Matrices & Determinant

As for everything else, so for a mathematical theory, beauty can be perceived but not explained..... Cayley Arthur

Introduction :

Any rectangular arrangement of numbers (real or complex) (or of real valued or complex valued expressions) is called a **matrix**. If a matrix has m rows and n columns then the **order** of matrix is written as m × n and we call it as order m by n

The general m × n matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1j} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2j} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & a_{i3} & \ldots\ldots & a_{ij} & \ldots\ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mj} & \ldots\ldots & a_{mn} \end{bmatrix}$$

where a_{ij} denote the element of i^{th} row & j^{th} column. The above matrix is usually denoted as $[a_{ij}]_{m \times n}$.

Notes :

(i) The elements $a_{11}, a_{22}, a_{33}, \ldots\ldots$ are called as **diagonal elements**. Their sum is called as **trace of A** denoted as tr(A)

(ii) Capital letters of English alphabets are used to denote matrices.

(iii) Order of a matrix : If a matrix has m rows and n columns, then we say that its order is "m by n", written as "m × n".

Example # 1 : Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$.

Solution : In general a 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

$a_{ij} = \frac{1}{2}|i - 3j|$, i = 1, 2, 3 and j = 1, 2

Therefore $a_{11} = \frac{1}{2}|1 - 3 \times 1| = 1$ $a_{12} = \frac{1}{2}|1 - 3 \times 2| = \frac{5}{2}$

$a_{21} = \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2}$ $a_{22} = \frac{1}{2}|2 - 3 \times 2| = 2$

$a_{31} = \frac{1}{2}|3 - 3 \times 1| = 0$ $a_{32} = \frac{1}{2}|3 - 3 \times 2| = \frac{3}{2}$

Hence the required matrix is given hy $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$

Types of Matrices :

Row matrix :

A matrix having only one row is called as row matrix (or row vector).General form of row matrix is $A = [a_{11}, a_{12}, a_{13}, \ldots, a_{1n}]$

This is a matrix of order "1 × n" (or a row matrix of order n)

Column matrix :

A matrix having only one column is called as column matrix (or column vector).

Column matrix is in the form $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix}$

This is a matrix of order "m × 1" (or a column matrix of order m)

Zero matrix :

$A = [a_{ij}]_{m \times n}$ is called a zero matrix, if $a_{ij} = 0 \ \forall$ i & j.

e.g. : (i) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Square matrix :

A matrix in which number of rows & columns are equal is called a square matrix. The general form of a square matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{bmatrix}$$ which we denote as $A = [a_{ij}]_n$.

This is a matrix of order "n × n" (or a square matrix of order n)

Diagonal matrix :

A square matrix $[a_{ij}]_n$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. (i.e., all the elements of the square matrix other than diagonal elements are zero)

Note : Diagonal matrix of order n is denoted as Diag $(a_{11}, a_{22}, \ldots\ldots a_{nn})$.

e.g. : (i) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$

Scalar matrix :

Scalar matrix is a diagonal matrix in which all the diagonal elements are same. $A = [a_{ij}]_n$ is a scalar matrix, if (i) $a_{ij} = 0$ for $i \neq j$ and (ii) $a_{ij} = k$ for $i = j$.

e.g. : (i) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

Unit matrix (Identity matrix) :

Unit matrix is a diagonal matrix in which all the diagonal elements are unity. Unit matrix of order 'n' is denoted by I_n (or I).

i.e. $A = [a_{ij}]_n$ is a unit matrix when $a_{ij} = 0$ for $i \neq j$ & $a_{ii} = 1$

eg. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Upper triangular matrix :

$A = [a_{ij}]_{m \times n}$ is said to be upper triangular, if $a_{ij} = 0$ for $i > j$ (i.e., all the elements below the diagonal elements are zero).

e.g. : (i) $\begin{bmatrix} a & b & c & d \\ 0 & x & y & z \\ 0 & 0 & u & v \end{bmatrix}$ (ii) $\begin{bmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & z \end{bmatrix}$

Lower triangular matrix :

$A = [a_{ij}]_{m \times n}$ is said to be a lower triangular matrix, if $a_{ij} = 0$ for $i < j$. (i.e., all the elements above the diagonal elements are zero.)

e.g. : (i) $\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ x & y & z \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ x & y & z & 0 \end{bmatrix}$

Comparable matrices : Two matrices A & B are said to be comparable, if they have the same order (i.e., number of rows of A & B are same and also the number of columns).

e.g. : (i) $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ are comparable

e.g. : (ii) $C = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ & $D = \begin{bmatrix} 3 & 0 \\ 4 & 1 \\ 2 & 3 \end{bmatrix}$ are not comparable

Equality of matrices :

Two matrices A and B are said to be equal if they are comparable and all the corresponding elements are equal.

Let $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{p \times q}$

$A = B$ iff (i) $m = p,\ n = q$
(ii) $a_{ij} = b_{ij}\ \forall\ i$ & j.

Example # 2 : Let $A = \begin{bmatrix} \sin\theta & 1/\sqrt{2} \\ -1/\sqrt{2} & \cos\theta \\ \cos\theta & \tan\theta \end{bmatrix}$ & $B = \begin{bmatrix} 1/\sqrt{2} & \sin\theta \\ \cos\theta & \cos\theta \\ \cos\theta & -1 \end{bmatrix}$. Find θ so that A = B.

Solution : By definition A & B are equal if they have the same order and all the corresponding elements are equal.

Thus we have $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$ & $\tan\theta = -1$

$\Rightarrow \quad \theta = (2n + 1)\pi - \frac{\pi}{4}$

Example # 3 : If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$, then find the values of a, b, c, x, y and z.

Solution : As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$x + 3 = 0 \qquad z + 4 = 6 \qquad 2y - 7 = 3y - 2$
$a - 1 = -3 \qquad 0 = 2c + 2 \qquad b - 3 = 2b + 4$

$\Rightarrow \quad a = -2,\ b = -7,\ c = -1,\ x = -3,\ y = -5,\ z = 2$

Multiplication of matrix by scalar :

Let λ be a scalar (real or complex number) & $A = [a_{ij}]_{m \times n}$ be a matrix. Thus the product λA is defined as $\lambda A = [b_{ij}]_{m \times n}$ where $b_{ij} = \lambda a_{ij}\ \forall\ i$ & j.

e.g. : $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix}$ & $-3A \equiv (-3)A = \begin{bmatrix} -6 & 3 & -9 & -15 \\ 0 & -6 & -3 & 9 \\ 0 & 0 & 3 & 6 \end{bmatrix}$

Note : If A is a scalar matrix, then $A = \lambda I$, where λ is a diagonal entry of A

Addition of matrices :

Let A and B be two matrices of same order (i.e. comparable matrices). Then A + B is defined to be.

$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$.
$= [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}\ \forall\ i$ & j.

e.g. : $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ -2 & -3 \\ 5 & 7 \end{bmatrix}$, $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 6 & 7 \end{bmatrix}$

Substraction of matrices :

Let A & B be two matrices of same order. Then A – B is defined as A + (– B) where – B is (– 1) B.

Properties of addition & scalar multiplication :

Consider all matrices of order m × n, whose elements are from a set F (F denote Q, R or C).
Let $M_{m \times n}(F)$ denote the set of all such matrices.

Then

(a) $A \in M_{m \times n}(F)$ & $B \in M_{m \times n}(F) \Rightarrow A + B \in M_{m \times n}(F)$
(b) $A + B = B + A$
(c) $(A + B) + C = A + (B + C)$
(d) $O = [o]_{m \times n}$ is the additive identity.
(e) For every $A \in M_{m \times n}(F)$, – A is the additive inverse.
(f) $\lambda(A + B) = \lambda A + \lambda B$
(g) $\lambda A = A\lambda$
(h) $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$

Example # 4 : IF $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that 2A + 3X = 5B

Solution : We have 2A + 3X = 5B.

$\Rightarrow \quad 3X = 5B - 2A$

$\Rightarrow \quad X = \frac{1}{3}(5B - 2A)$

$$\Rightarrow \quad X = \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$\Rightarrow \quad X = \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Multiplication of matrices :

Let A and B be two matrices such that the number of columns of A is same as number of rows of B. i.e., $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$.

Then $AB = [c_{ij}]_{m \times n}$ where $c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$, which is the dot product of i^{th} row vector of A and j^{th} column vector of B.

e.g. : $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 3 & 4 & 9 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$

Notes :

(1) The product AB is defined iff the number of columns of A is equal to the number of rows of B. A is called as premultiplier & B is called as post multiplier. AB is defined BA is defined.

(2) In general $AB \neq BA$, even when both the products are defined.

(3) $A(BC) = (AB)C$, whenever it is defined.

Properties of matrix multiplication :

Consider all square matrices of order 'n'. Let $M_n(F)$ denote the set of all square matrices of order n. (where F is Q, R or C). Then

(a) $A, B \in M_n(F) \Rightarrow AB \in M_n(F)$

(b) In general $AB \neq BA$

(c) $(AB)C = A(BC)$

(d) I_n, the identity matrix of order n, is the multiplicative identity.

$AI_n = A = I_nA \quad \forall A \in M_n(F)$

(e) For every non singular matrix A (i.e., $|A| \neq 0$) of $M_n(F)$ there exist a unique (particular) matrix $B \in M_n(F)$ so that $AB = I_n = BA$. In this case we say that A & B are multiplicative inverse of one another. In notations, we write $B = A^{-1}$ or $A = B^{-1}$.

(f) If λ is a scalar $(\lambda A)B = \lambda(AB) = A(\lambda B)$.

(g) $A(B + C) = AB + AC \quad \forall A, B, C \in M_n(F)$

(h) $(A + B)C = AC + BC \quad \forall A, B, C \in M_n(F)$.

Notes : (1) Let $A = [a_{ij}]_{m \times n}$. Then $AI_n = A$ & $I_m A = A$, where I_n & I_m are identity matrices of order n & m respectively.

(2) For a square matrix A, A^2 denotes AA, A^3 denotes AAA etc.

Example # 5 : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40I = O$

Solution : We have $A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

So $A^3 = AA^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

Now $A^3 - 23A - 40I = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 90-92+0 & 46-46+0 & 63-23-40 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Self practice problems :

(1) If $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, verify that $A(\alpha)\,A(\beta) = A(\alpha + \beta)$.

Hence show that in this case $A(\alpha).\ A(\beta) = A(\beta)\ .\ A(\alpha)$.

(2) Let $A = \begin{bmatrix} 4 & 6 & -1 \\ 3 & 0 & 2 \\ 1 & -2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$ and $C = [3\ \ 1\ \ 2]$.

Then which of the products ABC, ACB, BAC, BCA, CAB, CBA are defined. Calculate the product whichever is defined.

Answer (2) Only CAB is defined. $CAB = [25\ \ 100]$

Transpose of a matrix :

Let $A = [a_{ij}]_{m \times n}$. Then the transpose of A is denoted by A'(or A^T) and is defined as

$A' = [b_{ij}]_{n \times m}$ where $b_{ij} = a_{ji} \quad \forall\, i\ \&\ j$.

i.e. A' is obtained by rewriting all the rows of A as columns (or by rewriting all the columns of A as rows).

e.g. : $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \\ x & y & z & w \end{bmatrix}$, $A' = \begin{bmatrix} 1 & a & x \\ 2 & b & y \\ 3 & c & z \\ 4 & d & w \end{bmatrix}$

Results :

(i) For any matrix $A = [a_{ij}]_{m \times n}$, $(A')' = A$

(ii) Let λ be a scalar & A be a matrix. Then $(\lambda A)' = \lambda A'$

(iii) $(A + B)' = A' + B'$ & $(A - B)' = A' - B'$ for two comparable matrices A and B.

(iv) $(A_1 \pm A_2 \pm ..... \pm A_n)' = A_1' \pm A_2' \pm ..... \pm A_n'$, where A_i are comparable.

(v) Let $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$, then $(AB)' = B'A'$

(vi) $(A_1 A_2 ....A_n)' = A_n' . A_{n-1}' ....A_2' . A_1'$, provided the product is defined.

Symmetric & skew-symmetric matrix : A square matrix A is said to be symmetric if $A' = A$

i.e. Let $A = [a_{ij}]_n$. A is symmetric iff $a_{ij} = a_{ji}$ $\forall$ i & j. A square matrix A is said to be skew-symmetric if $A' = -A$

i.e. Let $A = [a_{ij}]_n$. A is skew-symmetric iff $a_{ij} = -a_{ji}$ $\forall$ i & j.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

$B = \begin{bmatrix} o & x & y \\ -x & o & z \\ -y & -z & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Notes :

(1) In a skew-symmetric matrix all the diagonal elements are zero.

$(\because a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0)$

(2) For any square matrix A, $A + A'$ is symmetric & $A - A'$ is skew-symmetric.

(3) Every square matrix can be uniqualy expressed as a sum of two square matrices of which one is symmetric and the other is skew-symmetric.

$A = B + C$, where $B = \frac{1}{2}(A + A')$ & $C = \frac{1}{2}(A - A')$.

Example # 6 : If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [1 \ 3 \ -6]$, verify that $(AB)' = B'A'$.

Solution : We have

$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [1 \ 3 \ -6]$

Then $AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} [1 \ 3 \ -6] = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$

Now $A' = [-2 \ 4 \ 5]$, $B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$

$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} [-2 \ 4 \ 5] = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$

Clearly $(AB)' = B'A'$

Example # 7 : Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution : Here $B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$

Let $P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 3 & 1 \\ -\frac{3}{2} & 1 & -3 \end{bmatrix}$

Now $P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, Let $Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$

Now $Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ -\frac{1}{2} & 0 & -3 \\ -\frac{5}{2} & 3 & 0 \end{bmatrix} = -Q$

Thus $Q = \frac{1}{2}(B - B')$ is a skew symmetric matrix.

Now $P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$

Thus, B is reresented as the sum of a symmetric and a skew symmetric matrix.

Thus, B is reresented as the sum of a symmetric and a skew symmetric matrix.

Example # 8 : Show that BAB' is symmetric or skew-symmetric according as A is symmetric or skew-symmetric (where B is any square matrix whose order is same as that of A).

Solution : Case - I
A is symmetric $\Rightarrow A' = A$
$(BAB')' = (B')'A'B' = BAB' \Rightarrow BAB'$ is symmetric.

Case - II
A is skew-symmetric $\Rightarrow A' = -A$
$(BAB')' = (B')'A'B'$
$= B(-A)B'$
$= -(BAB')$
$\Rightarrow$ BAB' is skew-symmetric

Self practice problems :

(3) For any square matrix A, show that $A'A$ & AA' are symmetric matrices.

(4) If A & B are symmetric matrices of same order, then show that AB + BA is symmetric and AB – BA is skew-symmetric.

Submatrix : Let A be a given matrix. The matrix obtained by deleting some rows or columns of A is called as submatrix of A.

eg. $A = \begin{bmatrix} a & b & c & d \\ x & y & z & w \\ p & q & r & s \end{bmatrix}$ Then $\begin{bmatrix} a & c \\ x & z \\ p & r \end{bmatrix}, \begin{bmatrix} a & b & d \\ p & q & s \end{bmatrix}, \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$ are all submatrices of A.

Determinant of a square matrix :

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix.

Let $A = [a]_{1\times1}$ be a 1×1 matrix. Determinant A is defined as $|A| = a$.

e.g. $A = [-3]_{1\times1}$ $|A| = -3$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A|$ is defined as $ad - bc$.

e.g. $A = \begin{bmatrix} 5 & 3 \\ -1 & 4 \end{bmatrix}$, $|A| = 23$

Minors & Cofactors :

Let Δ be a determinant. Then minor of element a_{ij}, denoted by M_{ij}, is defined as the determinant of the submatrix obtained by deleting i^{th} row & j^{th} column of Δ. Cofactor of element a_{ij}, denoted by C_{ij}, is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

e.g. 1 $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$M_{11} = d = C_{11}$
$M_{12} = c, C_{12} = -c$
$M_{21} = b, C_{21} = -b$
$M_{22} = a = C_{22}$

e.g. 2 $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

$M_{11} = \begin{vmatrix} q & r \\ y & z \end{vmatrix} = qz - yr = C_{11}$.

$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx, C_{23} = -(ay - bx) = bx - ay$ etc.

Determinant of any order :

Let $A = [a_{ij}]_n$ be a square matrix (n > 1). Determinant of A is defined as the sum of products of elements of any one row (or any one column) with corresponding cofactors.

e.g.1 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (using first row).

$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$ (using second column).

$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Transpose of a determinant : The transpose of a determinant is the determinant of transpose of the corresponding matrix.

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Properties of determinant :

(1) $|A| = |A'|$ for any square matrix A.

i.e. the value of a determinant remains unaltered, if the rows & columns are inter changed,

i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$

(2) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.

e.g. Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D_2 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D_2 = -D_1$

(3) Let λ be a scalar. Than $\lambda |A|$ is obtained by multiplying any one row (or any one column) of $|A|$ by λ

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $E = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $E = KD$

(4) $|AB| = |A||B|$.

(5) $|\lambda A| = \lambda^n |A|$, when $A = [a_{ij}]_n$.

(6) A skew-symmetric matrix of odd order has determinant value zero.

(7) If a determinant has all the elements zero in any row or column, then its value is zero,

i.e. $D = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

(8) If a determinant has any two rows (or columns) identical (or proportional), then its value is zero,

i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

(9) If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants, i.e.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(10) The value of a determinant is not altered by adding to the elements of any row (or column) a constant multiple of the corresponding elements of any other row (or column),

i.e. $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 + na_1 & b_3 + nb_1 & c_3 + nc_1 \end{vmatrix}$. Then $D_2 = D_1$

(11) Let $A = [a_{ij}]_n$. The sum of the products of elements of any row with corresponding cofactors of any other row is zero. (Similarly the sum of the products of elements of any column with corresponding cofactors of any other column is zero).

Example # 9 Simplify $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Solution : Let $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

$= (a + b + c) ((b - c) (a - b) - (c - a)^2)$
$= (a + b + c) (ab + bc - ca - b^2 - c^2 + 2ca - a^2)$
$= (a + b + c) (ab + bc + ca - a^2 - b^2 - c^2) \equiv 3abc - a^3 - b^3 - c^3$

Example # 10 Simplify $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Solution : Given determinant is equal to

$$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (a - b) (b - c) [ab^2 + abc + ac^2 + b^3 + b^2C + bc^2 - a^2b - a^2c - ab^2 - abc - b^3 - b^2c]$
$= (a - b) (b - c) [c(ab + bc + ca) - a(ab + bc + ca)]$
$= (a - b) (b - c) (c - a) (ab + bc + ca)$

Self practice problems

(5) Find the value of $\Delta = . \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$

(6) Simplify . $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$

(7) Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$.

(8) Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$ by using factor theorem .

Answers : (5) 0 (6) 0

Application of determinants : Following examples of short hand writing large expressions are:

(i) Area of a triangle whose vertices are (x_r, y_r); r = 1, 2, 3 is:

$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ If D = 0 then the three points are collinear.

(ii) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iii) The lines :
$a_1x + b_1y + c_1 = 0$........ (1)
$a_2x + b_2y + c_2 = 0$........ (2)
$a_3x + b_3y + c_3 = 0$........ (3)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Condition for the consistency of three simultaneous linear equations in 2 variables.

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$\begin{vmatrix} a & h & g \\ h & b & f \\ g & e & c \end{vmatrix}$

Singular & non singular matrix : A square matrix A is said to be singular or non-singular according as |A| is zero or non-zero respectively.

Cofactor matrix & adjoint matrix : Let $A = [a_{ij}]_n$ be a square matrix. The matrix obtained by replacing each element of A by corresponding cofactor is called as cofactor matrix of A, denoted as cofactor A. The transpose of cofactor matrix of A is called as adjoint of A, denoted as adj A.

i.e. if $A = [a_{ij}]_n$
then cofactor $A = [c_{ij}]_n$ when c_{ij} is the cofactor of a_{ij} $\forall$ i & j.
Adj $A = [d_{ij}]_n$ where $d_{ij} = c_{ji}$ $\forall$ i & j.

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Properties of cofactor A and adj A:

(a) $A \cdot \text{adj } A = |A|\, I_n = (\text{adj } A)\, A$ where $A = [a_{ij}]_n$.

(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A. In particular, for 3×3 matrix, $|\text{adj } A| = |A|^2$

(c) If A is a symmetric matrix, then adj A are also symmetric matrices.

(d) If A is singular, then adj A is also singular.

Example # 11 : For a 3×3 skew-symmetric matrix A, show that adj A is a symmetric matrix.

Solution : $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ $\qquad$ $\text{cof } A = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$

$\text{adj } A = (\text{cof } A)' = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$ which is symmetric.

Inverse of a matrix (reciprocal matrix) :

Let A be a non-singular matrix. Then the matrix $\frac{1}{|A|}\text{adj } A$ is the multiplicative inverse of A (we call it inverse of A) and is denoted by A^{-1}. We have $A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$

$\Rightarrow \quad A\left(\frac{1}{|A|}\text{adj } A\right) = I_n = \left(\frac{1}{|A|}\text{adj } A\right) A$, for A is non-singular

$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{adj } A.$

Remarks :

1. The necessary and sufficient condition for existence of inverse of A is that A is non-singular.
2. A^{-1} is always non-singular.
3. If $A = \text{dia}\,(a_{11}, a_{22}, \ldots, a_{nn})$ where $a_{ii} \neq 0\ \forall\ i$, then $A^{-1} = \text{diag}\,(a_{11}^{-1}, a_{22}^{-1}, \ldots, a_{nn}^{-1})$.
4. $(A^{-1})' = (A')^{-1}$ for any non-singular matrix A. Also $\text{adj}\,(A') = (\text{adj } A)'$.
5. $(A^{-1})^{-1} = A$ if A is non-singular.
6. Let k be a non-zero scalar & A be a non-singular matrix. Then $(kA)^{-1} = \frac{1}{k}A^{-1}$.
7. $|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$.
8. Let A be a non-singular matrix. Then $AB = AC \Rightarrow B = C$ & $BA = CA \Rightarrow B = C$.
9. A is non-singular and symmetric $\Rightarrow A^{-1}$ is symmetric.
10. $(AB)^{-1} = B^{-1}A^{-1}$ if A and B are non- singular.
11. In general $AB = \mathbf{0}$ does not imply $A = \mathbf{0}$ or $B = \mathbf{0}$. But if A is non-singular and $AB = \mathbf{0}$, then $B = \mathbf{0}$. Similarly B is non-singular and $AB = \mathbf{0} \Rightarrow A = \mathbf{0}$. Therefore, $AB = \mathbf{0} \Rightarrow$ either both are singular or one of them is $\mathbf{0}$.

Example # 12 : If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that $A \text{ adj } A = |A|\, I$. Also find A^{-1}

Solution : We have $|A| = 1\,(16-9) - 3\,(4-3) + 3\,(3-4) = 1 \neq 0$

Now $C_{11} = 7, C_{12} = -1, C_{13} = -1, C_{21} = -3, C_{22} = 1, C_{23} = 0, C_{31} = -3, C_{32} = 0, C_{33} = 1$

Therefore $\text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now $A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|.\, I$

Also $A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Example # 13 : Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Using the equation, find A^{-1}.

Solution : We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now $A^2 - 4A + I = 0$

Therefore $A\,A - 4A = -I$

or $AA(A^{-1}) - 4\,A\,A^{-1} = -I\,A^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A\,(A\,A^{-1}) - 4I = -A^{-1}$ $\qquad$ or $\qquad AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Example # 14 : For two non-singular matrices A & B, show that $\text{adj}\,(AB) = (\text{adj } B)\,(\text{adj } A)$

Solution : We have $(AB)\,(\text{adj}\,(AB)) = |AB|\, I_n$

$= |A|\,|B|\, I_n$

$A^{-1}\,(AB)(\text{adj}\,(AB)) = |A|\,|B|\,A^{-1}$

$\Rightarrow \quad B \text{ adj}\,(AB) = |B| \text{ adj } A \qquad \left(\because \quad A^{-1} = \frac{1}{|A|}\text{adj } A\right)$

$\Rightarrow \quad B^{-1} B \text{ adj}\,(AB) = |B|\, B^{-1} \text{ adj } A$

$\Rightarrow \quad \text{adj}\,(AB) = (\text{adj}B)\,(\text{adj } A)$

Self practice problems :

(9) If A is non-singular, show that adj (adj A) = $|A|^{n-2}$ A.

(10) Prove that adj (A^{-1}) = $(\text{adj } A)^{-1}$.

(11) For any square matrix A, show that |adj (adj A) | = $|A|^{(n-1)^2}$.

(12) If A and B are non-singular matrices, show that $(AB)^{-1} = B^{-1} A^{-1}$.

Elementary row transformation of matrix :

The following operations on a matrix are called as elementary row transformations.

(a) Interchanging two rows.

(b) Multiplications of all the elements of row by a nonzero scalar.

(c) Addition of constant multiple of a row to another row.

Note : Similar to above we have elementary column transformations also.

Remarks : Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write A ≈ B.

Finding inverse using Elementry operations

(i) Using row transformations :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations,

Step I : Write A = IA and

Step II : Apply a sequence of row operation on A = IA till we get, I = BA.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary row operations on the matrix equation X = AB, we will apply these row operatdions simultaneously on X and on the first matrix A of the product AB on RHS.

(ii) Using column transformations :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary column operations,

Step I : Write A = AI and

Step II : Apply a sequence of column operations on A = AI till we get, I = AB.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary column operations on the matrix equation X = AB, we will apply these row operatdions simultaneously on X and on the second matrix B of the product AB on RHS.

Example # 15 : Obtain the inverse of the matrix A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ using elementary operations.

Solution : Write A = IA, i.e. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ (applying $R_1 \leftrightarrow R_2$)

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \to R_3 - 3R_1$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_1 \to R_1 - 2R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \to R_3 + 5R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$ A (applying $R_3 \to \frac{1}{2} R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$ (Applying $R_1 \to R_1 + R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$ (Applying $R_2 \to R_2 - 2R_3$)

Hence $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

System of linear equations & matrices : Consider the system

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots + a_{2n} x_n = b_2$

....

$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots + a_{mn} x_n = b_n$.

|A| = 0 → singular

Let $A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ b_n \end{bmatrix}$.

Then the above system can be expressed in the matrix form as AX = B.

The system is said to be consistent if it has atleast one solution.

System of linear equations and matrix inverse:

If the above system consist of n equations in n unknowns, then we have AX = B where A is a square matrix.

Results :

(1) If A is non-singular, solution is given by $X = A^{-1}B$.

(2) If A is singular, (adj A) B = 0 and all the columns of A are not proportional, then the system has infinitely many solutions.

(3) If A is singular and (adj A) B ≠ 0, then the system has no solution (we say it is inconsistent).

Homogeneous system and matrix inverse :

If the above system is homogeneous, n equations in n unknowns, then in the matrix form it is $AX = O$

($\because$ in this case $b_1 = b_2 = \ldots\ldots b_n = 0$), where A is a square matrix

Results :

(1) If A is non-singular, the system has only the trivial solution (zero solution) $X = 0$

(2) If A is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non-trivial solutions

Example # 16 : Solve the system
$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$
using matrix inverse

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

Then the system is $AX = B$

$|A| = 6$. Hence A is non singular

$$\text{Cofactor } A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow x = 1,\ y = 2,\ z = 3$$

Self practice problems:

(13) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Find the inverse of A using |A| and adj A.

(14) Find real values of λ and μ so that the following systems has
(i) unique solution (ii) infinitely many solutions (iii) No solution
$$x + y + z = 6$$
$$x + 2y + 3z = 1$$
$$x + 2y + \lambda z = \mu$$

(15) Find λ so that the following homogeneous system have a non zero solution
$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z$$

Answers : (13) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$ (14) (i) $\lambda \neq 3$, $\mu \in R$ (ii) $\lambda = 3$, $\mu = 1$ (iii) $\lambda = 3$, $\mu \neq 1$ (15) $\lambda = 6$

Exercise-1

✎ Marked Questions may have for Revision Questions.

PART - I : SUBJECTIVE QUESTIONS

Section (A): Matrix, Algebra of Matrix, Transpose, symmetric and skew symmetric matrix

A-1. Construct a 3×2 matrix whose elements are given by $a_{ij} = 2i - j$.

A-2. If $\begin{bmatrix} x & y & 1 & z \\ 2x & y & 0 & w \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$, find x, y, z, w

A-3. Let $A + B + C = \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix}$, $4A + 2B + C = \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix}$ and $9A + 3B + C = \begin{bmatrix} 0 & -2 \\ 2 & 1 \end{bmatrix}$ then find A

A-4. If $A = \begin{bmatrix} 1 & 2 \\ [illegible] & -4 \\ [illegible] & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 2 \end{bmatrix}$, will AB be equal to BA. Also find AB & BA

A-5. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then show that $A^2 = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}$

A-6. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ show that $(I + A) = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

A-7. Given $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If $x \in R$. Then for what values of y, $F(x + y) = F(x)\,F(y)$.

A-8. Let $A = [a_{ij}]_{n \times n}$, where $a_{ij} = i^2 - j^2$. Show that A is skew-symmetric matrix

A-9. ✎ If $C = \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 7 & 9 \\ 4 & 2 & 8 \\ 6 & 5 & 3 \end{bmatrix}$, then trace of $C + C^3 + C^5 + \ldots\ldots + C^{99}$ is

Section (B) : Determinant of Matrix

B-1. If the minor of three-one element (i.e. M_{31}) in the determinant $\begin{vmatrix} 0 & 1 & \sec\alpha \\ \tan\alpha & -\sec\alpha & \tan\alpha \\ 1 & 0 & 1 \end{vmatrix}$ is 1 then find the value of α. $(0 \le \alpha \le \pi)$

B-2. Using the properties of determinants, evaluate:

(i) $\begin{vmatrix} 23 & 6 & 11 \\ 36 & 5 & 26 \\ 63 & 13 & 37 \end{vmatrix}$ (ii) $\begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}$

(iii) $\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix} + \begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix}$ (iv) $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$

B-3. Prove that :

(i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(ii) $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix} = -(a+b+c)(a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

(iv) If $\begin{vmatrix} 1 & a^2 & a^4 \\ 1 & b^2 & b^4 \\ 1 & c^2 & c^4 \end{vmatrix} = (a+b)(b+c)(c+a)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

B-4. If a, b, c are positive and are the p^{th}, q^{th}, r^{th} terms respectively of a G.P., show without expanding that, $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$.

B-5. Find the non – zero roots of the equation,

(i) $\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & c \end{vmatrix} = 0.$ (ii) $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$

x = y

B-6. If $S_r = \alpha^r + \beta^r + \gamma^r$ then show that $\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$.

B-7. Show that $\begin{vmatrix} a_1l_1+b_1m_1 & a_1l_2+b_1m_2 & a_1l_3+b_1m_3 \\ a_2l_1+b_2m_1 & a_2l_2+b_2m_2 & a_2l_3+b_2m_3 \\ a_3l_1+b_3m_1 & a_3l_2+b_3m_2 & a_3l_3+b_3m_3 \end{vmatrix} = 0$.

B-8. If $\begin{vmatrix} e^x & \sin x \\ \cos x & \ln(1+x) \end{vmatrix} = A + Bx + Cx^2 + \ldots\ldots$, then find the value of A and B.

Section (C) : Cofactor matrix, adj matrix and inverse of matrix

C-1. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and AB – CD = 0 find D.

C-2. (i) Prove that (adj adj A) = $|A|^{n-2}$ A
(ii) Find the value of |adj adj adj A| in terms of |A|

C-3. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

C-4. If A is a symmetric and B skew symmetric matrix and (A + B) is non-singular and $C = (A+B)^{-1}(A-B)$, then prove that
(i) $C^T(A+B)C = A + B$ (ii) $C^T(A-B)C = A - B$

Section (D) : Charactristic equation and system of equations

D-1. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ find a & b so that $A^2 + aA + bI = 0$. Hence find A^{-1} = $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$
a = −4 , b = 1

D-2. Find the total number of possible square matrix A of order 3 with all real entries, whose adjoint matrix B has characterstics polynomial equation as $\lambda^3 - \lambda^2 + \lambda + 1 = 0$.

D-3. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, show that $A^3 = (5A - I)(A - I)$

D-4. Apply Cramer's rule to solve the following simultaneous equations.
(i) $2x + y + 6z = 46$
$5x - 6y + 4z = 15$
$7x + 4y - 3z = 19$
(ii) $x + 2y + 3z = 2$
$x - y + z = 3$
$5x - 11y + z = 17$

D-5. Solve using Cramer's rule: $\frac{4}{x+5} + \frac{3}{y+7} = -1$ & $\frac{6}{x+5} - \frac{6}{y+7} = -5$.

D-6. Find those values of c for which the equations:
$2x + 3y = 3$
$(c+2)x + (c+4)y = c+6$
$(c+2)^2x + (c+4)^2y = (c+6)^2$ are consistent.
Also solve above equations for these values of c.

D-7. Solve the following systems of linear equations by matrix method.
(i) $2x - y + 3z = 8$, $-x + 2y + z = 4$, $3x + y - 4z = 0$
(ii) $x + y + z = 9$, $2x + 5y + 7z = 52$, $2x + y - z = 0$

D-8. Investigate for what values of λ, μ the simultaneous equations
$x + y + z = 6$; $x + 2y + 3z = 10$ & $x + 2y + \lambda z = \mu$ have;
(a) A unique solution (b) An infinite number of solutions. (c) No solution.

D-9. Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equations $x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$

D-10. Compute A^{-1}, if $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$. Hence solve the matrix equations

$$\begin{bmatrix} 3 & 0 & 3 \\ 2 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2y \\ z \\ 3y \end{bmatrix}.$$

D-11. Which of the following statement(s) is/are true :

S1 : The system of equations $\begin{matrix} 4x - 5y - 2z = 2 \\ 5x - 4y + 2z = 3 \\ 2x + 2y + 8z = 1 \end{matrix}$ is Inconsistent.

S2 : A matrix 'A' has 6 elements. The number of possible orders of A is 6.

S3 : For any 2 × 2 matrix A, if A (adjA) = $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then |A| = 10.

S4 : If A is skew symmetric, then B'AB is also skew symmetric.

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A): Matrix, Algebra of Matrix, Transpose, symmetric and skew symmetric matrix,

A-1. $\begin{bmatrix} x^2 + x & x \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -x+1 & x \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 5 & 1 \end{bmatrix}$ then x is equal to -

(A) – 1 (B) 2 (C) 1 (D) No value of x

A-2. If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$, then

(A) $AB = \begin{bmatrix} -5 & 8 & 0 \\ 0 & 4 & -2 \\ 3 & -9 & 6 \end{bmatrix}$ (B) AB = [– 2 – 1 4] (C) $AB = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (D) AB does not exist

A-3. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then B =

(A) $I\cos\theta + J\sin\theta$ (B) $I\cos\theta - J\sin\theta$ (C) $I\sin\theta + J\cos\theta$ (D) $-I\cos\theta + J\sin\theta$

A-4. In an upper triangular matrix $A = [a_{ij}]_{n \times n}$ the elements $a_{ij} = 0$ for elements below diag are zero

(A) i < j (B) i = j (C) i > j (D) i ≤ j

A-5. If A = diag (2, –1, 3), B = diag (–1, 3, 2), then A^2B =

(A) diag (5, 4, 11) (B) diag (– 4, 3, 18) (C) diag (3, 1, 8) (D) B

A-6. If A is a skew- symmetric matrix, then trace of A is

(A) 1 (B) – 1 (C) 0 diag elements = 0 (all) (D) none of these

A-7. Let $A = \begin{bmatrix} p & q \\ q & p \end{bmatrix}$ such that det(A) = r where p, q, r all prime numbers, then trace of A is equal to

(A) 6 (B) 5 (C) 2 (D) 3

$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

A-8. $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ and $(A^8 + A^6 + A^4 + A^2 + I)V = \begin{bmatrix} 31 \\ 62 \end{bmatrix}$. $A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(Where I is the (2 × 2) identity matrix), then the product of all elements of matrix V is

(A) 2 (B) 1 (C) 3 (D) –2

A-9. Let $A = \begin{bmatrix} 3x^2 \\ 1 \\ 6x \end{bmatrix}$, B = [a b c] and $C = \begin{bmatrix} (x+2)^2 & 5x^2 & 2x \\ 5x^2 & 2x & (x+1)^2 \\ 2x & (x+2)^2 & 5x^2 \end{bmatrix}$

Where a, b, c and x ∈ R, Given that tr (AB) = tr(C), then the value of (a + b + c).

(A) 7 (B) 2 (C) 1 (D) 4

Section (B) : Determinant of Matrix

B-1. If A and B are square matrices of order 3 such that |A| = – 1, |B| = 3, then |3AB| is equal to

(A) – 9 (B) – 81 (C) – 27 (D) 81

B-2. Let $A = \begin{bmatrix} \cos^{-1}x & \cos^{-1}y & \cos^{-1}z \\ \cos^{-1}y & \cos^{-1}z & \cos^{-1}x \\ \cos^{-1}z & \cos^{-1}x & \cos^{-1}y \end{bmatrix}$ such that |A| = 0, then maximum value of x + y + z is

(A) 3 (B) 0 (C) 1 (D) 2

B-3. The absolute value of the determinant $\begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$ is:

(A) $16\sqrt{2}$ (B) $8\sqrt{2}$ (C) 8 (D) None of these

B-4. If α, β & γ are the roots of the equation $x^3 + px + q = 0$ then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ = 0

(A) p (B) q (C) $p^2 - 2q$ (D) none

B-5. If a, b, c > 0 & x, y, z ∈ R then the determinant $\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^y + b^{-y})^2 & (b^y - b^{-y})^2 & 1 \\ (c^z + c^{-z})^2 & (c^z - c^{-z})^2 & 1 \end{vmatrix}$ = 0

(A) $a^xb^yc^z$ (B) $a^{-x}b^{-y}c^{-z}$ (C) $a^{2x}b^{2y}c^{2z}$ (D) zero

B-6. If a, b & c are non-zero real numbers then $D = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$ =

(A) abc (B) $a^2b^2c^2$ (C) bc + ca + ab (D) zero

B-7. The determinant $\begin{vmatrix} b_1 + c_1 & c_1 + a_1 & a_1 + b_1 \\ b_2 + c_2 & c_2 + a_2 & a_2 + b_2 \\ b_3 + c_3 & c_3 + a_3 & a_3 + b_3 \end{vmatrix}$ =

(A) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (B) $2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (C) $3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (D) none of these

B-8. If x, y, z ∈ R & $\Delta = \begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 3x & 7x+3y & 9x+7y+3z \end{vmatrix} = -16$ then value of x is

(A) – 2 (B) – 3 (C) 2 (D) 3

B-9. The determinant $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is :

(A) 0 (B) independent of θ (C) independent of ϕ (D) independent of θ & ϕ both

B-10. Let A be set of all determinants of order 3 with entries 0 or 1, B be the subset of A consisting of all determinants with value 1 and C be the subset of A consisting of all determinants with value –1. Then
STATEMENT -1 : The number of elements in set B is equal to number of elements in set C.
and
STATEMENT-2 : $(B \cap C) \subset A$

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

Section (C) : Cofactor matrix, adj matrix and inverse of matrix

C-1. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then adj A =

(A) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$

C-2. Identify statements S_1, S_2, S_3 in order for true(T)/false(F)

S_1 : If $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then adj A = A'

S_2 : If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

S_3 : If B is a non-singular matrix and A is a square matrix, then det $(B^{-1} AB)$ = det (A)

(A) TTF (B) FTT (C) TFT (D) TTT

C-3. If A, B are two n × n non-singular matrices, then

(A) AB is non-singular (B) AB is singular
(C) $(AB)^{-1} = A^{-1} B^{-1}$ (D) $(AB)^{-1}$ does not exist

C-4. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and X be a matrix such that A = BX, then X is equal to

(A) $\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (B) $\frac{1}{2}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (D) none of these

C-5. Let $\mathbf{A} = \begin{bmatrix} -1 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 1 \end{bmatrix}$ be a matrix, then (det **A**) x (adj $\mathbf{A}^{-1}$) is equal to

(A) $O_{3\times3}$ (B) I_3 (C) $\begin{bmatrix} -1 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & -3 & 1 \\ 3 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix}$

C-6. **STATEMENT-1 :** If $A = \begin{bmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+xc & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{bmatrix}$ and $B = \begin{bmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{bmatrix}$, then $|A| = |B|^2$.
STATEMENT-2 : If A^c is cofactor matrix of a square matrix A of order n then $|A^c| = |A|^{n-1}$.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true A^C
(E) Both STATEMENTS are false

Section (D) : Characteristic equation and system of equations

D-1. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ is a root of polynomial $x^3 - 6x^2 + 7x + k = 0$, then the value of k is

(A) 2 (B) 4 (C) –2 (D) 1

D-2 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (where bc ≠ 0) satisfies the equations $x^2 + k = 0$, then

(A) a + d = 0 & k = |A| (B) a – d = 0 & k = |A|
(C) a + d = 0 & k = –|A| (D) a + d ≠ 0 & k = |A|

D-3. If the system of equations x + 2y + 3z = 4, x + py + 2z = 3, x + 4y + μz = 3 has an infinite number of solutions and solution triplet is

(A) p = 2, μ = 3 and (5 – 4λ, λ – 1 , λ) (B) p = 2, μ = 4 and (5 – 4λ, , 2λ)
(C) 3 p = 2 μ and (5 – 4λ, λ – 1, 2λ) (D) p = 4, μ = 2 and (5 – 4λ, , λ)

D-4. Let λ and α be real. Find the set of all values of λ for which the system of linear equations have infinite solution ∀ real values of α.

λx + (sin α)y + (cos α) z = 0
x + (cos α)y + (sin α) z = 0
– x + (sin α) y + (cos α) z = 0

(A) $(-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$ (B) – 1
(C) $(-5, -\sqrt{2})$ (D) None of these

D-5. Let $A = \begin{bmatrix} a & 0 & b \\ 1 & e & 1 \\ c & 0 & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ where a,b, c, d, e ∈ {0, 1}
then number of such matrix A for which system of equation AX = 0 have unique solution.

(A) 16 (B) 6 (C) 5 (D) none

D-6. If the system of equations ax + y + z = 0,. x + by + z = 0 and x + y + cz = 0, where a, b, c ≠ 1, has a non–trivial solution, then the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is

(A) 1 (B) 2 (C) 3 (D) 4

PART - III : MATCH THE COLUMN

1. **Column I** | **Column II**

(A) $\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$ then x = (p) 2 B

(B) If A is a square matrix of order 3 × 3 and k is a scalar, then adj (kA) = k^m adj A, then m is (q) – 2

(C) If $A = \begin{bmatrix} 2 & \mu \\ \mu^2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} \gamma & 7 \\ 49 & \delta \end{bmatrix}$ here (A – B) is upper triangular matrix then number of possible values of μ are (r) 1 C

(D) If $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = k\,abc\,(a+b+c)^3$ then the value of k is (s) $-\frac{9}{8}$

2. **Column – I** | **Column – II**

$\text{adj}(2A^{-1}) = \frac{1}{2}\text{adj}\,A^{-1}$

(A) If A and B are square matrices of order 3 × 3, where |A| = 2 and |B| = 1, then $|(A^{-1})\,.\,\text{adj}\,(B^{-1})\,.\,\text{adj}\,(2A^{-1})|$ = (p) 7 B

(B) If A is a square matrix such that $A^2 = A$ and $(I + A)^3 = I + kA$, then k is equal to (q) 8 A

$A^{-1}.\text{adj}\,A^{-1} = |A^{-1}|.I$

(C) Matrix $\begin{bmatrix} a & b & (a\alpha - b) \\ b & c & (b\alpha - c) \\ 2 & 1 & 0 \end{bmatrix}$ is non invertible ($b^2 \neq ac$) if -2α is (r) 0

$\alpha = 1/2$

(D) If $A = [a_{ij}]_{3\times3}$ is a scalar matrix with $a_{11} = a_{22} = a_{33} = 2$ and $A(\text{adj}A) = kI_3$, then k is 8 (s) – 1 C

$A\,\text{adj}A = |A|.I$ $\quad \frac{\text{adj}\,X}{|X|} = X^{-1}$

Exercise-2

Marked Questions may have for Revision Questions.

PART - I : ONLY ONE OPTION CORRECT TYPE

1. Two matrices A and B have in total 6 different elements (none repeated) . How many different matrices A and B are possible such that product AB is defined.
(A) 5(6!) (B) 3(6!) (C) 12(6!) (D) 8(6!)

2. If AB = O for the matrices
$A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ then $\theta - \phi$ is
(A) an odd multiple of $\frac{\pi}{2}$ (B) an odd multiple of π
(C) an even multiple of $\frac{\pi}{2}$ (D) 0

3. If $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then value of X^n is, (where n is natural number)
(A) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (B) $\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$ (C) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (D) $\begin{bmatrix} 2n+1 & -4n \\ n & -(2n-1) \end{bmatrix}$

4. If A and B are two matrices such that AB = B and BA = A, then $A^2 + B^2$ =
(A) 2AB (B) 2BA (C) A + B (D) AB

5. Find number of all possible ordered sets of two (n × n) matrices **A** and **B** for which **AB** – **BA** = I
(A) infinite (B) n^2 (C) n! (D) zero

6. If B, C are square matrices of order n and if A = B + C, BC = CB, C^2 = O, then which of following is true for any positive integer N.
(A) $A^{N+1} = B^N (B + (N + 1) C)$ (B) $A^N = B^N (B + (N + 1) C)$
(C) $A^{N+1} = B (B + (N + 1) C)$ (D) $A^{N+1} = B^N (B + (N + 2) C)$

7. How many 3 × 3 skew symmetric matrices can be formed using numbers –2, –1, 1, 2, 3, 4, 0 (any number can be used any number of times but 0 can be used at most 3 times)
(A) 8 (B) 27 (C) 64 (D) 54

8. If A is a skew - symmetric matrix and n is an even positive integer, then A^n is
(A) a symmetric matrix (B) a skew-symmetric matrix
(C) a diagonal matrix (D) none of these

9. Number of 3 × 3 non symmetric matrix A such that $A^T = A^2 - I$ and $|A| \neq 0$, equals to
(A) 0 (B) 2 (C) 4 (D) Infinite

10. Matrix **A** is such that $\mathbf{A}^2 = 2\mathbf{A} - I$, where I is the identity matrix. Then for $n \geq 2$, $\mathbf{A}^n$ =
(A) $n\mathbf{A} - (n - 1)I$ (B) $n\mathbf{A} - I$ (C) $2^{n-1}\mathbf{A} - (n - 1)I$ (D) $2^{n-1}\mathbf{A} - I$

11. If $P = \begin{bmatrix} \frac{\sqrt3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt3}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$ and $x = P^TQ^{2005}P$, then x is equal to → JEE 2005
(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 4+2005\sqrt3 & 6015 \\ 2005 & 4-2005\sqrt3 \end{bmatrix}$
(C) $\frac{1}{4}\begin{bmatrix} 2+\sqrt3 & 1 \\ -1 & 2-\sqrt3 \end{bmatrix}$ (D) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt3 \\ 2+\sqrt3 & 2005 \end{bmatrix}$

12. Let $\Delta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$, then
(A) Δ is independent of θ (B) Δ is indepedent of ϕ
(C) Δ is a constant (D) none of these

13. $\Delta = \begin{vmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{vmatrix}$ is equal to
(A) $(a-b)^2(b-c)^2(c-a)^2$ (B) $2(a-b)(b-c)(c-a)$
(C) $4(a-b)(b-c)(c-a)$ (D) $(a+b+c)^3$

14. If $D = \begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$ then D =
(A) $1 + a^2 + b^2 + c^2$ (B) $a^2 + b^2 + c^2$ (C) $(a + b + c)^2$ (D) none

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15. Value of the $\Delta = \begin{vmatrix} a^3-x & a^4-x & a^5-x \\ a^5-x & a^6-x & a^7-x \\ a^7-x & a^8-x & a^9-x \end{vmatrix}$ is

(A) 0 (B) $(a^3-1)(a^6-1)(a^9-1)$
(C) $(a^3+1)(a^6+1)(a^9+1)$ (D) $a^{15}-1$

16. If $\Delta_1 = \begin{vmatrix} 2a & b & e \\ 2d & e & f \\ 4x & 2y & 2z \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} f & 2d & e \\ 2z & 4x & 2y \\ e & 2a & b \end{vmatrix}$, then the value of $\Delta_1 - \Delta_2$ is

(A) $x + \frac{y}{2} + z$ (B) 2 (C) 0 (D) 3

17. From the matrix equation AB = AC, we conclude B = C provided:
(A) A is singular (B) A is non-singular (C) A is symmetric (D) A is a square

18. Let $A = \begin{bmatrix} -2 & 7 & \sqrt{3} \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ and $A^4 = \lambda.I$, then λ is

(A) – 16 (B) 16 (C) 8 (D) –8

19. If A is 3 × 3 square matrix whose characterstic polynomial equations is $\lambda^3 - 3\lambda^2 + 4 = 0$ then trace of adjA is
(A) 0 (B) ~~3~~ (C) 4 (D) – 3

20. If a, b, c are non zeros, then the system of equations
$(\alpha + a)x + \alpha y + \alpha z = 0$
$\alpha x + (\alpha + b)y + \alpha z = 0$
$\alpha x + \alpha y + (\alpha + c)z = 0$
has a non-trivial solution if

NON – TRIVIAL SOLUTION → D = 0

(A) ~~$\alpha^{-1} = -(a^{-1} + b^{-1} + c^{-1})$~~ (B) $\alpha^{-1} = a + b + c$
(C) $\alpha + a + b + c = 1$ (D) none of these

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. Let X be the solution set of the equation
$A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the unit matrix and $X \subset N$ then the minimum value of
$\sum_x (\cos^x \theta + \sin^x \theta)$, $\theta \in R$ is :

2. If A is a diagonal matrix of order 3 × 3 is commutative with every square matrix of order 3 × 3 under multiplication and tr(A) = 12, then the value of |A| is : 64

3. A, is a (3×3) diagonal matrix having integral entries such that det(A) = 120, number of such matrices is 10n. Then n is :

$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ abc = 120

4. If $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \geq 0$, where a, b, $c \in R^+ - \{0\}$, then $\frac{a+b}{c}$ is

Conic MATRIX

5. If $a_1, a_2, a_3, 5, 4, a_6, a_7, a_8, a_9$ are in H.P. and $D = \begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$, then the value of **21D** is
(Where [.] represents, the greatest integer function)

D = 50/21

6. If $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = k(\alpha a + \beta b + \gamma c)^3$, then $(2\alpha + \beta - \gamma)^k$ is $(\alpha, \beta, \gamma, k \in z^+)$ 4

7. If A is a square matrix of order 3 and A′ denotes transpose of matrix A, A′ A = I and det A = 1, then det (A – I) must be equal to

orthogonal matrix

8. Suppose A is a matrix such that $A^2 = A$ and $(I + A)^6 = I + kA$, then k is 63

9. If $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = 64$, then (ab + bc + ac) is :

10. Let $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$ then the maximum value of f(x) is 6

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11. If $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N+1 \end{vmatrix}$ and $\sum_{n=1}^{N} U_n = \lambda \sum_{n=1}^{N} n^2$, then λ is

12. The absolute value of a for which system of equations, $a^3x + (a+1)^3y + (a+2)^3z = 0$, $ax + (a+1)y + (a+2)z = 0$, $x + y + z = 0$, has a non-zero solution is : 1

13. Consider the system of linear equations in x, y, z :
$(\sin 3\theta)x - y + z = 0$
$(\cos 2\theta)x + 4y + 3z = 0$
$2x + 7y + 7z = 0$
Number of values of $\theta \in (0, \pi)$ for which this system has non – trivial solution, is 2

14. The value of '2k' for which the set of equations $3x + ky - 2z = 0$, $x + ky + 3z = 0$, $2x + 3y - 4z = 0$ has a non – trivial solution over the set of rational is : 33 = 2k

15. $A_1 = [a_1]$

$A_2 = \begin{bmatrix} a_2 & a_3 \\ a_4 & a_5 \end{bmatrix}$

$A_3 = \begin{bmatrix} a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} \end{bmatrix}$ ....$A_n = [........]$

Where $a_r = [\log_2 r]$ ([.] denotes greatest integer). Then trace of A_{10}

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16. If $\left\{\frac{1}{2}(A - A' + I)\right\}^{-1} = \frac{2}{\lambda}\begin{bmatrix} \lambda - 13 & -\frac{\lambda}{3} & \frac{\lambda}{3} \\ -17 & 10 & -1 \\ 7 & -11 & 5 \end{bmatrix}$ for $A = \begin{bmatrix} -2 & 3 & 4 \\ 5 & -4 & -3 \\ 7 & 2 & 9 \end{bmatrix}$, then λ is : 3.9

17. Given $A = \begin{bmatrix} 2 & 0 & -\alpha \\ 5 & \alpha & 0 \\ 0 & \alpha & 3 \end{bmatrix}$ For $\alpha \in R - \{a, b\}$, A^{-1} exists and $A^{-1} = A^2 - 5bA + cI$, when $\alpha = 1$. The value of $a + 5b + c$ is :
$a = 0,\ b = \frac{6}{5},\ c = 11$

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. Which one of the following is wrong ?
(A) The elements on the main diagonal of a symmetric matrix are all zero
(B) The elements on the main diagonal of a skew - symmetric matrix are all zero
(C) For any square matrix A, A A′ is symmetric
(D) For any square matrix A, $(A + A')^2 = A^2 + (A')^2 + 2AA'$

2. Which of the following is true for matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$
(A) $A + 4I$ is a symmetric matrix
(B) $A^2 - 4A + 5I_2 = 0$
(C) $A - B$ is a diagonal matrix for any value of α if $B = \begin{bmatrix} \alpha & -1 \\ 2 & 5 \end{bmatrix}$
(D) $A - 4I$ is a skew symmetric matrix

3. Suppose a_1, a_2, a_3 are in A.P. and b_1, b_2, b_3 are in H.P. and let $\Delta = \begin{vmatrix} a_1 - b_1 & a_1 - b_2 & a_1 - b_3 \\ a_2 - b_1 & a_2 - b_2 & a_2 - b_3 \\ a_3 - b_1 & a_3 - b_2 & a_3 - b_3 \end{vmatrix}$, then
(A) Δ is independent of a_1, a_2, a_3,
(B) $a_1 - \Delta, a_2 - 2\Delta, a_3 - 3\Delta$ are in A.P.
(C) $b_1 + \Delta, b_2 + \Delta^2, b_3 + \Delta$ are in H.P.
(D) Δ is independent of b_1, b_2, b_3

4. Let $\theta = \frac{\pi}{5}$, $X = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, O is null maxtrix and I is an identity matrix of order 2 × 2, and if $I + X + X^2 + \ldots\ldots + X^n = O$, then n can be
(A) 9
(B) 19
(C) 4
(D) 29

5. If $\Delta = \begin{vmatrix} x & 2y - z & -z \\ y & 2x - z & -z \\ y & 2y - z & 2x - 2y - z \end{vmatrix}$, then
(A) $x - y$ is a factor of Δ
(B) $(x - y)^2$ is a factor of Δ
(C) $(x - y)^3$ is a factor of Δ
(D) Δ is independent of z

6. Let $a, b > 0$ and $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$, then
(A) $a + b - x$ is a factor of Δ
(B) $x^2 + (a + b)x + a^2 + b^2 - ab$ is a factor of Δ
(C) $\Delta = 0$ has three real roots if $a = b$
(D) $a + b + x$ is a factor of Δ

7. The determinant $\Delta = \begin{vmatrix} b & c & b\alpha + c \\ c & d & c\alpha + d \\ b\alpha + c & c\alpha + d & a\alpha^3 - c\alpha \end{vmatrix}$ is equal to zero if
(A) b, c, d are in A.P.
(B) b, c, d are in G.P.
(C) b, c, d are in H.P.
(D) α is a root of $ax^3 - bx^2 - 3cx - d = 0$

8. The determinant $\Delta = \begin{vmatrix} a^2(1+x) & ab & ac \\ ab & b^2(1+x) & bc \\ ac & bc & c^2(1+x) \end{vmatrix}$ is divisible by
(A) $x + 3$
(B) $(1 + x)^2$
(C) x^2
(D) $x^2 + 1$

9. If A is a non-singular matrix and A^T denotes the transpose of A, then:
(A) $|A| \neq |A^T|$
(B) $|A.\ A^T| = |A|^2$
(C) $|A^T.\ A| = |A^T|^2$
(D) $|A|+|A^T| \neq 0$

10. Let $f(x) = \begin{vmatrix} 2\sin x & \sin^2 x & 0 \\ 1 & 2\sin x & \sin^2 x \\ 0 & 1 & 2\sin x \end{vmatrix}$, then
(A) f(x) is independent of x
(B) $f'(\pi/2) = 0$
(C) $\int_{-\pi/2}^{\pi/2} f(x)dx = 0$
(D) tangent to the curve $y = f(x)$ at $x = 0$ is $y = 0$

11. Let $\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$, then
(A) $1 - x^3$ is a factor of Δ
(B) $(1 - x^3)^2$ is factor of Δ
(C) $\Delta(x) = 0$ has 4 real roots
(D) $\Delta'(1) = 0$

12. Let $f(x) = \begin{vmatrix} 1/x & \log x & x^n \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$, then (where $f^n(x)$ denotes n^{th} derivative of f(x))
(A) $f^n(1)$ is indepedent of a
(B) $f^n(1)$ is indepedent of n
(C) $f^n(1)$ depends on a and n
(D) $y = a(x - f^n(1))$ represents a straight line through the origin

13. If D is a determinant of order three and Δ is a determinant formed by the cofactors of determinant D ; then
(A) $\Delta = D^2$
(B) $D = 0$ implies $\Delta = 0$
(C) if $D = 27$, then Δ is perfect cube
(D) if $D = 27$, then Δ is perfect square

14. Let A, B, C, D be real matrices such that $A^T = BCD$; $B^T = CDA$; $C^T = DAB$ and $D^T = ABC$ for the matrix $M = ABCD$, then find M^{2016} ?
(A) M
(B) M^2
(C) M^3
(D) M^4

15. Let **A** and **B** be two 2 × 2 matrix with real entries. If **AB** = **O** and tr(**A**) = tr(**B**) = 0 then
(A) **A** and **B** are comutative w.r.t. operation of multiplication.
(B) **A** and **B** are not commutative w.r.t. operation of multiplication.
(C) **A** and **B** are both null matrices.
(D) **BA** = 0

16. If $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, then

(A) $|A| = 2$ (B) A is non-singular

(C) Adj. A = $\begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & -1 & 1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$ (D) A is skew symmetric matrix

17. If A and B are square matrices of order 3, then the true statement is/are (where I is unit matrix).
(A) det $(-A) = -\det A$
(B) If AB is singular then atleast one of A or B is singular
(C) det $(A + I) = 1 + \det A$
(D) det $(2A) = 2^3 \det A$

18. Let **M** be a 3×3 non-singular matrix with det(**M**)=4. If $\mathbf{M}^{-1}$ adj(adj **M**) = $\mathbf{k}^2 I$, then the value of '**k**' may be:
(A) +2 (B) 4 (C) –2 (D) –4

19. If AX = B where A is 3×3 and X and B are 3×1 matrices then which of the following is correct?
(A) If $|A| = 0$ then AX = B has infinite solutions
(B) If AX = B has infinite solutions then $|A| = 0$
(C) If (adj(A)) B = 0 and $|A| \neq 0$ then AX = B has unique solution
(D) If (adj(A)) B $\neq$ 0 & $|A| = 0$ then AX = B has no solution

PART - IV : COMPREHENSION

Comprehension # 1

Let 𝒜 be the set of all 3×3 symmetric matrices whose entries are 1,1,1,0,0,0,–1, –1, –1. B is one of the matrix in set 𝒜 and

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $U = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $V = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

1*. Number of such matrices B in set 𝒜 is λ, then λ lies in the interval
(A) (30, 40) (B) (38, 40) (C) (34, 38) (D) (25, 35)

2*. Number of matrices B such that equation BX = U has infinite solutions
(A) is at least 6 (B) is not more than 10 (C) lie between 8 to 16 (D) is zero.

3*. The equation BX = V
(A) is inconsistent for atleast 3 matrices B.
(B) is inconsistent for all matrices B.
(C) is inconsistent for at most 12 matrices B.
(D) has infinite number of solutions for at least 3 matrices B.

Comprehension # 2

Some special square matrices are defined as follows :

Nilpotent matrix : A square matrix A is said to be nilpotent (of order 2) if, $A^2 = O$. A square matrix is said to be nilpotent of order p, if p is the least positive integer such that $A^p = O$.

Idempotent matrix : A square matrix A is said to be idempotent if, $A^2 = A$.

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an idempotent matrix.

Involutory matrix : A square matrix A is said to be involutory if $A^2 = I$, I being the identity matrix.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix.

Orthogonal matrix : A square matrix A is said to be an orthogonal matrix if $A'A = I = AA'$.

4. If A and B are two square matrices such that AB = A & BA = B, then A & B are
(A) Idempotent matrices (B) Involutory matrices
(C) Orthogonal matrices (D) Nilpotent matrices

5. If the matrix $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal, then

(A) $\alpha = \pm \frac{1}{\sqrt{2}}$ (B) $\beta = \pm \frac{1}{\sqrt{6}}$ (C) $\gamma = \pm \frac{1}{\sqrt{3}}$ (D) all of these

6. The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is

(A) idempotent matrix (B) involutory matrix (C) nilpotent matrix (D) none of these

Comprehension # 3

Rank is a number associated with a matrix which is the highest order of non-singular sub matrix.

7. Rank of the matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -1 & 0 \\ 2 & -7 & 4 \end{bmatrix}$ is

(A) 1 (B) 2 (C) 3 (D) 0

8. If the matrix $A = \begin{bmatrix} y+a & b & c \\ a & y+b & c \\ a & b & y+c \end{bmatrix}$ has rank 3, then

(A) $y \neq (a + b + c)$ (B) $y \neq 1$
(C) $y = 0$ (D) $y \neq -(a + b + c)$ and $y \neq 0$

9. If A & B are two square matrices of order 3 such that rank of matrix AB is two, then
(A) A & B both are singular (B) A & B both are non-singular
(C) Atleast one of A & B is singular (D) Atleast one of A & B is non-singular

Exercise-3

Marked Questions may have for Revision Questions.
*** Marked Questions may have more than one correct option.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. Consider the system of equations **[IIT-JEE 2008, Paper-1, (3, – 1), 163]**

$$x - 2y + 3z = -1$$
$$-x + y - 2z = k$$
$$x - 3y + 4z = 1$$

STATEMENT - 1 : The system of equations has no solution for $k \neq 3$
and

STATEMENT-2 : The determinant $\begin{vmatrix} 1 & 3 & 1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 3$.

(A) STATEMENT-1 is True, STATEMENT-2 is True ; STATEMENT-2 is a correct explanation for STATEMENT-1
(B) STATEMENT-1 is True, STATEMENT-2 is True ; STATEMENT-2 is **NOT** a correct explanation for STATEMENT-1
(C) STATEMENT-1 is True, STATEMENT-2 is False
(D) STATEMENT-1 is False, STATEMENT-2 is True

Answers

EXERCISE-1

PART - I

Section (A) :

A-1. $\begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$ **A-2.** (x, y, z, w) = (1, 2, 4, 5)

A-3. $\begin{bmatrix} 2 & -1/2 \\ 4 & -1 \end{bmatrix}$

A-4. $AB = \begin{bmatrix} 18 & -11 & 10 \\ -16 & 47 & 10 \\ 62 & -23 & 42 \end{bmatrix}$, $BA = \begin{bmatrix} 49 & 24 \\ -7 & 58 \end{bmatrix}$

A-7. $y \in R$ **A-9.** Zero

Section (B)

B-1. $0, \frac{3\pi}{4}, \pi$

B-2. (i) 0   (ii) 0   (iii) 0   (iv) $5(3\sqrt{2} - 5\sqrt{3})$

B-5. (i) $x = -2\,b/a$ (ii) 4

B-8. A = 0, B = 0

Section (C) :

C-1. $\begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$ **C-2.** (ii) $|A|^{(n-1)^2}$

C-3. $\begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Section (D) :

D-1. $a = -4, b = 1, A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

D-2. 0

D-4. (i) x = 3, y = 4, z = 6

(ii) $x = -\frac{5k}{3} + \frac{8}{3}$, $y = -\frac{2k}{3} - \frac{1}{3}$, $z = k$, where $k \in R$

D-5. x = 7, y = 4

D-6. for c = 0, x = –3, y = 3; for c = – 10, $x = -\frac{1}{2}, y = \frac{4}{3}$

D-7. **(i)** x = 2, y = 2, z = 2 **(ii)** x = 1, y = 3, z = 5

D-8. **(a)** $\lambda \neq 3$ **(b)** $\lambda = 3, \mu = 10$
(c) $\lambda = 3, \mu \neq 10$

D-9. x = 3, y = – 2, z = – 1

D-10. x = 1, y = 2, z = 3, $A^{-1} = \frac{1}{17}\begin{bmatrix} 1 & 5 & 1 \\ 8 & 6 & -9 \\ 10 & -1 & -7 \end{bmatrix}$

D-11. S1, S3, S4

PART - II

Section (A)

A-1. (A)   **A-2.** (D)   **A-3.** (A)
A-4. (C)   **A-5.** (B)   **A-6.** (C)
A-7. (A)   **A-8.** (A)   **A-9.** (A)

Section (B)

B-1. (B)   **B-2.** (A)   **B-3.** (A)
B-4. (D)   **B-5.** (D)   **B-6.** (D)
B-7. (B)   **B-8.** (C)   **B-9.** (B)
B-10. (B)

Section (C)

C-1. (A)   **C-2.** (C)   **C-3.** (A)
C-4. (A)   **C-5.** (C)   **C-6.** (A)

Section (D)

D-1. (A)   **D-2** (A)   **D-3.** (D)
D-4. (B)   **D-5.** (B)   **D-6.** (A)

PART - III

1. (A) → (s), (B) → (p), (C) → (p), (D) → (p)
2. (A)→(q), (B)→(p), (C)→(s), (D)→(q)

EXERCISE-2

PART - I

1. (D)   **2.** (A)   **3.** (D)   **4.** (C)
5. (D)   **6.** (A)   **7.** (C)   **8.** (A)
9. (A)   **10.** (A)   **11.** (A)   **12.** (B)
13. (A)   **14.** (A)   **15.** (A)   **16.** (C)
17. (B)   **18.** (B)   **19.** (A)   **20.** (A)

PART - II

1. 2   **2.** 64   **3.** 36   **4.** 2
5. 50   **6.** 4   **7.** 0   **8.** 63
9. 4   **10.** 6   **11.** 2   **12.** 1
13. 2   **14.** 33   **15.** 80   **16.** $\lambda = 39$
17. 17

PART - III

1. (AD)   **2.** (BC)   **3.** (ABCD)   **4.** (ABD)
5. (AB)   **6.** (ABC)   **7.** (BD)   **8.** (AC)
9. (BCD)   **10.** (BCD)   **11.** (ABD)   **12.** (ABD)
13. (ABCD)   **14.** (BD)   **15.** (AD)   **16.** (BC)
17. (ABD)   **18.** (AC)   **19.** (BCD)

PART - IV

1. (AC)   **2.** (AC)   **3.** (AC)   **4.** (A)
5. (D)   **6.** (C)   **7.** (B)   **8.** (D)
9. (C)

EXERCISE-3

PART - I

1. (A)   **2.** (A)→(s),(B)→(p,q),(C)→(r),(D)→(p,q,s)
3. (A)   **4.** (B)   **5.** (B)   **6.** (D)
7. (C)   **8.** (D)   **9.** 3   **10.** 4
11. (C)   **12.** (D)   **13.** (A)   **14.** (B)
15. (A)   **16.** 9   **17.** (D)   **18.** (D)
19. (AD)   **20.** (CD)   **21.** (CD)
22. (AB)   **23.** (CD)   **24.** (BC)   **25.** (BC)
26. 2   **27.** (B)   **28.** (BCD)
29. (AC)   **30.** 1   **31.** (A)

PART - II

1. (3)   **2.** (1)   **3.** (3)   **4.** (3)
5. (4)   **6.** (3)   **7.** (4)   **8.** (1)
9. (3)   **10.** (3)   **11.** (2)   **12.** (3)
13. (2)   **14.** (2)   **15.** (4)   **16.** (1)
17. (4)   **18.** (4)   **19.** (3)   **20.** (2)
21. (2)   **22.** (1)   **23.** (4)   **24.** (4)
25. (3)   **26.** (3)   **27.** (1)   **27.** (4)
29. (2)