Geometry of SpaceTime - Einstein Theory of Gravity

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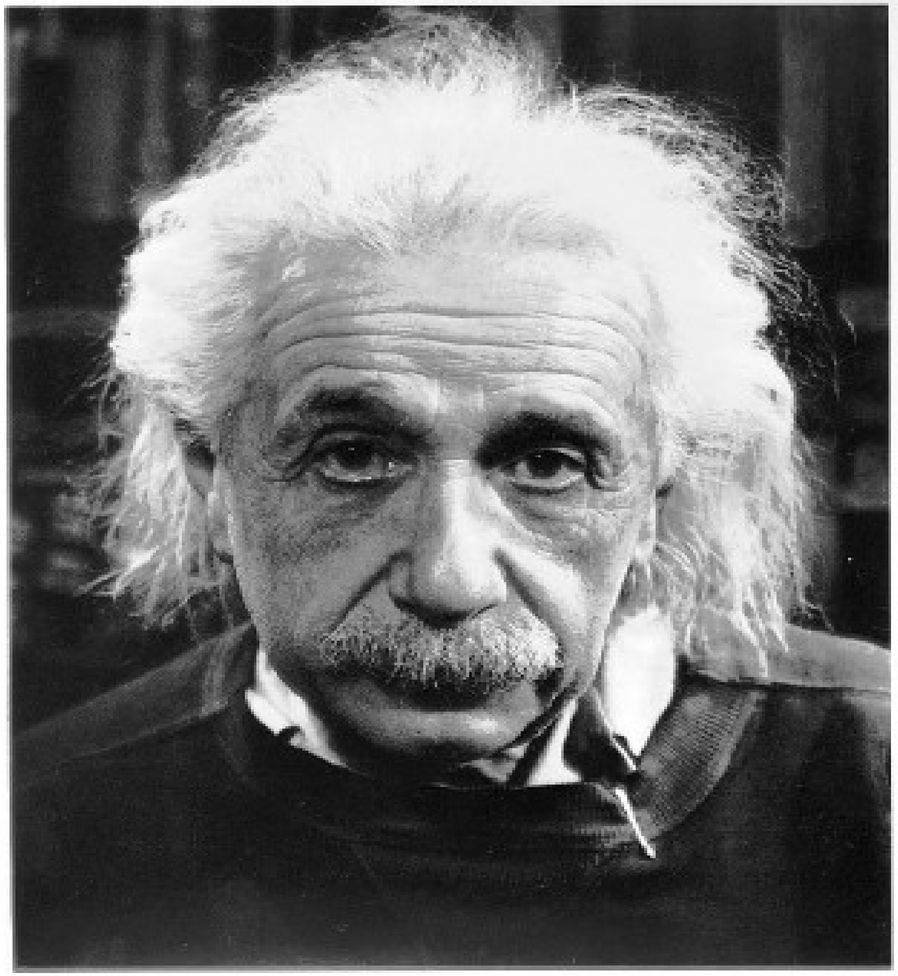
HD-SS2009-D2/2



"Things should be made as simple as possible,
but not any simpler."
- Albert Einstein

From SR to GR

- Lorentz transformations
- Effects in SR
- Minkowski spacetime
- Causal structure of SR
- Maxwell‘s theory in SR
- Hydrodynamics in SR
- How to incorporate gravity?
  ➔ Equivalence Principles
- SpaceTime as manifold of events with Levi-Civita connection

A Brief Review of Special Relativity

- **Special relativity (SR)** is the physical theory of measurement in inertial frames of reference proposed in 1905 by Albert Einstein (after considerable contributions of Hendrik Lorentz and Henri Poincaré) in the paper "On the Electrodynamics of Moving Bodies".
- It generalizes **Galileo's principle of relativity** – that all uniform motion is relative, and that there is no absolute and well-defined state of rest (no privileged reference frames) – from mechanics to all the laws of physics.
- In addition, special relativity incorporates the principle that the **speed of light** is the same for all inertial observers regardless of the state of motion of the source.

2 Simple Postulates (1905)

- “The laws of physics are the same in every inertial frame of reference”
  - *The Principle of Relativity*

- “The speed of light in vacuum is the same in all inertial frames of reference, and is independent of the motion of the source”
  - *Invariance of the speed of light*

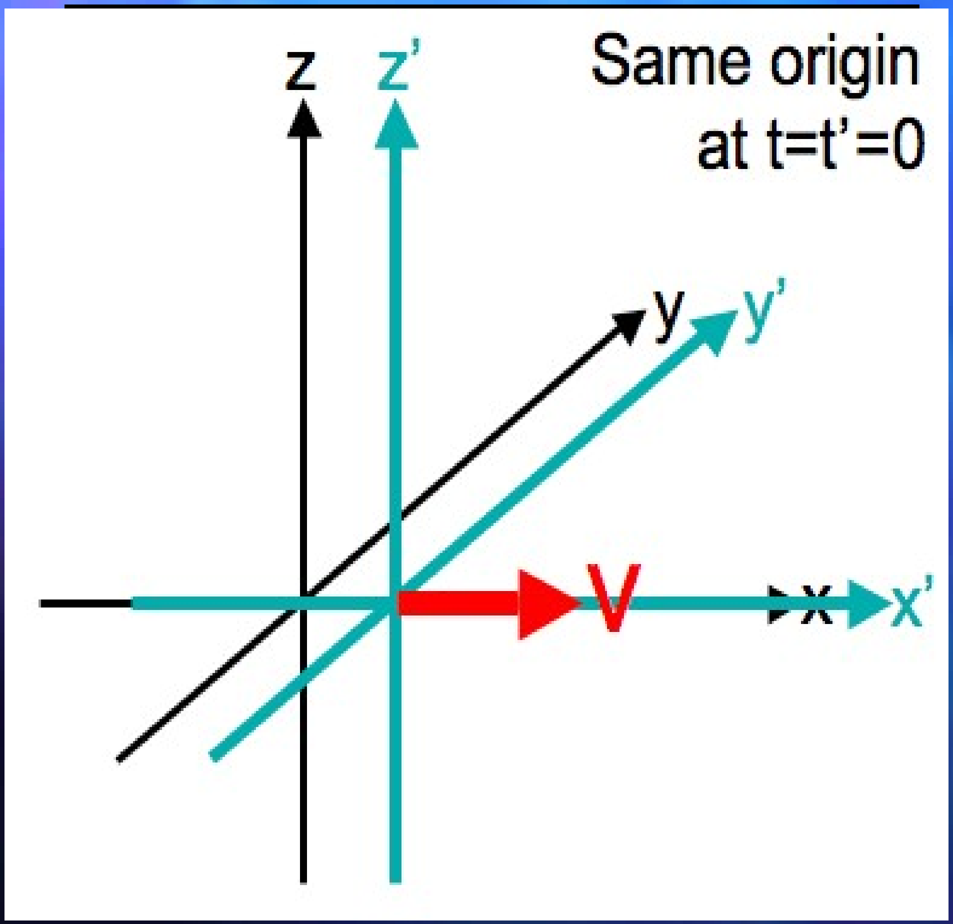
The Principle of Relativity

- This is a sweeping generalization of the principle of Galilean relativity, which refers only to the laws of <u>Newtonian</u> mechanics.
- The results of *any kind* of experiment performed in a laboratory at rest must be the same as when performed in a laboratory moving at a constant speed past the first one.
- No preferred inertial reference frame exists.
- It is impossible to detect absolute motion.

Consequences of Special Relativity

- Restricting the discussion to concepts of length, time, and simultaneity.
- In relativistic mechanics:
  - **There is no such thing as absolute length.**
  - **There is no such thing as absolute time.**
  - **Events at different locations that are observed to occur simultaneously in one frame are not observed to be simultaneous in another frame moving uniformly past the first.**

Basic Principles of Special Relativity



$$\begin{cases} t' = \gamma\left(t - vx\right) \\ x' = \gamma\left(x - vt\right) \\ y' = y \\ z' = z \end{cases}$$

Lorentz Transformations

- Vector notation for events ($\mu, \nu = 0,..,3$)

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

- Useful Four-Vectors

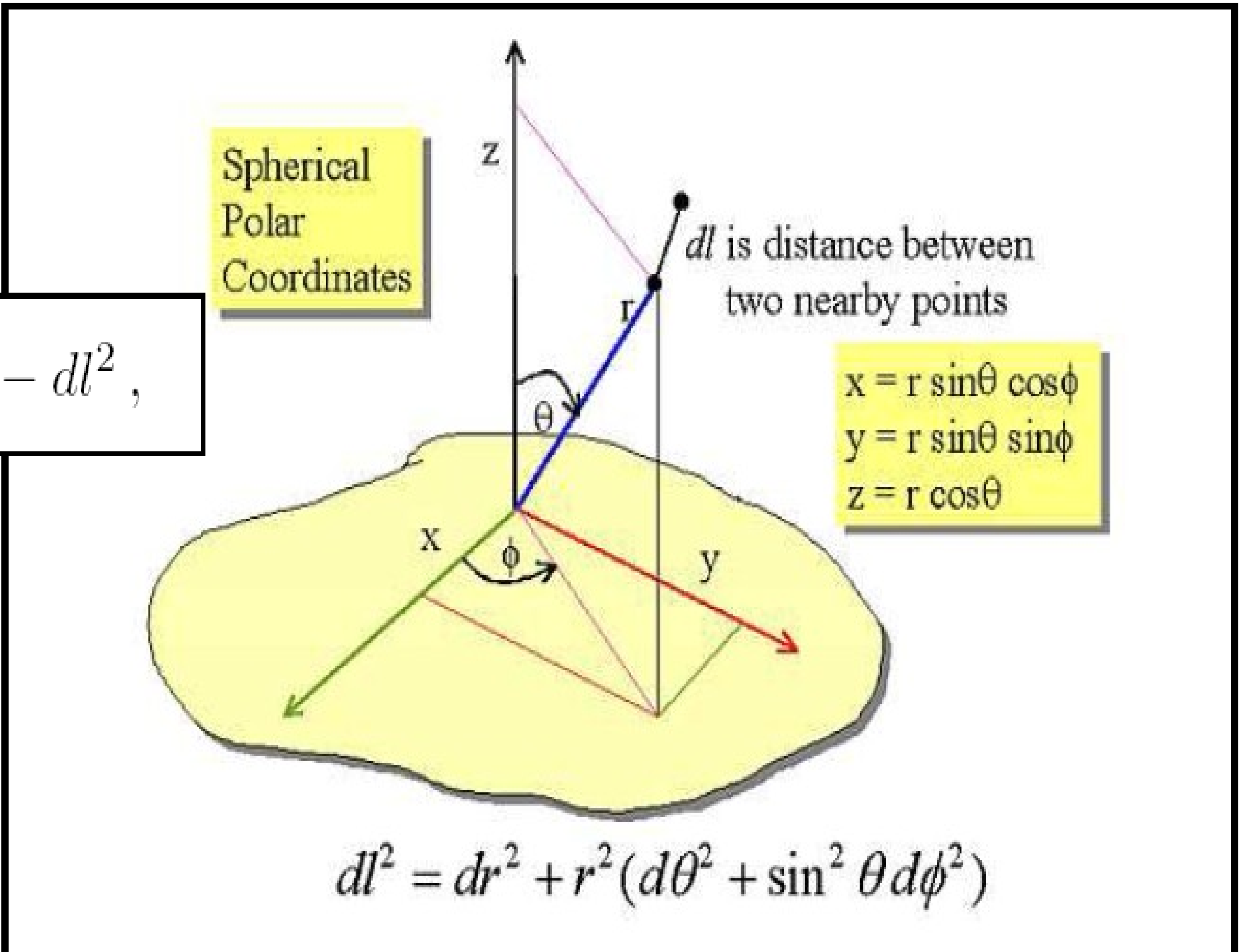
$$x^{\mu} = \begin{pmatrix} t \\ \mathbf{r} \end{pmatrix}$$

$$U^{\mu} = \frac{dx^{\mu}}{d\tau} = \begin{pmatrix} \gamma_{u} \\ \gamma_{u}\mathbf{u} \end{pmatrix} \qquad p^{\mu} = m\begin{pmatrix} \gamma_{u} \\ \gamma_{u}\mathbf{u} \end{pmatrix}$$

$$k^{\mu} = \begin{pmatrix} \omega \\ \mathbf{k} \end{pmatrix} \qquad p^{\mu} = \begin{pmatrix} \hbar\omega \\ \hbar\mathbf{k} \end{pmatrix}$$

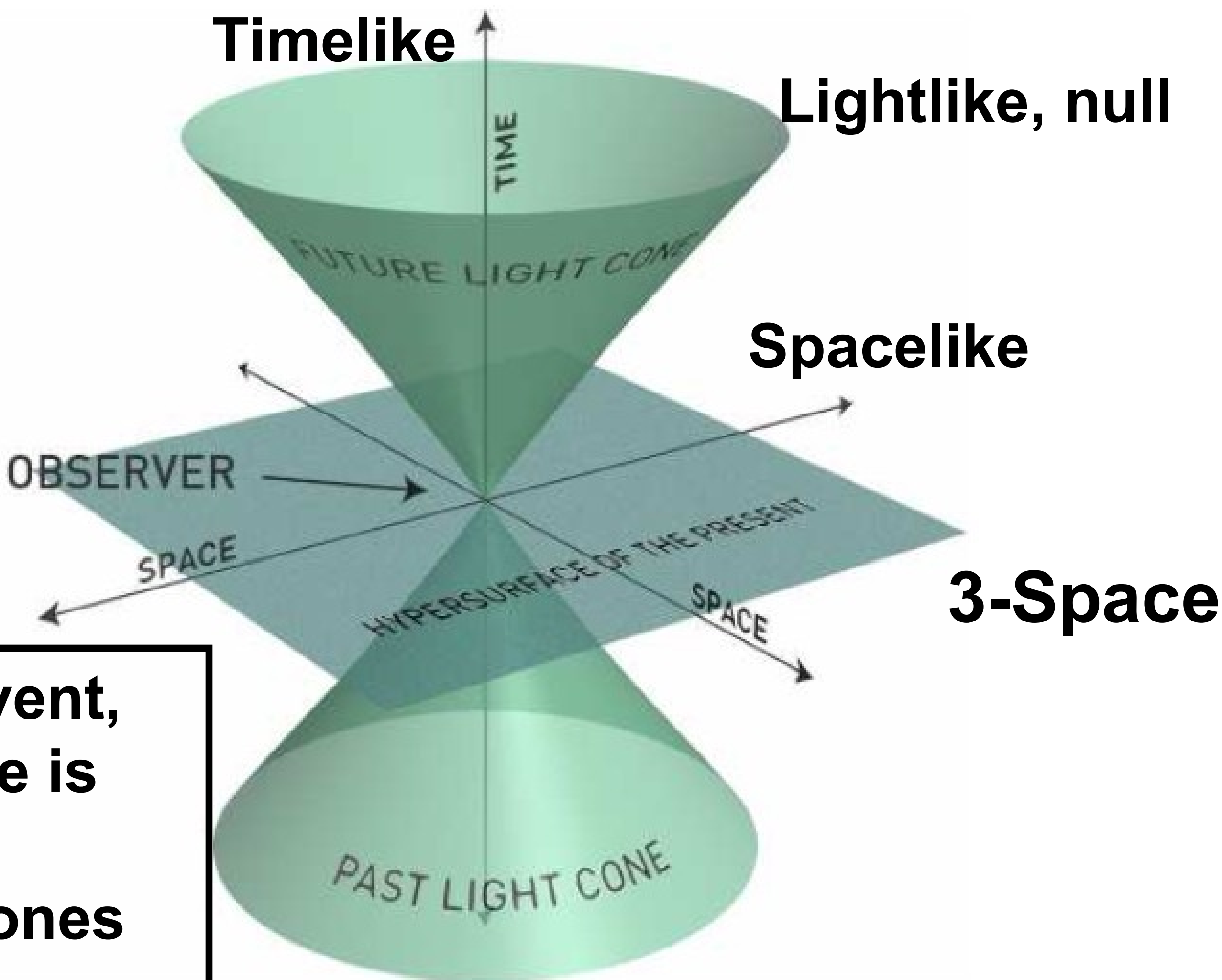
Minkowski Line Element

$$ds^2 = c^2\, dt^2 - dl^2\,,$$



Event:
(ct, x)
➔
Spacetime
= set of all events

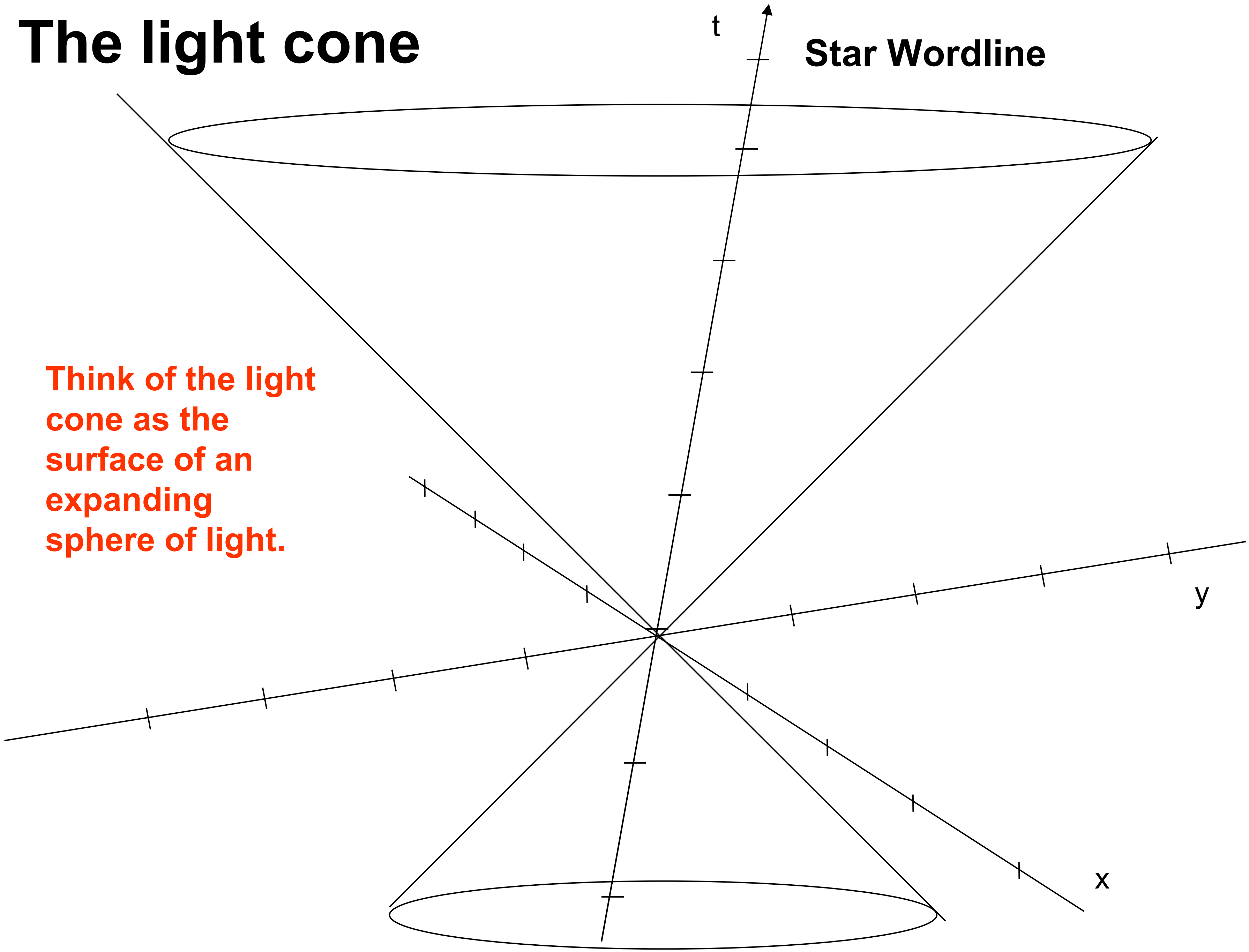
Causal Structure of SpaceTime



At each event, a lightcone is defined.

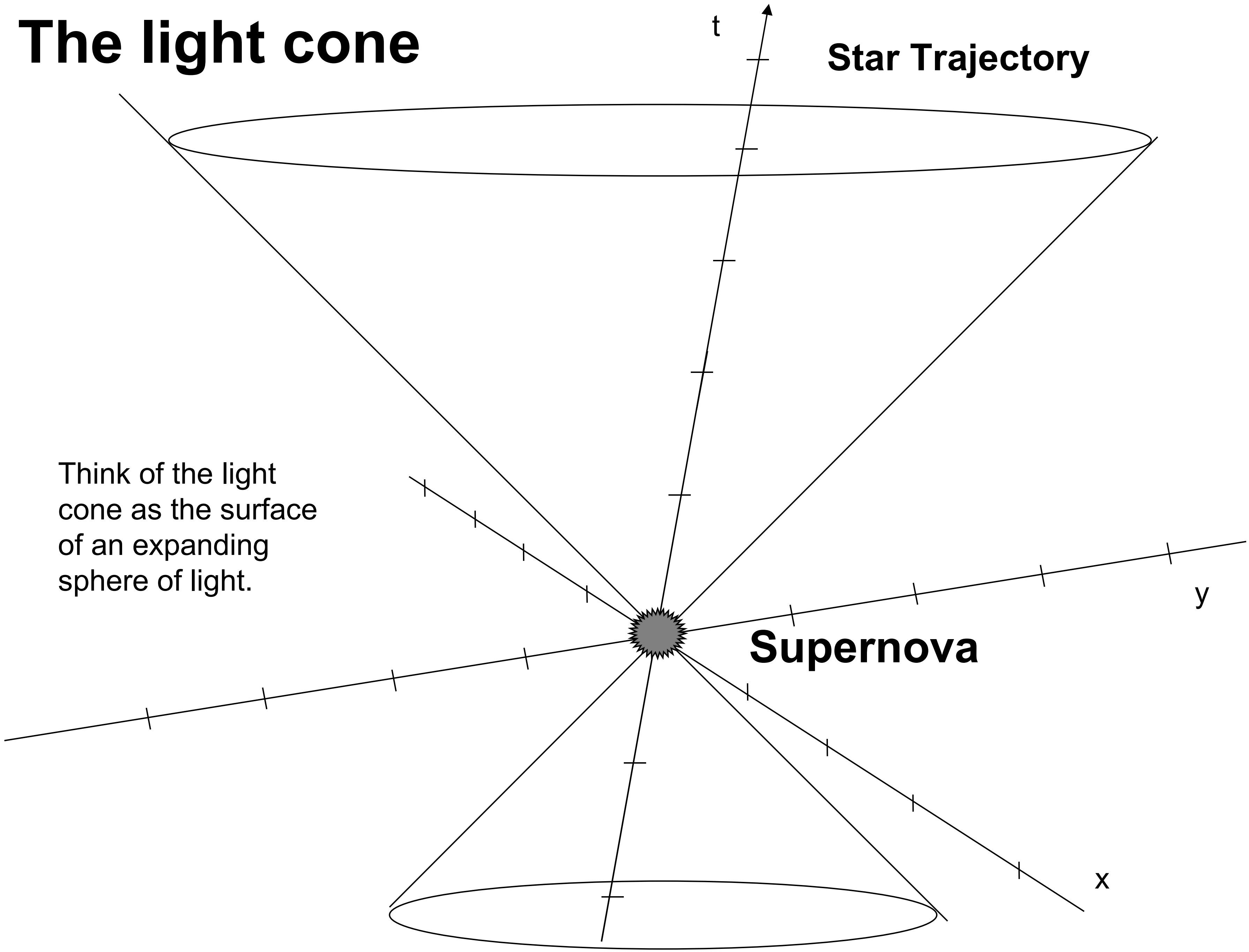
➔Light Cones are not curved.

The light cone



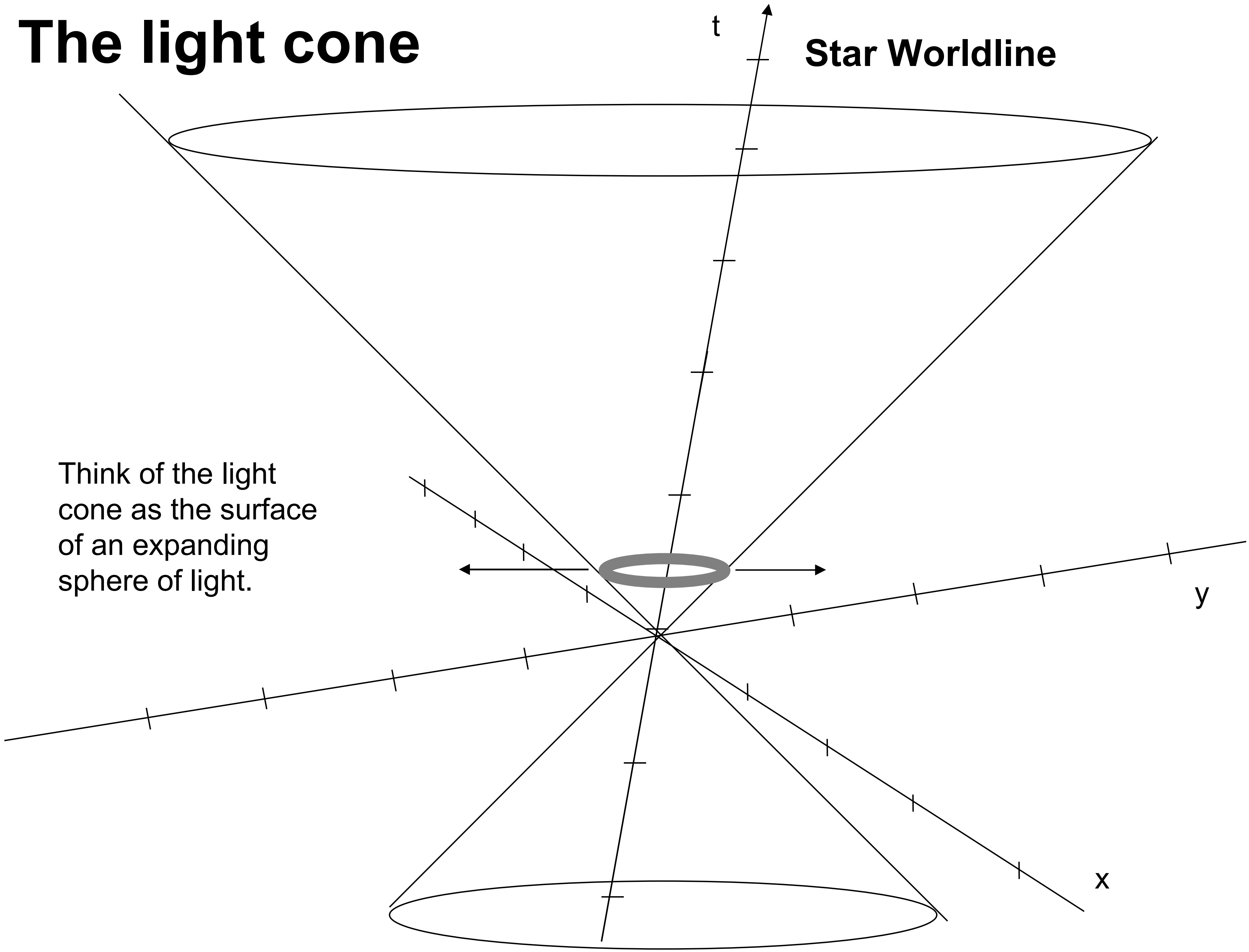
Think of the light cone as the surface of an expanding sphere of light.

The light cone



Think of the light cone as the surface of an expanding sphere of light.

The light cone



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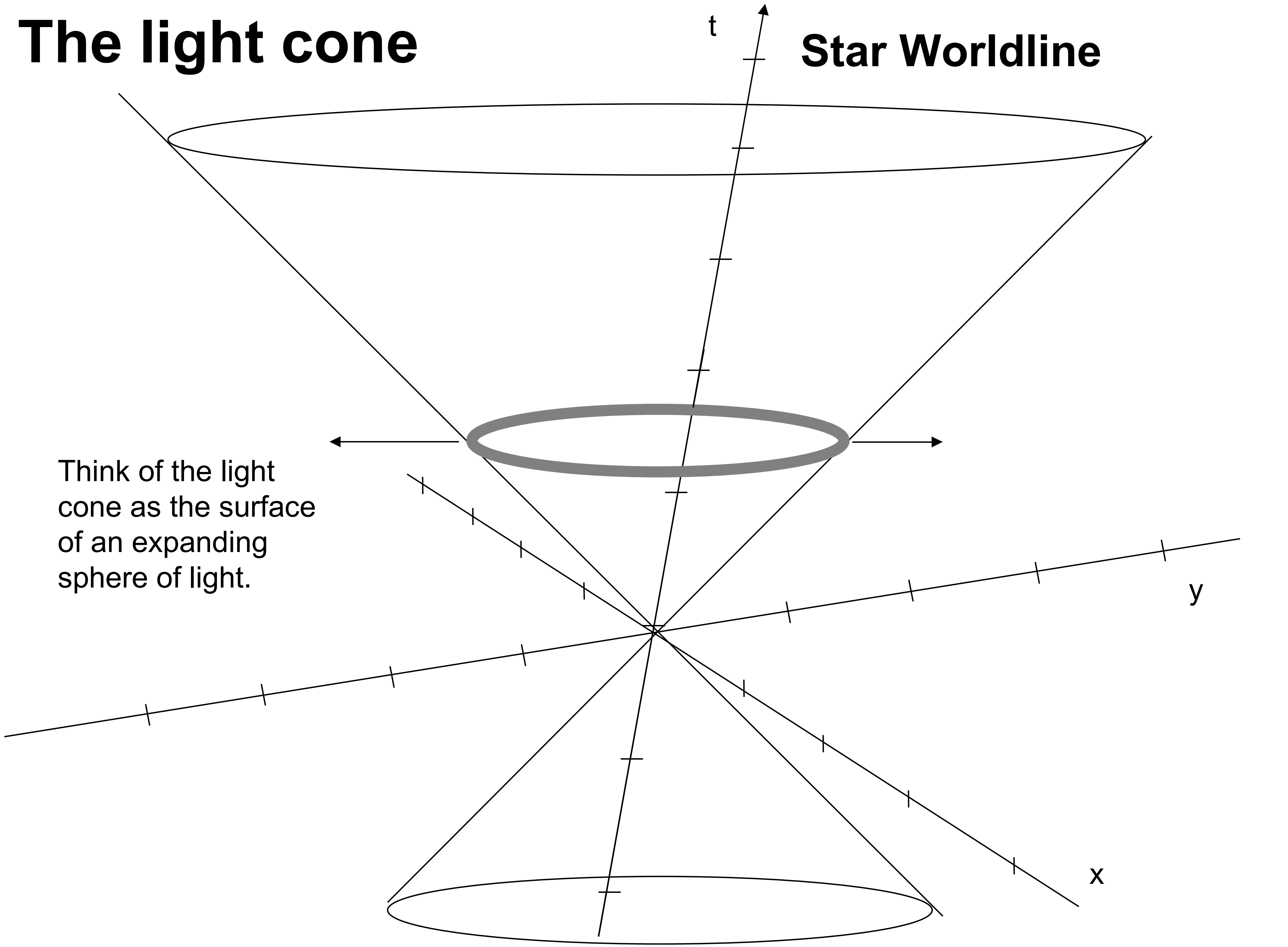

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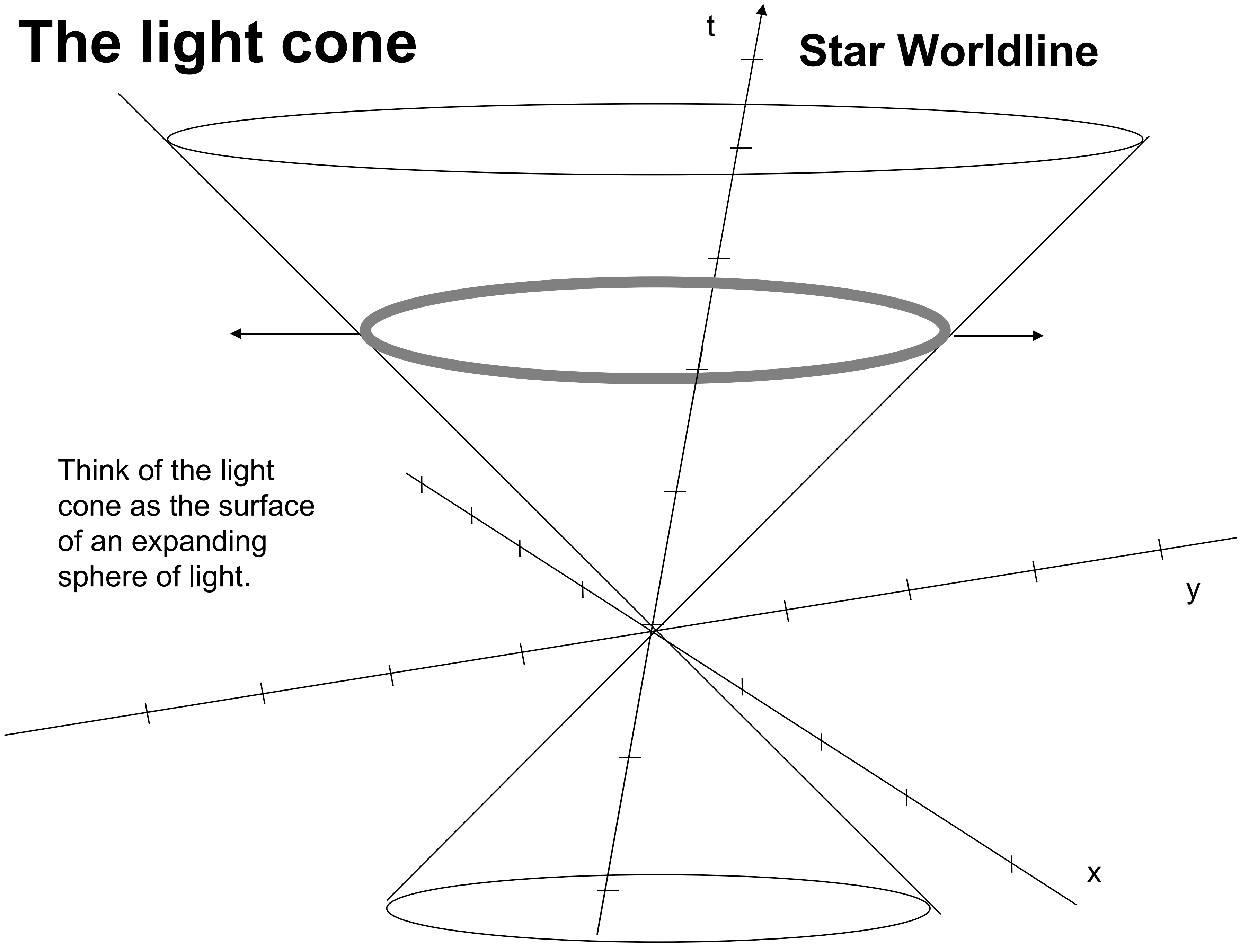
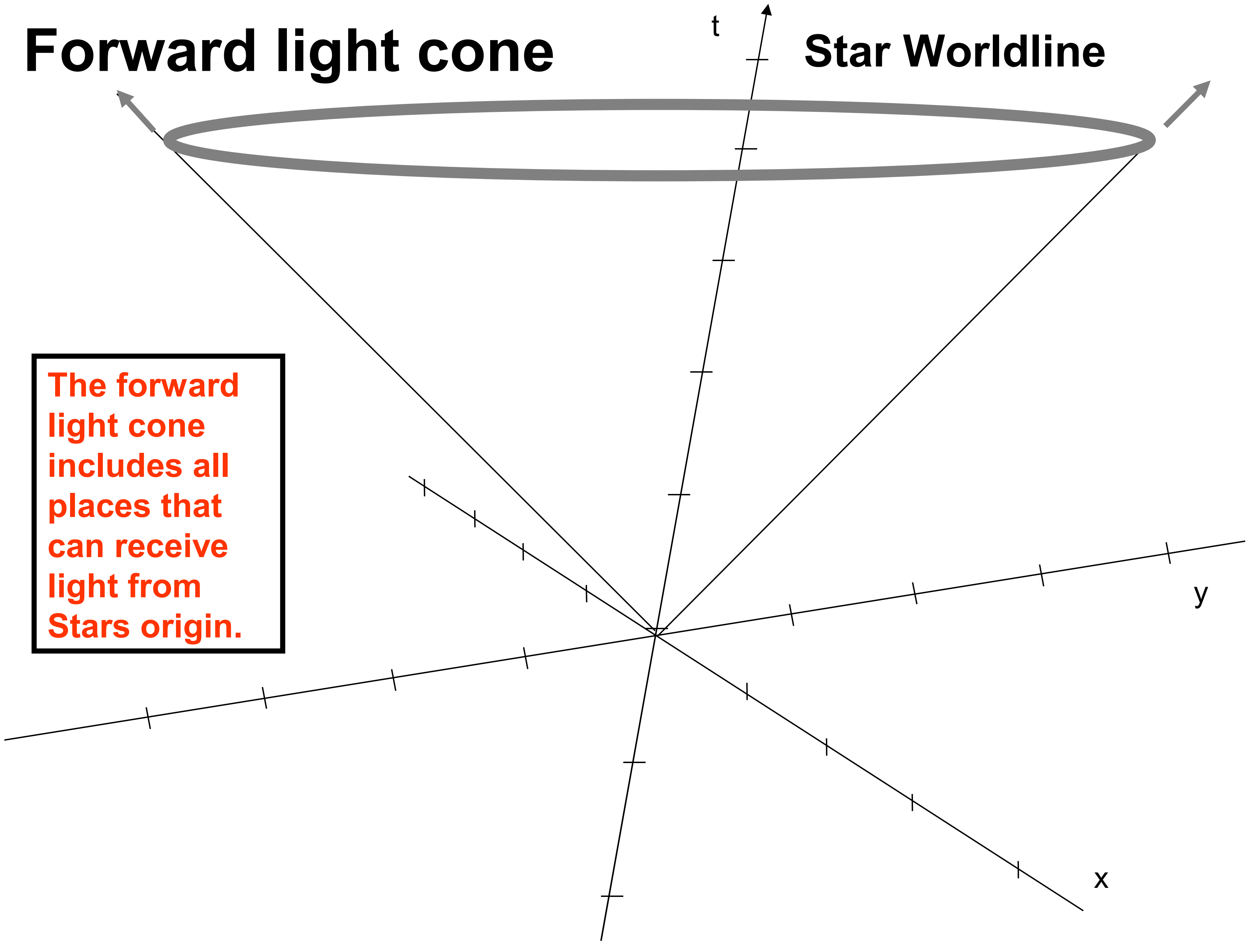
The light cone



Think of the light cone as the surface of an expanding sphere of light.

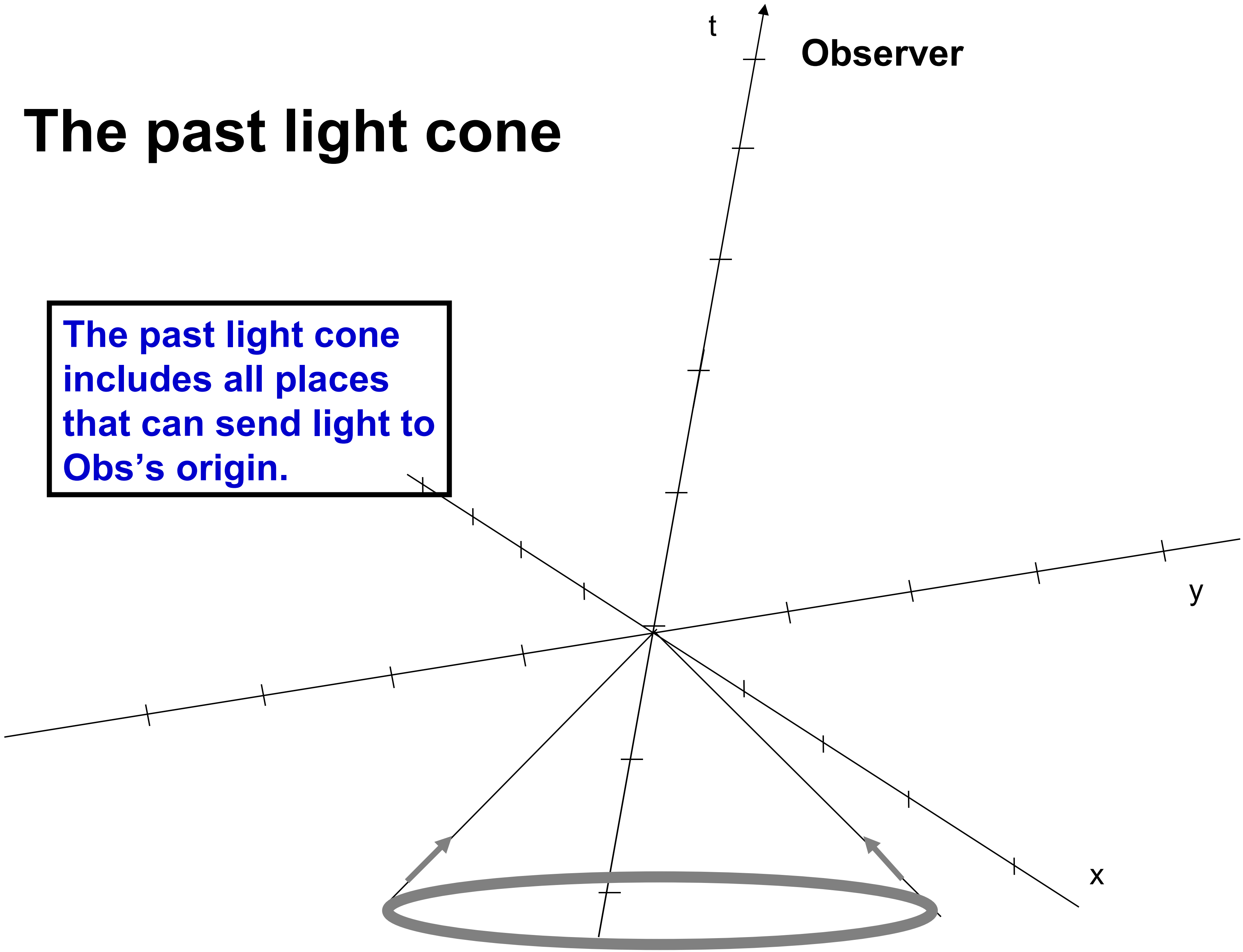
Forward light cone



The forward light cone includes all places that can receive light from Stars origin.

The past light cone

The past light cone includes all places that can send light to Obs's origin.



Invariance of Minkowski Metric

$$ds^2 = (dx)^T\, \eta\, (dx) = (dx')^T\, \eta\, (dx')$$

$$\boxed{\eta = \Lambda^T\, \eta\, \Lambda\,, \quad (dx') = \Lambda\, (dx)\,,\ (dx) = (dx^0, dx^i)^T}$$

The matrices which satisfy (2.13) are known as the **Lorentz transformations**; the set of them forms a group under matrix multiplication, known as the **Lorentz group**. There is a close analogy between this group and $O(3)$, the rotation group in three-dimensional space. The rotation group can be thought of as 3×3 matrices R which satisfy

$$\mathbf{I} = R^T \cdot \mathbf{I} \cdot R\,, \tag{2.14}$$

The Lorentz Group

$$\det(\Lambda^T) \cdot \det(\eta) \cdot \det(\Lambda) = \det(\eta)\,.$$

Since $\det(\eta) = -1$ and $\det(\Lambda^T) = \det(\Lambda)$, it follows

$$\boxed{\det{}^2(\Lambda) = +1 \quad , \quad \det(\Lambda) = \pm 1\,.}$$

If the 00–element of (2.13) is written out, it gives

$$(\Lambda^T \cdot \eta \cdot \Lambda)^0_0 = \eta_{00} = 1\,.$$

This can be written as

$\Lambda_0{}^0 > 1$, < -1

$$(\Lambda^0{}_0)^2 = 1 + (\Lambda^0{}_1)^2 + (\Lambda^0{}_2)^2 + (\Lambda^0{}_3)^2\,.$$

LT as Pseudo-Rotations

$$\begin{aligned} ct' &= ct \cosh\Phi - x \sinh\Phi \\ x' &= -ct \sinh\Phi + x \cosh\Phi\,. \end{aligned}$$

From this we see that the point defined by $x' = 0$ is moving with a velocity

$$\beta \equiv \frac{v}{c} = \frac{x}{ct} = \frac{\sinh\Phi}{\cosh\Phi} = \tanh\Phi\,.$$

To translate into more pedestrian notation, we can replace $\Phi = \tanh^{-1}(v/c)$

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\beta^2}} = \cosh\Phi \\ \gamma\beta &= \sinh\Phi \end{aligned}$$

Velocity Addition

Since Lorentz transformations form a group, performing two Lorentz transformations in the x–direction produces also a Lorentz tranformation. Using pseudo–rotations we find

$$\begin{aligned} ct' &= ct\cosh\Phi_1 - x\sinh\Phi_1 && (2.38)\\ x' &= -ct\sinh\Phi_1 + x\cosh\Phi_1 && (2.39)\\ ct'' &= ct'\cosh\Phi_2 - x'\sinh\Phi_2 = ct\cosh\Phi - x\sinh\Phi && (2.40)\\ x'' &= -ct'\sinh\Phi_2 + x'\cosh\Phi_2 = -ct\sinh\Phi + x\,cosh\Phi\,, && (2.41)\end{aligned}$$

with $\Phi = \Phi_1 + \Phi_2$. With $\tanh\Phi = v/c$ and the well–known theorem for the hyperbolic tangent

$$\tanh(\Phi_1 + \Phi_2) = \frac{\tanh\Phi_1 + \tanh\Phi_2}{1 + \tanh\Phi_1\tanh\Phi_2} \tag{2.42}$$

we obtain for the combined velocity

$$\boxed{v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}\,.} \tag{2.43}$$

Lorentz Tensors

- Tensors transform covariantly under LTs

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta}$$

$$T^{\alpha\beta}{}_{\nu} \Leftrightarrow A^{\alpha}B^{\beta}C_{\nu}$$

$$A_{,\mu} = \frac{\partial A}{\partial x^{\mu}} \qquad A^{,\mu} = \frac{\partial A}{\partial x_{\mu}}$$

Higher Rank Tensors

$$T^{\mu\nu} \rightarrow T'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}T^{\rho\sigma} .$$

$$\boxed{\boldsymbol{T} = \left(\rho c^2 + P\right) \underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} + P\,\boldsymbol{g}} ,$$

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & , \alpha\beta\gamma\delta \text{ even permutation of } 0123 \\ -1 & , \alpha\beta\gamma\delta \text{ odd permutation of } 0123 \\ 0 & , \text{otherwise} \end{cases}$$

$$\Lambda^{\alpha}{}_{\epsilon}\Lambda^{\beta}{}_{\zeta}\Lambda^{\gamma}{}_{\kappa}\Lambda^{\delta}{}_{\lambda}\,\epsilon^{\epsilon\zeta\kappa\lambda} = +\epsilon^{\alpha\beta\gamma\delta}$$

➔ **Dual Tensors**

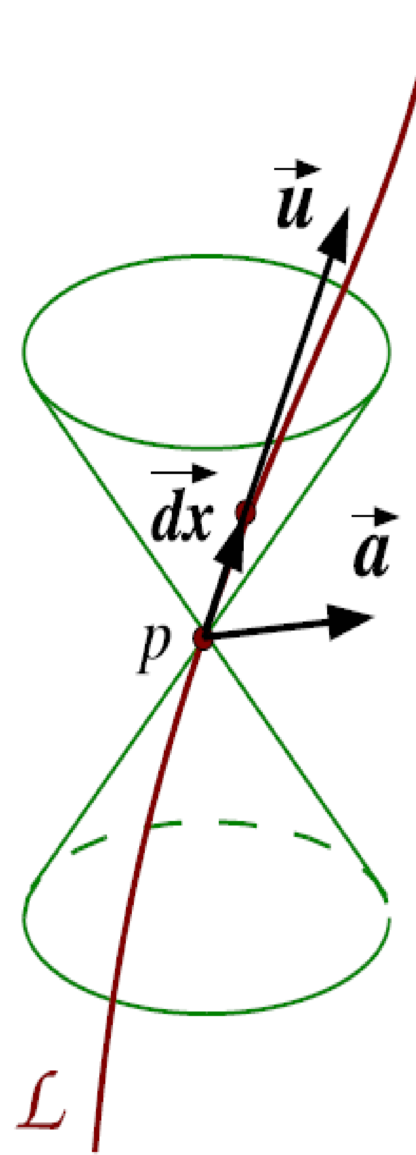
Worldline‘s 4-acceleration

The **4-acceleration** of the particle

$$\boxed{\vec{a} = \boldsymbol{\nabla}_{\vec{u}}\, \vec{u}}\,.$$

Since $\vec{u}$ is a unit vector, it follows that

$$\vec{u} \cdot \vec{a} = 0\,,$$



The Laws of Physics

- Write them as tensor equations (tensors are Lorentz covariant entities).
- **E** and **B** fields in Maxwell‘s theory e.g. are not covariant ➔ use Faraday tensor.

✂➔ Use conservation of energy and momentum.

✂➔ Derive field equations, if possible, from Lagrangians.

Lagrangian Field Theory

$$\mathcal{S} = \int \mathrm{d}^{D-1}x\,\mathrm{d}t\,\mathcal{L} = \int \mathrm{d}^{D-1}x\mathrm{d}t\left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

$$= \int \mathrm{d}^{D-1}x\,\mathrm{d}t\left[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}\delta^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}m^2\phi^2 - \sum_{n=3}^{\infty} g_n\phi^n\right]$$

Euler-Lagrange equations

$$\frac{\delta\mathcal{S}}{\delta\varphi} = -\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) + \frac{\partial\mathcal{L}}{\partial\varphi} = 0.$$

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + V'(\phi) = \partial_t^2\phi - \nabla^2\phi + V'(\phi) = 0$$

Hydrodynamic Equations Newtonian Euler Equations

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) &= 0, \\
\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}\right) &= -\vec{\nabla} p - \rho \vec{\nabla} \Phi + \kappa \rho \vec{F}_{GR}, \\
P &= K \rho^{1+\frac{1}{n}}, \ \text{(polytropic)} \\
\nabla^2 \Phi &= 4\pi G \rho, \\
\vec{F}_{GR} &= \vec{\nabla} \Phi_{GR}.
\end{aligned}
$$

3 Hydro Conservation Laws

Basic Equations without gravity

Mass conservation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \bullet \rho \vec{V} = 0$$

Momentum conservation:

$$\frac{\partial}{\partial t}\left(\rho \vec{V}\right) + \vec{\nabla} \bullet \left(\rho \vec{V}\vec{V} + P\overleftrightarrow{I}\right) = 0$$

Energy conservation:

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2 + \rho \varepsilon\right) + \vec{\nabla} \bullet \left[\rho \vec{V}\left(\tfrac{1}{2}V^2 + h\right)\right] = 0$$

Specific enthalpy

Decompose Energy-Momentum Tens

$$\boxed{\boldsymbol{T} = (\rho c^2 + P)\,\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} + P\,\boldsymbol{g}},$$

Observer velocity

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}_0 = -W$$

Total energy

$$E = W^2(\rho c^2 + P) - P\,.$$

$$E = \boldsymbol{T}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{u}}_0)$$

Momentum flux

$$\vec{\boldsymbol{S}} = W^2\left(\rho + \frac{P}{c^2}\right)\vec{\boldsymbol{U}}\,.$$

$$c\vec{\boldsymbol{S}} = -\boldsymbol{T}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_i)\,\vec{\boldsymbol{e}}_i$$

Stress tensor

$$S_{ij} = P\,\delta_{ij} + W^2\left(\rho + \frac{P}{c^2}\right)V^iV^j\,,$$

$$S_{ij} = \boldsymbol{T}(\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j)$$

Hydro Conservation Laws

state vector $\mathbf{U} = (D, S_i, \tau)^T$

primitive variables $\mathbf{P} = (\rho, v_1, v_2, v_3, P)^T$

$$\begin{aligned} D &= W\rho_0 \\ \vec{S} &= \rho_0 W^2 h \vec{v} \\ \tau &= \rho_0 W^2 h - P - D = E - D\,, \end{aligned}$$

$$\boxed{\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^i}{\partial x^i} = 0\,.}$$

Fluxes:

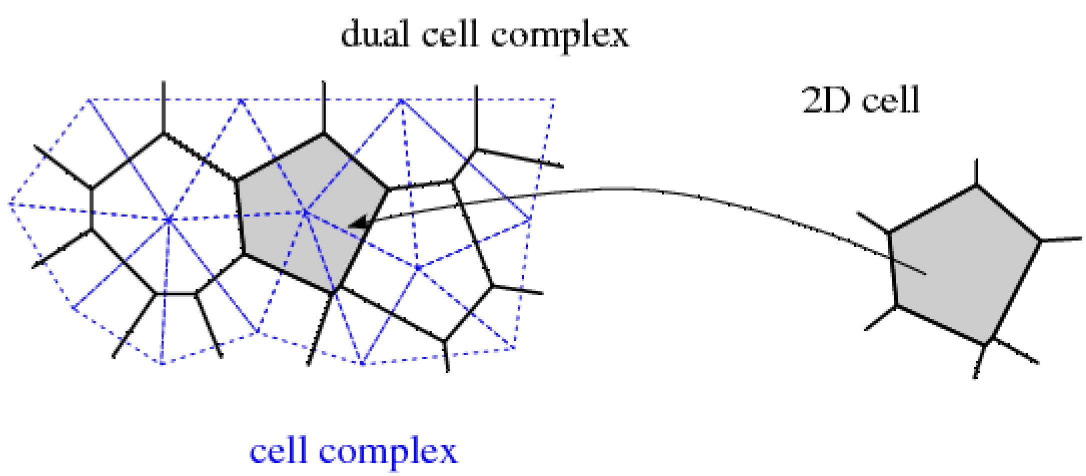
$$\mathbf{F}^i = \left(D v^i, S_j v^i + P \delta^i_j, (\tau + P) v^i \right).$$

Why Conservation Laws ?

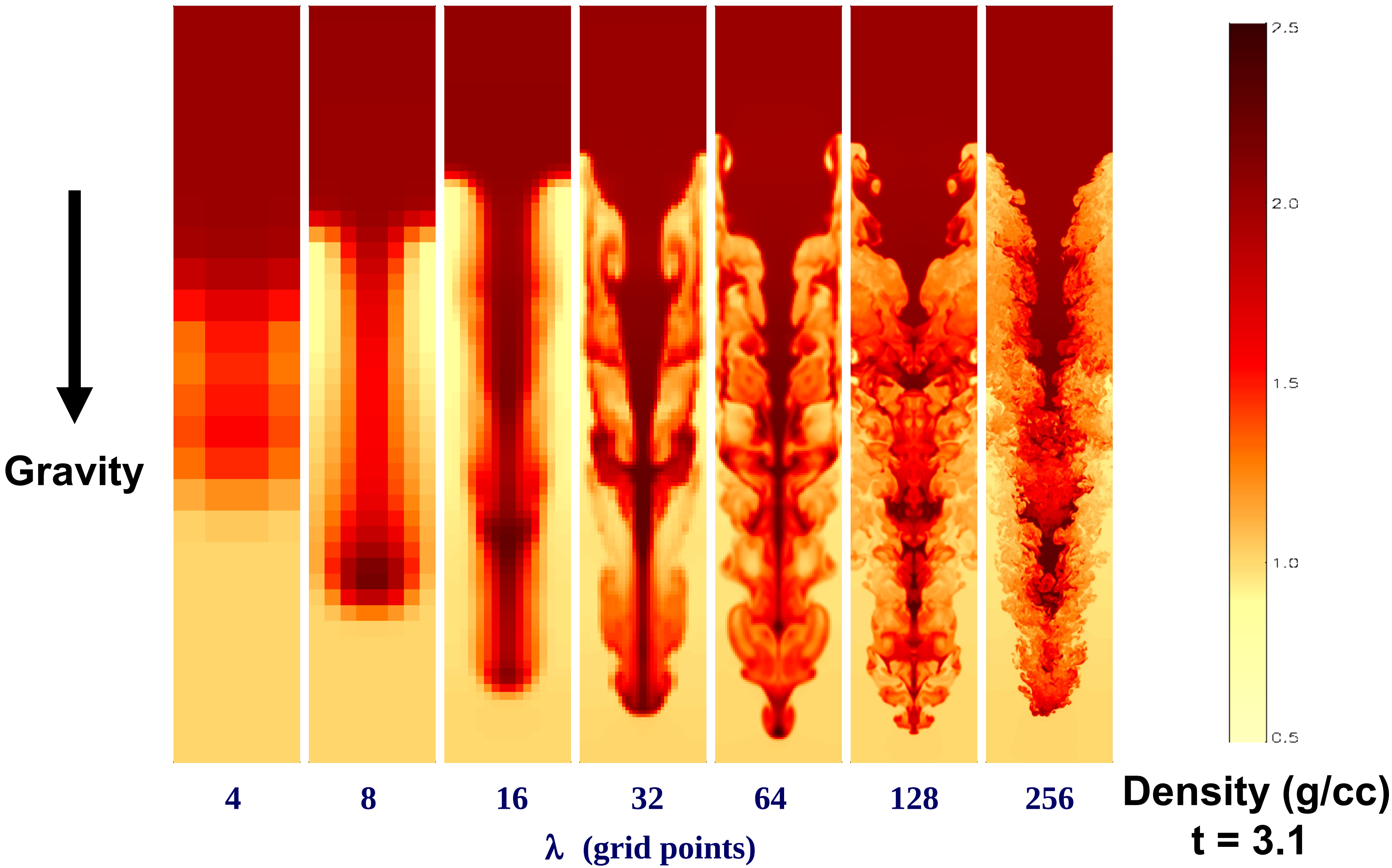
An integral conservation law asserts that the rate of change of the total amount of a quantity with density u in a fixed control volume V is equal to the total flux of the quantity through the boundary dV

$$\partial_t \int_V u \, dV + \int_{\partial V} f(u) \cdot d\mathbf{A} = 0$$

The integral conservation law is transferred to small control volumes.



3D Rayleigh-Taylor Instability



t = 3.1 sec

Codes: FLASH/PLUTO/…

Summary – Special Relativity

- Special Relativity is well established.
- Invariance of the speed of light is well tested, no preferred frame of reference in Minkowski
- Laws of physics are to be written in covariant way: ➔ Maxwell‘s theory with Faraday tens, ➔ Lagrangian field theories (scalar), ➔ Motion of a perfect fluid is written as a set of 3 conservation laws (numerically import).
- Open question: **How to include gravity?**

Μεδεις αγεωμέτρητος εισιτω
μον τήν στήγων.

Let none ignorant of geometry
enter my door.

*Legendary inscription over
the door of Plato's Academy*

Einstein's basic Idea: Gravity is Geometry, Geometry is Gravity

Steps to General Relativity

While special relativity overturned established ideas in physics, Einstein realised that it was incompatible with gravity

$$F_{12} = \frac{Gm_1m_2}{|\vec{r}_1(t) - \vec{r}_2(t)|^2}$$

The problem is positions according to who, and at what time?

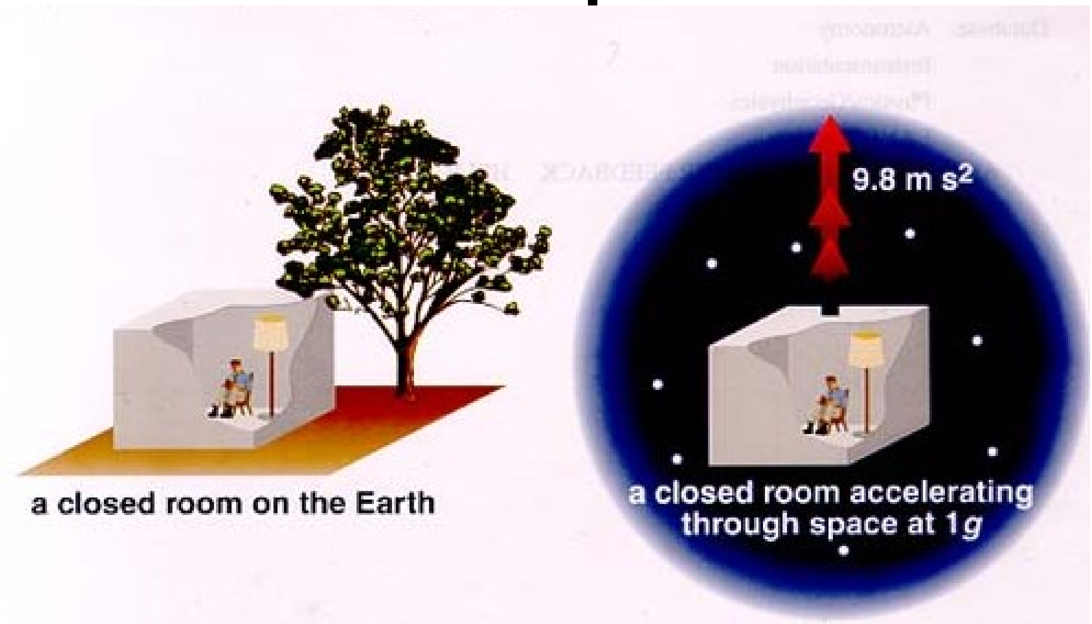
Einstein's revelation began with the realization of equivalence of gravitational and inertial mass. Simply put, all masses fall at the same rate. What happens if I drop a laboratory?

Clearly everything falls together. What Einstein realized is that effectively gravity has vanished.

Equivalence Principle

Einstein's "happiest thought" came from the realization he could take the equivalence principle further.

Simply put, Einstein reasoned that



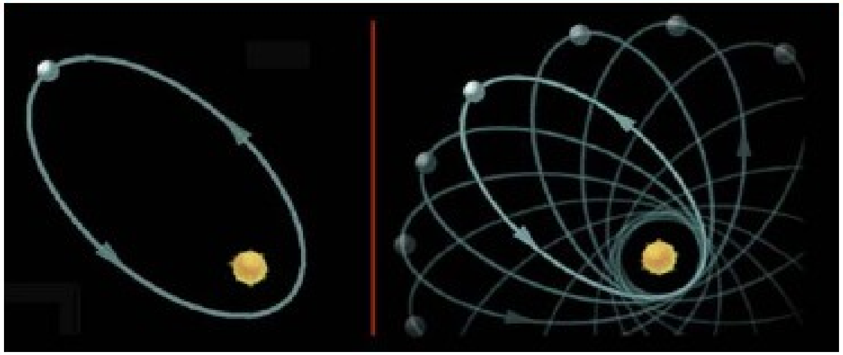
There is no experiment that can distinguish between uniform acceleration and a uniform gravitational field.

Triumph of Mathematical Astronomy in 19th Century

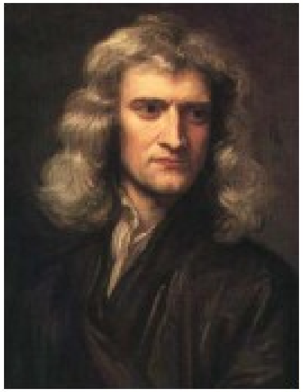
Discovery of Neptune: 1845

Urbain LeVerrier
(1811-1877)

- 1845: the search for Planet-X:
  - Anomaly in the Uranus' orbit → Neptune
  - Anomalous motion of Mercury → Vulcan



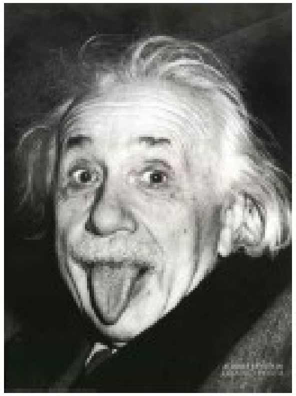
Newtonian Gravity | General Relativity



Sir Isaac Newton
(1643-1727)

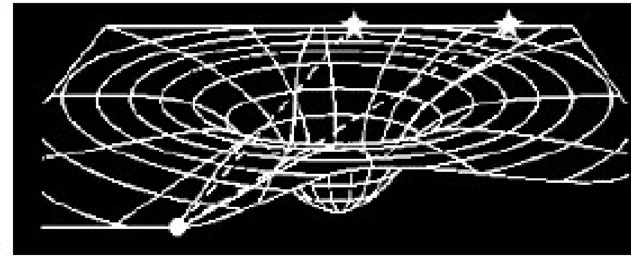
- Anomalous precession of Mercury's perihelion :
  - 43 arcsec/cy can not be explained by Newton's gravity
- Before publishing GR, in 1915, Einstein computed the expected perihelion precession of Mercury
  - When he got out 43 arcsec/cy – a new era just began!!

Almost in one year LeVerrier both confirmed the Newton's theory (Neptune) & cast doubt on it (Mercury's' anomaly).

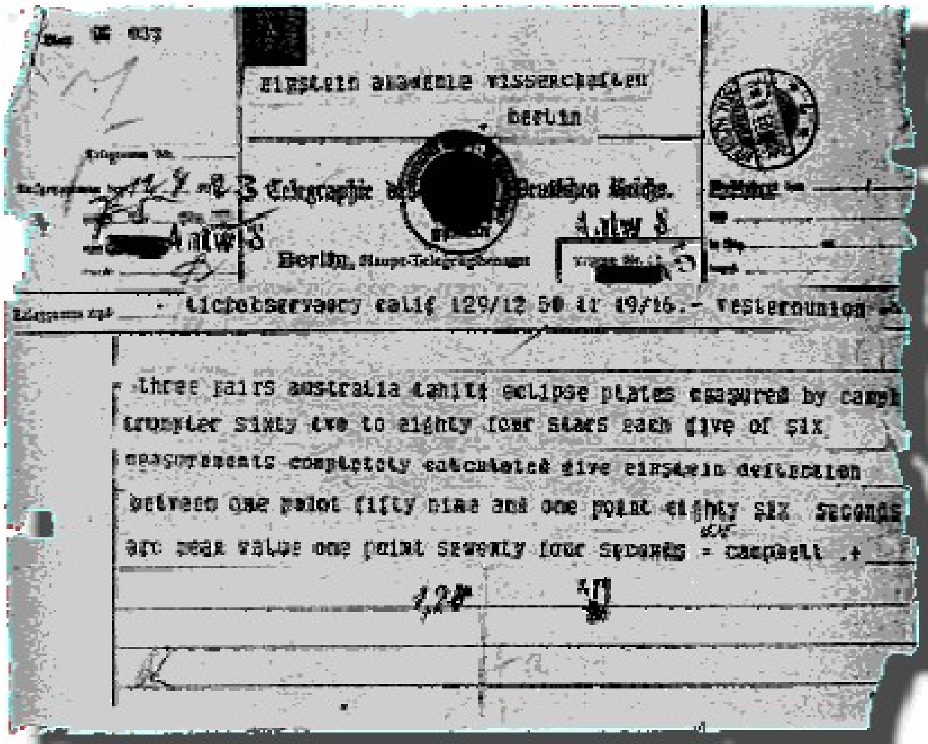


Albert Einstein
(1879-1955)

The First Test of General Theory of Relativity



Gravitational Deflection of Light: Solar Eclipse 1919



Einstein akademie wissenschaften berlin

Telegraphie des Deutschen Reichs.

Berlin

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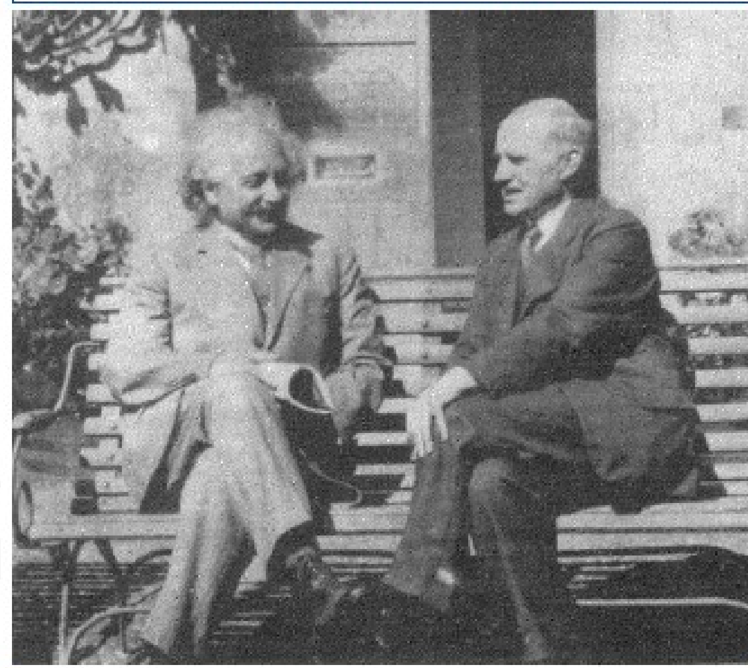
three pairs australia tahiti eclipse plates measured by campbell trumpler sixty five to eighty four stars each five of six measurements completely calculated give einstein deflection between one point fifty nine and one point eighty six seconds arc mean value one point seventy four seconds = campbell .+

Eddington's telegram to Einstein, 1919

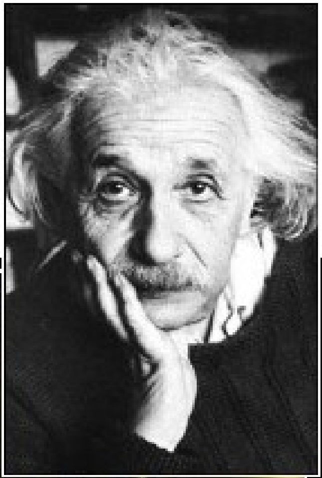
Possible outcomes in 1919:

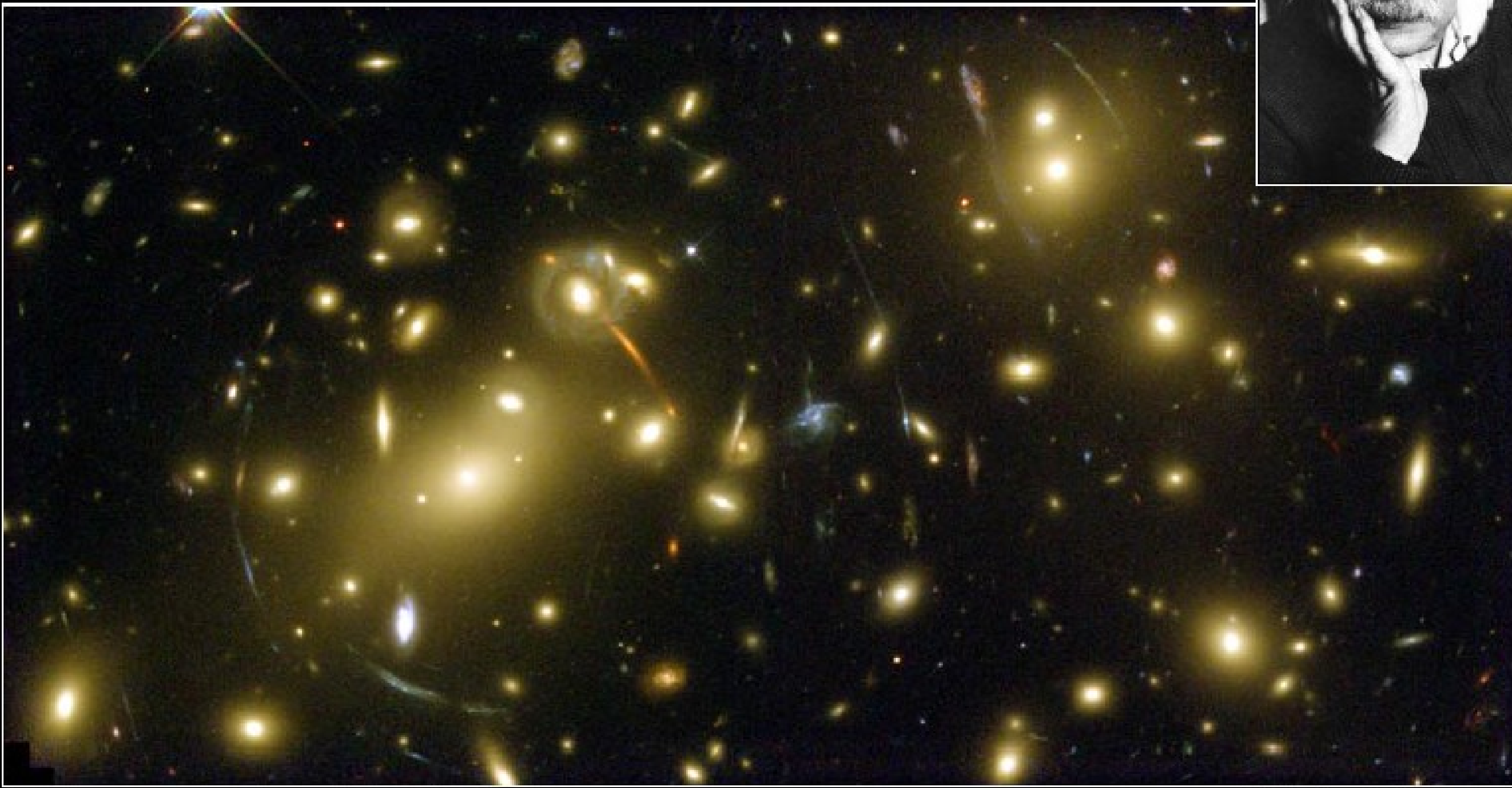
Deflection = 0;
Newton = 0.87 arcsec;
Einstein = 2 x Newton = 1.75 arcsec



Einstein and Eddington, Cambridge, 1930

Gravitational Deflection of Light is a Well-Known Effect Today





Galaxy Cluster Abell 2218 **HST • WFPC2**

NASA, A. Fruchter and the ERO Team (STScI) • STScI-PRC00-08

Geometry of SpaceTime

- What defines metric theories of gravity?
- Einstein‘s equivalence principle.
- Strong equivalence principle.
- SpaceTime as the manifold of events.
- Vector and tensor fields on manifolds.
- Connection and curvature on manifolds.
- Ricci, Einstein and Weyl tensors.
- p-forms on manifolds.

✂ ➔ Cartan formalism.

Metric Theories of Gravity

- **P1**: Space of all events is a 4-dimensional manifold endowed with a global symmetric metric field **g** (2-tensor) of signature (+---) or (-+++).
- **P2**: Gravity is related to the Levi-Civita **connection** on this manifold ➔ no torsion.
- **P3**: Trajectories of freely falling bodies (local inertial frames) are geodesics of that metric.
- **P4**: Any physical interaction (other than gravity) behaves in a local inertial frame as gravitation were absent (covariance).

Spacetime geometry is described by the metric $g_{\mu\nu}$. The curvature scalar $R[g_{\mu\nu}]$ is the most basic scalar quantity characterizing the curvature of spacetime at each point. The simplest action possible is thus

$$S = \frac{1}{16\pi G}\int R\, d^4x + S_{(\mathrm{matter})}$$

Varying with respect to $g_{\mu\nu}$ gives Einstein's equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}^{(\mathrm{matter})}$$

$G_{\mu\nu}$ is the Einstein tensor, characterizing curvature, and $T_{\mu\nu}$ is the energy-momentum tensor of matter.

Einstein's Gravity is Metric

General Relativity in a Nutshell

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$$

Einstein's Field Equations
couple matter to curvature

Trajectories of freely falling bodies (mass and massless)

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0$$

Geodesic Equation

Levi-Civita connection

Scalar-Tensor Gravity is Metric

Introduce a scalar field $\phi\,(x)$ that determines the strength of gravity. Einstein's equation

$$G_{\mu\nu} = 8\pi G T^{(\mathrm{m})}_{\mu\nu}$$

is replaced by

$$G_{\mu\nu} = f(\phi)\left[T^{(\mathrm{m})}_{\mu\nu} + T^{(\phi)}_{\mu\nu}\right]$$

variable “Newton's constant”

extra energy-momentum from ϕ

The new field $\phi\,(x)$ is an extra degree of freedom; an independently-propagating scalar particle (Brans and Dicke 1961; Quintessence models).

Brans-Dicke Theory is Metric

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\ \left(\phi R - \omega\frac{\partial_a\phi\partial^a\phi}{\phi} + \mathcal{L}_{\mathrm{M}}\right)$$

$$\Box\phi = \frac{8\pi}{3+2\omega}T$$

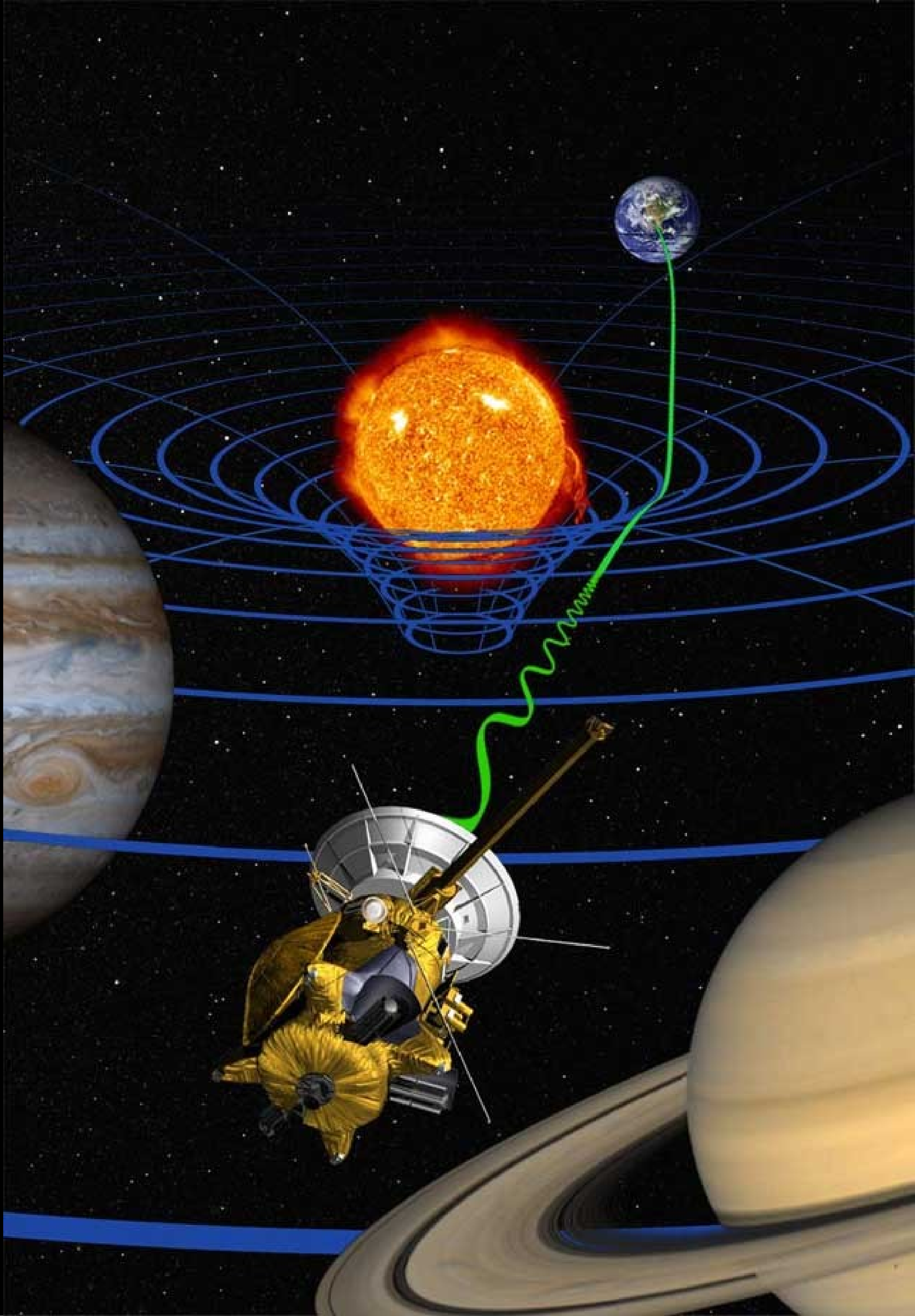
$$G_{ab} = \frac{8\pi}{\phi}T_{ab} + \frac{\omega}{\phi^2}(\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}\partial_c\phi\partial^c\phi) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi)$$

The new scalar ϕ is sourced by planets and the Sun, distorting the metric away from Schwarzschild. It can be tested many ways, e.g. from the time delay of signals from the Cassini mission.

Experiments constrain the “Brans-Dicke parameter” ω to be

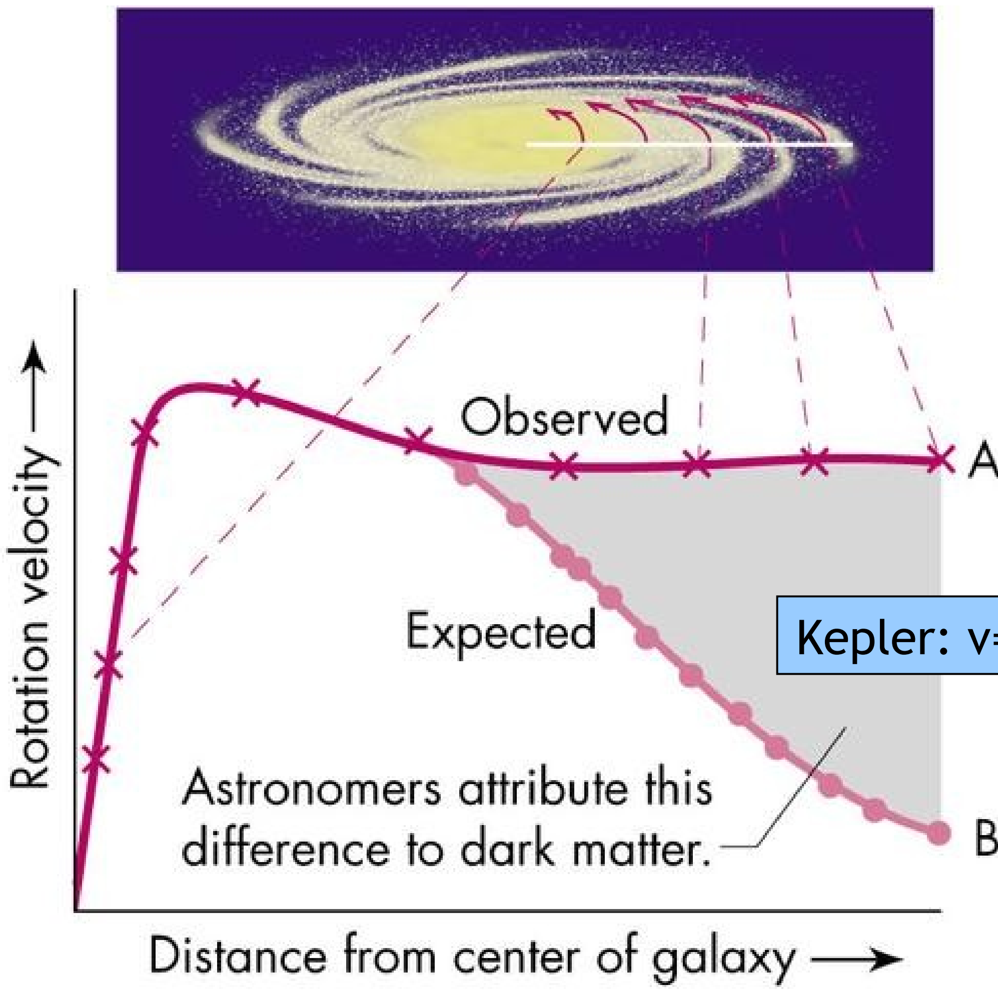
$$\omega > 40,000 ,$$

where $\omega = 1$ is GR.



Potential Wells are much deeper than can be explained with visible matter (➔ Modified Gravity?)

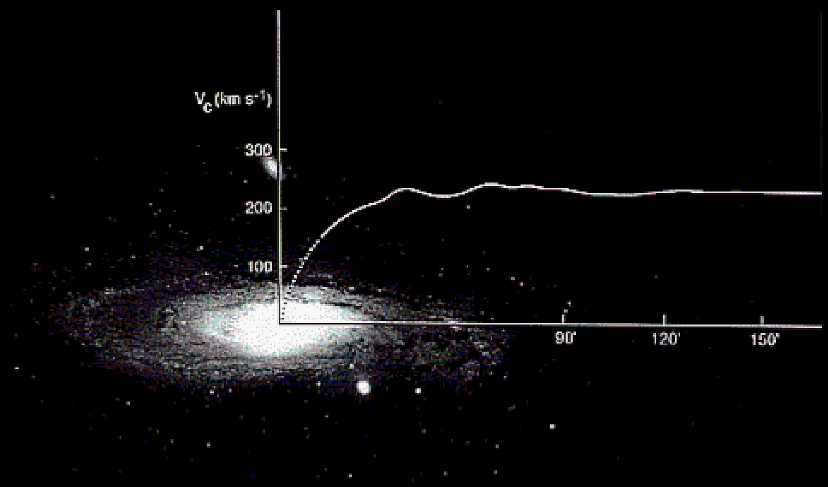
This has been measured for many years on galactic scales



Kepler: $v=[GM/R]^{1/2}$

Modified Newtonian Dynamics -- MOND

Milgrom (1984) noticed a remarkable fact: dark matter is only needed in galaxies once the acceleration due to gravity dips below $a_0 = 10^{-8}\ \mathrm{cm/s^2} \sim cH_0$.



He proposed a phenomenological force law, MOND, in which gravity falls off more slowly when it's weaker:

$$F \propto \begin{cases} 1/r^2, & a > a_0, \\ 1/r, & a < a_0. \end{cases}$$

Bekenstein (2004) introduced *TeVeS*, a relativistic version of MOND featuring the metric, a fixed-norm vector U_μ, scalar field ϕ, and Lagrange multipliers η and λ:

$$S = \frac{1}{16\pi G}\int d^4x\,(R + \mathcal{L}_U + \mathcal{L}_\phi)$$

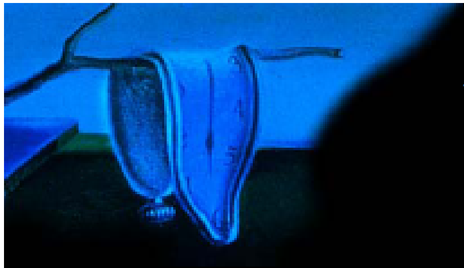
where

$$\mathcal{L}_U = -\frac{1}{2}KF^{\mu\nu}F_{\mu\nu} + \lambda(g^{\mu\nu}U_\mu U_\nu + 1)$$

$$\mathcal{L}_\phi = -\mu_0\eta(g^{\mu\nu} - U^\mu U^\nu)\partial_\mu\phi\partial_\nu\phi - V(\eta)$$

$$V(\eta) = \frac{3\mu_0}{128\pi l_B^2}\left[\eta(4 + 2\eta - 4\eta^2 + \eta^3) + 2\ln^2(\eta - 1)\right]$$

Not something you'd stumble upon by accident.



Theoretical landscape of early 1970s:
Competing Theories of Gravity

Newton 1686	Poincaré 1890	Einstein 1912	Nordstrøm 1912	Nordstrøm 1913
Einstein and Fokker 1914	Einstein 1916	Whitehead 1922	Cartan 1923	
Fierz and Pauli 1939	Birkhoff 1943	Milne 1948	Thiry 1948	Papapetrou 1954
Papapetrou 1954	Jordan 1955	Littlewood and Bergmann 1956	Brans and Dicke 1961	
Yilmaz 1962	Whitrow and Morduch 1965	Whitrow and Morduch 1965		
Kustaanheimo and Nuotio 1967	Deser and Laurent 1968	Page and Tupper 1968		
Bergmann 1968	Nordtvedt 1970	Bollini, Giambiagi and Tiomno 1970	Wagoner 1970	
Rosen 1971	Ni 1972	Ni 1972	Hellings and Nordtvedt 1972	Will and Nordtvedt 1972
Ni 1973	Yilmaz 1973	Lightman and Lee 1973	Lee, Lightman and Ni 1974	Rosen 1975
Belinfante and Swihart 1975	Lee et al. 1976	Bekenstein 1977	Barker 1978	Rastall 1979
Coleman 1983	Kaluza-Klein 1932	Overlooked (20th century)		

Einstein Equivalence Principle

- Einstein Equivalence Principle
  - Uniqueness of Free Fall
  - Local Lorentz Invariance
  - Local Position Invariance
- Metric Theory: Definition
  - Space-time is endowed with a symmetric metric
  - Trajectories of freely falling bodies are geodesics of that metric

$$\boxed{\text{Einstein Equivalence Principle}}$$
$$\Downarrow$$
$$\boxed{\text{Only Metric Theories viable}}$$

The Einstein Equivalence Principle (EEP)

- Test bodies fall with the same acceleration
  Weak Equivalence Principle (WEP)
- In a local freely falling frame, physics (non-gravitational) is independent of frame's velocity
  Local Lorentz Invariance (LLI)
- In a local freely falling frame, physics (non-gravitational) is independent of frame's location
  Local Position Invariance (LPI)

Universality of Free Fall

Test the Uniqueness of Free Fall
(a.k.a. the Weak Equivalence Principle):

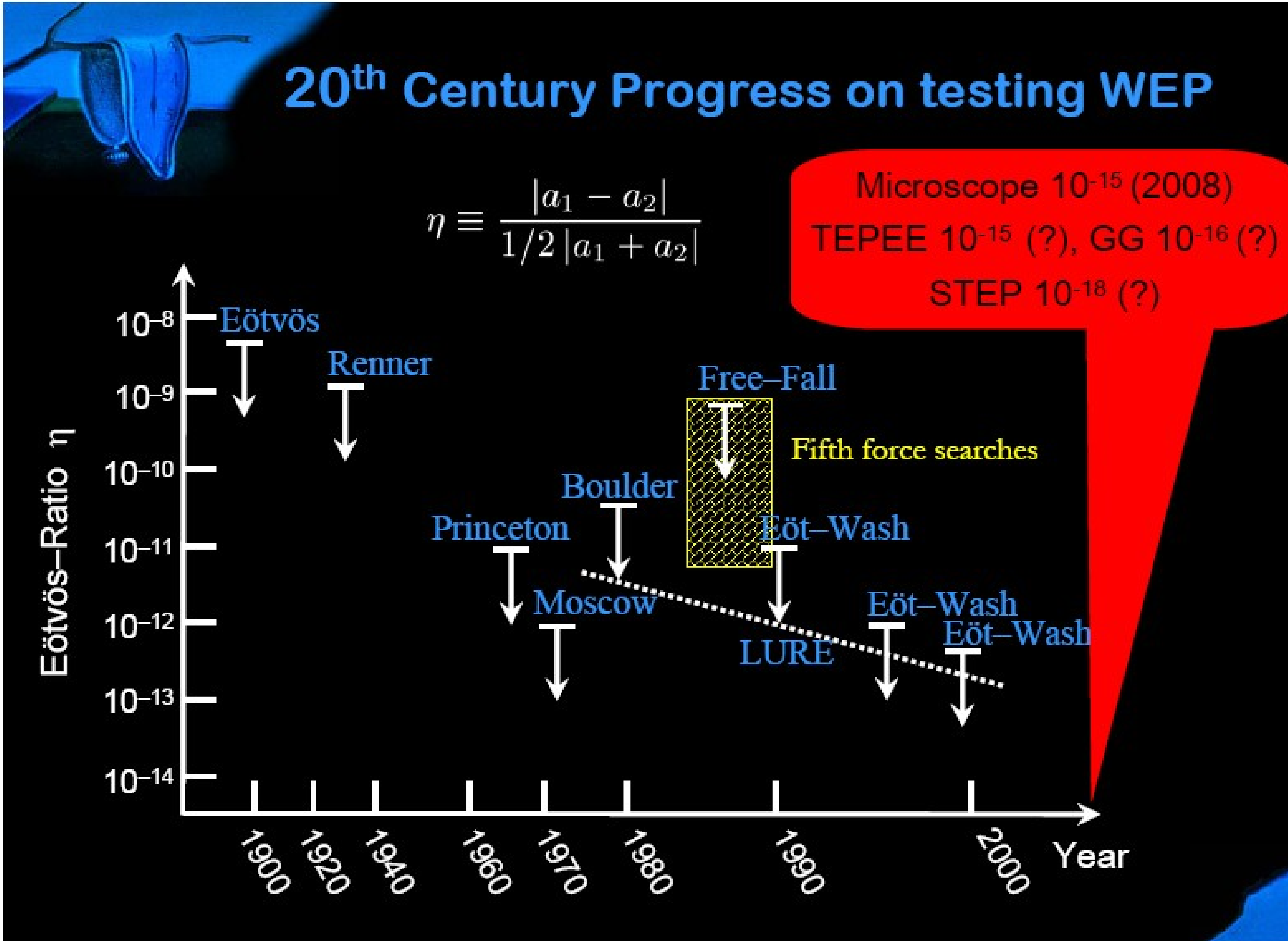
$$\vec{F} = m_i \, \vec{a} \quad = \quad m_g \, \vec{g}$$

$$\boxed{\Longrightarrow m_i \quad = \quad m_g}$$

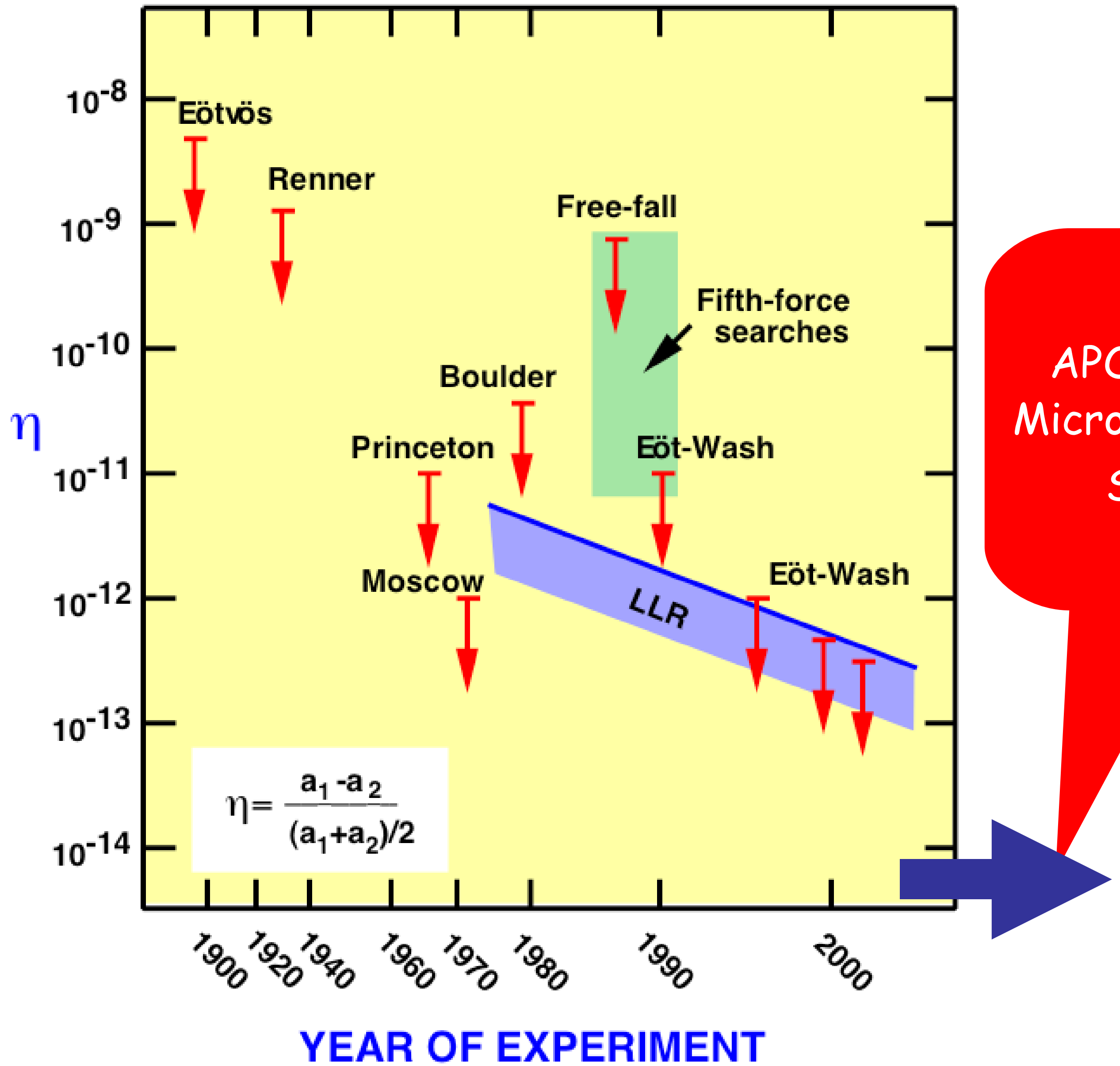
All bodies fall with the same acceleration

Define test parameter that signifying violation of WEP

$$\eta \equiv \frac{|a_1 - a_2|}{1/2\,|a_1 + a_2|}$$


20th Century Progress on testing WEP
$\eta \equiv \frac{|a_1 - a_2|}{1/2\,|a_1 + a_2|}$
Microscope 10-15 (2008)
TEPEE 10-15 (?), GG 10-16 (?)
STEP 10-18 (?)
Eötvös–Ratio η
10–8
10–9
10–10
10–11
10–12
10–13
10–14
Eötvös
Renner
Free–Fall
Fifth force searches
Boulder
Princeton
Eöt–Wash
Moscow
Eöt–Wash
Eöt–Wash
LURE
1900
1920
1940
1960
1970
1980
1990
2000
Year

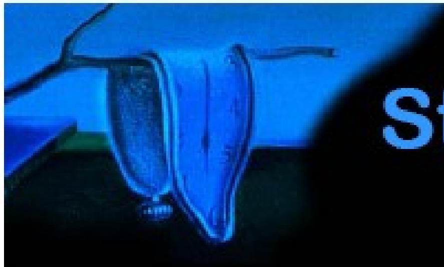
Tests of the Weak Equivalence Principle



APOLLO (LLR) 10^{-13}
Microscope 10^{-15}(2008)
STEP 10^{-18} (?)

Tests of Local Position Invariance

Constant	Limit (yr^{-1})	Z	Method
α	$<30 \times 10^{-16}$	0	Clock comparisons
	$<0.5 \times 10^{-16}$	0.15	Oklo reactor
	$<3.4 \times 10^{-16}$	0.45	^{187}Re decay
	$(6.4 \pm 1.4) \times 10^{-16}$	3.7	Quasar spectra
	$<1.2 \times 10^{-16}$	2.3	Quasar spectra
α_W	$<1 \times 10^{-11}$	0.15	Oklo reactor
	$<5 \times 10^{-12}$	10^9	BBN
m_e/m_p	$<3 \times 10^{-15}$	2-3	Quasar spectra



Strong Equivalence Principle

- Generalized Uniqueness of Free Fall:
  All bodies fall with the same acceleration
- Generalized Local Lorentz Invariance:
  All experiments are independent of the velocity of the local Lorentz frame
- Generalized Local Position Invariance
  All experiments are independent of where and when they are performed

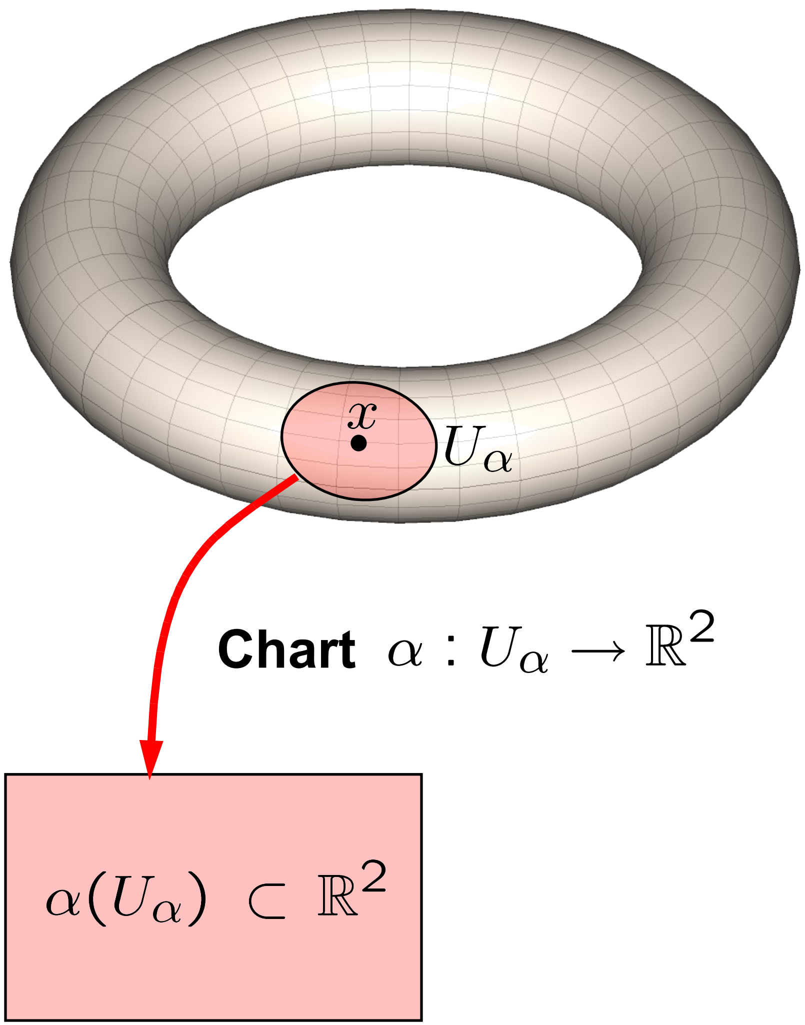
Strong Equivalence Principle (SEP)

- **At any event, always and everywhere, it is possible to choose a local inertial frame such that in a sufficiently small spacetime neighborhood all non-gravitational laws take on their familiar forms appropriate to the absence of gravity.**
- The moon (LLR), a neutron star, a Black Hole is an appropriate system. ➔ The moon follows a geodesic path within 1 cm accuracy.
- Is a galaxy a suitable such system?

From Minkowski to SpaceTime

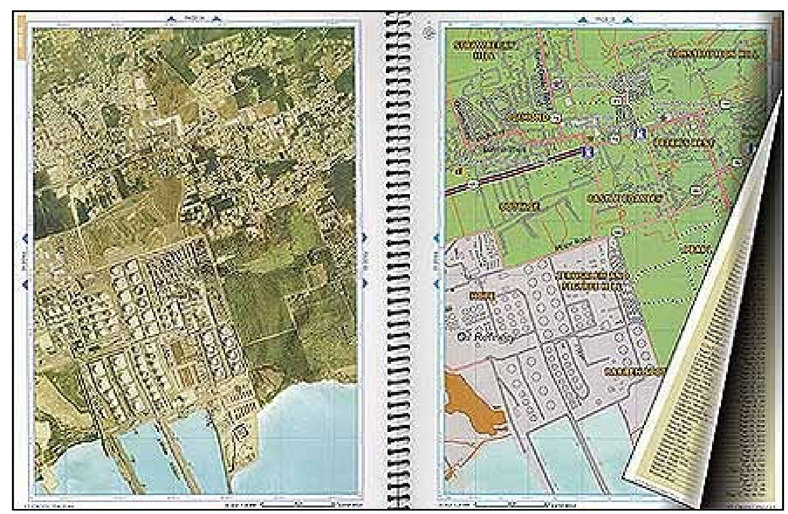
- Flat Minkowski space → set of events thought to form a manifold, can easily be generalized to higher dimensions (11 for string theories).
- We need the concept of a **metric** for distance measurements, geodesics and for the causal structure → concept of **pseudo-Riemannian manifolds**.

Charts and Atlases

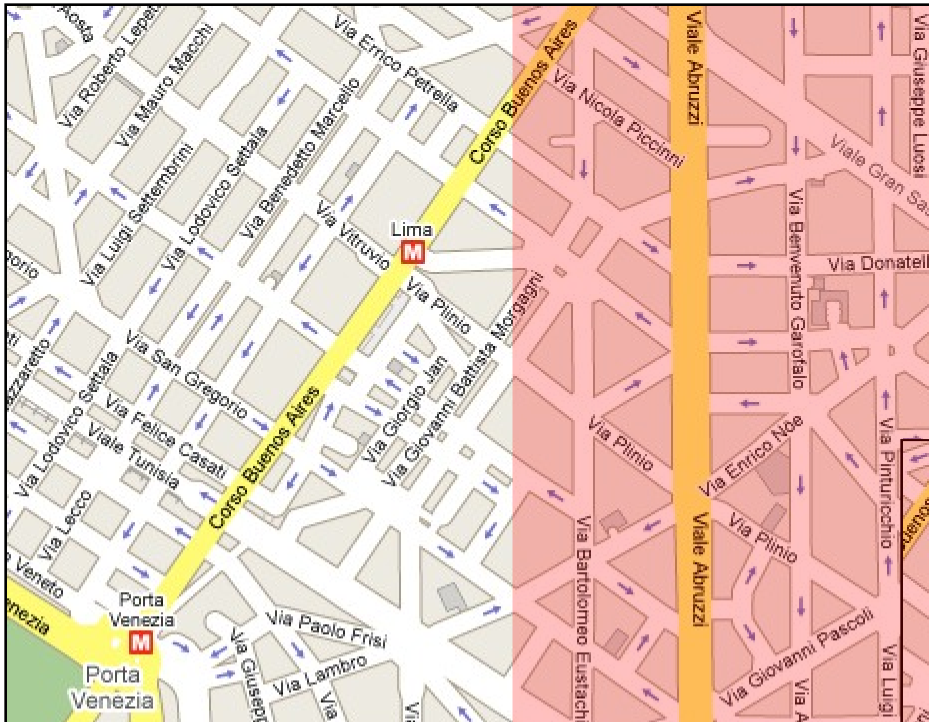


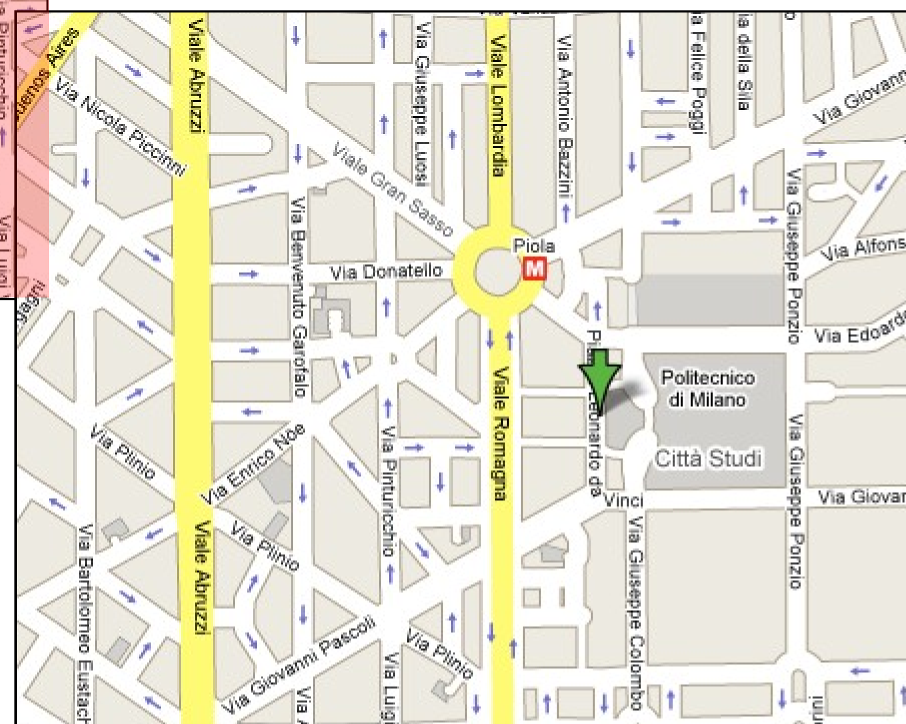
A homeomorphism $\alpha : U_\alpha \to \mathbb{R}^n$ from a neighborhood U_α of $x \in X$ to $\mathbb{R}^n$ is called a **chart**

A collection of charts whose domains cover the manifold is called an **atlas.**



Charts and Atlases



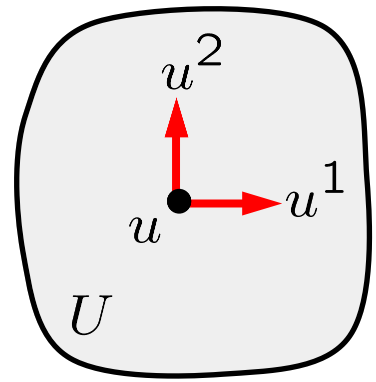


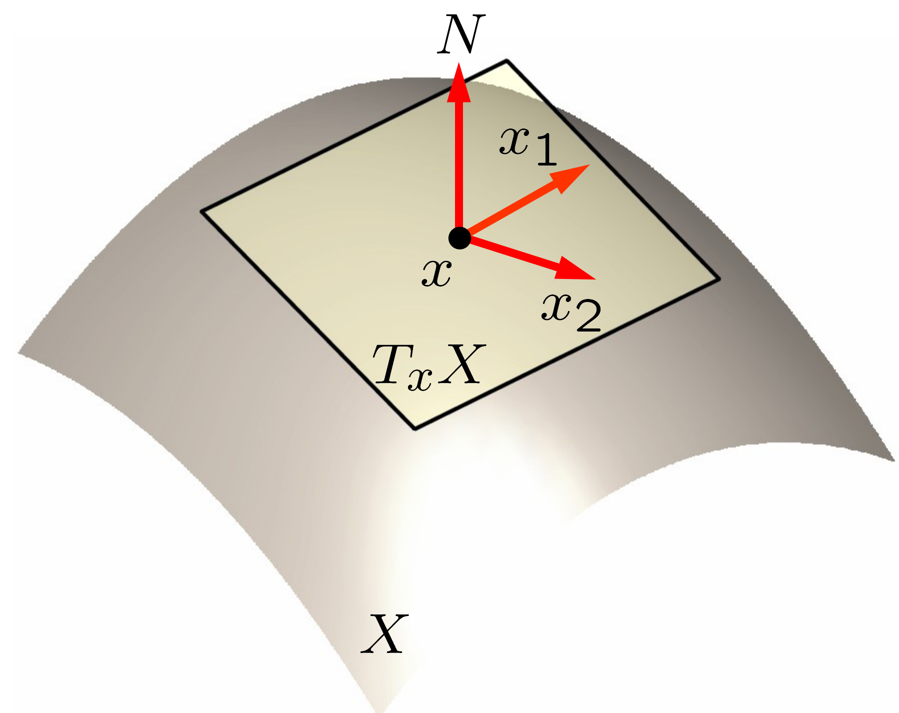
Tangent Plane & Normal

- At each point $u \in U$, we define **local system of coordinates**

$$x_1 = \frac{\partial x}{\partial u^1} \qquad x_2 = \frac{\partial x}{\partial u^2}$$

- A parametrization is **regular** if x_1 and x_2 are **linearly independent**.
- The plane $T_x X = \mathrm{span}\{x_1, x_2\}$ is **tangent plane** at $x = x(u)$.
- **Local Euclidean approximation** of the surface.
- $N \perp T_x X$ is the **normal** to surface.





Tangent Vectors and Coordinate Basis

Suppose we have a function on a manifold $f(x^\alpha)$, and a curve $x^\alpha(\sigma)$, we can define the derivative along the curve as

$$\frac{df}{d\sigma} = \lim_{\epsilon \to 0} \left[\frac{f(x^\alpha(\sigma+\epsilon)) - f(x^\alpha(\sigma))}{\epsilon} \right] = \frac{dx^\alpha}{d\sigma} \frac{\partial f}{\partial x^\alpha}$$

The vector ***t*** is the **tangent vector** to the curve and has the components

$$t^\alpha = \frac{dx^\alpha}{d\sigma}$$

And the **directional derivative** to be (basis vectors of T_x)

$$\boxed{\frac{d}{d\sigma} = t^\alpha \frac{\partial}{\partial x^\alpha} = \mathbf{t}}$$

Transforming Vectors

How do we transform the components of a vector from one coordinate system to another?

$$\mathbf{a} = a^{\alpha}\frac{\partial}{\partial x^{\alpha}} = \left(a^{\alpha}\frac{\partial x'^{\beta}}{\partial x^{\alpha}}\right)\frac{\partial}{\partial x'^{\beta}} \equiv a'^{\beta}\frac{\partial}{\partial x'^{\beta}}$$

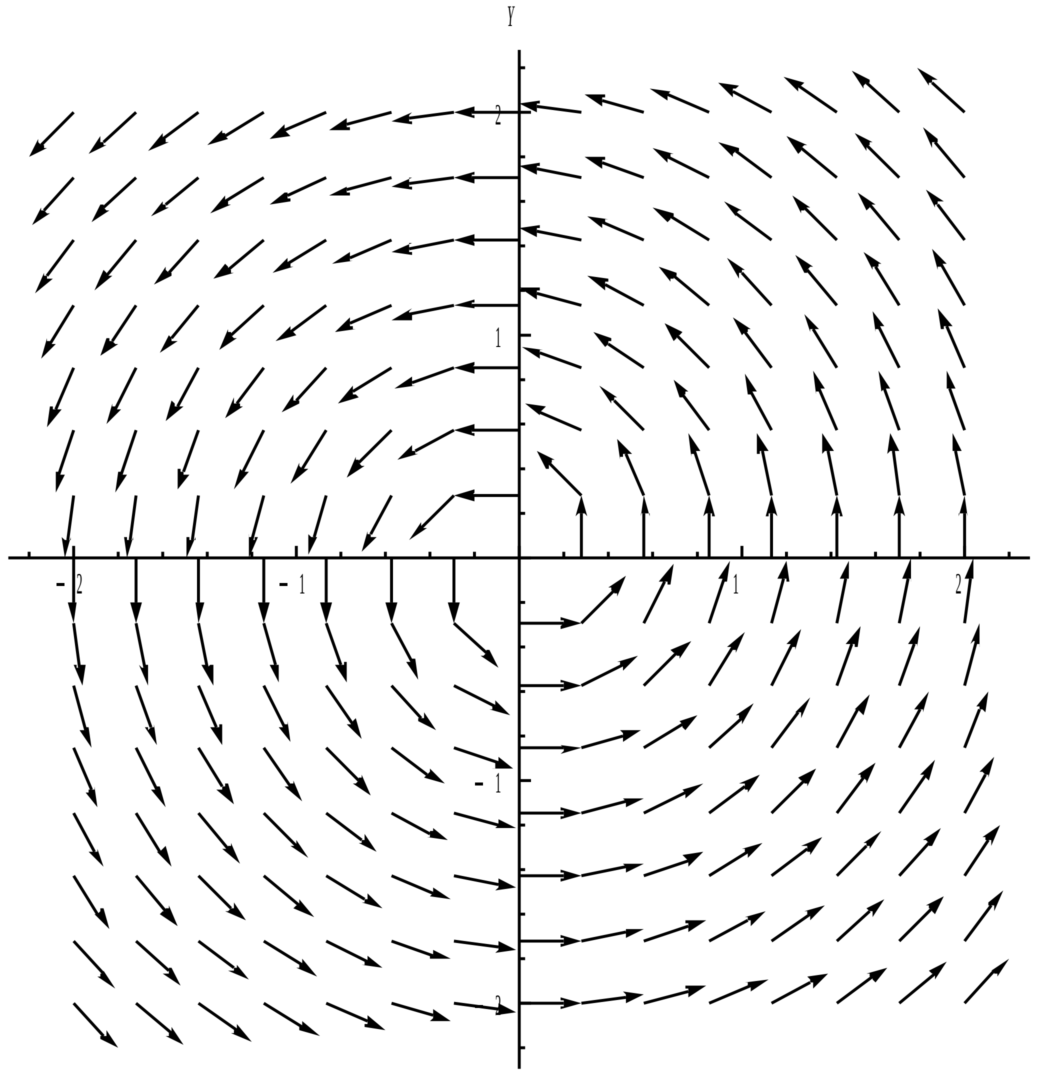
And so: $$a'^{\beta} = \left(\frac{\partial x'^{\beta}}{\partial x^{\alpha}}\right) a^{\alpha}$$

The inverse: $$a^{\beta} = \left(\frac{\partial x^{\beta}}{\partial x'^{\alpha}}\right) a'^{\alpha}$$

Vector Fields

- <u>Definition:</u>
- A *vector field* on $\mathbf{R}^2$ is a function *F* which assigns to each point (x,y) a 2-dimensional vector $\mathbf{V}(x,y)$.
- A vector field on a manifold M is a function which assigns to each point x a n-dimensional vector $\mathbf{X}(x)$ in T_xM.
- In a Riemannian manifold (M,g) we can choose as basis any orthonormal frame $\mathbf{e_a}$.

$$\mathbf{V}(x,y) = \langle -y,\ x \rangle / (x^2+y^2)^{1/2}$$



Dual Vectors – 1-Forms

A dual vector (or **covector or one-form**) is a linear map from a vector to a real number;

$$\omega(\mathbf{a}) = \omega_\alpha a^\alpha$$

Where ω_α are the components of the **covector.** As with vectors, we can express a covector in terms of dual basis vectors:

$$\omega = \omega_\alpha \mathbf{e}^\alpha$$

The basis $\{\mathbf{e}^\alpha\}$ is dual to the basis $\{\mathbf{e}_\alpha\}$ if:

$$e^\alpha(\mathbf{e}_\beta) \equiv \delta^\alpha_\beta$$

Example: One-Form

- Maxwell potential 1-form: $A = A_\mu \, dx^\mu$

The homogeneous eqs are satisfied automatically by setting

$$\mathbf{E} = -\nabla \phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Introducing the 2-form:

$$A_\mu = (\phi, -\mathbf{A})$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

2-Form

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \theta(x)$$

F is invariant under gauge transformation

General Tensor Fields

Just as a dual vector is a linear map from vectors to the reals, a general tensor can be defined the same way. Let

$$\Pi^r_s \equiv T^*_p M \times \cdots \times T^*_p M \times T_p M \times \cdots \times T_p M \tag{2.123}$$

represent the Cartesian product of r cotangent spaces and s tangent spaces at some point p on the manifold M, i.e. the space of ordered sets of r one–forms and s tangent vectors: $(\omega^1, ..., \omega^r, X_1, ..., X_s)$. We now consider a multilinear mapping T of Π^r_s to the reals. This mapping provides an association of any ordered set of r one–forms and s tangent vectors to the real number, $T(\omega^1, ..., \omega^r, X_1, ..., X_s)$. The condition that the map is multilinear requires that

$$\begin{aligned} T(\omega^1, ..., \omega^r, \alpha X + \beta Y, X_2, ..., X_s) \quad &= \\ \alpha T(\omega^1, ..., \omega^r, X, X_2, ..., X_s) \quad &+ \quad \beta T(\omega^1, ..., \omega^r, Y, X_2, ..., X_s) \end{aligned} \tag{2.124}$$

$$T \otimes S(\omega^1, \ldots, \omega^k, \ldots, \omega^{k+m}, X_1, \ldots, X_l, \ldots, X_{l+n}) \quad =$$
$$T(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l) \times S(\omega^{k+1}, \ldots, \omega^{k+m}, X_{l+1}, \ldots, X_{l+n} \quad) \quad .$$

Component of Tensors

The **components of a tensor** in a coordinate basis can be obtained by acting the tensor on the natural one–forms and vectors, here given in a 4D spacetime,

$$T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = T(dx^{\mu_1},\dots,dx^{\mu_k},\partial_{\nu_1},\dots,\partial_{\nu_l})\,. \tag{2.135}$$

This is equivalent to the expansion

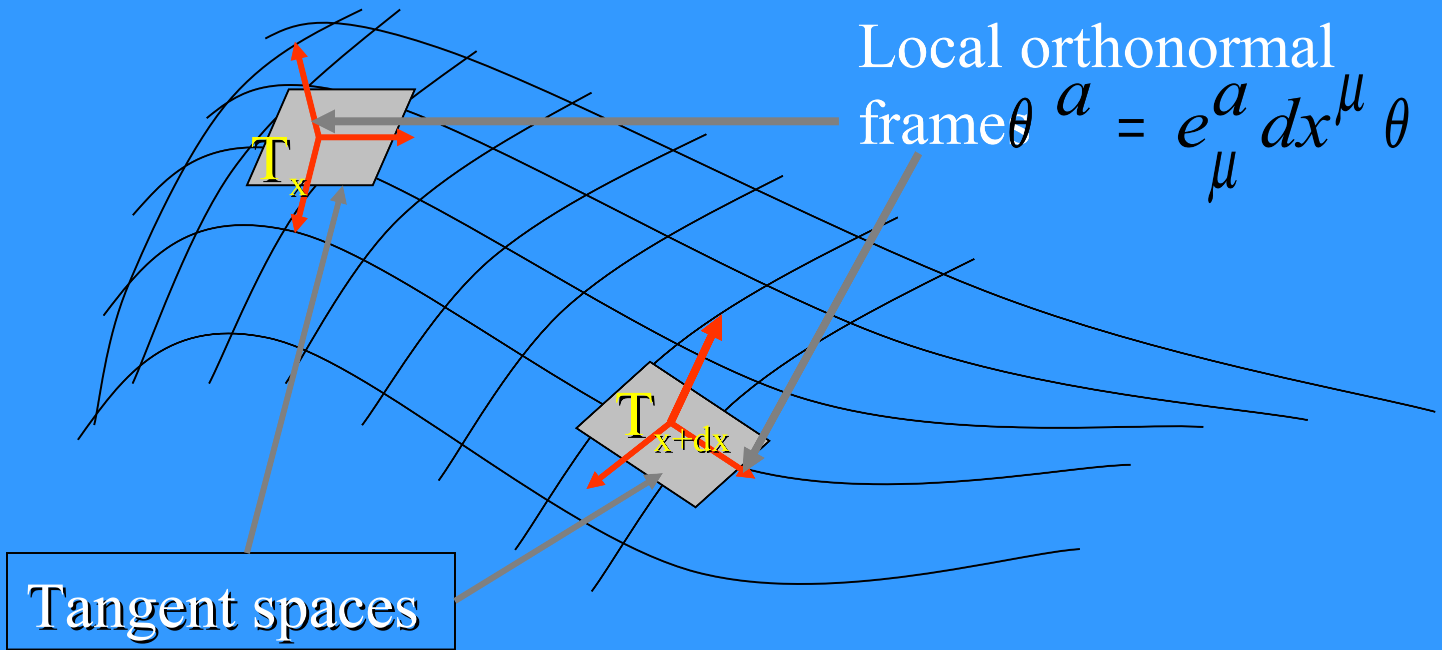
$$T = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_k}\otimes dx^{\nu_1}\otimes\cdots\otimes dx^{\nu_l}\,. \tag{2.136}$$

The transformation law for general tensors follows then the same pattern of replacing the Lorentz transformation matrix used in flat space with a matrix representing more general coordinate transformations

$$T^{\mu'_1\cdots\mu'_k}{}_{\nu'_1\cdots\nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}}\frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}}\cdots\frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}}\,T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\,. \tag{2.137}$$

Equivalence principle

Spacetime is a differentiable manifold endowed with a tangent space T_x, identical to Minkowski space, at each point.



$$\theta^a = e^a_\mu dx^\mu \theta$$

In this way, General Relativity *generalizes* Special Relativity.

The freedom to perform independent Lorentz rotations at each point in spacetime is a gauge symmetry [Utiyama (1955), Kibble (1961)...].

The Metric

The metric is central to studying relativity. In general;

$$ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta$$

The metric is symmetric and position dependent. The metric for flat spacetime in spherical polar coordinates is

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix}$$

The metric has 10 independent components, although there are 4 functions used in transforming coordinates, so really there are **6 independent functions in the metric**.

Local Inertial Frames

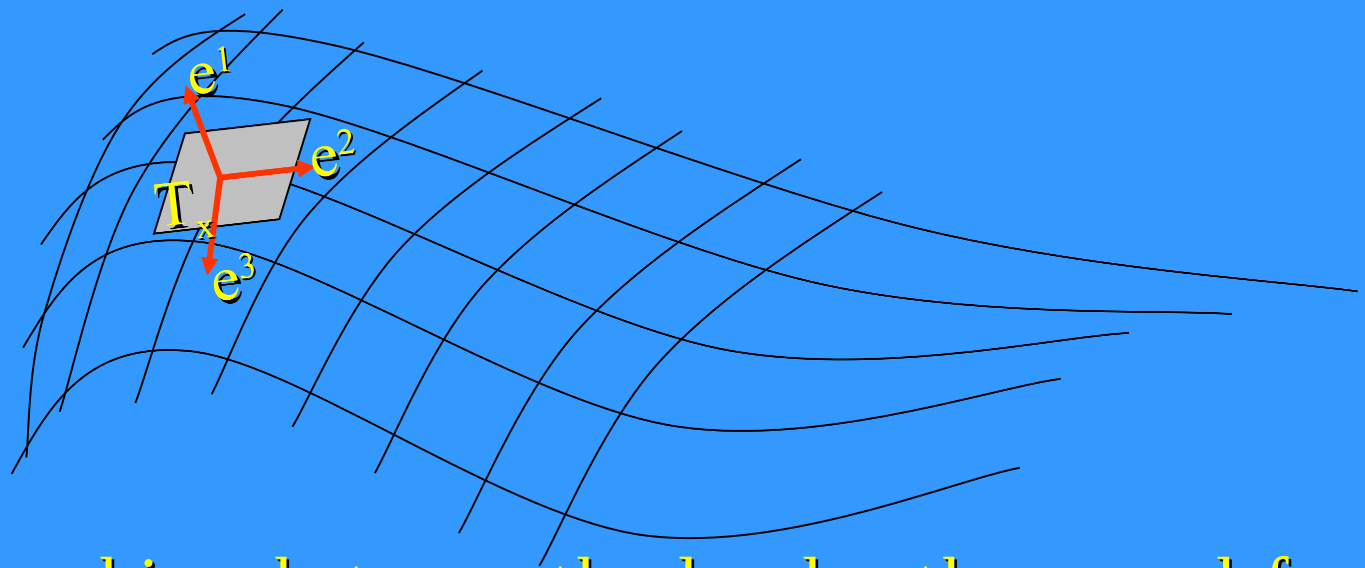
The equivalence principle states that the local properties of curved spacetime should be indistinguishable from flat spacetime.

Basically, this means that at a specific point in a general metric $g_{\alpha\beta}(x)$ we should be able to introduce a new coordinate such that;

$$g'_{\alpha\beta}(x'_P) = \eta_{\alpha\beta} \qquad \left.\frac{\partial g'_{\alpha\beta}}{\partial x'^{\gamma}}\right|_{x=x_P} = 0$$

So we have a locally flat piece of spacetime in which the rules of special relativity hold. This defines a **local inertial frame**.

Local Frames (Vielbein) and Metric



Isomorphism between the local orthonormal frame on the tangent space and the coordinate basis

Tetrad: $\theta^{a} = e^{a}_{\mu}(x)dx^{\mu} \equiv e^{a}$

This induces a metric structure for spacetime: $g_{\mu\nu}(x) = \eta_{\mathrm{ab}} e^{a}_{\mu}(x) e^{b}_{\nu}(x)$

Ex1: The Schwarzschild Metric

$$ds^2 = -\alpha^2\,dt^2 + dr^2/\alpha^2 + r^2\,d\Omega^2$$

$\alpha^2 = 1 - R_s/r$: Redshift factor

Schwarzschild Static CoFrames/Frames

$\theta^0 = \alpha\,dt$	$n = ???$
$\theta^1 = dr / \alpha$	$e_1 = ???$
$\theta^2 = r\,d\theta$	$e_2 = ???$
$\theta^3 = r \sin\theta\,d\phi$	$e_3 = ???$

Proper time of static observers (grav. Redshift)

$$\tau(r) = \tau_{\text{infinty}}\,(1-R_s/r)^{1/2}$$

Ex2: Metric of Rotating Stars

$$ds^2 = -\alpha^2\, dt^2 + R^2\, (d\phi - \omega\, dt)^2 + \exp 2\mu_1\, dr^2 + \exp 2\mu_2\, d\theta^2$$

Redshift factor

frame-dragging
$\beta^\phi = -\omega$

2-metric meridional plane

Bardeen Observers

$\theta^0 = \alpha\, dt$ $\qquad n = \frac{1}{\alpha}(\partial_t - \beta^i \partial_i)$

$\theta^1 = R\,(d\phi - \omega\, dt)$ $\qquad e_1 = ???$

$\theta^2 = \exp \mu_1\, dr$ $\qquad e_2 = ???$

$\theta^3 = \exp \mu_2\, d\theta$ $\qquad e_3 = ???$

Ex3: Gravitational Field of the Sun

For a slowly rotating spherical body like our Sun or the Earth, this metric can be expressed in terms of the Robertson parameters γ **and** β

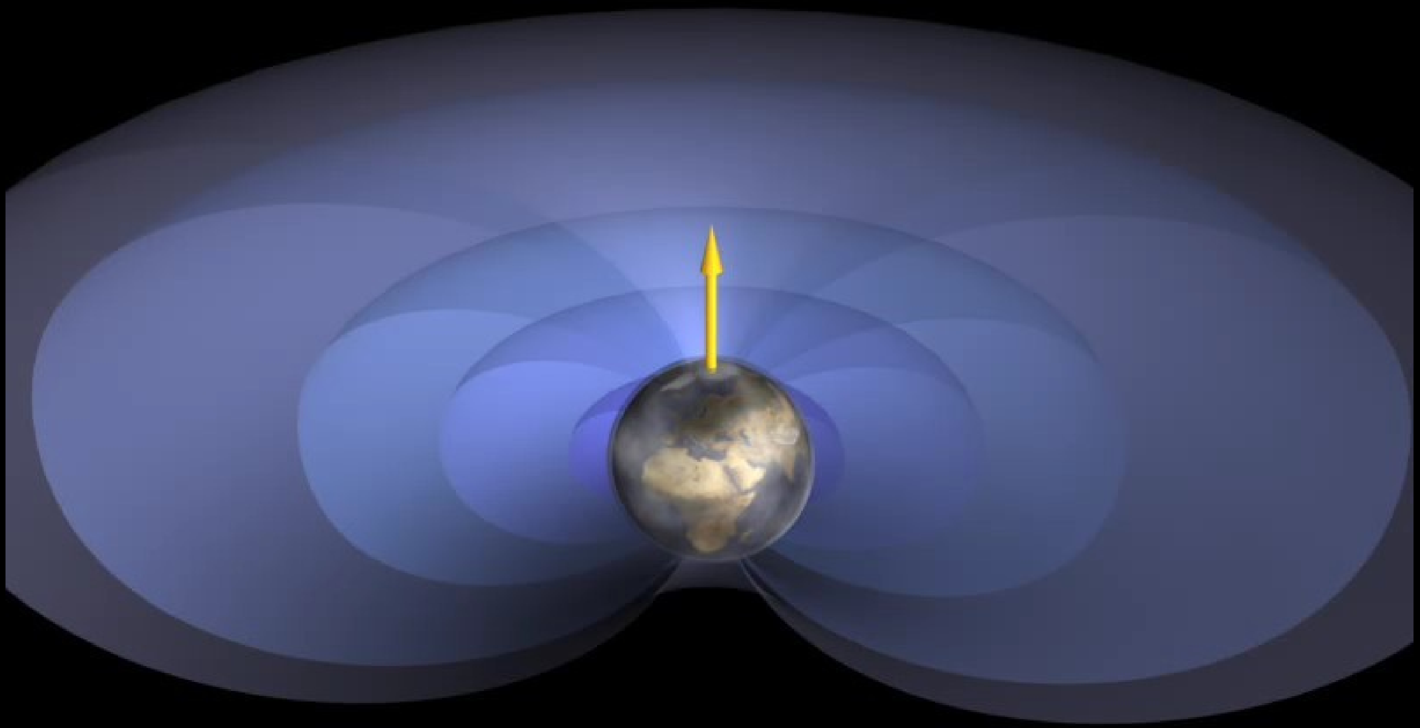
$$\boxed{ds^2 = -(1 - 2U + 2\beta U^2)dt^2 + (1 + 2\gamma U)\,\delta_{ik}\,dx^i\,dx^k - 2r^2\sin^2\theta\,\omega(r)\,dt\,d\phi\,.}$$

$$\omega(r) = \frac{2GJ}{c^2 r^3} \propto \frac{\Omega}{r^3}\,,$$

U **is the metric potential, which is the negati** **potential,**

$$U(\vec{x}) = \int \frac{G\rho(\vec{x}')\,d^3x'}{|\vec{x} - \vec{x}'|} \simeq \frac{GM}{c^2 r}\left[1 - J_2\left(\frac{R}{r}\right)^2 P_2(\cos\theta)\right] \qquad (69)$$

Earth Gravitomagnetic Field H similar to Magnetic Dipole



Origin of the Connection

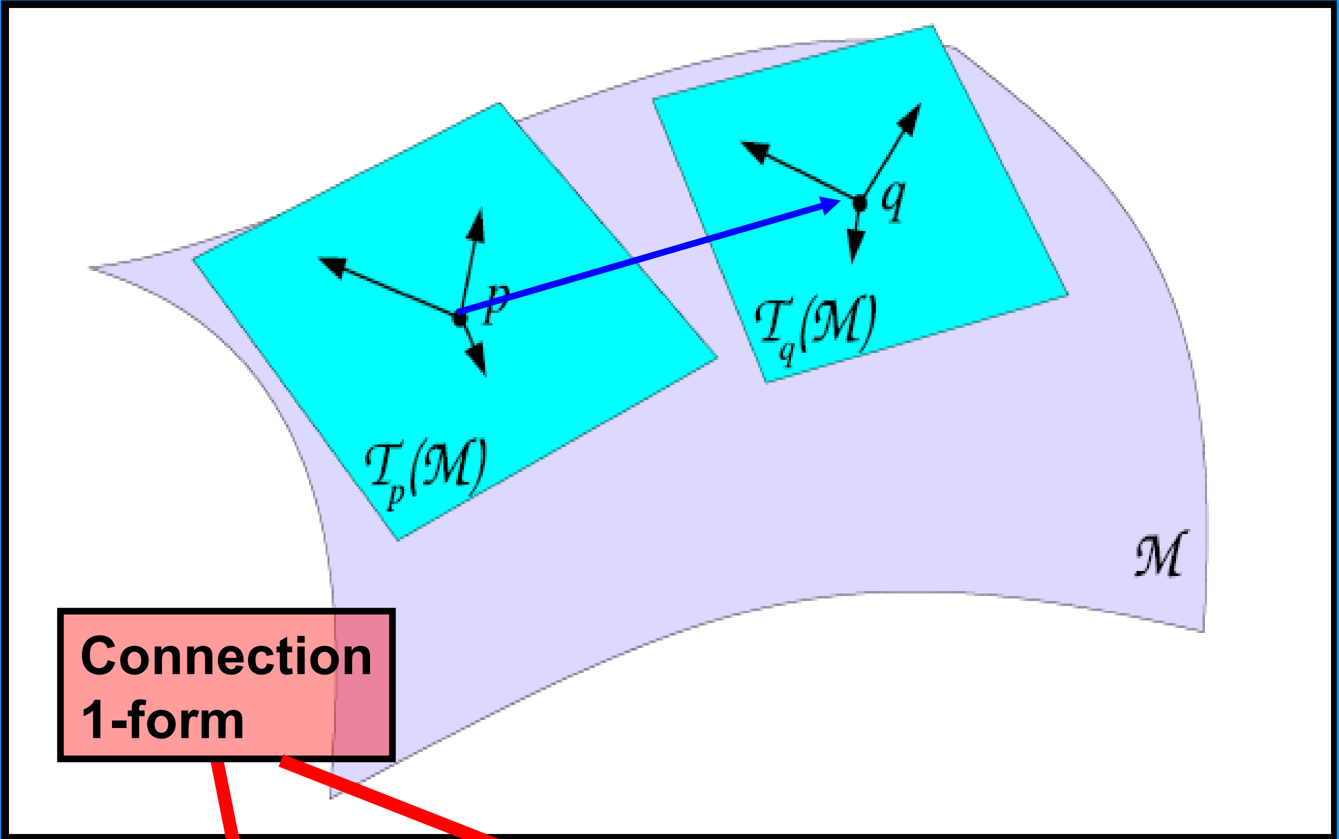
On a general manifold, there is no such natural identification between nearby tangent spaces, and so tangent vectors at nearby points cannot be compared in a well-defined way. The notion of an affine connection was introduced to remedy this problem by connecting nearby tangent spaces. The origins of this idea can be traced back to two main sources: surface theory and tensor calculus.

The second motivation for affine connections comes from the notion of a covariant derivative of vector fields. Before the advent of coordinate-independent methods, it was necessary to work with vector fields using their components in coordinate charts. These components can be differentiated, but the derivatives do not transform in a manageable way under changes of coordinates.

Affine Connection

- **Neighboring tangent spaces are not related**

✂ ➔ **need Connection**



$$\mathrm{e}_a(t+h) = \mathrm{e}_a(t) + \left[\omega(\gamma'(t))^1{}_a\,\mathrm{e}_1 + \ldots + \omega(\gamma'(t))^n{}_a\,\mathrm{e}_n\right] h\,.$$

Definition Let M be a smooth manifold. An *affine connection* on M is a differential operator, sending smooth vector fields X and Y to a smooth vector field $\nabla_X Y$, which satisfies the following conditions:

$$\boxed{\begin{array}{ll} \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z, & \nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z, \\ \nabla_{fX} Y = f \nabla_X Y, & \nabla_X (fY) = X[f] Y + f \nabla_X Y \end{array}}$$

for all smooth vector fields X, Y and Z and real-valued functions f on M. The vector field $\nabla_X Y$ is known as the *covariant derivative* of the vector field Y along X (with respect to the affine connection ∇). The *torsion tensor* T and the *curvature tensor* R of an affine connection ∇ are the operators sending smooth vector fields X, Y and Z on M to the smooth vector fields $T(X,Y)$ and $R(X,Y)Z$ given by

$$\boxed{\begin{array}{rcl} T(X,Y) & = & \nabla_X Y - \nabla_Y X - [X,Y], \\ R(X,Y)Z & = & \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{array}}$$

An affine connection ∇ on M is said to be *torsion-free* if its torsion tensor is everywhere zero (so that $\nabla_X Y - \nabla_Y X = [X,Y]$ for all smooth vector fields X and Y on M).

Covariant Derivative of Vector Fields

In components referred to any local coordinate system

$$\nabla_{\mathbf{X}}\mathbf{Y} = \nabla_{X^i \frac{\partial}{\partial x^i}} Y^j \frac{\partial}{\partial x^j} = X^i Y^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} + X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j}$$

if i , j are fixed $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$ **define a differentiable tensor field which is the derivative of** $\frac{\partial}{\partial x^j}$ **with respect to** $\frac{\partial}{\partial x^i}$ **and thus**

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = < \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, dx^k > \frac{\partial}{\partial x^k} := \Gamma_{ij}^{k} \frac{\partial}{\partial x^k}$$

Covariant Derivative of Vector Fields

$$\Gamma^{j}{}_{mk} \equiv \omega^{j}{}_{k}(\partial_m) .$$

The coefficients $\Gamma_{ij}^{k} = \Gamma_{ij}^{k}(p)$ are differentiable functions of the considered coordinates and are called **connection coefficients**.

Using these coefficients and the above expansion, in components, the covariant derivative of Y with respect to X can be written down as in an orthonormal frame:

$$\nabla_X Y = \left[X.Y^j + \omega^{j}{}_{k}(X)Y^k\right] \mathrm{e}_j .$$

$$Y^{j}{}_{;m} = Y^{j}{}_{,m} + \Gamma^{j}{}_{mk} Y^k$$

Covariant Derivative and Torsion

$$T^i_{jk} := \Gamma^i_{jk} - \Gamma^i_{kj}$$

define a tensor field which is represented in a coordinate system by

$$T(\nabla) = (\Gamma^i_{jk} - \Gamma^i_{kj})\frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k$$

This tensor field is antisymmetric in the covariant indices and is called **torsion tensor field** of the connection.

Connection 1-Forms

Since $\nabla_X \mathbf{e}_j$ is a tensor field of type (1,1), we must have a representation in the given basis

$$\boxed{\nabla_X \mathbf{e}_j = \omega^k_{\ j}(X)\, \mathbf{e}_k\,,} \tag{2.220}$$

where $\omega^k_{\ j}$ are one–forms. Accordingly, we may write

$$\nabla_X Y = (X.Y^j)\mathbf{e}_j + Y^j \omega^k_{\ j}(X)\mathbf{e}_k\,. \tag{2.221}$$

Let

$$\omega^k_{\ j}(\mathbf{e}_m) \equiv \omega^k_{\ mj} \tag{2.222}$$

be the coefficients of the expansion of $\omega^k_{\ j}$. **Elements of so(1,3)**

From here **we conclude that a connection ∇ is specified by the n^2 one–forms $\omega^k_{\ j}$ (called connection one–forms), i.e. by n^3 scalar fields $\omega^k_{\ mj}$**, where $n = \mathbf{Dim}(M)$. In a 4D space-time, a connection is given by 64 coefficients.

Covariant Derivative of 1-Forms

Covariant Derivative of One–Forms

We apply this rule to the special tensor product between a one–form α

$$\nabla_X(\alpha \otimes Y) = (\nabla_X \alpha) \otimes Y + \alpha \otimes (\nabla_X Y).$$

With contraction we can write this as

$$\nabla_X(\alpha(Y)) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y),$$

or as, since $\alpha(Y) = \alpha_j Y^j$ is a scalar,

$$\boxed{(\nabla_X \alpha)(Y) = X.\alpha(Y) - \alpha(\nabla_X Y).}$$

Covariant Derivative of 1-Forms

This leads us to the classical expression

$$\boxed{(\nabla_X \alpha)_j = X^k \alpha_{j;k} = X^k \alpha_{j,k} - \alpha_k \omega^k_{\ j}(X)\,.}$$

In a local coordinate basis, we use

$$\nabla_X(\partial_i) = X^k\, \nabla_{\partial_k}(\partial_i) = X^k\, \Gamma^l_{\ ki}\, \partial_l\,.$$

In addition we have according to (2.231)

$$(\nabla_X\, dx^j)(\partial_i) = X.< dx^j, \partial_i > - < dx^j, \nabla_X \partial_i >= -X^k\, \Gamma^j_{\ ki}\,,$$

or

$$\boxed{\nabla_X\, dx^j = -X^k\, \Gamma^j_{\ ki}\, dx^i\,.}$$

Covariant Derivative of Tens Fields

Finally, we give the local expression for this covariant derivative of a tensor field $S \in \Pi^p_q(M)$

$$S = S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}\, \partial_{i_1} \otimes \cdots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}\,. \tag{2.240}$$

With the above expressions for $\nabla_X(\partial_i)$ and $\nabla_X dx^j$ we obtain an expression for the components of ∇S denoted by $S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q ; k} \equiv \nabla_k S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$

$$\boxed{S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q ; k} = S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q , k} + \Gamma^{i_1}{}_{kl} S^{l i_2 \cdots i_p}{}_{j_1 \cdots j_q} + \cdots - \Gamma^{l}{}_{k j_1} S^{i_1 \cdots i_p}{}_{l j_2 \cdots j_q} - \cdots\,.} \tag{2.241}$$

The covariant derivatives of vector fields $X = \xi^i(x)\partial_i$ and one–forms $\alpha = \alpha_i(x)dx^i$ are then special cases

$$\xi^i{}_{;k} = \xi^i{}_{,k} + \Gamma^i{}_{km}\, \xi^m \tag{2.242}$$

$$\alpha_{i;k} = \alpha_{i,k} - \alpha_m\, \Gamma^m_{ki}\,. \tag{2.243}$$

Parallel Transport of Vector Fields

Parallel transport is supposed to be the curved-space generalization of the concept of "keeping the vector constant" as we move it along a path; similarly for a tensor of arbitrary rank. Given a curve $x^\mu(\lambda)$, the requirement of constancy of a tensor T along this curve in flat space is simply

$$\frac{dT}{d\lambda} \equiv u^\mu \frac{\partial T}{\partial x^\mu} = 0\,, \tag{2.244}$$

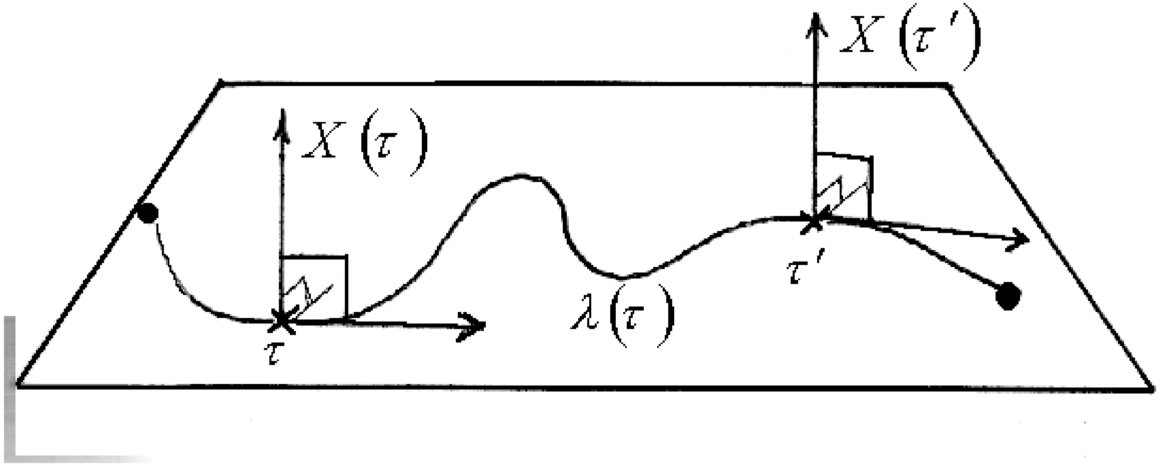
where $u^\mu = dx^\mu/d\lambda$ is the 4–velocity. We therefore define the covariant derivative along the path to be given by an operator

$$\frac{D}{d\lambda} \equiv u^\mu \nabla_\mu\,. \tag{2.245}$$

This is a well-defined tensor equation, since both the tangent vector $dx^\mu/d\lambda$ and the covariant derivative ∇T are tensors. This is known as the **equation of parallel transport**. For a vector it takes the form

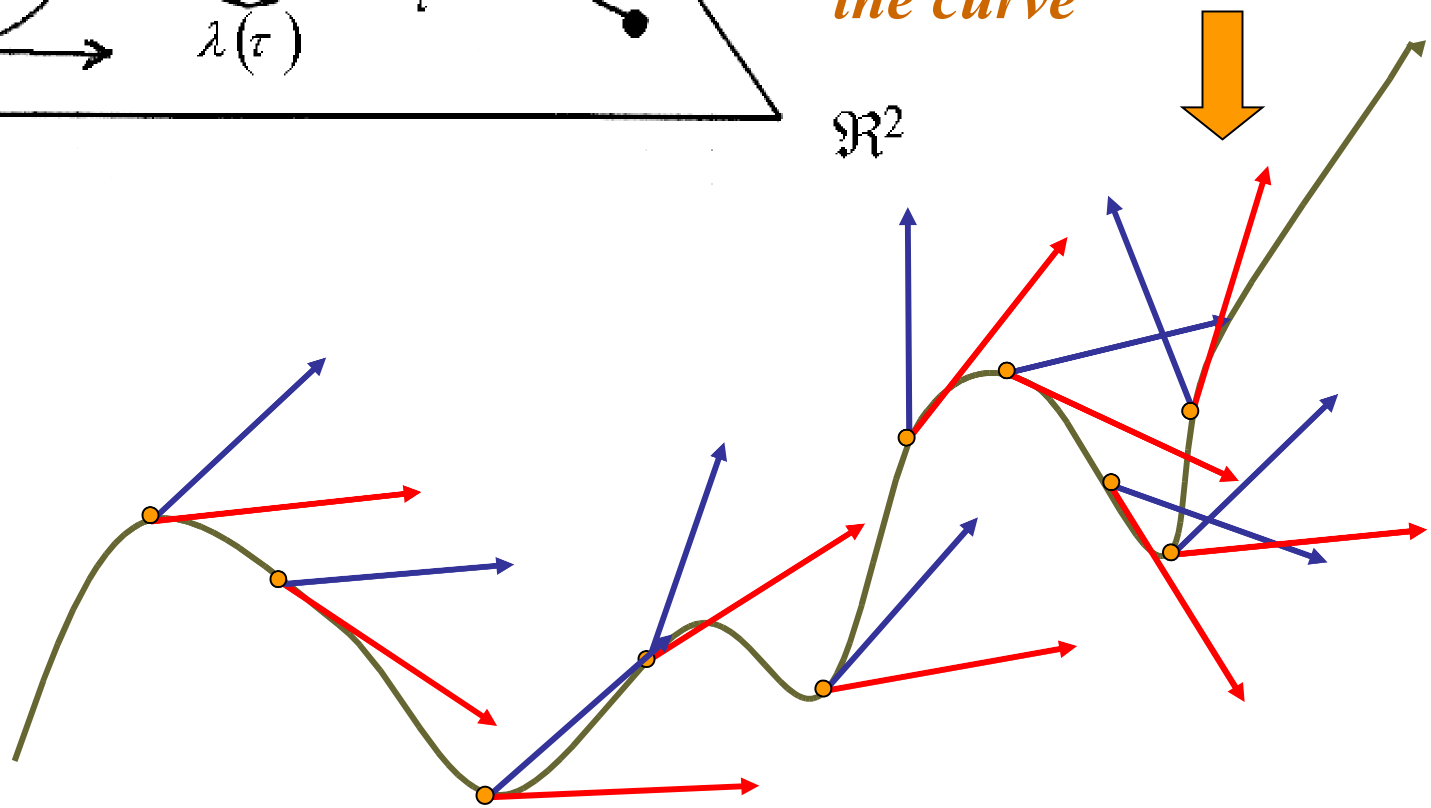
$$\frac{d}{d\lambda} V^\mu + u^\alpha\, \Gamma^\mu{}_{\alpha\beta} V^\beta = 0\,. \tag{2.246}$$

Parallel Transport



A vector field is parallel transported along a curve, when it mantains a constant angle with the tangent vector to the curve

$\Re^2$



Levi-Civita Connection

Theorem 4.3 *Let (M, g) be a Riemannian manifold. Then there exists a unique torsion-free affine connection ∇ on M compatible with the Riemannian metric g. This connection is characterized by the identity*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X[g(Y,Z)] + Y[g(X,Z)] - Z[g(X,Y)] \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned}$$

for all smooth vector fields X, Y and Z on M.

$$g_{mk}\Gamma^m_{ij} = \frac{1}{2}\left(g_{jk,i} + g_{ik,j} - g_{ij,k}\right).$$

$$\boxed{\Gamma^m_{ij} = \frac{1}{2} g^{mk}\left(g_{ki,j} + g_{kj,i} - g_{ij,k}\right).}$$

Divergence

$$\boxed{V^j_{\ ;j} = \frac{1}{\sqrt{g}}\,\partial_j(\sqrt{g}V^j)\,.}$$

This property can also be used to simplify the divergence of a (2,0)–tensor

$$T^{jk}_{\ \ ;k} = \partial_k T^{jk} + \Gamma^j_{\ jm} T^{mk} + \Gamma^k_{\ jm} T^{jm}\,.$$

If T is antisymmetric, the last term drops out

$$A^{jk}_{\ \ ;k} = \frac{1}{\sqrt{g}}\,\partial_k(\sqrt{g}A^{jk})\,.$$

➔ **Maxwell's equations**

If T is symmetric (the energy–momentum tensor e.g.), then

$$\boxed{T^{jk}_{\ \ ;k} = \frac{1}{\sqrt{g}}\,\partial_k(\sqrt{g}T^{jk}) + \Gamma^j_{\ km} T^{km}\,.}$$

➔ **Hydrodynamics**

Geodesics

Let (M, g) be a Riemannian manifold of dimension n, and let $\gamma \colon I \to M$ be a smooth curve in M, defined over some interval I in $\mathbb{R}$. We say that γ is a *geodesic* if and only if

$$\frac{D}{dt}\left(\frac{d\gamma(t)}{dt}\right) = 0.$$

Thus $\gamma \colon I \to U$ is a geodesic if and only if

$$\frac{d^2\gamma^i(t)}{dt^2} + \sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma^i_{jk}(\gamma(t)) \frac{d\gamma^j(t)}{dt}\frac{d\gamma^k(t)}{dt} = 0 \qquad (i = 1, 2, \ldots, n).$$

Length of Curves

Let (M, g) be a Riemannian manifold, and let $\gamma: [a, b] \to M$ be a smooth curve in M. The *length* $L(\gamma)$ of γ is defined by

$$L(\gamma) = \int_a^b |\gamma'(t)| \, dt,$$

where $|\gamma'(t)|^2 = g(\gamma'(t), \gamma'(t))$.

Theorem 6.4 *Let p and q be distinct points in a Riemannian manifold (M, g), and let $\gamma: [a, b] \to M$ be a piecewise smooth curve in M from p to q, parameterized by arclength. Suppose that the length of γ is less than or equal to the length of every other piecewise smooth curve from p to q. Then γ is a smooth (unbroken) geodesic in M.*

Riemann Curvature

The *Riemann curvature tensor* R of a Riemannian manifold (M, g) is given by the formula

$$R(W, Z, X, Y) = g(W, R(X, Y)Z),$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all smooth vector fields X, Y and Z on M. It follows immediately from Lemma 3.3 that the value of $R(W, Z, X, Y)$ at a point m of M depends only on the values W_m, Z_m, X_m and Y_m of the vector fields W, Z, X and Y at the point m.

Curvature Identities

Proposition 4.4 *Let (M, g) be a Riemannian manifold. The Riemann curvature tensor on M satisfies the following identities:—*

(i) $R(W, Z, X, Y) = -R(W, Z, Y, X)$,

(ii) $R(W, X, Y, Z) + R(W, Y, Z, X) + R(W, Z, X, Y) = 0$,

(iii) $R(W, Z, X, Y) = -R(Z, W, X, Y)$,

(iv) $R(W, Z, X, Y) = R(X, Y, W, Z)$

for all smooth vector fields X, Y, Z and W on M.

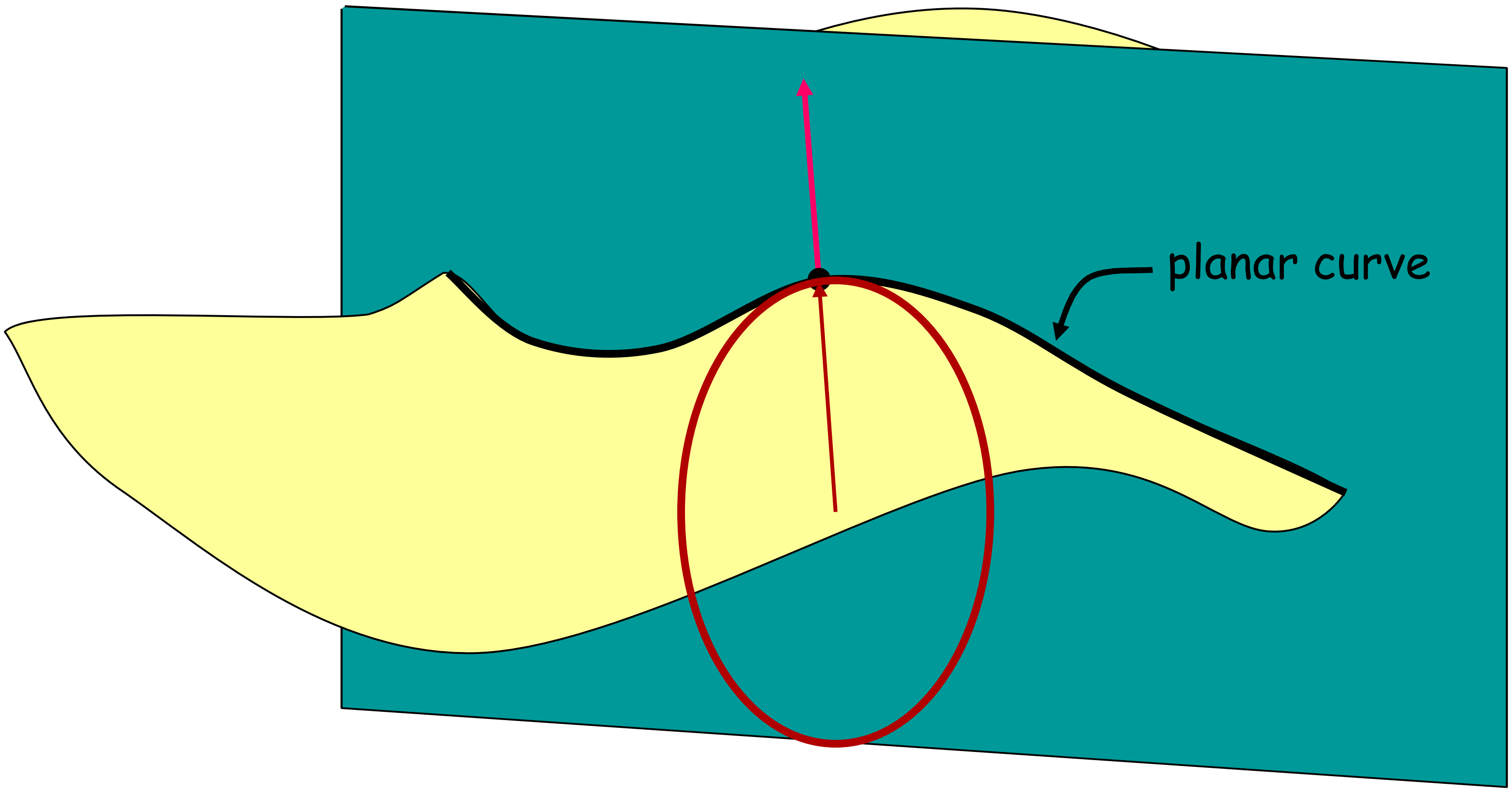
Sectional Curvature

Let m be point in M and let P be a two-dimensional vector subspace (plane) in the tangent space $T_m M$ to M at m. Let (E_1, E_2) be an orthonormal basis of P. We define the *sectional curvature* $K(P)$ of M in the plane P by the formula

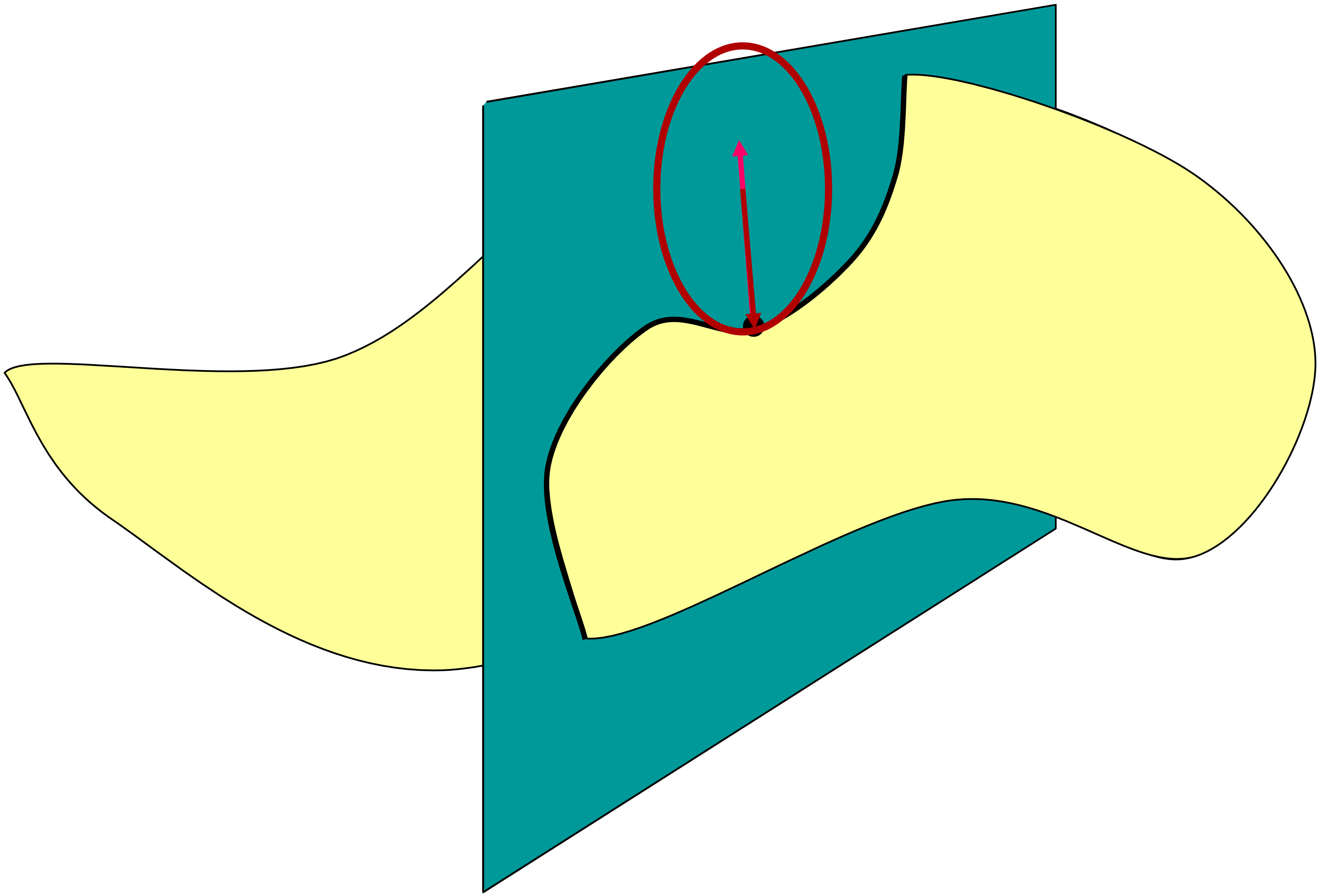
$$K(P) = R(E_1, E_2, E_1, E_2).$$

Lemma 4.5 *Let (M, g) be a Riemannian manifold, and let m be a point of M. Then the values of the sectional curvatures $K(P)$ for all planes P in the tangent space $T_m M$ to M at m determine the Riemann curvature tensor at m.*

Curvature of a surface



Curvature of a surface



Newtonian Gravity Tidal Forces

- In Newtonian theory we compare the evolution of two particles separated by a connecting vector $\eta^{\alpha}(t)$.
- Using Newton's equations and Taylor expanding the potential at point Q using the potential at P,

$$\ddot{\eta}^{\alpha} + K^{\alpha}{}_{\beta}\eta^{\beta} = 0$$

- Since $K^{a}{}_{b} = \partial^{a}\partial_{b}\phi$, $K^{a}{}_{a} = 0$, implies $\nabla^{2}\phi = 0$
- The equation of deviation contains within it the vacuum Poisson equation.
- The same analysis is extended to GR.

Newtonian Deviation

We can use Newtonian physics to study the motion of a pair of neighbouring particles in three space. In cartesian coordinates we have

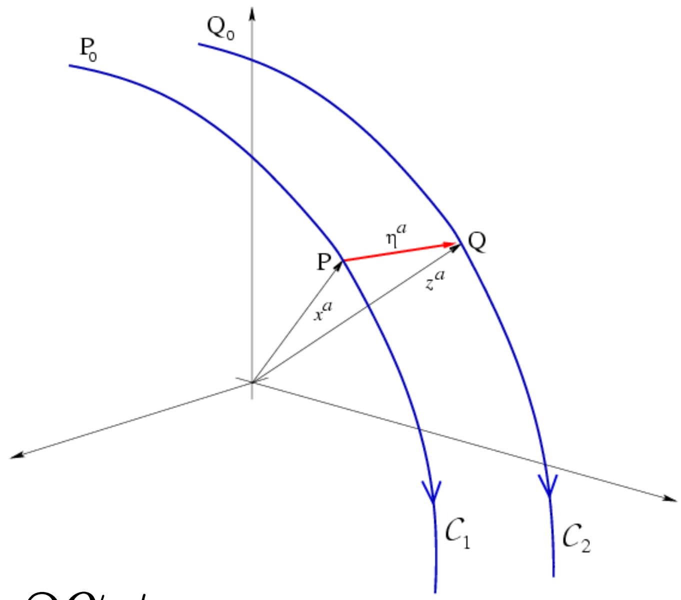
$$g_{\alpha\beta} = \delta_{\alpha\beta} = diag(1, 1, 1)$$

where the greek indicies run from 1... 3. Parameterized in terms of Newtonian time we can describe the particle motions as

$$\begin{aligned} x_1^\alpha &= x^\alpha(t) \\ x_2^\alpha &= x^\alpha(t) + \eta^\alpha(t) \end{aligned}$$

where $\eta^\alpha(t)$ is a small connecting vector.

Newtonian Deviation



Write the equations of motion a

$$\begin{aligned} \ddot{x}^\alpha &= -\partial^\alpha \phi \\ \ddot{x}^\alpha + \ddot{\eta}^\alpha &= -\partial^\alpha \phi - \eta^\beta \partial_\beta \partial^\alpha \phi \end{aligned}$$

where ϕ is the gravitational potential. The expression for the connecting vector is Taylor expanded about the path of the first particle (and so the derivatives are on the x^α path).

$$\partial^\alpha \phi = \delta^{\alpha\beta} \partial_\beta \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = grad(\phi)$$

GR Geodesic Deviation

Suppose we take a congruence of time-like geodesics labeled by their proper-time and a **selector parameter ν** i.e. $x^a = x^a(\tau, \nu)$. We can define a tangent vector and connecting vector ξ, such that

$$\begin{aligned} v^a &= dx^a/d\tau \\ \xi^a &= dx^a/d\nu \end{aligned}$$

We now need to make use of a Riemann tensor identity

$$\nabla_X(\nabla_Y Z^a) - \nabla_Y(\nabla_X Z^a) - \nabla_{[X,Y]} Z^a = R^a{}_{bcd} Z^b X^c Y^d$$

Remember: $\nabla_X Z^a = X^c \nabla_c Z^a$

Geodesic Deviation – Congruence of Geodesics

$$\mathcal{L}_{\mathbf{V}}\xi^{\beta} =$$

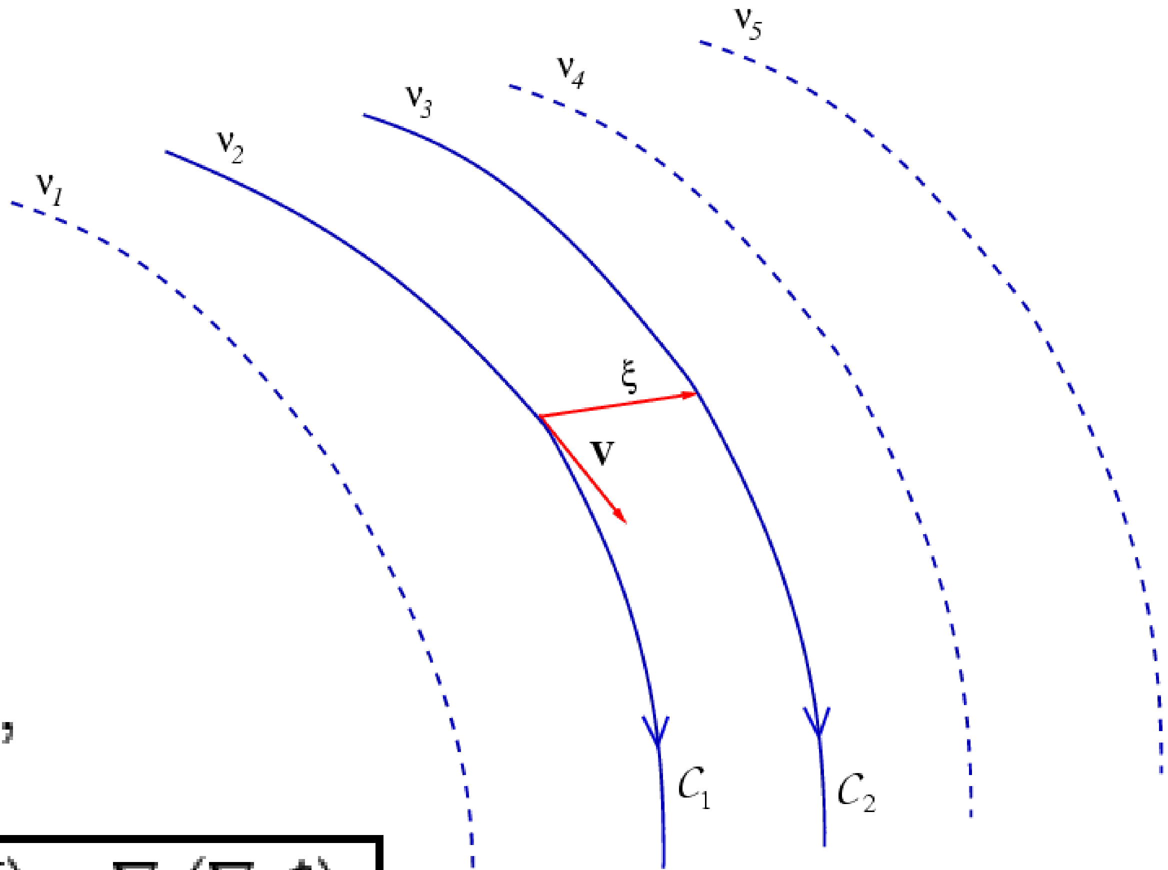
$$V^{\alpha}\partial_{\alpha}\xi^{\beta} - \xi^{\alpha}\partial_{\alpha}V^{\beta} =$$

$$\xi^{\beta}{}_{,\alpha}V^{\alpha} - V^{\beta}{}_{,\alpha}\xi^{\alpha} =$$

$$\xi^{\beta}{}_{;\alpha}V^{\alpha} - V^{\beta}{}_{;\alpha}\xi^{\alpha} =$$

$$\nabla_{\mathbf{V}}\xi^{\beta} - \nabla_{\boldsymbol{\xi}}V^{\beta} = 0\,,$$

$$\rightarrow \quad \boxed{\nabla_{\mathbf{V}}(\nabla_{\boldsymbol{\xi}}\mathbf{V}) = \nabla_{\mathbf{V}}(\nabla_{\mathbf{V}}\boldsymbol{\xi})\,.}$$



Equation of Geodetic Deviation

Setting $X^a = Z^a = v^a$ and $Y^a = \xi^a$ then

• the second term vanishes as v^a is tangent to the geodesic and is parallely transported ($\nabla_v v^a = 0$).

• the third term vanishes as the derivative with respect to τ and ν commute.

The result is

$$\boxed{\frac{D^2\xi^a}{D\tau^2} = \nabla_v \nabla_v \xi^a = R^a{}_{bcd} v^b v^c \xi^d}$$

Equation of Geodetic Deviation

Contracting with the spatial basis vectors results in a equation which is only dependent upon the spatial components of the orthogonal connecting vector

$$\begin{aligned}\frac{D^2\eta^\alpha}{D\tau^2} + K^\alpha{}_\beta\eta^\beta &= 0\\ K^\alpha{}_\beta &= -R^a{}_{bcd}e^\alpha{}_a v^b v^c e_\beta{}^d\end{aligned}$$

This form relates geometry to the physical separation of objects and will give the same results as the above Newtonian deviation equation.

Story So Far…

- **Equivalence principle implies special relativity is regained locally in a free-falling frame.**
- **Cannot distinguish locally a gravitational field from acceleration and hence we should treat gravity as an inertial force.**
- **Following SR we assume free particles follow time-like geodesics, with forces appearing though metric connections.**
- **The metric plays the role of a set of potentials. We can use these to determine a set of (tensorial) second order PDEs.**

Story So Far…

- **Genuine gravitational effects can be observed (nonlocally) where there is a variation in the field. This causes particles to move of converging/diverging geodesics described by the Riemann tensor via the geodesic deviation equation.**

- **The Riemann tensor involves second derivatives of the metric, and hence it may appear in the field equations (but, it has 20 components). There is one meaningful contraction of this tensor (Ricci tensor) which is related to the Einstein tensor and only has 10 components ➔ Ricci tensor.**

Bianchi Identities & Contractions

$\forall\ \nabla_a R_{debc} + \nabla_c R_{deab} + \nabla_b R_{deca} = 0$ ➔ hom. ED

- An important contraction is the **Ricci tensor** $R_{ab} = R^c{}_{acb}$
- Further contraction gives the **Ricci scalar**: $R = g_{ab} R^{ab} = R^a{}_a$
- These definitions lead to the **Einstein tensor**: $G_{ab} = R_{ab} - ½Rg_{ab}$
- Obeys the contracted Bianchi identity: $\nabla_b G_a{}^b = 0$.

The Ricci objects

From the Riemann tensor, we can define the Ricci tensor. It is defined through contraction of the tensor with Itself. This may seem strange, but if we have a mixed tensor, as we do here, this is a perfectly well-defined operation:

$$R^{a}{}_{bcd} \xrightarrow[\text{a and c}]{\text{Contraction over}} R^{a}{}_{bad} = R_{bd}$$

We may further contract the Ricci tensor, to the Ricci scalar. However, since the 2 indices are covariant, before we can contract, we have to raise one index. The metric helps us here to give:

$$g^{ab} R_{bd} = R^{a}{}_{d} \xrightarrow[\text{a and d}]{\text{Contraction Over}} R^{a}{}_{a} = R$$

The Riemann tensor R_{ijkl} is antisymmetric in both pairs of indices. By these symmetries, the only non–trivial contraction we can make is that leading to the **Ricci tensor** R_{ij} defined as

$$\boxed{R_{mn} = R^{j}{}_{mjn} = R_{nm}} \tag{2.309}$$

and its trace $R = R^m{}_m$, the **Ricci scalar**. Accordingly, it will be convenient to separate the Riemann tensor into trace–free parts and the Ricci part

$$\begin{aligned} C_{ijkl} = R_{ijkl} \quad & - \quad \frac{1}{n-2}\left[g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk}\right] \\ & + \quad \frac{1}{(n-1)(n-2)} R\left[g_{ik}g_{jl} - g_{il}g_{jk}\right]. \end{aligned} \tag{2.310}$$

This **Weyl tensor** C_{ijkl} has the same symmetries as the curvature tensor R_{ijkl}. The Weyl tensor is trace–free in the sense that

$$g^{jl}C_{ijkl} = 0\,. \tag{2.311}$$

The Weyl tensor is only defined, when the manifold carries a metric. The most important property is its conformal invariance against transformations of the type $g \to \Omega^2(x)g$. **In 4D, 10 components of the curvature tensor are in the Ricci tensor, while the other 10 components are given by the Weyl tensor.**

The Weyl Tensor

Now that we know what each term in the expression of the Weyl tensor is,

Components of the raised index Riemann tensor, $R_{abcd} = g_{ae}R^{e}{}_{bcd}$

The Metric

$$\boxed{C_{abcd}} = \boxed{R_{abcd}} - \frac{1}{n-2}\left(g_{ac}R_{db} - \boxed{g_{ad}}R_{cb} + g_{bd}R_{ca} - g_{bc}\boxed{R_{da}}\right) + \frac{1}{(n-1)(n-2)}\left(g_{ac}g_{db} - g_{ad}g_{cb}\right)\boxed{R}$$

The Ricci Tensor

The Ricci scalar

Components of the Weyl Tensor

Einstein's Field Equations 1915

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$$

Einstein Field Equations

Source: all types of matter

Trajectories of freely falling bodies are geodesics

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0$$

Geodesic Equation

Levi-Civita connection

Summary

- SpaceTime is the set of all events, it has the structure of a pseudo-Riemannian manifold with a **metric tensor field g**.
- Einstein's gravity assumes the connection to be metric, i.e. the **Levi-Civita connection**.
- Freely falling objects follow **geodesics** on this manifold, also self-gravitating ones (SEP).
- The Einstein tensor is coupled to the energy-momentum tensor of **all type of matter** in the spacetime (including fields and vacuum).

Spin Connection and Curvature

Exterior derivative: $d^2 = 0$

$$\beta = f(x)\,\alpha \;\rightarrow\; d\beta = df \wedge \alpha + f\, d\alpha$$

→ Cartan equations:

$$T^a = d\theta^a + \omega^a{}_b \wedge \theta^b \quad \text{: 1st struct equ}$$

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \text{ : 2nd struct equ}$$

Bianchi identities: $D = d + \omega\wedge$

$$DT^a = \Omega^a{}_b \wedge \theta^b \qquad D\Omega^a{}_b = 0$$