, 2)$ (C) $(-\infty, 1]$ (D) $(-\infty, 2]$

6. Which of the following pair of functions are identical ?

(A) $\sqrt{1 + \sin x}$, $\sin\frac{x}{2} + \cos\frac{x}{2}$ (B) $x, \frac{x^2}{x}$

(C) $\sqrt{x^2}, (\sqrt{x})^2$ (D) $\ell n\, x^3 + \ell n\, x^2$, $5\,\ell n\, x$

7. If domain of f(x) is $(-\infty, 0]$, then domain of $f(6\{x\}^2 - 5\{x\} + 1)$ is (where {.} represents fractional part function).

(A) $\bigcup_{n \in I}\left[n + \frac{1}{3}, n + \frac{1}{2}\right]$ (B) $(-\infty, 0)$ (C) $\bigcup_{n \in I}\left[n + \frac{1}{6}, n + 1\right]$ (D) $\bigcup_{n \in I}\left[n - \frac{1}{2}, n - \frac{1}{3}\right]$

8. Let f: $(e, \infty) \to R$ be defined by $f(x) = \ell n(\ell n(\ell n\, x))$, then

(A) f is one one but not onto (B) f is onto but not one - one
(C) f is one-one and onto (D) f is neither one-one nor onto

9. If $f(x) = 2[x] + \cos x$, then f: $R \to R$ is: (where [ ] denotes greatest integer function)

(A) one–one and onto (B) one–one and into
(C) many–one and into (D) many–one and onto

10. If $f : R \to R$ be a function such that $f(x) = \begin{cases} x|x| - 4 & ; x \in Q \\ x|x| - \sqrt{3} & ; x \notin Q \end{cases}$, then f(x) is

(A) one-one, onto (B) many one, onto (C) one-one, into (D) many one, into

11. $f(x) = |x - 1|$, $f: R^+ \to R$, $g(x) = e^x$, $g: [-1, \infty) \to R$. If the function fog (x) is defined, then its domain and range respectively are:

(A) $(0, \infty)$ and $[0, \infty)$ (B) $[-1, \infty)$ and $[0, \infty)$
(C) $[-1, \infty)$ and $\left[1 - \frac{1}{e}, \infty\right)$ (D) $[-1, \infty)$ and $\left[\frac{1}{e} - 1, \infty\right)$

12. Let $f : (2, 4) \to (1, 3)$ be a function defined by $f(x) = x - \left[\frac{x}{2}\right]$ (where [.] denotes the greatest integer function), then $f^{-1}(x)$ is equal to :

(A) 2x (B) $x + \left[\frac{x}{2}\right]$ (C) x + 1 (D) x – 1

13. The mapping $f : R \to R$ given by $f(x) = x^3 + ax^2 + bx + c$ is a bijection if

(A) $b^2 \le 3a$ (B) $a^2 \le 3b$ (C) $a^2 \ge 3b$ (D) $b^2 \ge 3a$

14. If the function f: $[1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$ then f^{-1} is

(A) $(1/2)^{x(x-1)}$ (B) $\frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$
(C) $\frac{1}{2}\left(1 - \sqrt{1 + 4\log_2 x}\right)$ (D) Not defined

15. Let $f : N \to N$, where $f(x) = x + (-1)^{x-1}$, then the inverse of f is.

(A) $f^{-1}(x) = x + (-1)^{x-1}$, $x \in N$ (B) $f^{-1}(x) = 3x + (-1)^{x-1}$, $x \in N$
(C) $f^{-1}(x) = x$, $x \in N$ (D) $f^{-1}(x) = (-1)^{x-1}$, $x \in N$

16. $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}x\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}x\right)$, $x \ne 0$ is equal to

(A) x (B) 2x (C) $\frac{2}{x}$ (D) $\frac{x}{2}$

17. The value of $\cot^{-1}\left\{\frac{\sqrt{1 - \sin x} + \sqrt{1 + \sin x}}{\sqrt{1 - \sin x} - \sqrt{1 + \sin x}}\right\}$, where $\frac{\pi}{2} < x < \pi$, is:

(A) $\pi - \frac{x}{2}$ (B) $\frac{\pi}{2} + \frac{\pi}{2}$ (C) $\frac{\pi}{2}$ (D) $2\pi - \frac{\pi}{2}$

18. The domain of the function $f(x) = \sin^{-1}\left(\frac{1+x^3}{2x^{3/2}}\right) + \sqrt{\sin(\sin x)} + \log_{(3\{x\}+1)}(x^2+1)$, where {.} represents fractional part function, is:

(A) $x \in \{1\}$ (B) $x \in R - \{1, -1\}$ (C) $x > 3, x \neq I$ (D) $x \in \phi$

19. The complete solution set of the inequality $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \leq 0$, where [.] denotes greatest integer function, is

(A) $(-\infty, \cot 3]$ (B) $[\cot 3, \cot 2]$ (C) $[\cot 3, \infty)$ (D) $(-\infty, \cot 2]$

20. The inequality $\sin^{-1}(\sin 5) > x^2 - 4x$ holds for

(A) $x \in \left(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi}\right)$ (B) $x > 2 + \sqrt{9-2\pi}$

(C) $x < 2 - \sqrt{9-2\pi}$ (D) $x \in \phi$

21. A function g(x) satisfies the following conditions

(i) Domain of g is $(-\infty, \infty)$ (ii) Range of g is $[-1, 7]$

(iii) g has a period π and (iv) $g(2) = 3$

Then which of the following may be possible.

(A) $g(x) = 3 + 4\sin(n\pi + 2x - 4),\ n \in I$ (B) $g(x) = \begin{cases} 3 & ;\ x = n\pi \\ 3+4\sin x & ;\ x \neq n\pi \end{cases}$

(C) $g(x) = 3 + 4\cos(n\pi + 2x - 4),\ n \in I$ (D) $g(x) = 3 - 8\sin(n\pi + 2x - 4),\ n \in I$

22. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\ldots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\ldots\right) = \frac{\pi}{2}$ for $0 < |x| < \sqrt{2}$, then x equals

(A) 1/2 (B) 1 (C) – 1/2 (D) –1

23. $\cot^{-1}\left(\sqrt{\cos\alpha}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right) = x$. then sin x is equal to -

(A) $\tan^2\left(\frac{\alpha}{2}\right)$ (B) $\cot^2\left(\frac{\alpha}{2}\right)$ (C) $\tan\alpha$ (D) $\cot\left(\frac{\alpha}{2}\right)$

24. The Inverse trigonometric equation $\sin^{-1}x = 2\sin^{-1}\alpha$, has a solution for

(A) $-\frac{\sqrt{3}}{2} < \alpha < \frac{\sqrt{3}}{2}$ (B) all real values of α (C) $|\alpha| \leq \frac{1}{\sqrt{2}}$ (D) $|\alpha| \geq \frac{1}{\sqrt{2}}$

25. If $f(x) = \cot^{-1}x$ $: R^+ \to \left(0, \frac{\pi}{2}\right)$

and $g(x) = 2x - x^2$ $: R \to R$. Then the range of the function f(g(x)) wherever define is

(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left(0, \frac{\pi}{4}\right)$ (C) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ (D) $\left(\frac{\pi}{4}\right)$

26. Given the functions $f(x) = e^{\cos^{-1}\left(\sin\left(x+\frac{\pi}{3}\right)\right)}$, $g(x) = \operatorname{cosec}^{-1}\left(\frac{4-2\cos x}{3}\right)$ and the function h(x) = f(x) defined only for those values of x, which are common to the domains of the functions f(x) and g(x). The range of the function h(x) is :

(A) $[e^{\frac{\pi}{6}}, e^{\pi}]$ (B) $[e^{-\frac{\pi}{6}}, e^{\pi}]$ (C) $(e^{\frac{\pi}{6}}, e^{\pi})$ (D) $[e^{-\frac{\pi}{6}}, e^{\frac{\pi}{6}}]$

12. Let $f : [-\sqrt{2}+1, \sqrt{2}+1] \rightarrow \left[\frac{-\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2}\right]$ be a function defined by $f(x) = \frac{1-x}{1+x^2}$.

If $f^{-1}(x) = \begin{cases} \frac{-1+\lambda\left(\sqrt{4x-4x^2+1}\right)}{2x}, & x \neq 0 \\ \mu, & x = 0 \end{cases}$, then $\lambda + \mu$ is.

13. The number of real solutions of the equation $x^3 + 1 = 2\sqrt[3]{2x-1}$, is :

14. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, where $-1 \le x, y, z \le 1$, then find the value of $x^2 + y^2 + z^2 + 2xyz$

15. The sum of absolute value of all possible values of x for which $\cos\tan^{-1}\sin\cot^{-1}x = \sqrt{\frac{226}{227}}$.

16. If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in N$, then the maximum value of 'n' is:

17. If $x \in (0, 1)$ and $f(x) = \sec\left\{\tan^{-1}\left(\frac{\sin(\cos^{-1}x)+\cos(\sin^{-1}x)}{\cos(\cos^{-1}x)+\sin(\sin^{-1}x)}\right)\right\}$, then $\sum_{r=2}^{10} f\left(\frac{1}{r}\right)$ is

18. If $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) = \frac{\pi}{4}$, then $\tan\theta$ is equal to

19. The number of real solutions of equation $\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x)$, $-10\pi \le x \le 10\pi$, is

20. The number of solution(s) of the equation, $\sin^{-1}x + \cos^{-1}(1-x) = \sin^{-1}(-x)$, is

21. Find the value of $3\sum_{n=1}^{\infty}\left\{\frac{1}{\pi}\sum_{k=1}^{\infty}\cot^{-1}\left(1+2\sqrt{\sum_{r=1}^{k}r^3}\right)\right\}^n$

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. If $f(x) = \sin \ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)$, then

(A) domain of f(x) is $(-2, 1)$
(B) domain of f(x) is $[-1, 1]$
(C) range of f(x) is $[-1, 1]$
(D) range of f(x) is $[-1, 1)$

2. D is domain and R is range of $f(x) = \sqrt{x-1} + 2\sqrt{3-x}$, then

(A) D : $[1, 3]$;
(B) D : $(-\infty, 1] \cup [3, \infty)$,
(C) R : $[1, \sqrt{3}]$
(D) R : $[\sqrt{2}, \sqrt{10}]$

3. If $[2\cos x] + [\sin x] = -3$, then the range of the function, $f(x) = \sin x + \sqrt{3}\cos x$ in $[0, 2\pi]$ lies in (where [ ] denotes greatest integer function)

(A) $[-\sqrt{3}, \sqrt{3})$
(B) $[-2, -\sqrt{3}]$
(C) $[-3, -1]$
(D) $[-2, -\sqrt{3})$

4. Let $D \equiv [-1, 1]$ is the domain of the following functions, state which of them are injective.

(A) $f(x) = \begin{cases} \tan^{-1}\frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$
(B) $g(x) = x^3$
(C) $h(x) = \sin 2x$
(D) $k(x) = \sin(\pi x/2)$

5. Let $f(x) = x^{135} + x^{125} - x^{115} + x^5 + 1$. If f(x) divided by $x^3 - x$, then the remainder is some function of x say g(x). Then g(x) is an :

(A) one-one function
(B) many one function
(C) into function
(D) onto function

6. The function $f : X \rightarrow Y$, defined by $f(x) = x^2 - 4x + 5$ is both one–one and onto if

(A) $X = [2, \infty)$ & $Y = [1, \infty)$
(B) $X = (-\infty, 2]$ & $Y = [1, \infty)$
(C) $X = [3, \infty)$ & $Y = [2, \infty)$
(D) $X = (-\infty, 2]$ & $Y = (1, \infty)$

7. $f : N \rightarrow N$ where $f(x) = x - (-1)^x$ then f is :

(A) one-one
(B) many-one
(C) onto
(D) into

8. Which one of the following pair of functions are **NOT** identical ?

(A) $e^{(\ell nx)/2}$ and $\sqrt{x}$
(B) tan (tanx) and cot (cotx)
(C) $\cos^2x + \sin^4x$ and $\sin^2x + \cos^4x$
(D) $\frac{|x|}{x}$ and sgn (x), where sgn(x) stands for signum function.

9. If the graph of the function $f(x) = \frac{a^x - 1}{x^n(a^x+1)}$ is symmetric about y-axis, then n is equal to:

(A) 1/5
(B) 1/3
(C) 1 / 4
(D) – 1 / 3

10. If $f(x) = \begin{cases} x^2 & x \le 1 \\ 1-x & x > 1 \end{cases}$ & composite function $h(x) = |f(x)| + f(x+2)$, then

(A) $h(x) = 2x^2 + 4x + 4$ $\forall x \le -1$
(B) $h(x) = x^2 + x + 1$ $\forall -1 < x \le 1$
(C) $h(x) = x^2 - x - 1$ $\forall -1 < x \le 1$
(D) $h(x) = -2$ $\forall x > 1$

11. Let $f(x) = \begin{cases} 0 & \text{for } x = 0 \\ x^2\sin\left(\frac{\pi}{x}\right) & \text{for } -1 < x < 1\ (x \neq 0) \\ x\,|x| & \text{for } x > 1 \text{ or } x < -1 \end{cases}$, then:

(A) f(x) is an odd function
(B) f(x) is an even function
(C) f(x) is neither odd nor even
(D) f ' (x) is an even function

12. If $f : [-2, 2] \rightarrow R$ where $f(x) = x^3 + \tan x + \left[\frac{x^2+1}{P}\right]$ is a odd function, then the value of parametric P, where [.] denotes the greatest integer function, can be

(A) $5 < P < 10$
(B) $P < 5$
(C) $P > 5$
(D) $P = 15$

13. If $f: R \rightarrow [-1, 1]$, where $f(x) = \sin\left(\frac{\pi}{2}[x]\right)$, (where [.] denotes the greatest integer function), then

(A) f (x) is onto
(B) f (x) is into
(C) f (x) is periodic
(D) f (x) is many one

14. If $f(x) = \dfrac{2x\,(\sin x + \tan x)}{2\left[\dfrac{x+2\pi}{\pi}\right] - 3}$ then it is, (where [.] denotes the greatest integer function)
(A) odd (B) Even (C) many one (D) one-one

15. Identify the statement(s) which is/are incorrect ?
(A) the function $f(x) = \sin x + \cos x$ is neither odd nor even
(B) the fundamental period of $f(x) = \cos(\sin x) + \cos(\cos x)$ is π
(C) the range of the function $f(x) = \cos(3\sin x)$ is $[-1, 1]$
(D) $f(x) = 0$ is a periodic function with period 2

16. If $F(x) = \dfrac{\sin \pi [x]}{\{x\}}$, then F (x) is: (where { . } denotes fractional part function and [ ] denotes greatest integer function and sgn (x) is a signum function)
(A) periodic with fundamental period 1 (B) even
(C) range is singleton (D) identical to $\text{sgn}\left(\text{sgn}\dfrac{\{x\}}{\sqrt{\{x\}}}\right) - 1$

17. Let $f : R \to R$ and $g : R \to R$ be two one-one and onto functions such that they are mirror images of each other about the line $y = a$. If $h(x) = f(x) + g(x)$, then h(x) is
(A) one-one (B) into (C) onto (D) many-one

18. Which of following pairs of functions are identical.
(A) $f(x) = e^{\ell n \sec^{-1} x}$ and $g(x) = \sec^{-1} x$
(B) $f(x) = \tan(\tan^{-1} x)$ and $g(x) = \cot(\cot^{-1} x)$
(C) $f(x) = \text{sgn}(x)$ and $g(x) = \text{sgn}(\text{sgn}(x))$
(D) $f(x) = \cot^2 x \cdot \cos^2 x$ and $g(x) = \cot^2 x - \cos^2 x$

19. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{3\pi}{2}$, then
(A) $x^{100} + y^{100} + z^{100} - \dfrac{9}{x^{101} + y^{101} + z^{101}} = 0$ (B) $x^{22} + y^{42} + z^{62} - x^{220} - y^{420} - z^{620} = 0$
(C) $x^{50} + y^{25} + z^{5} = 0$ (D) $\dfrac{x^{2008} + y^{2008} + z^{2008}}{(xyz)^{2009}} = 0$)

20. If $X = \text{cosec}\,\tan^{-1}\cos\cot^{-1}\sec\sin^{-1} a$ & $Y = \sec\cot^{-1}\sin\tan^{-1}\text{cosec}\cos^{-1} a$; where $0 \le a < 1$. Find the relation between X & Y. Then
(A) X = Y (B) $Y = \sqrt{3 - a^2}$ (C) $X \ne Y$ (D) X = 2Y

21. If α satisfies the inequation $x^2 - x - 2 > 0$, then a value exists for
(A) $\sin^{-1}\alpha$ (B) $\cos^{-1}\alpha$ (C) $\sec^{-1}\alpha$ (D) $\text{cosec}^{-1}\alpha$

22. For the function $f(x) = \ell n(\sin^{-1}\ell og_2 x)$,
(A) Domain is $\left[\dfrac{1}{2}, 2\right]$ (B) Range is $\left(-\infty, \ell n\dfrac{\pi}{2}\right]$
(C) Domain is (1, 2] (D) Range is R

23. In the following functions defined from [–1, 1] to [–1, 1] , then functions which are not bijective are
(A) $\sin(\sin^{-1}x)$ (B) $\dfrac{2}{\pi}\sin^{-1}(\sin x)$ (C) $(\text{sgn}\,x)\,\ell n\,e^x$ (D) $x^3\,\text{sgn}\,x$

24. The expression $\dfrac{1}{\sqrt{2}}\left\{\dfrac{\sin\cot^{-1}\cos\tan^{-1}t}{\cos\tan^{-1}\sin\cot^{-1}\sqrt{2}\,t}\right\}\cdot\left\{\sqrt{\dfrac{1+2t^2}{2+t^2}}\right\}$ can take the value
(A) 1/2 (B) –5 (C) 1 (D) 3/4

25. If $0 < x < 1$, then $\tan^{-1}\dfrac{\sqrt{1-x^2}}{1+x}$ is equal to:
(A) $\dfrac{1}{2}\cos^{-1}x$ (B) $\cos^{-1}\sqrt{\dfrac{1+x}{2}}$ (C) $\sin^{-1}\sqrt{\dfrac{1-x}{2}}$ (D) $\dfrac{1}{2}\tan^{-1}\sqrt{\dfrac{1+x}{1-x}}$

26. If $f(x) = \cos^{-1}x + \cos^{-1}\left\{\dfrac{x}{2} + \dfrac{1}{2}\sqrt{3-3x^2}\right\}$, then
(A) $f\left(\dfrac{2}{3}\right) = \dfrac{\pi}{3}$ (B) $f\left(\dfrac{2}{3}\right) = \dfrac{\pi}{2}$
(C) $f\left(\dfrac{1}{3}\right) = \dfrac{\pi}{3}$ (D) $f\left(\dfrac{1}{3}\right) = 2\cos^{-1}\dfrac{1}{3} - \dfrac{\pi}{3}$

27. $\displaystyle\sum_{n=1}^{\infty}\tan^{-1}\dfrac{4n}{n^4 - 2n^2 + 2}$ is equal to:
(A) $\tan^{-1}2 + \tan^{-1}3$ (B) $4\tan^{-1}1$ (C) $\pi/2$ (D) $\sec^{-1}(-\sqrt{2})$

28. If $\sin^2(2\cos^{-1}(\tan x)) = 1$ then x may be
(A) $x = \pi + \tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ (B) $x = \pi - \tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$
(C) $x = -\pi + \tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ (D) $x = -\pi - \tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

29. If $\sin^{-1}x + 2\cot^{-1}(y^2 - 2y) = 2\pi$, then
(A) $x + y = y^2$ (B) $x^2 = x + y$ (C) $y = y^2$ (D) $x^2 - x + y = y^2$

PART - IV : COMPREHENSION

Comprehension # 1

Given a function $f : A \to B$; where A = {1, 2, 3, 4, 5} and B = {6, 7, 8}

1. Find number of all such functions y = f(x) which are one-one ?
(A) 0 (B) 3^5 (C) ${}^5P_3$ (D) 5^3

2. Find number of all such functions y = f(x) which are onto
(A) 243 (B) 93 (C) 150 (D) none of these

3. The number of mappings of $g(x) : B \to A$ such that $g(i) \le g(j)$ whenever $i < j$ is
(A) 60 (B) 140 (C) 10 (D) 35

Comprehension # 2

If the period of a function f(x) is T. Then if we find the range of function over one period then it would be the complete range of the function.

4. Range of the function $f(x) = |\sin x\,|\cos x|| + \cos x\,|\sin x||$ is
(A) {0, 1} (B) [0, 1] (C) [0, 2] (D) [1, 2]

5. Range of the function $f(x) = \log_2 \left[3x - \left[x + \left[x + \left[x\right]\right]\right]\right]$ is
(where [·] is greatest integer function)
(A) [0, 1] (B) {0, 1} (C) (0, 1] (D) [0, 1)

6. If $f(x) = \frac{1}{x^2+1}$ and $g(x) = \sin \pi x + 8\left\{\frac{x}{2}\right\}$ where {·} denotes fractional part function then the range of f(g(x)) is
(A) R (B) [0, 1] (C) $\left(\frac{1}{65}, 1\right]$ (D) $\left[\frac{1}{65}, 1\right]$

Comprehension # 3

Let the domain and range of inverse circular functions are defined as follows

	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$

7. $\sin^{-1}x < \frac{3\pi}{4}$ then solution set of x is
(A) $\left(\frac{1}{\sqrt{2}}, 1\right]$ (B) $\left(-\frac{1}{\sqrt{2}}, -1\right]$ (C) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (D) none of these

8. If $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\text{cosec}^{-1}\,\text{cosec}\, x$ is
(A) $2\pi - x$ (B) $\pi + x$ (C) $\pi - x$ (D) $-\pi - x$

9. If $x \in [-1, 1]$, then range of $\tan^{-1}(-x)$ is
(A) $\left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$ (B) $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ (C) $[-\pi, 0]$ (D) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Exercise-3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1. If $0 < x < 1$, then $\sqrt{1+x^2}\,[\{x \cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$ **[IIT-JEE 2008, Paper-1, (3, –1), 82]**
(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

2. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \le 9x\}$ is **[IIT-JEE 2009, P-2, (4, –1), 80]**

3. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is **[IIT-JEE 2009, P-2, (4, –1), 80]**

4. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f(g(x))$, is **[IIT-JEE 2011, Paper-2, (3, –1), 80]**
(A) $\pm\sqrt{n\pi}$, $n \in \{0, 1, 2, ...\}$ (B) $\pm\sqrt{n\pi}$, $n \in \{1, 2, ...\}$
(C) $\frac{\pi}{2} + 2n\pi$, $n \in \{.....-2, -1, 0, 1, 2, ...\}$ (D) $2n\pi$, $n \in \{...., -2, -1, 0, 1, 2, ...\}$

5. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is **[IIT-JEE 2011, Paper-1, (4, 0), 80]**

6. The function $f : [0, 3] \to [1, 29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is **[IIT-JEE 2012, Paper-1, (3, –1), 70]**
(A) one-one and onto (B) onto but not one-one
(C) one-one but not onto (D) neither one-one nor onto

7*. Let $f : (-1, 1) \to IR$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ is (are) **[IIT-JEE 2012, Paper-2, (4, 0), 66]**
(A) $1 - \sqrt{\frac{3}{2}}$ (B) $1 + \sqrt{\frac{3}{2}}$ (C) $1 - \sqrt{\frac{2}{3}}$ (D) $1 + \sqrt{\frac{2}{3}}$

8. The value of $\cot\left(\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right)$ is **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**
(A) $\frac{23}{25}$ (B) $\frac{25}{23}$ (C) $\frac{23}{24}$ (D) $\frac{24}{23}$

9. Match List I with List II and select the correct answer using the code given below the lists :

	List - I		List - II
P.	$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)}\right)^2 + y^4\right)^{1/2}$ takes value	1.	$\frac{1}{2}\sqrt{\frac{5}{3}}$
Q.	If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then possible value of $\cos\frac{x-y}{2}$ is	2.	$\sqrt{2}$
R.	If $\cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$ then possible value of sec x is	3.	$\frac{1}{2}$
S.	If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$, $x \ne 0$, then possible value of x is	4.	1

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

Codes :

	P	Q	R	S
(A)	4	3	1	2
(B)	4	3	2	1
(C)	3	4	2	1
(D)	3	4	1	2

10. Let $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ be given by $f(x) = (\log(\sec x + \tan x))^3$. Then

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) f(x) is an odd function (B) f(x) is a one-one function
(C) f(x) is an onto function (D) f(x) is an even function

11. Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

12. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

(A) $\cos\beta > 0$ (B) $\sin\beta < 0$ (C) $\cos(\alpha+\beta) > 0$ (D) $\cos\alpha < 0$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$ where $Y = \{y \in N : y = 4x + 3$ for some $x \in N\}$. Its inverse is - **[AIEEE 2008 (3, –1), 105]**

(1) $g(y) = 4 + \frac{y+3}{4}$ (2) $g(y) = \frac{y+3}{4}$ (3) $g(y) = \frac{y-3}{4}$ (4) $g(y) = \frac{3y+3}{3}$

2. The value of $\cot\left(\text{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right)$ is **[AIEEE 2008 (3, –1), 105]**

(1) $\frac{3}{17}$ (2) $\frac{2}{17}$ (3) $\frac{5}{17}$ (4) $\frac{6}{17}$

6. The domain of the function $f(x) = \frac{1}{\sqrt{|x|-x}}$ is : **[AIEEE 2011, I, (4, –1), 120]**

(1) $(-\infty, \infty)$ (2) $(0, \infty)$ (3) $(-\infty, 0)$ (4) $(-\infty, \infty) - \{0\}$

4. Let f be a function defined by $f(x) = (x-1)^2 + 1, (x \ge 1)$. **[AIEEE 2011, II, (4, –1), 120]**
Statement - 1 : The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$.
Statement - 2 : f is a bijection and $f^{-1}(x) = 1 + \sqrt{x-1}, x \ge 1$.
(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(2) Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1
(3) Statement-1 is true, Statement-2 is false
(4) Statement-1 is false, Statement-2 is true .

5. If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then **[AIEEE - 2013, (4, –1),120]**
(1) $x = y = z$ (2) $2x = 3y = 6z$ (3) $6x = 3y = 2z$ (4) $6x = 4y = 3z$

6. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ equal to : **[JEE(Main) 2014, (4, – 1), 120]**

(1) $\frac{1}{1+\{g(x)\}^5}$ (2) $1 + \{g(x)\}^5$ (3) $1 + x^5$ (4) $5x^4$

7. Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is **[JEE(Main)2015,(4, –1),120]**

(1) $\frac{3x-x^3}{1-3x^2}$ (2) $\frac{3x+x^3}{1-3x^2}$ (3) $\frac{3x-x^3}{1+3x^2}$ (4) $\frac{3x+x^3}{1+3x^2}$

8. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x, x \ne 0$, and $S = \{x \in R : f(x) = f(-x)\}$; then S : **[JEE(Main) 2016, (4, – 1), 120]**

(1) contains exactly one element (2) contains exactly two elements.
(3) contains more than two elements. (4) is an empty set.

Answers

EXERCISE - 1

PART - I

Section (A) :

A-1. (i) yes (ii) no (iii) no (iv) no

A-2. (i) $R - \{-1, 1\}$
(ii) $2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2}, n \in I$
(iii) $(0, \infty)$ (iv) R (v) $[-2, 0) \cup (0, 1)$
(vi) $(2, 3]$ (vii) $(-\infty, -1] \cup [0, \infty)$
(viii) $\left(-\frac{1}{6}, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 6\right)$

A-3. (i) [0, 1] (ii) [– 1, 1] (iii) R (iv) ϕ
(v) $\bigcup_{n\in I}\left[n\pi, n\pi+\frac{\pi}{4}\right]$ (vi) $R - \{2n\pi\}, n \in I$
(vii) $(0, 1] \cup [4,5)$ (viii) (2, 3)

A-4. (i) $[0, \infty)$ (ii) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(iii) [0, 4] (iv) $\{-1, 1\}$

A-5. (i) Domain : R, Range : $\frac{1}{\sqrt{7}} \le y \le 1$
(ii) Domain : $N\cup\{0\}$, Range : {n! : n=0,1,2,...}
(iii) Domain $R - \{3\}$, Range : $R - \{6\}$
(iv) Domain : R, Range : {1}

A-6. (i) [0, 10] (ii) (0, 1] (iii) $\left(-\infty, \frac{49}{20}\right]$
(iv) [–4, 3] (v) [–1, 1] (vi) R
(vii) $n \in N$

A-7. (i) $(-\infty, 1]$ (ii) $\left[\frac{1}{\sqrt{16-1/\sqrt{2}}}, \infty\right)$
(iii) $\left[\frac{1}{3}, 1\right]$ (iv) $\left(-\infty, -\frac{1}{4}\right] \cup \left[-\frac{1}{20}, \infty\right)$
(v) $\left[\frac{1}{3}, 3\right]$ (vi) $\left[0, \frac{3}{\sqrt{2}}\right]$
(vii) $[4, \infty)$ (viii) [–11, 16] (ix) $\left[\frac{3}{4}, 1\right]$
(x) 1 (xi) $\left[1-\sin\sqrt{2}, 1+\sin\sqrt{2}\right]$

A-8. (i) many-one & into (ii) many-one & into
(iii) one-one & onto (iv) many-one & into
(v) one – one & into (vi) many-one & into
(vii) many-one & into (viii) many-one & onto
(ix) many-one & into

A-9. (i) bijective (injective as well as surjective)
(ii) neither surjective nor injective
(iii) surjective but not injective

A-10. (i) many-one & onto
(ii) many-one & into

A-11. (i) No (ii) Yes (iii) Yes (iv) No

A-12. n^n , n!

Section (B) :

B-1. (i) No (ii) No (iii) No (iv) No

B-2. $(2, \infty)$

B-4. $[fo(goh)](x) = [(fog)oh](x) = \sin^2\sqrt{x}$

B-5. (i) fog = x, x > 0 ; gof = x, $x \in R$
(ii) |sin x|, sin |x|
(iii) $\sin(x^2)$, $(\sin x)^2$
(iv) $\frac{3x^2-4x+2}{(x-1)^2}$, $\frac{x^2+2}{x^2+1}$

B-6. Domain : [1 , 2] ; Range : [ℓn2 , ℓn4)

B-7. $f(g(x)) = \begin{cases} 2-2x+x^2, & 0 \le x \le 1 \\ 2-x, & -1 \le x < 0 \end{cases}$

B-8. (i) $x \in R - \{0, 1\}$ (ii) $x \in R - \{-2, -1\}$
(iii) $x \in R - \left\{-\frac{3}{2}, -1\right\}$ (iv) $x \in R - \{-2, -1\}$

B-9. $fof(x) = \begin{cases} 3\sqrt{2}\,x & x \in Q-\{0\} \\ 3^2\,x & x \in Q^c \end{cases}$,

$fofof.....f(x) = \begin{cases} 3^{n-1}\sqrt{2}x & x \in Q-\{0\} \\ 3^n x & x \in Q^c \end{cases}$

B-10. $f(g(x)) = \begin{cases} x^2+1 & x \in [-1, 2] \\ 2x^2+1 & x \in (2, 3) \\ 2x+5 & x \in [3, 5] \end{cases}$

Section (C) :

C-1. (i) even, (ii) neither even nor odd
(iii) even, (iv) neither even nor odd
(v) even (vi) even

C-2. (i) neither even nor odd
(ii) even (iii) odd (iv) even

C-4. (i) 2π (ii) 2π (iii) 24
(iv) 70π (v) $\frac{2\pi}{3}$ (vi) 2π
(vii) $\pi/6$ (viii) 2π

Section (D) :

D-1. (i) f^{-1} Does not exists
(ii) $f^{-1} : R \to R ; f^{-1} = 7 + (4 - x^5)^{1/3}$
(iii) $f^{-1} : R \to R ; f^{-1} = \frac{e^x - e^{-x}}{2}$
(iv) $f^{-1}(x) = \frac{-1 + \sqrt{4x-3}}{2}$

D-2. $f^{-1} : R \to R, f^{-1}(x) = \frac{1}{2}\,\ell n\,(x + \sqrt{x^2+1})$

D-3. (a) $f^{-1}(x) = \begin{cases} \sqrt{-x} & x \le 0 \\ -\sqrt{x} & x > 0 \end{cases}$, 3
(b) $x = \frac{3-\sqrt{5}}{2}$

D-4. 1 **D-5.** $\alpha = 2$

Section (E) :

E-1. (i) $[-1, 1] - \{0\}$ (ii) $\left[-\frac{1}{3}, \frac{1}{2}\right]$ (iii) ϕ

E-2. (i) $(-\infty, \ell n\, \pi/2]$ (ii) $(0, \pi/2]$ (iii) $[0, \pi]$

E-3. (i) 1 (ii) $\frac{1}{\sqrt{3}}$ (iii) $\frac{\pi}{6}$

E-4. (i) $n\left(\frac{n+1}{2}\right)$

E-5. (i) $[-1, 0)$ (ii) $x > 1$
(iii) $(-\infty, \cot 3) \cup (\cot 2, \infty)$

Section (F) :

F-1. (i) $-\frac{\pi}{6}$ (ii) $-\frac{\pi}{3}$ (iii) $\frac{3\pi}{4}$ (iv) $\frac{\pi}{4}$

F-2. (i) $\pi - 4$ (ii) $4\pi - 10$ (iii) $2\pi - 6$
(iv) $4\pi - 10$ (v) $\frac{17\pi}{20}$

F-3. (i) 0 (ii) 1 (iii) $\frac{1}{2\sqrt{2}}$

Section (G) :

G-1. (i) $\frac{4}{5}$ (ii) $2\sqrt{2}$ (iii) $\frac{\sqrt{41}}{4}$ (iv) $\frac{63}{16}$
(v) $\frac{1+3\sqrt{5}}{8}$ (vi) $\frac{6-4\sqrt{5}}{15}$ (vii) 2
(viii) $\frac{\sqrt{5}}{3}$ **G-2.** $\frac{\pi}{2}$

Section (H) :

H-2. (i) $\pm\frac{1}{\sqrt{3}}$ (ii) $x = 3$

H-3. (i) $\pm\frac{1}{\sqrt{2}}$ (ii) $x = \frac{1}{2}$

Section (I) :

I-2. (i) $-\sin 1 < x \le 1$ (ii) $\cos 2 < x \le 1$
(iii) no solution

I-3. (i) $2\tan^{-1}x - \pi$ (ii) $\pi - 2\sin^{-1}x$
(iii) $2\pi - 2\cos^{-1}x$

I-4. $\frac{1+xy}{x-y}$

I-5. $B = [0, 4] ; f^{-1}(x) = \frac{1}{2}\left(\sin^{-1}\left(\frac{x-2}{2}\right) - \frac{\pi}{6}\right)$

I-6. (i) $\tan^{-1}(x+n) - \tan^{-1}x$ (ii) $\frac{\pi}{4}$ (iii) $\frac{\pi}{2}$

PART - II

Section (A) :

A-1.	(D)	A-2.	(A)	A-3.	(B)
A-4.	(B)	A-5.	(D)	A-6.	(A)
A-7.	(D)	A-8.	(B)	A-9.	(C)
A-10.	(B)	A-11.	(A)	A-12.	(D)
A-13.	(A)				

Section (B) :

B-1.	(A)	B-2.	(C)	B-3.	(B)

Section (C) :

C-1.	(B)	C-2.	(B)	C-3.	(B)
C-4.	(C)	C-5.	(D)	C-6.	(C)

Section (D) :

D-1.	(A)	D-2.	(A)	D-3.	(A)
D-4.	(A)	D-5.	(A)	D-6.	(B)

Section (E) :

E-1.	(A)	E-2.	(C)	E-3.	(D)
E-4.	(C)	E-5.	(D)	E-6.	(A)

Section (F) :

F-1.	(D)	F-2.	(B)	F-3.	(D)
F-4.	(B)	F-5.	(B)		

Section (G) :

G-1.	(A)	G-2.	(B)	G-3.	(C)
G-4.	(C)	G-5.	(A)		

Section (H) :

H-1.	(B)	H-2.	(C)	H-3.	(A)
H-4.	(D)	H-5.	(B)	H-6.	(C)

PART - III

1. (A)→(q,r), (B)→(q,r), (C)→(q), (D)→(s)
2. (A)→(q), (B)→(s), (C)→(p), (D)→(r),
3. (A)→(q,r), (B)→(q,r), (C)→(q,s),(D)→(q,r),
4. (A→ q ; B→ r,q ; C→ s ; D→ t)
5. (A)→(p), (B)→ (p), (C)→(p), (D)→(s)

EXERCISE - 2

PART - I

1.	(D)	2.	(B)	3.	(C)
4.	(D)	5.	(D)	6.	(D)
7.	(A)	8.	(C)	9.	(C)
10.	(D)	11.	(B)	12.	(C)
13.	(B)	14.	(B)	15.	(A)
16.	(C)	17.	(B)	18.	(D)
19.	(A)	20.	(A)	21.	(A)
22.	(B)	23.	(A)	24.	(C)
25.	(C)	26.	(A)		

PART - II

1.	17	2.	17	3.	0
4.	15	5.	2	6.	1
7.	34	8.	7	9.	20
10.	35	11.	22	12.	2
13.	3	14.	1	15.	30
16.	5	17.	54	18.	3
19.	20	20.	1	21.	1

PART - III

1.	(AC)	2.	(AD)	3.	(BCD)
4.	(BD)	5.	(AD)	6.	(ABC)
7.	(AC)	8.	(ABD)	9.	(ABD)
10.	(ACD)	11.	(AD)	12.	(ACD)
13.	(BCD)	14.	(AC)	15.	(BC)
16.	(ABCD)	17.	(BD)	18.	(BCD)
19.	(AB)	20.	(AB)	21.	(CD)
22.	(BC)	23.	(BCD)	24.	(AD)
25.	(ABC)	26.	(AD)	27.	(AD)
28.	(ABCD)	29.	(CD)		

PART - IV

1.	(A)	2.	(C)	3.	(D)
4.	(B)	5.	(B)	6.	(C)
7.	(A)	8.	(C)	9.	(B)

EXERCISE - 3

PART - I

1.	(C)	2.	7	3.	2
4.	(A)	5.	(1)	6.	(B)
7*.	(AB)	8.	(B)	9.	(B)
10.	(ABC)	11.	(3)	12.	(B,C,D)

PART - II

1.	(3)	2.	(4)	6.	(3)
4.	(1)	5.	(1)	6.	(2)
7.	(1)	8.	(2)		

High Level Problems (HLP)

SUBJECTIVE QUESTIONS

1. Find the domain of the function $f(x) = \sqrt{-\log_{\frac{x+4}{2}}\left(\log_2 \frac{2x-1}{3+x}\right)}$

2. Let $f(x) = (x^{12} - x^9 + x^4 - x + 1)^{-1/2}$. The domain of the function is :

3. Find the values of 'a' in the domain of the definition of the function, $f(a) = \sqrt{2a^2 - a}$ for which the roots of the equation, $x^2 + (a+1)x + (a-1) = 0$ lie between -2 & 1.

4. The domain of the function $f(x) = \sqrt{\dfrac{1}{(|x|-1)\cos^{-1}(2x+1) \cdot \tan 3x}}$ is:

5. Find domain of the following functions

(i) $f(x) = \sqrt{\log_{1/3}\log_4([x]^2 - 5)}$, where [.] denotes greatest integer function.

(ii) $f(x) = \dfrac{1}{[|x-1|] + [|12-x|] - 11}$, where [x] denotes the greatest integer not greater than x.

(iii) $f(x) = (x+0.5)^{\log_{(0.5+x)}\frac{x^2+2x-3}{4x^2-4x-3}}$

(iv) $f(x) = \dfrac{5}{\left[\frac{x-1}{2}\right]} - 3^{\sin^{-1}x^2} + \dfrac{(7x+1)!}{\sqrt{x+1}}$, where [.] denotes greatest integer function.

(v) $3^y + 2^{x^4} = 2^{4x^2-1}$

The range of the function $f(x) = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$, where [ ] is the greatest integer function, is:

Find the range of $f(x) = \dfrac{1}{2\{-x\}} - \{x\}$, (where {.} represents fractional part of x)

If $f : R \to R$; $f(x) = \dfrac{\sqrt{x^2+1} - 3x}{\sqrt{x^2+1} + x}$ then find the range of f(x).

If a function is defined as $f(x) = \sqrt{\log_{h(x)} g(x)}$, where $g(x) = |\sin x| + \sin x$, $h(x) = \sin x + \cos x$, $0 \le x \le \pi$. Then find the doman of f(x).

10. Find the domain and range of the following functions.

(i) $f(x) = \cos^{-1}\sqrt{\log_{[x]}\frac{|x|}{x}}$, where [.] denotes the greatest integer function .

(ii) $f(x) = \sqrt{\log_{1/2}\log_2[x^2+4x+5]}$ where [.] denotes the greatest integer function

(iii) $f(x) = \sin^{-1}\left[\log_2\left(\frac{x^2}{2}\right)\right]$, where [.] denotes greatest integer function .

(iv) $f(x) = \log_{[x-1]}\sin x$, where [] denotes greatest integer function .

(v) $f(x) = \tan^{-1}\left(\sqrt{[x]+[-x]}\right) + \sqrt{2-|x|} + \frac{1}{x^2}$, (where [ ] denotes greatest integer function)

11. If $f(x) = \dfrac{\sin^2 x + 4\sin x + 5}{2\sin^2 x + 8\sin x + 8}$, then range of f(x) is

12. Consider the function g(x) defined as $g(x)\left(x^{(2^{2011}-1)} - 1\right) = (x+1)(x^2+1)(x^4+1)\ldots\ldots\ldots(x^{2^{2010}}+1) - 1$ $(|x| \ne 1)$. Then the value of g(2) is equal to

13. It is given that f(x) is a function defined on N, satisfying f(1) = 1 and for any $x \in N$

$f(x+5) \ge f(x) + 5$ and $f(x+1) \le f(x) + 1$

If $g(x) = f(x) + 1 - x$, then g(2016) equals

14. Find the integral solutions to the equation $[x]\,[y] = x + y$. Show that all the non-integral solutions lie on exactly two lines. Determine these lines. Here [.] denotes greatest integer function.

15. Let $f(x) = Ax^2 + Bx + C$, where A, B, C are real numbers. Prove that if f(x) is an integer whenever x is integer, then the numbers 2A, A + B and C are all integers. Conversely, prove that if the numbers 2A, A + B and C are all integer then f(x) is an integer whenever x is an integer.

16. Let $g : R \to (0, \pi/3]$ is defined by $g(x) = \cos^{-1}\left(\dfrac{x^2-k}{1+x^2}\right)$. Then find the possible values of 'k' for which g is surjective.

17. Let $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ are the solutions of the equations $\cos x = x$, $\cos(\sin x) = x$ and $\sin(\cos x) = x$ respectively, then show that $\gamma < \alpha < \beta$

18. Let $f(x) = x(2-x)$, $0 \le x \le 2$. If the definition of 'f' is extended over the set, $R - [0, 2]$ by $f(x-2) = f(x)$, then prove that 'f' is a periodic function of period 2.

19. Let $f(x) = \log_2\log_3\log_4\log_5(\sin x + a^2)$. Find the set of values of a for which domain of f(x) is R.

20. The fundamental period of $\sin\frac{\pi}{4}[x] + \cos\frac{\pi x}{2} + \cos\frac{\pi}{3}[x]$, where [.] denotes the integral part of x, is.

21. $\tan^{-1}(\tan\theta) = \begin{cases} \pi+\theta, & -\frac{3\pi}{2} < \theta < -\frac{\pi}{2} \\ \theta, & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\pi+\theta, & \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}$, $\sin^{-1}(\sin\theta) = \begin{cases} -\pi-\theta, & -\frac{3\pi}{2} \le \theta < -\frac{\pi}{2} \\ \theta, & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \pi-\theta, & \frac{\pi}{2} < \theta \le \frac{3\pi}{2} \end{cases}$,

$\cos^{-1}(\cos\theta) = \begin{cases} -\theta, & -\pi \le \theta < 0 \\ \theta, & 0 \le \theta \le \pi \\ 2\pi-\theta, & \pi < \theta \le 2\pi \end{cases}$

Based on the above results, prove each of the following :

(i) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$ if $0 < x < 1$

(ii) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$ if $0 < x < 1$

(iii) $\cos^{-1} x = \pi + \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$ if $-1 < x < 0$

22. Express cot $(\text{cosec}^{-1}x)$ as an algebraic function of x.

23. Express $\sin^{-1}x$ in terms of (i) $\cos^{-1}\sqrt{1-x^2}$ (ii) $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ (iii) $\cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$

24. If $f(x) = \begin{bmatrix} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{bmatrix}$, then find $f^{-1}(x)$.

25. $\sin^{-1}\left(\dfrac{x^2}{4}+\dfrac{y^2}{9}\right) + \cos^{-1}\left(\dfrac{x}{2\sqrt{2}}+\dfrac{y}{3\sqrt{2}}-2\right)$ equals to :

26. If $\alpha = 2\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ & $\beta = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that $\alpha + \beta = \pi$. What the value of $\alpha + \beta$ will be if $x > 1$?

27. Solve $\{\cos^{-1} x\} + [\tan^{-1} x] = 0$ for real values of x. Where { . } and [.] are fractional part and greatest integer functions respectively.

28. Find the set of all real values of x satisfying the inequality $\sec^{-1}x > \tan^{-1}x$.

29. Find the solution of $\sin^{-1}\sqrt{\dfrac{x}{1+x}} - \sin^{-1}\dfrac{x-1}{x+1} = \sin^{-1}\dfrac{1}{\sqrt{1+x}}$.

30. (i) Find all positive integral solutions of the equation, $\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3$.

(ii) If 'k' be a positive integer, then show that the equation:
$\tan^{-1} x + \tan^{-1} y = \tan^{-1} k$ has no non–zero integral solution.

31. Determine the integral values of 'k' for which the system, $(\tan^{-1} x)^2 + (\cos^{-1} y)^2 = \pi^2 k$ and $\tan^{-1} x + \cos^{-1} y = \dfrac{\pi}{2}$ possess solution and find all the solutions.

32. Suppose X and Y are two sets and $f : X \to Y$ is a function. For a subset A of X, define f(A) to be the subset $\{f(a) : a \in A\}$ of Y. For a subset B of Y, define $f^{-1}(B)$ to be the subset $\{x \in X : f(x) \in B\}$ of X. Then prove the followings

(i) Statement "$f^{-1}(f(A)) = A$ for every $A \subset X$" is false

(ii) Statement "$f^{-1}(f(A)) = A$ for every $A \subset X$ if only if $f(X) = Y$" is false

(iii) Statement "$f(f^{-1}(B)) = B$ for every $B \subset Y$" is false

(iv) Statement "$f(f^{-1}(B)) = B$ for every $B \subset Y$ if only if $f(X) = Y$" is true

1. $(-4, -3) \cup (4, \infty)$ 2. $(-\infty, \infty)$ 3. $(-1/2, 0] \cup [1/2, 1)$ 4. $\left(-\frac{\pi}{6}, 0\right)$

5. (i) $[-3, -2) \cup [3, 4)$ (ii) $R - \{(0, 1) \cup \{1, 2, \ldots\ldots, 12\} \cup (12, 13)\}$

(iii) $\left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, \infty\right)$ (iv) $\left(\frac{n}{7}, n \in I, -1 \le n \le 6\right)$

(v). $\left(\frac{-\sqrt{3}-1}{\sqrt{2}}, \frac{-\sqrt{3}+1}{\sqrt{2}}\right) \cup \left(\frac{\sqrt{3}-1}{\sqrt{2}}, \frac{\sqrt{3}+1}{\sqrt{2}}\right)$

6. $\{\pi\}$ 7. $[\sqrt{2} - 1, \infty)$ 8. $(-1, \infty)$ 9. $\left[\frac{\pi}{6}, \frac{\pi}{2}\right)$

10. (i) D : $[2, \infty)$; R : $\{\pi/2\}$ (ii) D : $\left(-2-\sqrt{2}, -3\right] \cup, \left[-1, -2+\sqrt{2}\right)$; R $\{0\}]$

(iii) D : $\left(-\sqrt{8}, -1\right] \cup \left[1, \sqrt{8}\right)$; R : $\left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$

(iv) D : $[3, \pi) \cup \bigcup_{n=1}^{\infty} (2n\pi, 2n\pi + \pi)$; R : $(-\infty, 0]$ } (v) D : $\{-2, -1, 1, 2\}$; R : $\left\{\frac{1}{4}, 2\right\}$

11. $\left[\frac{5}{9}, 1\right]$ 12. 2 13. 1 14. Integral solution (0,0); (2,2). $x + y = 6$, $x + y = 0$

16. $k = -\frac{1}{2}$ 19. $a \in (-\infty, -\sqrt{626}) \cup (\sqrt{626}, \infty)$ 20. 24

22. $\cot(\operatorname{cosec}^{-1}x) = \begin{cases} -\sqrt{x^2-1} & \text{if } x \le -1 \\ \sqrt{x^2-1} & \text{if } x \ge 1 \end{cases}$

23. (i) $\sin^{-1}x = \begin{cases} -\cos^{-1}\sqrt{1-x^2}, & \text{if } -1 \le x < 0 \\ \cos^{-1}\sqrt{1-x^2} & \text{if } 0 \le x \le 1 \end{cases}$ $\begin{cases} -\cos^{-1}\sqrt{1-x^2}, & \text{यदि } -1 \le x < 0 \\ \cos^{-1}\sqrt{1-x^2} & \text{यदि } 0 \le x \le 1 \end{cases}$

(ii) $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$, for all $x \in (-1, 1)$ (iii) $\sin^{-1}x = \begin{cases} \cot^{-1}\frac{\sqrt{1-x^2}}{x} - \pi & \text{if } -1 \le x < 0 \\ \cot^{-1}\frac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \le 1 \end{cases}$

24. $f^{-1}(x) = \begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ \frac{x^2}{64}, & x > 16 \end{cases}$ 25. $\frac{3\pi}{2}$ 26. $-\pi$ 27. $\{1, \cos 1\}$

28. $\{x : x \in (-\infty, -1)\}$ 29. $x \ge 0$ 30. (i) Two solutions (1, 2) (2, 7)

31. $k = 1$, $x = \tan(1 - \sqrt{7})\frac{\pi}{4}$, $y = \cos(\sqrt{7} + 1)\frac{\pi}{4}$