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Principles of least  
action in quantum  
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THE PRINCIPLE OF LEAST ACTION  
IN QUANTUM MECHANICS.

by

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in Candidacy for the Degree  
of Doctor of Philosophy

Recommended for Acceptance by the  
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## I. Introduction.

Planck's discovery in 1900 of the quantum properties of light led to an enormously deeper understanding of the attributes and behaviour of matter, through the advent of the methods of quantum mechanics. When, however, these same methods are turned to the problem of light and the electromagnetic field great difficulties arise which have not been surmounted satisfactorily, so that Planck's observations still remain without a consistent fundamental interpretation.<sup>1</sup>

As is well known, the quantum electrodynamics that have been developed suffer from the difficulty that, taken literally, they predict infinite values for many experimental quantities which are obviously quite finite, such as, for example, the shift in energy of spectral lines due to interaction of the atom and the field. The classical field theory of Maxwell and Lorentz serves as the jumping-off point for this quantum electrodynamics. The latter theory, however, does not take over the ideas of classical theory concerning the internal structure of the electron, which ideas are so necessary to the classical theory to attain finite values for such quantities as the inertia of an electron. The researches of Dirac into the quantum properties of the electron have been so successful in interpreting such properties

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1. It is important to develop a satisfactory quantum electrodynamics also for another reason. At the present time theoretical physics is confronted with a number of fundamental unsolved problems dealing with the nucleus, the interactions of protons and neutrons, etc. In an attempt to tackle these, meson field theories have been set up in analogy to the electromagnetic field theory. But the analogy is unfortunately all too perfect; the infinite answers are all too prevalent and confusing.

as its spin and magnetic moment, and the existence of the positron, that it is hard to believe that it should be necessary in addition to attribute internal structure to it.

It has become, therefore, increasingly more evident that before a satisfactory quantum electrodynamics can be developed it will be necessary to develop a classical theory capable of describing charges without internal structure. Many of these have now been developed, but we will concern ourselves in this thesis with the theory of action at a distance worked out in 1941 by J. A. Wheeler and the author?

The new viewpoint pictures electrodynamic interaction as direct interaction at a distance between particles. The field then becomes a mathematical construction to aid in the solution of problems involving these interactions. The following principles are essential to the altered viewpoint:

- (1) The acceleration of a point charge is due only to the sum of its interactions with other charged particles. A charge does not act on itself.
- (2) The force of interaction which one charge exerts on another is calculated by means of the Lorentz force formula,  $F = e[F + \frac{v}{c} \times H]$ , in which the fields are the fields generated by the first charge according to Maxwell's equations.
- (3) The fundamental (microscopic) phenomena in nature are symmetrical with respect to interchange of past and future. This requires that the solution of Maxwell's equation to be used in computing the interactions is to be half the retarded plus half the advanced solution of Lienard and Wiechert.

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(2). Not published. See, however, Phys. Rev. 59, 683 (1941)

These principles, at first sight at such variance with elementary notions of causality, do in fact lead to essential agreement with the results of the more usual form of electrodynamics, and at the same time permit a consistent description of point charges and lead to a unique law of radiative damping. That this is the case has been shown in the work already referred to (see note 2). It is shown that these principles are equivalent to the equations of motion resulting from a principle of least action. The action function (due to Tetrode<sup>3</sup>, and, independently, to Fokker<sup>4</sup>) involves only the coordinates of the particles, no mention of fields being made. The field is therefore a derived concept, and cannot be pictured as analogous to the vibrations of some medium, with its own degrees of freedom (for example, the energy density is not necessarily positive.) Perhaps a word or two here as to what aspects of this theory make it a reasonable basis for a quantum theory of light would not be amiss.

When one attempts to list those phenomena which seem to indicate that light is quantized, the first type of phenomenon which comes to mind are like the photoelectric effect or the Compton effect. One is however, struck by the fact that since these phenomena deal with the interaction of light and matter their explanation may lie in the quantum aspects of matter, rather than requiring photons of light. This supposition is aided by the fact that if one solves the problem of an atom being perturbed

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(3). H. Tetrode, Zeits. f. Physik 10, 317 (1922).

(4). A. D. Fokker, Zeits. f. Physik 38, 386 (1929); Physica 9, 33 (1929); Physica 12, 145 (1932).

by a potential varying sinusoidally with the time, which would be the situation if matter were quantum mechanical and light classical one finds indeed that it will in all probability eject an electron whose energy shows an increase of  $h\nu$ , where  $\nu$  is the frequency of variation of the potential. In a similar way an electron perturbed by the potential of two beams of light of different frequencies and different directions will make transitions to a state in which its momentum and energy is changed by an amount just equal to that given by the formulas for the Compton effect, with one beam corresponding in direction and wavelength to the incoming photon and the other to the outgoing one. In fact, one may correctly calculate in this way the probabilities of absorption and induced emission of light by an atom.

When, however, we come to spontaneous emission and the mechanism of the production of light, we come much nearer to the real reason for the apparent necessity of photons. The fact that an atom emits spontaneously at all is impossible to explain by the simple picture given above. In empty space an atom emits light and yet there is no potential to perturb the system and so force it to make a transition. The explanation of modern quantum mechanical electrodynamics is that the atom is perturbed by the zero-point fluctuations of the quantized radiation field.

It is here that the theory of action at a distance gives us a different viewpoint. It says that an atom alone in empty space would, in fact, not radiate. Radiation is a consequence of the interaction with other atoms (namely, those in

the matter which absorbs the radiation). We are then led to the possibility that the spontaneous radiation of an atom in quantum mechanics also, may not be spontaneous at all, but induced by the interaction with other atoms, and that all of the apparent quantum properties of light and the existence of photons may be nothing more than the result of matter interacting with matter directly, and according to quantum mechanical laws.

An attempt to investigate this possibility and to find a quantum analogue of the theory of action at a distance, meets first the difficulty that it may not be correct to represent the field as a set of harmonic oscillators, each with its own degree of freedom, since the field in actuality is entirely determined by the particles. On the other hand, an attempt to deal quantum mechanically directly with the particles, which would seem to be the most satisfactory way to proceed, is faced with the circumstance that the equations of motion of the particles are expressed classically as a consequence of a principle of least action, and cannot, it appears, be expressed in Hamiltonian form.

For this reason a method of formulating a quantum analogue of systems for which no Hamiltonian, but rather a principle of least action, exists has been worked out. It is a description of this method which constitutes this thesis. Although the method was worked out with the express purpose of applying it to the theory of action at a distance, it is in fact independent of that theory, and is complete in itself. Nevertheless most of the illustrative examples will be taken from problems which arise in the action at a distance electrodynamics. In particular, the problem of the equivalence in quantum mechanics

of direct interaction and interaction through the agency of an intermediate harmonic oscillator will be discussed in detail. The solution of this problem is essential if one is going to be able to compare a theory which considers field oscillators as real mechanical and quantized systems, with a theory which considers the field as just a mathematical construction of classical electrodynamics required to simplify the discussion of the interactions between particles. On the other hand, no excuse need be given for including this problem, as its solution gives a very direct confirmation, which would otherwise be lacking, of the general utility and correctness of the proposed method of formulating the quantum analogue of systems with a least action principle.

The results of the application of these methods to quantum electrodynamics is not included in this thesis, but will be reserved for a future time when they shall have been more completely worked out. It has been the purpose of this introduction to indicate the motivation for the problems which are discussed herein. It is to be emphasized again that the work described here is complete in itself without regard to its application to electrodynamics, and it is this circumstance which makes it appear advisable to publish these results as an independent paper. One should therefore take the viewpoint that the present paper is concerned with the problem of finding a quantum mechanical description applicable to systems which in their classical analogue are expressible by a principle of least action, and not necessarily by Hamiltonian equations of motion.

The thesis is divided into two main parts. The first deals with the properties of classical systems satisfying a

principle of least action, while the second part contains the method of quantum mechanical description applicable to these systems. In the first part are also included some mathematical remarks about functionals. All of the analysis will apply to non-relativistic systems. The generalization to the relativistic case is not at present known.

## II. Least Action in Classical Mechanics.

### 1. The Concept of a Functional.

The mathematical concept of a functional will play a rather predominant role in what is to follow so that it seems advisable to begin at once by describing a few of the properties of functionals and the notation used in this paper in connection with them. No attempt is made at mathematical rigor.

To say  $F$  is a functional of the function  $f(\sigma)$  means that  $F$  is a number whose value depends on the form of the function  $f(\sigma)$  (where  $\sigma$  is just a parameter used to specify the form of  $f(\sigma)$ ). Thus,  $F = \int_{-\infty}^{\infty} f(\sigma)^2 e^{-\sigma^2} d\sigma$  (8.1) is a functional of  $f(\sigma)$  since it associates with every choice of the function  $f(\sigma)$  a number, namely the integral. Also, the area under a curve is a functional of the function representing the curve, since to each such function a number, the area, is associated. The expected value of the energy in quantum mechanics is a functional of the wave function. Again,  $F = f(0)$  (8.2) is a functional, which is especially simple because its value depends only on the value of the function  $f(\sigma)$  at the one point  $\sigma = 0$ .

We shall write, if  $F$  is a functional of  $f(\sigma)$ ,  $F[f(\sigma)]$ . A functional may have as its argument more than one function, or functions of more than one parameter, as,

$$F[x(t,s), y(t,s)] = \iint_{-\infty}^{\infty} x(t,s) y(t,s) \sin \omega s dt ds.$$

A functional  $F[f(\sigma)]$  may be looked upon as a function of an infinite number of variables, the variables being the value of the function  $f(\sigma)$  at each point  $\sigma$ . If the interval of the range of  $\sigma$  is divided up into a large number of points  $\sigma_i$ , and the value of the function at these points is  $f(\sigma_i) = f_i$ , say, then

approximately our functional may be written as a function of the variables  $q_i$ . Thus, in the case of equation (2) we could write, approximately,

$$F(\dots q_i \dots) = \sum_{i=-\infty}^{\infty} q_i^2 e^{-\epsilon^2 (\sigma_{ii} - \sigma_i)}.$$

We may define a process analogous to differentiation for our functionals. Suppose the function  $q(t)$  is altered slightly to  $q(t) + \lambda(t)$  by the addition of a small function  $\lambda(t)$ . From our approximate viewpoint we can say that each of the variables is changed from  $q_i$  to  $q_i + \lambda_i$ . The function is thereby changed by an amount

$$\sum_i \frac{\partial F(\dots q_i \dots)}{\partial q_i} \lambda_i.$$

In the case of a continuous number of variables, the sum becomes an integral and we may write, to the first order in  $\lambda$ ,

$$F[q(t) + \lambda(t)] - F[q(t)] = \int K(t) \lambda(t) dt. \quad (7.1)$$

where  $K(t)$  depends on  $F$ , and is what we shall call the functional derivative of  $F$  with respect to  $q$  at  $t$ , and shall symbolize,

with Eddington<sup>5</sup>, by  $\frac{\eta F[q(t)]}{\eta q(t)}$ . It is not simply  $\frac{\partial F(\dots q_i \dots)}{\partial q_i}$

as this is in general infinitesimal, but is rather the sum of these  $\frac{\partial F}{\partial q_i}$  over a short range of  $i$ , say from  $i+k$  to  $i-k$ , divided by the interval of the parameter,  $\sigma_{i+k} - \sigma_{i-k}$ .

Thus we write,

$$F[q(t) + \lambda(t)] = F[q(t)] + \int \frac{\eta F[q(t)]}{\eta q(t)} \lambda(t) dt + \text{higher order terms in } \lambda. \quad (9.2)$$

(5). A. S. Eddington, "The mathematical theory of relativity" (1923) p139.

For example, in equation (1) if we substitute  $g(t)$  for  $g$ , we obtain,

$$F[f+\lambda] = \int [g(t)^2 + 2g(t)\lambda(t) + \lambda(t)^2] e^{-\sigma^2} dt = \int g(t)^2 e^{-\sigma^2} dt + 2 \int g(t)\lambda(t) e^{-\sigma^2} dt + \text{higher terms in } \lambda.$$

Therefore, in this case, we have  $\frac{\partial F[g]}{\partial g(t)} = 2g(t)e^{-\sigma^2}$ .

In a similar way, if  $F[g(t)] = g(t)$ , then  $\frac{\partial F}{\partial g(t)} = s(t)$ , where  $s(t)$  is

Dirac's delta symbol, defined by  $\int s(t)f(t)dt = f(0)$  for any continuous function  $f$ .

The function  $g(t)$  for which  $\frac{\partial F}{\partial g(t)}$  is zero for all  $t$  is that function for which  $F$  is an extremum. For example, in classical mechanics the action,

$$\mathcal{A} = \int L(g(t), \dot{g}(t)) dt \quad (6.1).$$

is a functional of  $g(t)$ . Its functional derivative is,

$$\frac{\partial \mathcal{A}}{\partial g(t)} = -\frac{d}{dt} \left\{ \frac{\partial L(g(t), \dot{g}(t))}{\partial \dot{g}} \right\} + \frac{\partial L(g(t), \dot{g}(t))}{\partial g}. \quad (6.2)$$

If  $\mathcal{A}$  is an extremum the right hand side is zero.

## 2. The Principle of Least Action.

For most mechanical systems it is possible to find a functional,  $\mathcal{A}$ , called the action, which assigns a number to each possible mechanical path,  $g_1(t), g_2(t), \dots, g_N(t)$ , (we suppose  $N$  degrees of freedom, each with a coordinate  $g_n(t)$ , a function of a parameter (time)  $\sigma$ ) in such a manner that this number is an extremum for an actual path  $\bar{g}(t)$  which could arise in accordance with the laws of motion. Since this extremum often is a minimum this is called the principle of least action. It is

often convenient to use the principle itself, rather than the Newtonian equations of motion as the fundamental mechanical law. The form of the functional  $a[f_1(t) \dots f_N(t)]$  depends on the mechanical problem in question.

According to the principle of least action, then, if  $\delta[f_1(t) \dots f_N(t)]$  is the action functional, the equations of motion are N in number and are given by,

$$\frac{\delta a}{\delta f_1(t)} = 0, \frac{\delta a}{\delta f_2(t)} = 0, \dots, \frac{\delta a}{\delta f_N(t)} = 0 \quad (11.1)$$

(We shall often simply write  $\frac{\delta a}{\delta f(t)} = 0$ , as if there were only one variable). That is to say if all the derivatives of  $a$ , with respect to  $f_n(t)$ , computed for the functions  $f_n(t)$  are zero for all  $t$  and all  $n$ , then  $f_n(t)$  describes a possible mechanical motion for the system.

We have given an example, in equation (11.1), for the usual one dimensional problem when the action is the time integral of a Lagrangian (a function of position and velocity, only). As another example consider an action function arising in connection with the theory of action at a distance:

$$a = \int_{-\infty}^{\infty} \left\{ \frac{m(\dot{x}(t))^2}{2} - V(x(t)) + k^2 \dot{x}(t) \dot{x}(t+T_0) \right\} dt \quad (11.2)$$

It is approximately the action for a particle in a potential  $V(x)$ , and interacting with itself in a distant mirror by means of retarded and advanced waves. The time it takes for light to reach the mirror from the particle is assumed constant, and equal to  $T_0/2$ . The quantity  $k^2$  depends on the charge on the particle and its distance from the mirror. If we vary  $x(t)$  by a small amount,  $\lambda(t)$ , the consequent variation in  $a$  is,

$$\begin{aligned} \delta A &= \int_{t_0}^{\infty} \{ -m\ddot{x}(t) - V'(x(t)) \dot{x}(t) + k^2 \dot{x}(t) \dot{x}(t+t_0) + k^2 \dot{x}(t+t_0) \dot{x}(t) \} dt \\ &= \int_{t_0}^{\infty} \{ -m\ddot{x}(t) - V'(x(t)) - k^2 \dot{x}(t+t_0) - k^2 \dot{x}(t-t_0) \} \lambda(t) dt, \quad \text{or integrating by parts.} \end{aligned}$$

so that, according to our definition (7.2), we may write,

$$\frac{\eta A}{\eta x(t)} = -m\ddot{x}(t) - V'(x(t)) - k^2 \dot{x}(t+t_0) - k^2 \dot{x}(t-t_0) \quad (12.1)$$

The equation of motion of this system is obtained, according to (11.1) by setting  $\frac{\eta A}{\eta x(t)}$  equal to zero. It will be seen that the force acting at time  $t$  depends on the motion of the particle at other times than  $t$ . The equations of motion cannot be described directly in Hamiltonian form.

### 3. Conservation of Energy. Constants of the Motion.<sup>8</sup>

The problem we shall study in this section is that of determining to what extent the concepts of conservation of energy, momentum, etc., may be carried over to mechanical problems with a general form of action function. The usual principle of conservation of energy asserts that there is a function of positions at the time  $t$ , say, and of velocities of the particles whose value, for the actual motion of the particles, does not change with time. In our more general case however, the forces do not involve the positions of the particles only at one particular time, but usually a calculation of the forces requires a knowledge of the paths of the particles over some considerable range of time (see for example, eq. (12.1)). It is not possible in this case generally to find a constant of the motion which only involves the positions and velocities at one time.

For example, in the theory of action at a distance, the kinetic energy of the particles is not conserved. To find a

(8) This section is not essential to an understanding of the remainder of the paper.

conserved quantity one must add a term corresponding to the "energy in the field". The field, however, is a functional of the motion of the particles, so that it is possible to express this "field energy" in terms of the motion of the particles. For our simple example (11.2), account of the equations of motion (12.1), the quantity,

$$E(t) = \frac{m(\dot{x}(t))^2}{2} + V(x(t)) - k^2 \int_{t-T_0}^{t+T_0} \ddot{x}(r-T_0) \dot{x}(r) dr + k^2 \dot{x}(t) \dot{x}(t+T_0), \quad (13.1)$$

has, indeed, a zero derivative with respect to time. The first two terms represent the ordinary energy of the particles. The additional terms, representing the energy of interaction with the mirror (or rather, with itself) require a knowledge of the motion of the particle from the time  $t-T_0$  to  $t+T_0$ .

Can we really talk about conservation, when the quantity conserved depends on the path of the particles over considerable ranges of time? If the force acting on a particle be  $F(t)$  say, so that the particle satisfies the equation of motion  $m\ddot{x}(t) = F(t)$ , then it is perfectly clear that the integral,

$$I(t) = \int_{-\infty}^t [m\ddot{x}(t) - F(t)] \dot{x}(t) dt \quad (13.2)$$

has zero derivative with respect to  $t$ , when the path of the particle satisfies the equation of motion. Many such quantities having the same properties could easily be devised. We should not be inclined to say (13.2) actually represents a quantity of interest, in spite of its constancy.

The conservation of a physical quantity is of considerable interest because in solving problems it permits us to forget a great number of details. The conservation of energy can be derived from the laws of motion, but its value lies in the fact

that by the use of it certain broad aspects of a problem may be discussed, without going into the great detail that is often required by a direct use of the laws of motion.

To compute the quantity  $I(t)$ , of equation (12-2), for two different times,  $t_1$  and  $t_2$  that are far apart, in order to compare  $I(t_1)$  with  $I(t_2)$ , it is necessary to have detailed information of the path during the entire interval  $t_1$  to  $t_2$ . The value of  $I$  is equally sensitive to the character of the path for all times between  $t_1$  and  $t_2$ , even if these times lie very far apart. It is for this reason that the quantity  $I(t)$  is of little interest. If, however,  $F$  were to depend on  $x(t)$  only, so that it might be derived from a potential, (e.g.;  $F = -V'(x)$  ), then the integrand is a perfect differential, and may be integrated to become  $\frac{1}{2} m(x,t)^2 + V(x,t)$ . A comparison of  $I$  for two times,  $t_1$  and  $t_2$ , now depends only on the motion in the neighborhood of these times, all of the intermediate details being, so to speak, integrated out.

We therefore require two things if a quantity  $I(t)$  is to attract our attention as being dynamically important. The first is that it be conserved,  $I(t_1) = I(t_2)$ . The second is that  $I(t)$  should depend only locally on the path. That is to say, if one changes the path at some time  $t'$  in a certain (arbitrary) way, the change which is made in  $I(t)$  should decrease to zero as  $t'$  gets further and further from  $t$ . That is to say, we should like the condition  $\frac{\partial I(t)}{\partial t'}$   $\rightarrow 0$  as  $|t-t'| \rightarrow 0$  ( $\infty$  ; ) satisfied.<sup>6</sup>

(6). A more complete mathematical analysis than we include here is required to state rigorously just how fast it must approach zero as  $|t-t'|$  approaches infinity. The proofs stated herein are certainly valid if the quantities in (12-1) and (12-3) are assumed to become and remain equal to zero for values of  $|t-t'|$  greater than some finite one, no matter how large it may be.

The energy expression (15.1) satisfies this criterion, as we have already pointed out. Under what circumstances can we derive an analogous constant of the motion for a general action function?

We shall, in the first place, impose a condition on the equations of motion which seems to be necessary in order that an integral of the motion of the required type exist. In the equation  $\frac{\eta a}{\eta g(t)} = 0$ , which holds for an arbitrary time,  $t$ , we shall suppose that the influence of changing the path at time  $t'$  becomes less and less as  $|t-t'|$  approaches infinity. That is to say, we require,

$$\frac{\eta a}{\eta g(t)\eta g(t')} \rightarrow 0 \quad \text{as} \quad |t-t'| \rightarrow \infty \quad (15.1)$$

We next suppose that there exists a transformation (or rather, a continuous group of transformations) of coordinates, which we symbolize by  $g_m \rightarrow g_m + X_m(a)$  and which leaves the action invariant. (for example, the transformation may be a rotation). The transformation is to contain a parameter,  $a$ , and is to be a continuous function of  $a$ . For  $a$  equal to zero, the transformation should reduce to the identity, so that  $X_m(0) = 0$ . For very small  $a$  we may expand;  $X_m(a) = 0 + aY_m + \dots$ . That is to say, for infinitesimal  $a$ , if the coordinates  $g_m$  are changed to  $g_m + aY_m$  the action is left unchanged;

$$A[g_m(\sigma)] = A[g_m(\sigma) + aY_m(\sigma)] \quad (15.2)$$

For example, if the form of the action is unchanged if the particles take the same path at a later time, we may take,  $g_m(t) \rightarrow g_m(t+\tau)$ . In this case, for small  $a$ ,  $g_m(t) \rightarrow g_m(t) + a\dot{g}_m(t) + \dots$  so that  $Y_m = \dot{g}_m(t)$ .

For each such continuous set of transformations there

will be a constant of the motion. If the action is invariant with respect to change from  $f(t)$  to  $f(t+a)$ , then an energy will exist. If the action is invariant with respect to the translation of all the coordinates (rectangular coordinates, that is) by the same distance,  $a$ , then a momentum in the direction of the translation may be derived. For rotations around an axis through the angle,  $a$ , the corresponding constant of the motion is the angular momentum around that axis. We may show this connection between the groups of transformations and the constants of the motion, in the following way: For small  $a$ , from (15.2), we shall have,

$$\partial [f_n(\alpha)] = \partial [f_n(\alpha) + a Y_n(\alpha)]$$

Expanding the left side with respect to the change in the coordinates  $a Y_n(\alpha)$ , according to (9.2) to the first order in  $a$  we have,

$$\partial [f_n(\alpha) + a Y_n(\alpha)] = \partial [f_n(\alpha)] + a \sum_{m=1}^N \int_{-\infty}^{\infty} Y_m(t) \frac{na}{n f_m(t)} dt \quad (16.1)$$

and therefore, on account of (15.2),

$$\sum_{m=1}^N \int_{-\infty}^{\infty} Y_m(\alpha) \frac{na}{n f_m(\alpha)} d\alpha = 0 \quad (16.2)$$

Now consider the quantity,

$$I(T) = \sum_{m=1}^N \int_{-\infty}^T Y_m(\alpha) \frac{na}{n f_m(\alpha)} d\alpha \quad (16.3)$$

On account of (16.2) we can also write,

$$I(T) = - \sum_{m=1}^N \int_T^{\infty} Y_m(\alpha) \frac{na}{n f_m(\alpha)} d\alpha \quad (16.4)$$

Consider the derivative of  $I$  with respect to  $T$ ;  $\frac{dI(T)}{dT} = \sum_m Y_m(T) \frac{n_a}{n_{f_m}(T)}$

According to the equations of motion (16.1), this is seen to vanish, so that  $I(T)$  is independent of  $T$  for the real motion, and is therefore conserved. We must now prove, in order that it be acceptable as an important constant of the motion, that

$$\frac{nI(T)}{n_{f_m}(t)} \rightarrow 0 \quad \text{as } |T-t| \rightarrow \infty \text{ for any } m. \quad (17.1)$$

Suppose first that  $t > T$ . Let us compute  $\frac{nI(T)}{n_{f_m}(t)}$  directly from equation (16.3), obtaining,

$$\frac{nI(T)}{n_{f_m}(t)} = + \int_{-\infty}^T \sum_m \frac{nY_m(\sigma)}{n_{f_m}(\sigma)} \frac{n_a}{n_{f_m}(\sigma)} d\sigma + \int_{-\infty}^T \sum_m Y_m(\sigma) \frac{n^* a}{n_{f_m}(\sigma) n_{f_m}(\sigma)} d\sigma \quad (17.2)$$

Now we shall suppose that  $Y_m(\sigma)$  does not depend very much on values of  $f_m(\sigma)$  for times,  $t$ , far away from  $\sigma$ . That is to say we shall assume,

$$\frac{nY_m(\sigma)}{n_{f_m}(\sigma)} \rightarrow 0 \quad \text{as } |\sigma-t| \rightarrow \infty \quad (17.3)$$

In the first integral then, since  $t > T$ , and since only values of  $\sigma$  less than  $T$  appear in the integrand, for all such values,  $t-\sigma > t-T$ . As  $t-T$  approaches infinity, therefore, only terms in the first integral of (17.2) for which  $t-\sigma$  approaches infinity appear. We shall suppose that  $\frac{nY_m(\sigma)}{n_{f_m}(\sigma)}$  decreases sufficiently rapidly with increase in  $t-\sigma$  that the integral of it goes to zero as  $t-T$  becomes infinite. A similar analysis applies to the second integral of (17.2). Here the quantity  $\frac{n^* a}{n_{f_m}(\sigma) n_{f_m}(\sigma)}$  approaches zero because of our assumption

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(7). In fact, for all practical cases which come to mind (energy, momentum, angular momentum, corresponding to time displacement, translation, and rotation),  $\frac{nY_m(\sigma)}{n_{f_m}(\sigma)}$  is actually zero if  $\sigma \neq t$ .

(15.1), and we shall suppose this approach sufficiently rapid that the integral vanish in the limit.

Thus we have shown that  $\frac{\eta I(t)}{\eta f_m(t)} \rightarrow 0$  as  $t-T \rightarrow \infty$ .

To prove the corresponding relation for  $T-t \rightarrow \infty$  one may calculate  $\frac{\eta I(t)}{\eta f_m(t)}$  with  $t < T$  from (16.4), and proceed in exactly the same manner. In this way we can establish the required relation (17.1). This then shows that  $I(T)$  is an important quantity which is conserved.

A particularly important example is, of course, the energy expression. This is got by the transformation of displacing the time, as has already been mentioned, for which  $y_a(\sigma) = \dot{y}_m(\sigma)$ . The energy integral may therefore be expressed, according to (16.3) (we have changed the sign), as,

$$E(T) = - \int_{-\infty}^T \sum_{n=1}^N \dot{y}_m(\sigma) \frac{\eta a}{\eta f_n(\sigma)} d\sigma \quad (18.1)$$

In our example (11.2) we would get from this formula,

$$E(T) = - \int_{-\infty}^T \dot{x}(r) [-m\ddot{x}(r) - V'(x(r)) - k^2 \dot{x}(r + T_0) - k^2 \dot{x}(r - T_0)] dr$$

from which (18.1) has been derived by direct integration.

#### 4. Particles Interacting through an Intermediate Oscillator.

The problem we are going to discuss in this section, since it will give us a good example of a system for which only a principle of least action exists, is the following: Let us suppose we have two particles A and B which do not interact directly with each other, but there is a harmonic oscillator, O, with which both of the particles A and B interact. The harmonic oscillator, therefore serves as an intermediary by means of which particle A is influenced by the motion of particle B and vice versa.

In what way is this interaction through the intermediate oscillator equivalent to a direct interaction between the particles A and B, and can the motion of these particles, A, B, be expressed by means of a principle of least action, not involving the oscillator? (In the theory of electrodynamics this is the problem as to whether the interaction of particles through the intermediary of the field oscillators can also be expressed as a direct interaction at a distance.)

To make the problem precise, we let  $y(t)$  and  $z(t)$  represent coordinates of the particles A and B at the time  $t$ . Let the Lagrangians of the particles alone be designated by  $L_y$  and  $L_z$ . Let them each interact with the oscillator (with coordinate  $x(t)$ , Lagrangian  $\frac{m}{2}(x^2 - \omega^2 x^2)$ ) by means of a term in the Lagrangian for the entire system, which is of the form  $(I_y + I_x)x$ , where  $I_y$  is a function involving the coordinates of atom A only, and  $I_x$  is some function of the coordinates of B. (We have assumed the interaction linear in the coordinate of the oscillator.)

We then ask: If the action integral for  $y$ ,  $z$ ,  $x$ , is

$$\int [L_y + L_z + (\frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2}) + (I_y + I_x)x] dt \quad (1.1)$$

is it possible to find an action  $\mathcal{A}$ , a functional of  $y(t)$ ,  $z(t)$ , only, such that, as far as the motion of the particles A, B, are concerned, (i.e., for variations of  $y(t)$ ,  $z(t)$ ) the action  $\mathcal{A}$  is a minimum?

In the first place, since the actual motion of the particles A, B, depends not only on  $y$ ,  $z$ , initially (or at any other time) but also on the initial conditions satisfied by the oscillator, it is clear that  $\mathcal{A}$  is not determined absolutely, but the form that  $\mathcal{A}$  takes must have some dependence on the state of the

oscillator.

In the second place, since we are interested in an action principle for the particles, we must consider variations of the motion of these particles from the true motion. That is, we must consider dynamically impossible paths for these particles. We thus meet a new problem; when varying the motion of the particles A and B, what do we do about the oscillator? We cannot keep the entire motion of the oscillator fixed, for that would require having this entire motion directly expressed in the action integral, and we should be back where we started, with the action (111).

The answer to this question lies in the observation made above that the action must involve somehow some of the properties of the oscillator. In fact, since the oscillator has one degree of freedom it will require two numbers (e.g. position and velocity) to specify the state of the oscillator sufficiently accurately that the motion of the particles A and B is uniquely determined. Therefore in the action function for these particles, two parameters enter, which are arbitrary, and represent some properties of the motion of the oscillator. When the variation in the motion of the particles is taken, these quantities must be considered as constants, and thus, it is the properties of the oscillator described by these quantities which is considered fixed for the "impossible" motion of the particles. We should not be surprised to find that the action function for the particles depends on which properties of the oscillator are considered to be held fixed in the variation of y, and z. It is, on the other hand, somewhat unexpected that, as we shall see, not all possible conditions on the oscillator give rise to motions of y and z which are simply expressible in terms of

an action principle. Let us see how this works out in detail.

We shall assume that the functions  $I_y$  and  $I_z$  are zero for times,  $t$ , greater than  $T$  and less than 0. We shall also assume (for simplicity only) that  $I_y$  is a function of  $t$  and  $y(t)$  only, and does not depend on  $\dot{y}(t)$ . Similarly,  $I_z$  is to be independent of  $\dot{z}(t)$ . Then, from (19.1), the equation of motion of particle  $y$  is,

$$\frac{d}{dt} \left( \frac{\partial L_y}{\partial y} \right) - \frac{\partial L_y}{\partial \dot{y}} = \frac{\partial I_y}{\partial y} \cdot x(t) \quad (21.1) \text{, with a similar equation for } z.$$

That for the oscillator is,

$$m\ddot{x} + m\omega^2 x = [I_y(t) + I_z(t)] \cdot x \quad (21.2).$$

The solution of this last equation is, where we have written  $y = I_y + I_z$ ,

$$x(t) = x(0) \cos \omega t + \dot{x}(0) \frac{\sin \omega t}{\omega} + \frac{1}{m\omega} \int_0^t y(s) \sin \omega(t-s) ds \quad (21.3)$$

This may be expressed in other ways, for example,

$$x(t) = \frac{\sin \omega(T-t)}{\sin \omega T} \left[ x(0) - \frac{1}{m\omega} \int_0^T \sin \omega s y(s) ds \right] + \frac{\sin \omega t}{\sin \omega T} \left[ x(T) - \frac{1}{m\omega} \int_T^\infty \sin \omega(t-s) y(s) ds \right], \quad (21.4)$$

or again,

$$x(t) = \frac{1}{\sin \omega T} \left[ R_T \sin \omega t + R_0 \sin \omega(T-t) \right] + \frac{1}{2m\omega} \int_0^T \sin \omega(t-s) y(s) ds - \frac{1}{2m\omega} \int_T^\infty \sin \omega(t-s) y(s) ds, \quad (21.5)$$

$$\text{where we have written } R_T = \frac{1}{2} \left[ x(0) + x(T) \cos \omega T - \dot{x}(0) \frac{\sin \omega T}{\omega} \right] \quad (21.6)$$

$$\text{and } R_0 = \frac{1}{2} \left[ x(T) + x(0) \cos \omega T + \dot{x}(0) \frac{\sin \omega T}{\omega} \right] \quad (21.7)$$

It is seen that  $R_T$  is the mean of the coordinate of the oscillator at time  $T$  and what that coordinate would have been at this time if the oscillator had been free and started with its actual initial conditions. Similarly,  $R_0$  is the mean of the initial coordinate and what that coordinate would have had to be, were the oscillator free, to produce the actual final conditions at time  $T$ . Outside

the time range 0 to T the oscillator is, of course, simply a free oscillator.

These expressions for  $x(t)$  may be substituted into (21.1) and the corresponding equation for  $z$ , to obtain various expressions for the motion of the particles  $y$ ,  $z$ ; each expressed in terms of these particles and two parameters connected with the oscillator. For the expression (21.3) these parameters are  $x(0)$  and  $\dot{x}(0)$ ; for (21.4) they are  $x(0)$  and  $x(T)$ ; for (21.5) they are  $R_0$  and  $R_T$ . We should like to determine whether these expressions could be obtained from an action principle for  $y$  and  $z$  only.

If the action be  $A$ , then the expression (21.1) must be of the form  $\frac{\eta A}{\eta y(t)} = 0$ . That is to say, we must have,

$$\frac{\eta A}{\eta y(t)} = - \frac{d}{dt} \left( \frac{\partial L}{\partial y} \right) + \frac{\partial L}{\partial y_t} + \frac{\partial I_y}{\partial y} \cdot x(t) \quad (22.1)$$

We seek the

solution of this expression for each expression we may write for  $x(t)$ .

Now an equation such as (22.1) (which is really an infinite set of equations, one for each value of  $t$ ) does not always have a solution. One of the necessary requirements is, since

$$\frac{1}{\eta y(s)} \left( \frac{\eta A}{\eta y(t)} \right) = \frac{1}{\eta y(s)} \left( \frac{\eta A}{\eta y(s)} \right), \text{ that,}$$

$$\frac{1}{\eta y(s)} \left[ - \frac{d}{dt} \left( \frac{\partial L}{\partial y} \right) + \frac{\partial L}{\partial y_t} + \frac{\partial I_y}{\partial y} \cdot x(t) \right] = \frac{1}{\eta y(s)} \left[ - \frac{d}{ds} \left( \frac{\partial L}{\partial y} \right) + \frac{\partial L}{\partial y_s} + \frac{\partial I_y}{\partial y_s} \cdot x(s) \right].$$

This requires, therefore, if  $s \neq t$ , that  $x(t)$  satisfy,

$$\frac{\partial I_y}{\partial y} \frac{x(s)}{\eta y(s)} = \frac{\partial I_y}{\partial y} \frac{x(t)}{\eta y(t)} \quad (22.2)$$

For the expression (21.3), we have,

$$\frac{x(t)}{\eta y(s)} = \frac{1}{m\omega} \sin \omega(t-s) \cdot \frac{\partial I_y}{\partial y} \quad \text{if } s < t \quad (22.3)$$

$$= 0 \quad \text{if } s > t$$

With this expression equation (22.2) is not satisfied so that we may conclude that no simple action function will describe the motion of the particles A and B if the initial position and velocity of the oscillator are considered as fixed.

On the other hand, since for the expression (21.4) we get,

$$\frac{\eta x(t)}{\eta y(s)} = - \frac{\sin \omega(T-t) \sin \omega s}{m \omega \sin \omega T} \frac{\partial I_y}{\partial y} \Big|_s \quad \text{if } s < t \quad (23.1)$$

$$= - \frac{\sin \omega(T-s) \sin \omega t}{m \omega \sin \omega T} \frac{\partial I_y}{\partial x} \Big|_s \quad \text{if } s > t$$

, we may

conclude, since (23.1) satisfies (22.2), that an action does exist if the oscillator has a given initial and a given final position. In fact, we may solve equation (22.1) with  $x(t)$  expressed as in (21.4) and obtain, as an expression for the action,

$$A = \int_{t_0}^T [l_x + l_z] dt + \int_{t_0}^T \left[ \frac{\sin \omega(T-t)x(0) + \sin \omega t x(T)}{\sin \omega T} \right] r(t) dt - \frac{1}{m \omega \sin \omega T} \int_{t_0}^T dt \int_s^T ds \sin \omega(T-t) \sin \omega s Y(s) Y(t) \quad (23.2)$$

The motion of the particles is given by this action principle by requiring that it be a minimum for variations of  $y(t)$  and  $z(t)$ , where the quantities  $x(0)$  and  $x(T)$  are considered as fixed constants (for example, they might be zero).

In the case that  $x(t)$  is given by the expression (21.5) with  $R_0$  and  $R_T$  as constants, we find,

$$\frac{\eta x(t)}{\eta y(s)} = \frac{1}{2m\omega} \sin \omega(t-s) \frac{\partial I_y}{\partial y} \Big|_s \quad \text{if } s < t \quad (23.3)$$

$$= - \frac{1}{2m\omega} \sin \omega(t-s) \frac{\partial I_y}{\partial y} \Big|_s \quad \text{if } s > t$$

, so that the relation

(22.1) is satisfied for this case also. The action function for

24.

this case is,

$$A = \int_0^T [L_y + L_z] dt + \frac{1}{2m\omega} \int_0^T [R_T \sin \omega t + R_0 \sin \omega(T-t)] Y(t) dt + \\ + \frac{1}{2m\omega} \int_0^T \int_0^t \sin \omega(t-s) Y(t) Y(s) ds dt. \quad (24.1)$$

This action is of particular inasmuch as in the special case  $R_T = R_0 = 0$  the integrand no longer depends on  $T$ , and the limits of integration of 0 to  $T$  may be replaced without error by  $-\infty$  and  $\infty$  so that we obtain an action of the special form,

$$A = \int_{-\infty}^{\infty} [L_y + L_z] dt + \frac{1}{2m\omega} \int_{-\infty}^{\infty} \int_{-\infty}^t \sin \omega(t-s) Y(t) Y(s) ds dt. \quad (24.2)$$

If, now, after having passed to the limit  $T = \infty$ , we assume that  $Y(t)$  and  $L_y$ ,  $L_z$  do not depend on  $t$  explicitly, the substitution  $y(t) \rightarrow y(t+a)$  and  $z(t) \rightarrow z(t+a)$  does not alter the action function, so that an energy expression exists for this action. (In electrodynamics it leads to the half advanced plus half retarded interaction used in the action at a distance theory.)

Exactly the same formulas result if it is that  $Y(t)$  depends on the function  $y(t)$ , as any general functional. The action or the particles need not be the integral of a Lagrangian in the original form (19.1) either. If there are more than one intermediate oscillator, and if the oscillators are independent (ie., if no two of the oscillators interact directly) the expressions for the action, with the oscillators eliminated are similar to (23.2), (24.1) and (24.2), except that there is a sum of interaction terms, one for each oscillator. Thus, if the frequency of the  $j^{th}$  oscillator is  $\omega_j$ , its mass  $m_j$ , the interaction with the particles given by  $y_j$ , and there were  $N$  oscillators, (24.2), for example, would read,

$$\mathcal{A} = \int_{-\infty}^{\infty} [L_y + L_z] dt + \sum_{j=1}^N \frac{1}{2m_j \omega_j} \int_{-\infty}^{\infty} \int_{-\infty}^t \sin \omega_j(t-s) Y_j(s) Y_j(t) ds dt. \quad (25.1)$$

By compounding terms of the form (25.1) many different types of interaction may be obtained. For example, the interaction of equation (11.2) would result from (25.1) if we had only one particle x, (instead of y and z), and an infinite number of oscillators of unit mass ( $\frac{d}{dt} \omega \sin \omega t dw$ ) oscillators with frequency  $\omega$  in the range  $\omega$  to  $\omega + dw$ ), each interacting with the particle through the function  $Y_j(t) = \dot{x}/t$  (the same for each oscillator).

If we look a little more closely at the type of interaction of (24.2) we see that since  $Y(t) = I_y(t) + I_z(t)$  the interaction contains terms of the form  $I_y(t)I_y(s)$  and  $I_z(t)I_z(s)$  as well as  $I_y(t)I_z(s)$  and  $I_z(t)I_y(s)$ . Although the latter represent interactions between the particles A and B, the former type of term represents an interaction of particle A with itself, and particle B with itself, so to speak. What type of intermediate oscillator system will lead to interactions between particles, and to no interaction of a particle with itself?

This question is easily answered, when it is observed that if, in expression (25.1) for example, we take two oscillators,  $j = 1$ , and  $j = 2$ , so that  $\omega_1 = \omega_2$ ;  $m_1 = m_2 = m$ ;  $Y_1 = I_y + I_z$ ; and  $Y_2 = I_y - I_z$ , since,  $Y_1(s)Y_1(t) - Y_2(s)Y_2(t) = 2(I_y(s)I_z(t) + I_z(s)I_y(t))$ , the interaction (25.1) could be written as,

$$\mathcal{A} = \int_{-\infty}^{\infty} [L_y + L_z] dt + \frac{1}{m\omega} \int_{-\infty}^{\infty} \int_{-\infty}^t \sin \omega(t-s) [I_y(s)I_z(t) + I_z(s)I_y(t)] ds dt. \quad (25.2)$$

representing interactions of the particles with each other and having no "self-action" terms. Exactly the same procedure leads to the same results in the cases (23.2) and (24.1).

The original action, involving the oscillators, which leads to a form of this kind is, from what we have said,

$$\int_{-\infty}^{\infty} [L_y + L_z + (I_x + I_y)x_i + (I_z - I_y)x_z + \frac{m}{2}(\dot{x}_i^2 - \omega^2 x_i^2) - \frac{m}{2}(\dot{x}_z^2 - \omega^2 x_z^2)] dt$$

This may be written, by putting  $\eta_z = x_i + x_z$  and  $\eta_y = x_i - x_z$ ,

$$\int_{-\infty}^{\infty} [L_y + L_z + I_x \eta_z + I_y \eta_y + \frac{m}{2}(\dot{\eta}_y \dot{\eta}_z - \omega^2 \eta_y \eta_z)] dt.$$

This may be immediately generalized to the case where there are a number of particles  $y_k$ . The action,

$$\int_{-\infty}^{\infty} \left[ \sum_k (L_{y_k} + I_{y_k} \eta_{y_k}) + \sum_k \sum_{l \neq k} \frac{m}{2} (\dot{\eta}_{y_k} \dot{\eta}_{y_l} - \omega^2 \eta_{y_k} \eta_{y_l}) \right] dt$$

will lead to interactions only between pairs of particles  $k, l$ , no terms arising corresponding to the action of a particle on itself.

These action expressions will be of importance in the next part of the paper when we discuss them quantum mechanically.

Starting with a system with a Hamiltonian, we have, at least classically, found a corresponding non-Hamiltonian action

principle, by leaving out one member of the system. We have, in this, a way of checking a description which is intended as a generalization of quantum mechanics. We may start with a Hamiltonian system, where the quantum mechanics is well known, and show that by suitable elimination of the intermediate oscillator we get a system whose classical analogue obeys an action principle of the type (23.2) or (23.1). We do this on page 66.

### III. Least Action in Quantum Mechanics.

Classical mechanics is the limiting form of quantum mechanics when Planck's constant,  $\hbar$ , is considered as being vanishingly small. The classical system analogous to a quantum mechanical system (when such an analogue exists) may be mathematically exhibited most directly by letting  $\hbar$  approach zero in the quantum mechanical equations.

The inverse problem, that of determining a quantum mechanical description of a system whose classical mechanical behaviour is known, may not be so easily solved. Indeed, the solution cannot be expected to be unique; witness, for example, the Klein-Gordon equation and Dirac's equation for the relativistic behaviour of an electron, both of which have the same classical analogue and quite different quantum mechanical consequences.

There exist, however, very useful rules applicable when the classical equations may be put into Hamiltonian form, and conjugate coordinates and momenta may be defined. These rules, leading to Schrödinger's equations, or Heisenberg's matrix formulation are too well known to require description here.<sup>1</sup>

For classical systems, such as those described in the first part of this paper, which in general have no Hamiltonian form, and for which conjugate momenta and coordinates cannot be defined, no satisfactory method of quantization has been given. In fact, the usual formulations of quantum mechanics use concepts of Hamiltonian, and of momentum in such a direct

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(1) They are discussed very satisfactorily by Dirac in his book, "The Principles of Quantum Mechanics" (Oxford 1935) on page 88, and on page 119.

and fundamental way that it would seem almost impossible to do without them.

A formulation of quantum mechanics will be described here which does not require the ideas of a Hamiltonian or momentum operator for its expression. It has, as its central mathematical idea, the analogue of the action integral of classical mechanics. It is a solution to the problem of the quantization of the classical theory of least action described in the first part of the paper.

A generalization of quantum mechanics which is to apply to a wider range of classical systems must satisfy at least two conditions. First, in the limit as  $\hbar$  approaches zero the quantum mechanical equations must pass over into classical equations of motion applicable to systems in this widened range. And, second, they must be equivalent to the present formulations of quantum mechanics applicable to classical systems with Hamiltonians. The form of quantum mechanics to be described below does indeed satisfy both these criteria for systems for which a principle of least action exists. As an additional argument we shall show that the action principles arising in classical theory from the elimination of an intermediate harmonic oscillator arise in an analogous way in quantum mechanics.

1. The Lagrangian in Quantum Mechanics.

A description of the proposed formulation of quantum mechanics might best begin by recalling some remarks made by Dirac<sup>2</sup> concerning the analogue of the Lagrangian and the action in quantum mechanics. These remarks bear so directly on what is to follow and are so necessary for an understanding of it, that it is thought best to quote them in full, even though it results in a rather long quotation. Speaking of the transformation function  $(\psi'_t / \psi_t)$  connecting the representations referring to two different times  $t$  and  $T$ , he says, from the analogy between classical and quantum contact transformations .... we see that  $(\psi'_t / \psi_t)$  corresponds in the classical theory to  $e^{\frac{iS}{\hbar}}$ , where  $S$  is Hamilton's principal function for the time interval  $T$  to  $t$ , equal to the time-integral of the Lagrangian  $L$ ,

$$S = \int_t^T L dt \quad (21)$$

taking an infinitesimal time interval  $t$  to  $t+dt$ , we see that  $(\psi_{t+dt} / \psi_t)$  corresponds to  $e^{\frac{iLdt}{\hbar}}$ . This result gives probably the most fundamental quantum analogue for the classical Lagrangian function. It is preferable for the sake of analogy to consider the classical Lagrangian as a function of the coordinates at time  $t$  and the coordinates at time  $t+dt$ , instead of a function of the coordinates and velocities at time  $t$ .

There is an important action principle in classical mechanics concerning Hamilton's principal function (21). It says that this function remains stationary for small variations of the

(2). Dirac, "The Principles of Quantum Mechanics", p124 to 126.

jectory of the system which do not alter the end points, i.e. for variations of the q's at all intermediate times between T and t  $q_1$  and  $q_T$  fixed. Let us see what it corresponds to in the quantum theory.

Put

$$\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} L dt\right\} = \exp\left\{\frac{i}{\hbar} S(t_b, t_a)\right\} = B(t_b, t_a) \quad (22)$$

that  $B(t_b, t_a)$  corresponds to  $(q'_b/q'_a)$  in the quantum theory. Now use the time interval  $T \rightarrow t$  to be divided up into a large number of small time intervals  $T \rightarrow t_1, t_1 \rightarrow t_2, \dots, t_m \rightarrow t_{m+1}, t_m \rightarrow t$ , by introduction of a sequence of intermediate times  $t_1, t_2, \dots, t_m$ . Then

$$B(T, T) = B(t, t_m) B(t_m, t_{m-1}) \dots B(t_2, t_1) B(t_1, T) \quad (23)$$

corresponding quantum equation, which follows from the composition ... , is

$$(q'_T/q'_1) = \iiint \dots \int (q'_1/q'_{m+1}) dq'_{m+1} (q'_m/q'_{m-1}) dq'_{m-1} \dots (q'_1/q'_2) dq'_2 (q'_1/q'_1) \quad (24)$$

being written for  $q'_{t_n}$  for brevity. At first sight there does not seem to be any close correspondence between (23) and (24). We must, however, analyse the meaning of (23) rather more carefully. We regard each factor B as a function of the q's at the two ends of the time interval to which it refers. This makes the right-hand side of (23) a function, not only of  $q_T$  and  $q_1$ , but also of all intermediate q's. Equation (23) is valid only when we substitute the intermediate q's in its right-hand side their values for the trajectory, small variations in which leave  $S(t, T)$  stationary and therefore also, from (22), leave  $B(t, T)$  stationary. It is the process of substituting these values for the intermediate q's which corresponds to the integrations over all values for the intermediate q's. The quantum analogue of Hamilton's action principle

is thus absorbed in the composition law (24) and the classical requirement that the values of the intermediate q's shall make  $S(t, T)$  stationary corresponds to the condition in quantum mechanics that all values of the intermediate q's are important in proportion to their contribution to the integral in (24).

Let us see how (23) can be a limiting case of (24) for  $\hbar$  small. We must suppose the integrand in (24) to be of the form  $e^{iF/\hbar}$ , where F is a function of  $q'_1, q'_2, q'_3, \dots, q'_n$  which remains continuous as  $\hbar$  tends to zero, so that the integrand is a rapidly oscillating function when  $\hbar$  is small. The integral of such a rapidly oscillating function will be extremely small, except for the contribution arising from a region in the domain of integration where comparatively large variations in the  $q'_n$  produce only very small variations in F. Such a region must be the neighborhood of a point where F is stationary for small variations of the  $q'_n$ . Thus the integral in (24) is determined essentially by the value of the integrand at a point where the integrand is stationary for small variations of the intermediate q's, and so (24) goes over into (23)."

We may now point out that not only does  $(q'_{first}/q'_t)$  correspond to  $\exp \frac{i}{\hbar} [L(\frac{q'_{first}-q'_t}{\hbar}, q'_{first}) st]$ , where L( $\dot{q}, q$ ) is the Lagrangian for the classical system considered as a function of velocity and coordinate, but that often it is actually equal to it, within a normalization constant, in the limit as  $st$  approaches zero. That is, to terms or order  $st$ , if  $\sqrt{g(q)} dq$  is the volume element in q-space,

$$\psi(q'_{first}, t; st) = \int (q'_{first}/q'_t) \psi(q'_t, t) \sqrt{g(q'_t)} dq'_t = \int e^{\frac{i}{\hbar} st L(\frac{q'_{first}-q'_t}{\hbar}, q'_{first})} \psi(q'_t, t) \frac{\sqrt{g} dq'_t}{A(st)} \quad (31.1)$$

since  $q'_{first}$  and  $q'_t$  are just different variables it might be advantageous to avoid the subscripts, and to write;  $(Q = q'_{first}, q = q'_t)$

$$\int e^{\frac{i\epsilon t}{\hbar} L(\frac{q-q'}{\epsilon}, q)} \psi(q, t) \frac{\sqrt{g(q)}}{A(q)} dq = \psi(q, t+\epsilon t) \quad (32.1)$$

as  $\epsilon t \rightarrow 0$  to the first order in  $\epsilon t$ . This is an integral equation to determine the wave function  $\psi(q, t+\epsilon t)$  for the system at the time  $t+\epsilon t$  in terms of its value at time  $t$ . It therefore serves the same function as Schrodinger's equation, and indeed is equivalent to that equation if the normalization constant,  $A(q)$ , a function of  $\epsilon t$ , is chosen suitably.

To see how this comes about, we take the simplest case of a particle of mass  $m$  moving in one dimension in a force field of potential  $V(x)$ . Thus the classical Lagrangian function is

$L = \frac{1}{2} m \dot{x}^2 - V(x)$ . In accordance with equation (32.1), then, the wave function for this system must satisfy (where we have written  $\epsilon$  for  $\epsilon t$ ) for infinitesimal  $\epsilon$ , the equation,

$$\psi(x, t+\epsilon) = \int e^{\frac{i\epsilon}{\hbar} \left\{ \frac{m}{2\epsilon} (x-y)^2 - V(y) \right\}} \psi(y, t) \frac{dy}{A} \quad (32.2)$$

Let us substitute  $y = \eta + x$  in the integral;

$$\psi(x, t+\epsilon) = \int e^{\frac{i\epsilon}{\hbar} \left\{ \frac{m}{2\epsilon} \eta^2 - \epsilon V(\eta) \right\}} \psi(x+\eta, t) \frac{d\eta}{A} \quad (32.3)$$

Only values of  $\eta$  close to zero will contribute to the integral, because, for small  $\epsilon$ , other values of  $\eta$  make the exponential

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(3) One could just as well write  $L(\frac{q_{t+\epsilon}-q'_t}{\epsilon}, q'_t)$  for  $L(\frac{q_{t+\epsilon}-q'_t}{\epsilon}, q_{t+\epsilon})$ , the difference being of a higher order of smallness than we are concerned with.

oscillate so rapidly that there will arise little contribution to the integral. We are therefore led to expand  $\psi(x,t+\epsilon)$  in a Taylor series around  $\eta=0$ , obtaining, after rearranging the integral,

$$\psi(x,t+\epsilon) = \frac{e^{-\frac{i\epsilon}{\hbar} V(x)}}{A} \int e^{\frac{i\epsilon}{\hbar} \frac{m}{2\epsilon} \eta^2} [\psi(x,t) + \eta \frac{\partial \psi(x,t)}{\partial x} + \frac{\eta^2}{2} \frac{\partial^2 \psi(x,t)}{\partial x^2}, \dots] d\eta$$

Now,  $\int_{-\infty}^{\infty} e^{\frac{i\epsilon m}{\hbar} \eta^2} d\eta = \sqrt{\frac{2\pi\hbar\epsilon i}{m}}$  (see Pierces integral tables 487),

and by differentiating both sides with respect to  $m$ , one may show

$$\int_{-\infty}^{\infty} \eta^2 \cdot e^{\frac{i\epsilon m}{\hbar} \eta^2} d\eta = \sqrt{\frac{2\pi\hbar\epsilon i}{m}} \frac{\pi\epsilon i}{m}$$

The integral with  $\eta$  in the integrand is zero since it is the integral of an odd function. Therefore,

$$\psi(x,t+\epsilon) = \frac{\sqrt{\frac{2\pi\hbar\epsilon i}{m}}}{A} e^{-\frac{i\epsilon}{\hbar} V(x)} \left\{ \psi(x,t) + \frac{\hbar\epsilon i}{m} \frac{\partial^2 \psi}{\partial x^2} + \text{terms in } \epsilon^3 \text{ etc.} \right\} \quad (33.1)$$

The left hand side of this, for very small  $\epsilon$  approaches  $\psi(x,t)$  so that for the equality to hold we must choose,

$$A = \sqrt{\frac{2\pi\hbar\epsilon i}{m}} \quad (33.2)$$

Expanding the both sides of (33.1) in powers of  $\epsilon$  up to the first, we find,

$$\psi(x,t) + \epsilon \frac{\partial \psi(x,t)}{\partial t} = \psi(x,t) - \frac{i\epsilon}{\hbar} V(x) \psi(x,t) + \frac{\hbar\epsilon i}{2m} \frac{\partial^2 \psi}{\partial x^2},$$

and therefore,  $-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi$

which is just Schrodinger's equation for the system in question.

This confirms the remark that equation (32.1) is equivalent to Schrodinger's differential equation for the wave function  $\psi$ . Thus given a classical system describable by a Lagrangian which is a function of velocities and coordinates only, a quantum mechanical description of an analogous system may be written down directly, without first working out a Hamiltonian.

If the problem involves more than one particle, or particles with more than one degree of freedom, equation (32.1) remains formally the same, if  $q$  and  $Q$  are considered to represent entire sets of coordinates, and  $\int \sqrt{g} d^4q$  represents the volume integral over the space of these coordinates. The form of Schrodinger's equation which will be arrived at will be definite and will not suffer from the type of ambiguity one finds if one tries to substitute  $\frac{\hbar^2}{8\pi^2}$  for  $p_q$  in the classical Hamiltonian.

It will be important to consider the consequences of (32.1) when applied to connect the wave function at one time to its value at some finite later time.

Suppose we know the wave function  $\psi(q_0, t_0)$  at time  $t_0$ , and desire it at time  $T$ . Dividing the time interval up into a very large number of small time intervals,  $t_0$  to  $t_1$ ,  $t_1$  to  $t_2$ , ...,  $t_{n-1}$  to  $T$ , and applying relation (32.1) to each time interval we obtain, if we write  $q_i$  for the coordinate associated with time  $t_i$ ,

$$\psi(q_{in}, t_{in}) \approx \int e^{\frac{i}{\hbar} \int L(\frac{q_{in}-q_i}{t_{in}-t_i}, q_{in}) \cdot (t_{in}-t_i)} \cdot \psi(q_i, t_i) \frac{\sqrt{g(q_i)}}{A(t_{in}-t_i)} dq_i \quad (34.1)$$

(4). We have, of course only proved the equivalence in a very special case. It is apparent, however, that the proof may be readily extended to any Lagrangian which is a quadratic function of the velocities, in addition to, perhaps, some terms linear in the velocity arising from magnetic fields. The equivalence will be shown in a more general manner later.

Thus, by induction we find the relation, (35.1)

$$\psi(q, t) \approx \int \dots \int \psi^*(q_0) \left\{ \frac{1}{\pi} \sum_{i=0}^m \left[ L \left( \frac{q_{i+1} - q_i}{t_{m+1} - t_i}, q_{i+1} \right) \cdot (t_{i+1} - t_i) \right] \right\} \cdot \psi(q_0, t_0) \frac{\sqrt{g_0} dt_0 \sqrt{g_1} dq_1 \dots \sqrt{g_m} dq_m}{A(t_0 - t_0) \cdot A(t_1 - t_0) \dots A(t_m - t_0)}$$

where in the sum we shall write Q for  $q_{m+1}$ , and T for  $t_{m+1}$ . In the limit as we take finer and finer subdivisions of the interval  $t_0$  to T and thus make an ever increasing number of successive integrations, the expression on the right side of (35.1) becomes equal to  $\psi(q, t)$ . The sum in the exponential resembles  $\int_{t_0}^T L(q, \dot{q}) dt$  with the integral written as a Riemann sum.

In a similar manner we can compute  $\psi(q_0, t_0)$  in terms of the wave function at the later time  $T = t_{m+1}$ , by the equation;

$$\psi^*(q_0, t_0) = \int \dots \int \psi^*(q_{m+1}, t_{m+1}) \psi^* \left\{ \frac{1}{\pi} \sum_{i=0}^m \left[ L \left( \frac{q_{i+1} - q_i}{t_{m+1} - t_i}, q_{i+1} \right) \cdot (t_{i+1} - t_i) \right] \right\} \frac{\sqrt{g_{m+1}} dq_{m+1} \dots \sqrt{g_0} dq_0}{A(t_{m+1} - t_0) \dots A(t_0 - t_0)} \quad (35.2)$$

## 2. The Calculation of Matrix Elements in the Language of a Lagrangian.

Suppose we wish to compute the average value of some function  $f(q)$  of the coordinates at the time  $t_0$ , which we shall call  $\langle f(q_0) \rangle$ :

$$\langle f(q_0) \rangle = \int \psi^*(q_0, t_0) f(q_0) \psi(q_0, t_0) \sqrt{g_0} dq_0 \quad (35.3)$$

Let us try to express this in terms of the wave function at some more future time,  $t = t_{m+1}$ , by equation (35.2), and in terms of the wave function  $\psi(q_m, t_m)$  at some earlier time  $t = t_m$  by an equation analogous to (35.1) (we shall let negative indices stand for times earlier than  $t_0$ ). We obtain

$$\langle f(q_i) \rangle = \iiint \int f^*(q_{m+1}, t_{m+1}) \exp \left\{ \frac{i}{\hbar} \sum_{i=m+1}^n \left[ L \left( \frac{q_{i+1}-q_i}{t_{i+1}-t_i}, t_{i+1} \right) \cdot (t_{i+1}-t_i) \right] \right\} \cdot f(q_i) \cdot$$

$$\cdot \psi(q_m, t_m) \cdot \frac{\sqrt{g} dq_m \dots \sqrt{g} dq_3 \sqrt{g} df_1 \dots \sqrt{g} df_{m-1}}{A(t_{m+1}-t_m) \dots A(t_2-t_1) \dots A(t_{m+1}-t_m)} \quad (36.1)$$

We shall be dealing with expressions of this form to a considerable extent, so that we shall make some remarks concerning it. In the first place, we will always take  $t_m$  to be a fixed time  $t_1$  very far in the past, and  $t_{m+1}$  to be  $T_2$ , a time very far in the future. Secondly, we may generalize somewhat by taking the wave function at  $T_2$  arbitrarily, say  $\chi$ , and not necessarily the same as what the wave function  $\psi$  chosen at time  $t_1$  would become at this later time  $T_2$ . In this way we shall have an expression more like a matrix element than an average. We shall use the symbol  $\langle \chi | f(q_i) | \psi \rangle$  to represent this quantity. We shall continue to call it simply the average of  $f$  even though it more closely resembles a matrix element. We may, of course calculate quantities such as  $\langle \chi | f(q_j) | \psi \rangle$ , which correspond to the average value of  $f(q_j)$  at the slightly later time,  $t_j$ , merely by replacing the  $f(q_j)$  appearing in the integral on the right hand side of (36.1) with  $f(q_j)$ . Thus, we may find the time rate of change of the average value of  $f(q_j)$ , as,

$$\frac{d}{dt} \langle \chi | f(q_j) | \psi \rangle = \frac{\langle \chi | f(q_j) | \psi \rangle - \langle \chi | f(q_j) | \psi \rangle}{t_j - t_0} = \langle \chi | \frac{f(q_j) - f(q_j)}{t_j - t_0} | \psi \rangle \quad (36.2)$$

where the last expression means that we are to replace  $f(q_j)$  in the integral of (36.1) by  $\frac{f(q_j) - f(q_j)}{t_j - t_0}$ . We may say then, that the symbol  $\langle \chi | f(q_j) | \psi \rangle$  means that the quantity inside is to be multiplied by an exponential of the form of equation (36.1), by a wave function

$\psi$  at  $t_i$ , and one,  $\psi$ , at  $T_1$ , and integrated over all coordinates. Finally, the limit as the subdivisions of time become finer and finer is to be taken. We shall, for a while, disregard which wave functions are put into the expression, and shall simply write,  $\langle |f| \rangle$ , but we shall discuss this later. In this way we can define the average of  $F[\psi(t)]$  where  $F$  is any functional<sup>5</sup> at all of  $\psi(t)$ . We need merely to express the functional approximately as a function of the values,  $f_i$ , of  $\psi$  at the points  $t_i$ , place this function in the integral of (36.1) and pass to the limit.

### 3. The Equations of Motion in Lagrangian Form.

Let us now consider some functional, which, expressed in terms of  $f_i$ , the values of  $\psi$  at the times  $t_i$ , is  $F(f_i)$ , that is to say, a function of  $\dots f_1, g_0, f_1, g_1, \dots$  etc. Let us calculate  $\langle \left| \frac{\partial F}{\partial f_k} \right| \rangle$ . Replacing  $f$  by  $\frac{\partial \psi}{\partial t}$  in (36.1) we see that we may integrate by parts with respect to  $\psi_{ik}$ . The integrated part may be assumed to vanish, for if we consider that the integrations over the other  $\psi$ 's were performed the remaining integrand would be similar to the square of the wave function at time  $t_k$ , which presumably vanishes at infinity. We find after integrating by parts, an expression similar to (36.1) but having a different term for  $f(g_0)$ ; namely we find,

$$\left\langle \left| \frac{1}{\sqrt{f_k}} \frac{\partial (f_k \psi_{ik}) \cdot F}{\partial f_k} \right| \right\rangle = -\frac{i}{\hbar} \left\langle \left| F \cdot \frac{\partial}{\partial f_k} \left\{ \sum_{i=1}^m \left[ L \left( \frac{f_{i+1}-f_i}{t_{i+1}-t_i}, \psi_{iw} \right) \cdot (t_{i+1}-t_i) \right] \right\} \right| \right\rangle \quad (37.1)$$

Thus performing the indicated differentiation we obtain,

$$\begin{aligned} \left\langle \left| \frac{1}{\sqrt{f_k}} \frac{\partial (f_k F)}{\partial f_k} \right| \right\rangle &= \\ &= \frac{i}{\hbar} \left\langle \left| F \left\{ L_i \left[ \frac{q_{iw} - f_i}{t_{i+1} - t_i}, q_{iw} \right] - L_i \left( \frac{f_k - q_{iw}}{t_k - t_{i+1}}, q_{iw} \right) - (t_k - t_{i+1}) \cdot L_i \left( \frac{f_k - f_{k-1}}{t_k - t_{k-1}}, q_{iw} \right) \right\} \right| \right\rangle \quad (37.2) \end{aligned}$$

(5). Properties of functionals and our notation with regard to them is described in the first section (p8) of the paper.

where we have written  $L_f$  for the function  $\frac{\partial L}{\partial f}$ , and  $L_f$  for  $\frac{\partial L}{\partial f}$ . The expression in {} may be remembered most easily by noting that its limit for infinitesimal subdivisions is

$$\left[ \frac{d}{dt} \left( \frac{\partial L}{\partial f} \right) - \frac{\partial L}{\partial f} \right] dt$$

This relation (37.2) is fundamental in that, when compared to corresponding expressions in the usual form of quantum mechanics, it contains, as we shall see, in one equation, both the equations of motion and the commutation rules for p and q. The way this comes about can be seen most clearly by applying equation (37.2) to the simple example with  $L = m \frac{x^2}{t} - V(x)$ . It then reads, ( $\hbar = 1$ )

$$\langle \left| \frac{\partial F}{\partial x_k} \right| \rangle = \frac{i}{\hbar} \langle \left| F \cdot \left\{ m \left( \frac{x_{k+1} - x_k}{t_{k+1} - t_k} \right) - m \left( \frac{x_k - x_{k-1}}{t_k - t_{k-1}} \right) - (t_k - t_{k-1}) V'(x_k) \right\} \right| \rangle \quad (38.1)$$

If  $F = x_k$  then (38.1) becomes;

$$\langle \left| \text{II} \right| \rangle = \frac{i}{\hbar} \langle \left| m \left( \frac{x_{k+1} - x_k}{t_{k+1} - t_k} \right) x_k - x_k m \left( \frac{x_k - x_{k-1}}{t_k - t_{k-1}} \right) - x_k (t_k - t_{k-1}) V'(x_k) \right| \rangle$$

In the limit of fine subdivision of the time, since as  $t_k - t_{k-1} \rightarrow 0$  the last term becomes unimportant, we can write, supposing as usual that in all these equations the limit is to be taken,

$$\langle \left| \left( m \frac{x_{k+1} - x_k}{t_{k+1} - t_k} x_k - x_k \left( m \frac{x_k - x_{k-1}}{t_k - t_{k-1}} \right) \right) \right| \rangle = \frac{i}{\hbar} \langle \text{III} \rangle \quad (39.2)$$

This is equivalent to the statement, in the ordinary notation of quantum mechanics, that the average value of  $pq - qp$  is equal to the average value of  $\frac{1}{i} I$ . The order of the factors in the usual mechanics here shows up as the order of the terms in time. (Exact relations by which formulas of the notation of equation (39.2) can be translated into relations in the more usual notation will be given in a later section (page 42)).

Again by equation (37.2) for  $F = x_{k+1}$ , say, we find,

$$\left\langle \left| X_{n+3} \left( m \frac{x_{n+1} - x_n}{t_{n+1} - t_n} \right) - X_{n+3} \left( m \frac{x_n - x_{n-1}}{t_n - t_{n-1}} \right) \right| \right\rangle = 0 \quad \text{in the limit. (38.1)}$$

Thus, the difference of two successive momentum measurements followed by a position measurement, multiplied and averaged, is infinitesimal since the two successive momentum measurements give the same result - but if the position measurement occurs between the momentum measurements, in time, the result is no longer small, the position measurement having disturbed the momentum between the times the momentum was measured. (Of course, these quantities cannot actually be looked upon as averages of quantities in the classical sense because of the  $i$  in the expressions.)

In equation (38.2) we may replace  $\left\langle \left| X_n \cdot m \left( \frac{x_n - x_{n-1}}{t_n - t_{n-1}} \right) \right| \right\rangle$  by its value a moment later, namely,  $\left\langle \left| X_{n+1} \cdot m \left( \frac{x_n - x_{n-1}}{t_n - t_{n-1}} \right) \right| \right\rangle$  without changing the value of the expression by a finite amount. Thus (38.2) may be rearranged to read,

$$\left\langle \left| (x_{n+1} - x_n)^2 \right| \right\rangle = - \frac{2k}{m\hbar} (t_{n+1} - t_n) \cdot \left\langle \left| \right| \right\rangle \quad (38.2)$$

This describes the well known fact that a wave packet spreads parabolically in time from a point, and that although the average value of the displacement of a particle in the time  $dt$  is  $vdt$ , where  $v$  is the mean velocity, the mean value of the square of this displacement is not of order  $dt^2$ , but only of order  $dt$ . We mention it here to point out that although

the form for average velocity is, from equation (36.2),

$\left\langle \left| \frac{x_{n+1} - x_n}{t_{n+1} - t_n} \right| \right\rangle$ , the average kinetic energy, for example, must

be written  $\left\langle \left| \frac{m}{2} \left( \frac{x_{n+1} - x_n}{t_{n+1} - t_n} \cdot \frac{x_n - x_{n-1}}{t_n - t_{n-1}} \right) \right| \right\rangle$  rather than  $\left\langle \left| \frac{m}{2} \left( \frac{x_{n+1} - x_n}{t_{n+1} - t_n} \right)^2 \right| \right\rangle$ .

The latter is infinite.

If, in (38.1), we had chosen for  $F$  the expression  $G_1 X_n G_2$ ,

where  $G_i$  is any function of the coordinates,  $x_i$ , belonging to times  $t_i$  later than  $t_n$  ( $t_i > t_n$ ), and  $G_1$  is any function of the coordinates belonging to times earlier than  $t_n$ , we would have found, in place of equation (38.2), the relation,

$$\left\langle \left| G_i \left[ \left( m \frac{x_{n+1} - x_n}{t_{n+1} - t_n} \right) \cdot x_n - x_n \cdot \left( m \frac{x_n - x_{n-1}}{t_n - t_{n-1}} \right) \right] G_1 \right| \right\rangle = \frac{1}{\epsilon} \langle [G_i, G_1] \rangle \quad (40.1)$$

This is equivalent to the usual relation among averages,

$$\langle [G_i, (p\dot{x} - q\dot{p})] G_1 \rangle = \frac{1}{\epsilon} \langle [G_i, G_1] \rangle$$

Since  $G_i$  and  $G_1$  are arbitrary functions of their coordinates, we may think of equation (40.1) as equivalent to the operator equation  $p\dot{q} - q\dot{p} = \frac{1}{\epsilon}$

Replacing  $F$  simply by  $G_i G_1$  with  $G_i$ ,  $G_1$ , defined as before, equation (38.1) becomes after dividing through by  $(t_n - t_{n-1})$ , since the left side is zero,

$$\left\langle \left| G_i \left[ \frac{m \left( \frac{x_{n+1} - x_n}{t_{n+1} - t_n} \right) - m \left( \frac{x_n - x_{n-1}}{t_n - t_{n-1}} \right)}{t_n - t_{n-1}} - V'(x_n) \right] G_1 \right| \right\rangle = 0 \quad (40.2)$$

This is equivalent to the operator equation, in the usual notation of quantum mechanics, which is the quantum analogue of Newton's law of motion, namely,  $m\ddot{x} - V'x = 0$ . This law and the commutation rules are of course equivalent to the commutation rules and the rules  $HF - FH = \frac{1}{\epsilon} F$  with  $H = \frac{p^2}{2m} + V(x)$  in the usual formulation. Thus equations (40.1) and (40.2) state all that is needed to completely solve the problem for this system, and hence equation (38.1), or its generalization, equation (37.2), from which they can be derived is all that is required.

Even though we are going to extend this to problems for which no Hamiltonian exists, it may still be of interest to

describe the Hamiltonian from our point of view, when one does exist. Let us consider the average of any functional  $F(g_i)$ . To calculate the rate of change with time of this quantity we may use a relation analogous to (36.2). Another method is as follows; Suppose the variables  $g_i$  which appear in  $F$  are limited to indices between the times  $t_i$  and  $t_{i+1}$ . That is to say,  $F$  is a functional of only the variables  $g_{i-1}$  to  $g_{i+1}$ . (In the limit as our subdivisions in time become infinite,  $i, i'$  may become infinite, but we want  $t_i$  and  $t_{i'}$  to remain bounded, so that  $F$  covers only a finite span of time). Now, if in our expression (36.1) for the average value of  $F$  the values of the times  $t_i$  for  $i$  between  $l$  and  $l'$  were increased by a constant small amount  $\delta$  it would be equivalent to calculating our  $F$  at a time  $\delta$  later, with, however, the same wave functions kept fixed at fixed times,  $t_l$  and  $t_{l'}$ . This will give us just  $\langle |F| \rangle + \delta \frac{d}{dt} \langle |F| \rangle$  to the first order in  $\delta$ . Therefore,  $\frac{d}{dt} \langle |F| \rangle$  is the derivative with respect to  $\delta$  of the quantity we get by so augmenting the time variables. To compute the indicated quantities we look at equation (36.1) and notice that if all the times were altered as indicated, the only change made in the formula would be to replace  $t_{i-1}, t_i$ , by  $t_{i-1}, t_i - \delta$  and  $t_{i+1}, t_{i+2}$  by  $t_{i+1}, t_{i+2} + \delta$ . Doing this, and taking the derivative with respect to  $\delta$ , we find,

$$\begin{aligned} \frac{d}{dt} \langle |F| \rangle &= \frac{i}{\hbar} \left\{ \left\{ L_g \left( \frac{q_i - q_{i-1}}{t_i - t_{i-1}}, g_i \right) \cdot \frac{q_{i+1} - q_i}{t_{i+1} - t_i} - L \left( \frac{q_i - q_{i-1}}{t_i - t_{i-1}}, g_i \right) + \frac{\hbar}{c} \alpha (t_i - t_{i-1}) \right\} F - \right. \\ &\quad \left. - F \left\{ L_g \left( \frac{q_{i+1} - q_i}{t_{i+1} - t_i}, g_{i+1} \right) \cdot \frac{q_{i+2} - q_{i+1}}{t_{i+2} - t_{i+1}} - L \left( \frac{q_{i+1} - q_i}{t_{i+1} - t_i}, g_{i+1} \right) + \frac{\hbar}{c} \alpha (t_{i+1} - t_i) \right\} \right\} \end{aligned}$$

where  $\alpha(t) = \frac{1}{dt(t)} \ln A(t)$ . Equation (41.1) only applies if  $F$  contains only coordinates between  $t_1$  and  $t_{n+1}$ , and does not involve the time explicitly. If  $F$  does involve the time term  $\sum_{i=1}^n \left\langle \left| \frac{\partial F}{\partial t_i} \right| \right\rangle$  should be added to the right side.

We may compare (41.1) to the usual relation  $\frac{d}{dt} \langle F \rangle = \frac{i}{\hbar} \langle HF - HF \rangle$  and see that the analogue of  $H$  is the expression in the {} in this equation. The terms  $\frac{i}{\hbar} \alpha(t_{n+1} - t_1)$  arise from the differentiation of the normalizing factor,  $A$ . They serve to keep the expression for  $H$  finite in the limit of infinitesimal subdivision, in spite of the fact that it may contain terms of the form  $\left\langle \left( \frac{q_1 - q_{n+1}}{t_{n+1} - t_1} \right)^2 \right\rangle$  whose magnitude we have already discussed. For example, in view of (39.2) for the simple Lagrangian  $\frac{m\dot{x}^2}{2} - V(x)$ , it is seen that  $\alpha$  must be  $\frac{1}{2(t_{n+1} - t_1)}$  and that, therefore,  $A(t_{n+1} - t_1) = \text{const.} \sqrt{t_{n+1} - t_1}$ , as we have already found for this case. (Eq. 33.2)

#### Translation to the Ordinary Notation of Quantum Mechanics.

What we have been doing so far is no more than to reexpress ordinary quantum mechanics in a somewhat different language. In the next few pages we shall require this altered language in order to describe the generalization we are to make to systems without a simple Lagrangian function of coordinates and velocities. Before we do this, it is perhaps worth while to show how the relations we have derived up to this point, for systems possessing a Hamiltonian,  $H$ , say, may be most readily translated into the more usual notation.

The usual expression for the wave function at time  $t_1$ , in terms of the wave function at time  $t$ , is given by means of

which, of course, is correct. The left side of equation (37.1) when translated with the aid of equations (43.4) and (43.5) does indeed vanish in the limit as the division of time into intervals becomes finer and finer.

As a further example we may mention the equivalence between  $\langle \tilde{L}(x_i, f(x_i)) \rangle$  and the average of  $e^{\frac{i}{\hbar} H(T_i - t)} [ \frac{1}{2\pi} (p f(x_i) + f(x_i) p) ] e^{\frac{i}{\hbar} H(t - T_i)}$  (44.1)

as may be seen most easily by considering the average of

$\frac{x_{i+1} - x_{i-1}}{t_{i+1} - t_{i-1}} f(x_i)$  according to obvious generalizations of equations (43.4) and (43.5).

### 5. The Generalization to any Action Function.

We now make the generalization to the case when the classical action need not be of the form  $A = \int L(f, g) dt$ , but is some other more general functional of  $f(t)$ . In equation (36.1), as has already been remarked, the phase of the exponential is just  $\frac{i}{\hbar} \int L(f, g) dt$  written as a Riemann sum due to our subdivision of the time into finite, but small, intervals. The obvious suggestion is, then, to replace this exponent by  $\frac{i}{\hbar}$  times the more general action. The action must of course first be expressed in an approximate way in terms of  $f_i, t_i$ , in such a way that as the subdivision becomes finer and finer it more nearly approaches the action expressed as a functional of  $f(t)$ .

In order to get a clearer idea of what this will lead to, let us choose a simple action function to keep in mind, for which no Hamiltonian exists. We may take,

$$A = \int_{-\infty}^{\infty} \left\{ \frac{m \dot{x}(t)^2}{2} - V(x(t)) + k^2 \dot{x}(t) \dot{x}(t + T_0) \right\} dt \quad (44.2)$$

which is an approximate action function for a particle in a potential

$v(x)$  and which also interacts with itself in a mirror, by means of half advanced and half retarded waves, exactly as in (11.2).

In the expression of equation (36.1) the integral of  $L$  extends only over a finite time range  $T_1$  to  $T_2$ . Our action (44.2) is meaningless for a finite time range. In fact if we were to integrate over the range from  $T_1$  to  $T_2$  the action might still depend on values of  $x(t)$  outside this range.

This difficulty may be circumvented by altering our mechanical problem. We may assume that at a certain very large positive time  $T_2$ , and at a large negative time  $T_1$ , all of the interactions (e.g., the charges) have gone to zero and the particles are just a set of free particles (or at least their motion is describable by a Lagrangian). We may then put wave functions,  $\chi$  and  $\psi$ , for these times, when the particles are free, into (36.1). (We might then suppose that the motion in the actual problem may be a limit of the motion as these times  $T_1$  and  $T_2$  move out to infinity). We therefore compute, by analogy to (36.1), the quantity,

$$\langle \chi | F | \psi \rangle = \int \chi^* f_{T_1} \frac{d\chi}{dt} \left\{ \frac{i}{\hbar} A(f_n - p_1, p_1, p_2, p_3, f_n - q_n) \right\} \cdot F(-p_1, p_2, \dots) \psi(f_{T_2}) \frac{\sqrt{g} df_{T_2} \dots \sqrt{g} df_{T_1}}{A(p_1 - t_n) \dots A(t_{n+1} - T_1)} \quad (45.1)$$

In the limit as  $\hbar \rightarrow 0$  this will lead to the classical action principle,  $\delta S = 0$ , in the way outlined by Dirac (see page 29.) since nothing is altered to invalidate that argument.

We may obtain the fundamental relation of our quantum mechanics, analogous to equation (37.1), by integrating the formula for the average of  $\frac{\partial \sqrt{g} F}{\partial q_n}$  by parts, to obtain,

$$\left\langle \chi \left| \frac{1}{\sqrt{g} f_{T_2}} \frac{\partial (\sqrt{g} f_{T_2}) F}{\partial q_n} \right| \psi \right\rangle = -\frac{i}{\hbar} \left\langle \chi | F \cdot \frac{\partial A}{\partial q_n} | \psi \right\rangle \quad (45.2)$$

As we have remarked above, this contains the analogue of the equations of motion as well as the quantum commutation rules.

### o. Conservation of Energy. Constants of the Motion.

Because of the importance in ordinary quantum mechanics of operators which correspond to classical constants of motion, we shall mention briefly the analogue of these operators in our generalized formulation. Since these are not needed for the remainder of the paper, they have not been studied in detail.

The notation will be as in the classical case described in section 3, of the first part of the paper. The general discussion given there applies equally well in this case, so that we shall not repeat it. We will suppose, for simplicity that there is only one coordinate,  $q(t)$ , instead of the  $N$  coordinates  $q_i(t)$ .

From the equations of motion (45.2) we can verify directly that,

$$\begin{aligned} & \left\langle \psi \left| \sum_{t_1, t_2}^{\bar{t}_1, \bar{t}_2} \left[ g_i \frac{\partial \mathcal{A}}{\partial q_i} + \frac{\hbar}{i} \sqrt{g_i} \frac{\partial \sqrt{g_i}}{\partial q_i} g_i + \frac{\hbar}{i} \frac{\partial g_i}{\partial q_i} \right] \right| \psi \right\rangle \\ &= -\frac{\hbar}{i} \left\langle \psi \left| \sum_{t_1, t_2}^{\bar{t}_1, \bar{t}_2} g_i \frac{\partial \mathcal{A}}{\partial q_i} \right| \psi \right\rangle \end{aligned} \quad (46.1)$$

If  $\bar{t}_1$  and  $\bar{t}_2$  are far apart, we can suppose that, as we have proved in the classical case, if the action is invariant with respect to the transformation  $q \rightarrow q + \alpha y$ , the expression on the left of (46.1) multiplying  $\mathcal{A}$ , can be expressed as a difference  $I_{\bar{t}_2} - I_{\bar{t}_1}$ , where the functional  $I_{\bar{t}_2}$  involves the coordinates in the neighbourhood of  $\bar{t}_2$ , and  $I_{\bar{t}_1}$  involves, in the same way, coordinates for times in the neighbourhood of  $\bar{t}_1$ .

If the expression,  $\mathcal{I}$ , involves coordinates for only a finite range around some time,  $\bar{t}_0$ , and  $\bar{t}_1$  and  $\bar{t}_2$  are outside this range ( $\bar{t}_2 > \bar{t}_0 > \bar{t}_1$ ) then the right side is independent of  $\bar{t}_1$  and  $\bar{t}_2$ . This is the analogue of the fact that  $I$  is a constant of the motion. In this case also, the left side becomes analogous to the quantity  $IF - FI$ , and the right side is just a differential operation performed on  $F$ . The differential operation is characteristic of the group of transformations from which  $I$  is derived.

Thus, for a displacement in the  $x$  direction, the differential operator is  $\frac{\partial}{\partial x}$ , the corresponding constant of the motion is momentum in the  $x$  direction, and we have the analogue of the operator equation  $p_x F - F p_x = \frac{1}{\epsilon} \frac{\partial F}{\partial x}$ . More accurately, we have the analogue of the equation,

$$e^{\frac{i}{\epsilon} H(\bar{t}_1 - \bar{t}_0)} h e^{-\frac{i}{\epsilon} H(\bar{t}_2 - \bar{t}_0)} F - F e^{\frac{i}{\epsilon} H(\bar{t}_2 - \bar{t}_0)} h e^{-\frac{i}{\epsilon} H(\bar{t}_1 - \bar{t}_0)} = \frac{i}{\epsilon} \frac{\partial F}{\partial x} \text{ for all } \bar{t}_1 \text{ and } \bar{t}_2.$$

For the time-displacement the differential operator is  $\frac{d}{dt} - \frac{p}{\epsilon}$  and the constant is the negative of the energy, so that we have the analogue of  $HF - FH = -\frac{1}{\epsilon} \frac{df}{dt} + \frac{1}{\epsilon} \frac{\partial F}{\partial t}$ . If  $F$  does not depend explicitly on the time, we can write the right side in the usual way, as  $i \dot{F}$ .

For the energy expression, classically  $y(t) = \dot{g}(t)$ .

The formula (41.1) can be deduced from (46.1) most simply by writing for  $y_i$  the form,  $y_i = \frac{1}{2} \left[ \frac{g_{i+} - g_i}{t_{i+} - t_i} + \frac{g_i - g_{i-}}{t_i - t_{i-}} \right]$ . in the case that the coordinates are rectangular. The method described in connection with (41.1) (page 41) of augmenting all the times between  $\bar{t}_1$  and  $\bar{t}_2$  by a fixed amount  $\delta$  may be applied here also to obtain an alternative expression for the energy.

## 7. The Role of the Wave Function.

The problem discussed in this section is that of the existence of a wave function for times between  $T_1$  and  $T_2$ .

It is to be noted, that in view of the relation (45.1) it is no longer possible to express the formula for the averages in some such simple form as  $\int \phi''(f_0) F(f_0) d\lambda f_0$ . Suppose that  $F$  is especially simple so that it contains only the coordinate  $q_0$ . (For example, perhaps we want to compute the average value of  $q_0$  itself.) According to equation (45.1), it may be expressed in the form,

$\int \rho(f_0) F(f_0) \sqrt{g_0} df_0$  (48.1), where  $\rho(f_0)$  is the result of integrating the integrand of (45.1) with respect to every variable  $q_i$  except  $q_0$ . This is to be compared to the usual expression when a Hamiltonian exists,  $\int \phi''(f_0) F(f_0) \psi_1(f_0) \sqrt{g_0} df_0$ . (48.2). They would be equivalent if  $\rho(f_0)$  could be expressed as the product of two functions  $\phi_1(f_0)$  and  $\phi_2(f_0)$  in a natural and useful way. In general, however the integral of  $e^{i\hbar \frac{\dot{L}}{L} t}$  over all the variables except  $q_0$  cannot be so expressed. For the particular case that  $L$  is the integral of an ordinary Lagrangian function of velocity and position, the exponential can be broken up into two factors;

$$e^{i\hbar \frac{\dot{L}}{L} t} \left\{ L\left(\frac{q_1 - q_{1*}}{t_{1*} - t_{1*}}, q_1\right) (t_{1*} - t_{1*}) + L\left(\frac{q_{1*} - q_{2*}}{t_{2*} - t_{1*}}, q_{1*}\right) \cdot (t_{2*} - t_{1*}) + \dots \right\} \quad (48.1)$$

and,

$$e^{i\hbar \frac{\dot{L}}{L} t} \left\{ L\left(\frac{q_1 - q_{1*}}{t_{1*} - t_{1*}}, q_{1*}\right) \cdot (t_{1*} - t_{1*}) + L\left(\frac{q_{1*} - q_{2*}}{t_{2*} - t_{1*}}, q_{1*}\right) \cdot (t_{2*} - t_{1*}) + \dots \right\}$$

These factors contain only the variable  $q_0$  in common, so that when the integrations on the other variables are performed in expression (45.1), the result remains factorable. The quantity  $\phi_1(f_0)$  is the result of integrating the first factor, and is expressed in a form exactly similar to (48.1). The quantity  $\phi_2(f_0)$  is expressed

in terms of  $\chi'$  by an equation like one obtains from (35.1) by taking the complex conjugate. (e.g., 35.2).

We can take the viewpoint, then, that the wave function is just a mathematical construction, useful under certain particular conditions to analyze the problem presented by the more generalized quantum mechanical equations (45.1) and (45.2), but not generally applicable. It is not unreasonable that it should be impossible to find a quantity like a wave function, which has the property of describing the state of the system at one moment, and from which the state at other moments may be derived. In the more complicated mechanical systems (e.g., the example, (44.2)) the state of motion of a system at a particular time is not enough to determine in a simple manner the way that the system will change in time. It is also necessary to know the behaviour of the system at other times; information which a wave function is not designed to furnish. An interesting, and at present unsolved, question is whether there exists a quantity analogous to a wave function for these more general systems, and which reduces to the ordinary wave function in the case that the action is the integral of a Lagrangian. That such exist is, of course, not at all necessary. Quantum mechanics can be worked entirely without a wave function, by speaking of matrices and expectation values only. In practice, however, the wave function is a great convenience, and dominates most of our thought in quantum mechanics. For this reason we shall find it especially convenient, in interpreting the physical meaning of the theory, to assume our mechanical system is such that, no matter how complex between the time  $T_1$  and  $T_2$ , outside of this range the action is the integral of a Lagrangian. In this way we may speak

of the state of the system at times  $T_1$  and  $T_2$ , at least, and represent it by a wave function. This will enable us to describe the meaning of the new generalization in terms with which we are already familiar. This we do in the next section.

### 8. Transition Probabilities.

We shall suppose, as suggested above, that our action has the form of the integral of a Lagrangian for times  $T_2$  or later, and for times  $T_1$  or earlier, but that it is arbitrary in between. In this way, we may speak of the state of the system at time  $T_1$  as being given by a wave function  $\psi$ , and of the state of the system at time  $T_2$ , by a wave function  $\chi$ . We can then make the physical assumption that the probability that the system in state  $\psi$  at time  $T_1$  will be found, at the time  $T_2$ , in the state  $\chi$  is the square of the absolute value of the quantity  $\langle \chi | \psi \rangle$ .

The quantity may be defined by the expression (45.1) with  $F$  replaced simply by unity.

We can define other physical quantities in terms of this, by determining the changes in this probability, or rather in the quantity  $\langle \chi | \psi \rangle$ , produced by perturbations of the motion.

We shall indicate by a subscript the action for which the quantity  $\langle \chi | \psi \rangle$  is calculated, by writing  $\langle \chi | \psi \rangle_a$  if the action is  $a$ . Suppose the action is slightly altered (in the interval  $T_1$  to  $T_2$ ) to become  $a + \epsilon g$  where  $\epsilon$  is a very small parameter. From the form of equation (45.1) we would have,

$$\langle \chi | \psi \rangle_{a+\epsilon g} = \langle \chi | e^{\frac{i\epsilon g}{\hbar}} | \psi \rangle_a \quad (50.1)$$

so that if  $\epsilon$  is small enough to insure convergence, we may write,

$$\langle X|I|\psi\rangle_{at\epsilon q} = \langle X|I|\psi\rangle_a + \frac{i\epsilon}{\pi} \langle X|g|I\rangle_a - \frac{\epsilon^2}{2\pi^2} \langle X|g^2|I\rangle_a - \dots \quad (51.1)$$

We can therefore interpret  $\langle X|g|I\rangle_a$  by saying it is

$\frac{d}{d\epsilon} (\langle X|I|\psi\rangle_{at\epsilon q})$  at  $\epsilon = 0$ . It should be emphasized here that  $\langle X|g|I\rangle_a$  where  $g$  and  $I$  are any two functionals cannot in general be written in a way analogous to a matrix product (e.g.,  $\sum_n \langle X|g|p_n\rangle \langle p_n|I|\psi\rangle$ ) as can be done in the usual mechanics. (This is because  $g$  and  $I$  may overlap in time, and so neither be before the other.) The term  $\langle X|g|I\rangle_a$  may either be interpreted as  $\langle X|I|\psi\rangle_a$  with  $g$  replaced by  $g_I$ , or, alternatively, as the first order change in  $\langle X|I|\psi\rangle_a$  on changing  $a$  to  $a+\epsilon g$  (see equation (51.2)).

We have, incidentally, derived a perturbation equation (51.1), which may be easily generalized (change  $a$  to  $a+rI$  in (51.1), differentiate both sides with respect to  $r$ , and set  $r=0$ ) to read,

$$\langle X|I|\psi\rangle_{at\epsilon q} = \langle X|I|\psi\rangle_a + \frac{i\epsilon}{\pi} \langle X|g|I\rangle_a - \frac{\epsilon^2}{i\pi} \langle X|g^2|I\rangle_a + \dots \quad (51.2)$$

This permits us to express the average of a functional for one action function in terms of averages of other functionals for a slightly different action. For some particular problems, such as, for example, the electrodynamic one, it may turn out that the action may be considered as the sum of two terms,  $a_0 + a_1$ , the first expressible as the integral of a Lagrangian, while the second, not so expressible, may be considered as a small perturbation. Equation (51.2) then permits the actual matrix elements to be expressed in terms of the matrix elements with the Lagrangian action,  $a_0$ , alone. Since for this action,  $a_0$ , the

problem is comparatively simple because wave functions can be defined, the relation (51.2) will serve as a practical method for solving problems in these cases.

Perturbations may also be considered as producing transitions. Suppose the state or the system at the early time  $t_1$  was  $\psi$ . Let us choose some state  $X$ , at the time  $T_2$ , and ask for the probability, with the perturbed action  $A + \epsilon \mathcal{F}$ , that the system will be found in this state at this time. It is just

$|\langle X | \psi \rangle_{\text{pert}}|^2$ . We shall further suppose that  $X$  is so chosen that, were it not for the perturbation,  $\epsilon \mathcal{F}$ , the system would have no chance of being found in the state  $X$ ; that is,  $\langle X | \psi \rangle_a = 0$ . Therefore from equation (51.1), to the order  $\epsilon^2$  the probability that the system originally in state  $\psi$ , is found in the state  $X$  at the time  $T_2$ , due to the perturbation - i.e., the transition probability - is just,

$$\frac{1}{\hbar^2} |\langle X | \epsilon \mathcal{F} | \psi \rangle_a|^2 \quad (52.1)$$

For the special case of a simple perturbing potential acting for a time 0 to  $T$ , we have  $\epsilon \mathcal{F} = - \int_0^T V dt$ , so that our transition probability becomes the more usual expression (Compare, Dirac, "The Principles of Quantum Mechanics", p.177, Eq.21.)

$$\frac{1}{\hbar^2} |\langle X | \int_0^T V dt | \psi \rangle_a|^2$$

It is of interest to notice that if we are given that the wave function at time  $T_1$  is  $\psi$ , although we cannot trace the behaviour of the function through the interval  $T_1$  to  $T_2$ , we can, nevertheless, answer the question, "What will the wave function be at the time  $T_2$ , and later times?" (Of course,

If we know the wave function at the time  $T_2$  we can find it later, since it satisfies a Schrödinger equation from that time on.) If we call the wave function at time  $T_2$ ,  $\psi_{T_2}$ , and expand it in terms of a complete set of orthonormal wave functions  $\chi_m$  at that time, say  $\psi_{T_2} = \sum a_m \chi_m$ , the coefficients  $a_n$  will be just  $\langle \chi_m | \psi_{T_2} \rangle_a$ . Therefore, we have,

$$\psi_{T_2} = \sum_m \chi_m \langle \chi_m | \psi_{T_2} \rangle_a.$$

On account of the form (45.1) for calculating  $\langle \chi_m | \psi_{T_2} \rangle_a$  we find that this may also be expressed as,

(45.1)

$$\psi_{T_2}(q) = \int \psi_{T_2}^*(q) \{ A(q, q_{m+1}, \dots, q_0, q_1, \dots, q_{T_1}) \} \cdot \psi_{T_2}(q_{T_1}) \cdot \frac{\sqrt{g} dq_{m+1}}{A} \dots \frac{\sqrt{g} dq_1}{A} \frac{dq_0}{A}.$$

In the limit when the subdivisions in time become infinitely fine, where  $A(q, q_{m+1}, \dots, q_0, q_1, \dots, q_{T_1})$  is that part of the action functional applicable for times from  $T_1$  to  $T_2$ . (Compare 35.1).

#### 9. Expectation Values for Observables

The physical interpretation which is given in the above section, although the only consistent one available, is rather unsatisfactory. This is because the interpretation requires the concept of states representable by a wave function, while we have pointed out that such a representation is in general impossible. We are therefore forced to alter our mechanical problem so that the action has a simple form at large future and past times, so that we may speak of a wave function at these times, at least. This difficulty also finds itself reflected in the mathematical formulation of the operations performed in calculating average values from the equation (45.1). We have not defined precisely what is to be done when the action does not become simple at times far from the present.

One possibility that suggests itself is to devise some sort of limiting process so that the interpretation of the last section could be used, and the limit taken as  $T_1 \rightarrow -\infty$  and  $T_2 \rightarrow +\infty$ . The author has made several attempts in this direction but they all appear artificial, having mathematical, rather than physical, content.

An alternative possibility is to avoid the mention of wave functions altogether, and use, as the fundamental physical concept, the expectation value of a quantity, rather than a transition probability. The work done in this connection, which is presented in this section, is admittedly very incomplete and the results tentative. It is included because many of the formulas derived would seem to be of value, and the author believes that the solution to the problem of physical interpretation will lie somewhere in this direction.

In ordinary quantum mechanics the matrix element of an operator  $A$ , between two states  $\psi_m$  and  $\psi_n$ , is given by,

$$A_{mn} = \int \psi_m^* A \psi_n dVol.$$

The expected value for the quantity represented by the operator  $A$ , for the state represented by the wave function  $\psi_n$  is,

$A_{nn} = \int \psi_n^* A \psi_n dVol.$  In an exactly similar way, we have used our definition (45.1) for a matrix element of a functional  $F$  between the state whose wave function at the time  $T_2$  is  $\chi$ , and a state whose wave function at the time  $T_1$  is  $\psi$ . To compute the expected value of the functional whose wave function at the time  $T_1$  is  $\psi_{T_1}$ , we calculate  $\langle \psi_{T_1} | F | \psi_{T_1} \rangle$ , where  $\psi_{T_1}$  is given in terms of  $\psi_{T_1}$  by the equation (53.1).

Another important quantity in quantum mechanics is the trace

of a matrix,<sup>6</sup>  $\text{Tr}[A] = \sum_{mn} A_{mn}$ . It measures the relative (unnormalized) average expected value when a priori each state  $\chi_m$  is considered as equally likely. We shall speak of it simply as the expectation of A.

Let us suppose A is an operator which has only certain particular eigenvalues  $a_n$ , so that  $A \chi_n = a_n \chi_n$ , for some set of functions  $\chi_n$ . Let us also suppose  $F_{a_n}(x)$  is a function of x which is zero unless  $x = a_n$ , and  $F_{a_n}(a_n) = 1$ . Then let us find the trace,  $\text{Tr}[BF_{a_m}(A)]$ . ( $F_{a_n}/A$  is a projection operator). The matrix  $B \cdot F_{a_m}(A)$  has for its k,l element, in a representation with the functions  $\chi_n$ ,  $[B F_{a_m}(A)]_{kl} = \sum_m B_{km} [F_{a_m}(A)]_{ml}$ . But, since A is diagonal in this representation,  $[F_{a_m}(A)]_{ml} = 0$  if  $m \neq l$ , and equals  $F_{a_m}(a_m)$  otherwise. Therefore  $[B F_{a_m}(A)]_{kl} = B_{km} F_{a_m}(a_m)$ . Now,  $F_{a_m}(a_l) = 0$ , unless  $l = m$  so that we find,

$\text{Tr}[BF_{a_m}(A)] = B_{mm}$  That is to say, the trace of  $BF_{a_m}(A)$  is the expected value of B for the state for which the quantity A has the value  $a_m$ .

In a like manner it is not hard to show that,  $\text{Tr}[F_{b_n}(B) \cdot F_{a_m}(A)]$  is equal to the probability that the quantity B will be found to have the value  $b_n$  in a state where it is known that A has the value  $a_m$  (neglecting degeneracies).

These examples are given to remind the reader of the fact that by means of the concept of trace all of the important physical concepts can be derived. What corresponds to taking a

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(6). See J. vNeumann, "Mathematische Grundlage der Quantenmechanik" (1932), p 93, ff.

trace in our form of quantum mechanics?

As we can see from equations (45.1) and (53.1) the expression for  $\langle \psi_n | \mathcal{G} | \psi_n \rangle$  can be written in the form,

$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi_n^*(\mathbf{q}) \psi_n(\mathbf{q}') d\mathbf{q} d\mathbf{q}'$  where  $\rho(\mathbf{q}, \mathbf{q}')$  is given by a complicated expression obtained by substituting (53.1) into (45.1). That is to say, what corresponds to the diagonal element  $A_{nn}$  may be written in our case as  $\int \rho(\mathbf{q}, \mathbf{q}') \psi_n^*(\mathbf{q}) \psi_n(\mathbf{q}') d\mathbf{q} d\mathbf{q}'$ . The sum of the diagonal elements, and therefore the trace, corresponds to,

$\sum_n \int \rho(\mathbf{q}, \mathbf{q}') \psi_n^*(\mathbf{q}) \psi_n(\mathbf{q}') d\mathbf{q} d\mathbf{q}'$ . As is well known, however, the sum over all  $n$  of  $\psi_n^*(\mathbf{q}) \psi_n(\mathbf{q}')$  is just  $\delta(\mathbf{q} - \mathbf{q}')$ , so that what corresponds to the trace of  $\mathcal{G}$  is  $\int \rho(\mathbf{q}, \mathbf{q}') d\mathbf{q}$ . We are therefore led to consider the quantity,

$$\begin{aligned} T_n \langle \mathcal{G} \rangle &= \int \exp \left\{ -\frac{i}{\hbar} S \left[ q_1, q_{m+1}, \dots, q_{m+n}, q_n \right] \right\} \cdot \exp \left\{ \frac{i}{\hbar} S \left[ q_1, q_m, \dots, q_{m+1}, q_n \right] \right\} \cdot \\ &\quad \cdot g_1(q_1, q_0) \cdot \sqrt{g_{q_1}} dq_1 \cdot \sqrt{g_{q_n}} dq_n \cdot \frac{\int g_{q_{m+1}} dq_{m+1}}{A_{m+1}} \cdot \frac{\int g_{q_m} dq_m}{A_m} \cdots \frac{\int g_{q_2} dq_2}{A_2} \cdot \frac{\int g_{q_1} dq_1}{A_1}. \end{aligned} \quad (56.1)$$

The passage to the limit of infinitely fine subdivisions is implied, as usual. Since now there are no wave functions, and therefore nothing special about the times  $T_1$  and  $T_2$ , we can consider the true trace to be the limit of the  $T_n \langle \mathcal{G} \rangle$ , defined in (56.1), for a sequence of mechanical systems each of which has an action identical to the true one for successively longer time intervals. (The problem of convergence is ever present.)

The trace defined in (56.1) is identical to the usual trace of quantum mechanics if  $\mathcal{G}$  is a function of one coordinate (e.g.,  $q_0$ ) only, and the action is the time integral of a Lagrangian. It lacks, however, in the general case, one

important property, and it is this that makes the results of this section so uncertain. The trace of an arbitrary functional is not always a real number!

We lack some condition on the functionals which we are to place into (56/), in order to obtain a real value, so that we can say that the functional represents some real observable quantity, the expectation of which is the trace. That is to say, we lack the condition on a functional that it represent an observable, analogous to the condition in ordinary quantum mechanics that an operator, to represent an observable, must be Hermitian. The correct criterion is not known to the author. The most obvious suggestions revolve around the generalization to (56/) which is obtained by letting  $\mathcal{F}$  be a function of the  $q'$  variables, as well as of  $q$ . A real trace is obtained if  $\mathcal{F}$  is any function symmetrical with respect to interchange of each  $q$  with the corresponding  $q'$  (ie; if  $\mathcal{F}(\dots q_1, f_0 \dots; \dots q'_1, f'_0 \dots) = \mathcal{F}(\dots q'_1, f_0 \dots; \dots q_1, f'_0 \dots)$ ). For example  $\mathcal{F}$  might be a function of  $\frac{1}{2}(q_j + q'_j)$  only. This symmetry condition may be all that is necessary to insure that the functional correspond to a real observable. The product, and the sum of two such symmetrical functionals is again symmetrical.

We pass from the general problem of a criterion for any functional to that of trying to determine the form of certain special functionals which we should like to identify with special observables (in particular, projection operators). Let us first try to find the  $\mathcal{F}$  that we are to place in (56/) so that the resulting trace is the probability that  $q$  at the time  $\bar{t}_2$  (i.e.,  $q(\bar{t}_2)$ ) has the value  $b$  if it is known that  $q$  at the time  $\bar{t}_1$  has the value  $a$ . If  $\mathcal{L}$  is the integral of a Lagrangian the answer is simply  $\mathcal{F} = \delta(q_{\bar{t}_1} - b)\delta(q_{\bar{t}_2} - a)$ , as can be immediately verified.

On the other hand, I have not succeeded in prove that the trace of this quantity is in general real. The trace of,

$$\delta\left(\frac{q_{t_1} + q'_{t_1}}{2} - b\right) \cdot \delta\left(\frac{q_{t_1} + q'_{t_1}}{2} - a\right)$$

is, however,

real and gives the same value to the desired probability if the action is an integral of a Lagrangian.

We may therefore tentatively assume that,

$$T_h \left\langle \delta\left(\frac{q_{t_1} + q'_{t_1}}{2} - b\right) \cdot \delta\left(\frac{q_{t_1} + q'_{t_1}}{2} - a\right) \right\rangle \cdot db \quad \text{gives the}$$

relative probability that, if  $q$  has the value  $a$  at the time  $\bar{t}_1$ , a measurement of  $q$  at the time  $\bar{t}_2$  will lead to  $b$ , in the range  $db$ .

(The absolute probability may be gotten by dividing by

$$T_h \left\langle \delta\left(\frac{q_{t_1} + q'_{t_1}}{2} - a\right) \right\rangle \cdot db$$

We shall likewise assume that

$$T_h \left\langle \delta\left(\frac{1}{\epsilon} \left[ \frac{q_{t_1+\epsilon} + q'_{t_1+\epsilon}}{2} - \frac{q_{t_1} + q'_{t_1}}{2} \right] - v \right) \cdot \delta\left(\frac{q_{t_1} + q'_{t_1}}{2} - a\right) \right\rangle dv \quad \text{gives}$$

the relative probability that, if  $q$  has the value  $a$  at the time  $\bar{t}_1$ , a measurement of velocity at the time  $\bar{t}_2$  will lead to  $v$  in the range  $dv$ . That this gives the correct answer, in the case of a Lagrangian action which involves the coordinate  $q$  in the kinetic energy term as  $\frac{1}{2}mq^2$ , is shown below.

In a similar way we can define the probabilities for any quantities involving linear combinations of the coordinates (and that includes velocities, accelerations, etc.). For example, the probability that the difference in the coordinate  $q$  at the time  $\bar{t}_3$  and its value at the time  $\bar{t}_2$  is between  $b$  and  $b+db$ , when it is known that at the time  $\bar{t}_1$  the velocity plus  $c$  times the position is  $a$ , is given by the trace of,

$$\delta\left(\frac{q_{t_1} + q'_{t_1}}{2} - \frac{q_{t_2} + q'_{t_2}}{2} - b\right) \cdot \delta\left(\frac{q_{t_1} + q'_{t_1}}{2\epsilon} - \frac{q_{t_2} + q'_{t_2}}{2\epsilon} + c(q_{t_1} + q'_{t_1}) - a\right) \cdot db$$

(This has been checked for a free harmonic oscillator). It is possible that this is even true if  $\bar{t}_3$  is before  $\bar{t}_1$ , and  $\bar{t}_2$  after  $\bar{t}_1$ .

We shall now show that  $\int_{\bar{t}_1} \left( \delta \left( \frac{1}{\epsilon} \left[ \frac{q_{t_1+\epsilon} + q'_{t_1-\epsilon}}{2} - \frac{q_{t_1} + q'_{t_1}}{2} \right] - V \right) \cdot \delta \right)$

where  $\delta$  involves times earlier than  $t_2$  does agree with the usual form for finding the probability of a given momentum,  $mv$ , if the action is the integral of a Lagrangian, say  $\frac{1}{2}m\dot{q}^2 - V(q)$ .

In the expression (56.1) the integral on  $q_{T_2}$  can be immediately performed, inasmuch as, (see 33.2)

$$\frac{1}{A} \int_{q_{T_2}} \epsilon^{\frac{i\epsilon}{\hbar} \left[ \frac{m}{\epsilon} \left( \frac{q_{T_2} - q_m}{\epsilon} \right)^2 - V(q_{T_2}) \right]} \cdot e^{-\frac{i\epsilon}{\hbar} \left[ \frac{m}{\epsilon} \left( \frac{q_{T_2} - q'_m}{\epsilon} \right)^2 - V(q'_{T_2}) \right]} = \delta(q_{T_2} - q'_m) \cdot A^2$$

Hence the integral on  $q'_m$  means merely replacing  $q'_m$  by  $q_m$ . We have thus again the same expression as (56.1) with one term integrated from the end. We can repeat this process many times, telescoping the Lagrangians, until we come to the term  $q_{T_2} + \epsilon$ . (That is to say, we could have taken  $T_2$  equal to  $t_2 + \epsilon$  without loss of generality). Let us suppose also that we have integrated all variables following  $q_{T_2}$  and  $q'_{T_2}$  and the net result is  $\rho(q_{T_2}, q'_{T_2})$ . (It can be expressed in this form for any  $\epsilon$  satisfying our conditions in virtue of the form of the action.). That is to say, we must calculate,

$$\int \delta \left( \frac{1}{\epsilon} \left[ \frac{q_{t_1+\epsilon} - q_{t_1} + q'_{t_1}}{2} \right] - V \right) \cdot e^{-\frac{i\epsilon}{\hbar} \left[ \frac{m}{\epsilon} \left( \frac{q_{t_1+\epsilon} - q'_{t_1}}{\epsilon} \right)^2 - V(q_{t_1+\epsilon}) \right]} \cdot e^{+\frac{i\epsilon}{\hbar} \left[ \frac{m}{\epsilon} \left( \frac{q_{t_1+\epsilon} - q_{t_1}}{\epsilon} \right)^2 - V(q_{t_1+\epsilon}) \right]} \cdot \rho(q'_{t_1}, q_{t_1}) \cdot dq_{t_1+\epsilon} \cdot \frac{dq'_{t_1}}{\sqrt{\frac{m\epsilon}{\hbar}}} \cdot \frac{dq_{t_1}}{\sqrt{\frac{m\epsilon}{\hbar}}}$$

The phases of the exponentials, when combined, reduce simply to

$$\text{i.e. } \frac{m}{\hbar} \left[ \left( \frac{\delta_{t_1+\epsilon} - \delta_{t_1}}{\epsilon} \right)^2 - \left( \frac{\delta_{t_1+\epsilon} - \delta'_{t_1}}{\epsilon} \right)^2 \right].$$

The integral on the  $\delta$  function, over  $q_{t_1+\epsilon}$ , requires that one substitute for this quantity  $f_{t_1+\epsilon} = \frac{f_{t_1} + f_{t_1'}}{2} + iE$ , and multiply by  $\epsilon$ , to obtain, in the final result,

$$\int \frac{m}{2\pi\hbar} e^{-\frac{imy}{\hbar} q_{t_1}} e^{\frac{imv}{\hbar} q'_{t_1}} \rho(q_{t_1}, q'_{t_1}) dq_{t_1} dq'_{t_1}.$$

This agrees with the usual expression for the probability of a given momentum  $p$ , if  $p = mv$ . (The extra factor of normalization,  $m$ , comes from the fact that  $dp = mdv$ .)

#### 10. Application to the Forced Harmonic Oscillator.

As a special problem, because we shall need the results in the next section, we consider, from the point of view of the modified quantum mechanics, the problem of the forced harmonic oscillator; that is, an oscillator interacting with another system. This problem, when the oscillator is interacting with a Lagrangian system, can of course be handled by the usual methods of quantum mechanics. We shall see, however, that the added power of looking at all the times at once, so to speak, which arises in such equations as (45/1), has some advantages. With a wave function, the oscillator and the interacting system are so firmly interlocked, mathematically, that it is hard to study the properties of the oscillator without, at the same time, solving for the motion of the interacting system. We shall be able here, however, to solve that half of the problem which involves the oscillator, without solving the entire problem.

If  $x$  is the coordinate of our oscillator, the action

is of the form,

$$a = a_0 + \int dt \left\{ \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2} + Y(t)X \right\} \quad (61.1)$$

where  $a_0$  is the action of the other particles of the system of which the oscillator is a part, and  $Y(t)X$  is the interaction of the oscillator with the rest of the system. Thus, if we symbolize the coordinates of the rest of the system by  $Q$ ,  $Y(t)$  is some functional of  $Q$ . (We might also contemplate that the oscillator is not interacting with any other quantum mechanical system, but is simply acted on by a forcing potential. In this case,  $Y(t)$  is a simple function of  $t$ , and represents the force acting on the oscillator at time  $t$ .) We shall suppose that  $Y(t)$  is zero for times outside the range 0 to  $T$ , and shall compute matrix elements of the form  $\langle X_T | / | Y_0 \rangle_a$  where  $Y_0$  is a wave function at time  $t=0$ , (it involves  $Q$  as well as  $x$ ), and  $X_T$  is a wave function at time  $t=T$ . Writing this in more detail, by equation (45.1) it is,

$$\langle X_T | / | Y_0 \rangle_a = \int X_T(Q_m, X_m) d\Gamma \frac{1}{A} \left\{ Q_0, Q_1, \dots \right\} + \sum_{i=0}^{m-1} \left[ \frac{m}{2} \left( \frac{X_{i+1} - X_i}{t_{i+1} - t_i} \right)^2 - \frac{m\omega^2 x_i^2}{2} + Y_i X_i \right] [E_{i+1} - t_i] \}.$$

$$\cdot Y_0(Q_0, X_0) \frac{\sqrt{g dQ_m} \dots \sqrt{g dQ_0}}{A_m \dots A_1} \frac{dx_m \cdot dx_{m-1} \dots dx_0}{\sqrt{\frac{2\pi i k}{m} (t_m - t_{m-1})} \dots \sqrt{\frac{2\pi i k}{m} (t_1 - t_0)}}$$

(61.2)

Where the  $A_m$  are the normalizing constants appropriate to the action  $a_0[Q_i]$ , and  $Q_i$  is the variable  $Q$  at time  $t_i$ ,  $x_i$  the variable  $x$  at the time  $t_i$ , and  $Y_i = Y(t_i)$ , a function, perhaps, of the  $Q_i$ . We are to set  $t_m = T$  and  $t_0 = 0$ .

Inasmuch as the integrand is a quadratic function of the  $x_i$  (for  $i \neq 0$  or  $m$ ) we may actually perform the integrations over

these  $x_i$ , and leave the integrations on  $Q_i$  to be performed later. We will simplify the work by taking all the intervals,  $t_{i+1} - t_i$ , equal, and equal to  $\epsilon$ . We are therefore led to consider the quantity,

$$\tilde{J}_r(x_m, x_0; T) = \lim_{\epsilon \rightarrow 0} \prod_{mE+T}^M \left\{ \frac{i\epsilon}{\pi} \sum_{i=0}^{m-1} \left[ \frac{m}{2} \left( \frac{x_{i+1} - x_i}{\epsilon} \right)^2 - \frac{m\omega^2 x_i^2}{2} + Y_i x_i \right] \right\} \frac{dx_{m-1}}{\sqrt{2\pi x_i}} \dots \frac{dx_0}{\sqrt{2\pi x_0}} \quad (62.1)$$

We shall perform the integrations one after the other, starting with  $x_1$ , then  $x_2$ , etc. We shall determine the result by a recursive method. We can guess, that after the integrations from  $x_1$  to  $x_{i-1}$  have been performed, the integrand will depend, except for a factor involving other  $x$ 's, quadratically on the variable  $x_i$ , according to the following form;

$$A_i e^{\frac{i}{\pi} (\alpha_i x_i^2 + \beta_i x_i x_{i+1} + \delta_i x_i + \eta_i)} \frac{dx_i}{\sqrt{\frac{2\pi x_i}{m}}} \quad (62.2)$$

where  $A_i, \delta_i, \eta_i$ :

are constants, independent of  $x_1, x_{i+1}$ , etc., which are to be found. The integral on  $x_i$  may now be performed, by writing the exponent as,

$$\frac{i}{\pi} \alpha_i \left[ x_i + \frac{\beta_i x_{i+1} + \delta_i}{2\alpha_i} \right]^2 - \frac{i}{\pi} \frac{(\beta_i x_{i+1} + \delta_i)^2}{4\alpha_i} + \frac{i}{\pi} \eta_i \quad \text{changing the}$$

variable  $x_i$  to  $x_i + \frac{\beta_i x_{i+1} + \delta_i}{2\alpha_i}$  and using  $\int e^{\frac{i}{\pi} y^2} dy = \sqrt{\pi\alpha_i}$ .

We obtain,  $\frac{A_i}{\sqrt{2\pi\alpha_i}} e^{\frac{i}{\pi} \left( \eta_i - \frac{(\beta_i x_{i+1} + \delta_i)^2}{4\alpha_i} \right)}$ . Multiplying this

by the term  $e^{\frac{i}{\pi} \left( \frac{2m x_{i+1}^2}{2\epsilon} - \frac{m x_i x_{i+1}}{\epsilon} - \frac{m\omega^2}{2} x_{i+1}^2 + E x_i x_{i+1} \right)}$  giving

the part of the exponential in (62.1) depending on  $x_{i+1}$ , we find that after the integration on  $x_i$  is performed the dependence of

the integral on  $x_{i+1}$  is,

$$\frac{A_i}{\epsilon \omega \alpha_i} \int_{x_{i+1}/m}^{\infty} \left\{ \frac{2m x_{i+1}}{2\epsilon} - \frac{m x_{i+1} x_{i+2}}{\epsilon} - \frac{m \omega^2}{2} x_{i+1}^2 + Y_{i+1} E x_{i+1} + \eta_i - \frac{\beta_i^2 x_{i+1}}{4d_i} - \frac{\beta_i \delta_i}{2d_i} x_{i+1} - \frac{\delta_i^2}{4d_i} \right\} \frac{dx_{i+1}}{\sqrt{m}}$$

This is again of the form (62.2), so that our guess is self-consistent, if we set,

$$A_{i+1} = \sqrt{\frac{m}{2\epsilon \alpha_i}} \cdot A_i \quad (63.1)$$

$$\delta_{i+1} = E Y_{i+1} - \frac{2\delta_i \beta_i}{4d_i} \quad (63.4)$$

$$\xi_{i+1} = \frac{m}{\epsilon} - \frac{\beta_i^2}{4d_i} - \frac{m \omega^2 \epsilon}{2} \quad (63.2)$$

$$\eta_{i+1} = \eta_i - \frac{\delta_i}{4d_i} \quad (63.5)$$

$$\alpha_{i+1} = -\frac{m}{\epsilon} \quad (63.3)$$

We note, therefore,  $\beta_i$  is a constant  $-\frac{m}{2\epsilon}$ . We shall solve the other equations in the limit  $\epsilon \rightarrow 0$  under the assumption (which will be self-consistent, and is therefore correct) that

$\alpha_i - \frac{m}{2\epsilon}$ ,  $\delta_i$ ,  $\eta_i$ ,  $A_i$  are all finite.

Replacing (63.3) in (63.2) and setting  $\alpha = \frac{m}{2\epsilon} + \lambda$  (63.6),

find,

$$\lambda_{i+1} = \frac{m}{2\epsilon} - \frac{m^2}{4\epsilon(\frac{m}{2\epsilon} + \lambda)} - \frac{m \omega^2 \epsilon}{2}$$

Expanding the fraction  $\frac{1}{1 + \frac{m}{2\epsilon} \lambda}$

by the series  $1 - \frac{m}{2\epsilon} \lambda + \frac{4\epsilon^2}{m} \lambda^2$

and keeping no further terms,

find,  $\lambda_{i+1} - \lambda_i = -\frac{2\epsilon}{m} \lambda_i^2 - \frac{m \omega^2}{2} \epsilon$ . As  $\epsilon \rightarrow 0$ , then,  $\lambda_i$  may be considered as a function of  $t_i$ , so that dividing both sides by  $\epsilon$ , in the limit we may write,  $\frac{d\lambda}{dt} = -\frac{2}{m} \lambda^2 - \frac{m \omega^2}{2} \quad (63.7)$

This has the solution  $\lambda = \frac{m \omega}{2} \operatorname{crt} \omega(t + \text{const.}) \quad (63.8)$  Since, for small  $t$ , (e.g.,  $t = \epsilon$ ),  $\lambda$  is  $\frac{m}{\epsilon}$ ,  $\lambda$  must approach  $\frac{m}{2\epsilon}$ . This it does if the const. is zero in (63.8). Therefore,

$$\alpha = \frac{m}{2\epsilon} + \lambda = \frac{m}{2\epsilon} + \frac{m \omega}{2} \operatorname{crt} \omega T \quad (63.9)$$

place this in (63.1), yielding,  $A_{it1} = \frac{1}{\sqrt{1-\frac{m\omega}{\epsilon} \cot \omega t}} \approx A_i (1 - \frac{m\omega}{\epsilon} \cot \omega t)$  64

for small  $\epsilon$ . In the limit then,  $\frac{dA}{dt} = -A \cdot \frac{\omega}{\epsilon} \cot \omega t$

this implies  $A = \frac{\text{const.}}{\sqrt{\sin \omega t}}$ . Since for  $t$  of the order  $\epsilon$ ,  $A$  is

$\frac{1}{\sqrt{m}}$ , the constant must be  $\sqrt{\frac{m\omega}{\epsilon}}$  so that we find,

$$A = \sqrt{\frac{m\omega}{2\pi i t \sin \omega t}} \quad (64.1)$$

Putting our values of  $\alpha$  and  $\beta$  into (63.4), find,

$\dot{x}_{it1} = \epsilon Y_{it1} + \frac{m\delta_i}{2\epsilon(\frac{m}{\epsilon} + \frac{m\omega \cot \omega t}{\epsilon})}$ , which leads to the differential equation,

$\frac{dx}{dt} = Y - \delta \cdot \omega \cot \omega t$ . This equation has the general solution,

$$\delta = \frac{1}{\sin \omega t} \int_0^t Y(s) \sin \omega s ds + \frac{\text{const}}{\sin \omega t}$$

Since, for small values of  $t$ , of order  $\epsilon$ ,  $\delta$  approaches  $-\frac{m}{\epsilon} x_0$ , the constant here is  $-m\omega x_0$ . Therefore,

$$\delta = -\frac{m\omega x_0}{\sin \omega t} + \frac{1}{\sin \omega t} \int_0^t Y(s) \sin \omega s ds \quad (64.2)$$

Replacing  $\alpha$  by its leading term,  $\frac{m}{2\epsilon}$ , in equation (63.5) we obtain, in the limit, the equation,  $\frac{dy}{dt} = -\frac{s^2}{2m}$

Thus, we find,  $y = -\int \frac{[s(t)]^2}{2m} dt$  (64.3), and is subject to the

condition that as  $t \rightarrow \epsilon$ ,  $y \rightarrow \frac{m}{2\epsilon} x_0^2$ . We may now utilize these results to compute  $G(x_m, x_0; T)$ .

Since it is clear from its mode of formation that,

$$G\left(\frac{m}{2\epsilon} x_m^2 - \frac{m}{\epsilon} x_m x_{m+1}\right) G_{ij}(x_m, x_j; T) = A_m e^{i(\delta_m x_m^2 + \gamma_m x_m x_{m+1} + \eta_{mm})}$$

we may calculate  $G$  in the limit as  $\epsilon \rightarrow 0$ ,  $m\epsilon \rightarrow T$ , from our

expressions (63.1), (63.3), (64.2), (64.3) and (64.1). After a little algebraical rearrangement, it may be written in the rather convenient form (we have replaced  $x_m$  by  $x$ , and  $x_0$  by  $x'$ ),

$$G_r(x, x'; T) = G_0(x-a, x'-b; T) \cdot \exp \frac{i}{2m\omega} \left\{ \int_0^T \int_0^t Y(s) Y(t) \sin \omega t ds dt + m\omega s \sin \omega t \cdot a b \right\} \quad (65.1)$$

where  $a = \frac{1}{m\omega \sin \omega T} \int_0^T Y(t) \cos \omega t dt \quad (65.2)$

and

$$b = \frac{1}{m\omega \sin \omega T} \int_0^T Y(t) \cos \omega(T-t) dt \quad (65.3)$$

and  $G_0(y, y'; T)$  is the value to which  $G_r(y, y'; T)$  reduces when  $\gamma=0$ , and is the well known generating function for the unperturbed harmonic oscillator;

$$G_0(y, y'; T) = \sqrt{\frac{m\omega}{2\pi i k \sin \omega T}} \exp \frac{m\omega i}{2k \sin \omega T} \left\{ (y^2 + (y')^2) \cos \omega T - 2yy' \right\} \quad (65.4)$$

Going back to equation (61.2), we may now express the average in question by the simplified expression,

$$\langle Y_r || Y_0 \rangle_a = \int Y_r(Q_m, x) e^{\frac{i}{\hbar} A_0 [.. g ..]} \cdot G_r(x, x'; T) Y_0(Q_0, x') dx dx'. \frac{\sqrt{g d Q_m} \dots \sqrt{g d Q_0}}{A_m \dots A_1} \quad (65.5)$$

We shall need this formula in the next section. It is the analogue, in a sense, of the solution (21.3,4,5) of the equations of motion for the oscillator, in the classical case. Here, as there, the solution for the motion of the oscillator is expressed in terms of the interacting system, without it actually being necessary to solve for the motion of this interacting system.

### 11. Particles Interacting through an Intermediate Oscillator.

The problem which we discuss in this section is the quantum analogue of the problem discussed in section 4 of the first part of the paper. Given two atoms A and B, each of which interacts with an oscillator O, to what extent can the motion of the oscillator be disregarded and the atoms be considered as interacting directly? This problem has been solved in a special case by Fermi<sup>7</sup> who has shown that the oscillators of the electromagnetic field which represent longitudinal waves could be eliminated from the Hamiltonian, provided an additional term be added representing instantaneous Coulomb interactions between the particles. Our problem is analogous to his except that in the general case, as we can see from the classical analogue, we shall expect that the interactions will not be instantaneous, and hence not expressible in Hamiltonian form.

Drawing on the classical analogue we shall expect that the system with the oscillator is not equivalent to the system without the oscillator for all possible motions of the oscillator, but only for those for which some property (e.g., the initial and final position) of the oscillator is fixed. These properties, in the cases discussed, are not properties of the system at just one time, so we will not expect to find the equivalence simply by specifying the state of the oscillator at a certain time, by means of a particular wave function. It is for this reason that the ordinary methods of quantum mechanics do not suffice to solve this problem.

The natural question to ask, working from analogy with the classical system, is: "What is the expected value of  $\mathcal{F}$ , a

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(7) E. Fermi, Rev. of Mod. Phys. 4, (1932) p/31.

functional involving the particles only, if it is known that at time 0, the position of the oscillator was  $x(0) = \alpha$ , and that at the time T it was  $x(T) = \beta$ ? If we can show that the answer to this question is, with a given fixed  $\alpha$  and  $\beta$ , the same as the expectation of  $\mathcal{F}$  calculated by a formula of exactly the form (56.1) for an action principle involving the particles alone, then we shall have found the conditions under which a direct interaction can be represented as acting through an intermediate oscillator.

That is, we should like to satisfy, (see p. 58 ff.)

$$\text{Tr} \left\langle \mathcal{F} \cdot \delta \left( \frac{x_T + x'_T}{2} - \beta \right) \cdot \delta \left( \frac{x_0 + x'_0}{2} - \alpha \right) \right\rangle_{A, B + a} = \text{const.} \cdot \text{Tr} \langle \mathcal{F} \rangle_{A, B} \quad (67.1)$$

where the trace on the left is computed for the action of the particles and the oscillator, and that on the right only involves the particles. (The constant appears because we are interested only in relative expectations, and can normalize the trace later.)

We shall simplify matters by supposing the particles are represented by a set of coordinates, which we symbolize by Q, that the action for the particles is  $A_0$ , that the oscillator has the coordinate  $x(t)$ , with the Lagrangian  $\frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2$ , and that the Lagrangian of the interaction is  $I(t)x(t)$  where  $I(t)$  is some functional of  $Q(t)$ . Let the action for the particles with the oscillator eliminated be  $A_0 + \int f$ , where  $f$ , a functional of Q only, the action of interaction, is to be found to satisfy (67.1).

If (67.1) is to be satisfied for all arbitrary functionals,  $\mathcal{F}$ , we must have, on account of (56.1),

$$\int \left[ \frac{1}{2} \hbar \frac{i}{\kappa} \left[ Q_0 + \int_0^T T(H) X(t) dt + \int_0^T \frac{1}{2} (x^{(2)} - \omega x^{(1)}) dt \right] \cdot \frac{1}{2} \hbar \frac{i}{\kappa} \left[ Q'_0 + \int_0^T T'(H) X(t) dt + \int_0^T \frac{1}{2} (x'^{(2)} - \omega' x'^{(1)}) dt \right] \right] \cdot S\left(\frac{x_T + x'_T}{2} - \beta\right) S\left(\frac{x_0 + x'_0}{2} - \alpha\right) dx_T dx_{T'} \frac{dx \dots dx}{A \dots A} \frac{dx' \dots dx'}{A' \dots A'} \quad (68.1)$$

where  $Q'$ ,  $\gamma'$ , and  $J'$  are <sup>the same</sup> <sub>but</sub> functionals of a different variable (namely,  $Q'$ ) than  $Q$ ,  $\gamma$ , and  $J$ , and the integrations over the  $x$ 's are to be performed as prescribed in detail in (56.1). We can divide out the  $e^{\frac{i}{\kappa}(Q_0 - Q'_0)}$  from each side, of course. Our main problem is to show that the left side, when integrated over all the coordinates of the oscillator, can be expressed as a product of an exponential function of  $Q'$  only with an exponential of minus the same function of  $Q$  only. This will be by no means generally true, and is a consequence of our special choice of  $\delta$  functions on the left of (68.1).

Since as far as the variables  $x$  are concerned, the action is Lagrangian, the integration over all  $x$  and  $x'$  from  $x_{T_2}$  to  $x_T$  can be immediately performed in the way indicated at the end of section 9 (see page 59). In a like manner the integrations from  $x_{T_1}$  to  $x_0$  may be performed. The integrations on the  $x$ -primes from  $x'_{0+\epsilon}$  to  $x'_{T-\epsilon}$  can next be performed according to the method of the last section, and they yeild  $G_{\gamma'}^*(x_T, x_0; T)$ . Those on the corresponding intermediate  $x$ 's result in  $G_{\gamma'}(x_T, x_0; T)$ . Thus we must prove that,

$$\int \int S(x_T - \beta) S(x_0 - \alpha) G_{\gamma'}^*(x_T, x_0; T) G_{\gamma'}(x_T, x_0; T) dx_T dx_0 = \text{const. } e^{-\frac{i}{\kappa} \int \int \frac{i}{\kappa} f' f} \quad (68.2)$$

The left side is simply  $G_{\gamma'}(\beta, \alpha; T) G_{\gamma'}(\beta, \alpha; T)$ , and by substituting  $\beta, \alpha$  in the formula (65.14) we can find that this left

side is of the form of the right side, if we take const. =  $\frac{mw}{\sin \omega T}$   
and  $J$  equal to

$$J = \frac{mw \cot \omega T}{2} [(\beta - b)^2 + (\alpha - a)^2] - \frac{mw}{\sin \omega T} (\beta - b)(\alpha - a) + \frac{1}{\sin \omega} \int \int_{0}^T Y(t) Y(s) \sin \omega (t-s) dt ds + \frac{mw \sin \omega T}{2} ab.$$

with  $a$  and  $b$  as in (6.5.2).  $J'$  is then the corresponding thing  
with  $Y'$  replacing  $Y$  everywhere. After a little rearranging, we get,

$$\begin{aligned} J &= \int_0^T \frac{\alpha \sin \omega(T-t) + \beta \sin \omega t}{\sin \omega T} i(t) dt - \frac{1}{mw \sin \omega T} \int_0^T \int_0^t \sin \omega(T-t) \sin \omega s Y(s) Y(t) ds dt \\ &\quad + \frac{mw \cot \omega T}{2} [\beta^2 + \alpha^2] - \frac{mw}{\sin \omega T} \beta \alpha. \end{aligned} \quad (6.9.1)$$

This is identical to the expression (2.12) obtained for the action  
in the corresponding classical problem in which the initial value  
of  $x$  is held at  $x(0)=\alpha$ , and the final value at  $x(T)=\beta$ . The  
added constant term has, of course, no significance. (It will cancel  
a corresponding term in  $J'$ ).

Thus, the particles with the action of interaction  $J$ ,  
may be replaced by a system with an intermediate oscillator,  
provided that, in calculating the expectation of any functional  
of the particles, it is calculated under the conditions that the  
oscillator's initial position is known to be  $\alpha$  and its final  
position is known to be  $\beta$ . It is to be noted that we have  
not proved that, in general, the system with the oscillator is  
equivalent to one without, for that is not true. The equivalence  
only holds if the oscillator is known to satisfy certain conditions.

For the other example, (2.5), of conditions leading to  
an action principle, which we have considered in the classical  
case, we are led, here, to ask the question: "What is the expected  
value of  $J$  if it is known that for the oscillator,

$$\frac{1}{2} [x(0) + x(T) \cos \omega T - \dot{x}(0) \frac{\sin \omega T}{\omega}] = R_0$$

and that,  $\frac{1}{2} [x(T) + x(0) \cos \omega T + \dot{x}(0) \frac{\sin \omega T}{\omega}] = R_T$  ?

To answer this question we must try to satisfy an equation analogous to (68.1) but with  $\delta(\frac{x_r+x_r'}{2}-\beta) \cdot \delta(\frac{x_e+x_e'}{2}-\alpha)$  replaced by,

$$\delta\left(\frac{1}{4}\left\{x_0+x_0' + (x_r+x_r') \cos \omega T - \frac{(x_{re}+x_{re}')}{\epsilon \omega} \sin \omega T\right\} - R_0\right).$$

$$\cdot \delta\left(\frac{1}{4}\left\{x_r+x_r' + (x_e+x_e') \cos \omega T + \frac{(x_{re}+x_{re}')}{\epsilon \omega} \sin \omega T\right\} - R_T\right). \quad (70.1)$$

In this case the integrations on  $x$  and  $x'$  can only be performed from  $x_{T_2}$  to  $x_{T+\epsilon}$ , leaving an extra Lagrangian factor, and those from  $x_{T_1}$  can proceed only to  $x_{-\epsilon}$ . The integrations between  $x_0$  and  $x_T$  can be performed as before, giving G functions. The result is that we must find an  $\mathcal{J}$  to satisfy:

$$\int \left\{ dx_{re} \cdot e^{-\frac{i\epsilon}{\lambda} \left[ \frac{m}{\epsilon} \left( \frac{x_{re}-x_r}{\epsilon} \right)^2 - \frac{m\omega^2}{\epsilon} x_{re}^2 \right]} \right. \cdot \left. \frac{dx_r}{\sqrt{\frac{2\pi m\epsilon}{\lambda}}} \cdot G_r(x_r, x_e; T) \cdot \frac{dx_e}{\sqrt{\frac{2\pi m\epsilon}{\lambda}}} \cdot e^{-\frac{i\epsilon}{\lambda} \left[ \frac{m}{\epsilon} \left( \frac{x_e-x_r}{\epsilon} \right)^2 - \frac{m\omega^2}{\epsilon} x_e^2 \right]} \right\} \cdot$$

$$\cdot \delta\left(\frac{1}{4}\left\{x_0+x_0' + (x_r+x_r') \cos \omega T - \frac{2x_{re}-(x_r+x_e)}{\epsilon \omega} \sin \omega T\right\} - R_0\right) \cdot \delta\left(\frac{1}{4}\left\{x_r+x_r' + (x_e+x_e') \cos \omega T + \frac{(x_r+x_e)-2x_e}{\epsilon \omega} \sin \omega T\right\} - R_T\right).$$

$$\cdot e^{-\frac{i\epsilon}{\lambda} \left[ \frac{m}{\epsilon} \left( \frac{x_{re}-x_r}{\epsilon} \right)^2 - \frac{m\omega^2}{\epsilon} x_{re}^2 \right]} \cdot \frac{dx_r}{\sqrt{\frac{2\pi m\epsilon}{\lambda}}} \cdot G_r(x_r, x_e; T) \cdot \frac{dx_e}{\sqrt{\frac{2\pi m\epsilon}{\lambda}}} \cdot e^{-\frac{i\epsilon}{\lambda} \left[ \frac{m}{\epsilon} \left( \frac{x_e-x_r}{\epsilon} \right)^2 - \frac{m\omega^2}{\epsilon} x_e^2 \right]} \cdot dx_{-e} \} \\ = \text{const. } e^{-\frac{i\epsilon}{\lambda} \mathcal{J}' \frac{i\epsilon}{\lambda} \mathcal{J}}. \quad (70.2)$$

Since the integration of this complicated expression is perfectly straightforward, we shall not include it here. (It is best to integrate with respect to  $dx_{re}$  and  $dx_e$  first. The term  $e^{+\frac{i\epsilon}{\lambda} \frac{m}{\epsilon} (x_e'/\epsilon)^2 - \frac{i\epsilon}{\lambda} \frac{m\omega^2}{\epsilon} x_e'^2}$  can be neglected because it is of no importance in the limit as  $\epsilon \rightarrow 0$ .) The result is that the left side may be made equal to the right side by choosing the constant as  $\frac{2\pi \epsilon m\omega}{\sin \omega T}$

and taking  $\mathcal{J}$  to be,

$$\mathcal{J} = \int_0^T \frac{R_0 \sin \omega(t-t) + R_1 \sin \omega t}{\sin \omega T} Y(t) dt + \frac{1}{2m\omega} \int_0^T \int_0^t \sin \omega(t-s) Y(t) Y(s) dt ds, \quad (71.1)$$

with a similar expression for  $\mathcal{J}'$ , obtained by replacing  $Y$  by  $Y'$ . This is again in agreement with the classical result.

To answer the question: "What is the expectation of if it is known that initially the position of the oscillator is  $w$ , and its velocity is  $v$ ?", we must try to satisfy an equation analogous to (68.1), but with  $\delta\left(\frac{x_0+x_0'}{z} - \beta\right) \cdot \delta\left(\frac{x_0+x_0'}{z} - \alpha\right)$  replaced by

$$\delta\left(\frac{x_0+x_0'}{z} - w\right) \cdot \delta\left(\frac{x_0+x_0'}{z} - \frac{x_{-c}+x_c'}{z} - v\right). \quad \text{If this is done, however, and the integrations are carried out, the left side of (68.1) becomes,}$$

$$\frac{m}{2\pi\hbar} \exp\left(\frac{i}{\hbar m\omega} \left\{ \int_0^T (Y(t) - Y'(t)) (mV \sin \omega t + \omega W \cos \omega t) dt + \int_0^T \int_0^t (Y(t) - Y'(t)) (Y(s) + Y'(s)) \sin \omega(t-s) ds \right\} \right). \quad (71.2)$$

It is seen that (68.1) cannot be satisfied by any choice of  $\mathcal{J}$ , there now appear in the exponential (71.2) cross terms between  $Y$  and  $Y'$ , such as  $Y(t)Y'(s)$ . This corresponds to the classical finding, that no action exists in case the initial position and velocity are held constant.

These results serve as a confirmation of our formal generalization to systems without a Hamiltonian. They have obvious application to electrodynamics into which we will, however, not go here.

We should like to make a remark, before closing this section, about equation (71.2). Even though it does not lead to a system which can be expressed by a quantum mechanical least action principle, it is nevertheless correct, of course, that to find the average of the functional  $\mathcal{J}$  we must multiply  $\mathcal{J}$  by the

expression (71.2) times  $\exp \frac{i}{\hbar} (Q - Q')$  and integrate over all the  $Q$  and  $Q'$ . That is to say, the expected value of  $\mathcal{J}$  for this system is obtained in a way analogous to (56.1), except that the phase of the exponential, which in (56.1) is of the form  $\frac{i}{\hbar} \{ a(Q) - a(Q') \}$ , is now of the form  $\frac{i}{\hbar} B(Q, Q')$ , where the quantity  $B$  involves the expression in (71.2).

What should be the behaviour of the system, described in this way, in the classical limit as  $\hbar \rightarrow 0$ ? Dirac's argument (see page 29) in this case leads to the conclusion that only those values of  $Q$ , and of  $Q'$ , will be of importance which satisfy both,

$$\frac{\partial B(Q, Q')}{\partial Q(t)} = 0 \quad \text{and} \quad \frac{\partial B(Q, Q')}{\partial Q'(t)} = 0$$

Inasmuch as  $B(Q, Q') = -B(Q', Q)$ , the second of these equations results from the first by interchange of the  $Q$  and  $Q'$ . Therefore one solution of these equations would have the property that  $Q(t) = Q'(t)$  where  $Q(t)$  satisfies,

$$\left. \frac{\partial B(Q, Q')}{\partial Q(t)} \right|_{Q'(t)=Q(t)} = 0 \quad (72.1)$$

With  $B$  as given by (71.2), this leads immediately to the classical equation got by substituting for  $x(t)$  from (21.3) into (21.1), and replacing  $x(0)$  by  $w$ , and  $\dot{x}(0)$  by  $v$ . This suggests a way of quantizing systems which classically do not satisfy a simple principle of least action, but we shall not investigate this here.

## 12. Conclusion.

We have presented, in the foregoing pages, a generalization of quantum mechanics applicable to a system whose classical analogue is described by a principle of least action. It is important to emphasize, however, some of the difficulties and limitations of the description presented here.

One of the most important limitations has already been discussed. The interpretation of the formulas from the physical point of view is rather unsatisfactory. The interpretation in terms of the concept of transition probability requires our altering the mechanical system, and our speaking of states of the system at times very far from the present. The interpretation in terms of expectations, which avoids this difficulty, is incomplete, since the criterion that a functional represent a real physical observable is lacking. It is possible that an analysis of the theory of measurements is required here. A concept such as the "reduction of the wave packet" is not directly applicable, for in the mathematics we must describe the system for all times, and if a measurement is going to be made in the interval of interest, this fact must be put somehow into the equations from the start. Summarizing: a physical interpretation should be sought which does not refer to the behaviour of the system at times very far distant from a present time of interest.

A point of vagueness is the normalization factor,  $A$ . No rule has been given to determine it for a given action expression. This question is related to the difficult mathematical question as to the conditions under which the limiting process of subdividing the time scale, required by relations such as (45.1), actually converges.

The problem of the form that relativistic quantum mechanics, and the Dirac equation, take from this point of view remains unsolved. Attempts to substitute, for the action, the classical relativistic form (integral of proper time) have met with difficulties associated with the fact that the square root involved becomes imaginary for certain values of the coordinates over which the action is integrated.

The final test of any physical theory lies, of course, in experiment. No comparison to experiment has been made in the paper. The author hopes to apply these methods to quantum electrodynamics. It is only out of some such direct application that an experimental comparison can be made.

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**ABSTRACT:**

**THE PRINCIPLE OF LEAST ACTION IN QUANTUM MECHANICS.**

A generalization of quantum mechanics is given in which the central mathematical concept is the analogue of the action in classical mechanics. It is therefore applicable to mechanical systems whose equations of motion cannot be put into Hamiltonian form. It is only required that some form of least action principle be available.

It is shown that if the action is the time integral of a function of velocity and position (that is, if a Lagrangian exists), the generalization reduces to the usual form of quantum mechanics. In the classical limit, the quantum equations go over into the corresponding classical ones, with the same action function.

As a special problem, because of its application to electrodynamics, and because the results serve as a confirmation of the proposed generalization, the interaction of two systems through the agency of an intermediate harmonic oscillator is discussed in detail. It is shown that in quantum mechanics, just as in classical mechanics, under certain circumstances the oscillator can be completely eliminated, its place being taken by a direct, but, in general, not instantaneous, interaction between the two systems.

The work is non-relativistic throughout.

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May 4, 1942.  
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