Answers

EXERCISE - 1

PART - I

Section (A) :

A-1. (i) $\left(\frac{2}{x}\right)^5 - 5\left(\frac{2}{x}\right)^3 + 10\left(\frac{2}{x}\right) - 10\left(\frac{x}{2}\right) + 5\left(\frac{x}{2}\right)^3 - \left(\frac{x}{2}\right)^5$

(ii) $y^8 + 8y^5 + 24y^2 + \frac{32}{y} + \frac{16}{y^4}$

A-2. $n = 9$ **A-3.** 7

A-4. (i) ${}^9C_3$ (ii) $-2^7 \cdot {}^{12}C_7$

A-5. ${}^{11}C_5 \frac{a^6}{b^5}$, ${}^{11}C_6 \frac{a^5}{b^6}$, $ab = 1$

A-6. $\frac{17}{54}$ **A-7.** (i) 171 (ii) –438

A-8 15

Section (B) :

B-1. (i) $-\frac{35x}{y}, \frac{35y}{x}$ (ii) $(-1)^n \frac{(2n)!}{n!\,n!} x^n$

B-3. (i) 4 (iii) 3, 03, 803

B-4. 101^{50}

B-5. (i) $T_4 = -455 \times 3^{12}$ and $T_5 = 455 \times 3^{12}$
(ii) 22

B-6. (i) T_4 (ii) T_5, T_6 (iii) T_5 (iv) T_6

Section (D) :

D-1. $\frac{15015}{16}$

D-2. (i) 142 (ii) – 197

D-4. (i) 280 (ii) 2^5

D-5. 20

PART - II

Section (A) :

A-1.	(C)	**A-2.**	(C)	**A-3.**	(A)
A-4.	(B)	**A-5.**	(A)	**A-6.**	(A)
A-7.	(B)	**A-8.**	(C)	**A-9.**	(C)
A-10.	(B)				

Section (B) :

B-1.	(B)	**B-2.**	(C)	**B-3.**	(D)
B-4.	(A)	**B-5.**	(A)	**B-6.**	(A)
B-7.	(A)				

Section (C) :

C-1.	(B)	**C-2.**	(C)	**C-3.**	(C)
C-4.	(B)				

Section (D) :

D-1. (D) **D-2.** (D) **D-3.** (A)

PART - III

1. (A) → (q, s), (B) → (q), (C) → (s), (D) → (p, s)

EXERCISE - 2

PART - I

1.	(B)	**2.**	(A)	**3.**	(C)
4.	(C)	**5.**	(B)	**6.**	(D)
7.	(A)	**8.**	(D)	**9.**	(C)
10.	(B)	**11.**	(B)	**12.**	(B)
13.	(C)	**14.**	(C)	**15.**	(A)
16.	(B)	**17.**	(D)	**18.**	(A)
19.	(A)	**20.**	(C)		

PART - II

1.	k = 11	**2.**	10	**3.**	2
4.	5	**5.**	3^5	**6.**	2
7.	2	**8.**	50	**9.**	3
10.	n = 12	**11.**	2	**12.**	1
13.	12	**14.**	1	**15.**	3
16.	3	**17.**	9	**18.**	2
19.	5	**20.**	0	**21.**	2

PART - III

1.	(ABCD)	**2.**	(CD)	**3.**	(AC)
4.	(AC)	**5.**	(CD)	**6.**	(AC)
7.	(ACD)	**8.**	(AC)	**9.**	(ABC)
10.	(AB)	**11.**	(BC)	**12.**	(BD)
13.	(AD)				

PART - IV

1.	(A)	**2.**	(B)	**3.**	(A)
4.	(D)	**5.**	(C)	**6.**	(C)
7.	(A)	**8.**	(ACD)	**9.**	(ABC)

EXERCISE - 3

PART - I

1.	(C)	**2.**	(D)	**4.**	(B)
5.	(C)	**6.**	(D)	**8.**	(D)
9.	(B)	**10.**	(D)	**11.**	6
12.	(C)	**13.**	8	**14.**	5

PART - II

1.	(4)	**2.**	(2)	**3.**	(3)
4.	(3)	**5.**	(2)	**6.**	(1)
7.	(2)	**8.**	(1)	**9.**	(2)
10.	(3)	**11.**	(3)	**12.**	(2)
13.	(1)	**14.**	(2)	**15.**	(3)
16.	(1)	**17.**	(3)	**18.**	(2)
19.	(1)	**20.**	**(3) or Bonus**		

Comprehension # 2 (Q. No. 4 to 6)

Let P be a product given by $P = (x + a_1)(x + a_2) \ldots\ldots\ldots (x + a_n)$

and Let $S_1 = a_1 + a_2 + \ldots\ldots + a_n = \sum_{i=1}^{n} a_i$, $S_2 = \sum\sum_{i<j} a_i.a_j$, $S_3 = \sum\sum\sum_{i<j<k} a_i.a_j.a_k$ and so on,

then it can be shown that $P = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots\ldots\ldots + S_n$.

4. The coefficient of x^8 in the expression $(2 + x)^2 (3 + x)^3 (4 + x)^4$ must be
(A) 26 (B) 27 (C) 28 (D) 29

5. The coefficient of x^{203} in the expression $(x - 1)(x^2 - 2)(x^3 - 3) \ldots\ldots\ldots (x^{20} - 20)$ must be
(A) 11 (B) 12 (C) 13 (D) 15

6. The coefficient of x^{98} in the expression of $(x - 1)(x - 2) \ldots\ldots\ldots (x - 100)$ must be
(A) $1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2$
(B) $(1 + 2 + 3 + \ldots\ldots + 100)^2 - (1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2)$
(C) $\frac{1}{2}[(1 + 2 + 3 + \ldots\ldots + 100)^2 - (1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2)]$
(D) None of these

6.92 approx

$7 - 4\sqrt{3} < 1$

Comprehension # 3 (Q.No. 7 to 9)

Let $(7 + 4\sqrt{3})^n = I + f = {}^nC_0.7^n + {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + \ldots\ldots$ ....(i)

where I & f are its integral and fractional parts respectively.

It means $0 < f < 1$

~4
1.73
4
6.92

Now, $0 < 7 - 4\sqrt{3} < 1 \Rightarrow 0 < (7 - 4\sqrt{3})^n < 1$

Let $(7 - 4\sqrt{3})^n = f' = {}^nC_0.7^n - {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + \ldots\ldots$ ....(ii)

$\Rightarrow 0 < f' < 1$

Adding (i) and (ii) (so that irrational terms cancelled out)

$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$

$= 2\,[{}^nC_0\,7^n + {}^nC_2\,7^{n-2}(4\sqrt{3})^2 + \ldots\ldots\ldots]$

$I + f + f'$ = even integer $\Rightarrow$ ($f + f'$ must be an integer)

$0 < f + f' < 2 \Rightarrow f + f' = 1$

with help of above analysis answer the following questions

7. If $(3\sqrt{3} + 5)^n = p + f$, where p is an integer and f is a proper fraction, then find the value of $(3\sqrt{3} - 5)^n$, $n \in N$, is
(A) $1 - f$, if n is even (B) f, if n is even (C) $1 - f$, if n is odd (D) f, if n is odd

8. If $(9 + \sqrt{80})^n = I + f$, where I, n are integers and $0 < f < 1$, then :
(A) I is an odd integer (B) I is an even integer
(C) $(I + f)(1 - f) = 1$ (D) $1 - f = (9 - \sqrt{80})^n$

9. The integer just above $(\sqrt{3} + 1)^{2n}$ is, for all $n \in N$.
(A) divisible by 2^n (B) divisible by 2^{n+1} (C) divisible by 8 (D) divisible by 16

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$

(A) the number of irrational terms is 19
(B) middle term is irrational
(C) the number of rational terms is 2
(D) 9th term is rational

2. The coefficient of x^4 in $\left(\frac{1+x}{1-x}\right)^2$, $|x| < 1$, is

(A) 4 (B) –4 (C) $10 + {}^4C_2$ (D) 16

3. $7^9 + 9^7$ is divisible by :

(A) 16 (B) 24 (C) 64 (D) 72

4. The sum of the series $\sum_{r=1}^{n}(-1)^{r-1}.\ {}^nC_r(a-r)$ is equal to :

(A) 5 if a = 5 (B) –5 if a = 5 (C) –5 if a = –5 (D) 5 if a = –5

5. Let $a_n = \frac{1000^n}{n!}$ for $n \in N$, then a_n is greatest, when

(A) n = 997 (B) n = 998 (C) n = 999 (D) n = 1000

6. ${}^nC_0 - 2.3\,{}^nC_1 + 3.3^2\,{}^nC_2 - 4.3^3\,{}^nC_3 + \ldots\ldots\ldots + (-1)^n (n+1)\,{}^nC_n\,3^n$ is equal to

(A) $2^n\left(\frac{3n}{2}+1\right)$ if n is even
(B) $2^n\left(n+\frac{3}{2}\right)$ if n is even
(C) $-2^n\left(\frac{3n}{2}+1\right)$ if n is odd
(D) $2^n\left(n+\frac{3}{2}\right)$ if n is odd

7. Element in set of values of r for which, ${}^{18}C_{r-2} + 2.\ {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ is :

(A) 9 (B) 5 (C) 7 (D) 10

8. The expansion of $(3x + 2)^{-1/2}$ is valid in ascending powers of x, if x lies in the interval.

(A) (0, 2/3) (B) (–3/2, 3/2) (C) (–2/3, 2/3) (D) $(-\infty, -3/2) \cup (3/2, \infty)$

9. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{20}x^{20}$, then :

(A) $a_1 = 20$ (B) $a_2 = 210$ (C) $a_4 = 8085$ (D) $a_{20} = 2^2.\ 3^7.\ 7$

10. In the expansion of $(x + y + z)^{25}$

(A) every term is of the form ${}^{25}C_r.\ {}^rC_k.\ x^{25-r}.\ y^{r-k}.\ z^k$
(B) the coefficient of $x^8 y^9 z^9$ is 0
(C) the number of terms is 325
(D) none of these

11. If $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_{40}x^{40}$, then $a_0 + a_2 + a_4 \ldots\ldots. + a_{38}$ is equal to :

(A) $2^{19}(2^{30} + 1)$ (B) $2^{19}(2^{20} - 1)$ (C) $2^{39} - 2^{19}$ (D) $2^{39} + 2^{19}$

12. $n^n\left(\frac{n+1}{2}\right)^{2n}$ is $(n \in N)$

(A) Less than $\left(\frac{n+1}{2}\right)^3$
(B) Greater than or equal to $\left(\frac{n+1}{2}\right)^3$
(C) Less than $(n!)^3$
(D) Greater than or equal to $(n!)^3$.

13. If recursion polynomials $P_k(x)$ are defined as $P_1(x) = (x-2)^2$, $P_2(x) = ((x-2)^2-2)^2$
$P_3(x) = ((x-2)^2 - 2)^2 - 2)^2$ .... (In general $P_k(x) = (P_{k-1}(x) - 2)^2$, then the constant term in $P_k(x)$ is

(A) 4 (B) 2 (C) 16 (D) a perfect square

PART - IV : COMPREHENSION

Comprehension # 1 (Q. No. 1 to 3)

Consider, sum of the series $\sum\sum_{0\leq i<j\leq n} f(i)\,f(j)$ In the given summation, i and j are not independent. In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)\,f(j) = \sum_{i=1}^{n}\left(f(i)\left(\sum_{j=1}^{n} f(j)\right)\right)$ i and j are independent. In this summation, three types of terms occur, those when i < j, i > j and i = j.
Also, sum of terms when i < j is equal to the sum of the terms when i > j if f(i) and f(j) are symmetrical. So, in that case

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum\sum_{0\leq i<j\leq n} f(i)f(j) + \sum\sum_{0\leq i<j\leq n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$= 2\sum\sum_{0\leq i<j\leq n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\Rightarrow \sum\sum_{0\leq i<j\leq n} f(i)f(j) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} f(i)f(j) - \sum\sum_{i=j} f(i)f(j)}{2}$$

When f(i) and f(j) are not symmetrical, we find the sum by listing all the terms.

1. $\sum\sum_{0\leq i<j\leq n} {}^nC_i\ {}^nC_j$ is equal to

(A) $\frac{2^{2n} - {}^{2n}C_n}{2}$ (B) $\frac{2^{2n} + {}^{2n}C_n}{2}$ (C) $\frac{2^{2n} - {}^nC_n}{2}$ (D) $\frac{2^{2n} + {}^nC_n}{2}$

2. $\sum_{m=0}^{n}\sum_{p=0}^{m} {}^nC_m.\ {}^mC_p$ is equal to

(A) $2^n–1$ (B) 3^n (C) $3^n–1$ (D) 2^n

3. $\sum\sum_{0\leq i\leq j\leq n}\left({}^nC_i + {}^nC_j\right)$

(A) $(n+2)2^n$ (B) $(n+1)2^n$ (C) $(n-1)2^n$ (D) $(n+1)2^n–1$

16. If $(1 + x + x^2 + x^3)^5 = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots + a_{15}x^{15}$, then a_{10} equals to :
(A) 99 (B) 101 (C) 100 (D) 110

17. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, the value of $\sum_{r=0}^{n} \frac{n-2r}{{}^nC_r}$ is :

(A) $\frac{n}{2}a_n$ (B) $\frac{1}{4}a_n$ (C) na_n (D) 0

18. The sum of: $3.{}^nC_0 - 8.{}^nC_1 + 13.{}^nC_2 - 18.{}^nC_3 + \ldots$ upto $(n+1)$ terms is $(n \geq 2)$:
(A) zero (B) 1 (C) 2 (D) none of these

19. If $\sum_{r=0}^{n-1}\left(\frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}}\right)^3 = \frac{4}{5}$ then n =

(A) 4 (B) 6 (C) 8 (D) None of these

20. The number of terms in the expansion of $\left(x^2 + 1 + \frac{1}{x^2}\right)^n$, $n \in N$, is :

(A) 2n (B) 3n (C) 2n + 1 (D) 3n + 1

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. If $\frac{1}{1!10!} + \frac{1}{2!9!} + \frac{1}{3!10!} + \ldots\ldots + \frac{1}{1!10!} = \frac{2}{k!}(2^{k-1} - 1)$ then find the value of k.

2. If the 6^{th} term in the expansion of $\left[\frac{1}{x^{8/3}} + x^2\log_{10}x\right]^8$ is 5600, then x =

3. The number of values of 'x' for which the fourth term in the expansion, $\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}} + \frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$ is 336, is :

4. If second, third and fourth terms in the expansion of $(x + a)^n$ are 240, 720 and 1080 respectively, then n is equal to

5. Let the co-efficients of x^n in $(1 + x)^{2n}$ & $(1 + x)^{2n-1}$ be P & Q respectively, then $\left(\frac{P+Q}{Q}\right)^5 =$

6. In the expansion of $\left(3^{\frac{-x}{4}} + 3^{\frac{5x}{4}}\right)^n$, the sum of the binomial coefficients is 256 and four times the term with greatest binomial coefficient exceeds the square of the third term by 21n, then find 4x.

7. If $\sum_{k=1}^{19}\frac{(-2)^k}{k!(19-k)!} = \frac{-\lambda}{19!}$ then find λ.

8. The value of p, for which coefficient of x^{50} in the expression
$(1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + \ldots\ldots + 1001x^{1000}$ is equal to ${}^{1002}C_p$, is :

9. If {x} denotes the fractional part of 'x', then $82\left\{\frac{3^{1001}}{82}\right\} =$

10. The index 'n' of the binomial $\left(\frac{x}{5} + \frac{2}{5}\right)^n$ if the only 9^{th} term of the expansion has numerically the greatest coefficient $(n \in N)$, is :

11. The number of values of 'r' satisfying the equation, ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r}$ is :

12. Find the value of
${}^6C_0 \cdot {}^{12}C_6 - {}^6C_1\,{}^{11}C_6 + {}^6C_2\,{}^{10}C_6 + {}^6C_2\,{}^{10}C_6 - {}^6C_3 \cdot {}^9C_6 + {}^6C_4 \cdot {}^8C_6 - {}^6C_5 \cdot {}^7C_6 + {}^6C_6 \cdot {}^6C_6$

13. If n is a positive integer & $C_k = {}^nC_k$, find the value of $\left(\sum_{k=1}^{n}\frac{k^3}{n(n+1)^2.(n+2)}\left(\frac{C_k}{C_{k-1}}\right)^2\right)^{-1}$ is :

14. The value of the expression $\left(\sum_{r=0}^{10}{}^{10}C_r\right)\left(\sum_{K=0}^{10}(-1)^K\frac{{}^{10}C_K}{2^K}\right)$ is :

15. The value of λ if $\sum_{m=97}^{100}{}^{100}C_m \cdot {}^mC_{97} = 2^{\lambda} \cdot {}^{100}C_{97}$, is :

16. If $(1 + x + x^2 + \ldots + x^p)^6 = a_0 + a_1x + a_2x^2 + \ldots + a_{6p}x^{6p}$, then the value of :
$\frac{1}{p(p+1)^6}[a_1 + 2a_2 + 3a_3 + \ldots. + 6p\,a_{6p}]$ is :

17. If $({}^{2n}C_1)^2 + 2.({}^{2n}C_2)^2 + 3.({}^{2n}C_3)^2 + \ldots + 2n.({}^{2n}C_{2n})^2 = 18 \cdot {}^{4n-1}C_{2n-1}$, then n is :

18. If $\sum_{r=0}^{n}\frac{2r+3}{r+1}.{}^nC_r = \frac{(n+k).2^{n+1}-1}{n+1}$ then 'k' is

19. If $\sum_{r=0}^{n}\frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{1}{a(n+b)}$, then a + b is

20. $\sum_{k=1}^{3n} 6\,{}^nC_{2k-1}(-3)^k$ is equal to :

21. If x is very large as compare to y, then the value of k in $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = 1 + \frac{y^2}{kx^2}$

Section (D) : Negative & fractional index, Multinomial theorem

D-1. If $|x| < 1$, then the co-efficient of x^n in the expansion of $(1 + x + x^2 + x^3 + \ldots)^2$ is
(A) n (B) n – 1 (C) n + 2 (D) n + 1

D-2. The co–efficient of x^4 in the expansion of $(1 - x + 2x^2)^{12}$ is: 4
(A) ${}^{12}C_3$ (B) ${}^{13}C_3$ (C) ${}^{14}C_4$ (D) ${}^{12}C_3 + 3\,{}^{13}C_3 + {}^{14}C_4$

D-3. If $(1 + x)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{10}x^{10}$, then value of $(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ is 4
(A) 2^{10} (B) 2 (C) 2^{20} (D) None of these

PART - III : MATCH THE COLUMN

1.

	Column – I		Column – II
(A)	If $(r + 1)^{th}$ term is the first negative term in the expansion of $(1 + x)^{7/2}$, then the value of r (where $0 < x < 1$) is 5	(p)	divisible by 2 D
(B)	If the sum of the co-efficients in the expansion of $(1 + 2x)^n$ is 6561, and T_r is the greatest term in the expansion for $x = 1/2$ then r is n = 8,	(q)	divisible by 5 A
(C)	${}^nC_r$ is divisible by n, $(1 < r < n)$ if n is	(r)	divisible by 10
(D)	The coefficient of x^4 in the expression $(1 + 2x + 3x^2 + 4x^3 + \ldots\text{up to } \infty)^{1/2}$ is c, $(c \in N)$, then c + 1 (where $\lvert x \rvert < 1$) is c = 1, c + 1 = 2	(s)	a prime number A G D

$4+1-1$ C_4 $= {}^4C_4 = 1$

Exercise-2

Marked Questions may have for Revision Questions.

PART - I : ONLY ONE OPTION CORRECT TYPE

1. In the expansion of $\left(3\sqrt{\frac{a}{b}} + 3\sqrt{\frac{b}{\sqrt{a}}}\right)^{21}$, the term containing same powers of a & b is TRICKY ${}^nC_r = {}^nC_{n-r}$
(A) 11^{th} term (B) 13^{th} term (C) 12^{th} term (D) 6^{th} term

2. Consider the following statements :
S_1 : Number of dissimilar terms in the expansion of $(1 + x + x^2 + x^3)^n$ is $3n + 1$
S_2 : $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3) \ldots (1 + x + x^2 + \ldots + x^{100})$ when written in the ascending power of x then the highest exponent of x is 5000.
S_3 : $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^nC_{r+1}$
S_4 : If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$, then $a_0 + a_2 + a_4 + \ldots + a_{2n} = \frac{3^n - 1}{2}$ $\left(\frac{3^n+1}{2}\right)$
State, in order, whether S_1, S_2, S_3, S_4 are true or false
(A) TFTF (B) TTTT (C) FFFF (D) FTFT

3. If $\frac{{}^nC_r + 4\,{}^nC_{r+1} + 6\,{}^nC_{r+2} + 4\,{}^nC_{r+3} + {}^nC_{r+4}}{{}^nC_r + 3\,{}^nC_{r+1} + 3\,{}^nC_{r+2} + {}^nC_{r+3}} = \frac{n+k}{r+k}$ then the value of k is :
(A) 1 (B) 2 (C) 4 (D) 5

4. The co-efficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + \ldots + (1 + x)^{30}$ is :
(A) ${}^{51}C_5$ (B) ${}^9C_5$ (C) ${}^{31}C_6 - {}^{21}C_6$ (D) ${}^{30}C_5 + {}^{20}C_5$

5. The coefficient of x^{52} in the expansion $\sum_{m=0}^{100} {}^{100}C_m (x - 3)^{100-m} . 2^m$ is :
(A) ${}^{100}C_{47}$ (B) ${}^{100}C_{48}$ (C) $-{}^{100}C_{52}$ (D) $-{}^{100}C_{100}$

6. The sum of the coefficients of all the integral powers of x in the expansion of $(1 + 2\sqrt{x})^{40}$ is :
(A) $3^{40} + 1$ (B) $3^{40} - 1$ (C) $\frac{1}{2}(3^{40} - 1)$ (D) $\frac{1}{2}(3^{40} + 1)$

7. $\sum_{r=0}^{n} (-1)^r\, {}^nC_r . \frac{(1 + r\ln 10)}{(1 + \ln 10^n)^r} =$
(A) 0 (B) 1/2 (C) 1 (D) None of these

8. The coefficient of the term independent of x in the expansion of $\left(\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \frac{x-1}{x - x^{\frac{1}{2}}}\right)^{10}$ is :
(A) 70 (B) 112 (C) 105 (D) 210

9. Coefficient of x^{n-1} in the expansion of, $(x + 3)^n + (x + 3)^{n-1}(x + 2) + (x + 3)^{n-2}(x + 2)^2 + \ldots + (x + 2)^n$ is :
(A) ${}^{n+1}C_2(3)$ (B) ${}^{n-1}C_2(5)$ (C) ${}^{n+1}C_2(5)$ (D) ${}^nC_2(5)$

10. Let $f(n) = 10^n + 3.4^{n+2} + 5$, $n \in N$. The greatest value of the integer which divides f(n) for all n is :
(A) 27 (B) 9 (C) 3 (D) None of these

11. If $(1 + x)^n = \sum_{r=0}^{n} a_r x^r$ and $b_r = 1 + \frac{a_r}{a_{r-1}}$ and $\prod_{r=1}^{n} b_r = \frac{(101)^{100}}{100!}$, then n equals to :
(A) 99 (B) 100 (C) 101 (D) 102

12. Number of rational terms in the expansion of $(1 + \sqrt{2} + \sqrt{5})^6$ is:
(A) 7 (B) 10 (C) 6 (D) 8

13. If $S = {}^{404}C_4 - {}^4C_1 . {}^{303}C_4 + {}^4C_2 . {}^{202}C_5 - {}^4C_3 . {}^{101}C_4 = (101)^k$ then k equals to :
(A) 1 (B) 2 (C) 4 (D) 6

14. ${}^{10}C_0^{\,2} - {}^{10}C_1^{\,2} + {}^{10}C_2^{\,2} - \ldots - ({}^{10}C_9)^2 + ({}^{10}C_{10})^2 =$
(A) 0 (B) $({}^{10}C_5)^2$ (C) $-{}^{10}C_5$ (D) $2\,{}^9C_5$

15. The sum $\sum_{r=0}^{n} (r + 1)\, C_r^2$ is equal to :
(A) $\frac{(n+2)(2n-1)!}{n!(n-1)!}$ (B) $\frac{(n+2)(2n+1)!}{n!(n-1)!}$ (C) $\frac{(n+2)(2n+1)!}{n!(n+1)!}$ (D) $\frac{(n+2)(2n-1)!}{n!(n+1)!}$

D-3. Assuming ' x ' to be so small that x^2 and higher powers of ' x ' can be neglected, show that,

$$\frac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}} \text{ is approximately equal to, } 1-\frac{305}{96}x.$$

D-4. (i) Find the coefficient of $a^5 b^4 c^7$ in the expansion of $(bc + ca + ab)^8$.
(ii) Sum of coefficients of odd powers of x in expansion of $(9x^2 + x - 8)^6$

D-5. Find the coefficient of x^7 in $(1 - 2x + x^3)^5$.

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : General Term & Coefficient of x^k in $(ax + b)^n$

A-1. The $(m + 1)^{th}$ term of $\left(\frac{x}{y}+\frac{y}{x}\right)^{2m+1}$ is:
(A) independent of x
(B) a constant
(C) depends on the ratio x/y and m
(D) none of these

A-2. The total number of distinct terms in the expansion of, $(x + a)^{100} + (x - a)^{100}$ after simplification is :
(A) 50 (B) 202 (C) 51 (D) none of these

A-3. The value of, $\frac{18^3+7^3+3.18.7.25}{3^6+6.243.2+15.81.4+20.27.8+15.9.16+6.3.32+64}$ is :
(A) 1 (B) 2 (C) 3 (D) none

A-4. In the expansion of $\left(3-\sqrt{\frac{17}{4}+3\sqrt{2}}\right)^{15}$ the 11th term is a :
(A) positive integer
(B) positive irrational number
(C) negative integer
(D) negative irrational number.

A-5. If the second term of the expansion $\left[a^{1/13}+\frac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$, then the value of $\frac{^nC_3}{^nC_2}$ is:
(A) 4 (B) 3 (C) 12 (D) 6

A-6. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$, the number of terms free from radicals is:
(A) 730 (B) 729 (C) 725 (D) 750

A-7. The value of m, for which the coefficients of the $(2m + 1)^{th}$ and $(4m + 5)^{th}$ terms in the expansion of $(1 + x)^{10}$ are equal, is
(A) 3 (B) 1 (C) 5 (D) 8

A-8. The co-efficient of x in the expansion of $(1 - 2x^3 + 3x^5)\left(1+\frac{1}{x}\right)^8$ is :
(A) 56 (B) 65 (C) 154 (D) 62

A-9. Given that the term of the expansion $(x^{1/3} - x^{-1/2})^{15}$ which does not contain x is 5 m, where m ∈ N, then m=
(A) 1100 (B) 1010 (C) 1001 (D) 1002

A-10. The term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^4\left(x+\frac{1}{x}\right)^3$ is:
(A) – 3 (B) 0 (C) 1 (D) 3

Section (B) : Middle term, Remainder & Numerically/Algebrically Greatest terms

B-1. If k ∈ R and the middle term of $\left(\frac{k}{2}+2\right)^8$ is 1120, then value of k is:
(A) 3 (B) 2 (C) – 3 (D) – 4

B-2. The remainder when 2^{2003} is divided by 17 is :
(A) 1 (B) 2 (C) 8 (D) 7

B-3. The last two digits of the number 3^{400} are:
(A) 81 (B) 43 (C) 29 (D) 01

B-4. The last three digits in 10 ! are :
(A) 800 (B) 700 (C) 500 (D) 600

B-5. The value of $\sum_{r=1}^{10} r.\frac{^nC_r}{^nC_{r-1}}$ is equal to
(A) 5(2n – 9) (B) 10 n (C) 9 (n – 4) (D) n – 2

B-6. $\sum_{r=0}^{n-1}\frac{^nC_r}{^nC_r+{^nC_{r+1}}} =$
(A) $\frac{n}{2}$ (B) $\frac{n+1}{2}$ (C) $(n+1)\frac{n}{2}$ (D) $\frac{n(n-1)}{2(n+1)}$

B-7. Find numerically greatest term in the expansion of $(2 + 3x)^9$, when x = 3/2.
(A) $^9C_6 . 2^9 . (3/2)^{12}$ (B) $^9C_3 . 2^9 . (3/2)^6$ (C) $^9C_5 . 2^9 . (3/2)^{10}$ (D) $^9C_4 . 2^9 . (3/2)^8$

Section (C) : Summation of series, Variable upper index & Product of binomial coefficients

C-1. $\frac{^{11}C_0}{1}+\frac{^{11}C_1}{2}+\frac{^{11}C_2}{3}+......+\frac{^{11}C_{10}}{11} =$
(A) $\frac{2^{11}-1}{11}$ (B) $\frac{2^{11}-1}{6}$ (C) $\frac{3^{11}-1}{11}$ (D) $\frac{3^{11}-1}{6}$

C-2. The value of $\frac{C_0}{1.3}-\frac{C_1}{2.3}+\frac{C_2}{3.3}-\frac{C_3}{4.3}+.........+(-1)^n\frac{C_n}{(n+1).3}$ is :
(A) $\frac{3}{n+1}$ (B) $\frac{n+1}{3}$ (C) $\frac{1}{3(n+1)}$ (D) none of these

C-3. The value of the expression $^{47}C_4 + \sum_{j=1}^{5} {^{52-j}C_3}$ is equal to: $^{52}C_4$
(A) $^{47}C_5$ (B) $^{52}C_5$ (C) $^{52}C_4$ (D) $^{49}C_4$

C-4. The value of $\binom{50}{0}\binom{50}{1}+\binom{50}{1}\binom{50}{2}+..........+\binom{50}{49}\binom{50}{50}$ is, where $^nC_r = \binom{n}{r}$
(A) $\binom{100}{50}$ (B) $\binom{100}{51}$ (C) $\binom{50}{25}$ (D) $\binom{50}{25}^2$

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Exercise-1

✍ Marked Questions may have for Revision Questions.

PART - I : SUBJECTIVE QUESTIONS

Section (A) : General Term & Coefficient of x^k in $(ax+b)^n$

A-1. Expand the following :

(i) $\left(\frac{2}{x}-\frac{x}{2}\right)^5$, $(x \neq 0)$ (ii) $\left(y^2+\frac{2}{y}\right)^4$, $(y \neq 0)$

A-2. In the binomial expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of the 7th term from the begining to the 7th term from the end is 1 : 6 ; find n. 9

A-3. Find the degree of the polynomial $\left(x+(x^3-1)^{\frac{1}{2}}\right)^5+\left(x-(x^3-1)^{\frac{1}{2}}\right)^5$. 7

A-4. Find the coefficient of

(i) x^6y^3 in $(x+y)^9$ (ii) $a^5 b^7$ in $(a-2b)^{12}$

A-5. Find the co-efficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ and of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ and find the relation between 'a' & 'b' so that these co-efficients are equal. (where a, b ≠ 0).

A-6. Find the term independent of 'x' in the expansion of the expression,

$(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

A-7. ✍ (i) Find the coefficient of x^5 in $(1+2x)^6(1-x)^7$. 9

(ii) Find the coefficient of x^4 in $(1+2x)^4(2-x)^5$. HW

A-8. In the expansion of $\left(x^3-\frac{1}{x^2}\right)^n$, n ∈ N, if the sum of the coefficients of x^5 and x^{10} is 0, then n is : HW

Section (B) : Middle term, Remainder & Numerically/Algebrically Greatest terms

B-1. Find the middle term(s) in the expansion of

(i) $\left(\frac{x}{y}-\frac{y}{x}\right)^7$ (ii) $(1-2x+x^2)^n$

B-2. Prove that the co-efficient of the middle term in the expansion of $(1+x)^{2n}$ is equal to the sum of the co-efficients of middle terms in the expansion of $(1+x)^{2n-1}$.

B-3. (i) Find the remainder when 7^{98} is divided by 5

(ii) Using binomial theorem prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25

(iii) Find the last digit, last two digits and last three digits of the number $(27)^{27}$.

B-4. ✍ Which is larger : $(99^{50}+100^{50})$ or $(101)^{50}$.

B-5. (i) Find numerically greatest term(s) in the expansion of $(3-5x)^{15}$ when $x=\frac{1}{5}$

(ii) Which term is the numerically greatest term in the expansion of $(2x+5y)^{34}$, when x = 3 & y = 2 ?

B-6. ✍ Find the term in the expansion of $(2x-5)^6$ which have

(i) Greatest binomial coefficient (ii) Greatest numerical coefficient
(iii) Algebrically greatest coefficient (iv) Algebrically least coefficient

Section (C) : Summation of series, Variable upper index & Product of binomial coefficients

C-1. If $C_0, C_1, C_2, \ldots C_n$ are the binomial coefficients in the expansion of $(1+x)^n$ then prove that :

(i) ✍ $\frac{(3.2-1)}{2}C_1+\frac{3^2.2^2-1}{2^2}C_2+\frac{3^3.2^3-1}{2^3}C_3+\ldots\ldots+\frac{3^n.2^n-1}{2^n}C_n=\frac{2^{3n}-3^n}{2^n}$

(ii) $\frac{C_1}{C_0}+2\frac{C_2}{C_1}+3\frac{C_3}{C_2}+\ldots\ldots+n\frac{C_n}{C_{n-1}}=\frac{n(n+1)}{2}$

(iii) $(C_0+C_1)(C_1+C_2)(C_2+C_3)(C_3+C_4)\ldots\ldots(C_{n-1}+C_n)=\frac{C_0C_1C_2\ldots\ldots C_{n-1}(n+1)^n}{n!}$.

(iv) $C_0-2C_1+3C_2-4C_3+\ldots+(-1)^n(n+1)C_n=0$

(v) $4C_0+\frac{4^2}{2}.C_1+\frac{4^3}{3}C_2+\ldots\ldots+\frac{4^{n+1}}{n+1}C_n=\frac{5^{n+1}-1}{n+1}$

(vi) $\frac{2^2.C_0}{1.2}+\frac{2^3.C_1}{2.3}+\frac{2^4.C_2}{3.4}+\ldots\ldots+\frac{2^{n+2}.C_n}{(n+1)(n+2)}=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}$

C-2. ✍ $C_0+\frac{2^2.C_1}{2}+\frac{2^3.C_2}{3}+\frac{2^4.C_3}{4}+\ldots\ldots\frac{2^{n+1}.C_n}{n+1}=\frac{3^{n+1}-1}{n+1}$

C-3. Prove that ${}^nC_r+{}^{n-1}C_r+{}^{n-2}C_r+\ldots\ldots+{}^rC_r={}^{n+1}C_{r+1}$

C-4. If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots\ldots+C_nx^n$, prove that

(i) $C_0C_3+C_1C_4+\ldots\ldots+C_{n-3}C_n=\frac{(2n)!}{(n+3)!(n-3)!}$

(ii) $C_0C_r+C_1C_{r+1}+\ldots\ldots+C_{n-r}C_n=\frac{(2n)!}{(n+r)!(n-r)!}$

(iii) ✍ $C_0^2-C_1^2+C_2^2-C_3^2+\ldots\ldots+(-1)^nC_n^2=0$ or $(-1)^{n/2}C_{n/2}$ according as n is odd or even.

Section (D) : Negative & fractional index, Multinomial theorem

D-1. ✍ Find the co-efficient of x^6 in the expansion of $(1-2x)^{-5/2}$.

D-2. ✍ (i) Find the coeff of x^{12} in $\frac{4+2x-x^2}{(1+x)^3}$

(ii) Find the coeff of x^{100} in $\frac{3-5x}{(1-x)^2}$

Example-18 : If x is so small such that its square and higher powers may be neglected, then find the value of $\dfrac{(1-2x)^{1/3}+(1+5x)^{-3/2}}{(9+x)^{1/2}}$

Solution : $\dfrac{(1-2x)^{1/3}+(1+5x)^{-3/2}}{(9+x)^{1/2}} = \dfrac{1-\frac{2}{3}x+1-\frac{15x}{2}}{3\left(1+\frac{x}{9}\right)^{1/2}} = \dfrac{1}{3}\left(2-\dfrac{49}{6}x\right)\left(1+\dfrac{x}{9}\right)^{-1/2}$

$= \dfrac{1}{3}\left(2-\dfrac{49}{6}x\right)\left(1-\dfrac{x}{18}\right) = \dfrac{1}{2}\left(2-\dfrac{x}{9}-\dfrac{49}{6}x\right) = 1-\dfrac{x}{18}-\dfrac{49}{12}x = 1-\dfrac{149}{36}x$

Self practice problems :

(13) Find the possible set of values of x for which expansion of $(3-2x)^{1/2}$ is valid in ascending powers of x.

(14) If $y = \dfrac{2}{5}+\dfrac{1.3}{2!}\left(\dfrac{2}{5}\right)^2+\dfrac{1.3.5}{3!}\left(\dfrac{2}{5}\right)^3+\ldots\ldots\ldots\ldots$, then find the value of y^2+2y

(15) The coefficient of x^{50} in $\dfrac{2-3x}{(1-x)^3}$ is

(A) 500 (B) 1000 (C) –1173 (D) 1173

Ans. (13) $x \in \left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$ (14) 4 (15) C

Multinomial theorem :

As we know the Binomial Theorem –

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r = \sum_{r=0}^{n} \frac{n!}{(n-r)!\,r!} x^{n-r} y^r$$

putting $n-r = r_1$, $r = r_2$

therefore, $(x+y)^n = \displaystyle\sum_{r_1+r_2=n} \frac{n!}{r_1!\ r_2!} x^{r_1} . y^{r_2}$

Total number of terms in the expansion of $(x+y)^n$ is equal to number of non-negative integral solution of $r_1+r_2 = n$ i.e. ${}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n+1$

In the same fashion we can write the multinomial theorem

$$(x_1+x_2+x_3+\ldots\ldots\ldots\ldots x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \frac{n!}{r_1!\,r_2!\ldots r_k!} x_1^{r_1} . x_2^{r_2} \ldots x_k^{r_k}$$

Here total number of terms in the expansion of $(x_1+x_2+\ldots\ldots\ldots+x_k)^n$ is equal to number of non-negative integral solution of $r_1+r_2+\ldots\ldots\ldots+r_k = n$ i.e. ${}^{n+k-1}C_{k-1}$

${}^{n+k-1}C_{k-1}$

Example # 19 : Find the coefficient of $a^2b^3c^4d$ in the expansion of $(a-b-c+d)^{10}$

Solution : $(a-b-c+d)^{10} = \displaystyle\sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!\,r_2!\,r_3!\,r_4!} (a)^{r_1}(-b)^{r_2}(-c)^{r_3}(d)^{r_4}$

we want to get $a^2 b^3 c^4 d$ this implies that $r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 1$

$\therefore$ coeff. of $a^2 b^3 c^4 d$ is $\dfrac{(10)!}{2!\,3!\,4!\,1!}(-1)^3(-1)^4 = -12600$

Example # 20 : In the expansion of $\left(1+x+\dfrac{7}{x}\right)^{11}$, find the term independent of x.

Solution : $\left(1+x+\dfrac{7}{x}\right)^{11} = \displaystyle\sum_{r_1+r_2+r_3=11} \frac{(11)!}{r_1!\,r_2!\,r_3!} (1)^{r_1}(x)^{r_2}\left(\frac{7}{x}\right)^{r_3}$

The exponent 11 is to be divided among the base variables 1, x and $\dfrac{7}{x}$ in such a way so that we get x^0.

Therefore, possible set of values of (r_1, r_2, r_3) are (11, 0, 0), (9, 1, 1), (7, 2, 2), (5, 3, 3), (3, 4, 4), (1, 5, 5)

Hence the required term is

$$\frac{(11)!}{(11)!}(7^0) + \frac{(11)!}{9!\,1!\,1!}7^1 + \frac{(11)!}{7!\,2!\,2!}7^2 + \frac{(11)!}{5!\,3!\,3!}7^3 + \frac{(11)!}{3!\,4!\,4!}7^4 + \frac{(11)!}{1!\,5!\,5!}7^5$$

$$= 1 + \frac{(11)!}{9!\,2!}\cdot\frac{2!}{1!\,1!}7^1 + \frac{(11)!}{7!\,4!}\cdot\frac{4!}{2!\,2!}7^2 + \frac{(11)!}{5!\,6!}\cdot\frac{6!}{3!\,3!}7^3$$

$$+ \frac{(11)!}{3!\,8!}\cdot\frac{8!}{4!\,4!}7^4 + \frac{(11)!}{1!\,10!}\cdot\frac{(10)!}{5!\,5!}7^5$$

$$= 1 + {}^{11}C_2 . {}^2C_1 . 7^1 + {}^{11}C_4 . {}^4C_2 . 7^2 + {}^{11}C_6 . {}^6C_3 . 7^3 + {}^{11}C_8 . {}^8C_4 . 7^4 + {}^{11}C_{10} . {}^{10}C_5 . 7^5$$

$$= 1 + \sum_{r=1}^{5} {}^{11}C_{2r} . {}^{2r}C_r . 7^r$$

Self practice problems :

(16) The number of terms in the expansion of $(a+b+c+d+e)^n$ is

(A) ${}^{n+4}C_4$ (B) ${}^{n+3}C_n$ (C) ${}^{n+5}C_n$ (D) n + 1

(17) Find the coefficient of $x^2 y^3 z^1$ in the expansion of $(x-2y-3z)^7$

(18) Find the coefficient of x^{17} in $(2x^2-x-3)^9$

Ans. (16) A (17) $\dfrac{7!}{2!\,3!\,1!}24$ (18) 2304

$${}^{n+5-1}C_{5-1} = {}^{n+4}C_4$$

$\boxed{r_1+r_2+r_3+\text{—}\ r_k = n}$

no of ~~ways~~ non-ve integral solutions $= {}^{n+k-1}C_{k-1}$

Example # 15 : Find the summation of the following series –

(i) ${}^{m}C_0 + {}^{m+1}C_1 + {}^{m+2}C_2 + \dots\dots\dots + {}^{n}C_m$

(ii) ${}^{n}C_3 + 2 \,.\, {}^{n+1}C_3 + 3 \,.\, {}^{n+2}C_3 + \dots\dots + n \,.\, {}^{2n-1}C_3$

Solution : (i) **I Method :** Using property, ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

${}^{m}C_0 + {}^{m+1}C_1 + {}^{m+2}C_2 + \dots\dots\dots + {}^{n}C_m$

${}^{m}C_m + {}^{m+1}C_m + {}^{m+2}C_m + \dots\dots\dots + {}^{n}C_m$

$= \underbrace{{}^{m+1}C_{m+1} + {}^{m+1}C_m} + {}^{m+2}C_m + \dots\dots\dots + {}^{n}C_m \quad \{\because {}^{m}C_m = {}^{m+1}C_{m+1}\}$

$= \underbrace{{}^{m+2}C_{m+1} + {}^{m+2}C_m} + \dots\dots\dots + {}^{n}C_m$

$= {}^{m+3}C_{m+1} + \dots\dots\dots + {}^{n}C_m = {}^{n}C_{m+1} + {}^{n}C_m = {}^{n+1}C_{m+1}$

II Method

${}^{m}C_m + {}^{m+1}C_m + {}^{m+2}C_m + \dots\dots + {}^{n}C_m$

The above series can be obtained by writing the coefficient of x^m in

$(1 + x)^m + (1 + x)^{m+1} + \dots\dots + (1 + x)^n$

Let $S = (1 + x)^m + (1 + x)^{m+1} + \dots\dots\dots + (1 + x)^n$

$$= \frac{(1+x)^m\left[(1+x)^{n-m+1}-1\right]}{x} = \frac{(1+x)^{n+1}-(1+x)^m}{x}$$

$$= \text{coefficient of } x^m \text{ in } \frac{(1+x)^{n+1}}{x} - \frac{(1+x)^m}{x} = {}^{n+1}C_{m+1} + 0 = {}^{n+1}C_{m+1}$$

(ii) ${}^{n}C_3 + 2 \,.\, {}^{n+1}C_3 + 3 \,.\, {}^{n+2}C_3 + \dots\dots + n \,.\, {}^{2n-1}C_3$

The above series can be obatined by writing the coefficient of x^3 in

$(1 + x)^n + 2 \,.\, (1 + x)^{n+1} + 3 \,.\, (1 + x)^{n+2} + \dots\dots + n \,.\, (1 + x)^{2n-1}$

Let $S = (1 + x)^n + 2 \,.\, (1 + x)^{n+1} + 3 \,.\, (1 + x)^{n+2} + \dots\dots + n\,(1 + x)^{2n-1}$ .....(i)

$(1 + x)S = (1 + x)^{n+1} + 2\,(1 + x)^{n+2} + \dots\dots + (n-1)(1 + x)^{2n-1} + n(1 + x)^{2n}$ ....(ii)

Subtracting (ii) from (i)

$-xS = (1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n+2} + \dots\dots + (1 + x)^{2n-1} - n(1 + x)^{2n}$

$$= \frac{(1+x)^n\left[(1+x)^n-1\right]}{x} - n(1+x)^{2n}$$

$$S = \frac{-(1+x)^{2n}+(1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

$x^3 : S$ (coefficient of x^3 in S)

$$x^3 : \frac{-(1+x)^{2n}+(1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

Hence, required summation of the series is $-{}^{2n}C_5 + {}^{n}C_5 + n \,.\, {}^{2n}C_4$

Example # 16 : Prove that $C_1 - C_3 + C_5 - \dots\dots = 2^{n/2} \sin\frac{n\pi}{4}$.

Solution : Consider the expansion $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \dots\dots + C_n x^n$ ....(i)

putting $x = -i$ in (i) we get

$(1 - i)^n = C_0 - C_1 i - C_2 + C_3 i + C_4 + \dots\dots (-1)^n C_n i^n$

or $2^{n/2}\left[\cos\left(-\frac{n\pi}{4}\right) + i\sin\left(-\frac{n\pi}{4}\right)\right] = (C_0 - C_2 + C_4 - \dots\dots) - i\,(C_1 - C_3 + C_5 - \dots\dots)$ ....(ii)

Equating the imaginary part in (ii) we get $C_1 - C_3 + C_5 - \dots\dots = 2^{n/2} \sin\frac{n\pi}{4}$.

Self practice problems :

(12) Prove the following

(i) $5C_0 + 7C_1 + 9C_2 + \dots\dots\dots + (2n + 5)\,C_n = 2^n\,(n + 5)$

(ii) $4C_0 + \frac{4^2}{2} \,.\, C_1 + \frac{4^3}{3}\,C_2 + \dots\dots\dots + \frac{4^{n+1}}{n+1}\,C_n = \frac{5^{n+1}-1}{n+1}$

(iii) ${}^{n}C_0 \,.\, {}^{n+1}C_n + {}^{n}C_1 \,.\, {}^{n}C_{n-1} + {}^{n}C_2 \,.\, {}^{n-1}C_{n-2} + \dots\dots + {}^{n}C_n \,.\, {}^{1}C_0 = 2^{n-1}\,(n + 2)$

(iv) ${}^{2}C_2 + {}^{3}C_2 + \dots\dots + {}^{n}C_2 = {}^{n+1}C_3$

Binomial theorem for negative and fractional indices :

If $n \in R$, then

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\dots\dots$$
$$\dots\dots\dots + \frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}x^r + \dots\dots\dots \infty.$$

Remarks

(i) The above expansion is valid for any rational number other than a whole number if $|x| < 1$.

(ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite, and the symbol ${}^{n}C_r$ cannot be used to denote the coefficient of the general term.

(iii) The first term must be unity in the expansion, when index 'n' is a negative integer or fraction

$$(x + y)^n = \begin{cases} x^n\left(1+\frac{y}{x}\right)^n = x^n\left\{1+n.\frac{y}{x}+\frac{n(n-1)}{2!}\left(\frac{y}{x}\right)^2+\dots\right\} & \text{if } \left|\frac{y}{x}\right|<1 \\ y^n\left(1+\frac{x}{y}\right)^n = y^n\left\{1+n.\frac{x}{y}+\frac{n(n-1)}{2!}\left(\frac{x}{y}\right)^2+\dots\right\} & \text{if } \left|\frac{x}{y}\right|<1 \end{cases}$$

(iv) The general term in the expansion of $(1 + x)^n$ is $T_{r+1} = \frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}x^r$

(v) When 'n' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$ (x^2 and higher powers of x can be neglected)

(vi) Expansions to be remembered ($|x| < 1$)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots\dots + (-1)^r x^r + \dots\dots\infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots\dots + x^r + \dots\dots\infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots\dots + (-1)^r (r + 1)\, x^r + \dots\dots\infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots\dots + (r + 1)x^r + \dots\dots \infty$

Example # 17 : Prove that the coefficient of x^r in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$

Solution: $(r + 1)^{th}$ term in the expansion of $(1 - x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)\dots\dots(-n-r+1)}{r!}(-x)^r$$

$$= (-1)^r\frac{n(n+1)(n+2)\dots\dots(n+r-1)}{r!}(-x)^r = \frac{n(n+1)(n+2)\dots\dots(n+r-1)}{r!}x^r$$

$$= \frac{(n-1)!\,n(n+1)\dots\dots(n+r-1)}{(n-1)!\,r!}x^r$$ Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)!\,r!} = {}^{n+r-1}C_r$ **Proved**

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Properties of binomial coefficients :

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_rx^r + \ldots\ldots\ldots + C_nx^n$ ......(1)

where C_r denotes ${}^nC_r$

(1) The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n

Putting x = 1 in (1)

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n \qquad \ldots\ldots(2)$$

or $$\sum_{r=0}^{n} {}^nC_r = 2^n$$

(2) Again putting x = –1 in (1), we get

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots\ldots + (-1)^n\,{}^nC_n = 0 \qquad \ldots\ldots(3)$$

or $$\sum_{r=0}^{n} (-1)^r\,{}^nC_r = 0$$

(3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

from (2) and (3)

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots\ldots = 2^{n-1}$$

(4) Sum of two consecutive binomial coefficients

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

L.H.S. $$= {}^nC_r + {}^nC_{r-1} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\,\frac{(n+1)}{r(n-r+1)}$$

$$= \frac{(n+1)!}{(n-r+1)!\,r!} = {}^{n+1}C_r = \text{R.H.S.}$$

(5) Ratio of two consecutive binomial coefficients

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

(6) $${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\,{}^{n-2}C_{r-2} = \ldots\ldots\ldots = \frac{n(n-1)(n-2)\ldots\ldots(n-(r-1))}{r(r-1)(r-2)\ldots\ldots 2.1}$$

Example # 13 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + c_nx^n$, then show that

(i) $C_0 + 4C_1 + 4^2C_2 + \ldots\ldots\ldots + 4^n C_n = 5^n$.

(ii) $3C_0 + 5C_1 + 7.\,C_2 + \ldots\ldots + (2n + 3)\,C_n = 2^n (n + 3)$.

(iii) $$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots\ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

Solution : (i) $(1 + x)^n = C_0 + C_1\,x + C_2x^2 + \ldots\ldots\ldots + C_nx^n$

put x = 4

$C_0 + 4C_1 + 4^2C_2 + \ldots\ldots\ldots + 4^n C_n = 5^n$.

(ii) L.H.S. $= 3C_0 + 5C_1 + 7.\,C_2 + \ldots\ldots + (2n + 3)\,C_n$

$$= \sum_{r=0}^{n}(2r+3)\,.\,{}^nC_r = 2\sum_{r=0}^{n} r.\,{}^nC_r + 3\sum_{r=0}^{n}{}^nC_r$$

$$= 2n\sum_{r=1}^{n}{}^{n-1}C_{r-1} + 3\sum_{r=0}^{n}{}^nC_r = 2n\,.\,2^{n-1} + 3.\,2^n = 2^n(n + 3) \qquad \text{RHS}$$

(iii) **I Method : By Summation**

L.H.S. $$= C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots\ldots\ldots + \frac{C_n}{n+1}$$

$$= \sum_{r=0}^{n}\,.\,\frac{{}^nC_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n}\,.\,{}^{n+1}C_{r+1} \qquad \left\{\frac{n+1}{r+1}.\,{}^nC_r = {}^{n+1}C_{r+1}\right\} = \frac{2^{n+1}-1}{n+1} \text{ R.H.S.}$$

II Method : By Integration

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_n\,x^n$.

Integrating both sides, within the limits 0 to 1.

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots\ldots + C_n\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \left(C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1}\right) - 0$$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots\ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1} \text{ Proved}$$

Example # 14 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, then prove that

(i) $C_0C_1 + C_1C_2 + C_2C_3 + \ldots\ldots\ldots + C_{n-1}\,C_n = {}^{2n}C_{n-1}$ or ${}^{2n}C_{n+1}$

(ii) $1^2.\,C_1^{\,2} + 2^2.\,C_2^{\,2} + 3^2.\,C_3^{\,2} + \ldots\ldots\ldots + n^2C_n^{\,2} = n^2.\,{}^{2n-2}C_{n-1}$

Solution : (i) $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_n\,x^n$. ....(i)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_nx^0$ ....(ii)

Multiplying (i) and (ii)

$(C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_nx^n)\,(C_0x^n + C_1x^{n-1} + \ldots\ldots\ldots + C_nx^0) = (1 + x)^{2n}$

Comparing coefficient of x^{n-1},

$C_0C_1 + C_1C_2 + C_2C_3 + \ldots\ldots\ldots + C_{n-1}\,C_n = {}^{2n}C_{n-1}$ or ${}^{2n}C_{n+1}$

(ii) $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_n\,x^n$. ....(i)

differentiating w.r.t x......

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots\ldots + nC_n\,x^{n-1}$.

multiplying by x.......

$n\,x(1 + x)^{n-1} = C_1x + 2C_2x^2 + 3C_3x^3 + \ldots\ldots\ldots + nC_n\,x^n$

Now differentiate w.r.t. x.....

$n(1 + x)^{n-1} + n\,(n-1)x.(1+x)^{n-2} = 1^2C_1 + 2^2C_2x + 3^2C_3x^2 + \ldots\ldots + n^2C_nx^{n-1}$ ....(ii)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_nx^0$ ....(iii)

multiplying (ii) & (iii) and comparing the cofficient of x^{n-1}

$$1^2.\,C_1^{\,2} + 2^2.\,C_2^{\,2} + 3^2.\,C_3^{\,2} + \ldots\ldots\ldots + n^2C_n^{\,2} = n\left({}^{2n-1}C_{n-1} - {}^{2n-2}C_{n-2}\right) + n^2\,{}^{2n-2}C_{n-2}$$

$$= n^2\,{}^{2n-2}C_{n-1} = \text{R.H.S.}$$

Case - II

When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is not an integer (Let its integral part be m), then

(i) $T_{r+1} > T_r$ when $r < \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ (r = 1, 2, 3,........, m–1, m)

i.e. $T_2 > T_1$, $T_3 > T_2$, ..., $T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when $r > \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ (r = m + 1, m + 2, ...n)

i.e. $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, ..., $T_{n+1} < T_n$

Conclusion :

When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Note : (i) In any binomial expansion, the middle term(s) has greatest binomial coefficient.
In the expansion of $(a + b)^n$

If n	No. of greatest binomial coefficient	Greatest binomial coefficient
Even	1	${}^nC_{n/2}$
Odd	2	${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$

(Values of both these coefficients are equal)

(ii) In order to obtain the term having numerically greatest coefficient, put a = b = 1, and proceed as discussed above.

Example # 8 : Find the numerically greatest term in the expansion of $(7 - 3x)^{25}$ when $x = \dfrac{1}{3}$.

Solution : $m = \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|} = \dfrac{25+1}{1+\left|\dfrac{7}{-1}\right|} = \dfrac{26}{8}$

[m] = 3 ([m] denotes GIF)

$\therefore$ T_4 is numerically greatest term

Self practice problems :

(3) Find the term independent of x in $\left(x^2 - \dfrac{3}{x}\right)^9$

(4) The sum of all rational terms in the expansion of $(3^{1/7} + 5^{1/2})^{14}$ is
(A) 3^2 (B) $3^2 + 5^7$ (C) $3^7 + 5^2$ (D) 5^7

(5) Find the coefficient of x^{-2} in $(1 + x^2 + x^4)\left(1 - \dfrac{1}{x^2}\right)^{18}$

(6) Find the middle term(s) in the expansion of $(1 + 3x + 3x^2 + x^3)^{2n}$

(7) Find the numerically greatest term in the expansion of $(2 + 5x)^{21}$ when $x = \dfrac{2}{5}$.

Ans. (3) 28.3^7 (4) B (5) –681
(6) ${}^{6n}C_{3n} \cdot x^{3n}$ (7) $T_{11} = T_{12} = {}^{21}C_{10}\ 2^{21}$

Example # 9 : Show that $7^n + 5$ is divisible by 6, where n is a positive integer.

Solution : $7^n + 5 = (1 + 6)^n + 5$
$= {}^nC_0 + {}^nC_1 \cdot 6 + {}^nC_2 \cdot 6^2 + ... + {}^nC_n 6^n + 5.$
$= 6. C_1 + 6^2. C_2 + ... + C_n . 6^n + 6.$
$= 6\lambda$, where λ is a positive integer
Hence, $7^n + 5$ is divisible by 6.

Example # 10 : What is the remainder when 7^{81} is divided by 5.

Solution : $7^{81} = 7.7^{80} = 7. (49)^{40} = 7 (50 - 1)^{40}$
$= 7 [{}^{40}C_0 (50)^{40} - {}^{40}C_1 (50)^{39} + ... - {}^{40}C_{39} (50)^1 + {}^{40}C_{40} (50)^0]$
= 5(k) + 7(where k is a positive integer)
= 5 (k + 1) + 2
Hence, remainder is 2.

Example # 11 : Find the last digit of the number $(13)^{12}$.

Solution : $(13)^{12} = (169)^6 = (170 - 1)^6$
$= {}^6C_0 (170)^6 - {}^6C_1 (170)^5 + ... - {}^6C_5 (170)^1 + {}^6C_6 (170)^0$
Hence, last digit is 1

Note : We can also conclude that last three digits are 481.

Example-12 : Which number is larger $(1.1)^{100000}$ or 10,000 ?

Solution : By Binomial Theorem
$(1.1)^{100000} = (1 + 0.1)^{100000}$
$= 1 + {}^{100000}C_1 (0.1) +$ other positive terms
$= 1 + 100000 \times 0.1 +$ other positive terms
$= 1 + 10000 +$ other positive terms
Hence $(1.1)^{100000} > 10,000$

Self practice problems :

(8) If n is a positive integer, then show that $6^n - 5n - 1$ is divisible by 25.

(9) What is the remainder when 3^{257} is divided by 80 .

(10) Find the last digit, last two digits and last three digits of the number $(81)^{25}$

(11) Which number is larger $(1.3)^{2000}$ or 600

Ans. (9) 3 (10) 1, 01, 001 (11) $(1.3)^{2000}$.

Some standard expansions :

(i) Consider the expansion

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + ...+ {}^nC_r x^{n-r} y^r + ...+ {}^nC_n x^0 y^n \quad ....(i)$$

(ii) Now replace $y \to -y$ we get

$$(x - y)^n = \sum_{r=0}^{n} {}^nC_r (-1)^r x^{n-r} y^r = {}^nC_0 x^n y^0 - {}^nC_1 x^{n-1} y^1 + ...+ {}^nC_r (-1)^r x^{n-r} y^r + ...+ {}^nC_n (-1)^n x^0 y^n \quad ....(ii)$$

(iii) Adding (i) & (ii), we get
$(x + y)^n + (x - y)^n = 2[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + ...]$

(iv) Subtracting (ii) from (i), we get
$(x + y)^n - (x - y)^n = 2[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 + ...]$

Example # 3 : The number of dissimilar terms in the expansion of $(1 + x^4 - 2x^2)^{15}$ is
(A) 21 (B) 31 (C) 41 (D) 61

Solution : $(1 - x^2)^{30}$
Therefore number of dissimilar terms = **31.**

General term :

$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + \ldots\ldots\ldots + {}^nC_r\, x^{n-r} y^r + \ldots\ldots\ldots + {}^nC_n\, x^0 y^n$
$(r + 1)^{th}$ term is called general term and denoted by T_{r+1}.
$T_{r+1} = {}^nC_r\, x^{n-r} y^r$

Note : The r^{th} term from the end is equal to the $(n - r + 2)^{th}$ term from the begining, i.e. ${}^nC_{n-r+1}\, x^{r-1} y^{n-r+1}$

Example # 4 : Find (i) 15^{th} term of $(2x - 3y)^{20}$ (ii) 4^{th} term of $\left(\frac{3x}{5} - y\right)^7$

Solution : (i) $T_{14+1} = {}^{20}C_{14}\,(2x)^6(-3y)^{14} = {}^{20}C_{14}\, 2^6 3^{14} x^6.y^{14}$

(ii) $T_{3+1} = {}^7C_3\left(\frac{3x}{5}\right)^4(-y)^3 = -{}^7C_3\left(\frac{3}{5}\right)^4 x^4y^3$

Example # 5 : Find the number of rational terms in the expansion of $\left(2^{\frac{1}{3}} + 3^{\frac{1}{5}}\right)^{600}$

Solution : The general term in the expansion of $\left(2^{\frac{1}{3}} + 3^{\frac{1}{5}}\right)^{600}$ is

$$T_{r+1} = {}^{600}C_r\left(2^{\frac{1}{3}}\right)^{600-r}\left(3^{\frac{1}{5}}\right)^r = {}^{600}C_r\; 2^{\frac{600-r}{3}}\; 3^{\frac{r}{5}}$$

The above term will be rational if exponent of 3 and 2 are integers

It means $\frac{600-r}{3}$ and $\frac{r}{5}$ must be integers

The possible set of values of r is {0, 15, 30,45............,600}
Hence, number of rational terms is 41

Middle term(s) :

(a) If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)^{th}$ term.

(b) If n is odd, there are two middle terms, which are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

Example # 6 : Find the middle term(s) in the expansion of

(i) $(1 + 2x)^{12}$ (ii) $\left(2y - \frac{y^2}{2}\right)^{11}$

Solution : (i) $(1 + 2x)^{12}$

Here, n is even, therefore middle term is $\left(\frac{12+2}{2}\right)^{th}$ term.

It means T_7 is middle term $T_7 = {}^{12}C_6\,(2x)^6$

(ii) $\left(2y - \frac{y^2}{2}\right)^{11}$

Here, n is odd therefore, middle terms are $\left(\frac{11+1}{2}\right)^{th}$ & $\left(\frac{11+1}{2}+1\right)^{th}$.
It means T_6 & T_7 are middle terms

$T_6 = {}^{11}C_5\,(2y)^6\left(-\frac{y^2}{2}\right)^5 = -2\;{}^{11}C_5\, y^{16} \Rightarrow T_7 = {}^{11}C_6\,(2y)^5\left(-\frac{y^2}{2}\right)^6 = \frac{{}^{11}C_6}{2}y^{17}$

Example # 7 : Find term which is independent of x in $\left(x^2 - \frac{1}{x^6}\right)^{16}$

Solution : $T_{r+1} = {}^{16}C_r\,(x^2)^{16-r}\left(-\frac{1}{x^6}\right)^r$

For term to be independent of x, exponent of x should be 0
$32 - 2r = 6r \Rightarrow r = 4 \quad \therefore T_5$ is independent of x.

Numerically greatest term in the expansion of $(a + b)^n$, $n \in N$

Binomial expansion of $(a + b)^n$ is as follows : –
$(a + b)^n = {}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1}b^1 + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_r\, a^{n-r}b^r + \ldots\ldots + {}^nC_n\, a^0 b^n$
If we put certain values of a and b in RHS, then each term of binomial expansion will have certain value. The term having numerically greatest value is said to be numerically greatest term.
Let T_r and T_{r+1} be the r^{th} and $(r + 1)^{th}$ terms respectively
$T_r = {}^nC_{r-1}\, a^{n-(r-1)}\, b^{r-1}$
$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Now, $\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}}\frac{a^{n-r}\,b^r}{a^{n-r+1}b^{r-1}}\right| = \frac{n-r+1}{r}\cdot\left|\frac{b}{a}\right|$

Consider $\left|\frac{T_{r+1}}{T_r}\right| \geq 1$

$\left(\frac{n-r+1}{r}\right)\left|\frac{b}{a}\right| \geq 1 \Rightarrow \frac{n+1}{r} - 1 \geq \left|\frac{a}{b}\right| \Rightarrow r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$

Case - I When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ $(r = 1, 2, 3 \ldots., m - 1)$
i.e. $T_2 > T_1, T_3 > T_2, \ldots\ldots, T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when $r = m$
i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when $r > m$ $(r = m + 1, m + 2, \ldots\ldots\ldots n)$
i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, \ldots\ldots\ldots T_{n+1} < T_n$

Conclusion :

When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Binomial Theorem

"Obvious" is the most dangerous word in mathematics......... Bell, Eric Temple

Binomial expression :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x + y$, $x^2y + \frac{1}{xy^2}$, $3 - x$, $\sqrt{x^2+1} + \frac{1}{(x^3+1)^{1/3}}$ etc.

Terminology used in binomial theorem :

Factorial notation : $\underline{|n}$ or $n!$ is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3.2.1 & ; \text{ if } n \in N \\ 1 & ; \text{ if } n = 0 \end{cases}$$

Note : $n! = n\,.\,(n-1)!$; $n \in N$

Mathematical meaning of ^nC_r : The term ^nC_r denotes number of combinations of r things choosen from n distinct things mathematically, $^nC_r = \frac{n!}{(n-r)!\,r!}$, $n \in N$, $r \in W$, $0 \le r \le n$

Note : Other symbols of of ^nC_r are $\binom{n}{r}$ and $C(n, r)$.

Properties related to ^nC_r :

(i) $^nC_r = {^nC_{n-r}}$

Note : If $^nC_x = {^nC_y}$ $\Rightarrow$ Either $x = y$ or $x + y = n$

(ii) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(iii) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(iv) $^nC_r = \frac{n}{r}\,{^{n-1}C_{r-1}} = \frac{n(n-1)}{r(r-1)}\,{^{n-2}C_{r-2}} = ... = \frac{n(n-1)(n-2).........(n-(r-1))}{r(r-1)(r-2).......2.1}$

(v) If n and r are relatively prime, then ^nC_r is divisible by n. But converse is not necessarily true.

Statement of binomial theorem :

$(a + b)^n = {^nC_0}\,a^nb^0 + {^nC_1}\,a^{n-1}b^1 + {^nC_2}\,a^{n-2}b^2 + ... + {^nC_r}\,a^{n-r}b^r + ... + {^nC_n}\,a^0b^n$

where $n \in N$

or $(a + b)^n = \sum_{r=0}^{n} {^nC_r}\,a^{n-r}b^r$

Note : If we put a = 1 and b = x in the above binomial expansion, then

or $(1 + x)^n = {^nC_0} + {^nC_1}\,x + {^nC_2}\,x^2 + ... + {^nC_r}\,x^r + ... + {^nC_n}\,x^n$

or $(1 + x)^n = \sum_{r=0}^{n} {^nC_r}\,x^r$

Example # 1 : Expand the following binomials :

(i) $(x+\sqrt{2})^5$ (ii) $\left(1-\frac{3x^2}{2}\right)^4$

Solution : (i) $(x+\sqrt{2})^5 = {^5C_0}x^5 + {^5C_1}x^4\left(\sqrt{2}\right) + {^5C_2}x^3\left(\sqrt{2}\right)^2 + {^5C_3}x^2\left(\sqrt{2}\right)^3 + {^5C_4}x\left(\sqrt{2}\right)^4 + {^5C_5}\left(\sqrt{2}\right)^5$

$= x^5 + 5\sqrt{2}\,x^4 + 20x^3 + 20\sqrt{2}\,x^2 + 20x + 4\sqrt{2}$

(ii) $\left(1-\frac{3x^2}{2}\right)^4 = {^4C_0} + {^4C_1}\left(-\frac{3x^2}{2}\right) + {^4C_2}\left(-\frac{3x^2}{2}\right)^2 + {^4C_3}\left(-\frac{3x^2}{2}\right)^3 + {^4C_4}\left(-\frac{3x^2}{2}\right)^4$

$= 1 - 6x^2 + \frac{27}{2}x^4 - \frac{27}{2}x^6 + \frac{81}{16}x^8$

Example # 2 : Expand the binomial $\left(\frac{2}{x}+x\right)^{10}$ up to four terms

Solution : $\left(\frac{2}{x}+x\right)^{10} = {^{10}C_0}\left(\frac{2}{x}\right)^{10} + {^{10}C_1}\left(\frac{2}{x}\right)^9 x + {^{10}C_2}\left(\frac{2}{x}\right)^8 x^2 + {^{10}C_3}\left(\frac{2}{x}\right)^7 x^3 + ...$

Self practice problems

(1) Write the first three terms in the expansion of $\left(2-\frac{y}{3}\right)^6$.

(2) Expand the binomial $\left(\frac{x^2}{3}+\frac{3}{x}\right)^5$.

Ans. (1) $64 - 64y + \frac{80}{3}y^2$ (2) $\frac{x^{10}}{243} + \frac{5}{27}x^7 + \frac{10}{3}x^4 + 30x + \frac{135}{x^2} + \frac{243}{x^5}$.

Observations :

(i) The number of terms in the binomial expansion $(a + b)^n$ is $n + 1$.
(ii) The sum of the indices of a and b in each term is n.
(iii) The binomial coefficients ($^nC_0$, $^nC_1$ ....^nC_n) of the terms equidistant from the beginning and the end are equal, i.e. $^nC_0 = {^nC_n}$, $^nC_1 = {^nC_{n-1}}$ etc. $\{\because {^nC_r} = {^nC_{n-r}}\}$
(iv) The binomial coefficient can be remembered with the help of the following pascal's Triangle (also known as Meru Prastra provided by Pingla)

Index of the binomial	The binomial coefficient
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Regarding Pascal's Triangle, we note the following :
(a) Each row of the triangle begins with 1 and ends with 1.
(b) Any entry in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right.