

Complex Numbers

The shortest path between two truths in the real domain passes through the complex domain.Hadamard, Jacques

The complex number system

Indian mathematician Mahavira (850 A.D.) was first to mention in his work 'Ganitasara Sangraha'; 'As in nature of things a negative (quantity) is not a square (quantity), it has, therefore, no square root'. Hence there is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$.

A symbol $\sqrt{-1}$, denoted by letter i was introduced by Swiss Mathematician, Leonhard Euler (1707-1783) in 1748 to provide solutions of equation $x^2 + 1 = 0$. i was regarded as a fictitious or imaginary number which could be manipulated algebraically like an ordinary real number, except that its square was -1 . The letter i was used to denote $\sqrt{-1}$, possibly because i is the first letter of the Latin word 'imaginarius'.

To permit solutions of such polynomial equations, the set of complex numbers is introduced. We can consider a complex number as having the form $a + bi$ where a and b are real numbers. It is denoted by z i.e. $z = a + ib$. ' a ' is called as real part of z which is denoted by $(\text{Re } z)$ and ' b ' is called as imaginary part of z , which is denoted by $(\text{Im } z)$.

Any complex number is :

- (i) Purely real, if $b = 0$;
- (ii) Imaginary, if $b \neq 0$.
- (iii) Purely imaginary, if $a = 0$

Note :

- (a) The set R of real numbers is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.
- (b) Zero is purely real as well as purely imaginary but not imaginary.
- (c) $i = \sqrt{-1}$ is called the imaginary unit.
Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.
- (d) $\sqrt{a} \sqrt{b} = \sqrt{ab}$ only if atleast one of a or b is non-negative.
- (e) If $z = a + ib$, then $a - ib$ is called complex conjugate of z and written as $\bar{z} = a - ib$
- (f) Real numbers satisfy order relations whereas imaginary numbers do not satisfy order relations i.e. $i > 0$, $3 + i < 2$ are meaningless.

Self Practice Problems

- (1) Write the following as complex number
 - (i) $\sqrt{-16}$
 - (ii) \sqrt{x} , ($x > 0$)
 - (iii) $-b + \sqrt{-4ac}$, ($a, c > 0$)
- (2) Write the following as complex number
 - (i) \sqrt{x} ($x < 0$)
 - (ii) roots of $x^2 - (2 \cos \theta)x + 1 = 0$

- Answers :**
- | | | | |
|-----|---------------------|--|--------------------------|
| (1) | (i) $0 + 4i$ | (ii) $\sqrt{x} + 0i$ | (iii) $-b + i\sqrt{4ac}$ |
| (2) | (i) $0 + i\sqrt{x}$ | (ii) $\cos \theta + i \sin \theta$, $\cos \theta - i \sin \theta$ | |

Complex Numbers

Algebraic Operations:

Fundamental operations with complex numbers

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by -1 when it occurs.

1. Addition $(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$
2. Subtraction $(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$
3. Multiplication $(a + bi)(c + di) = ac + adi + bci + bd^2 = (ac - bd) + (ad + bc)i$
4. Division $\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac - adi + bci - bd^2}{c^2 - d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$

Inequalities in imaginary numbers are not defined. There is no validity if we say that imaginary number is positive or negative. e.g., $z > 0, 4 + 2i < 2 + 4i$ are meaningless.

In real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$ however in complex numbers,

$z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

Example # 1 : Find the multiplicative inverse of $4 + 3i$.

Solution : Let z be the multiplicative inverse of $4 + 3i$ then

$$z = \frac{1}{4+3i} \times \frac{4-3i}{4-3i} = \frac{4-3i}{16+9} = \frac{4-3i}{25}. \quad \text{Ans. } \frac{4-3i}{25}$$

Self Practice Problem :

(3) Simplify $i^n + i^{n+1} + i^{n+2} + i^{n+3}$, $n \in \mathbb{N}$. Ans. 0

Equality In Complex Number:

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal respectively i.e., $z_1 = z_2 \Leftrightarrow \operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

Example # 2 : Find the value of x and y for which $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2iy)$, where $x, y \in \mathbb{R}$

Solution : $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2iy)$, where $x, y \in \mathbb{R}$
 $\Rightarrow x^4 - 3x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$
and $2x - y = -5 + 2y$
 $2x + 5 = 3y$

when $x = 2 \Rightarrow y = 3$
and $x = -2 \Rightarrow y = 1/3$ Ans. (2,3) or (-2,1/3)

Example # 3 : Find the value of expression $x^4 + 4x^3 + 5x^2 + 2x + 3$, when $x = -1 + i$.

Solution : $x = -1 + i$
 $(x + 1)^2 = i^2$

$$\begin{aligned} x^2 + 2x + 2 &= 0 \\ \text{now, } x^4 + 4x^3 + 5x^2 + 2x + 3 &= (x^2 + 2x + 2)(x^2 + 2x - 1) + 5 = 5 \end{aligned}$$

Example # 4 : Find the square root of $-21 - 20i$

Solution : Let $x + iy = \sqrt{-21 - 20i}$

$$\begin{aligned} (x + iy)^2 &= -21 - 20i \\ x^2 - y^2 &= -21 \\ xy &= -10 \\ \text{From (i) & (ii)} \\ x^2 - 4 &\Rightarrow x = \pm 2 \\ \text{when } x = 2, y = -5 \text{ and } x = -2, y = 5 \\ x + iy &= (2 - 15) \text{ or } (-2 + 15) \end{aligned}$$

Complex Numbers

Self Practice Problem

- (4) Solve for $z : \bar{z} = iz^2$
(5) Given that $x, y \in \mathbb{R}$, solve : $4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i$

Answers : (4) $\pm \frac{\sqrt{3}}{2} - \frac{1}{2}i, 0, i$ (5) $x = K, y = \frac{3K}{2}, K \in \mathbb{R}$

Representation of a complex number :

Early in the 19th century, Karl Friedrich Gauss (1777 – 1855) and William Rowan Hamilton (1805 – 1865) independently and almost simultaneously proposed idea of defining complex number as ordered pair of real numbers. i.e. $a + ib = (a, b)$. To each complex number there corresponds one and only one point in plane, and conversely to each point in the plane there corresponds one and only one complex number. Because of this we often refer to the complex number z as the point Z .

Cartesian Form (Geometric Representation) : Every complex number $z = x + iy$ can be represented by a point on the Cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y) .



Length OP is called modulus of the complex number which is denoted by $|z|$ & θ is called argument or amplitude.

$$|z| = \sqrt{x^2 + y^2} \text{ and } \tan \theta = \left(\frac{y}{x} \right) \text{ (angle made by OP with positive x-axis)}$$

Note : Argument of a complex number is a many valued function. If θ is the argument of a complex number then $2n\pi + \theta$; $n \in \mathbb{Z}$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

- (i) The unique value of θ such that $-\pi < \theta \leq \pi$ is called the principal value of the argument.
- (ii) Unless otherwise stated, $\operatorname{arg} z$ implies principal value of the argument.
- (iii) By specifying the modulus & argument a complex number is defined completely. For the complex number $0 + 0i$ the argument is not defined and this is the only complex number which is only given by its modulus.

(b) Trigonometric/Polar Representation :

$z = r(\cos \theta + i \sin \theta)$ where $|z| = r$, $\operatorname{arg} z = \theta$; $\bar{z} = r(\cos \theta - i \sin \theta)$

Note : $\cos \theta + i \sin \theta$ is also written as $CiS \theta$

(c) Euler's Formula :

$$z = re^{i\theta}, |z| = 1, \operatorname{arg} z = \theta$$

$$\bar{z} = re^{-i\theta}$$

Proof of this formula is beyond scope of present discussion. A heuristic proof serving as motivation for this formula is by considering expansion.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\text{put } x = i\theta$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right) = \cos \theta + i \sin \theta$$

Note : If θ is real then $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$; $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

(d)

Vectorial Representation:

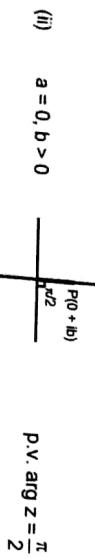
Every complex number can be considered as the position vector of a point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

Argument of a Complex Number:

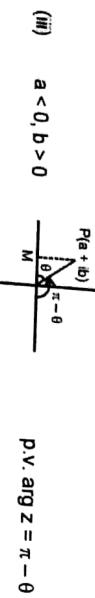
Argument of a non-zero complex number $P(z)$ is denoted and defined by $\arg(z) = \text{angle which } OP \text{ makes with the positive direction of real axis.}$
 If $|OP| = |z| = r$ and $\arg(z) = \theta$, then obviously $z = r(\cos\theta + i\sin\theta)$, called the polar form of z .
 'Argument of z' would mean principal argument of z (i.e. argument lying in $(-\pi, \pi]$ unless the context requires otherwise. Thus argument of a complex number $z = a + bi = r(\cos\theta + i\sin\theta)$ is the value of θ satisfying $r\cos\theta = a$ and $r\sin\theta = b$. Let $\theta = \tan^{-1} \left| \frac{b}{a} \right|$.



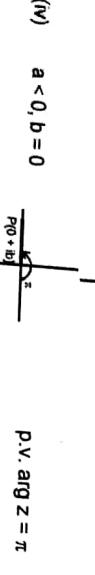
$$\text{p.v. arg } z = \theta$$



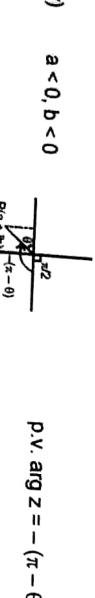
$$\text{p.v. arg } z = \frac{\pi}{2}$$



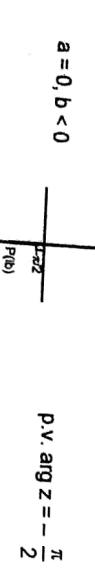
$$\text{p.v. arg } z = \pi - \theta$$



$$\text{p.v. arg } z = \pi$$



$$\text{p.v. arg } z = -(\pi - \theta)$$



$$\text{p.v. arg } z = -\frac{\pi}{2}$$



$$\text{p.v. arg } z = \theta$$

(viii)

 $a > 0, b = 0$ 

$$\text{p.v. arg } z = 0$$

Example # 5 : Solve for z if $(\bar{z})^2 + 2|z| = 0$.

Solution : Let $z = x + iy$

$$\Rightarrow x^2 - y^2 - 2ixy + 2\sqrt{x^2 + y^2} = 0 \Rightarrow x^2 - y^2 + 2\sqrt{x^2 + y^2} = 0 \text{ and } 2xy = 0$$

$$\text{when } x = 0, \Rightarrow -y^2 + 2|y| = 0 \Rightarrow y = 0, 2, -2$$

$$\Rightarrow z = 0, 2i, -2i$$

$$\text{when } y = 0 \Rightarrow x^2 + 2|x| = 0$$

$$\Rightarrow x = 0 \Rightarrow z = 0$$

$$\text{Ans. } z = 0, 2i, -2i$$

Example # 6 : Find the modulus and principal argument of complex number $z = 1 + i \tan \alpha$, $\pi < \alpha < \frac{3\pi}{2}$

Solution : $|z| = \sqrt{1 + \tan^2 \alpha} = |\sec \alpha| = -\sec \alpha$, where $\pi < \alpha < \frac{3\pi}{2}$

$$\arg(z) = \tan^{-1} \left| \frac{\tan \alpha}{1} \right| = \tan^{-1} (\tan \alpha) = \alpha - \pi$$

$$\text{Ans. } -\sec \alpha, \alpha - \pi$$

Self Practice Problems

(6) Find the principal argument and $|z|$. If $z = \frac{(2+i)(3-4i)}{3+i}$

(7) Find the $|z|$ and principal argument of the complex number $z = -8(\cos 310^\circ - i \sin 310^\circ)$

Answers : (6) $-\pi/4, \frac{5\sqrt{2}}{2}$ (7) $8, -130^\circ$

Demoivre's Theorem:**Case I****Statement :**

If n is any integer then

$$(i) (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

$$(ii) (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)(\cos \theta_3 + i \sin \theta_3) \dots (\cos \theta_n + i \sin \theta_n)$$

$$= \cos(\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n)$$

Case II

Statement : If $p, q \in Z$ and $q \neq 0$ then

$$(\cos \theta + i \sin \theta)^{p/q} = \cos \left(\frac{2k\pi + p\theta}{q} \right) + i \sin \left(\frac{2k\pi + p\theta}{q} \right)$$

where $k = 0, 1, 2, 3, \dots, q-1$

Note : Continued product of the roots of a complex quantity should be determined using theory of equations.

Self practice problems :

(8) Prove the identities :

$$(a) \cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta,$$

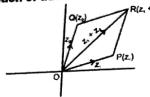
$$(b) (\sin 5\theta)/(\sin \theta) = 16 \cos^4 \theta - 12 \cos^2 \theta + 1, \text{ if } \theta \neq 0, \pm \pi, \pm 2\pi \dots$$

Complex Numbers

- (9) Prove that identities
 (a) $\sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$, (b) $\cos^4 \theta = \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}$

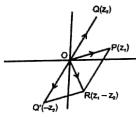
Geometrical Representation of Fundamental Operations :

(i) Geometrical representation of addition.



If two points P and Q represent complex numbers z_1 and z_2 respectively in the Argand plane, then the sum $z_1 + z_2$ is represented by the extremity R of the diagonal OR of parallelogram OPRQ having OP and OQ as two adjacent sides.

(ii) Geometric representation of subtraction.



(iii) Modulus and argument of multiplication of two complex numbers.

Theorem : For any two complex numbers z_1, z_2 we have $|z_1 z_2| = |z_1| |z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

Proof : $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \Rightarrow |z_1 z_2| = |z_1| |z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

i.e. to multiply two complex numbers, we multiply their absolute values and add their arguments.

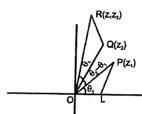
Note : (i) $P.V. \arg(z_1 z_2) \neq P.V. \arg(z_1) + P.V. \arg(z_2)$

$$(ii) |z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|$$

$$(iii) \arg(z_1 z_2 \dots z_n) = \arg z_1 + \arg z_2 + \dots + \arg z_n$$

(iv) Geometrical representation of multiplication of complex numbers.

Let P, Q be represented by $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$ respectively. To find point R representing complex number $z_1 z_2$, we take a point L on real axis such that $OL = 1$ and draw triangle OQR similar to triangle OLP. Therefore



$$\frac{OR}{OQ} = \frac{OP}{OL} \Rightarrow OR = OP \cdot OQ \quad \text{i.e.} \quad OR = r_1 r_2 \quad \text{and} \quad \angle QOR = \theta_1$$

$$\hat{\angle}OR = \hat{\angle}OP + \hat{\angle}POQ + \hat{\angle}QOR = \theta_1 + \theta_2 - \theta_1 + \theta_1 = \theta_1 + \theta_2$$

$$\text{Hence, } R \text{ is represented by } z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

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(v) Modulus and argument of division of two complex numbers.

Theorem : If z_1 and z_2 ($\neq 0$) are two complex numbers, then $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

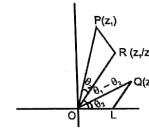
Note : $P.V. \arg\left(\frac{z_1}{z_2}\right) \neq P.V. \arg(z_1) - P.V. \arg(z_2)$

(vi) Geometrical representation of the division of complex numbers.

Let P, Q be represented by $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$ respectively. To find point R representing complex number $\frac{z_1}{z_2}$, we take a point L on real axis such that $OL = 1$ and draw a triangle OPR similar to OQL.

$$\text{Therefore } \frac{OP}{OQ} = \frac{OR}{OL} \Rightarrow OR = \frac{r_1}{r_2}$$

$$\text{and } \hat{\angle}OR = \hat{\angle}OP - \hat{\angle}OP = \theta_1 - \theta_2$$



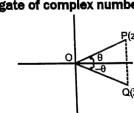
$$\text{Hence, } R \text{ is represented by } \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Conjugate of a complex Number :

Conjugate of a complex number $z = a + ib$ is denoted and defined by $\bar{z} = a - ib$.

In a complex number if we replace i by $-i$, we get conjugate of the complex number. \bar{z} is the mirror image of z about real axis on Argand's Plane.

Geometrical representation of conjugate of complex number.



$$|z| = |\bar{z}|$$

$$\arg(\bar{z}) = -\arg(z)$$

$$\text{General value of } \arg(\bar{z}) = 2\pi - P.V. \arg(z)$$

Properties

$$(i) \text{ If } z = x + iy, \text{ then } x = \frac{z + \bar{z}}{2}, y = \frac{z - \bar{z}}{2i}$$

$$(ii) z = \bar{z} \Leftrightarrow z \text{ is purely real}$$

$$(iii) z + \bar{z} = 0 \Leftrightarrow z \text{ is purely imaginary}$$

$$(iv) \text{ Relation between modulus and conjugate. } |z|^2 = z \bar{z}$$

$$(v) \bar{\bar{z}} = z$$

$$(vi) (z_1 \pm z_2) = \bar{z}_1 \pm \bar{z}_2$$

$$(vii) \overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \text{ In general } (\bar{z}^n) = (\bar{z})^n$$

$$(viii) \overline{\left(\frac{z_1}{z_2} \right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad (z_2 \neq 0)$$

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Theorem : Imaginary roots of polynomial equations with real coefficients occur in conjugate pairs

Proof : If z_0 is a root of $a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0$,

$a_0, a_1, \dots, a_n \in \mathbb{R}$, then $a_0 z_0^n + a_1 z_0^{n-1} + \dots + a_{n-1} z_0 + a_n = 0$

By using property (vi) and (vii) we have $a_0 \bar{z}_0^n + a_1 \bar{z}_0^{n-1} + \dots + a_{n-1} \bar{z}_0 + a_n = 0$

$\Rightarrow \bar{z}_0$ is also a root.

Note : If $w = f(z)$, then $\bar{w} = f(\bar{z})$

Theorem : $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1 \bar{z}_2 + \bar{z}_1 z_2) = |z_1|^2 + |z_2|^2 \pm 2 \operatorname{Re}(z_1 \bar{z}_2)$
 $= |z_1|^2 + |z_2|^2 \pm 2 |z_1| |z_2| \cos(\theta_1 - \theta_2)$

Example # 7 : If $\frac{z-1}{z+1}$ is purely imaginary, then prove that $|z| = 1$

$$\begin{aligned} \text{Solution : } \operatorname{Re}\left(\frac{z-1}{z+1}\right) &= 0 \quad \Rightarrow \quad \frac{z-1}{z+1} + \left(\frac{\bar{z}-1}{\bar{z}+1}\right) = 0 \\ &\Rightarrow \frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1} = 0 \\ &\Rightarrow \frac{z\bar{z}-z-\bar{z}+1}{z\bar{z}+z-\bar{z}+1} = 0 \\ &\Rightarrow z\bar{z} = 1 \quad \Rightarrow \quad |z|^2 = 1 \quad \Rightarrow \quad |z| = 1 \text{ Hence proved} \end{aligned}$$

Self Practice Problem

(10) If $|z + \alpha| > |\bar{z}z + 1|$ and $|\alpha| > 1$, then show that $|z| < 1$.

(11) If $z = x + iy$ and $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, then show that $f(z) = \bar{z}^2 + 2iz$

(12) If $x + iy = \sqrt{\frac{a+ib}{c+id}}$ prove that $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$

Distance, Triangular Inequality

If $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, then distance between points z_1, z_2 in argand plane is

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

In triangle OAC

$$OC \leq OA + AC$$

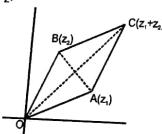
$$OA \leq AC + OC$$

$$AC \leq OA + OC$$

using these inequalities we have $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

Similarly from triangle OAB

we have $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$



Note :

(a) $||z_1| - |z_2|| = |z_1 + z_2|, |z_1 - z_2| = |z_1| + |z_2|$ iff origin, z_1 and z_2 are collinear and origin lies between z_1 and z_2 .

(b) $|z_1 + z_2| = |z_1| + |z_2|, ||z_1| - |z_2|| = |z_1 - z_2|$ iff origin, z_1 and z_2 are collinear and z_1 and z_2 lie on the same side of origin.

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Example # 8 : If $|z - 5 - 7i| = 9$, then find the greatest and least values of $|z - 2 - 3i|$.

Solution : We have $9 = |z - (5 + 7i)|$ = distance between z and $5 + 7i$.
 Thus locus of z is the circle of radius 9 and centre at $5 + 7i$. For such a z (on the circle), we have to find its greatest and least distance as from $2 + 3i$, which obviously 14 and 4.

Example # 9 : Find the minimum value of $|z| + |z - 2|$

Solution : $|z| + |z - 2| \geq |z + 2 - z|$

$$|z| + |z - 2| \geq 2$$

Example # 10 : If $\theta_i \in [\pi/6, \pi/3]$, $i = 1, 2, 3, 4, 5$, and $z^4 \cos \theta_1 + z^3 \cos \theta_2 + z^2 \cos \theta_3 + z \cos \theta_4 + \cos \theta_5 = 2\sqrt{3}$,

then show that $|z| > \frac{3}{4}$

Solution : Given that $\cos \theta_1 \cdot z^4 + \cos \theta_2 \cdot z^3 + \cos \theta_3 \cdot z^2 + \cos \theta_4 \cdot z + \cos \theta_5 = 2\sqrt{3}$

or $|\cos \theta_1 \cdot z^4 + \cos \theta_2 \cdot z^3 + \cos \theta_3 \cdot z^2 + \cos \theta_4 \cdot z + \cos \theta_5| = 2\sqrt{3}$

$$2\sqrt{3} \leq |\cos \theta_1 \cdot z^4| + |\cos \theta_2 \cdot z^3| + |\cos \theta_3 \cdot z^2| + |\cos \theta_4 \cdot z| + |\cos \theta_5|$$

$$\therefore \theta_i \in [\pi/6, \pi/3]$$

$$\therefore \frac{1}{2} \leq \cos \theta_i \leq \frac{\sqrt{3}}{2}$$

$$2\sqrt{3} \leq \frac{\sqrt{3}}{2} |z|^4 + \frac{\sqrt{3}}{2} |z|^3 + \frac{\sqrt{3}}{2} |z|^2 + \frac{\sqrt{3}}{2} |z| + \frac{\sqrt{3}}{2}$$

$$3 \leq |z|^4 + |z|^3 + |z|^2 + |z|$$

Case I : If $|z| \geq 1$, then above result is automatically true

Case II : If $|z| < 1$, then

$$3 < |z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + \dots \dots \dots \infty$$

$$3 < \frac{|z|}{1-|z|} \Rightarrow 3 - 3|z| < |z| \Rightarrow |z| > \frac{3}{4} \text{ Hence by both cases, } |z| > \frac{3}{4}$$

Example # 11 : If $|z - \frac{3}{z}| = 2$, then find maximum and minimum value of $|z|$.

$$\left| |z| - \frac{3}{|z|} \right| \leq \left| z - \frac{3}{z} \right|$$

Let $|z| = r$

$$\left| r - \frac{3}{r} \right| \leq 2 \Rightarrow -2 \leq r - \frac{3}{r} \leq 2$$

$$r^2 + 2r - 3 \geq 0 \quad \dots \dots \text{(i)} \quad \text{and} \quad r^2 - 2r - 3 \leq 0 \quad \dots \dots \text{(ii)}$$

$$\Rightarrow r \in [1, 3]$$

from (i) and (ii)

$$|z|_{\max} = 3 \text{ and } |z|_{\min} = 1.$$

Self Practice Problem

(13) $|z - 3| < 1$ and $|z - 4i| > M$ then find the positive real value of M for which there exist at least one complex number z satisfying both the equation.

Complex Numbers

(14) If z lies on circle $|z| = 2$, then show that $\left| \frac{1}{z^4 - 4z^2 + 3} \right| \leq \frac{1}{3}$

Answers : (13) $M \in (0, 6)$

Important results :

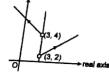
(i) $\arg z = \theta$ represents points (non-zero) on ray 

emanating from origin making an angle θ with positive direction of real axis

(ii) $\arg(z - z_1) = \theta$ represents points ($>z_1$) on ray emanating from z_1 making an angle θ with positive direction of real axis 

Example # 12: Solve for z , which satisfy $\text{Arg}(z - 3 - 2i) = \frac{\pi}{6}$ and $\text{Arg}(z - 3 - 4i) = \frac{2\pi}{3}$.

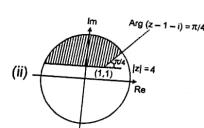
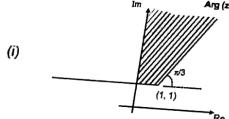
Solution : From the figure, it is clear that there is no z , which satisfy both ray



Example # 13: Sketch the region given by

(i) $\pi/2 \geq \text{Arg}(z - 1 - i) \geq \pi/3$

(ii) $|z| \leq 4$ & $\text{Arg}(z - i - 1) > \pi/4$



Self Practice Problems

(15) Sketch the region given by

(i) $|\text{Arg}(z - i - 2)| < \pi/4$

(ii) $\text{Arg}(z + 1 - i) \leq \pi/6$

(16) Consider the region $|z - 4 - 3i| \leq 3$. Find the point in the region which has

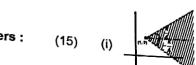
(i) $\max |z|$

(ii) $\min |z|$

(iii) $\max \arg(z)$

(iv) $\min \arg(z)$

Complex Numbers

Answers : (15) (i) 

(16) (i) $\frac{32}{5} + i\frac{24}{5}$

(ii) $\frac{8}{5} + i\frac{6}{5}$

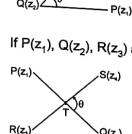
(iii) $\frac{28}{25} + i\frac{96}{25}$

(iv) $4 + 0i$

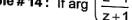
Rotation theorem :

(i) If $P(z_1)$ and $Q(z_2)$ are two complex numbers such that $|z_1| = |z_2|$, then $z_2 = z_1 e^{i\theta}$ where $\theta = \angle POQ$

(ii) If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then $\left(\frac{z_3 - z_2}{z_1 - z_2} \right) = \left| \frac{z_3 - z_2}{z_1 - z_2} \right| e^{i\theta}$



(iii) If $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers and $\angle STQ = \theta$, then $\left| \frac{z_3 - z_4}{z_1 - z_2} \right| e^{i\theta}$



Example # 14 : If $\arg \left(\frac{z-1}{z+1} \right) = \frac{\pi}{4}$ then interpret the locus.

Solution : $\arg \left(\frac{z-1}{z+1} \right) = \frac{\pi}{4} \Rightarrow \arg \left(\frac{1-z}{-1-z} \right) = \frac{\pi}{4}$

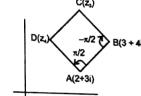
Here $\arg \left(\frac{1-z}{-1-z} \right)$ represents the angle between lines joining -1 and z , and 1 and z . As this angle is constant, the locus of z will be a larger segment of circle. (angle in a segment is constant).

Example # 15 : If $A(2+3i)$ and $B(3+4i)$ are two vertices of a square ABCD (take in anticlock wise order) then find C and D.

Solution : Let affix of C and D are z_3 and z_4 respectively.

Considering $\angle DAB = 90^\circ$ and $AD = AB$

$$\text{we get } \frac{z_4 - (2+3i)}{(3+4i) - (2+3i)} = \frac{AD}{AB} e^{i\frac{\pi}{2}}$$



$$\Rightarrow z_4 - (2+3i) = (1+i)i \quad \Rightarrow \quad z_4 = 2+3i+i-1 = 1+4i$$

$$\text{and } \frac{z_3 - (3+4i)}{(2+3i) - (3+4i)} = \frac{CB}{AB} e^{-i\frac{\pi}{2}} \quad \Rightarrow \quad z_3 = 3+4i-(1+i)(-i)$$

$$\Rightarrow z_3 = 3+4i+i-1 = 2+5i$$

Complex Numbers

Self Practice Problems

- (17) Let ABC be an isosceles triangle inscribed in the circle $|z|=r$ with $AB=AC$. If z_1, z_2, z_3 represent the points A, B, C respectively, show that $z_1 z_2 z_3 = z_1^2$
- (18) Check that $z_1 z_2$ and $z_2 z_3$ are parallel or, not
where,

$$\begin{aligned} z_1 &= 1+i & z_2 &= 4+2i \\ z_2 &= 2-i & z_3 &= 1-i \end{aligned}$$
- (19) P is a point on the argand diagram on the circle with OP as diameter, two point Q and R are taken such that $\angle POQ = \angle QOR = \theta$. If O is the origin and P, Q, R are represented by complex z_1, z_2, z_3 respectively then show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2 \theta$
- (20) If a, b, c, u, v, w are complex numbers representing the vertices of two triangles such that $c = (1-r)a + rb, w = (1-r)u + rv$ where r is a complex number show that the two triangles are similar.

Answers : (18) $z_1 z_2$ and $z_2 z_3$ are not parallel.

Cube Root of Unity :

- (i) The cube roots of unity are $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$.
- (ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$ where $r \in \mathbb{Z}$ but is not the multiple of 3.
- (iii) In polar form the cube roots of unity are:

$$\cos 0 + i \sin 0, \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}, \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$$
- (iv) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.
- (v) The following factorisation should be remembered:

$$(a, b, c \in \mathbb{R} \text{ and } \omega \text{ is the cube root of unity})$$

$$a^3 - b^3 = (a-b)(a-\omega b)(a-\omega^2 b); \quad x^2 + x + 1 = (x-\omega)(x-\omega^2)$$

$$a^3 + b^3 = (a+b)(a+\omega b)(a+\omega^2 b); \quad a^2 + ab + b^2 = (a-b\omega)(a-b\omega^2)$$

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$$

Example # 16 : Find the value of $\omega^{200} + \omega^{198} + \omega^{193}$.
 Solution : $\omega^{200} + \omega^{198} + \omega^{193}$
 $\omega^2 + 1 + \omega = 0$.

Example # 17 If W is an imaginary cube root of unity then find the value of $\frac{1}{1+2w} + \frac{1}{2+w} - \frac{1}{1+w}$

Solution :
$$\begin{aligned} \frac{1}{1+w+w} + \frac{1}{1+(1+w)} - \frac{1}{1+w} &= \frac{1}{-w^2+w} + \frac{1}{1-w^2} - \frac{1}{-w^2} \\ &= \frac{1}{w(1-w)} + \frac{1}{(1-w)^2} - \frac{1}{w^2} = \frac{w(1+w) + w^2 + 1 - w^2}{w^2(1-w^2)} = \frac{1+w+w^2}{w^2(1-w^2)} = 0 \\ \text{Ans. 0} \end{aligned}$$

Self Practice Problem

- (21) Find $\sum_{r=0}^{100} (1 + \omega^r + \omega^{2r})$
- (22) It is given that n is an odd integer greater than three, but n is not a multiple of 3. Prove that $x^3 + x^2 + x$ is a factor of $(x+1)^n - x^n - 1$

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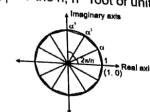
Complex Numbers

- (23) If $x = a + bi, y = a\alpha + b\beta, z = a\beta + b\alpha$ where α, β are imaginary cube roots of unity show that $xyz = a^3 + b^3$
- (24) If $x^2 - x + 1 = 0$, then find the value of $\sum_{n=1}^5 \left(x^n + \frac{1}{x^n} \right)^2$

Answers : (21) 102 (24) 8

n^{th} Roots of Unity :

If $1, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-1}$ are the n^{th} root of unity then :



- (i) They are in G.P. with common ratio $e^{i(2\pi/n)}$
 (ii) $1^p + \alpha_1^p + \alpha_2^p + \dots + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n
 $= n$ if p is an integral multiple of n
 (iii) $(1 - \alpha_1)(1 - \alpha_2) \dots (1 - \alpha_{n-1}) = n$
 $(1 + \alpha_1)(1 + \alpha_2) \dots (1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.
 (iv) $1, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-1} = 1$ or -1 according as n is odd or even.

Example # 18 : Find the roots of the equation $z^5 = -32i$, whose real part is negative.
 Solution : $z^5 = -32i$

$$z^5 = 2^5 e^{i(4n-1)\frac{\pi}{2}}, \quad n = 0, 1, 2, 3, 4.$$

$$z = 2e^{i(4n-1)\frac{\pi}{10}}$$

$z = 2e^{-\frac{\pi}{10}}, 2e^{\frac{3\pi}{10}}, 2e^{\frac{7\pi}{10}}, 2e^{\frac{11\pi}{10}}, 2e^{\frac{15\pi}{10}}$ roots with negative real part are $2e^{\frac{7\pi}{10}}, 2e^{\frac{11\pi}{10}}$.

Example # 19 : Find the value $\sum_{k=1}^6 \left(\sin \frac{2\pi k}{7} - \cos \frac{2\pi k}{7} \right)$

Solution :
$$\begin{aligned} \sum_{k=1}^6 \left(\sin \frac{2\pi k}{7} \right) - \sum_{k=1}^6 \left(\cos \frac{2\pi k}{7} \right) &= \sum_{k=0}^6 \sin \frac{2\pi k}{7} - \sum_{k=0}^6 \cos \frac{2\pi k}{7} + 1 \\ &= \sum_{k=0}^6 \text{(Sum of imaginary part of seven seventh roots of unity)} \\ &- \sum_{k=0}^6 \text{(Sum of real part of seven seventh roots of unity)} + 1 = 0 - 0 + 1 = 1 \end{aligned}$$

Self Practice Problems

- (25) If $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the fifth roots of unity then find $\sum_{i=1}^4 \frac{1}{2 - \alpha_i}$

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Complex Numbers

(26) If α, β, γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$ and ω is a complex cube root of unity then prove that $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = 3\omega^2$

(27) Find all values of $(-256)^{1/4}$ Interpret the result geometrically.

Answers : (25) $\frac{49}{31}$
(27) $4 \left[\cos\left(\frac{2r+1}{4}\right)\pi + i \sin\left(\frac{2r+1}{4}\right)\pi \right], r = 0, 1, 2, 3;$ vertices of a square in a circle of radius 4 & centre (0, 0)

The Sum Of The Following Series Should Be Remembered :

$$(i) \cos \theta + \cos 2\theta + \cos 3\theta + \dots + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\frac{n+1}{2}\theta\right).$$

$$(ii) \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\frac{n+1}{2}\theta\right).$$

Note : If $\theta = (2n/\pi)$ then the sum of the above series vanishes.

Logarithm Of A Complex Quantity :

$$\text{Log}_e(\alpha + i\beta) = \frac{1}{2} \text{Log}_e(\alpha^2 + \beta^2) + i \left(2n\pi + \tan^{-1} \frac{\beta}{\alpha} \right) \text{ where } n \in \mathbb{Z}.$$

Example # 20 : Find the value of

- (i) $\log(1 + \sqrt{3}i)$ Ans. $\log 2 + i(2n\pi + \frac{\pi}{3})$
- (ii) $\log(-1)$ Ans. $i(2n+1)\pi$
- (iii) 2^i Ans. $\cos(\ln 2) + i \sin(\ln 2) = e^{i(\ln 2)}$
- (iv) i^i Ans. $e^{-(4n+1)\frac{\pi}{2}}$
- (v) $|(1+i)^i|$ Ans. $e^{-(8n+1)\frac{\pi}{4}}$
- (vi) $\arg((1+i)^i)$ Ans. $\frac{1}{2}i\ln(2)$.

Solution : (i) $\log(1 + \sqrt{3}i) = \log\left(2 e^{i\left(\frac{\pi}{3} + 2n\pi\right)}\right) = \log 2 + i\left(\frac{\pi}{3} + 2n\pi\right)$

(ii) $\log(-1) = \log 1 + i(2n\pi + \pi) = i\pi(2n+1)$

(iii) $2^i = e^{i\ln 2} = [\cos(\ln 2) + i \sin(\ln 2)]$

(iv) $z = i^i \Rightarrow \log z = i \log i \Rightarrow \log z = -(4n+1)\pi/2 \Rightarrow \log i = i(4n+1)\pi/2$

(v) $|(1+i)^i| \Rightarrow z = (1+i)^i \Rightarrow \log z = i \log(1+i)$

$$\begin{aligned} &\Rightarrow \log(1+i) = \log \sqrt{2} + i(2n\pi + \pi/4) \\ &\Rightarrow \log z = i \log \sqrt{2} + i(2n\pi + \pi/4) \\ &\Rightarrow |z| = e^{-(8n+1)\pi/4} \end{aligned}$$

(vi) $\arg((1+i)^i)$

$z = (1+i)^i$

$\log z = i \log(1+i) \Rightarrow \log(1+i) = \log \sqrt{2} + i(2n\pi + \pi/4)$

$\log z = i \log(1+i) = i \log \sqrt{2} - (2n\pi + \pi/4) \Rightarrow z = e^{i \log \sqrt{2} - (2n\pi + \pi/4)}$

$\theta = \log \sqrt{2}$

Complex Numbers

Self Practice Problem

(28) Find the real part of $\cos(1+i)$

Geometrical Properties :

Answers : $\frac{1+e^2}{2e} \cos 1$

Section formula

If z_1 and z_2 are affixes of the two points P and Q respectively and point C divides the line segment joining P and Q internally in the ratio m : n then affix z of C is given by

$$z = \frac{mz_2 + nz_1}{m+n}$$
 where $m, n > 0$

If C divides PQ in the ratio m : n externally then $z = \frac{mz_2 - nz_1}{m-n}$

Note : If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a+b+c=0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2, z_3 are collinear.

(1) If the vertices A, B, C of a Δ are represented by complex numbers z_1, z_2, z_3 respectively and a, b, c are the length of sides then,

(i) Centroid of the $\Delta ABC = \frac{z_1 + z_2 + z_3}{3}$

(ii) Orthocentre of the $\Delta ABC =$

$$\frac{(a \sec A)z_1 + (b \sec B)z_2 + (c \sec C)z_3}{a \sec A + b \sec B + c \sec C}$$

(iii) Incentre of the $\Delta ABC = (az_1 + bz_2 + cz_3) / (a+b+c)$.

(iv) Circumcentre of the $\Delta ABC =$

$$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) / (\sin 2A + \sin 2B + \sin 2C)$$

(2) $\text{amp}(z) = \theta$ is a ray emanating from the origin inclined at an angle θ to the positive x-axis.

(3) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(4) The equation of a line joining z_1 & z_2 is given by, $z = z_1 + t(z_2 - z_1)$ where t is a real parameter.

(5) $z = z_1(1+it)$ where t is a real parameter is a line through the point z_1 & perpendicular to the line joining z_1 to the origin.

(6) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0. \text{ This is also the condition for three complex numbers } z, z_1, z_2 \text{ to be collinear. The above equation on manipulating, takes the form } \bar{a}z + a\bar{z} + r = 0 \text{ where } r \text{ is real and } a \text{ is a non zero complex constant.}$$

(7) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is $\arg \frac{z - z_2}{z - z_1} = \pm \frac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.

Complex Numbers

- (8) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is the number $\frac{z_3-z_1}{z_3-z_2}, \frac{z_4-z_2}{z_4-z_1}$ should be real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)} = \frac{(z-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$.
- (9) $\arg\left(\frac{z-z_1}{z-z_2}\right) = \theta$ represent (i) a line segment if $\theta = \pi$
 (ii) Pair of ray if $\theta = 0$ (iii) a part of circle, if $0 < \theta < \pi$.
- (10) If $|z-z_1| + |z-z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose foci are z_1 & z_2
- (11) If $|z-z_1| = k$ where $k \in (0, 1) \cup (1, \infty)$, then locus of z is circle.
- (12) If $||z-z_1| - |z-z_2|| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose foci are z_1 & z_2 .

Match the following columns :

Column - I	Column - II
(i) If $ z-3+2i + z+i = 0$, then locus of z represents	(i) circle
(ii) If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, then locus of z represents...	(ii) Straight line
(iii) If $ z-8-2i + z-5-6i = 5$ then locus of z represents	(iii) Ellipse
(iv) If $\arg\left(\frac{z-3+4i}{z+2-5i}\right) = \frac{5\pi}{6}$, then locus of z represents	(iv) Hyperbola
(v) If $ z-1 + z+i = 10$ then locus of z represents	(v) Major Arc
(vi) If $ z-3+i - z+2-i = 1$ then locus of z represents	(vi) Minor arc
(vii) If $ z-3i = 25$	(vii) Perpendicular bisector of a line segment
(viii) If $\arg\left(\frac{z-3+5i}{z+i}\right) = \pi$	(viii) Line segment

MISCELLANEOUS EXAMPLES

Example # 21 : If z_1 and z_2 are two complex numbers and $c > 0$, then prove that

$$|z_1 + z_2|^2 \leq (1+c)|z_1|^2 + (1+c^{-1})|z_2|^2$$

Solution : We have to prove : $|z_1 + z_2|^2 \leq (1+c)|z_1|^2 + (1+c^{-1})|z_2|^2$

$$\text{i.e. } |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + \bar{z}_1 z_2 \leq (1+c)|z_1|^2 + (1+c^{-1})|z_2|^2$$

$$\text{or } z_1 \bar{z}_2 + \bar{z}_1 z_2 \leq c|z_1|^2 + c^{-1}|z_2|^2 \quad \text{or } c|z_1|^2 + \frac{1}{c}|z_2|^2 - z_1 \bar{z}_2 - \bar{z}_1 z_2 \geq 0$$

(using $\operatorname{Re}(z_1 \bar{z}_2) \leq |z_1| |z_2|$) or $\left(\sqrt{c} |z_1| - \frac{1}{\sqrt{c}} |z_2|\right)^2 \geq 0$ which is always true.

Complex Numbers

Example # 22 : If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\operatorname{Re}(z_1 \bar{z}_2) = 0$, then show that the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies the following

Solution : (i) $|w_1| = 1$ (ii) $|w_2| = 1$ (iii) $\operatorname{Re}(w_1 \bar{w}_2) = 0$

$$\begin{aligned} a &= \cos \theta, b = \sin \theta \\ c &= \cos \phi, d = \sin \phi \\ \operatorname{Re}(z_1 \bar{z}_2) &= 0 \Rightarrow \theta - \phi = \frac{n\pi}{2} \quad n = \pm 1 \Rightarrow c = \sin \theta, d = -\cos \theta \\ \Rightarrow w_1 &= \cos \theta + i \sin \theta \\ w_2 &= \sin \theta - i \cos \theta \\ \Rightarrow |w_1| &= 1, |w_2| = 1 \\ w_1 \bar{w}_2 &= \cos \theta \sin \phi - \sin \theta \cos \phi + i(\sin^2 \theta - \cos^2 \theta) = -i \cos 2\theta \\ \Rightarrow \operatorname{Re}(w_1 \bar{w}_2) &= 0 \end{aligned}$$

Example # 23 : Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then show that $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

Solution : $z_1 = r(\cos \theta + i \sin \theta), -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$z_2 = r(\cos \phi + i \sin \phi), -\pi < \phi < 0$

$$\Rightarrow \frac{z_1+z_2}{z_1-z_2} = -i \cot\left(\frac{\theta-\phi}{2}\right), -\frac{\pi}{4} < \frac{\theta-\phi}{2} < \frac{3\pi}{4}$$

Hence purely imaginary.

Example # 24 : If z_1, z_2 & z_3 are the affixes of three points A, B & C respectively and satisfy the condition $|z_1 - z_2| = |z_1| + |z_2|$ and $|(2-i)z_1 + iz_3| = |z_1| + |(1-i)z_1 + iz_3|$ then prove that $\triangle ABC$ is a right angled.

Solution : $|z_1 - z_2| = |z_1| + |z_2|$

$\Rightarrow z_1, z_2$ and origin will be collinear and z_1, z_2 will be opposite side of origin

Similarly $|(2-i)z_1 + iz_3| = |z_1| + |(1-i)z_1 + iz_3|$

$\Rightarrow z_1$ and $(1-i)z_1 + iz_3 = z_4$ say, are collinear with origin and lies on same side of origin. Let $z_4 = \lambda z_1, \lambda \neq 0$

then $(1-i)z_1 + iz_3 = \lambda z_1$

$$\Rightarrow i(z_3 - z_1) = (\lambda - 1)z_1 \Rightarrow \frac{(z_3 - z_1)}{-z_1} = (\lambda - 1)i \Rightarrow \frac{z_3 - z_1}{0 - z_1} = m e^{i\theta}, m = \lambda - 1$$

$\Rightarrow z_3 - z_1$ is perpendicular to the vector $0 - z_1$.

i.e. also z_2 is on line joining origin and z_1

so we can say the triangle formed by z_1, z_2 and z_3 is right angled.

Exercise-1

Marked Questions may have for Revision Questions.

PART - I : SUBJECTIVE QUESTIONS**Section (A) : Algebra of Complex Numbers and Its Representation and Demoivre's Theorem**A-1. Find the real values of x and y for which the following equation is satisfied :

$$(i) \frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$$

$$(ii) \frac{x}{1+2i} + \frac{y}{3+2i} = \frac{5+6i}{8i-1}$$

$$(iii) x^2 - y^2 - i(2xy + 2x - 3y) = 2i$$

A-2. Find the real values of x and y for which the following equation is satisfied :

$$(i) (2+3i)x^2 - (3-2i)y = 2x - 3y + 5i$$

$$(ii) 4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i$$

A-3. (i) Find the real values of x and y for which $z_1 = 9y^2 - 4 - 10ix$ and $z_2 = 8y^2 - 20i$ are conjugate complex of each other.
(ii) Find the value of $x^4 - x^3 + x^2 + 3x - 5$ if $x = 2 + 3i$

A-4. Find

$$(i) \text{the square root of } 7 + 24i \quad (ii) \sqrt{i} + \sqrt{-i}$$

A-5. Solve the following for z :
 $z^2 - (3-2i)z = (5i-5)$ A-6. Simplify and express the result in the form of $a + bi$:

$$(i) -i(9+6i)(2-i)^{-1} \quad (ii) \left(\frac{4i^3 - i}{2i+1} \right)^2$$

$$(iii) \frac{1}{(1-\cos\theta)+2i\sin\theta} \quad (iv) (\sqrt{3}+i)e^{-i\frac{\pi}{6}}$$

A-7. Convert the following complex numbers in Eulers form

$$(i) z = -\pi \quad (ii) z = 5i \quad (iii) z = -\sqrt{3} - i \quad (iv) z = -2 \left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5} \right)$$

A-8. Find the modulus, argument and the principal argument of the complex numbers.

$$(i) z = 1 + \cos \frac{18\pi}{25} + i \sin \frac{18\pi}{25} \quad (ii) z = -2(\cos 30^\circ + i \sin 30^\circ)$$

$$(iii) (\tan 1 - i)^2 \quad (iv) \frac{i-1}{i(1-\cos \frac{2\pi}{5}) + \sin \frac{2\pi}{5}}$$

A-9. Dividing polynomial $f(z)$ by $z - i$, we get the remainder i and dividing it by $z + i$, we get the remainder $1 + i$. Find the remainder upon the division of $f(z)$ by $z^2 + 1$.A-10. If $(\sqrt{3} + i)^{100} = 2^{20}(a + ib)$, then find
(i) $a^2 + b^2$ (ii) b A-11. If n is a positive integer, prove the following

$$(i) (1 + \cos \theta + i \sin \theta)^n + (1 + \cos \theta - i \sin \theta)^n = 2^{n+1} \cos \frac{\theta}{2} \cos \frac{n\theta}{2}$$

$$(ii) (1 + i)^n + (1 - i)^n = 2^{2^{n-1}} \cdot \cos \frac{n\pi}{4}$$

A-12. Show that $e^{2im} \left(\frac{i \cot \theta + 1}{i \cot \theta - 1} \right)^m = 1$.A-13. If $x_i = \cos \left(\frac{\pi}{3^i} \right) + i \sin \left(\frac{\pi}{3^i} \right)$, prove that $x_1 x_2 x_3 \dots$ upto infinity = i .**Section (B) : Argument / Modulus / Conjugate Properties and Triangle Inequality**B-1. If $z = x + iy$ is a complex number such that $z = (a + ib)^2$ then
(i) find \bar{z}
(ii) show that $x^2 + y^2 = (a^2 + b^2)^2$ B-2. If z_1 and z_2 are conjugate to each other, then find $\arg(-z_2)$.B-3. If $z \neq -1$ is a complex number such that $\frac{z-1}{z+1}$ is purely imaginary, then find $|z|$ B-4. If $|z-2| = 2|z-1|$, where z is a complex number, prove $|z|^2 = \frac{4}{3} \operatorname{Re}(z)$ using
(i) polar form of z , (ii) $z = x + iy$, (iii) modulus, conjugate propertiesB-5. For any two complex numbers z_1, z_2 and any two real numbers a, b show that
 $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ B-6. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$ then prove that $\left| \frac{1-z_1\bar{z}_2}{z_1-z_2} \right| < 1$.B-7. If $k > 0$, $|z| = |w| = k$ and $\alpha = \frac{z - \bar{w}}{k^2 + zw}$, then find $\operatorname{Re}(\alpha)$.B-8. (i) If $w = \frac{z+i}{z-i}$ is purely real then find $\arg z$.(ii) If $w = \frac{z+4i}{z+2i}$ is purely imaginary then find $|z+3i|$.B-9. If $a = e^{ia}$, $b = e^{ib}$, $c = e^{ic}$ and $\cos \alpha + \cos \beta + \cos \gamma = 0 = \sin \alpha + \sin \beta + \sin \gamma$, then prove the following
(i) $a + b + c = 0$ (ii) $ab + bc + ca = 0$
(iii) $a^2 + b^2 + c^2 = 0$ (iv) $\sum \cos 2\alpha = \sum \sin 2\alpha$ B-10. If $|z-1| + |z+1| = 1$ then find range of principle argument of z .

Complex Numbers

Section (C) : Geometry of Complex Number and Rotation Theorem

- C-1. If $|z - 2 + i| = 2$, then find the greatest and least value of $|z|$.
- C-2. If $|z + 3| \leq 3$ then find minimum and maximum values of
 (i) $|z|$ (ii) $|z - 1|$ (iii) $|z + 1|$
- C-3. Interpret the following locus in $z \in C$.
 (i) $1 < |z - 2i| < 3$ (ii) $\operatorname{Im}(z) \geq 1$
 (iii) $\operatorname{Arg}(z - 3 - 4i) = \pi/3$ (iv) $\operatorname{Re}\left(\frac{z+2i}{iz+2}\right) \leq 4$ ($z \neq 2i$)
- C-4. If O is origin and affixes of P, Q, R are respectively $z_1, iz, z + iz$. Locate the points on complex plane. If $\Delta PQR = 200$ then find
 (i) $|z|$ (ii) sides of quadrilateral OPRQ
- C-5. Let z_1, z_2, z_3 are three pair wise distinct complex numbers and t_1, t_2, t_3 are non-negative real numbers such that $t_1 + t_2 + t_3 = 1$. Prove that the complex number $z = t_1 z_1 + t_2 z_2 + t_3 z_3$ lies inside a triangle with vertices z_1, z_2, z_3 or on its boundary.
- C-6. The three vertices of a triangle are represented by the complex numbers, 0, z_1 and z_2 . If the triangle is equilateral, then show that $z_1^2 + z_2^2 = z_1 z_2$. Further if z_0 is circumcentre then prove that $z_1^2 + z_2^2 = 3z_0^2$.
- C-7. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further, assume that the origin, z_1 and z_2 form an equilateral triangle. Then show that $a^2 = 3b$.
- C-8. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $(z - z_1)/(z - z_2)$ is $\pi/4$, then find the length of arc of the locus.
- C-9. Let I: $\operatorname{Arg}\left(\frac{z-8i}{z+6}\right) = \pm \frac{\pi}{2}$
 II: $\operatorname{Re}\left(\frac{z-8i}{z+6}\right) = 0$
 Show that locus of z in I or II lies on $x^2 + y^2 + 6x - 8y = 0$. Hence show that locus of z can also be represented by $\frac{z-8i}{z+6} + \frac{\bar{z}+8i}{\bar{z}+6} = 0$. Further if locus of z is expressed as $|z + 3 - 4i| = R$, then find R .
- C-10. Show that $z\bar{z} + (4 - 3i)z + (4 + 3i)\bar{z} + 5 = 0$ represents circle. Hence find centre and radius.
- C-11. If z_1 & z_2 are two complex numbers & if $\operatorname{arg}\frac{z_1 + z_2}{z_1 - z_2} = \frac{\pi}{2}$ but $|z_1 + z_2| \neq |z_1 - z_2|$ then identify the figure formed by the points represented by 0, z_1 , z_2 & $z_1 + z_2$.

Section (D) : Cube root and n^{th} Root of Unity.

- D-1. If $\omega (\neq 1)$ be a cube root of unity and $(1 + \omega^4)^n = (1 + \omega^2)^n$ then find the least positive integral value of n
- D-2. When the polynomial $5x^3 + Mx + N$ is divided by $x^2 + x + 1$, the remainder is 0. Then find $M + N$.

Complex Numbers

D-3. Show that $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \dots$ to $2n$ factors = 2^{2n}

D-4. Let ω is non-real root of $x^3 = 1$

(i) If $P = \omega^n$, ($n \in \mathbb{N}$) and

$$Q = (\omega^m C_0 + \omega^{2m} C_1 + \dots) + (\omega^n C_1 + \omega^{2n} C_4 + \dots) \omega + (\omega^m C_2 + \omega^{2m} C_5 + \dots) \omega^2 \text{ then find } \frac{P}{Q}$$

(ii) If $P = 1 - \frac{\omega}{2} - \frac{\omega^2}{4} - \frac{\omega^3}{8} \dots$ upto ∞ terms and $Q = \frac{1 - \omega^2}{2}$ then find value of PQ .

D-5. If $x = 1 + i\sqrt{3}$; $y = 1 - i\sqrt{3}$ and $z = 2$, then prove that $x^p + y^p = z^p$ for every prime $p > 3$

D-6. Solve $(z - 1)^4 - 16 = 0$. Find sum of roots. Locate roots, sum of roots and centroid of polygon formed by roots in complex plane.

D-7. Find the value(s) of the following

$$(i) \left(\frac{1 + \sqrt{-3}}{2} \right)^3 \quad (ii) \left(\frac{1 + \sqrt{-3}}{2} \right)^{3/4}$$

Hence find continued product if two or more distinct values exists.

D-8. If $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of $x^5 - 1 = 0$, then find the value

$$\text{of } \frac{\omega - \alpha_1}{\omega^2 - \alpha_1} \cdot \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} \cdot \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} \cdot \frac{\omega - \alpha_4}{\omega^2 - \alpha_4} \text{ (where } \omega \text{ is imaginary cube root of unity.)}$$

D-9. $a = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$ then find the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^6$

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : Algebra of Complex Numbers and Its Representation and Demoivre's Theorem

- A-1. If z is a complex number such that $|z| = 4$ and $\operatorname{arg}(z) = \frac{5\pi}{6}$, then z is equal to
 (A) $-2\sqrt{3} + 2i$ (B) $2\sqrt{3} + i$ (C) $2\sqrt{3} - 2i$ (D) $-\sqrt{3} + i$
- A-2. The complex numbers $\sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other, for
 (A) $x = n\pi$ (B) $x = 0$ (C) $x = \frac{n\pi}{2}$ (D) no value of x
- A-3. The least value of n ($n \in \mathbb{N}$), for which $\left(\frac{1+i}{1-i}\right)^n$ is real, is
 (A) 1 (B) 2 (C) 3 (D) 4
- A-4. In G.P. the first term & common ratio are both $\frac{1}{2}(\sqrt{3} + i)$, then the modulus of n^{th} term is :
 (A) 1 (B) 2^n (C) 4^n (D) 3^n
- A-5. If $z = (3 + 7i)(p + iq)$, where $p, q \in \mathbb{R} - \{0\}$, is purely imaginary, then minimum value of $|z|^2$ is
 (A) 0 (B) 58 (C) $\frac{3364}{3}$ (D) 3364

Complex Numbers

- A-6. If $z = x + iy$ and $z^{1/3} = a - ib$ then $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$ where $k =$
 (A) 1 (B) 2 (C) 3 (D) 4
- A-7. If $z = \frac{\pi}{4}(1+i)^4 \left(\frac{1-\sqrt{5}i}{\sqrt{5}+i} + \frac{\sqrt{5}-i}{1+\sqrt{5}i} \right)$, then $\left(\frac{|z|}{\operatorname{amp}(z)} \right)$ equals
 (A) 1 (B) π (C) 3π (D) 4
- A-8. The set of values of $a \in \mathbb{R}$ for which $x^2 + i(a-1)x + 5 = 0$ will have a pair of conjugate imaginary roots is
 (A) R (B) $\{1\}$ (C) $\{a : a^2 - 2a + 21 > 0\}$ (D) $\{0\}$
- A-9. Let z is a complex number satisfying the equation, $z^3 - (3+i)z + m + 2i = 0$, where $m \in \mathbb{R}$. Suppose the equation has a real root α , then find the value of $\alpha^4 + m^4$
 (A) 32 (B) 16 (C) 8 (D) 64
- A-10. The expression $\frac{(1+it\alpha)^n}{(1-it\alpha)^n} - \frac{1+it\alpha}{1-it\alpha}$ when simplified reduces to :
 (A) zero (B) $2 \sin n\alpha$ (C) $2 \cos n\alpha$ (D) none
- A-11. If $(\cos\theta + i \sin\theta)(\cos 2\theta + i \sin 2\theta) \dots (\cos n\theta + i \sin n\theta) = 1$, then the value of θ is
 (A) $\frac{3m\pi}{n(n+1)}$, $m \in \mathbb{Z}$ (B) $\frac{2m\pi}{n(n+1)}$, $m \in \mathbb{Z}$ (C) $\frac{4m\pi}{n(n+1)}$, $m \in \mathbb{Z}$ (D) $\frac{m\pi}{n(n+1)}$, $m \in \mathbb{Z}$
- A-12. Let principle argument of complex number be re-defined between $(\pi, 3\pi]$, then sum of principle arguments of roots of equation $z^6 + z^3 + 1 = 0$ is
 (A) 0 (B) 3π (C) 6π (D) 12π

Section (B) : Argument / Modulus / Conjugate Properties and Triangle Inequality

- B-1. If $|z| = 1$ and $\omega = \frac{z-1}{z+1}$ (where $z \neq -1$), the $\operatorname{Re}(\omega)$ is
 (A) 0 (B) $-\frac{1}{|z+1|^2}$ (C) $\frac{|z|}{|z+1|} \cdot \frac{1}{|z+1|^2}$ (D) $\frac{\sqrt{2}}{|z+1|^2}$
- B-2. If $a^2 + b^2 = 1$, then $\frac{(1+b+ia)}{(1+b-ia)} =$
 (A) 1 (B) 2 (C) $b+ia$ (D) $a+ib$
- B-3. If $(2+i)(2+2i)(2+3i) \dots (2+ni) = x+iy$, then the value of $5.8.13. \dots (4+n^2)$
 (A) $(x^2 + y^2)$ (B) $\sqrt{(x^2 + y^2)}$ (C) $2(x^2 + y^2)$ (D) $(x+y)$
- B-4. If $z = x + iy$ satisfies $\operatorname{amp}(z-1) = \operatorname{amp}(z+3)$ then the value of $(x-1) : y$ is equal to
 (A) $2 : 1$ (B) $1 : 3$ (C) $-1 : 3$ (D) does not exist
- B-5. If $(1+i)z = (1-i)\bar{z}$ then z is
 (A) $t(1-i)$, $t \in \mathbb{R}$ (B) $t(1+i)$, $t \in \mathbb{R}$ (C) $\frac{t}{1+i}$, $t \in \mathbb{R}^*$ (D) $\frac{t}{1-i}$, $t \in \mathbb{R}^*$

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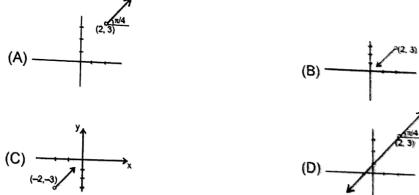
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Complex Numbers

- B-6. Let z and ω be two non-zero complex numbers such that $|z| = |\omega|$ and $\arg z = \pi - \arg \omega$, then z equals
 (A) ω (B) $-\omega$ (C) $\bar{\omega}$ (D) $i\bar{\omega}$
- B-7. If z_1 and z_2 are two non-zero complex numbers such that $\left| \frac{z_1}{z_2} \right| = 2$ and $\arg(z_1 z_2) = \frac{3\pi}{2}$, then $\frac{z_1}{z_2}$ is equal to
 (A) 2 (B) -2 (C) -2i (D) 2i
- B-8. Number of complex numbers z such that $|z| = 1$ and $|z/\bar{z} - \bar{z}/z| = 1$ is
 (A) 4 (B) 6 (C) 8 (D) more than 8
- B-9. If $|z_1| = |z_2|$ and $\arg(z_1/z_2) = \pi$, then $z_1 + z_2$ is equal to
 (A) 1 (B) 3 (C) 0 (D) more than 8
- B-10. The number of solutions of the system of equations $\operatorname{Re}(z^2) = 0$, $|z| = 2$ is
 (A) 4 (B) 3 (C) 0 (D) 2
- B-11. If $|z-1| = |z|^2 + 1$, then z lies on :
 (A) the real axis (B) the imaginary axis (C) a circle (D) an ellipse
- B-12. If $|z-2| + |z-2| \geq ||z| - |z-2||$, then locus of z is
 (A) circle (B) line segment (C) point (D) complete x-y plane
- B-13. The minimum value of $|3z-3| + |2z-4|$ equal to
 (A) 1 (B) 2 (C) 3 (D) 4

Section (C) : Geometry of Complex Number and Rotation Theorem

- C-1. The complex number $z = x + iy$ which satisfy the equation $\frac{|z-5i|}{|z+5i|} = 1$ lie on:
 (A) the x-axis (B) the straight line $y = 5$
 (C) a circle passing through the origin (D) the y-axis
- C-2. The inequality $|z-4| < |z-2|$ represents :
 (A) $\operatorname{Re}(z) > 0$ (B) $\operatorname{Re}(z) < 0$ (C) $\operatorname{Re}(z) > 2$ (D) $\operatorname{Re}(z) > 3$
- C-3. If $\operatorname{Arg}(z-2-3i) = \frac{\pi}{4}$, then the locus of z is



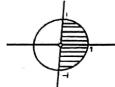
- C-4. The system of equations $\begin{cases} |z+1-i| = 2 \\ \operatorname{Re} z \geq 1 \end{cases}$, where z is a complex number has :
 (A) no solution (B) exactly one solution
 (C) two distinct solutions (D) infinite solution

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Complex Numbers

C-5. The locus of z which lies in shaded region is best represented by



- (A) $|z| \leq 1, -\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}$
 (B) $|z| = 1, -\frac{\pi}{2} \leq \arg z \leq 0$
 (C) $|z| \geq 0, 0 \leq \arg z \leq \frac{\pi}{2}$
 (D) $|z| \leq 1, -\frac{\pi}{2} \leq \arg z \leq \pi$

C-6. The equation $|z - 1|^2 + |z + 1|^2 = 2$ represents

- (A) a circle of radius '1'
 (B) a straight line
 (C) the ordered pair $(0, 0)$
 (D) None of these

C-7. If $\arg\left(\frac{z-2}{z-4}\right) = \frac{\pi}{3}$ then locus of z is :

- (A) equilateral triangle
 (B) arc of circle
 (C) arc of ellipse
 (D) two rays making angle $\frac{\pi}{3}$ between them

C-8. The region of Argand diagram defined by $|z - 1| + |z + 1| \leq 4$ is :

- (A) interior of an ellipse
 (B) exterior of a circle
 (C) interior and boundary of an ellipse
 (D) exterior of ellipse

C-9. The vector $z = -4 + 5i$ is turned counter clockwise through an angle of 180° & stretched 1.5 times. The complex number corresponding to the newly obtained vector is :

- (A) $-6 - \frac{15}{2}i$
 (B) $-6 + \frac{15}{2}i$
 (C) $6 + \frac{15}{2}i$
 (D) $-6 - \frac{15}{2}i$

C-10. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if :

- (A) $z_1 + z_4 = z_2 + z_3$
 (B) $z_1 + z_3 = z_2 + z_4$
 (C) $z_1 + z_2 = z_3 + z_4$
 (D) $z_1 z_3 = z_2 z_4$

C-11. Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C and $(z_1 - z_2)^2 = k(z_1 - z_3)(z_2 - z_3)$, then find k

- (A) 1
 (B) 2
 (C) 3
 (D) -2

C-12. If z_1, z_2, z_3 are vertices of an equilateral triangle inscribed in the circle $|z| = 2$ and if $z_1 = 1 + i\sqrt{3}$, then

- (A) $z_2 = -2, z_3 = 1 + i\sqrt{3}$
 (B) $z_2 = 2, z_3 = 1 - i\sqrt{3}$
 (C) $z_2 = -2, z_3 = 1 - i\sqrt{3}$
 (D) $z_2 = 1 - i\sqrt{3}, z_3 = -1 - i\sqrt{3}$

Section (D) : Cube root of unity and n^{th} Root of Unity.

D-1. Let z_1 and z_2 be two non real complex cube roots of unity and $|z - z_1|^2 + |z - z_2|^2 = \lambda$ be the equation of a circle with z_1, z_2 as ends of a diameter then the value of λ is

- (A) 4
 (B) 3
 (C) 2
 (D) $\sqrt{2}$

D-2. If $x = a + b + c, y = a\alpha + b\beta + c$ and $z = a\beta + b\alpha + c$, where α and β are imaginary cube roots of unity,

- (A) $2(a^2 + b^2 + c^2)$
 (B) $2(a^2 - b^2 - c^2)$
 (C) $a^2 + b^2 + c^2 - 3abc$
 (D) $a^2 - b^2 - c^2$

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Complex Numbers

D-3. If $1, \omega, \omega^2$ are the cube roots of unity, then $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & 1 & \omega^n \\ \omega^{2n} & \omega^n & 1 \end{vmatrix}$ is equal to

- (A) 0
 (B) 1
 (C) ω
 (D) ω^2

D-4. If $x^2 + x + 1 = 0$, then the numerical value of $\left(\frac{x+1}{x}\right)^2 + \left(\frac{x^2+1}{x^2}\right)^2 + \left(\frac{x^3+1}{x^3}\right)^2 + \left(\frac{x^4+1}{x^4}\right)^2 + \dots + \left(\frac{x^{27}+1}{x^{27}}\right)^2$ is equal to

- (A) 54
 (B) 36
 (C) 27
 (D) 18

D-5. If $a = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \dots$, $b = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \dots$, $c = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \dots$, then find $a^3 + b^3 + c^3 - 3abc$.

- (A) 1
 (B) 2
 (C) 3
 (D) 4

D-6. If equation $(z - 1)^n = 1$ ($n \in \mathbb{N}$) has solutions, then n can be :

- (A) 2
 (B) 3
 (C) 6
 (D) 9

D-7. If α is non real and $\alpha = \sqrt[3]{1}$ then the value of $2^{i(1+\alpha+\alpha^2+\alpha^3-\alpha^{-1})}$ is equal to

- (A) 4
 (B) 2
 (C) 1
 (D) 8

D-8. If $\alpha = e^{i\theta/11}$ then Real $(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$ equals to :

- (A)
 (B) 1
 (C) $-\frac{1}{2}$
 (D) -1

PART - III : MATCH THE COLUMN

1. Match the column

Column - I
(Complex number Z)

Column - II
(Principal argument of Z)

$$(A) Z = \frac{(1+i)^5(1+\sqrt{3})^2}{-2i(-\sqrt{3}+i)}$$

(p) π

$$(B) Z = \sin \frac{6\pi}{5} + i \left(1 + \cos \frac{6\pi}{5}\right)$$

(q) $-\frac{7\pi}{18}$

$$(C) Z = 1 + \cos \left(\frac{11\pi}{9}\right) + i \sin \left(\frac{11\pi}{9}\right)$$

(r) $\frac{9\pi}{10}$

$$(D) Z = \sin x \sin(x-60) \sin(x+60)$$

where $x \in \left[0, \frac{\pi}{3}\right]$ and $x \in \mathbb{R}$

(s) $-\frac{5\pi}{12}$

(t) 0



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Complex Numbers

2. **Column I**

- (A) $z^4 - 1 = 0$
 (B) $z^4 + 1 = 0$
 (C) $iz^4 + 1 = 0$
 (D) $iz^4 - 1 = 0$

3. Which of the condition/ conditions in column II are satisfied by the quadrilateral formed by z_1, z_2, z_3, z_4 in order given in column I?

Column - I

- (A) Parallelogram
 (B) Rectangle
 (C) Rhombus
 (D) Square

**Column II
(one of the values of z)**

- (P) $z = \cos \frac{\pi}{8} + i \sin \frac{\pi}{8}$
 (Q) $z = \cos \frac{\pi}{8} - i \sin \frac{\pi}{8}$
 (R) $z = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$
 (S) $z = \cos 0 + i \sin 0$
 (T) $\frac{z_1 - z_3}{z_3 - z_4}$ is purely imaginary
 (U) $\frac{z_1 - z_2}{z_2 - z_4}$ is real
 (V) $|z_1 - z_4| = |z_2 - z_3|$
 (W) $|z_1 - z_3| = |z_2 - z_4|$

4. Let z_1 lies on $|z| = 1$ and z_2 lies on $|z| = 2$.

Column - I

- (A) Maximum value of $|z_1 + z_2|$
 (B) Minimum value of $|z_1 - z_2|$
 (C) Minimum value of $|2z_1 + 3z_2|$
 (D) Maximum value of $|z_1 - 2z_2|$

Column - II

- (P) 3
 (Q) 1
 (R) 4
 (S) 5

八

Complex Numbers

3. If $z_1 = -3 + 5i$; $z_2 = -5 - 3i$ and z is a complex number lying on the line segment joining

- (A) $-\frac{3\pi}{4}$ (B) $-\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{5\pi}{6}$

4. If $(a+ib)^k = \alpha + i\beta$, then $(b+ia)^k$ is equal to

- (A) $\beta + i\alpha$ (B) $\alpha - i\beta$ (C) $\beta - i\alpha$ (D) $-\alpha - i\beta$

5. Let z be non real number such that $\frac{1+z+z^2}{1-z+z^2} \in \mathbb{R}$, then value of $7|z|$ is

- (A) 1 (B) 3 (C) 5 (D) 7

6. If $|z_1| = 2$, $|z_2| = 3$, $|z_3| = 4$ and $|z_1 + z_2 + z_3| = 2$, then the value of $|4z_1z_3 + 9z_2z_3 + 16z_1z_2|$

- (A) 24 (B) 48 (C) 96 (D) 120

7. If $|z_1 - 1| < 1$, $|z_2 - 2| < 2$, $|z_3 - 3| < 3$, then $|z_1 + z_2 + z_3|$

- (A) is less than 6 (B) is more than 12 (C) is more than 3 (D) lies between 6 and 12

8. Let $O = (0, 0)$; $A = (3, 0)$; $B = (0, -1)$ and $C = (3, 2)$, then minimum value of $|z| + |z-3| + |z+1| + |z-2|$ occur at

- (A) intersection point of AB and CO (B) intersection point of AC and BO

- (C) intersection point of CB and AO (D) mean of O, A, B, C

9. Given z is a complex number with modulus 1. Then the equation $((1+i\alpha)/(1-i\alpha))^k = z$ in 'a' has

- (A) all roots real and distinct (B) two real and two imaginary

- (C) three roots real and one imaginary (D) one root real and three imaginary

10. Let A, B, C represent the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle ABC lies at the origin, then the orthocentre is represented by the complex

- (A) $z_1 + z_2 - z_3$ (B) $z_2 + z_3 - z_1$ (C) $z_3 + z_1 - z_2$ (D) $z_1 + z_2 + z_3$

11. The real values of the parameter 'a' for which at least one complex number $z = x + iy$ satisfies both the equality $|z - ai| = a + 4$ and the inequality $|z - 2| < 1$.

- (A) $\left(-\frac{21}{10}, -\frac{5}{6}\right)$ (B) $\left(-\frac{7}{2}, -\frac{5}{6}\right)$ (C) $\left(\frac{5}{6}, \frac{7}{2}\right)$ (D) $\left(-\frac{21}{10}, \frac{7}{2}\right)$

12. The points of intersection of the two curves $|z - 3| = 2$ and $|z| = 2$ in an argand plane are:

- (A) $\frac{1}{2}(7 \pm i\sqrt{3})$ (B) $\frac{1}{2}(3 \pm i\sqrt{7})$ (C) $\frac{3}{2} \pm i\sqrt{\frac{7}{2}}$ (D) $\frac{7}{2} \pm i\sqrt{\frac{3}{2}}$

13. The equation of the radical axis of the two circles represented by the equations,

- $|z - 2| = 3$ and $|z - 2 - 3i| = 4$ on the complex plane is :

- (A) $3iz - 3i\bar{z} - 2 = 0$ (B) $3iz - 3i\bar{z} + 2 = 0$ (C) $iz - i\bar{z} + 1 = 0$ (D) $2iz - 2i\bar{z} + 3 = 0$

14. If $\log_{1/2} \left(\frac{|z-1|+4}{3|z-1|-2} \right) > 1$, then the locus of z is

- (A) Exterior to circle with center $1 + i0$ and radius 10
 (B) Interior to circle with center $1 + i0$ and radius 10
 (C) Circle with center $1 + i0$ and radius 10
 (D) Circle with center $2 + i0$ and radius 10



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Complex Numbers

15. Points z_1 & z_2 are adjacent vertices of a regular octagon. The vertex z_3 adjacent to z_2 ($z_3 \neq z_1$) is represented by :
- $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 + z_2)$
 - $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 - z_2)$
 - $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_2 - z_1)$
 - none of these
16. If $p = a + b\omega + c\omega^2$, $q = b + c\omega + a\omega^2$ and $r = c + a\omega + b\omega^2$ where $a, b, c \neq 0$ and ω is the non-real complex cube root of unity, then :
- $p + q + r = a + b + c$
 - $p^2 + q^2 + r^2 = 2(pq + qr + rp)$
 - $p^2 + q^2 + r^2 = a^2 + b^2 + c^2$
 - None of these
17. The points $z_1 = 3 + \sqrt{3}i$ and $z_2 = 2\sqrt{3} + 6i$ are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors z_1 and z_2 is :
- $\frac{(3+2\sqrt{3})}{2} + \frac{\sqrt{3}+2}{2}i$
 - $z = 5 + 5i$
 - $z = -1 - i$
 - none
18. Let ω be the non real cube root of unity which satisfy the equation $h(x) = 0$ where $h(x) = x f(x^3) + x^2 g(x^3)$. If $h(x)$ is polynomial with real coefficient then which statement is incorrect.
- $f(1) = 0$
 - $g(1) = 0$
 - $h(1) = 0$
 - $(D) g(1) \neq f(1)$
19. If $1, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-1}$ be the n^{th} roots of unity, then the value of $\sin \frac{\pi}{n}, \sin \frac{2\pi}{n}, \sin \frac{3\pi}{n}, \dots, \sin \frac{(n-1)\pi}{n}$ equals
- $\frac{n}{2^n}$
 - $\frac{n}{2^{n-1}}$
 - $\frac{n+1}{2^{n-1}}$
 - $\frac{n}{2^{n+1}}$

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. If a and b are positive integer such that $N = (a + ib)^3 - 107i$ is a positive integer then find the value of $\frac{N}{2}$
2. Let z, w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and $\arg zw = \pi$. If $\operatorname{Re}(z) < 0$ and principal arg $z = \frac{a\pi}{b}$ then find the value of $a + b$. (where a & b are co-prime natural numbers)
3. If $x = 9^{1/3} 9^{1/9} 9^{1/27} \dots \infty$, $y = 4^{1/3} 4^{-1/9} 4^{1/27} \dots \infty$, and $z = \sum_{r=1}^{\infty} (1+i)^{-r}$ and principal argument of $P = (x + yz)$ is $-\tan^{-1}\left(\frac{\sqrt{a}}{b}\right)$ then determine $a^2 + b^2$. (where a & b are co-prime natural numbers)
4. $z_1, z_2 \in \mathbb{C}$ and $z_1^2 + z_2^2 \in \mathbb{R}$,
 $z_1(z_1^2 - 3z_2^2) = 2$, $z_2(3z_1^2 - z_2^2) = 1$
If $z_1^2 + z_2^2 = \lambda$ then determine λ^2
5. Let $|z| = 2$ and $w = \frac{z+1}{z-1}$ where $z, w \in \mathbb{C}$ (where \mathbb{C} is the set of complex numbers), then find product of maximum and minimum value of $|w|$.

Complex Numbers

6. A function ' Ψ ' is defined by $f(z) = (4 + i)z^2 + az + \gamma$ for all complex number z , where a and γ are complex numbers if $f(1)$ and $f(i)$ are both real and the smallest possible values of $|a| + |\gamma|$ is p then determine p^2 .
7. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$, and $\arg(z) - \arg(\omega) = \frac{\pi}{2}$, then find the value of $5i\bar{z}\omega$.
8. Number of complex number satisfying $|z| = \max\{|z-1|, |z+1|\}$.
9. If z_1 & z_2 both satisfy the relation, $z + \bar{z} = 2|z-1|$ and $\arg(z_1 - z_2) = \frac{\pi}{4}$, then find the imaginary part of $(z_1 + z_2)$.
10. If $a_1, a_2, a_3, \dots, a_n, A_1, A_2, A_3, \dots, A_n, k$ are all real numbers and number of imaginary roots of the equation $\frac{A_1^2}{x-a_1} + \frac{A_2^2}{x-a_2} + \dots + \frac{A_n^2}{x-a_n} = k$ is α . Then find the value of $\alpha + 15$.
11. How many complex number z such that $|z| < \frac{1}{3}$ and $\sum_{i=1}^n a_i z^i = 1$ where $|a_i| < 2$.
12. If a variable circle S touches $S_1 : |z - z_1| = 7$ internally and $S_2 : |z - z_2| = 4$ externally while the curves S_1 & S_2 touch internally to each other, (z_1, z_2) . If the eccentricity of the locus of the centre of the curve S is 'e' find the value of $11e$.
13. Given that, $|z - 1| = 1$, where 'z' is a point on the argand plane. $\frac{z-2}{2z} = \alpha i \tan(\arg z)$. Then determine $\frac{1}{\alpha^2}$.
14. Area of the region formed by $|z| \leq 4$ & $-\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{3}$ on the Argand diagram is expressed in the form of $\frac{a\pi}{b}$. Then find the value of ab (where a & b are co-prime natural number)
15. The points A, B, C represent the complex numbers z_1, z_2, z_3 respectively on a complex plane & the angle B & C of the triangle ABC are each equal to $\frac{1}{2}(\pi - \alpha)$. If $(z_2 - z_3)^2 = \lambda(z_3 - z_1)(z_1 - z_2) \sin^2 \frac{\alpha}{2}$ then determine λ .
16. If ω and ω^2 are the non-real cube roots of unity and $a, b, c \in \mathbb{R}$ such that $\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} = 2\omega^2$ and $\frac{1}{a+\omega^2} + \frac{1}{b+\omega^2} + \frac{1}{c+\omega^2} = 2\omega$. If $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = \lambda$ then determine λ^4 .
17. If $L = \lim_{n \rightarrow \infty} \left[\frac{n}{(1-n\omega)(1-n\omega^2)} + \frac{n}{(2-n\omega)(2-n\omega^2)} + \dots + \frac{n}{(n-n\omega)(n-n\omega^2)} \right]$ then find the value of $\frac{\pi}{\sqrt{3}L}$ is equal to (where ω is non real cube root of unity)

Complex Numbers

18. The value of $\sum_{k=1}^6 \left(\sin \frac{2\pi k}{7} - i \cos \frac{2\pi k}{7} \right) = \alpha$ then find α^4
19. Let $Z = \left(e^{i \frac{2\pi}{15}} \right)^7$. If $\arg \left(\frac{1+Z_1 + Z_2 + Z_3 + \dots + Z_7}{1+Z_8 + Z_9 + Z_{10} + \dots + Z_{14}} \right) = \frac{ax}{b}$, then $b-a$ equals. (where a & b are co-prime natural number)
20. If A_1, A_2, \dots, A_n be the vertices of an n -sided regular polygon such that $\frac{1}{A_1 A_2} = \frac{1}{A_1 A_3} + \frac{1}{A_1 A_4}$, then find the value of n

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ ($a, b, c, d \in \mathbb{R}$) has 4 non real roots, two with sum $3+4i$ and the other two with product $13+i$.
 (A) $a = 51$ (B) $a = -6$ (C) $c = -70$ (D) $d = 170$
2. The quadratic equation $z^2 + (p+ip')z + q + iq' = 0$; where p, p', q, q' are all real.
 (A) if the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$.
 (B) if the equation has two equal roots then $pp' = 2q'$.
 (C) if the equation has two equal roots then $p^2 - p'^2 = 4q$.
 (D) if the equation has one real root then $p^2 - pp'q' + q'^2 = 0$.
3. The value of $i^n + i^{-n}$ for $i = \sqrt{-1}$ and $n \in \mathbb{N}$ is :
 (A) $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$ (B) $\frac{(1-i)^{2n}}{2^n} + \frac{2^n}{(1-i)^{2n}}$ (C) $\frac{(1+i)^{2n}}{2^n} + \frac{2^n}{(1+i)^{2n}}$ (D) $\frac{2^n}{(1+i)^{2n}} + \frac{2^n}{(1-i)^{2n}}$
4. If $\text{amp}(z_1 z_2) = 0$ and $|z_1| = |z_2| = 1$, then
 (A) $z_1 + z_2 = 0$ (B) $z_1 z_2 = 1$ (C) $z_1 = \bar{z}_2$ (D) $z_1 = z_2$
5. Let z_1 and z_2 are two complex numbers such that $(1-i)z_1 = 2z_2$ and $\arg(z_1 z_2) = \frac{\pi}{2}$, then $\arg(z_2)$ is equal to
 (A) $3\pi/8$ (B) $\pi/8$ (C) $5\pi/8$ (D) $-7\pi/8$
6. If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ (where z_1 and z_2 are non-zero complex numbers), then
 (A) $\frac{z_1}{z_2}$ is purely real (B) $\frac{z_1}{z_2}$ is purely imaginary
 (C) $z_1 \bar{z}_2 + z_2 \bar{z}_1 = 0$ (D) $\text{amp} \frac{z_1}{z_2}$ may be equal to $\frac{\pi}{2}$
7. a, b, c are real numbers in the polynomial, $P(z) = 2z^4 + az^3 + bz^2 + cz + 3$. If two roots of the equation $P(z) = 0$ are 2 and i . Then which of the following are true.
 (A) $a = -\frac{11}{2}$ (B) $b = 5$ (C) $c = -\frac{11}{2}$ (D) $a = -11$

Complex Numbers

8. If $Z = \frac{(1+i)(1+2i)(1+3i)\dots(1+ni)}{(1-i)(2-i)(3-i)\dots(n-i)}$, $n \in \mathbb{N}$ then principal argument of Z can be
 (A) 0 (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{2}$ (D) π
9. For complex numbers z and w , if $|z|^2 w - |w|^2 z = z - w$. Which of the following are true :
 (A) $z = w$ (B) $z \bar{w} = 1$ (C) $z = w + 2$ (D) $\bar{z} w = 1$
10. If z satisfies the inequality $|z - 1 - 2i| \leq 1$, then which of the following are true.
 (A) maximum value of $|z| = \sqrt{5} + 1$ (B) minimum value of $|z| = \sqrt{5} - 1$
 (C) maximum value of $\arg(z) = \pi/2$ (D) minimum value of $\arg(z) = \tan^{-1}\left(\frac{3}{4}\right)$
11. The curve represented by $z = \frac{3}{2 + \cos \theta + i \sin \theta}$, $\theta \in [0, 2\pi]$
 (A) never meets the imaginary axis (B) meets the real axis in exactly two points
 (C) has maximum value of $|z|$ as 3 (D) has minimum value of $|z|$ as 1
12. POQ is a straight line through the origin O. P and Q represent the complex number $a + ib$ and $c + id$ respectively and $OP = OQ$. Then which of the following are true :
 (A) $|a + ib| = |c + id|$ (B) $a + c = b + d$
 (C) $\arg(a + ib) = \arg(c + id)$ (D) none of these
13. Let $i = \sqrt{-1}$. Define a sequence of complex number by $z_1 = 0$, $z_{n+1} = z_n^2 + i$ for $n \geq 1$. Then which of the following are true.
 (A) $|z_{2050}| = \sqrt{3}$ (B) $|z_{207}| = \sqrt{2}$ (C) $|z_{201}| = 1$ (D) $|z_{211}| = \sqrt{2}$
14. If $|z_1| = |z_2| = \dots = |z_n| = 1$ then which of the following are true.
 (A) $\bar{z}_1 = \frac{1}{z_1}$
 (B) $|z_1 + z_2 + \dots + z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right|$
 (C) Centroid of polygon with $2n$ vertices $z_1, z_2, \dots, z_n, \frac{1}{z_1}, \frac{1}{z_2}, \dots, \frac{1}{z_n}$ (need not be in order) lies on real axis.
 (D) Centroid of polygon with $2n$ vertices $z_1, z_2, \dots, z_n, \frac{1}{z_1}, \frac{1}{z_2}, \dots, \frac{1}{z_n}$ (need not be in order) lies on imaginary axis.
15. If $2 \cos \theta = x + \frac{1}{x}$ and $2 \cos \phi = y + \frac{1}{y}$, then
 (A) $x^n + \frac{1}{x^n} = 2 \cos(n\theta)$, $n \in \mathbb{Z}$ (B) $\frac{x}{y} + \frac{y}{x} = 2 \cos(\theta - \phi)$
 (C) $xy + \frac{1}{xy} = 2 \cos(\theta + \phi)$ (D) $x^m y^n + \frac{1}{x^m y^n} = 2 \cos(m\theta + n\phi)$, $m, n \in \mathbb{Z}$

Complex Numbers

16. If $\left| \frac{z-\alpha}{z-\beta} \right| = k$, $k > 0$ where, $z = x + iy$ and $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ are fixed complex numbers. Then which of the following are true?
- If $k \neq 1$ then locus is a circle whose centre is $\left(\frac{k^2\beta - \alpha}{k^2 - 1} \right)$
 - If $k \neq 1$ then locus is a circle whose radius is $\left| \frac{k(\alpha - \beta)}{1 - k^2} \right|$
 - If $k = 1$ then locus is perpendicular bisector of line joining $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$
 - If $k \neq 1$ then locus is a circle whose centre is $\left(\frac{k^2\alpha - \beta}{k^2 - 1} \right)$

17. The equation $|z+i| - |z-i| = k$ represents
 (A) a hyperbola if $0 < k < 2$
 (B) a pair of ray if $k > 2$
 (C) a straight line if $k = 0$
 (D) a pair of ray if $k = 2$

18. The equation $|z-i| + |z+i| = k$, $k > 0$, can represent
 (A) an ellipse if $k > 2$
 (B) line segment if $k = 2$
 (C) an ellipse if $k = 5$
 (D) no locus if $k = 1$

19. If $|z_1| = |z_2| = |z_3| = 1$ and z_1, z_2, z_3 are represented by the vertices of an equilateral triangle then
 (A) $z_1 + z_2 + z_3 = 0$
 (B) $z_1 z_2 z_3 = 1$
 (C) $z_1 z_2 + z_2 z_3 + z_3 z_1 = 0$
 (D) $z_1^2 + z_2^2 + z_3^2 = 2z_1 z_2 z_3$

20. Let z_1, z_2, z_3 be three distinct complex numbers satisfying, $|z_1 - 1| = |z_2 - 1| = |z_3 - 1| = 1$. Let A, B & C be the points representing vertices of equilateral triangle in the Argand plane corresponding to z_1, z_2 and z_3 respectively. Which of the following are true?
 (A) $z_1 + z_2 + z_3 = 3$
 (B) $z_1^2 + z_2^2 + z_3^2 = 3$
 (C) area of triangle ABC = $\frac{3\sqrt{3}}{4}$
 (D) $z_1 z_2 + z_2 z_3 + z_3 z_1 = 1$

21. If $1, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-1}$ be the n^{th} roots of unity, then which of the following are true?
 (A) $\frac{1}{1-\alpha_1} + \frac{1}{1-\alpha_2} + \dots + \frac{1}{1-\alpha_{n-1}} = \frac{n-1}{2}$
 (B) $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\dots(1-\alpha_{n-1}) = n$.
 (C) $(2-\alpha_1)(2-\alpha_2)(2-\alpha_3)\dots(2-\alpha_{n-1}) = 2^{n-1}$
 (D) $\frac{1}{1-\alpha_1} + \frac{1}{1-\alpha_2} + \dots + \frac{1}{1-\alpha_{n-1}} = \frac{n}{2}$

22. Which of the following are true.
 (A) $\cos x + {}^n C_1 \cos 2x + {}^n C_2 \cos 3x + \dots + {}^n C_n \cos (n+1)x = 2^n \cdot \cos^n \frac{x}{2} \cdot \cos \left(\frac{n+2}{2} \right)x$
 (B) $\sin x + {}^n C_1 \sin 2x + {}^n C_2 \sin 3x + \dots + {}^n C_n \sin (n+1)x = 2^n \cdot \cos^n \frac{x}{2} \cdot \sin \left(\frac{n+2}{2} \right)x$
 (C) $1 + {}^n C_1 \cos x + {}^n C_2 \cos 2x + \dots + {}^n C_n \cos nx = 2^n \cdot \cos^n \frac{x}{2} \cdot \cos \left(\frac{nx}{2} \right)$
 (D) ${}^n C_1 \sin x + {}^n C_2 \sin 2x + \dots + {}^n C_n \sin nx = 2^n \cdot \cos^n \frac{x}{2} \cdot \sin \left(\frac{nx}{2} \right)$

Complex Numbers

23. If α, β, γ are distinct roots of $x^3 - 3x^2 + 3x + 7 = 0$ and ω is non-real cube root of unity, then the value of $\frac{\alpha-1}{\beta-1} \cdot \frac{\beta-1}{\gamma-1} \cdot \frac{\gamma-1}{\alpha-1}$ can be equal to:
 (A) ω^2 (B) $2\omega^2$ (C) $3\omega^2$ (D) 3ω
24. If z is a complex number then the equation $z^2 + z|z| + |z|^2 = 0$ is satisfied by
 (A) $z = k\omega$ where $k \in R$
 (B) $z = k\omega^2$ where k is non negative real
 (C) $z = k\omega$ where k is positive real
 (D) $z = k\omega^2$ where $k \in R$.
25. If α is imaginary n^{th} (n ≥ 3) root of unity. Which of the following are true.
 (A) $\sum_{r=1}^{n-1} (n-r) \alpha^r = \frac{n\alpha}{1-\alpha}$
 (B) $\sum_{r=1}^{n-1} (n-r) \sin \frac{2r\pi}{n} = \frac{n}{2} \cot \frac{\pi}{n}$
 (C) $\sum_{r=1}^{n-1} (n-r) \cos \frac{2r\pi}{n} = -\frac{n}{2}$
 (D) $\sum_{r=1}^{n-1} (n-r) \alpha^r = \frac{n}{1-\alpha}$
26. Which of the following is true
 (A) roots of the equation $z^{10} - z^6 - 992 = 0$ with real part positive = 5
 (B) roots of the equation $z^{10} - z^6 - 992 = 0$ with real part negative = 5
 (C) roots of the equation $z^{10} - z^6 - 992 = 0$ with imaginary part non-negative = 6
 (D) roots of the equation $z^{10} - z^6 - 992 = 0$ with imaginary part negative = 4

PART - IV : COMPREHENSION

Comprehension # 1

Let $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$. For sum of series $C_0 + C_1 + C_2 + \dots$, put $x = 1$. For sum of series $C_0 + C_2 + C_4 + C_6 + \dots$, or $C_1 + C_3 + C_5 + \dots$, add or subtract equations obtained by putting $x = 1$ and $x = -1$.
 For sum of series $C_0 + C_3 + C_6 + \dots$ or $C_1 + C_4 + C_7 + \dots$ or $C_2 + C_5 + C_8 + \dots$ we substitute $x = 1$, $x = \omega$, $x = \omega^2$ and add or manipulate results.
 Similarly, if suffixes differ by 'p' then we substitute p^{th} roots of unity and add.

1. $C_0 + C_3 + C_6 + C_9 + \dots =$
 (A) $\frac{1}{3} \left[2^n - 2 \cos \frac{n\pi}{3} \right]$ (B) $\frac{1}{3} \left[2^n + 2 \cos \frac{n\pi}{3} \right]$ (C) $\frac{1}{3} \left[2^n - 2 \sin \frac{n\pi}{3} \right]$ (D) $\frac{1}{3} \left[2^n + 2 \sin \frac{n\pi}{3} \right]$
2. $C_1 + C_5 + C_9 + \dots =$
 (A) $\frac{1}{4} \left[2^n - 2^{n/2} \cos \frac{n\pi}{4} \right]$ (B) $\frac{1}{4} \left[2^n + 2^{n/2} \cos \frac{n\pi}{4} \right]$
 (C) $\frac{1}{4} \left[2^n - 2^{n/2} \sin \frac{n\pi}{4} \right]$ (D) $\frac{1}{4} \left[2^n + 2^{n/2} \sin \frac{n\pi}{4} \right]$

Comprehension # 2

As we know $e^{i\theta} = \cos\theta + i\sin\theta$ and $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$
 Let $\alpha, \beta, \gamma \in R$ such that $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$

3. $\Sigma \sin(\alpha + \beta) = \Sigma \cos(\alpha + \beta) =$
 (A) 0 (B) $3\cos\alpha\cos\beta\cos\gamma$ (C) $3\cos(\alpha + \beta + \gamma)$ (D) 3

Complex Numbers

Exercise-3

Marked Questions may have for Review

PART - I : JEE (ADVANCED)

* Marked Questions may have more than one correct answer.

Complex Numbers

4. $\sum \cos(2\alpha - \beta - \gamma)$
 (A) 0 (B) $3\cos\alpha \cos\beta \cos\gamma$ (C) $3 \cos(\alpha + \beta + \gamma)$ (D) 3
5. $\sum \cos 3\alpha =$
 (A) 0 (B) $3\cos\alpha \cos\beta \cos\gamma$ (C) $3 \cos(\alpha + \beta + \gamma)$ (D) 3
6. If $\theta \in R$ then $\frac{\sum \cos^3(\theta + \alpha)}{\prod \cos(\theta + \alpha)} =$
 (A) 0 (B) $3\cos\alpha \cos\beta \cos\gamma$ (C) $3 \cos(\alpha + \beta + \gamma)$ (D) 3

Comprehension # 3

ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. Let the points D and M represent complex numbers $1+i$ and $2-i$ respectively. If θ is arbitrary real, then $z = re^{i\theta}$, $R_1 \leq r \leq R_2$ lies in annular region formed by concentric circles $|z| = R_1$, $|z| = R_2$.

7. A possible representation of point A is
 (A) $3 - \frac{i}{2}$ (B) $3 + \frac{i}{2}$ (C) $1 + \frac{3}{2}i$ (D) $3 - \frac{3}{2}i$
8. If z is any point on segment DM then $w = e^{iz}$ lies in annular region formed by concentric circles.
 (A) $|w|_{\min} = 1$, $|w|_{\max} = 2$ (B) $|w|_{\min} = \frac{1}{e}$, $|w|_{\max} = e$
 (C) $|w|_{\min} = \frac{1}{e^2}$, $|w|_{\max} = e^2$ (D) $|w|_{\min} = \frac{1}{2}$, $|w|_{\max} = 1$

Comprehension # 4

Logarithm of a complex number is given by

$$\begin{aligned}\log_e(x+iy) &= \log_e(|z|e^{i\theta}) \\ &= \log_e|z| + \log_e e^{i\theta} \\ &= \log_e|z| + i\theta \\ &= \log_e \sqrt{x^2 + y^2} + i \arg(z)\end{aligned}$$

$$\therefore \log_e(z) = \log_e|z| + i \arg(z)$$

In general $\log_e(x+iy) = \frac{1}{2} \log_e(x^2 + y^2) + i \left(2n\pi + \tan^{-1} \frac{y}{x} \right)$ where $n \in \mathbb{Z}$.

9. Write $\log_e(1 + \sqrt{3}i)$ in $(a + ib)$ form
 (A) $\log_e 2 + i(2n\pi + \frac{\pi}{3})$ (B) $\log_e 3 + i(n\pi + \frac{\pi}{3})$
 (C) $\log_e 2 + i(2n\pi + \frac{\pi}{6})$ (D) $\log_e 2 + i(2n\pi - \frac{\pi}{3})$

10. Find the real part of $(1-i)^{-1}$.

- (A) $e^{i\pi/4 + 2n\pi} \cos\left(\frac{1}{2} \log_e 2\right)$ (B) $e^{-i\pi/4 + 2n\pi} \cos\left(\frac{1}{2} \log_e 2\right)$
 (C) $e^{-i\pi/4 + 2n\pi} \cos(\log_e 2)$ (D) $e^{-i\pi/2 + 2n\pi} \cos\left(\frac{1}{2} \log_e 2\right)$

Let A, B, C be three sets of complex numbers

$$A = \{z : \operatorname{Im} z \geq 1\}$$

$$B = \{z : |z - 2| \leq 1\}$$

$$C = \{z : \operatorname{Re}(z) \geq 1 - i\}$$

1. The number of elements in the set $A \cap B \cap C$.
 (A) 0 (B) 1
2. Let z be any point in $A \cap B \cap C$. Then, $|z|$
 (A) 25 and 29 (B) 30 and 34
3. Let z be any point $A \cap B \cap C$ and let $w = z^2$.
 (A) -6 and 3 (B) -3 and 6
4. Let $z = \cos \theta + i \sin \theta$. Then the value of
 (A) $\frac{1}{\sin^2 \theta}$ (B) $\frac{1}{3 \sin^2 \theta}$

5. Let $z = x + iy$ be a complex number whose vertices are the roots of the equation $\bar{z}^2 = 1$.
 (A) 48 (B) 32

6. Let z_1 and z_2 be two distinct complex numbers such that $0 < t < 1$. If $\operatorname{Arg}(w)$ denotes the principal value of $\arg(w)$, then
 (A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
 (C) $\left| \frac{z - z_1}{z_2 - z_1} \cdot \frac{\bar{z}_2 - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \right| = 0$

7. Let ω be the complex number $\cos \theta + i \sin \theta$ satisfying
 (B) $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$

Complex Numbers

ANSWERS

EXERCISE - 1

PART - I

Section (A) :

A-1. (i) $3, -1$ (ii) $x = 1$ and $y = 2$,
(iii) $(-2, 2)$ or $\left(-\frac{2}{3}, -\frac{2}{3}\right)$

A-2. (i) $(1, 1) \left(0, \frac{5}{2}\right)$ (ii) $x=K, y=\frac{3K}{2}, K \in \mathbb{R}$

A-3. (i) $[-2, 2); (-2, -2)]$ (ii) $[-77 + 108i]$

A-4. (i) $\pm(4+3i)$ (ii) $\pm\sqrt{2} + 0i$ or $0 \pm \sqrt{2}i$

A-5. $z = (2+i)$ or $(1-3i)$

A-6. (i) $\frac{21}{5} - \frac{12}{5}i$ (ii) $3+4i$

(iii) $\frac{1}{2(1+3\cos^2 \theta)} + i \frac{-\cot \theta}{2(1+3\cos^2 \theta)}$ (iv) 2

A-7. (i) $\pi e^{i\pi}$ (ii) $5e^{i\pi}$

(iii) $2e^{-\frac{5\pi}{6}}c$ (iv) $2e^{-\frac{4\pi}{5}}$

A-8. (i) $|z| = 2 \cos \frac{9\pi}{25}$ Principal Arg $z = \frac{9\pi}{25}$

$\arg z = \frac{9\pi}{25} + 2k\pi, k \in \mathbb{Z}$

(ii) Modulus = 2, Arg = $2k\pi - \frac{5\pi}{6}$,

$k \in \mathbb{I}$, Principal Arg = $-\frac{5\pi}{6}$

(iii) Modulus = $\sec^2 1$, arg = $2k\pi + (2-\pi)$,
Principal Arg = $(2-\pi)$

(iv) Modulus = $\frac{1}{\sqrt{2}}$ cosec $\frac{\pi}{5}$,

$\arg z = 2k\pi + \frac{11\pi}{20}$, Principal Arg = $\frac{11\pi}{20}$

A-9. $\frac{iz}{2} + \frac{1}{2} + i$

A-10. (i) 4 (ii) $\sqrt{3}$

Section (B) :

B-1. (i) $(a-ib)^2$ B-2. π

B-3. 1 B-7. 0 B-8. (i) $\pm \frac{\pi}{2}$ (ii) 1

B-10. $\arg z \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]$

Section (C) :

C-1. $\sqrt{5} + 2 & \sqrt{5} - 2$

C-2. (i) 0, 6 (ii) 1, 7 (iii) 0, 5

C-3. (i) The region between the concentric circles with centre at $(0, 2)$ & radii 1 and 3 units

(ii) The part of the complex plane on or above the line $y = 1$

(iii) a ray emanating from the point $(3+4i)$ directed away from the origin & having

equation, $\sqrt{3}x - y + 4 - 3\sqrt{3} = 0, x > 3$

(iv) Region outside or on the circle with

centre $\frac{1}{2} + 2i$ and radius $\frac{1}{2}$

Complex Numbers

C-4. (i) $|z| = 20$ (ii) $OP = OQ = PR = QR = 20$

C-8. $9\frac{\pi}{\sqrt{2}}$ C-9. 5

C-10. $-4 - 3i, 2\sqrt{5}$

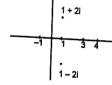
C-11. a rhombus but not a square

Section (D) :

D-1. 3 D-2. -5 D-3. 4^n

D-4. (i) 1 (ii) 1

D-6. $z = -1, 3, 1-2i, 1+2i$



Sum = 4
centroid = 1

D-7. (i) -1

(ii) $e^{(6n+1)\frac{\pi}{4}i}, n = 0, 1, 2, 3$.

Continued product = 1

D-8. ω D-9. $x^2 + x + 2 = 0$

PART - II

Section (A) :

A-1. (A) A-2. (D) A-3. (B)

A-4. (A) A-5. (D) A-6. (D)

A-7. (D) A-8. (B) A-9. (A)

A-10. (A) A-11. (C) A-12. (D)

Section (B) :

B-1. (A) B-2. (C) B-3. (A)

B-4. (D)	B-5. (A)	B-6. (D)
B-7. (D)	B-8. (C)	B-9. (C)
B-10. (A)	B-11. (B)	B-12. (D)
B-13. (B)		

Section (C) :

C-1. (A)	C-2. (D)	C-3. (A)
C-4. (B)	C-5. (A)	C-6. (C)
C-7. (B)	C-8. (C)	C-9. (A)
C-10. (B)	C-11. (B)	C-12. (C)

Section (D) :

D-1. (B)	D-2. (C)	D-3. (A)
D-4. (A)	D-5. (A)	D-6. (C)
D-7. (A)	D-8. (C)	

PART - III

1. (A) \rightarrow (s), (B) \rightarrow (r), (C) \rightarrow (q), (D) \rightarrow (p)

2. A \rightarrow s; B \rightarrow r; C \rightarrow p; D \rightarrow q.

3. a \rightarrow p, r; b \rightarrow p, q, r, t; c \rightarrow p, r, s; d \rightarrow p, q, r, s, t.

4. (A) \rightarrow (p), (B) \rightarrow (q), (C) \rightarrow (r), (D) \rightarrow (s)

EXERCISE - 2

PART - I

1. (B) 2. (A) 3. (D)

4. (A) 5. (D) 6. (B)

7. (C) 8. (C) 9. (A)

10. (D) 11. (A) 12. (B)

13. (B) 14. (A) 15. (C)

16. (C) 17. (B) 18. (D)

19. (B)

Section (B) :

B-1. (A) B-2. (C) B-3. (A)

Corporate Office : CG Tower, A-46 & 52, IPIA, Near City Mall, Jhalawar Road, Kota (Raj.) - 324005
Website : www.resonance.ac.in | E-mail : contact@resonance.ac.in
Toll Free : 1800 258 5555 | CIN: U80302RJ2007PLC024029

Corporate Office : CG Tower, A-46 & 52, IPIA, Near City Mall, Jhalawar Road, Kota (Raj.) - 324005
Website : www.resonance.ac.in | E-mail : contact@resonance.ac.in
Toll Free : 1800 258 5555 | CIN: U80302RJ2007PLC024029

ADVNC - 40

PART - II

1. 99 2. 7 3. 13
 4. 25 5. 1 6. 2
 7. 5 8. 0 9. 2
 10. 15 11. 0 12. 3
 13. 16 14. 60 15. 4
 16. 16 17. 3 18. 1
 19. 1 20. 7

PART - III

1. (ABCD) 2. (ABC) 3. (BD)
 4. (BC) 5. (BD) 6. (BCD)
 7. (ABC) 8. (ABCD) 9. (ABD)
 10. (ABCD) 11. (ABCD) 12. (AB)
 13. (BCD) 14. (ABC) 15. (ABCD)
 16. (ABC) 17. (ACD) 18. (ABCD)
 19. (ACD) 20. (ABC) 21. (ABC)
 22. (ABCD) 23. (CD) 24. (BC)
 25. (ABC) 26. (ABCD)

PART - IV

1. (B) 2. (D) 3. (A)
 4. (D) 5. (C) 6. (D)
 7. (A) 8. (B) 9. (A)
 10. (B)

EXERCISE**PART -**

1. (B) 2. (C)
 4. (D) 5. (A)
 7. 1
 8. (A)-(q,r), (B)-(p), (C)-
 9. (5)
 10. Bonus ($w = e^{i\pi/3}$ is because of this the integer.) (if $w =$ then answer c)
 11. (D) 12. (C)
 14. (BCD) 15. (B)
 17. (C) 18. 4
 20. (ACD) 21. (BD)

PART - II

1. (3) 2. (1)
 4. (1) 5. (2)
 7. (3) 8. (4)
 10. (3) 11. (1)