Published Works Of Srinivasa Ramanujan

Srinivasa Ramanujan (born December 22, 1887, Erode, India — died April 26, 1920, Kumbakonam) was an Indian mathematician who lived during the British Rule in India. Though he had almost no formal training in pure mathematics, he made substantial contributions to the theory of numbers include pioneering discoveries of the properties of the partition function.

Srinivasa Ramanujan

Born	22 December 1887 Erode, Madras Presidency, British India (present-day Tamil Nadu, India)
Died	26 April 1920 (aged 32) Kumbakonam, Madras Presidency, British India (present-day Tamil Nadu, India)
Residence	• Kumbakonam, Madras Presidency, British India (present-day Tamil Nadu, India) • Madras, Madras Presidency, British India (present-day Chennai, Tamil Nadu, India) • London, England, United Kingdom of Great Britain and Ireland (present-day United Kingdom)
Nationality	Indian

Education	Government Arts College (no degree) Pachaiyappa's College (no degree) Trinity College, Cambridge (BSc, 1916)
Known for	Landau–Ramanujan constant Mock theta functions Ramanujan conjecture Ramanujan prime Ramanujan–Soldner constant Ramanujan theta function Ramanujan's sum Rogers–Ramanujan identities Ramanujan's master theorem Ramanujan–Sato series
Awards	Fellow of the Royal Society
Scientific career	
Fields	Mathematics
Institutions	Trinity College, Cambridge
Thesis	Highly Composite Numbers (1916)
Academic advisors	G. H. Hardy J. E. Littlewood
Influences	G. S. Carr
Influenced	G. H. Hardy
Signature	

Some properties of Bernoulli's numbers

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1. Let the well-known expansion of $x \cot x$ (*vide* Edwards' *Differential Calculus*, §149) be written in the form

$$x \cot x = 1 - \frac{B_2}{2!}(2x)^2 - \frac{B_4}{4!}(2x)^4 - \frac{B_6}{6!}(2x)^6 - \cdots, \tag{1}$$

from which we infer that B_0 may be supposed to be -1. Now

$$\begin{aligned}
\cot x &= \frac{\cos x}{\sin x} = \frac{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots}{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots} \\
&= \frac{\sin 2x}{1 - \cos 2x} = \frac{\dfrac{2x}{1!} - \dfrac{(2x)^3}{3!} + \dfrac{(2x)^5}{5!} - \cdots}{\dfrac{(2x)^2}{2!} - \dfrac{(2x)^4}{4!} + \dfrac{(2x)^6}{6!} - \cdots} \\
&= \frac{1 + \cos 2x}{\sin 2x} = \frac{2 - \dfrac{(2x)^2}{2!} + \dfrac{(2x)^4}{4!} - \dfrac{(2x)^6}{6!} + \cdots}{2x - \dfrac{(2x)^3}{3!} + \dfrac{(2x)^5}{5!} - \dfrac{(2x)^7}{7!} + \cdots}
\end{aligned}$$

Multiplying both sides in each of the above three relations by the deno minator of the right-hand side and equating the coefficients of x^n on both sides, we can write the results thus:

$$c_1 \frac{B_{n-1}}{2} - c_3 \frac{B_{n-3}}{2^3} + c_5 \frac{B_{n-5}}{2^5} - \cdots + \frac{(-1)^{\frac{1}{2}(n-1)}}{2^n} B_0 + \frac{n}{2^n}(-1)^{\frac{1}{2}(n-1)} = 0, \tag{2}$$

where n is any odd integer;

$$c_2 B_{n-2} - c_4 B_{n-4} + c_6 B_{n-6} - \cdots + (-1)^{\frac{1}{2}(n-2)} B_0 + \frac{n}{2}(-1)^{\frac{1}{2}(n-2)} = 0, \tag{3}$$

where n is any even integer;

$$c_1 B_{n-1} - c_3 B_{n-3} + c_5 B_{n-5} - \cdots + (-1)^{\frac{1}{2}(n-1)} B_0 + \frac{n}{2}(-1)^{\frac{1}{2}(n-1)} = 0, \tag{4}$$

where n is any odd integer greater than unity.

From any one of (2), (3), (4) we can calculate the B's. But as n becomes greater and greater the calculation will get tedious. So we shall try to find simpler methods.

2. We know

$$(x\cot x)^2 = -x^2\left(1+\frac{d\cot x}{dx}\right).$$

Using (1) and equating the coefficients of x^n on both sides, and simplifying, we have

$$\tfrac{1}{2}(n+1)B_n = c_2B_2B_{n-2} + c_4B_4B_{n-4} + c_6B_6B_{n-6} + \cdots,$$

the last term being $c_{\frac{n}{2}-1}B_{\frac{n}{2}-1}B_{\frac{n}{2}+1}$ or $\frac{1}{2}c_{\frac{n}{2}}(B_{\frac{n}{2}})^2$ according as $\frac{n}{2}$ is odd or even. . . . (5)
A similar result can be obtained by equating the coefficients of x^n in the identity

$$\frac{d\tan x}{dx} = 1+\tan^2 x.$$

3. Again

$$\begin{aligned}-\tfrac{1}{2}x(\cot\tfrac{1}{2}x+\coth\tfrac{1}{2}x) &= -\tfrac{1}{2}x(\cot\tfrac{1}{2}x+i\cot\tfrac{1}{2}ix)\\ &= 2\left\{B_0+B_4\frac{x^4}{4!}+B_8\frac{x^8}{8!}+\cdots\right\},\end{aligned}$$

by using (1). The expression may also be written

$$\begin{aligned}-\tfrac{1}{2}x\frac{(\cos\frac{1}{2}x\sin\frac{1}{2}ix+i\sin\frac{1}{2}x\cos\frac{1}{2}ix)}{\sin\frac{1}{2}x\sin\frac{1}{2}ix} &= -\tfrac{1}{2}x\frac{(1+i)\sin\frac{1}{2}x(1+i)-(1-i)\sin\frac{1}{2}x(1-i)}{\cos\frac{1}{2}x(1-i)-\cos\frac{1}{2}x(1+i)}\\ &= -x\frac{\dfrac{x}{1!}-\dfrac{x^5}{2^2\cdot 5!}+\dfrac{x^9}{2^4\cdot 9!}-\cdots}{\dfrac{x^2}{2!}-\dfrac{x^6}{2^2\cdot 6!}+\dfrac{x^{10}}{2^4\cdot 10!}-\cdots}\end{aligned}$$

by expanding the numerator and the denominator, and simplifying by De Moivre's theorem. Hence

$$2\left(B_0+B_4\frac{x^4}{4!}+B_8\frac{x^8}{8!}+\cdots\right) = -x\frac{\dfrac{x}{1!}-\dfrac{x^5}{2^2\cdot 5!}+\cdots}{\dfrac{x^2}{2!}-\dfrac{x^6}{2^2\cdot 6!}+\cdots}. \tag{6}$$

Similarly

$$\begin{aligned}-\tfrac{1}{2}x\left(\cot\tfrac{1}{2}x-\coth\tfrac{1}{2}x\right) &= 2\left(B_2\frac{x^2}{2!}+B_6\frac{x^6}{6!}+B_{10}\frac{x^{10}}{10!}+\cdots\right)\\ &= x\frac{\frac{1}{2}(1-i)\sin\frac{1}{2}x(1+i)-\frac{1}{2}(1+i)\sin\frac{1}{2}x(1-i)}{\cos\frac{1}{2}x(1+i)-\cos\frac{1}{2}x(1-i)}\\ &= x\frac{\dfrac{x^3}{2\cdot 3!}-\dfrac{x^7}{2^3\cdot 7!}+\dfrac{x^{11}}{2^5\cdot 11!}+\cdots}{\dfrac{x^2}{2!}-\dfrac{x^6}{2^2\cdot 6!}+\dfrac{x^{10}}{2^4\cdot 10!}-\cdots}.\end{aligned} \tag{7}$$

Proceeding as in §1 we have, if n is an even integer greater than 2,

$$c_2\frac{B_{n-2}}{2} - c_6\frac{B_{n-6}}{2^3} + c_{10}\frac{B_{n-10}}{2^5} - \cdots + \frac{n}{2^{\frac{1}{2}(n+2)}}(-1)^{\frac{1}{4}n} \quad \text{or} \quad \frac{n}{2^{\frac{1}{2}(n+2)}}(-1)^{\frac{1}{4}(n-2)} = 0, \tag{8}$$

according as n or $n-2$ is a multiple of 4.
Analogous results can be obtained from $\tan\frac{1}{2}x \pm \tanh\frac{1}{2}x$.
In (2), (3) and (4) there are $\frac{1}{2}n$ terms, while in (5) and (8) there are $\frac{1}{4}n$ or $\frac{1}{4}(n-2)$ terms. Thus B_n can be found from only half of the previous B's.

4. A still simpler method can be deduced from the following identities.
If 1, ω, ω^2 be the three cube roots of unity, then

$$4\sin x \sin x\omega \sin x\omega^2 = -(\sin 2x + \sin 2x\omega + \sin 2x\omega^2),$$

as may easily be verified.
By logarithmic differentiation, we have

$$\cot x + \omega\cot x\omega + \omega^2\cot x\omega^2 = 2\frac{\cos 2x + \omega\cos 2x\omega + \omega^2\cos 2x\omega^2}{\sin 2x + \sin 2x\omega + \sin 2x\omega^2}.$$

Writing $\frac{1}{2}x$ for x,

$$-\tfrac{1}{2}x(\cot\tfrac{1}{2}x + \omega\cot\tfrac{1}{2}x\omega + \omega^2\cot\tfrac{1}{2}x\omega^2) = -x\frac{\cos x + \omega\cos x\omega + \omega^2\cos x\omega^2}{\sin x + \sin x\omega + \sin x\omega^2}$$

and, proceeding as in §3, we get

$$3\left(B_0 + B_6\frac{x^6}{6!} + B_{12}\frac{x^{12}}{12!} + \cdots\right) = -x\frac{\dfrac{x^2}{2!} - \dfrac{x^8}{8!} + \dfrac{x^{14}}{14!} - \cdots}{\dfrac{x^3}{3!} - \dfrac{x^9}{9!} + \dfrac{x^{15}}{15!} - \cdots}. \tag{9}$$

Again

$$\cot\tfrac{1}{2}x\omega - \cot\tfrac{1}{2}x\omega^2 = \frac{\cos x\omega^2 - \cos x\omega}{2\sin\frac{1}{2}x\sin\frac{1}{2}x\omega\sin\frac{1}{2}x\omega^2} = \frac{2(\cos x\omega - \cos x\omega^2)}{\sin x + \sin x\omega + \sin x\omega^2}.$$

Multiplying both sides by $-\frac{1}{2}x(\omega^2-\omega)$ and adding to the corresponding sides of the previous result, we have

$$-\tfrac{1}{2}x(\cot\tfrac{1}{2}x + \omega^2\cot\tfrac{1}{2}x\omega + \omega\cot\tfrac{1}{2}x\omega^2) = -x\frac{\cos x + \omega^2\cos x\omega + \omega\cos x\omega^2}{\sin x + \sin x\omega + \sin x\omega^2}.$$

Hence, as before,

$$3\left(B_2\frac{x^2}{2!} + B_8\frac{x^8}{8!} + B_{14}\frac{x^{14}}{14!} + \cdots\right) = x\frac{\dfrac{x^4}{4!} - \dfrac{x^{10}}{10!} + \dfrac{x^{16}}{16!} - \cdots}{\dfrac{x^3}{3!} - \dfrac{x^9}{9!} + \dfrac{x^{15}}{15!} - \cdots}. \tag{10}$$

Similarly

$$-x(\cot \tfrac{1}{2}x + \cot \tfrac{1}{2}x\omega + \cot \tfrac{1}{2}x\omega^2) = x\frac{\cos x + \cos x\omega + \cos x\omega^2 - 3}{\sin x + \sin x\omega + \sin x\omega^2},$$

and therefore

$$6\left(B_4\frac{x^4}{4!} + B_{10}\frac{x^{10}}{10!} + B_{16}\frac{x^{16}}{16!} + \cdots\right) = x\frac{\dfrac{x^6}{6!} - \dfrac{x^{12}}{12!} + \dfrac{x^{18}}{18!} - \cdots}{\dfrac{x^3}{3!} - \dfrac{x^9}{9!} + \dfrac{x^{15}}{15!} - \cdots}. \tag{11}$$

Multiplying up and equating coefficients in (9), (10) and (11) as usual, we have,

$$c_3B_{n-3} - c_9B_{n-9} + c_{15}B_{n-15} - \cdots = 0, \tag{12}$$

the last term being $\frac{1}{6}n(-1)^{\frac{1}{6}(n-1)}$, $\frac{1}{3}n(-1)^{\frac{1}{6}(n+1)}$, $\frac{1}{3}n(-1)^{\frac{1}{6}(n-3)}$.
Again, dividing both sides in (10) by x and differentiating, we have

$$3\left(B_2\frac{1}{2!} + 7B_8\frac{x^6}{8!} + 13B_{14}\frac{x^{12}}{14!} + \cdots\right) = \frac{d}{dx}\left(\frac{\dfrac{x^4}{4!} - \dfrac{x^{10}}{10!} + \dfrac{x^{16}}{16!} - \cdots}{\dfrac{x^3}{3!} - \dfrac{x^9}{9!} + \dfrac{x^{15}}{15!} - \cdots}\right)$$
$$= 1 - \frac{\dfrac{x^2}{2!} - \dfrac{x^8}{8!} + \dfrac{x^{14}}{14!} - \cdots}{\dfrac{x^3}{3!} - \dfrac{x^9}{9!} + \dfrac{x^{15}}{15!} - \cdots} \cdot \frac{\dfrac{x^4}{4!} - \dfrac{x^{10}}{10!} + \dfrac{x^{16}}{16!} - \cdots}{\dfrac{x^3}{3!} - \dfrac{x^9}{9!} + \dfrac{x^{15}}{15!} - \cdots}.$$

Hence by (9) and (10),

$$3\left(B_2\frac{x^2}{2!} + 7B_8\frac{x^8}{8!} + 13B_{14}\frac{x^{14}}{14!} + \cdots\right)$$
$$= x^2 + 9\left(B_0 + B_6\frac{x^6}{6!} + B_{12}\frac{x^{12}}{12!} + \cdots\right)\left(B_2\frac{x^2}{2!} + B_8\frac{x^8}{8!} + B_{14}\frac{x^{14}}{14!} + \cdots\right).$$

Equating the coefficients of x^n we have, if $n > 2$ and $n - 2$ is a multiple of 6,

$$\tfrac{1}{3}(n+2)B_n = c_6B_{n-6}B_6 + c_{12}B_{n-12}B_{12} + c_{18}B_{n-18}B_{18} + \cdots. \tag{13}$$

From (12) the B's can be calculated very quickly and (13) may prove useful in checking the calculations. The number of terms is one-third of that in (4); thus B_{24} is found from B_{18}, B_{12} and B_6.

5. We shall see later on how the B's can be obtained from their properties only. But to know these properties, it will be convenient to calculate a few B's by substituting $3, 5, 7, 9, \cdots,$

for n in succession in (12). Thus

$$B_0 = -1;\quad B_2 = \tfrac{1}{6};\quad B_4 = \tfrac{1}{30};\quad B_6 = \tfrac{1}{42};\quad B_8 - \tfrac{1}{3}B_2 = -\tfrac{1}{45};$$

$$B_{10} - \tfrac{5}{2}B_4 = -\tfrac{1}{132};\quad B_{12} - 11B_6 = -\tfrac{4}{455};\quad B_{14} - \tfrac{143}{4}B_8 + \frac{B_2}{5} = \tfrac{1}{120};$$

$$B_{16} - \tfrac{286}{3}B_{10} + 4B_4 = \tfrac{1}{306};\quad B_{18} - 221B_{12} + \tfrac{204}{5}B_6 = \tfrac{3}{665};$$

$$B_{20} - \tfrac{3230}{7}B_{14} + \tfrac{1938}{7}B_8 - \frac{B_2}{7} = -\tfrac{1}{231};$$

$$B_{22} - \tfrac{3553}{4}B_{16} + \tfrac{7106}{5}B_{10} - \tfrac{11}{2}B_4 = -\tfrac{1}{552};$$

and so on. Hence we have finally the following values:

$$B_2 = \tfrac{1}{6};\quad B_4 = \tfrac{1}{30};\quad B_6 = \tfrac{1}{42};\quad B_8 = \tfrac{1}{30};\quad B_{10} = \tfrac{5}{66};\quad B_{12} = \tfrac{691}{2730};\quad B_{14} = \tfrac{7}{6};$$

$$B_{16} = \tfrac{3617}{510};\quad B_{18} = \tfrac{43867}{798};\quad B_{20} = \tfrac{174611}{330};\quad B_{22} = \tfrac{854513}{138};\quad B_{24} = \tfrac{236364091}{2730};$$

$$B_{26} = \tfrac{8553103}{6};\quad B_{28} = \tfrac{23749461029}{870};\quad B_{30} = \tfrac{8615841276005}{14322};$$

$$B_{32} = \tfrac{7709321041217}{510};\quad B_{34} = \tfrac{2577687858367}{6};\quad B_{36} = \tfrac{26315271553053477373}{1919190};$$

$$B_{38} = \tfrac{2929993913841559}{6};\quad B_{40} = \tfrac{261082718496449122051}{13530};\cdots, B_\infty = \infty.$$

6. It will be observed* that, if n is even but not equal to zero,
(i) B_n is a fraction and the numerator of B_n/n in its lowest terms is a prime number, (14)
(ii) the denominator of B_n contains each of the factors 2 and 3 once and only once, (15)
(iii) $2^n(2^n - 1)B_n/n$ is an integer and consequently $2(2^n - 1)B_n$ is an *odd* integer. (16)

From (16) it can easily be shewn that the denominator of $2(2^n - 1)B_n/n$ in its lowest terms is the greatest power of 2 which divides n; and consequently, if n is not a multiple of 4, then $4(2^n - 1)B_n/n$ is an odd integer. ...(17)

It follows from (14) that the numerator of B_n in its lowest terms is divisible by the greatest measure of n prime to the denominator, and the quotient is a prime number. ...(18)

Examples: (a) 2 and 3 are the only prime factors of 12, 24 and 36 and they are found in the denominators of B_{12}, B_{24} and B_{36} and their numerators are prime numbers.

*See §12 below

(b) 11 is not found in the denominator of B_{22}, and hence its numerator is divisible by 11; similarly, the numerators of B_{26}, B_{34}, B_{38} are divisible by 13, 17, 19, respectively and the quotients in all cases are prime numbers.
(c) 5 is found in the denominator of B_{20} and not in that of B_{30}, and consequently the numerator of B_{30} is divisible by 5 while that of B_{20} is a prime number. Thus we may say that if a prime number appearing in n is not found in the denominator it will appear in the numerator, and *vice versa.*

7. Next, let us consider the denominators.
All the denominators are divisible by 6; those of $B_4, B_8, B_{12}, \cdots$ by 5; those of $B_6, B_{12}, B_{18}, \cdots$ by 7; those of $B_{10}, B_{20}, B_{30}, \cdots$ by 11; but those of $B_8, B_{16}, B_{24}, \cdots$ are *not* divisible by 9; and those of $B_{14}, B_{28}, \cdots$ are *not* divisible by 15. Hence we may infer that:
the denominator of B_n is the continued product of prime numbers which are the next numbers (in the natural order) to the factors of n (including unity and the number itself) ...(19)
As an example take the denominator of B_{24}. Write all the factors of 24, viz. 1, 2, 3, 4, 6, 8, 12, 24. The next numbers to these are 2, 3, 4, 5, 7, 9, 13, 25. Strike out the *composite* numbers and we have the prime numbers 2, 3, 5, 7, 13. And the denominator of B_{24} is the product of 2, 3, 5, 7, 13, i.e., 2730.
It is unnecessary to write the *odd* factors of n except unity, as the next numbers to these are even and hence composite.
The following are some further examples:

Even factors of n and unity		Denominator of B_n
B_2	$\cdots$ 1, 2, $\cdots$ $\cdots$ $\cdots$ $\cdots$	$2 \cdot 3 = 6$
B_6	$\cdots$ 1, 2, 6 $\cdots$ $\cdots$ $\cdots$ $\cdots$	$2 \cdot 3 \cdot 7 = 42$
B_{12}	$\cdots$ 1, 2, 4, 6, 12 $\cdots$ $\cdots$ $\cdots$	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = 2730$
B_{20}	$\cdots$ 1, 2, 4, 10, 20 $\cdots$ $\cdots$ $\cdots$	$2 \cdot 3 \cdot 5 \cdot 11 = 330$
B_{30}	$\cdots$ 1, 2, 6, 10, 30 $\cdots$ $\cdots$ $\cdots$	$2 \cdot 3 \cdot 7 \cdot 11 \cdot 31 = 14322$
B_{42}	$\cdots$ 1, 2, 6, 14, 42 $\cdots$ $\cdots$ $\cdots$	$2 \cdot 3 \cdot 7 \cdot 43 = 1806$
B_{56}	$\cdots$ 1, 2, 4, 8, 14, 28, 56 $\cdots$ $\cdots$	$2 \cdot 3 \cdot 5 \cdot 29 = 870$
B_{72}	$\cdots$ 1, 2, 4, 6, 8, 12, 18, 24, 36, 72 $\cdots$	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 = 140100870$
B_{90}	$\cdots$ 1, 2, 6, 10, 18, 30, 90 $\cdots$ $\cdots$	$2 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 31 = 272118$
B_{110}	$\cdots$ 1, 2, 10, 22, 110 $\cdots$ $\cdots$ $\cdots$	$2 \cdot 3 \cdot 11 \cdot 23 = 1518$

8. Again taking the fractional part of any B and splitting it into partial fractions, we see that:
the fractional part of $B_n = (-1)^{\frac{1}{2}n}$\{the sum of the reciprocals of the prime factors of the denominator of B_n\} $- (-1)^{\frac{1}{2}n}$. (20)

$$\text{Thus the fractional part of } B_{16} \;=\; \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{17} - 1 = \tfrac{47}{510};$$

that of $B_{22} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{23} = \frac{17}{138}$;
that of $B_{28} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{29} - 1 = \frac{59}{870}$; and so on.

9. It can be inferred from (20) that:
If G be the G.C.M. and L the L.C.M. of the denominators of B_m and B_n, then L/G is the denominator of $B_m - (-1)^{\frac{1}{2}(m-n)}B_n$, and hence, if the denominators of B_m and B_n are equal, then $B_m - (-1)^{\frac{1}{2}(m-n)}B_n$ is an integer. (21)
Example: $B_{24} - B_{12}$ and $B_{32} - B_{16}$ are integers, while the denominator of $B_{10} + B_{20}$ is 5.
It will be observed that:
(1) If n is a multiple of 4, then the numerator of $B_n - \frac{1}{30}$ in its lowest terms is divisible by 20; but if n is not a multiple of 4 then that of $\frac{B_n}{n} - \frac{1}{12}$ in its lowest terms is divisible by 5; (22)
(2) If n is any integer, then

$$2(2^{4n+2} - 1)\frac{B_{4n+2}}{2n+1}, \quad 2(2^{8n+4} - 1)\frac{B_{8n+4}}{2n+1}, \quad 2(2^{8n+4} - 1)\frac{B_{16n+8}}{2n+1}$$

are integers of the form $30p + 1$. (23)

10. If a B is known to lie between certain limits, then it is possible to find its exact value from the above properties.
Suppose we know that B_{22} lies between 6084 and 6244; its exact value can be found as follows.
The fractional part of $B_{22} = \frac{17}{138}$ by (20), also B_{22} is divisible by 11 by (18). And by (22) $B_{22} - \frac{11}{6}$ must be divisible by 5. To satisfy these conditions B_{23} must be either $6137\frac{17}{138}$ or $6192\frac{17}{138}$.
But according to (18) the numerator of B_{22} should be a *prime* number after it is divided by 11; and consequently B_{22} must be equal to $6192\frac{17}{138}$ or $\frac{854513}{138}$, since the numerator of $6137\frac{17}{138}$ is divisible not only by 11 but also by 7 and 17.

11. It is known (Edward's *Differential calculus*, Ch. v, Ex.29) that

$$B_n = \frac{2 \cdot n!}{(2\pi)^n}\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \cdots\right),$$

or

$$\frac{2 \cdot n!}{(2\pi)^n} = B_n\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right) \tag{24}$$

where $2, 3, 5, \ldots$ are prime numbers.
Also

$$\frac{B_n}{2n} = \int_0^\infty \frac{x^{n-1}}{e^{2\pi x} - 1}dx. \tag{25}$$

For

$$\int_0^\infty \frac{x^{n-1}}{e^{2\pi x}-1}dx = \int_0^\infty x^{n-1}(e^{-2\pi x}+e^{-4\pi x}+\cdots)dx$$
$$= \frac{(n-1)!}{(2\pi)^n}\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\right) = \frac{B_n}{2n}$$

by (24). In a similar manner

$$\int_0^\infty \frac{x^n}{(e^{\pi x}-e^{-\pi x})^2}dx = \frac{B_n}{4\pi},$$

and

$$\int_0^\infty x^{n-2}\log(1-e^{-2\pi x})dx = -\frac{\pi B_n}{n(n-1)}. \tag{26}$$

Take logarithms of both sides in (24) and write for $\log_e n!$ the well known expansion of $\log_e \Gamma(n+1)$, as in Carr's *Synopsis*, viz.

$$(n+\tfrac{1}{2})\log n - n + \tfrac{1}{2}\log 2\pi + \frac{B_2}{1\cdot 2n} - \frac{B_4}{3\cdot 4n^3} + \frac{B_6}{5\cdot 6n^5} - \cdots$$
$$- (-1)^p \frac{B_{2p}\theta}{(2p-1)2pn^{2p-1}}, \tag{27}$$

where $0<\theta<1$, and where

$$\begin{aligned}
\frac{B_{2p}\theta}{(2p-1)2pn^{2p-1}} &= \frac{B_{2p}}{(2p-1)2pn^{2p-1}} - \frac{B_{2p+2}}{(2p+1)(2p+2)n^{2p+1}} + \cdots \\
&= -\frac{1}{\pi}\int_0^\infty \frac{x^{2p-2}}{n^{2p-1}}\log(1-e^{-2\pi x})dx + \frac{1}{\pi}\int_0^\infty \frac{x^{2p}}{n^{2p+1}}\log(1-e^{-2\pi x})dx - \cdots \\
&= -\frac{1}{\pi}\int_0^\infty \left(\frac{x^{2p-2}}{n^{2p-1}} - \frac{x^{2p}}{n^{2p+1}} + \cdots\right)\log(1-e^{-2\pi x})dx \\
&= -\frac{1}{\pi}\int_0^\infty \frac{x^{2p-2}}{n^{2p-3}(n^2+x^2)}\log(1-e^{-2\pi x})dx \\
&= -\int_0^\infty \frac{x^{2p-2}\log(1-e^{-2\pi nx})}{\pi(1+x^2)}dx.
\end{aligned}$$

We can find the integral part of B_n, and since the fractional part can be found, as shewn in §8, the exact value of B_n is known. Unless the calculation is made to depend upon the values of $\log_{10} e, \log_e 10, \pi, \ldots,$ which are known to a great number of decimal places, we should have to find the logarithms of certain numbers whose values are not found in the tables to as many places of decimals as we require. Such difficulties are removed by the method given in §13.

12. Results (14) to (17), (20) and (21) can be obtained as follows. We have

$$\begin{aligned}\frac{1}{2x^2} &+ \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \cdots \\ &- \frac{1}{x} - \frac{1}{6(x^3-x)} + \frac{1}{5(x^5-x)} + \frac{1}{7(x^7-x)} + \frac{1}{11(x^{11}-x)} + \cdots \\ &= \frac{1}{x^{15}} - \frac{7}{x^{17}} + \frac{55}{x^{19}} - \frac{529}{x^{21}} + \cdots \end{aligned} \tag{28}$$

where 5, 7, 11, 13, are prime numbers above 3. If we can prove that the left-hand side of (28) can be expanded in ascending powers of $1/x$ with integral coefficients, then (20) and (21) are at once deduced as follows.
From (27) we have

$$\begin{aligned}\frac{d^2 \log\Gamma(n+1)}{dn^2} &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \cdots \\ &= \frac{1}{n} - \frac{1}{2n^2} + \frac{B_2}{n^3} - \frac{B_4}{n^5} + \frac{B_6}{n^7} - \frac{B_8}{n^9} + \cdots - (-)^p \frac{B_{2p}\theta}{n^{2p+1}}, \end{aligned} \tag{29}$$

where

$$\begin{aligned}\frac{B_{2p}\theta}{n^{2p+1}} &= \frac{B_{2p}}{n^{2p+1}} - \frac{B_{2p+2}}{n^{2p+3}} + \cdots \\ &= 4\pi \int_0^\infty \frac{x^{2p}}{n^{2p+1}(e^{\pi x} - e^{-\pi x})^2} dx - 4\pi \int_0^\infty \frac{x^{2p+2}}{n^{2p+3}(e^{\pi x} - e^{-\pi x})^2} dx + \cdots \\ &= \pi \int_0^\infty \left(\frac{x^{2p}}{n^{2p+1}} - \frac{x^{2p+2}}{n^{2p+3}} + \cdots \right) \frac{dx}{\sinh^2 \pi x} \\ &= \pi \int_0^\infty \frac{x^{2p}}{n^{2p-1}(n^2+x^2)} \frac{dx}{\sinh^2 \pi x} = \int_0^\infty \frac{\pi x^{2p}}{(1+x^2)\sinh^2 \pi n x} dx.\end{aligned}$$

Substituting the result of (29) in (28) we see that

$$\frac{B_2}{x^3} - \frac{B_4}{x^5} + \frac{B_6}{x^7} - \cdots - \frac{1}{6(x^3-x)} + \frac{1}{5(x^5-x)} + \frac{1}{7(x^7-x)} + \frac{1}{11(x^{11}-x)} + \cdots,$$

where $5, 7, 11, \ldots$ are prime numbers, can be expanded in ascending powers of $1/x$ with integral coefficients.
Therefore $B_2 - \frac{1}{6}$, $-B_4 - \frac{1}{6} + \frac{1}{5}$, $B_6 - \frac{1}{6} + \frac{1}{7}$, $-B_8 - \frac{1}{6} + \frac{1}{5}$, $B_{10} - \frac{1}{6} + \frac{1}{11}, \cdots$, which are the coefficients of $1/x^3, 1/x^5, 1/x^7, \cdots$, are integers.
Writing $\frac{1}{2} + \frac{1}{3} - 1$ for $-\frac{1}{6}$ we get the results of (20) and (21).
Again changing n to $\frac{1}{2}n$ in (29), and subtracting half of the result from (29), we have

$$\begin{aligned}\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} - \cdots = \frac{1}{2n^2} - \frac{(2^2-1)B_2}{n^3} + \frac{(2^4-1)B_4}{n^5} \\ -\frac{(2^6-1)B_6}{n^7} + \cdots + (-1)^p(2^{2p}-1)\frac{B_{2p}\theta}{n^{2p+1}},\end{aligned} \tag{30}$$

where $0 < \theta < 1$, and also, by (29),

$$(2^{2p}-1)\frac{B_{2p}\theta}{n^{2p+1}} = \int_0^\infty \frac{\pi x^{2p}\cosh \pi n x}{(1+x^2)\sinh^2 \pi n x}dx.$$

Thus we see that, if we can prove that twice the left hand side of (30) can be expanded in ascending powers of $1/n$ with integral coefficients, then the second part of (16) is at once proved.
Again from (27) we have

$$\begin{aligned}\frac{d\log\Gamma(n+1)}{dn} &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \frac{1}{n} - \gamma \\ &= \log n + \frac{1}{2n} - \frac{B_2}{2n^2} + \frac{B_4}{4n^4} - \frac{B_6}{6n^6} + \frac{B_8}{8n^8} - \cdots + (-1)^p\frac{B_{2p}\theta}{2pn^{2p}},\end{aligned} \tag{31}$$

where $0 < \theta < 1$; and also, by (25),

$$\frac{B_{2p}\theta}{2pn^{2p}} = \int_0^\infty \frac{2x^{2p-1}}{(1+x^2)(e^{2\pi n x}-1)}dx,$$

from which it can easily be shewn that

$$\begin{aligned}&\frac{1}{n+2} - \frac{1}{n+4} + \frac{1}{n+6} - \frac{1}{n+8} + \frac{1}{n+10} - \cdots \\ &\quad = \frac{1}{2n} - 2(2^2-1)\frac{B_2}{2n^2} + 2^3(2^4-1)\frac{B_4}{4n^4} - 2^5(2^6-1)\frac{B_6}{6n^6} + \cdots \\ &\quad\quad +(-1)^p 2^{2p-1}(2^{2p}-1)\frac{B_{2p}\theta}{2pn^{2p}} - \cdots,\end{aligned} \tag{32}$$

where $0 < \theta < 1$; and also, by (31),

$$2^{2p-1}(2^{2p}-1)\frac{B_{2p}\theta}{2pn^{2p}} = \int_0^\infty \frac{x^{2p-1}}{2(1+x^2)\sinh\frac{1}{2}(\pi n x)}dx.$$

From the above theorem we see that, if we can prove that

$$2\left(\frac{1}{n+2} - \frac{1}{n+4} + \frac{1}{n+6} - \cdots\right),$$

can be expanded in ascending powers of $1/n$ with integral coefficients, then the first part of (16) at once follows.

13. The first few digits, and the number of digits in the integral part as well as in the numerator of B_n, can be found from the approximate formula:

$$\log_{10} B_n = (n+\tfrac{1}{2})\log_{10} n - 1.2324743503n + 0.700120,$$

the true value being greater by about $0.0362/n$ when n is great. (33)
This formula is proved as follows: taking logarithms of both sides in (24),

$$\log_e B_n = (n+\tfrac{1}{2})\log_e n - n(1+\log_e 2\pi) + \tfrac{1}{2}\log_e 8\pi$$

nearly. Multiplying both sides by $\log_{10} e$ or .4342944819, and reducing, we can get the result.

14. Changing n to $n-2$ in (24) and taking the ratio of the two results, we have

$$B_n = \frac{n(n-1)}{4\pi^2}B_{n-2}\left(1-\frac{2^2-1}{2^n-1}\right)\left(1-\frac{3^2-1}{3^n-1}\right)\left(1-\frac{5^2-1}{5^n-1}\right)\cdots, \tag{34}$$

where $2, 3, 5, \ldots$ are prime numbers.
Hence we see that $\frac{B_n}{B_{n-2}}$ approaches $\frac{n(n-1)}{4\pi^2}$ very rapidly as n becomes greater and greater. (35)
From the value of π, viz. 3.14159, 26535, 89793, 23846, 26433, 83279, 50288, 41971, 69399, $\ldots$, the integral part of any B can be found from the previous B; and from the integral part the exact value can at once be written by help of (20) as follows:

Approximate ratio of any B to previous B			lies between*				Hence the exact value is	
B_2	$\cdots$	$\cdots$	0	and	1	$\cdots$	$1-\frac{1}{2}-\frac{1}{3}$	$=\frac{1}{6}$
B_4	$=\frac{3\cdot4}{4\pi^2}B_2$	$\cdots$	0	and	1	$\cdots$	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-1$	$=\frac{1}{30}$
B_6	$=\frac{5\cdot6}{4\pi^2}B_4$	$\cdots$	0	and	1	$\cdots$	$1-\frac{1}{2}-\frac{1}{3}-\frac{1}{7}$	$=\frac{1}{42}$
B_8	$=\frac{7\cdot8}{4\pi^2}B_6$	$\cdots$	0	and	1	$\cdots$	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-1$	$=\frac{1}{30}$
B_{10}	$=\frac{9\cdot10}{4\pi^2}B_8$	$\cdots$	0	and	1	$\cdots$	$1-\frac{1}{2}-\frac{1}{3}-\frac{1}{11}$	$=\frac{5}{66}$
B_{12}	$=\frac{11\cdot12}{4\pi^2}B_{10}$	$\cdots$	0	and	1	$\cdots$	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}-1$	$=\frac{691}{2730}$
B_{14}	$=\frac{13\cdot14}{4\pi^2}B_{12}$	$\cdots$	0	and	2	$\cdots$	$2-\frac{1}{2}-\frac{1}{3}$	$=\frac{7}{6}$
B_{16}	$=\frac{15\cdot16}{4\pi^2}B_{14}$	$\cdots$	7	and	8	$\cdots$	$6+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{17}$	$=\frac{3617}{510}$
B_{18}	$=\frac{17\cdot18}{4\pi^2}B_{16}$	$\cdots$	54	and	55	$\cdots$	$56-\frac{1}{2}-\frac{1}{3}-\frac{1}{7}-\frac{1}{19}$	$=\frac{43867}{798}$
B_{20}	$=\frac{19\cdot20}{4\pi^2}B_{18}$	$\cdots$	529	and	530	$\cdots$	$528+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{11}$	$=\frac{174611}{330}$
$\cdots$	$\cdots$	$\cdots$						

15. From the preceding theorems we know some of the properties of B_n for all positive even values of n. As an example let us take $B_{444} = N/D$.
The fractional part of B_{444} is $\frac{23975417}{90709710}$ by (20). The numerator of B_{444} is divisible by 37 and the quotient is a prime number by (18). Again $\log_{10} B_{444} = 630\cdot2433$, nearly, by (33). Therefore the integral part of B_{444} contains 631 digits, the first 4 digits being 1751. Again

$$\log_{10} N = \log_{10} B_{444} + \log_{10} D = 630\cdot2433 + \log_{10} 90709710 = 638\cdot2010$$

nearly. Therefore N contains 639 digits, the first four digits being 1588.
Again the numerator of $B_{444} - \frac{1}{30}$ is divisible by 20; that is to say, if $[B_{444}]$ is the integral part of B_{444}, $[B_{444}] + \frac{23975417}{90709710} - \frac{1}{30} = [B_{444}] + \frac{698392}{3023657}$ has a numerator divisible by 20. Therefore the integral part of B_{444} ends with 4 and also the figure next to the last is even. Hence N ends with 57 and also the third figure from the last is even.

16. Instead of starting with $\cot x$ as in §§ 2 and 3, we may start with $\tan x$ or $\operatorname{cosec} x$ and get other similar results.
Thus

$$\begin{aligned}(i)\ \tfrac{4}{3}B_n(2^n-1) &= c_6B_{n-6}(2^{n-6}-1) - c_{12}B_{n-12}(2^{n-12}-1) + \cdots \\ &+\ \tfrac{1}{3}n(-1)^{\frac{1}{6}(n-2)} \text{ or } \tfrac{1}{6}n(-1)^{\frac{1}{6}(n-4)} \text{ or } \tfrac{1}{3}n(-1)^{\frac{1}{6}(n-6)}\cdots, \qquad (36)\end{aligned}$$

*These integral limits are got from a rough calculation of any B from the preceding B by the formula given in the first column.

$$(ii)\ c_3\left(1-\frac{1}{2^{n-4}}\right)B_{n-3}-c_9\left(1-\frac{1}{2^{n-10}}\right)B_{n-9}+c_{15}\left(1-\frac{1}{2^{n-16}}\right)B_{n-15}-\cdots$$
$$=\tfrac{2}{3}\cdot\frac{n}{2^n}\{3^{\frac{1}{2}(n-1)}+(-1)^{\frac{1}{6}(n-3)}\ \text{or}\ (-1)^{\frac{1}{6}(n+1)}\ \text{or}\ (-1)^{\frac{1}{6}(n+5)}\}. \tag{37}$$

17. The formulæ obtained in §§1, 3, 4 may be called the one interval, two interval and three interval formula respectively. The p interval formulæ can be got by taking the pth roots of unity or of i according as p is odd or even.
For example, let us take the fifth roots of unity $(1,\alpha,\alpha^2,\alpha^3,\alpha^4)$, and find the 5 interval formulæ.
Let

$$\begin{aligned}\phi(x) &= \sin x+\sin x\alpha+\sin x\alpha^2+\sin x\alpha^3+\sin x\alpha^4\\ &= 5\left(\frac{x^5}{5!}-\frac{x^{15}}{15!}+\frac{x^{25}}{25!}-\cdots\right).\end{aligned}$$

Then it can easily be shewn that

$$\begin{aligned}16\sin x\sin x\alpha\sin x\alpha^2\sin x\alpha^3\sin x\alpha^4 &= \phi(2x)-\phi\{2x(\alpha+\alpha^4)\}-\phi\{2x(\alpha^2+\alpha^3)\}\\ &= \phi(2x)+\phi\{x(1+\sqrt{5})\}+\phi\{x(1-\sqrt{5})\}.\end{aligned}$$

Taking logarithms and differentiating both sides, we have

$$5\left(B_0+B_{10}\frac{x^{10}}{10!}+B_{20}\frac{x^{20}}{20!}+B_{30}\frac{x^{30}}{30!}+\cdots\right)=-x\frac{\frac{x^4}{4!}(1+\alpha_5)-\frac{x^{14}}{14!}(1+\alpha_{15})+\cdots}{\frac{x^5}{5!}(1+\alpha_5)-\frac{x^{15}}{15!}(1+\alpha_{15})+\cdots}, \tag{38}$$

where

$$\alpha_n=\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n,\quad \text{so that } \alpha_n\alpha_m=\alpha_{n+m}+(-1)^n\alpha_{m-n}.$$

Similarly

$$\text{(i)}\ 5\left(B_2\frac{x^2}{2!}+B_{12}\frac{x^{12}}{12!}+B_{22}\frac{x^{22}}{22!}+\cdots\right)=x\frac{\frac{x^6}{6!}(1+\alpha_7)-\frac{x^{16}}{16!}(1+\alpha_{17})+\cdots}{\frac{x^5}{5!}(1+\alpha_5)-\frac{x^{15}}{15!}(1+\alpha_{15})+\cdots}, \tag{39}$$

$$\text{(ii)}\ 5\left(B_4\frac{x^4}{4!}+B_{14}\frac{x^{14}}{14!}+B_{24}\frac{x^{24}}{24!}+\cdots\right)=x\frac{\frac{x^8}{8!}(\alpha_7-1)-\frac{x^{18}}{18!}(\alpha_{17}-1)+\cdots}{\frac{x^5}{5!}(1+\alpha_5)-\frac{x^{15}}{15!}(1+\alpha_{15})+\cdots}, \tag{40}$$

(iii) $10\left(B_6\frac{x^6}{6!}+B_{16}\frac{x^{16}}{16!}+B_{26}\frac{x^{26}}{26!}+\cdots\right)$

$$=x\frac{\frac{x^{10}}{10!}(\alpha_{10}-3)-\frac{x^{20}}{20!}(\alpha_{20}-3)+\frac{x^{30}}{30!}(\alpha_{30}-3)-\cdots}{\frac{x^5}{5!}(1+\alpha_5)-\frac{x^{15}}{15!}(1+\alpha_{15})+\frac{x^{25}}{25!}(1+\alpha_{25})-\cdots}, \tag{41}$$

(iv) $5\left(B_8\frac{x^8}{8!}+B_{18}\frac{x^{18}}{18!}+B_{28}\frac{x^{28}}{28!}+\cdots\right)$

$$=x\frac{\frac{x^{12}}{12!}(\alpha_{11}-1)-\frac{x^{22}}{22!}(\alpha_{21}-1)+\frac{x^{32}}{32!}(\alpha_{31}-1)-\cdots}{\frac{x^5}{5!}(1+\alpha_5)-\frac{x^{15}}{15!}(1+\alpha_{15})+\frac{x^{25}}{25!}(1+\alpha_{25})-\cdots}. \tag{42}$$

Again from

$$16\cos x\cos x\alpha\cos x\alpha^2\cos x\alpha^3\cos x\alpha^4$$
$$=1+\Psi[2x]+\Psi[2x(\alpha+\alpha^4)]+\Psi[2x(\alpha^2+\alpha^3)]$$

where

$$\begin{aligned}\Psi(x) &= \cos x+\cos x\alpha+\cos x\alpha^2+\cos\alpha^3+\cos x\alpha^4\\ &= 5\left(1-\frac{x^{10}}{10!}+\frac{x^{20}}{20!}-\frac{x^{30}}{30!}+\cdots\right),\end{aligned}$$

and similar relations, we can get many other identities.

18. The four interval formulæ can be got from the following identities: If

$$a_n=\left(1+\frac{1}{\sqrt{2}}\right)^n+\left(1-\frac{1}{\sqrt{2}}\right)^n \quad\text{and}\quad b_n=\left(1+\frac{1}{\sqrt{2}}\right)^n-\left(1-\frac{1}{\sqrt{2}}\right)^n,$$

so that $a_{m+n}=a_ma_n-a_{m-n}/2^n$; and $b_{m+n}=a_mb_n+b_{m-n}/2^n$; then:

(i) $4\left\{B_0+B_8\frac{x^8}{8!}+B_{16}\frac{x^{16}}{16!}+B_{24}\frac{x^{24}}{24!}+\cdots\right\}$

$$=-x\frac{\frac{x^3}{3!}a_2-\frac{x^{11}}{11!}a_6+\frac{x^{19}}{19!}a_{10}-\cdots}{\frac{x^4}{4!}a_2-\frac{x^{12}}{12!}a_6+\frac{x^{20}}{20!}a_{10}-\cdots}, \tag{43}$$

(ii)
$$4\left\{B_2\frac{x^2}{2!}+B_{10}\frac{x^{10}}{10!}+B_{18}\frac{x^{18}}{18!}+\cdots\right\}$$
$$=-x\frac{\dfrac{x^5}{5!}a_3-\dfrac{x^{13}}{13!}a_7+\dfrac{x^{21}}{21!}a_{11}-\cdots}{\dfrac{x^4}{4!}a_2-\dfrac{x^{12}}{12!}a_6+\dfrac{x^{20}}{20!}a_{10}-\cdots}, \tag{44}$$

(iii)
$$4\sqrt{2}\left\{B_4\frac{x^4}{4!}+B_{12}\frac{x^{12}}{12!}+B_{20}\frac{x^{20}}{20!}+\cdots\right\}$$
$$=x\frac{\dfrac{x^7}{7!}b_3-\dfrac{x^{15}}{15!}b_7+\dfrac{x^{23}}{23!}b_{11}-\cdots}{\dfrac{x^4}{4!}a_2-\dfrac{x^{12}}{12!}a_6+\dfrac{x^{20}}{20!}a_{10}-\cdots}, \tag{45}$$

(iv)
$$4\sqrt{2}\left\{B_6\frac{x^6}{6!}+B_{14}\frac{x^{14}}{14!}+B_{22}\frac{x^{22}}{22!}+\cdots\right\}$$
$$=x\frac{\dfrac{x^9}{9!}b_4-\dfrac{x^{17}}{17!}b_8+\dfrac{x^{25}}{25!}b_{12}-\cdots}{\dfrac{x^4}{4!}a_2-\dfrac{x^{12}}{12!}a_6+\dfrac{x^{20}}{20!}a_{10}-\cdots}. \tag{46}$$

On question 330 of Professor Sanjana

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1. Prof. Sanjana remarks that it is not easy to evaluate the series

$$\frac{1}{1^n}+\frac{1}{2}\frac{1}{3^n}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^n}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{7^n}+\cdots \quad \text{ad} \quad \text{inf.},$$

if $n>3$. In attempting to sum the series for all values of n, I have arrived at the following results:

Let

$$\begin{aligned} f(p) &= \frac{1}{1+p}+\frac{1}{2}\frac{1}{3+p}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5+p}+\cdots \\ &= \int_0^1 x^p\left(1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\cdots\right)dx \\ &= \int_0^1 \frac{x^p}{\sqrt{1-x^2}}dx = \tfrac{1}{2}\int_0^1 x^{\frac{1}{2}(p-1)}(1-x)^{-\frac{1}{2}}dx \\ &= \tfrac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma(\frac{1}{2})}{\Gamma\left(\frac{p+2}{2}\right)} = \frac{\pi^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}. \end{aligned}$$

But

$$\Gamma\left(\tfrac{p+1}{2}\right)=\frac{\pi^{\frac{1}{2}}}{2^p}\frac{\Gamma(p+1)}{\Gamma\left(\frac{p+2}{2}\right)}$$

(*vide* Williamson, *Integral Calculus*, p. 164).

Therefore

$$f(p)=\frac{\pi}{2^{p+1}}\frac{\Gamma(p+1)}{\{\Gamma\left(\frac{p+2}{2}\right)\}^2}.$$

Therefore

$$\begin{aligned} \log\{f(p)\} &= \log(\tfrac{1}{2}\pi)-p\log 2 \\ &\quad +\frac{p^2}{2}\left(1-\frac{1}{2}\right)S_2-\frac{p^3}{3}\left(1-\frac{1}{2^2}\right)S_3+\cdots, \end{aligned} \tag{1}$$

where $S_n\equiv\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots$ ad inf. (*vide* Carr's *Synopsis*, 2295).

Again, by expanding $f(p)$ in ascending powers of p, we have

$$\begin{aligned} f(p) &= \left(1+\frac{1}{2}\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5}+\cdots\right)-p\left(1+\frac{1}{2}\frac{1}{3^2}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^2}+\cdots\right) \\ &\quad +p^2\left(1+\frac{1}{2}\frac{1}{3^3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^3}+\cdots\right)-\cdots \\ &= \frac{\pi}{2}\{\phi(0)-p\phi(1)+p^2\phi(2)-p^3\phi(3)+\cdots\}, \end{aligned}$$

where

$$\frac{1}{1^{n+1}}+\frac{1}{2}\frac{1}{3^{n+1}}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^{n+1}}+\cdots\equiv\frac{\pi}{2}\phi(n).$$

Hence (1) may be written

$$\begin{aligned}\log\tfrac{1}{2}\pi\quad &+\quad \log\{\phi(0)-p\cdot\phi(1)+p^2\cdot\phi(2)-\cdots\}\\ &=\quad \log(\tfrac{1}{2}\pi)-p\log 2+\frac{p^2}{2}\left(1-\frac{1}{2}\right)S_2-\frac{p^3}{3}\left(1-\frac{1}{2^2}\right)S_3+\cdots\\ &=\quad \log(\tfrac{1}{2}\pi)-p\sigma_1+\frac{p^2}{2}\sigma_2-\frac{p^3}{3}\sigma_3+\cdots,\end{aligned}$$

where

$$\sigma_n\equiv 1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\cdots.$$

Differentiating with respect to p, and equating the coefficients of p^{n-1}, we have

$$n\phi(n)\equiv\sigma_1\phi(n-1)+\sigma_2\phi(n-2)+\sigma_3\phi(n-3)+\cdots\quad\text{to } n \text{ terms.}$$

Thus we see that

$$\begin{aligned}\frac{\pi}{2}\phi(0)\equiv 1+\frac{1}{2}\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5}+\cdots &= \frac{\pi}{2},\\ \frac{\pi}{2}\phi(1)\equiv 1+\frac{1}{2}\frac{1}{3^2}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^2}+\cdots &= \frac{\pi}{2}(\log 2),\\ \frac{\pi}{2}\phi(2)\equiv 1+\frac{1}{2}\frac{1}{3^3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^3}+\cdots &= \frac{\pi^3}{48}+\frac{\pi}{4}(\log 2)^2,\\ \frac{\pi}{2}\phi(3)\equiv 1+\frac{1}{2}\frac{1}{3^4}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^4}+\cdots &= \frac{\pi^3}{48}\log 2+\frac{\pi^3}{12}(\log 2)^3+\frac{\pi}{6}\sigma_3\\ &= \frac{\pi^3}{48}\log 2+\frac{\pi^3}{12}(\log 2)^3+\frac{\pi}{8}S_3,\end{aligned}$$

and so on.

2. More generally, consider the series

$$\frac{1}{b^n}-\frac{a}{1!}\frac{1}{(b+1)^n}+\frac{a(a-1)}{2!}\frac{1}{(b+2)^n}-\cdots.$$

Writing

$$\frac{\Gamma(b)\Gamma(a+1)}{\Gamma(a+b+1)}\phi(n-1)$$

for this, and taking the identity

$$\begin{aligned}&\frac{1}{b+p}-\frac{a}{1!}\frac{1}{b+1+p}+\frac{a(a-1)}{2!}\frac{1}{b+2+p}-\cdots\\ &=\int_0^1 x^{b+p-1}(1-x)^a dx=\frac{\Gamma(b+p)\Gamma(a+1)}{\Gamma(a+b+p+1)}\ ,\end{aligned}$$

we find

$$n\phi(n) = \sigma_1\phi(n-1) + \sigma_2\phi(n-2) + \sigma_3\phi(n-3) + \cdots \text{ to } n \text{ terms},$$

where

$$\sigma_n \equiv \frac{1}{b^n} - \frac{1}{(a+b+1)^n} + \frac{1}{(b+1)^n} - \frac{1}{(a+b+2)^n} + \cdots \ .$$

Examples: Put $a = -\frac{1}{2}, b = \frac{1}{4}$. Then we see that

$$(i) \quad 1 + \frac{1}{2}\frac{1}{5} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{9} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13} + \cdots = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}},$$

$$(ii) \quad 1 + \frac{1}{2}\frac{1}{5^2} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{9^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13^2} + \cdots = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}}\frac{\pi}{4},$$

$$(iii) \quad 1 + \frac{1}{2}\frac{1}{5^3} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{9^3} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13^3} + \cdots = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}}\left\{\frac{\pi^2}{32} + \frac{1}{2}S_2'\right\},$$

$$(iv) \quad 1 + \frac{1}{2}\frac{1}{5^4} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{9^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13^4} + \cdots = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}}\left\{\frac{5\pi^3}{384} + \frac{\pi}{8}S_2' + \frac{1}{3}S_3'\right\},$$

where $S_r' = \dfrac{1}{1^r} - \dfrac{1}{3^r} + \dfrac{1}{5^r} - \dfrac{1}{7^r} + \cdots$.

Note on a set of simultaneous equations*

Journal of the Indian Mathematical Society, IV, 1912, 94 – 96

1. Consider the equations

$$\begin{array}{rcl}
x_1 + x_2 + x_3 + & \cdots & +x_n = a_1, \\
x_1y_1 + x_2y_2 + x_3y_3 + & \cdots & +x_ny_n = a_2, \\
x_1y_1^2 + x_2y_2^2 + x_3y_3^2 + & \cdots & +x_ny_n^2 = a_3, \\
x_1y_1^3 + x_2y_2^3 + x_3y_3^3 + & \cdots & +x_ny_n^3 = a_4, \\
\cdots \quad \cdots \quad \cdots \quad \cdots & \cdots & \cdots \quad \cdots \\
x_1y_1^{2n-1} + x_2y_2^{2n-1} + x_3y_3^{2n-1} + & \cdots & +x_ny_n^{2n-1} = a_{2n},
\end{array}$$

where $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_n$ are $2n$ unknown quantities.
Now, let us take the expression

$$\phi(\theta) \equiv \frac{x_1}{1-\theta y_1} + \frac{x_2}{1-\theta y_2} + \frac{x_3}{1-\theta y_3} + \cdots + \frac{x_n}{1-\theta y_n} \tag{1}$$

and expand it in ascending powers of θ. Then we see that the expression is equal to

$$a_1 + a_2\theta + a_3\theta^2 + \cdots + a_{2n}\theta^{2n-1} + \cdots. \tag{2}$$

But (1), when simplified, will have for its numerator an expression of the $(n-1)$th degree in θ, and for its denominator an expression of the nth degree in θ.
Thus we may suppose that

$$\begin{aligned}
\phi(\theta) &= \frac{A_1 + A_2\theta + A_3\theta^2 + \cdots + A_n\theta^{n-1}}{1 + B_1\theta + B_2\theta^2 + B_3\theta^3 + \cdots + B_n\theta^n} \\
&= a_1 + a_2\theta + a_3\theta^2 + \cdots a_{2n}\theta^{2n-1} + \cdots;
\end{aligned} \tag{3}$$

and so $\qquad (1 + B_1\theta + \cdots)(a_1 + a_2\theta + \cdots) = A_1 + A_2\theta + \cdots.$
Equating the coefficients of like powers of θ, we have

$$\begin{aligned}
A_1 &= a_1, \\
A_2 &= a_2 + a_1B_1, \\
A_3 &= a_3 + a_2B_1 + a_1B_2, \\
A_n &= a_n + a_{n-1}B_1 + a_{n-2}B_2 + \cdots + a_1B_{n-1}, \\
0 &= a_{n+1} + a_nB_1 + \cdots + a_1B_n, \\
0 &= a_{n+2} + a_{n+1}B_1 + \cdots + a_2B_n, \\
0 &= a_{n+3} + a_{n+2}B_1 + \cdots + a_3B_n, \\
\vdots & \qquad\qquad \vdots \\
0 &= a_{2n} + a_{2n-1}B_1 + \cdots + a_nB_n.
\end{aligned}$$

*For a solution, by determinants, of a similar set of equations, see Burnside and Panton, *Theory of Equations*, Vol, II, p.106, Ex.3. [Editor, *J.Indian Math. Soc.*]

From these $B_1, B_2, \ldots B_n$ can easily be found, and since $A_1, A_2, \ldots, A_n$ depend upon these values they can also be found.
Now, splitting (3) into partial fractions in the form

$$\frac{p_1}{1-q_1\theta}+\frac{p_2}{1-q_2\theta}+\frac{p_3}{1-q_3\theta}+\cdots+\frac{p_n}{1-q_n\theta},$$

and comparing with (1), we see that

$$\begin{aligned} x_1 &= p_1, \quad y_1 = q_1; \\ x_2 &= p_2, \quad y_2 = q_2; \\ x_3 &= p_3, \quad y_3 = q_3; \\ &\cdots \qquad \cdots \end{aligned}$$

2. As an example we may solve the equations:

$$\begin{aligned} x+y+z+u+v &= 2, \\ px+qy+rz+su+tv &= 3, \\ p^2x+q^2y+r^2z+s^2u+t^2v &= 16, \\ p^3x+q^3y+r^3z+s^3u+t^3v &= 31, \\ p^4x+q^4y+r^4z+s^4u+t^4v &= 103, \\ p^5x+q^5y+r^5z+s^5u+t^5v &= 235, \\ p^6x+q^6y+r^6z+s^6u+t^6v &= 674, \\ p^7x+q^7y+r^7z+s^7u+t^7v &= 1669, \\ p^8x+q^8y+r^8z+s^8u+t^8v &= 4526, \\ p^9x+q^9y+r^9z+s^9u+t^9v &= 11595, \end{aligned}$$

where $x, y, z, u, v, p, q, r, s, t$ are the unknowns. Proceeding as before, we have

$$\frac{x}{1-\theta p}+\frac{y}{1-\theta q}+\frac{z}{1-\theta r}+\frac{u}{1-\theta s}+\frac{v}{1-\theta t}$$

$$= 2+3\theta+16\theta^2+31\theta^3+103\theta^4+235\theta^5+674\theta^6+1669\theta^7+4526\theta^8+11595\theta^9+\cdots$$

By the method of indeterminate coefficients, this can be shewn to be equal to

$$\frac{2+\theta+3\theta^2+2\theta^3+\theta^4}{1-\theta-5\theta^2+\theta^3+3\theta^4-\theta^5}.$$

Splitting this into partial fractions, we get the values of the unknowns, as follows :

$$\begin{array}{ll|ll} x = & -\frac{3}{5}\,, & p = & -1 \\ y = & \frac{18+\sqrt{5}}{10}, & q = & \frac{3+\sqrt{5}}{2} \\ z = & \frac{18-\sqrt{5}}{10}, & r = & \frac{3-\sqrt{5}}{2} \\ u = & -\frac{8+\sqrt{5}}{2\sqrt{5}}, & s = & \frac{\sqrt{5}-1}{2}, \\ v = & \frac{8-\sqrt{5}}{2\sqrt{5}}, & t = & -\frac{\sqrt{5}+1}{2}. \end{array}$$

Irregular numbers

Journal of the Indian Mathematical Society, V, 1913, 105 – 106

1. Let $a_2, a_3, a_5, a_7, \ldots$ denote numbers less than unity, where the subscripts 2,3,5,7, ... are the series of prime numbers. Then

$$\begin{aligned}\frac{1}{1-a_2}\cdot\frac{1}{1-a_3}\cdot\frac{1}{1-a_5}\cdots &= 1+a_2+a_3+a_2\cdot a_2+a_5 \\ &\quad +a_2\cdot a_3+a_7+a_2\cdot a_2\cdot a_2+a_3\cdot a_3+\ldots, \qquad (1)\end{aligned}$$

the terms being so arranged that the products obtained by multiplying the subscripts are the series of natural numbers 2, 3, 4, 5, 6, 7, 8, 9, ...
The above result is easily got if we remember that the natural numbers are formed by multiplying primes and their powers.

2. Similarly, we have

$$\begin{aligned}\frac{1}{1+a_2}\cdot\frac{1}{1+a_3}\cdot\frac{1}{1+a_5}\cdots &= 1-a_2-a_3+a_2\cdot a_2-a_5 \\ &\quad +a_2\cdot a_3-a_7-a_2\cdot a_2\cdot a_2+a_3\cdot a_3+\ldots, \qquad (2)\end{aligned}$$

where the sign is negative whenever a term contains an *odd* number of prime subscripts.

3. Put $a_2 = 1/2^n, a_3 = 1/3^n, a_5 = 1/5^n, \ldots$ in (1), and we get

$$\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\cdots = \frac{1}{S_n}, \qquad (3)$$

where S_n denotes $1/1^n + 1/2^n + 1/3^n + 1/4^n + \ldots$.
Changing n into $2n$ in (3) and dividing by the original, we obtain

$$\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\cdots = \frac{S_n}{S_{2n}}. \qquad (4)$$

Examples:

$$(i) \quad \left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\cdots = \frac{15}{\pi^2}, \qquad (5)$$

$$(ii) \quad \left(1+\frac{1}{2^4}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{5^4}\right)\cdots = \frac{105}{\pi^4}, \qquad (6)$$

since

$$S_2 = \pi^2/6, S_4 = \pi^4/90, S_8 = \pi^8/9450.$$

4. Subtract (2) from (1) and put $a_2 = 2^{-n} \dots$; then

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{11^n} + \frac{1}{12^n} + \dots = \frac{S_n^2 - S_{2n}}{2S_n},$$

where the numbers 2, 3, 5, 7, 8, ... contain an *odd* number of prime divisors. Examples:

$$(i) \quad \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \cdots = \frac{\pi^2}{20}, \tag{7}$$

$$(ii) \quad \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{8^4} + \cdots = \frac{\pi^4}{1260}. \tag{8}$$

5. Again (2,3,5,7, ... being the prime numbers)

$$\begin{aligned}(1 + a_2)(1 + a_3)(1 + a_5)(1 + a_7) \cdots \quad &= \quad 1 + a_2 + a_3 + a_5 \\ &\quad + a_2 \cdot a_3 + a_7 + a_2 \cdot a_5 + a_{11} + a_{13} + \cdots, \end{aligned} \tag{9}$$

where the product of the subscripts in any term is a natural number containing *dissimilar* prime divisors; and

$$(1 - a_2)(1 - a_3)(1 - a_5)(1 - a_7) \dots = 1 - a_2 - a_3 - a_5 + a_2 \cdot a_3 - a_7, \tag{10}$$

where the signs are negative whenever the number of factors is odd.

6. Replacing as before $a_2, a_3, a_5, \dots$ by the values given in § 3 and using (4), we deduce that

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \dots = \frac{S_n}{S_{2n}}, \tag{11}$$

where 2, 3, 5, 6, 7, ... are the numbers containing *dissimilar* prime divisors.

7. Also taking half the difference between (3) and (4),

$$\begin{aligned}\frac{1}{2^n} + \frac{1}{3^n} \quad &+ \quad \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \frac{1}{19^n} + \frac{1}{23^n} + \frac{1}{29^n} \\ &+ \quad \frac{1}{30^n} + \frac{1}{31^n} + \dots = \frac{S_n^2 - S_{2n}}{2S_nS_{2n}}, \end{aligned} \tag{12}$$

where 2, 3, 5, ... are numbers containing an *odd* number of *dissimilar* prime divisors.

Examples:

$$(i)\quad \frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{9}{2\pi^2}, \tag{13}$$

$$(ii)\quad \frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{15}{2\pi^4}. \tag{14}$$

8. Subtracting (11) from S_n, we have

$$\frac{1}{4^n}+\frac{1}{8^n}+\frac{1}{9^n}+\frac{1}{12^n}+\ldots=\frac{S_n(S_{2n}-1)}{S_{2n}}, \tag{15}$$

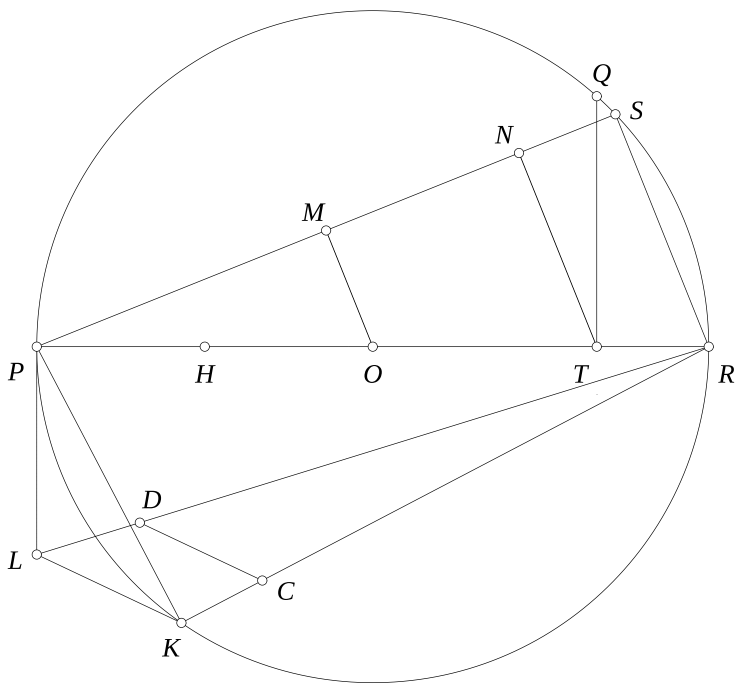
where 4, 8, 9, ... are composite numbers having *at least* two *equal* prime divisors.

Squaring the circle

Journal of the Indian Mathematical Society, V, 1913, 132

Let PQR be a circle with center O, of which a diameter is PR. Bisect PO at H and let T be the point of trisection of OR nearer R. Draw TQ perpendicular to PR and place the chord $RS = TQ$.

Join PS, and draw OM and TN parallel to RS. Place a chord $PK = PM$, and draw the tangent $PL = MN$. Join RL, RK and KL. Cut off $RC = RH$. Draw CD parallel to KL, meeting RL at D.



Then the square on RD will be equal to the circle PQR approximately. For

$$RS^2 = \frac{5}{36}d^2,$$

where d is the diameter of the circle. Therefore

$$PS^2 = \frac{31}{36}d^2.$$

But PL and PK are equal to MN and PM respectively. Therefore

$$PK^2 = \frac{31}{144}d^2, \quad \text{and} \quad PL^2 = \frac{31}{324}d^2.$$

Hence

$$RK^2 = PR^2 - PK^2 = \frac{113}{144}d^2,$$

and

$$RL^2 = PR^2 + PL^2 = \frac{355}{324}d^2.$$

But

$$\frac{RK}{RL} = \frac{RC}{RD} = \frac{3}{2}\sqrt{\frac{113}{355}},$$

and

$$RC = \frac{3}{4}d.$$

Therefore

$$RD = \frac{d}{2}\sqrt{\frac{355}{113}} = r\sqrt{\pi}, \quad \text{very} \quad \text{nearly}$$

Note: If the area of the circle be 140,000 square miles, then RD is greater than the true length by about an inch.

Modular equations and approximations to π

Quarterly Journal of Mathematics, XLV, 1914, 350 – 372

1. If we suppose that

$$(1+e^{-\pi\sqrt{n}})(1+e^{-3\pi\sqrt{n}})(1+e^{-5\pi\sqrt{n}})\cdots = 2^{\frac{1}{4}}e^{-\pi\sqrt{n}/24}G_n \tag{1}$$

and

$$(1-e^{-\pi\sqrt{n}})(1-e^{-3\pi\sqrt{n}})(1-e^{-5\pi\sqrt{n}})\cdots = 2^{\frac{1}{4}}e^{-\pi\sqrt{n}/24}g_n, \tag{2}$$

then G_n and g_n can always be expressed as roots of algebraical equations when n is any rational number. For we know that

$$(1+q)(1+q^3)(1+q^5)\ldots = 2^{\frac{1}{6}}q^{\frac{1}{24}}(kk')^{-\frac{1}{12}} \tag{3}$$

and

$$(1-q)(1-q^3)(1-q^5)\cdots = 2^{\frac{1}{6}}q^{\frac{1}{24}}k^{-\frac{1}{12}}k'^{\frac{1}{6}}. \tag{4}$$

Now the relation between the moduli k and l, which makes

$$n\frac{K'}{K} = \frac{L'}{L},$$

where $n = r/s, r$ and s being positive integers, is expressed by the modular equation of the rsth degree. If we suppose that $k = l', k' = l$, so that $K = L', K' = L$, then

$$q = e^{-\pi L'/L} = e^{-\pi\sqrt{n}},$$

and the corresponding value of k may be found by the solution of an algebraical equation. From (1), (2), (3) and (4) it may easily be deduced that

$$g_{4n} = 2^{\frac{1}{4}}g_nG_n, \tag{5}$$

$$G_n = G_{1/n},\ 1/g_n = g_{4/n}, \tag{6}$$

$$(g_nG_n)^8(G_n^8 - g_n^8) = \frac{1}{4}. \tag{7}$$

I shall consider only integral values of n. It follows from (7) that we need consider only one of G_n or g_n for any given value of n; and from (5) that we may suppose n not divisible by 4. It is most convenient to consider g_n when n is even, and G_n when n is odd.

2. Suppose then that n is odd. The values of G_n and g_{2n} are got from the same modular equation. For example, let us take the modular equation of the 5th degree, viz.

$$\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 = 2\left(u^2v^2 - \frac{1}{u^2v^2}\right), \tag{8}$$

where

$$2^{\frac{1}{4}}q^{\frac{1}{24}}u = (1+q)(1+q^3)(1+q^5)\cdots$$

and

$$2^{\frac{1}{4}}q^{\frac{5}{24}}v = (1+q^5)(1+q^{15})(1+q^{25})\cdots$$

By changing q to $-q$ the above equation may also be written as

$$\left(\frac{v}{u}\right)^3 - \left(\frac{u}{v}\right)^3 = 2\left(u^2v^2 + \frac{1}{u^2v^2}\right), \tag{9}$$

where

$$2^{\frac{1}{4}}q^{\frac{1}{24}}u = (1-q)(1-q^3)(1-q^5)\cdots$$

and

$$2^{\frac{1}{4}}q^{\frac{5}{24}}v = (1-q^5)(1-q^{15})(1-q^{25})\cdots$$

If we put $q = e^{-\pi/\sqrt{5}}$ in (8), so that $u = G_{\frac{1}{5}}$ and $v = G_5$, and hence $u = v$, we see that

$$v^4 - v^{-4} = 1.$$

Hence

$$v^4 = \frac{1+\sqrt{5}}{2}, \quad G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{4}}.$$

Similarly, by putting $q = e^{-\pi\sqrt{\frac{2}{5}}}$, so that $u = g_{\frac{2}{5}}$ and $v = g_{10}$, and hence $u = 1/v$, we see that

$$v^6 - v^{-6} = 4.$$

Hence

$$v^2 = \frac{1+\sqrt{5}}{2}, \; g_{10} = \sqrt{\frac{1+\sqrt{5}}{2}}.$$

Similarly it can be shewn that

$$\begin{aligned}
G_9 &= \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{\frac{1}{3}}, \quad g_{18} = (\sqrt{2}+\sqrt{3})^{\frac{1}{3}}, \\
G_{17} &= \sqrt{\left(\frac{5+\sqrt{17}}{8}\right)} + \sqrt{\left(\frac{\sqrt{17}-3}{8}\right)}, \\
g_{34} &= \sqrt{\left(\frac{7+\sqrt{17}}{8}\right)} + \sqrt{\left(\frac{\sqrt{17}-1}{8}\right)},
\end{aligned}$$

and so on.

3. In order to obtain approximations for π we take logarithms of (1) and (2). Thus

$$\left.\begin{aligned} \pi &= \tfrac{24}{\sqrt{n}}\log(2^{\frac{1}{4}}G_n) \\ \pi &= \tfrac{24}{\sqrt{n}}\log(2^{\frac{1}{4}}g_n) \end{aligned}\right\}, \tag{10}$$

approximately, the error being nearly $\frac{24}{\sqrt{n}}e^{-\pi\sqrt{n}}$ in both cases. These equations may also be written as

$$e^{\pi\sqrt{n}/24} = 2^{\frac{1}{4}}G_n, \quad e^{\pi\sqrt{n}/24} = 2^{\frac{1}{4}}g_n \tag{11}$$

In those cases in which G_n^{12} and g_n^{12} are simple quadratic surds we may use the forms

$$(G_n^{12} + G_n^{-12})^{\frac{1}{12}}, \quad (g_n^{12} + g_n^{-12})^{\frac{1}{12}},$$

instead of G_n and g_n, for we have

$$g_n^{12} = \frac{1}{8}e^{\frac{1}{2}\pi\sqrt{n}} - \frac{3}{2}e^{-\frac{1}{2}\pi\sqrt{n}},$$

approximately, and so

$$g_n^{12} + g_n^{-12} = \frac{1}{8}e^{\frac{1}{2}\pi\sqrt{n}} + \frac{13}{2}e^{-\frac{1}{2}\pi\sqrt{n}},$$

approximately, so that

$$\pi = \frac{2}{\sqrt{n}}\log\{8(g_n^{12} + g_n^{-12})\}, \tag{12}$$

the error being about $\frac{104}{\sqrt{n}}e^{-\pi\sqrt{n}}$, which is of the same order as the error in the formulæ (10). The formula (12) often leads to simpler results. Thus the second of formulæ (10) gives

$$e^{\pi\sqrt{18}/24} = 2^{\frac{1}{4}}g_{18}$$

or

$$e^{\frac{1}{4}\pi\sqrt{18}} = 10\sqrt{2} + 8\sqrt{3}.$$

But if we use the formula (12), or

$$e^{\pi\sqrt{n}/24} = 2^{\frac{1}{4}}(g_n^{12} + g_n^{-12})^{\frac{1}{12}},$$

we get a simpler form, viz.

$$e^{\frac{1}{8}\pi\sqrt{18}} = 2\sqrt{7}.$$

4. The values of g_{2n} and G_n are obtained from the same equation. The approximation by means of g_{2n} is preferable to that by G_n for the following reasons.

(a) It is more accurate. Thus the error when we use G_{65} contains a factor $e^{-\pi\sqrt{65}}$, whereas that when we use g_{130} contains a factor $e^{-\pi\sqrt{130}}$.
(b) For many values of n, g_{2n} is simpler in form than G_n; thus

$$g_{130} = \sqrt{\left\{(2+\sqrt{5})\left(\frac{3+\sqrt{13}}{2}\right)\right\}},$$

while

$$G_{65} = \left\{\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{13}}{2}\right)\right\}^{\frac{1}{4}} \sqrt{\left\{\sqrt{\left(\frac{9+\sqrt{65}}{8}\right)} + \sqrt{\left(\frac{1+\sqrt{65}}{8}\right)}\right\}}.$$

(c) For many values of n, g_{2n} involves quadratic surds only, even when G_n is a root of an equation of higher order. Thus G_{23}, G_{29}, G_{31} are roots of cubic equations, G_{47}, G_{79} are those of quintic equations, and G_{71} is that of a septic equation, while $g_{46}, g_{58}, g_{62}, g_{94}, g_{142}$ and g_{158} are all expressible by quadratic surds.

5. Since G_n and g_n can be expressed as roots of algebraical equations with rational coefficients, the same is true of G_n^{24} or g_n^{24}. So let us suppose that

$$1 = ag_n^{-24} - bg_n^{-48} + \cdots,$$

or

$$g_n^{24} = a - bg_n^{-24} + \cdots.$$

But we know that

$$\begin{aligned} 64e^{-\pi\sqrt{n}}g_n^{24} &= 1 - 24e^{-\pi\sqrt{n}} + 276e^{-2\pi\sqrt{n}} - \cdots, \\ 64g_n^{24} &= e^{\pi\sqrt{n}} - 24 + 276e^{-\pi\sqrt{n}} - \cdots, \\ 64a - 64bg_n^{-24} + \cdots &= e^{\pi\sqrt{n}} - 24 + 276e^{-\pi\sqrt{n}} - \cdots, \\ 64a - 4096be^{-\pi\sqrt{n}} + \cdots &= e^{\pi\sqrt{n}} - 24 + 276e^{-\pi\sqrt{n}} - \cdots, \end{aligned}$$

that is

$$e^{\pi\sqrt{n}} = (64a + 24) - (4096b + 276)e^{-\pi\sqrt{n}} + \cdots \qquad (13)$$

Similarly, if

$$1 = aG_n^{-24} - bG_n^{-48} + \cdots,$$

then

$$e^{\pi\sqrt{n}} = (64a - 24) - (4096b + 276)e^{-\pi\sqrt{n}} + \cdots \qquad (14)$$

From (13) and (14) we can find whether $e^{\pi\sqrt{n}}$ is very nearly an integer for given values of n, and ascertain also the number of 9's or 0's in the decimal part. But if G_n and g_n be simple quadratic surds we may work independently as follows. We have, for example,

$$g_{22} = \sqrt{(1+\sqrt{2})}.$$

Hence

$$\begin{aligned} 64g_{22}^{24} &= e^{\pi\sqrt{22}} - 24 + 276e^{-\pi\sqrt{22}} - \cdots, \\ 64g_{22}^{-24} &= 4096e^{-\pi\sqrt{22}} + \cdots, \end{aligned}$$

so that

$$64(g_{22}^{24} + g_{22}^{-24}) = e^{\pi\sqrt{22}} - 24 + 4372e^{-\pi\sqrt{22}} + \cdots = 64\{(1+\sqrt{2})^{12} + (1-\sqrt{2})^{12}\}.$$

Hence

$$e^{\pi\sqrt{22}} = 2508951.9982\ldots.$$

Again

$$G_{37} = (6+\sqrt{37})^{\frac{1}{4}},$$

$$\begin{aligned} 64G_{37}^{24} &= e^{\pi\sqrt{37}} + 24 + 276e^{-\pi\sqrt{37}} + \cdots, \\ 64G_{37}^{-24} &= 4096e^{-\pi\sqrt{37}} - \cdots, \end{aligned}$$

so that

$$64(G_{37}^{24} + G_{37}^{-24}) = e^{\pi\sqrt{37}} + 24 + 4372e^{-\pi\sqrt{37}} - \cdots = 64\{(6+\sqrt{37})^{6} + (6-\sqrt{37})^{6}\}.$$

Hence

$$e^{\pi\sqrt{37}} = 199148647.999978\ldots.$$

Similarly, from

$$g_{58} = \sqrt{\left(\frac{5+\sqrt{29}}{2}\right)},$$

we obtain

$$64(g_{58}^{24} + g_{58}^{-24}) = e^{\pi\sqrt{58}} - 24 + 4372e^{-\pi\sqrt{58}} + \cdots = 64\left\{\left(\frac{5+\sqrt{29}}{2}\right)^{12} + \left(\frac{5-\sqrt{29}}{2}\right)^{12}\right\}.$$

Hence

$$e^{\pi\sqrt{58}} = 24591257751.99999982\ldots.$$

6. I have calculated the values of G_n and g_n for a large number of values of n. Many of these results are equivalent to results given by Weber; for example,

$$G_{13}^4 = \frac{3+\sqrt{13}}{2}, \qquad G_{25} = \frac{1+\sqrt{5}}{2},$$

$$g_{30}^6 = (2+\sqrt{5})(3+\sqrt{10}), \quad G_{37}^4 = 6+\sqrt{37},$$

$$G_{49} = \frac{7^{\frac{1}{4}} + \sqrt{(4+\sqrt{7})}}{2}, \quad g_{58}^2 = \frac{5+\sqrt{29}}{2},$$

$$g_{70}^2 = \frac{(3+\sqrt{5})(1+\sqrt{2})}{2},$$

$$G_{73} = \sqrt{\left(\frac{9+\sqrt{73}}{8}\right)} + \sqrt{\left(\frac{1+\sqrt{73}}{8}\right)},$$

$$G_{85} = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{9+\sqrt{85}}{2}\right)^{\frac{1}{4}},$$

$$G_{97} = \sqrt{\left(\frac{13+\sqrt{97}}{8}\right)} + \sqrt{\left(\frac{5+\sqrt{97}}{8}\right)},$$

$$g_{190}^2 = (2+\sqrt{5})(3+\sqrt{10}),$$

$$G_{385}^2 = \frac{1}{8}(3+\sqrt{11})(\sqrt{5}+\sqrt{7})(\sqrt{7}+\sqrt{11})(3+\sqrt{5}),$$

and so on. I have also many results not given by Weber. I give a complete table of new results. In Weber's notation, $G_n = 2^{-\frac{1}{4}} f\{\sqrt{(-n)}\}$ and $g_n = 2^{-\frac{1}{4}} f_1\{\sqrt{(-n)}\}$.

TABLE I

$$g_{62} + \frac{1}{g_{62}} = \frac{1}{2}\{\sqrt{(1+\sqrt{2})} + \sqrt{(9+5\sqrt{2})}\},$$

$$G_{65}^2 = \sqrt{\left\{\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{13}}{2}\right)\right\}}\left\{\sqrt{\left(\frac{1+\sqrt{65}}{8}\right)} + \sqrt{\left(\frac{9+\sqrt{65}}{8}\right)}\right\},$$

$$g_{66}^2 = \sqrt{(\sqrt{2}+\sqrt{3})}(7\sqrt{2}+3\sqrt{11})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{7+\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{\sqrt{33}-1}{8}\right)}\right\},$$

$$G_{69}^2 = (3\sqrt{3}+\sqrt{23})^{\frac{1}{4}}\left(\frac{5+\sqrt{23}}{4}\right)^{\frac{1}{6}}\left\{\sqrt{\left(\frac{6+3\sqrt{3}}{4}\right)}+\sqrt{\left(\frac{2+3\sqrt{3}}{4}\right)}\right\},$$

$$G_{77}^2 = \{\frac{1}{2}(\sqrt{7}+\sqrt{11})(8+3\sqrt{7})\}^{\frac{1}{4}}\left\{\sqrt{\left(\frac{6+\sqrt{11}}{4}\right)}+\sqrt{\left(\frac{2+\sqrt{11}}{4}\right)}\right\},$$

$$G_{81}^3 = \frac{(2\sqrt{3}+2)^{\frac{1}{3}}+1}{(2\sqrt{3}-2)^{\frac{1}{3}}-1},$$

$$g_{90} = \{(2+\sqrt{5})(\sqrt{5}+\sqrt{6})\}^{\frac{1}{6}}\left\{\sqrt{\left(\frac{3+\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{\sqrt{6}-1}{4}\right)}\right\},$$

$$g_{94} + \frac{1}{g_{94}} = \frac{1}{2}\{\sqrt{(7+\sqrt{2})}+\sqrt{(7+5\sqrt{2})}\},$$

$$g_{98} + \frac{1}{g_{98}} = \frac{1}{2}\{\sqrt{2}+\sqrt{(14+4\sqrt{14})}\},$$

$$g_{114}^2 = \sqrt{(\sqrt{2}+\sqrt{3})}(3\sqrt{2}+\sqrt{19})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{23+3\sqrt{57}}{8}\right)}+\sqrt{\left(\frac{15+3\sqrt{57}}{8}\right)}\right\},$$

$$G_{117} = \frac{1}{2}\left(\frac{3+\sqrt{13}}{2}\right)^{\frac{1}{4}}(2\sqrt{3}+\sqrt{13})^{\frac{1}{6}}\{3^{\frac{1}{4}}+\sqrt{(4+\sqrt{3})}\},$$

$$\left.\begin{aligned} G_{121} + \frac{1}{G_{121}} &= \left(\frac{11}{2}\right)^{\frac{1}{6}}\left\{\left(3+\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}+\left(3-\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}\right\} \\ \frac{1}{G_{121}} &= \tfrac{1}{3\sqrt{2}}[(11-3\sqrt{11})^{\frac{1}{3}}\{(3\sqrt{11}+3\sqrt{3}-4)^{\frac{1}{3}}+(3\sqrt{11}-3\sqrt{3}-4)^{\frac{1}{3}}\}-2] \end{aligned}\right\},$$

$$g_{126} = \sqrt{\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)}(\sqrt{6}+\sqrt{7})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{3+\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{\sqrt{2}-1}{4}\right)}\right\}^2,$$

$$g_{138}^2 = \sqrt{\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)}(78\sqrt{2}+23\sqrt{23})^{\frac{1}{6}}\times\left\{\sqrt{\left(\frac{5+2\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{1+2\sqrt{6}}{4}\right)}\right\},$$

$$G_{141}^2 = (4\sqrt{3}+\sqrt{47})^{\frac{1}{4}}\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)^{\frac{1}{6}}\left\{\sqrt{\left(\frac{18+9\sqrt{3}}{4}\right)}+\sqrt{\left(\frac{14+9\sqrt{3}}{4}\right)}\right\},$$

$$G_{145}^2 = \sqrt{\left\{\frac{(2+\sqrt{5})(5+\sqrt{29})}{2}\right\}}\left\{\sqrt{\left(\frac{17+\sqrt{145}}{8}\right)}+\sqrt{\left(\frac{9+\sqrt{145}}{8}\right)}\right\},$$

$$\frac{1}{G_{147}} = 2^{-\frac{1}{12}}\left[\frac{1}{2}+\frac{1}{\sqrt{3}}\left\{\sqrt{\left(\frac{7}{4}\right)-(28)^{\frac{1}{6}}}\right\}\right],$$

$$\begin{aligned} G_{153} &= \left\{\sqrt{\left(\frac{5+\sqrt{17}}{8}\right)}+\sqrt{\left(\frac{\sqrt{17}-3}{8}\right)}\right\}^2 \\ &\qquad \times\left\{\sqrt{\left(\frac{37+9\sqrt{17}}{4}\right)}+\sqrt{\left(\frac{33+9\sqrt{17}}{4}\right)}\right\}^{\frac{1}{3}}, \end{aligned}$$

$$\begin{aligned} g_{154}^2 &= \sqrt{\left\{(2\sqrt{2}+\sqrt{7})\left(\frac{\sqrt{7}+\sqrt{11}}{2}\right)\right\}} \\ &\qquad \times\left\{\sqrt{\left(\frac{13+2\sqrt{22}}{4}\right)}+\sqrt{\left(\frac{9+2\sqrt{22}}{4}\right)}\right\}, \end{aligned}$$

$$g_{158} + \frac{1}{g_{158}} = \frac{1}{2}\{\sqrt{(9+\sqrt{2})}+\sqrt{(17+13\sqrt{2})}\},$$

$$\left.\begin{aligned} G_{169} + \frac{1}{G_{169}} &= \left(\frac{13}{4}\right)^{\frac{1}{6}}\left\{\left(1+\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}+\left(1-\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}\right\}^2 \\ \frac{1}{G_{169}} &= \frac{1}{3}\left[(\sqrt{13}-2)+\left(\frac{13-3\sqrt{13}}{2}\right)^{\frac{1}{3}}\right. \\ &\qquad \left.\times\left\{\left(3\sqrt{3}-\frac{11-\sqrt{13}}{2}\right)^{\frac{1}{3}}-\left(3\sqrt{3}+\frac{11-\sqrt{13}}{2}\right)^{\frac{1}{3}}\right\}\right] \end{aligned}\right\},$$

$$g_{198} = \sqrt{(1+\sqrt{2})}(4\sqrt{2}+\sqrt{33})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{9+\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{1+\sqrt{33}}{8}\right)}\right\},$$

$$G_{205} = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3\sqrt{5}+\sqrt{41}}{2}\right)^{\frac{1}{4}}\left\{\sqrt{\left(\frac{7+\sqrt{41}}{8}\right)}+\sqrt{\left(\frac{\sqrt{41}-1}{8}\right)}\right\},$$

$$
\begin{aligned}
G_{213}^{2} &= (5\sqrt{3}+\sqrt{71})^{\frac{1}{4}}\left(\frac{59+7\sqrt{71}}{4}\right)^{\frac{1}{6}} \\
&\qquad \times\left\{\sqrt{\left(\frac{21+12\sqrt{3}}{2}\right)}+\sqrt{\left(\frac{19+12\sqrt{3}}{2}\right)}\right\}, \\
G_{217}^{2} &= \left\{\sqrt{\left(\frac{9+4\sqrt{7}}{2}\right)}+\sqrt{\left(\frac{11+4\sqrt{7}}{2}\right)}\right\} \\
&\qquad \times\left\{\sqrt{\left(\frac{12+5\sqrt{7}}{4}\right)}+\sqrt{\left(\frac{16+5\sqrt{7}}{4}\right)}\right\}, \\
G_{225} &= \left(\frac{1+\sqrt{5}}{4}\right)(2+\sqrt{3})^{\frac{1}{3}}\{\sqrt{(4+\sqrt{15})}+15^{\frac{1}{4}}\}, \\
g_{238} &= \left\{\sqrt{\left(\frac{1+2\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{5+2\sqrt{2}}{4}\right)}\right\} \\
&\qquad \times\left\{\sqrt{\left(\frac{1+3\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{2}}{4}\right)}\right\}, \\
G_{265}^{2} &= \sqrt{\left\{(2+\sqrt{5})\left(\frac{7+\sqrt{53}}{2}\right)\right\}}\left\{\sqrt{\left(\frac{89+5\sqrt{265}}{8}\right)}+\sqrt{\left(\frac{81+5\sqrt{265}}{8}\right)}\right\}, \\
G_{289} &= \left[\sqrt{\left\{\frac{17+\sqrt{17}+17^{\frac{1}{4}}(5+\sqrt{17})}{16}\right\}}+\sqrt{\left\{\frac{1+\sqrt{17}+17^{\frac{1}{4}}(5+\sqrt{17})}{16}\right\}}\right]^{2}, \\
G_{301}^{2} &= \left\{(8+3\sqrt{7})\left(\frac{23\sqrt{43}+57\sqrt{7}}{2}\right)\right\}^{\frac{1}{4}} \\
&\qquad \times\left\{\sqrt{\left(\frac{46+7\sqrt{43}}{4}\right)}+\sqrt{\left(\frac{42+7\sqrt{43}}{4}\right)}\right\},
\end{aligned}
$$

$$g_{310} = \left(\frac{1+\sqrt{5}}{2}\right)\sqrt{(1+\sqrt{2})}\left\{\sqrt{\left(\frac{7+2\sqrt{10}}{4}\right)}+\sqrt{\left(\frac{3+2\sqrt{10}}{4}\right)}\right\},$$

$$\left.\begin{aligned} G_{325} &= \left(\frac{3+\sqrt{13}}{2}\right)^{\frac{1}{4}} t, \quad \text{where} \\ & t^3+t^2\left(\frac{1-\sqrt{13}}{2}\right)^2+t\left(\frac{1+\sqrt{13}}{2}\right)^2+1 \\ & \qquad = \sqrt{5}\left\{t^3-t^2\left(\tfrac{1+\sqrt{13}}{2}\right)+t\left(\tfrac{1-\sqrt{13}}{2}\right)-1\right\} \end{aligned}\right\},$$

$$G_{333} = \frac{1}{2}(6+\sqrt{37})^{\frac{1}{4}}(7\sqrt{3}+2\sqrt{37})^{\frac{1}{6}}\{\sqrt{(7+2\sqrt{3})}+\sqrt{(3+2\sqrt{3})}\},$$

$$\left.\begin{aligned} G_{363} &= 2^{\frac{5}{12}} t, \quad \text{where} \\ & 2t^3-t^2\{(4+\sqrt{33})+\sqrt{(11+2\sqrt{33})}\} \\ & \qquad -t\{1+\sqrt{(11+2\sqrt{33})}\}-1=0 \end{aligned}\right\},$$

$$G_{441}^2 = \left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)(2+\sqrt{3})^{\frac{1}{3}}\left\{\frac{2+\sqrt{7}+\sqrt{(7+4\sqrt{7})}}{2}\right\}\left\{\frac{\sqrt{(3+\sqrt{7})}+(6\sqrt{7})^{\frac{1}{4}}}{\sqrt{(3+\sqrt{7})}-(6\sqrt{7})^{\frac{1}{4}}}\right\},$$

$$G_{445} = \sqrt{(2+\sqrt{5})}\left(\frac{21+\sqrt{445}}{2}\right)^{\frac{1}{4}}\sqrt{\left\{\left(\frac{13+\sqrt{89}}{8}\right)+\sqrt{\left(\frac{5+\sqrt{89}}{8}\right)}\right\}},$$

$$\begin{aligned} G_{465}^2 &= \sqrt{\left\{(2+\sqrt{3})\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3\sqrt{3}+\sqrt{31}}{2}\right)\right\}}(5\sqrt{5}+2\sqrt{31})^{\frac{1}{6}} \\ & \quad \times\left\{\sqrt{\left(\frac{2+\sqrt{31}}{4}\right)}+\sqrt{\left(\frac{6+\sqrt{31}}{4}\right)}\right\} \\ & \quad \times\left\{\sqrt{\left(\frac{11+2\sqrt{31}}{2}\right)}+\sqrt{\left(\frac{13+2\sqrt{31}}{2}\right)}\right\}, \end{aligned}$$

$$
\begin{aligned}
G_{505}^2 &= (2+\sqrt{5})\sqrt{\left\{\left(\frac{1+\sqrt{5}}{2}\right)(10+\sqrt{101})\right\}} \\
&\qquad \times \left\{\left(\frac{5\sqrt{5}+\sqrt{101}}{4}\right)+\sqrt{\left(\frac{105+\sqrt{505}}{8}\right)}\right\}, \\
g_{522} &= \sqrt{\left(\frac{5+\sqrt{29}}{2}\right)}(5\sqrt{29}+11\sqrt{6})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}, \\
G_{553}^2 &= \left\{\sqrt{\left(\frac{96+11\sqrt{79}}{4}\right)}+\sqrt{\left(\frac{100+11\sqrt{79}}{4}\right)}\right\} \\
&\qquad \times \left\{\sqrt{\left(\frac{141+16\sqrt{79}}{2}\right)}+\sqrt{\left(\frac{143+16\sqrt{79}}{2}\right)}\right\}, \\
g_{630} &= (\sqrt{14}+\sqrt{15})^{\frac{1}{6}}\sqrt{\left\{(1+\sqrt{2})\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)\right\}} \\
&\qquad \times \left\{\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}+2}{4}\right)}+\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}-2}{4}\right)}\right\} \\
&\qquad \times \left\{\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}+4}{8}\right)}+\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}-4}{8}\right)}\right\}, \\
G_{765}^2 &= \left(\frac{3+\sqrt{5}}{2}\right)(16+\sqrt{255})^{\frac{1}{6}}\sqrt{\left\{(4+\sqrt{15})\left(\frac{9+\sqrt{85}}{2}\right)\right\}} \\
&\qquad \times \left\{\sqrt{\left(\frac{6+\sqrt{51}}{4}\right)}+\sqrt{\left(\frac{10+\sqrt{51}}{4}\right)}\right\} \\
&\qquad \times \left\{\sqrt{\left(\frac{18+3\sqrt{51}}{4}\right)}+\sqrt{\left(\frac{22+3\sqrt{51}}{4}\right)}\right\},
\end{aligned}
$$

$$
\begin{aligned}
G_{777}^2 &= \sqrt{\left\{(2+\sqrt{3})(6+\sqrt{37})\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)\right\}}(246\sqrt{7}+107\sqrt{37})^{\frac{1}{6}} \\
&\qquad \times \left\{\sqrt{\left(\frac{6+3\sqrt{7}}{4}\right)}+\sqrt{\left(\frac{10+3\sqrt{7}}{4}\right)}\right\} \\
&\qquad \times \left\{\sqrt{\left(\frac{15+6\sqrt{7}}{2}\right)}+\sqrt{\left(\frac{17+6\sqrt{7}}{2}\right)}\right\},
\end{aligned}
$$

$$
\begin{aligned}
G_{1225} &= \left(\frac{1+\sqrt{5}}{2}\right)(6+\sqrt{35})^{\frac{1}{4}}\left\{\frac{7^{\frac{1}{4}}+\sqrt{(4+\sqrt{7})}}{2}\right\}^{\frac{3}{2}} \\
&\qquad \times \left[\sqrt{\left\{\frac{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{(10\sqrt{7})}}{8}\right\}}\right. \\
&\qquad \left. +\sqrt{\left\{\frac{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{(10\sqrt{7})}}{8}\right\}}\;\right],
\end{aligned}
$$

$$
\begin{aligned}
G_{1353}^2 &= \sqrt{\left\{(3+\sqrt{11})(5+3\sqrt{3})\left(\frac{11+\sqrt{123}}{2}\right)\right\}} \\
&\qquad \times \left(\frac{6817+321\sqrt{451}}{4}\right)^{\frac{1}{6}} \\
&\qquad \times \left\{\sqrt{\left(\frac{17+3\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{25+3\sqrt{33}}{8}\right)}\right\} \\
&\qquad \times \left\{\sqrt{\left(\frac{561+99\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{569+99\sqrt{33}}{8}\right)}\right\},
\end{aligned}
$$

$$
G_{1645}^2 = (2+\sqrt{5})\sqrt{\left\{(3+\sqrt{7})\left(\frac{7+\sqrt{47}}{2}\right)\right\}}\left(\frac{73\sqrt{5}+9\sqrt{329}}{2}\right)^{\frac{1}{4}}
$$

$$\times\left\{\sqrt{\left(\frac{119+7\sqrt{329}}{8}\right)}+\sqrt{\left(\frac{127+7\sqrt{329}}{8}\right)}\right\}$$

$$\times\left\{\sqrt{\left(\frac{743+41\sqrt{329}}{8}\right)}+\sqrt{\left(\frac{751+41\sqrt{329}}{8}\right)}\right\}.$$

7. Hence we deduce the following approximate formulæ

TABLE II

$$\begin{aligned}
e^{\frac{1}{8}\pi\sqrt{18}} &= 2\sqrt{7},\quad e^{\pi\sqrt{22/12}}=2+\sqrt{2},\quad e^{\frac{1}{4}\pi\sqrt{30}}=20\sqrt{3}+16\sqrt{6},\\
e^{\frac{1}{4}\pi\sqrt{34}} &= 12(4+\sqrt{17}),\quad e^{\frac{1}{2}\pi\sqrt{46}}=144(147+104\sqrt{2})\\
e^{\frac{1}{4}\pi\sqrt{42}} &= 84+32\sqrt{6},\quad e^{\pi\sqrt{58/12}}=\frac{5+\sqrt{29}}{\sqrt{2}},\\
e^{\frac{1}{4}\pi\sqrt{70}} &= 60\sqrt{35}+96\sqrt{14},\quad e^{\frac{1}{4}\pi\sqrt{78}}=300\sqrt{3}+208\sqrt{6},\\
e^{\pi\sqrt{55/24}} &= \frac{1+\sqrt{(3+2\sqrt{5})}}{\sqrt{2}},\quad e^{\frac{1}{4}\pi\sqrt{102}}=800\sqrt{3}+196\sqrt{51},\\
e^{\frac{1}{4}\pi\sqrt{130}} &= 12(323+40\sqrt{65}),\quad e^{\pi\sqrt{190/12}}=(2\sqrt{2}+\sqrt{10})(3+\sqrt{10}),\\
\pi &= \frac{12}{\sqrt{130}}\log\left\{\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right\},\\
\pi &= \frac{24}{\sqrt{142}}\log\left\{\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right\},\\
\pi &= \frac{12}{\sqrt{190}}\log\{(2\sqrt{2}+\sqrt{10})(3+\sqrt{10})\},\\
\pi &= \frac{12}{\sqrt{310}}\log[\frac{1}{4}(3+\sqrt{5})(2+\sqrt{2})\{(5+2\sqrt{10})+\sqrt{(61+20\sqrt{10})}\}],\\
\pi &= \frac{4}{\sqrt{522}}\log\left[\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)^3(5\sqrt{29}+11\sqrt{6})\right.
\end{aligned}$$

$$\times\left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}^6\Bigg].$$

The last five formulæ are correct to 15, 16, 18, 22 and 31 places of decimals respectively.

8. Thus we have seen how to approximate to π by means of logarithms of surds. I shall now shew how to obtain approximations in terms of surds only. If

$$n\frac{K'}{K}=\frac{L'}{L},$$

we have

$$\frac{ndk}{kk'^2K^2}=\frac{dl}{ll'^2L^2}.$$

But, by means of the modular equation connecting k and l, we can express dk/dl as an algebraic function of k, a function moreover in which all coefficients which occur are algebraic numbers. Again,

$$q=e^{-\pi K'/K},\quad q^n=e^{-\pi L'/L},$$

$$\frac{q^{\frac{1}{12}}(1-q^2)(1-q^4)(1-q^6)\cdots}{q^{\frac{1}{12}n}(1-q^{2n})(1-q^{4n})(1-q^{6n})\cdots}=\left(\frac{kk'}{ll'}\right)^{\frac{1}{6}}\sqrt{\left(\frac{K}{L}\right)}. \tag{15}$$

Differentiating this equation logarithmically, and using the formula

$$\frac{dq}{dk}=\frac{\pi^2 q}{2kk'^2K^2},$$

we see that

$$n\left\{1-24\left(\frac{q^{2n}}{1-q^{2n}}+\frac{2q^{4n}}{1-q^{4n}}+\cdots\right)\right\}-\left\{1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\cdots\right)\right\}$$
$$=\frac{KL}{\pi^2}A(k), \tag{16}$$

where $A(k)$ denotes an algebraic function of the special class described above. I shall use the letter A generally to denote a function of this type.
Now, if we put $k=l'$ and $k'=l$ in (16), we have

$$n\left\{1-24\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\cdots\right)\right\}$$
$$-\left\{1-24\left(\frac{1}{e^{2\pi/\sqrt{n}}-1}+\frac{2}{e^{4\pi/\sqrt{n}}-1}+\cdots\right)\right\}=\left(\frac{K}{\pi}\right)^2A(k). \tag{17}$$

The algebraic function $A(k)$ of course assumes a purely numerical form when we substitute the value of k deduced from the modular equation. But by substituting $k = l'$ and $k' = l$ in (15) we have

$$\begin{aligned} &n^{\frac{1}{4}} e^{-\pi\sqrt{n}/12}(1-e^{-2\pi\sqrt{n}})(1-e^{-4\pi\sqrt{n}})(1-e^{-6\pi\sqrt{n}})\cdots \\ &= e^{-\pi/(12\sqrt{n})}(1-e^{-2\pi/\sqrt{n}})(1-e^{-4\pi/\sqrt{n}})(1-e^{-6\pi/\sqrt{n}})\cdots \end{aligned}$$

Differentiating the above equation logarithmically we have

$$\begin{aligned} &n\left\{1-24\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\cdots\right)\right\} \\ &\quad+\left\{1-24\left(\frac{1}{e^{2\pi/\sqrt{n}}-1}+\frac{2}{e^{4\pi/\sqrt{n}}-1}+\cdots\right)\right\}=\frac{6\sqrt{n}}{\pi}. \end{aligned} \tag{18}$$

Now, adding (17) and (18), we have

$$1-\frac{3}{\pi\sqrt{n}}-24\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\cdots\right)=\left(\frac{K}{\pi}\right)^2 A(k). \tag{19}$$

But it is known that

$$1-24\left(\frac{q}{1+q}+\frac{3q^3}{1+q^3}+\frac{5q^5}{1+q^5}+\cdots\right)=\left(\frac{2K}{\pi}\right)^2(1-2k^2),$$

so that

$$1-24\left(\frac{1}{e^{\pi\sqrt{n}}+1}+\frac{3}{e^{3\pi\sqrt{n}}+1}+\cdots\right)=\left(\frac{K}{\pi}\right)^2 A(k). \tag{20}$$

Hence, dividing (19) by (20), we have

$$\frac{1-\frac{3}{\pi\sqrt{n}}-24\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\cdots\right)}{1-24\left(\frac{1}{e^{\pi\sqrt{n}}+1}+\frac{3}{e^{3\pi\sqrt{n}}+1}+\cdots\right)}=R, \tag{21}$$

where R can always be expressed in radicals if n is any rational number. Hence we have

$$\pi=\frac{3}{(1-R)\sqrt{n}}, \tag{22}$$

nearly, the error being about $8\pi e^{-\pi\sqrt{n}}(\pi\sqrt{n}-3)$.

9. We may get a still closer approximation from the following results. It is known that

$$1+240\sum_{r=1}^{r=\infty}\frac{r^3q^{2r}}{1-q^{2r}}=\left(\frac{2K}{\pi}\right)^4(1-k^2k'^2),$$

and also that

$$1 - 504\sum_{r=1}^{r=\infty}\frac{r^5 q^{2r}}{1-q^{2r}} = \left(\frac{2K}{\pi}\right)^6 (1-2K^2)(1+\frac{1}{2}k^2k'^2).$$

Hence, from (19), we see that

$$\left\{1 - \frac{3}{\pi\sqrt{n}} - 24\sum_{r=1}^{r=\infty}\frac{r}{e^{2\pi r\sqrt{n}}-1}\right\}\left\{1 + 240\sum_{r=1}^{r=\infty}\frac{r^3}{e^{2\pi r\sqrt{n}}-1}\right\}$$
$$= R'\left\{1 - 504\sum_{r=1}^{r=\infty}\frac{r^5}{e^{2\pi r\sqrt{n}}-1}\right\}, \qquad (23)$$

where R' can always be expressed in radicals for any rational value of n. Hence

$$\pi = \frac{3}{(1-R')\sqrt{n}}, \qquad (24)$$

nearly, the error being about $24\pi(10\pi\sqrt{n} - 31)e^{-2\pi\sqrt{n}}$
It will be seen that the error in (24) is much less than that in (22), if n is at all large.

10. In order to find R and R' the series in (16) must be calculated in finite terms. I shall give the final results for a few values of n.

Table III

$$q = e^{-\pi K'/K}, \quad q^n = e^{-\pi L'/L},$$

$$f(q) = n\left(1 - 24\sum_1^\infty \frac{q^{2mn}}{1-q^{2mn}}\right) - \left(1 - 24\sum_1^\infty \frac{q^{2m}}{1-q^{2m}}\right),$$

$$f(2) = \frac{4KL}{\pi^2}(k'+l),$$

$$f(3) = \frac{4KL}{\pi^2}(1+kl+k'l'),$$

$$f(4) = \frac{4KL}{\pi^2}(\sqrt{k'}+\sqrt{l})^2,$$

$$f(5) = \frac{4KL}{\pi^2}(3+kl+k'l')\sqrt{\left(\frac{1+kl+k'l'}{2}\right)},$$

$$f(7) = \frac{12KL}{\pi^2}(1+kl+k'l'),$$

$$f(11) = \frac{8KL}{\pi^2}\{2(1+kl+k'l') + \sqrt{(kl)} + \sqrt{(k'l')} - \sqrt{(kk'll')}\},$$

$$\begin{aligned}
f(15) &= \frac{4KL}{\pi^2}[\{1+(kl)^{\frac{1}{4}}+(k'l')^{\frac{1}{4}}\}^4-\{1+kl+k'l'\}], \\
f(17) &= \frac{4KL}{\pi^2}\sqrt{\{}44(1+k^2l^2+k'^2l'^2)+168(kl+k'l'-kk'll') \\
&\qquad -102(1-kl-k'l')(4kk'll')^{\frac{1}{3}}-192(4kk'll')^{\frac{2}{3}}\}, \\
f(19) &= \frac{24KL}{\pi^2}\{(1+kl+k'l')+\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\}, \\
f(23) &= \frac{4KL}{\pi^2}[11(1+kl+k'l')-16(4kk'll')^{\frac{1}{6}}\{1+\sqrt{(kl)}+\sqrt{(k'l')}\}-20(4kk'll')^{\frac{1}{3}}], \\
f(31) &= \frac{12KL}{\pi^2}[3(1+kl+k'l')+4\{\sqrt{(kl)}+\sqrt{(k'l')}+\sqrt{(kk'll')}\} \\
&\qquad -4(kk'll')^{\frac{1}{4}}\{1+(kl)^{\frac{1}{4}}+(k'l')^{\frac{1}{4}}\}], \\
f(35) &= \frac{4KL}{\pi^2}[2\{\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\} \\
&\qquad +(4kk'll')^{-\frac{1}{6}}\{1-\sqrt{(kl)}-\sqrt{(k'l')}\}^3].
\end{aligned}$$

Thus the sum of the series (19) can be found in finite terms, when $n = 2,3,4,5,\ldots$, from the equations in Table III. We can use the same table to find the sum of (19) when $n = 9, 25, 49, \ldots$; but then we have also to use the equation

$$\frac{3}{\pi} = 1 - 24\left(\frac{1}{e^{2\pi}-1}+\frac{2}{e^{4\pi}-1}+\frac{3}{e^{6\pi}-1}+\cdots\right),$$

which is got by putting $k = k' = 1/\sqrt{2}$ and $n = 1$ in (18).
Similarly we can find the sum of (19) when $n = 21, 33, 57, 93, \ldots$, by combining the values of $f(3)$ and $f(7)$, $f(3)$ and $f(11)$, and so on, obtained from Table III.

11. The errors in (22) and (24) being about

$$8\pi e^{-\pi\sqrt{n}}(\pi\sqrt{n}-3), \quad 24\pi(10\pi\sqrt{n}-31)e^{-2\pi\sqrt{n}},$$

we cannot expect a high degree of approximation for small values of n. Thus, if we put $n = 7, 9, 16$, and 25 in (24), we get

$$\begin{aligned}
\frac{19}{16}\sqrt{7} &= 3.14180\ldots, \\
\frac{7}{3}\left(1+\frac{\sqrt{3}}{5}\right) &= 3.14162\ldots, \\
\frac{99}{80}\left(\frac{7}{7-3\sqrt{2}}\right) &= 3.14159274\ldots,
\end{aligned}$$

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) = 3.14159265380\ldots,$$

while

$$\pi = 3.14159265358\ldots.$$

But if we put $n = 25$ in (22), we get only

$$\frac{9}{5}+\sqrt{\frac{9}{5}} = 3.14164\ldots.$$

12. Another curious approximation to π is

$$\left(9^2+\frac{19^2}{22}\right)^{\frac{1}{4}} = 3.14159265262\ldots.$$

This value was obtained empirically, and it has no connection with the preceding theory.

The actual value of π, which I have used for purposes of calculation, is

$$\frac{355}{113}\left(1-\frac{.0003}{3533}\right) = 3.1415926535897943\ldots,$$

which is greater than π by about 10^{-15}. This is obtained by simply taking the reciprocal of $1-(113\pi/355)$.

In this connection it may be interesting to note the following simple geometrical constructions for π. The first merely gives the ordinary value $355/113$. The second gives the value $(9^2+19^2/22)^{\frac{1}{4}}$ mentioned above.

(1) Let AB (Fig.1) be a diameter of a circle whose centre is O. Bisect AO at M and trisect OB at T. Draw TP perpendicular to AB and meeting the circumference at P. Draw a chord BQ equal to PT and join AQ. Draw OS and TR parallel to BQ and meeting AQ at S and R respectively. Draw a chord AD equal to AS and a tangent $AC = RS$. Join BC, BD, and CD; cut off $BE = BM$, and draw EX, parallel to CD, meeting BC at X.

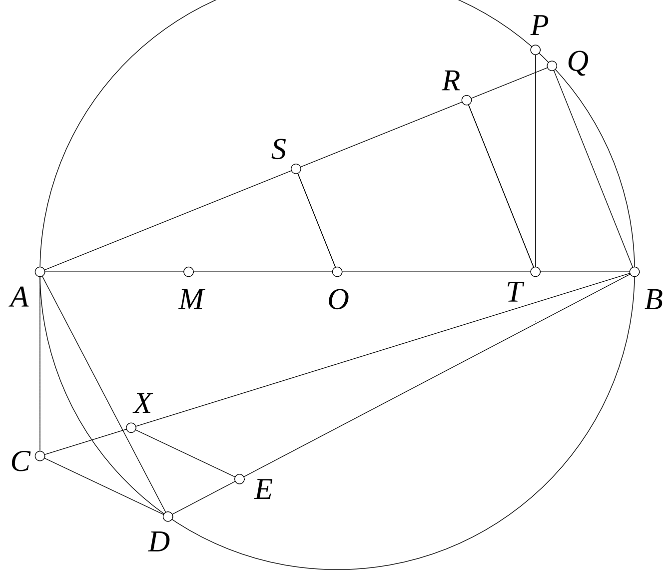


Fig. 1.

Then the square on BX is very nearly equal to the area of the circle, the error being less than a tenth of an inch when the diameter is 40 miles long.

(2) Let AB (Fig.2) be a diameter of a circle whose centre is O. Bisect the arc ACB at C and trisect AO at T. Join BC and cut off from it CM and MN equal to AT. Join AM and AN and cut off from the latter AP equal to AM. Through P draw PQ parallel to MN and meeting AM at Q. Join OQ and through T draw TR, parallel to OQ and meeting AQ at R. Draw AS perpendicular to AO and equal to AR, and join OS.

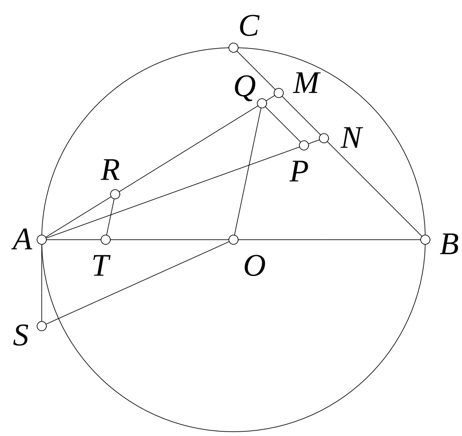


Fig. 2.

Then the mean proportional between OS and OB will be very nearly equal to a sixth of the circumference, the error being less than a twelfth of an inch when the diameter is 8000 miles long.

13. I shall conclude this paper by giving a few series for $1/\pi$.
It is known that, when $k \leq 1/\sqrt{2}$,

$$\left(\frac{2K}{\pi}\right)^2 = 1 + \left(\frac{1}{2}\right)^3 (2kk')^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^3 (2kk')^4 + \cdots \tag{25}$$

Hence we have

$$\begin{aligned} &q^{\frac{1}{3}}(1-q^2)^4(1-q^4)^4(1-q^6)^4\cdots \\ &\quad = \left(\frac{1}{4}kk'\right)^{\frac{2}{3}} \left\{1 + \left(\frac{1}{2}\right)^3 (2kk')^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^3 (2kk')^4 + \cdots\right\}. \end{aligned} \tag{26}$$

Differentiating both sides in (26) logarithmically with respect to k, we can easily shew that

$$\begin{aligned} &1 - 24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \cdots\right) \\ &\quad = (1-2k^2)\left\{1 + 4\left(\frac{1}{2}\right)^3 (2kk')^2 + 7\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 (2kk')^4 + \cdots\right\}. \end{aligned} \tag{27}$$

But it follows from (19) that, when $q = e^{-\pi\sqrt{n}}, n$ being a rational number, the left-hand side of (27) can be expressed in the form

$$A\left(\frac{2K}{\pi}\right)^2 + \frac{B}{\pi},$$

where A and B are algebraic numbers expressible by surds. Combining (25) and (27) in such a way as to eliminate the term $(2K/\pi)^2$, we are left with a series for $1/\pi$. Thus, for example,

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{19}{4^3}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \cdots, \qquad (q = e^{-\pi\sqrt{3}}, 2kk' = \frac{1}{2}), \tag{28}$$

$$\frac{16}{\pi} = 5 + \frac{47}{64}\left(\frac{1}{2}\right)^3 + \frac{89}{64^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{131}{64^3}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \cdots, \qquad (q = e^{-\pi\sqrt{7}}, 2kk' = \frac{1}{8}), \tag{29}$$

$$\frac{32}{\pi} = (5\sqrt{5}-1) + \frac{47\sqrt{5}+29}{64}\left(\frac{1}{2}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^8$$

$$+\frac{89\sqrt{5}+59}{64^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^{16}+\cdots,$$

$$\left[q=e^{-\pi\sqrt{15}}, 2kk'=\frac{1}{8}\left(\frac{\sqrt{5}-1}{2}\right)\right]; \tag{30}$$

here $5\sqrt{5}-1, 47\sqrt{5}+29, 89\sqrt{5}+59, \ldots$ are in arithmetical progression.

14. The ordinary modular equations express the relations which hold between k and l when $nK'/K = L'/L$, or $q^n = Q$, where

$$q = e^{-\pi K'/K}, \quad Q = e^{-\pi L'/L},$$

$$K = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots.$$

There are corresponding theories in which q is replaced by one or other of the functions

$$q_1 = e^{-\pi K_1'\sqrt{2}/K_1}, q_2 = e^{-2\pi K_2'/(K_2\sqrt{3})}, q_3 = e^{-2\pi K_3'/K_3},$$

where

$$\begin{aligned}
K_1 &= 1+\frac{1\cdot 3}{4^2}k^2+\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}k^4+\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{4^2\cdot 8^2\cdot 12^2}k^6+\cdots,\\
K_2 &= 1+\frac{1\cdot 2}{3^2}k^2+\frac{1\cdot 2\cdot 4\cdot 5}{3^2\cdot 6^2}k^4+\frac{1\cdot 2\cdot 4\cdot 5\cdot 7\cdot 8}{3^2\cdot 6^2\cdot 9^2}k^6+\cdots\\
K_3 &= 1+\frac{1\cdot 5}{6^2}k^2+\frac{1\cdot 5\cdot 7\cdot 11}{6^2\cdot 12^2}k^4+\frac{1\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17}{6^2\cdot 12^2\cdot 18^2}k^6+\cdots.
\end{aligned}$$

From these theories we can deduce further series for $1/\pi$, such as

$$\frac{27}{4\pi} = 2+17\frac{1}{2}\frac{1}{3}\frac{2}{3}\left(\frac{2}{27}\right)+32\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 4}{3\cdot 6}\frac{2\cdot 5}{3\cdot 6}\left(\frac{2}{27}\right)^2+\cdots, \tag{31}$$

$$\frac{15\sqrt{3}}{2\pi} = 4+37\frac{1}{2}\frac{1}{3}\frac{2}{3}\left(\frac{4}{125}\right)+70\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 4}{3\cdot 6}\frac{2\cdot 5}{3\cdot 6}\left(\frac{4}{125}\right)^2+\cdots, \tag{32}$$

$$\frac{5\sqrt{5}}{2\pi\sqrt{3}} = 1+12\frac{1}{2}\frac{1}{6}\frac{5}{6}\left(\frac{4}{125}\right)+23\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 7}{6\cdot 12}\frac{5\cdot 11}{6\cdot 12}\left(\frac{4}{125}\right)^2+\cdots, \tag{33}$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}} = 8+141\frac{1}{2}\frac{1}{6}\frac{5}{6}\left(\frac{4}{85}\right)^3+274\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 7}{6\cdot 12}\frac{5\cdot 11}{6\cdot 12}\left(\frac{4}{85}\right)^6+\cdots, \tag{34}$$

$$\frac{4}{\pi} = \frac{3}{2} - \frac{23}{2^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{43}{2^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots, \tag{35}$$

$$\frac{4}{\pi\sqrt{3}} = \frac{3}{4} - \frac{31}{3\cdot 4^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{59}{3^2\cdot 4^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots, \tag{36}$$

$$\frac{4}{\pi} = \frac{23}{18} - \frac{283}{18^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{543}{18^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots, \tag{37}$$

$$\frac{4}{\pi\sqrt{5}} = \frac{41}{72} - \frac{685}{5\cdot 72^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{1329}{5^2\cdot 72^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots, \tag{38}$$

$$\frac{4}{\pi} = \frac{1123}{882} - \frac{22583}{882^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{44043}{882^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots, \tag{39}$$

$$\frac{2\sqrt{3}}{\pi} = 1 + \frac{9}{9}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{17}{9^2}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots, \tag{40}$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1}{9} + \frac{11}{9^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{21}{9^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots, \tag{41}$$

$$\frac{1}{3\pi\sqrt{3}} = \frac{3}{49} + \frac{43}{49^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{83}{49^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots, \tag{42}$$

$$\frac{2}{\pi\sqrt{11}} = \frac{19}{99} + \frac{299}{99^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{579}{99^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots, \tag{43}$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{53883}{99^{10}}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots. \tag{44}$$

In all these series the first factors in each term form an arithmetical progression; e.g. 2, 17, 32, 47, ..., in (31), and 4, 37, 70, 103, ... , in (32). The first two series belong to the theory of q_2, the next two to that of q_3, as the rest to that of q_1.

The last series (44) is extremely rapidly convergent. Thus, taking only the first term, we see that

$$\frac{1103}{99^2} = .11253953678\ldots,$$

$$\frac{1}{2\pi\sqrt{2}} = .11253953951\ldots.$$

15. In concluding this paper I have to remark that the series

$$1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\frac{3q^6}{1-q^6}+\cdots\right),$$

which has been discussed in §§ 8-13, is very closely connected with the perimeter of an ellipse whose eccentricity is k. For, if a and b be the semi-major and the semi-minor axes, it is known that

$$p=2\pi a\left\{1-\frac{1}{2^2}k^2-\frac{1^2\cdot 3}{2^2\cdot 4^2}k^4-\frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^4\cdot 6^2}k^6-\cdots\right\}, \tag{45}$$

where p is the perimeter and k the eccentricity. It can easily be seen from (45) that

$$p=4ak'^2\left\{K+k\frac{dK}{dk}\right\}. \tag{46}$$

But, taking the equation

$$q^{\frac{1}{12}}(1-q^2)(1-q^4)(1-q^6)\cdots=(2kk')^{\frac{1}{6}}\sqrt{(K/\pi)},$$

and differentiating both sides logarithmically with respect to k, and combining the result with (46) in such a way as to eliminate dK/dk, we can shew that

$$p=\frac{4a}{3K}\left[K^2(1+k'^2)+(\tfrac{1}{2}\pi)^2\left\{1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\cdots\right)\right\}\right]. \tag{47}$$

But we have shewn already that the right-hand side of(47) can be expressed in terms of K if $q=e^{-\pi\sqrt{n}}$, where n is any rational number. It can also be shewn that K can be expressed in terms of Γ-functions if q be of the forms $e^{-\pi n}$, $e^{-\pi n\sqrt{2}}$ and $e^{-\pi n\sqrt{3}}$, where n is rational. Thus, for example, we have

$$\left.\begin{aligned}
&k=\sin\tfrac{\pi}{4}, \qquad q=e^{-\pi},\\
&\qquad p=a\sqrt{\left(\tfrac{\pi}{2}\right)}\left\{\frac{\Gamma(\frac14)}{\Gamma(\frac34)}+\frac{\Gamma(\frac34)}{\Gamma(\frac54)}\right\},\\
&k=\tan\tfrac{\pi}{8}, \qquad q=e^{-\pi\sqrt{2}},\\
&\qquad p=a\sqrt{\left(\tfrac{\pi}{4}\right)}\left\{\frac{\Gamma(\frac18)}{\Gamma(\frac58)}+\frac{\Gamma(\frac58)}{\Gamma(\frac98)}\right\},\\
&k=\sin\tfrac{\pi}{12}, \qquad q=e^{-\pi\sqrt{3}},\\
&\qquad p=a\sqrt{\left(\tfrac{\pi}{\sqrt{3}}\right)}\left\{\left(1+\tfrac{1}{\sqrt{3}}\right)\frac{\Gamma(\frac13)}{\Gamma(\frac56)}+2\frac{\Gamma(\frac56)}{\Gamma(\frac13)}\right\},\\
&\tfrac{b}{a}=\tan^2\tfrac{\pi}{8}, \qquad q=e^{-2\pi}\\
&\qquad p=(a+b)\sqrt{\left(\tfrac{\pi}{2}\right)}\left\{\tfrac12\frac{\Gamma(\frac14)}{\Gamma(\frac34)}+\frac{\Gamma(\frac34)}{\Gamma(\frac54)}\right\},
\end{aligned}\right\} \tag{48}$$

and so on.

16. The following approximations for p were obtained empirically:

$$p = \pi[3(a+b) - \sqrt{\{(a+3b)(3a+b)\}} + \epsilon], \tag{49}$$

where ϵ is about $ak^{12}/1048576$;

$$p = \pi\left\{(a+b) + \frac{3(a-b)^2}{10(a+b) + \sqrt{(a^2+14ab+b^2)}} + \epsilon\right\}, \tag{50}$$

where ϵ is about $3ak^{20}/68719476736$.

On the integral $\int_0^x \frac{\tan^{-1} t}{t}\, dt$

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1. Let

$$\phi(x) = \int_0^x \frac{\tan^{-1} t}{t}\, dt. \tag{1}$$

Then it is easy to see that

$$\phi(x) + \phi(-x) = 0; \tag{2}$$

and that

$$\phi(x) = \frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \ldots \tag{3}$$

provided that $|x| \le 1$.
Changing t into $1/t$ in (1), we obtain

$$\phi(x) - \phi\left(\frac{1}{x}\right) = \frac{1}{2}\pi \log x, \tag{4}$$

provided that the real part of x is positive.
The results in the following two sections can be very easily proved by differentiating both sides with respect to x.

2. If $0 < x < \frac{1}{2}\pi$, then

$$\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \ldots = \phi(\tan x) - x \log(\tan x). \tag{5}$$

If, in particular, we put $x = \frac{1}{8}\pi$ and $\frac{1}{12}\pi$ in (5), we obtain

$$\frac{1}{1^2} + \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \sqrt{2}\phi(\sqrt{2} - 1) + \frac{\pi}{4\sqrt{2}} \log(1 + \sqrt{2}); \tag{6}$$

and

$$2\phi(1) = 3\phi(2 - \sqrt{3}) + \frac{1}{4}\pi \log(2 + \sqrt{3}). \tag{7}$$

If $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, then

$$2\phi\left(\tan\frac{x}{2}\right) = \sin x + \frac{2}{3}\frac{\sin^3 x}{3} + \frac{2\cdot 4}{3\cdot 5}\frac{\sin^5 x}{5} + \cdots. \tag{8}$$

If $0 < x < \frac{1}{2}\pi$, then

$$\frac{\sin x}{1^2}\cos x + \frac{\sin 2x}{2^2}\cos^2 x + \frac{\sin 3x}{3^2}\cos^3 x + \cdots$$
$$= \phi(\tan x) + \frac{1}{2}\pi\log\cos x - x\log\sin x; \tag{9}$$

and

$$\frac{\cos x + \sin x}{1^2} + \frac{1}{2}\frac{\cos^3 x + \sin^3 x}{3^2} + \frac{1\cdot 3}{2\cdot 4}\frac{\cos^5 x + \sin^5 x}{5^2} + \cdots$$
$$= \phi(\tan x) + \frac{1}{2}\pi\log(2\cos x). \tag{10}$$

If $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ and a be any number such that

$$|(1-a)\sin x| \leq 1, \quad \left|\left(1-\frac{1}{a}\right)\cos x\right| \leq 1,$$

then

$$\frac{\sin x}{1^2}\left(1-\frac{1}{a}\right)\cos x + \frac{\sin 2x}{2^2}\left(1-\frac{1}{a}\right)^2\cos^2 x + \frac{\sin 3x}{3^2}\left(1-\frac{1}{a}\right)^3\cos^3 x + \cdots$$
$$+\frac{\sin(x+\frac{1}{2}\pi)}{1^2}(1-a)\sin x - \frac{\sin 2(x+\frac{1}{2}\pi)}{2^2}(1-a)^2\sin^2 x + \cdots$$
$$= \phi(\tan x) - \phi(a\tan x) + x\log a. \tag{11}$$

3. Let $R(x)$ and $I(x)$ denote the real and the imaginary parts of x respectively. Then, if $-1 < R(x) < 1$,

$$\log\left(1-\frac{x^2}{1^2}\right) - 3\log\left(1-\frac{x^2}{3^2}\right) + 5\log\left(1-\frac{x^2}{5^2}\right) - \cdots$$
$$= \frac{4}{\pi}[\phi(1) - \phi\{\tan\frac{1}{4}\pi(1-x)\}] + \log\tan\frac{1}{4}\pi(1-x). \tag{12}$$

Putting $x = \frac{2}{3}$ in (12) and using (7), we obtain

$$\left(1-\frac{4}{3^2}\right)\left(1-\frac{4}{9^2}\right)^{-3}\left(1-\frac{4}{15^2}\right)^5\left(1-\frac{4}{21^2}\right)^{-7}\left(1-\frac{4}{27^2}\right)^9\cdots = (2-\sqrt{3})^{\frac{2}{3}}e^n,$$

where

$$n = \frac{4}{3\pi}\phi(1) \tag{13}$$

Again, subtracting $\log(1-x)$ from both sides in (12) and making $x \to 1$, we obtain

$$\left(1-\frac{1}{3^2}\right)^{-3}\left(1-\frac{1}{5^2}\right)^{5}\left(1-\frac{1}{7^2}\right)^{-7}\left(1-\frac{1}{9^2}\right)^{9}\cdots = \frac{\pi}{8}e^{3n}. \tag{14}$$

If $-1 < I(x) < 1$, then

$$\log\left(1+\frac{x^2}{1^2}\right) - 3\log\left(1+\frac{x^2}{3^2}\right) + 5\log\left(1+\frac{x^2}{5^2}\right) - \cdots$$
$$= \frac{4}{\pi}\{\phi(1) - \phi(e^{-\frac{1}{2}\pi x})\} - 2x\tan^{-1} e^{-\frac{1}{2}\pi x}. \tag{15}$$

From this and (7) we see that, if $\frac{1}{2}\pi x = \log(2+\sqrt{3})$, then

$$\left(1+\frac{x^2}{1^2}\right)\left(1+\frac{x^2}{3^2}\right)^{-3}\left(1+\frac{x^2}{5^2}\right)^{5}\left(1+\frac{x^2}{7^2}\right)^{-7}\cdots = e^n, \tag{16}$$

where n is the same as in (13).
It follows at once from (12) and (15) that, if $-1 < R(\beta) < 1, \quad -1 < I(\alpha) < 1$, then

$$e^{\frac{1}{2}\pi\alpha\beta} = \left(\frac{1^2+\alpha^2}{1^2-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^5\left(\frac{7^2-\beta^2}{7^2+\alpha^2}\right)^7\cdots, \tag{17}$$

provided that $\cosh\frac{1}{2}\pi\alpha = \sec\frac{1}{2}\pi\beta$.

4. Now changing x into $2x(1+i)$ in (15), we have

$$\log\left(1+\frac{8ix^2}{1^2}\right) - 3\log\left(1+\frac{8ix^2}{3^2}\right) + 5\log\left(1+\frac{8ix^2}{5^2}\right) - \cdots$$
$$= \frac{4}{\pi}\phi(1) - 4x(1+i)\tan^{-1} e^{-\pi x(1+i)} - \frac{4}{\pi}\left\{\frac{1}{1^2}e^{-\pi x(1+i)} - \frac{1}{3^2}e^{-3\pi x(1+i)} + \dots\right\}.$$

Equating real and imaginary parts we see that, if x is positive, then

$$\log\left(1+\frac{64x^4}{1^4}\right) - 3\log\left(1+\frac{64x^4}{3^4}\right) + 5\log\left(1+\frac{64x^4}{5^4}\right) - \cdots$$
$$= \frac{8}{\pi}\phi\ (1) - 2x\log\left(\frac{\cosh\pi x + \sin\pi x}{\cosh\pi x - \sin\pi x}\right) - 4x\tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)$$
$$-\frac{8}{\pi}\left\{\frac{\cos\pi x}{1^2}e^{-\pi x} - \frac{\cos 3\pi x}{3^2}e^{-3\pi x} + \frac{\cos 5\pi x}{5^2}e^{-5\pi x} - \cdots\right\}; \tag{18}$$

and

$$\tan^{-1}\frac{8x^2}{1^2} - 3\tan^{-1}\frac{8x^2}{3^2} + 5\tan^{-1}\frac{8x^2}{5^2} - \cdots$$
$$= x\log\left(\frac{\cosh\pi x + \sin\pi x}{\cosh\pi x - \sin\pi x}\right) - 2x\tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)$$
$$+ \frac{4}{\pi}\left\{\frac{\sin\pi x}{1^2}e^{-\pi x} - \frac{\sin 3\pi x}{3^2}e^{-3\pi x} + \frac{\sin 5\pi x}{5^2}e^{-5\pi x} - \cdots\right\}. \tag{19}$$

It follows from (18) that, if n is a positive odd integer, then

$$\left(1+\frac{4n^4}{1^4}\right)\left(1+\frac{4n^4}{3^4}\right)^{-3}\left(1+\frac{4n^4}{5^4}\right)^{5}\left(1+\frac{4n^4}{7^4}\right)^{-7}\cdots$$
$$= e^{\frac{8}{\pi}\phi(1)}\left(\frac{1-e^{-\frac{1}{2}\pi n}}{1+e^{-\frac{1}{2}\pi n}}\right)^{2n\cos\frac{1}{2}(n-1)\pi}, \tag{20}$$

and, if n is any even integer, then

$$\left(1+\frac{4n^4}{1^4}\right)\left(1+\frac{4n^4}{3^4}\right)^{-3}\left(1+\frac{4n^4}{5^4}\right)^{5}\left(1+\frac{4n^4}{7^4}\right)^{-7}\cdots$$
$$= \exp\left\{\frac{8}{\pi}\phi(1) - \frac{8}{\pi}(-1)^{\frac{1}{2}n}[\phi(e^{-\frac{1}{2}\pi n}) + \frac{1}{2}\pi n\tan^{-1}e^{-\frac{1}{2}\pi n}]\right\}. \tag{21}$$

Similarly from (19) we see that, if n is any positive odd integer, then

$$\tan^{-1}\frac{2n^2}{1^2} - 3tan^{-1}\frac{2n^2}{3^2} + 5\tan^{-1}\frac{2n^2}{5^2} - \cdots$$
$$= \frac{4}{\pi}(-1)^{\frac{1}{2}(n-1)}\left\{\frac{\pi n}{4}\log\left(\frac{1+e^{-\frac{1}{2}\pi n}}{1-e^{-\frac{1}{2}\pi n}}\right) + \frac{1}{1^2}e^{-\frac{1}{2}\pi n} + \frac{1}{3^2}e^{-\frac{3}{2}\pi n} + \frac{1}{5^2}e^{-\frac{5}{2}\pi n} + \cdots\right\}; \tag{22}$$

and, if n is a positive even integer, then

$$\tan^{-1}\frac{2n^2}{1^2} - 3\tan^{-1}\frac{2n^2}{3^2} + 5\tan^{-1}\frac{2n^2}{5^2} - \ldots = 2n(-1)^{\frac{1}{2}n-1}\tan^{-1}e^{-\frac{1}{2}\pi n}. \tag{23}$$

In this connection it may be interesting to note that

$$\tan^{-1}e^{-\frac{1}{2}\pi n} = \frac{\pi}{4} - \left(\tan^{-1}\frac{n}{1} - \tan^{-1}\frac{n}{3} + \tan^{-1}\frac{n}{5} - \ldots\right) \tag{24}$$

for all real values of n.

5. Remembering that $\frac{\pi}{4\cosh\pi x} = \frac{1}{1^2+4x^2} - \frac{3}{3^2+4x^2} + \frac{5}{5^2+4x^2} - \ldots$ we have

$$\frac{\pi}{4}\sum_1^\infty \frac{1}{n^2\cosh\pi nx} = \sum_{n=1}^{n=\infty}\left\{\frac{1}{n^2(1^2+4n^2x^2)} - \frac{3}{n^2(3^2+4n^2x^2)} + \cdots\right\}$$

$$= \frac{\pi^3}{8}\left(\frac{1}{3}+\frac{x^2}{2}\right) - \pi x \left(\frac{\coth\frac{\pi}{2x}}{1^2} - \frac{\coth\frac{3\pi}{2x}}{3^2} + \frac{\coth\frac{5\pi}{2x}}{5^2} - \cdots\right). \tag{25}$$

That is to say, if α and β are real and $\alpha\beta = \pi^2$, then

$$\begin{aligned}&\phi(1) + 2\phi(e^{-\alpha}) + 2\phi(e^{-2\alpha}) + 2\phi(e^{-3\alpha}) + \cdots \\ &\quad = \frac{\pi}{8}\left(\frac{\alpha}{3}+\frac{\beta}{2}\right) - \frac{\pi}{4\beta}\left\{\frac{1}{1^2\cosh\beta} + \frac{1}{2^2\cosh 2\beta} + \cdots\right\}.\end{aligned} \tag{26}$$

If, in particular, we put $\alpha = \beta = \pi$ in (26), we obtain

$$\begin{aligned}\phi(1) &= \frac{5\pi^2}{48} - 2\left\{\frac{1}{1^2(e^{\pi}-1)} - \frac{1}{3^2(e^{3\pi}-1)} + \frac{1}{5^2(e^{5\pi}-1)}\cdots\right\} \\ &\quad -\frac{1}{2}\left\{\frac{1}{(1^2e^{\pi}+e^{-\pi})} + \frac{1}{2^2(e^{2\pi}+e^{-2\pi})} + \frac{1}{3^2(e^{3\pi}+e^{-3\pi})} + \cdots\right\} \\ &= .9159655942,\end{aligned} \tag{27}$$

approximately.

On the number of divisors of a number

Journal of the Indian Mathematical Society, VII, 1915, 131 – 133

1. If δ be a divisor of N, then there is a conjugate divisor δ' such that $\delta\delta' = N$. Thus we see that
the number of divisors from 1 to $\sqrt{N}$ is equal to the number of divisors from $\sqrt{N}$ to N. (1)
From this it evidently follows that

$$d(N) < 2\sqrt{N}, \tag{2}$$

where $d(N)$ denotes the number of divisors of N (including unity and the number itself). This is only a trivial result, as all the numbers from 1 to $\sqrt{N}$ cannot be divisors of N. So let us try to find the best possible superior limit for $d(N)$ by using purely elementary reasoning.

2. First let us consider the case in which all the prime divisors of N are known. Let

$$N = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdots p_n^{a_n},$$

where $p_1, p_2, p_3 \dots p_n$ are a given set of n primes . Then it is easy to see that

$$d(N) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots (1 + a_n). \tag{3}$$

But

$$\begin{aligned} &\frac{1}{n}\{(1 + a_1) \log p_1 + (1 + a_2) \log p_2 + \cdots + (1 + a_n) \log p_n\} \\ &\quad > \{(1 + a_1)(1 + a_2) \cdots (1 + a_n) \log p_1 \log p_2 \cdots \log p_n\}^{\frac{1}{n}}, \end{aligned} \tag{4}$$

since the arithmetic mean of unequal positive numbers is always greater than their geometric mean. Hence

$$\frac{1}{n}\{\log p_1 + \log p_2 + \cdots + \log p_n + \log N\} > \{\log p_1 \log p_2 \cdots \log p_n \cdot d(N)\}^{\frac{1}{n}}.$$

In other words

$$d(N) < \frac{\left\{\frac{1}{n}\log(p_1 p_2 p_3 \cdots p_n N)\right\}^n}{\log p_1 \log p_2 \log p_3 \cdots \log p_n}, \tag{5}$$

for all values of N whose prime divisors are $p_1, p_2, p_3, \dots, p_n$.

3. Next let us consider the case in which only the number of prime divisors of N is known. Let

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n},$$

where n is a given number; and let

$$N' = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdots p^{a_n},$$

where p is the natural nth prime. Then it is evident that

$$N' \leq N; \tag{6}$$

and

$$d(N') = d(N). \tag{7}$$

But

$$d(N') < \frac{\left\{\frac{1}{n}\log(2 \cdot 3 \cdot 5 \cdots p \cdot N')\right\}^n}{\log 2 \log 3 \log 5 \cdots \log p}, \tag{8}$$

by virtue of (5). It follows from (6) to (8) that, if p be the natural nth prime, then

$$d(N) < \frac{\left\{\frac{1}{n}\log(2 \cdot 3 \cdot 5 \cdots p \cdot N)\right\}^n}{\log 2 \log 3 \log 5 \cdots \log p}, \tag{9}$$

for all values of N having n prime divisors.

4. Finally, let us consider the case in which nothing is known about N. Any integer N can be written in the form

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots,$$

where $a_\lambda \geq 0$. Now let

$$x^h = 2, \tag{10}$$

where h is any positive number. Then we have

$$\frac{d(N)}{N^h} = \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{3^{ha_3}} \cdot \frac{1+a_5}{5^{ha_5}} \tag{11}$$

But from (10) we see that, if q be any prime greater than x, then

$$\frac{1+a_q}{q^{ha_q}} \leq \frac{1+a_q}{x^{ha_q}} = \frac{1+a_q}{2^{a_q}} \leq 1. \tag{12}$$

It follows from (11) and (12) that, if p be the largest prime not exceeding x, then

$$\begin{aligned} \frac{d(N)}{N^h} &\leq \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{3^{ha_3}} \cdot \frac{1+a_5}{5^{ha_5}} \cdots \frac{1+a_p}{p^{ha_p}} \\ &\leq \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{2^{ha_3}} \cdot \frac{1+a_5}{2^{ha_5}} \cdots \frac{1+a_p}{2^{ha_p}}. \end{aligned} \tag{13}$$

But it is easy to shew that the maximum value of $(1+a)2^{-ha}$ for the variable a is $\frac{2^h}{he\log 2}$. Hence

$$\frac{d(N)}{N^h} \le \left(\frac{2^h}{he\log 2}\right)^{\omega(x)}, \tag{14}$$

where $\omega(x)$ denotes the number of primes not exceeding x. But from (10) we have

$$h = \frac{\log 2}{\log x}.$$

Substituting this in (14), we obtain

$$d(N) \le N^{\frac{\log 2}{\log x}} \left\{\frac{2^{\frac{\log 2}{\log x}} \log x}{e(\log 2)^2}\right\}^{\omega(x)}. \tag{15}$$

But it is easy to verify that, if $x \ge 6.05$, then

$$2^h < e(\log 2)^2.$$

From this and (15) it follows that, if $x \ge 6.05$, then

$$d(N) < 2^{(\log N)/(\log x)}(\log x)^{\omega(x)} \tag{16}$$

for all values of N, $\omega(x)$ being the number of primes not exceeding x.

5. The symbol "O" is used in the following sense:

$$\phi(x) = O\{(\Psi(x)\}$$

means that there is a positive constant K such that

$$\left|\frac{\phi(x)}{\Psi(x)}\right| \le K$$

for all sufficiently large values of x (see Hardy, *Orders of Infinity*, pp. 5 *et seq.*). For example:

$$5x = O(x); \frac{1}{2}x = O(x); x\sin x = O(x); \sqrt{x} = O(x); \log x = O(x);$$

but $$x^2 \ne O(x);\ x\log x \ne O(x).$$

Hence it is obvious that

$$\omega(x) = O(x). \tag{17}$$

Now, let us suppose that

$$x = \frac{\log N}{(\log\log N)^2}$$

in (16). Then we have

$$\log x = \log\log N + O(\log\log\log N);$$

and so

$$\frac{\log N}{\log x} = \frac{\log N}{\log\log N} + O\left\{\frac{\log N \log\log\log N}{(\log\log N)^2}\right\}. \tag{18}$$

Again

$$\omega(x)\log\log x = O(x\log\log x) = O\left\{\frac{\log N \log\log\log N}{(\log\log N)^2}\right\}. \tag{19}$$

It follows from (16), (18) and (19), that

$$\log d(N) < \frac{\log 2 \log N}{\log\log N} + O\left\{\frac{\log N \log\log\log N}{(\log\log N)^2}\right\} \tag{20}$$

for all sufficiently large values of N.

On the sum of the square roots of the first n natural numbers

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1. Let

$$\phi_1(n) = \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} - \left(C_1 + \tfrac{2}{3}n\sqrt{n} + \tfrac{1}{2}\sqrt{n}\right) - \tfrac{1}{6}\sum_{\nu=0}^{\nu=\infty}\{\sqrt{(n+\nu)} + \sqrt{(n+\nu+1)}\}^{-3},$$

where C_1 is a constant such that $\phi_1(1) = 0$. Then we see that

$$\phi_1(n) - \phi_1(n+1) = -\sqrt{(n+1)} + \left[\tfrac{2}{3}(n+1)\sqrt{(n+1)} + \tfrac{1}{2}\sqrt{(n+1)}\right] - (\tfrac{2}{3}n\sqrt{n} + \tfrac{1}{2}\sqrt{n}) + \tfrac{1}{6}\{\sqrt{n} - \sqrt{(n+1)}\}^3 = 0.$$

But $\phi_1(1) = 0$. Hence $\phi_1(n) = 0$ for all values of n. That is to say

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \cdots + \sqrt{n} = C_1 + \tfrac{2}{3}n\sqrt{n} + \tfrac{1}{2}\sqrt{n} + \tfrac{1}{6}\left[\left\{\sqrt{n} + \sqrt{(n+1)}\right\}^{-3} + \left\{\sqrt{(n+1)} + \sqrt{(n+2)}\right\}^{-3} + \left\{\sqrt{(n+2)} + \sqrt{(n+3)}\right\}^{-3} + \cdots\right]. \tag{1}$$

But it is known that

$$C_1 = \frac{1}{4\pi}\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots\right). \tag{2}$$

Putting $n = 1$ in (1) and using (2), we obtain

$$2\pi\left\{\frac{1}{(\sqrt{1})^3} + \frac{1}{(\sqrt{1}+\sqrt{2})^3} + \frac{1}{(\sqrt{2}+\sqrt{3})^3} + \frac{1}{(\sqrt{3}+\sqrt{4})^3} + \cdots\right\} = 3\left\{\frac{1}{(\sqrt{1})^3} + \frac{1}{(\sqrt{2})^3} + \frac{1}{(\sqrt{3})^3} + \frac{1}{(\sqrt{4})^3} + \cdots\right\}. \tag{3}$$

2. Again let

$$\phi_2(n) = 1\sqrt{1} + 2\sqrt{2} \ldots + n\sqrt{n} - \left(C_2 + \tfrac{2}{5}n^2\sqrt{n} + \tfrac{1}{2}n\sqrt{n} + \tfrac{1}{8}\sqrt{n}\right) - \tfrac{1}{40}\sum_{\nu=0}^{\nu=\infty}[\sqrt{(n+\nu)} + \sqrt{(n+\nu+1)}]^{-5},$$

where C_2 is a constant such that $\phi_2(1) = 0$. Then we have

$$\begin{aligned}\phi_2(n) - \phi_2(n+1) = -(n+1)\sqrt{(n+1)} \\ +\{\tfrac{2}{5}(n+1)^2\sqrt{(n+1)} + \tfrac{1}{2}(n+1)\sqrt{(n+1)} + \tfrac{1}{8}\sqrt{(n+1)}\} \\ -\{\tfrac{2}{5}n^2\sqrt{n} + \tfrac{1}{2}n\sqrt{n} + \tfrac{1}{8}\sqrt{n}\} + \tfrac{1}{40}\{\sqrt{n} - \sqrt{(n+1)}\}^5 = 0.\end{aligned}$$

But $\phi_2(1) = 0$. Hence $\phi_2(n) = 0$. In other words

$$\begin{aligned}1\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \cdots n\sqrt{n} = C_2 + \tfrac{2}{5}n^2\sqrt{n} + \tfrac{1}{2}n\sqrt{n} + \tfrac{1}{8}\sqrt{n} + \tfrac{1}{40}\Big[\{\sqrt{n} + \sqrt{(n+1)}\}^{-5} \\ +\{\sqrt{(n+1)} + \sqrt{(n+2)}\}^{-5} + \{\sqrt{(n+2)} + \sqrt{(n+3)}\}^{-5} + \cdots\Big]. \quad (4)\end{aligned}$$

But it is known that

$$C_2 = -\frac{3}{16\pi^2}\left(\frac{1}{1^2\sqrt{1}} + \frac{1}{2^2\sqrt{2}} + \frac{1}{3^2\sqrt{3}} + \cdots\right). \tag{5}$$

It is easy to see from (4) and (5) that

$$\begin{aligned}2\pi^2\left\{\frac{1}{(\sqrt{1})^5} + \frac{1}{(\sqrt{1}+\sqrt{2})^5} + \frac{1}{(\sqrt{2}+\sqrt{3})^5} + \frac{1}{(\sqrt{3}+\sqrt{4})^5} + \cdots\right\} \\ = 15\left\{\frac{1}{(\sqrt{1})^5} + \frac{1}{(\sqrt{2})^5} + \frac{1}{(\sqrt{3})^5} + \frac{1}{(\sqrt{4})^5} + \cdots\right\}. \quad (6)\end{aligned}$$

3. The corresponding results for higher powers are not so neat as the previous ones. Thus for example

$$\begin{aligned}1^2\sqrt{1} + 2^2\sqrt{2} + 3^2\sqrt{3} + \cdots + n^2\sqrt{n} = C_3 + \sqrt{n}(\tfrac{2}{7}n^3 + \tfrac{1}{2}n^2 + \tfrac{5}{24}n) \\ -\tfrac{1}{96}[\{\sqrt{n} + \sqrt{(n+1)}\}^{-3} + \{\sqrt{(n+1)} + \sqrt{(n+2)}\}^{-3} + \cdots] \\ +\tfrac{1}{224}\Big[\{\sqrt{n} + \sqrt{(n+1)}\}^{-7} + \{\sqrt{(n+1)} + \sqrt{(n+2)}\}^{-7} \\ +\{\sqrt{(n+2)} + \sqrt{(n+3)}\}^{-7} + \cdots\Big]; \quad (7)\end{aligned}$$

$$\begin{aligned}1^3\sqrt{1} + 2^3\sqrt{2} + \cdots + n^3\sqrt{n} = C_4 + \sqrt{n}(\tfrac{2}{9}n^4 + \tfrac{1}{2}n^3 + \tfrac{7}{24}n^2 - \tfrac{7}{384}) \\ -\tfrac{1}{192}\Big[\{\sqrt{n} + \sqrt{(n+1)}\}^{-5} + \{\sqrt{(n+1)} + \sqrt{(n+2)}\}^{-5} + \cdots\Big] \\ +\tfrac{1}{1152}[\{\sqrt{n} + \sqrt{(n+1)}\}^{-9} + \{\sqrt{(n+1)} + \sqrt{(n+2)}\}^{-9} + \cdots]; \quad (8)\end{aligned}$$

and so on.

The constants $C_3, C_4, \ldots$ can be ascertained from the well-known result that *the constant in the approximate summation of the series* $1^{r-1} + 2^{r-1} + 3^{r-1} + \cdots + n^{r-1}$ is

$$\frac{2\Gamma(r)}{(2\pi)^r}\left(\frac{1}{1^r}+\frac{1}{2^r}+\frac{1}{3^r}+\frac{1}{4^r}+\cdots\right)\cos\tfrac{1}{2}\pi r, \tag{9}$$

provided that the real part of r is greater than 1.

4. Similarly we can shew, by induction, that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} = C_0 + 2\sqrt{n} + \frac{1}{2\sqrt{n}}$$
$$-\tfrac{1}{2}\left\{\frac{\{\sqrt{n}+\sqrt{(n+1)}\}^{-3}}{\sqrt{\{n(n+1)\}}}+\frac{\{\sqrt{(n+1)}+\sqrt{(n+2)^{-3}}\}}{\sqrt{\{(n+1)(n+2)\}}}+\cdots\right\}, \tag{10}$$

The value of C_0 can be determined as follows: from (10) we have

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{(2n)}}-2\sqrt{(2n)}\to C_0, \tag{11}$$

as $n \to \infty$. Also

$$2\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{6}}+\cdots+\frac{1}{\sqrt{(2n)}}\right)-2\sqrt{(2n)}\to C_0\sqrt{2}, \tag{12}$$

as $n \to \infty$.
Now subtracting (12) from (11) we see that

$$\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots-\frac{1}{\sqrt{(2n)}}\to C_0(1-\sqrt{2}),\ \text{ as } n\to\infty.$$

That is to say

$$C_0 = -(1+\sqrt{2})\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots\right). \tag{13}$$

We can also shew, by induction, that

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n} = C_1 + \tfrac{2}{3}n\sqrt{n}+\tfrac{1}{2}\sqrt{n}+\frac{1}{24\sqrt{n}}$$
$$-\tfrac{1}{24}\left[\frac{\{\sqrt{n}+\sqrt{(n+1)}\}^{-5}}{\sqrt{\{n(n+1)\}}}+\frac{\{\sqrt{(n+1)}+\sqrt{(n+2)}\}^{-5}}{\sqrt{\{(n+1)(n+2)\}}}+\cdots\right]. \tag{14}$$

The asymptotic expansion of $\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}$ for large values of n can be shewn to be

$$C_1+\tfrac{2}{3}n\sqrt{n}+\tfrac{1}{2}\sqrt{n}+\frac{1}{\sqrt{n}}\left(\tfrac{1}{24}-\frac{1}{1920n^2}+\frac{1}{9216n^4}-\cdots\right), \tag{15}$$

by using the Euler-Maclaurin sum formula.

On the product $\prod_{n=0}^{n=\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$

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1. Let

$$\phi(\alpha,\beta)=\left\{1+\left(\frac{\alpha+\beta}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{2+\alpha}\right)^3\right\}. \tag{1}$$

It is easy to see that

$$\begin{aligned}&\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{n+\beta}\right)^3\right\}\\ &\qquad=\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{\beta+2\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}\left[1-\left\{\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]\\ &\qquad\quad\times\left[1-\left\{\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right];\end{aligned} \tag{2}$$

$$\prod_{n=1}^{n=\infty}\left\{\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{\beta+2\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}\right\}=\frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}; \tag{3}$$

and

$$\begin{aligned}&\prod_{n=1}^{n=\infty}\left[1-\left\{\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]\left[1-\left\{\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]\\ &\qquad=\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}.\end{aligned} \tag{4}$$

It follows from (1) - (4) that

$$\begin{aligned}&\phi(\alpha,\beta)\phi(\beta,\alpha)\\ &\qquad=\frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}\left\{\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}\right\}.\end{aligned} \tag{5}$$

But it is evident that, if $\alpha-\beta$ be any integer, then $\phi(\alpha,\beta)/\phi(\beta,\alpha)$ can be expressed in finite terms. From this and (5) it follows that $\phi(\alpha,\beta)$ can be expressed in finite terms, if $\alpha-\beta$ be any integer. That is to say

$$\left\{1+\left(\frac{x}{a}\right)^3\right\}\left\{1+\left(\frac{x}{a+d}\right)^3\right\}\left\{1+\left(\frac{x}{a+2d}\right)^3\right\}\cdots$$

can be expressed in finite terms if $x-2a$ be a multiple of d.

2. Suppose now that $\alpha=\beta$ in (5). We obtain

$$\left\{1+\left(\frac{2\alpha}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{2+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{3+\alpha}\right)^3\right\}\cdots$$
$$=\frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(1+3\alpha)}\frac{\sinh\pi\alpha\sqrt{3}}{\pi\alpha\sqrt{3}}. \tag{6}$$

Similarly, putting $\beta=\alpha+1$ in (5), we obtain

$$\left\{1+\left(\frac{2\alpha+1}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha+1}{2+\alpha}\right)^3\right\}\cdots$$
$$=\frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(2+3\alpha)}\frac{\cosh\pi(\frac{1}{2}+\alpha)\sqrt{3}}{\pi}. \tag{7}$$

Again, since

$$\left\{1+\left(\frac{\alpha}{n}\right)^3\right\}\left\{1+3\left(\frac{\alpha}{2n+\alpha}\right)^2\right\}=\frac{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\alpha^2}{n^2}+\frac{\alpha^4}{n^4}\right)}{\left(1+\frac{\alpha}{2n}\right)^2},$$

it is easy to see that

$$\left[\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\cdots\right]\left[\left\{1+3\left(\frac{\alpha}{2+\alpha}\right)^2\right\}\left\{1+3\left(\frac{\alpha}{4+\alpha}\right)^2\right\}\cdots\right]$$
$$=\frac{\Gamma(\frac{1}{2}\alpha)}{\Gamma\{\frac{1}{2}(1+\alpha)\}}\left(\frac{\cosh\pi\alpha\sqrt{3}-\cos\pi\alpha}{2^{\alpha+2}\pi\alpha\sqrt{\pi}}\right). \tag{8}$$

3. It is known that, if the real part of α is positive, then

$$\log\Gamma(\alpha)=(\alpha-\tfrac{1}{2})\log\alpha-\alpha+\tfrac{1}{2}\log 2\pi+2\int_0^\infty\frac{\tan^{-1}(x/\alpha)}{e^{2\pi x}-1}dx. \tag{9}$$

From this we can shew that, if the real part of α is positive, then

$$\tfrac{1}{2}\log 2\pi\alpha + \frac{\pi\alpha}{\sqrt{3}} + \log\left\{\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\left(1+\frac{\alpha^3}{3^3}\right)\cdots\right\}$$
$$= \log\left(\frac{\cosh\pi\alpha\sqrt{3} - \cos\pi\alpha}{\pi\alpha}\right) + 2\int_0^\infty \frac{\tan^{-1}(x/\alpha)^3}{e^{2\pi x}-1}dx. \tag{10}$$

From this and the previous section it follows that

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{2\pi n x}-1}dx$$

can be expressed in finite terms if n is a positive integer. Thus, for example,

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{2\pi x}-1}dx = \frac{1}{4}\log 2\pi - \frac{\pi}{4\sqrt{3}} - \tfrac{1}{2}\log(1+e^{-\pi\sqrt{3}}); \tag{11}$$

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{4\pi x}-1}dx = \frac{1}{8}\log 12\pi - \frac{\pi}{4\sqrt{3}} - \frac{1}{4}\log(1-e^{-2\pi\sqrt{3}}); \tag{12}$$

and so on.

4. It is also easy to see that

$$\frac{1^2}{1^3+n^3} - \frac{2^2}{2^3+n^3} + \frac{3^2}{3^3+n^3} - \frac{4^2}{4^3+n^3} + \cdots$$
$$= \frac{1}{3}\left(\frac{1}{1+n} - \frac{1}{2+n} + \frac{1}{3+n} - \frac{1}{4+n} + \ldots\right)$$
$$+ \frac{4}{3}\left\{\frac{2-n}{(2-n)^2+3n^2} - \frac{4-n}{(4-n)^2+3n^2} + \frac{6-n}{(6-n)^2+3n^2} - \ldots\right\}. \tag{13}$$

Since

$$\frac{\pi}{4\cosh\frac{1}{2}\pi x} = \frac{1}{1^2+x^2} - \frac{3}{3^2+x^2} + \frac{5}{5^2+x^2} - \ldots,$$

it is clear that the left-hand side of (13) can be expressed in finite terms if n is any odd integer. For example,

$$\frac{1^2}{1^3+1} - \frac{2^2}{2^3+1} + \frac{3^2}{3^3+1} - \frac{4^2}{4^3+1} + \ldots = \frac{1}{3}(1 - \log 2 + \pi \operatorname{sech} \tfrac{1}{2}\pi\sqrt{3}). \tag{14}$$

The corresponding integral in this case is

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\frac{dx}{n^6+x^6} = \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{2x^2}+\sum_{\nu=1}^{\nu=\infty}\frac{(-1)^\nu}{\nu^2+x^2}\right\}\frac{x^6 dx}{n^6+x^6}$$
$$= \frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}+\dots\right)$$
$$-\frac{4}{3}\left\{\frac{n+2}{(n+2)^2+3n^2}-\frac{n+4}{(n+4)^2+3n^2}+\frac{n+6}{(n+6)^2+3n^2}-\dots\right\}; \tag{15}$$

and so the integral on the left-hand side of (15) can be expressed in finite terms if n is any odd integer. For example,

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\frac{dx}{1+x^6} = \frac{1}{3}(\log 2 - 1 + \pi \operatorname{sech} \tfrac{1}{2}\pi\sqrt{3}). \tag{16}$$

Some definite integrals

Messenger of Mathematics, XLIV, 1915, 10 – 18

1. Consider the integral

$$\int_0^\infty \frac{\cos 2mx\ dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\{1+x^2/(a+1)^2\}\cdots},$$

where m and a are positive.
It can be easily proved that

$$\left\{1-\left(\frac{t}{a}\right)^2\right\}\left\{1-\left(\frac{t}{a+1}\right)^2\right\}\left\{1-\left(\frac{t}{a+2}\right)^2\right\}\cdots\left\{1-\left(\frac{t}{a+n-1}\right)^2\right\}$$
$$=\frac{\Gamma(a+n-t)\Gamma(a+n+t)\{\Gamma(a)\}^2}{\Gamma(a-t)\Gamma(a+t)\{\Gamma(a+n)\}^2},$$

where n is any positive integer. Hence, by splitting

$$\frac{1}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\cdots\{1+x^2/(a+n-1)^2\}}$$

into partial fractions, we see that it is equal to

$$\frac{2\Gamma(2a)\{\Gamma(a+n)\}^2}{\{\Gamma(a)\}^2\Gamma(n)\Gamma(2a+n)}\left\{\frac{a}{a^2+x^2}-\frac{2a}{1!}\frac{n-1}{n+2a}\frac{a+1}{(a+1)^2+x^2}\right.$$
$$\left.+\frac{2a(2a+1)}{2!}\frac{(n-1)(n-2)}{(n+2a)(n+2a+1)}\frac{a+2}{(a+2)^2+x^2}-\cdots\right\}.$$

Multiplying both sides by $\cos 2mx$ and integrating from 0 to ∞ with respect to x, we have

$$\int_0^\infty \frac{\cos 2mx\ dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\cdots\{1+x^2/(a+n-1)^2\}}$$
$$=\frac{\pi\Gamma(2a)\{\Gamma(a+n)\}^2}{\{\Gamma(a)\}^2\Gamma(n)\Gamma(2a+n)}\left\{e^{-2am}-\frac{2a}{1!}\frac{n-1}{n+2a}e^{-2(a+1)m}+\cdots\right\}.$$

The limit of the right-hand side, as $n\to\infty$, is

$$\frac{\pi\Gamma(2a)}{\{\Gamma(a)\}^2}\left\{e^{-2am}-\frac{2a}{1!}e^{-2(a+1)m}+\frac{2a(2a+1)}{2!}e^{-2(a+2)m}-\cdots\right\}$$

$$= \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)}\operatorname{sech}^{2a} m.$$

Hence

$$\int_0^\infty \frac{\cos 2mx\ dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\cdots} = \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)}\operatorname{sech}^{2a} m. \tag{1}$$

Since

$$\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\cdots = \frac{\{\Gamma(a)\}^2}{\Gamma(a+ix)\Gamma(a-ix)},$$

the formula (1) is equivalent to

$$\int_0^\infty |\Gamma(a+ix)|^2 \cos 2mx\ dx = \tfrac{1}{2}\sqrt{\pi}\Gamma(a)\Gamma(a+\tfrac{1}{2})\operatorname{sech}^{2a} m. \tag{2}$$

2. In a similar manner we can prove that

$$\int_0^\infty \left(\frac{1+x^2/b^2}{1+x^2/a^2}\right)\left(\frac{1+x^2/(b+1)^2}{1+x^2/(a+1)^2}\right)\left(\frac{1+x^2/(b+2)^2}{1+x^2/(a+2)^2}\right)\cdots \cos mx\ dx$$

$$= \frac{\pi\Gamma(2a)\{\Gamma(b)\}^2}{\{\Gamma(a)\}^2\Gamma(b+a)\Gamma(b-a)}\left\{e^{-am} - \frac{2a}{1!}\frac{b-a-1}{b+a}e^{-(a+2)m}\right.$$

$$\left. + \frac{2a(2a+1)}{2!}\frac{(b-a-1)(b-a-2)}{(b+a)(b+a+1)}e^{-(a+2)m} - \cdots\right\},$$

where m is positive and $0 < a < b$. When $0 < a < b - \frac{1}{2}$, the integral and the series remain convergent for $m = 0$, and we obtain the formulæ

$$\int_0^\infty \left(\frac{1+x^2/b^2}{1+x^2/a^2}\right)\left(\frac{1+x^2/(b+1)^2}{1+x^2/(a+1)^2}\right)\left(\frac{1+x^2/(b+2)^2}{1+x^2/(a+2)^2}\right)\cdots\ dx$$

$$= \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a+\frac{1}{2})\Gamma(b)\Gamma(b-a-\frac{1}{2})}{\Gamma(a)\Gamma(b-\frac{1}{2})\Gamma(b-a)}, \tag{3}$$

$$\int_0^\infty \left|\frac{\Gamma(a+ix)}{\Gamma(b+ix)}\right|^2\ dx = \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a)\Gamma(a+\frac{1}{2})\Gamma(b-a-\frac{1}{2})}{\Gamma(b-\frac{1}{2})\Gamma(b)\Gamma(b-a)}. \tag{4}$$

If $a_1, a_2, a_3, \ldots, a_n$ be n positive numbers in arithmetical progression, then

$$\int_0^\infty \frac{dx}{(a_1^2+x^2)(a_2^2+x^2)(a_3^2+x^2)\cdots(a_n^2+x^2)}$$

is a particular case of the above integral, and its value can be written down at once. Thus, for example, by putting $a = \frac{11}{10}$ and $b = \frac{61}{10}$, we obtain

$$\int_0^\infty \frac{dx}{(x^2+11^2)(x^2+21^2)(x^2+31^2)(x^2+41^2)(x^2+51^2)}$$
$$= \frac{5\pi}{12\cdot 13\cdot 16\cdot 17\cdot 18\cdot 22\cdot 23\cdot 24\cdot 31\cdot 32\cdot 41}.$$

3. It follows at once from equation (1), by applying Fourier's theorem

$$\int_0^\infty \cos ny dy \int_0^\infty \phi(x)\cos xy\; dx = \tfrac{1}{2}\pi\phi(n),$$

that, when a and n are positive,

$$\begin{aligned}\int_0^\infty &\operatorname{sech}^{2a} x \cos 2nx\; dx \\ &= \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a)}{\Gamma(a+\frac{1}{2})}\frac{1}{\{1+n^2/a^2\}\{1+n^2/(a+1)^2\}\{1+n^2/(a+2)^2\}\cdots} \\ &= \tfrac{1}{2}\sqrt{\pi}\frac{|\Gamma(a+in)|^2}{\Gamma(a)\Gamma(a+\frac{1}{2})}. \qquad (5)\end{aligned}$$

Hence the function

$$\phi(a) = \int_0^\infty \operatorname{sech}^a x \cos nx\; dx \quad (0 < a < 2)$$

satisfies the functional equation

$$\phi(a)\phi(2-a) = \frac{\pi \sin \pi a}{2(1-a)(\cosh \pi n - \cos \pi a)}.$$

4. Let

$$\int_a^b f(x)F(nx)\; dx = \Psi(n),$$

and

$$\int_{\alpha}^{\beta} \phi(x)F(nx)\ dx = \chi(n).$$

Then, if we suppose the functions f, ϕ, and F to be such that the order of integration is indifferent, we have

$$\begin{aligned}\int_{a}^{b} f(x)\chi(nx)\ dx &= \int_{\alpha}^{\beta} dy \int_{a}^{b} f(x)\phi(y)F(nxy)\ dx \\ &= \int_{\alpha}^{\beta} \phi(y)\Psi(ny)dy. \qquad (6)\end{aligned}$$

A number of curious relations between definite integrals may be deduced from this result. We have, for example, the formulæ

$$\int_{0}^{\infty} \frac{\cos 2nx}{\cosh \pi x}\ dx = \frac{1}{2\cosh n}, \qquad (7)$$

$$\int_{0}^{\infty} \frac{\cos 2nx\ dx}{1 + 2\cosh \frac{2}{3}\pi x} = \frac{\sqrt{3}}{2(1 + 2\cosh 2n)}, \qquad (8)$$

$$\int_{0}^{\infty} e^{-x^2} \cos 2nx\ dx = \tfrac{1}{2}\sqrt{\pi}e^{-n^2}. \qquad (9)$$

By applying the general result (6) to the integrals (7) and (8), we obtain

$$\sqrt{3}\int_{0}^{\infty} \frac{dx}{\cosh \pi x(1 + 2\cosh 2nx)} = \int_{0}^{\infty} \frac{dx}{\cosh nx(1 + 2\cosh \frac{2}{3}\pi x)};$$

or, in other words, if $\alpha\beta = \frac{3}{4}\pi^2$, then

$$\begin{aligned}&\sqrt{\alpha}\int_{0}^{\infty} \frac{dx}{\cosh \alpha x(1 + 2\cosh \pi x)} \\ &\quad = \sqrt{\beta}\int_{0}^{\infty} \frac{dx}{\cosh \beta x(1 + 2\cosh \pi x)}. \qquad (10)\end{aligned}$$

In the same way, from (8) and (9), we obtain

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}\,dx}{1+2\cosh\alpha x} = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}\,dx}{1+2\cosh\beta x}, \tag{11}$$

with the condition $\alpha\beta = \frac{4}{3}\pi$; and, from (7) and (9),

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{\cosh\alpha x}\,dx = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{\cosh\beta x}\,dx, \tag{12}$$

with the conditions $\alpha\beta = \pi$. *
Similarly, by taking the two integrals

$$\int_0^\infty \frac{\sin nx}{e^{2\pi x}-1}\,dx = \tfrac{1}{2}\left(\frac{1}{e^n-1}+\tfrac{1}{2}-\frac{1}{n}\right),$$

and

$$\int_0^\infty xe^{-x^2}\sin nx\,dx = \frac{1}{4}\sqrt{\pi}ne^{-\frac{1}{4}n^2},$$

we can prove that, if $\alpha\beta = \pi^2$, then

$$\frac{1}{\sqrt[4]{\alpha}}\left\{1+2\alpha\int_0^\infty \frac{e^{-\alpha x}}{e^{2\pi\sqrt{x}}-1}\,dx\right\}$$
$$= \frac{1}{\sqrt[4]{\beta}}\left\{1+2\beta\int_0^\infty \frac{e^{-\beta x}}{e^{2\pi\sqrt{x}}-1}\,dx\right\}; \tag{13}$$

and so on.

5. Suppose now that a, b and n are positive, and

$$\int_0^\infty \phi(a,x) \begin{matrix}\cos\\ \sin\end{matrix}\, nx\,dx = \Psi(a,n). \tag{14}$$

Then, if the conditions of Fourier's double integral theorem are satisfied, we have

$$\int_0^\infty \Psi(b,x) \begin{matrix}\cos\\ \sin\end{matrix}\, nx\,dx = \tfrac{1}{2}\pi\phi(b,n). \tag{15}$$

*Formulæ equivalent to (11) and (12) were given by Hardy, *Quarterly Journal*, XXXV, p.193

Applying the formula (6) to (14) and (15), we obtain

$$\tfrac{1}{2}\pi \int_0^\infty \phi(a,x)\phi(b,nx)\ dx = \int_0^\infty \Psi(b,x)\Psi(a,nx)\ dx. \tag{16}$$

Thus, when $a = b$, we have the formula

$$\tfrac{1}{2}\pi \int_0^\infty \phi(x)\phi(nx)\ dx = \int_0^\infty \Psi(x)\Psi(nx)\ dx,$$

where

$$\psi(t) = \int_0^\infty \phi(x) \begin{matrix}\cos\\ \sin\end{matrix}\ tx\ dx;$$

and, in particular, if $n = 1$, then

$$\tfrac{1}{2}\pi \int_0^\infty \{\phi(x)\}^2\ dx = \int_0^\infty \{\Psi(x)\}^2\ dx.$$

if

$$\phi(a,x) = \frac{1}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\cdots} \quad (a > 0),$$

then, by (1),

$$\Psi(a,x) = \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)}\operatorname{sech}^{2a}\tfrac{1}{2}x.$$

Hence, by (16),

$$\int_0^\infty \phi(a,x)\phi(b.x)\ dx = \frac{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}{2\Gamma(a)\Gamma(b)} \int_0^\infty \operatorname{sech}^{2a+2b}\tfrac{1}{2}x\ dx;$$

and so

$$\int_0^\infty \frac{dx}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\cdots\{1 + x^2/b^2\}\{1 + x^2/(b+1)^2\}\cdots}$$
$$= \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})\Gamma(a+b)}{\Gamma(a)\Gamma(b)\Gamma(a+b+\frac{1}{2})}, \tag{17}$$

a and b being positive: or

$$\int_0^\infty |\Gamma(a+ix)\Gamma(b+ix)|^2\ dx = \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a)\Gamma(a+\frac{1}{2})\Gamma(b)\Gamma(b+\frac{1}{2})\Gamma(a+b)}{\Gamma(a+b+\frac{1}{2})}. \tag{18}$$

As particular cases of the above result, we have, when $b=1$,

$$\int_0^\infty \frac{x}{\sin h\pi x}\frac{dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\cdots}=\frac{a}{2(1+2a)};$$

when $b=2$,

$$\int_0^\infty \frac{x^3}{\sin h\pi x}\frac{dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\cdots}=\frac{a^2}{2(1+2a)(3+2a)};$$

and so on. Since $\Pi\{1+x^2/(a+n)^2\}$ can be expressed in finite terms by means of hyperbolic functions when $2a$ is an integer, we can deduce a large number of special formulæ from the preceding results.

6. Another curious formula is the following. If $0<r<1, n>0$, and $0<a<r^{n-1}$, then

$$\int_0^\infty \frac{(1+arx)(1+ar^2x)\cdots}{(1+x)(1+rx)(1+r^2x)\cdots}x^{n-1}\,dx$$
$$=\frac{\pi}{\sin n\pi}\prod_{m=1}^{m=\infty}\frac{(1-r^{m-n})(1-ar^m)}{(1-r^m)(1-ar^{m-n})}, \tag{19}$$

unless n is an integer or a is of the form r^p, where p is a positive integer.
If $a=r^p$, the formula reduces to

$$\int_0^\infty \frac{x^{n-1}\,dx}{(1+x)(1+rx)\cdots(1+r^px)}$$
$$=\frac{\pi}{\sin n\pi}\frac{(1-r^{1-n})(1-r^{2-n})\cdots(1-r^{p-n})}{(1-r)(1-r^2)\cdots(1-r^p)}. \tag{20}$$

If n is an integer, the value of the integral is in any case

$$-\frac{\log r}{1-a}\frac{(1-r)(1-r^2)\cdots(1-r^{n-1})}{(r-a)(r^2-a)\cdots(r^{n-1}-a)}.$$

My own proofs of the above results make use of a general formula, the truth of which depends on conditions which I have not yet investigated completely. A direct proof depending on Cauchy's theorem will be found in Mr Hardy's note which follows this paper. The final formula used in Mr Hardy's proof can be proved as follows. Let

$$f(t)=\prod_{m=0}^{m=\infty}\left(\frac{1-btx^m}{1-atx^m}\right)=A_0+A_1t+A_2t^2+\cdots.$$

Then it is evident that

$$(1-at)f(t) = (1-bt)f(tx).$$

That is

$$(1-at)(A_0 + A_1 t + A_2 t^2 + \cdots) = (1-bt)(A_0 + A_1 tx + A_2 t^2 x^2 + \cdots).$$

Equating the coefficients of t^n, we obtain

$$A_n = A_{n-1}\frac{1-bx^{n-1}}{1-x^n};$$

and A_0 is evidently 1. Hence we have

$$f(t) = 1 + t\frac{a-b}{1-x} + t^2\frac{(a-b)(a-bx)}{(1-x)(1-x^2)} + \cdots. \tag{21}$$

7. As a particular case of (19), we have, when $a=0$,

$$\int_0^\infty \frac{x^{n-1}\,dx}{(1+x)(1+rx)(1+r^2x)\cdots} = \frac{\pi}{\sin n\pi}\frac{1-r^{1-n}}{1-r}\frac{1-r^{2-n}}{1-r^2}\cdots. \tag{22}$$

When n is an integer, the value of the integral reduces to

$$-r^{-\frac{1}{2}n(n-1)}(1-r)(1-r^2)\cdots(1-r^{n-1})\log r.$$

When we put $n=\frac{1}{2}$ in (19), we have

$$\int_0^\infty \frac{1}{1+x^2}\frac{1+ar^2x^2}{1+r^2x^2}\frac{1+ar^4x^2}{1+r^4x^2}\cdots\;dx$$

$$= \tfrac{1}{2}\pi\frac{1-ar^2}{1-r^2}\frac{1-ar^4}{1-r^4}\cdots\frac{1-r}{1-ar}\frac{1-r^3}{1-ar^3}\cdots. \tag{23}$$

If, in particular, $n=\frac{1}{2}$ in (22), or $a=0$ in (23), then

$$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\cdots}$$

$$= \tfrac{1}{2}\pi\frac{1-r}{1-r^2}\frac{1-r^3}{1-r^4}\frac{1-r^5}{1-r^6}\cdots = \frac{\pi}{2(1+r+r^3+r^6+r^{10}+\cdots)}, \tag{24}$$

the nth term in the denominator being $r^{\frac{1}{2}n(n-1)}$. Thus, for example, when $r=e^{-5\pi}$, we have

$$\int_0^\infty \frac{dx}{(1+x^2)(1+e^{-10\pi}x^2)(1+e^{-20x^2})\cdots}$$

$$= \frac{\pi}{2(1 + e^{-5\pi} + e^{-15\pi} + e^{-30\pi} + \cdots)}$$
$$= \pi^{\frac{3}{4}} \Gamma\left(\tfrac{3}{4}\right) \sqrt{5} \sqrt[8]{2} \tfrac{1}{2}(1 + \sqrt[4]{5}) \{\tfrac{1}{2}(1 + \sqrt{5})\}^{\frac{1}{2}} e^{-5\pi/8}.$$

Similarly

$$\int_0^\infty \frac{dx}{(1 + x^2)(1 + e^{-20\pi} x^2)(1 + e^{-40\pi} x^2) \cdots}$$
$$= \pi^{\frac{3}{4}} \Gamma\left(\tfrac{3}{4}\right) \sqrt{5} \sqrt[4]{2} \tfrac{1}{2}(1 + \sqrt[4]{5})^2 \{\tfrac{1}{2}(1 + \sqrt{5})\}^{\frac{5}{2}} e^{-5\pi/4};$$

and

$$\int_0^\infty \frac{dx}{(1 + x^2)(1 + .001x^2)(1 + .00001x^2) \cdots} = \tfrac{1}{2}\pi \frac{10}{11} \frac{1110}{1111} \frac{111110}{111111} \cdots$$
$$= \frac{\pi}{2.202\ 002\ 000\ 200\ 002\ 000\ 002 \ldots}.$$

Some definite integrals connected with Gauss's sums

Messenger of Mathematics, XLIV, 1915, 75 – 85

1. If n is real and positive, and $|I(t)|$, where $I(t)$ is the imaginary part of t, is less than either n or 1, we have

$$\begin{aligned}\int_0^\infty \frac{\cos \pi tx}{\cosh \pi x} e^{-i\pi nx^2}\, dx &= 2\int_0^\infty\int_0^\infty \frac{\cos \pi tx \cos 2\pi xy}{\cosh \pi y} e^{-i\pi nx^2}\, dxdy \\ &= \sqrt{n}\, \exp\left\{-\tfrac{1}{4}i\pi\left(1-\frac{t^2}{n}\right)\right\}\int_0^\infty \frac{\cos \pi tx}{\cosh \pi nx} e^{i\pi nx^2}\, dx. \qquad (1)\end{aligned}$$

When $n = 1$ the above formula reduces to

$$\int_0^\infty \frac{\cos \pi tx}{\cosh \pi x} \sin \pi x^2\, dx = \tan\{\tfrac{1}{8}\pi(1-t^2)\}\int_0^\infty \frac{\cos \pi tx}{\cosh \pi x} \cos \pi x^2\, dx. \qquad (2)$$

if $t = 0$, and

$$\left.\begin{aligned}\phi(n) &= \int_0^\infty \frac{\cos \pi nx^2}{\cosh \pi x}\, dx, \\ \Psi(n) &= \int_0^\infty \frac{\sin \pi nx^2}{\cosh \pi x}\, dx,\end{aligned}\right\} \qquad (3)$$

then

$$\left.\begin{aligned}\phi(n) &= \sqrt{\left(\tfrac{2}{n}\right)\Psi\left(\tfrac{1}{n}\right)+\Psi(n)}, \\ \Psi(n) &= \sqrt{\left(\tfrac{2}{n}\right)\phi\left(\tfrac{1}{n}\right)-\phi(n)}.\end{aligned}\right\} \qquad (3')$$

Similarly, if $\frac{1}{2}\sqrt{3}|I(t)|$ is less than either 1 or n, we have

$$\begin{aligned}&\int_0^\infty \frac{\cos \pi tx}{1+2\cosh(2\pi x/\sqrt{3})} e^{i\pi nx^2}\, dx \\ &= \sqrt{n}\, \exp\left\{-\tfrac{1}{4}i\pi\left(1-\frac{t^2}{n}\right)\right\}\int_0^\infty \frac{\cos \pi tx}{1+2\cosh(2\pi nx/\sqrt{3})} e^{-i\pi nx^2}\, dx. \qquad (4)\end{aligned}$$

If in (4) we suppose $n=1$, we obtain

$$\int_0^\infty \frac{\cos\pi tx\sin\pi x^2}{1+2\cosh(2\pi x/\sqrt{3})}\,dx = \tan\{\tfrac{1}{8}\pi(1-t^2)\}\int_0^\infty \frac{\cos\pi tx\cos\pi x^2}{1+2\cosh(2\pi x/\sqrt{3})}\,dx; \tag{5}$$

and if $t=0$, and

$$\left.\begin{aligned}\phi(n) &= \int_0^\infty \frac{\cos\pi nx^2}{1+2\cosh(2\pi x/\sqrt{3})}\,dx,\\ \Psi(n) &= \int_0^\infty \frac{\sin\pi nx^2}{1+2\cosh(2\pi x/\sqrt{3})}\,dx,\end{aligned}\right\} \tag{6}$$

then

$$\left.\begin{aligned}\phi(n) &= \sqrt{\left(\tfrac{2}{n}\right)}\,\Psi\left(\tfrac{1}{n}\right)+\Psi(n),\\ \Psi(n) &= \sqrt{\left(\tfrac{2}{n}\right)}\,\phi\left(\tfrac{1}{n}\right)-\phi(n).\end{aligned}\right\} \tag{6$'$}$$

In a similar manner we can prove that

$$\int_0^\infty \frac{\sin\pi tx}{\tanh\pi x}e^{-i\pi nx^2}\,dx = -\sqrt{n}\,\exp\left\{\tfrac{1}{4}i\pi\left(1+\frac{t^2}{n}\right)\right\}\int_0^\infty \frac{\sin\pi tx}{\tanh\pi nx}e^{i\pi nx^2}\,dx. \tag{7}$$

If we put $n=1$ in (7), we obtain

$$\int_0^\infty \frac{\sin\pi tx}{\tanh\pi x}\cos\pi x^2\,dx = \tan\{\frac{1}{8}\pi(1+t^2)\}\int_0^\infty \frac{\sin\pi tx}{\tanh\pi x}\sin\pi x^2\,dx. \tag{8}$$

Now

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\int_0^\infty \frac{\sin atx}{\tanh bx}e^{icx^2}\,dx &= \lim_{t\to 0}\frac{1}{t}\int_0^\infty \frac{2\sin atx}{e^{2bx}-1}e^{icx^2}\,dx + \lim_{t\to 0}\int_0^\infty \frac{\sin atx}{t}e^{icx^2}\,dx\\ &= \int_0^\infty \frac{ae^{icx}}{e^{2b\sqrt{x}}-1}\,dx + \frac{ia}{2c}.\end{aligned} \tag{9}$$

Hence, dividing both sides of (7) by t, and making $t\to 0$, we obtain the result corresponding to (3) and (6), viz.: if

$$\left.\begin{aligned}\phi(n) &= \int_0^\infty \frac{\cos\pi nx}{e^{2\pi\sqrt{x}}-1}\,dx,\\ \Psi(n) &= \frac{1}{2\pi n}+\int_0^\infty \frac{\sin\pi nx}{e^{2\pi\sqrt{x}}-1}\,dx,\end{aligned}\right\} \tag{10}$$

then

$$\left.\begin{aligned}\phi(n) &= \tfrac{1}{n}\sqrt{\left(\tfrac{2}{n}\right)\Psi\left(\tfrac{1}{n}\right) - \Psi(n)},\\ \Psi(n) &= \tfrac{1}{n}\sqrt{\left(\tfrac{2}{n}\right)\phi\left(\tfrac{1}{n}\right) + \phi(n)}.\end{aligned}\right\} \tag{10'}$$

2. I shall now shew that the integral (1) may be expressed in finite terms for all rational values of n. Consider the integral

$$J(t) = \int_0^\infty \frac{\cos tx}{\cosh\frac{1}{2}\pi x}\frac{dx}{a^2+x^2}.$$

If $R(a)$ and t are positive, we have

$$\begin{aligned} J(t) &= \frac{4}{\pi}\int_0^\infty \sum_{r=0}^{r=\infty}\frac{(-1)^r(2r+1)}{x^2+(2r+1)^2}\frac{\cos tx}{a^2+x^2}\,dx \\ &= 2\sum_{r=0}^{r=\infty}\frac{(-1)^r}{a^2-(2r+1)^2}\left\{e^{-(2r+1)t} - \frac{1}{a}(2r+1)e^{-at}\right\} \\ &= \frac{\pi e^{-at}}{2a\cos\frac{1}{2}\pi a} + 2\sum_{r=0}^{r=\infty}\frac{(-1)^r e^{-(2r+1)t}}{a^2-(2r+1)^2}, \end{aligned} \tag{11}$$

and it is easy to see that this last equation remains true when t is complex, provided $R(t) > 0$ and $|I(t)| \le \frac{1}{2}\pi$. Thus the integral $J(t)$ can be expressed in finite terms for all rational values of a. Thus, for example, we have

$$\left.\begin{aligned} \int_0^\infty \frac{\cos tx}{\cosh\frac{1}{2}\pi x}\frac{dx}{1+x^2} &= \cosh t\log(2\cosh t) - t\sinh t, \\ \int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\frac{dx}{1+x^2} &= 2\cosh t - (e^{2t}\tan^{-1}e^{-t} + e^{-2t}\tan^{-1}e^{t}), \end{aligned}\right\} \tag{12}$$

and so on. Now let

$$F(n) = \int_0^\infty \frac{\cos 2tx}{\cosh \pi x}e^{-i\pi n x^2}\,dx. \tag{13}$$

Then, if $R(a) > 0$,

$$\int_0^\infty e^{-an}F(n)dn = \int_0^\infty \frac{\cos 2tx}{\cosh\pi x}\frac{dx}{a+i\pi x^2}. \tag{14}$$

Now let

$$f(n) = \sum_{r=0}^{r=\infty} (-1)^r \exp\{-(2r+1)t + \tfrac{1}{4}(2r+1)^2 i\pi n\}$$
$$+ \frac{1}{\sqrt{n}} \exp\left\{-i\left(\tfrac{1}{4}\pi - \frac{t^2}{\pi n}\right)\right\} \sum_{r=0}^{r=\infty} (-1)^r \exp\left\{-(2r+1)\frac{t}{n} \ -\tfrac{1}{4}(2r+1)^2 \frac{i\pi}{n}\right\}. \quad (15)$$

Then

$$\begin{aligned} \int_0^\infty e^{-an} f(n) dn &= \sum_{r=0}^{r=\infty} \frac{(-1)^r e^{-(2r+1)t}}{a - \frac{1}{4}(2r+1)^2 i\pi} + \sqrt{\left(\frac{\pi}{2a}\right)} \frac{\exp\{-\sqrt{(2a/\pi)}(1-i)t\}}{(1+i)\cosh\{(1+i)\sqrt{(\frac{1}{2}\pi a)}\}} \\ &= \int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \frac{dx}{a + i\pi x^2}, \end{aligned} \quad (16)$$

in virtue of (11); and therefore

$$\int_0^\infty e^{-an}\{F(n) - f(n)\} dn = 0. \quad (17)$$

Now it is known that, if $\phi(n)$ is continuous and

$$\int_0^\infty e^{-an} \phi(n) dn = 0,$$

for all positive values of a (or even only for an infinity of such values in arithmetical progression), then

$$\phi(n) = 0,$$

for all positive values of n. Hence

$$F(n) = f(n). \quad (18)$$

Equating the real and imaginary parts in (13) and (15) we have

$$\int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \cos \pi n x^2 \ dx = \left\{ e^{-t} \cos \frac{\pi n}{4} - e^{-3t} \cos \frac{9\pi n}{4} + e^{-5t} \cos \frac{25\pi n}{4} - \cdots \right\}$$
$$+ \frac{1}{\sqrt{n}} \left\{ e^{-t/n} \cos\left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{\pi}{4n}\right) - e^{-3t/n} \cos\left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{9\pi}{4n}\right) + \cdots \right\}, \quad (19)$$

$$\int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \sin \pi n x^2 \, dx = -\left\{ e^{-t} \sin \frac{\pi n}{4} - e^{-3t} \sin \frac{9\pi n}{4} + e^{-5t} \sin \frac{25\pi n}{4} - \cdots \right\}$$
$$+ \frac{1}{\sqrt{n}} \left\{ e^{-t/n} \sin\left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{\pi}{4n} \right) - e^{-3t/n} \sin\left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{9\pi}{4n} \right) + \cdots \right\}. \qquad (20)$$

We can verify the results (18), (19) and (20) by means of the equation (1). This equation can be expressed as a functional equation in $F(n)$, nd it is easy to see that $f(n)$ satisfies the same equation.

The right-hand side of these equations can be expressed in finite terms if n is any rational number. For let $n = a/b$, where a and b are any two positive integers and one of them is odd. Then the results (19) and (20) reduce to

$$2\cosh bt \int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \cos\left(\frac{\pi a x^2}{b} \right) \, dx$$
$$= [\cosh\{(1-b)t\} \cos(\pi a/4b) - \cosh\{(3-b)t\} \cos(9\pi a/4b)$$
$$+ \cosh\{(5-b)t\} \cos(25\pi a/4b) - \cdots \text{ to } b \text{ terms}]$$
$$+ \sqrt{\left(\frac{b}{a}\right)} \left[\cosh\left\{ \left(1 - \frac{1}{a}\right) bt \right\} \cos\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{\pi b}{4a} \right) \right.$$
$$\left. - \cosh\left\{ \left(1 - \frac{3}{a}\right) bt \right\} \cos\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{9\pi b}{4a} \right) + \cdots \text{ to } a \text{ terms} \right], \qquad (21)$$

$$2\cosh bt \int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \sin\left(\frac{\pi a x^2}{b} \right) \, dx$$
$$= -[\cosh\{(1-b)t\} \sin(\pi a/4b) - \cosh\{(3-b)t\} \sin(9\pi a/4b)$$
$$+ \cosh\{(5-b)t\} \sin(25\pi a/4b) - \cdots \text{ to } b \text{ terms}]$$
$$+ \sqrt{\left(\frac{b}{a}\right)} \left[\cosh\left\{ \left(1 - \frac{1}{a}\right) bt \right\} \sin\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{\pi b}{4a} \right) \right.$$
$$\left. - \cosh\left\{ \left(1 - \frac{3}{a}\right) bt \right\} \sin\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{9\pi b}{4a} \right) + \cdots \text{ to } a \text{ terms} \right]. \qquad (22)$$

Thus, for example, we have, when $a = 1$ and $b = 1$,

$$\int_0^\infty \frac{\cos \pi x^2}{\cosh \pi x} \cos 2\pi tx \, dx = \frac{1 + \sqrt{2} \sin \pi t^2}{2\sqrt{2} \cosh \pi t}, \qquad (23)$$

$$\int_0^\infty \frac{\sin \pi x^2}{\cosh \pi x} \cos 2\pi t x \, dx = \frac{-1 + \sqrt{2}\cos \pi t^2}{2\sqrt{2}\cosh \pi t}. \tag{24}$$

It is easy to verify that (23) and (24) satisfy the relation (2).
The values of the integrals

$$\int_0^\infty \frac{\cos \pi n x^2}{\cosh \pi x}\, dx, \qquad \int_0^\infty \frac{\sin \pi n x^2}{\cosh \pi x}\, dx$$

can be obtained easily from the preceding results by putting $t = 0$, and need no special discussion. By successive differentiations of the results (19) and (20) with respect to t and n, we can evaluate the integrals

$$\left.\begin{aligned} &\int_0^\infty x^{2m-1} \frac{\sin tx}{\cosh \pi x} \begin{matrix}\cos\\ \sin\end{matrix}\ \pi n x^2 \, dx, \\ &\int_0^\infty x^{2m} \frac{\cos tx}{\cosh \pi x} \begin{matrix}\cos\\ \sin\end{matrix}\ \pi n x^2 \, dx, \end{aligned}\right\} \tag{25}$$

for all rational values of n and all positive integral values of m. Thus, for example, we have

$$\left.\begin{aligned} &\int_0^\infty x^2 \frac{\cos \pi x^2}{\cosh \pi x}\, dx = \frac{1}{8\sqrt{2}} - \frac{1}{4\pi}, \\ &\int_0^\infty x^2 \frac{\sin \pi x^2}{\cosh \pi x}\, dx = \frac{1}{8} - \frac{1}{8\sqrt{2}}. \end{aligned}\right\} \tag{26}$$

3. We can get many interesting results by applying the theory of Cauchy's reciprocal functions to the preceding results. It is known that, if

$$\int_0^\infty \phi(x) \cos knx \, dx = \Psi(n), \tag{27}$$

then (i) $\frac{1}{2}\alpha\{\frac{1}{2}\phi(0) + \phi(\alpha) + \phi(2\alpha) + \phi(3\alpha) + \cdots\}$
$$= \tfrac{1}{2}\Psi(0) + \Psi(\beta) + \Psi(2\beta) + \Psi(3\beta) + \cdots, \tag{27}$$
with the condition $\alpha\beta = 2\pi/k$;
(ii) $\alpha\sqrt{2}\{\phi(\alpha) - \phi(3\alpha) - \phi(5\alpha) + \phi(7\alpha) + \phi(9\alpha) - \cdots\}$
$$= \Psi(\beta) - \Psi(3\beta) - \Psi(5\beta) + \Psi(7\beta) + \Psi(9\beta) - \cdots, \tag{27}$$
with the condition $\alpha\beta = \pi/4k$;

(iii) $\alpha\sqrt{3}\{\phi(\alpha) - \phi(5\alpha) - \phi(7\alpha) + \phi(11\alpha) + \phi(13\alpha) - \cdots\}$

$$= \Psi(\beta) - \Psi(5\beta) - \Psi(7\beta) + \Psi(11\beta) + \Psi(13\beta) - \cdots, \tag{27}$$

with the condition $\alpha\beta = \pi/6k$, where 1, 5, 7, 11, 13, ... are the odd natural numbers without the multiples of 3.

There are of course corresponding results for the function

$$\int_0^\infty \phi(x) \sin knx \, dx = \Psi(n), \tag{28}$$

such as

$$\alpha\{\phi(\alpha) - \phi(3\alpha) + \phi(5\alpha) - \cdots\} = \Psi(\beta) - \Psi(3\beta) + \Psi(5\beta) - \cdots,$$

with the condition $\alpha\beta = \pi/2k$.

Thus from (23) and (27) (i) we obtain the following results. If

$$F(\alpha, \beta) = \sqrt{\alpha}\left\{\frac{1}{2} + \sum_{r=1}^{r=\infty} \frac{\cos r^2\pi\alpha^2}{\cosh r\pi\alpha}\right\} - \sqrt{\beta}\sum_{r=1}^{r=\infty} \frac{\sin r^2\pi\beta^2}{\cosh r\pi\beta}, \tag{29}$$

then

$$F(\alpha, \beta) = F(\beta, \alpha) = \sqrt{(2\alpha)}\{\frac{1}{2} + e^{-\pi\alpha} + e^{-4\pi\alpha} + e^{-9\pi\alpha} + \cdots\}^2,$$

provided that $\alpha\beta = 1$.

4. If, instead of starting with the integral (11), we start with the corresponding sine integral, we can shew that, when $R(a)$ and $R(t)$ are positive and $|I(t)| \leq \pi$,

$$\int_0^\infty \frac{\sin tx}{\sinh \pi x} \frac{dx}{a^2 + x^2} = \frac{1}{2a^2} - \frac{\pi e^{-at}}{2a \sin \pi a} + \sum_{r=1}^{r=\infty} \frac{(-1)^r e^{-rt}}{a^2 - r^2}. \tag{30}$$

Hence the above integral can be expressed in finite terms for all rational values of a. For example, we have

$$\int_0^\infty \frac{\sin tx}{\sinh \frac{1}{2}\pi x} \frac{dx}{1 + x^2} = e^t \tan^{-1} e^{-t} - e^{-t} \tan^{-1} e^t. \tag{31}$$

From (30) we can deduce that

$$\int_0^\infty \frac{\sin 2tx}{\sinh \pi x} e^{-i\pi nx^2} \, dx = \frac{1}{2} - e^{-2t+i\pi n} + e^{-4t+4i\pi n} - e^{-6t+9i\pi n} + \cdots$$

$$-\frac{1}{\sqrt{n}} \exp\left\{\left(\tfrac{1}{4}\pi + \frac{t^2}{\pi n}\right) i\right\} \{e^{-(t+\frac{1}{4}i\pi)/n} + e^{-(3t+\frac{9}{4}i\pi)/n} + \cdots\}, \tag{32}$$

$R(t)$ being positive and $|I(t)| \leq \frac{1}{2}\pi$. The right-hand side can be expressed in finite terms for all rational values of n. Thus, for example, we have

$$\int_0^\infty \frac{\cos \pi x^2}{\sinh \pi x} \sin 2\pi tx \; dx = \frac{\cosh \pi t - \cos \pi t^2}{2 \sinh \pi t}, \tag{33}$$

$$\int_0^\infty \frac{\sin \pi x^2}{\sinh \pi x} \sin 2\pi tx \; dx = \frac{\sin \pi t^2}{2 \sinh \pi t}, \tag{34}$$

and so on.

Applying the formula (28) to (33) and (34), we have, when $\alpha\beta = \frac{1}{4}$,

$$\left.\begin{aligned} \sqrt{\alpha}\sum_{r=0}^{r=\infty}(-1)^r \frac{\cos\{(2r+1)^2\pi\alpha^2\}}{\sinh\{(2r+1)\pi\alpha\}} &+ \sqrt{\beta}\sum_{r=0}^{r=\infty}(-1)^r \frac{\cos\{(2r+1)^2\pi\beta^2\}}{\sinh\{(2r+1)\pi\beta\}} \\ &= 2\sqrt{\alpha}\{\frac{1}{2} + e^{-2\pi\alpha} + e^{-8\pi\alpha} + e^{-18\pi\alpha} + \cdots\}^2; \\ \sqrt{\alpha}\sum_{r=0}^{r=\infty}(-1)^r \frac{\sin\{(2r+1)^2\pi\alpha^2\}}{\sinh\{(2r+1)\pi\alpha\}} &= \sqrt{\beta}\sum_{r=0}^{r=\infty}(-1)^r \frac{\sin\{(2r+1)^2\pi\beta^2\}}{\sinh\{(2r+1)\pi\beta\}}. \end{aligned}\right\} \tag{35}$$

By successive differentiation of (32) with respect to t and n we can evaluate the integrals

$$\left.\begin{aligned} &\int_0^\infty x^{2m-1} \frac{\cos tx}{\sinh \pi x} \begin{matrix}\cos \\ \sin\end{matrix} \; \pi n x^2 \; dx, \\ &\int_0^\infty x^{2m} \frac{\sin tx}{\sinh \pi x} \begin{matrix}\cos \\ \sin\end{matrix} \; \pi n x^2 \; dx \end{aligned}\right\} \tag{36}$$

for all rational values of n and all positive integral values of m. Thus, for example, we have

$$\left.\begin{aligned} &\int_0^\infty x \frac{\cos \pi x^2}{\sinh \pi x} \; dx = \frac{1}{8}, && \int_0^\infty x \frac{\sin \pi x^2}{\sinh \pi x} \; dx = \frac{1}{4\pi}, \\ &\int_0^\infty x^3 \frac{\cos \pi x^2}{\sinh \pi x} \; dx = \frac{1}{16}\left(\frac{1}{4} - \frac{3}{\pi^2}\right), && \int_0^\infty x^3 \frac{\sin \pi x^2}{\sinh \pi x} \; dx = \frac{1}{16\pi}, \end{aligned}\right\} \tag{37}$$

and so on.

The denominators of the integrands in (25) and (36) are $\cosh \pi x$ and $\sinh \pi x$. Similar integrals having the denominators of their integrands equal to

$$\prod_1^r \cosh \pi a_r x \sinh \pi b_r x$$

can be evaluated, if a_r and b_r are rational, by splitting up the integrand into partial fractions.

5. The preceding formulæ may be generalised. Thus it may be shewn that, if $R(a)$ and $R(t)$ are positive, $|I(t)| \leq \pi$, and $-1 < R(\theta) < 1$, then

$$\sin\pi\theta \int_0^\infty \frac{\cos tx}{\cosh\pi x + \cos\pi\theta}\frac{dx}{a^2+x^2} = \frac{\pi}{2a}\frac{e^{-at}\sin\pi\theta}{\cos\pi a + \cos\pi\theta} + \sum_{r=0}^{r=\infty}\left\{\frac{e^{-(2r+1-\theta)t}}{a^2-(2r+1-\theta)^2} - \frac{e^{-(2r+1+\theta)t}}{a^2-(2r+1+\theta)^2}\right\}. \tag{38}$$

From (38) it can be deduced that, if n and $R(t)$ are positive, $|I(t)| \leq \pi$, and $-1 < \theta < 1$, then

$$\begin{aligned}\sin\pi\theta \int_0^\infty \frac{\cos tx}{\cosh\pi x + \cos\pi\theta} e^{-i\pi nx^2}\, dx &= \sum_{r=0}^{r=\infty}\{e^{-(2r+1-\theta)t+(2r+1-\theta)2i\pi n} - e^{-(2r+1+\theta)t+(2r+1+\theta)2i\pi n}\} \\ &+ \frac{1}{\sqrt{n}}\exp\left\{-\tfrac{1}{4}i\left(\pi - \frac{t^2}{\pi n}\right)\right\}\sum_{r=1}^{r=\infty}(-1)^{r-1}\sin r\pi\theta e^{-(2rt+r^2 i\pi)/4n}.\end{aligned} \tag{39}$$

The right-hand side can be expressed in finite terms if n and θ are rational. In particular, when $\theta = \frac{1}{3}$, we have

$$\begin{aligned}\int_0^\infty \frac{\cos tx}{1+2\cosh(2\pi x/\sqrt{3})} e^{-i\pi nx^2}\, dx &= \frac{1}{2}\{e^{-\frac{1}{3}(t\sqrt{3}-i\pi n)} - e^{-\frac{1}{3}(2t\sqrt{3}-4i\pi n)} + e^{-\frac{1}{3}(4t\sqrt{3}-16i\pi n)} - \cdots\} \\ &+ \frac{1}{2\sqrt{n}}\exp\left\{-\tfrac{1}{4}i\left(\pi - \frac{t^2}{\pi n}\right)\right\} \\ &\quad \{e^{-(t\sqrt{3}+i\pi)/3n} - e^{-(2t\sqrt{3}+4i\pi)/3n} + e^{-(4t\sqrt{3}+16i\pi)/3n} - \cdots\},\end{aligned} \tag{40}$$

where 1,2,4,5, ... are the natural numbers without the multiples of 3.

As an example, when $n = 1$, we have

$$\left.\begin{aligned} \int_0^\infty \frac{\cos \pi x^2 \cos \pi t x}{1 + 2\cosh(2\pi x/\sqrt{3})}\, dx &= \frac{1 - 2\sin\{(\pi - 3\pi t^2)/12\}}{8\cosh(\pi t/\sqrt{3}) - 4}, \\ \int_0^\infty \frac{\sin \pi x^2 \cos \pi t x}{1 + 2\cosh(2\pi x/\sqrt{3})}\, dx &= \frac{-\sqrt{3} + 2\cos\{(\pi - 3\pi t^2)/12\}}{8\cosh(\pi t/\sqrt{3}) - 4}. \end{aligned}\right\} \tag{41}$$

6. The formula (32) assumes a neat and elegant form when t is changed to $t + \frac{1}{2}i\pi$. We have then

$$\begin{aligned} &\int_0^\infty \frac{\sin tx}{\tanh \pi x} e^{-i\pi n x^2}\, dx \qquad (n > 0, t > 0) \\ &\quad = \{\tfrac{1}{2} + e^{-t+i\pi n} + e^{-2t+4i\pi n} + e^{-3t+9i\pi n} + \cdots\} \\ &\qquad - \frac{1}{\sqrt{n}} \exp\left\{\tfrac{1}{4}i\left(\pi + \frac{t^2}{\pi n}\right)\right\} \{\tfrac{1}{2} + e^{-(t+i\pi)/n} + e^{-(2t+4i\pi)/n} + \cdots\}. \end{aligned} \tag{42}$$

In particular, when $n = 1$, we have

$$\left.\begin{aligned} \int_0^\infty \frac{\cos \pi x^2}{\tanh \pi x} \sin 2\pi t x\, dx &= \frac{1}{2} \tanh \pi t\{1 - \cos(\tfrac{1}{4}\pi + \pi t^2)\}, \\ \int_0^\infty \frac{\sin \pi x^2}{\tanh \pi x} \sin 2\pi t x\, dx &= \frac{1}{2} \tanh \pi t \sin(\tfrac{1}{4}\pi + \pi t^2). \end{aligned}\right\} \tag{43}$$

We shall now consider an important special case of (42). It can easily be seen from (9) that the left-hand side of (42), when divided by t, tends to

$$\int_0^\infty \frac{\cos \pi n x}{e^{2\pi\sqrt{x}} - 1}\, dx - i\left\{\frac{1}{2\pi n} + \int_0^\infty \frac{\sin \pi n x}{e^{2\pi\sqrt{x}} - 1}\, dx\right\} \tag{44}$$

as $t \to 0$. But the limit of the right-hand side of (42) divided by t can be found when n is rational. Let then $n = a/b$, where a and b are any two positive integers, and let

$$\phi(n) = \int_0^\infty \frac{\cos \pi n x}{e^{2\pi\sqrt{x}} - 1}\, dx, \quad \Psi(n) = \frac{1}{2\pi n} + \int_0^\infty \frac{\sin \pi n x}{e^{2\pi\sqrt{x}} - 1}\, dx.$$

The relation between $\phi(n)$ and $\Psi(n)$ has been stated already in (10′). From (42) and (44) it can easily be deduced that, if a and b are both odd, then

$$\left.\begin{aligned}\phi\left(\frac{a}{b}\right) &= \tfrac{1}{4}\sum_{r=1}^{r=b}(b-2r)\cos\left(\frac{r^2\pi a}{b}\right) - \frac{b}{4a}\sqrt{\left(\frac{b}{a}\right)}\sum_{r=1}^{r=a}(a-2r)\sin\left(\tfrac{1}{4}\pi+\frac{r^2 b\pi}{a}\right),\\ \Psi\left(\frac{a}{b}\right) &= -\tfrac{1}{4}\sum_{r=1}^{r=b}(b-2r)\sin\left(\frac{r^2\pi a}{b}\right) + \frac{b}{4a}\sqrt{\left(\frac{b}{a}\right)}\sum_{r=1}^{r=a}(a-2r)\cos\left(\tfrac{1}{4}\pi+\frac{r^2\pi b}{a}\right),\end{aligned}\right\} \tag{45}$$

It can easily be seen that these satisfy the relation (10′). Similarly, when one of a and b is odd and the other even, it can be shewn that

$$\left.\begin{aligned}\phi\left(\tfrac{a}{b}\right) &= \tfrac{\sigma}{4\pi a\sqrt{a}} - \tfrac{1}{2}\textstyle\sum_{r=1}^{r=b} r\left(1-\tfrac{r}{b}\right)\cos\left(\tfrac{r^2\pi a}{b}\right)\\ &\quad + \tfrac{b}{2a}\sqrt{\left(\tfrac{b}{a}\right)}\textstyle\sum_{r=1}^{r=a} r\left(1-\tfrac{r}{a}\right)\sin\left(\tfrac{1}{4}\pi+\tfrac{r^2\pi b}{a}\right),\\ \Psi\left(\tfrac{a}{b}\right) &= \tfrac{\sigma'}{4\pi a\sqrt{a}} + \tfrac{1}{2}\textstyle\sum_{r=1}^{r=b} r\left(1-\tfrac{r}{b}\right)\sin\left(\tfrac{r^2\pi a}{b}\right)\\ &\quad - \tfrac{b}{2a}\sqrt{\left(\tfrac{b}{a}\right)}\textstyle\sum_{r=1}^{r=a} r\left(1-\tfrac{r}{a}\right)\cos\left(\tfrac{1}{4}\pi+\tfrac{r^2\pi b}{a}\right),\end{aligned}\right\} \tag{46}$$

where

$$\left.\begin{aligned}\sigma &= \sqrt{b}\textstyle\sum_1^a \cos\left(\tfrac{1}{4}\pi+\tfrac{r^2\pi b}{a}\right) = \sqrt{a}\textstyle\sum_1^b \sin\left(\tfrac{r^2\pi a}{b}\right),\\ \sigma' &= \sqrt{b}\textstyle\sum_1^a \sin\left(\tfrac{1}{4}\pi+\tfrac{r^2\pi b}{a}\right) = \sqrt{a}\textstyle\sum_1^b \cos\left(\tfrac{r^2\pi a}{b}\right).\end{aligned}\right\} \tag{47}$$

Thus, for example, we have

$$\left.\begin{aligned}&\phi(0) = \tfrac{1}{12}, \phi(1) = \tfrac{2-\sqrt{2}}{8}, \phi(2) = \tfrac{1}{16}, \phi(4) = \tfrac{3-\sqrt{2}}{32},\\ &\phi(6) = \tfrac{13-4\sqrt{3}}{144}, \phi\left(\tfrac{1}{2}\right) = \tfrac{1}{4\pi}, \phi\left(\tfrac{2}{5}\right) = \tfrac{8-3\sqrt{5}}{16},\end{aligned}\right\} \tag{48}$$

and so on.

By differentiating (42) with respect to n, we can evaluate the integrals

$$\int_0^\infty \frac{x^m}{e^{2\pi\sqrt{x}}-1} \begin{matrix}\cos\\ \sin\end{matrix}\ \pi nx\ dx \tag{49}$$

for all rational values of n and positive integral values of m. Thus, for example, we have

$$\left.\begin{aligned} \int_0^\infty \frac{x cos \frac{1}{2}\pi x}{e^{2\pi\sqrt{x}}-1}\,dx &= \frac{13-4\pi}{8\pi^2}, \\ \int_0^\infty \frac{x\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}\,dx &= \tfrac{1}{64}\left(\frac{1}{2}-\frac{3}{\pi}+\frac{5}{\pi^2}\right), \\ \int_0^\infty \frac{x^2\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}\,dx &= \tfrac{1}{256}\left(1-\frac{5}{\pi}+\frac{5}{\pi^2}\right), \end{aligned}\right\} \tag{50}$$

and so on.

Summation of a certain series

Messenger of Mathematics, XLIV, 1915, 157 – 160

1. Let

$$\begin{aligned}\Phi(s,x) &= \sum_{n=0}^{n=\infty}\{\sqrt{(x+n)}+\sqrt{(x+n+1)}\}^{-s}\\ &= \sum_{n=0}^{n=\infty}\{\sqrt{(x+n+1)}-\sqrt{(x+n)}\}^{s}.\end{aligned}$$

The object of this paper is to give a finite expression of $\Phi(s,0)$ in terms of Riemann ζ-functions, when s is an odd integer greater than 1.
Let $\zeta(s,x)$, where $x>0$, denote the function expressed by the series

$$x^{-s}+(x+1)^{-s}+(x+2)^{-s}+\cdots,$$

and its analytical continuations. Then

$$\zeta(s,1)=\zeta(s), \quad \zeta(s,\tfrac{1}{2})=(2^s-1)\zeta(s), \tag{1}$$

where $\zeta(s)$ is the Riemann ζ-function;

$$\zeta(s,x)-\zeta(s,x+1)=x^{-s}; \tag{2}$$

$$\left.\begin{aligned}&1^s+2^s+3^s+\cdots+n^s=\zeta(-s)-\zeta(-s,n+1),\\ &1^s+3^s+5^s+\cdots+(2n-1)^s=(1-2^s)\zeta(-s)-2^s\zeta(-s,n+\tfrac{1}{2})\end{aligned}\right\}, \tag{3}$$

if n is a positive integer; and

$$\begin{aligned}\lim_{x\to\infty}\Bigg\{\zeta(s,x)-\tfrac{1}{2}x^{-s}+\Bigg(&\frac{x^{1-s}}{1-s}-B_2\frac{s}{2!}x^{-s-1}+B_4\frac{s(s+1)(s+2)}{4!}x^{-s-3}\\ &-B_6\frac{s(s+1)(s+2)(s+3)(s+4)}{6!}x^{-s-5}+\cdots \text{ to } n \text{ terms}\Bigg)\Bigg\}=0,\end{aligned} \tag{4}$$

if n is a positive integer, $-(2n-1)<s<1$, and $B_2=\frac{1}{6}, B_4=\frac{1}{30}$, $B_6=\frac{1}{42}$, $B_8=\frac{1}{30},\ldots,$ are Bernoulli's numbers.
Suppose now that

$$\Psi(x)=6\zeta(-\tfrac{1}{2},x)+(4x-3)\sqrt{x}+\Phi(3,x).$$

Then from (2) we see that

$$\Psi(x)-\Psi(x+1) = 6\sqrt{x}+(4x-3)\sqrt{x}-(4x+1)\sqrt{(x+1)}+\{\sqrt{(x+1)}-\sqrt{x}\}^3=0;$$

and from (4) that $\Psi(x) \to 0$ as $x \to \infty$. It follows that $\Psi(x) = 0$. That is to say,

$$6\zeta(-\tfrac{1}{2}, x) + (4x - 3)\sqrt{x} + \Phi(3, x) = 0. \tag{5}$$

Similarly, we can shew that

$$40\zeta(-\frac{3}{2}, x) + (16x^2 - 20x + 5)\sqrt{x} + \Phi(5, x) = 0. \tag{6}$$

2. Remembering the functional equation satisfied by $\zeta(s)$, viz.,

$$\zeta(1 - s) = 2(2\pi)^{-s}\Gamma(s)\zeta(s)\cos\tfrac{1}{2}\pi s, \tag{7}$$

we see from (3) and (5) that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} = \frac{2}{3}n^{\frac{3}{2}} + \tfrac{1}{2}\sqrt{n} - \frac{1}{4\pi}\zeta(\frac{3}{2}) + \frac{1}{6}\Phi(3, n); \tag{8}$$

and

$$\begin{aligned}&\sqrt{1} + \sqrt{3} + \sqrt{5} + \cdots + \sqrt{(2n - 1)}\\ &\quad = \frac{1}{3}(2n - 1)^{\frac{3}{2}} + \tfrac{1}{2}\sqrt{(2n - 1)} + \frac{\sqrt{2} - 1}{4\pi}\zeta(\frac{3}{2}) + \frac{1}{3\sqrt{2}}\Phi(3, n - \tfrac{1}{2}).\end{aligned} \tag{9}$$

Similarly from (6), we have

$$\begin{aligned}&1\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \cdots + n\sqrt{n}\\ &\quad = \frac{2}{5}n^{\frac{5}{2}} + \tfrac{1}{2}n^{\frac{3}{2}} + \frac{1}{8}\sqrt{n} - \frac{3}{16\pi^2}\zeta(\frac{5}{2}) + \frac{1}{40}\Phi(5, n);\end{aligned} \tag{10}$$

and

$$\begin{aligned}&1\sqrt{1} + 3\sqrt{3} + 5\sqrt{5} + \cdots + (2n - 1)\sqrt{(2n - 1)}\\ &\quad = \frac{1}{5}(2n - 1)^{\frac{5}{2}} + \tfrac{1}{2}(2n - 1)^{\frac{3}{2}} + \frac{1}{4}\sqrt{(2n - 1)}\\ &\qquad + \frac{3(2\sqrt{2} - 1}{16\pi^2}\zeta(\tfrac{5}{2}) + \frac{1}{10\sqrt{2}}\Phi(5, n - \tfrac{1}{2}).\end{aligned} \tag{11}$$

It also follows from (5) and (6) that

$$\begin{aligned}&\sqrt{(a + d)} + \sqrt{(a + 2d)} + \sqrt{(a + 3d)} + \cdots + \sqrt{(a + nd)}\\ &\quad = C + \frac{2}{3d}(a + nd)^{\frac{3}{2}} + \tfrac{1}{2}\sqrt{(a + nd)} + \frac{1}{6}\sqrt{d}\Phi(3, n + a/d);\end{aligned} \tag{12}$$

and

$$(a + d)^{\frac{3}{2}} + (a + 2d)^{\frac{3}{2}} + (a + 3d)^{\frac{3}{2}} + \cdots + (a + nd)^{\frac{3}{2}}$$

$$= C' + \frac{2}{5d}(a+nd)^{\frac{5}{2}} + \tfrac{1}{2}(a+nd)^{\frac{3}{2}} + \frac{1}{8}d\sqrt{(a+nd)} + \frac{1}{40}d\sqrt{d}\Phi(5, n+a/d), \quad (13)$$

where C and C' are independent of n.

Putting $n = 1$ in (8) and (10), we obtain

$$\Phi(3, 0) = \frac{3}{2\pi}\zeta\ (\frac{3}{2}), \quad \Phi(5, 0) = \frac{15}{2\pi^2}\zeta\ (\frac{5}{2}). \quad (14)$$

3. The preceding results may be generalised as follows. If s be an odd integer greater than 1, then

$$\begin{aligned}
&\Phi(s, x) + \tfrac{1}{2}\{\sqrt{x} + \sqrt{(x-1)}\}^s + \tfrac{1}{2}\{\sqrt{x} - \sqrt{(x-1)}\}^s \\
&\quad + \frac{s}{1!}2^{s-2}\zeta(1 - \tfrac{1}{2}s, x) + \frac{s(s-4)(s-5)}{3!}2^{s-6}\zeta(3 - \tfrac{1}{2}s, x) \\
&\quad + \frac{s(s-6)(s-7)(s-8)(s-9)}{5!}2^{s-10}\zeta(5 - \tfrac{1}{2}s, x) \\
&\quad + \frac{s(s-8)(s-9)(s-10)(s-11)(s-12)(s-13)}{7!}2^{s-14} \\
&\qquad \times\zeta(7 - \tfrac{1}{2}s, x) + \cdots \ \text{ to } \ [\tfrac{1}{4}(s+1)] \ \text{ terms } \ = 0, \quad (15)
\end{aligned}$$

where $[x]$ denotes, as usual, the integral part of x. This can be proved by induction, using the formula

$$\begin{aligned}
&\{\sqrt{x} + \sqrt{(x \pm 1)}\}^s + \{\sqrt{x} - \sqrt{(x \pm 1)}\}^s \\
&\quad = (2\sqrt{x})^s \pm \frac{s}{1!}(2\sqrt{x})^{s-2} + \frac{s(s-3)}{2!}(2\sqrt{x})^{s-4} \\
&\qquad \pm \frac{s(s-4)(s-5)}{3!}(2\sqrt{x})^{s-6} + \cdots \ \text{ to } \ [1 + \tfrac{1}{2}s] \ \text{ terms}, \quad (16)
\end{aligned}$$

which is true for all positive integral values of s.

Similarly, we can shew that if s is a positive even integer, then

$$\begin{aligned}
&\frac{s}{1!}2^{s-2}\{\zeta(1 - \tfrac{1}{2}s) - \zeta(1 - \tfrac{1}{2}s, x)\} \\
&\quad + \frac{s(s-4)(s-5)}{3!}2^{s-6}\{\zeta(3 - \tfrac{1}{2}s) - \zeta(3 - \tfrac{1}{2}s, x)\} \\
&\quad + \frac{s(s-6)(s-7)(s-8)(s-9)}{5!}2^{s-10}\{\zeta(5 - \tfrac{1}{2}s) - \zeta(5 - \tfrac{1}{2}s, x)\} \\
&\quad + \cdots \ \text{ to } \ [\tfrac{1}{4}(s+2)] \ \text{ terms} \\
&\qquad = \tfrac{1}{2}\{\sqrt{x} + \sqrt{(x-1)}\}^s + \tfrac{1}{2}\{\sqrt{x} - \sqrt{(x-1)}\}^s - 1. \quad (17)
\end{aligned}$$

Now, remembering (7) and putting $x = 1$ in (15), we obtain

$$\begin{aligned}\Phi(s,0) = -\frac{s}{\sqrt{2}}\pi^{-\frac{1}{2}(1+s)}\cos\tfrac{1}{4}\pi s\{&1\cdot 3\cdot 5\cdots(s-2)\pi\zeta(\tfrac{1}{2}s)\\ &-3\cdot 5\cdot 7\cdots(s-4)\tfrac{1}{2}(s-5)\tfrac{1}{3}\pi^3\zeta(\tfrac{1}{2}s-2)\\ &+5\cdot 7\cdot 9\cdots(s-6)\tfrac{1}{2}(s-7)\tfrac{1}{4}(s-9)\tfrac{1}{5}\pi^5\zeta(\tfrac{1}{2}s-4)\\ &-7\cdot 9\cdot 11\cdots(s-8)\tfrac{1}{2}(s-9)\tfrac{1}{4}(s-11)\tfrac{1}{6}(s-13)\tfrac{1}{7}\pi^7\zeta(\tfrac{1}{2}s-6)\\ &+9\cdot 11\cdot 13\cdots(s-10)\tfrac{1}{2}(s-11)\tfrac{1}{4}(s-13)\tfrac{1}{6}(s-15)\tfrac{1}{8}(s-17)\\ &\times\tfrac{1}{9}\pi^9\zeta(\tfrac{1}{2}s-8)-\cdots \quad\text{to}\quad [\tfrac{1}{4}(s+1)] \quad\text{terms}\quad\},\end{aligned} \tag{18}$$

If s is an odd integer greater than 1. Similarly, putting $x = \frac{1}{2}$ in (15), we can express $\Phi(s, \frac{1}{2})$ in terms of ζ-functions, if s is an odd integer greater than 1.

4. It is also easy to shew that, if

$$\Psi(s,x) = \sum_{n=0}^{n=\infty}\frac{\{\sqrt{(x+n)}+\sqrt{(x+n+1)}\}^{-s}}{\sqrt{\{(x+n)(x+n+1)\}}},$$

then

$$\begin{aligned}\Psi(s,x) &- \tfrac{1}{2}\frac{\{\sqrt{x}+\sqrt{(x-1)}\}^s-\{\sqrt{x}-\sqrt{(x-1)}\}^s}{\sqrt{\{x(x-1)\}}}\\ &= \frac{s-2}{1!}2^{s-2}\zeta(2-\tfrac{1}{2}s,x)+\frac{(s-4)(s-5)(s-6)}{3!}2^{s-6}\zeta(4-\tfrac{1}{2}s,x)\\ &\quad+\frac{(s-6)(s-7)(s-8)(s-9)(s-10)}{5!}2^{s-10}\zeta(6-\tfrac{1}{2}s,x)\\ &\quad+\cdots \quad\text{to}\quad [\tfrac{1}{4}(s+1)] \quad\text{terms},\end{aligned} \tag{19}$$

provided that s is a positive odd integer. For example

$$\left.\begin{aligned}\Psi(1,x) &= \tfrac{1}{\sqrt{x}},\\ \Psi(3,x) &= 4\sqrt{x}-\tfrac{1}{\sqrt{x}}+2\zeta(\tfrac{1}{2},x),\\ \Psi(5,x) &= 16x\sqrt{x}-12\sqrt{x}+\tfrac{1}{\sqrt{x}}+24\zeta(-\tfrac{1}{2},x),\end{aligned}\right\} \tag{20}$$

and so on.

New expressions for Riemann's functions $\xi(s)$ and $\Xi(t)$

Quarterly Journal of Mathematics, XLVI, 1915, 253 – 260

1. The principal object of this paper is to prove that if the real parts of α and β are positive, and $\alpha\beta = \pi^2$, and t is real, then

$$\begin{aligned}
&\alpha^{-\frac{1}{4}}\left\{\frac{1}{1+t^2} + 4\alpha\int_0^\infty \left(\frac{1}{3^2+t^2} - \frac{\alpha}{1!}\frac{x^2}{7^2+t^2} + \frac{\alpha^2}{2!}\frac{x^4}{11^2+t^2} - \cdots\right)\frac{x\,dx}{e^{2\pi x}-1}\right\}\\
&\quad - \beta^{-\frac{1}{4}}\left\{\frac{1}{1+t^2} + 4\beta\int_0^\infty \left(\frac{1}{3^2+t^2} - \frac{\beta}{1!}\frac{x^2}{7^2+t^2} + \frac{\beta^2}{2!}\frac{x^4}{11^2+t^2} - \cdots\right)\frac{x\,dx}{e^{2\pi x}-1}\right\}\\
&\quad = \frac{\pi^{-\frac{3}{4}}}{4t}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\sin\left(\frac{t}{8}\log\frac{\beta}{\alpha}\right). \qquad (1)
\end{aligned}$$

Consider the integral

$$J(u) = \int_0^\infty \frac{xe^{-\pi u x^2}}{e^{2\pi x}-1}\,dx,$$

where the real part of u is positive. Since

$$\int_0^\infty \frac{\sin \pi n x}{e^{\pi x}-1}\,dx = \frac{1}{e^{2\pi n}-1} + \frac{1}{2} - \frac{1}{2\pi n},$$

we have

$$\begin{aligned}
J(u) + \frac{1}{4\pi u} - \frac{1}{4\pi\sqrt{u}} &= \int_0^\infty xe^{-\pi u x^2}\left(\frac{1}{e^{2\pi x}-1} + \frac{1}{2} - \frac{1}{2\pi x}\right)\,dx\\
&= \int_0^\infty\int_0^\infty xe^{-\pi u x^2}\frac{\sin \pi x y}{e^{\pi y}-1}\,dx\,dy = u^{-\frac{3}{2}}\int_0^\infty \frac{xe^{-\pi x^2/u}}{e^{2\pi x}-1}\,dx; \qquad (2)
\end{aligned}$$

and so

$$J(u) - \frac{1}{4\pi\sqrt{u}} = u^{-\frac{3}{2}}\int_0^\infty xe^{-\pi x^2/u}\left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x}\right)\,dx. \qquad (3)$$

Suppose now that $s = \sigma + it$, where $0 < \sigma < 1$. Then, from (3), we have

$$\int_0^1 u^{\frac{1}{2}(s-1)} \left\{ J(nu) - \frac{1}{4\pi\sqrt{(nu)}} \right\} du$$

$$= n^{-\frac{3}{2}} \int_0^1 u^{\frac{1}{2}(s-4)} \, du \int_0^\infty x e^{-\pi x^2/nu} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) dx. \tag{4}$$

Changing u into $1/v$, we obtain

$$n^{-\frac{3}{2}} \int_1^\infty v^{-\frac{1}{2}s} \, dv \int_0^\infty x e^{-\pi v x^2/n} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) dx$$

$$= n^{-\frac{3}{2}} \left\{ \int_0^\infty v^{-\frac{1}{2}s} \, dv \int_0^\infty x e^{-\pi v x^2/n} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) dx \right.$$

$$\left. - \int_0^1 v^{-\frac{1}{2}s} \, dv \int_0^\infty x e^{-\pi v x^2/n} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) dx \right\}.$$

But

$$\int_0^1 u^{\frac{1}{2}(s-1)} \left\{ J(nu) - \frac{1}{4\pi\sqrt{(nu)}} \right\} du$$

$$= \int_0^1 u^{\frac{1}{2}(s-1)} J(nu) \, du - \frac{1}{2\pi s\sqrt{n}}$$

$$= -\frac{1}{2\pi s\sqrt{n}} + \int_0^1 u^{\frac{1}{2}(s-1)} \, du \int_0^\infty \frac{x e^{-\pi n u x^2}}{e^{2\pi x}-1} \, dx$$

$$= -\frac{1}{2\pi s\sqrt{n}} + \int_0^\infty \frac{x \, dx}{e^{2\pi x}-1} \int_0^1 u^{\frac{1}{2}(s-1)} e^{-\pi n u x^2} \, du$$

$$= -\frac{1}{2\pi s\sqrt{n}} + 2\int_0^\infty \left\{ \frac{1}{1+s} - \frac{\pi n x^2}{1!(3+s)} + \frac{(\pi n x^2)^2}{2!(5+s)} - \cdots \right\} \frac{x \, dx}{e^{2\pi x}-1}. \tag{5}$$

Also

$$n^{-\frac{3}{2}} \int_0^\infty v^{-\frac{1}{2}s} \, dv \int_0^\infty x e^{-\pi v x^2/n} \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x} \right) dx$$

$$= n^{-\frac{3}{2}} \int_0^\infty x \left(\frac{1}{e^{2\pi x} - 1} - \frac{1}{2\pi x} \right) \, dx \int_0^\infty v^{-\frac{1}{2}s} e^{-\pi v x^2/n} \, dv$$

$$= \pi^{\frac{1}{2}(s-2)} n^{-\frac{1}{2}(s+1)} \Gamma(1 - \frac{1}{2}s) \int_0^\infty x^{s-1} \left(\frac{1}{e^{2\pi x} - 1} - \frac{1}{2\pi x} \right) \, dx$$

$$= -\frac{n^{-\frac{1}{2}(s+1)}}{4\pi\sqrt{\pi}} \Gamma\left(-\frac{s}{2}\right) \Gamma\left(\frac{s-1}{2}\right) \xi(s), \tag{6}$$

where

$$\xi(s) = (s-1)\Gamma(1 + \frac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s).$$

Finally

$$n^{-\frac{3}{2}} \int_0^1 v^{-\frac{1}{2}s} \, dv \int_0^\infty x e^{-\pi v x^2/n} \left(\frac{1}{e^{2\pi x} - 1} - \frac{1}{2\pi x} \right) \, dx$$

$$= -\frac{n^{-\frac{3}{2}}}{2\pi} \int_0^1 v^{-\frac{1}{2}s} \, dv \int_0^\infty e^{-\pi v x^2/n} \, dx + n^{-\frac{3}{2}} \int_0^1 v^{-\frac{1}{2}s} \, dv \int_0^\infty \frac{x e^{-\pi v x^2/n}}{e^{2\pi x} - 1} \, dx$$

$$= -\frac{1}{4\pi n} \int_0^1 v^{-\frac{1}{2}(1+s)} \, dv + n^{-\frac{3}{2}} \int_0^\infty \frac{x \, dx}{e^{2\pi x} - 1} \int_0^1 v^{-\frac{1}{2}s} e^{-\pi v x^2/n} \, dv$$

$$= -\frac{1}{2\pi n(1-s)} + 2n^{-\frac{3}{2}} \int_0^\infty \left\{ \frac{1}{2-s} - \frac{\pi x^2/n}{1!(4-s)} + \frac{(\pi x^2/n)^2}{2!(6-s)} - \cdots \right\} \frac{x \, dx}{e^{2\pi x} - 1}. \tag{7}$$

All the inversions of the order of integration, effected in the preceding argument, are easily justified, since every integral remains convergent when the subject of integration is replaced by its modulus.

It follows from (4) - (7) that, if the real parts of α and β are positive, and $\alpha\beta = \pi^2$, then

$$\alpha^{-\frac{1}{4}} \left\{ \frac{1}{1-s} - 4\alpha \int_0^\infty \left(\frac{1}{1+s} - \frac{\alpha}{1!} \frac{x^2}{3+s} + \frac{\alpha^2}{2!} \frac{x^4}{5+s} - \cdots \right) \frac{x \, dx}{e^{2\pi x} - 1} \right\}$$

$$+\beta^{-\frac{1}{4}} \left\{ \frac{1}{s} - 4\beta \int_0^\infty \left(\frac{1}{2-s} - \frac{\beta}{1!} \frac{x^2}{4-s} + \frac{\beta^2}{2!} \frac{x^4}{6-s} - \cdots \right) \frac{x \, dx}{e^{2\pi x} - 1} \right\}$$

$$= \frac{1}{2} \pi^{-\frac{3}{4}} \left(\frac{\alpha}{\beta} \right)^{\frac{1}{8} - \frac{1}{4}s} \Gamma\left(-\frac{s}{2}\right) \Gamma\left(\frac{s-1}{2}\right) \xi(s). \tag{8}$$

Changing s to $\frac{1}{2}(1+it)$ in (8), and writing as usual

$$\xi(\frac{1}{2}+\frac{1}{2}it)=\Xi(\frac{1}{2}t),$$

and equating the real and imaginary parts, we obtain the formula (1), and also the formula

$$\begin{aligned}\alpha^{-\frac{1}{4}}&\left\{\frac{1}{1+t^2}-4\alpha\int_0^\infty\left(\frac{3}{3^2+t^2}-\frac{\alpha}{1!}\frac{7x^2}{7^2+t^2}+\frac{\alpha^2}{2!}\frac{11x^4}{11^2+t^2}-\cdots\right)\frac{x\,dx}{e^{2\pi x}-1}\right\}\\ &+\beta^{-\frac{1}{4}}\left\{\frac{1}{1+t^2}-4\beta\int_0^\infty\left(\frac{3}{3^2+t^2}-\frac{\beta}{1!}\frac{7x^2}{7^2+t^2}+\frac{\beta^2}{2!}\frac{11x^4}{11^2+t^2}-\cdots\right)\frac{x\,dx}{e^{2\pi x}-1}\right\}\\ &=\frac{1}{4}\pi^{-\frac{3}{4}}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi(\frac{1}{2}t)\cos\left(\frac{t}{8}\log\frac{\alpha}{\beta}\right).\end{aligned}\tag{9}$$

2. We have proved (8) on the assumption that $0<\sigma<1$. But it can be shewn that the formula is true for all values of s other than integral values.
Suppose first that $-1>\sigma<0$. The formula (3) is equivalent to

$$J(u)=u^{-\frac{3}{2}}\int_0^\infty xe^{-\pi x^2/u}\left(\frac{1}{e^{2\pi x}-1}-\frac{1}{2\pi x}+\frac{1}{2}\right)\,dx.\tag{10}$$

Using this formula as we used (3) in the previous section, we can shew that (8) is true in the strip $-1<\sigma<0$ also. In the right-hand side of (3), the first term in the expansion of $1/(e^{2\pi x}-1)$, viz. $1/(2\pi x)$, is removed, and in that of (10) two terms are removed. By considering the corresponding formulæ in which more and more terms in the expansion of $1/(e^{2\pi x}-1)$, viz.

$$\frac{1}{2\pi x}-\frac{1}{2}+\frac{\pi x}{6}-\frac{\pi^3x^3}{90}+\frac{\pi^5x^5}{945}-\frac{\pi^7x^7}{9450}+\frac{\pi^9x^9}{93555}-\cdots,$$

are removed,we can shew that the formula (8) is true in the strips $-2<\sigma<1, -3<\sigma<-2$, and so on. That it is also true in the strips $1<\sigma<2, 2<\sigma<3, \ldots$ is easily deduced from the functional equation $\xi(s)=\xi(1-s)$.
The formula also holds on the lines which divide the strips, except at the special points $s=k$, where k is an integer. This follows at once from the continuity of $\xi(s)$ and the uniform convergence of the integrals in question.

3. As a particular case of (9) we have, when $\alpha=\beta=\pi$,

$$\frac{1}{1+t^2}-4\pi\int_0^\infty\left(\frac{3}{3^2+t^2}-\frac{\pi}{1!}\frac{7x^2}{7^2+t^2}+\frac{\pi^2}{2!}\frac{11x^4}{11^2+t^2}-\cdots\right)\frac{x\,dx}{e^{2\pi x}-1}$$

$$= \frac{1}{8\sqrt{\pi}}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi(\frac{1}{2}t). \qquad (11)$$

But the left-hand side of (11) is equal to

$$\int_0^\infty \left\{ e^{-z} - 4\pi \int_0^\infty \left(e^{-3z} - \frac{\pi x^2}{1!}e^{-7z} + \frac{\pi^2 x^4}{2!}e^{-11z} - \cdots \right) \frac{x\ dx}{e^{2\pi x}-1} \right\} \cos tz\ dz.$$

Hence,

$$\int_0^\infty \left\{ e^{-z} - 4\pi \int_0^\infty \frac{xe^{-3z-\pi x^2 e^{-4z}}}{e^{2\pi x}-1}\ dx \right\} \cos tz\ dz$$
$$= \frac{1}{8\sqrt{\pi}}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi(\frac{1}{2}t). \qquad (12)$$

It follows from this and Fourier's theorem that

$$e^{-n} - 4\pi e^{-3n} \int_0^\infty \frac{xe^{-\pi x^2 e^{-4n}}}{e^{2\pi x}-1}\ dx$$
$$= \frac{1}{4\pi\sqrt{\pi}} \int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi(\frac{1}{2}t)\cos nt\ dt. \qquad (13)$$

But it is easily seen from (2) that, if α and β are positive and $\alpha\beta = \pi^2$, then

$$\alpha^{-\frac{1}{4}} \left\{ 1 + 4\alpha \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\ dx \right\} = \beta^{-\frac{1}{4}} \left\{ 1 + 4\beta \int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\ dx \right\}. \qquad (14)$$

From this it follows that the left-hand side of (13) is an even function of n. and so the formula (13) is true for all real values of n.

4. It can easily be shewn that, if $\alpha\beta = 4\pi^2$ and $R(s)$, where $R(s)$ is the real part of s, is greater than -1, then

$$\frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\alpha^{\frac{1}{2}(s-1)} + \frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\alpha^{\frac{1}{2}(s+1)}$$
$$+ \alpha^{\frac{1}{2}(s+1)} \int_0^\infty \int_0^\infty \frac{x^s \sin\alpha xy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\ dx\ dy$$
$$= \frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\beta^{\frac{1}{2}(s-1)} + \frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\beta^{\frac{1}{2}(s+1)}$$

$$+\beta^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty \frac{x^s \sin\beta xy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\,dx\,dy. \tag{15}$$

From this we can shew, by arguments similar to those of §§ 1- 2, that if $\alpha\beta = 4\pi^2$ and $R(s) > -1$ then

$$\frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\frac{\alpha^{\frac{1}{2}(s-1)}}{s-1-t}+\frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\frac{\alpha^{\frac{1}{2}(s+1)}}{s+1-t}+\alpha^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty\left\{\frac{\alpha xy}{1!(s+3-t)}-\frac{(\alpha xy)^3}{3!(s+7-t)}\right.$$

$$\left.+\frac{(\alpha xy)^5}{5!(s+11-t)}-\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}+\frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\frac{\beta^{\frac{1}{2}(s-1)}}{s-1+t}+\frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\frac{\beta^{\frac{1}{2}(s+1)}}{(s+1+t)}$$

$$+\beta^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty\left\{\frac{\beta xy}{1!(s+3+t)}-\frac{(\beta xy)^3}{3!(s+7+t)}+\frac{(\beta xy)^5}{5!(s+11+t)}-\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$

$$=\left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}t}\frac{2^{\frac{1}{2}(s-3)}\Gamma\{\frac{1}{4}(s-1+t)\}\Gamma\{\frac{1}{4}(s-1-t)\}}{\pi\quad (s+1)^2-t^2}\times\xi\left(\frac{1+s+t}{2}\right)\xi\left(\frac{1+s-t}{2}\right). \tag{16}$$

From this we deduce that, if $\alpha\beta = 4\pi^2$ and $R(s) > -1$, then

$$\frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\frac{s-1}{(s-1)^2+t^2}\{\alpha^{\frac{1}{2}(s-1)}+\beta^{\frac{1}{2}(s-1)}\}+\frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\frac{s+1}{(s+1)^2+t^2}\{\alpha^{\frac{1}{2}(s+1)}+\beta^{\frac{1}{2}(s+1)}\}$$

$$+\alpha^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty\left\{\frac{\alpha xy}{1!}\frac{s+3}{(s+3)^2+t^2}-\frac{(\alpha xy)^3}{3!}\frac{s+7}{(s+7)^2+t^2}+\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$

$$+\beta^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty\left\{\frac{\beta xy}{1!}\frac{s+3}{(s+3)^2+t^2}-\frac{(\beta xy)^3}{3!}\frac{s+7}{(s+7)^2+t^2}+\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$

$$=\frac{2^{\frac{1}{2}(s-3)}}{\pi}\frac{\Gamma\{\frac{1}{4}(s-1+it)\}\Gamma\{\frac{1}{4}(s-1-it)\}}{(s+1)^2+t^2}$$

$$\times\;\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\cos\left(\frac{1}{4}t\log\frac{\alpha}{\beta}\right); \tag{17}$$

and

$$\frac{\zeta(1-s)}{4\cos\frac{1}{2}\pi s}\frac{1}{(s-1)^2+t^2}\{\alpha^{\frac{1}{2}(s-1)}-\beta^{\frac{1}{2}(s-1)}\}+\frac{\zeta(-s)}{8\sin\frac{1}{2}\pi s}\frac{1}{(s+1)^2+t^2}\{\alpha^{\frac{1}{2}(s+1)}-\beta^{\frac{1}{2}(s+1)}\}$$

$$+\alpha^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty\left\{\frac{\alpha xy}{1!}\frac{1}{(s+3)^2+t^2}-\frac{(\alpha xy)^3}{3!}\frac{1}{(s+7)^2+t^2}+\cdots\right\}\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$

$$-\beta^{\frac{1}{2}(s+1)}\int_0^\infty\int_0^\infty\left\{\frac{\beta xy}{1!}\frac{1}{(s+3)^2+t^2}-\frac{(\beta xy)^3}{3!}\frac{1}{(s+7)^2+t^2}+\cdots\right\}\frac{x^s\ dx\ dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$
$$=\frac{2^{\frac{1}{2}(s-3)}}{\pi}\frac{\Gamma\{\frac{1}{4}(s-1+it)\}\Gamma\{\frac{1}{4}(s-1-it)\}}{(s+1)^2+t^2}$$
$$\times\ \Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\sin\left(\frac{1}{4}t\log\frac{\alpha}{\beta}\right). \tag{18}$$

5. Proceeding as in § 3 we can shew that, if n is real, and

$$F(n)=\int_0^\infty\frac{\Gamma\{\frac{1}{4}(s-1+it)\}\Gamma\{\frac{1}{4}(s-1-it)\}}{(s+1)^2+t^2}\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\cos ntdt,$$

then, if $R(s)>1$,

$$F(n)=\frac{1}{8}(4\pi)^{-\frac{1}{2}(s-3)}\left\{\int_0^\infty\frac{x^s\ dx}{(e^{xe^n}-1)(e^{xe^{-n}-1})}-2\Gamma(s)\zeta(s)\cosh n(1-s)\right\}; \tag{19}$$

if $-1<R(s)<1$,

$$F(n)=\frac{1}{8}(4\pi)^{-\frac{1}{2}(s-3)}\int_0^\infty x^s\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)\ dx; \tag{20}$$

if $-3<R(s)<-1$,

$$F(n)=\frac{1}{8}(4\pi)^{-\frac{1}{2}(s-3)}\left\{\int_0^\infty x^s\left\{\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}+\frac{1}{2}\right)\right.$$
$$\left.\times\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}+\frac{1}{2}\right)\ dx-\Gamma(1+s)\zeta(1+s)\cosh n(1+s)\right\}; \tag{21}$$

and so on. If, in particular, we put $s=0$ in (20), we obtain

$$\int_0^\infty\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\{\Xi(\frac{1}{2}t)\}^2\frac{\cos nt}{1+t^2}\ dt$$
$$=\pi\sqrt{\pi}\int_0^\infty\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)\ dx. \tag{22}$$

Highly composite numbers

Proceedings of the London Mathematical Society, 2, XIV, 1915, 347 – 409

CONTENTS

This paper, as it appeared orginally, is not complete. Since the *London Mathematical Society* was in some financial difficulty at that time, Ramanujan had to suppress part of what he had written in order to save expense. The unpublished part of the manuscript with annotation has been reproduced: Highly Composite Numbers by Srinivasa Ramanujan, Jean-Louis Nicolas, Guy Robin, *The Ramanujan Journal*, Volume 1, Issue 2, 1997, pp. 119 – 153.

I.
Introduction and Summary of Results

1. The number $d(N)$ of divisors of N varies with extreme irregularity as N tends to infinity, tending itself to infinity or remaining small according to the form of N. In this paper I prove a large number of results which add a good deal to our knowledge of the behaviour of $d(N)$.

It was proved by Dirichlet [*] that

$$\frac{d(1)+d(2)+d(3)+\cdots+d(N)}{N} = \log N + 2\gamma - 1 + O\left(\frac{1}{\sqrt{N}}\right) \; ^{\dagger}$$

where γ is the Eulerian constant. Voronöi [‡] and Landau [§] have shewn that the error term may be replaced by $O(N^{-\frac{2}{3}+\epsilon})$, or indeed $O(N^{-\frac{2}{3}}\log N)$. It seems not unlikely that the real value of the error is of the form $O(N^{-\frac{3}{4}+\epsilon})$, but this is as yet unproved. Mr. Hardy has, however, shewn recently [¶] that the equation

$$\frac{d(1)+d(2)+d(3)+\cdots d(N)}{N} = \log N + 2\gamma - 1 + o(N^{-\frac{3}{4}})$$

is certainly false. He has also proved that

$$d(1)+d(2))+\cdots+d(N-1)+\tfrac{1}{2}d(N)) - N\log N - (2\gamma-1)N - \tfrac{1}{4}$$
$$= \sqrt{N}\sum_{1}^{\infty}\frac{d(n)}{\sqrt{n}}[H_1\{4\pi\sqrt{(nN)}\} - Y_1\{4\pi\sqrt{(nN)}\}],$$

where Y_n is the ordinary second solution of Bessel's equation, and

$$H_1(x) = \frac{2}{\pi}\int_1^{\infty}\frac{we^{-xw}dw}{\sqrt{(w^2-1)}};$$

and that the series on the right-hand side is the sum of the series

$$\frac{N^{\frac{1}{4}}}{\pi\sqrt{2}}\sum_{1}^{\infty}\frac{d(n)}{n^{\frac{3}{4}}}\cos\{4\pi\sqrt{(nN)} - \tfrac{1}{4}\pi\},$$

and an absolutely and uniformly convergent series.

---

[*] *Werke*, Vol.2, p.49.

[†] $f = O(\phi)$ means that a constant exists such that $|f| < K\phi$: $f = o(\phi)$ means that $f/\phi \to 0$.

[‡] *Crelle's Journal*, Vol. 126. p. 241.

[§] *Göttinger Nachrichten*, 1912.

[¶] *Comptes Rendus* May 10, 1915.

The "average" order of $d(N))$ is thus known with considerable accuracy. In this paper I consider, not the average order of $d(N))$, but its maximum order. This problem has been much less studied. It is obvious that

$$d(N) < 2\sqrt{N}.$$

It was shewn by Wigert* that

$$d(N) < 2^{\frac{\log N}{\log\log N}(1+\epsilon)} \tag{i}$$

for all positive values of ϵ and all sufficiently large values of N, and that

$$d(N) > 2^{\frac{\log N}{\log\log N}(1-\epsilon)} \tag{ii}$$

for an infinity of values of N. From (i) it follows in particular that

$$d(N) < N^{\delta}$$

for all positive values of δ and all sufficiently large values of N.
Wigert proves (i) by purely elementary reasoning, but uses the "Prime Number Theorem"†to prove (ii). This is, however, unnecessary, the inequality (ii) being also capable of elementary proof. In § 5 I shew, by elementary reasoning, that

$$d(N) < 2^{\frac{\log N}{\log\log N}+O\frac{\log N}{(\log\log N)^2}}$$

for all values of N, and

$$d(N) > 2^{\frac{\log N}{\log\log N}+O\frac{\log N}{(\log\log N)^2}}$$

for an infinity of values of N. I also shew later on that, if we assume all known results concerning the distribution of primes, then

$$d(N) < 2^{Li(\log N)+O[\log N e^{-a\sqrt{(\log\log N)}}]}$$

for all values of N, and

$$d(N) > 2^{Li(\log N)+O[\log N e^{-a\sqrt{(\log\log N)}}]}$$

for an infinity of values of N, where a is a positive constant.
I then adopt a different point of view, I define a highly composite number as a number whose number of divisors exceeds that of all its predecessors. Writing such a number in the form

$$N = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p^{a_p},$$

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†The theorem that $\pi(x) \sim \frac{x}{\log x}$, $\pi(x)$ being the number of primes not exceeding x.

I prove that

$$a_2 \geq a_3 \geq a_5 \geq \cdots \geq a_p,$$

and that

$$a_p = 1,$$

for all highly composite values of N except 4 and 36.
I then go on to prove that the indices near the beginning form a decreasing sequence in the stricter sense, i.e., that

$$a_2 > a_3 > a_5 > \cdots > a_\lambda,$$

where λ is a certain function of p.
Near the end groups of equal indices may occur, and I prove that there are actually groups of indices equal to

$$1, 2, 3, 4, \ldots, \mu,$$

where μ again is a certain function of p. I also prove that if λ is fairly small in comparison with p, then

$$a_\lambda \log \lambda \sim \frac{\log p}{\log 2};$$

and that the later indices can be assigned with an error of at most unity.
I prove also that two successive highly composite numbers are asymptotically equivalent, i.e., that the ratio of two consecutive such numbers tends to unity. These are the most striking results. More precise ones will be found in the body of the paper. These results give us a fairly accurate idea of the structure of a highly composite number.
I then select from the general aggregate of highly composite numbers a special set which I call "superior highly composite numbers". I determine completely the general form of all such numbers, and I shew how a combination of the idea of a superior highly composite number with the assumption of the truth of the Riemann hypothesis concerning the roots of the ζ-function leads to even more precise results concerning the maximum order of $d(N)$. These results naturally differ from all which precede in that they depend on the truth of a hitherto unproved hypothesis.

II.
Elementary Results concerning the Order of $d(N)$.

2. Let $d(N)$ denote the number of divisors of N, and let

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n}, \tag{1}$$

where $p_1, p_2, p_3, \ldots, p_n$ are a given set of n primes. Then

$$d(N) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots (1 + a_n). \tag{2}$$

From (1) we see that

$$\begin{aligned}&(1/n)\log(p_1p_2p_3\cdots p_nN)\\&\quad=(1/n)\{(1+a_1)\log p_1+(1+a_2)\log p_2+\cdots+(1+a_n)\log p_n\}\\&\quad>\{(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_n)\log p_1\log p_2\cdots\log p_n\}^{1/n}.\end{aligned}$$

Hence we have

$$d(N)<\frac{\{(1/n)\log(p_1p_2p_3\cdots p_nN)\}^n}{\log p_1\log p_2\log p_3\cdots\log p_n}, \tag{3}$$

for all values of N.
We shall now consider how near to this limit it is possible to make $d(N)$ by choice of the indices $a_1, a_2, a_3, \ldots, a_n$. Let us suppose that

$$1+a_m=v\frac{\log p_n}{\log p_m}+\epsilon_m(m=1,2,3,\ldots,n), \tag{4}$$

where v is a large integer and $-\frac{1}{2}<\epsilon_m<\frac{1}{2}$. Then, from (4), it is evident that

$$\epsilon_n=0. \tag{5}$$

Hence, by a well-known theorem due to Dirichlet*, it is possible to choose values of v as large as we please and such that

$$|\epsilon_1|<\epsilon,\ |\epsilon_2|<\epsilon,\ |\epsilon_3|<\epsilon,\ldots,|\epsilon_{n-1}|<\epsilon, \tag{6}$$

where $\epsilon\le v^{-1/(n-1)}$. Now let

$$t=v\log p_n,\quad \delta_m=\epsilon_m\log p_m. \tag{7}$$

Then from (1), (4) and (7), we have

$$\log(p_1p_2p_3\cdots p_nN)=nt+\sum_1^n\delta_m. \tag{8}$$

Similarly, from (2), (4) and (7) we see that

$$\begin{aligned}d(N)&=\frac{(t+\delta_1)(t+\delta_2)\cdots(t+\delta_n)}{\log p_1\log p_2\log p_3\cdots\log p_n}\\&=\frac{t^n\exp\left\{\frac{\sum\delta_m}{t}-\frac{\sum\delta_m^2}{2t^2}+\frac{\sum\delta_m^3}{3t^3}-\cdots\right\}}{\log p_1\log p_2\log p_3\cdots\log p_n}\end{aligned}$$

* *Werke*, Vol. 1, p. 635.

$$= \quad \left(t + \frac{\sum \delta_m}{n}\right)^n \frac{\exp\left\{-\dfrac{n\sum\delta_m^2 - (\sum\delta_m)^2}{2nt^2} + \dfrac{n^2\sum\delta_m^3 - (\sum\delta_m)^3}{3n^2t^3} - \cdots\right\}}{\log p_1 \log p_2 \log p_3 \cdots \log p_n}$$

$$= \quad \frac{\{(1/n)\log(p_1p_2p_3\cdots p_nN)\}^n}{\log p_1 \log p_2 \cdots \log p_n}$$

$$\left[1 - \tfrac{1}{2}(\log N)^{-2}\left\{n^2\sum\delta_m^2 - n\left(\sum\delta_m\right)^2\right\} + \cdots\right], \tag{9}$$

in virtue of (8)). From (6), (7) and (9) it follows that it is possible to choose the indices $a_1, a_2, \ldots, a_n$ so that

$$d(N) = \frac{\{(1/n)\log(p_1p_2p_3\cdots p_nN)\}^n}{\log p_1 \log p_2 \cdots \log p_n}\{1 - O(\log N)^{-2n/(n-1)}\}, \tag{10}$$

where the symbol O has its ordinary meaning.

The following examples shew how close an approximation to $d(N)$ may be given by the right-hand side of (3). If

$$N = 2^{72} \cdot 7^{25},$$

then, according to (3), we have

$$d(N) < 1898.00000685\ldots; \tag{11}$$

and as a matter of fact $d(N) = 1898$. Similarly, taking

$$N = 2^{568} \cdot 3^{358},$$

we have, by (3),

$$d(N)) < 204271.000000372\ldots; \tag{12}$$

while the actual value of $d(N)$ is 204271. In a similar manner, when

$$N = 2^{64} \cdot 3^{40} \cdot 5^{27},$$

we have, by (3),

$$d(N) < 74620.00412\ldots; \tag{13}$$

while actually

$$d(N) = 74620.$$

3. Now let us suppose that, while the number n of different prime factors of N remains fixed, the primes p_ν, as well as the indices a_ν, are allowed to vary. It is evident that $d(N)$, considered as a function of N, is greatest when the primes p_ν are the first n primes, say $2, 3, 5, \ldots, p$, where p is the nth prime. It therefore follows from (3) that

$$d(N) < \frac{\{(1/n)\log(2\cdot 3\cdot 5\cdots p\cdot N)\}^n}{\log 2\log 3\log 5\cdots\log p}, \tag{14}$$

and from (10) that it is possible to choose the indices so that

$$d(N) = \frac{\{(1/n)\log(2\cdot 3\cdot 5\cdots p\cdot N)\}^n}{\log 2\log 3\log 5\cdots\log p}\{1 - O(\log N)^{-2n/(n-1)}\}. \tag{15}$$

4. Before we proceed to consider the most general case, in which nothing is known about N, we must prove certain preliminary results. Let $\pi(x)$ denote the number of primes not exceeding x, and let

$$\vartheta(x) = \log 2 + \log 3 + \log 5 + \cdots + \log p,$$

and

$$\varpi(x) = \log 2\cdot\log 3\cdot\log 5\cdots\log p,$$

where p is the largest prime not greater than x; also let $\phi(t)$ be a function of t such that $\phi'(t)$ is continuous between 2 and x. Then

$$\begin{aligned}
\int_2^x \pi(t)\phi'(t)dt &= \int_2^3 \phi'(t)dt + 2\int_3^5 \phi'(t)dt + 3\int_5^7 \phi'(t)dt \\
&\quad + 4\int_7^{11} \phi'(t)dt + \cdots + \pi(x)\int_p^x \phi'(t)dt \\
&= \{\phi(3) - \phi(2)\} + 2\{\phi(5) - \phi(3)\} + 3\{\phi(7) - \phi(5)\} \\
&\quad + 4\{\phi(11) - \phi(7)\} + \cdots + \pi(x)\{\phi(x) - \phi(p)\} \\
&= \pi(x))\phi(x) - \{\phi(2) + \phi(3) + \phi(5) + \cdots + \phi(p)\}.
\end{aligned} \tag{16}$$

As an example let us suppose that $\phi(t) = \log t$. Then we have

$$\pi(x)\log x - \vartheta(x) = \int_2^x \frac{\pi(t)}{t}dt. \tag{17}$$

Again let us suppose that $\phi(t) = \log\log t$. Then we see that

$$\pi(x)\log\log x - \log\varpi(x) = \int_2^x \frac{\pi(t)}{t\log t}dt. \tag{18}$$

But

$$\int_2^x \frac{\pi(t)}{t\log t}dt = \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt + \int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}dt\right) du.$$

Hence we have

$$\begin{aligned}&\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)}\right\} - \log\varpi(x)\\ &\quad = \pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\} + \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt + \int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}dt\right) du. \qquad (19)\end{aligned}$$

But

$$\begin{aligned}\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\} &= \pi(x)\log\left\{1 - \frac{\pi(x)\log x - \vartheta(x)}{\pi(x)\log x}\right\}\\ &= \pi(x)\log\left\{1 - \frac{1}{\pi(x)\log x}\int_2^x \frac{\pi(t)}{t}dt\right\}\\ &< -\frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt;\end{aligned}$$

and so

$$\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\} + \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt < 0. \qquad (20)$$

Again,

$$\begin{aligned}\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\} &= -\pi(x)\log\left\{1 + \frac{\pi(x)\log x - \vartheta(x)}{\vartheta(x)}\right\}\\ &= -\pi(x)\log\left\{1 + \frac{1}{\vartheta(x)}\int_2^x \frac{\pi(t)}{t}dt\right\} > -\frac{\pi(x)}{\vartheta(x)}\int_2^x \frac{\pi(t)}{t}dt;\end{aligned}$$

and so

$$\begin{aligned}\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\} + \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt &> -\frac{\pi(x)\log x - \vartheta(x)}{\vartheta(x)\log x}\int_2^x \frac{\pi(t)}{t}dt\\ &= -\frac{1}{\vartheta(x)\log x}\left\{\int_2^x \frac{\pi(t)}{t}dt\right\}^2. \qquad (21)\end{aligned}$$

It follows from (19), (20) and (21) that

$$\begin{aligned}\int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}\,dt\right)du \quad &> \quad \pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)}\right\} - \log\varpi(x)\\ &> \quad \int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}\,dt\right)du\\ &\qquad -\frac{1}{\vartheta(x)\log x}\left\{\int_2^x \frac{\pi(t)}{t}\,dt\right\}^2.\end{aligned}$$

Now it is easily proved by elementary methods* that

$$\pi(x) = O\left(\frac{x}{\log x}\right), \quad \frac{1}{\vartheta(x)} = O\left(\frac{1}{x}\right);$$

and so

$$\int_2^x \frac{\pi(t)}{t}dt = O\left(\frac{x}{\log x}\right).$$

Hence

$$\int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}dt\right)du = \int_2^x O\left\{\frac{1}{(\log u)^3}\right\}du = O\left\{\frac{x}{(\log x)^3}\right\};$$

and

$$\frac{1}{\vartheta(x)\log x}\left\{\int_2^x \frac{\pi(t)}{t}dt\right\}^2 = \frac{1}{\vartheta(x)\log x}O\left\{\frac{x^2}{(\log x)^2}\right\} = O\left\{\frac{x}{(\log x)^3}\right\}.$$

Hence we see that

$$\frac{\{\vartheta(x)/\pi(x)\}^{\pi(x)}}{\varpi(x)} = e^{O[x/(\log x)^3]}. \tag{22}$$

5. We proceed to consider the case in which nothing is known about N. Let

$$N' = 2^{a_1}\cdot 3^{a_2}\cdot 5^{a_3}\cdots p^{a_n}.$$

Then it is evident that $d(N) = d(N')$, and that

$$\vartheta(p) \le \log N' \le \log N. \tag{23}$$

*See Landau, *Handbuch*, pp. 71 *et seq.*

It follows from (3) that

$$
\begin{aligned}
d(N) &= d(N') < \frac{1}{\varpi(p)}\left\{\frac{\vartheta(p)+\log N'}{\pi(p)}\right\}^{\pi(p)} \\
&\le \left\{1+\frac{\log N}{\vartheta(p)}\right\}^{\pi(p)} \frac{\{\vartheta(p)/\pi(p)\}^{\pi(p)}}{\varpi(p)} \\
&= \left\{1+\frac{\log N}{\vartheta(p)}\right\}^{\pi(p)} e^{O[p/(\log p)^3]} = \left\{1+\frac{\log N}{\vartheta(p)}\right\}^{\pi(p)+O[p/(\log p)^3]}, \qquad (24)
\end{aligned}
$$

in virtue of (22) and (23). But from (17) we know that

$$\pi(p)\log p - \vartheta(p) = O\left(\frac{p}{\log p}\right);$$

and so

$$\vartheta(p) = \pi(p)\{\log p + O(1)\} = \pi(p)\{\log\vartheta(p) + O(1)\}.$$

Hence

$$\pi(p) = \vartheta(p)\left\{\frac{1}{\log\vartheta(p)} + O\frac{1}{\{\log\vartheta(p)\}^2}\right\}. \qquad (25)$$

It follows from (24) and (25) that

$$d(N) \le \left\{1+\frac{\log N}{\vartheta(p)}\right\}^{\frac{\vartheta(p)}{\log\vartheta(p)}+O\frac{\vartheta(p)}{[\log\vartheta(p)]^2}}.$$

Writing t instead of $\vartheta(p)$, we have

$$d(N) \le \left(1+\frac{\log N}{t}\right)^{\frac{t}{\log t}+O\frac{t}{(\log t)^2}}; \qquad (26)$$

and from (23) we have

$$t \le \log N. \qquad (27)$$

Now, if N is a function of t, the order of the right-hand side of (26), considered as a function of N, is increased when N is decreased in comparison with t, and decreased when N is increased in comparison with t. Thus the most unfavourable hypothesis is that N, considered as a function of t, is as small as is compatible with the relation (27). We may therefore write $\log N$ for t in (26). Hence

$$d(N) < 2^{\frac{\log N}{\log\log N}+O\frac{\log N}{(\log\log N)^2}}, \qquad (28)$$

for all values of N*

The inequality (28) has been proved by purely elementary reasoning. We have not assumed, for example, the prime number theorem, expressed by the relation

$$\pi(x) \sim \frac{x}{\log x}\,.^{\dagger}$$

We can also, without assuming this theorem, shew that the right-hand side of (28) is actually the order of $d(N)$ for an infinity of values of N. Let us suppose that

$$N = 2\cdot 3\cdot 5\cdot 7\cdots p.$$

Then

$$d(N) = 2^{\pi(p)} = 2^{\frac{t}{\log t} + O\frac{t}{(\log t)^2}},$$

in virtue of (25). Since $\log N = \vartheta(p) = t$, we see that

$$d(N) = 2^{\frac{\log N}{\log\log N} + O\frac{\log N}{(\log\log N)^2}},$$

for an infinity of values of N. Hence the maximum order of $d(N)$ is

$$2^{\frac{\log N}{\log\log N} + O\frac{\log N}{(\log\log N)^2}}.$$

III.
The Structure of Highly Composite Numbers.

6. A number N may be said to be a highly composite number, if $d(N') < d(N)$ for all values of N' less than N. It is easy to see from the definition that, if N is highly composite and $d(N') > d(N)$, then there is at least one highly composite number M, such that

$$N < M \leq N'. \tag{29}$$

* If we assume *nothing* about $\pi(x)$, we can shew that

$$d(N) < 2^{\frac{\log N}{\log\log N} + O\frac{\log N \log\log\log N}{(\log\log N)^2}}.$$

If we assume the prime number theorem, and nothing more, we can shew that

$$d(N) < 2^{\frac{\log N}{\log\log N} + [1+O(1)]\frac{\log N}{(\log\log N)^2}}.$$

If we assume that

$$\pi(x) = \frac{x}{\log x} + O\frac{x}{(\log x)^2},$$

we can shew that

$$d(N) < 2^{\frac{\log N}{\log\log N} + \frac{\log N}{(\log\log N)^2} + O\frac{\log N}{(\log\log N)^3}}.$$

† $\phi(x) \sim \Psi(x)$ means that $\phi(x)/\Psi(x) \to 1$ as $x \to \infty$.

if N and N' are consecutive highly composite numbers, then $d(M) \leq d(N)$ for all values of M between N and N'. It is obvious that

$$d(N) < d(2N) \tag{30}$$

for all values of N. It follows from (29) and (30) that, if N is highly composite, then there is at least one highly composite number M such that $N < M \leq 2N$. That is to say, there is at least one highly composite number N, such that

$$x < N \leq 2x, \tag{31}$$

if $x \geq 1$.

7. I do not know of any method for determining consecutive highly composite numbers except by trial. The following table gives the consecutive highly composite values of N, and the corresponding values of $d(N)$ and $dd(N)$, up to $d(N) = 10080$.
The numbers marked with the asterisk in the table are called superior highly composite numbers. Their definition and properties will be found in §§ 32, 33.

$dd(N)$	$d(N)$	N
2	$2 = 2$	$*2 = 2$
2	$3 = 3$	$4 = 2^2$
3	$4 = 2^2$	$*6 = 2 \cdot 3$
4	$6 = 2 \cdot 3$	$*12 = 2^2 \cdot 3$
4	$8 = 2^3$	$24 = 2^3 \cdot 3$
3	$9 = 3^2$	$36 = 2^2 \cdot 3^2$
4	$10 = 2 \cdot 5$	$48 = 2^4 \cdot 3$
6	$12 = 2^2 \cdot 3$	$*60 = 2^2 \cdot 3 \cdot 5$
5	$16 = 2^4$	$*120 = 2^3 \cdot 3 \cdot 5$
6	$18 = 2 \cdot 3^2$	$180 = 2^2 \cdot 3^2 \cdot 5$
6	$20 = 2^2 \cdot 5$	$240 = 2^4 \cdot 3 \cdot 5$
8	$24 = 2^3 \cdot 3$	$*360 = 2^3 \cdot 3^2 \cdot 5$
8	$30 = 2 \cdot 3 \cdot 5$	$720 = 2^4 \cdot 3^2 \cdot 5$
6	$32 = 2^5$	$840 = 2^3 \cdot 3 \cdot 5 \cdot 7$
9	$36 = 2^2 \cdot 3^2$	$1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$
8	$40 = 2^3 \cdot 5$	$1680 = 2^4 \cdot 3 \cdot 5 \cdot 7$
10	$48 = 2^4 \cdot 3$	$*2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$
12	$60 = 2^2 \cdot 3 \cdot 5$	$*5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$
7	$64 = 2^6$	$7560 = 2^3 \cdot 3^3 \cdot 5 \cdot 7$
12	$72 = 2^3 \cdot 3^2$	$10080 = 2^5 \cdot 3^2 \cdot 5 \cdot 7$
10	$80 = 2^4 \cdot 5$	$15120 = 2^4 \cdot 3^3 \cdot 5 \cdot 7$
12	$84 = 2^2 \cdot 3 \cdot 7$	$20160 = 2^6 \cdot 3^2 \cdot 5 \cdot 7$
12	$90 = 2 \cdot 3^2 \cdot 5$	$25200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7$

$dd(N)$	$d(N)$	N
12	$96 = 2^5 \cdot 3$	$27720 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
9	$100 = 2^2 \cdot 5^2$	$45360 = 2^4 \cdot 3^4 \cdot 5 \cdot 7$
12	$108 = 2^2 \cdot 3^3$	$50400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7$
16	$120 = 2^3 \cdot 3 \cdot 5$	$*55440 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
8	$128 = 2^7$	$83160 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
15	$144 = 2^4 \cdot 3^2$	$110880 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
12	$160 = 2^5 \cdot 5$	$166320 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
16	$168 = 2^3 \cdot 3 \cdot 7$	$221760 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
18	$180 = 2^2 \cdot 3^2 \cdot 5$	$277200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$
14	$192 = 2^6 \cdot 3$	$332640 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
12	$200 = 2^3 \cdot 5^2$	$498960 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$
16	$216 = 2^3 \cdot 3^3$	$554400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$
12	$224 = 2^5 \cdot 7$	$665280 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
20	$240 = 2^4 \cdot 3 \cdot 5$	$\ast 720720 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
9	$256 = 2^8$	$1081080 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
18	$288 = 2^5 \cdot 3^2$	$\ast 1441440 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
14	$320 = 2^6 \cdot 5$	$2162160 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
20	$336 = 2^4 \cdot 3 \cdot 7$	$2882880 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
24	$360 = 2^3 \cdot 3^2 \cdot 5$	$3603600 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
16	$384 = 2^7 \cdot 3$	$*4324320 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
15	$400 = 2^4 \cdot 5^2$	$6486480 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
20	$432 = 2^4 \cdot 3^3$	$7207200 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
14	$448 = 2^6 \cdot 7$	$8648640 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
24	$480 = 2^5 \cdot 3 \cdot 5$	$10810800 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
24	$504 = 2^3 \cdot 3^2 \cdot 7$	$14414400 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
10	$512 = 2^9$	$17297280 = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
21	$576 = 2^6 \cdot 3^2$	$\ast 21621600 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
24	$600 = 2^3 \cdot 3 \cdot 5^2$	$32432400 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
16	$640 = 2^7 \cdot 5$	$36756720 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
24	$672 = 2^5 \cdot 3 \cdot 7$	$43243200 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
30	$720 = 2^4 \cdot 3^2 \cdot 5$	$61261200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
18	$768 = 2^8 \cdot 3$	$73513440 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
18	$800 = 2^5 \cdot 5^2$	$110270160 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
24	$864 = 2^5 \cdot 3^3$	$122522400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
16	$896 = 2^7 \cdot 7$	$147026880 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
28	$960 = 2^6 \cdot 3 \cdot 5$	$183783600 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
30	$1008 = 2^4 \cdot 3^2 \cdot 7$	$245044800 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
11	$1024 = 2^{10}$	$294053760 = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
24	$1152 = 2^7 \cdot 3^2$	$\ast 367567200 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
30	$1200 = 2^4 \cdot 3 \cdot 5^2$	$551350800 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$

$dd(N)$	$d(N)$	N		
18	$1280 = 2^8 \cdot 5$	698377680	$=$	$2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
28	$1344 = 2^6 \cdot 3 \cdot 7$	735134400	$=$	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
36	$1440 = 2^5 \cdot 3^2 \cdot 5$	1102701600	$=$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
20	$1536 = 2^9 \cdot 3$	1396755360	$=$	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
21	$1600 = 2^6 \cdot 5^2$	2095133040	$=$	$2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
40	$1680 = 2^4 \cdot 3 \cdot 5 \cdot 7$	2205403200	$=$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
28	$1728 = 2^6 \cdot 3^3$	2327925600	$=$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
18	$1792 = 2^8 \cdot 7$	2793510720	$=$	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
32	$1920 = 2^7 \cdot 3 \cdot 5$	3491888400	$=$	$2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$2016 = 2^5 \cdot 3^2 \cdot 7$	4655851200	$=$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
12	$2048 = 2^{11}$	5587021440	$=$	$2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
27	$2304 = 2^8 \cdot 3^2$	$*$6983776800	$=$	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$2400 = 2^5 \cdot 3 \cdot 5^2$	10475665200	$=$	$2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
32	$2688 = 2^7 \cdot 3 \cdot 7$	$*$13967553600	$=$	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
42	$2880 = 2^6 \cdot 3^2 \cdot 5$	20951330400	$=$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
22	$3072 = 2^{10} \cdot 3$	27935107200	$=$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
48	$3360 = 2^5 \cdot 3 \cdot 5 \cdot 7$	41902660800	$=$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
32	$3456 = 2^7 \cdot 3^3$	48886437600	$=$	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
20	$3584 = 2^9 \cdot 7$	64250746560	$=$	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
45	$3600 = 2^4 \cdot 3^2 \cdot 5^2$	73329656400	$=$	$2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$3840 = 2^8 \cdot 3 \cdot 5$	80313433200	$=$	$2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
42	$4032 = 2^6 \cdot 3^2 \cdot 7$	97772875200	$=$	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
13	$4096 = 2^{12}$	128501483120	$=$	$2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
48	$4320 = 2^5 \cdot 3^3 \cdot 5$	146659312800	$=$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
30	$4608 = 2^9 \cdot 3^2$	160626866400	$=$	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
42	$4800 = 2^6 \cdot 3 \cdot 5^2$	240940299600	$=$	$2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
60	$5040 = 7 \cdot 5 \cdot 3^2 \cdot 2^4$	293318625600	$=$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$5376 = 2^8 \cdot 3 \cdot 7$	*321253732800	$=$	$2^6 \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
48	$5760 = 2^7 \cdot 3^2 \cdot 5$	481880599200	$=$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
24	$6144 = 2^{11} \cdot 3$	642507465600	$=$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
56	$6720 = 2^6 \cdot 3 \cdot 5 \cdot 7$	963761198400	$=$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
36	$6912 = 2^8 \cdot 3^3$	1124388064800	$=$	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
22	$7168 = 2^{10} \cdot 7$	1606268664000	$=$	$2^6 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
54	$7200 = 2^5 \cdot 3^2 \cdot 5^2$	1686582097200	$=$	$2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
40	$7680 = 2^9 \cdot 3 \cdot 5$	1927522396800	$=$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
48	$8064 = 2^7 \cdot 3^2 \cdot 7$	$*$2248776129600	$=$	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
14	$8192 = 2^{13}$	3212537328000	$=$	$2^7 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
56	$8640 = 2^6 \cdot 3^3 \cdot 5$	3373164194400	$=$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
33	$9216 = 2^{10} \cdot 3^2$	4497552259200	$=$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
72	$10080 = 2^5 \cdot 3^2 \cdot 5 \cdot 7$	6746328388800	$=$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$

8. Now let us consider what must be the nature of N in order that N should be a highly composite number. In the first place it must be of the form

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \cdots p_1^{a_{p_1}},$$

where

$$a_2 \geq a_3 \geq a_5 \geq \cdots \geq a_{p_1} \geq 1. \tag{32}$$

This follows at once from the fact that

$$d(\varpi_2^{a_2} \varpi_3^{a_3} \varpi_5^{a_5} \cdots \varpi_{p_1}^{a_{p_1}}) = d(2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p_1^{a_{p_1}}),$$

for all prime values of $\varpi_2, \varpi_3, \varpi_5, \ldots, \varpi_{p_1}$.
It follows from the definition that, if N is highly composite and $N' < N$, then $d(N')$ must be less that $d(N)$. For example, $\frac{5}{6}N < N$, and so $d(\frac{5}{6}N) < d(N)$. Hence

$$\left(1 + \frac{1}{a_2}\right)\left(1 + \frac{1}{a_3}\right) > \left(1 + \frac{1}{1 + a_5}\right),$$

provided that N is a multiple of 3.
It is convenient to write

$$a_\lambda = 0 \quad (\lambda > p_1). \tag{33}$$

Thus if N is not a multiple of 5 then a_5 should be considered as 0.
Again, a_{p_1} must be less than or equal to 2 for all values of p_1. For let P_1 be the prime next above p_1. Then it can be shewn that $P_1 < p_1^2$ for all values of p_1.*
Now, if a_{p_1} is greater than 2, let

$$N' = \frac{NP_1}{P_1^2}.$$

Then N' is an integer less than N, and so $d(N') < d(N)$. Hence

$$(1 + a_{p_1}) > 2(a_{p_1} - 1),$$

or

$$3 > a_{p_1},$$

*It can be proved by elementary methods that, if $x \geq 1$, there is at least one prime p such that $x < p \leq 2x$. This result is known as Bertrand's Postulate: for a proof, see Landau, *Handbuch*, p. 89. It follows at once that $P_1 < p_1^2$, if $p_1 > 2$; and the inequality is obviously true when $p_1 = 2$. Some similar results used later in this and the next section may be proved in the same kind of way. It is for some purposes sufficient to know that there is always a prime p such that $x < p < 3x$, and the proof of this is easier than that of Bertrand's Postulate. These inequalities are enough, for example, to shew that

$$\log P_1 = \log p_1 + O(1).$$

which contradicts our hypothesis. Hence

$$a_{p_1} \leq 2, \tag{34}$$

for all values of p_1.

Now let $p_1'', p_1', p_1, P_1, P_1'$ be consecutive primes in ascending order. Then, if $p_1 \geq 5, a_{p_1''}$ must be less than or equal to 4. For, if this were not so, we could suppose that

$$N' = \frac{NP_1}{(p_1'')^3}.$$

But it can easily be shewn that, if $p_1 \geq 5$, then

$$(p_1'')^3 > P_1;$$

and so $N' < N$ and $d(N') < d(N)$. Hence

$$(1 + a_{p_1''}) > 2(a_{p_1''} - 2). \tag{35}$$

But since $a_{p_1''} \geq 5$, it is evident that

$$(1 + a_{p_1''}) \leq 2(a_{p_1''} - 2),$$

which contradicts (35); therefore, if $p_1 \geq 5$, then

$$a_{p_1''} \leq 4. \tag{36}$$

Now let

$$N' = \frac{Np_1''P_1}{p_1'p_1}.$$

It is easy to verify that, if $5 \leq p_1 \leq 19$, then

$$p_1'p_1 > p_1''P_1;$$

and so $N' < N$ and $d(N') < d(N)$. Hence

$$(1 + a_{p_1})(1 + a_{p_1'})(1 + a_{p_1''}) > 2a_{p_1}a_{p_1'}(2 + a_{p_1''}),$$

or

$$\left(1 + \frac{1}{a_{p_1}}\right)\left(1 + \frac{1}{a_{p_1'}}\right) > 2\left(1 + \frac{1}{1 + a_{p_1''}}\right),$$

But from (36) we know that $1 + a_{p_1''} \leq 5$. Hence

$$\left(1 + \frac{1}{a_{p_1}}\right)\left(1 + \frac{1}{a_{p_1'}}\right) > 2\tfrac{2}{5}. \tag{37}$$

From this it follows that $a_{p_1} = 1$. For, if $a_{p_1} \geq 2$, then

$$\left(1 + \frac{1}{a_{p_1}}\right)\left(1 + \frac{1}{a_{p_1'}}\right) \leq 2\tfrac{1}{4},$$

in virtue of (32). This contradicts (37). Hence, if $5 \leq p_1 \leq 19$, then

$$a_{p_1} = 1. \tag{38}$$

Next let

$$N' = NP_1P_1'/(p_1p_1'p_1'').$$

It can easily be shewn that, if $p_1 \geq 11$, then

$$P_1P_1' < p_1p_1'p_1'';$$

and so $N' < N$ and $d(N') < d(N)$. Hence

$$(1 + a_{p_1})(1 + a_{p_1'})(1 + a_{p_1''}) > 4a_{p_1}a_{p_1'}a_{p_1''},$$

or

$$\left(1 + \frac{1}{a_{p_1}}\right)\left(1 + \frac{1}{a_{p_1'}}\right)\left(1 + \frac{1}{a_{p_1''}}\right) > 4. \tag{39}$$

From this we infer that a_{p_1} must be 1. For, if $a_{p_1} \geq 2$, it follows from (32) that

$$\left(1 + \frac{1}{a_{p_1}}\right)\left(1 + \frac{1}{a_{p_1'}}\right)\left(1 + \frac{1}{a_{p_1''}}\right) \leq 3\tfrac{3}{8},$$

which contradicts (39). Hence we see that, if $p_1 \geq 11$, then

$$a_{p_1} = 1. \tag{40}$$

It follows from (38) and (40) that, if $p_1 \geq 5$, then

$$a_{p_1} = 1. \tag{41}$$

But if $p_1 = 2$ or 3, then from (34) it is clear that

$$a_{p_1} = 1 \quad \text{or} \quad 2. \tag{42}$$

It follows that $a_{p_1} = 1$ for all highly composite numbers, except for 2^2, and perhaps for certain numbers of the form $2^a \cdot 3^2$. In the latter case $a \geq 2$. It is easy to shew that, if $a \geq 3, 2^a.3^2$ cannot be highly composite. For if we suppose that

$$N' = 2^{a-1} \cdot 3 \cdot 5,$$

then it is evident that $N' < N$ and $d(N') < d(N)$, and so

$$3(1+a) > 4a,$$

or

$$a < 3.$$

Hence it is clear that a cannot have any other value except 2. Moreover we can see by actual trial that 2^2 and $2^2 \cdot 3^2$ are highly composite. *Hence*

$$a_{p_1} = 1 \tag{43}$$

for all highly composite values of N save 4 and 36, when

$$a_{p_1} = 2.$$

Hereafter when we use this result it is to be understood that 4 and 36 are exceptions.

9. It follows from (32) and (43) that N must be of the form

$$\begin{array}{rc} & 2\cdot 3\cdot 5\cdot 7\cdots p_1 \\ \times & 2\cdot 3\cdot 5\cdot 7\cdots p_2 \\ \times & 2\cdot 3\cdot 5\cdots p_3 \\ \times & \ldots, \end{array} \tag{44}$$

where $p_1 > p_2 \geq p_3 \geq p_4 \geq \cdots$ and the number of rows is a_2.
Let P_r be the prime next above p_r, so that

$$\log P_r = \log p_r + O(1), \tag{45}$$

in virtue of Bertrand's Postulate. Then it is evident that

$$a_{p_r} \geq r, \quad a_{P_r} \leq r - 1; \tag{46}$$

and so

$$a_{P_r} \leq a_{p_r} - 1. \tag{47}$$

It is to be understood that

$$a_{P_1} = 0, \tag{48}$$

in virtue of (33).
It is clear from the form of (44) that r can never exceed a_2, and that

$$p_{a_\lambda} = \lambda. \tag{49}$$

10. Now let

$$N' = \frac{N}{\nu}\lambda^{[\log\nu/\log\lambda]*},$$

where $\nu \leq p_1$ so that N' is an integer. Then it is evident that $N' < N$ and $d(N') < d(N)$, and so

$$(1+a_\nu)(a+a_\lambda) > a_\nu\left(1+a_\lambda+\left[\frac{\log\nu}{\log\lambda}\right]\right),$$

or

$$1+a_\lambda > a_\nu\left[\frac{\log\nu}{\log\lambda}\right]. \tag{50}$$

Since the right-hand side vanishes when $\nu > p_1$, we see that (50) is true for all values of λ and ν †.

Again let

$$N' = N_{\mu\lambda}^{-1-[\log\mu/\log\lambda]},$$

where $[\log\mu/\log\lambda] < a_\lambda$, so that N' is an integer. Then it is evident that $N' < N$ and $d(N') < d(N)$, and so

$$(1+a_\mu)(1+a_\lambda) > (2+a_\mu)\left(a_\lambda-\left[\frac{\log\mu}{\log\lambda}\right]\right). \tag{51}$$

Since the right-hand side is less than or equal to 0 when

$$a_\lambda \leq [\log\mu/\log\lambda],$$

we see that (51) is true for all values of λ and μ. From (51) it evidently follows that

$$(1+a_\lambda) < (2+a_\mu)\left[\frac{\log(\lambda\mu)}{\log\lambda}\right]. \tag{52}$$

From (50) and (52) it is clear that

$$a_\nu\left[\frac{\log\nu}{\log\lambda}\right] \leq a_\lambda \leq a_\mu + (2+a_\mu)\left[\frac{\log\mu}{\log\lambda}\right], \tag{53}$$

for all values of λ, μ and ν.

Now let us suppose that $\nu = p_1$ and $\mu = P_1$, so that $a_\nu = 1$ and $a_\mu = 0$. Then we see that

$$\left[\frac{\log p_1}{\log\lambda}\right] \leq a_\lambda \leq 2\left[\frac{\log P_1}{\log\lambda}\right], \tag{54}$$

for all values of λ. Thus, for example, we have

$$p_1 = 3, \quad 1 \leq a_2 \leq 4;$$

* $[x]$ denotes as usual the integral part of x.

† That is to say all prime values of λ and ν, since λ in a_λ is be definition prime.

$$p_1 = 5, \quad 2 \leq a_2 \leq 4;$$
$$p_1 = 7, \quad 2 \leq a_2 \leq 6;$$
$$p_1 = 11, \quad 3 \leq a_2 \leq 6;$$

and so on. It follows from (54) that, if $\lambda \leq p_1$, then

$$a_\lambda \log \lambda = O(\log p_1), \quad a_\lambda \log \lambda \neq o(\log p_1). \tag{55}$$

11. Again let

$$N' = N_\lambda^{[\sqrt{\{(1+a_\lambda+a_\mu)\log\mu/\log\lambda\}}]} \mu^{-1-[\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}}]},$$

and let us assume for the moment that

$$a_\mu > \sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}},$$

in order that N' may be an integer. Then $N' < N$ and $d(N') < d(N)$, and so

$$\begin{aligned}(1+a_\lambda)(1+a_\mu) > \quad & \{1 + a_\lambda + [\sqrt{\{(1+a_\lambda+a_\mu)\log\mu/\log\lambda\}}]\} \\ & \times\{a_\mu - [\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}}]\} \\ > \quad & \{a_\lambda + \sqrt{\{(1+a_\lambda+a_\mu)\log\mu/\log\lambda\}}\} \\ & \times\{a_\mu - \sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}}\}. \end{aligned} \tag{56}$$

It is evident that the right-hand side of (56) becomes negative when

$$a_\mu < \sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}},$$

while the left-hand side remains positive, and so the result is still true. Hence

$$a_\mu \log\mu - a_\lambda \log\lambda < 2\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda\log\mu\}}, \tag{57}$$

for all values of λ and μ. Interchanging λ and μ in (57), we obtain

$$a_\lambda \log\lambda - a_\mu \log\mu < 2\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda\log\mu\}}. \tag{58}$$

From (57) and (58) it evidently follows that

$$|a_\lambda \log\lambda - a_\mu \log\mu| < 2\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda\log\mu\}}, \tag{59}$$

for all values of λ and μ. It follows from this and (55) that, if λ and μ are neither greater than p_1, then

$$a_\lambda \log\lambda - a_\mu \log\mu = O\sqrt{\{\log p_1 \log(\lambda\mu)\}}, \tag{60}$$

and so that, if $\log\lambda = o(\log p_1)$, then

$$a_2 \log 2 \sim a_3 \log 3 \sim a_5 \log 5 \sim \cdots \sim a_\lambda \log \lambda. \tag{61}$$

12. It can easily be shewn by elementary algebra that, if x, y, m and n are not negative, and if

$$|x - y| < 2\sqrt{(mx + ny + mn)},$$

then

$$\left.\begin{array}{l} |\sqrt{(x+n)} - \sqrt{(y+m)}| < \sqrt{(m+n)}; \\ |\sqrt{(x+n)} - \sqrt{(m+n)}| < \sqrt{(y+m)}. \end{array}\right\} \tag{62}$$

From (62) and (59) it follows that

$$|\sqrt{\{(1+a_\lambda)\log\lambda\}} - \sqrt{\{(1+a_\mu)\log\mu\}}| < \sqrt{\{\log(\lambda\mu)\}}, \tag{63}$$

and

$$|\sqrt{\{(1+a_\lambda)\log\lambda\}} - \sqrt{\{\log(\lambda\mu)\}}| < \sqrt{\{(1+a_\mu)\log\mu\}}, \tag{64}$$

for all values of λ and μ. If, in particular, we put $\mu = 2$ in (63), we obtain

$$\begin{aligned} \sqrt{\{(1+a_2)\log 2\}} - \sqrt{\{\log(2\lambda)\}} &< \sqrt{\{(1+a_\lambda)\log\lambda\}} \\ &< \sqrt{\{(1+a_2)\log 2\}} + \sqrt{\{\log(2\lambda)\}}, \end{aligned} \tag{65}$$

for all values of λ. Again, from (63), we have

$$(1+a_\lambda)\log\lambda < (\sqrt{\{(1+a_\nu)\log\nu\}} + \sqrt{\{\log(\lambda\nu)\}})^2,$$

or

$$a_\lambda \log\lambda < (1+a_\nu)\log\nu + \log\nu + 2\sqrt{\{(1+a_\nu)\log\nu\log(\lambda\nu)\}}. \tag{66}$$

Now let us suppose that $\lambda \leq \mu$. Then, from (66), it follows that

$$\begin{aligned} a_\lambda \log\lambda + \log\mu &< (1+a_\nu)\log\nu + \log(\mu\nu) + 2\sqrt{\{(1+a_\nu)\log\nu\log(\lambda\nu)\}} \\ &\leq (1+a_\nu)\log\nu + \log(\mu\nu) + 2\sqrt{\{(1+a_\nu)\log\nu\log(\mu\nu)\}} \\ &= \{\sqrt{\{(1+a_\nu)\log\nu\}} + \sqrt{\log(\mu\nu)}\}^2, \end{aligned} \tag{67}$$

with the condition that $\lambda \leq \mu$. Similarly we can shew that

$$a_\lambda \log\lambda + \log\mu > \{\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)}\}^2, \tag{67'}$$

with the condition that $\lambda \le \mu$.

13. Now let

$$N' = \frac{N}{\lambda} 2^{[\log\lambda/\{\pi(\mu)\log 2\}]} 3^{[\log\lambda/\{\pi(\mu)\log 3\}]} \cdots \mu^{[\log\lambda/\{\pi(\mu)\log\mu\}]},$$

where $\pi(\mu)\log\mu < \log\lambda \le \log p_1$. Then it is evident that N' is an integer less than N, and so $d(N') < d(N)$. Hence

$$\begin{aligned}&\left(1+\frac{1}{a_\lambda}\right)(1+a_2)(1+a_3)(1+a_5)\cdots(1+a_\mu)\\ &\qquad > \left\{a_2 + \frac{\log\lambda}{\pi(\mu)\log 2}\right\}\left\{a_3 + \frac{\log\lambda}{\pi(\mu)\log 3}\right\}\cdots\left\{a_\mu + \frac{\log\lambda}{\mu(\mu)\log\mu}\right\};\end{aligned}$$

that is

$$\begin{aligned}&\left\{a_2\log 2 + \frac{\log\lambda}{\pi(\mu)}\right\}\left\{a_3\log 3 + \frac{\log\lambda}{\pi(\mu)}\right\}\cdots\left\{a_\mu\log\mu + \frac{\log\lambda}{\pi(\mu)}\right\}\\ &\qquad < \left(1+\frac{1}{a_\lambda}\right)(a_2\log 2+\log 2)(a_3\log 3+\log 3)\cdots(a_\mu\log\mu+\log\mu)\\ &\qquad \le \left(1+\frac{1}{a_\lambda}\right)(a_2\log 2+\log\mu)(a_3\log 3+\log\mu)\cdots(a_\mu\log\mu+\log\mu).\end{aligned}$$

In other words

$$\begin{aligned}&\left(1+\frac{1}{a_\lambda}\right)\\ &\quad > \left\{1+\frac{\frac{\log\lambda}{\pi(\mu)}-\log\mu}{a_2\log 2+\log\mu}\right\}\left\{1+\frac{\frac{\log\lambda}{\pi(\mu)}-\log\mu}{a_3\log 3+\log\mu}\right\}\cdots\left\{1+\frac{\frac{\log\lambda}{\pi(\mu)}-\log\mu}{a_\mu\log\mu+\log\mu}\right\}\\ &\quad > \left\{1+\frac{\frac{\log\lambda}{\pi(\mu)}-\log\mu}{\{\sqrt{\{(1+a_\nu)\log\nu\}}+\sqrt{\log(\mu\nu)}\}^2}\right\}^{\pi(\mu)},\end{aligned} \tag{68}$$

w here ν is any prime, in virtue of (67). From (68) it follows that

$$\sqrt{\{(1+a_\nu)\log\nu\}} + \sqrt{\log(\mu\nu)} > \sqrt{\left\{\frac{\frac{\log\lambda}{\pi(\mu)}-\log\mu}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)}-1}\right\}}, \tag{69}$$

provided that $\pi(\mu)\log\mu < \log\lambda \le \log p_1$.

14. Again let

$$N' = N\lambda 2^{-1-[\log\lambda/\{\pi(\mu)\log 2\}]} 3^{-1-[\log\lambda/\{\pi(\mu)\log 3\}]} \cdots \mu^{-1-[\log\lambda/\{\pi(\mu)\log\mu\}]},$$

where $\mu \le p_1$ and $\lambda > \mu$. Let us assume for the moment that

$$a_\kappa \log\kappa > \frac{\log\lambda}{\pi(\mu)},$$

for all values of κ less than or equal to μ, so that N' is an integer. Then, by arguments similar to those of the previous section, we can shew that

$$\frac{1+a_\lambda}{2+a_\lambda} > \left\{1 - \frac{\frac{\log\lambda}{\pi(\mu)} + \log\mu}{\{\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)}\}^2}\right\}^{\pi(\mu)}. \tag{70}$$

From this it follows that

$$|\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)}| < \sqrt{\left\{\frac{\frac{\log\lambda}{\pi(\mu)} + \log\mu}{1 - \left(\frac{1+a_\lambda}{2+a_\lambda}\right)^{1/\pi(\mu)}}\right\}}, \tag{71}$$

provided that $\mu \le p_1$ and $\mu < \lambda$. The condition that

$$a_\kappa \log\kappa > \{\log\lambda/\pi(\mu)\}$$

is unnecessary because we know from (67′) that

$$|\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)}| < \sqrt{(a_\kappa\log\kappa + \log\mu)} \le \sqrt{\left\{\frac{\log\lambda}{\pi(\mu)} + \log\mu\right\}}, \tag{72}$$

when

$$a_\kappa \log\kappa \le \{\log\lambda/\pi(\mu)\},$$

and the last term in (72) is evidently less than the right-hand side of (71).

15. We shall consider in this and the following sections some important deductions from the preceding formulæ. Putting $\nu = 2$ in (69) and (71), we obtain

$$\sqrt{\{(1+a_2)\log 2\}} > \sqrt{\left\{\frac{\frac{\log\lambda}{\pi(\mu)} - \log\mu}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1}\right\}} - \sqrt{\log(2\mu)}, \tag{73}$$

provided that $\pi(\mu)\log\mu < \log\lambda \le \log p_1$, and

$$\sqrt{\{(1+a_2)\log 2\}} < \sqrt{\left\{\frac{\frac{\log\lambda}{\pi(\mu)} + \log\mu}{1 - \left(\frac{1+a_\lambda}{2+a_\lambda}\right)^{1/\pi(\mu)}}\right\}} + \sqrt{\log(2\mu)}, \tag{74}$$

provided that $\mu \le p_1$, and $\mu < \lambda$. Now supposing that $\lambda = p_1$ in (73), and $\lambda = P_1$ in (74), we obtain

$$\sqrt{\{(1+a_2)\log 2\}} > \sqrt{\left\{\frac{\frac{\log p_1}{\pi(\mu)} - \log\mu}{2^{1/\pi(\mu)} - 1}\right\}} - \sqrt{\log(2\mu)}, \tag{75}$$

provided that $\pi(\mu)\log\mu < \log p_1$, and

$$\sqrt{\{(1+a_2)\log 2\}} < \sqrt{\left\{\frac{\frac{\log P_1}{\pi(\mu)} + \log\mu}{1 - 2^{-1/\pi(\mu)}}\right\}} + \sqrt{\log(2\mu)}, \tag{76}$$

provided that $\mu \le p_1$. In (75) and (76) μ can be so chosen as to obtain the best possible inequality for a_2. If p_1 is too small, we may abandon this result in favour of

$$\left[\frac{\log p_1}{\log 2}\right] \le a_2 \le 2\left[\frac{\log P_1}{\log 2}\right], \tag{77}$$

which is obtained from (54) by putting $\lambda = 2$.
After having obtained in this way what information we can about a_2, we may use (73) and (74) to obtain information about a_λ. Here also we have to choose μ so as to obtain the best possible inequality for a_λ. But if λ is too small we may, instead of this, use

$$\begin{aligned}\sqrt{\{(1+a_2)\log 2\}} - \sqrt{\log(2\lambda)} \quad &< \quad \sqrt{\{(1+a_\lambda)log\lambda\}} \\ &< \quad \sqrt{\{(1+a_2)\log 2\}} + \sqrt{\log(2\lambda)},\end{aligned} \tag{78}$$

which is obtained by putting $\mu = 2$ in (63).

16. Now let us consider the order of a_2. From (73) it is evident that, if $\pi(\mu)\log\mu < \log\lambda \le \log p_1$, then

$$(1+a_2)\log 2 + \log(2\mu) + 2\sqrt{\{(1+a_2)\log 2\log(2\mu)\}} > \frac{\frac{\log\lambda}{\pi(\mu)} - \log\mu}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1} \tag{79}$$

But we know that for positive values of x,

$$\frac{1}{e^x - 1} = \frac{1}{x} + O(1), \quad \frac{1}{e^x - 1} = O\left(\frac{1}{x}\right).$$

Hence

$$\frac{\log\lambda}{\pi(\mu)}\frac{1}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1} \quad = \quad \frac{\log\lambda}{\pi(\mu)}\left\{\frac{\pi(\mu)}{\log\left(1+\frac{1}{a_\lambda}\right)} + O(1)\right\}$$

$$= \frac{\log\lambda}{\log\left(1+\frac{1}{a_\lambda}\right)} + O\left\{\frac{\log\lambda}{\pi(\mu)}\right\};$$

and

$$\frac{\log\mu}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)}-1} = O\left\{\frac{\pi(\mu)\log\mu}{\log\left(1+\frac{1}{a_\lambda}\right)}\right\} = O(\mu a_\lambda).$$

Again from (55) we know that $a_2 = O(\log p_1)$. Hence (79) may be written as

$$\begin{aligned} a_2\log 2 &+ O\sqrt{(\log p_1\log\mu)+O(\log\mu)} \\ &\geq \frac{\log\lambda}{\log\left(1+\frac{1}{a_\lambda}\right)} + O\left\{\frac{\log\lambda}{\pi(\mu)}\right\} + O(\mu a_\lambda). \end{aligned} \tag{80}$$

But

$$\log\mu = O(\mu a_\lambda),$$

$$\mu a_\lambda = \frac{\mu}{\log\lambda}\cdot a_\lambda\log\lambda = O\left(\frac{\mu\log p_1}{\log\lambda}\right),$$

$$\frac{\log\lambda}{\pi(\mu)} = O\left\{\frac{\log\lambda\log\mu}{\mu}\right\}.$$

Again

$$\frac{\log\lambda\log\mu}{\mu} + \frac{\mu\log p_1}{\log\lambda} > 2\sqrt{(\log p_1\log\mu)};$$

and so

$$\sqrt{(\log p_1\log\mu)} = O\left(\frac{\log\lambda\log\mu}{\mu} + \frac{\mu\log p_1}{\log\lambda}\right).$$

Hence (80) may be replaced by

$$a_2\log 2 \geq \frac{\log\lambda}{\log\left(1+\frac{1}{a_\lambda}\right)} + O\left(\frac{\log\lambda\log\mu}{\mu} + \frac{\mu\log p_1}{\log\lambda}\right), \tag{81}$$

provided that $\pi(\mu)\log\mu < \log\lambda \leq \log p_1$. Similarly, from (74), we can shew that

$$a_2\log 2 \leq \frac{\log\lambda}{\log\left(1+\frac{1}{1+a_\lambda}\right)} + O\left(\frac{\log\lambda\log\mu}{\mu} + \frac{\mu\log p_1}{\log\lambda}\right), \tag{82}$$

provided that $\mu \leq p_1$ and $\mu < \lambda$. Now supposing that $\lambda = p_1$ in (81), and $\lambda = P_1$ in (82), and also that

$$\mu = O\sqrt{(\log p_1\log\log p_1)}, \mu \neq o\sqrt{(\log p_1\log\log p_1)},^*$$

* $f \neq o(\phi)$ is to be understood as meaning that $|f| > K\phi$, where K is a constant, and $f \neq O(\phi)$ as meaning that $|f|/\phi \to \infty$. They are *not* the mere nagations of $f = o(\phi)$ and $f = O(\phi)$.

we obtain

$$\left.\begin{aligned} a_2 \log 2 &\geq \tfrac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}, \\ a_2 \log 2 &\leq \tfrac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}. \end{aligned}\right\} \tag{83}$$

From (83) it evidently follows that

$$a_2 \log 2 = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}. \tag{84}$$

And it follows from this and (60) that if $\lambda \leq p_1$ then

$$a_\lambda \log \lambda = \frac{\log p_1}{\log 2} + O\{\sqrt{\log p_1 \log \lambda)} + \sqrt{(\log p_1 \log\log p_1)}\}. \tag{85}$$

Hence, if $\log \lambda = o(\log p_1)$, we have

$$a_2 \log 2 \sim a_3 \log 3 \sim a_5 \log 5 \sim \cdots \sim a_\lambda \log \lambda \sim \frac{\log p_1}{\log 2}. \tag{86}$$

17. The relations (86) give us information about the order of a_λ when λ is sufficiently small compared to p_1, in fact, when λ is of the form p_1^ϵ, where $\epsilon \to 0$. Such values of λ constitute but a small part of its total range of variation, and it is clear that further formulæ must be proved before we can gain an adequate idea of the general behaviour of a_λ. From (81), (82) and (84) it follows that

$$\left.\begin{aligned} \frac{\log \lambda}{\log\left(1 + \frac{1}{a_\lambda}\right)} &\leq \frac{\log p_1}{\log 2} + O\left\{\frac{\log \lambda \log \mu}{\mu} + \frac{\mu \log p_1}{\log \lambda} + \sqrt{(\log p_1 \log\log p_1)}\right\}, \\ \frac{\log \lambda}{\log\left(1 + \frac{1}{1+a_\lambda}\right)} &\geq \frac{\log p_1}{\log 2} + O\left\{\frac{\log \lambda \log \mu}{\mu} + \frac{\mu \log p_1}{\log \lambda} + \sqrt{(\log p_1 \log\log p_1)}\right\}, \end{aligned}\right\} \tag{87}$$

provided that $\pi(\mu) \log \mu \log \lambda \leq \log p_1$. From this we can easily shew that if

$$\pi(\mu) \log \mu \log \lambda \leq \log p_1$$

then

$$\left.\begin{aligned} a_\lambda &\leq (2^{\log \lambda / \log p_1} - 1)^{-1} + O\left\{\frac{\log \mu}{\mu} + \frac{\mu \log p_1}{(\log \lambda)^2} + \frac{\sqrt{(\log p_1 \log\log p_1)}}{\log \lambda}\right\}, \\ a_\lambda &\geq (2^{\log \lambda / \log p_1} - 1)^{-1} - 1 + O\left\{\frac{\log \mu}{\mu} + \frac{\mu \log p_1}{(\log \lambda)^2} + \frac{\sqrt{(\log p_1 \log\log p_1)}}{\log \lambda}\right\}. \end{aligned}\right\} \tag{88}$$

Now let us suppose that

$$\log\lambda \neq o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)}.$$

Then we can choose μ so that

$$\mu = O\left\{\log\lambda\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\},$$

$$\mu \neq o\left\{\log\lambda\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\}.$$

Now it is clear that $\log\mu = O(\log\log p_1)$, and so

$$\frac{\log\mu}{\mu} = O\left(\frac{\log\log p_1}{\mu}\right) = O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\};$$

and

$$\frac{\mu\log p_1}{(\log\lambda)^2} = O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}.$$

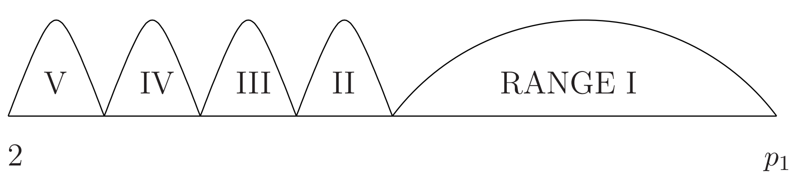
From this and (88) it follows that, if

$$\log\lambda \neq o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)},$$

then

$$\left.\begin{aligned} a_\lambda &\leq (2^{\log\lambda/\log p_1} - 1)^{-1} + O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}, \\ a_\lambda &\geq (2^{\log\lambda/\log p_1} - 1)^{-1} + O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}. \end{aligned}\right\} \tag{89}$$

Now we shall divide that primes from 2 to p_1 into five ranges thus



$$(lp_1)^\kappa \quad e\{\kappa(lp)^{\frac{1}{3}}\}e\left\{\kappa\left(\frac{lp}{l_2p}\right)^{\frac{1}{2}}\right\}e\{\kappa(lpl_2p)^{\frac{1}{2}}\}$$

We shall use the inequalities (89) to specify the behavior of a_λ in ranges I and II, and the formula (85) in ranges IV and V. Range III we shall deal with differently, by a different choice of μ in the inequalities (88). We can easily see that each result in the following sections gives the most information in its particular range.

18. Range I:

$$\log\lambda \neq O\sqrt{(\log p_1 \log\log p_1)}.^*$$

Let

$$\Lambda = [(2^{\log\lambda/\log p_1} - 1)^{-1}],$$

and let

$$(2^{\log\lambda/\log p_1} - 1)^{-1} + \epsilon_\lambda,$$

where $-\frac{1}{2} < \epsilon_\lambda < \frac{1}{2}$, be an integer, so that

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = \Lambda + 1 - \epsilon_\lambda \tag{90}$$

when $\epsilon_\lambda > 0$, and

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = \Lambda - \epsilon_\lambda \tag{91}$$

when $\epsilon_\lambda < 0$. By our supposition we have

$$\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda} = o(1). \tag{92}$$

First let us consider the case in which

$$\epsilon_\lambda \neq O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\},$$

so that

$$\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda} = o(\epsilon_\lambda). \tag{93}$$

It follows from (89), (90), and (93) that, if $\epsilon_\lambda > 0$, then

$$\left.\begin{aligned} a_\lambda &\leq \Lambda + 1 - \epsilon_\lambda + o(\epsilon_\lambda), \\ a_\lambda &\geq \Lambda - \epsilon_\lambda + o(\epsilon_\lambda). \end{aligned}\right\} \tag{94}$$

---

*We can with a little trouble replace all equations of the type $f = O(\phi)$ which occur by inequalities of the type $|f| < K\phi$, with definite numerical constants. This would enable us to extend all the different ranges a little. For example, an equation true for

$$\log\lambda \neq O\sqrt{(\log p_1)}$$

would be replaced by an inequality true for $\log\lambda > K\sqrt{(\log p_1)}$, where K is definite constant, and similarly $\log\lambda = o\sqrt{(\log p_1)}$ would be replaced by $\log\lambda < k\sqrt{(\log p_1)}$.

Since $0 < \epsilon_\lambda < \frac{1}{2}$, and a_λ and Λ are integers, it follows from (94) that

$$a_\lambda \leq \Lambda, \quad a_\lambda > \Lambda - 1. \tag{95}$$

Hence

$$a_\lambda = \Lambda. \tag{96}$$

Similarly from (89) (91) and (93) we see that, if $\epsilon_\lambda < 0$, then

$$\left.\begin{aligned} a_\lambda &\leq \Lambda - \epsilon_\lambda + o(\epsilon_\lambda), \\ a_\lambda &\geq \Lambda - 1 - \epsilon_\lambda + o(\epsilon_\lambda). \end{aligned}\right\} \tag{97}$$

Since $-\frac{1}{2} < \epsilon_\lambda < 0$, it follows from (97) that the inequalities (95), and therefore the equation (96), still hold. Hence (96) holds whenever

$$\epsilon_\lambda \neq O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log \lambda}\right\}. \tag{98}$$

In particular it holds whenever

$$\epsilon_\lambda \neq 0(1), \tag{99}$$

Now let us consider the case in which

$$\epsilon_\lambda = 0\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log \lambda}\right\}, \tag{100}$$

so that $\epsilon_\lambda = o(1)$, in virtue of (92). It follows from this and (89) and (90) that, if $\epsilon_\lambda > 0$, then

$$\left.\begin{aligned} a_\lambda &\leq \Lambda + 1 + o(1), \\ a_\lambda &\geq \Lambda + o(1). \end{aligned}\right\} \tag{101}$$

Hence

$$a_\lambda \leq \Lambda + 1, \quad a_\lambda \geq \Lambda;$$

and so

$$a_\lambda = \Lambda \ \text{ or } \ \Lambda + 1. \tag{102}$$

Similarly from (89), (91), and (100), we see that, if $\epsilon_\lambda < 0$, then

$$\left.\begin{aligned} a_\lambda &\leq \Lambda + o(1), \\ a_\lambda &\geq \Lambda - 1 + o(1). \end{aligned}\right\} \tag{103}$$

Hence

$$a_\lambda \leq \Lambda, \quad a_\lambda \geq \Lambda - 1;$$

and so

$$a_\lambda = \Lambda \quad \text{or} \quad \Lambda - 1. \tag{104}$$

For example, let us suppose that it is required to find a_λ when $\lambda \sim p_1^{\frac{1}{8}}$. We have

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = (2^{1/8} - 1)^{-1} + o(1) = 11.048\cdots + o(1).$$

It is evident that $\Lambda = 11$ and $\epsilon_\lambda \neq o(1)$. Hence $a_\lambda = 11$.

19. The results in the previous section may be rewritten with slight modifications, in order that the transition of a_λ from one value to another may be more clearly expressed. Let

$$\lambda = p_1^{\frac{\log(1+1/x)}{\log 2}}, \tag{105}$$

and let $x + \epsilon_x$, where $-\frac{1}{2} < \epsilon_x < \frac{1}{2}$, be an integer. Then the range of x which we are now considering is

$$x = o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)}, \tag{106}$$

and the results of the previous section may be stated as follows. If

$$\epsilon_x \neq O\left\{x\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\}, \tag{107}$$

then

$$a_\lambda = [x]. \tag{108}$$

As a particular case of this we have

$$a_\lambda = [x],$$

when $\epsilon_x \neq o(1)$. But if

$$\epsilon_x = O\left\{x\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\}, \tag{109}$$

then when $\epsilon_x > 0$

$$a_\lambda = [x] \quad \text{or} \quad [x+1]; \tag{110}$$

and when $\epsilon_x < 0$

$$a_\lambda = [x] \quad \text{or} \quad [x-1]. \tag{110$'$}$$

20. Range II:

$$\left.\begin{aligned} &\log\lambda = O\sqrt{(\log p_1 \log\log p_1)}, \\ &\log\lambda \neq o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)}. \end{aligned}\right\}$$

From (89) it follows that

$$a_\lambda = (2^{\log\lambda/\log p_1} - 1)^{-1} + O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}. \tag{111}$$

But

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = \frac{\log p_1}{\log 2 \log\lambda} + O(1).$$

Hence

$$a_\lambda \log\lambda = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}. \tag{112}$$

As an example we may suppose that

$$\lambda \sim e^{\sqrt{(\log p_1)}}.$$

Then from (112) it follows that

$$a_\lambda = \frac{\sqrt{(\log p_1)}}{\log 2} + O\sqrt{(\log\log p_1)}.$$

21. Range III:

$$\left.\begin{aligned} &\log\lambda = O\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)}, \\ &\log\lambda \neq o(\log p_1)^{\frac{1}{3}}. \end{aligned}\right\}$$

Let us suppose that $\mu = O(1)$ in (88). Then we see that

$$a_\lambda = \frac{\log p_1}{\log 2 \log\lambda} + O(1) + O\left\{\frac{\log\mu}{\mu} + \frac{\mu\log p_1}{(\log\lambda)^2} + \frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}, \tag{113}$$

or

$$a_\lambda \log\lambda = \frac{\log p_1}{\log 2} + O\left\{\frac{\log\mu\log\lambda}{\mu} + \frac{\mu\log p_1}{\log\lambda} + \sqrt{(\log p_1 \log\log p_1)}\right\}. \tag{114}$$

Now

$$\frac{\log\mu\log\lambda}{\mu} = O(\log\lambda) = o\left(\frac{\log p_1}{\log\lambda}\right),$$

$$\frac{\mu\log p_1}{\log\lambda} = O\left(\frac{\log p_1}{\log\lambda}\right),$$

$$\sqrt{(\log p_1 \log\log p_1)} = O\left(\frac{\log p_1}{\log\lambda}\right).$$

Hence

$$a_\lambda \log\lambda = \frac{\log p_1}{\log 2} + O\left(\frac{\log p_1}{\log\lambda}\right). \tag{115}$$

For example, when

$$\lambda \sim e^{(\log p_1)^{\frac{3}{8}}},$$

we have

$$a_\lambda = \frac{(\log p_1)^{\frac{5}{8}}}{\log 2} + O(\log p_1)^{\frac{1}{4}}.$$

22. Range IV:

$$\left.\begin{aligned} &\log\lambda = O(\log p_1)^{\frac{1}{3}}, \\ &\log\lambda \neq o(\log\log p_1). \end{aligned}\right\}$$

In this case it follows from (85) that

$$a_\lambda \log\lambda = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\lambda)}. \tag{116}$$

As an example in this range, when we suppose that

$$\lambda \sim e^{(\log p_1)^{\frac{1}{4}}},$$

we obtain from (116)

$$a_\lambda = \frac{(\log p_1)^{\frac{3}{4}}}{\log 2} + O(\log p_1)^{\frac{3}{8}}.$$

23. Range V: $\log\lambda = O(\log\log p_1)$.

From (85) it follows that

$$a_\lambda \log \lambda = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}. \tag{117}$$

For example, we may suppose that

$$\lambda \sim e^{\sqrt{(\log\log p_1)}}.$$

Then

$$a_\lambda = \frac{\log p_1}{\log 2\sqrt{(\log\log p_1)}} + O\sqrt{(\log p_1)}.$$

24. Let λ' be the prime next below λ, so that $\lambda' \leq \lambda - 1$. Then it follows from (63) that

$$\sqrt{\{(1+a_{\lambda'})\log \lambda'\}} - \sqrt{\{(1+a_\lambda)\log \lambda\}} > -\sqrt{\log(\lambda\lambda')}. \tag{118}$$

Hence

$$\sqrt{\{(1+a_{\lambda'}\log(\lambda-1)\}} - \sqrt{\{(1+a_\lambda)\log \lambda\}} > -\sqrt{\{2\log \lambda\}}. \tag{119}$$

But

$$\log(\lambda - 1) < \log \lambda - \frac{1}{\lambda} < \log \lambda \left(1 - \frac{1}{2\lambda \log \lambda}\right)^2;$$

and so (119) may be replaced by

$$\sqrt{(1+a_{\lambda'})} - \sqrt{(1+a_\lambda)} > \frac{\sqrt{(1+a_{\lambda'})}}{2\lambda \log \lambda} - \sqrt{2}. \tag{120}$$

But from (54) we know that

$$1 + a_{\lambda'} \geq 1 + \left[\frac{\log p_1}{\log \lambda'}\right] > \frac{\log p_1}{\log \lambda'} > \frac{\log p_1}{\log \lambda}.$$

From this and (120) it follows that

$$\sqrt{(1+a_{\lambda'})} - \sqrt{(1+a_\lambda)} > \frac{\sqrt{(\log p_1)}}{2\lambda(\log \lambda)^{\frac{3}{2}}} - \sqrt{2}. \tag{121}$$

Now let us suppose that $\lambda^2(\log \lambda)^3 < \frac{1}{8}\log p_1$. Then, from (121), we have

$$\sqrt{(1+a_{\lambda'})} - \sqrt{(1+a_\lambda)} > 0,$$

or

$$a_{\lambda'} > a_\lambda. \tag{122}$$

From (122) it follows that, if $\lambda^2(\log\lambda)^3 < \frac{1}{8}\log p_1$, then

$$a_2 > a_3 > a_5 > a_7 > \cdots > a_\lambda. \tag{123}$$

In other words, *in a large highly composite number*

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \cdots p_1,$$

the indices comparatively near the beginning form a decreasing sequence in the strict sense which forbids equality. Later on groups of equal indices will in general occur.

To sum up, we have obtained fairly accurate information about a_λ for all possible values of λ. The range I is by far the most extensive, and throughout this range a_λ is known with an error never exceeding 1. The formulæ (86) hold throughout a range which includes all the remaining ranges II - V, and a considerable part of I as well, while we have obtained more precise formulæ for each individual range II-V.

25. Now let us consider the nature of p_r. It is evident that r cannot exceed a_2; i.e., r cannot exceed

$$\frac{\log p_1}{(\log 2)^2} + O\sqrt{(\log p_1 \log\log p_1)}. \tag{124}$$

From (55) it evidently follows that

$$\left.\begin{aligned} a_{p_r}\log p_r &= O(\log p_1), \\ a_{p_r}\log p_r &\neq o(\log p_1); \end{aligned}\right\} \tag{125}$$

$$\left.\begin{aligned} (1+a_{P_r})\log p_r &= O(\log p_1), \\ (1+a_{P_r})\log p_r &\neq o(\log p_1). \end{aligned}\right\} \tag{126}$$

But from (46) we know that

$$\left.\begin{aligned} &a_{p_r}\log p_r \geq r\log p_r, \\ &(1+a_{P_r})\log p_r \leq r\log p_r. \end{aligned}\right\} \tag{127}$$

From (125) - (127) it follows that

$$\left.\begin{aligned} r\log p_r &= O(\log p_1), \\ r\log p_r &\neq o(\log p_1); \end{aligned}\right\} \tag{128}$$

and

$$\left.\begin{aligned} a_{p_r} &= O(r), \\ a_{p_r} &\neq o(r). \end{aligned}\right\} \tag{129}$$

26. Supposing that $\lambda = p_r$ in (81) and $\lambda = P_r$ in (82), and remembering (128), we see that, if $r\mu = o(\log p_1)$, then

$$\log\left(1+\frac{1}{a_{p_r}}\right) \geq \frac{\log p_r}{a_2 \log 2}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}, \tag{130}$$

and

$$\log\left(1+\frac{1}{a+a_{P_r}}\right) \leq \frac{\log P_r}{a_2 \log 2}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}. \tag{131}$$

But, from (47), we have

$$\log\left(1+\frac{1}{a_{p_r}}\right) \leq \log\left(1+\frac{1}{1+a_{P_r}}\right).$$

Also we know that

$$\log P_r = \log p_r + O(1) = \log p_r\left\{1+O\left(\frac{1}{\log p_r}\right)\right\} = \log p_r\left\{1+O\left(\frac{r}{\log p_1}\right)\right\}.$$

Hence (131) may be replaced by

$$\log\left(1+\frac{1}{a_{p_r}}\right) \leq \frac{\log p_r}{a_2 \log 2}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}. \tag{132}$$

From (130) and (132) it is evident that

$$\log\left(1+\frac{1}{a_{p_r}}\right) = \frac{\log p_r}{a_1 \log 2}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}. \tag{133}$$

In a similar manner

$$\log\left(1+\frac{1}{1+a_{P_r}}\right) = \frac{\log p_r}{a_2 \log 2}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}. \tag{134}$$

Now supposing that

$$\left.\begin{aligned} r\mu &= o(\log p_1), \\ r\mu &\neq O(\log\mu), \end{aligned}\right\} \tag{135}$$

and dividing (134) by (133), we have

$$\frac{\log\left(1+\frac{1}{1+a_{P_r}}\right)}{\log\left(1+\frac{1}{a_{p_r}}\right)} = 1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right),$$

or

$$1+\frac{1}{1+a_{P_r}} = 1+\frac{1}{a_{p_r}}+O\left\{\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)/a_{p_r}\right\},$$

that is

$$\frac{1}{1+a_{P_r}} = \frac{1}{a_{p_r}}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}.$$

Hence

$$a_{p_r} = a_{P_r}+1+O\left(\frac{\log\mu}{\mu}+\frac{r^2\mu}{\log p_1}\right), \tag{136}$$

in virtue of (129). But $a_{P_r} \leq r-1$, and so

$$a_{p_r} \leq r+O\left(\frac{\log\mu}{\mu}+\frac{r^2\mu}{\log p_1}\right). \tag{137}$$

But we know that $a_{p_r} \geq r$. Hence it is clear that

$$a_{p_r} = r+O\left(\frac{\log\mu}{\mu}+\frac{r^2\mu}{\log p_1}\right). \tag{138}$$

From this and (136) it follows that

$$a_{P_r} = r-1+O\left(\frac{\log\mu}{\mu}+\frac{r^2\mu}{\log p_1}\right), \tag{139}$$

provided that the conditions (135) are satisfied.
Now let us suppose that $r = o\sqrt{(\log p_1)}$. Then we can choose μ such that $r^2\mu = o(\log p_1)$ and $\mu \neq O(1)$. Consequently we have

$$\frac{\log\mu}{\mu} = o(1), \quad \frac{r^2\mu}{\log p_1} = o(1);$$

and so it follows from (138) and (139) that

$$a_{p_r} = 1+a_{P_r} = r, \tag{140}$$

provided that $r = o\sqrt{(\log p_1)}$. From this it is clear that, if $r = o\sqrt{(\log p_1)}$, then

$$p_1 > p_2 > p_3 > p_4 > \cdots > p_r. \tag{141}$$

In other words, *in a large highly composite number*

$$2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdots p_1,$$

the indices comparatively near the end form a sequence of the type

$$\ldots 5 \ldots 4 \ldots 3 \ldots 2 \ldots 1.$$

Near the beginning gaps in the indices will in general occur.

Again, let us suppose that $r = o(\log p_1), r \neq o\sqrt{(\log p_1)}$, and $\mu = O(1)$ in (138) and (139). Then we see that

$$\left.\begin{aligned} a_{p_r} &= r + O\left(\frac{r^2}{\log p_1}\right), \\ a_{P_r} &= r + O\left(\frac{r^2}{\log p_1}\right); \end{aligned}\right\} \tag{142}$$

provided that $r = o(\log p_1)$ and $r \neq o\sqrt{(\log p_1)}$. But when $r \neq o(\log p_1)$, we shall use the general result, viz.,

$$\left.\begin{aligned} a_{p_r} &= O(r), \quad a_{p_r} \neq o(r), \\ a_{P_r} &= O(r), \quad a_{P_r} \neq o(r), \end{aligned}\right\} \tag{143}$$

which is true for all values of r except 1.

27. It follows from (87) and (128) that

$$\left.\begin{aligned} \frac{\log p_r}{\log\left(1 + \frac{1}{a_{p_r}}\right)} &\leq \frac{\log p_1}{\log 2} + O\left\{\frac{\log p_1 \log \mu}{r\mu} + r\mu + \sqrt{(\log p_1 \log\log p_1)}\right\}, \\ \frac{\log P_r}{\log\left(1 + \frac{1}{1+a_{P_r}}\right)} &\geq \frac{\log p_1}{\log 2} + O\left\{\frac{\log p_1 \log \mu}{r\mu} + r\mu + \sqrt{(\log p_1 \log\log p_1)}\right\}, \end{aligned}\right\} \tag{144}$$

with the condition that $r\mu = o(\log p_1)$. From this it can easily be shewn, by arguments similar to those used in the beginning of the previous section, that

$$\frac{\log p_r}{\log(1 + 1/r)} = \frac{\log p_1}{\log 2} + O\left\{\frac{\log p_1 \log \mu}{r\mu} + r\mu + \sqrt{(\log p_1 \log\log p_1)}\right\}, \tag{145}$$

provided that $r\mu = o(\log p_1)$.
Now let us suppose that $r = o(\log p_1)$; then we can choose μ such that

$$\mu = o\left(\frac{\log p_1}{r}\right), \quad \mu \neq O(1).$$

Consequently $r\mu = o(\log p_1)$ and $\log \mu = o(\mu)$, and so

$$\frac{\log p_1 \log \mu}{r\mu} = o(\log p_1).$$

From these relations and (145) it follows that, if $r = o(\log p_1)$, then

$$\frac{\log p_r}{\log(1 + 1/r)} \sim \frac{\log p_1}{\log 2}; \tag{146}$$

that is to say that, if $r = o(\log p_1)$, *then*

$$\frac{\log p_1}{\log 2} \sim \frac{\log p_2}{\log(1+\frac{1}{2})} \sim \frac{\log p_3}{\log(1+\frac{1}{3})} \sim \cdots \sim \frac{\log p_r}{\log(1+1/r)}. \tag{147}$$

Again let us suppose that $r = O\sqrt{(\log p_1 \log\log p_1)}$ in (145). Then it is possible to choose μ such that

$$\left.\begin{array}{l} r\mu = O\sqrt{(\log p_1 \log\log p_1)}, \\ r\mu \neq o\sqrt{(\log p_1 \log\log p_1)}. \end{array}\right\} \tag{148}$$

It is evident that $\log\mu = O(\log\log p_1)$, and so

$$\frac{\log p_1 \log\mu}{r\mu} = O\left(\frac{\log p_1 \log\log p_1}{r\mu}\right) = O\sqrt{(\log p_1 \log\log p_1)},$$

in virtue of (148). Hence

$$\frac{\log p_r}{\log(1+1/r)} = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}, \tag{149}$$

provided that

$$r = O\sqrt{(\log p_1 \log\log p_1)}.$$

Now let us suppose that $r = o(\log p_1), r \neq o\sqrt{(\log p_1 \log\log p_1)}$ and $\mu = O(1)$, in (145). Then it is evident that

$$\log p_1 = O(r^2), \quad \sqrt{(\log p_1 \log\log p_1} = O(r),$$

and

$$\frac{\log p_1 \log\mu}{r\mu} = O\left(\frac{\log p_1)}{r}\right) = O(r).$$

Hence we see that

$$\frac{\log p_r}{\log(1+1/r)} = \frac{\log p_1}{\log 2} + O(r), \tag{150}$$

if

$$r = o(\log p_1), \quad r \neq o\sqrt{(\log p_1 \log\log p_1)}.$$

But, if $r \neq o(\log p_1)$, we see from (128) that

$$\left.\begin{array}{l} \dfrac{\log p_r}{\log(1+1/r)} = O(\log p_1), \\[2ex] \dfrac{\log p_r}{\log(1+1/r)} \neq o(\log p_1). \end{array}\right\} \tag{151}$$

From (150) and (151) it follows that, if $r \neq o\sqrt{(\log p_1 \log\log p_1)}$, then

$$\frac{\log p_r}{\log(1+1/r)} = \frac{\log p_1}{\log 2} + O(r); \tag{152}$$

and from (149) and (152) that, if $r = o(\log p_1)$, then

$$\frac{\log p_r}{\log(1+1/r)} \sim \frac{\log p_1}{\log 2},$$

in agreement with (147). This result will, in general, fail for the largest possible values of r, which are of order $\log p_1$.

It must be remembered that all the results involving p_1 may be written in terms of N, since $p_1 = O(\log N)$ and $p_1 \neq o(\log N)$, and consequently

$$\log p_1 = \log\log N + O(1). \tag{153}$$

28. We shall now prove that successive highly composite numbers are asymptotically equivalent. Let m and n be any two positive integers which are prime to each other, such that

$$\log mn = o(\log p_1) = o(\log\log N); \tag{154}$$

and let

$$\frac{m}{n} = 2^{\delta_2} \cdot 3^{\delta_3} \cdot 5^{\delta_5} \cdots \wp^{\delta_\wp}. \tag{155}$$

Then it is evident that

$$mn = 2^{|\delta_2|} \cdot 3^{|\delta_3|} \cdot 5^{|\delta_5|} \cdots \wp^{|\delta_\wp|}. \tag{156}$$

Hence

$$\delta_\lambda \log\lambda = O(\log mn) = o(\log p_1) = o(a_\lambda \log\lambda); \tag{157}$$

so that $\delta_\lambda = o(a_\lambda)$.

Now

$$d\left(\frac{m}{n}N\right) = d(N)\left(1 + \frac{\delta_2}{1+a_2}\right)\left(1 + \frac{\delta_3}{1+a_3}\right)\cdots\left(1 + \frac{\delta_\wp}{1+a_\wp}\right). \tag{158}$$

But, from (60), we know that

$$a_\lambda \log\lambda = a_2 \log 2 + O\sqrt{(\log p_1 \log\lambda)}.$$

Hence

$$1 + \frac{\delta_\lambda}{1+a_\lambda} \quad = \quad 1 + \frac{\delta_\lambda \log\lambda}{a_2 \log 2} + O\left\{|\delta_\lambda|\left(\frac{\log\lambda}{\log p_1}\right)^{\frac{3}{2}}\right\}$$

$$\begin{aligned}
&= 1+\frac{\delta_\lambda \log\lambda}{a_2\log 2}+O\left\{|\delta_\lambda|\frac{\log\lambda}{\log p_1}\sqrt{\left(\frac{\log\wp}{\log p_1}\right)}\right\}\\
&= \exp\left\{\frac{\delta_\lambda \log\lambda}{a_2\log 2}+O\frac{|\delta_\lambda|\log\lambda}{\log p_1}\sqrt{\left(\frac{\log\wp}{\log p_1}\right)}+O\left(\frac{\delta_\lambda\log\lambda}{\log p_1}\right)^2\right\}\\
&= \exp\left\{\frac{\delta_\lambda \log\lambda}{a_2\log 2}+O\frac{|\delta_\lambda|\log\lambda}{\log p_1}\sqrt{\left(\frac{\log mn}{\log p_1}\right)}\right\}. \qquad (159)
\end{aligned}$$

It follows from (155), (156), (158) and (159) that

$$\begin{aligned}
d\left(\frac{m}{n}N\right) &= d(N)\exp\left\{\frac{\delta_2\log 2+\delta_3\log 3+\cdots+\delta_\wp\log\wp}{a_2\log 2}\right.\\
&\qquad \left.+O\frac{|\delta_2|\log 2+|\delta_3|\log 3+\cdots+|\delta_\wp|\log\wp}{\log p_1}\sqrt{\left(\frac{\log mn}{\log p_1}\right)}\right\}\\
&= d(N)e^{\frac{\log(m/n)}{a_2\log 2}+O\left(\frac{\log mn}{\log p_1}\right)^{\frac{3}{2}}}\\
&= d(N)e^{\frac{1}{a_2\log 2}\left\{\log\frac{m}{n}+O\log mn\sqrt{\left(\frac{\log mn}{\log p_1}\right)}\right\}}. \qquad (160)
\end{aligned}$$

Putting $m=n+1$, we see that, if

$$\log n = o(\log p_1) = o(\log\log N),$$

then

$$\begin{aligned}
d\left\{N\left(1+\frac{1}{n}\right)\right\} &= d(N)e^{\frac{1}{a_2\log 2}\left\{\log\left(1+\frac{1}{n}\right)+O\left(\log n\sqrt{\frac{\log n}{\log p_1}}\right)\right\}}\\
&= d(N)\left(1+\frac{1}{n}\right)^{\frac{1+O\left\{n\log n\sqrt{\left(\frac{\log n}{\log\log N}\right)}\right\}}{a_2\log 2}} \qquad (161)
\end{aligned}$$

Now it is possible to choose n such that

$$n(\log n)^{\frac{3}{2}} \neq o\sqrt{(\log\log N)},$$

and

$$1+O\left\{n\log n\sqrt{\left(\frac{\log n}{\log\log N}\right)}\right\}>0;$$

that is to say

$$d\left\{N\left(1+\frac{1}{n}\right)\right\} > d(N). \qquad (162)$$

From this and (29) it follows that, if N *is a highly composite number, then the next highly composite number is of the form*

$$N + O\left\{\frac{N(\log\log\log N)^{\frac{3}{2}}}{\sqrt{(\log\log N)}}\right\} \tag{163}$$

Hence the ratio of two consecutive highly composite numbers tends to unity.
It follows from (163) that the number of highly composite numbers not exceeding x is not of the form

$$o\left\{\frac{\log x\sqrt{(\log\log x)}}{(\log\log\log x)^{\frac{3}{2}}}\right\}.$$

29. Now let us consider the nature of $d(N)$ for highly composite values of N. From (44) we see that

$$d(N) = 2^{\pi(p_1)-\pi(p_2)} \cdot 3^{\pi(p_2)-\pi(p_3)} \cdot 4^{\pi(p_3)-\pi(p_4)} \cdots (1+a_2). \tag{164}$$

From this it follows that

$$d(N) = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots \varpi^{a_\varpi}, \tag{165}$$

where ϖ is the largest prime not exceeding $1 + a_2$; and

$$\alpha_\lambda = \pi(p_{\lambda-1}) + O(p_\lambda). \tag{166}$$

It also follows that, if $\wp_1, \wp_2, \wp_3, \ldots, \wp_\lambda$ are a given set of primes, then a number $\bar{\mu}$ can be found such that the equation

$$d(N) = \wp_1^{\beta_1} \cdot \wp_2^{\beta_2} \cdot \wp_3^{\beta_3} \cdots \wp_\mu^{\beta_\mu} \cdots \wp_\lambda^{\beta_\lambda}$$

is impossible if N is a highly composite number and $\beta_\mu > \bar{\mu}$. We may state this roughly by saying that as N (a highly composite number) tends to infinity, then, not merely in N itself, but also in $d(N)$, the number of prime factors, as well as the indices, must tend to infinity. In particular such an equation as

$$d(N) = k \cdot 2^m, \tag{167}$$

where k is fixed, becomes impossible when m exceeds a certain limit depending on k.
It is easily seen from (153), (164), and (165) that

$$\left.\begin{aligned} \varpi &= O(a_2) = O(\log p_1) = O(\log\log N) = O\{\log\log d(N)\}, \\ \varpi &\neq o(a_2) = o(\log p_1) = o(\log\log N) = o\{\log\log d(N)\}. \end{aligned}\right\} * \tag{168}$$

It follows from (147) that if $\lambda = o(\log p_1)$ then

$$\frac{\log \alpha_2}{\log(1-\frac{1}{2})} \sim \frac{\log \alpha_3}{\log(1-\frac{1}{3})} \sim \frac{\log \alpha_5}{\log(1-\frac{1}{5})} \sim \cdots \frac{\log \alpha_\lambda}{\log(1-\frac{1}{\lambda})}. \tag{169}$$

Similarly, from (149), it follows that if $\lambda = O\sqrt{(\log p_1 \log\log\log p_1)}$ then

$$\frac{\log(1+\alpha_\lambda)}{\log(1-1/\lambda)} = -\frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log\log p_1)}. \tag{170}$$

Again, from (152), we see that if $\lambda \neq o\sqrt{(\log p_1 \log\log\log p_1)}$ then

$$\frac{\log(1+\alpha_\lambda)}{\log(1-1/\lambda)} = -\frac{\log p_1}{\log 2} + O(\lambda). \tag{171}$$

In the left-hand side we cannot write α_λ instead of $1+\alpha_\lambda$, as α_λ may be zero for a few values of λ.
From (165) and (170) we can shew that

$$\log d(N) = \alpha_2 \log 2 + O(\alpha_3), \quad \log d(N) \neq \alpha_2 \log 2 + o(\alpha_3);$$

and so

$$\log d(N) = \alpha_2 \log 2 + e^{\frac{\log\frac{3}{2}}{\log 2}\log p_1 + O\sqrt{(\log p_1 \log\log\log p_1)}} \tag{172}$$

But from (163) we see that

$$\log\log d(N) = \log p_1 + O(\log\log p_1).$$

From this and (172) it follows that

$$a_2 \log 2 = \log d(N) - \{\log d(N)\}^{\frac{\log\frac{3}{2}}{\log 2} + O\sqrt{\left\{\frac{\log\log\log d(N)}{\log\log d(N)}\right\}}}. \tag{173}$$

30. Now we shall consider the order of $dd(N)$ for highly composite values of N. It follows from (165) that

$$\log dd(N) = \log(1+\alpha_2) + \log(1+\alpha_3) + \cdots + \log(1+\alpha_\varpi). \tag{174}$$

Now let $\lambda, \lambda', \lambda'', \ldots$ be consecutive primes in ascending order, and let

$$\lambda = O\sqrt{(\log p_1 \log\log\log p_1)},$$

*More precisely $\varpi \sim a_2$. But this involves the assumption that two consecutive primes are asymptotically equivalent. This follows at once from the prime number theorem. It appears probable that such a result cannot really be as deep as the prime number theorem, but nobody has succeeded up to now in proving it by elementary reasoning.

$$\lambda \neq o\sqrt{(\log p_1 \log\log p_1)}.$$

Then, from (174), we have

$$\begin{aligned}\log dd(N) &= \log(1+\alpha_2)+\log(1+\alpha_3)+\cdots+\log(1+\alpha_\lambda)\\ &+\ \log(1+\alpha_{\lambda'})+\log(1+a_{\lambda''})+\cdots+\log(1+\alpha_\varpi).\end{aligned} \tag{175}$$

But, from (170), we have

$$\begin{aligned}\log(1+\alpha_2) &+ \log(1+\alpha_3)+\cdots+\log(1+\alpha_\lambda)\\ &= -\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})(1-\tfrac{1}{5})\cdots\left(1-\frac{1}{\lambda}\right)\right\}\\ &+ O\sqrt{(\log p_1\log\log p_1)}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{\lambda}\right)\right\}.\end{aligned} \tag{176}$$

It can be shewn, without assuming the prime number theorem*, that

$$-\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})(1-\tfrac{1}{5})\cdots\left(1-\frac{1}{p}\right)\right\}=\log\log p+\gamma+O\left(\frac{1}{\log p}\right), \tag{177}$$

where γ is the Eulerian constant. Hence

$$\log\left\{(1-\tfrac{1}{2}(1-\tfrac{1}{3})(1-\tfrac{1}{5})\cdots\left(1-\frac{1}{p}\right)\right\}=O(\log\log p).$$

From this and (176) it follows that

$$\begin{aligned}\log(1+\alpha_2) &+ \log(1+\alpha_3)+\cdots+\log(1+\alpha_\lambda)\\ &= -\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{\lambda}\right)\right\}\\ &+ O\{\sqrt{(\log p_1\log\log p_1)}\log\log\lambda\}\\ &= -\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{\lambda}\right)\right\}\\ &+ O\{\sqrt{(\log p_1\log\log p_1)}\log\log\log p_1\}.\end{aligned} \tag{178}$$

Again, from (152), we see that

$$\begin{aligned}&\log(1+\alpha_{\lambda'})+\log(1+\alpha_{\lambda''})+\cdots+\log(1+\alpha_\varpi)\\ &\quad=-\frac{\log p_1}{\log 2}\log\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda''}\right)\cdots\left(1-\frac{1}{\varpi}\right)\right\}\end{aligned}$$

*See Landau, *Handbuch*, p.139.

$$
\begin{aligned}
&+O\left\{\lambda'\log\left(1-\frac{1}{\lambda'}\right)+\lambda''\log\left(1-\frac{1}{\lambda''}\right)+\cdots+\varpi\log\left(1-\frac{1}{\varpi}\right)\right\}\\
&=-\frac{\log p_1}{\log 2}\log\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda'}\right)\cdots\left(1-\frac{1}{\varpi}\right)\right\}+O\{\pi(\varpi)-\pi(\lambda)\}\\
&=-\frac{\log p_1}{\log 2}\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda''}\right)\cdots\left(1-\frac{1}{\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right). \qquad (179)
\end{aligned}
$$

From (175), (178) and (179) it follows that

$$
\begin{aligned}
\log dd(N) &= -\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{\lambda}\right)\right\}\\
&\quad +O\{\sqrt{(\log p_1\log\log p_1)}\log\log\log p_1\}\\
&\quad -\frac{\log p_1}{\log 2}\log\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda''}\right)\cdots\left(1-\frac{1}{\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)\\
&= -\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)\\
&= \frac{\log p_1}{\log 2}\left\{\log\log\varpi+\gamma+O\left(\frac{1}{\log\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)\\
&= \frac{\log p_1}{\log 2}\left\{\log\log\log p_1+\gamma+O\left(\frac{1}{\log\log p_1}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)\\
&= \frac{\log\log N}{\log 2}\left\{\log\log\log\log N+\gamma+O\left(\frac{1}{\log\log\log N}\right)\right\}, \qquad (180)
\end{aligned}
$$

in virtue of (177), (168), and (163). Hence, if N is a highly composite number, then

$$dd(N)=(\log N)^{\frac{1}{\log 2}\left\{\log\log\log\log N+\gamma+O\left(\frac{1}{\log\log\log N}\right)\right\}}. \qquad (181)$$

31. It may be interesting to note that, as far as the table is constructed,

$$
\begin{aligned}
&2, 2^2, 2^3, \ldots, 2^{13},\quad 3, 3\cdot 2, 3\cdot 2^2, \ldots, 3\cdot 2^{11},\quad 5\cdot 2, 5\cdot 2^2, \ldots, 5\cdot 2^8,\\
&\qquad 7\cdot 2^5, 7\cdot 2^6, \ldots, 7\cdot 2^{10},\quad 9, 9\cdot 2, 9\cdot 2^2, \ldots, 9\cdot 2^{10},
\end{aligned}
$$

and so on, occur as values of $d(N)$. But we know from § 29 that $k\cdot 2^m$ cannot be the value of $d(N)$ for sufficiently large values of m; and so numbers of the form $k\cdot 2^m$ which occur as the value of $d(N)$ in the table must disappear sooner or later when the table is extended. Thus numbers of the form $5\cdot 2^m$ have begun to disappear in the table itself. The powers of 2 disappear at any rate from 2^{18} onwards. The least number having 2^{18} divisors is

$$2^7\cdot 3^3\cdot 5^3\cdot 7\cdot 11\cdot 13\cdots 41\cdot 43,$$

while the smaller number, viz.,

$$2^8 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdots 41$$

has a larger number of divisors. viz. $135 \cdot 2^{11}$. The numbers of the form $7 \cdot 2^m$ disappear at least from $7 \cdot 2^{13}$ onwards. The least number having $7 \cdot 2^{13}$ divisors is

$$2^6 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdots 31 \cdot 37,$$

while the smaller number, viz.

$$2^9 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdots 31$$

has a larger number of divisors, viz. $225 \cdot 2^8$.

IV
Superior Highly Composite Numbers

32. A number N may be said to be a superior highly composite number if there is a positive number ϵ such that

$$\frac{d(N)}{N^\epsilon} \geq \frac{d(N')}{(N')^\epsilon}, \tag{182}$$

for all values of N' less that N, and

$$\frac{d(N)}{N^\epsilon} > \frac{d(N')}{(N')^\epsilon} \tag{183}$$

for all values of N' greater that N.
All superior highly composite numbers are also highly composite. For, if $N' < N$, it follows from (182) that

$$d(N) \geq d(N') \left(\frac{N}{N'}\right)^\epsilon > d(N');$$

and so N is highly composite.

33. Now let us consider what must be the nature of N in order that it should be a superior highly composite number. In the first place it must be of the form

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p_1^a, \tag{184}$$

or of the form

$$\begin{array}{rl} & 2 \cdot 3 \cdot 5 \cdot 7 \cdots p_1 \\ \times & 2 \cdot 3 \cdot 5 \cdot 7 \cdots p_2 \end{array}$$

$$\begin{aligned} &\times \quad 2\cdot 3\cdot 5\cdots p_3 \\ &\times \quad \ldots\ldots: \end{aligned}$$

i.e. it must satisfy the conditions for a highly composite number. Now let

$$N' = N/\lambda,$$

where $\lambda \leq p_1$. Then from (182) it follows that

$$\frac{1+a_\lambda}{\lambda^{\epsilon a_\lambda}} \geq \frac{a_\lambda}{\lambda^{\epsilon(a_\lambda - 1)}},$$

or

$$\lambda^\epsilon \leq \left(1 + \frac{1}{a_\lambda}\right). \tag{185}$$

Again let

$$N' - N_\lambda.$$

Then, from (183), we see that

$$\frac{1+a_\lambda}{\lambda^{\epsilon a_\lambda}} > \frac{2+a_\lambda}{\lambda^{\epsilon(a_\lambda + 1)}},$$

or

$$\lambda^\epsilon > \left(1 + \frac{1}{1+a_\lambda}\right). \tag{186}$$

Now supposing that $\lambda = p_1$ in (185) and $\lambda = P_1$ in (186), we obtain

$$\frac{\log 2}{\log P_1} < \epsilon \leq \frac{\log 2}{\log p_1}. \tag{187}$$

Now let us suppose that $\epsilon = 1/x$. Then, from (187), we have

$$p_1 \leq 2^x < P_1. \tag{188}$$

That is, p_1 is the largest prime not exceeding 2^x. It follows from (185) that

$$a_\lambda \leq (\lambda^{1/x} - 1)^{-1}. \tag{189}$$

Similarly, from (186),

$$a_\lambda > (\lambda^{1/x} - 1)^{-1} - 1. \tag{190}$$

From (189) and (190) it is clear that

$$a_\lambda = [\lambda^{1/x} - 1)^{-1}]. \tag{191}$$

Hence N is of the form

$$2^{[(2^{1/x}-1)^{-1}]}\cdot 3^{[(3^{1/x}-1)^{-1}]}\cdot 5^{[(5^{1/x}-1)^{-1}]}\dots p_1, \tag{192}$$

where p_1 is the largest prime not exceeding 2^x.

34. Now let us suppose that $\lambda = p_r$ in (189). Then

$$a_{p_r} \leq (p_r^{1/x}-1)^{-1}.$$

But we know that $r \leq a_{pr}$. Hence

$$r \leq (p_r^{1/x}-1)^{-1},$$

or

$$p_r \leq \left(1+\frac{1}{r}\right)^x. \tag{193}$$

Similarly by supposing that $\lambda = P_r$ in (190), we see that

$$a_{P_r} > (P_r^{1/x}-1)^{-1}-1.$$

But we know that $r-1 \geq ap_r$. Hence

$$r > (P_r^{1/x}-1)^{-1}$$

or

$$P_r > \left(1+\frac{1}{r}\right)^x. \tag{194}$$

From (193) and (194) it is clear that p_r is the largest prime not exceeding $(1+1/r)^x$. Hence N is of the form

$$\begin{array}{ll} & 2\cdot 3\cdot 5\cdot 7\cdots p_1 \\ \times & 2\cdot 3\cdot 5\cdot 7\cdots p_2 \\ \times & 2\cdot 3\cdot 5\cdots p_3 \\ \times & \dots\dots\dots, \end{array} \tag{195}$$

where p_1 is the largest prime not greater than $2^x, p_2$ is the largest prime not greater than $\left(\frac{3}{2}\right)^x$, and so on. In other words N is of the form

$$e^{\vartheta(2^x)+\vartheta(\frac{3}{2})^x+\vartheta(\frac{4}{3})^x+\cdots}, \tag{196}$$

and $d(N)$ is of the form

$$2^{\pi(2^x)}\cdot(\tfrac{3}{2})^{\pi(\frac{3}{2})^x}\cdot(\tfrac{4}{3})^{\pi(\frac{4}{3})^x}\cdots. \tag{197}$$

Thus to every value of x not less than 1 correspondence one, and only one, value of N.

35. Since

$$\frac{d(N)}{N^{1/x}} \geq \frac{d(N')}{(N')^{1/x}},$$

for all values of N', it follows from (196) and (197) that

$$d(N) \leq N^{1/x} \frac{2^{\pi(2^x)}}{e^{(1/x)\vartheta(2^x)}} \frac{(\frac{3}{2})^{\pi(\frac{3}{2})^x}}{e^{(1/x)\vartheta(\frac{3}{2})^x}} \frac{(\frac{4}{3})^{\pi(\frac{4}{3})^x}}{e^{(1/x)\vartheta(\frac{4}{3})^x}} \cdots, \tag{198}$$

for all values of N and x; and $d(N)$ is equal to the right-hand side when

$$N = e^{\vartheta(2^x)+\vartheta(\frac{3}{2})^x+\vartheta(\frac{4}{3})^x+\cdots}. \tag{199}$$

Thus, for example, putting $x = 2$, 3, 4 in (198), we obtain

$$\left.\begin{aligned} &d(N) \leq \sqrt{(3N)}, \\ &d(N) \leq 8(3N/35)^{\frac{1}{3}}, \\ &d(N) \leq 96(3N/50050)^{\frac{1}{4}}, \end{aligned}\right\} \tag{200}$$

for all values of N; and $d(N) = \sqrt{(3N)}$ when $N = 2^2 \cdot 3; d(N) = 8(3N/35)^{\frac{1}{3}}$ when $N = 2^3 \cdot 3^2 \cdot 5 \cdot 7; d(N) = 96(3N/50050)^{\frac{1}{4}}$ when

$$N = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13.$$

36. M and N are consecutive superior highly composite numbers if there are no superior highly composite numbers between M and N.

From (195) and (196) it is easily seen that, if M and N are any two superior highly composite numbers, and if $M > N$, then M is a multiple of N; and also that, if M and N are two consecutive superior highly composite numbers, and if $M > N$, then M/N is a prime number. From this it follows that consecutive superior highly composite numbers are of the form

$$\pi_1, \quad \pi_1\pi_2, \quad \pi_1\pi_2\pi_3, \quad \pi_1\pi_2\pi_3\pi_4, \quad \ldots, \tag{201}$$

where $\pi_1, \pi_2, \pi_3 \ldots$ are primes. In order to determine $\pi_1, \pi_2, \ldots$ we proceed as follows. Let x_1' be the smallest value of x such that $[2^x]$ is prime x_2' the smallest value of x such that $[\left(\frac{3}{2}\right)^x]$ is prime, and so on; and let $x_1, x_2, \ldots$ be the numbers $x_1', x_2' \ldots$ arranged in order of magnitude. Then π_n is the prime corresponding to x_n, and

$$N = \pi_1\pi_2\pi_3 \cdots \pi_n, \tag{202}$$

if $x_n \leq x < x_{n+1}$.

37. From the preceding results we see that the number of superior highly composite numbers not exceeding

$$e^{\vartheta(2^x)+\vartheta(\frac{3}{2})^x+\vartheta(\frac{4}{3})^x+\cdots} \tag{203}$$

is

$$\pi(2^x) + \pi(\tfrac{3}{2})^x + \pi(\tfrac{4}{3})^x + \cdots.$$

In other words if $x_n \le x < x_{n+1}$ then

$$n = \pi(2^x) + \pi(\tfrac{3}{2})^x + \pi(\tfrac{4}{3})^x + \cdots. \tag{204}$$

It follows from (192) and (202) that, of the primes $\pi_1, \pi_2, \pi_3, \ldots, \pi_n$, the number of primes which are equal to a given prime ϖ is equal to

$$[(\varpi^{1/x} - 1)^{-1}]. \tag{205}$$

Further, the greatest of the primes $\pi_1, \pi_2, \pi_3 \ldots, \pi_n$ is the largest prime not greater than 2^x, and is asymptotically equivalent to the natural nth prime, in virtue of (204).

The following table gives the values of π_n and x_n for the first 50 values of n, that is till x_n reaches very nearly 7.

$\pi_1 = 2$	$x_1 = \frac{\log 2}{\log 2} = 1$
$\pi_2 = 3$	$x_2 = \frac{\log 3}{\log 2} = 1.5849\cdots$
$\pi_3 = 2$	$x_3 = \frac{\log 2}{\log(\frac{3}{2})} = 1.7095\cdots$
$\pi_4 = 5$	$x_4 = \frac{\log 5}{\log 2} = 2.3219\ldots$
$\pi_5 = 2$	$x_5 = \frac{\log 2}{\log(\frac{4}{3})} = 2.4094\ldots$

$$\pi_6 = 3 \qquad x_6 = \frac{\log 3}{\log(\frac{3}{2})} = 2.7095\ldots$$

$$\pi_7 = 7 \qquad x_7 = \frac{\log 7}{\log 2} = 2.8073\ldots$$

$$\pi_8 = 2 \qquad x_8 = \frac{\log 2}{\log(\frac{5}{4})} = 3.1062\ldots$$

$$\pi_9 = 11 \qquad x_9 = \frac{\log 11}{\log 2} = 3.4594\ldots$$

$$\pi_{10} = 13 \qquad x_{10} = \frac{\log 13}{\log 2} = 3.7004\ldots$$

$$\pi_{11} = 2 \qquad x_{11} = \frac{\log 2}{\log(\frac{6}{5})} = 3.8017\ldots$$

$$\pi_{12} = 3 \qquad x_{12} = \frac{\log 3}{\log(\frac{4}{3})} = 3.8188\ldots$$

$$\pi_{13} = 5 \qquad x_{13} = \frac{\log 5}{\log(\frac{3}{2})} = 3.9693\cdots$$

$$\pi_{14} = 17 \qquad x_{14} = \frac{\log 17}{\log 2} = 4.0874\ldots$$

$$\pi_{15} = 19 \qquad x_{15} = \frac{\log 19}{\log 2} = 4.2479\ldots$$

$$\pi_{16} = 2 \qquad x_{16} = \frac{\log 2}{\log(\frac{7}{6})} = 4.4965\ldots$$

$$\pi_{17} = 23 \qquad x_{17} = \frac{\log 23}{\log 2} = 4.5235\ldots$$

$$\pi_{18} = 7 \qquad x_{18} = \frac{\log 7}{\log(\frac{3}{2})} = 4.7992\ldots$$

$$\pi_{19} = 29 \qquad x_{19} = \frac{\log 29}{\log 2} = 4.8579\ldots$$

$$\pi_{20} = 3 \qquad x_{20} = \frac{\log 3}{\log(\frac{5}{4})} = 4.9233\ldots$$

$$\pi_{21} = 31 \qquad x_{21} = \frac{\log 31}{\log 2} = 4.9541\ldots$$

$$\pi_{22} = 2 \qquad x_{22} = \frac{\log 2}{\log(\frac{8}{7})} = 5.1908\ldots$$

$$\pi_{23} = 37 \qquad x_{23} = \frac{\log 37}{\log 2} = 5.2094\ldots$$

$$\pi_{24} = 41 \qquad x_{24} = \frac{\log 41}{\log 2} = 5.3575\ldots$$

$$\pi_{25} = 43 \qquad x_{25} = \frac{\log 43}{\log 2} = 5.4262\ldots$$

$$\pi_{26} = 47 \qquad x_{26} = \frac{\log 47}{\log 2} = 5.5545\ldots$$

$$\pi_{27} = 5 \qquad x_{27} = \tfrac{\log 5}{\log(\frac{4}{3})} = 5.5945\ldots$$

$$\pi_{28} = 53 \qquad x_{28} = \tfrac{\log 53}{\log 2} = 5.7279\ldots$$

$$\pi_{29} = 59 \qquad x_{29} = \tfrac{\log 59}{\log 2} = 5.8826\ldots$$

$$\pi_{30} = 2 \qquad x_{30} = \tfrac{\log 2}{\log(\frac{9}{8})} = 5.8849\ldots$$

$$\pi_{31} = 11 \qquad x_{31} = \tfrac{\log 11}{\log(\frac{3}{2})} = 5.9139\ldots$$

$$\pi_{32} = 61 \qquad x_{32} = \tfrac{\log 61}{\log 2} = 5.9307\ldots$$

$$\pi_{33} = 3 \qquad x_{33} = \tfrac{\log 3}{\log(\frac{6}{5})} = 6.0256\ldots$$

$$\pi_{34} = 67 \qquad x_{34} = \tfrac{\log 67}{\log 2} = 6.0660\ldots$$

$$\pi_{35} = 71 \qquad x_{35} = \tfrac{\log 71}{\log 2} = 6.1497\ldots$$

$$\pi_{36} = 73 \qquad x_{36} = \tfrac{\log 73}{\log 2} = 6.1898\ldots$$

$$\pi_{37} = 79 \qquad x_{37} = \tfrac{\log 79}{\log 2} = 6.3037\ldots$$

$$\pi_{38} = 13 \qquad x_{38} = \tfrac{\log 13}{\log(\frac{3}{2})} = 6.3259\ldots$$

$$\pi_{39} = 83 \qquad x_{39} = \tfrac{\log 83}{\log 2} = 6.3750\ldots$$

$$\pi_{40} = 89 \qquad x_{40} = \tfrac{\log 89}{\log 2} = 6.4757\ldots$$

$$\pi_{41} = 2 \qquad x_{41} = \tfrac{\log 2}{\log(\frac{10}{9})} = 6.5790\ldots$$

$$\pi_{42} = 97 \qquad x_{42} = \tfrac{\log 97}{\log 2} = 6.5999\ldots$$

$$\pi_{43} = 101 \qquad x_{43} = \tfrac{\log 101}{\log 2} = 6.6582\ldots$$

$$\pi_{44} = 103 \qquad x_{44} = \tfrac{\log 103}{\log 2} = 6.6724\ldots$$

$$\pi_{45} = 107 \qquad x_{45} = \tfrac{\log 107}{\log 2} = 6.7414\ldots$$

$$\pi_{46} = 7 \qquad x_{46} = \tfrac{\log 7}{\log(\frac{4}{3})} = 6.7641\ldots$$

$$\pi_{47} = 109 \qquad x_{47} = \tfrac{\log 109}{\log 2} = 6.7681\ldots$$

$$\pi_{48} = 113 \qquad x_{48} = \tfrac{\log 113}{\log 2} = 6.8201\ldots$$

$$\pi_{49} = 17 \qquad x_{49} = \tfrac{\log 17}{\log(\frac{3}{2})} = 6.9875\ldots$$

$$\pi_{50} = 127 \qquad x_{50} = \tfrac{\log 127}{\log 2} = 6.9886\ldots$$

38. It follows from (17) and (198) that $\log d(N) \leq F(x)$, where

$$F(x) = \frac{1}{x}\log N + \frac{1}{x}\left\{\int_2^{2^x} \frac{\pi(t)}{t}dt + \int_2^{(\frac{3}{2})^x} \frac{\pi(t)}{t}dt + \int_2^{(\frac{4}{3})^x} \frac{\pi(t)}{t}dt + \cdots\right\}, \tag{206}$$

for all values of N and x. In order to obtain the best possible upper limit for $\log d(N)$, we must choose x so as to make the right-hand side a minimum.

The function $F(x)$ is obviously continuous unless $(1 + 1r)^x = p$, where r is a positive integer and p a prime. It is easily seen to be continuous even then, and so continuous without exception. Also

$$\begin{aligned} F'(x) &= -\frac{1}{x^2}\log N - \frac{1}{x^2}\left\{\int_2^{2^x} \frac{\pi(t)}{t}dt + \int_2^{(\frac{3}{2})^x} \frac{\pi(t)}{t}dt + \cdots\right\} \\ &\qquad + \frac{1}{x}\{\pi(2^x)\log 2 + \pi(\tfrac{3}{2})^x \log \tfrac{3}{2} + \cdots\} \\ &= \frac{1}{x^2}\{\vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + \vartheta(\tfrac{4}{3})^x + \cdots - \log N\}, \end{aligned} \tag{207}$$

unless $(1 + 1/r)^x = p$, in virtue of (17).

Thus we see that $F(x)$ is continuous, and $F'(x)$ exists and is continuous except at certain isolated points. The sign of $F'(x)$, where it exists, is that of

$$\vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + \vartheta(\tfrac{4}{3})^x + \cdots - \log N,$$

and

$$\vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + \vartheta(\tfrac{4}{3})^x + \cdots,$$

is a monotonic function. Thus $F'(x)$ is first negative and then positive, changing sign once only, and so $F(x)$ has a unique minimum. Thus $F(x)$ is a minimum when x is a function of N defined by the inequalities

$$\vartheta(2^v) + \vartheta(\tfrac{3}{2})^v + \vartheta(\tfrac{4}{3})^v + \cdots \left\{\begin{array}{l} < \log N (y < x) \\ > \log N (y > x) \end{array}\right\}. \tag{208}$$

Now let $D(N)$ be a function of N such that

$$D(N) = 2^{\pi(2^x)}(\tfrac{3}{2})^{\pi(\frac{3}{2})^x}(\tfrac{4}{3})^{\pi(\frac{4}{3})^x}\cdots, \tag{209}$$

where x is the function of N defined by the inequalities (208). Then, from (198), we see that

$$d(N) \leq D(N), \tag{210}$$

for all values of N; and $d(N) = D(N)$ for all superior highly composite values of N. Hence $D(N)$ is the maximum order of $d(N)$. In other words, $d(N)$ will attain its maximum order when N is a superior highly composite number.

V.
Application to the Order of $d(N)$.

39. The most precise result known concerning the distribution of the prime numbers is that

$$\left.\begin{aligned} \pi(x) &= Li(x) + O(xe^{-a\sqrt{\log x}}), \\ \vartheta(x) &= x + O(xe^{-a\sqrt{\log x}}), \end{aligned}\right\} \tag{211}$$

where

$$Li(x) = \int^x \frac{dt}{\log t}$$

and a is a positive constant.
In order to find the maximum order of $d(N)$ we have merely to determine the order of $D(N)$ from the equations (208) and (209). Now, from (208), we have

$$\log N = \vartheta(2^x) + O(\tfrac{3}{2})^x = \vartheta(2^x) + o(2^{2x/3});$$

and so

$$\vartheta(2^x) = \log N + o(\log N)^{\frac{2}{3}}; \tag{212}$$

and similarly from (209) we have

$$\pi(2^x) = \frac{\log D(N)}{\log 2} + o(\log N)^{\frac{2}{3}}. \tag{213}$$

It follows from (211) - (213) that the maximum order of $d(N)$ is

$$2^{Li(\log N)+O[\log Ne^{-a\sqrt{(\log\log N)}}]}. \tag{214}$$

It does not seen to be possible to obtain an upper limit for $d(N)$ notably more precise than (214) without assuming results concerning the distribution of primes which depend on hitherto unproved properties of the Riemann ζ-function.

40. We shall now assume that the "Riemann hypothesis" concerning the ζ-function is true, i.e., that all the complex roots of $\zeta(s)$ have their real part equal to $\frac{1}{2}$. Then it is known that

$$\vartheta(x) = x - \sqrt{x} - \sum \frac{x^\rho}{\rho} + O(x^{\frac{1}{3}}), \tag{215}$$

where ρ is a complex root of $\zeta(s)$, and that

$$\pi(x) = Li(x) - \tfrac{1}{2}Li(\sqrt{x}) - \sum Li(x^\rho) + O(x^{\frac{1}{3}})$$

$$= Li(x) - \frac{\sqrt{x}}{\log x} - \frac{2\sqrt{x}}{(\log x)^2} - \frac{1}{\log x}\sum\frac{x^\rho}{\rho} - \frac{1}{(\log x)^2}\sum\frac{x^\rho}{\rho^2} + O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\}, \quad (216)$$

since $\sum \frac{x^\rho}{\rho^k}$ is absolutely convergent when $k > 1$. Also it is known that

$$\sum\frac{x^\rho}{\rho} = O\{\sqrt{x(\log x)^2}\}; \quad (217)$$

and so

$$\vartheta(x) - x = O\{\sqrt{x(\log x)^2}\}. \quad (218)$$

From (215) and (216) it is clear that

$$\pi(x) = Li(x) + \frac{\vartheta(x) - x}{\log x} - R(x) + O\left\{\frac{\sqrt{x}}{\log x)^3}\right\}, \quad (219)$$

where

$$R(x) = \frac{2\sqrt{x} + \sum\frac{x^\rho}{\rho^2}}{(\log x)^2} \quad (220)$$

But it follows from Taylor's theorem and (218) that

$$Li\vartheta(x) - Li(x) = \frac{\vartheta(x) - x}{\log x} + O(\log x)^2, \quad (221)$$

and from (219) and (221) it follows that

$$\pi(x) = Li\vartheta(x) - R(x) + O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\}. \quad (222)$$

41. It follows from the functional equation satisfied by $\zeta(s)$, viz.,

$$(2\pi)^{-s}\Gamma(s)\zeta(s)\cos\tfrac{1}{2}\pi s = \tfrac{1}{2}\zeta(1-s), \quad (223)$$

that

$$(1-s)\pi^{-\frac{1}{4}\sqrt{s}}\Gamma\left(\frac{1+\sqrt{s}}{4}\right)\zeta\left(\frac{1+\sqrt{s}}{2}\right)$$

is an integral function of s whose apparent order is less than 1, and hence is equal to

$$\Gamma(\tfrac{1}{4})\zeta(\tfrac{1}{2})\prod\left\{1 - \tfrac{s}{(2\rho-1)^2}\right\},$$

[where ρ runs through the complex roots of $\zeta(s)$ whose imaginary parts are positive]. From this we can easily deduce that

$$s(1+s)\pi^{-\frac{1+s}{2}}\Gamma\left(\frac{1+s}{2}\right)\zeta(1+s) = \prod\left(1 + \frac{s}{\rho}\right), \quad (224)$$

[where ρ now runs through all the roots]. Subtracting 1 from both sides, dividing the result by s, and then making $s \to 0$, we obtain

$$\sum \frac{1}{\rho} = 1 + \tfrac{1}{2}(\gamma - \log 4\pi), \tag{225}$$

where γ is the Eulerian constant. Hence we see that

$$\begin{aligned} \left|\sum \frac{x^\rho}{\rho^2}\right| \leq \sum \left|\frac{x^\rho}{\rho^2}\right| = \sqrt{x}\sum \frac{1}{\rho(1-\rho)} = \sqrt{x}\sum\left(\frac{1}{\rho} + \frac{1}{1-\rho}\right) \\ = 2\sqrt{x}\sum \frac{1}{\rho} = \sqrt{x}(2 + \gamma - \log 4\pi). \end{aligned} \tag{226}$$

It follows from (220) and (226) that

$$(\log 4\pi - \gamma)\sqrt{x} \leq R(x)(\log x)^2 \leq (4 + \gamma - \log 4\pi)\sqrt{x}. \tag{227}$$

It can easily be verified that

$$\left.\begin{array}{l} \log 4\pi - \gamma = 1.954, \\ 4 + \gamma - \log 4\pi = 2.046 \end{array}\right\} \tag{228}$$

approximately.

42. Now

$$R(x) = \frac{2\sqrt{x} + S(x)}{(\log x)^2},$$

where

$$S(x) = \sum \frac{x^\rho}{\rho^2};$$

so that, considering $R(x)$ as a function of a continuous variable, we have

$$\begin{aligned} R'(x) &= \frac{1}{\sqrt{x}(\log x)^2} - \frac{4\sqrt{x} + 2S(x)}{x(\log x)^3} + \frac{S'(x)}{(\log x)^2} \\ &= \frac{S'(x)}{(\log x)^2} + O\left\{\frac{1}{\sqrt{x}(\log x)^2}\right\}, \end{aligned}$$

for all values of x for which $S(x)$ possesses a differential coefficient.
Now the derived series of $S(x)$, viz.,

$$\bar{S}(x) = \frac{1}{x}\sum \frac{x^\rho}{\rho},$$

is uniformly convergent throughout any interval of positive values of x which does not include any value of x of the form $x = p^m$; and $S(x)$ is continuous for all values of x. It follows that

$$S(x_1) - S(x_2) = \int_{x_1}^{x_2} \bar{S}(x)dx,$$

for all positive values of x_1 and x_2, and that $S(x)$ possesses a derivative

$$S'(x) = \bar{S}(x),$$

whenever x is not of the form p^m. Also

$$\bar{S}(x) = O\left\{\frac{(\log x)^2}{\sqrt{x}}\right\}.$$

Hence

$$R(x+h) = R(x) + \int_x^{x+h} O\left(\frac{1}{\sqrt{t}}\right) dt = R(x) + O\left(\frac{h}{\sqrt{x}}\right). \tag{229}$$

43. Now

$$\begin{aligned} \log N &= \vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + O(\tfrac{4}{3})^x \\ &= \vartheta(2^x) + (\tfrac{3}{2})^x + O\{x^2(\tfrac{3}{2})^{\frac{1}{2}x}\} + O(\tfrac{4}{3})^x \\ &= \vartheta(2^x) + (\tfrac{3}{2})^x + O(2^{5x/12}). \end{aligned}$$

Similarly $\log D(N) = \log 2 \cdot \pi(2^x) + \log(\frac{3}{2}) Li(\frac{3}{2})^x + O(2^{5x/12})$. Writing X for 2^x, we have

$$\left.\begin{aligned} &\log N = \vartheta(X) + X^{\log(\frac{3}{2})/\log 2} + O\{(\log N)^{\frac{5}{12}}\}; \\ &\log D(N) = \log 2.\pi(X) + \log(\tfrac{3}{2}) Li\{X^{\log(\frac{3}{2})/\log 2}\} + O(X^{\frac{5}{12}}. \end{aligned}\right\} \tag{230}$$

It follows that

$$\log N = X + O[X^{\log(\frac{3}{2}/\log 2}];$$

and so

$$X = \log N + O[(\log N)^{\log(\frac{3}{2})/\log 2}]. \tag{231}$$

Again, from (230) and (231), it follows that

$$\log N = \vartheta(X) + (\log N)^{\log(\frac{3}{2})/\log 2} + O\{(\log N)^{\frac{5}{12}}\}, \tag{232}$$

and

$$\begin{aligned} \log D(N) &= \log 2.\pi(X) + \log(\tfrac{3}{2}) Li(\log N)^{\log(\frac{3}{2})/\log 2} + O\{(\log N)^{\frac{5}{12}}\} \\ &= \log 2\left\{Li\vartheta(X) - R(X) + O\left[\frac{\sqrt{X}}{(\log X)^3}\right]\right\} \\ &\quad + \log(\tfrac{3}{2}) Li\{(\log N)^{\log(\frac{3}{2})/\log 2}\} + O\{(\log N)^{\frac{5}{12}}\}, \end{aligned} \tag{233}$$

in virtue of (222). From (231) and (233) it evidently follows that

$$\begin{aligned}\log D(N) &= \log 2\cdot Li\vartheta(X) - \log 2\cdot R(X) + \log(\tfrac{3}{2})Li\{(\log N)^{\log(\frac{3}{2})/\log 2}\}\\ &\quad +O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}\\ &= \log 2\cdot Li\{\log N - (\log N)^{\log(\frac{3}{2})/\log 2} + O(\log N)^{\frac{5}{12}}\}\\ &\quad -\log 2\cdot R\{\log N + O(\log N)^{\log(\frac{3}{2}/\log 2}\}\\ &\quad +\log(\tfrac{3}{2})Li\{(\log N)^{\log(\frac{3}{2})/\log 2}\} + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}, \qquad (234)\end{aligned}$$

in virtue of (231) and (232). But

$$\begin{aligned}&Li\{\log N - (\log N)^{|log(\frac{3}{2})/\log 2}\} + O(\log N)^{\frac{5}{12}}\}\\ &\quad = Li(\log N) - \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} + O\left\{\frac{(\log N)^{\frac{5}{12}}}{\log\log N}\right\} + O\left\{\frac{(\log N)^{\{2\log(\frac{3}{2})/\log 2\}-1}}{(\log\log N)^2}\right\}\\ &\quad = Li(\log N) - \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} + O(\log N)^{\frac{5}{12}};\end{aligned}$$

and

$$\begin{aligned}R\{\log N + O(\log N)^{\log(\frac{3}{2})/\log 2}\} &= R(\log N) + O\{(\log N)^{\{\log(\frac{3}{2})/\log 2\}-\frac{1}{2}}\}\\ &= R(\log N) + O(\log N)^{\frac{1}{10}},\end{aligned}$$

in virtue of (229). Hence (234) may be replaced by

$$\begin{aligned}\log D(N) &= \log 2\cdot Li(\log N) + \log(\tfrac{3}{2})Li\{(\log n)^{\log(\frac{3}{2})\log 2}\}\\ &\quad -\log 2\frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} - \log 2.R(\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}. \qquad (235)\end{aligned}$$

That is to say the maximum order of $d(N)$ is

$$2^{Li(\log N)+\phi(N)}, \qquad (236)$$

where

$$\begin{aligned}\phi(N) &= \frac{\log(\frac{3}{2})}{\log 2}Li\{(\log N)^{\log(\frac{3}{2})/\log 2}\} - \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} - R(\log N)\\ &\quad +O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.\end{aligned}$$

This order is actually attained for an infinity of values of N.

44. We can now find the order of the number of superior highly composite numbers not exceeding a given number N. Let N' be the smallest superior highly composite number greater than N, and let

$$N' = e^{\vartheta(2^x)+\vartheta(\frac{3}{2})^x+\vartheta(\frac{4}{3})^x+\cdots}.$$

Then, from § 37, we know that

$$2N \le N' \le 2^x N, \tag{237}$$

so that $N' = 0(N \log N)$; and also that the number of superior highly composite numbers not exceeding N' is

$$n = \pi(2^x) + \pi(\tfrac{3}{2})^x + \pi(\tfrac{4}{3})^x + \cdots.$$

By arguments similar to those of the previous section we can shew that

$$\begin{aligned} n \;\;=\;\; & Li(\log N) + Li(\log N)^{\log(\frac{3}{2})/\log 2} - \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} - R(\log N) \\ & + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}. \end{aligned} \tag{238}$$

It is easy to see from § 37 that, if the largest superior highly composite number not exceeding N is

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p^{a_p},$$

then the number of superior highly composite numbers not exceeding N is the sum of all the indices, viz,

$$a_2 + a_3 + a_5 + \cdots + a_p$$

45. Proceeding as in § 28, we can shew that, if N is a superior highly composite number and m and n are any two positive integers such that [n is a divisor of N, and]

$$\log mn = o(\log\log N),$$

then

$$d\left(\frac{m}{n}N\right) = d(N)2^{\frac{\log(m/n)}{\log\log N} + o\left(\frac{\log mn}{\log\log N}\right)^2}. \tag{239}$$

From this we can easily shew that the next highly composite number is of the form

$$N + O\left\{\frac{N(\log\log\log N)^2}{\log\log N}\right\}. \tag{240}$$

Again, let us S and S be any two consecutive superior highly composite numbers, and let

$$S = e^{\vartheta(2^x)+\vartheta(\frac{3}{2})+\vartheta(\frac{4}{3})^x+\cdots}$$

Then it follows from § 35 that

$$d(N) < \left(\frac{N}{S}\right)^{1/x} d(S), \tag{241}$$

for all values of N except S and s'. Now, if S be the nth superior highly composite number, so that

$$x_n \leq x < x_{n+1},$$

where x_n is the same as in § 36, we see that

$$d(N) < \left(\frac{N}{S}\right)^{1/x_n} d(S), \tag{241$'$}$$

for all values of N except S and s'. If N is S or S', then the inequality becomes an equality. It follows from § 36 that $d(S) \leq 2d(S')$. Hence, if N be highly composite and $S' < N < S$, so that $dS' < d(N) < d(S)$, then

$$\tfrac{1}{2}d(S) < d(N) < d(S'), \quad d(S') < d(N) < 2d(S').$$

From this it is easy to see that the order (236) is actually attained by $d(N)$ whenever N is a highly composite number. But it may also be attained when N is not a highly composite number. For example, if

$$N = (2 \cdot 3 \cdot 5 \cdots p_1) \times (2 \cdot 3 \cdot 5 \cdots p_2),$$

where p_1 is the largest prime not greater than 2^x, and p_2 the largest prime not greater than $(\frac{3}{2})^x$, it is easily seen that $d(N)$ attains the order (236): and N is not highly composite.

VI.
Special Forms of N.

46. In §§-38 we have indirectly solved the following problem: to find the relations which must hold between $x_1, x_2, x_3, \ldots$ in order that

$$2^{\pi(x_1)} \cdot (\tfrac{3}{2})^{x(x_2)} \cdot (\tfrac{4}{3})^{\pi(x_3)} \cdots$$

may be a maximum, when it is given that

$$\vartheta(x_1) + \vartheta(x_2) + \vartheta(x_3) + \cdots$$

is a fixed number. The relations which we obtained are

$$\frac{\log 2}{\log x_1} = \frac{\log(\frac{3}{2})}{\log x_2} = \frac{\log(\frac{4}{3})}{\log x_3} = \cdots.$$

This suggests the following more general problem. If N is an integer of the form

$$e^{c_1\vartheta(x_1) + c_2\vartheta(x_2) + c_3\vartheta(x_3) + \cdots}, \tag{242}$$

where $c_1, c_2, c_3, \ldots$ are any given positive integers, it is required to find the nature of N, that is to say the relations which hold between $x_1, x_2, x_3, \ldots$, when $d(N)$ is of maximum order. From (242) we see that

$$d(N) = (1+c_1)^{\pi(x_1)} \left(\frac{1+c_1+c_2}{1+c_1}\right)^{\pi(x_2)} \left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{\pi(x_3)} \tag{242$'$}$$

If we define the "superior" numbers of the class (242) by the inequalities

$$\frac{d(N)}{N^\epsilon} \geq \frac{d(N')}{(N')^\epsilon},$$

for all values of N' less than N, and

$$\frac{d(N)}{N^\epsilon} > \frac{d(N')}{(N')^\epsilon},$$

for all values of N' greater than N, N and N' in the two inequalities being of the form (242), and proceed as in § 33, we can shew that

$$d(N) \leq N^{1/x} \frac{(1+c_1)^{\pi\{(1+c_1)^{x/c_1}\}}}{e^{(c_1/x)\vartheta\{(1+c_1)^{x/c_1}\}}} \frac{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{\pi\left\{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{x/c_2}\right\}}}{e^{(c_2/x)\vartheta\left\{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{x/c_2}\right\}}} \cdots, \tag{243}$$

for all values of x, and for all values of N of the form (242). From we can shew, by arguments similar to those of § 38, that N must be of the form

$$e^{c_1\vartheta\{(1+c_1)^{x/c_1}\} + c_2\vartheta\left\{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{x/c_2}\right\} + c_3\vartheta\left\{\left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{x/c_3}\right\} + \cdots}, \tag{244}$$

and $d(N)$ of the form

$$(1+c_1)^{\pi\left\{(1+c_1)^{\frac{x}{c_1}}\right\}} \left(\frac{1+c_1+c_2}{1+c_1}\right)^{\pi\left\{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{\frac{x}{c_2}}\right\}} \left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{\pi\left\{\left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{\frac{x}{c_3}}\right\}} \tag{244$'$}$$

From (244) and (244') we can find the maximum order of $d(N)$, as in § 43.

47. We shall now consider the order of $d(N)$ for some special forms of N. The simplest case is that in which N is of the form

$$2 \cdot 3 \cdot 5 \cdot 7 \cdots p;$$

so that

$$\log N = \vartheta(p),$$

and

$$d(N) = 2^{\pi(p)}.$$

It is easy to shew that

$$d(N) = 2^{Li(\log N) - R(\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}}. \tag{245}$$

In this case $d(N)$ is exactly a power of 2, and this naturally suggests the question: what is the maximum order of $d(N)$ when $d(N)$ is exactly a power of 2?

It is evident that, if $d(N)$ is a power of 2, the indices of the prime divisors of N cannot be any other numbers except 1,3,7,15,31, ...; an so in order that $d(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta(x_1) + 2\vartheta(x_2) + 4\vartheta(x_3) + 8\vartheta(x_4) + \cdots},$$

and $d(N)$ of the form

$$2^{\pi(x_1) + \pi(x_2) + \pi(x_3) + \cdots}.$$

It follows from § 46 that, in order that $d(N)$ should be of maximum order N, must be of the form

$$e^{\vartheta(x) + 2\vartheta(\sqrt{x}) + 4\vartheta(x^{\frac{1}{4}}) + 8\vartheta(x^{\frac{1}{8}}) + \cdots}, \tag{246}$$

and $d(N)$ of the form

$$2^{\pi(x) + \pi(\sqrt{x}) + \pi(x^{\frac{1}{4}}) + \pi(x^{\frac{1}{8}}) + \cdots}. \tag{247}$$

Hence the maximum order of $d(N)$ can easily be shewn to be

$$2^{Li(\log N) + \frac{4\sqrt{(\log N)}}{(\log\log N)^2} - R(\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}}. \tag{248}$$

It is easily seen from (246) that the least number having 2^n divisors is

$$2 \cdot 3 \cdot 4 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 16 \cdot 17 \cdot 19 \cdot 23 \cdot 25 \cdot 29 \cdots \quad \text{to } n \text{ factors}, \tag{249}$$

where 2, 3, 4, 5, 7, ... are the natural primes, their squares, fourth powers and so on, arranged according to order of magnitude.

48. We have seen that the last indices of the prime divisors of N must be 1, if $d(N)$ is of maximum order. Now we shall consider the maximum order of $d(N)$ when the indices of the prime divisors of N are never less than an integer n. In the first place, in order that $d(N)$ should be of maximum order, N must be of the form

$$e^{n\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots},$$

and $d(N)$ of the form

$$(1+n)^{\pi(x_1)}\left(\frac{2+n}{1+n}\right)^{\pi(x_2)}\left(\frac{3+n}{2+n}\right)^{\pi(x_3)}\cdots.$$

It follows from §46 that N must be of the form

$$e^{n\vartheta\{(1+n)^{x/n}\}+\vartheta\left\{\left(\frac{2+n}{1+n}\right)^x\right\}+\vartheta\left\{\left(\frac{3+n}{2+n}\right)^x\right\}+\cdots}, \tag{250}$$

and $d(N)$ of the form

$$(1+n)^{\pi\{(1+n)^{x/n}\}}\left(\frac{2+n}{1+n}\right)^{\pi\left\{\left(\frac{1+n}{1+n}\right)^x\right\}}\left(\frac{3+n}{2+n}\right)^{\pi\left\{\left(\frac{3+n}{2+n}\right)^x\right\}}\cdots \tag{251}$$

Then, by arguments similar to those of § 43, we can shew that the maximum order of $d(N)$ is

$$(n+1)^{Li\{(1/n)\log N\}+\phi(N)}, \tag{252}$$

where

$$\begin{aligned}\phi(N) \quad &= \quad \left\{\frac{\log(n+2)}{\log(n+1)}-1\right\}Li\left\{\left(\frac{1}{n}\log N\right)^{n\frac{\log(n+2)}{\log(n-1)}-n}\right\}\\ &\quad -\frac{\left(\frac{1}{n}\log N\right)^{n\frac{\log(n+2)}{\log(n-1)}-n}}{n\log\left(\frac{1}{n}\log N\right)}-R\left(\frac{1}{n}\log N\right)+)\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.\end{aligned}$$

if $n \geq 3$, it is easy to verify that

$$n\frac{\log(n+2)}{\log(n+1)}-n<\tfrac{1}{2};$$

and so (252) reduces to

$$(n+1)^{Li\{(1/n)\log N\}-R\{(1/n)\log N\}+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}}, \tag{253}$$

provided that $n \geq 3$.

49. Let us next consider the maximum order of $d(N)$ when N is a perfect nth power. In order that $d(N)$ should be of maximum order, N must be of the form

$$e^{n\vartheta(x_2)+n\vartheta(x_2)+n\vartheta(x_3)+\cdots}$$

and $d(N)$ of the form

$$(1+n)^{\pi(x_1)}\left(\frac{1+2n}{1+n}\right)^{\pi(x_2)}\left(\frac{1+3n}{1+2n}\right)^{\pi(x_3)}\cdots.$$

It follows from § 46 that N must be of the form

$$e^{n\vartheta\{(1+n)^x\}+n\vartheta\left\{\left(\frac{1+2n}{1+n}\right)^x\right\}+n\vartheta\left\{\left(\frac{1+3n}{1+2n}\right)^x\right\}+\cdots}, \tag{254}$$

and $d(N)$ of the form

$$(1+n)^{pi\{(1+n)^x\}}\left(\frac{1+2n}{1+n}\right)^{\pi\left\{\left(\frac{1+2n}{1+n}\right)^x\right\}}\left(\frac{1+3n}{1+2n}\right)^{\pi\left\{\left(\frac{1+3n}{1+2n}\right)^x\right\}}\cdots \tag{255}$$

Hence we can shew that the maximum order of $d(N)$ is

$$(n+1)^{Li\{(1/n)\log N\}-R\{(1/n)\log N\}+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}}, \tag{256}$$

provided that $n>1$.

50. Let $l(N)$ denote the least common multiple of the first N natural numbers. Then it can easily be shewn that

$$l(N)=2^{[\log N\log 2]}\cdot 3^{[\log N/\log 3]}\cdot 5^{[\log N/\log 5]}\cdots p, \tag{257}$$

where p is the largest prime not greater than N. From this we can shew that

$$l(N)=e^{\vartheta(N)+\vartheta(\sqrt{N})+\vartheta(N^{\frac{1}{3}})+\vartheta(N^{\frac{1}{4}})+\cdots}; \tag{258}$$

and so

$$d\{l(N)\}=2^{\pi(N)}(\tfrac{3}{2})^{\pi(N)}(\tfrac{4}{3})^{\pi(N^{\frac{1}{3}})}\cdots. \tag{259}$$

From (258) and (259) we can shew that, if N is of the form $l(M)$, then

$$d(N)=2^{Li(\log N)+\phi(N)}, \tag{260}$$

where

$$\phi(N)=\frac{\log(\frac{9}{8})}{\log 2}\frac{\sqrt{(\log N)}}{\log\log N}+\frac{4\log(\frac{3}{2})}{(\log\log N)^2}-R(\log N)+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.$$

It follows from (258) that

$$l(N) = e^{N+O\{\sqrt{N(\log)^2}\}}; \tag{261}$$

and from (259) that

$$d\{l(N)\} = 2^{Li(n)+O(\sqrt{N\log N})}. \tag{262}$$

51. Finally, we shall consider the number of divisors of $N!$ It is easily seen that

$$N! = 2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdots p^{a_p}, \tag{263}$$

where p is the largest prime not greater than N, and

$$a_\lambda = \left[\frac{N}{\lambda}\right] + \left[\frac{N}{\lambda^2}\right] + \left[\frac{N}{\lambda^3}\right] + \cdots.$$

It is evident that the primes greater than $\frac{1}{2}N$ and not exceeding N appear once in $N!$, the primes greater than $\frac{1}{3}N$ and not exceeding $\frac{1}{2}N$ appear twice, and so on up to those greater than $N/[\sqrt{N}]$ and not exceeding $N/(\sqrt{N}]-1)$, appearing $[\sqrt{N}]-1$ times*. The indices of the smaller primes cannot be specified so simply. Hence it is clear that

$$N! = e^{\vartheta(N)+\vartheta(\frac{1}{2}N)+\vartheta(\frac{1}{3}N)+\cdots+\vartheta\left(\frac{N}{[\sqrt{N}]-1}\right)} \times 2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdots \varpi^{a_\varpi}, \tag{264}$$

where ϖ is the largest prime not greater than $\sqrt{N}$, and

$$a_\lambda - 1 + [\sqrt{N}] = \left[\frac{N}{\lambda}\right] + \left[\frac{N}{\lambda^2}\right] + \left[\frac{N}{\lambda^3}\right] + \cdots.$$

From (264) we see that

$$\begin{aligned} d(N!) &= 2^{\pi(N)}(\tfrac{3}{2})^{\pi(\frac{1}{2}N)}(\tfrac{4}{3})^{\pi(\frac{1}{3}N)}\cdots \text{ to } [\sqrt{N}]-1 \text{ factors} \\ &\quad \times e^{O\{\log(1+a_2)+\log(1+a_3)+\cdots+\log(1+a_\varpi)\}} \\ &= 2^{\pi(N)}(\tfrac{3}{2})^{\pi(\frac{1}{2}N)}(\tfrac{4}{3})^{\pi(\frac{1}{3}N)}\cdots \text{ to } [\sqrt{N}]-1 \text{ factors} \\ &\quad \times e^{O\{\varpi\log(1+a_2)\}} \\ &= 2^{Li(N)}(\tfrac{3}{2})^{Li(\frac{1}{2}N)}(\tfrac{4}{3})^{Li(\frac{1}{3}N)}\cdots \text{ to } [\sqrt{N}] \text{ factors} \\ &\quad \times e^{O(\sqrt{N\log N})}. \end{aligned} \tag{265}$$

Since

$$Li(N) = \frac{N}{\log N} + O\left\{\frac{N}{(\log N)^2}\right\},$$

*Strictly speaking, this is true only when $N \geq 4$.

we see that

$$d(N!) = C^{\frac{N}{\log N} + O\left\{\frac{N}{(\log N)^2}\right\}}, \tag{266}$$

where

$$C = (1+1)^1 (1+\tfrac{1}{2})^{\frac{1}{2}} (1+\tfrac{1}{3})^{\frac{1}{3}} (1+\tfrac{1}{4})^{\frac{1}{4}} \cdots.$$

From this we can easily deduce that, if N is of the form $M!$, then

$$d(N) = C^{\frac{\log N}{(\log\log N)^2} + \frac{2\log N \log\log\log N}{(\log\log N)^3} + O\left\{\frac{\log N}{(\log\log N)^3}\right\}}, \tag{267}$$

where C is the same constant as in (266).

52. It is interesting in this connection to shew how, by considering numbers of certain special forms, we can obtain lower limits for the maximum orders of the iterated functions $dd(n)$ and $ddd(n)$. By supposing that

$$N = 2^{2-1} \cdot 3^{3-1} \cdots p^{p-1},$$

we can shew that

$$dd(n) > 4^{\frac{\sqrt{(2\log n)}}{\log\log n}} \tag{268}$$

for an infinity of values of n. By supposing that

$$N = 2^{2^{a_2}-1} \cdot 3^{3^{a_3}-1} \cdots p^{p^{a_p}-1},$$

where

$$a_\lambda = \left[\frac{\log p}{\log \lambda}\right] - 1$$

we can shew that

$$ddd(n) > (\log n)^{\log\log\log\log n} \tag{269}$$

for an infinity of values of n.

On certain infinite series

Messenger of Mathematics, XLV, 1916, 11 – 15

1. This paper is merely a continuation of the paper on "Some definite integrals" published in this *Journal**. It deals with some series which resemble those definite integrals not merely in form but in many other respects. In each case there is a functional relation. In the case of the integrals there are special values of a parameter for which the integrals may be evaluated in finite terms. In the case of the series the corresponding results involve elliptic functions.

2. It can be shewn, by the theory of residues, that if α and β are real and $\alpha\beta = \frac{1}{4}\pi^2$, then

$$\begin{aligned} & \frac{\alpha}{(\alpha+t)\cosh\alpha} - \frac{3\alpha}{(9\alpha+t)\cosh 3\alpha} + \frac{5\alpha}{(25\alpha+t)\cosh 5\alpha} - \cdots \\ + \quad & \frac{\beta}{(\beta-t)\cosh\beta} - \frac{3\beta}{(9\beta-t)\cosh 3\beta} + \frac{5\beta}{(25\beta-t)\cosh 5\beta} - \cdots \\ = \quad & \frac{\pi}{4\cos\sqrt{(\alpha t)}\cosh\sqrt{(\beta t)}}. \end{aligned} \tag{1}$$

Now let

$$\begin{aligned} F(n) \quad = \quad & \left\{\frac{\alpha e^{in\alpha}}{\cosh\alpha} - \frac{3\alpha e^{9in\alpha}}{\cosh 3\alpha} + \frac{5\alpha e^{25in\alpha}}{\cosh 5\alpha} - \cdots\right\} \\ & - \left\{\frac{\beta e^{-in\beta}}{\cosh\beta} - \frac{3\beta e^{-9in\beta}}{\cosh 3\beta} + \frac{5\beta e^{-25in\beta}}{\cosh 5\beta} - \cdots\right\} \end{aligned} \tag{2}$$

Then we see that, if t is positive,

$$\int_0^\infty e^{-2tn} F(n)dn = \frac{\pi i}{4\cosh\{(1-i)\sqrt{(\alpha t)}\}\cosh\{(1+i)\sqrt{(\beta t)}\}} \tag{3}$$

in virtue of (1). Again, let

$$\begin{aligned} f(n) = -\frac{1}{2n}\sqrt{\left(\frac{\pi}{2n}\right)} \sum\sum (-1)^{\frac{1}{2}(\mu+\nu)} \{\mu(1+i)\sqrt{\alpha} - \nu(1-i)\sqrt{\beta}\} \\ \times e^{-(\pi\mu\nu - i\mu^2\alpha + i\nu^2\beta)/4n} \qquad (\mu = 1, 3, 5, \ldots; \nu = 1, 3, 5, \ldots). \end{aligned} \tag{4}$$

*[No. 11 of this volume (pp. 66 – 74); see also No.12 (pp. 75 – 86).]

Then it is easy to shew that

$$\int_0^\infty e^{-2tn} f(n)dn = \frac{\pi i}{4\cosh\{(1-i)\sqrt{(\alpha t)}\}\cosh\{(1+i)\sqrt{(\beta t)}\}}. \tag{5}$$

Hence, by a theorem due to Lerch*, we obtain

$$F(n) = f(n) \tag{6}$$

for all positive values of n, provided that $\alpha\beta = \frac{1}{4}\pi^2$. In particular, when $\alpha = \beta = \frac{1}{2}\pi$, we have

$$\begin{aligned}
&\frac{\sin\frac{1}{2}\pi n}{\cosh\frac{1}{2}\pi} - \frac{3\sin\frac{9}{2}\pi n}{\cosh\frac{3}{2}\pi} + \frac{5\sin\frac{25}{2}\pi n}{\cosh\frac{5}{2}\pi} - \cdots \\
&\quad = -\frac{1}{4n\sqrt{n}}\sum\sum(-1)^{\frac{1}{2}(\mu+\nu)}e^{-\pi\mu\nu/4n} \\
&\qquad \left[(\mu+\nu)\cos\frac{\pi(\mu^2-\nu^2)}{8n} + (\mu-\nu)\sin\frac{\pi(\mu^2-\nu^2)}{8n}\right] \\
&\qquad\qquad (\mu = 1, 3, 5, \ldots; \nu = 1, 3, 5, \ldots)
\end{aligned} \tag{7}$$

for all positive values of n. As particular cases of (7), we have

$$\begin{aligned}
&\frac{\sin(\frac{1}{2}\pi/a)}{\cosh\frac{1}{2}\pi} - \frac{3\sin(\frac{9}{2}\pi/a)}{\cosh\frac{3}{2}\pi} + \frac{5\sin(\frac{25}{2}\pi/a)}{\cosh\frac{5}{2}\pi} - \cdots \\
&\quad = \frac{1}{4}a\sqrt{a}\left(\frac{1}{\cosh\frac{1}{4}\pi a} - \frac{3}{\cosh\frac{3}{4}\pi a} + \frac{5}{\cosh\frac{5}{4}\pi a} - \cdots\right) \\
&\quad = \frac{1}{2}a\sqrt{a}(e^{-\frac{1}{16}\pi a} - e^{-\frac{9}{16}\pi a} - e^{-\frac{25}{16}\pi a} + e^{-\frac{49}{16}\pi a} + \cdots)^4,
\end{aligned} \tag{8}$$

if a is a positive even integer; and

$$\begin{aligned}
&\frac{\sin(\frac{1}{2}\pi/a)}{\cosh\frac{1}{2}\pi} - \frac{3\sin(\frac{9}{2}\pi/a)}{\cosh\frac{3}{2}\pi} + \cdots \\
&\quad = \frac{1}{2}a\sqrt{a}\left(\frac{\cosh\frac{1}{4}\pi a}{\cosh\frac{1}{2}\pi a} + \frac{3\cosh\frac{3}{4}\pi a}{\cosh\frac{3}{2}\pi a} - \frac{5\cosh\frac{5}{4}\pi a}{\cosh\frac{5}{2}\pi a} - \frac{7\cosh\frac{7}{4}\pi a}{\cosh\frac{7}{2}\pi a} + \cdots\right),
\end{aligned} \tag{9}$$

if a is a positive odd integer; and so on.

3. It is also easy to shew that if $\alpha\beta = \pi^2$, then

$$\left\{\frac{\alpha}{(\alpha+t)\sinh\alpha} - \frac{2\alpha}{(4\alpha+t)\sinh 2\alpha} + \frac{3\alpha}{(9\alpha+t)\sinh 3\alpha} - \cdots\right\}$$

* See Mr. Hardy's note at the end of my previous paper [*Messenger of Mathematics*, XLIV, pp. 18 – 21].

$$-\left\{\frac{\beta}{(\beta-t)\sinh\beta}-\frac{2\beta}{(4\beta-t)\sinh 2\beta}+\frac{3\beta}{(9\beta-t)\sinh 3\beta}-\cdots\right\}$$
$$=\frac{1}{2t}-\frac{\pi}{2\sin\sqrt{(\alpha t)}\sinh\sqrt{(\beta t)}}. \tag{10}$$

From this we can deduce, as in the previous section, that if $\alpha\beta=\pi^2$, then

$$\frac{\alpha e^{in\alpha}}{\sinh\alpha}-\frac{2\alpha e^{4in\alpha}}{\sinh 2\alpha}+\frac{3\alpha e^{9in\alpha}}{\sinh 3\alpha}-\cdots+\frac{\beta e^{-in\beta}}{\sinh\beta}-\frac{2\beta e^{-4in\beta}}{\sinh 2\beta}+\frac{3\beta e^{-9in\beta}}{\sinh 3\beta}-\cdots$$
$$=\frac{1}{2}-\frac{1}{n}\sqrt{\left(\frac{\pi}{2n}\right)}\times\sum\sum\{\mu(1-i)\sqrt{\alpha}+\nu(1+i)\sqrt{\beta}\}e^{-(2\pi\mu\nu-i\mu^2\alpha+i\nu^2\beta)/4n}$$
$$(\mu=1,3,5,\ldots;\nu=1,3,5,\ldots) \tag{11}$$

for all positive values of n. If, in particular, we put $\alpha=\beta=\pi$, we obtain

$$\frac{1}{4\pi}-\frac{\cos\pi n}{\sinh\pi}+\frac{2\cos 4\pi n}{\sinh 2\pi}-\frac{3\cos 9\pi n}{\sinh 3\pi}+\cdots$$
$$=\frac{1}{2n\sqrt{(2n)}}\sum\sum e^{-\pi\mu\nu/2n}\left\{(\mu+\nu)\cos\frac{\pi(\mu^2-\nu^2)}{4n}+(\mu-\nu)\sin\frac{\pi(\mu^2-\nu^2)}{4n}\right\}$$
$$(\mu=1,3,5,\ldots;\nu=1,3,5,\ldots) \tag{12}$$

for all positive values of n. Thus, for example, we have

$$\frac{1}{4\pi}-\frac{\cos(2\pi/a)}{\sinh\pi}+\frac{2\cos(8\pi/a)}{\sinh 2\pi}-\frac{3\cos(18\pi/a)}{\sinh 3\pi}+\cdots$$
$$=\frac{1}{8}a\sqrt{a}\left(\frac{1}{\sinh\frac{1}{4}\pi a}+\frac{3}{\sinh\frac{3}{4}\pi a}+\frac{5}{\sinh\frac{5}{4}\pi a}+\ldots\right)$$
$$=\frac{1}{4}a\sqrt{a}(e^{-\frac{1}{16}\pi a}+e^{-\frac{9}{16}\pi a}+e^{-\frac{25}{16}\pi a}+e^{-\frac{49}{16}\pi a}+\cdots)^4, \tag{13}$$

if a is a positive even integer; and

$$\frac{1}{4\pi}-\frac{\cosh(2\pi/a)}{\sinh\pi}+\frac{2\cosh(8\pi/a)}{\sinh 2\pi}-\cdots$$
$$=\frac{1}{4}a\sqrt{a}\left(\frac{\sinh\frac{1}{4}\pi a}{\cosh\frac{1}{2}\pi a}-\frac{3\sinh\frac{3}{4}\pi a}{\cosh\frac{3}{2}\pi a}-\frac{5\sinh\frac{5}{4}\pi a}{\cosh\frac{5}{2}\pi a}+\frac{7\sinh\frac{7}{4}\pi a}{\cosh\frac{7}{2}\pi a}+\cdots\right) \tag{14}$$

if a is a positive odd integer.

4. In a similar manner we can shew that, if $\alpha\beta=\pi^2$, then

$$\frac{\alpha e^{in\alpha}}{e^{2\alpha}-1}+\frac{2\alpha e^{4in\alpha}}{e^{4\alpha}-1}+\frac{3\alpha e^{9in\alpha}}{e^{6\alpha}-1}+\cdots+\frac{\beta e^{-in\beta}}{e^{2\beta}-1}+\frac{2\beta e^{-4in\beta}}{e^{4\beta}-1}+\frac{3\beta e^{-9in\beta}}{e^{6\beta}-1}+\cdots$$

$$= \alpha \int_0^\infty \frac{xe^{-in\alpha x^2}}{e^{2\pi x}-1}dx\,^* + \beta \int_0^\infty \frac{xe^{in\beta x^2}}{e^{2\pi x}-1}dx - \frac{1}{4}$$
$$+\frac{1}{n}\sqrt{\left(\frac{\pi}{2n}\right)}\sum_{\mu=1}^\infty\sum_{\nu=1}^\infty\{\mu(1-i)\sqrt{\alpha}+\nu(1+i)\sqrt{\beta}\}e^{-(2\pi\mu\nu-i\mu^2\alpha+i\nu^2\beta)/n} \tag{15}$$

for all positive values of n. Putting $\alpha=\beta=\pi$ in (15) we see that, if $n>0$, then

$$\frac{1}{8\pi}+\frac{\cos\pi n}{e^{2\pi}-1}+\frac{2\cos 4\pi n}{e^{4\pi}-1}+\frac{3\cos 9\pi n}{e^{6\pi}-1}+\cdots$$
$$=\int_0^\infty \frac{x\cos\pi n x^2}{e^{2\pi x}-1}dx+\frac{1}{2n\sqrt{(2n)}}\sum_{\mu=1}^\infty\sum_{\nu=1}^\infty e^{-2\pi\mu\nu/n}$$
$$\times\left[(\mu+\nu)\cos\left\{\frac{\pi(\mu^2-\nu^2)}{n}\right\}+(\mu-\nu)\sin\left\{\frac{\pi(\mu^2-\nu^2)}{n}\right\}\right]. \tag{16}$$

As particular cases of (16) we have

$$\frac{1}{8\pi}+\frac{\cos(\pi/a)}{e^{2\pi}-1}+\frac{2\cos(4\pi/a)}{e^{4\pi}-1}+\frac{3\cos(9\pi/a)}{e^{6\pi}-1}+\cdots$$
$$=\int_0^\infty \frac{x\cos(\pi x^2/a)}{e^{2\pi x}-1}dx+a\sqrt{(\tfrac{1}{2}a)}\left(\frac{1}{e^{2\pi a}-1}+\frac{2}{e^{4\pi a}-1}+\frac{3}{e^{6\pi a}-1}+\cdots\right), \tag{17}$$

If a is a positive even integer;

$$\frac{1}{8\pi}+\frac{\cos(\pi/a)}{e^{2\pi}-1}+\frac{2\cos(4\pi/a)}{e^{4\pi}-1}+\frac{3\cos(9\pi/a)}{e^{6\pi}-1}+\cdots$$
$$=\int_0^\infty \frac{x\cos(\pi x^2/a)}{e^{2\pi x}-1}dx+a\sqrt{(\tfrac{1}{2}a)}\left(\frac{1}{e^{2\pi a}+1}-\frac{2}{e^{4\pi a}+1}+\frac{3}{e^{6\pi a}+1}-\cdots\right), \tag{18}$$

if a is a positive odd integer; and

$$\frac{1}{8\pi}+\frac{\cos(2\pi/a)}{e^{2\pi}-1}+\frac{2\cos(8\pi/a)}{e^{4\pi}-1}+\frac{3\cos(18\pi/a)}{e^{6\pi}-1}+\cdots=\int_0^\infty\frac{x\cos(2\pi x^2/a)}{e^{2\pi x}-1}dx+\tfrac{1}{4}a\sqrt{a}S$$
$$\text{where } S=\sum_{n=1}^\infty\frac{(-1)^{n-1}n}{e^{n\pi a}+(-1)^n} \text{ or } S=\sum_{n=1}^\infty\frac{n}{e^{n\pi a}+(-1)^{n-1}} \tag{19}$$

according as $a\equiv 1$ or $a\equiv 3 \pmod 4$.

*I shewed in my former paper [No.12 of the present volume] that this integral can be calculated in finite terms whenever $n\alpha$ is a rational multiple of π.

It may be interesting to note that different functions dealt with in this paper have the same asymptotic expansion for small values of n. For example, the two different functions

$$\frac{1}{8\pi}+\frac{\cos n}{e^{2\pi}-1}+\frac{2\cos 4n}{e^{4\pi}-1}+\frac{3\cos 9n}{e^{6\pi}-1}+\cdots$$

and

$$\int_0^\infty \frac{x\cos nx^2}{e^{2\pi x}-1}dx$$

have the same asymptotic expansion, viz.

$$\frac{1}{24}-\frac{n^2}{1008}+\frac{n^4}{6336}-\frac{n^6}{17280}+\cdots^{*} \tag{20}$$

*This series (in spite of the appearance of the first few terms) diverges for all values of n.

Some formulæ in the analytic theory of numbers

Messenger of Mathematics, XLV, 1916, 81 – 84

I have found the following formulæ incidentally in the course of other investigations. None of them seem to be of particular importance, nor does their proof involve the use of any new ideas, but some of them are so curious that they seem to be worth printing. I denote by $d(x)$ the number of divisors of x, if x is an integer, and zero otherwise, and by $\zeta(s)$ the Riemann Zeta-function.

$$\text{(A)} \qquad \frac{\zeta^4(s)}{\zeta(2s)} = 1^{-s}d^2(1) + 2^{-s}d^2(2) + 3^{-s}d^2(3) + \cdots, \tag{1}$$

$$\frac{\eta^4(s)}{(1-2^{-2s})\zeta(2s)} = 1^{-s}d^2(1) - 3^{-s}d^2(3) + 5^{-s}d^2(5) - \cdots, \tag{2}$$

where

$$\eta(s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \cdots \ .$$

$$\text{(B)} \qquad d^2(1) + d^2(2) + d^2(3) + \ldots + d^2(n)$$
$$= An(\log n)^3 + Bn(\log n)^2 + Cn\log n + Dn + O(n^{\frac{3}{5}+\epsilon}),^* \tag{3}$$

where

$$A = \frac{1}{\pi^2},\ B = \frac{12\gamma - 3}{\pi^2} - \frac{36}{\pi^4}\zeta'(2),$$

γ is Euler's constant, C, D more complicated constants, and ϵ any positive number.

$$\text{(C)} \qquad d^3\left(\frac{n}{1}\right) + d^3\left(\frac{n}{3}\right) + d^3\left(\frac{n}{2}\right) + \cdots$$
$$= \left\{d\left(\frac{n}{1}\right) + d\left(\frac{n}{2}\right) + d\left(\frac{n}{3}\right) + \cdots\right\}^2,^\dagger \tag{4}$$

$$\sum_1^\infty n^{-s}d^r(n) = \{\zeta(s)\}^{2^r}\phi(s), \tag{5}$$

where $\phi(s)$ is absolutely convergent for $R(s) > \frac{1}{2}$, and in particular

$$\sum_1^\infty \frac{1}{n^s d(n)} = \prod_p \left\{p^s \log\left(\frac{1}{1-p^{-s}}\right)\right\} = \sqrt{\{\zeta(s)\}}\phi(s). \tag{6}$$

---

*If we assume the Riemann hypothesis, the error term here is of the form $O(n^{\frac{1}{2}+\epsilon})$.

†Mr Hardy has pointed out to me that this formula has been given already by Liouville, *Journal de Mathématiques*, Ser.2, Vol. II (1857), p.393.

$$\text{(D)} \qquad \frac{1}{d(1)} + \frac{1}{d(2)} + \frac{1}{d(3)} + \cdots + \frac{1}{d(n)}$$
$$= n\left\{\frac{A_1}{(\log n)^{\frac{1}{2}}} + \frac{A_2}{(\log n)^{\frac{3}{2}}} + \cdots + \frac{A_r}{(\log n)^{r-\frac{1}{2}}} + O\frac{1}{(\log n)^{r+\frac{1}{2}}}\right\}, \tag{7}$$

where
$$A_1 = \frac{1}{\sqrt{\pi}}\prod_p \left\{\sqrt{(p^2-p)}\log\left(\frac{p}{p-1}\right)\right\}$$

and $A_2, A_3, \ldots A_r$ are more complicated constants.
More generally

$$d^s(1) + d^s(2) + d^s(3) + \ldots + d^s(n)$$
$$= n\{A_1(\log n)^{2^s-1} + A_2(\log n)^{2^s-2} + \ldots + A_{2^s}\} + O(n^{\frac{1}{2}+\epsilon}),^* \tag{8}$$

if 2^s is an integer, and

$$d^s(1) + d^s(2) + d^s(3) + \ldots + d^s(n)$$
$$= n\left\{A_1(\log n)^{2^s-1} + A_2(\log n)^{2^s-2} + \ldots + \frac{A_{r+2^s}}{(\log n)^r} + O\left[\frac{1}{(\log n)^{r+1}}\right]\right\}, \tag{9}$$

if 2^s is not an integer, the A's being constants.

$$(E) \qquad d(1)d(2)d(3)\ldots d(n) = 2^{n(\log\log n + C)+\phi(n)}, \tag{10}$$

where
$$C = \gamma + \sum_2^\infty \left\{\log_2\left(1+\frac{1}{\nu}\right) - \frac{1}{\nu}\right\}(2^{-\nu} + 3^{-\nu} + 5^{-\nu} + \ldots).$$

Here $2, 3, 5, \ldots$ are the primes and

$$\frac{\phi(n)}{n} = \frac{\gamma-1}{\log n} + \frac{1!}{(\log n)^2}(\gamma+\gamma_1-1) + \frac{2!}{(\log n)^3}(\gamma+\gamma_1+\gamma_2-1) + \ldots$$
$$+\frac{(r-1)!}{(\log n)^r}(\gamma+\gamma_1+\gamma_2+\ldots+\gamma_{r-1}-1) + O\left\{\frac{1}{(\log n)^{r+1}}\right\},$$

where
$$\zeta(1+s) = \frac{1}{s} + \gamma - \gamma_1 s + \gamma_2 s^2 - \gamma_3 s^3 + \ldots$$

or
$$r!\gamma_r = \lim_{\nu\to\infty}\left\{(\log 1)^r + \frac{1}{2}(\log 2)^r + \ldots + \frac{1}{\nu}(\log \nu)^r - \frac{1}{r+1}(\log \nu)^{r+1}\right\}.$$

---

*Assuming the Riemann hypothesis.

$$\text{(F)} \qquad d(uv) = \sum_1^\infty \mu(n) d\left(\frac{u}{n}\right) d\left(\frac{v}{n}\right) = \sum \mu(\delta) d\left(\frac{u}{\delta}\right) d\left(\frac{v}{\delta}\right), \tag{11}$$

where δ is a common factor of u and v, and

$$\frac{1}{\zeta(s)} = \sum_1^\infty \frac{\mu(n)}{n^s}.$$

$$\text{(G)} \qquad \text{If} \quad D_v(n) = d(v) + d(2v) + \ldots + d(nv),$$

we have

$$D_v(n) = \sum \mu(\delta) d\left(\frac{v}{\delta}\right) D_1\left(\frac{n}{\delta}\right), \tag{12}$$

where δ is a divisor of v, and

$$D_v(n) = \alpha(v) n(\log n + 2\gamma - 1) + \beta(v) n + \Delta_v(n), \tag{13}$$

where

$$\sum_1^\infty \frac{\alpha(\nu)}{\nu^s} = \frac{\zeta^2(s)}{\zeta(1+s)}, \sum_1^\infty \frac{\beta(\nu)}{\nu^s} = -\frac{\zeta^2(s)\zeta'(1+s)}{\zeta^2(1+s)},$$

and

$$\Delta_v(n) = O(n^{\frac{1}{3}} \log n) \ ^*$$

$$\begin{aligned} (H) \qquad & d(v+c) + d(2v+c) + d(3v+c) + \ldots + d(nv+c) \\ & = \alpha_c(v) n(\log n + 2\gamma - 1) + \beta_c(v) n \Delta_{v,c}(n), \end{aligned} \tag{14}$$

where

$$\sum_1^\infty \frac{\alpha_c(\nu)}{\nu^s} = \frac{\zeta(s)\sigma_{-s}(|c|)}{\zeta(1+s)},$$

$$\sum_1^\infty \frac{\beta_c(\nu)}{\nu^s} = \frac{\zeta(s)\sigma_{-s}(|c|)}{\zeta(1+s)} \left\{ \frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1+s)}{\zeta(1+s)} + \frac{\sigma_{-s}{}'(|c|)}{\sigma_{-s}(|c|)} \right\},$$

$\sigma_s(n)$ being the sum of the sth powers of the divisors of n and $\sigma'_s(n)$ the derivative of $\sigma_s(n)$ with respect to s, and

$$\Delta_{v,c}(n) = O(n^{\frac{1}{3}} \log n).^\dagger$$

(I) The formulæ(1) and (2) are special cases of

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}$$

---

*It seems not unlikely that $\Delta_v(n)$ is of the form $O(n^{\frac{1}{4}+\epsilon})$. Mr Hardy has recently shewn that $\Delta_1(n)$ is not of the form $o\{(n \log n)^{\frac{1}{4}} \log\log n\}$. The same is true in this case also.

†It is very likely that the order of $\Delta_{v,c}(n)$ is the same as that of $\Delta_1(n)$.

$$= 1^{-s}\sigma_a(1)\sigma_b(1) + 2^{-s}\sigma_a(2)\sigma_b(2) + 3^{-s}\sigma_a(3)\sigma_b(3) + \cdots; \tag{15}$$

$$\frac{\eta(s)\eta(s-a)\eta(s-b)\eta(s-a-b)}{(1-2^{-2s+a+b})\zeta(2s-a-b)}$$
$$= 1^{-s}\sigma_a(1)\sigma_b(1) - 3^{-s}\sigma_a(3)\sigma_b(3) + 5^{-s}\sigma_a(5)\sigma_b(5) - \cdots \tag{16}$$

It is possible to find an approximate formula for the general sum

$$\sigma_a(1)\sigma_b(1) + \sigma_a(2)\sigma_b(2) + \ldots + \sigma_a(n)\sigma_b(n). \tag{17}$$

The general formula is complicated, The most interesting cases are $a = 0, b = 0$, when the formula is (3); $a = 0, b = 1$, when it is

$$\frac{\pi^4 n^2}{72\zeta(3)}(\log n + 2c) + nE(n), \tag{18}$$

where

$$c = \gamma - \frac{1}{4} + \frac{\zeta'(2)}{\zeta(2)} - \frac{\zeta'(3)}{\zeta(3)},$$

and the order of $E(n)$ is the same as that of $\Delta_1(n)$; and $a = 1, b = 1$, when it is

$$\frac{5}{6}n^3\zeta(3) + E(n), \tag{19}$$

where

$$E(n) = O\{n^2(\log n)^2\},\ E(n) \neq o(n^2 \log n).$$

(J) If $s > 0$, then

$$\sigma_s(1)\sigma_s(2)\sigma_s(3)\sigma_s(4)\ldots\sigma_s(n) = \theta c^n (n!)^s, \tag{20}$$

where

$$1 > \theta > (1-2^{-s})(1-3^{-s})(1-5^{-s})\ldots(1-\varpi^{-s}),$$

ϖ is the greatest prime not exceeding n, and

$$c = \prod_p \left\{ \left(\frac{p^{2s}-1}{p^{2s}-p^s}\right)^{1/p} \left(\frac{p^{3s}-1}{p^{3s}-p^s}\right)^{1/p^2} \left(\frac{p^{4s}-1}{p^{4s}-p^s}\right)^{1/p^3} \cdots \right\}.$$

(K) If $(\frac{1}{2} + q + q^4 + q^9 + q^{16} + \cdots)^2 = \frac{1}{4} + \sum_1^\infty r(n)q^n$,
so that

$$\zeta(s)\eta(s) = \sum_1^\infty r(n)n^{-s},$$

then

$$\frac{\zeta^2(s)\eta^2(s)}{(1+2^{-s})\zeta(2s)} = 1^{-s}r^2(1) + 2^{-s}r^2(2) + 3^{-s}r^2(3) + \cdots \tag{21}$$

$$r^2(1) + r^2(2) + r^2(3) + \cdots + r^2(n) = \frac{n}{4}(\log n + C) + O(n^{\frac{3}{5}+\epsilon}), \tag{22}$$

where

$$C = 4\gamma - 1 + \frac{1}{3}\log 2 - \log \pi + 4\log \Gamma(\frac{3}{4}) - \frac{12}{\pi^2}\zeta'(2).$$

These formulæ are analogous to (1) and (3).

On certain arithmetical functions

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1. Let $\sigma_s(n)$ denote the sum of the sth powers of the divisors of n (including 1 and n), and let

$$\sigma_s(0) = \tfrac{1}{2}\zeta(-s),$$

where $\zeta(s)$ is the Riemann Zeta-function. Further let

$$\sum_{r,s}(n) = \sigma_r(0)\sigma_s(n) + \sigma_r(1)\sigma_s(n-1) + \cdots + \sigma_r(n)\sigma_s(0). \tag{1}$$

In this paper I prove that

$$\sum_{r,s}(n) = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n)$$
$$+\frac{\zeta(1-r)+\zeta(1-s)}{r+s}n\sigma_{r+s-1}(n) + O\{n^{\frac{2}{3}(r+s+1)}\}, \tag{2}$$

whenever r and s are positive odd integers. I also prove that there is no error term on the right-hand side of (2) in the following nine cases: $r=1, s=1; r=1, s=3; r=1, s=5; r=1, s=7; r=1, s=11, r=3, s=3; r=3, s=5; r=3,, s=9; r=5, s=7$. That is to say $\sum_{r,s}(n)$ has a finite expression in terms of $\sigma_{r+s+1}(n)$ and $\sigma_{r+s-1}(n)$ in these nine cases; but for other values of r and s it involves other arithmetical functions as well.
It appears probable, from the empirical results I obtain in §§ 18-23, that the error term on the right-hand side of (2) is of the form

$$O\{n^{\frac{1}{2}(r+s+1+\epsilon)}\}, \tag{3}$$

where ϵ is any positive number, and not of the form

$$o\{n^{\frac{1}{2}(r+s+1)}\}. \tag{4}$$

But all I can prove rigorously is (i) that the error is of the form

$$O\{n^{\frac{2}{3}(r+s+1)}\}$$

in all cases, (ii) that it is of the form

$$O\{n^{\frac{2}{3}(r+s+\frac{3}{4})}\} \tag{5}$$

if $r+s$ is of the form $6m$, (iii) that it is of the form

$$O\{n^{\frac{2}{3}(r+s+\frac{1}{2})}\} \tag{6}$$

if $r+s$ is of the form $6m+4$, and (iv) that it is not of the form

$$o\{n^{\frac{1}{2}(r+s)}\}. \tag{7}$$

It follows from (2) that, if r and s are positive odd integers, then

$$\sum_{r,s}(n) \sim \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n). \tag{8}$$

It seems very likely that (8) is true for all positive values of r and s, but this I am at present unable to prove.

2. If $\sum_{r,s}(n)/\sigma_{r+s+1}(n)$ tends to a limit, then the limit must be

$$\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}.$$

For then

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sum_{r,s}(n)}{\sigma_{r+s+1}(n)} &= \lim_{n\to\infty}\frac{\sum_{r,s}(1)+\sum_{r,s}(2)+\cdots+\sum_{r,s}(n)}{\sigma_{r+s+1}(1)+\sigma_{r+s+1}(2)+\cdots+\sigma_{r+s+1}(n)}\\
&= \lim_{x\to 1}\frac{\sum_{r,s}(0)+\sum_{r,s}(1)x+\sum_{r,s}(2)x^2+\cdots}{\sigma_{r+s+1}(0)+\sigma_{r+s+1}(1)x+\sigma_{r+s+1}(2)x^2+\cdots}\\
&= \lim_{x\to 1}\frac{S_rS_s}{S_{r+s+1}},
\end{aligned}$$

where

$$S_r = \tfrac{1}{2}\zeta(-r)+\frac{1^r x}{1-x}+\frac{2^r x^2}{1-x^2}+\frac{3^r x^3}{1-x^3}+\cdots \tag{9}$$

Now it is known that, if $r>0$, then

$$S_r \sim \frac{\Gamma(r+1)\zeta(r+1)}{(1-x)^{r+1}}, \tag{10}$$

as $x\to 1$ *. Hence we obtain the result stated.

3. It is easy to see that

$$\begin{aligned}
&\sigma_r(1)+\sigma_r(2)+\sigma_r(3)+\cdots+\sigma_r(n)\\
&\quad = u_1+u_2+u_3+u_4+\cdots+u_n,
\end{aligned}$$

*Knopp, *Dissertation* (Berlin, 1907), p.34.

where

$$u_t = 1^r + 2^r + 3^r + \cdots + \left[\frac{n}{t}\right]^r.$$

From this it is easy to deduce that

$$\sigma_r(1) + \sigma_r(2) + \cdots + \sigma_r(n) \sim \frac{n^{r+1}}{r+1}\zeta(r+1)\ ^* \tag{11}$$

and

$$\sigma_r(1)(n-1)^s + \sigma_r(2)(n-2)^s + \cdots + \sigma_r(n-1)1^s \sim \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\zeta(r+1)n^{r+s+1},$$

provided $r > 0, s \geq 0$. Now

$$\sigma_s(n) > n^s,$$

and

$$\sigma_s(n) < n^s(1^{-s} + 2^{-s} + 3^{-s} + \cdots) = n^s\zeta(s).$$

From these inequalities and (1) it follows that

$$\underline{\lim}\frac{\sum_{r,s}(n)}{n^{r+s+1}} \geq \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\zeta(r+1), \tag{12}$$

if $r > 0$ and $s \geq 0$; and

$$\overline{\lim}\frac{\sum_{r,s}(n)}{n^{r+s+1}} \leq \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\zeta(r+1)\zeta(s), \tag{13}$$

if $r > 0$ and $s > 1$. Thus $n^{-r-s-1}\sum_{r,s}(n)$ oscillates between limits included in the interval

$$\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\zeta(r+1),\ \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\zeta(r+1)\zeta(s).$$

On the other hand $n^{-r-s-1}\sigma_{r+s+1}(n)$ oscillates between 1 and $\zeta(r+s+1)$, assuming values as near as we please to either of these limits. The formula (8) shews that the actual limits of indetermination of $n^{-r-s-1}\sum_{r,s}(n)$ are

$$\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)},$$

$$\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)\zeta(r+s+1)}{\zeta(r+s+2)}. \tag{14}$$

Naturally

$$\zeta(r+1) < \frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)} < \frac{\zeta(r+1)\zeta(s+1)\zeta(r+s+1)}{\zeta(r+s+2)} < \zeta(r+1)\zeta(s)\ .^\dagger$$

---

*(10) follows from this as an immediate corollary.

What is remarkable about the formula (8) is that it shews the asymptotic equality of two functions neither of which itself increases in a regular manner.

4. It is easy to see that, if n is a positive integer, then

$$\cot\tfrac{1}{2}\theta\sin n\theta = 1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos(n-1)\theta + \cos n\theta.$$

Suppose now that

$$\begin{aligned}&\left(\tfrac{1}{4}\cot\tfrac{1}{2}\theta + \frac{x\sin\theta}{1-x} + \frac{x^2\sin 2\theta}{1-x^2} + \frac{x^3\sin 3\theta}{1-x^3} + \cdots\right)^2\\ &= (\tfrac{1}{4}\cot\tfrac{1}{2}\theta)^2 + C_0 + C_1\cos\theta + C_2\cos 2\theta + C_3\cos 3\theta + \cdots,\end{aligned}$$

where C_n is independent of θ. Then we have

$$\begin{aligned} C_0 &= \tfrac{1}{2}\left(\frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} + \cdots\right)\\ &+ \tfrac{1}{2}\left\{\left(\frac{x}{1-x}\right)^2 + \left(\frac{x^2}{1-x^2}\right)^2 + \left(\frac{x^3}{1-x^3}\right)^2 + \cdots\right\}\\ &= \tfrac{1}{2}\left\{\frac{x}{(1-x)^2} + \frac{x^2}{(1-x^2)^2} + \frac{x^3}{(1-x^3)^2} + \cdots\right\}\\ &= \tfrac{1}{2}\left\{\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \cdots\right\}. \end{aligned} \tag{15}$$

Again

$$\begin{aligned} C_n &= \tfrac{1}{2}\frac{x^n}{1-x^n} + \frac{x^{n+1}}{1-x^{n+1}} + \frac{x^{n+2}}{1-x^{n+2}} + \frac{x^{n+3}}{1-x^{n+3}} + \cdots\\ &+ \frac{x}{1-x}\cdot\frac{x^{n+1}}{1-x^{n+1}} + \frac{x^2}{1-x^2}\cdot\frac{x^{n+2}}{1-x^{n+2}} + \frac{x^3}{1-x^3}\cdot\frac{x^{n+3}}{1-x^{n+3}} + \cdots\\ &- \tfrac{1}{2}\left\{\frac{x}{1-x}\cdot\frac{x^{n-1}}{1-x^{n-1}} + \frac{x^2}{1-x^2}\cdot\frac{x^{n-2}}{1-x^{n-2}} + \cdots + \frac{x^{n-1}}{1-x^{n-1}}\cdot\frac{x}{1-x}\right\}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{C_n}{x^n}(1-x^n) &= \tfrac{1}{2} + \left(\frac{x}{1-x} - \frac{x^{n+1}}{1-x^{n+1}}\right) + \left(\frac{x^2}{1-x^2} - \frac{x^{n+2}}{1-x^{n+2}}\right) + \cdots\\ &- \tfrac{1}{2}\left\{\left(1 + \frac{x}{1-x} + \frac{x^{n-1}}{1-x^{n-1}}\right) + \left(1 + \frac{x^2}{1-x^2} + \frac{x^{n-2}}{1-x^{n-2}}\right)\right. \end{aligned}$$

[†] For example when $r = 1$ and $s = 9$ this inequality becomes $1.64493\ldots < 1.64616\ldots < 1.64697\ldots < 1.64823\ldots$.

$$+\cdots+\left(1+\frac{x^{n-1}}{1-x^{n-1}}+\frac{x}{1-x}\right)\Bigg\}$$

$$= \quad \frac{1}{1-x^n}-\frac{n}{2}.$$

That is to say

$$C_n=\frac{x^n}{(1-x^n)^2}-\frac{nx^n}{2(1-x^n)}. \tag{16}$$

It follows that

$$\begin{aligned}&\left(\tfrac{1}{4}\cot\tfrac{1}{2}\theta+\frac{x\sin\theta}{1-x}+\frac{x^2\sin 2\theta}{1-x^2}+\frac{x^3\sin 3\theta}{1-x^3}+\cdots\right)^2\\&=\left(\tfrac{1}{4}\cot\tfrac{1}{2}\theta\right)^2+\frac{x\cos\theta}{(1-x)^2}+\frac{x^2\cos 2\theta}{(1-x^2)^2}+\frac{x^3\cos 3\theta}{(1-x^3)^2}+\cdots\\&+\tfrac{1}{2}\left\{\frac{x}{1-x}(1-\cos\theta)+\frac{2x^2}{1-x^2}(1-\cos 2\theta)+\frac{3x^3}{1-x^3}(1-\cos 3\theta)+\cdots\right\}.\end{aligned} \tag{17}$$

Similarly, using the equation

$$\begin{aligned}&\cot^2\tfrac{1}{2}\theta(1-\cos n\theta)=\\&\quad(2n-1)+4(n-1)\cos\theta+4(n-2)\cos 2\theta+\cdots+4\cos(n-1)\theta+\cos n\theta,\end{aligned}$$

we can shew that

$$\begin{aligned}&\Bigg\{\tfrac{1}{8}\cot^2\tfrac{1}{2}\theta+\tfrac{1}{12}+\frac{x}{1-x}(1-\cos\theta)+\frac{2x^2}{1-x^2}(1-\cos 2\theta)+\\&\qquad\frac{3x^3}{1-x^3}(1-\cos 3\theta)+\cdots\Bigg\}^2=\left(\tfrac{1}{8}\cot^2\tfrac{1}{2}\theta+\tfrac{1}{12}\right)^2\\&+\tfrac{1}{12}\left\{\frac{1^3x}{1-x}(5+\cos\theta)+\frac{2^3x^2}{1-x^2}(5+\cos 2\theta)+\frac{3^3x^3}{1-x^3}(5+\cos 3\theta)+\cdots\right\}.\end{aligned} \tag{18}$$

For example, putting $\theta=\frac{2}{3}\pi$ and $\theta=\frac{1}{2}\pi$ in (17), we obtain

$$\begin{aligned}&\left(\tfrac{1}{6}+\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\frac{x^5}{1-x^5}+\cdots\right)^2\\&=\tfrac{1}{36}+\tfrac{1}{3}\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{4x^4}{1-x^4}+\frac{5x^5}{1-x^5}+\cdots\right),\end{aligned} \tag{19}$$

where $1,2,4,5,\ldots$ are the natural numbers without the multiples of 3; and

$$\left(\tfrac{1}{4}+\frac{x}{1-x}-\frac{x^3}{1-x^3}+\frac{x^5}{1-x^5}-\frac{x^7}{1-x^7}+\cdots\right)^2$$

$$= \tfrac{1}{16} + \tfrac{1}{2}\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{5x^5}{1-x^5} + \cdots\right), \tag{20}$$

where $1, 2, 3, 5, \ldots$ are the natural numbers without the multiples of 4.

5. It follows from (18) that

$$\left(\frac{1}{2\theta^2} + \frac{\theta^2}{2!}S_3 - \frac{\theta^4}{4!}S_5 + \frac{\theta^6}{6!}S_7 - \cdots\right)^2$$
$$= \frac{1}{4\theta^4} + \tfrac{1}{2}S_3 - \tfrac{1}{12}\left(\frac{\theta^2}{2!}S_5 - \frac{\theta^4}{4!}S_7 + \frac{\theta^6}{6!}S_9 - \cdots\right), \tag{21}$$

where S_r is the same as in (9). Equating the coefficients of θ^n in both sides in (21), we obtain

$$\frac{(n-2)(n+5)}{12(n+1)(n+2)}S_{n+3} = \binom{n}{2}S_3S_{n-1} + \binom{n}{4}S_5S_{n-8} +$$
$$\binom{n}{6}S_7S_{n-5} + \cdots + \binom{n}{n-2}S_{n-1}S_3, \tag{22}$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!},$$

if n is an even integer greater than 2.

Let us now suppose that

$$\Phi_{r,s}(x) = \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} m^r n^s x^{mn}, \tag{23}$$

so that

$$\Phi_{r,s}(x) = \Phi_{s,r}(x),$$

and

$$\left.\begin{aligned} \Phi_{0,s}(x) &= \frac{1^s x}{1-x} + \frac{2^s x^2}{1-x^2} + \frac{3^s x^3}{1-x^3} + \cdots = S_s - \tfrac{1}{2}\zeta(-s), \\ \Phi_{1,s}(x) &= \frac{1^s x}{(1-x)^2} + \frac{2^s x^2}{(1-x^2)^2} + \frac{3^s x^3}{(1-x^3)^2} + \cdots \end{aligned}\right\}. \tag{24}$$

Further let

$$\left.\begin{aligned} P &= -24S_1 = 1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \cdots\right)^*, \\ Q &= 240S_3 = 1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \cdots\right), \\ R &= -540S_5 = 1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \frac{3^5x^3}{1-x^3} + \cdots\right) \end{aligned}\right\}. \tag{25}$$

The putting $n = 4, 6, 8, \ldots$ in (22) we obtain the results contained in the following table.

TABLE 1

1. $1 - 24\Phi_{0,1}(x) = P.$
2. $1 + 240\Phi_{0,3}(x) = Q.$
3. $1 - 504\Phi_{0,5}(x) = R.$
4. $1 + 480\Phi_{0,7}(x) = Q^2.$
5. $1 - 264\Phi_{0,9}(x) = QR.$
6. $691 + 65520\Phi_{0,11}(x) = 441Q^3 + 250R^2.$
7. $1 - 24\Phi_{0,13}(x) = Q^2 R.$
8. $3617 + 16320\Phi_{0,15}(x) = 1617Q^4 + 2000QR^2.$
9. $43867 - 28728\Phi_{0,17}(x) = 38367Q^3 R + 5500R^3.$
10. $174611 + 13200\Phi_{0,19}(x) = 53361Q^5 + 121250Q^2 R^2.$
11. $77683 - 552\Phi_{0,21}(x) = 57183Q^4 R + 20500QR^3.$
12. $236364091 + 131040\Phi_{0,23}(x) = 49679091Q^6 + 176400000Q^3 R^2$ $+10285000R^4.$
13. $657931 - 24\Phi_{0,25}(x) = 392931Q^5 R + 265000Q^2 R^3.$
14. $3392780147 + 6960\Phi_{0,27}(x) = 489693897Q^7 + 2507636250Q^4 R^2$ $+395450000QR^4.$
15. $1723168255201 - 171864\Phi_{0,29}(x) = 815806500201Q^6 R$ $+881340705000Q^3 R^3 + 26021050000R^5.$
16. $7709321041217 + 32640\Phi_{0,31}(x) = 764412173217Q^8$ $+5323905468000Q^5 R^2 + 1621003400000Q^2 R^4.$

In general

$$\tfrac{1}{2}\zeta(-s) + \Phi_{0,s}(x) = \sum K_{m,n} Q^m R^n, \tag{26}$$

where $K_{m,n}$ is a constant and m and n are positive integers (including zero) satisfying the equation

$$4m + 6n = s + 1.$$

This is easily proved by induction, using (22).

*If $x = q^2$, then in the notation of elliptic functions

$$\begin{aligned} P &= \frac{12\eta\omega}{\pi^2} = \left(\frac{2K}{\pi}\right)^2 \left(\frac{3E}{K} + k^2 - 2\right), \\ Q &= \frac{12g_2\omega^4}{\pi^4} = \left(\frac{2K}{\pi}\right)^4 (1 - k^2 + k^4), \\ R &= \frac{216g_3\omega^6}{\pi^6} = \left(\frac{2K}{\pi}\right)^6 (1 + k^2)(1 - 2k^2)(1 - \tfrac{1}{2}k^2). \end{aligned}$$

6. Again from (17) we have

$$\left(\frac{1}{2\theta}+\frac{\theta}{1!}S_1-\frac{\theta^3}{3!}S_3+\frac{\theta^5}{5!}S_5-\cdots\right)^2$$
$$=\frac{1}{4\theta^2}+S_1-\frac{\theta^2}{2!}\Phi_{1,2}(x)+\frac{\theta^4}{4!}\Phi_{1,4}(x)-\frac{\theta^6}{6!}\Phi_{1,6}(x)+\cdots$$
$$+\tfrac{1}{2}\left(\frac{\theta^2}{2!}S_3-\frac{\theta^4}{4!}S_5+\frac{\theta^6}{6!}S_7-\cdots\right). \tag{27}$$

Equating the coefficients of θ^n in both sides in (27) we obtain

$$\frac{n+3}{2(n+1)}S_{n+1}-\Phi_{1,n}(x)=\binom{n}{1}S_1S_{n-1}+\binom{n}{3}S_3S_{n-3}+$$
$$\binom{n}{5}S_5S_{n-5}+\cdots+\binom{n}{n-1}S_{n-1}S_1, \tag{28}$$

if n is a positive even integer. From this we deduce the results contained in Table II.

TABLE II

1. $288\Phi_{1,2}(x)=Q-P^2$.
2. $720\Phi_{1,4}(x)=PQ-R$.
3. $1008\Phi_{1,6}(x)=Q^2-PR$.
4. $720\Phi_{1,8}(x)=Q(PQ-R)$.
5. $1584\Phi_{1,10}(x)=3Q^3+2R^2-5PQR$.
6. $65520\Phi_{1,12}(x)=P(441Q^3+250R^2)-691Q^2R$.
7. $144\Phi_{1,14}(x)=Q(3Q^3+4R^2-7PQR)$.

In general

$$\Phi_{1,s}(x)=\sum K_{l,m,n}P^lQ^mR^n, \tag{29}$$

where $l\le 2$ and $2l+4m+6n=s+2$. This is easily proved by induction, using (28).

7. We have

$$\left.\begin{aligned}
x\frac{dP}{dx} &= -\ 24\Phi_{1,2}(x)=\frac{P^2-Q}{12},\\
x\frac{dQ}{dx} &= \ \ 240\Phi_{1,4}(x)=\frac{PQ-R}{3},\\
x\frac{dR}{dx} &= -\ 504\Phi_{1,6}(x)=\frac{PR-Q^2}{2}
\end{aligned}\right\} \tag{30}$$

Suppose now that $r<s$ and that $r+s$ is even. Then

$$\Phi_{r,s}(x)=\left(x\frac{d}{dx}\right)^r\Phi_{0,s-r}(x), \tag{31}$$

and $\Phi_{0,s-r}(x)$ is a polynomial in Q and R. Also

$$x\frac{dP}{dx},\ x\frac{dQ}{dx},\ x\frac{dR}{dx}$$

are polynomials in P, Q and R. Hence $\Phi_{r,s}(x)$ is a polynomial in P, Q and R. Thus we deduce the results contained in Table III.

TABLE III

1. $1728\Phi_{2,3}(x) = 3PQ - 2R - P^3.$
2. $1728\Phi_{2,5}(x) = P^2Q - 2PR + Q^2.$
3. $1728\Phi_{2,7}(x) = 2PQ^2 - P^2R - QR.$
4. $8640\Phi_{2,9}(x) = 9P^2Q^2 - 18PQR + 5Q^3 + 4R^2.$
5. $1728\Phi_{2,11}(x) = 6PQ^3 - 5P^2QR + 4PR^2 - 5Q^2R.$
6. $6912\Phi_{3,4}(x) = 6P^2Q - 8PR + 3Q^2 - P^4.$
7. $3456\Phi_{3,6}(x) = P^3Q - 3P^2R + 3PQ^2 - QR.$
8. $5184\Phi_{3,8}(x) = 6P^2Q^2 - 2P^3R - 6PQR + Q^3 + R^2.$
9. $20736\Phi_{4,5}(x) = 15PQ^2 - 20P^2R + 10P^3Q - 4QR - P^5.$
10. $41472\Phi_{4,7}(x) = 7(P^4Q - 4P^3R + 6P^2Q^2 - 4PQR) + 3Q^3 + 4R^2.$

In general

$$\Phi_{r,s}(x) = \sum K_{l,m,n}P^lQ^mR^n, \tag{32}$$

where $l-1$ does not exceed the smaller of r and s and

$$2l + 4m + 6n = r + s + 1.$$

The results contained in these three tables are of course really results in the theory of elliptic functions. For example Q and R are substantially the invariants g_2 and g_3, and the formulæ of Table I are equivalent to the formulæ which express the coefficients in the series

$$\wp(u) = \frac{1}{u^2} + \frac{g_2u^2}{20} + \frac{g_3u^4}{28} + \frac{g_2^2u^6}{1200} + \frac{3g_2g_3u^8}{6160} + \cdots$$

in terms of g_2 and g_3. The elementary proof of these formulæ given in the preceding sections seems to be of some interest in itself.

8. In what follows we shall require to know the form of $\Phi_{1,s}(x)$ more precisely than is shewn by the formula (29).
We have

$$\tfrac{1}{2}\zeta(-s) + \Phi_{0,s}(x) = \sum K_{m,n}Q^mR^n, \tag{33}$$

where s is an odd integer greater than 1 and $4m + 6n = s + 1$. Also

$$x\frac{d}{dx}(Q^mR^n) = \left(\frac{m}{3} + \frac{n}{2}\right)PQ^mR^n - \left(\frac{m}{3}Q^{m-1}R^{n+1} + \frac{n}{3}Q^{m+2}R^{n-1}\right). \tag{34}$$

Differentiating (33) and using (34) we obtain

$$\Phi_{1,s+1}(x) = \tfrac{1}{12}(s+1)P\{\tfrac{1}{2}\zeta(-s) + \Phi_{0,s}(x)\} + \sum K_{m,n}Q^m R^n, \tag{35}$$

where s is an odd integer greater than 1 and $4m + 6n = s + 3$. But when $s = 1$ we have

$$\Phi_{1,2}(x) = \frac{Q - P^2}{288}. \tag{36}$$

9. Suppose now that

$$\begin{aligned} F_{r,s}(x) &= \{\tfrac{1}{2}\zeta(-r) + \Phi_{0,r}(x)\}\{\tfrac{1}{2}\zeta(-s) + \Phi_{0,s}(x)\} \\ -\frac{\zeta(1-r)+\zeta(1-s)}{r+s}\Phi_{1,r+s}(x) &- \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)} \\ &\times\{\tfrac{1}{2}\zeta(-r-s-1) + \Phi_{0,r+s+1}(x)\}. \end{aligned} \tag{37}$$

Then it follows from (33), (35) and (36) that, if r and s are positive odd integers,

$$F_{r,s}(x) = \sum K_{m,n}Q^m R^n, \tag{38}$$

where

$$4m + 6n = r + s + 2.$$

But it is easy to see, from the functional equation satisfied by $\zeta(s)$, viz.

$$(2\pi)^{-s}\Gamma(s)\zeta(s)\cos\tfrac{1}{2}\pi s = \tfrac{1}{2}\zeta(1-s), \tag{39}$$

that

$$F_{r,s}(0) = 0. \tag{40}$$

Hence $Q^3 - R^2$ is a factor of the right-hand side in (38), that is to say

$$F_{r,s}(x) = (Q^3 - R^2)\sum K_{m,n}Q^m R^n, \tag{41}$$

where

$$4m + 6n = r + s - 10.$$

10. It is easy to deduce from (30) that

$$x\frac{d}{dx}\log(Q^3 - R^2) = P. \tag{42}$$

But it is obvious that

$$P = x\frac{d}{dx}\log[x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}]; \tag{43}$$

and the coefficient of x in $Q^3-R^2=1728$. Hence

$$Q^3-R^2=1728x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}. \tag{44}$$

But it is known that

$$\begin{aligned}&\{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots\}^3\\ &\quad=1-3x+5x^3-7x^6+9x^{10}-\cdots\end{aligned} \tag{45}$$

Hence

$$Q^3-R^2=1728x(1-3x+5x^3-7x^6+\cdots)^8. \tag{46}$$

The coefficient of $x^{\nu-1}$ in $1-3x+5x^3-\cdots$ is numerically less than $\sqrt{(8\nu)}$, and the coefficient of x^ν in Q^3-R^2 is therefore numerically less than that of x^ν in

$$1728x\{\sqrt{(8\nu)}(1+x+x^3+x^6+\cdots)\}^8.$$

But

$$x(1+x+x^3+x^6+\cdots)^8=\frac{1^3x}{1-x^2}+\frac{2^3x^2}{1-x^4}+\frac{3^3x^3}{1-x^6}+\cdots, \tag{47}$$

and the coefficient of x^ν in the right-hand side is positive and less than

$$\nu^3\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots\right).$$

Hence the coefficient of x^ν in Q^3-R^2 is of the form

$$\nu^4O(\nu^3)=O(\nu^7).$$

That is to say

$$Q^3-R^2=\sum O(\nu^7)x^\nu. \tag{48}$$

Differentiating (48) and using (42) we obtain

$$P(Q^3-R^2)=\sum O(\nu^8)x^\nu. \tag{49}$$

Differentiating this again with respect to x we have

$$A(P^2-Q)(Q^3-R^2)+BQ(Q^3-R^2)=\sum O(\nu^9)x^\nu,$$

where A and B are constants. But

$$P^2-Q=-288\Phi_{1,2}(x)=-288\left\{\frac{1^2x}{(1-x)^2}+\frac{2^2x^2}{(1-x^2)^2}+\cdots\right\},$$

and the coefficient of x^ν in the right-hand side is a constant multiple of $\nu\sigma_1(\nu)$. Hence

$$\begin{aligned}(P^2 - Q)(Q^3 - R^2) &= \sum O\nu\sigma_1(\nu)x^\nu \sum O(\nu^7)x^\nu \\ &= \sum O(\nu^8)\{\sigma_1(1) + \sigma_1(2) + \cdots \\ &\quad +\sigma_1(\nu)\}x^\nu = \sum O(\nu^{10})x^\nu,\end{aligned}$$

and so

$$Q(Q^3 - R^2) = \sum O(\nu^{10})x^\nu. \tag{50}$$

Differentiating this again with respect to x and using arguments similar to those used above, we deduce

$$R(Q^3 - R^2) = \sum O(\nu^{12})x^\nu. \tag{51}$$

Suppose now that m and n are any two positive integers including zero, and that $m + n$ is not zero. Then

$$\begin{aligned}Q^mR^n(Q^3 - R^2) &= Q(Q^3 - R^2)Q^{m-1}R^n \\ &= \sum O(\nu^{10})x^\nu\{\sum O(\nu^3)x^\nu\}^{m-1}\{\sum O(\nu^5)x^\nu\}^n \\ &= \sum O(\nu^{10})x^\nu \sum O(\nu^{4m-5})x^\nu \sum O(\nu^{6n-1})x^\nu \\ &= \sum O(\nu^{4m+6n+6})x^\nu,\end{aligned}$$

If m is not zero, Similarly we can shew that

$$\begin{aligned}Q^mR^n(Q^3 - R^2) &= R(Q^3 - R^2)Q^mR^{n-1} \\ &= \sum O(\nu^{4m+6n+6})x^\nu,\end{aligned}$$

if n is not zero. Therefore in any case

$$(Q^3 - R^2)Q^mR^n = \sum O(\nu^{4m+6n+6})x^\nu. \tag{52}$$

11. Now let r and s be any two positive odd integers including zero. Then, when $r + s$ is equal to 2,4,6,8 or 12, there are no values of m and n satisfying the relation

$$4m + 6n = r + s - 10$$

in (41); consequently in these cases

$$F_{r,s}(x) = 0. \tag{53}$$

When $r+s=10, m$ and n must both be zero, and this result does not apply; but it follows from (41) and (48) that

$$F_{r,s}(x) = \sum O(\nu^7)x^\nu. \tag{54}$$

And when $r+s \geq 14$ it follows from (52) that

$$F_{r,s}(x) = \sum O(\nu^{r+s-4})x^\nu. \tag{55}$$

Equating the coefficients of x^ν in both sides in (53), (54) and (55) we obtain

$$\begin{aligned}\sum_{r,s}(n) = {} & \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n) \\ & + \frac{\zeta(1-r)+\zeta(1-s)}{r+s}n\sigma_{r+s-1}(n) + E_{r,s}(n),\end{aligned} \tag{56}$$

where

$$\begin{aligned} E_{r,s}(n) &= 0, & r+s &= 2,4,6,8,12; \\ E_{r,s}(n) &= O(n^7), & r+s &= 10; \\ E_{r,s}(n) &= O(n^{r+s-4}), & r+s &\geq 14. \end{aligned}$$

Since $\sigma_{r+s+1}(n)$ is of order n^{r+s+1}, it follows that in all cases

$$\sum_{r,s}(n) \sim \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n). \tag{57}$$

The following table gives the values of $\sum_{r,s}(n)$ when $r+s=2,4,6,8,12$.

TABLE IV

1. $\sum_{1,1}(n) = \frac{5\sigma_3(n)-6n\sigma_1(n)}{12}$.

2. $\sum_{1,3}(n) = \frac{7\sigma_5(n)-10n\sigma_3(n)}{80}$.

3. $\sum_{3,3}(n) = \frac{\sigma_7(n)}{120}$.

4. $\sum_{1,5}(n) = \frac{10\sigma_7(n) - 21n\sigma_5(n)}{252}$.

5. $\sum_{3,5}(n) = \frac{11\sigma_9(n)}{5040}$.

6. $\sum_{1,7}(n) = \frac{11\sigma_9(n) - 30n\sigma_7(n)}{480}$.

7. $\sum_{5,7}(n) = \frac{\sigma_{13}(n)}{10080}$.

8. $\sum_{3,9}(n) = \frac{\sigma_{13}(n)}{2640}$.

9. $\sum_{1,11}(n) = \frac{691\sigma_{13}(n) - 2730n\sigma_{11}(n)}{65520}$.

12. In this connection it may be interesting to note that

$$\begin{aligned}\sigma_1(1)\sigma_3(n) + \sigma_1(3)\sigma_3(n-1) + \sigma_1(5)\sigma_3(n-2) + \cdots \\ + \sigma_1(2n+1)\sigma_3(0) = \tfrac{1}{240}\sigma_5(2n+1).\end{aligned} \tag{58}$$

This formula may be deduced from the identity

$$\begin{aligned}\frac{1^5 x}{1-x} + \frac{3^5 x^2}{1-x^3} + \frac{5^5 x^3}{1-x^5} + \cdots \\ = Q\left(\frac{x}{1-x} + \frac{3x^2}{1-x^3} + \frac{5x^3}{1-x^5} + \cdots\right),\end{aligned} \tag{59}$$

which can be proved by means of the theory of elliptic functions or by elementary methods.

13. More precise results concerning the order of $E_{r,s}(n)$ can be deduced from the theory of elliptic functions. Let

$$x = q^2.$$

Then we have

$$\left.\begin{array}{rcl} Q &=& \phi^8(q)\{1-(kk')^2\} \\ R &=& \phi^{12}(q)(k'^2 - k^2)\{1 + \frac{1}{2}(kk')^2\} \\ &=& \phi^{12}(q)\{1 + \frac{1}{2}(kk')^2\}\sqrt{\{1-(2kk')^2\}} \end{array}\right\}, \tag{60}$$

where $\phi(q) = 1 + 2q + 2q^4 + 2q^9 + \cdots$

But, if

$$f(q) = q^{\frac{1}{24}}(1-q)(1-q^2)(1-q^3)\cdots,$$

then we know that

$$\left.\begin{aligned}
2^{\frac{1}{6}}f(q) &= k^{\frac{1}{12}}k'^{\frac{1}{3}}\phi(q) \\
2^{\frac{1}{6}}f(-q) &= (kk')^{\frac{1}{12}}\phi(q) \\
2^{\frac{1}{3}}f(q^2) &= (kk')^{\frac{1}{6}}\phi(q) \\
2^{\frac{2}{3}}f(q^4) &= k^{\frac{1}{3}}k'^{\frac{1}{12}}\phi(q)
\end{aligned}\right\} \tag{61}$$

It follows from (41), (60) and (61) that, if $r+s$ is of the form $4m+2$, but not equal to 2 or to 6, then

$$F_{r,s}(q^2) = \frac{f^{4(r+s-4)}(-q)}{f^{2(r+s-10)}(q^2)} \sum_1^{\frac{1}{4}(r+s-6)} K_n \frac{f^{24n}(q^2)}{f^{24n}(-q)}, \tag{62}$$

and if $r+s$ is of the form $4m$, but not equal to 4, 8 or 12, then

$$F_{r,s}(q^2) = \frac{f^{4(r+s-6)}(-q)}{f^{2(r+s-10)}(q^2)}\{f^8(q) - 16f^8(q^4)\} \sum_1^{\frac{1}{4}(r+s-8)} K_n \frac{f^{24n}(q^2)}{f^{24n}(-q)}, \tag{63}$$

when K_n depends on r and s only. Hence it is easy to see that in all cases $F_{r,s}(q^2)$ can be expressed as

$$\begin{aligned}
\sum K_{a,b,c,d,e,h,k}\{f^3(-q)\}^a \left\{\frac{f^5(-q)}{f^2(q^2)}\right\}^b \left\{\frac{f^5(q^2)}{f^2(-q)}\right\}^c \left\{\frac{f^5(q)}{f^2(q^2)}f^3(q)\right\}^d& \\
\times \left\{\frac{f^5(q^4)}{f^2(q^2)}f^3(q^4)\right\}^e f^h(-q)f^k(q^2),&
\end{aligned} \tag{64}$$

where a, b, c, d, e, h, k are zero or positive integers such that

$$\begin{aligned}
a+b+c+2(d+e) &= [\tfrac{2}{3}(r+s+2)], \\
h+k &= 2(r+s+2) - 3[\tfrac{2}{3}(r+s+2)],
\end{aligned}$$

and $[x]$ denotes as usual the greatest integer in x. But

$$\left.\begin{aligned}
f(q) &= q^{\frac{1^2}{24}} - q^{\frac{5^2}{24}} - q^{\frac{7^2}{24}} + q^{\frac{11^2}{24}} + \cdots \\
f^3(q) &= q^{\frac{1^2}{8}} - 3q^{\frac{3^2}{8}} + 5q^{\frac{5^2}{8}} - 7q^{\frac{7^2}{8}} + \cdots \\
\frac{f^5(q)}{f^2(q^2)} &= q^{\frac{1^2}{24}} - 5q^{\frac{5^2}{24}} + 7q^{\frac{7^2}{24}} - 11q^{\frac{11^2}{24}} + \cdots \\
\frac{f^5(q^2)}{f^2(-q)} &= q^{\frac{1^2}{3}} - 2q^{\frac{2^2}{3}} + 4q^{\frac{4^2}{3}} - 5q^{\frac{5^2}{3}} + \cdots
\end{aligned}\right\}, \tag{65}$$

where $1, 2, 4, 5, \cdots$ are the natural numbers without the multiples of 3, and $1, 5, 7, 11, \cdots$ are the natural odd numbers without the multiples of 3.
Hence it is easy to see that

$$n^{-\frac{1}{2}(a+b+c)-d-e}E_{r,s}(n)$$

is not of higher order than the coefficient of q^{2n} in

$$\phi^a(q^{\frac{1}{8}})\phi^b(q^{\frac{1}{24}})\phi^c(q^{\frac{1}{3}})\{\phi(q^{\frac{1}{24}})\phi(q^{\frac{1}{8}})\}^d\{\phi(q^{\frac{2}{3}})\phi(q^{\frac{1}{2}})\}^e\phi^h(q^{\frac{1}{24}})\phi^k(q^{\frac{1}{12}}),$$

or the coefficient of q^{48n} in

$$\phi^{a+d}(q^3)\phi^{b+d+h}(q)\phi^c(q^8)\phi^e(q^{16})\phi^e(q^{12})\phi^k(q^2).$$

But the coefficient of q^ν in $\phi^2(q^2)$ cannot exceed that of q^ν in $\phi^2(q)$, since

$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2); \tag{66}$$

and it is evident that the coefficient of q^ν in $\phi(q^{4\lambda})$ cannot exceed that of q^ν in $\phi(q^\lambda)$. Hence it follows that

$$n^{-\frac{1}{2}[\frac{2}{3}(r+s+2)]}E_{r,s}(n)$$

is not of higher order than the coefficient of q^{48n} in

$$\phi^A(q)\phi^B(q^3)\phi^C(q^2),$$

where A, B, C are zero or positive integers such that

$$A + B + C = 2(r + s + 2) - 2[\tfrac{2}{3}(r + s + 2)],$$

and C is 0 or 1.
Now, if $r + s \geq 14$, we have

$$A + B + C \geq 12,$$

and so

$$A + B \geq 11.$$

Therefore one at least of A and B is greater than 5. But

$$\phi^6(q) = \sum_0^\infty O(\nu^2)q^\nu.^* \tag{67}$$

Hence it is easily deduced that

$$\phi^A(q)\phi^B(q^3)\phi^C(q^2) = \sum O\{\nu^{\frac{1}{2}(A+B+C)-1}\}q^\nu. \tag{68}$$

*See §§24–25.

It follows that

$$E_{r,s}(n) = O\{n^{r+s-\frac{1}{2}[\frac{2}{3}(r+s-1)]}\}, \tag{69}$$

If $r + s \geq 14$. We have already shewn in § 11 that, if $r + s = 10$, then

$$E_{r,s}(n) = O(n^7). \tag{70}$$

This agrees with (69). Thus we see that in all cases

$$E_{r,s}(n) = O\{n^{\frac{2}{3}(r+s+1)}\}; \tag{71}$$

and that, if $r + s$ is of the form $6m$, then

$$E_{r,s}(n) = O\{n^{\frac{2}{3}(r+s+\frac{3}{4})}\}, \tag{72}$$

and if of the form $6m + 4$, then

$$E_{r,s}(n) = O\{n^{\frac{2}{3}(r+s+\frac{1}{2})}\}. \tag{73}$$

14. I shall now prove that the order of $E_{r,s}(n)$ is not less than that of $n^{\frac{1}{2}(r+s)}$. In order to prove this result I shall follow the method used by Messrs Hardy and Littlewood in their paper "Some problems of Diophantine approximation" (II) *.

Let

$$q = e^{\pi i \tau}, q' = e^{\pi i T},$$

where

$$T = \frac{c + d\tau}{a + b\tau},$$

and

$$ad - bc = 1.$$

Also let

$$V = \frac{v}{a + b\tau}.$$

Then we have

$$\omega\sqrt{v}e^{\pi ibvV}\vartheta_1(v, \tau) = \sqrt{V}\vartheta_1(V, T), \tag{74}$$

where ω is an eighth root of unity and

$$\vartheta_1(v, \tau) = 2\sin\pi v \cdot q^{\frac{1}{4}}\Pi_1^{\infty}(1 - q^{2n})(1 - 2q^{2n}\cos 2\pi v + q^{4n}). \tag{75}$$

* *Acta Mathematica*, Vol. XXXVII, pp. 193 – 238.

From (75) we have

$$\log \vartheta_1(v,\tau) = \log(2, \sin \pi v) + \tfrac{1}{4}\log q - \sum_1^\infty \frac{q^{2n}(1+2\cos 2n\pi v)}{n(1-q^{2n})}. \tag{76}$$

It follows from (74) and (76) that

$$\begin{aligned} &\log \sin \pi v + \tfrac{1}{2}\log v + \tfrac{1}{4}\log q + \log \omega - \sum_1^\infty \frac{q^{2n}(1+2\cos 2n\pi v)}{n(1-q^{2n})} \\ &= \log \sin \pi V + \tfrac{1}{2}\log V + \tfrac{1}{4}\log q' - \pi i b v V - \sum_1^\infty \frac{{q'}^{2n}(1+2\cos 2n\pi V)}{n(1-{q'}^{2n})}. \end{aligned} \tag{77}$$

Equating the coefficients of v^{8+1} on the two sides of (77), we obtain

$$\begin{aligned} &(a+b\tau)^{s+1}\left\{\tfrac{1}{2}\zeta(-s) + \frac{1^s q^2}{1-q^2} + \frac{2^s q^4}{1-q^4} + \frac{3^s q^6}{1-q^6} + \cdots\right\} \\ &= \tfrac{1}{2}\zeta(-s) + \frac{1^s q'^2}{1-q'^2} + \frac{2^s q'^4}{1-q'^4} + \frac{3^s q'^6}{1-q'^6} + \cdots, \end{aligned} \tag{78}$$

provided that s is an odd integer greater than 1. If, in particular,we put $s = 3$ and $s = 5$ in (78) we obtain

$$\begin{aligned} &(a+b\tau)^4\left\{1 + 240\left(\frac{1^3 q^2}{1-q^2} + \frac{2^3 q^4}{1-q^4} + \frac{3^3 q^6}{1-q^6} + \cdots\right)\right\} \\ &= \left\{1 + 240\left(\frac{1^3 q'^2}{1-q'^2} + \frac{2^3 q'^4}{1-q'^4} + \frac{3^3 q'^6}{1-q'^6} + \cdots\right)\right\}, \end{aligned} \tag{79}$$

and

$$\begin{aligned} &(a+b\tau)^6\left\{1 - 504\left(\frac{1^5 q^2}{1-q^2} + \frac{2^5 q^4}{1-q^4} + \frac{3^5 q^6}{1-q^6} + \cdots\right)\right\} \\ &= \left\{1 - 504\left(\frac{1^5 q'^2}{1-q'^2} + \frac{2^5 q'^4}{1-q'^4} + \frac{3^5 q'^6}{1-q'^6} + \cdots\right)\right\}. \end{aligned} \tag{80}$$

It follows from (38), (79) and (80) that

$$(a+b\tau)^{r+s+2} F_{r,s}(q^2) = F_{r,s}(q'^2). \tag{81}$$

It can easily be seen from (56) and (37) that

$$F_{r,s}(x) = \sum_1^\infty E_{r,s}(n) x^n. \tag{82}$$

Hence

$$(a+b\tau)^{r+s+2}\sum_{1}^{\infty}E_{r,s}(n)q^{2n}=\sum_{1}^{\infty}E_{r,s}(n)q'^{2n}. \tag{83}$$

It is important to observe that

$$E_{r,s}(1)=\frac{\zeta(-r)+\zeta(-s)}{2}-\frac{\zeta(1-r)+\zeta(1-s)}{r+s}$$
$$-\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\neq 0, \tag{84}$$

if $r+s$ is not equal to 2,4,6,8 or 12. This is easily proved by the help of the equation (39).

15. Now let

$$\tau=u+iy, t=e^{-\pi y}(u>0,y>0,0<t<1),$$

so that

$$q=e^{\pi iu-\pi y}=te^{\pi iu};$$

and let us suppose that p_n/q_n is a convergent to

$$u=\frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+\cdots},$$

so that

$$\eta_n=p_{n-1}q_n-p_nq_{n-1}=\pm 1.$$

Further, let us suppose that

$$a=p_n, \qquad b=-q_n,$$
$$c=\eta_n p_{n-1}, \quad d=-\eta_n q_{n-1},$$

so that

$$ad-bc=\eta_n^2=1.$$

Furthermore, let

$$y=1/(q_nq'_{n+1}),$$

where

$$q'_{n+1}=a'_{n+1}q_n+q_{n-1},$$

and a'_{n+1} is the complete quotient corresponding to a_{n+1}.
Then we have

$$|a+b\tau|=|p_n-q_nu-iq_ny|=\frac{|\pm 1-i|}{q'_{n+1}}=\frac{\sqrt{2}}{q'_{n+1}}, \tag{85}$$

and

$$|q'|=e^{-\pi\lambda},$$

where

$$\lambda = \mathbf{I}(T) = \mathbf{I}\left(\frac{c+d\tau}{a+b\tau}\right) + \mathbf{I}\left\{\frac{d}{b} - \frac{1}{b(a+b\tau)}\right\}$$
$$= \frac{y}{(1/q'_{n+1})^2 + qn^2y^2} = \frac{q'_{n+1}}{2q_n}, \tag{86}$$

and $\mathbf{I}(T)$ is the imaginary part of T. It follows from (83), (85) and (86) that

$$|\sum_1^\infty E_{r,s}(n)q^{2n}| = \left(\frac{q'_{n+1}}{\sqrt{2}}\right)^{r+s+2} |\sum_1^\infty E_{r,s}(n)q'^{2n}|$$
$$\geq \left(\frac{q'_{n+1}}{\sqrt{2}}\right)^{r+s+2} \{|E_{r,s}(1)|e^{-2\pi\lambda} - |E_{r,s}(2)e^{-4\pi\lambda} - |E_{r,s}(3)|e^{-6\pi\lambda} - \cdots\}. \tag{87}$$

We can choose a number λ_0, depending only on r and s, such that

$$|E_{r,s}(1)|e^{-2\pi\lambda} > 2\{|E_{r,s}(2)|e^{-4\pi\lambda} + |E_{r,s}(3)|e^{-6\pi\lambda} + \cdots\}$$

for $\lambda \geq \lambda_0$. Let us suppose $\lambda_0 > 10$. Let us also suppose that the continued fraction for u satisfies the condition

$$4\lambda_0 q_n > q'_{n+1} > 2\lambda_0 q_n \tag{88}$$

for an infinity of values of n. Then

$$\left|\sum_1^\infty E_{r,s}(n)q^{2n}\right| \geq \tfrac{1}{2}|E_{r,s}(1)| \left(\frac{q'_{n+1}}{\sqrt{2}}\right)^{r+s+2} e^{-4\pi\lambda_0} > K(q'_{n+1})^{r+s+2}, \tag{89}$$

where K depends on r and s only. Also

$$q_n q'_{n+1} = 1/y,$$

$$q'_{n+1} > \frac{1}{\sqrt{y}} = \sqrt{\left\{\frac{\pi}{\log(1/t)}\right\}} > \frac{K}{\sqrt{(1-t)}}.$$

It follows that, if u is an irrational number such that the condition (88) is satisfied for an infinity of values of n, then

$$\left|\sum_1^\infty E_{r,s}(n)q^{2n}\right| > K(1-t)^{-\frac{1}{2}(r+s+2)} \tag{90}$$

for an infinity of values of t tending to unity. But if we had

$$E_{r,s}(n) = o\{n^{\frac{1}{2}(r+s)}\}$$

then we should have

$$|\sum_{1}^{\infty} E_{r,s}(n)q^{2n}| = o\{(1-t)^{-\frac{1}{2}(r+s+2)}\},$$

which contradicts (90). It follows that the error term in $\sum_{r,s}(n)$ is not of the form

$$o\{n^{\frac{1}{2}(r+s)}\}. \tag{91}$$

The arithmetical function $\tau(n)$.

16. We have seen that

$$E_{r,s}(n) = 0,$$

if $r+s$ is equal to 2,4,6,8, or 12. In these cases $\sum_{r,s}(n)$ has a finite expression in terms of $\sigma_{r+s+1}(n)$ and $\sigma_{r+s-1}(n)$. In other cases $\sum_{r,s}(n)$ involves other arithmetical functions as well. The simplest of these is the function $\tau(n)$ defined by

$$\sum_{1}^{\infty} \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}. \tag{92}$$

These cases arise when $r+s$ has one of the values 10, 14, 16, 18, 20 or 24.
Suppose that $r+s$ has one of these values. Then

$$\frac{1728\sum_1^\infty E_{r,s}(n)x^n}{(Q^3-R^2)E_{r,s}(1)}$$

is, by (41) and (82), equal to the corresponding one of the functions

$$1, Q, R, Q^2, QR, Q^2R.$$

In other words

$$\sum_{1}^{\infty} E_{r,s}(n)x^n = E_{r,s}(1)\sum_{1}^{\infty}\tau(n)x^n$$
$$\left\{1+\frac{2}{\zeta(11-r-s)}\sum_{1}^{\infty} n^{r+s-11}\frac{x^n}{1-x^n}\right\}. \tag{93}$$

We thus deduce the formulæ

$$E_{r,s}(n) = E_{r,s}(1)\tau(n), \tag{94}$$

if $r+s=10$; and

$$\sigma_{r+s-11}(0)E_{r,s}(n) = E_{r,s}(1)\{\sigma_{r+s-11}(0)\tau(n)$$

$$+\sigma_{r+s-11}(1)\tau(n-1)+\cdots+\sigma_{r+s-11}(n-1)\tau(1)\}, \tag{95}$$

if $r+s$ is equal to 14, 16, 18, 20 or 24. It follows from (94) and (95) that, if $r+s=r'+s'$, then

$$E_{r,s}(n)E_{r',s'}(1)=E_{r,s}(1)E_{r',s'}(n), \tag{96}$$

and in general

$$E_{r,s}(m)E_{r',s'}(n)=E_{r,s}(n)E_{r',s'}(m), \tag{97}$$

when $r+s$ has one of the values in question. The different cases in which $r+s$ has the same value are therefore not fundamentally distinct.

17. The values of $\tau(n)$ may be calculated as follows: differentiating (92) logarithmically with respect to x, we obtain

$$\sum_1^\infty n\tau(n)x^n = P\sum_1^\infty \tau(n)x^n. \tag{98}$$

Equating the coefficients of x^n in both sides in (98),we have

$$\tau(n)=\frac{24}{1-n}\{\sigma_1(1)\tau(n-1)+\sigma_1(2)\tau(n-2)+\cdots+\sigma_1(n-1)\tau(1)\}. \tag{99}$$

If, instead of starting with (92), we start with

$$\sum_1^\infty \tau(n)x^n = x(1-3x+5x^3-7x^6+\cdots)^8,$$

we can shew that

$$\begin{aligned}&(n-1)\tau(n)-3(n-10)\tau(n-1)+5(n-28)\tau(n-3)-7\\&(n-55)\tau(n-6)+\cdots \text{ to } [\tfrac{1}{2}\{1+\sqrt{(8n-7)}\}] \text{ terms } =0,\end{aligned} \tag{100}$$

where the rth term of the sequence 0,1,3,6, ... is $\frac{1}{2}r(r-1)$, and the rth term of the sequence 1,10,28,55, ... is $1+\frac{9}{2}r(r-1)$. We thus obtain the values of $\tau(n)$ in the following table.

TABLE V

n	$\tau(n)$	n	$\tau(n)$
1	$+1$	16	$+987136$
2	-24	17	-6905934
3	$+252$	18	$+2727432$
4	-1472	19	$+10661420$
5	$+4830$	20	-7109760
6	-6048	21	-4219488
7	-16744	22	-12830688
8	$+84480$	23	$+18643272$
9	-113643	24	$+21288960$
10	-115920	25	-25499225
11	$+534612$	26	$+13865712$
12	-370944	27	-73279080
13	-577738	28	$+24647168$
14	$+401856$	29	$+128406630$
15	$+1217160$	30	-29211840

18. Let us consider more particularly the case in which $r + s = 10$. The order of $E_{r,s}(n)$ is then the same as that of $\tau(n)$. The determination of this order is a problem interesting in itself. We have proved that $E_{r,s}(n)$, and therefore $\tau(n)$, is of the form $O(n^7)$ and not of the form $o(n^5)$. There is reason for supposing that $\tau(n)$ is of the form $O(n^{\frac{11}{2}+\epsilon})$ and not of the form $o(n^{\frac{11}{2}})$. For it appears that

$$\sum_1^\infty \frac{\tau(n)}{n^t} = \prod_p \frac{1}{1 - \tau(p)p^{-t} + p^{11-2t}}. \tag{101}$$

This assertion is equivalent to the assertion that, if

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_r^{a_r},$$

where $p_1, p_2, \ldots, p_r$ are the prime divisors of n, then

$$n^{-\frac{11}{2}}\tau(n) = \frac{\sin(1+a_1)\theta_{p_1}}{\sin\theta_{p_1}} \frac{\sin(1+a_2)\theta_{p_2}}{\sin\theta_{p_2}} \cdots \frac{\sin(1+a_r)\theta_{p_r}}{\sin\theta_{p_r}}, \tag{102}$$

where

$$\cos\theta_p = \tfrac{1}{2}p^{-\frac{11}{2}}\tau(p).$$

It would follow that, if n and n' are prime to each other, we must have

$$\tau(nn') = \tau(n)\tau(n'). \tag{103}$$

Let us suppose that (102) is true, and also that (as appears to be highly probable)

$$\{2\tau(p)\}^2 \leq p^{11}, \tag{104}$$

so that θ_p is real. Then it follows from (102) that

$$n^{-\frac{11}{2}}|\tau(n)| \leq (1+a_1)(1+a_2)\cdots(1+a_r),$$

that is to say

$$|\tau(n)| \leq n^{\frac{11}{2}} d(n), \tag{105}$$

where $d(n)$ denotes the number of divisors of n.
Now let us suppose that $n = p^a$, so that

$$n^{-\frac{11}{2}}\tau(n) = \frac{\sin(1+a)\theta_p}{\sin\theta_p}.$$

Then we can choose a as large as we please and such that

$$\left|\frac{\sin(1+a)\theta_p}{\sin\theta_p}\right| \geq 1.$$

Hence

$$|\tau(n)| \geq n^{\frac{11}{2}} \tag{106}$$

for an infinity of values of n.

19. It should be observed that precisely similar questions arise with regard to the arithmetical function $\Psi(n)$ defined by

$$\sum_0^\infty \Psi(n)x^n = f^{a_1}(x^{c_1})f^{a_2}(x^{c_2})\cdots f^{a_r}(x^{c_r}), \tag{107}$$

where

$$f(x) = x^{\frac{1}{24}}(1-x)(1-x^2)(1-x^3)\cdots,$$

the a's and c's are integers, the latter being positive,

$$\tfrac{1}{24}(a_1c_1 - a_2c_2 + \cdots + a_rc_r)$$

is equal to 0 or 1, and

$$l\left(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \cdots + \frac{a_r}{c_r}\right),$$

where l is the least common multiple of $c_1, c_2, \ldots, c_r$, is equal to 0 or to a divisor of 24. The arithmetical functions $\chi(n), P(n), \chi_4(n), \Omega(n)$ and $\Theta(n)$, studied by Dr. Glaisher in the *Quarterly Journal*, Vols. XXXVI-XXXVIII, are of this type. Thus

$$\sum_1^\infty \chi(n)x^n = f^6(x^4),$$

$$\begin{aligned}
\sum_1^\infty P(n)x^n &= f^4(x^2)f^4(x^4), \\
\sum_1^\infty \chi_4(n)x^n &= f^4(x)f^2(x^2)f^4(x^4), \\
\sum_1^\infty \Omega(n)x^n &= f^{12}(x^2), \\
\sum_1^\infty \Theta(n)x^n &= f^8(x)f^8(x^2).
\end{aligned}$$

20. The results (101) and (104) may be written as

$$\sum_1^\infty \frac{E_{r,s}(n)}{n^t} = E_{r,s}(1)\prod_p \frac{1}{1-2c_p p^{-t}+p^{r+s+1-2t}}, \tag{108}$$

where

$$c_p^2 \le p^{r+s+1},$$

and

$$2c_p E_{r,s}(1) = E_{r,s}(p).$$

It seems probable that the result (108) is true not only for $r+s=10$ but also when $r+s$ is equal to 14, 16, 18, 20 or 24, and that

$$\left|\frac{E_{r,s}(n)}{E_{r,s}(1)}\right| \le n^{\frac{1}{2}(r+s+1)}d(n) \tag{109}$$

for all values of n, and

$$\left|\frac{E_{r,s}(n)}{E_{r,s}(1)}\right| \ge n^{\frac{1}{2}(r+s+1)} \tag{110}$$

for an infinity of values of n. If this be so, then

$$E_{r,s}(n) = O\{n^{\frac{1}{2}(r+s+1+\epsilon)}\}, E_{r,s}(n) \ne o\{n^{\frac{1}{2}(r+s+1)}\}. \tag{111}$$

And it seems very likely that these equations hold generally, whenever r and s are positive odd integers.

21. It is of some interest to see what confirmation of these conjectures can be found from a study of the coefficients in the expansion of

$$x\{(1-x^{24/\alpha})(1-x^{48/\alpha})(1-x^{72/\alpha})\cdots\}^a = \sum_1^\infty \Psi_\alpha(n)x^n,$$

where α is a divisor of 24. When $\alpha = 1$ and $\alpha = 3$ we know the actual value of $\Psi_\alpha(n)$. For we have

$$\sum_1^\infty \Psi_1(n)x^n = x^{1^2} - x^{5^2} - x^{7^2} + x^{11^2} + x^{13^2} - x^{17^2} - \cdots, \tag{112}$$

where 1, 5, 7, 11, ... the natural odd numbers without the multiples of 3; and

$$\sum_1^\infty \Psi_3(n)x^n = x^{1^2} - 3x^{3^2} + 5x^{5^2} - 7x^{7^2} + \cdots \tag{113}$$

The corresponding Dirichlet's series are

$$\sum_1^\infty \frac{\Psi_1(n)}{n^s} = \frac{1}{(1+5^{-2s})(1+7^{-2s})(1-11^{-2s})(1-13^{-2s})\cdots}, \tag{114}$$

where 5, 7, 11, 13, ... are the primes greater than 3, those of the form $12n \pm 5$ having the plus sign and those of the form $12n \pm 1$ the minus sign; and

$$\sum_1^\infty \frac{\Psi_3(n)}{n^s} = \frac{1}{(1+3^{1-2s})(1-5^{1-2s})(1+7^{1-2s})(1+11^{1-2s})\cdots} \tag{115}$$

where 3, 5, 7, 11, ... are the odd primes, those of the form $4n-1$ having the plus sign and those of the form $4n+1$ the minus sign.

It is easy to see that

$$|\Psi_1(n)| \le 1, \quad |\Psi_3(n)| \le \sqrt{n} \tag{116}$$

for all values of n, and

$$|\Psi_1(n)| = 1, \quad |\Psi_3(n)| = \sqrt{n} \tag{117}$$

for an infinity of values of n.

The next simplest case is that in which $\alpha = 2$. In this case it appears that

$$\sum_1^\infty \frac{\Psi_2(n)}{n^s} = \Pi_1\Pi_2, \tag{118}$$

where

$$\Pi_1 = \frac{1}{(1+5^{-2s})(1-7^{-2s})(1-11^{-2s})(1+17^{-2s})\cdots},$$

5, 7, 11, ... being the primes of the forms $12n-1$ and $12n \pm 5$, those of the form $12n+5$ having the plus sign and the rest the minus sign; and

$$\Pi_2 = \frac{1}{(1+13^{-s})^2(1-37^{-s})^2(1-61^{-s})^2(1+73^{-s})^2\cdots},$$

13, 37, 61, … being the primes of the form $12n+1$, those of the form $m^2+(6n-3)^2$ having the plus sign and those of the form $m^2+(6n)^2$ the minus sign.
This is equivalent to the assertion that if

$$n = (5^{a_5} \cdot 7^{a_7} \cdot 11^{a_{11}} \cdot 17^{a_{17}} \cdots)^2 13^{a_{13}} \cdot 37^{a_{37}} \cdot 61^{a_{61}} \cdot 73^{a_{73}} \cdots,$$

where a_p is zero or a positive integer, then

$$\Psi_2(n) = (-1)^{a_5+a_{13}+a_{17}+a_{29}+a_{41}+\cdots}(1+a_{13})(1+a_{37})(1+a_{61})\cdots, \tag{119}$$

where 5, 13, 17, 29, … are the primes of the form $4n+1$, excluding those of the form $m^2+(6n)^2$; and that otherwise

$$\Psi_2(n) = 0. \tag{120}$$

It follows that

$$|\Psi_2(n)| \leq d(n) \tag{121}$$

for all values of n, and

$$|\Psi_2(n)| \geq 1 \tag{122}$$

for an infinity of values of n. These results are easily proved to be actually true.

22. I have investigated also the cases in which α has one of the values 4, 6, 8 or 12. Thus for example, when $\alpha = 6$, I find

$$\sum_1^\infty \frac{\Psi_6(n)}{n^s} = \Pi_1\Pi_2,^* \tag{123}$$

where

$$\Pi_1 = \frac{1}{(1-3^{2-2s})(1-7^{2-2s})(1-11^{2-2s})\cdots},$$

3, 7, 11, … being the primes of the form $4n-1$; and

$$\Pi_2 = \frac{1}{(1-2c_5\cdot 5^{-s}+5^{2-2s})(1-2c_{13}\cdot 13^{-s}+13^{2-2s})\cdots},$$

5, 13, 17, … being the primes of the form $4n+1$, and $c_p = u^2-(2v)^2$, where u and v are the unique pair of positive integers for which $p = u^2+(2v)^2$. This is equivalent to the assertion that if

$$n = (3^{a_3} \cdot 7^{a_7} \cdot 11^{a_{11}} \cdots)^2 \cdot 5^{a_5} \cdot 13^{a_{13}} \cdot 17^{a_{17}} \cdots,$$

*$\Psi_6(n)$ is Dr. Glaisher's $\lambda(n)$.

then

$$\frac{\Psi_6(n)}{n} = \frac{\sin(1+a_5)\theta_5}{\sin\theta_5} \cdot \frac{\sin(1+a_{13})\theta_{13}}{\sin\theta_{13}} \cdot \frac{\sin(1+a_{17})\theta_{17}}{\sin\theta_{17}} \cdots, \tag{124}$$

where

$$\tan\tfrac{1}{2}\theta_p = \frac{u}{2v} \quad (0 < \theta_p < \pi),$$

and that otherwise $\Psi_6(n) = 0$. From these results it would follow that

$$|\Psi_6(n)| \leq nd(n) \tag{125}$$

for all values of n, and

$$|\Psi_6(n)| \geq n \tag{126}$$

for an infinity of values of n. What can actually be proved to be true is that

$$|\Psi_6(n) < 2nd(n)$$

for all values of n, and

$$|\Psi_6(n)| \geq n$$

for an infinity of values of n.

23. In the case in which $\alpha = 4$ I find that, if

$$n = (5^{a_5} \cdot 11^{a_{11}} \cdot 17^{a_{17}} \cdots)^2 \cdot 7^{a_7} \cdot 13^{a_{13}} \cdot 19^{a_{19}} \cdots,$$

where 5, 11, 17, ... are the primes of the form $6m-1$ and 7, 13, 19, ... are those of the form $6m+1$, then

$$\frac{\Psi_4(n)}{\sqrt{n}} = (-1)^{a_5+a_{11}+a_{17}+\cdots} \frac{\sin(1+a_7)\theta_7}{\sin\theta_7} \cdot \frac{\sin(1+a_{13})\theta_{13}}{\sin\theta_{13}} \cdots, \tag{127}$$

where

$$\tan\theta_p = \frac{u\sqrt{3}}{1 \pm 3v} \quad (0 < \theta_p < \pi),$$

and u and v are the unique pair of positive integers for which $p = 3u^2 + (1 \pm 3v)^2$; and that $\Psi_4(n) = 0$ for other values.
In the case in which $\alpha = 8$ I find that, if

$$n = (2^{a_2} \cdot 5^{a_5} \cdot 11^{a_{11}} \cdots)^2 \cdot 7^{a_7} \cdot 13^{a_{13}} \cdot 19^{a_{19}} \cdots,$$

where 2, 5, 11, ... are the primes of the form $3m-1$ and 7, 13, 19, ... are those of the form $6m+1$, then

$$\frac{\Psi_8(n)}{n\sqrt{n}} = (-1)^{a_2+a_5+a_{11}+\cdots} \frac{\sin 3(1+a_7)\theta_7}{\sin 3\theta_7} \cdot \frac{\sin 3(1+a_{13})\theta_{13}}{\sin 3\theta_{13}} \cdots, \tag{128}$$

where θ_p is the same as in (127); and that $\Psi_8(n) = 0$ for other values.
The case in which $\alpha = 12$ will be considered in § 28.
In short, such evidence as I have been able to find, while not conclusive, points to the truth of the results conjectured in § 18.

24. Analysis similar to that of the preceding sections may be applied to some interesting arithmetical functions of a different kind. Let

$$\phi^s(q) = 1 + 2\sum_1^\infty r_s(n)q^n, \tag{129}$$

where

$$\phi(q) = 1 + 2q + 2q^4 + 2q^9 + \cdots,$$

so that $r_s(n)$ is the number of representations of n as the sum of s squares. Further let

$$\begin{aligned}\sum_1^\infty \delta_2(n)q^n &= 2\left(\frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \cdots\right)\\ &= 2\left(\frac{q}{1+q^2} + \frac{q^2}{1+q^4} + \frac{q^3}{1+q^6} + \cdots\right);\end{aligned} \tag{130}$$

$$(2^s-1)B_s\sum_1^\infty \delta_{2s}(n)q^n = s\left(\frac{1^{s-1}q}{1+q} + \frac{2^{s-1}q^2}{1-q^2} + \frac{3^{s-1}q^3}{1+q^3} + \cdots\right), \tag{131}$$

when s is a multiple of 4;

$$(2^s-1)B_s\sum_1^\infty \delta_{2s}(n)q^n = s\left(\frac{1^{s-1}q}{1-q} + \frac{2^{s-1}q^2}{1+q^2} + \frac{3^{s-1}q^3}{1-q^3} + \cdots\right), \tag{132}$$

when $s+2$ is a multiple of 4;

$$\begin{aligned}E_s\sum_1^\infty \delta_{2s}(n)q^n = 2^s&\left(\frac{1^{s-1}q}{1+q^2} + \frac{2^{s-1}q^2}{1+q^4} + \frac{3^{s-1}q^3}{1+q^6} + \cdots\right)\\ &+2\left(\frac{1^{s-1}q}{1-q}\frac{3^{s-1}q^3}{1-q^3} + \frac{5^{s-1}q^5}{1-q^5} - \cdots\right),\end{aligned} \tag{133}$$

when $s-1$ is a multiple of 4;

$$\begin{aligned}E_s\sum_1^\infty \delta_{2s}(n)q^n = 2^s&\left(\frac{1^{s-1}q}{1+q^2} + \frac{2^{s-1}q^2}{1+q^4} + \frac{3^{s-1}q^3}{1+q^6} + \cdots\right)\\ &-2\left(\frac{1^{s-1}q}{1-q} - \frac{3^{s-1}q^3}{1-q^3} + \frac{5^{s-1}q^5}{1-q^5} - \cdots\right),\end{aligned} \tag{134}$$

when $s+1$ is a multiple of 4. In these formulæ

$$B_2 = \tfrac{1}{6}, B_4 = \tfrac{1}{30}, B_6 = \tfrac{1}{42}, B_8 = \tfrac{1}{30}, B_{10} = \tfrac{5}{66}, \ldots$$

are Bernoulli's numbers, and

$$E_1 = 1, E_3 = 1, E_5 = 5, E_7 = 61, E_9 = 1385, \ldots$$

are Euler's numbers. Then $\delta_{2s}(n)$ is in all cases an arithmetical function depending on the real divisors of n; thus, for example, when $s+2$ is a multiple of 4, we have

$$(2^s - 1)B_s\delta_{2s}(n) = s\{\sigma_{s-1}(n) - 2^s\sigma_{s-1}(\tfrac{1}{4}n)\}, \tag{135}$$

where $\sigma_s(x)$ should be considered as equal to zero if x is not an integer.
Now let

$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n). \tag{136}$$

Then I can prove (see § 26) that

$$e_{2s}(n) = 0 \tag{137}$$

if $s = 1, 2, 3, 4$ and that

$$e_{2s}(n) = O(n^{s-1-\frac{1}{2}[\frac{2}{3}s]+\epsilon}) \tag{138}$$

for all positive integral values of s. But it is easy to see that, if $s \geq 3$, then

$$Hn^{s-1} < \delta_{2s}(n) < Kn^{s-1}, \tag{139}$$

where H and K are positive constants. It follows that

$$r_{2s}(n) \sim \delta_{2s}(n) \tag{140}$$

for all positive integral values of s.
It appears probable, from the empirical results I obtain at the end of this paper, that

$$e_{2s}(n) = O\{n^{\frac{1}{2}(s-1)+\epsilon}\} \tag{141}$$

for all positive integral values of s; and that

$$e_{2s}(n) \neq o\{n^{\frac{1}{2}(s-1)}\} \tag{142}$$

if $s \geq 5$. But all that I can actually prove is that

$$e_{2s}(n) = O(n^{s-1-\frac{1}{2}[\frac{2}{3}s]}) \tag{143}$$

if $s \geq 9$ and that

$$e_{2s}(n) \neq o(n^{\frac{1}{2}s-1}) \tag{144}$$

if $s \geq 5$.

25. Let

$$f_{2s}(q) = \sum_1^\infty e_{2s}(n)q^n = \sum_1^\infty \{r_{2s}(n) - \delta_{2s}(n)\}q^n. \tag{145}$$

Then it can be shewn by the theory of elliptic functions that

$$f_{2s}(q) = \phi^{2s}(q) \sum_{1 \leq n \leq \frac{1}{4}(s-1)} K_n (kk')^{2n}, \tag{146}$$

that is to say that

$$f_{2s}(q) = \frac{f^{4s}(-q)}{f^{2s}(q^2)} \sum_{1 \leq n \leq \frac{1}{4}(s-1)} K_n \frac{f^{24n}(q^2)}{f^{24n}(-q)}, \tag{147}$$

where $\phi(q)$ and $f(q)$ are the same as in § 13. We thus obtain the results contained in the following table.

TABLE VI

1. $f_2(q) = 0, \quad f_4(q) = 0, \quad f_6(q) = 0, \quad f_8(q) = 0.$
2. $5f_{10}(q) = 16\frac{f^{14}(q^2)}{f^4(-q)}, \quad f_{12}(q) = 8f^{12}(q^2).$
3. $61f_{14}(q) = 728f^4(-q)f^{10}(q^2), \quad 17f_{16}(q) = 256f^8(-q)f^8(q^2).$
4. $1385f_{18}(q) = 24416f^{12}(-q)f^6(q^2) - 256\frac{f^{30}(q^2)}{f^{12}(-q)}.$
5. $31f_{20}(q) = 616f^{16}(-q)f^4(q^2) - 128\frac{f^{28}(q^2)}{f^8(-q)}.$
6. $50521f_{22}(q) = 1103272f^{20}(-q)f^2(q^2) - 821888\frac{f^{26}(q^2)}{f^4(-q)}.$
7. $691f_{24}(q) = 16576f^{24}(-q) - 32768f^{24}(q^2).$

It follows from the last formula of Table VI that

$$\tfrac{691}{64}e_{24}(n) = (-1)^{n-1}259\tau(n) - 512\tau(\tfrac{1}{2}n), \tag{148}$$

where $\tau(n)$ is the same as in § 16, and $\tau(x)$ should be considered as equal to zero if x is not an integer.

Results equivalent to 1,2,3,4 of Table VI were given by Dr. Glaisher in the *Quarterly Journal*, Vol. XXXVIII. The arithmetical functions called by him

$$\chi_4(n),\ \Omega(n),\ W(n),\ \Theta(n),\ U(n)$$

are the coefficients of q^n in

$$\frac{f^{14}(q^2)}{f^4(-q)}, f^{12}(q^2), f^4(-q)f^{10}(q^2), f^8(q)f^8(q^2), f^{12}(-q)f^6(q^2).$$

He gave reduction formulæ for these functions and observed how the functions which I call $e_{10}(n), e_{12}(n)$ and $e_{16}(n)$ can be defined by means of the complex divisors of n. It is very likely that $\tau(n)$ is also capable of such a definition.

26. Now let us consider the order of $e_{2s}(n)$. It is easy to see from (147) that $f_{2s}(q)$ can be expressed in the form

$$\sum K_{a,b,c,h,k}\{f^3(-q)\}^a \left\{\frac{f^5(-q)}{f^2(q^2)}\right\}^b \left\{\frac{f^5(q^2)}{f^2(-q)}\right\}^c f^h(-q)f^k(q^2), \tag{149}$$

where a, b, c, h, k are zero or positive integers, such that

$$a+b+c = [\tfrac{2}{3}s],\ \ h+k = 2s-3[\tfrac{2}{3}s].$$

Proceeding as in § 13 we can easily shew that

$$n^{-\frac{1}{2}[\frac{2}{3}s]}e_{2s}(n)$$

cannot be of higher order than the coefficient of q^{24n} in

$$\phi^A(q)\phi^B(q^3)\phi^C(q^2), \tag{150}$$

where C is 0 or 1 and

$$A+B+C = 2s-2[\tfrac{2}{3}s].$$

Now, if $s \geq 5, A+B+C \geq 4$; and so $A+B \geq 3$. Hence one at least of A and B is greater than 1. But we know that

$$\phi^2(q) = \sum O(\nu^\epsilon)q^\nu.$$

It follows that the coefficient of q^{24n} in (150) is of order not exceeding

$$n^{\frac{1}{2}(A+B+C)-1+\epsilon}.$$

Thus

$$e_{2s}(n) = O(n^{s-1-\frac{1}{2}[\frac{2}{3}s]+\epsilon}) \tag{151}$$

for all positive integral values of s.

27. When $s \geq 9$ we can obtain a slightly more precise result.
If $s \geq 16$ we have $A+B+C \geq 12$; and so $A+B \geq 11$. Hence one at least of A and B is greater than 5. But

$$\phi^6(q) = \sum O(\nu^2)q^\nu.$$

It follows that the coefficient of q^{24n} in (150) is of order not exceeding

$$n^{\frac{1}{2}(A+B+C)-1},$$

or that

$$e_{2s}(n) = O(n^{s-1-\frac{1}{2}[\frac{2}{3}s]}), \tag{152}$$

if $s \geq 16$. We can easily shew that (152) is true when $9 \leq s \leq 16$ considering all the cases separately, using the identities.

$$\begin{aligned}
&f^{12}(-q)f^6(q^2) = \{f^3(-q)\}^4, \{f^3(q^2)\}^2,\\
&\frac{f^{30}(q^2)}{f^{12}(-q)} = \left\{\frac{f^5(q^2)}{f^2(-q)}\right\}^6,\\
&f^{16}(-q)f^4(q^2) = \left\{\frac{f^5(-q)}{f^2(q^2)}\right\}^4 \left\{\frac{f^5(q^2)}{f^2(-q)}\right\}^2 f^2(q^2),\\
&\frac{f^{28}(q^2)}{f^8(-q)} = \left\{\frac{f^5(q^2)}{f^2(-q)}\right\}^4 \{f^3(q^2)\}^2 f^2(q^2), \cdots,
\end{aligned}$$

and proceeding as in the previous two sections.
The argument of §§ 14-15 may also be applied to the function $e_{2s}(n)$. We find that

$$e_{2s}(n) \neq o(n^{\frac{1}{2}s-1}). \tag{153}$$

I leave the proof to the reader.

28. There is reason to suppose that

$$\left.\begin{aligned}
&e_{2s}(n) = O\{n^{\frac{1}{2}(s-1+\epsilon)}\}\\
&e_{2s}(n) \neq o\{n^{\frac{1}{2}(s-1)}\}
\end{aligned}\right\}, \tag{154}$$

if $s \geq 5$. I find, for example, that

$$\sum_1^\infty \frac{e_{10}(n)}{n^s} = \frac{e_{10}(1)}{1+2^{2-s}}\Pi_1\Pi_2, \tag{155}$$

where

$$\Pi_1 = \frac{1}{(1-3^{4-2s})(1-7^{4-2s})(1-11^{4-2s})\cdots},$$

3, 7, 11, ... being the primes of the form $4n-1$, and

$$\Pi_2 = \frac{1}{(1-2c_5 \cdot 5^{-s}+5^{4-2s})(1-2c_{13}\cdot 13^{-s}+13^{4-2s})\cdots},$$

5, 13, 17, ... being the primes of the form $4n+1$, and

$$c_p = u^2 - (4v)^2,$$

where u and v are the unique pair of positive integers satisfying the equation

$$u^2 + (4v)^2 = p^2.$$

The equation (155) is equivalent to the assertion that, if

$$n = (3^{a_3}\cdot 7^{a_7}\cdot 11^{a_{11}}\cdots)^2 \cdot 2^{a_2}\cdot 5^{a_5}\cdot 13^{a_{13}}\cdots,$$

where a_p is zero or a positive integer, then

$$\frac{e_{10}(n)}{n^2 e_{10}(1)} = (-1)^{a_2}\frac{\sin 4(1+a_5)\theta_5}{\sin 4\theta_5}\cdot\frac{\sin 4(1+a_{13})\theta_{13}}{\sin 4\theta_{13}}\cdots, \tag{156}$$

where

$$\tan\theta_p = \frac{u}{v} \quad (0<\theta_p<\tfrac{1}{2}\pi),$$

u and v being integers satisfying the equation $u^2+v^2=p$; and $e_{10}(n)=0$ otherwise. If this is true then we should have

$$\left|\frac{e_{10}(n)}{e_{10}(1)}\right| \le n^2 d(n) \tag{157}$$

for all values of n, and

$$\left|\frac{e_{10}(n)}{e_{10}(1)}\right| \ge n^2 \tag{158}$$

for an infinity of values of n. In this case we can prove that, if n is the square of a prime of the form $4m-1$, then

$$\left|\frac{e_{10}(n)}{e_{10}(1)}\right| = n^2.$$

Similarly I find that

$$\sum_1^\infty \frac{e_{12}(n)}{n^s} = e_{12}(1)\prod_p\left(\frac{1}{1+2c_p\cdot p^{-s}+p^{5-2s}}\right), \tag{159}$$

p being an odd prime and $c_p^2 \leq p^5$. From this it would follow that

$$\left|\frac{e_{12}(n)}{e_{12}(1)}\right| \leq n^{\frac{5}{2}} d(n) \tag{160}$$

for all values of n, and

$$\left|\frac{e_{12}(n)}{e_{12}(1)}\right| \geq n^{\frac{5}{2}} \tag{161}$$

for an infinity of values of n.
Finally I find that

$$\sum_1^{\infty} \frac{e_{16}(n)}{n^s} = \frac{e_{16}(1)}{1+2^{3-s}} \prod_p \left(\frac{1}{1+2c_p \cdot p^{-s} + p^{7-2s}}\right), \tag{162}$$

p being an odd prime and $c_p^2 \leq p^7$. From this it would follow that

$$\left|\frac{e_{16}(n)}{e_{16}(1)}\right| \leq n^{\frac{7}{2}} d(n) \tag{163}$$

for all values of n, and

$$\left|\frac{e_{16}(n)}{e_{16}(1)}\right| \geq n^{\frac{7}{2}} \tag{164}$$

for an infinity of values of n.
In the case in which $2s = 24$ we have

$$\tfrac{691}{64} e_{24}(n) = (-1)^{n-1} 259\tau(n) - 512\tau(\tfrac{1}{2}n).$$

I have already stated the reasons for supposing that

$$|\tau(n)| \leq n^{\frac{11}{2}} d(n)$$

for all values of n, and

$$|\tau(n)| \geq n^{\frac{11}{2}}$$

for an infinity of values of n.

A series for Euler's constant γ

Messenger of Mathematics, XLVI, 1917, 73 – 80

1. In a paper recently published in this Journal (Vol.XLIV, pp. 1 – 10), Dr. Glaisher proves a number of formulæ of the type

$$\gamma = 1 - 2\left(\frac{S_3}{3\cdot 4} + \frac{S_5}{5\cdot 6} + \frac{S_7}{7\cdot 8} + \cdots\right),$$

where

$$S_n = 1^{-n} + 2^{-n} + 3^{-n} + 4^{-n} + \cdots,$$

and conjectures the existence of a general formula

$$\gamma = \lambda_r - (r+1)(r+2)\cdots(2r)$$
$$\times\left\{\frac{S_3}{3(r+3)(r+4)\cdots(2r+2)} + \frac{S_5}{5(r+5)(r+6)\cdots(2r+4)} + \cdots\right\},$$

where λ_r is a rational number. I propose now to prove the general formula of which Dr. Glaisher's are particular cases: this formula is itself a particular case of still more general formulæ.

2. Let r and t be any two positive numbers. Then

$$\int_0^1 x^{r-1}(1-x)^{t-1}\log\Gamma(1-x)\,dx = \int_0^1 x^{t-1}(1-x)^{r-1}\log\Gamma(x)\,dx$$
$$= \int_0^1 x^{t-1}(1-x)^{r-1}\log\Gamma(1+x)\,dx - \int_0^1 x^{t-1}(1-x)^{r-1}\log x\,dx \tag{1}$$

But

$$\int_0^1 x^{r-1}(1-x)^{t-1}\log\Gamma(1-x)\,dx$$
$$= \int_0^1 x^{r-1}(1-x)^{t-1}\left\{\gamma x + S_2\frac{x^2}{2} + S_3\frac{x^3}{3} + \cdots\right\}\,dx$$
$$= \frac{\Gamma(1+r)\Gamma(t)}{\Gamma(1+r+t)}\gamma + \frac{\Gamma(2+r)\Gamma(t)}{\Gamma(2+r+t)}\frac{S_2}{2} + \frac{\Gamma(3+r)\Gamma(t)}{\Gamma(3+r+t)}\frac{S_3}{3} + \cdots \tag{2}$$

Similarly

$$\int_0^1 x^{t-1}(1-x)^{r-1}\log\Gamma(1+x)\ dx$$

$$= -\frac{\Gamma(1+t)\Gamma(r)}{\Gamma(1+r+t)}\gamma + \frac{\Gamma(2+t)\Gamma(r)}{\Gamma(2+r+t)}\frac{S_2}{2} - \frac{\Gamma(3+t)\Gamma(r)}{\Gamma(3+r+t)}\frac{S_3}{3} + \cdots \tag{2'}$$

And also

$$\begin{aligned}\int_0^1 x^{t-1}(1-x)^{r-1}\log x\ dx &= \frac{d}{dt}\int_0^1 x^{t-1}(1-x)^{r-1}\ dx = \frac{d}{dt}\left\{\frac{\Gamma(t)\Gamma(r)}{\Gamma(r+t)}\right\}\\ &= \frac{\Gamma(r)\Gamma(t)}{\Gamma(r+t)}\left\{\frac{\Gamma'(t)}{\Gamma(t)} - \frac{\Gamma'(r+t)}{\Gamma(r+t)}\right\}\\ &= -\frac{\Gamma(r)\Gamma(t)}{\Gamma(r+t)}\int_0^1 x^{t-1}\frac{1-x^r}{1-x}\ dx \end{aligned}\tag{3}$$

It follows from (1)–(3) that, if r and t are positive, then

$$\begin{aligned}&\frac{r}{1(r+t)}\gamma + \frac{r(r+1)}{2(r+t)(r+t+1)}S_2 + \frac{r(r+1)(r+2)}{3(r+t)(r+t+1)(r+t+2)}S_3 + \cdots\\ &\quad + \frac{t}{1(r+t)}\gamma - \frac{t(t+1)}{2(r+t)(r+t+1)}S_2 + \frac{t(t+1)(t+2)}{3(r+t)(r+t+1)(r+t+2)}S_3 - \cdots\\ &\quad = \int_0^1 \frac{x^{t-1}(1-x^r)}{1-x}\ dx \end{aligned}\tag{4}$$

Now, interchanging r and t in (4), and taking the sum and the difference of the two results, we see that, if r and t are positive, then

$$\frac{r+t}{1(r+t)}\gamma + \frac{r(r+1)(r+2)+t(t+1)(t+2)}{3(r+t)(r+t+1)(r+t+2)}S_3 + \cdots = \frac{1}{2}\int_0^1 \frac{x^{r-1}+x^{t-1}-2x^{r+t-1}}{1-x}\ dx; \tag{5}$$

and

$$\frac{r(r+1)-t(t+1)}{2(r+t)(r+t+1)}S_2 + \frac{r(r+1)(r+2)(r+3)-t(t+1)(t+2)(t+3)}{4(r+t)(r+t+1)(r+t+2)(r+t+3)}S_4 + \cdots = \frac{1}{2}\int_0^1 \frac{x^{t-1}-x^{r-1}}{1-x}\ dx. \tag{6}$$

The right-hand sides of (5) and (6) can be expressed in finite terms if r and t are rational. If, in particular, r and t are integers, then

$$\int_0^1 \frac{x^{r-1}+x^{t-1}-2x^{r+t-1}}{1-x}\,dx = \frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\cdots+\frac{1}{r+t-1}$$
$$+\frac{1}{t}+\frac{1}{t+1}+\frac{1}{t+2}+\cdots+\frac{1}{r+t-1};$$

and

$$\int_0^1 \frac{x^{t-1}-x^{r-1}}{1-x}\,dx = \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{r-1}\right)$$
$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{t-1}\right).$$

3. Let us now suppose that $t=r$ in (5). Then it is clear that

$$\gamma+\frac{(r+1)(r+2)}{3(2r+1)(2r+2)}S_3+\frac{(r+1)(r+2)(r+3)(r+4)}{5(2r+1)(2r+2)(2r+3)(2r+4)}S_5+\cdots$$
$$=\int_0^1 \frac{x^{r-1}(1-x^r)}{1-x}\,dx = \int_0^1 \frac{1+x^{2r-1}}{1+x}\,dx, \tag{7}$$

if $r>0$. If we suppose, in (7), that r is an integer, we obtain the formula conjectured by Dr Glaisher, the value of λ_r being

$$\int_0^1 \frac{1+x^{2r-1}}{1+x}\,dx = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2r-1}.$$

Again, dividing both sides in (6) by $r-t$ and making $t\to r$, we see that, if $r>0$, then

$$\frac{r+1}{2(2r+1)}\left(\frac{1}{r}+\frac{1}{r+1}\right)S_2$$
$$+\frac{(r+1)(r+2)(r+3)}{4(2r+1)(2r+2)(2r+3)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\frac{1}{r+3}\right)S_4+\cdots$$
$$=-\int_0^1 \frac{x^{r-1}\log x}{1-x}\,dx = \frac{1}{r^2}+\frac{1}{(r+1)^2}+\frac{1}{(r+2)^2}+\frac{1}{(r+3)^2}+ \tag{8}$$

Thus for example we have

$$\frac{\pi^2}{12} = (1+\tfrac{1}{2})\frac{S_2}{2\cdots 3} + (1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})\frac{S_4}{4\cdot 5}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6})\frac{S_6}{6\cdot 7}+\cdots.$$

4. If we start with the integral

$$\int_0^1 x^{r-1}(1-x)^{t-1}\log\Gamma\left(1-\frac{x}{2}\right)\,dx,$$

and proceed as in § 2, we can shew that, if r and t are positive, then

$$\begin{aligned}&\frac{r}{1(r+t)}S_1' + \frac{r(r+1)}{2(r+t)(r+t+1)}S_2' + \frac{r(r+1)(r+2)}{3(r+t)(t+t+1)(r+t+2)}S_3' + \cdots \\ &\quad -\frac{t}{1(r+t)}S_1' + \frac{t(t+1)}{2(r+t)(r+t+1)}S_2' - \frac{t(t+1)(t+2)}{3(r+t)(r+t+1)(r+t+2)}S_3' + \cdots \\ &\quad = \int_0^1 \frac{x^{t-1}(1-x^r)}{1-x}\,dx - \log\frac{\pi}{2},\end{aligned} \tag{9}$$

where

$$S_n' = 1^{-n} - 2^{-n} + 3^{-n} - 4^{-n} + \cdots$$

From (9) we can easily deduce that, if r and t are positive, then

$$\begin{aligned}&\frac{r(r+1)+t(t+1)}{2(r+t)(r+t+1)}S_2' \\ &\quad + \frac{r(r+1)(r+2)(r+3)+t(t+1)(t+2)(t+3)}{4(r+t)(r+t+1)(r+t+2)(r+t+3)}S_4' + \cdots \\ &\quad = \frac{1}{2}\int_0^1 \frac{x^{r-1}+x^{t-1}-2x^{r+t-1}}{1-x}\,dx - \log\frac{\pi}{2};\end{aligned} \tag{10}$$

and

$$\frac{r-t}{1(r+t)}S_1' + \frac{r(r+1)(r+2)-t(t+1)(t+2)}{3(r+t)(r+t+1)(r+t+2)}S_3' + \cdots = \frac{1}{2}\int_0^1 \frac{x^{t-1}-x^{r-1}}{1-x}\,dx. \tag{11}$$

As particular cases of (10) and (11), we have

$$\log\frac{\pi}{2} + \frac{r+1}{2(2r+1)}S_2' + \frac{(r+1)(r+2)(r+3)}{4(2r+1)(2r+2)(2r+3)}S_4' + \cdots = \int_0^1 \frac{1+x^{2r-1}}{1+x}\,dx; \tag{12}$$

and

$$\begin{aligned}
&\frac{1}{r}S_1' + \frac{(r+1)(r+2)}{3(2r+1)(2r+2)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}\right)S_3' \\
&\quad + \frac{(r+1)(r+2)(r+3)(r+4)}{5(2r+1)(2r+2)(2r+3)(2r+4)} \\
&\qquad \times \left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\frac{1}{r+3}+\frac{1}{r+4}\right)S_5' + \cdots \\
&\quad = \frac{1}{r^2}+\frac{1}{(r+1)^2}+\frac{1}{(r+2)^2}+\cdots,
\end{aligned} \tag{13}$$

provided that $r > 0$. Thus for example we have

$$1 = \log\frac{\pi}{2} + 2\left(\frac{S_2'}{2.3}+\frac{S_4'}{4.5}+\frac{S_6'}{6.7}+\cdots\right);$$

$$\frac{\pi^2}{12} = \frac{S_1'}{1.2} + \frac{S_3'}{3.4}(1+\frac{1}{2}+\frac{1}{3}) + \frac{S_5'}{5.6}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}) + \cdots.$$

5. The preceding results may be generalised as follows. Let $\zeta(s,x)$ denote the function represented by the series

$$x^{-s} + (x+1)^{-s} + (x+2)^{-s} + (x+3)^{-s} + \cdots (x > 0)$$

and its analytical continuations, so that $\zeta(s,1) = \zeta(s)$ and $\zeta(s,\frac{1}{2}) = (2^s - 1)\zeta(s), \zeta(s)$ being the Riemann ζ-function. Then

$$\begin{aligned}
&\int_0^1 x^{r-1}(1-x)^{t-1}\zeta(s,1-x)\,dx = \int_0^1 x^{t-1}(1-x)^{r-1}\zeta(s,x)\,dx \\
&= \int_0^1 x^{t-1}(1-x)^{r-1}\zeta(s,1+x)\,dx + \int_0^1 x^{t-s-1}(1-x)^{r-1}\,dx,
\end{aligned} \tag{14}$$

provided that r and t are positive. But we know that, if $|x| < 1$, then

$$\zeta(s,1-x) = \zeta(s) + \frac{s}{1!}\zeta(s+1)x + \frac{s(s+1)}{2!}\zeta(s+2)x^2 + \cdots; \tag{15}$$

and that

$$\int_0^1 x^{t-s-1}(1-x)^{r-1}\,dx = \frac{\Gamma(t-s)\Gamma(r)}{\Gamma(r-s+t)}, \tag{16}$$

provided that $t > s$. It follows from (14)–(16) that, if r and t are positive and $t > s$, then

$$\begin{aligned}&\left\{\zeta(s)+\frac{s}{1!}\frac{r}{r+t}\zeta(s+1)+\frac{s(s+1)}{2!}\frac{r(r+1)}{(r+t)(r+t+1)}\zeta(s+2)+\cdots\right\}\\&\quad-\left\{\zeta(s)-\frac{s}{1!}\frac{t}{r+t}\zeta(s+1)+\frac{s(s+1)}{2!}\frac{t(t+1)}{(r+t)(r+t+1)}\zeta(s+2)-\cdots\right\}\\&\quad=\frac{\Gamma(r+t)\Gamma(t-s)}{\Gamma(t)\Gamma(r-s+t)}.\end{aligned}\tag{17}$$

As particular cases of (17), we have

$$\begin{aligned}&\frac{s}{1!}\frac{r+t}{r+t}\zeta(s+1)+\frac{s(s+1)(s+2)}{3!}\frac{r(r+1)(r+2)+t(t+1)(t+2)}{(r+t)(r+t+1)(r+t+2)}\zeta(s+3)+\cdots\\&\qquad=\frac{1}{2}\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)}+\frac{\Gamma(r-s)}{\Gamma(r)}\right\},\end{aligned}\tag{18}$$

and

$$\begin{aligned}&\frac{s(s+1)}{2!}\frac{r(r+1)-t(t+1)}{(r+t)(r+t+1)}\zeta(s+2)\\&\quad+\frac{s(s+1)(s+2)(s+3)}{4!}\frac{r(r+1)(r+2)(r+3)-t(t+1)(t+2)(t+3)}{(r+t)(r+t+1)(r+t+2)(r+t+3)}\zeta(s+4)+\cdots\\&\quad=\frac{1}{2}\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)}-\frac{\Gamma(r-s)}{\Gamma(r)}\right\},\end{aligned}\tag{19}$$

provided that r and t are positive and greater than s. From (18) and (19) we deduce that, if r is positive and greater than s, then

$$\begin{aligned}&\frac{s}{1!}\frac{r}{2r}\zeta(s+1)+\frac{s(s+1)(s+2)}{3!}\frac{r(r+1)(r+2)}{2r(2r+1)(2r+2)}\zeta(s+3)+\cdots\\&\qquad=\frac{1}{2}\frac{\Gamma(2r)\Gamma(r-s)}{\Gamma(r)\Gamma(2r-s)},\end{aligned}\tag{20}$$

and

$$\begin{aligned}&\frac{s(s+1)}{2!}\frac{r(r+1)}{2r(2r+1)}\left(\frac{1}{r}+\frac{1}{r+1}\right)\zeta(s+2)\\&+\frac{s(s+1)(s+2)(s+3)}{4!}\frac{r(r+1)(r+2)(r+3)}{2r(2r+1)(2r+2)(2r+3)}\\&\times\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\frac{1}{r+3}\right)\zeta(s+4)+\cdots\end{aligned}$$

$$= \frac{1}{2}\frac{\Gamma(2r)\Gamma(r-s)}{\Gamma(r)\Gamma(2r-s)}\int_0^1 \frac{x^{r-s-1}(1-x^s)}{1-x}\,dx. \tag{21}$$

6. If we start with the integral

$$\int_0^1 x^{r-1}(1-x)^{t-1}\zeta\left(s, 1-\frac{x}{2}\right)\,dx,$$

and proceed as in § 5, we can shew that, if r and t are positive and $t > s$, then

$$\begin{aligned}
&\zeta_1(s) + \frac{s}{1!}\frac{r}{r+t}\zeta_1(s+1)\frac{s(s+1)}{2!}\frac{r(r+1)}{(r+t)(r+t+1)}\zeta_1(s+2) + \cdots \\
&+\zeta_1(s) - \frac{s}{1!}\frac{t}{r+t}\zeta_1(s+1) + \frac{s(s+1)}{2!}\frac{t(t+1)}{(r+t)(r+t+1)}\zeta_1(s+2) - \cdots \\
&\qquad = \frac{\Gamma(r+t)\Gamma(t-s)}{\Gamma(t)\Gamma(r-s+t)},
\end{aligned} \tag{22}$$

where $\zeta_1(s)$ is the function represented by the series

$$1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots$$

and its analytical continuations. From (22) we deduce that, if r and t are positive and greater than s, then

$$\begin{aligned}
&(1+1)\zeta_1(s) + \frac{s(s+1)}{2!}\frac{r(r+1)+t(t+1)}{(r+t)(r+t+1)}\zeta_1(s+2) + \cdots \\
&\qquad = \frac{1}{2}\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)} + \frac{\Gamma(r-s)}{\Gamma(r)}\right\};
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\frac{s}{1!}\frac{r-t}{r+t}\zeta_1(s+1) \\
+\;&\frac{s(s+1)(s+2)}{3!}\frac{r(r+1)(r+2)-t(t+1)(t+2)}{(r+t)(r+t+1)(r+t+2)}\zeta_1(s+3) + \cdots \\
=\;&\frac{1}{2}\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)} - \frac{\Gamma(r-s)}{\Gamma(r)}\right\}.
\end{aligned} \tag{24}$$

As particular cases of (23) and (24), we have

$$\zeta_1(s) + \frac{s(s+1)}{2!}\frac{r(r+1)}{2r(2r+1)}\zeta_1(s+2) + \cdots = \frac{1}{2}\frac{\Gamma(2r)\Gamma(r-s)}{\Gamma(r)\Gamma(2r-s)}, \tag{25}$$

and

$$\begin{aligned}\frac{s}{1!}\frac{r}{2r}\frac{1}{r}\zeta_1(s+1) \quad &+ \quad \frac{s(s+1)(s+2)}{3!}\frac{r(r+1)(r+2)}{2r(2r+1)(2r+2)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}\right)\zeta_1(s+3)+\cdots \\ &= \quad \frac{1}{2}\frac{\Gamma(2r)\Gamma(r-s)}{\Gamma(r)\Gamma(2r-s)}\int_0^1 \frac{s^{r-s-1}(1-x^s)}{1-x}\,dx, \qquad (26)\end{aligned}$$

provided that r is positive and greater than s.

On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$

Proceedings of the Cambridge Philosophical Society, XIX, 1917, 11 – 21

1. It is well known that all positive integers can be expressed as the sum of four squares. This naturally suggests the question: *For what positive integral values of* a, b, c, d, *can all positive integers be expressed in the form*

$$ax^2 + by^2 + cz^2 + du^2? \tag{1.1}$$

I prove in this paper that there are only 55 sets of values of a, b, c, d for which this is true. The more general problem of finding all sets of values of a, b, c, d for which all integers *with a finite number of exceptions* can be expressed in the form (1.1), is much more difficult and interesting. I have considered only very special cases of this problem, with two variables instead of four; namely, the cases in which (1.1) has one of the special forms

$$a(x^2 + y^2 + z^2) + bu^2, \tag{1.2}$$

and

$$a(x^2 + y^2) + b(z^2 + u^2). \tag{1.3}$$

These two cases are comparatively easy to discuss. In this paper I give the discussion of (1.2) only, reserving that of (1.3) for another paper.

2. Let us begin with the first problem. We can suppose, without loss of generality, that

$$a \le b \le c \le d. \tag{2.1}$$

if $a > 1$, then 1 cannot be expressed in the form (1.1); and so

$$a = 1 \tag{2.2}$$

If $b > 2$, then 2 is an exception; and so

$$1 \le b \le 2. \tag{2.3}$$

We have therefore only to consider the two cases in which (1.1) has one or other of the forms

$$x^2 + y^2 + cz^2 + du^2,\ x^2 + 2y^2 + cz^2 + du^2.$$

In the first case, If $c > 3$, then 3 is an exception; and so

$$1 \le c \le 3. \tag{2.31}$$

In the second case, if $c > 5$, then 5 is an exception; and so

$$2 \leq c \leq 5. \tag{2.32}$$

We can now distinguish 7 possible cases.

$$(2.41) \qquad x^2 + y^2 + z^2 + du^2.$$

If $d > 7, 7$ is an exception; and so

$$1 \leq d \leq 7. \tag{2.411}$$

$$(2.42) \qquad x^2 + y^2 + 2z^2 + du^2.$$

If $d > 14, 14$ is an exception; and so

$$2 \leq d \leq 14. \tag{2.421}$$

$$(2.43) \qquad x^2 + y^2 + 3z^2 + du^2.$$

If $d > 6, 6$ is an exception; and so

$$3 \leq d \leq 6. \tag{2.431}$$

$$(2.44) \qquad x^2 + 2y^2 + 2z^2 + du^2.$$

If $d > 7, 7$ is an exception; and so

$$2 \leq d \leq 7. \tag{2.441}$$

$$(2.45) \qquad x^2 + 2y^2 + 3z^2 + du^2.$$

If $d > 10, 10$ is an exception; and so

$$3 \leq d \leq 10. \tag{2.451}$$

$$(2.46) \qquad x^2 + 2y^2 + 4z^2 + du^2.$$

If $d > 14, 14$ is an exception; and so

$$4 \leq d \leq 14. \tag{2.461}$$

$$(2.47) \qquad x^2 + 2y^2 + 5z^2 + du^2.$$

If $d > 10, 10$ is an exception; and so

$$5 \leq d \leq 10. \tag{2.471}$$

We have thus eliminated all possible sets of values of a, b, c, d, except the following 55*:

*L. E. Dickson (*Bulletin of the Ameraican Math. Soc.* [vol XXXIII (1927), pp. 63 – 70]) has pointed out that Ramanujan has overlooked the fact that (1, 2, 5, 5) does not represent 15. Consequently, there are only 54 forms.

$1, 1, 1, 1$	$1, 2, 3, 5$	$1, 2, 4, 8$
$1, 1, 1, 2$	$1, 2, 4, 5$	$1, 2, 5, 8$
$1, 1, 2, 2$	$1, 2, 5, 5$	$1, 1, 2, 9$
$1, 2, 2, 2$	$1, 1, 1, 6$	$1, 2, 3, 9$
$1, 1, 1, 3$	$1, 1, 2, 6$	$1, 2, 4, 9$
$1, 1, 2, 3$	$1, 2, 2, 6$	$1, 2, 5, 9$
$1, 2, 2, 3$	$1, 1, 3, 6$	$1, 1, 2, 10$
$1, 1, 3, 3$	$1, 2, 3, 6$	$1, 2, 3, 10$
$1, 2, 3, 3$	$1, 2, 4, 6$	$1, 2, 4, 10$
$1, 1, 1, 4$	$1, 2, 5, 6$	$1, 2, 5, 10$
$1, 1, 2, 4$	$1, 1, 1, 7$	$1, 1, 2, 11$
$1, 2, 2, 4$	$1, 1, 2, 7$	$1, 2, 4, 11$
$1, 1, 3, 4$	$1, 2, 2, 7$	$1, 1, 2, 12$
$1, 2, 3, 4$	$1, 2, 3, 7$	$1, 2, 4, 12$
$1, 2, 4, 4$	$1, 2, 4, 7$	$1, 1, 2, 13$
$1, 1, 1, 5$	$1, 2, 5, 7$	$1, 2, 4, 13$
$1, 1, 2, 5$	$1, 1, 2, 8$	$1, 1, 2, 14$
$1, 2, 2, 5$	$1, 2, 3, 8$	$1, 2, 4, 14$
$1, 1, 3, 5$		

Of these 55 forms, the 12 forms

$1, 1, 1, 2$	$1, 1, 2, 4$	$1, 2, 4, 8$
$1, 1, 2, 2$	$1, 2, 2, 4$	$1, 1, 3, 3$
$1, 2, 2, 2$	$1, 2, 4, 4$	$1, 2, 3, 6$
$1, 1, 1, 4$	$1, 1, 2, 8$	$1, 2, 5, 10$

have been already considered by Liouville and Pepin*.

3. I shall now prove that all integers can be expressed in each of the 55 forms. In order to prove this we shall consider the seven cases (2.41)–(2.47) of the previous section separately. We shall require the following results concerning ternary quadratic arithmetical forms.

*There are a large number of short notes by Liouville in Vols. V–VIII of the second series of his Journal. See also Pepin, *ibid.*, Ser.4, Vol. VI, pp.1 – 67. The object of the work of Liouville and Pepin is rather different from mine, viz., to determine, in a number of special cases, explicit formulæ for the number of representations, in terms of other arithmetical functions.

The necessary and sufficient condition that a number *cannot* be expressed in the form

$$x^2 + y^2 + z^2 \tag{3.1}$$

is that it should be of the form

$$4^\lambda(8\mu+7), \quad (\lambda = 0, 1, 2, \cdots, \mu = 0, 1, 2, \cdots). \tag{3.11}$$

Similarly the necessary and sufficient conditions that a number *cannot* be expressed in the forms

$$x^2 + y^2 + 2z^2, \tag{3.2}$$

$$x^2 + y^2 + 3z^2, \tag{3.3}$$

$$x^2 + 2y^2 + 2z^2, \tag{3.4}$$

$$x^2 + 2y^2 + 3z^2, \tag{3.5}$$

$$x^2 + 2y^2 + 4z^2, \tag{3.6}$$

$$x^2 + 2y^2 + 5z^2, \tag{3.7}$$

are that it should be of the forms

$$4^\lambda(16\mu+14), \tag{3.21}$$

$$9^\lambda(9\mu+6), \tag{3.31}$$

$$4^\lambda(8\mu+7), \tag{3.41}$$

$$4^\lambda(16\mu+10), \tag{3.51}$$

$$4^\lambda(16\mu+14), \tag{3.61}$$

$$25^\lambda(25\mu+10) \text{ or } 25^\lambda(25\mu+15).^* \tag{3.71}$$

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*Results (3.11)–(3.71) may tempt us to suppose that there are similar simple results for the form $ax^2 + by^2 + cz^2$, whatever are the values of a, b, c. It appears, however, that in most cases there are no such simple results. For instance, the numbers which are not of the form $x^2 + 2y^2 + 10z^2$ are those belonging to one or other of the *four* classes

$$25^\lambda(8\mu+7),\ 25^\lambda(25\mu+5),\ 25^\lambda(25\mu+15),\ 25^\lambda(25\mu+20).$$

Here some of the numbers of the first class belong also to one of the next three classes.

Again, the even numbers which are not of the form $x^2 + y^2 + 10z^2$ are the numbers

$$4^\lambda(16\mu+6),$$

while the odd numbers that are not of the form, viz.

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, \cdots$$

do not seem to obey any simple law.

The result concerning $x^2 + y^2 + z^2$ is due to Cauchy: for a proof see. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, p.550. The other results can be proved in an analogous manner. The form $x^2 + y^2 + 2z^2$ has been considered by Lebesgue, and the form $x^2 + y^2 + 3z^2$ by Dirichlet. For references see Bachmann, *Zahlentheorie*, Vol,IV, p.149.

4. We proceed to consider the seven cases (2.41)–(2.47). In the first case we have to shew that any number N can be expressed in the form

$$N = x^2 + y^2 + z^2 + du^2, \tag{4.1}$$

d being any integer between 1 and 7 inclusive.
If N is not of the form $4^\lambda(8\mu + 7)$, we can satisfy (4.1) with $u = 0$. We may therefore suppose that $N = 4^\lambda(8\mu + 7)$.
First, suppose that d has one of the values 1,2,4,5,6. Take $u = 2^\lambda$. Then the number

$$N - du^2 = 4^\lambda(8\mu + 7 - d)$$

is plainly not of the form $4^\lambda(8\mu + 7)$, and is therefore expressible in the form $x^2 + y^2 + z^2$.
Next, let $d = 3$. If $\mu = 0$, take $u = 2^\lambda$. Then

$$N - du^2 = 4^{\lambda+1}.$$

If $\mu \geq 1$, take $u = 2^{\lambda+1}$. Then

$$N - du^2 = 4^\lambda(8\mu - 5).$$

I have succeeded in finding a law in the following six simple cases:

$$x^2 + y^2 + 4z^2,$$
$$x^2 + y^2 + 5z^2,$$
$$x^2 + y^2 + 6z^2,$$
$$x^2 + y^2 + 8z^2,$$
$$x^2 + 2y^2 + 6z^2,$$
$$x^2 + 2y^2 + 8z^2.$$

The numbers which are not of these forms are the numbers

$$4^\lambda(8\mu + 7) \text{ or } 8\mu + 3,$$
$$4^\lambda(8\mu + 3),$$
$$9^\lambda(9\mu + 3),$$
$$4^\lambda(16\mu + 14),\ 16\mu + 6, \text{ or } 4\mu + 3,$$
$$4^\lambda(8\mu + 5),$$
$$4^\lambda(8\mu + 7) \text{ or } 8\mu + 5.$$

In neither of these cases is $N - du^2$ of the form $4^\lambda(8\mu + 7)$, and therefore in either case it can be expressed in the form $x^2 + y^2 + z^2$.
Finally, let $d = 7$. If μ is equal to 0,1, or 2, take $u = 2^\lambda$. Then $N - du^2$ is equal to $0, 2 \cdot 4^{\lambda+1}$, or $4^{\lambda+2}$. If $\mu \geq 3$, take $u = 2^{\lambda+1}$. Then

$$N - du^2 = 4^\lambda(8\mu - 21).$$

Therefore in either case $N - du^2$ can be expressed in the form $x^2 + y^2 + z^2$.
Thus in all cases N is expressible in the form (4.1). Similarly we can dispose of the remaining cases, with the help of the results stated in § 3. Thus in discussing (2.42) we use the theorem that every number not of the form (3.21) can be expressed in the form (3.2). The proofs differ only in detail, and it is not worth while to state them at length.

5. We have seen that all integers without any exception can be expressed in the form

$$m(x^2 + y^2 + z^2) + nu^2, \tag{5.1}$$

when

$$m = 1, \ \ 1 \leq n \leq 7,$$

and

$$m = 2, \ \ n = 1.$$

We shall now consider the values of m and n for which all integers *with a finite number of exceptions* can be expressed in the form (5.1).
In the first place m must be 1 or 2. For, if $m > 2$, we can choose an integer ν so that

$$nu^2 \not\equiv \nu (\mathrm{mod}\, m)$$

for all values of u. Then

$$\frac{(m\mu + \nu) - nu^2}{m},$$

where μ is any positive integer, is not an integer; and so $m\mu + \nu$ can certainly not be expressed in the form (5.1).
We have therefore only to consider the two cases in which m is 1 or 2. First let us consider the form

$$x^2 + y^2 + z^2 + nu^2. \tag{5.2}$$

I shall shew that, when n has any of the values

$$1, 4, 9, 17, 25, 36, 68, 100, \tag{5.21}$$

or is of any of the forms

$$4k + 2, \ 4k + 3, \ 8k + 5, \ 16k + 12, \ 32k + 20, \tag{5.22}$$

then all integers save a finite number, and in fact all integers from $4n$ onwards at any rate, can be expressed in the form (5.2); but that for the remaining values of n there is an infinity of integers which cannot be expressed in the form required.

In proving the first result we need obviously only consider numbers of the form $4^{\lambda}(8\mu+7)$ greater than n, since otherwise we may take $u=0$. The numbers of this form less than n are plainly among the exceptions.

6. I shall consider the various cases which may arise in order of simplicity.

$$(6.1) \qquad n \equiv 0 (\bmod 8).$$

There are an infinity of exceptions. For suppose that

$$N = 8\mu + 7.$$

Then the number

$$N - nu^2 \equiv 7 (\bmod 8)$$

cannot be expressed in the form $x^2+y^2+z^2$.

$$(6.2) \qquad n \equiv 2 (\bmod 4).$$

There is only a finite number of exceptions. In proving this we may suppose that $N = 4^{\lambda}(8\mu+7)$. Take $u=1$. Then the number

$$N - nu^2 = 4^{\lambda}(8\mu+7) - n$$

is congruent to $1, 2, 5$ or 6 to modulus 8, and so can be expressed in the form $x^2+y^2+z^2$. Hence the only numbers which cannot be expressed in the form (5.2) in this case are the numbers of the form $4^{\lambda}(8\mu+7)$ not exceeding n.

$$(6.3) \qquad n \equiv 5 (\bmod 8).$$

There is only a finite number of exceptions. We may suppose again that $N = 4^{\lambda}(8\mu+7)$. First, let $\lambda \neq 1$. Take $u=1$. Then

$$N - nu^2 = 4^{\lambda}(8\mu+7) - n \equiv 2 \text{ or } 3 (\bmod 8).$$

If $\lambda = 1$ we cannot take $u=1$, since

$$N - n \equiv 7 (\bmod 8);$$

so we take $u=2$. Then

$$N - nu^2 = 4^{\lambda}(8\mu+7) - 4n \equiv 8 (\bmod 32).$$

In either of these cases $N - nu^2$ is of the form $x^2+y^2+z^2$.

Hence the only numbers which cannot be expressed in the form (5.2) are those of the form $4^\lambda(8\mu+7)$ not exceeding n, and those of the form $4(8\mu+7)$ lying between n and $4n$.

$$(6.4) \qquad n \equiv 3 (\bmod 4).$$

There is only a finite number of exceptions. Take

$$N = 4^\lambda(8\mu+7).$$

if $\lambda \geq 1$, take $u = 1$. Then

$$N - nu^2 \equiv 1 \text{ or } 5 (\bmod 8).$$

if $\lambda = 0$, take $u = 2$. Then

$$N - nu^2 \equiv 3 (\bmod 8).$$

In either case the proof is completed as before.

In order to determine precisely which are the exceptional numbers, we must consider more particularly the numbers between n and $4n$ for which $\lambda = 0$. For these u must be 1, and

$$N - nu^2 \equiv 0 (\bmod 4).$$

But the numbers which are multiples of 4 and which cannot be expressed in the form $x^2 + y^2 + z^2$ are the numbers

$$4^\kappa(8\nu+7), \ (\kappa = 1, 2, 3, \ldots, \nu = 0, 1, 2, 3, \ldots).$$

The exceptions required are therefore those of the numbers

$$n + 4^\kappa(8\nu+7) \tag{6.41}$$

which lie between n and $4n$ and are of the form

$$8\mu + 7 \qquad (6.42).$$

Now in order that (6.41) may be of the form (6.42), κ must be 1 if n is of the form $8k+3$, and κ may have any of the values 2, 3, 4, ... if n is of the form $8k+7$. Thus the only numbers which cannot be expressed in the form (5.2), in this case, are those of the form $4^\lambda(8\mu+7)$ less than n and those of the form

$$n + 4^\kappa(8\nu+7), \quad (\nu = 0, 1, 2, 3, \cdots),$$

lying between n and $4n$, where $\kappa = 1$ if n is of the form $8k+3$, and $\kappa > 1$ if n is of the form $8k+7$.

$$(6.5) \qquad n \equiv 1 (\bmod 8).$$

In this case we have to prove that

(i) if $n \geq 33$, there is an infinity of integers which cannot be expressed in the form (5.2);

(ii) if n is $1, 9, 17$, or 25, there is only a finite number of exceptions.
In order to prove (i) suppose that $N = 7.4^\lambda$. Then obviously u cannot be zero. But if u is not zero u^2 is always of the form $4^\kappa(8\nu + 1)$. Hence

$$N - nu^2 = 7 \cdot 4^\lambda - n \cdot 4^\kappa(8\nu + 1).$$

Since $n \geq 33, \lambda$ must be greater than or equal to $\kappa + 2$, to ensure that the right-hand side shall not be negative. Hence

$$N - nu^2 = 4^\kappa(8k + 7),$$

where

$$k = 14 \cdot 4^{\lambda-\kappa-2} - n\nu - \frac{1}{8}(n + 7)$$

is an integer; and so $N - nu^2$ is not of the form $x^2 + y^2 + z^2$.
In order to prove (ii) we may suppose, as usual, that

$$N = 4^\lambda(8\mu + 7).$$

If $\lambda = 0$, take $u = 1$. Then

$$N - nu^2 = 8\mu + 7 - n \equiv 6 \pmod 8.$$

If $\lambda \geq 1$, take $u = 2^{\lambda-1}$. Then

$$N - nu^2 = 4^{\lambda-1}(8k + 3),$$

where

$$k = 4(\mu + 1) - \frac{1}{8}(n + 7).$$

In either case the proof may be completed as before. Thus the only numbers which cannot be expressed in the form (5.2), in this case, are those of the form $8\mu + 7$ not exceeding n. In other words, there is no exception when $n = 1$; 7 is the only exception when $n = 9$; 7 and 15 are the only exceptions when $n = 17$; 7, 15 and 23 are the only exceptions when $n = 25$.

(6.6) $\quad n \equiv 4 (\mathrm{mod}\, 32).$

By arguments similar to those used in (6.5), we can shew that
(i) if $n \geq 132$, there is an infinity of integers which cannot be expressed in the form (5.2);
(ii) if n is equal to 4, 36, 68, or 100, there is only a finite number of exceptions, namely the numbers of the form $4^\lambda(8\mu + 7)$ not exceeding n.

(6.7) $\quad n \equiv 20 (\mathrm{mod}\, 32).$

By arguments similar to those used in (6.3), we can shew that the only numbers which cannot be expressed in the form (5.2) are those of the form $4^\lambda(8\mu+7)$ not exceeding n, and those of the form $4^2(8\mu + 7)$ lying between n and $4n$.

(6.8) $\quad n \equiv 12 (\mathrm{mod}\, 16).$

By arguments similar to those used in (6.4), we can shew that the only numbers which cannot be expressed in the form (5.2) are those of the form $4^\lambda(8\mu+7)$ less than n, and those of the form

$$n+4^\kappa(8\nu+7), \quad (\nu=0,1,2,3,\ldots),$$

lying between n and $4n$ where $\kappa=2$ if n is of the form $4(8k+3)$ and $\kappa>2$ if n is of the form $4(8k+7)$.

We have thus completed the discussion of the form (5.2), and determined the exceptional values of N precisely whenever they are finite in number.

7. We shall proceed to consider the form

$$2(x^2+y^2+z^2)+nu^2. \tag{7.1}$$

In the first place n must be odd; otherwise the odd numbers cannot be expressed in this form. Suppose then that n is odd. I shall shew that all integers save a finite number can be expressed in the form (7.1); and that the numbers which cannot be so expressed are

(i) the odd numbers less than n,

(ii) the numbers of the form $4^\lambda(16\mu+14)$ less that $4n$,

(iii) the numbers of the form $n+4^\lambda(16\mu+14)$ greater than n less than $9n$,

(iv) the numbers of the form

$$cn+4^\kappa(16\nu+14), \qquad (\nu=0,1,2,3,\ldots),$$

greater than $9n$ and less than $25n$, where $c=1$ if $n\equiv 1 \pmod 4$, $c=9$ if $n\equiv 3 \pmod 4$, $\kappa=2$ if $n^2\equiv 1 \pmod{16}$, and $\kappa>2$ if $n^2\equiv 9 \pmod{16}$.

First, let us suppose N even. Then, since n is odd and N is even, it is clear that u must be even. Suppose then that

$$u=2v,\ \ N=2M.$$

We have to shew that M can be expressed in the form

$$x^2+y^2+z^2+2nv^2. \tag{7.2}$$

Since $2n\equiv 2 \pmod 4$, it follows from (6.2) that all integers except those which are less than $2n$ and of the form $4^\lambda(8\mu+7)$ can be expressed in the form (7.2). Hence the only *even* integers which cannot be expressed in the form (7.1) are those of the form $4^\lambda(16\mu+14)$ less than $4n$.

This completes the discussion of the case in which N is even. If N is odd the discussion is more difficult. In the first place, all odd numbers less than n are plainly among the exceptions. Secondly, since n and N are both odd, u must also be odd. We can therefore suppose that

$$N=n+2M,\ \ u^2=1+8\Delta,$$

where Δ is an integer of the form $\frac{1}{2}k(k+1)$, so that Δ may assume the values 0, 1, 3, 6, ... And we have to consider whether $n + 2M$ can be expressed in the form

$$2(x^2 + y^2 + z^2) + n(1 + 8\Delta),$$

or M in the form

$$x^2 + y^2 + z^2 + 4n\Delta. \tag{7.3}$$

If M is not of the form $4^{\lambda}(8\mu + 7)$, we can take $\Delta = 0$. If it is of this form, and less than $4n$, it is plainly an exception. These numbers give rise to the exceptions specified in (iii) of section 7. We may therefore suppose that M is of the form $4^{\lambda}(8\mu+7)$ and greater than $4n$.

8. In order to complete the discussion, we must consider the three cases in which $n \equiv 1 \pmod 8$, $n \equiv 5 \pmod 8$, and $n \equiv 3 \pmod 4$ separately.

(8.1) $n \equiv 1 \mod 8$).

If λ is equal to 0,1, or 2, take $\Delta = 1$. Then

$$M - 4n\Delta = 4^{\lambda}(8\mu + 7) - 4n$$

is of one of the forms

$$8\nu + 3, \;\; 4(8\nu + 3), \;\; 4(8\nu + 6).$$

If $\lambda \geq 3$ we cannot take $\Delta = 1$, since $M - 4n\Delta$ assumes the form $4(8\nu + 7)$; so we take $\Delta = 3$. Then

$$M - 4n = \Delta 4^{\lambda}(8\mu + 7) - 12n$$

is of the form $4(8\nu+5)$. In either of these cases $M - 4n\Delta$ is of the form $x^2+y^2+z^2$. Hence the only values of M, other than those already specified which cannot be expressed in the form (7.3), are those of the form

$$4^{\kappa}(8\nu + 7), \quad (\nu = 0, 1, 2, \ldots, \kappa > 2),$$

lying between $4n$ and $12n$. In other words, the only numbers greater than $9n$ which cannot be expressed in the form (7.1), in this case, are the numbers of the form

$$n + 4^{\kappa}(8\nu + 7), \quad (\nu = 0, 1, 2, \ldots, \kappa > 2),$$

lying between $9n$ and $25n$.

(8.2) $n \equiv 5 \mod 8$).

if $\lambda \neq 2$, take $\Delta = 1$. Then

$$M - 4n\Delta = 4^{\lambda}(8\mu + 7) - 4n$$

is of one of the forms

$$8\nu + 3, \;\; 4(8\nu + 2), \;\; 4(8\nu + 3).$$

If $\lambda = 2$, we cannot take $\Delta = 1$, since $M - 4n\Delta$ assumes the form $4(8\nu + 7)$; so we take $\Delta = 3$. Then

$$M - 4n\Delta = 4^{\lambda}(8\mu + 7) - 12n$$

is of the form $4(8\nu+5)$. In either of these cases $M - 4n\Delta$ is of the form $x^2+y^2+z^2$. Hence the only values of M, other than those already specified, which cannot be expressed in the form (7.3), are those of the form $16(8\mu+7)$ lying between $4n$ and $12n$. In other words, the only numbers greater than $9n$ which cannot be expressed in the form (7.1), in this case, are the numbers of the form $n + 4^2(16\mu + 14)$ lying between $9n$ and $25n$.

(8.3) $n \equiv 3 \pmod 4$.

If $\lambda \neq 1$, take $\Delta = 1$. Then

$$M - 4n\Delta = 4^{\lambda}(8\mu + 7) - 4n$$

is of one of the forms

$$8\nu + 3, \quad 4(4\nu + 1).$$

If $\lambda = 1$, take $\Delta = 3$. Then

$$M - 4n\Delta = 4(8\mu + 7) - 12n$$

is of the form $4(4\nu + 2)$. In either of these cases $M - 4n\Delta$ is of the form $x^2 + y^2 + z^2$. This completes the proof that there is only a finite number of exceptions. In order to determine what they are in this case, we have to consider the values of M, between $4n$ and $12n$, for which $\Delta = 1$ and

$$M - 4n\Delta = 4(8\mu + 7 - n) \equiv 0 \text{ mod } 16).$$

But the numbers which are multiples of 16 and which cannot be expressed in the form $x^2 + y^2 + z^2$ are the numbers

$$4^{\kappa}(8\nu + 7), \quad (\kappa = 2, 3, 4, \ldots, \nu = 0, 1, 2, \ldots).$$

The exceptional values of M required are therefore those of the numbers

$$4n + 4^{\kappa}(8\nu + 7) \tag{8.31}$$

which lie between $4n$ and $12n$ and are of the form

$$4(8\mu + 7). \tag{8.32}$$

But in order that (8.31) may be of the form (8.32), κ must be 2 if n is of the form $8k + 3$, and κ may have any of the values 3, 4, 5, … if n is of the form $8k + 7$. It follows that the only numbers greater than $9n$ which cannot be expressed in the form (7.1), in this case, are the numbers of the form

$$9n + 4^{\kappa}(16\nu + 14), \quad (\nu = 0, 1, 2, \ldots),$$

lying between $9n$ and $25n$, where $\kappa = 2$ if n is of the form $8k + 3$, and $\kappa > 2$ if n is of the form $8k + 7$.

This completes the proof of the results stated in section 7.

On certain trigonometrical sums and their applications in the theory of numbers

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1. The trigonometrical sums with which this paper is concerned are of the type

$$c_s(n) = \sum_{\lambda} \cos \frac{2\pi\lambda n}{s},$$

where λ is prime to s and not greater than s. It is plain that

$$c_s(n) = \sum \alpha^n,$$

where α is a primitive root of the equation

$$x^s - 1 = 0.$$

These sums are obviously of very great interest, and a few of their properties have been discussed already*. But, so far as I know, they have never been considered from the point of view which I adopt in this paper; and I believe that all the results which it contains are new.

My principal object is to obtain expressions for a variety of well-known arithmetical functions of n in the form of a series

$$\sum_s a_s c_s(n).$$

A typical formula is

$$\sigma(n) = \frac{\pi^2 n}{6} \left\{ \frac{c_1(n)}{1^2} + \frac{c_2(n)}{2^2} + \frac{c_3(n)}{3^2} + \cdots \right\},$$

where $\sigma(n)$ is the sum of the divisors of n. I give two distinct methods for the proof of this and a large variety of similar formulæ. The majority of my formulæ are "elementary" in the technical sense of the word — they can (that is to say) be proved by a combination of processes involving only finite algebra and simple general theorems concerning infinite series. These are however some which are of a "deeper" character, and can only be proved by means of theorems which seem to depend essentially on the theory of analytic functions. A typical formula of this class is

$$c_1(n) + \tfrac{1}{2}c_2(n) + \tfrac{1}{3}c_3(n) + \cdots = 0,$$

*See, e.g., Dirichlet-Dedekind, *Vorlesungen über Zahlentheorie*, ed. 4, Supplement VII, pp. 360 – 370.

a formula which depends upon, and is indeed substantially equivalent to, the "Prime Number Theorem" of Hadamard and de la Vallée Poussin.

Many of my formulæ are intimately connected with those of my previous paper "On certain arithmetical functions", published in 1916 in these *Transactions**. They are also connected (in a manner pointed out in § 15) with a joint paper by Mr Hardy and myself, "Asymptotic Formulæ in Combinatory Analysis", in course of publication in the *Proceedings of the London Mathematical Society*†.

2. Let $F(u, v)$ be any function of u and v, and let

$$D(n) = \sum_{\delta} F(\delta, \delta'), \tag{2.1}$$

where δ is a divisor of n and $\delta\delta' = n$. For instance

$$D(1) = F(1,1); \qquad D(2) = F(1,2) + F(2,1);$$

$$D(3) = F(1,3) + F(3,1); \quad D(4) = F(1,4) + F(2,2) + F(4,1);$$

$$D(5) = F(1,5) + F(5,1); \quad D(6) = F(1,6) + F(2,3) + F(3,2) + F(6,1); \ \ldots\ldots$$

It is clear that $D(n)$ may also be expressed in the form

$$D(n) = \sum_{\delta} F(\delta', \delta). \tag{2.2}$$

Suppose now that

$$\eta_s(n) = \sum_{0}^{s-1} \cos\frac{2\pi\nu n}{s}, \tag{2.3}$$

so that $\eta_s(n) = s$ if s is a divisor of n and $\eta_s(n) = 0$ otherwise. Then

$$D(n) = \sum_{1}^{t} \frac{1}{\nu}\eta_\nu(n) F\left(\nu, \frac{n}{\nu}\right), \ddagger \tag{2.4}$$

where t is any number not less than n. Now let

$$c_s(n) = \sum_{\lambda} \cos\frac{2\pi\lambda n}{s}, \tag{2.5}$$

where λ is prime to s and does not exceed s; e.g.

$$c_1(n) = 1; \quad c_2(n) = \cos n\pi; \quad c_3(n) = 2\cos\tfrac{2}{3}n\pi;$$

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*[No.18 of this volume].

†[No. 36 of this volume].

‡$\sum_{1}^{t}$ is to be understood as meaning $\sum_{1}^{[t]}$, where $[t]$ denotes as usual the greatest integer in t.

$$c_4(n) = 2\cos\tfrac{1}{2}n\pi; \quad c_5(n) = 2\cos\tfrac{2}{5}n\pi + 2\cos\tfrac{4}{5}n\pi;$$
$$c_6(n) = 2\cos\tfrac{1}{3}n\pi; \quad c_7(n) = 2\cos\tfrac{2}{7}n\pi + 2\cos\tfrac{4}{7}n\pi + 2\cos\tfrac{6}{7}n\pi;$$
$$c_8(n) = 2\cos\tfrac{1}{4}n\pi + 2\cos\tfrac{3}{4}n\pi; \quad c_9(n) = 2\cos\tfrac{2}{9}n\pi + 2\cos\tfrac{4}{9}n\pi + 2\cos\tfrac{8}{9}n\pi;$$
$$c_{10}(n) = 2\cos\tfrac{1}{5}n\pi + 2\cos\tfrac{3}{5}n\pi; \ldots.$$

It follows from (2.3) and (2.5) that

(2.6) $$\eta_s(n) = \sum_\delta c_\delta(n),$$

where δ is a divisor of s; and hence* that

(2.7) $$c_s(n) = \sum_\delta \mu(\delta')\eta_\delta(n),$$

where δ is a divisor of $s, \delta\delta' = s$, and

(2.8) $$\sum \frac{\mu(\nu)}{\nu^s} = \frac{1}{\zeta(s)},$$

$\zeta(s)$ being the Riemann Zeta-function. In particular

$$c_1(n) = \eta_1(n);\ c_2(n) = \eta_2(n) - \eta_1(n);\ c_3(n) = \eta_3(n) - \eta_1(n);$$
$$c_4(n) = \eta_4(n) - \eta_2(n);\ c_5(n) = \eta_5(n) - \eta_1(n); \ldots\ldots$$

But from (2.3) we know that $\eta_\delta(n) = 0$ if δ is not a divisor of n; and so we can suppose that, in (2.7), δ is a common divisor of n and s. It follows that

$$|c_s(n)| \le \sum \delta,$$

where δ is a divisor of n; so that

(2.9) $$c_\nu(n) = O(1)$$

if n is fixed and $\nu \to \infty$. Since

$$\eta_s(n) = \eta_s(n+s); \quad c_s(n) = c_s(n+s),$$

the values of $c_s(n)$ for $n = 1, 2, 3, \ldots$ can be shewn conveniently by writing

$$c_1(n) = \overline{1}; \quad c_2(n) = \overline{-1, 1}; \quad c_3(n) = \overline{-1, -1, 2};$$
$$c_4(n) = \overline{0, -2, 0, 2}; \quad c_5(n) = \overline{-1, -1, -1, -1, 4};$$

*See Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 577.

$$c_6(n) = \overline{1, -1, -2, -1, 1, 2} : \quad c_7(n) = \overline{-1, -1, -1, -1, -1, -1, 6};$$
$$c_8(n) = \overline{0, 0, 0, -4, 0, 0, 0, 4}; \quad c_9(n) = \overline{0, 0, -3, 0, 0, -3, 0, 0, 6};$$
$$c_{10}(n) = \overline{1, -1, 1, -1, -4, -1, 1, -1, 1, 4}; \ldots\ldots$$

the meaning of the third formula, for example, being that $c_3(1) = -1, c_3(2) = -1, c_3(3) = 2$, and that these values are then repeated periodically.

It is plain that we have also

(2.91)
$$c_\nu(n) = O(1).$$

when ν is fixed and $n \to \infty$.

3. Substituting (2.6) in (2.4) and collecting the coefficients of $c_1(n)$, $c_2(n)$, $c_3(n)\ldots$, we find that

(3.1)
$$D(n) = c_1(n) \sum_1^t \frac{1}{\nu} F\left(\nu, \frac{n}{\nu}\right) + c_2(n) \sum_1^{\frac{1}{2}t} \frac{1}{2\nu} F\left(2\nu, \frac{n}{2\nu}\right) + c_3(n) \sum_1^{\frac{1}{3}t} \frac{1}{3\nu} F\left(3\nu, \frac{n}{3\nu}\right) + \cdots,$$

where t is any number not less than n. If we use (2.2) instead of (2.1) we obtain another expression, viz.

(3.2)
$$D(n) = c_1(n) \sum_1^t \frac{1}{\nu} F\left(\frac{n}{\nu}, \nu\right) + c_2(n) \sum_1^{\frac{1}{2}t} \frac{1}{2\nu} F\left(\frac{n}{2\nu}, 2\nu\right), + c_3(n) \sum_1^{\frac{1}{3}t} \frac{1}{3\nu} F\left(\frac{n}{3\nu}, 3\nu\right) + \cdots,$$

where t is any number not less that n.

Suppose now that

$$F_1(u, v) = F(u, v) \log u, \ \ F_2(u, v) = F(u, v) \log v.$$

Then we have

$$\begin{aligned} D(n) \log n &= \sum_\delta F(\delta, \delta') \log n = \sum_\delta F(\delta, \delta') \log(\delta\delta') \\ &= \sum_\delta F_1(\delta, \delta') + \sum_\delta F_2(\delta, \delta'), \end{aligned}$$

where δ is a divisor of n and $\delta\delta' = n$.

Now for $\sum_\delta F_1(\delta,\delta')$ we shall write the expression corresponding to (3.1) and for $\sum_\delta F_2(\delta,\delta')$ the expression corresponding to (3.2). Then we have

$$D(n)\log n = c_1(n)\sum_1^r \frac{\log\nu}{\nu}F\left(\nu,\frac{n}{\nu}\right)+c_2(n)\sum_1^{\frac12 r}\frac{\log 2\nu}{2\nu}F\left(2\nu,\frac{n}{2\nu}\right) \tag{3.3}$$
$$+\ c_3(n)\sum_1^{\frac13 r}\frac{\log 3\nu}{3\nu}F\left(3\nu,\frac{n}{3\nu}\right)+\cdots+c_1(n)\sum_1^t\frac{\log\nu}{\nu}F\left(\frac{n}{\nu},\nu\right)$$
$$+\ c_2(n)\sum_1^{\frac12 t}\frac{\log 2\nu}{2\nu}F\left(\frac{n}{2\nu},2\nu\right)+c_3(n)\sum_1^{\frac13 t}\frac{\log 3\nu}{3\nu}F\left(\frac{n}{3\nu},3\nu\right)+\cdots,$$

where r and t are any two numbers not less than n. If, in particular, $F(u,v)=F(v,u)$, then (3.3) reduces to

$$\tfrac12 D(n)\log n = c_1(n)\sum_1^t\frac{\log\nu}{\nu}F\left(\nu,\frac{n}{\nu}\right) \tag{3.4}$$
$$+\,c_2(n)\sum_1^{\frac12 t}\frac{\log 2\nu}{2\nu}F\left(2\nu,\frac{n}{2\nu}\right)+c_3(n)\sum_1^{\frac13 t}\frac{\log 3\nu}{3\nu}F\left(3\nu,\frac{n}{3\nu}\right)+\cdots,$$

where t is any number not less than n.

4. We may also write $D(n)$ in the form

$$D(n)=\sum_{\delta=1}^{u}F(\delta,\delta')+\sum_{\delta=1}^{r}F(\delta',\delta), \tag{4.1}$$

where δ is a divisor of $n, \delta\delta'=n$, and u,v are may two positive numbers such that $uv=n$, it being understood that, if u and v are both integral, a term $F(u,v)$ is to be subtracted from the right-hand side. Hence (with the same conventions)

$$D(n)=\sum_1^u\frac{1}{\nu}\eta_\nu(n)F\left(\nu,\frac{n}{\nu}\right)+\sum_1^v\frac{1}{\nu}\eta_\nu(n)F\left(\frac{n}{\nu},\nu\right),$$

Applying to this formula transformations similar to those of §3, we obtain

$$D(n) \quad = \quad c_1(n)\sum_1^u\frac{1}{\nu}F\left(\nu,\frac{n}{\nu}\right)+c_2(n)\sum_1^{\frac12 u}\frac{1}{2\nu}F\left(2\nu,\frac{n}{2\nu}\right)+\cdots \tag{4.2}$$
$$+\,c_1(n)\sum_1^v\frac{1}{\nu}F\left(\frac{n}{\nu},\nu\right)+c_2(n)\sum_1^{\frac12 v}\frac{1}{2\nu}F\left(\frac{n}{2\nu},2\nu\right)+\cdots,$$

where u and v are positive numbers such that $uv = n$. If u and v are integers then a term $F(u, v)$ should be subtracted from the right-hand side.
If we suppose that $0 < u \leq 1$ then (4.2) reduces to (3.2), and if $0 < v \leq 1$ it reduces to (3.1). Another particular case of interest is that in which $u = v$. Then

$$\begin{aligned} D(n) &= c_1(n) \sum_1^{\sqrt{n}} \frac{1}{\nu} \left\{ F\left(\nu, \frac{n}{\nu}\right) + F\left(\frac{n}{\nu}, \nu\right) \right\} \\ &\quad + c_2(n) \sum_1^{\frac{1}{2}\sqrt{n}} \frac{1}{2\nu} \left\{ F\left(2\nu, \frac{n}{2\nu}\right) + F\left(\frac{n}{2\nu}, 2\nu\right) \right\} + \cdots . \end{aligned} \tag{4.3}$$

If n is a perfect square then $F(\sqrt{n}, \sqrt{n})$ should be subtracted from the right hand side.

5. We shall now consider some special forms of these general equations. Suppose that $F(u, v) = v^s$, so that $D(n)$ is the sum $\sigma_s(n)$ of the sth powers of the divisors of n. Then from (3.1) and (3.2) we have

$$\frac{\sigma_s(n)}{n^s} = c_1(n) \sum_1^{t} \frac{1}{\nu^{s+1}} + c_2(n) \sum_1^{\frac{1}{2}t} \frac{1}{(2\nu)^{s+1}} + c_3(n) \sum_1^{\frac{1}{3}t} \frac{1}{(3\nu)^{s+1}} + \cdots , \tag{5.1}$$

$$\sigma_s(n) = c_1(n) \sum_1^{t} \nu^{s-1} + c_2(n) \sum_1^{\frac{1}{2}t} (2\nu)^{s-1} + c_3(n) \sum_1^{\frac{1}{3}t} (3\nu)^{s-1} + \cdots , \tag{5.2}$$

where t is any number not less than n: from (3.3)

$$\begin{aligned} \sigma_s(n) \log n &= c_1(n) \sum_1^{r} \nu^{s-1} \log \nu + c_2(n) \sum_1^{\frac{1}{2}r} (2\nu)^{s-1} \log 2\nu + \cdots \\ &\quad + n^s \left\{ c_1(n) \sum_1^{t} \frac{\log \nu}{\nu^{s+1}} + c_2(n) \sum_1^{\frac{1}{2}t} \frac{\log 2\nu}{(2\nu)^{s+1}} + \cdots \right\} , \end{aligned} \tag{5.3}$$

where r and t are two numbers not less than n: and from (4.2)

$$\begin{aligned} \sigma_s(n) &= c_1(n) \sum_1^{u} \nu^{s-1} + c_2(n) \sum_1^{\frac{1}{2}u} (2\nu)^{s-1} + c_3(n) \sum_1^{\frac{1}{3}u} (3\nu)^{s-1} + \cdots \\ &\quad + n^s \left\{ c_1(n) \sum_1^{v} \frac{1}{\nu^{s+1}} + c_2(n) \sum_1^{\frac{1}{2}v} \frac{1}{(2\nu)^{s+1}} + c_3(n) \sum_1^{\frac{1}{3}v} \frac{1}{(3\nu)^{s+1}} + \cdots \right\} , \end{aligned} \tag{5.4}$$

where $uv = n$. If u and v are integers then u^s should be subtracted from the right-hand side.
Let $d(n) = \sigma_0(n)$ denote the number of divisors of n and $\sigma(n) = \sigma_1(n)$ the sum of the divisors of n. Then from (5.1) – (5.4) we obtain

$$d(n) = c_1(n)\sum_1^{t}\frac{1}{\nu} + c_2(n)\sum_1^{\frac{1}{2}t}\frac{1}{2\nu} + c_3(n)\sum_1^{\frac{1}{3}t}\frac{1}{3\nu} + \cdots, \tag{5.5}$$

$$\sigma(n) = c_1(n)[t] + c_2(n)[\tfrac{1}{2}t] + c_3(n)[\tfrac{1}{3}t] + \cdots, \tag{5.6}$$

$$\tfrac{1}{2}d(n)\log n = c_1(n)\sum_1^{t}\frac{\log\nu}{\nu} + c_2(n)\sum_1^{\frac{1}{2}t}\frac{\log 2\nu}{2\nu} + c_3(n)\sum_1^{\frac{1}{3}t}\frac{\log 3\nu}{3\nu} + \cdots, \tag{5.7}$$

$$\begin{aligned} d(n) \quad = \quad & c_1(n)\left\{\sum_1^{u}\frac{1}{\nu} + \sum_1^{v}\frac{1}{\nu}\right\} + c_2(n)\left\{\sum_1^{\frac{1}{2}u}\frac{1}{2\nu} + \sum_1^{\frac{1}{2}v}\frac{1}{2\nu}\right\} \\ & + c_3(n)\left\{\sum_1^{\frac{1}{3}u}\frac{1}{3\nu} + \sum_1^{\frac{1}{3}v}\frac{1}{3\nu}\right\} + \cdots \end{aligned} \tag{5.8}$$

where $t \geq n$ and $uv = n$. If u and v are integers then 1 should be subtracted from the right-hand side of (5.8). Putting $u = v = \sqrt{n}$ in (5.8) we obtain

$$\tfrac{1}{2}d(n) = c_1(n)\sum_1^{\sqrt{n}}\frac{1}{\nu} + c_2(n)\sum_1^{\frac{1}{2}\sqrt{n}}\frac{1}{2\nu} + c_3(n)\sum_1^{\frac{1}{3}\sqrt{n}}\frac{1}{3\nu} + \cdots, \tag{5.9}$$

unless n is a perfect square, when $\frac{1}{2}$ should be subtracted from the right-hand side. It may be interesting to note that, if we replace the left-hand side in (5.9) by

$$\left[\tfrac{1}{2} + \tfrac{1}{2}d(n)\right],$$

then the formula is true without exception.

6. So far our work has been based on elementary formal transformations, and no questions of convergence have arisen. We shall now consider the equation (5.1) more carefully. Let us suppose that $s > 0$. Then

$$\sum_1^{t/k}\frac{1}{(k\nu)^{s+1}} = \sum_1^{\infty}\frac{1}{(k\nu)^{s+1}} + O\left(\frac{1}{kt^s}\right) = \frac{1}{k^{s+1}}\zeta(s+1) + O\left(\frac{1}{kt^s}\right).$$

The number of terms in the right-hand side of (5.1) is $[t]$. Also we know that $c_\nu(n) = O(1)$ as $n \to \infty$. Hence

$$\frac{\sigma_s(n)}{n^s} = \zeta(s+1)\sum_{\nu=1}^{t}\frac{c_\nu(n)}{\nu^{s+1}} + O\left\{\frac{1}{t^s}\sum_{\nu=1}^{t}\frac{1}{\nu}\right\} = \zeta(s+1)\sum_{1}^{\infty}\frac{c_\nu(n)}{\nu^{s+1}} + O\left(\frac{\log t}{t^s}\right).$$

Making $t \to \infty$, we obtain

$$\sigma_s(n) = n^s\zeta(s+1)\left\{\frac{c_1(n)}{1^{s+1}} + \frac{c_2(n)}{2^{s+1}} + \frac{c_3(n)}{3^{s+1}} + \cdots\right\}, \tag{6.1}$$

if $s > 0$. Similarly, if we make $t \to \infty$ in (5.3), we obtain

$$\begin{aligned}\sigma_s(n)\log n = c_1(n)\sum_{1}^{r}\nu^{s-1}\log\nu + c_2(n)\sum_{1}^{\frac{1}{2}r}(2\nu)^{s-1}\log 2\nu + \cdots \\ +n^s\left\{c_1(n)\sum_{1}^{\infty}\frac{\log\nu}{\nu^{s+1}} + c_2(n)\sum_{1}^{\infty}\frac{\log 2\nu}{(2\nu)^{s+1}} + \cdots\right\}.\end{aligned}$$

But

$$\sum_{1}^{\infty}\frac{\log k\nu}{(k\nu)^{s+1}} = \frac{\log k}{k^{s+1}}\zeta(s+1) - \frac{1}{k^{s+1}}\zeta'(s+1).$$

It follows from this and (6.1) that

$$\begin{aligned}\sigma_s(n)\left\{\frac{\zeta'(s+1)}{\zeta(s+1)} + \log n\right\} &= c_1(n)\sum_{1}^{t}\nu^{s-1}\log\nu \\ &+ c_2(n)\sum_{1}^{\frac{1}{2}t}(2\nu)^{s-1}\log 2\nu + \cdots \\ &+ n^s\zeta(s+1)\left\{\frac{c_1(n)\log 1}{1^{s+1}} + \frac{c_2(n)\log 2}{2^{s+1}} + \frac{c_3(n)\log 3}{3^{s+1}} + \cdots\right\},\end{aligned} \tag{6.2}$$

where $s > 0$ and $t \geq n$. Putting $s = 1$ in (6.1) and (6.2) we obtain

$$\sigma(n) = \frac{\pi^2}{6}n\left\{\frac{c_1(n)}{1^2} + \frac{c_2(n)}{2^2} + \frac{c_3(n)}{3^2} + \cdots\right\}, \tag{6.3}$$

$$\begin{aligned}\sigma(n)\left\{\frac{\zeta'(2)}{\zeta(2)} + \log n\right\} \quad = \quad & \frac{\pi^2}{6}n\left\{\frac{c_1(n)}{1^2}\log 1 + \frac{c_2(n)}{2^2}\log 2 + \cdots\right\} \\ & + c_1(n)[t]\log 1 + c_2(n)[\tfrac{1}{2}t]\log 2 + \cdots \\ & + c_1(n)\log[t]! + c_2(n)\log[\tfrac{1}{2}t]! + \cdots,\end{aligned} \tag{6.4}$$

where $t \geq n$.

7. Since

$$\sigma_s(n) = n^s \sigma_{-s}(n), \tag{7.1}$$

we may write (6.1) in the form

$$\frac{\sigma_{-s}(n)}{\zeta(s+1)} = \frac{c_1(n)}{1^{s+1}} + \frac{c_2(n)}{2^{s+1}} + \frac{c_3(n)}{3^{s+1}} + \cdots, \tag{7.2}$$

where $s > 0$. This result has been proved by purely elementary methods. But in order to know whether the right-hand side of (7.2) is convergent or not for values of s less than or equal to zero we require the help of theorems which have only been established by transcendental methods.

Now the right-hand side of (7.2) is an ordinary Dirichlet's series for

$$\sigma_{-s}(n) \times \frac{1}{\zeta(s+1)}.$$

The first factor is a finite Dirichlet's series and so an absolutely convergent Dirichlet's series. It follows that the right-hand side of (7.2) is convergent whenever the Dirichlet's series for $1/\zeta(s+1)$, viz.

$$\sum \frac{\mu(n)}{n^{1+s}}, \tag{7.3}$$

is convergent. But it is known* that the series (7.3) is convergent when $s = 0$ and that its sum is 0.

It follows from this that

$$c_1(n) + \tfrac{1}{2}c_2(n) + \tfrac{1}{3}c_3(n) + \cdots = 0. \tag{7.4}$$

Nothing is known about the convergence of (7.3) when $-\frac{1}{2} < s < 0$. But with the assumption of the truth of the hitherto unproved Riemann hypothesis it has been proved† that (7.3) is convergent when $s > -\frac{1}{2}$. With this assumption we see that (7.2) is true when $s > -\frac{1}{2}$. In other words, if $-\frac{1}{2} < s < \frac{1}{2}$, then

$$\begin{aligned} \sigma_s(n) &= \zeta(1-s)\left\{\frac{c_1(n)}{1^{1-s}} + \frac{c_2(n)}{2^{1-s}} + \frac{c_3(n)}{3^{1-s}} + \cdots\right\} \\ &= n^s\zeta(1+s)\left\{\frac{c_1(n)}{1^{1+s}} + \frac{c_2(n)}{2^{1+s}} + \frac{c_3(n)}{3^{1+s}} + \cdots\right\}. \end{aligned} \tag{7.5}$$

*Landau, *Handbuch*, p. 591.

†Littlewood, *Comptes Rendus*, 29 Jan. 1912.

8. It is known* that all the series obtained from (7.3) by term-by-term differentiation with respect to s are convergent when $s = 0$, and it is obvious that the derivatives of $\sigma_{-s}(n)$ with respect to s are all finite Dirichlet's series and so absolutely convergent. It follows that all the derivatives of the right-hand side of (7.2) are convergent when $s = 0$; and so we can equate the coefficients of like powers of s from the two sides of (7.2). Now

$$\frac{1}{\zeta(s+1)} = s - \gamma s^2 + \cdots, \tag{8.1}$$

where γ is Euler's constant. And

$$\sigma_{-s}(n) = \sum_{\delta} \delta^{-s} = \sum_{\delta} 1 - s\sum_{\delta} \log\delta + \cdots,$$

where δ is a divisor of n. But

$$\sum_{\delta} \log\delta = \sum_{\delta} \log\delta' = \tfrac{1}{2}\sum_{\delta} \log(\delta\delta') = \tfrac{1}{2}d(n)\log n,$$

where $\delta\delta' = n$. Hence

$$\sigma_{-s}(n) = d(n) - \tfrac{1}{2}sd(n)\log n + \cdots. \tag{8.2}$$

Now equating the coefficients of s and s^2 from the two sides of (7.2) and using (8.1) and (8.2), we obtain

$$c_1(n)\log 1 + \tfrac{1}{2}c_2(n)\log 2 + \tfrac{1}{3}c_3(n)\log 3 + \cdots = -d(n), \tag{8.3}$$

$$c_1(n)(\log 1)^2 + \tfrac{1}{2}c_2(n)(\log 2)^2 + \tfrac{1}{3}c_3(n)(\log 3)^2 + \cdots = -d(n)(2\gamma + \log n). \tag{8.4}$$

9. I shall now find an expression of the same kind for $\phi(n)$, the number of numbers prime to and not exceeding n. Let $p_1, p_2, p_3, \ldots$ be the prime divisors of n, and let

$$\phi_s(n) = n^s(1 - p_1^{-s})(1 - p_2^{-s})(1 - p_3^{-s})\cdots, \tag{9.1}$$

so that $\phi_1(n) = \phi(n)$. Suppose that

$$F(u.v) = \mu(u)v^s.$$

Then it is easy to see that

$$D(n) = \phi_s(n).$$

Hence, from (3.1), we have

$$\frac{\phi_s(n)}{n^s} = c_1(n)\sum_{1}^{t} \frac{\mu(\nu)}{\nu^{s+1}} + c_2(n)\sum_{1}^{\frac{1}{2}t} \frac{\mu(2\nu)}{(2\nu)^{s+1}} + \cdots, \tag{9.2}$$

*Landau, *Handbuch*, p.594

where t is any number not less than n. If $s > 0$ we can make $t \to \infty$, as in §6. Then we have

$$\frac{\phi_s(n)}{n^s} = c_1(n)\sum_1^\infty \frac{\mu(\nu)}{\nu^{s+1}} + c_2(n)\sum_1^\infty \frac{\mu(2\nu)}{(2\nu)^{s+1}} + \cdots. \tag{9.3}$$

But it can easily be shewn that

$$\sum_1^\infty \frac{\mu(n\nu)}{\nu^s} = \frac{\mu(n)}{\zeta(s)(1-p_1^{-s})(1-p_2^{-s})(1-p_3^{-s})\cdots}, \tag{9.4}$$

where $p_1, p_2, p_3, \ldots$ are the prime divisors of n. In other words

$$\sum_1^\infty \frac{\mu(n\nu)}{\nu^s} = \frac{\mu(n)n^s}{\phi_s(n)\zeta(s)}. \tag{9.5}$$

It follows from (9.3) and (9.5) that

$$\frac{\phi_s(n)\zeta(s+1)}{n^s} = \frac{\mu(1)c_1(n)}{\phi_{s+1}(1)} + \frac{\mu(2)c_2(n)}{\phi_{s+1}(2)} + \frac{\mu(3)c_3(n)}{\phi_{s+1}(3)} + \cdots. \tag{9.6}$$

In particular

$$\begin{aligned}\frac{\pi^2}{6n}\phi(n) = c_1(n) &- \frac{c_2(n)}{2^2-1} - \frac{c_3(n)}{3^2-1} - \frac{c_5(n)}{5^2-1} \\ &+ \frac{c_6(n)}{(2^2-1)(3^2-1)} - \frac{c_7(n)}{7^2-1} + \frac{c_{10}(n)}{(2^2-1)(5^2-1)} - \cdots.\end{aligned} \tag{9.7}$$

10. I shall now consider an application of the main formulæ to the problem of the number of representations of a number as the sum of 2, 4, 6, 8, … squares. We require the following preliminary results.

(1) Let

$$\sum D(n)x^n = X_1 = \frac{1^{s-1}x}{1+x} + \frac{2^{s-1}x^2}{1-x^2} + \frac{3^{s-1}x^3}{1+x^3} + \cdots. \tag{10.1}$$

We shall choose

$$\begin{aligned} F(u,v) &= v^{s-1}, & u &\equiv 1 \pmod 2, \\ F(u,v) &= -v^{s-1}, & u &\equiv 2 \pmod 4, \\ F(u,v) &= (2^s-1)v^{s-1}, & u &\equiv 0 \pmod 4. \end{aligned}$$

Then from (3.1) we can shew, by arguments similar to those used in §6, that

$$(10.11)\qquad \begin{aligned}D(n) = n^{s-1}(1^{-s}+3^{-s}+5^{-s}+\cdots)\{1^{-s}c_1(n)+2^{-s}c_4(n)+3^{-s}c_3(n)\\ +4^{-s}c_8(n)5^{-s}c_5(n)+6^{-s}c_{12}(n)+7^{-s}c_7(n)+8^{-s}c_{16}(n)+\cdots\}.\end{aligned}$$

if $s > 1$.

(2) Let

$$(10.2)\qquad \sum D(n)x^n = X_2 = \frac{1^{s-1}x}{1-x}+\frac{2^{s-1}x^2}{1+x^2}+\frac{3^{s-1}x^3}{1-x^3}+\cdots.$$

We shall choose

$$\begin{aligned}F(u,v)&=v^{s-1}, && u\equiv 1(\bmod 2),\\ F(u,v)&=v^{s-1}, && u\equiv 2(\bmod 4),\\ F(u,v)&=(1-2^s)v^{s-1}, && u\equiv 0(\bmod 4).\end{aligned}$$

Then we obtain as before

$$(10.21)\qquad \begin{aligned}D(n) = n^{s-1}(1^{-s}+3^{-s}+5^{-s}+\cdots)\{1^{-s}c_1(n)-2^{-s}c_4(n)+3^{-s}c_3(n)\\ -4^{-s}c_8(n)+5^{-s}c_5(n)-6^{-s}c_{12}(n)+7^{-s}c_7(n)-8^{-s}c_{16}(n)+\cdots\}.\end{aligned}$$

(3) Let

$$(10.3)\qquad \sum D(n)x^n = X_3 = \frac{1^{s-1}x}{1+x^2}+\frac{2^{s-1}x^2}{1+x^4}+\frac{3^{s-1}x^3}{1+x^6}+\cdots.$$

We shall choose

$$\begin{aligned}F(u,v)&=0, && u\equiv 0(\bmod 2),\\ F(u,v)&=v^{s-1}, && u\equiv 1(\bmod 4),\\ F(u,v)&=-v^{s-1}, && u\equiv 3(\bmod 4).\end{aligned}$$

Then we obtain as before

$$(10.31)\qquad D(n) = n^{s-1}(1^{-s}-3^{-s}+5^{-s}-\cdots)\{1^{-s}c_1(n)-3^{-s}c_3(n)+5^{-s}c_5(n)-\cdots\}.$$

(4) We shall also require a similar formula for the function $D(n)$ defined by

$$(10.4)\qquad \sum D(n)x^n = X_4 = \frac{1^{s-1}x}{1-x}-\frac{3^{s-1}x^3}{1-x^3}+\frac{5^{s-1}x^5}{1-x^5}-\cdots.$$

The formula required is not a direct consequence of the preceding analysis, but if, instead of starting with the function

$$c_r(n) = \sum_\lambda \cos\frac{2\pi n\lambda}{r},$$

we start with the function

$$s_r(n) = \sum_\lambda (-1)^{\frac{1}{2}(\lambda-1)} \sin \frac{2\pi n\lambda}{r},$$

where λ is prime to r and does not exceed r, and proceed as in §§2–3, we can shew that

$$D(n) = \tfrac{1}{2} n^{s-1}(1^{-s} - 3^{-s} + 5^{-s} - \cdots)\{1^{-s}s_4(n) + 2^{-s}s_8(n) + 3^{-s}s_{12}(n) + \cdots\}. \tag{10.41}$$

It should be observed that there is a correspondence between $c_r(n)$ and the ordinary ζ-function of the one hand and $s_r(n)$ and the function

$$\eta(s) = 1^{-s} - 3^{-s} + 5^{-s} - \cdots$$

on the other. It is possible to define an infinity of systems of trigonometrical sums such as $c_r(n), s_r(n)$, each corresponding to one of the general class of "L-functions*" of which $\zeta(s)$ and $\eta(s)$ are the simplest members.

We have shewn that (10.31) and (10.41) are true when $s > 1$. But if we assume that the Dirichlet's series for $1/\eta(s)$ is convergent when $s = 1$, a result which is precisely of the same depth as the prime number theorem and has only been established by transcendental methods, then we can shew by arguments similar to those of §7 that (10.31) and (10.41) are true when $s = 1$.

11. I have shewn elsewhere† that if s is a positive integer and

$$1 + \sum r_s(n)x^n = (1 + 2x + 2x^4 + 2x^9 + \cdots)^s,$$

then

$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n),$$

where $e_{2s}(n) = 0$ when $s = 1, 2, 3$ or 4 and is of lower order ‡ than $\delta_{2s}(n)$ in all cases; that if s is a multiple of 4 then

$$(1^{-s} + 3^{-s} + 5^{-s} + \cdots) \sum \delta_{2s}(n)x^n = \frac{\pi^s}{(s-1)!}X_1; \tag{11.1}$$

if s is of the form $4k + 2$ then

$$(1^{-s} + 3^{-s} + 5^{-s} + \cdots) \sum \delta_{2s}(n)x^n = \frac{\pi^s}{(s-1)!}X_2; \tag{11.2}$$

if s is of the form $4k + 1$ then

$$(1^{-s} - 3^{-s} + 5^{-s} - \cdots) \sum \delta_{2s}(n)x^n = \frac{\pi^s}{(s-1)!}(X_3 + 2^{1-s}X_4); \tag{11.3}$$

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*See Landau, *Handbuch*, pp. 414 *et seq.*

†*Transactions of the Cambridge Philosophical Society* Vol.XXII, 1916, pp. 159 – 184. [No.18 of this volume; see in particular §§ 24 – 28, pp. 202 – 208].

‡For a more precise result concerning the order of $e_{2s}(n)$ see § 15.

except when $s=1$; and if s is of the form $4k+3$ then

$$(11.4) \qquad (1^{-s}-3^{-s}+5^{-s}-\cdots)\sum \delta_{2s}(n)x^n = \frac{\pi^s}{(s-1)!}(X_3 - 2^{1-s}X_4),$$

X_1, X_2, X_3, X_4 being the same as in §10.
In the case in which $s=1$ it is well known that

$$(11.5) \qquad \begin{aligned} \sum \delta_2(n)x^n &= 4\left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \cdots\right) \\ &= 4\left(\frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \frac{x^3}{1+x^6} + \cdots\right) \end{aligned}$$

It follows from §10 that, if s is a multiple of 4 then

$$(11.11) \qquad \begin{aligned} \delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}&\{1^{-s}c_1(n) + 2^{-s}c_4(n) + 3^{-s}c_3(n) + 4^{-s}c_8(n) \\ &+5^{-s}c_5(n) + 6^{-s}c_{12}(n) + 7^{-s}c_7(n) + 8^{-s}c_{16}(n) + \cdots\}; \end{aligned}$$

if s is of the form $4k+2$ then

$$(11.21) \qquad \begin{aligned} \delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}&\{1^{-s}c_1(n) - 2^{-s}c_4(n) + 3^{-s}c_3(n) - 4^{-s}c_8(n) \\ &+5^{-s}c_5(n) - 6^{-s}c_{12}(n) + 7^{-s}c_7(n) - 8^{-s}c_{16}(n) + \cdots\}; \end{aligned}$$

if s is of the form $4k+1$ then

$$(11.31) \qquad \begin{aligned} \delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}&\{1^{-s}c_1(n) + 2^{-s}s_4(n) - 3^{-s}c_3(n) + 4^{-s}s_8(n) \\ &+5^{-s}c_5(n) + 6^{-s}s_{12}(n) - 7^{-s}c_7(n) + 8^{-s}s_{16}(n) + \cdots\}; \end{aligned}$$

except when $s=1$; and if s is of the form $4k+3$ then

$$(11.41) \qquad \begin{aligned} \delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}&\{1^{-s}c_1(n) - 2^{-s}s_4(n) - 3^{-s}c_3(n) - 4^{-s}s_8(n) \\ &+5^{-s}c_5(n) - 6^{-s}s_{12}(n) - 7^{-s}c_7(n) - 8^{-s}s_{16}(n) + \cdots\}; \end{aligned}$$

From (11.5) and the remarks at the end of the previous section, it follows that

$$(11.51) \qquad \begin{aligned} r_2(n) = \delta_2(n) &= \pi\{c_1(n) - \tfrac{1}{3}c_3(n) + \tfrac{1}{5}c_5(n) - \cdots\} \\ &= \pi\{\tfrac{1}{2}s_4(n) + \tfrac{1}{4}s_8(n) + \tfrac{1}{6}s_{12}(n) + \cdots\}, \end{aligned}$$

but this of course not such an elementary result as the preceding ones.

We can combine all the formulæ (11.11) – (11.41) in one by writing

$$\begin{aligned}\delta_{2s}(n) &= \frac{\pi^s n^{s-1}}{(s-1)!}\{1^{-s}\mathbf{c_1}(n) + 2^{-s}\mathbf{c_4}(n) + 3^{-s}\mathbf{c_3}(n) + 4^{-s}\mathbf{c_8}(n) \\ &+5^{-s}\mathbf{c_5}(n) + 6^{-s}\mathbf{c_{12}}(n) + 7^{-s}\mathbf{c_7}(n) + 8^{-s}\mathbf{c_{16}}(n) + \cdots\},\end{aligned} \tag{11.6}$$

where s is an integer greater than 1 and

$$\mathbf{c_r}(n) = c_r(n)\cos\tfrac{1}{2}\pi s(r-1) - s_r(n)\sin\tfrac{1}{2}\pi s(r-1).$$

12. We can obtain analogous results concerning the number of representations of a number as the sum of 2, 4, 6, 8, ... triangular numbers. Equation (147) of my former paper* is equivalent to

$$\begin{aligned}(1-2x+2x^4-2x^9+\cdots)^{2s} &= 1 + \sum_1^\infty \delta_{2s}(n)(-x)^n \\ &+ \frac{f^{4s}(x)}{f^{2s}(x^2)} \sum_{1\le n\le \frac14(s-1)} K_n(-x)^n \frac{f^{24n}(x^2)}{f^{24n}(x)},\end{aligned} \tag{12.1}$$

where K_n is a constant and

$$f(x) = (1-x)(1-x^2)(1-x^3)\cdots.$$

Suppose now that

$$x = e^{-\pi\alpha},\ \ x' = e^{-2\pi/\alpha}.$$

Then we know that

$$\sqrt{\alpha}(1-2x+2x^4-2x^9+\cdots) = 2x'^{\frac18}(1+x'+x'^3+x'^6+\cdots), \tag{12.2}$$

$$\sqrt{(\tfrac12\alpha)}x^{\frac{1}{24}}f(x) = x'^{\frac{1}{12}}f(x'^2),\ \ \sqrt{\alpha}x^{\frac{1}{12}}f(x^2) = x'^{\frac{1}{24}}f(x'). \tag{12.3}$$

Finally $1+\sum_1^\infty \delta_{2s}(n)(-x)^n$ can be expressed in powers of x' by using the formula

$$\begin{aligned}&\alpha^s\left\{\tfrac12\zeta(1-2s) + \frac{1^{2s-1}}{e^{2\alpha}-1} + \frac{2^{2s-1}}{e^{4\alpha}-1} + \frac{3^{2s-1}}{e^{6\alpha}-1} + \cdots\right\} \\ &= (-\beta)^s\left\{\tfrac12\zeta(1-2s) + \frac{1^{2s-1}}{e^{2\beta}-1} + \frac{2^{2s-1}}{e^{4\beta}-1} + \frac{3^{2s-1}}{e^{6\beta}-1} + \cdots\right\},\end{aligned} \tag{12.4}$$

** Loc. cit.*, p.181 [p. 204 of this volume].

where $\alpha\beta = \pi^2$ and s is an integer greater than 1; and

$$(12.5)\qquad \begin{aligned} &(2\alpha)^{s+\frac{1}{2}}\left\{\frac{1^{2s}}{e^{\alpha}+e^{-\alpha}}+\frac{2^{2s}}{e^{2\alpha}+e^{-2\alpha}}+\frac{3^{2s}}{e^{3\alpha}+e^{-3\alpha}}+\cdots\right\}\\ &=(-\beta)^s\sqrt{(2\beta)}\left\{\tfrac{1}{2}\eta(-2s)+\frac{1^{2s}}{e^{\beta}-1}-\frac{3^{2s}}{e^{3\beta}-1}+\frac{5^{2s}}{e^{5\beta}-1}-\cdots\right\}, \end{aligned}$$

where $\alpha\beta = \pi^2$, s is any positive integer, and $\eta(s)$ is the function represented by the series $1^{-s} - 3^{-s} + 5^{-s} - \cdots$ and its analytical continuations.

It follows from all these formulæ that, if s is a positive integer and

$$(12.6)\qquad (1+x+x^3+x^6+\cdots)^{2s} = \sum r'_{2s}(n)x^n = \sum \delta'_{2s}(n)x^n + \sum e'_{2s}(n)x^n,$$

then

$$\sum e'_{2s}(n)x^n = \frac{f^{4s}(x^2)}{f^{2s}(x)}\sum_{1\le n\le\frac{1}{4}(s-1)} K_n(-x)^{-n}\frac{f^{24n}(x)}{f^{24n}(x^2)},$$

where K_n is a constant, and $f(x)$ is the same as in (12.1);

$$(12.61)\qquad (1^{-s}+3^{-s}+5^{-s}+\cdots)\sum\delta'_{2s}(n)x^n = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\left(\frac{1^{s-1}x}{1-x^2}+\frac{2^{s-1}x^2}{1-x^4}+\frac{3^{s-1}x^3}{1-x^6}+\cdots\right)$$

if s is a multiple of 4;

$$(12.62)\qquad (1^{-s}+3^{-s}+5^{-s}+\cdots)\sum\delta'_{2s}(n)x^n = \frac{2(\frac{1}{4}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\left(\frac{1^{s-1}x^{\frac{1}{2}}}{1-x}+\frac{3^{s-1}x^{\frac{3}{2}}}{1-x^3}+\frac{5^{s-1}x^{\frac{5}{2}}}{1-x^5}+\cdots\right)$$

if s is of the form $4k+2$;

$$(12.63)\qquad \begin{aligned} &(1^{-s}-3^{-s}+5^{-s}-\cdots)\sum\delta'_{2s}(n)x^n = \frac{2(\frac{1}{8}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\\ &\times\left(\frac{1^{s-1}x^{\frac{1}{4}}}{1+x^{\frac{1}{2}}}+\frac{3^{s-1}x^{\frac{3}{4}}}{1+x^{\frac{3}{2}}}+\frac{5^{-s}x^{\frac{5}{4}}}{1+x^{\frac{5}{2}}}+\cdots+\frac{1^{s-1}x^{\frac{1}{4}}}{1-x^{\frac{1}{2}}}-\frac{3^{s-1}x^{\frac{3}{4}}}{1-x^{\frac{3}{2}}}+\frac{5^{s-1}x^{\frac{5}{4}}}{1-x^{\frac{5}{2}}}-\cdots\right) \end{aligned}$$

if s is of the form $4k+1$ (except when $s = 1$); and

$$(12.64)\qquad \begin{aligned} &(1^{-s}-3^{-s}+5^{-s}-\cdots)\sum\delta'_{2s}(n)x^n = \frac{2(\frac{1}{8}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\\ &\times\left(\frac{1^{s-1}x^{\frac{1}{4}}}{1+x^{\frac{1}{2}}}+\frac{3^{s-1}x^{\frac{3}{4}}}{1+x^{\frac{3}{2}}}+\frac{5^{-s}x^{\frac{5}{4}}}{1+x^{\frac{5}{2}}}+\cdots-\frac{1^{s-1}x^{\frac{1}{4}}}{1-x^{\frac{1}{2}}}+\frac{3^{s-1}x^{\frac{3}{4}}}{1-x^{\frac{3}{2}}}-\frac{5^{-s}x^{\frac{5}{4}}}{1-x^{\frac{5}{2}}}+\cdots\right) \end{aligned}$$

if s is of the form $4k+3$. In the case in which $s=1$ we have

$$\begin{aligned}\sum \delta_2'(n)x^n &= x^{-\frac{1}{4}}\left(\frac{x^{\frac{1}{4}}}{1+x^{\frac{1}{2}}}+\frac{x^{\frac{3}{4}}}{1+x^{\frac{3}{2}}}+\frac{x^{\frac{5}{4}}}{1+x^{\frac{5}{2}}}+\cdots\right)\\ &= x^{-\frac{1}{4}}\left(\frac{x^{\frac{1}{4}}}{1+x^{\frac{1}{2}}}-\frac{x^{\frac{3}{4}}}{1+x^{\frac{3}{2}}}+\frac{x^{\frac{5}{4}}}{1+x^{\frac{5}{2}}}-\cdots\right).\end{aligned} \tag{12.65}$$

It is easy to see that the principal results proved about $e_{2s}(n)$ in my former paper are also true of $e_{2s}'(n)$, and in particular that

$$e_{2s}'(n)=0$$

when $s=1,2,3$ or 4, and

$$r_{2s}'(n)\sim\delta_{2s}'(n)$$

for all values of s.

13. It follows from (12.62) that, if s is of the form $4k+2$, then

$$(1^{-s}+3^{-s}+5^{-s}+\cdots)\delta_{2s}'(n)$$

is the coefficient of x^n in

$$\frac{2(\frac{1}{4}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\left(\frac{1^{s-1}x^{\frac{1}{2}}}{1-x}+\frac{2^{s-1}x}{1-x^2}+\frac{3^{s-1}x^{\frac{3}{2}}}{1-x^3}+\cdots\right). \tag{13.1}$$

Similarly from (12.63) and (12.64) it follows that, if s is an odd integer greater than 1, then $(1^{-s}-3^{-s}+5^{-s}-\cdots)\delta_{2s}'(n)$ is the coefficient of x^n in

$$\frac{4(\frac{1}{8}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\left(\frac{1^{s-1}x^{\frac{1}{4}}}{1+x^{\frac{1}{2}}}+\frac{2^{s-1}x^{\frac{1}{2}}}{1+x}+\frac{3^{s-1}x^{\frac{3}{4}}}{1+x^{\frac{3}{2}}}+\cdots\right). \tag{13.2}$$

Now by applying our main formulæ to (12.61) and (13.1) and (13.2) we obtain:

$$\begin{aligned}\delta_{2s}'(n) = {} & \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n+\tfrac{1}{4}s)^{s-1}\\ & \{1^{-s}c_1(n+\tfrac{1}{4}s)+3^{-s}c_3(n+\tfrac{1}{4}s)+5^{-s}c_5(n+\tfrac{1}{4}s)+\cdots\}\end{aligned} \tag{13.3}$$

if s is a multiple of 4;

$$\begin{aligned}\delta_{2s}'(n) = {} & \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n+\tfrac{1}{4}s)^{s-1}\\ & \{1^{-s}c_1(2n+\tfrac{1}{2}s)+3^{-s}c_3(2n+\tfrac{1}{2}s)+5^{-s}c_5(2n+\tfrac{1}{2}s)+\cdots\}\end{aligned} \tag{13.4}$$

if s is twice an odd number; and

$$\delta'_{2s}(n) = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n+\tfrac{1}{4}s)^{s-1} \tag{13.5}$$
$$\{1^{-s}c_1(4n+s) - 3^{-s}c_3(4n+s) + 5^{-s}c_5(4n+s) - \cdots\}$$

if s is an odd number greater than 1.

Since the coefficient of x^n in $(1+x+x^3+\cdots)^2$ is that of x^{4n+1} in

$$\left(\tfrac{1}{2}+x+x^4+\cdots\right)^2,$$

it follows from (11.51) that

$$r'_2(n) = \delta'_2(n) = \frac{\pi}{4}\{c_1(4n+1) - \tfrac{1}{3}c_3(4n+1) + \tfrac{1}{5}c_5(4n+1) - \cdots\}. \tag{13.6}$$

This result however depends on the fact that the Dirichlet's series for $1/\eta(s)$ is convergent when $s = 1$.

14. The preceding formulæ for $\sigma_s(n), \delta_{2s}(n), \delta'_{2s}(n)$ may be arrived at by another method. We understand by

$$\frac{\sin n\pi}{k\sin(n\pi/k)} \tag{14.1}$$

the limit of

$$\frac{\sin x\pi}{k\sin(x\pi/k)}$$

when $x \to n$. It is easy to see that, if n and k are positive integers, and k odd, then (14.1) is equal to 1 if k is a divisor of n and to 0 otherwise.

When k is even we have (with similar conventions)

$$\frac{\sin n\pi}{k\tan(n\pi/k)} = 1 \text{ or } 0 \tag{14.2}$$

according as k is a divisor of n or not. It follows that

$$\begin{aligned}\sigma_{s-1}(n) = \quad & n^{s-1}\left\{1^{-s}\left(\frac{\sin n\pi}{\sin n\pi}\right) + 2^{-s}\left(\frac{\sin n\pi}{\tan\frac{1}{2}n\pi}\right)\right. \\ & \left. +3^{-s}\left(\frac{\sin n\pi}{\sin\frac{1}{3}n\pi}\right) + 4^{-s}\left(\frac{\sin n\pi}{\tan\frac{1}{4}n\pi}\right) + \cdots\right\}.\end{aligned} \tag{14.3}$$

Similarly from the definitions of $\delta_{2s}(n)$ and $\delta'_{2s}(n)$ we find that

$$\{1^{-s} + (-3)^{-s} + 5^{-s} + (-7)^{-s} + \cdots\}\delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}\left\{1^{-s}\left(\frac{\sin n\pi}{\sin n\pi}\right)\right. \tag{14.4}$$
$$\left. +2^{-s}\left(\frac{\sin n\pi}{\sin(\frac{1}{2}n\pi+\frac{1}{2}s\pi)}\right) +3^{-s}\left(\frac{\sin n\pi}{\sin(\frac{1}{3}n\pi+s\pi)}\right) + 4^{-s}\left(\frac{\sin n\pi}{\sin(\frac{1}{4}n\pi+\frac{3}{2}s\pi)}\right) + \cdots\right\}$$

if s is an integer greater than 1;

$$\begin{aligned} r_2(n) = \delta_2(n) &= 4\left\{\left(\frac{\sin n\pi}{\sin n\pi}\right) - \tfrac{1}{3}\left(\frac{\sin n\pi}{\sin \frac{1}{3}n\pi}\right) + \tfrac{1}{5}\left(\frac{\sin n\pi}{\sin \frac{1}{5}n\pi}\right) - \cdots\right\} \\ &= 4\left\{\tfrac{1}{2}\left(\frac{\sin n\pi}{\cos \frac{1}{2}n\pi}\right) - \tfrac{1}{4}\left(\frac{\sin n\pi}{\cos \frac{1}{4}n\pi}\right) + \tfrac{1}{6}\left(\frac{\sin n\pi}{\cos \frac{1}{6}n\pi}\right) - \cdots\right\}; \end{aligned} \tag{14.5}$$

$$\begin{aligned} &(1^{-s} + 3^{-s} + 5^{-s} + \cdots)\delta'_{2s}(n) = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n + \tfrac{1}{4}s)^{s-1} \\ &\left\{1^{-s}\left(\frac{\sin(n+\frac{1}{4}s)\pi}{\sin(n+\frac{1}{4}s)\pi}\right) + 3^{-s}\left(\frac{\sin(n+\frac{1}{4}s)\pi}{\sin\frac{1}{3}(n+\frac{1}{4}s)\pi}\right)\right. \\ &\left. +5^{-s}\left(\frac{\sin(n+\frac{1}{4}s)\pi}{\sin\frac{1}{5}(n+\frac{1}{4}s)\pi}\right) + \cdots\right\} \end{aligned} \tag{14.6}$$

if s is a multiple of 4;

$$\begin{aligned} &(1^{-s} + 3^{-s} + 5^{-s} + \cdots)\delta'_{2s}(n) = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n + \tfrac{1}{4}s)^{s-1} \\ &\left\{1^{-s}\left(\frac{\sin(2n+\frac{1}{2}s)\pi}{\sin(2n+\frac{1}{2}s)\pi}\right) + 3^{-s}\left(\frac{\sin(2n+\frac{1}{2}s)\pi}{\sin\frac{1}{3}(2n+\frac{1}{2}s)\pi}\right)\right. \\ &\left. +5^{-s}\left(\frac{\sin(2n+\frac{1}{2}s)\pi}{\sin\frac{1}{5}(2n+\frac{1}{2}s)\pi}\right) + \cdots\right\} \end{aligned} \tag{14.7}$$

if s is twice an odd number;

(14.8)

$$\begin{aligned} (1^{-s} - 3^{-s} + 5^{-s} - \cdots)\delta'_{2s}(n) = \; &\frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n + \tfrac{1}{4}s)^{s-1} \\ &\left\{1^{-s}\left(\frac{\sin(4n+s)\pi}{\sin(4n+s)\pi}\right) - 3^{-s}\left(\frac{\sin(4n+s)\pi}{\sin\frac{1}{3}(4n+s)\pi}\right)\right. \\ &\left. +5^{-s}\left(\frac{\sin(4n+s)\pi}{\sin\frac{1}{5}(4n+s)\pi}\right) - \cdots\right\} \end{aligned}$$

if s is an odd number greater than 1; and

$$\begin{aligned} &r'_2(n) = \delta'_2(n) = \left(\frac{\sin(4n+1)\pi}{\sin(4n+1)\pi}\right) - \tfrac{1}{3}\left(\frac{\sin(4n+1)\pi}{\sin\frac{1}{3}(4n+1)\pi}\right) \\ &+\tfrac{1}{5}\left(\frac{\sin(4n+1)\pi}{\sin\frac{1}{5}(4n+1)\pi}\right) - \cdots\Bigg\}. \end{aligned} \tag{14.9}$$

In all these equations the series on the right hand are finite Dirichlet's series and therefore absolutely convergent.

But the series (14.3) is (as easily shewn by actual multiplication) the product of the two series

$$1^{-s}c_1(n)+2^{-s}c_2(n)+\cdots$$

and

$$n^{s-1}(1^{-s}+2^{-s}+3^{-s}+\cdots).$$

We thus obtain an alternative proof of the formulæ (7.5). Similarly taking the previous expression of $\delta_{2s}(n)$. viz. the right-hand side of (11.6), and multiplying it by the series

$$1^{-s}+(-3)^{-s}+5^{-s}+(-7)^{-s}+\cdots$$

we can shew that the product is actually the right-hand side of (14.4). The formulæ for $\delta'_{2s}(n)$ can be disposed off similarly.

15. The formulæ which I have found are closely connected with a method used for another purpose by Mr. Hardy and myself*. The function

$$(1+2x+2x^4+2x^9+\cdots)^{2s}=\sum r_{2s}(n)x^n \tag{15.1}$$

has every point of the unit circle as a singular point. If x approaches a "rational point" $\exp(-2p\pi i/q)$ on the circle, the function behaves roughly like

$$\frac{\pi^s(\omega_{p,q})^s}{\{-(2p\pi i/q)-\log x\}^s}, \tag{15.2}$$

where $\omega_{p,q}=1,0$, or -1 according as q is of the form $4k+1, 4k+2$ or $4k+3$, while if q is of the form $4k$ then $\omega_{p,q}=-2i$ or $2i$ according as p is of the form $4k+1$ or $4k+3$.

Following the argument of our paper referred to, we can construct simple functions of x which are regular except at one point of the circle of convergence, and there behave in a manner very similar to that of the function (15.1); for example at the point $\exp(-2p\pi i/q)$ such a function is

$$\frac{\pi^s(\omega_{p,q})^s}{(s-1)!}\sum_1^\infty n^{s-1}e^{2np\pi i/q}x^n. \tag{15.3}$$

The method which we used, with particular reference to the function

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}=\sum p(n)x^n, \tag{15.4}$$

* "Asymptotic formulæ in Combinatory Analysis", *Proc. London Math. Soc.*, Ser.2, Vol. XVII, 1918, pp. 75 – 115 [No.36 of this volume].

was to approximate to the coefficients by means of a sum of a large number of the coefficients of these auxiliary functions. This method leads, in the present problem, to formulæ of the type

$$r_{2s}(n) = \delta_{2s}(n) + O(n^{\frac{1}{2}s}),$$

the first term on the right-hand side presenting itself precisely in the form of the series (11.11) etc.

It is a very interesting problem to determine in such cases whether the approximate formula gives an exact representation of such an arithmetical function. The results proved here shew that, in the case of $r_{2s}(n)$, this is in general not so. The formula represents not $r_{2s}(n)$ but (except when $s = 1$) its dominant term $\delta_{2s}(n)$, which is equal to $r_{2s}(n)$ only when $s = 1, 2, 3$ or 4. When $s = 1$ the formula gives $2\delta_2(n)$*.

16. We shall now consider the sum

$$\sigma_s(1) + \sigma_s(2) + \cdots + \sigma_s(n). \tag{16.1}$$

Suppose that

$$\begin{aligned} T_r(n) &= \tfrac{1}{2}\sum_{\lambda}\left(\frac{\sin\{(2n+1)\pi\lambda/r\}}{\sin(\pi\lambda/r)} - 1\right), \\ U_r(n) &= \tfrac{1}{2}\sum_{\lambda}\frac{\sin\{(2n+1)\pi\lambda/r\}}{\sin(\pi\lambda/r)}, \end{aligned} \tag{16.2}$$

where λ is prime to r and does not exceed r, so that

$$T_r(n) = c_r(1) + c_r(2) + \cdots + c_r(n)$$

and

$$U_r(n) = T_r(n) + \tfrac{1}{2}\phi(r),$$

where $\phi(n)$ is the same as in §9. Since $c_r(n) = O(1)$ as $r \to \infty$, it follows that

$$T_r(n) = O(1), \quad U_r(n) = O(r), \tag{16.21}$$

as $r \to \infty$. It follows from (7.5) that, if $s > 0$, then

$$\begin{aligned} \sigma_{-s}(1) + \sigma_{-s}(2) + \cdots + \sigma_{-s}(n) = \;& \zeta(s+1) \\ & \left\{n + \frac{T_2(n)}{2^{s+1}} + \frac{T_3(n)}{3^{s+1}} + \frac{T_4(n)}{4^{s+1}} + \cdots\right\}. \end{aligned} \tag{16.3}$$

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*The method is also applicable to the problem of the representation of a number by the sum of an *odd* number of squares, and gives an exact result when the number of squares is 3, 5, or 7. See G.H. Hardy, "On the representation of a number as the sum of any number of squares, and in particular of five or seven," *Proc. London Math. Soc. (Records of proceedings at meetings,* March 1918). A fuller account of this paper will appear shortly in the *Proceedings of the National Academy of Sciences* (Washington, D.C.) [*loc. cit.*, Vol.IV, 1918, 189-193].

Since

$$\sum_1^\infty \frac{\phi(n)}{\nu^{s+1}} = \frac{\zeta(s)}{\zeta(s+1)}$$

if $s > 1$, (16.3) can be written as

(16.31)

$$\sigma_{-s}(1)\sigma_{-s}(2) + \cdots + \sigma_{-s}(n) = \zeta(s+1) \left\{n + \tfrac{1}{2} + \frac{U_2(n)}{2^{s+1}} + \frac{U_3(n)}{3^{s+1}} + \frac{U_4(n)}{4^{s+1}} + \cdots\right\} - \tfrac{1}{2}\zeta(s),$$

if $s > 1$. Similarly from (8.3), (8.4) and (11.51) we obtain

$$\begin{aligned} &d(1) + d(2) + \cdots + d(n) \\ &= -\tfrac{1}{2}T_2(n)\log 2 - \tfrac{1}{3}T_3(n)\log 3 - \tfrac{1}{4}T_4(n)\log 4 - \cdots, \end{aligned} \tag{16.4}$$

$$\begin{aligned} &d(1)\log 1 + d(2)\log 2 + \cdots + d(n)\log n \\ &= \tfrac{1}{2}T_2(n)\{2\gamma\log 2 - (\log 2)^2\} + \tfrac{1}{3}T_3(n)\{2\gamma\log 3 - (\log 3)^2\} + \cdots, \end{aligned} \tag{16.5}$$

$$r_2(1) + r_2(2) + \cdots + r_2(n) = \pi\{n - \tfrac{1}{3}T_3(n) + \tfrac{1}{5}T_5(n) - \tfrac{1}{7}T_7(n) + \cdots\}. \tag{16.6}$$

Suppose now that

$$T_{r,s}(n) = \sum_\lambda \left(1^s \cos\frac{2\pi\lambda}{r} + 2^s\cos\frac{4\pi\lambda}{r} + \cdots + n^s\cos\frac{2n\pi\lambda}{r}\right),$$

where λ is prime to r and does not exceed r, so that

$$T_{r,s}(n) = 1^s c_r(1) + 2^s c_r(2) + \cdots + n^s c_r(n).$$

Then it follows from (7.5) that

$$\begin{aligned} &\sigma_s(1) + \sigma_s(2) + \cdots + \sigma_s(n) \\ &= \zeta(s+1)\left\{(1^s + 2^s + \cdots + n^s) + \frac{T_{2,s}(n)}{2^{s+1}} + \frac{T_{3,s}(n)}{3^{s+1}} + \frac{T_{4,s}(n)}{4^{s+1}} + \cdots\right\} \end{aligned} \tag{16.7}$$

if $s > 0$. Putting $s = 1$ in (16.3) and (16.7), we find that

$$\begin{aligned} &(n-1)\sigma_{-1}(1) + (n-2)\sigma_{-1}(2) + \cdots + (n-n)\sigma_{-1}(n) \\ &= \frac{\pi^2}{6}\left\{\frac{n(n-1)}{2} + \frac{\nu_2(n)}{2^2} + \frac{\nu_3(n)}{3^2} + \frac{\nu_4(n)}{4^2} + \cdots\right\}, \end{aligned} \tag{16.8}$$

where

$$\nu_r(n) = \tfrac{1}{2}\sum_\lambda\left\{\frac{\sin^2(\pi n\lambda/r)}{\sin^2(\pi\lambda/r)} - n\right\},$$

λ being prime to r and not exceeding r.

It has been proved by Wigert *, by less elementary methods, that the left-hand side of (16.8) is equal to

$$\frac{\pi^2}{12}n^2 - \tfrac{1}{2}n(\gamma - 1 + \log 2n\pi) - \tfrac{1}{24} + \frac{\sqrt{n}}{2\pi}\sum_{1}^{\infty}\frac{\sigma_{-1}(\nu)}{\sqrt{\nu}}J_1\{4\pi\sqrt{(\nu n)}\}, \tag{16.9}$$

where J_1 is the ordinary Bessel's function.

17. We shall now find a relation between the functions (16.1) and (16.3) which enables us to determine the behaviour of the former for large values of n. It is easily shewn that this function is equal to

$$\sum_{\nu=1}^{\sqrt{n}}\left(1^s + 2^s + 3^s + \cdots + \left[\frac{n}{\nu}\right]^s\right) + \sum_{\nu=1}^{\sqrt{n}}\nu^s\left[\frac{n}{\nu}\right] - [\sqrt{n}]\sum_{\nu=1}^{\sqrt{n}}\nu^s. \tag{17.1}$$

Now

$$1^s + 2^s + \cdots + k^s = \zeta(-s) + \frac{(k+\frac{1}{2})^{s+1}}{s+1} + O(k^{s-1})$$

for all values of s, it being understood that

$$\zeta(-s) + \frac{(k+\frac{1}{2})^{s+1}}{s+1}$$

denotes $\gamma + \log(k+\frac{1}{2})$ when $s = -1$. Let

$$\left[\frac{n}{\nu}\right] = \frac{n}{\nu} - \tfrac{1}{2} + \epsilon_\nu, \quad [\sqrt{n}] = t = \sqrt{n} - \tfrac{1}{2} + \epsilon.$$

Then we have

$$1^s + 2^s + \cdots + \left[\frac{n}{\nu}\right]^s = \zeta(-s) + \frac{1}{s+1}\left(\frac{n}{\nu}\right)^{s+1} + \epsilon_\nu\left(\frac{n}{\nu}\right)^s + O\left(\frac{n^{s-1}}{\nu^{s-1}}\right)$$

and

$$\nu^s\left[\frac{n}{\nu}\right] = n\nu^{s-1} - \tfrac{1}{2}\nu^s + \epsilon_\nu\nu^s.$$

It follows from these equations and (17.1) that

$$\begin{aligned}\sigma_s(1) + \sigma_s(2) + \cdots + \sigma_s(n) = & \sum_{\nu=1}^{t}\left\{\zeta(-s) + \frac{1}{s+1}\left(\frac{n}{\nu}\right)^{s+1} + n\nu^{s-1}\right.\\ & \left. + \epsilon_\nu\left(\tfrac{n}{\nu}\right)^s + \epsilon_\nu\nu^s - (\sqrt{n}+\epsilon)\nu^s + O\left(\tfrac{n^{s-1}}{\nu^{s-1}}\right)\right\}.\end{aligned} \tag{17.2}$$

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* *Acta Mathematica,* Vol.XXXVII, 1914, pp. 113 – 140(p.140).

Changing s to $-s$ in (17.2) we have

(17.21)
$$\begin{aligned}&n^s\{\sigma_{-s}(1)+\sigma_{-s}(2)+\cdots+\sigma_{-s}(n)\}\\&=\sum_{\nu=1}^{t}\left\{n^s\zeta(s)+\frac{n\nu^{s-1}}{1-s}+\left(\frac{n}{\nu}\right)^{s+1}+\epsilon_\nu\nu^s+\epsilon_\nu\left(\frac{n}{\nu}\right)^s\right.\\&\qquad\left.-(\sqrt{n}+\epsilon)\left(\frac{n}{\nu}\right)^s+O\left(\frac{\nu^{s+1}}{n}\right)\right\}.\end{aligned}$$

It follows that

(17.3)
$$\begin{aligned}&n^s\{\sigma_{-s}(1)+\sigma_{-s}(2)+\cdots+\sigma_{-s}(n)\}-\{\sigma_s(1)+\sigma_s(2)+\cdots+\sigma_s(n)\}\\&=\sum_{\nu=1}^{t}\left\{n^s\zeta(s)-\zeta(-s)+\frac{s}{1+s}\left(\frac{n}{\nu}\right)^{s+1}+\frac{s}{1-s}n\nu^{s-1}+(\sqrt{n+\epsilon})\nu^s\right.\\&\qquad\left.-(\sqrt{n}+\epsilon)\left(\frac{n}{\nu}\right)^s+O\left(\frac{n^{s-1}}{\nu^{s-1}}+\frac{\nu^{s+1}}{n}\right)\right\}.\end{aligned}$$

Suppose now that $s>0$. Then, since ν varies from 1 to t, it is obvious that

$$\frac{\nu^{s+1}}{n}<\frac{n^{s-1}}{\nu^{s-1}}$$

and so

$$O\left(\frac{\nu^{s+1}}{n}\right)=O\left(\frac{n^{s+1}}{\nu^{s-1}}\right).$$

. Also

$$\sum_{\nu=1}^{t}\{n^s\zeta(s)-\zeta(-s)\}=(\sqrt{n}-\tfrac{1}{2}+\epsilon)\{n^s\zeta(s)-\zeta(-s)\};$$

$$\sum_{\nu=1}^{t}\frac{s}{1+s}\left(\frac{n}{\nu}\right)^{s+1}=\frac{sn^{s+1}}{1+s}\zeta(1+s)-\frac{n^{s+1}}{s+1}(\sqrt{n}+\epsilon)^{-s}+O(n^{\frac{1}{2}s});$$

$$\sum_{\nu=1}^{t}\frac{s}{1-s}n\nu^{s-1}=\frac{ns}{1-s}\zeta(1-s)+\frac{n}{1-s}(\sqrt{n}+\epsilon)^s+O(n^{\frac{1}{2}s});$$

$$\sum_{\nu=1}^{t}(\sqrt{n}+\epsilon)\nu^s=(\sqrt{n}+\epsilon)\zeta(-s)+\frac{(\sqrt{n}+\epsilon)^{2+s}}{1+s}+O(n^{\frac{1}{2}s});$$

$$\sum_{\nu=1}^{t}(\sqrt{n}+\epsilon)\left(\frac{n}{\nu}\right)^s=n^s(\sqrt{n}+\epsilon)\zeta(s)+\frac{n^s}{1-s}(\sqrt{n}+\epsilon)^{2-s}+O(n^{\frac{1}{2}s});$$

and

$$\sum_{\nu=1}^{t}O\left(\frac{n^{s-1}}{\nu^{s-1}}\right)=O(m),$$

where

$$m = n^{\frac{1}{2}s}\ (s < 2),\ m = n \log n (s = 2),\ m = n^{s-1}\ (s > 2). \tag{17.4}$$

It follows that the right-hand side of (17.3) is equal to

$$\frac{sn^{1+s}}{1+s}\zeta(1+s) + \frac{sn}{1-s}\zeta(1-s) - \tfrac{1}{2}n^s\zeta(s) + \frac{(\sqrt{n}+\epsilon)^{2+s} - n^{s+1}(\sqrt{n}+\epsilon)^{-s}}{1+s}$$
$$+\frac{n(\sqrt{n}+\epsilon)^{s} - n^{s}(\sqrt{n}+\epsilon)^{2-s}}{1-s} + O(m).$$

But

$$\frac{(\sqrt{n}+\epsilon)^{2+s} - n^{s+1}(\sqrt{n}+\epsilon)^{-s}}{1+s} = 2\epsilon n^{\frac{1}{2}(1+s)} + O(n^{\frac{1}{2}s});$$
$$\frac{n(\sqrt{n}+\epsilon)^{s} - n^{s}(\sqrt{n}+\epsilon)^{2-s}}{1-s} = -2\epsilon n^{\frac{1}{2}(1+s)} + O(n^{\frac{1}{2}s}).$$

It follows that

$$\begin{aligned} &\sigma_s(1) + \sigma_s(2) + \cdots + \sigma_s(n) = n^s\{\sigma_{-s}(1) + \sigma_{-s}(2) + \cdots + \sigma_{-s}(n)\} \\ &-\frac{sn^{1+s}}{1+s}\zeta(1+s) + \tfrac{1}{2}n^s\zeta(s) - \frac{sn}{1-s}\zeta(1-s) + O(m) \end{aligned} \tag{17.5}$$

if $s > 0, m$ being the same as in (17.4). If $s = 1$, (17.5) reduces to

$$\begin{aligned} &(n-1)\sigma_{-1}(1) + (n-2)\sigma_{-1}(2) + \cdots + (n-n)\sigma_{-1}(n) \\ &= \tfrac{\pi^2}{12}n^2 - \tfrac{1}{2}n(\gamma - 1 + \log 2n\pi) + O(\sqrt{n})^*. \end{aligned} \tag{17.6}$$

From (16.2) and (17.5) it follows that

$$\begin{aligned} \sigma_s(1) + \sigma_s(2) + \cdots + \sigma_s(n) &= \frac{n^{1+s}}{1+s}\zeta(1+s) + \tfrac{1}{2}n^s\zeta(s) \\ &+\frac{sn}{s-1}\zeta(1-s) + n^s\zeta(1+s)\left\{\frac{T_2(n)}{2^{s+1}} + \frac{T_3(n)}{3^{s+1}} + \frac{T_4(n)}{4^{s+1}} + \cdots\right\} + O(m), \end{aligned} \tag{17.7}$$

for all positive values of s. If $s > 1$, the right-hand side can be written as

$$\begin{aligned} &\frac{ns}{s-1}\zeta(1-s) + n^s\zeta(1+s) \\ &\left\{\frac{n}{1+s} + \frac{1}{2} + \frac{U_2(n)}{2^{s+1}} + \frac{U_3(n)}{3^{s+1}} + \frac{U_4(n)}{4^{s+1}} + \cdots\right\} + O(m). \end{aligned} \tag{17.8}$$

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*This result has been proved by Landau. See his report on Wigert's memoir in the *Göttingische gelehrte Anzeigen*, 1915, pp. 377 – 414 (p.402). Landau has also, by a more transcendental method, replaced $O(\sqrt{n})$ by $O(n^{\frac{2}{5}})$ (*loc.cit.*, p.414).

Putting $s = 1$ in (17.7) we obtain

$$\begin{aligned}\sigma_1(1) + \sigma_1(2) + \cdots + \sigma_1(n) &= \frac{\pi^2}{12}n^2 + \tfrac{1}{2}n(\gamma - 1 + \log 2n\pi) \\ &+ \frac{\pi^2 n}{6}\left\{\frac{T_2(n)}{2^2} + \frac{T_3(n)}{3^2} + \frac{T_4(n)}{4^2} + \cdots\right\} + O(\sqrt{n}).\end{aligned} \tag{17.9}$$

Additional note to §7 (*May* 1, 1918).

From (7.2) it follows that

$$\frac{1}{\zeta(r)}\{1^{-s}\sigma_{1-r}(1) + 2^{-s}\sigma_{1-r}(2) + \cdots\} = 1^{-s}\sum_1^\infty m^{-r}c_m(1) + 2^{-s}\sum_1^\infty m^{-r}c_m(2) + \cdots,$$

or

$$\frac{\zeta(s)\zeta(r+s-1)}{\zeta(r)} = \sum_1^\infty\sum_1^\infty \frac{c_m(n)}{m^r n^s},$$

from which we deduce

$$\zeta(s)\sum_\delta \mu(\delta)\delta'^{1-s} = \frac{c_m(1)}{1^s} + \frac{c_m(2)}{2^s} + \frac{c_m(3)}{3^s} + \cdots,$$

δ being a divisor of m and δ' its conjugate. The series on the right-hand side is convergent for $s > 0$ (except when $m = 1$, when it reduces to the ordinary series for $\zeta(s)$).
When $s = 1, m > 1$ we have to replace the left-hand side by its limit as $s \to 1$. We find that

$$c_m(1) + \tfrac{1}{2}c_m(2) + \tfrac{1}{3}c_m(3) + \cdots = -\Lambda(m), \tag{18}$$

$\Lambda(m)$ being the well-known arithmetical function which is equal to $\log p$ if m is a power of a prime p and to zero otherwise.

Some definite integrals

Proceedings of the London Mathematical Society, 2, XVII, 1918,
Records for 17 Jan. 1918

Typical formulæ are:

$$\int_{-\infty}^{\infty} \frac{e^{nix}dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = \frac{(2\cos\frac{1}{2}n)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{\frac{1}{2}n(\beta-\alpha)i} \text{ (or 0)}, \tag{1}$$

$$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} e^{nix}dx = \pm\frac{2\pi i(2\sin\frac{1}{2}N)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} e^{-n\alpha i+\frac{1}{2}(\pi-N)(\beta-\alpha-1)i} \text{ (or 0)}, \tag{2}$$

$$\int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x)e^{nix}dx$$

$$= \frac{2\pi i\Gamma(\alpha+\beta)}{(2\sin\frac{1}{2}N)^{\alpha+\beta}} e^{\frac{1}{2}n(\beta-\alpha)i}\left[\epsilon_n(\beta)e^{k\pi(\alpha+\beta)i} - \epsilon_n(-\alpha)e^{-k\pi(\alpha+\beta)i}\right]. \tag{3}$$

Here n is real, $n = 2k\pi + N(0 \le N < 2\pi)$ in (2), and $n = (2k-1)\pi + N(0 \le N < 2\pi)$ in (3). In (1) the zero value is to be taken if $|n| \ge \pi$, the non-zero value otherwise. In (2) α must be complex: the zero value is to be taken if n and $\Im(\alpha)$ have the same sign, the positive sign if $n \ge 0$ and $\Im(\alpha) < 0$, and the negative sign if $n \le 0$ and $\Im(\alpha) > 0$. In (3) α and β must both be complex; and $\epsilon_n(\zeta)$ is 0,1, or -1 according as (i) $\pi - n$ and $\Im(\zeta)$ have the same sign, (ii) $n \le \pi$ and $\Im(\zeta) < 0$, (iii) $n \ge \pi$ and $\Im(\zeta) > 0$.

The convergence conditions are, in general,

(1) $\Re(\alpha+\beta) > 1$,   (2) $\Re(\alpha-\beta) < 0$,   (3)$\Re(\alpha+\beta) < 1$.

But there are certain special cases in which a more stringent condition is required.

A formula of a different character, deduced from (1), is

$$\int_{-\infty}^{\infty} \frac{J_{\alpha+x}(\lambda)}{\lambda^{\alpha+x}} \frac{J_{\beta-x}(\mu)}{\mu^{\beta-x}} e^{nix}dx = \left(\frac{2\cos\frac{1}{2}n}{\Omega}\right)^{\frac{1}{2}(\alpha+\beta)}$$

$$e^{\frac{1}{2}n(\beta-\alpha)i} J_{\alpha+\beta}\{\sqrt{(2\Omega\cos\tfrac{1}{2}n)}\} \text{ (or 0)}.$$

Here

$$\Omega = \lambda^2 e^{\frac{1}{2}ni} + \mu^2 e^{-\frac{1}{2}ni};$$

the zero value is to be taken if $|n|+ \geq \pi$, the non-zero value otherwise; and the condition of convergence is, in general, that

$$\Re(\alpha+\beta) > -1.$$

The formulæ include a large number of interesting special cases, such as

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} &= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}, \\
\int_{0}^{\infty} \frac{\sin \pi x dx}{x(x^2-1^2)(x^2-2^2)\cdots(x^2-k^2)} &= (-1)^k \frac{2^{2k-1}\pi}{(2k)!}, \\
\int_{-\infty}^{\infty} J_{\alpha+x}(\lambda) J_{\beta-x}(\lambda) dx &= J_{\alpha+\beta}(2\lambda).
\end{aligned}$$

The formula

$$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}$$
$$= \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)},$$

may also be mentioned : it holds, in general, if

$$\Re(\alpha+\beta+\gamma+\delta) > 3.$$

A fuller account of these formulæ will be published in the *Quarterly Journal of Mathematics**

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*[See No.27 of this volume]

Some definite integrals

Journal of the Indian Mathematical Society, XI, 1919, 81 – 87

I have shewn elsewhere* that the definite integrals

$$\phi_w(t) = \int\limits_0^\infty \frac{\cos \pi tx}{\cosh \pi x} e^{-\pi w x^2} dx,$$

$$\psi_w(t) = \int\limits_0^\infty \frac{\sin \pi tx}{\sinh \pi x} e^{-\pi w x^2} dx$$

can be evaluated in finite terms if w is any rational multiple of i.
In this paper I shall shew, by a much simpler method, that these integrals can be evaluated not only for these values but also for many other values of t and w.
Now we have

$$\begin{aligned} \phi_w(t) &= 2\int_0^\infty \int_0^\infty \frac{\cos 2\pi xz}{\cosh \pi z} \cos \pi tx e^{-\pi w x^2} dx dz \\ &= \frac{e^{-\frac{1}{4}\pi t^2 w'}}{\sqrt{w}} \int_0^\infty \frac{\cosh \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx \end{aligned}$$

where w' stands for $1/w$.
It follows that

$$\phi_w(t) = \frac{1}{\sqrt{w}} e^{-\frac{1}{4}\pi t^2 w'} \phi_{w'}(itw'). \tag{1}$$

Again

$$\begin{aligned} \phi_w(t+w) &= \frac{1}{\sqrt{w}} e^{-\frac{1}{4}\pi (t+w)^2 w'} \\ &\quad \times \int_0^\infty \frac{\cosh(\pi tx/w)\cosh \pi x + \sinh \pi tx/w \sinh \pi x}{\cosh \pi x} e^{-\pi x^2/w} dx \\ &= \frac{1}{\sqrt{w}} e^{-\frac{1}{4}\pi (t+w)^2/w} \\ &\quad \times \left\{ \tfrac{1}{2}\sqrt{w} e^{\frac{1}{4}\pi t^2/w} + 2\int_0^\infty \int_0^\infty \frac{\sin 2\pi xz}{\sinh \pi z} \sinh \frac{\pi tx}{w} e^{-\pi x^2/w} dx dz \right\} \\ &= \frac{1}{\sqrt{w}} e^{-\frac{1}{4}\pi (t+w)^2/w} \end{aligned}$$

* *Messenger of Mathematics,* Vol,44, 1915, pp. 75 – 85 [No.12 of this volume].

$$\times \left\{ \tfrac{1}{2}\sqrt{w}e^{\frac{1}{4}\pi t^2/w} + \sqrt{w}e^{\frac{1}{4}\pi t^2/w} \int_0^\infty \frac{\sin \pi t x}{\sinh \pi x} e^{-\pi w x^2} dx \right\}.$$

In other words

$$e^{\frac{1}{4}\pi t^2/w}\{\tfrac{1}{2} + \psi_w(t)\} = e^{\frac{1}{4}\pi(t+w)^2/w}\phi_w(t+w). \tag{2}$$

It is obvious that

$$\left.\begin{array}{rcl} \phi_w(t) & = & \phi_w(-t) \\ \psi_w(t) & = & -\psi_w(-t) \end{array}\right\}. \tag{3}$$

From (1), (2) and (3) we easily find that

$$\tfrac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\frac{1}{4}\pi t^2/w} \left\{ \tfrac{1}{2} - \psi_{w'}\left(\frac{it}{w} + i\right) \right\}. \tag{4}$$

It is easy to see that

$$\phi_w(i) = \frac{1}{2\sqrt{w}}; \quad \psi_w(i) = \frac{i}{2\sqrt{w}}; \quad \phi_w(w) = \tfrac{1}{2}e^{-\frac{1}{4}\pi w};$$

$$\tfrac{1}{2} - \psi_w(w) = e^{-\frac{1}{4}\pi w}\phi_w(0); \quad \phi_w(w \pm i) = \left(\frac{1}{2\sqrt{w}} + \frac{i}{2}\right) e^{-\frac{1}{4}\pi w};$$

$$\psi_w(w \pm i) = \tfrac{1}{2} \pm \frac{i}{2\sqrt{w}} e^{-\frac{1}{4}\pi w}; \quad \phi_w(\tfrac{1}{2}w) + \psi_w(\tfrac{1}{2}w) = \tfrac{1}{2}.$$

Again we see that

$$\phi_w(t+i) + \phi_w(t-i) = \frac{1}{\sqrt{w}} e^{-\frac{1}{4}\pi t^2/w}; \tag{5}$$

and

$$\psi_w(t+i) - \psi_w(t-i) = \frac{i}{\sqrt{w}} e^{-\frac{1}{4}\pi t^2/w}. \tag{6}$$

From (1) and (5) we deduce that

$$e^{\frac{1}{4}\pi(t+w)^2/w}\phi_w(t+w) + e^{\frac{1}{4}\pi(t-w)^2/w}\phi_w(t-w) = e^{\frac{1}{4}\pi t^2/w}. \tag{7}$$

Similarly from (4) and (6) we obtain

$$e^{\frac{1}{4}\pi(t+w)^2/w}\left\{\tfrac{1}{2} - \psi_w(t+w)\right\} = e^{\frac{1}{4}\pi(t-w)^2/w}\{\tfrac{1}{2} + \psi_w(t-w)\}. \tag{8}$$

It is easy to deduce from (5) that if n is a positive integer, then

$$\phi_w(t) + (-1)^{n+1}\phi_w(t \pm 2ni)$$

$$= \tfrac{1}{\sqrt{w}}\left\{e^{-\frac14\pi(t\pm i)^2/w} - e^{-\frac14\pi(t\pm 3i)^2/w} + e^{-\frac14\pi(t\pm 5i)^2/w} - \cdots \text{ to } n \text{ terms}\right\}. \tag{9}$$

Similarly from (6) we have

$$\begin{aligned}&\psi_w(t) - \psi_w(t \pm 2ni)\\ &= \mp\tfrac{i}{\sqrt{w}}\left\{e^{-\frac14\pi(t+i)^2/w} + e^{-\frac14\pi(t+3i)^2/w} + e^{-\frac14\pi(t+5i)^2/w} + \cdots \text{ to } n \text{ terms}\right\}.\end{aligned} \tag{10}$$

Again from (7) we have

$$\begin{aligned}&e^{\frac14\pi t^2/w}\phi_w(t) + (-1)^{n+1}e^{\frac14\pi(t+2nw)^2/w}\phi_w(t+2nw)\\ &\quad= e^{\frac14\pi(t+w)^2/w} - e^{\frac14\pi(t+3w)^2/w} + e^{\frac14\pi(t+5w)^2/w} - \cdots \ \text{ to } n \text{ terms};\end{aligned} \tag{11}$$

and from (8)

$$\begin{aligned}&e^{\frac14\pi t^2/w}\{\tfrac12 + \psi_w(t)\} + (-1)^{n+1}e^{\frac14\pi(t+2nw)^2/w}\{\tfrac12 + \psi_w(t+2nw)\}\\ &= e^{\frac14\pi(t+2w)^2/w} - e^{\frac14\pi(t+4w)^2/w} + e^{\frac14\pi(t+6w)^2/w} - \cdots \ \text{ to } n \text{ terms.}\end{aligned} \tag{12}$$

Now, combining (9) and (11), we deduce that, if m and n are positive integers and $s = t + 2mw \pm 2ni$, then

$$\begin{aligned}&\phi_w(s) + (-1)^{(m+1)(n+1)}e^{-\frac12\pi m(s+t)}\phi_w(t)\\ &= e^{-\frac14\pi s^2/w}\left\{e^{\frac14\pi(s-w)^2/w} - e^{\frac14\pi(s-3w)^2/w} + e^{\frac14\pi(s-5w)^2/w} - \cdots \text{ to } m \text{ terms}\right\}\\ &\quad + \frac{(-1)^{(m+1)(n+1)}}{\sqrt{w}}e^{-\frac12\pi m(s+t)}\\ &\quad \times\left\{e^{-\frac14\pi(t\pm i)^2/w} - e^{-\frac14\pi(t\pm 3i)^2/w} + e^{-\frac14\pi(t\pm 5i)^2/w} - \cdots \text{ to } n \text{ terms}\right\}.\end{aligned} \tag{13}$$

Similarly, combining (10) and (12), we obtain

$$\begin{aligned}&\tfrac12 - \psi_w(s) + (-1)^{mn+m+1}e^{-\frac12\pi m(s+t)}\{\tfrac12 - \psi_w(t)\}\\ &= e^{-\frac14\pi s^2/w}\left\{e^{\frac14\pi(s-2w)^2/w} - e^{\frac14\pi(s-4w)^2/w} + e^{\frac14\pi(s-6w)^2/w} - \cdots \text{ to } m \text{ terms}\right\}\\ &\quad \pm(-1)^{mn+m+1}\frac{i}{\sqrt{w}}e^{-\frac12\pi m(s+t)}\\ &\quad \times\left\{e^{-\frac14\pi(t\pm i)^2/w} + e^{-\frac14\pi(t\pm 3i)^2/w} + e^{-\frac14\pi(t\pm 5i)^2/w} + \cdots \text{ to } n \text{ terms}\right\},\end{aligned} \tag{14}$$

where s and t have the same relation as in (13).

Suppose now that $s = t$ in (13) and (14). Then we see that, if $w = in/m$, then

$$\begin{aligned}
&\phi_w(t)\{1+(-1)^{(m+1)(n+1)}e^{-\pi mt}\}\\
&\quad = e^{-\frac{1}{4}\pi t^2/w}\left\{e^{\frac{1}{4}\pi(t-w)^2/w} - e^{\frac{1}{4}\pi(t-3w)^2/w} + e^{\frac{1}{4}\pi(t-5w)^2/w} - \cdots \text{ to } m \text{ terms}\right\}\\
&\qquad + \frac{(-1)^{(m+1)(n+1)}}{\sqrt{w}}e^{-\pi mt}\left\{e^{-\frac{1}{4}\pi(t-i)^2/w} - e^{-\frac{1}{4}\pi(t-3i)^2/w} + \cdots \text{ to } n \text{ terms}\right\}; \qquad (15)
\end{aligned}$$

$$\begin{aligned}
&\{\tfrac{1}{2}-\psi_w(t)\}\{1+(-1)^{mn+m+1}e^{-\pi mt}\}\\
&\quad = e^{-\frac{1}{4}\pi t^2/w}\left\{e^{\frac{1}{4}\pi(t-2w)^2/w} - e^{\frac{1}{4}\pi(t-4w)^2/w} + \cdots \text{ to } m \text{ terms}\right\}\\
&\qquad + (-1)^{mn+m}\frac{i}{\sqrt{w}}e^{-\pi mt}\left\{e^{-\frac{1}{4}\pi(t-i)^2/w} - e^{-\frac{1}{4}\pi(t-3i)^2/w} + \cdots \text{ to } n \text{ terms}\right\}. \qquad (16)
\end{aligned}$$

where $\sqrt{w}$ should be taken as

$$e^{\frac{1}{4}\pi i}\sqrt{\left(\frac{n}{m}\right)}.$$

In (15) and (16) there is no loss of generality in supposing that one of the two numbers m and n is odd.

Now equating the real and imaginary parts in (15), we deduce that, if m and n are positive integers of which one is odd, then

$$\begin{aligned}
&2\cosh nt\int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\cos\left(\frac{\pi m x^2}{n}\right)dx\\
&= [\cosh\{(1-n)t\}\cos(\pi m/4n) - \cosh\{(3-n)t\}\cos(9\pi m/4n) + \cdots \text{ to } n \text{ terms}]\\
&\quad + \sqrt{\left(\frac{n}{m}\right)}\left[\cosh\left\{\left(1-\frac{1}{m}\right)nt\right\}\cos\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{\pi n}{4m}\right)\right.\\
&\quad \left. - \cosh\left\{\left(1-\frac{3}{m}\right)nt\right\}\cos\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{9\pi n}{4m}\right) + \cdots \text{ to } m \text{ terms}\right]; \qquad (17)
\end{aligned}$$

and

$$\begin{aligned}
&2\cosh nt\int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\sin\left(\frac{\pi m x^2}{n}\right)dx\\
&= -[\cosh\{(1-n)t\}\sin(\pi m/4n) - \cosh\{(3-n)t\}\sin(9\pi m/4n)\\
&\quad + \cosh\{(5-n)t\}\sin(25\pi/4n) - \cdots \text{ to } n \text{ terms}]\\
&\quad + \sqrt{\left(\frac{n}{m}\right)}\left[\cosh\left\{\left(1-\frac{1}{m}\right)nt\right\}\sin\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{\pi n}{4m}\right)\right.\\
&\quad \left. - \cosh\left\{\left(1-\frac{3}{m}\right)nt\right\}\sin\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{9\pi n}{\pi m}\right) + \cdots \text{ to } n \text{ terms}\right]. \qquad (18)
\end{aligned}$$

Equating the real and imaginary parts in (16), we can find similar expressions for the integrals

$$\int_0^\infty \frac{\sin tx}{\sinh \pi x} \sin\left(\frac{\pi m x^2}{n}\right) dx, \int_0^\infty \frac{\sin tx}{\sinh \pi x} \cos\left(\frac{\pi m x^2}{n}\right) dx.$$

From these formulæ we can evaluate a number of definite integrals, such as

$$\begin{aligned}
\int_0^\infty \frac{\cos 2\pi t x}{\cosh \pi x} \cos \pi x^2 dx &= \frac{1+\sqrt{2}\sin \pi t^2}{2\sqrt{2}\cosh \pi t}, \\
\int_0^\infty \frac{\cos 2\pi t x}{\cosh \pi x} \sin \pi x^2 dx &= \frac{-1+\sqrt{2}\cos \pi t^2}{2\sqrt{2}\cosh \pi t}, \\
\int_0^\infty \frac{\sin 2\pi t x}{\sinh \pi x} \cos \pi x^2 dx &= \frac{\cosh \pi t - \cos \pi t^2}{2\sinh \pi t}, \\
\int_0^\infty \frac{\sin 2\pi t x}{\sinh \pi x} \sin \pi x^2 dx &= \frac{\sin \pi t^2}{2\sinh \pi t},
\end{aligned}$$

and so on.

Again supposing that $s = -t$ in (13), we deduce that if $t = mw \pm ni$, where m and n are positive integers of which one at least is odd, then

$$\begin{aligned}
\phi_w(t) = \tfrac{1}{2} e^{-\frac{1}{4}\pi t^2/w} &\left\{ e^{\frac{1}{4}\pi(t-w)^2/w} - e^{\frac{1}{4}\pi(t-3w)^2/w} + \cdots \text{ to } m \text{ terms} \right\} \\
&+ \frac{1}{2\sqrt{w}} \left\{ e^{-\frac{1}{4}\pi(t\mp i)^2/w} - e^{-\frac{1}{4}\pi(t\mp 3i)^2/w} + \cdots \text{ to } n \text{ terms} \right\}. \qquad (19)
\end{aligned}$$

This formula is not true when both m and n are even.

If $t = mw \pm ni$, where m and n are both even, then

$$\begin{aligned}
&\phi_w(t) + (-1)^{(1+\frac{1}{2}m)(1+\frac{1}{2}n)} e^{-\frac{1}{4}\pi m t} \phi_w(0) \\
&= e^{-\frac{1}{4}\pi t^2/w} \left\{ e^{\frac{1}{4}\pi(t-w)^2/w} - e^{\frac{1}{4}\pi(t-3w)^2/w} + \cdots \text{ to } \tfrac{1}{2}m \text{ terms} \right\} \\
&\quad + \frac{(-1)^{1+\frac{1}{2}m)(1+\frac{1}{2}n)}}{\sqrt{w}} e^{-\frac{1}{4}\pi m t} \left\{ e^{\frac{1}{4}\pi/w} - e^{\frac{9}{4}\pi/w} + e^{\frac{25}{4}\pi/w} - \cdots \text{ to } \tfrac{1}{2}n \text{ terms} \right\}. \qquad (20)
\end{aligned}$$

This is easily obtained by putting $t = 0$ and then changing s to t in (13). Similarly from (14) we deduce that if $t = mw \pm ni$, where m and n are both even, or both odd, or m is even and n is odd, then

$$\begin{aligned}
\psi_w(t) &= -\tfrac{1}{2} e^{-\frac{1}{4}\pi t^2/w} \left\{ e^{\frac{1}{4}\pi(t-2w)^2/w} - e^{\frac{1}{4}\pi(t-4w)^2/w} + \cdots \text{ to } m \text{ terms} \right\} \\
&\quad \pm \frac{i}{2\sqrt{w}} \left\{ e^{-\frac{1}{4}\pi(t\mp i)^2/w} + e^{-\frac{1}{4}\pi(t\mp 3i)^2/w} + \cdots \text{ to } n \text{ terms} \right\}. \qquad (21)
\end{aligned}$$

If $t = mw \pm ni$, where m is odd and n is even, then

$$
\begin{aligned}
&\tfrac{1}{2} - \psi_w(t) + \{(-1)^{1+\frac{1}{4}(m-1)(n+2)} e^{-\frac{1}{4}\pi\{(m-1)t+mw\}} \phi_w(0) \\
&= \quad e^{-\frac{1}{4}\pi t^2/w} \left\{ e^{\frac{1}{4}\pi(t-2w)^2/w} - e^{\frac{1}{4}\pi(t-4w)^2/w} \cdots + \text{ to } \tfrac{1}{2}(m-1) \text{ terms} \right\} \\
&\quad\ \pm(-1)^{1+\frac{1}{4}(m-1)(n+2)} \frac{i}{\sqrt{w}} e^{-\frac{1}{4}\pi(m-1)(t+w)} \\
&\quad\ \times \left\{ e^{-\frac{1}{4}\pi(w\pm i)^2/w} - e^{-\frac{1}{4}\pi(w\pm 3i)^2/w} + \cdots \text{ to } \tfrac{1}{2}n \text{ terms} \right\}. \qquad (22)
\end{aligned}
$$

This is obtained by putting $t = w$ in (14). A number of definite integrals such as the following can be evaluated with the help of the above formulæ:

$$
\begin{aligned}
\int_0^\infty \frac{\cos \pi t x}{\cosh \pi x} e^{-\pi(t+i)x^2} dx &= \frac{1+i}{2\sqrt{2}} e^{-\frac{1}{4}\pi t} \left\{ 1 - \frac{i}{\sqrt{(t+i)}} \right\}, \\
\int_0^\infty \frac{\sin \pi t x}{\sinh \pi x} e^{-\pi(t+i)x^2} dx &= \frac{1}{2} - \frac{1+i}{2\sqrt{2}} \cdot \frac{e^{-\frac{1}{4}\pi t}}{\sqrt{(t+i)}},
\end{aligned}
$$

and so on.

A proof of Bertrand's postulate

Journal of the Indian Mathematical Society, XI, 1919, 181 – 182

1. Landau in his *Handbuch,* pp. 89 – 92, gives a proof of a theorem the truth of which was conjectured by Bertrand: namely that there is at least one prime p such that $x < p \leq 2x$, if $x \geq 1$. Landau's proof is substantially the same as that given by Tschebyschef. The following is a much simpler one.
Let $\nu(x)$ denote the sum of the logarithms of all the primes not exceeding x and let

$$\Psi(x) = \nu(x) + \nu(x^{\frac{1}{2}}) + \nu(x^{\frac{1}{3}}) + \cdots, \tag{1}$$

$$\log[x]! = \Psi(x) + \Psi(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x) + \cdots, \tag{2}$$

where $[x]$ denotes as usual the greatest integer in x.
From (1) we have

$$\Psi(x) - 2\Psi(\sqrt{x}) = \nu(x) - \nu(x^{\frac{1}{2}}) + \nu(x^{\frac{1}{3}}) - \cdots, \tag{3}$$

and from (2)

$$\log[x]! - 2\log[\tfrac{1}{2}x]! = \Psi(x) - \Psi(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x) - \cdots. \tag{4}$$

Now remembering that $\nu(x)$ and $\Psi(x)$ are steadily increasing functions, we find from (3) and (4) that

$$\Psi(x) - 2\Psi(\sqrt{x}) \leq \nu(x) \leq \Psi(x); \tag{5}$$

and

$$\Psi(x) - \Psi(\tfrac{1}{2}x) \leq \log[x]! - 2\log[\tfrac{1}{2}x]! \leq \Psi(x) - \Psi(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x). \tag{6}$$

But it is easy to see that

$$\begin{aligned} \log\Gamma(x) - 2\log\Gamma(\tfrac{1}{2}x + \tfrac{1}{2}) &\leq \log[x]! - 2\log[\tfrac{1}{2}x]! \\ &\leq \log\Gamma(x+1) - 2\log\Gamma(\tfrac{1}{2}x + \tfrac{1}{2}). \end{aligned} \tag{7}$$

Now using Stirling's approximation we deduce from (7) that

$$\log[x]! - 2\log[\tfrac{1}{2}x]! < \tfrac{3}{4}x, \text{ if } x > 0; \tag{8}$$

and

$$\log[x]! - 2\log[\tfrac{1}{2}x]! > \tfrac{2}{3}x, \text{ if } x > 300. \tag{9}$$

It follows from (6), (8) and (9) that

$$\Psi(x) - \Psi(\tfrac{1}{2}x) < \tfrac{3}{4}x, \text{ if } x > 0; \tag{10}$$

and

$$\Psi(x) - \Psi(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x) > \tfrac{2}{3}x, \text{ if } x > 300. \tag{11}$$

Now changing x to $\frac{1}{2}x, \frac{1}{4}x, \frac{1}{8}x, \ldots$ in (10) and adding up all the results, we obtain

$$\Psi(x) < \tfrac{3}{2}x, \text{ if } x > 0. \tag{12}$$

Again we have

$$\begin{aligned}\Psi(x) - \Psi(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x) &\le \nu(x) + 2\Psi(\sqrt{x}) - \nu(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x)\\ &< \nu(x) - \nu(\tfrac{1}{2}x) + \tfrac{1}{2}x + 3\sqrt{x},\end{aligned} \tag{13}$$

in virtue of (5) and (12).
It follows from (11) and (13) that

$$\nu(x) - \nu(\tfrac{1}{2}x) > \tfrac{1}{6}x - 3\sqrt{x}, \text{ if } x > 300. \tag{14}$$

But it is obvious that $\frac{1}{6}x - 3\sqrt{x} \ge 0$, if $x \ge 324$. Hence

$$\nu(2x) - \nu(x) > 0, \text{ if } x \ge 162. \tag{15}$$

In other words there is at least one prime between x and $2x$ if $x \ge 162$. Thus Bertrand's Postulate is proved for all values of x not less than 162; and, by actual verification, we find that it is true for smaller values.

2. Let $\pi(x)$ denote the number of primes not exceeding x. Then, since $\pi(x) - \pi(\frac{1}{2}x)$ is the number of primes between x and $\frac{1}{2}x$, and $\nu(x) - \nu(\frac{1}{2}x)$ is the sum of logarithms of primes between x and $\frac{1}{2}x$, it is obvious that

$$\nu(x) - \nu(\tfrac{1}{2}x) \le \{\pi(x) - \pi(\tfrac{1}{2}x)\} \log x, \tag{16}$$

for all values of x. It follows from (14) and (16) that

$$\pi(x) - \pi(\tfrac{1}{2}x) > \frac{1}{\log x}(\tfrac{1}{6}x - 3\sqrt{x}), \text{ if } x > 300. \tag{17}$$

From this we easily deduce that

$$\pi(x) - \pi(\tfrac{1}{2}x) \ge 1, 2, 3, 4, 5, \ldots, \; if \; x \ge 2, 11, 17, 29, 41, \ldots, \tag{18}$$

respectively.

Some properties of $p(n)$, the number of partitions of n[*]

Proceedings of the Cambridge Philosophical Society, XIX, 1919, 207 – 210

1. A recent paper by Mr Hardy and myself[†] contains a table, calculated by Major MacMahon, of the values of $p(n)$, the number of unrestricted partitions of n, for all values of n from 1 to 200. On studying the numbers in this table I observed a number of curious congruence properties, apparently satisfied by $p(n)$. Thus

(1)	$p(4)$,	$p(9)$,	$p(14)$,	$p(19)$,	$\cdots$	$\equiv 0$	(mod 5),
(2)	$p(5)$,	$p(12)$,	$p(19)$,	$p(26)$,	$\cdots$	$\equiv 0$	(mod 7),
(3)	$p(6)$,	$p(17)$,	$p(28)$,	$p(39)$,	$\cdots$	$\equiv 0$	(mod 11),
(4)	$p(24)$,	$p(49)$,	$p(74)$,	$p(99)$,	$\cdots$	$\equiv 0$	(mod 25),
(5)	$p(19)$,	$p(54)$,	$p(89)$,	$p(124)$,	$\cdots$	$\equiv 0$	(mod 35),
(6)	$p(47)$,	$p(96)$,	$p(145)$,	$p(194)$,	$\cdots$	$\equiv 0$	(mod 49),
(7)	$p(39)$,	$p(94)$,	$p(149)$,	$\cdots$		$\equiv 0$	(mod 55),
(8)	$p(61)$,	$p(138)$,	$\cdots$			$\equiv 0$	(mod 77),
(9)	$p(116)$,	$\cdots$				$\equiv 0$	(mod 121),
(10)	$p(99)$,	$\cdots$				$\equiv 0$	(mod 125).

From these data I conjectured the truth of the following theorem: if $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \pmod{\delta}$ *then*

$$p(\lambda),\ p(\lambda+\delta),\ p(\lambda+2\delta), \cdots \equiv 0 \pmod{\delta}.$$

This theorem is supported by all the available evidence; but I have not yet been able to find a general proof.

I have, however, found quite simple proofs of the theorems expressed by (1) and (2), viz.

$$p(5m+4) \equiv 0 \pmod{5} \tag{1}$$

and

$$p(7m+5) \equiv 0 \pmod{7}. \tag{2}$$

From these

$$p(35m+19) \equiv 0 \pmod{35} \tag{5}$$

follows at once as a corollary. These proofs I give in § 2 and § 3. I can also prove

$$p(25n+24) \equiv 0 \pmod{25} \tag{4}$$

[*] [See also Ramanujan's posthumous paper "Congruence properties of partitions" in the *Math. Zeitschrift*, [No.30 of this volume].

[†] G. H. Hardy and S.Ramanujan, "Asymptotic formulæ in Combinatory Analysis," *Proc. London Math. Soc.*, Ser. 2, Vol. XVII, 1918, pp. 75 – 115 (Table IV, pp. 114 – 115) [No.36 of this volume].

and

(6), $$p(49n+47) \equiv 0 \pmod{49},$$

but only in a more recondite way, which I sketch in § 4.

2. Proof of (1). We have

(11) $$\begin{aligned} x\{(1-x)(1-x^2)(1-x^3)\cdots\}^4 \\ &= x(1-3x+5x^3-7x^6+\cdots)(1-x-x^2+x^5+\cdots) \\ &= \sum(-1)^{\mu+\nu}(2\mu+1)x^{1+\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(3\nu+1)}, \end{aligned}$$

the summation extending from $\mu=0$ to $\mu=\infty$ and from $\nu=-\infty$ to $\nu=\infty$.
Now if

$$1+\tfrac{1}{2}\mu(\mu+1)+\tfrac{1}{2}\nu(3\nu+1) \equiv 0 \pmod{5},$$

then

$$8+4\mu(\mu+1)+4\nu(3\nu+1) \equiv 0 \pmod{5},$$

and therefore

(12) $$(2\mu+1)^2+2(\nu+1)^2 \equiv 0 \pmod{5}.$$

But $(2\mu+1)^2$ is congruent to 0,1 or 4, and $2(\nu+1)^2$ to 0, 2, or 3. Hence it follows from (12) that $2\mu+1$ and $\nu+1$ are both multiples of 5. That is to say, the coefficient of x^{5n} in (11) is a multiple of 5.
Again, all the coefficients in $(1-x)^{-5}$ are multiples of 5, except those of $1, x^5, x^{10}, \ldots,$ which are congruent to 1: that is to say

$$\frac{1}{(1-x)^5} \equiv \frac{1}{1-x^5} \pmod{5},$$

or

$$\frac{1-x^5}{(1-x)^5} \equiv 1 \pmod{5}.$$

Thus all the coefficients in

$$\frac{(1-x^5)(1-x^{10})(1-x^{15})\cdots}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^5}$$

(except the first) are multiples of 5. Hence the coefficient of x^{5n} in

$$\frac{x(1-x^5)(1-x^{10})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} = x\{(1-x)(1-x^2)\cdots\}^4\frac{(1-x^5)(1-x^{10})\cdots}{\{(1-x)(1-x^2)\cdots\}^5}$$

is a multiple of 5. And hence, finally, the coefficient of x^{5n} in

$$\frac{x}{(1-x)(1-x^2)(1-x^3)\cdots}$$

is a multiple of 5; which proves (1).

3. Proof of (2). The proof of (2) is very similar. We have

$$\begin{aligned}(13)\qquad & x^2\{(1-x)(1-x^2)(1-x^3)\cdots\}^6 \\ &= x^2(1-3x+5x^3-7x^6+\cdots)^2 \\ &= \sum(-1)^{\mu+\nu}(2\mu+1)(2\nu+1)x^{2+\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(\nu+1)},\end{aligned}$$

the summation now extending from 0 to ∞ for both μ and ν. If

$$2+\tfrac{1}{2}\mu(\mu+1)+\tfrac{1}{2}\nu(\nu+1)\equiv 0 \pmod 7,$$

then

$$16+4\mu(\mu+1)+4\nu(\nu+1)\equiv \quad \pmod 7,$$

$$(2\mu+1)^2+(2\nu+1)^2\equiv 0 \pmod 7,$$

and $2\mu+1$ and $2\nu+1$ are both divisible by 7. Thus the coefficient of x^{7n} in (13) is divisible by 49.

Again, all the coefficients in

$$\frac{(1-x^7)(1-x^{14})(1-x^{21})\cdots}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^7}$$

(except the first) are multiples of 7. Hence (arguing as in § 2) we see that the coefficient of x^{7n} in

$$\frac{x^2}{(1-x)(1-x^2)(1-x^3)\cdots}$$

is a multiple of 7; which proves (2). As I have already pointed out, (5) is a corollary.

4. The proofs of (4) and (6) are more intricate, and in order to give them I have to consider a much more difficult problem. viz. that of expressing

$$p(\lambda)+p(\lambda+\delta)x+p(\lambda+2\delta)x^2+\cdots$$

in terms of Theta-functions, in such a manner as to exhibit explicitly the common factors of the coefficients, if such common factors exist. I shall content myself with sketching the method of proof, reserving any detailed discussion of it for another paper.

It can be shewn that

$$\begin{aligned}(14)\qquad & \frac{(1-x^5)(1-x^{10})(1-x^{15})\cdots}{(1-x^{\frac{1}{5}})(1-x^{\frac{2}{5}})(1-x^{\frac{3}{5}})\cdots} = \frac{1}{\xi^{-1}-x^{\frac{1}{5}}-\xi x^{\frac{2}{5}}} \\ &= \frac{\xi^{-4}-3x\xi+x^{\frac{1}{5}}(\xi^{-3}+2x\xi^2)+x^{\frac{2}{5}}(2\xi^{-2}-x\xi^3)+x^{\frac{3}{5}}(3\xi^{-1}+x\xi^4)+5x^{\frac{4}{5}}}{\xi^{-5}-11x-x^2\xi^5},\end{aligned}$$

where

$$\xi = \frac{(1-x)(1-x^4)(1-x^6)(1-x^9)\cdots}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)\cdots},$$

the indices of the powers of x, in both numerator and denominator of ξ, forming two arithmetical progressions with common difference 5. It follows that

$$(15) \quad (1-x^5)(1-x^{10})(1-x^{15})\cdots\{p(4)+p(9)x+p(14)x^2+\cdots\} = \frac{5}{\xi^{-5}-11x-x^2\xi^5}\ .$$

Again, if in (14) we substitute $\omega x^{\frac{1}{5}}, \omega^2 x^{\frac{1}{5}}, \omega^3 x^{\frac{1}{5}}$, and $\omega^4 x^{\frac{1}{5}}$, where $\omega^5 = 1$, for $x^{\frac{1}{5}}$, and multiply the resulting five equations, we obtain

$$(16) \qquad \left\{\frac{(1-x^5)(1-x^{10})(1-x^{15})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}\right\}^6 = \frac{1}{\xi^{-5}-11x-x^2\xi^5}.$$

From (15) and (16) we deduce

$$(17) \qquad p(4)+p(9)x+p(14)x^2+\cdots = 5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6};$$

from which it appears directly that $p(5m+4)$ is divisible by 5.

The corresponding formula involving 7 is

$$(18) \qquad p(5)+p(12)x+p(19)x^2+\cdots = 7\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^3}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^4} + 49x\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^7}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^8},$$

which shews that $p(7m+5)$ is divisible by 7.

From (16) it follows that

$$\frac{p(4)x+p(9)x^2+p(14)x^3+\cdots}{5\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^4} = \frac{x}{(1-x)(1-x^2)(1-x^3)\cdots}\,\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^5}.$$

As the coefficient of x^{5n} on the right-hand side is a multiple of 5, it follows that $p(25m+24)$ is divisible by 25.

Similarly

$$\frac{p(5)x+p(12)x^2+p(19)x^3+\cdots}{7\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^2} = x(1-3x+5x^3-7x^6+\cdots)\frac{(1-x^7)(1-x^{14})\cdots}{\{(1-x)(1-x^2)\cdots\}^7} + 7x^2\frac{\{(1-x^7)(1-x^{14})\cdots\}^5}{\{(1-x)(1-x^2)\cdots\}^8};$$

from which it follows that $p(49m+47)$ is divisible by 49.

Another proof of (1) and (2) has been found by Mr. H. B. C. Darling, to whom my conjecture had been communicated by Major MacMahon. This proof will also be published in these *Proceedings*. I have since found proofs of (3), (7) and (8).

Proof of certain identities in combinatory analysis

Proceedings of the Cambridge Philosophical Society, XIX, 1919,214 – 216

Let

$$\begin{aligned} G(x) &= 1+\sum_{1}^{\infty}(-1)^{\nu}x^{2\nu}q^{\frac{1}{2}\nu(5\nu-1)}(1-xq^{2\nu})\frac{(1-xq)(1-xq^2)\cdots(1-xq^{\nu-1})}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^{\nu})} \\ &= 1-x^2q^2(1-xq^2)\frac{1}{1-q}+x^4q^9(1-xq^4)\frac{1-xq}{(1-q)(1-q^2)}-\cdots. \end{aligned} \tag{1}$$

If we write

$$1-xq^{2\nu}=1-q^{\nu}+q^{\nu}(1-xq^{\nu}),$$

every term in (1) is split up into two parts. Associating the second part of each term with the first part of the succeeding term, we obtain

$$G(x)=(1-x^2q^2)-x^2q^3(1-x^2q^6)\frac{1-xq}{1-q}+x^4q^{11}(1-x^2q^{10})\frac{(1-xq)(1-xq^2)}{(1-q)(1-q^2)}-\cdots. \tag{2}$$

Now consider

$$H(x)=\frac{G(x)}{1-xq}-G(xq). \tag{3}$$

Substituting for the first term from (2) and for the second term from (1), we obtain

$$\begin{aligned} H(x) &= xq-\frac{x^2q^3}{1-q}\{(1-q)+xq^4(1-xq^2)\}+\frac{x^4q^{11}(1-xq^2)}{(1-q)(1-q^2)}\{(1-q^2)+xq^7(1-xq^3)\} \\ &\quad -\frac{x^6q^{24}(1-xq^2)(1-xq^3)}{(1-q)(1-q^2)(1-q^3)}\{(1-q^3)+xq^{10}(1-xq^4)\}+\cdots. \end{aligned}$$

Associating, as before, the second part of each term with the first part of the succeeding term, we obtain

$$\begin{aligned} H(x) &= xq(1-xq^2)\left\{1-x^2q^6(1-xq^4)\frac{1}{1-q}+x^4q^{17}(1-xq^6)\frac{1-xq^3}{(1-q)(1-q^2)}\right. \\ &\quad \left.-x^6q^{33}(1-xq^8)\frac{(1-xq^3)(1-xq^4)}{(1-q)(1-q^2)(1-q^3)+}\cdots\right\} \\ &= xq(1-xq^2)G(xq^2). \end{aligned} \tag{4}$$

If now we write

$$K(x)=\frac{G(x)}{(1-xq)G(xq)},$$

we obtain, from (3) and (4),

$$K(x) = 1 + \frac{xq}{K(xq)},$$

and so

$$K(x) = 1 + \frac{xq}{1+}\,\frac{xq^2}{1+}\,\frac{xq^3}{1+\cdots}. \tag{5}$$

In particular we have

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+\cdots} = \frac{1}{K(1)} = \frac{(1-q)G(q)}{G(1)}; \tag{6}$$

or

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+\cdots} = \frac{1-q-q^4+q^7+q^{13}-\cdots}{1-q^2-q^3+q^9+q^{11}-\cdots}. \tag{7}$$

This equation may also be written in the form

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+\cdots} = \frac{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\cdots}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})\cdots}. \tag{8}$$

If we write

$$F(x) = \frac{G(x)}{(1-xq)(1-xq^2)(1-xq^3)\cdots},$$

then (4) becomes

$$F(x) = F(xq) + xqF(xq^2),$$

from which it readily follows that

$$F(x) = 1 + \frac{xq}{1-q} + \frac{x^2q^4}{(1-q)(1-q^2)} + \frac{x^3q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots. \tag{9}$$

In particular we have

$$\begin{aligned} 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \cdots &= \frac{G(1)}{(1-q)(1-q^2)(1-q^3)+\cdots} \\ &= \frac{1-q^2-q^3+q^9+q^{11}-\cdots}{(1-q)(1-q^2)(1-q^3)\cdots} \\ &= \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\cdots}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} 1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \cdots &= \frac{(1-q)G(q)}{(1-q)(1-q^2)(1-q^3)}\cdots \\ &= \frac{1-q-q^4+q^7+q^{13}-\cdots}{(1-q)(1-q^2)(1-q^3)\cdots} \\ &= \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})\cdots}. \end{aligned} \tag{11}$$

A class of definite integrals

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1. It is well known that

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\cos x)^m e^{inx}\, dx = \frac{\pi}{2^m} \frac{\Gamma(1+m)}{\Gamma\{1+\frac{1}{2}(m+n)\}\Gamma\{1+\frac{1}{2}(m-n)\}} \tag{1.1}$$

if $R(m) > -1$. It follows from this and Fourier's Theorem that, if n is any real number except $\pm\pi$ and $R(\alpha+\beta) > 1$, or if $n = \pm\pi$ and $R(\alpha+\beta) > 2$, then

$$\int_{-\infty}^{\infty} \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\, dx = \frac{(2\cos\frac{1}{2}n)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{\frac{1}{2}in(\beta-\alpha)} \quad \textit{or} \quad 0, \tag{1.2}$$

according as $|n| < \pi$ or $|n| \geq \pi$. In particular we have

$$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \tag{1.21}$$

if $R(\alpha+\beta) > 1$; and

$$\int_{0}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\alpha-x)} = \frac{2^{2\alpha-3}}{\Gamma(2\alpha-1)} \tag{1.22}$$

if $R(\alpha) > \frac{1}{2}$. If α is an integer $n+1$, (1.22) reduces to

$$\int_{0}^{\infty} \frac{\sin\pi x}{x\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}}\, dx = \frac{\pi}{2}\frac{2^{2n}(n!)^2}{(2n)!}. \tag{1.23}$$

Again, if m is a positive integer, we have

$$\frac{\sin m\pi x}{\sin\pi x} = \begin{cases} 1 + 2\cos 2\pi x + 2\cos 4\pi x + \cdots \quad \text{to } \frac{1}{2}(m+1) \text{ terms} \qquad \text{or} \\ 2\cos\pi x + 2\cos 3\pi x + \cdots \quad \text{to } \frac{1}{2}m \text{ terms} \end{cases},$$

according as m is odd or even. It follows from this and (1.2) that, if $R(\alpha) > 1$,

$$\int_{0}^{\infty} \frac{\sin m\pi x}{\sin\pi x} \frac{dx}{\Gamma(\alpha+x)\Gamma(\alpha-x)} = \frac{2^{2\alpha-3}}{\Gamma(2\alpha-1)} \quad \textit{or} \quad 0,$$

according as m is odd or even. Hence, if m and n are positive integers, we have

$$\int_{0}^{\infty} \frac{\sin m\pi x}{x\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}}\, dx = \frac{\pi}{2}\frac{2^{2n}(n!)^2}{(2n)!} \;\; \textit{or} \;\; 0, \tag{1.24}$$

according as m is odd or even. From this we easily deduce that, if l, m, and n are positive integers,

(1.25)
$$\int_0^\infty \frac{(\sin m\pi x)^{2l+1}}{x\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}}\,dx = 2^{2(n-l)-1}\frac{(2l)!}{(2n)!}\left(\frac{n!}{l!}\right)^2 \pi \ \textit{or}\ 0,$$

according as m is odd or even. It follows that

(1.26)
$$\int_0^\infty \frac{(\sin m\pi x)^{2n+1}}{x\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}}\,dx = \frac{\pi}{2} \ \textit{or}\ 0,$$

according as m is odd or even. Similarly we can shew that

(1.27)
$$\int_0^\infty \frac{(\sin m\pi x)^{2l}}{\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}}\,dx = 0$$

for all positive integral values of l, m, and n.

In this connection it is interesting to note the following results:

(i) If $R(\alpha+\beta) > 1$ and $R(\gamma+\delta) > 1$, then

(1.3)
$$\Gamma(\alpha+\beta-1)\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(2\cos\pi x)^{\gamma+\delta-2}e^{i\pi(\gamma-\delta)x}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx$$
$$= \Gamma(\gamma+\delta-1)\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(2\cos\pi x)^{\alpha+\beta-2}e^{i\pi(\alpha-\beta)x}}{\Gamma(\gamma+x)\Gamma(\delta-x)}\,dx.$$

This is easily proved by writing

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cos\pi z)^{\alpha+\beta-2}e^{i\pi z(\alpha-\beta+2x)}dz$$

instead of

$$\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha+x)\Gamma(\beta-x)}$$

in the left-hand side of (1.3).

(ii) If m and n are integers of which one is odd and the other even, and $m \geq 0$ and $R(\alpha+\beta) > 2$, then

(1.4)
$$(\alpha+\beta-2)\int_\xi^\infty \frac{(\cos\pi x)^m e^{in\pi x}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \int_\xi^{\xi+1} \frac{(\cos\pi x)^m e^{in\pi x}}{\Gamma(\alpha-1+x)\Gamma(\beta-x)}\,dx$$

where ξ is any real number. This is proved as follows: Suppose that the left-hand side, minus the right-hand side, is $f(\xi)$. Then

$$f'(\xi) = -\frac{(\alpha+\beta-2)(\cos\pi\xi)^m}{\Gamma(\alpha+\xi)\Gamma(\beta-\xi)}e^{in\pi\xi}$$
$$-\frac{(-1)^{m+n}(\cos\pi\xi)^m e^{in\pi\xi}}{\Gamma(\alpha+\xi)\Gamma(\beta-\xi-1)} + \frac{(\cos\pi\xi)^m e^{in\pi\xi}}{\Gamma(\alpha-1+\xi)\Gamma(\beta-\xi)} = 0.$$

Hence $f(\xi)$ is a constant which is easily seen to be zero.

2. Before proceeding further I shall give a few general rules for generalising the results in the previous and the following sections. If

$$f_r(\zeta) = \int_\xi^\eta \frac{(x+\epsilon_1)(x+\epsilon_2)\cdots(x+\epsilon_r)}{\Gamma(\zeta \pm x)} F(x)\, dx,$$

where r is zero or a positive integer, and the ϵ's, ξ, η, ζ and $F(x)$ are all arbitrary, then it is easy to see that

$$f_{r+1}(\zeta+1) = \pm\{f_r(\zeta) - (\zeta \mp \epsilon_{r+1}) f_r(\zeta+1)\}, \tag{2.1}$$

provided the necessary convergence conditions are satisfied.
Similarly if

$$f_r(\zeta) = \int_\xi^\eta (x+\epsilon_1)(x+\epsilon_2)\cdots(x+\epsilon_r)\Gamma(\zeta \pm x) F(x)\, dx,$$

then

$$f_{r+1}(\zeta) = \pm\{f_r(\zeta+1) - (\zeta \mp \epsilon_{r+1}) f_r(\zeta)\}. \tag{2.2}$$

Thus we see that, if $f_0(\zeta)$ is known, $f_r(\zeta)$ can be easily determined.
Suppose now that $P(x)$ is a polynomial of the rth degree and N any integer greater than or equal to r. Let D, E and Δ denote the usual operators so that

$$E = 1 + \Delta = e^D.$$

Then, if

$$f(\zeta) = \int_\xi^\eta \frac{F(x)}{\Gamma(\zeta \pm x)}\, dx,$$

$$\int_\xi^\eta \frac{P(x)F(x)}{\Gamma(\zeta \pm x)}\, dx = \sum_0^N \frac{f(\zeta-\nu)}{\nu!} (\pm\Delta)^\nu P\left\{-\tfrac{1}{2}\nu \pm \left(1 - \zeta + \tfrac{1}{2}\nu\right)\right\}, \tag{2.3}$$

as easily seen by replacing $P(x)$ by

$$(1 \pm \Delta E^{-\frac{1}{2} \pm \frac{1}{2}})^{\zeta \pm x - 1} P\{\pm(1-\zeta)\}.$$

Similarly using the equation

$$P(x) = (1 \mp \Delta E^{-\frac{1}{2} \mp \frac{1}{2}})^{-(\zeta \pm x)} P(\pm\zeta),$$

we find that, if

$$f(\zeta) = \int_\xi^\eta \Gamma(\zeta \pm x) F(x)\, dx,$$

then

$$\int_\xi^\eta \Gamma(\zeta \pm x)P(x)F(x)\,dx = \sum_0^N \frac{f(\zeta+\nu)}{\nu!}(\pm\Delta)^\nu P\left\{-\tfrac{1}{2}\nu \mp \left(\zeta + \tfrac{1}{2}\nu\right)\right\}. \tag{2.4}$$

As an illustration let us apply (2.3) to (1.2). We find that if n is any real number except $\pm\pi$, and $R(\alpha+\beta) > 1+r$, or if $n = \pm\pi$, and $R(\alpha+\beta) > 2+r$, and N is any positive integer greater than or equal to r, where r is the degree of the polynomial $P(x)$, then

$$\int_{-\infty}^\infty \frac{P(x)e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \begin{cases} 0 \qquad \text{or} \\ \displaystyle\sum_0^N \frac{k_\nu}{\nu!}\frac{(2\cos\frac{1}{2}n)^{\alpha+\beta-\nu-2}}{\Gamma(\alpha+\beta-\nu-1)}e^{\frac{1}{2}in(\beta-\alpha)} \end{cases}, \tag{2.5}$$

according as $|n| \geq \pi$ or $|n| < \pi$, k_ν being either $e^{\frac{1}{2}in\nu}\Delta^\nu P(1-\alpha)$ or

$$e^{-\frac{1}{2}in\nu}(-\Delta)^\nu P(\beta-\nu-1).$$

It is immaterial which value of k_ν we take.

If

$$P(x) = \frac{\Gamma(\zeta_1+x)}{\Gamma(\zeta_2+x)},$$

where $\zeta_1 - \zeta_2$ is a positive integer, then it is well known that

$$\Delta^\nu P(x) = \frac{(\zeta_1-\zeta_2)!}{(\zeta_1-\zeta_2-\nu)!}\frac{\Gamma(\zeta_1+x)}{\Gamma(\zeta_2+x+\nu)}. \tag{2.6}$$

This affords a very good example for the previous formulæ.

It is easy to see that (1.2) can be restated as

$$\int_{-\infty}^\infty \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = 0 \tag{2.7}$$

or

$$\begin{aligned}\int_{-\infty}^\infty \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx &= \sum_1^\infty \frac{e^{in(\nu-\alpha)}}{\Gamma(\nu)\Gamma(\alpha+\beta-\nu)} \\ &= \sum_1^\infty \frac{e^{in(\beta-\nu)}}{\Gamma(\nu)\Gamma(\alpha+\beta-\nu)},\end{aligned} \tag{2.71}$$

according as $|n| \geq \pi$ or $|n| < \pi$ (n being of course real). But

$$\int_{-\infty}^\infty \frac{P(x)e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \int_{-\infty}^\infty \frac{e^{(D+in)x}P(0)}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx.$$

Hence, if the conditions stated for (2.5) are satisfied,

$$\int_{-\infty}^{\infty} \frac{P(x)e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = 0 \tag{2.8}$$

or

$$\int_{-\infty}^{\infty} \frac{P(x)e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \sum_{1}^{\infty} \frac{e^{in(\nu-\alpha)}P(\nu-\alpha)}{\Gamma(\nu)\Gamma(\alpha+\beta-\nu)} = \sum_{1}^{\infty} \frac{e^{in(\beta-\nu)}P(\beta-\nu)}{\Gamma(\nu)\Gamma(\alpha+\beta-\nu)}, \tag{2.81}$$

according as $|n| \geq \pi$ or $|n| < \pi$.

3. We shall now consider an important extension of (1.2). Let $[x]$ denote the greatest integer not exceeding x, so that (e.g.) $[-5\frac{1}{2}] = -6$. Let us agree further that

$$\sum_{\mu}^{\nu} = 0,$$

if $\nu < \mu$. Then if n and s are real and $\phi(z)$ is a function that can be expanded in the form

$$\sum_{-\infty}^{\infty} C_\nu z^\nu$$

when $|z| = 1$, we have

$$\int_{-\infty}^{\infty} \frac{\phi(e^{isx})e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \sum C_\nu \frac{\{2\cos\frac{1}{2}(n+\nu s)\}^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{\frac{1}{2}i(\beta-\alpha)(n+\nu s)}, \tag{3.1}$$

where the summation is bounded by

$$-\left[\frac{\pi}{|s|} + \frac{n}{s}\right] \leq \nu \leq \left[\frac{\pi}{|s|} - \frac{n}{s}\right],$$

provided that either

(i) $\pi + n$ and $\pi - n$ are not multiples of s, and $R(\alpha+\beta) > 1$,

or (ii) $\pi + n$ and $\pi - n$ are multiples of s, and $R(\alpha+\beta) > 2$,

or (iii) $C_{(\pi-n)/s} = 0$ and $C_{-(\pi+n)/s} = 0$, whenever one or the other or both of the suffixes of C happen to be integral, and $R(\alpha+\beta) > 1$,

or (iv) $\pi + n$ and $\pi - n$ are multiples of s, $\alpha + \beta$ is an integer greater than 1, and $C_{(\pi-n)/s} = e^{2i\pi\alpha} C_{-(\pi+n)/s}$.

The formula (3.1) is easily obtained by substituting the series for ϕ and integrating term-by-term, using (1.2).

It should be remembered that (3.1) is not true if n or s ceases to be real, though the integral may be convergent. In such cases, generally, the integral cannot be evaluated in finite terms.

The following integrals can be evaluated at once, with the help of (3.1):

$$\int_{-\infty}^{\infty} \frac{dx}{(pe^{imx}+qe^{inx})\Gamma(\alpha+x)\Gamma(\beta-x)}, \tag{3.11}$$

where m and n are real and $|p| \neq |q|$;

$$\int_{-\infty}^{\infty} \frac{\left(\begin{matrix}\cos\\ \sin\end{matrix}\ sx\right)^m e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx, \tag{3.12}$$

where n and s are real and m is a positive integer;

$$\int_{-\infty}^{\infty} \frac{(1+\epsilon e^{isx})^m e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx, \tag{3.13}$$

where m is real and $|\epsilon| < 1$;

$$\int_{-\infty}^{\infty} \frac{e^{p\cos sx+iq\sin sx+inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx. \tag{3.14}$$

For instance the value of (3.14) is

$$\sum \left(\frac{q+p}{q-p}\right)^{\frac{1}{2}\nu} J_\nu\left\{\sqrt{(q^2-p^2)}\right\} \frac{\left\{2\cos\frac{1}{2}(n+\nu s)\right\}^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{\frac{1}{2}i(\beta-\alpha)(n+\nu s)},$$

where $J_\nu(x)$ is the ordinary Bessel function of the νth order, and $\{(q+p)/(q-p)\}^{\frac{1}{2}\nu}$ should be interpreted so that the first term in the expansion of

$$\{(q+p)/(q-p)\}^{\frac{1}{2}\nu} J_\nu\left\{\sqrt{(q^2-p^2)}\right\}$$

is

$$\frac{\left\{\frac{1}{2}(p+q)\right\}^\nu}{\nu!}.$$

Putting $s = 2\pi$ in (3.1) we obtain the following corollary. If ϕ is the same function as in (3.1), and n is any real number, then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\phi(e^{2i\pi x})}{\Gamma(\alpha+x)\Gamma(\beta-x)} e^{inx}\,dx \\ = C_\lambda \frac{\left\{2\cos\left(\frac{1}{2}n-\pi[(\pi+n)/2\pi]\right)\right\}^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{i(\beta-\alpha)\left\{\frac{1}{2}n-\pi[(\pi+n)/2\pi]\right\}},\end{aligned} \tag{3.2}$$

where

$$\lambda = -\left[\frac{\pi + n}{2\pi}\right],$$

provided that either

(i) n is not an odd multiple of π, and $R(\alpha+\beta) > 1$,

or (ii) n is an odd multiple of π , and $R(\alpha+\beta) > 2$,

or (iii) n is an odd multiple of π, $C_{(\pi-n)/2\pi} = 0$ and $C_{-(\pi+n)/2\pi} = 0$, and $R(\alpha+\beta) > 1$,

or (iv) n is an odd multiple of π, $\alpha+\beta$ is an integer greater than 1, and $C_{(\pi-n)/2\pi} = e^{2i\pi\alpha} C_{-(\pi+n)/2\pi}$.

Thus we see that the value of each integrals (3.12) – (3.14), when $s = 2\pi$, reduces to a single term.

The next section will be denoted to the application of (3.2) in evaluating some special integrals.

4. Suppose that α is not real and

$$\phi(z) = 1 + e^{-2i\pi\alpha} z + e^{-4i\pi\alpha} z^2 + \cdots \quad (I(\alpha) < 0),$$

and
$$\phi(z) = -e^{2i\pi\alpha} z^{-1} - e^{4i\pi\alpha} z^{-2} + \cdots \quad (I(\alpha) > 0),$$

so that $\phi(z)$ is convergent when $|z| = 1$. Then it is easy to see that

$$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} e^{inx}\, dx = 2i\pi e^{-i\pi\alpha} \int_{-\infty}^{\infty} \frac{\phi(e^{2i\pi x})}{\Gamma(1-\alpha+x)\Gamma(\beta-x)} e^{ix(\pi-n)}\, dx.$$

It follows from (3.2) that if α is not real, n real, and

(i) n is neither 0 nor any multiple of 2π, and $R(\alpha-\beta) < 0$,

or (ii) n has the same sign as $I(\alpha)$ and $R(\alpha-\beta) < 0$,

or (iii) n is 0, or a multiple of 2π, having the sign opposite to that of $I(\alpha)$, and $R(\alpha-\beta) < -1$,

or (iv) n is not 0 and $\alpha-\beta$ is a negative integer, then

$$\text{(4.1)} \qquad \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} e^{inx}\, dx = 0 \qquad \text{or}$$

$$= \pm \frac{2i\pi}{\Gamma(\beta-\alpha)} \left\{ 2\cos\left(\frac{\pi-n}{2} + \pi\left[\frac{n}{2\pi}\right]\right)\right\}^{\beta-\alpha-1} \times \exp\left\{-in\alpha + i(\beta-\alpha-1)\left(\frac{\pi-n}{2} + \pi\left[\frac{n}{2\pi}\right]\right)\right\},$$

the zero value being taken when n and $I(\alpha)$ have the same sign, the plus sign when $n \geq 0$ and $I(\alpha) < 0$, and the minus sign when $n \leq 0$ and $I(\alpha) > 0$.

As a particular cases of (4.1) we have

$$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\,dx = 0, \tag{4.11}$$

if α is not real and $R(\alpha-\beta) < -1$. If α is not real, n real, and (i) r is a positive integer, *or* (ii) $r = 0$ and $n \neq 0$, then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{inx}\,dx}{(x+\alpha)(x+\alpha+1)\cdots(x+\alpha+r)} &= 0 \\ \text{or } &\pm \frac{2i\pi}{r!}\left(2\sin\tfrac{1}{2}n\right)^r e^{\frac{1}{2}ir(\pi-n)-in\alpha},\end{aligned} \tag{4.12}$$

the different values being selected as in (4.1).
Similarly we can shew that if α and β are not real and n is real, and

(i) n is not an odd multiple of π and $R(\alpha+\beta) < 1$,

or (ii) n is an odd multiple of π which has either the sign of $I(\beta)$ or the sign opposite to that of $I(\alpha)$, and $R(\alpha+\beta) < 0$,

or (iii) n is an odd multiple of π which has neither the sign of $I(\beta)$ nor the sign opposite to that of $I(\alpha)$, and $R(\alpha+\beta) < 1$, then

$$\begin{aligned}\int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x)e^{inx}\,dx &= \frac{2i\pi\Gamma(\alpha+\beta)}{|2\cos\frac{1}{2}n|^{\alpha+\beta}} e^{\frac{1}{2}in(\beta-\alpha)} \\ \times \Bigg(\eta_n(\beta)\exp\left\{i\pi(\alpha+\beta)\left[\frac{(\pi+n)}{2\pi}\right]\right\} &- \eta_n(-\alpha)\exp\left\{-i\pi(\alpha+\beta)\left[\frac{(\pi+n)}{2\pi}\right]\right\}\Bigg),\end{aligned} \tag{4.2}$$

where $\eta_n(\zeta)$ is equal to 0 when $\pi - n$ and $I(\zeta)$ have the same sign, to 1 when $n \leq \pi$ and $I(\zeta) < 0$, and to -1 when $n \geq \pi$ and $I(\zeta) > 0$.
It should be remembered that for real values of n

$$\left|2\cos\tfrac{1}{2}n\right| = 2\cos\left(\tfrac{1}{2}n - \pi\left[\frac{(\pi+n)}{2\pi}\right]\right).$$

It follows, in particular, that if α and β are not real, and $R(\alpha+\beta) < 1$, then

$$\int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x)\,dx = 0 \quad \textit{or} \quad \pm\, 2^{1-\alpha-\beta} i\pi\Gamma(\alpha+\beta), \tag{4.21}$$

the zero value being chosen when $I(\alpha)$ and $I(\beta)$ have different signs, the plus sign when $I(\alpha)$ and $I(\beta)$ are both negative, and the minus sign when $I(\alpha)$ and $I(\beta)$ are both positive.
The following results can either be deduced from (1.5) or be proved independently in the same way as (1.4).

If n is zero or any multiple of 2π, ξ real, α any number except the real numbers less than or equal to $-\xi$, and β is any number such that $R(\beta-\alpha)>1$, then

$$(\beta-\alpha-1)\int_{\xi}^{\infty}\frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}e^{inx}\,dx=\int_{\xi}^{\xi+1}\frac{\Gamma(\alpha+x)}{\Gamma(\beta-1+x)}e^{inx}\,dx. \tag{4.3}$$

If n is any odd multiple of π, α and ξ are the same as in (4.3), and β is not real and $R(\alpha+\beta)<0$, then

$$(\alpha+\beta)\int_{\xi}^{\infty}\Gamma(\alpha+x)\Gamma(\beta-x)e^{inx}\,dx=\int_{\xi}^{\xi+1}\Gamma(\alpha+x)\Gamma(\beta+1-x)e^{inx}\,dx. \tag{4.4}$$

5. We now proceed to consider an application of (1.2) to some other functions. Suppose that $U_s(x)$, $V_s(x)$ and $W_s(x)$ are many-valued functions of x defined by

$$U_s(x)=\frac{u_0x^s}{\Gamma(1+s)}+\frac{u_1x^{s+\epsilon}}{\Gamma(1+s+\epsilon)}+\frac{u_2x^{s+2\epsilon}}{\Gamma(1+s+2\epsilon)}+\cdots,$$

$$V_s(x)=\frac{v_0x^s}{\Gamma(1+s)}+\frac{v_1x^{s+\epsilon}}{\Gamma(1+s+\epsilon)}+\frac{v_2x^{s+2\epsilon}}{\Gamma(1+s+2\epsilon)}+\cdots,$$

$$W_s(x)=\frac{w_0x^s}{\Gamma(1+s)}+\frac{w_1x^{s+\epsilon}}{\Gamma(1+s+\epsilon)}+\frac{w_2x^{s+2\epsilon}}{\Gamma(1+s+2\epsilon)}+\cdots,$$

where $R(\epsilon)\geq 0$, the u's, v's and w's are any numbers connected by the relation to be found by equating the coefficients of the various powers of k in the equation

$$w_0+w_1k+w_2k^2+\cdots=(u_0+u_1k+u_2k^2+\cdots)(v_0+v_1k+v_2k^2+\cdots)^*,$$

and the series $U_s(x)$, $V_s(x)$ and $W_s(x)$ are convergent at least for the values of s and x that appear in the equation (5.2).

The functions U, V and W are many valued. If $|x/y|=1$ and $|\arg(x/y)|<\pi$, then one value of $\arg(x+y)$ is given by the equation

$$\arg x+\arg y=2\arg(x+y). \tag{5.1}$$

If we choose $\arg x$ and $\arg y$ arbitrarily, and agree that

$$x^{s+\mu\epsilon}=\exp\{(s+\mu\epsilon)(\log|x|+i\arg x)\},$$

and that of $y^{s+\mu\epsilon}$ and $(x+y)^{s+\mu\epsilon}$ are to be interpreted similarly, that value of $\arg(x+y)$ being chosen which is given by (5.1), then a definite branch of W is associated with any arbitrary pair of branches of U and V.

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*These series need not, of course, be convergent for any value of k.

If α, β, x, y are any numbers such that $|x/y| = 1$, and $R(\alpha+\beta) > 0$ when $|\arg(x/y)| = \pi$ and $R(\alpha+\beta) > -1$ otherwise, then

$$\int_{-\infty}^{\infty} U_{\alpha+\xi}(x) V_{\beta-\xi}(y)(x/y)^{\xi} d\xi = 0 \quad \textit{or} \quad W_{\alpha+\beta}(x+y), \tag{5.2}$$

according as $|\arg(x/y)| \geq \pi$ or $|\arg(x/y)| < \pi$, whatever be the branches of $U(x)$ and $V(y)$, provided that the corresponding branch of $W(x+y)$ is fixed in accordance with the conventions explained above. This is proved as follows. Suppose that

$$x = te^{\frac{1}{2}in}, \quad y = te^{-\frac{1}{2}in},$$

where t is arbitrary and n is any real number. Then the integral becomes

$$\int_{-\infty}^{\infty} U_{\alpha+\xi}(te^{\frac{1}{2}in}) V_{\beta-\xi}(te^{-\frac{1}{2}in}) e^{in\xi} d\xi.$$

If we expand the integral in powers of t, and integrate term by term with the help of (1.2) and then make use of the relations between the u's, v's and w's, the result will be

$$W_{\alpha+\beta}\left(2t\cos\tfrac{1}{2}n\right) = W_{\alpha+\beta}(x+y),$$

or zero, according to the conditions stated with regard to (5.2).
In particular, if $R(\alpha+\beta) > -1$, we have

$$\int_{-\infty}^{\infty} U_{\alpha+\xi}(x) V_{\beta-\xi}(x) d\xi = W_{\alpha+\beta}(2x). \tag{5.21}$$

Suppose now that $G_s(p, x)$ is a many-valued function of x defined by

$$G_s(p,x) = \frac{x^s}{\Gamma(s+1)} - \frac{p}{1!}\frac{x^{s+1}}{\Gamma(s+2)} + \frac{p(p+1)}{2!}\frac{x^{s+2}}{\Gamma(s+3)} - \cdots.$$

Then it follows from (5.2) that if α, β, x, y are any numbers such that $|x/y| = 1$, and $R(\alpha+\beta) > 0$ when $|\arg(x/y)| = \pi$ and $R(\alpha+\beta) > -1$ otherwise, we have

$$\int_{-\infty}^{\infty} G_{\alpha+\xi}(p,x) G_{\beta-\xi}(q,y)(x/y)^{\xi} d\xi = 0 \quad \textit{or} \quad G_{\alpha+\beta}(p+q, x+y), \tag{5.3}$$

according as $|\arg(x/y)| \geq \pi$ or $< \pi$, whatever be the branches of $G(p,x)$ and $G(q,y)$, provided that the corresponding branch of $G(p+q, x+y)$ is chosen according to our former convention. If, in particular, $R(\alpha+\beta) > -1$, then

$$\int_{-\infty}^{\infty} G_{\alpha+\xi}(p,x) G_{\beta-\xi}(q,x) d\xi = G_{\alpha+\beta}(p+q, 2x). \tag{5.31}$$

It may be interesting to note that the right-hand sides of (5.3) and (5.31) are of the form $G_{\alpha+\beta}(p+q,z)$, which reduces to

$$\frac{z^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

when $p=-q$, becoming independent of p or q.

The ordinary Bessel's functions are particular cases of the function $G_s(p,x)$. Hence we have the following particular results. If n is real, and $R(\alpha+\beta)>0$ when $n=\pm\pi$ and $R(\alpha+\beta)>-1$ otherwise, then

$$(5.4)\qquad \int_{-\infty}^{\infty}\frac{J_{\alpha+\xi}(x)}{x^{\alpha+\xi}}\frac{J_{\beta-\xi}(y)}{y^{\beta-\xi}}e^{in\xi}d\xi=0$$
$$\text{or}\quad\left(\frac{2\cos\frac{1}{2}n}{x^2e^{-\frac{1}{2}in}+y^2e^{\frac{1}{2}in}}\right)^{\frac{1}{2}(\alpha+\beta)}e^{\frac{1}{2}in(\beta-\alpha)}J_{\alpha+\beta}\left[\sqrt{2\cos\tfrac{1}{2}n\left(x^2e^{-\frac{1}{2}in}+y^2e^{\frac{1}{2}in}\right)}\right],$$

according as $|n|\geq\pi$ or $<\pi$. If n is real, and $R(\alpha+\beta)>0$ when $n=\pm\pi$ and $R(\alpha+\beta)>-1$ otherwise, then

$$(5.41)\qquad \int_{-\infty}^{\infty}J_{\alpha+\xi}(x)J_{\beta-\xi}(x)e^{in\xi}d\xi=0\quad\textit{or}\quad e^{\frac{1}{2}in(\beta-\alpha)}J_{\alpha+\beta}\left(2x\cos\tfrac{1}{2}n\right),$$

according as $|n|\geq\pi$ or $<\pi$. If $R(\alpha+\beta)>-1$, then

$$(5.42)\qquad \int_{-\infty}^{\infty}\frac{J_{\alpha+\xi}(x)}{x^{\alpha+\xi}}\frac{J_{\beta-\xi}(y)}{y^{\beta-\xi}}d\xi=\frac{J_{\alpha+\beta}\left\{\sqrt{(2x^2+2y^2)}\right\}}{\left(\frac{1}{2}x^2+\frac{1}{2}y^2\right)^{\frac{1}{2}(\alpha+\beta)}}$$

and

$$(5.43)\qquad \int_{-\infty}^{\infty}J_{\alpha+\xi}(x)J_{\beta-\xi}(x)d\xi=J_{\alpha+\beta}(2x).$$

6. We shall now consider some special cases of the integral

$$(6.1)\qquad \int_{-\infty}^{\infty}\frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+lx)\Gamma(\delta-lx)}\,dx,$$

l and n being real numbers.

Replacing $1/\{\Gamma(\gamma+lx)\Gamma(\delta-lx)\}$ by

$$\frac{1}{\pi\Gamma(\gamma+\delta-1)}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(2\cos z)^{\gamma+\delta-2}e^{-iz(\gamma-\delta+2lx)}dz,$$

it follows from (1.2) that (6.1) is equal to

$$\text{(6.11)}\qquad \frac{1}{\pi\Gamma(\alpha+\beta-1)\Gamma(\gamma+\delta-1)}\int_u^v \left\{2\cos\left(\tfrac{1}{2}n-lz\right)\right\}^{\alpha+\beta-2}(2\cos z)^{\gamma+\delta-2} \times \exp\left\{i(\beta-\alpha)\left(\tfrac{1}{2}n-lz\right)+i(\delta-\gamma)z\right\}dz,$$

where u and v are the lower and upper extremities of the common part of the intervals

$$-\tfrac{1}{2}\pi < z < \tfrac{1}{2}\pi, \quad -\tfrac{1}{2}\pi < \tfrac{1}{2}n - lz < \tfrac{1}{2}\pi.$$

If the intervals do not overlap, the value of (6.1) is zero. It is easy to see that if

$$\text{(6.12)}\qquad |n| \geq \pi(1+|l|),$$

the intervals do not overlap; and that, if they do overlap and $l>0$, then

$$u = \left|\frac{\pi}{4}+\frac{n-\pi}{4l}\right| - \left|\frac{\pi}{4}-\frac{n-\pi}{4l}\right|,$$

and

$$v = \left|\frac{\pi}{4}+\frac{n+\pi}{4l}\right| - \left|\frac{\pi}{4}-\frac{n+\pi}{4l}\right|.$$

It should also be observed that, though (6.11) may not be convergent for all the values of α,β,γ and δ for which (6.1) is convergent, yet we may evaluate (6.11) when it is convergent, and so obtain a formula for (6.1) which may be extended, by the theory of analytic continuation, to all values of the parameters for which the integral converges.
From (6.12) we see that, if l and n are any real numbers such that $|n| \geq \pi(1+|l|)$, then

$$\text{(6.2)}\qquad \int_{-\infty}^{\infty}\frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+lx)\Gamma(\delta-lx)}\,dx = 0,$$

provided that (i) $R(\alpha+\beta+\gamma+\delta) > 2$ when $|n| > \pi(1+|l|)$,
and (ii) $R(\alpha+\beta+\gamma+\delta) > 3$ when $|n| = \pi(1+|l|)$.

7. Suppose now that $l=1$ and $n=0$; then (6.11) reduces to

$$\frac{1}{\pi\Gamma(\alpha+\beta-1)\Gamma(\gamma+\delta-1)}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(2\cos z)^{\alpha+\beta+\gamma+\delta-4}e^{iz(\alpha-\beta-\gamma+\delta)}dz,$$

which is easily evaluated by the help of (1.1). Hence

$$\text{(7.1)}\qquad \int_{-\infty}^{\infty}\frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)},$$

provided that (i) $R(\alpha+\beta+\gamma+\delta)>3$, *or* (ii) $2(\alpha-\gamma)$ and $2(\beta-\delta)$ are odd integers and $R(\alpha+\beta+\gamma+\delta)>2$.

It should be noted that the formula fails when $\alpha+\beta+\gamma+\delta=3$ and $2(\alpha-\gamma)$ and $2(\beta-\delta)$ are odd integers. The value of the integral in this case is some times $1/2\pi$ and some times $-1/2\pi$. The value to be selected may be fixed as follows. It is easy to see that, in this case, one and only one of the numbers $\alpha+\beta-1, \beta+\gamma-1, \gamma+\delta-1$ and $\delta+\alpha-1$ will be an integer less than or equal to zero. If $\alpha+\beta-1$ or $\beta+\gamma-1$ happens to be such a number, then the value of the integral is $\pm 1/2\pi$, according as $2(\beta-\delta)\equiv \mp 1 \pmod 4$. But if $\gamma+\delta-1$ or $\delta+\alpha-1$ happens to be such a number, the value of the integral is $\pm 1/2\pi$, according as $2(\beta-\delta)\equiv \pm 1 \pmod 4$.

As particular cases of (7.1), we have

$$\int_{-\infty}^{\infty} \frac{1}{\{\Gamma(\alpha+x)\Gamma(\beta-x)\}^2}\,dx = \frac{\Gamma(2\alpha+2\beta-3)}{\{\Gamma(\alpha+\beta-1)\}^4}, \tag{7.11}$$

provided that $R(\alpha+\beta)>\frac{3}{2}$,

$$\int_0^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\alpha-x)\Gamma(\beta+x)\Gamma(\beta-x)} = \frac{\Gamma(2\alpha+2\beta-3)}{2\Gamma(2\alpha-1)\Gamma(2\beta-1)\{\Gamma(\alpha+\beta-1)\}^2}, \tag{7.12}$$

provided that (i) $R(\alpha+\beta)>\frac{3}{2}$, *or* (ii) $2(\alpha-\beta)$ is an odd integer and $R(\alpha+\beta)>1$. If $2(\alpha+\beta)=3$ and $2(\alpha-\beta)$ is an odd integer, then the value of the integral (7.12), when $\alpha\ge 1$, is $\pm 1/2\pi$, according as $2(\alpha-\beta)\equiv\pm 1 \pmod 4$, and when $\alpha<1$ it is $\pm 1/2\pi$, according as $2(\alpha-\beta)\equiv\mp 1 \pmod 4$.

Putting $\alpha=\beta$ in (7.11) or in (7.12), we obtain

$$\int_0^{\infty} \frac{dx}{\{\Gamma(\alpha+x)\Gamma(\alpha-x)\}^2} = \frac{\Gamma(4\alpha-3)}{2\{\Gamma(2\alpha-1)\}^4}, \tag{7.13}$$

if $R(\alpha)>\frac{3}{4}$. Suppose again that $l=1, n=\pi$, and $\alpha+\delta=\beta+\gamma$. Then (6.11) reduces to

$$\frac{e^{\frac{1}{2}i\pi(\beta-\alpha)}}{\pi\Gamma(\alpha+\beta-1)\Gamma(\gamma+\delta-1)}\int_0^{\frac{1}{2}\pi} (2\sin z)^{\alpha+\beta-2}\,(2\cos z)^{\gamma+\delta-2}dz.$$

Hence we see that

(7.2)

$$\int_{-\infty}^{\infty} \frac{e^{\pm i\pi x}\,dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{e^{\pm\frac{1}{2}i\pi(\beta-\alpha)}}{2\Gamma\left\{\frac{1}{2}(\alpha+\beta)\right\}\Gamma\left\{\frac{1}{2}(\gamma+\delta)\right\}\Gamma(\alpha+\delta-1)},$$

if $\alpha+\delta=\beta+\gamma$ and $R(\alpha+\beta+\gamma+\delta)>2$. In particular

$$\int_{-\infty}^{\infty} \frac{e^{\pm i\pi x}}{\{\Gamma(\alpha+x)\Gamma(\beta-x)\}^2}\,dx = \frac{e^{\pm\frac{1}{2}i\pi(\beta-\alpha)}}{2\Gamma(\alpha+\beta-1)\left[\Gamma\left\{\frac{1}{2}(\alpha+\beta)\right\}\right]^2}, \tag{7.21}$$

if $R(\alpha+\beta)>1$, and

$$\int_0^\infty \frac{\cos \pi x}{\{\Gamma(\alpha+x)\Gamma(\alpha-x)\}^2}\,dx = \frac{1}{4\Gamma(2\alpha-1)\{\Gamma(\alpha)\}^2}, \tag{7.22}$$

if $R(\alpha)>\frac{1}{2}$.

8. It follows from (6.2), (7.1), and (7.2) that, if

$$\phi(z)=\sum_{-\infty}^{\infty} c_{2\nu}z^{2\nu}, \quad \psi(z)=\sum_{-\infty}^{\infty} c_{2\nu+1}z^{2\nu+1},$$

the series being convergent when $|z|=1$, then

$$\int_{-\infty}^{\infty} \frac{\phi(e^{i\pi x})}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\,dx \tag{8.1}$$
$$= \frac{c_0\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)},$$

provided that (i)$R(\alpha+\beta+\gamma+\delta)>3$ *or* (ii)$R(\alpha+\beta+\gamma+\delta)>2$ and

$$c_2e^{i\pi(\beta+\delta)}+c_{-2}e^{-i\pi(\beta+\delta)}=2c_0\cos\pi(\beta-\delta),$$
$$c_2e^{-i\pi(\alpha+\gamma)}+c_{-2}e^{i\pi(\alpha+\gamma)}=2c_0\cos\pi(\alpha-\gamma);$$

and that

$$\int_{-\infty}^{\infty} \frac{\psi(e^{i\pi x})}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\,dx \tag{8.2}$$
$$= \frac{c_1e^{\frac{1}{2}i\pi(\beta-\alpha)}+c_{-1}e^{-\frac{1}{2}i\pi(\beta-\alpha)}}{2\Gamma\left\{\frac{1}{2}(\alpha+\beta)\right\}\Gamma\left\{\frac{1}{2}(\gamma+\delta)\right\}\Gamma(\alpha+\delta-1)},$$

provided that $\alpha+\delta=\beta+\gamma$ and $R(\alpha+\beta+\gamma+\delta)>2$.
If $\alpha+\delta-\beta-\gamma$ is an integer other than zero, it is possible to evaluate the integrals (7.2) and (8.2) in finite terms, but not as a single term.
The following integrals can be evaluated as a single term, with the help of (8.1) and (8.2), whenever they are convergent:

$$\int_{-\infty}^{\infty} \frac{\Gamma(\delta+x)e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)}\,dx, \tag{8.3}$$

where (i) n is an odd multiple of π, *or* (ii) n is an even multiple of π and $\alpha+1=\beta+\gamma+\delta$;

$$\int_{-\infty}^{\infty} \frac{\Gamma(\gamma+x)\Gamma(\delta\pm x)}{\Gamma(\alpha+x)\Gamma(\beta\pm x)}e^{inx}\,dx, \tag{8.4}$$

where (i) n is an even multiple of π, *or* (ii) n is an odd multiple of π and $\alpha+\delta=\beta+\gamma$;

$$\int_{-\infty}^{\infty}\frac{\Gamma(\beta+x)\Gamma(\gamma-x)\Gamma(\delta+x)}{\Gamma(\alpha+x)}e^{inx}\,dx, \tag{8.5}$$

where (i) n is an odd multiple of π, *or* (ii) n is an even multiple of π and $\alpha+\beta+\gamma=1+\delta$;

$$\int_{-\infty}^{\infty}\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)e^{inx}\,dx, \tag{8.6}$$

where (i) n is an even multiple of π, *or* (ii) n is an odd multiple of π and $\alpha+\delta=\beta+\gamma$. Thus for instance, if δ is not real, $\alpha+1=\beta+\gamma+\delta$, and $R(\alpha+\beta+\gamma-\delta)>1$, then

$$\int_{-\infty}^{\infty}\frac{\Gamma(\delta+x)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)}\,dx = \frac{\pi e^{\pm\frac{1}{2}i\pi(\delta-\gamma)}}{\Gamma(\alpha-\delta)\Gamma\left\{\frac{1}{2}(\alpha+\beta)\right\}\Gamma\left\{\frac{1}{2}(\gamma-\delta+1)\right\}}, \tag{8.31}$$

according as $I(\delta)$ is positive or negative; and if γ and δ are not real and $R(\alpha+\beta-\gamma-\delta)>1$, then

$$\int_{-\infty}^{\infty}\frac{\Gamma(\gamma+x)\Gamma(\delta+x)}{\Gamma(\alpha+x)\Gamma(\beta+x)}\,dx = 0 \quad \text{or} \quad \pm\frac{2i\pi^2}{\sin\pi(\gamma-\delta)}\frac{\Gamma(\alpha+\beta-\gamma-\delta-1)}{\Gamma(\alpha-\gamma)\Gamma(\alpha-\delta)\Gamma(\beta-\gamma)\Gamma(\beta-\delta)}, \tag{8.41}$$

the zero value being taken when $I(\gamma)$ and $I(\delta)$ have the same sign, the plus sign when $I(\gamma)>0$ and $I(\delta)<0$, and the minus sign when $I(\gamma)<0$ and $I(\delta)>0$.

9. The following results are easily obtained with the help of (6.11). If $2(\alpha-\beta)=\gamma-\delta$ and $R(\alpha+\beta+\gamma+\delta)>3$, then

$$\int_{-\infty}^{\infty}\frac{e^{\pm i\pi x}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+2x)\Gamma(\delta-2x)}\,dx = \frac{2^{\alpha+\beta+\gamma+\delta-5}e^{\pm\frac{1}{2}i\pi(\beta-\alpha)}\Gamma\left\{\frac{1}{2}(\alpha+\beta+\gamma+\delta-3)\right\}}{\sqrt{\pi}\,\Gamma\left\{\frac{1}{2}(\alpha+\beta)\right\}\Gamma(\gamma+\delta-1)\Gamma(2\alpha+\delta-2)}. \tag{9.1}$$

If $R(\alpha+\beta)>\frac{3}{2}$, then

$$\int_{0}^{\infty}\frac{\cos\pi x}{\Gamma(\alpha+x)\Gamma(\alpha-x)\Gamma(\beta+2x)\Gamma(\beta-2x)}\,dx = \frac{2^{2\alpha+2\beta-6}\Gamma\left(\alpha+\beta-\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(\alpha)\Gamma(2\beta-1)\Gamma(2\alpha+\beta-2)}. \tag{9.11}$$

If $\alpha+\beta+\gamma+\delta=4$, then

$$(9.12)\qquad \int_{-\infty}^{\infty} \frac{\cos \pi(x+\beta+\gamma)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+2x)\Gamma(\delta-2x)}\,dx = \frac{1}{2\Gamma(\gamma+\delta-1)\Gamma(2\alpha+\delta-2)\Gamma(2\beta+\gamma-2)}.$$

If $2(\alpha-\beta)=\gamma-\delta+k$, where k is ± 1 or ± 2, then

$$(9.2)\qquad \int_{-\infty}^{\infty} \frac{\sin \pi(2x+\alpha-\beta)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+2x)\Gamma(\delta-2x)}\,dx = \pm\frac{2^{2\alpha-\gamma-3}}{\sqrt{\pi}\,\Gamma\left(\beta+\gamma-\alpha+\frac{1}{2}\right)\Gamma(2\alpha+\delta-2)},$$

provided that $R(\alpha+\beta+\gamma+\delta)>2$.
If $3(\alpha-\beta)=\gamma-\delta+k$, where k is ± 1 or ± 2, then

$$(9.3)\qquad \int_{-\infty}^{\infty} \frac{\sin \pi(2x+\alpha-\beta)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+3x)\Gamma(\delta-3x)}\,dx = \pm\frac{3^{3\alpha+\delta-4}\Gamma(2\alpha-\beta+\delta-2)}{4\pi\,\Gamma(\gamma+\delta-1)\Gamma(3\alpha+\delta-3)},$$

provided that (i) $R(\alpha+\beta+\gamma+\delta)>3$, *or* (ii)$\beta+\gamma-2\alpha$ is integral and $R(\alpha+\beta+\gamma+\delta)>2$. In (9.2) and (9.3) the plus sign or minus sign on the right-hand side is to be taken according as k is positive or negative. If k is an integer other than ± 1 or ± 2, the integrals in (9.2) and (9.3) can still be evaluated in finite terms, but in a less simple form.

10. In this connection, it may be interesting to note that, if n is an even multiple of π, and $\alpha+\beta+\gamma+\delta=4$, then

$$(10.1)\qquad (\alpha+\beta-2)(\beta+\gamma-2)\int_{\xi}^{\infty} \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\,dx = \int_{\xi}^{\xi+1} \frac{e^{inx}}{\Gamma(\alpha-1+x)\Gamma(\beta-x)\Gamma(\gamma-1+x)\Gamma(\delta-x)}\,dx$$

for all real values of ξ. The proof of this is the same as that of (1.4).
Finally I may mention the formula

$$(10.2)\qquad \int_{-\infty}^{\infty} J_{\alpha+\xi}(x)J_{\beta-\xi}(x)J_{\gamma+\xi}(x)J_{\delta-\xi}(x)d\xi = \left(\tfrac{1}{2}x\right)^{\alpha+\beta+\gamma+\delta} \times\sum_{\nu=1}^{\nu=\infty} \frac{\left(-\frac{1}{4}x^2\right)^{\nu-1}\{\Gamma(\alpha+\beta+\gamma+\delta+2\nu-1)\}^2}{\Gamma(\nu)\Gamma(\alpha+\beta+\gamma+\delta+\nu)\Gamma(\alpha+\beta+\nu)\Gamma(\beta+\gamma+\nu)\Gamma(\gamma+\delta+\nu)\Gamma(\delta+\alpha+\nu)},$$

which holds if (i) $R(\alpha+\beta+\gamma+\delta)>-1$, *or* (ii) $2(\alpha-\gamma)$ and $2(\beta-\delta)$ are odd integers and $R(\alpha+\beta+\gamma+\delta)>-2$.

Congruence properties of partitions

Proceedings of the London Mathematical Society, 2, XVIII, 1920,
Records for 13 March 1919

In a paper published recently in the *Proceedings of the Cambridge Philosophical Society*[*], I proved a number of arithmetical properties of $p(n)$, the number of unrestricted partitions of n, and in particular that

$$p(5n+4) \equiv 0 \pmod{5},$$

and

$$p(7n+5) \equiv 0 \pmod{7}.$$

Alternative proofs of these two theorems were found afterwards by Mr. H. B. C. Darling[†]. I have since found another method which enables me to prove all these properties and a variety of others, of which the most striking is

$$p(11n+6) \equiv 0 \pmod{11}.$$

There are also corresponding properties in which the moduli are powers of 5, 7, or 11; thus

$$p(25n+24) \equiv 0 \pmod{25},$$

$$p(49n+19), \quad p(49n+33), \quad p(49n+40), \quad p(49n+47) \equiv 0 \pmod{49},$$

$$p(121n+116) \equiv 0 \pmod{121}.$$

It appears that there are no equally simple properties for any moduli involving primes other than these three.

The function $\tau(n)$ defined by the equation

$$\sum_{1}^{\infty} \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24},$$

also possesses very remarkable arithmetical properties. Thus

$$\tau(5n) \equiv 0 \pmod{5},$$

$$\tau(7n), \quad \tau(7n+3), \quad \tau(7n+5), \quad \tau(7n+6) \equiv 0 \pmod{7},$$

while

$$\tau(23n+\nu) \equiv 0 \pmod{23},$$

if ν is any one of the numbers

$$5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22.$$

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[*] Vol. XIX, 1919, pp. 207 – 210 [No. 25 of this volume; see also No. 30] .

[†] *Ibid.*, pp. 217, 218.

Algebraic relations between certain infinite products

Proceedings of the London Mathematical Society, 2, XVIII, 1920,
Records for 13 March 1919

It was proved by Prof. L. J. Rogers* that

$$\begin{aligned} G(x) &= 1+\frac{1}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\frac{x^9}{(1-x)(1-x^2)(1-x^3)}+\cdots \\ &= \frac{1}{(1-x)(1-x^6)(1-x^{11})}\cdots\times\frac{1}{(1-x^4)(1-x^9)(1-x^{14})\cdots}, \end{aligned}$$

and

$$\begin{aligned} H(x) &= 1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)(1-x^2)}+\frac{x^{12}}{(1-x)(1-x^2)(1-x^3)}+\cdots \\ &= \frac{1}{(1-x^2)(1-x^7)(1-x^{12})}\cdots\times\frac{1}{(1-x^3)(1-x^8)(1-x^{13})\cdots}. \end{aligned}$$

Simpler proofs were afterwards found Prof. Rogers and myself.†
I have now found an algebraic relation between $G(x)$ and $H(x)$, viz.:

$$H(x)\{G(x)\}^{11}-x^2G(x)\{H(x)\}^{11}=1+11x\{G(x)H(x)\}^6.$$

Another noteworthy formula is

$$H(x)G(x^{11})-x^2G(x)H(x^{11})=1.$$

Each of these formulæ is the simplest of a large class.

* *Proc. London Math. Soc.*, Ser. 1, Vol. XXV, 1894, pp. 318 – 343.

† *Proc. Camb. Phil. Soc.*, Vol. XIX, 1919, pp. 211 – 216. A short account of the history of the theorems is given by Mr. Hardy in a note attached to this paper. [For Ramanujan's proofs see No. 26 of this volume.]

Congruence properties of partitions

Mathematische Zeitschrift, IX, 1921, 147 – 153

[Extracted from the manuscripts of the author by G. H. Hardy]*

1. Let

$$P = 1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \cdots\right), \tag{1.11}$$

$$Q = 1 + 240\left(\frac{x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \cdots\right), \tag{1.12}$$

$$R = 1 - 504\left(\frac{x}{1-x} + \frac{2^5x^2}{1-x^2} + \frac{3^5x^3}{1-x^3} + \cdots\right), \tag{1.13}$$

$$f(x) = (1-x)(1-x^2)(1-x^3)\cdots. \tag{1.2}$$

Then it is well known that

$$f(x) = 1 - x - x^2 + x^5 + x^7 - \cdots = 1 + \sum_{n=1}^{\infty}(-1)^n\left(x^{\frac{1}{2}n(3n-1)} + x^{\frac{1}{2}n(3n+1)}\right), \tag{1.3}$$

*Srinivasa Ramanujan, Fellow of Trinity College, Cambridge, and of the Royal Society of London, died in India on 26 April, 1920, aged 32. The manuscript from which this note is derived is a sequel to a short memoir "Some properties of $p(n)$, the number of partitions of n," *Proceedings of the Cambridge Philosophical Society*, Vol.XIX (1919), 207-210 [No.25 of this volume]. In this memoir Ramanujan proves that

$$p(5n+4) \equiv 0 \pmod 5$$

and

$$p(7n+5) \equiv 0 \pmod 7,$$

and states without proof a number of further congruences to moduli of the form $5^a7^b11^c$ of which the most striking is

$$p(11n+6) \equiv 0 \pmod{11}.$$

Here now proofs are given of the first two congruences, and the first published proof of the third.

The manuscript contains a large number of further results. It is very incomplete, and will require very careful editing before it can be published in full. I have taken from it the three simplest and most striking results, as a short but characteristic example of the work of a man who was beyond question one of the most remarkable mathematicians of his time.

I have adhered to Ramanujan's notation, and followed his manuscript as closely as I can. A few insertions of my own are marked by brackets. The most substantial of these is in § 5, where Ramanujan's manuscript omits the proof of (5.4). Whether I have reconstructed his argument correctly I cannot say.

The references given in the footnotes to "Ramanujan" are to his memoir "On certain arithmetical functions," *Transactions of the Cambridge Philosophical Society*, Vol. XXII, No.9 (1916), 159-184 [No.18 of this volume].

$$(1.4)\qquad Q^3 - R^2 = 1728x(f(x))^{24}.$$

Further, let

$$(1.51)\qquad \Phi_{r,s}(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s x^{mn} = \sum_{n=1}^{\infty} n^r \sigma_{s-r}(n) x^n,$$

where $\sigma_k(n)$ is the sum of the kth powers of the divisors of n; so that

$$(1.52)\qquad \Phi_{0,s}(x) = \frac{x}{1-x} + \frac{2^s x^2}{1-x^2} + \frac{3^s x^3}{1-x^3} + \cdots,$$

and in particular

$$(1.53)\qquad P = 1 - 24\Phi_{0,1}(x),\ Q = 1 + 240\Phi_{0,3}(x),\ R = 1 - 504\Phi_{0,5}(x).$$

Then [it may be deduced from the theory of the elliptic modular functions, and has been shewn by the author in a direct and elementary manner *, that, when $r+s$ is odd, and $r < s$, $\Phi_{r,s}(x)$ is expressible as a polynomial in P, Q, and R, in the form

$$\Phi_{r,s}(x) = \sum k_{l,m,n} P^l Q^m R^n,$$

where

$$l - 1 \leq \text{ Min } (r,s),\ 2l + 4m + 6n = r + s + 1.$$

In particular †]

$$(1.61)\qquad Q^2 = 1 + 480\Phi_{0,7}(x) = 1 + 480\left(\frac{x}{1-x} + \frac{2^7 x^2}{1-x^2} + \cdots\right),$$

$$(1.62)\qquad QR = 1 - 264\Phi_{0,9}(x) = 1 - 264\left(\frac{x}{1-x} + \frac{2^9 x^2}{1-x^2} + \cdots\right),$$

$$(1.63)\qquad \begin{aligned} 441Q^3 + 250R^2 &= 691 + 65520\Phi_{0,11}(x) \\ &= 691 + 65520\left(\frac{x}{1-x} + \frac{2^{11} x^2}{1-x^2} + \cdots\right), \end{aligned}$$

*Ramanujan, p. 165 [pp. 181 – 183].

†Ramanujan, pp. 163 – 165 [pp. 180 – 181] (Tables I to III). Ramanujan carried the calculation of formulæ of this kind to considerable lengths, the formula of Table I being

$$\begin{aligned} 7709321041217 + 32640\Phi_{0,31}(x) &= 764412173217Q^8 \\ &+5323905468000Q^5R^2 + 1621003400000Q^2R^4. \end{aligned}$$

It is worth while to quote one such formula; for it is impossible to understand Ramanujan without realising his love of numbers for their own sake.

$$Q - P^2 = 288\Phi_{1,2}(x), \tag{1.71}$$

$$PQ - R = 720\Phi_{1,4}(x), \tag{1.72}$$

$$Q^2 - PR = 1008\Phi_{1,6}(x), \tag{1.73}$$

$$Q(PQ - R) = 720\Phi_{1,8}(x), \tag{1.74}$$

$$3PQ - 2R - P^3 = 1728\Phi_{2,3}(x), \tag{1.81}$$

$$P^2Q - 2PR + Q^2 = 1728\Phi_{2,5}(x), \tag{1.82}$$

$$2PQ^2 - P^2R - QR = 1728\Phi_{2,7}(x), \tag{1.83}$$

$$6P^2Q - 8PR + 3Q^2 - P^4 = 6912\Phi_{3,4}(x), \tag{1.91}$$

$$P^3Q - 3P^2R + 3PQ^2 - QR = 3456\Phi_{3,6}(x), \tag{1.92}$$

$$15PQ^2 - 20P^2R + 10P^3Q - 4QR - P^5 = 20736\Phi_{4,5}(x). \tag{1.93}$$

Modulus 5

2. We denote generally by J an integral power-series in x whose coefficients are integers. It is obvious from (1.12) that

$$Q = 1 + 5J.$$

Also $n^5 - n \equiv 0 \pmod 5$, and so, from (1.11) and (1.13),

$$R = P + 5J.$$

Hence

$$Q^3 - R^2 = Q(1 + 5J)^2 - (P + 5J)^2 = Q - P^2 + 5J.$$

Using (1.4), (1.71), and (1.51), we obtain

$$1728x(f(x))^{24} = 288\sum_{n=1}^{\infty} n\sigma_1(n)x^n + 5J. \tag{2.1}$$

Also

$$(1-x)^{25} = 1 - x^{25} + 5J,$$

and so

$$(f(x))^{25} = f(x^{25}) + 5J,$$

$$(f(x))^{24} = \frac{f(x^{25})}{f(x)} + 5J. \tag{2.2}$$

But

$$\frac{1}{f(x)} = 1 + p(1)x + p(2)x^2 + \cdots,$$

and therefore, by (2.1) and (2.2),

$$\begin{aligned} &1728xf(x^{25})(1 + p(1)x + p(2)x^2 + \cdots) \\ &= 1728x\frac{f(x^{25})}{f(x)} = 1728x(f(x))^{24} + 5J \\ &= 288\sum_{n=1}^{\infty} n\sigma_1(n)x^n + 5J. \end{aligned} \tag{2.3}$$

Multiplying by 2, rejecting multiples of 5, and replacing $f(x^{25})$ by its expansion given by (1.3), we obtain

$$\begin{aligned} (x - x^{26} - x^{51} + x^{126} + \cdots)&(1 + p(1)x + p(2)x^2 + \cdots) \\ &= \sum_{n=1}^{\infty} n\sigma_1(n)x^n + 5J. \end{aligned}$$

Hence

$$\begin{aligned} &p(n-1) - p(n-26) - p(n-51) + p(n-126) + p(n-176) \\ &-p(n-301) - \cdots \equiv n\sigma_1(n) \pmod 5, \end{aligned} \tag{2.4}$$

the numbers 1,26, 51, ... being the numbers of the forms

$$\frac{25}{2}n(3n-1)+1,\ \frac{25}{2}n(3n+1)+1,$$

or, what is the same thing, of the forms

$$\frac{1}{2}(5n-1)(15n-2),\ \frac{1}{2}(5n+1)(15n+2).$$

In particular it follows from (2.3) that

$$p(5m-1) \equiv 0 \pmod 5. \tag{2.5}$$

Modulus 7

3. It is obvious from (1.13) that

$$R = 1 + 7J.$$

Also $n^7 - n \equiv 0 \pmod 7$, and so, from (1.11) and (1.61),

$$Q^2 = P + 7J.$$

Hence

$$\begin{aligned}(Q^3 - R^2)^2 = (PQ - 1 + 7J)^2 &= P^2Q^2 - 2PQ + 1 + 7J \\ &= P^2 - 2PQ + R + 7J.\end{aligned}$$

But, from (1.72) and (1.81),

$$\begin{aligned}P^3 - 2PQ + R &= 144\sum_{n=1}^{\infty}(5n\sigma_3(n) - 12n^2\sigma_1(n))x^n \\ &= \sum_{n=1}^{\infty}(n^2\sigma_1(n) - n\sigma_3(n))x^n + 7J.\end{aligned}$$

And therefore

$$(Q^3 - R^2)^2 = \sum_{n=1}^{\infty}(n^2\sigma_1(n) - n\sigma_3(n))x^n + 7J. \tag{3.1}$$

Again (by the same argument which led to (2.2)) we have

$$(f(x))^{48} = \frac{f(x^{49})}{f(x)} + 7J. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$\begin{aligned}x^2\frac{f(x^{49})}{f(x)} &= x^2(f(x))^{48} + 7J = 1728^2x^2(f(x))^{48} + 7J \\ &= (Q^3 - R^2)^2 + 7J \\ &= \textstyle\sum_{n=1}^{\infty}(n^2\sigma_1(n) - n\sigma_3(n))x^n + 7J.\end{aligned} \tag{3.3}$$

From (3.3) it follows (just as (2.4) and (2.5) followed from (2.3)) that

$$\begin{aligned} p(n-2) &- p(n-51) - p(n-100) + p(n-247) + p(n-345) \\ &- p(n-590) - \cdots \equiv n^2\sigma_1(n) - n\sigma_3(n) \pmod{7}, \end{aligned} \tag{3.4}$$

the numbers, 2, 51, 100, ... being those of the forms

$$\frac{1}{2}(7n-1)(21n-4),\ \frac{1}{2}(7n+1)(21n+4);$$

and that

$$p(7m-2) \equiv 0 \pmod{7}. \tag{3.5}$$

Modulus 11.

4. It is obvious from (1.62) that

$$QR = 1 + 11J. \tag{4.1}$$

Also $n^{11} - n \equiv 0 \pmod{11}$, and so, from (1.11) and (1.63),

$$\begin{aligned} Q^3 - 3R^2 &= 441Q^3 + 250R^2 + 11J \\ &= 691 + 65520\left(\frac{x}{1-x} + \frac{2^{11}x^2}{1-x^2} + \cdots\right) + 11J \\ &= -2 + 48\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \cdots\right) + 11J \\ &= -2P + 11J. \end{aligned} \tag{4.2}$$

It is easily deduced that

$$\begin{aligned} (Q^3 - R^2)^5 &= (Q^3 - 3R^2)^5 - Q(Q^3 - 3R^2)^3 \\ &\quad -R(Q^3 - 3R^2)^2 + 6QR + 11J \\ &= P^5 - 3P^3Q - 4P^2R + 6QR + 11J. \end{aligned} \tag{4.3}$$

[For

$$\begin{aligned} & (Q^3 - 3R^2)^5 - Q(Q^3 - 3R^2)^3 - R(Q^3 - 3R^2)^2 + 6QR \\ = \ & (Q^3 - 3R^2)^5 - Q^3R^2(Q^3 - 3R^2)^3 - Q^3R^4(Q^3 - 3R^2)^2 + 6Q^6R^6 + 11J \\ = \ & Q^{15} - 16Q^{12}R^2 + 98Q^9R^4 - 285Q^6R^6 + 423Q^3R^8 - 243R^{10} + 11J \\ = \ & (Q^3 - R^2)^5 + 11J \end{aligned}$$

by (4.1), and (4.3) then follows from (4.2).]

Again, [if we multiply (1.74), (1.83), (1.92), and (1.93) by -1, 3, -4, and -1, and add, we obtain, on rejecting multiples of 11,]

$$P^5 - 3P^3Q - 4P^2R + 6QR = -5\Phi_{1,8} + 3\Phi_{2,7} + 3\Phi_{3,6} - \Phi_{4,5} + 11J;$$

and from this and (4.3) follows

$$\begin{aligned}(Q^3 - R^2)^5 &= -\textstyle\sum_{n=1}^{\infty}(5n\sigma_7(n) - 3n^2\sigma_3(n) - 3n^3\sigma_5(n) \\ &\quad + n^4\sigma_1(n))x^n + 11J.\end{aligned} \tag{4.4}$$

But (by the same argument which led to (2.2) and (3.2)) we have

$$(f(x))^{120} = \frac{f(x^{121})}{f(x)} + 11J. \tag{4.5}$$

From (4.4) and (4.5)

$$\begin{aligned}x^5\frac{f(x^{121})}{f(x)} &= x^5(f(x))^{120} + 11J = 1728^5x^5(f(x))^{120} + 11J \\ &= (Q^3 - R^2)^5 + 11J \\ &= -\sum_{n=1}^{\infty}(5n\sigma_7(n) - 3n^2\sigma_5(n) - 3n^3\sigma_3(n) + n^4\sigma_1(n))x^n + 11J.\end{aligned}$$

It now follows as before that

$$\begin{aligned}p(n-5) &- p(n-126) - p(n-247) + p(n-610) + p(n-852) \\ &- p(n-1457) - \cdots \equiv -n^4\sigma_1(n) + 3n^3\sigma_3(n) + 3n^2\sigma_5(n) \\ &- 5n\sigma_7(n) \pmod{11},\end{aligned} \tag{4.6}$$

5, 126, 247, ... being the numbers of the forms

$$\frac{1}{2}(11n-2)(33n-5),\ \frac{1}{2}(11n+2)(33n+5);$$

and in particular that

$$p(11m-5) \equiv 0 \pmod{11}. \tag{4.7}$$

5. If we are only concerned to prove (4.7), it is not necessary to assume quite so much. Let us write ϑ for the operation $x\frac{d}{dx}$. Then* we have

$$\vartheta P = \frac{1}{12}(P^2 - Q), \tag{5.11}$$

*Ramanujan, p.165 [pp. 181].

(5.12) $$\vartheta Q=\frac{1}{3}(PQ-R),$$

(5.13) $$\vartheta R=\frac{1}{2}(PR-Q^2).$$

From these equations we deduce [by straightforward calculation

$$\begin{aligned}864\vartheta^4 P &= P^5-10P^3Q-15PQ^2+20P^2R+4QR,\\ 72\vartheta^3 Q &= 5P^3Q+15PQ^2-15P^2R-5QR,\\ 24\vartheta^2 R &= -14PQ^2+7P^2R+7QR.\end{aligned}$$

The left-hand side of each of these equations is of the form

$$x\frac{dJ}{dx}.$$

Multiplying by 1,8, and 2, adding and rejecting multiples of 11, we find

(5.2) $$P^5-3P^3Q+2P^2R=x\frac{dJ}{dx}+11J.$$

We have also, by (5.11),

$$6P^2R-6QR=72xR\frac{dP}{dx}.$$

But, differentiating (4.2), and using (4.1), we obtain

$$\begin{aligned}72xR\frac{dP}{dx} &= 36xR\left(-3Q^2\frac{dQ}{dx}+6R\frac{dR}{dx}\right)+11J\\ &= -108xQ\frac{dQ}{dx}+216xR^2\frac{dR}{dx}+11J\\ &= x\frac{dJ}{dx}+11J.\end{aligned}$$

Hence

(5.3) $$6P^2R-6QR=x\frac{dJ}{dx}+11J.$$

From (5.2) and (5.3) we deduce

$$P^5-3P^3Q-4P^2R+6QR=x\frac{dJ}{dx}+11J,$$

and from (4.3)]

(5.4) $$(Q^3-R^2)^5=x\frac{dJ}{dx}+11J.$$

Finally, from (4.5) and (5.4),

$$\begin{aligned}x^5\frac{f(x^{121})}{f(x)} &= x^5(f(x))^{120}+11J=(Q^3-R^2)^5+11J\\ &= x\frac{dJ}{dx}+11J.\end{aligned}$$

As the coefficient of x^{11m} on the right-hand side is a multiple of 11, (4.7) follows immediately.

Papers written in collaboration with G. H. Hardy

Papers 31 to 37

Une formule asymptotique pour le nombre des partitions de n

Comptes Rendus, 2 Jan. 1917

1. Les divers Problèmes de la théorie de la partition des nombres ont été étudiés surtout par les mathématiciens anglais, Cayley, Sylvester et Macmahon*, qui les ont abordés d'un point de vue purement algébrique. Ces auteurs n'y ont fait aucune application des méthodes de la théorie des fonctions, de sorte qu'on ne trouve pas, dans la théorie en question, de formules asymptotiques, telles qu'on en rencontre, par exemple, dans la théorie des nombres premiers. Il nous semble donc que les résultats que nous allons faire connaître peuvent présenter quelque nouveauté.

2. Nous nous sommes occupés surtout de la fonction $p(n)$, nombre des partitions de n. On a

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = \sum_0^\infty p(n)x^n \quad (|x| < 1).$$

Nous avons pensé d'abord à faire usage de quelque théorème de caractère *Taubérien*: on désigne ainsi les théorèmes réciproques du théorème classique d'Abel et de ses généralisations. A cette catégorie appartient l'énoncé suivant:

Soit $g(x) = \sum a_n x^n$ une série de puissances à coefficients POSITIFS*, telle qu'on ait*

$$\log g(x) \sim \frac{A}{1-x},$$

quand x tend vers un par valeurs positives. Alors on a

$$\log s_n = \log (a_0 + a_1 + \cdots a_n) \sim 2\sqrt{(An)},$$

quand n tend vers l'infini†.

En posant $g(x) = (1-x)f(x)$, on a

$$A = \frac{\pi^2}{6};$$

et nous en tirons

$$p(n) = e^{\pi\sqrt{\left(\frac{2}{3}n\right)}(1+\epsilon)}, \tag{1}$$

où ϵ tend vers zéro avec $1/n$.

*Voir le grand traité *Combinatory Analysis* de M. P. A. Macmahon (Cambridge, 1915-16).

†Nous avons donné des généralisations étendues de ce théorẽme dans un mémoire qui doit paraître dans un autre recueil. [The paper referred to is No. 34 of this volume; see, in particular, pp. 314 – 322.]

3. Pour pousser l'approximation plus loin, il faut recourir au théorème de Cauchy. Des formules

$$p(n) = \frac{1}{2\pi i}\int \frac{f(x)}{x^{n+1}}\,dx,$$

avec un chemin d'intégration convenable intérieur au cercle re rayon un, et

$$f(x) = \frac{x^{\frac{1}{24}}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)}\exp\left(\frac{\pi^2}{6\log(1/x)}\right) f\left\{\exp\left(-\frac{4\pi^2}{\log(1/x)}\right)\right\} \tag{2}$$

(fournie par la théorie de la transformation linéaire des fonctions elliptiques), nous avons tiré, en premier lieu, la formule vraiment asymptotique

$$p(n) \sim P(n) = \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\left(\frac{2}{3}n\right)}}. \tag{3}$$

On a

$$\begin{array}{llll} p(10) = 42, & p(20) = 627, & p(50) = 204226, & p(80) = 15796476; \\ P(10) = 48, & P(20) = 692, & P(50) = 217590, & P(80) = 16606781. \end{array}$$

Les valeurs correspondantes de $P(n) : p(n)$ sont

$$1.145; 1.104; 1.065; 1.051 :$$

la valeur approximative est toujours en excès.

4. Mais nous avòns abouti plus tard à des résultats beaucoup plus satisfaisants. Nous considérons la fonction

$$F(x) = \frac{1}{\pi\sqrt{2}}\sum_{1}^{\infty}\frac{d}{dn}\left\{\frac{\cosh[\pi\sqrt{\{\frac{2}{3}(n-\frac{1}{24})\}}]-1}{\sqrt{(n-\frac{1}{24})}}\right\}x^n. \tag{4}$$

En faisant usage des formules sommatoires que démontre M. E. Lindelöf dans son beau livre *Le calcul des résidus*, on trouve aisément que $F(x)$ (on parle, il va sans dire, de la branche principale) a pour seul point singulier le point $x = 1$, et que la fonction

$$F(x) - \frac{x^{\frac{1}{24}}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)}\left[\exp\left\{\frac{\pi^2}{6\log(1/x)}\right\}-1\right]$$

est régulière pour $x = 1$. On est conduit naturellement à appliquer le théorème de Cauchy à la fonction $f(x) - F(x)$, et l'on trouve

$$p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\frac{e^{\pi\sqrt{\{\frac{2}{3}(n-\frac{1}{24})\}}}}{\sqrt{(n-\frac{1}{24})}} + O(e^{k\sqrt{n}}) = Q(n) + O(e^{k\sqrt{n}}), \tag{5}$$

où k désigne un nombre quelconque supérieur à $\pi/\sqrt{6}$. L'apporoximation, pour des valeurs assez grandes de n, est très bonne: on trouve, en effet,

$$\begin{array}{lll} p(61) = 1121505, & p(62) = 1300156, & p(63) = 1505499; \\ Q(61) = 1121539, & Q(62) = 1300121, & Q(63) = 1505536. \end{array}$$

La valeur approximative est, pour les valeurs suffsamment grandes de n, alternativement en excès et en défaut.

5. On peut pousser ces calculs beaucoup plus loin. On forme des foncttions, analogues à $F(x)$, qui présentent, pour les valeurs

$$x = -1, e^{\frac{2}{3}\pi i}, e^{-\frac{2}{3}\pi i}, i, -i, e^{\frac{2}{5}\pi i}, \dots,$$

des singularités d'un type très analogue à celles que présente $f(x)$. On soustrait alors de $f(x)$ une somme d'un nombre fini convenable de ces fonctions. On trouve ainsi, par exemple,

$$\begin{aligned} p(n) \; = \; & \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} \frac{e^{\pi\sqrt{\{\frac{2}{3}(n-\frac{1}{24})\}}}}{\sqrt{(n-\frac{1}{24})}} + \frac{(-1)^n}{2\pi} \frac{d}{dn} \frac{e^{\frac{1}{2}\pi\sqrt{\{\frac{2}{3}(n-\frac{1}{24})\}}}}{\sqrt{(n-\frac{1}{24})}} \\ & + \frac{\sqrt{3}}{\pi\sqrt{2}} \cos\left(\frac{2n\pi}{3} - \frac{\pi}{18}\right) \frac{d}{dn} \frac{e^{\frac{1}{3}\pi\sqrt{\{\frac{2}{3}(n-\frac{1}{24})\}}}}{\sqrt{(n-\frac{1}{24})}} + O(e^{k\sqrt{n}}), \end{aligned} \tag{6}$$

où k désigne un nombre quelconque plus grand que $\frac{1}{4}\pi\sqrt{\frac{2}{3}}$.

Proof that almost all numbers n are composed of about $\log\log n$ prime factors

Proceedings of the London Mathematical Society, 2, XVI,1917,
Records for 14 Dec. 1916

A number n is described in popular language as a *round* number if it is composed of a considerable number of comparatively small factors: thus $1200 = 2^4 \cdot 3 \cdot 5^2$ would generally be said to be a round number. It is a matter of common experience that round numbers are exceedingly rare. The fact may be verified by anybody who will make a practice of factorising numbers, such as the numbers of taxi-cab or railway carriages, which are presented to his attention in moments of leisure. The object of this paper* is to provide the mathematical explanation of this phenomenon.

Let $\pi_\nu(x)$ denote the number of numbers which do not exceed x and are formed of exactly ν prime factors. There is an ambiguity in this definition, for we may count multiple factors multiply or not. But the results are substantially the same on either interpretation.

It has been proved by Landau† that

$$\pi_\nu(x) \sim \frac{x}{\log\, x} \frac{(\log\, \log\, x)^{\nu-1}}{(\nu-1)!}, \tag{1}$$

as $x \to \infty$, for every fixed value of ν. It is moreover obvious that

$$[x] = \pi_1(x) + \pi_2(x) + \pi_3(x) + \cdots, \tag{2}$$

and

$$x = \frac{x}{\log\, x}\left\{1 + \log\, \log\, x + \frac{(\log\, \log\, x)^2}{2!} + \cdots\right\}. \tag{3}$$

Landau's result shews that there is certain correspondence between the terms of the series (2) and (3). The correspondence is far from exact. The first series is finite, for it is obvious that $\pi_\nu(x) = 0$ if $\nu >$ (log x/log 2; and the second is infinite. But it is reasonable to anticipate a correspondence accurate enough to throw considerable light on the distribution of the numbers less than x in respect of number of their prime factors.

The greatest term of the series (3) occurs when ν is about $\log\, \log\, x$. And if we consider the block of terms for which

$$\log\, \log\, x - \phi(x)\sqrt{(\log\, \log\, x)} < \nu < \log\, \log\, x + \phi(x)\sqrt{(\log\, \log\, x)}, \tag{4}$$

*The paper has been published in the *Quarterly Journal of Mathematics*, Vol. XLVIII, pp. 76 – 92[No. 35 of this volume].

†See *Handbuch der Lehre von Verteilung der Primzahalen*, pp. 203 – 213.

where $\phi(x)$ is any function of x which tends to infinity with x, we find without difficulty that it is these terms which contribute almost all the sum of the series: the ratio of their sum to that of the remaining terms tends to infinity with x.

In this paper we shew that the same conclusion holds for the series (2). Let us consider all numbers n which do not exceed x, and denote by x_P the number of them which possess a property $P(n, x)$: this property may be a function of both n and x, or of one variable only. If then $x_P/x \to 1$ when $x \to \infty$, we say that *almost all numbers less than x possess the property P*. And if P is a function of n only, we say simply that *almost all numbers possess the property P*. This being so, we prove the following theorems.

1. *Almost all numbers n less than x are formed of more than*

$$\log\log x - \phi(x)\sqrt{(\log\log x)}$$

and less than

$$\log\log x + \phi(x)\sqrt{(\log\log x)}$$

prime factors.

2. *Almost all numbers n are formed of more than*

$$\log\log n - \phi(n)\sqrt{(\log\log n)}$$

and less than

$$\log\log n + \phi(n)\sqrt{(\log\log n)}$$

prime factors.

In these theorems ϕ is any function of x (or n) which tends to infinity with its argument: and either theorem is true in whichever manner the factors of n are counted. The only serious difficulty in the proof lies in replacing Landau's asymptotic relations (1) by inequalities valid for all values of ν and x.

Since log log n tends to infinity with extreme slowness, the theorems are fully sufficient to explain the observations which suggested them.

Asymptotic formulæ in combinatory analysis

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Records for 1 *March* 1917

A preliminary account of some of the contents of this paper* appeared in the *Comptes Rendus* of January 2nd, 1917. The paper contains a full discussion and proof of the results there stated. The asymptotic formula for $p(n)$, the number of unrestricted partitions of n, of which only the first three terms were given, is completed; and it is shewn that, by taking a number of terms of order $\sqrt{n}$, the *exact* value of $p(n)$ can be obtained for all sufficiently large values of n. Some account is also given of actual or possible applications of the method used to other problems in Combinatory Analysis or the Analytic Theory of Numbers.

* [No. 36 of this volume.]

Asymptotic formulæ for the distribution of integers of various types*

Proceedings of the London Mathematical Society, 2, XVI, 1917, 112 – 132

1. *Statement of the problem.*

1.1 We denote by q a number of the form

$$2^{a_2} 3^{a_3} 5^{a_5} \cdots p^{a_p}, \tag{1.11}$$

where $2, 3, 5, \ldots, p$ are primes and

$$a_2 \geq a_3 \geq a_5 \geq \ldots \geq a_p; \tag{1.111}$$

and by $Q(x)$ the number of such numbers which do not exceed x: and our problem is that of determining the order of $Q(x)$. We prove that

$$Q(x) = \exp\left[\{l + o(l)\}\frac{2\pi}{\sqrt{3}}\sqrt{\left(\frac{\log x}{\log\log x}\right)}\right], \tag{1.12}$$

that is to say that to every positive ϵ corresponds an $x_0 = x_0(\epsilon)$, such that

$$\left(\frac{2\pi}{\sqrt{3}} - \epsilon\right)\sqrt{\left(\frac{\log x}{\log\log x}\right)} < \log Q(x) < \left(\frac{2\pi}{\sqrt{3}} + \epsilon\right)\sqrt{\left(\frac{\log x}{\log\log x}\right)}, \tag{1.121}$$

for $x > x_0$. The function $Q(x)$ is thus of higher order than any power of $\log x$, but of lower order than any power of x.

The interest of the problem is threefold. In the first place the result itself, and the method by which it is obtained, are curious and interesting in themselves. Secondly, the method of proof is one which, as we shew at the end of the paper, may be applied to a whole class of problems in the analytic theory of numbers: it enables us, for example, to find asymptotic formulæ for the number of partitions of n into positive integers, or into different positive integers, or into primes. Finally , the class of numbers q includes as a sub-class the "highly composite" numbers recently studied by Mr.Ramanujan in an elaborate memoir in these *Proceedings*† The problem of determining, with any precision, the number $H(x)$ of highly

*This paper was originally communicated under the title "A problem in the Analytic Theory of Numbers."

†Ramanujan, "Highly Composite Numbers," *Proc. London Math. Soc.*, Ser.2, Vol.XIV 1915, pp. 347 – 409 [No. 15 of this volume].

composite numbers not exceeding x appears to be one of extreme difficulty. Mr Ramanujan has proved, by elementary methods, that the order of $H(x)$ is at any rate greater than that of $\log x$*: but it is still uncertain whether or not the order of $H(x)$ is greater than that of any power of $\log x$. In order to apply transcendental methods to this problem, it would be necessary to study the properties of the function

$$\mathfrak{H}(s) = \sum \frac{1}{h^s},$$

where h is a highly composite number, and we have not been able to make any progress in this direction. It is therefore very desirable to study the distribution of wider classes of numbers which include the highly composite numbers and possess some at any rate of their characteristic properties. The simplest and most natural such class is that of the numbers q; and here progress is comparatively easy, since the function

$$\mathfrak{Q}(s) = \sum \frac{1}{q^s} \tag{1.13}$$

possesses a product expression analogous to Euler's product expression for $\zeta(s)$, viz.

$$\mathfrak{Q}(s) = \prod_1^\infty \left(\frac{1}{1 - l_n^{-s}}\right), \tag{1.14}$$

where $l_n = 2 \cdot 3 \cdot 5 \cdots p_n$ is the product of the first n primes.
We have not been able to apply to this problem the methods, depending on the theory of functions of a complex variable, by which the Prime Number Theorem was proved. The function $\mathfrak{Q}(s)$ has the line $\sigma = 0$† as a line of essential singularities, and we are not able to obtain sufficiently accurate information concerning the nature of these singularities. But it is easy enough to determine the behaviour of $\mathfrak{Q}(s)$ as a function of the *real* variable s; and it proves sufficient for our purpose to determine an asymptotic formula for $\mathfrak{Q}(s)$ when $s \to 0$, and then to apply a "Tauberian" theorem similar to those proved by Messrs Hardy and Littlewood in a series of papers published in these *Proceedings* and elsewhere‡.
This "Tauberian" theorem is in itself of considerable interest as being (so far as we are aware) the first such theorem which deals with functions or sequences tending to infinity more rapidly than any power of the variable.

2. *Elementary results.*

2.1 Let us consider, before proceeding further, what information concerning the order of $Q(x)$ can be obtained by purely elementary methods.

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*As great as that of $\dfrac{\log x\sqrt{(\log\log x)}}{(\log\log\log x)^{\frac{3}{2}}}$: see p.385 of his memoir [p. 139 of this volume].

†We write as usual $s = \sigma + it$.

‡See, in particular, Hardy and Littlewood, "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive," *Proc. London Math. Soc.,* Ser.2, Vol XIII, 1914, pp. 174 – 191; and "Some theorems concerning Dirichlet's series," *Messenger of Mathematics*, Vol.XLIII, 1914, pp. 134 – 147.

Let

$$l_n = 2 \cdot 3 \cdot 5 \cdots p_n = e^{\vartheta(p_n)}\,, \tag{2.11}$$

where $\vartheta(x)$ is Tschebyshef's function

$$\vartheta(x) = \sum_{p \le x} \log p\,.$$

The class of numbers q is plainly identical with the class of numbers of the form

$$l_1^{b_1} l_2^{b_2} \cdots l_n^{b_n}\,, \tag{2.12}$$

where $$b_1 \ge 0,\ b_2 \ge 0,\ \ldots, b_n \ge 0.$$
Now every b can be expressed in one and only one way in the form

$$b_i = c_{i,m} 2^m + c_{i,m-1} 2^{m-1} + \cdots + c_{i,0}\,, \tag{2.13}$$

where every c is equal to zero or to unity. We have therefore

$$q = \prod_{i=1}^{n} \left(l_i^{\sum\limits_{j=0}^{m} c_{i,j} 2^j} \right) = \prod_{j=0}^{m} \prod_{i=1}^{n} l_i^{c_{i,j} 2^j} = \prod_{j=0}^{m} r_j^{2^j}\,, \tag{2.14}$$

say, where

$$r_j = l_1^{c_{1,j}} l_2^{c_{2,j}} \cdots l_n^{c_{n,j}}\,. \tag{2.141}$$

Let r denote , generally, a number of the form

$$r = l_1^{\,c_1} l_2^{\,c_2} \cdots l_n^{\,c_n}\,, \tag{2.15}$$

where every c is zero or unity : and $R(x)$ the number of such numbers which do not exceed x. If $q \le x$, we have

$$r_0 \le x,\ \ r_1^2 \le x,\ \ r_2^4 \le x,\ \ldots$$

The number of possible values of r_0, in formula (2.14), cannot therefore exceed $R(x)$; the number of possible values of r_1 cannot exceed $R(x^{\frac{1}{2}})$; and so on. The total number of values of q can therefore not exceed

$$S(x) = R(x) R(x^{\frac{1}{2}}) R(x^{\frac{1}{4}}) \cdots R(x^{2^{-\varpi}})\,, \tag{2.16}$$

where ϖ is the largest number such that

$$x^{2^{-\varpi}} \ge 2, \quad x \ge 2^{2^{\varpi}}\,. \tag{2.161}$$

Thus

$$Q(x) \le S(x). \tag{2.17}$$

2.2 We denote by f and g the largest numbers such that

$$l_f \le x\,, \tag{2.211}$$

$$l_1 l_2 \cdots l_g \le x\,. \tag{2.212}$$

It is known* (and may be proved by elementary methods) that constants A and B exist, such that

$$\vartheta(x) \ge Ax \quad (x \ge 2)\,, \tag{2.221}$$

and

$$p_n \ge Bn \log n \quad (n \ge 1)\,. \tag{2.222}$$

We have therefore $$e^{Ap} f \le x\,,$$

$$f \log f = O(\log x)\,,$$

$$\log f = O(\log\log x)\,; \tag{2.23}$$

and $$\sum_1^g \vartheta(p_\nu) \le \log x\,, \quad \sum_1^g p_\nu = O(\log x)\,,$$

$$\sum_1^g \nu \log \nu = O(\log x)\,, g^2 \log g = O(\log x)\,,$$

$$g = O\sqrt{\left(\frac{\log x}{\log\log x}\right)}\,. \tag{2.24}$$

But it is easy to obtain an upper bound for $R(x)$ in terms of f and g. The number of numbers $l_1, l_2, \ldots,$ not exceeding x, is not greater than f; the number of products not exceeding x, of pairs of such numbers, is *a fortiori* not greater than $\frac{1}{2}f(f-1)$; and so on. Thus

$$R(x) \le f + \frac{f(f-1)}{2!} + \frac{f(f-1)(f-2)}{3!} + \cdots\,,$$

where the summation need be extended to g terms only, since

$$l_1 l_2 \cdots l_g l_{g+1} > x\,.$$

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*See Landau, *Handbuch*, pp. 79, 83, 214.

A fortiori, we have

$$R(x) \leq 1 + f + \frac{f^2}{2!} + \cdots + \frac{f^g}{g!} < (1+f)^g = e^{g \log(1+f)}\,.$$

Thus

$$R(x) = e^{O(g \log f)} = e^{O(\sqrt{\log x \log\log x})}\,, \tag{2.25}$$

by (2.23) and (2.24). Finally, since

$$\log \sqrt{x} \log\log \sqrt{x} < \frac{1}{2} \log x \log\log x\,,$$

it follows from (2.16) and (2.17) that

$$\begin{aligned} Q(x) &= \exp\left[O\left\{\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{\varpi}}\right)\sqrt{(\log x \log\log x)}\right\}\right] \\ &= e^{O\{\sqrt{(\log x \log\log x)}\}}. \end{aligned} \tag{2.26}$$

2.3 A *lower* bound for $Q(x)$ may be found as follows. If g is defined as in 2.2, we have

$$l_1 l_2 \cdots l_g \leq x < l_1 l_2 \cdots l_g l_{g+1}\,.$$

It follows from the analysis of 2.2 that

$$l_{g+1} = e^{\vartheta(p_{g+1})} = e^{O(g \log g)}\,,$$

and

$$l_1 l_2 \cdots l_g = \exp\left\{\sum_1^g \vartheta(p_\nu)\right\} = e^{O(g^2 \log g)}.$$

Thus

$$x < e^{O(g^2 \log g)};$$

which is only possible if g is greater than a constant positive multiple of

$$\sqrt{\left(\frac{\log x}{\log\log x}\right)}.$$

Now the numbers $l_1, l_2, \ldots, l_g$ can be combined in 2^g different ways, and each such combination gives a number q not greater than x. Thus

$$Q(x) \geq 2^g > \exp\left\{K\sqrt{\left(\frac{\log x}{\log\log x}\right)}\right\}\,, \tag{2.31}$$

where K is a positive constant. From (2.26) and (2.31) it follows that there are positive constants K and L such that

$$K\sqrt{\left(\frac{\log x}{\log\log x}\right)} < \log Q(x) < L\sqrt{(\log x \log\log x)}\,. \tag{2.32}$$

The inequalities (2.32) give a fairly accurate idea as to the order of magnitude of $Q(x)$. But they are much less precise than the inequalities (1.121). To obtain these requires the use of less elementary methods.

3. *The behaviour of $\mathfrak{Q}(s)$ when $s \to 0$ by positive values.*

3.1. From the fact, already used in 2.1, that the class of numbers q is identical with the class of numbers of the form (2.12), it follows at once that

$$\mathfrak{Q}(s) = \sum \frac{1}{q^s} = \prod_1^\infty \left(\frac{1}{1 - l_n^{-s}}\right). \tag{1.14}$$

Both series and product are absolutely convergent for $\sigma > 0$, and

$$\log \mathfrak{Q}(s) = \phi(s) + \frac{1}{2}\phi(2s) + \frac{1}{3}\phi(3s) + \cdots, \tag{3.11}$$

where

$$\phi(s) = \sum_1^\infty l_n^{-s}. \tag{3.111}$$

We have also

$$\begin{aligned} \phi(s) &= \frac{1-2^{-s}}{2^s - 1} + 2^{-s}\frac{1-3^{-s}}{3^s-1} + 2^{-s}3^{-s}\frac{1-5^{-s}}{5^s-1} + \cdots \\ &= \frac{1}{2^s-1} + 2^{-s}\left(\frac{1}{3^s-1} - \frac{1}{2^s-1}\right) + 2^{-s}3^{-s}\left(\frac{1}{5^s-1} - \frac{1}{3^s-1}\right) + \cdots \\ &= \frac{1}{2^s-1} \sum_1^\infty e^{-s\vartheta(p_n)} \int_{p_n}^{p_{n+1}} \frac{d}{dx}\left(\frac{1}{x^s-1}\right)\, dx \\ &= \frac{1}{2^s-1} - s\int_2^\infty \frac{x^{s-1}}{(x^s-1)^2} e^{-s\vartheta(x)}\, dx. \end{aligned} \tag{3.12}$$

3.2. Lemma. *If $x > 1$, $s > 0$, then*

$$\frac{1}{(s\log x)^2} - \frac{1}{12} < \frac{x^s}{(x^s-1)^2} < \frac{1}{(s\log x)^2}. \tag{3.21}$$

Write $x^s = e^u$: then we have to prove that

$$\frac{1}{u^2} - \frac{1}{12} < \frac{e^u}{(e^u-1)^2} < \frac{1}{u^2} \tag{3.22}$$

for all positive values of u ; or (writing w for $\frac{1}{2}u$) that

$$\frac{1}{w^2} - \frac{1}{3} < \frac{1}{\sinh^2 w} < \frac{1}{w^2} \tag{3.23}$$

for all positive values of w. But it is easy to prove that the function

$$f(w) = \frac{1}{w^2} - \frac{1}{\sinh^2 w}$$

is a steadily decreasing function of w, and that its limit when $w \to 0$ is $\frac{1}{3}$; and this establishes the truth of the lemma.

3.3. We have therefore

$$\phi(s) = \frac{1}{2^s - 1} - \phi_1(s) = \frac{1}{s \log 2} - \phi_1(s) + O(1)\,, \tag{3.31}$$

where

$$\frac{1}{s}\int_2^\infty \left\{\frac{1}{(\log x)^2} - \frac{s^2}{12}\right\} e^{-s\vartheta(x)}\frac{dx}{x} < \phi_1(s) < \frac{1}{s}\int_2^\infty \frac{e^{-s\vartheta(x)}}{(\log x)^2}\frac{dx}{x}\,. \tag{3.311}$$

From the second of these inequalities, and (2.221), it follows that

$$\begin{aligned}\phi_1(s) < \frac{1}{s}\int_2^\infty \frac{e^{-Asx}}{(\log x)^2}\frac{dx}{x} &= \frac{e^{-2As}}{s\log 2} - A\int_2^\infty \frac{e^{-Asx}}{\log x}\,dx \\ &= \frac{1}{s\log 2} - A\int_2^\infty \frac{e^{-Asx}}{\log x}dx + O(1)\,;\end{aligned}$$

and so that

$$\phi(s) > A\int_2^\infty \frac{e^{-Asx}}{\log x}dx + O(1)\,. \tag{3.32}$$

On the other hand there is a positive constant B, such that

$$\vartheta(x) < Bx \quad (x \geq 2)^*\,.$$

Thus

$$\begin{aligned}\phi_1(s) > \frac{1}{s}\int_2^\infty \left\{\frac{1}{(\log x)^2} - \frac{s^2}{12}\right\} e^{-Bsx}\frac{dx}{x} \quad &= \quad \frac{1}{s}\int_2^\infty \frac{e^{-Bsx}}{(\log x)^2}\frac{dx}{x} + O(1) \\ &= \quad \frac{1}{s\log 2} - B\int_2^\infty \frac{e^{-Bsx}}{\log x}dx + O(1)\,;\end{aligned}$$

*Landau, *Handbuch, loc.cit*

and so

$$\phi(s) < B\int_2^\infty \frac{e^{-Bsx}}{\log x}dx + O(1)\,. \tag{3.33}$$

3.4. Lemma. *If H is any positive number, then*

$$J(s) = H\int_2^\infty \frac{e^{-Hsx}}{\log x}dx \sim \frac{1}{s\log(1/s)}\,,$$

when $s \to 0$.
Given any positive number ϵ, we can choose ξ and X, so that

$$\int_0^\xi He^{-Hx}dx < \epsilon\,, \quad \int_X^\infty He^{-Hx}dx < \epsilon\,.$$

Now

$$\begin{aligned} s\log\left(\frac{1}{s}\right)J(s) &= \int_{2s}^\infty \frac{He^{-Hu}\log(1/s)}{\log u + \log(1/s)}du = \int_{2s}^{\sqrt{s}} + \int_{\sqrt{s}}^\xi + \int_\xi^X + \int_X^\infty \\ &= j_1(s) + j_2(s) + j_3(s) + j_4(s)\,, \end{aligned}$$

say. And we have

$$\begin{aligned} 0 &< j_1(s) < \frac{\log(1/s)}{\log 2}\int_{2s}^{\sqrt{s}} He^{-Hu}du = O\{\sqrt{s}\log(1/s)\} = o(1)\,, \\ 0 &< j_2(s) < 2\int_0^\xi He^{-Hu}du < 2\epsilon\,, \\ & j_3(s) = \int_\xi^X He^{-Hu}du + o(1)\,, \\ 0 &< j_4(s) < \int_X^\infty He^{-Hu}du < \epsilon\,; \end{aligned}$$

and so

$$\begin{aligned} \left|1 - s\log\left(\frac{1}{s}\right)J(s)\right| &= \left|\int_0^\infty He^{-Hu}du - j_1(s) - j_2(s) - j_3(s) - j_4(s)\right| \\ &< 5\epsilon + o(1) < 6\epsilon\,, \end{aligned}$$

for all sufficiently small values of s.

3.5. From (3.32), (3.33), and lemma just proved, it follows that

$$\phi(s) = \sum l_n^{-s} \sim \frac{1}{s\log(1/s)}\,. \tag{3.51}$$

From this formula we can deduce an asymptotic formula for $\log \mathfrak{Q}(s)$. We choose N so that

$$\sum_{N<n} \frac{1}{n^2} < \epsilon, \tag{3.52}$$

and we write

$$\begin{aligned} \log \mathfrak{Q}(s) = \sum \frac{1}{n} \phi(ns) &= \sum_{1 \le n \le N} + \sum_{N<n<1/\sqrt{s}} + \sum_{1/\sqrt{s} \le n \le 1/s} + \sum_{1/s<n} \\ &= \Phi_1(s) + \Phi_2(s) + \Phi_3(s) + \Phi_4(s)\,, \end{aligned} \tag{3.53}$$

say.
In the first place

$$\Phi_1(s) = \frac{1+o(1)}{s\log(1/s)} \sum_1^N \frac{1}{n^2}\,. \tag{3.541}$$

In the second place

$$\phi(ns) = O\left\{\frac{1}{ns\log(1/ns)}\right\},$$

and

$$\log(1/ns) > \frac{1}{2}\log(1/s),$$

if $N < n < 1/\sqrt{s}$. It follows that a constant K exists such that

$$\Phi_2(s) < \frac{K}{s\log(1/s)} \sum_{N<n} \frac{1}{n^2} < \frac{K\epsilon}{s\log(1/s)}\,. \tag{3.542}$$

Thirdly, $\sqrt{s} \le ns \le 1$ in $\Phi_3(s)$, and a constant L exists such that

$$\phi(ns) < \frac{L}{\sqrt{s}\log(1/s)}\,.$$

Thus

$$\Phi_3(s) < \frac{L}{\sqrt{s}\log(1/s)} \sum_1^{1/s} \frac{1}{n} < \frac{2L}{\sqrt{s}}\,, \tag{3.543}$$

for all sufficiently small values of s.
Finally, in $\Phi_4(s)$ we have $ns > 1$, and a constant M exists such that

$$\phi(ns) < M 2^{-ns}\,.$$

Thus

$$\Phi_4(s) < M \sum_{1/s<n} \frac{2^{-ns}}{n} < sM \sum_{1/s<n} 2^{-ns} < \frac{sM}{1-2^{-s}} = O(1)\,. \tag{3.544}$$

From (3.53), (3.541)–(3.544), and (3.52) it follows that

$$\begin{aligned}\log \mathfrak{Q}(s) = {} & \frac{1}{s\log(1/s)}\left[\{1+o(1)\}\left(\frac{\pi^2}{6}+\rho\right)+\rho'\right] \\ & + O\left\{\frac{1}{\sqrt{s}\log(1/s)}\right\} + O(1)\,,\end{aligned} \tag{3.55}$$

where

$$|\rho| < \epsilon\,,\ |\rho'| < K\epsilon.$$

Thus

$$\log \mathfrak{Q}(s) \sim \frac{\pi^2}{6s\log(1/s)}\,, \tag{3.56}$$

or

$$\mathfrak{Q}(s) = \exp\left[\{1+o(1)\}\frac{\pi^2}{6s\log(1/s)}\right]. \tag{3.57}$$

4. *A Tauberian theorem.*

4.1. The passage from (3.57) to (1.12) depends upon a theorem of the "Tauberian" type.

THEOREM A. *Suppose that*

(1) $\lambda_1 \geq 0\,,\quad \lambda_n > \lambda_{n-1}\,,\quad \lambda_n \to \infty\,;$

(2) $\lambda_n/\lambda_{n-1} \to 1\,;$

(3) $a_n \geq 0\,;$

(4) $A > 0\ , a > 0\,;$

(5) $\sum a_n e^{-\lambda_n s}$ *is convergent for* $s > 0$;

(6) $f(s) = \sum a_n e^{-\lambda_n s} = \exp\left[\{1+o(1)\}As^{-\alpha}\left\{\log\frac{1}{s}\right\}^{-\beta}\right]$,

when $s \to 0$. *Then*

$$A_n = a_1 + a_2 + \cdots + a_n = \exp[\{1+o(1)\}B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)}]\,,$$

where

$$B = A^{1/(1+\alpha)}\alpha^{-\alpha/(1+\alpha)}(1+\alpha)^{1+[\beta/(1+\alpha)]},$$

when $n \to \infty$.

We are given that

$$(4.11) \qquad (1-\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} < \log f(s) < (1+\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta},$$

for every positive δ and all sufficiently small values of s; and we have to shew that

$$(4.12) \qquad (1-\epsilon)B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)} < \log A_n < (1+\epsilon)B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)},$$

for every positive ϵ and all sufficiently large values of n.

In the argument which follows we shall be dealing with three variables, δ, s and n (or m), the two latter variables being connected by an equation or by inequalities, and with an auxiliary parameter ζ. We shall use the letter η, without a suffix, to denote generally a function of δ, s and n (or m)*, which is not the same in different formulæ , but in all cases tends to zero when δ and s tend to zero and n (or m) to infinity; so that, given any positive ϵ, we have

$$0 < |\eta| < \epsilon,$$

for $\qquad 0 < \delta < \delta_0, \quad 0 < s < s_0, \quad n > n_0.$

We shall use the symbol η_ζ to denote a function *of ζ only* which tends to zero with ζ, so that

$$0 < |\eta_\zeta| < \epsilon,$$

if ζ is small enough. It is to be understood that the choice of a ζ to satisfy certain conditions is in all cases prior to that of δ, s and n (or m). Finally, we use the letters $H, K, \ldots$ to denote positive numbers independent of these variables *and of* ζ.

The *second* of the inequalities (4.12) is very easily proved. For

$$(4.131) \qquad A_n e^{-\lambda_n s} < a_1 e^{-\lambda_1 s} + a_2 e^{-\lambda_2 s} + \cdots + a_n e^{-\lambda_n s} < f(s) < \exp\left\{(1+\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\},$$

$$(4.132) \qquad A_n < \exp\chi,$$

where

$$(4.1321) \qquad \chi = (1+\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} + \lambda_n s.$$

*η may, of course, in some cases be a function of some of these variables only.

We can choose a value of s, corresponding to every large value of n, such that

$$(1-\delta)A\alpha s^{-1-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} < \lambda_n < (1+\delta)A\alpha s^{-1-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}. \tag{4.14}$$

From these inequalities we deduce, by an elementary process of approximation,

$$\begin{aligned}(1-\eta)(A\alpha)^{-1/(1+\alpha)}\lambda_n^{1/(1+\alpha)}\left(\log\frac{1}{s}\right)^{\beta/(1+\alpha)}& \\ < \frac{1}{s} < (1+\eta)(A\alpha)^{-1/(1+\alpha)}\lambda_n^{1/(1+\alpha)}&\left(\log\frac{1}{s}\right)^{\beta/(1+\alpha)},\end{aligned} \tag{4.151}$$

$$\frac{1-\eta}{1+\alpha}\log\lambda_n < \log\frac{1}{s} < \frac{1+\eta}{1+\alpha}\log\lambda_n, \tag{4.152}$$

$$\begin{aligned}(1-\eta)\frac{\alpha B}{1+\alpha}\lambda_n^{-1/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)}& \\ < s < (1+\eta)\frac{\alpha B}{1+\alpha}\lambda_n^{-1/(1+\alpha)}&(\log\lambda_n)^{-\beta/(1+\alpha)},\end{aligned} \tag{4.153}$$

$$\chi < (1+\eta)B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)}. \tag{4.154}$$

We have therefore

$$\log A_n < (1+\epsilon)B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)}, \tag{4.16}$$

for every positive ϵ and all sufficiently large values of n*.

4.2. We have

$$\begin{aligned}f(s) &= \sum a_n e^{-\lambda_n s} = \sum A_n(e^{-\lambda_n s} - \epsilon^{-\lambda_{n+1}s}) \\ &= s\sum_1^\infty A_n\int_{\lambda_n}^{\lambda_{n+1}} e^{-sx}dx = s\int_0^\infty \mathcal{A}(x)e^{-sx}dx,\end{aligned} \tag{4.21}$$

where $\mathcal{A}(x)$ is the discontinuous function defined by

$$\mathcal{A}(x) = A_n \quad (\lambda_n \le x < \lambda_{n+1})^\dagger,$$

*We use the second inequality (4.12) in the proof of the first. It would be sufficient for our purpose to begin by proving a result cruder than (4.16), with any constant K on the right-hand side instead of $(1+\epsilon)B$. But it is equally easy to obtain the more precise inequality. Compare the argument in the second of the two papers by Hardy and Littlewood quoted on p.114 [p. 306 of this volume] (pp. 143 *et seq.*)

†Compare Hardy and Riesz, "The General Theory of Dirichlet's Series," *Cambridge Tracts in Mathematics*, No.18, 1915, p.24.

so that, by (4.16),

$$\log \mathcal{A}(x) < (1+\epsilon) B x^{\alpha/(1+\alpha)} (\log x)^{-\beta/(1+\alpha)} \tag{4.22}$$

for every positive ϵ and all sufficiently large values of x. We have therefore

$$\begin{aligned} \exp\left\{(1-\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}\right\} &< s\int_0^\infty \mathcal{A}(x) e^{-sx} dx \\ &< \exp\left\{(1+\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}\right\} \end{aligned} \tag{4.23}$$

for every positive δ and all sufficiently small values of s.

We define λ_x, a steadily increasing and continuous function of the continuous variable x, by the equation

$$\lambda_x = \lambda_n + (x-n)(\lambda_{n+1} - \lambda_n) \quad (n \le x \le n+1)\,.$$

We can then choose m so that

$$\frac{1}{s} = \frac{1+\alpha}{\alpha B} \lambda_m^{1/(1+\alpha)} (\log \lambda_m)^{\beta/(1+\alpha)}\,. \tag{4.24}$$

We shall now shew that the limits of the integral in (4.23) may be replaced by $(1-\zeta)\lambda_m$ and $(1+\zeta)\lambda_m$, where ζ is an arbitrary positive number less than unity.

We write

$$\begin{aligned} J(s) &= s\int_0^\infty \mathcal{A}(x) e^{-sx} dx \\ &= s\left\{\int_0^{\lambda_m/H} + \int_{\lambda_m/H}^{(1-\zeta)\lambda_m} + \int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} + \int_{(1+\zeta)\lambda_m}^{H\lambda_m} + \int_{H\lambda_m}^{\infty}\right\} \\ &= J_1 + J_2 + J_3 + J_4 + J_5\,, \end{aligned} \tag{4.25}$$

where H is a constant, in any case greater than 1, and large enough to satisfy certain further conditions which will appear in a moment; and we proceed to shew that J_1, J_2, J_4 and J_5 are negligible in comparison with the exponentials which occur in (4.23), and so in comparison with J_3.

4.3 The integrals J_1 and J_5 are easily disposed of. In the first place we have

$$\begin{aligned} J_1 = s\int_0^{\lambda_m/H} \mathcal{A}(x) e^{-sx} dx &< \mathcal{A}\left(\frac{\lambda_m}{H}\right) \\ &< \exp\left\{(1+\delta) B \left(\frac{\lambda_m}{H}\right)^{\alpha/(1+\alpha)} \left(\log \frac{\lambda_m}{H}\right)^{-\beta/(1+\alpha)}\right\}, \end{aligned} \tag{4.31}$$

by (4.22)*. It will be found, by a straightforward calculation, that this expression is less than

$$\exp\left\{(1+\eta)A(1+\alpha)H^{-\alpha/(1+\alpha)}s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}, \tag{4.32}$$

and is therefore certainly negligible if H is sufficiently large.
Thus J_1 is negligible. To prove that J_5 is negligible we prove first that

$$sx > 4Bx^{\alpha/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)},$$

if $x > H\lambda_m$ and H is large enough†. It follows that

$$\begin{aligned} J_5 = s\int_{H\lambda_m}^{\infty}\mathcal{A}(x)e^{-sx}dx &< s\int_{H\lambda_m}^{\infty}\exp\{(1+\delta)Bx^{\alpha/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)} - sx\}dx \\ &< s\int_0^{\infty}e^{-\frac{1}{2}sx}dx = \frac{1}{2}, \end{aligned}$$

and is therefore negligible.

4.4 The integrals J_2 and J_4 may be discussed in practically the same way, and we may confine ourselves to the latter. We have

$$J_4(s) = s\int_{(1+\zeta)\lambda_m}^{H\lambda_m}\mathcal{A}(x)e^{-sx}dx < s\int_{(1+\zeta)\lambda_m}^{H\lambda_m}e^{\psi}dx, \tag{4.41}$$

where

$$\psi = (1+\delta)Bx^{\alpha/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)} - sx. \tag{4.411}$$

The maximum of the function ψ occurs for $x = x_0$, where

$$\frac{1}{s} = (1+\eta)\frac{1+\alpha}{\alpha B}x_0^{1/(1+\alpha)}(\log x_0)^{\beta/(1+\alpha)}. \tag{4.42}$$

From this equation, and (4.24), it plainly results that

$$(1-\eta)\lambda_m < x_0 < (1+\eta)\lambda_m, \tag{4.43}$$

and that x_0 falls (when δ and s are small enough) between $(1-\zeta)\lambda_m$ and $(1+\zeta)\lambda_m$.
Let us write $x = x_0 + \xi$ in J_4. Then

$$\psi(x) = \psi(x_0) + \frac{1}{2}(1+\delta)B\xi^2\frac{d^2}{dx_1^2}\{x_1^{\alpha/(1+\alpha)}(\log x_1)^{-\beta/(1+\alpha)}\},$$

*With δ in the place of ϵ.

†We suppress the details of the calculation, which is quite straightforward.

where $x_0 < x_1 < x$ and *a fortiori*

$$(1-\zeta)\lambda_m < x_1 < H\lambda_m\,.$$

It follows that

$$\frac{d^2}{dx_1^2}\{x_1^{\alpha/(1+\alpha)}(\log x_1)^{-\beta/(1+\alpha)}\} < -K\lambda_m^{\alpha/(1+\alpha)-2}(\log\lambda_m)^{-\beta/(1+\alpha)}. \tag{4.44}$$

On the other hand, an easy calculation shews that

$$(1-\eta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} < \psi(x_0) < (1+\eta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}. \tag{4.45}$$

Thus

$$\begin{aligned} J_4 &< \exp\left\{(1+\eta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} \\ &\quad\times\int_{(\zeta-\eta)\lambda_m}^{\infty}\exp\{-L\xi^2\lambda_m^{\alpha/(1+\alpha)-2}(\log\lambda_m)^{-\beta/(1+\alpha)}\}d\xi \\ &< \exp\left\{(1+\eta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} - M\zeta^2\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\right\} \\ &< \exp\left\{(1+\eta-N\zeta^2)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}. \end{aligned} \tag{4.46}$$

Since ζ is independent of δ and s, this inequality shews that J_4 is negligible; and a similar argument may be applied to J_2.

4.5 We may therefore replace the inequalities (4.23) by

$$\begin{aligned} &\exp\left\{(1-\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} \\ &\qquad< s\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m}\mathcal{A}(x)e^{-sx}dx < \exp\left\{(1+\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}. \end{aligned} \tag{4.51}$$

Since $\mathcal{A}(x)$ is a steadily increasing function of x, it follows that

$$\exp\left\{(1-\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} < s\mathcal{A}\{(1+\zeta)\lambda_m\}\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m}e^{-sx}dx\,, \tag{4.521}$$

$$\exp\left\{(1+\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} > s\mathcal{A}\{(1-\zeta)\lambda_m\}\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m}e^{-sx}dx\,; \tag{4.522}$$

or

$$(e^{\zeta s\lambda_m} - e^{-\zeta s\lambda_m})\mathcal{A}\{(1-\zeta)\lambda_m\} < \exp\left\{(1+\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} + \lambda_m s\right\}, \tag{4.531}$$

$$(e^{\zeta s\lambda_m} - e^{-\zeta s\lambda_m})\mathcal{A}\{(1+\zeta)\lambda_m\} > \exp\left\{(1-\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} + \lambda_m s\right\}. \tag{4.532}$$

If we substitute for s, in terms of λ_m, in the right-hand sides of (4.531) and (4.532), we obtain expressions of the form

$$\exp\{(1+\eta)B\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$$

On the other hand

$$e^{\zeta s\lambda_m} - e^{-\zeta s\lambda_m}$$

is of the form

$$\exp\{\eta_\zeta\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$$

We have thus

$$A\{(1-\zeta)\lambda_m\} < \exp\{(1+\eta_\zeta+\eta)B\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\}, \tag{4.541}$$

$$A\{(1+\zeta)\lambda_m\} > \exp\{(1-\eta_\zeta-\eta)B\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\}. \tag{4.542}$$

Now let ν be any number such that

$$(1-\zeta)\lambda_m \le \lambda_\nu \le (1+\zeta)\lambda_m. \tag{4.55}$$

Since $\lambda_n/\lambda_{n-1} \to 1$, it is clear that *all* numbers n from a certain point onwards will fall among the numbers ν. It follows from (4.541) and (4.542) that

$$\begin{aligned}\exp\{(1-\eta_\zeta-\eta)(1-\eta_\zeta)B\lambda_\nu^{\alpha/(1+\alpha)}(\log\lambda_\nu)^{-\beta/(1+\alpha)}\} &< A(\lambda_\nu)\\ &< \exp\{(1+\eta_\zeta+\eta)(1+\eta_\zeta)B\lambda_\nu^{\alpha/(1+\alpha)}(\log\lambda_\nu)^{-\beta/(1+\alpha)}\};\end{aligned} \tag{4.56}$$

and therefore that, given ϵ we can choose first ζ and then n_0 so that

$$\begin{aligned}\exp\{(1-\epsilon)B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)}\} &< A(\lambda_n)\\ &< \exp\{(1+\epsilon)B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)}\},\end{aligned} \tag{4.57}$$

for $n \ge n_0$. This completes the proof of the theorem.

4.6 There is of course a corresponding "Abelian" theorem, which we content ourselves with enunciating. This theorem is naturally not limited by the restriction that the coefficients a_n are positive.

Theorem B. *Suppose that*

(1) $\lambda_1 \geq 0\,, \quad \lambda_n > \lambda_{n-1}\,, \quad \lambda_n \to \infty\,;$

(2) $\lambda_n/\lambda_{n-1} \to 1\,;$

(3) $A > 0\ , 0 < \alpha < 1\,;$

(4) $A_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n = \exp\left[\{1 + o(1)\}A\lambda_n^\alpha(\log\lambda_n)^{-\beta}\right]\,,$

when $n \to \infty$. Then the series $\sum a_n e^{-\lambda_n s}$ is convergent for $s > 0$, and

$$f(s) = \sum a_n e^{-\lambda_n s} = \exp[\{1 + o(1)\}Bs^{-\alpha/(1-\alpha)}\left(\log\frac{1}{s}\right)^{-\beta/(1-\alpha)}]\,,$$

where

$$B = A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)^{1+[\beta/(1-\alpha)]},$$

when $s \to 0$.

The proof of this theorem, which is naturally easier than that of the correlative Tauberian theorem, should present no difficulty to anyone who has followed the analysis which precedes.

4.7 The simplest and most interesting cases of Theorems A and B are those in which

$$\lambda_n = n\,, \quad \beta = 0\,.$$

It is then convenient to write x for e^{-s}. We thus obtain

Theorem C. *If $A > 0$, $0 < \alpha < 1$, and*

$$\log A_n = \log(a_1 + a_2 + \cdots + a_n) \sim An^\alpha\,,$$

then the series $\sum a_n x^n$ is convergent for $|x| < 1$, and

$$\log f(x) = \log(\sum a_n x^n) \sim B(1-x)^{-\alpha/(1-\alpha)}\,,$$

where

$$B = (1-\alpha)\alpha^{\alpha/(1-\alpha)}A^{1/(1-\alpha)},$$

when $x \to 1$ by real values.

If the coefficients are positive the converse inference is also correct. That is to say, if

$$A > 0\,, \alpha > 0\,,$$

and

$$\log f(x) \sim A(1-x)^{-\alpha}\,,$$

then

$$\log A_n \sim B n^{\alpha/(1+\alpha)},$$

where

$$B = (1+\alpha)\alpha^{-\alpha/(1+\alpha)} A^{1/(1+\alpha)}.$$

5. *Application to our problem, and to other problems in the Theory of Numbers.*

5.1 We proved in 3 that

$$\log \mathfrak{Q}(s) \sim \frac{\pi^2}{6s\log(1/s)}. \tag{3.56}$$

In Theorem A take

$$\lambda_n = \log n\,, \quad A = \frac{\pi^2}{6}\,, \quad \alpha = 1\,, \quad \beta = 1\,.$$

Then all the conditions of the theorem are satisfied. And A_n is $Q(n)$, the number of numbers q not exceeding n. We have therefore

$$\log Q(n) \sim B\sqrt{\left(\frac{\log n}{\log\log n}\right)}, \tag{5.11}$$

where

$$B = 2^{\frac{3}{2}}\sqrt{\left(\frac{\pi^2}{6}\right)} = \frac{2\pi}{\sqrt{3}}. \tag{5.12}$$

5.2 The method which we have followed in solving this problem is one capable of many other interesting applications.

Suppose, for example, that $R_r(n)$ is the number of ways in which n can be represented as the sum of any number of r^{th} powers of positive integers*.

We shall prove that

$$\log R_r(n) \sim (r+1)\left\{\frac{1}{r}\Gamma\left(\frac{1}{r}+1\right)\zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)} n^{1/(r+1)}. \tag{5.21}$$

*Thus $28 = 3^3+1^3 = 3\cdot 2^3 + 4\cdot 1^3 = 2\cdot 2^3 + 12\cdot 1^3 = 2^3 + 20\cdot 1^3 = 28\cdot 1^3$: and $R_3(28) = 5$. The *order* of the powers is supposed to be indifferent, so that (e.g.) 3^3+1^3 and 1^3+3^3 are not reckoned as separate representations.

In particular, if $P(n) = R_1(n)$ is the number of partitions of n, then

$$\log P(n) \sim \pi\sqrt{\frac{2n}{3}}\,. \tag{5.22}$$

We need only sketch the proof, which is in principle similar to the main proof of this paper. We have

$$\sum_1^\infty R_r(n)e^{-ns} = \Pi_1^\infty\left(\frac{1}{1-e^{-s\nu^r}}\right),$$

and so

$$f(s) = \sum_1^\infty \{R_r(n) - R_r(n-1)\}e^{-ns} = \Pi_2^\infty\left(\frac{1}{1-e^{-s\nu^r}}\right).^* \tag{5.23}$$

It is obvious that $R_r(n)$ increases with n and that all the coefficients in $f(s)$ are positive. Again,

$$\begin{aligned}\log f(s) &= \sum_2^\infty \log\left(\frac{1}{1-e^{-s\nu^r}}\right) = \sum_2^\infty (e^{-s\nu^r} + \frac{1}{2}e^{-2s\nu^r} + \cdots) \\ &= \sum_{k=1}^\infty \frac{1}{k}\phi(ks)\,,\end{aligned} \tag{5.24}$$

where

$$\phi(s) = \sum_2^\infty e^{-s\nu^r}\,. \tag{5.241}$$

But

$$\phi(s) \sim \Gamma\left(\frac{1}{r}+1\right)s^{-1/r}\,, \tag{5.25}$$

when $s \to 0$; and we can deduce, by an argument similar to that of 3.5, that

$$\log f(s) \sim \Gamma\left(\frac{1}{r}+1\right)\zeta\left(\frac{1}{r}+1\right)s^{-1/r}\,. \tag{5.26}$$

We now obtain (5.21) by an application of Theorem A, taking

$$\lambda_n = n\,,\quad \alpha = \frac{1}{r}\,,\quad \beta = 0\,,\quad A = \Gamma\left(\frac{1}{r}+1\right)\zeta\left(\frac{1}{r}+1\right).$$

---

* $R_r(0)$ is to be interpreted as zero.

In a similar manner we can shew that, if $S(n)$ is the number of partitions of n into *different* positive integers, so that

$$\begin{aligned}\sum S(n)e^{-ns} &= (1+e^{-s})(1+e^{-2s})(1+e^{-3s})+\cdots \\ &= \frac{1}{(1-e^{-s})(1-e^{-3s})(1-e^{-5s})\cdots},\end{aligned}$$

then

$$\log S(n) \sim \pi\sqrt{\frac{n}{3}}\,; \tag{5.27}$$

that if $T_r(n)$ is the number of representations of n as the sum of r^{th} powers of *primes*, then

$$\log T_r(n) \sim (r+1)\left\{\Gamma\left(\frac{1}{r}+2\right)\zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)} n^{1/(r+1)}(\log n)^{-r/(r+1)}\,; \tag{5.28}$$

and, in particular, that if $T(n) = T_1(n)$ is the number of partitions of n into primes, then

$$\log T(n) \sim \frac{2\pi}{\sqrt{3}}\sqrt{\left(\frac{n}{\log n}\right)}\,. \tag{5.281}$$

Finally, we can shew that if r and s are positive integers, $a > 0$, and $0 \leq b \leq 1$, and

$$\sum \phi(n)x^n = \frac{\{(1+ax)(1+ax^2)(1+ax^3)\cdots\}^r}{\{(1+bx)(1+bx^2)(1+bx^3)\cdots\}^s}\,, \tag{5.291}$$

then

$$\log \phi(n) \sim 2\sqrt{(cn)}\,, \tag{5.292}$$

where

$$c = r\int_0^a \frac{\log(1+t)}{t}dt - s\int_0^b \frac{\log(1-t)}{t}dt\,. \tag{5.2921}$$

In particular, if $a = 1, b = 1$, and $r = s$, we have

$$\sum \phi(n)x^n = (1-2x+2x^4-2x^9+\cdots)^{-r}\,, \tag{5.293}$$

$$\log \phi(n) \sim \pi\sqrt{(rn)}\,. \tag{5.294}$$

[*Added March 28th, 1917.* – Since this paper was written M.G.Valiron ("Sur la croissance du module maximum des séries entières," *Bulletin de la Société mathématique de France*, Vol.XLIV, 1916, pp. 45 – 64) has published a number of very interesting theorems concerning power-series which are more or less directly related to ours. M.Valiron considers power-series only, and his point of view is different from ours, in some respects more restricted and in others more general.

He proves in particular that *the necessary and sufficient conditions that*

$$\log M(r) \sim \frac{A}{(1-r)^{\alpha}},$$

where $M(r)$ *is the maximum modulus of* $f(x) = \sum a_n x^n$ *for* $|x| = r$, *are that*

$$\log |a_n| < (1+\epsilon)(1+\alpha)A^{1/(1+\alpha)} \left(\frac{n}{\alpha}\right)^{\alpha/(1+\alpha)}$$

for $n > n_0(\epsilon)$, *and*

$$\log |a_n| > (1-\epsilon_p)(1+\alpha)A^{1/(1+\alpha)} \left(\frac{n}{\alpha}\right)^{\alpha/(1+\alpha)}$$

for $n = n_p\,(p = 1, 2, 3, \ldots)$, *where* $n_{p+1}/n_p \to 1$ *and* $\epsilon_p \to 0$ *as* $p \to \infty$.

M.Valiron refers to previous, but less general or less precise, results given by Borel (*Leçons sur les séries à termes posítifs*, 1902, Ch. V) and by Wiman ("Über dem Zusammenhang zwischen dem Maximal-betrage einer analytischen Funktion und dem grössten Gliede der zugehörigen Taylor'schen Reihe," *Acta Mathematica*, Vol.XXXVII, 1914, pp. 305 – 326). We may add a reference to Le Roy, "Valeurs asymptotiques de certaines séries procédant suivant les puissances entières et positives d'une variable réelle," *Bulletin des sciences mathématiques*, Ser.2, Vol. XXIV, 1900, pp.245 – 268.

We have more recently obtained results concerning $P(n)$, the number of partitions of n, far more precise than (5.22). A preliminary account of these researches has appeared, under the title "Une formule asymptotique pour le nombre des partitions de n," in the *Comptes Rendus* of January 2^{nd}, 1917*; and a fuller account has been presented to the Society. See *Records of Proceedings at Meetings*, March 1^{st}, 1917†.]

*[No. 31 of this volume.]

†[No. 33 of this volume; see also No. 36.]

The normal number of prime factors of a number n

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I.

Statement of the problem.

1.1. The problem with which we are concerned in this paper may be stated roughly as follows: *What is the normal degree of compositeness of a number n?* We shall prove a number of theorems the general result of which is to shew that n is, as a rule, composed of about $\log \log n$ factors.

These statements are vague, and we must define our problem more precisely before we proceed further.

1.2. There are two ways in which it is natural to measure the "degree of compositeness" of n, viz.

(1) by its number of *divisors,*
(2) by its number of *prime factors.*

In this paper we adopt the second point of view. A distinction arises according as multiple factors are or are not counted multiply. We shall denote the number of *different* prime factors by $f(n)$, and the total number of prime factors by $F(n)$, so that (e.g.)

$$f(2^3 3^2 5) = 3, \ F(2^3 3^2 5) = 6.$$

With regard to these functions (or any other arithmetical functions of n) four questions naturally suggest themselves.

(1) In the first place we may ask what is the *minimum* order of the function considered. We wish to determine an elementary function of n, with as low a rate of increase as possible, such that (e.g.)

$$f(n) \leq \phi(n)$$

for an infinity of values of n. This question is, for the functions now under consideration, trivial; for it is plain that

$$f(n) = F(n) = 1$$

when n is a prime.

(2) Secondly, we may ask what is the *maximum* order of the function. This question also is trivial for $F(n)$: we have

$$F(n) = \frac{\log n}{\log\ 2}$$

whenever n is of the form 2^k; and there is no $\phi(n)$ of slower increase which satisfies the conditions. The answer is less obvious in the case of $f(n)$: but, supposing n to be the product of the first k primes, we can shew (by purely elementary reasoning) that, if ϵ is any positive number, we have

$$f(n) < (1+\epsilon)\frac{\log\ n}{\log\log\ n}$$

for all sufficiently large values of n, and

$$f(n) > (1-\epsilon)\frac{\log\ n}{\log\log\ n}$$

for an infinity of values; so that the maximum order of $f(n)$ is

$$\frac{\log\ n}{\log\log\ n}.$$

It is worth mentioning, for the sake of comparison, that the minimum order of $d(n)$, the number of *divisors* of n, is 2, while the maximum order lies between

$$2^{(1-\epsilon)\log n/\log\log n}, \quad 2^{(1+\epsilon)\log n/\log\log n}$$

for every positive value of ϵ.*

(3) Thirdly, we may ask what is the *average* order of the function. It is well known – to return for a moment to the theory of $d(n)$ – that

$$d(1)+d(2)+\cdots+d(n) \sim n\log n. \tag{1.21}$$

This result, indeed, is almost trivial, and far more is known; it is known in fact that

$$d(1)+d(2)+\cdots+d(n) = n\log n+(2\gamma-1)n+O(n^{\frac{1}{3}}\log n), \tag{1.22}$$

this result being one of the deepest in the analytic theory of numbers. It would be natural, then, to say that the average order of $d(n)$ is $\log n$.

A similar problem presents itself for $f(n)$ and $F(n)$; and it is easy to shew that, in this sense, *the average order of* $f(n)$ *and* $F(n)$ *is* $\log\log n$. In fact it may be shewn, by purely elementary methods, that

$$f(1)+f(2)+\cdots+f(n) = n\log\log n+An+O\left(\frac{n}{\log\ n}\right), \tag{1.23}$$

$$F(1)+F(2)+\cdots+F(n) = n\log\log n+Bn+O\left(\frac{n}{\log\ n}\right), \tag{1.24}$$

where A and B are certain constants.

*Wigret, "Sur l'ordre de grandeur du nombre des diviseurs d'un entier," *Arkiv for matematik,* Vol. III (1907), No. 18, pp. 1 – 9. See Ramanujan, "Highly Composite Numbers," *Proc. London math. soc.,* Ser. 2, Vol. XIV (1915), pp. 347 – 409 [No. 15 of this volume] for more precise results.

This problem, however, we shall dismiss for the present, as results still more precise than (1.23) and (1.24) can be found by transcendental methods.
(4) Fourthly, we may ask what is the *normal* order of the function. This phrase requires a little more explanation.
Suppose that $N(x)$ is the number of numbers, not exceeding x, which possess a certain property P. This property may be a function of n only, or of x only, of both n and x: we shall be concerned only with cases in which it is a function of one variable alone. And suppose further that

$$N(x) \sim x \tag{1.25}$$

when $x \to \infty$. Then we shall say, if P is a function of n only, that *almost all numbers* possess the property, and, if P is a function of x only, that *almost all numbers less than* x possess the property. Thus, to take a trivial example, almost all numbers are composite.
If then $g(n)$ is an arithmetical function of n and $\phi(n)$ an elementary increasing function, and if, for every positive ϵ, we have

$$(1-\epsilon)\phi(n) < g(n) < (1+\epsilon)\phi(n) \tag{1.26}$$

for almost all values of n, we shall say that *the normal order of* $g(n)$ *is* $\phi(n)$.
It is in no way necessary that a function should possess either a determinate average order or a determinate normal order, or that one should be determinate when the other is, or that, if both are determinate, they should be the same. But we shall find that each of the functions $f(n)$ and $F(n)$ has both the average order and the normal order $\log\log n$.
The definitions just given may be modified so as to apply only to numbers of a special class. If $Q(x)$ is the number of numbers of the class not exceeding x, and $N(x)$ the number of *these* numbers which possess the property P, and

$$N(x) \sim Q(x), \tag{1.27}$$

then we shall say that almost all of the numbers (or almost all of the numbers less than x) possess the property.

II.
"Quadratfrei" numbers.

2.1. It is most convenient to begin by confining our attention to *quadratfrei* numbers, numbers, that is to say, which contain no prime factor raised to a power higher than the first. It is well known that the number $Q(x)$ of such numbers, not exceeding x, satisfies the relation

$$Q(x) \sim \frac{6}{\pi^2} x \text{ *} \tag{2.11}$$

We shall denote by $\pi_\nu(x)$ the number of numbers which are products of just ν different prime factors and do not exceed x, so that $\pi_1(x) = \pi(x)$ is the number of primes not exceeding x, and

$$\sum_1^\infty \pi_\nu(x) = Q(x).$$

It has been shewn by Landau† that

$$\pi_\nu(x) \sim \frac{x}{\log x} \frac{(\log\log x)^{\nu-1}}{(\nu-1)!} \tag{2.12}$$

for every fixed value of ν. In what follows we shall require, not an asymptotic equality, but an inequality satisfied for all values of ν and x.

2.2. Lemma A. *There are absolute constants C and K such that*

$$\pi_{\nu+1}(x) < \frac{K_x}{\log x} \frac{(\log\log x + C)^\nu}{\nu!} \tag{2.21}$$

for

$$\nu = 0, 1, 2, \dots, \ x \geq 2.$$

The inequality (2.21) is certainly true when $\nu = 0$, whatever the value of C. It is known (and may be proved by elementary methods‡) that

$$\sum_{p \leq x} \frac{1}{p} < \log\log x + B \tag{2.221}$$

and

$$\sum_{p \leq x} \frac{\log p}{p} < H \log x, \tag{2.222}$$

where B and H are constants. We shall prove by induction that (2.21) is true if

$$C > B + H. \tag{2.223}$$

*Landau, *Handbuch*, p. 581.

†Landau, *Handbuch*, pp. 203 *et seq.*

‡In applying (2.21) we assume that $x/p \geq 2$. If $x/p < 2, \pi_\nu(x/p)$ is zero.

Consider the numbers which do not exceed x and are comprised in the table

$$\begin{aligned} 2 \cdot p_1 \cdot p_2 \cdots p_\nu, \\ 3 \cdot p_1 \cdot p_2 \cdots p_\nu, \\ 5 \cdot p_1 \cdot p_2 \cdots p_\nu, \\ \cdots\cdots, \\ P \cdot p_1 \cdot p_2 \cdots p_\nu, \end{aligned}$$

where $p_1, p_2, \ldots, p_\nu$ are, in each row, different primes arranged in ascending order of magnitude, and where $p_\nu \geq 2$ in the first row, $p_\nu \geq 3$ in second, $p_\nu \geq 5$ in the third, and so on. It is plain that $P \leq \sqrt{x}$. The total number of numbers in the table is plainly not greater than

$$\sum_{p^2 \leq x} \pi_\nu \left(\frac{x}{p}\right).$$

If now $\omega_1, \omega_2, \ldots$ are primes and

$$\omega_1 < \omega_2 \cdots < \omega_{\nu+1},\ \omega_1 \omega_2 \cdots \omega_{\nu+1 \leq x},$$

the number $\omega_1 \omega_2 \cdots \omega_{\nu+1}$ will occur at least ν times in the table, once in the row in which the first figure is ω_1 once in that in which it is $\omega_2, \ldots$, once in that in which it is ω_ν. We have therefore

$$\nu \pi_{\nu+1}(x) \leq \sum_{p^2 \leq x} \pi_\nu \left(\frac{x}{p}\right). \tag{2.23}$$

Assuming, then, that (2.21) is true when ν is replaced by $\nu - 1$, we obtain*

$$\begin{aligned} \pi_{\nu+1}(x) &< \frac{Kx}{\nu!} \sum_{p^2 \leq x} \frac{1}{p \log(x/p)} \left\{\log\log\left(\frac{x}{p}\right) + C\right\}^{\nu-1} \\ &< \frac{Kx(\log\log x + C)^{\nu-1}}{\nu!} \sum_{p^2 \leq x} \frac{1}{p \log(x/p)}. \end{aligned} \tag{2.24}$$

But

$$\frac{1}{\log x - \log p} = \frac{1}{\log x} + \frac{\log p}{(\log x)^2}\left\{1 + \frac{\log p}{\log x} + \cdots\right\} \leq \frac{1}{\log x} + \frac{2\log p}{(\log x)^2}, \tag{2.25}$$

since $\log p \leq \frac{1}{2} \log x$; and so

$$\begin{aligned} \sum_{p^2 \leq x} \frac{1}{p \log(x/p)} &\leq \frac{1}{\log x} \sum_{p^2 \leq x} \frac{1}{p} + \frac{2}{(\log x)^2} \sum_{p^2 \leq x} \frac{\log p}{p} \\ &< \frac{\log\log x + B}{\log x} + \frac{H}{\log x} < \frac{\log\log x + C}{\log x}. \end{aligned} \tag{2.26}$$

*It may be well to observe explicitly that $\log\log 2 + C$ is positive.

Substituting in (2.24), we obtain (2.21).

2.3. We can now prove one of our main theorems.

THEOREM A. *Suppose that ϕ is a function of x which tends steadily to infinity with x. Then*

$$\log\log x - \phi\surd(\log\log x) < f(n) < \log\log x + \phi\surd(\log\log x) \tag{2.31}$$

for almost all quadratfrei *numbers n less than x.*
It is plainly enough to prove that

$$S_1 = \sum_{\nu < llx - \phi\sqrt{(llx)}} \pi_{\nu+1}(x) = o(x), \tag{2.321}$$

$$S_2 = \sum_{\nu > llx + \phi\sqrt{(llx)}} \pi_{\nu+1}(x) = o(x).^* \tag{2.322}$$

It will be sufficient to consider one of the two sums S_1, S_2, say the latter; the discussion of S_1 proceeds on the same lines and is a little simpler. We have

$$S_2 < K\frac{x}{\log x} \sum_{\nu > llx + \phi\surd(llx)} \frac{(llx + C)^\nu}{\nu!}. \tag{2.33}$$

Write

$$\log\log x + C = \xi.$$

Then the condition that

$$\nu > \log\log x + \phi\sqrt{\log\log x},$$

where ϕ is some function of x which tends steadily to infinity with x, is plainly equivalent to the condition that

$$\nu > \xi + \Psi\surd\xi,$$

where Ψ is some function of ξ which tends steadily to infinity with ξ; and so what we have to prove is that

$$S = \sum_{\nu > \xi + \Psi\surd\xi} \frac{\xi^\nu}{\nu!} = o(e^\xi). \tag{2.34}$$

We choose a positive number δ so small that

$$\frac{\delta}{2\cdot 3} + \frac{\delta^2}{3\cdot 4} + \frac{\delta^3}{4\cdot 5} + \cdots < \tfrac{1}{4}, \tag{2.35}$$

---

*We shall sometimes write $lx, llx, \ldots$, instead of $\log x, \log\log x, \ldots$, in order to shorten our formulæ.

and write

$$S = \sum_{\xi+\Psi\sqrt{\xi}<\nu\leq(1+\delta)\xi} \frac{\xi^\nu}{\nu!} + \sum_{\nu>(1+\delta)\xi} \frac{\xi^\nu}{\nu!} = S' + S'', \tag{2.36}$$

say. In the first place we have

$$\begin{aligned} S'' &< \frac{\xi_1^\nu}{\nu_1!}\left\{1 + \frac{\xi}{\nu_1+1} + \frac{\xi^2}{(\nu_1+1)(\nu_1+2)} + \cdots\right\} \\ &< \frac{\xi_1^\nu}{\nu_1!}\left\{1 + \frac{1}{1+\delta} + \frac{1}{(1+\delta)^2} + \cdots\right\} = \frac{1+\delta}{\delta}\frac{\xi^{\nu_1}}{\nu_1!}, \end{aligned} \tag{2.371}$$

where ν_1 is the smallest integer greater than $(1+\delta)\xi$. It follows that

$$\begin{aligned} S'' &< \frac{K}{\delta\sqrt{\nu_1}} e^{\nu_1(\log\xi - \log\nu_1 + 1)} \\ &< \frac{K}{\delta\sqrt{\xi}} e^{\Delta\xi}, \end{aligned} \tag{2.372}$$

where the K's are absolute constants and

$$\Delta = (1+\delta)\log(1+\delta) - \delta.$$

And since

$$(1+\delta)\log(1+\delta) - \delta > (1+\delta)(\delta - \tfrac{1}{2}\delta^2) - \delta = \tfrac{1}{2}\delta^2(1-\delta) = \eta,$$

say, we obtain

$$S'' < \frac{K}{\delta\sqrt{\xi}} e^{(1-\eta)\xi}, \tag{2.373}$$

where η is positive: so that

$$S'' = o(e^\xi). \tag{2.374}$$

In S', we write $\nu = \xi + \mu$, so that $\Psi\sqrt{\xi} < \mu \leq \delta\xi$. Then

$$\begin{aligned} \frac{\xi^\nu}{\nu!} &= \frac{\xi^{\xi+\mu}}{(\xi+\mu)!} \\ &< \frac{K}{\sqrt{\xi}} \exp\{(\xi+\mu)\log\xi - (\xi+\mu)\log(\xi+\mu) + \xi + \mu\} \\ &= \frac{K}{\sqrt{\xi}} \exp\left\{(\xi+\mu)\left(1 - \frac{\mu}{\xi} + \frac{\mu^2}{2\xi^2} - \cdots\right)\right\} \\ &< \frac{K}{\sqrt{\xi}} e^{\xi - (\mu^2/4\xi)}.^* \end{aligned} \tag{2.381}$$

Thus

$$S' = O\left(\frac{e^{\xi}}{\sqrt{\xi}} \sum_{\Psi\sqrt{\xi}}^{\delta\xi} e^{-\mu^2/4\xi}\right)^{*} \tag{2.382}$$
$$= O\left\{\frac{e^{\xi}}{\sqrt{\xi}} \int_{\Psi\sqrt{\xi}}^{\infty} e^{-t^2/4\xi} dt\right\}$$
$$= O\left(e^{\xi} \int_{\Psi}^{\infty} e^{-u^2} du\right) = o(e^{\xi}),$$

since $\Psi \to \infty$. From (2.36), (2.374), and (2.382) follows the truth of (2.34), and so that of (2.322). As (2.321) may be proved in the same manner, the proof of Theorem A is completed.

2.4. Theorem A$'$. If ϕ *is a function of* n *which tends steadily to infinity with* n, *then almost all* quadratfrei *numbers have between*

$$\log\log n - \phi\sqrt{\log\log n} \text{ and } \log\log n + \phi\sqrt{\log\log n}$$

prime factors.

This theorem is a simple corollary of Theorem A. Consider the numbers not exceeding x. We may plainly neglect numbers less than $\sqrt{x}$; so that

$$\tfrac{1}{2}\log x \leq \log n \leq \log x,$$

$$\log\log x - \log 2 \leq \log\log n \leq \log\log x. \tag{2.41}$$

Given ϕ, we can determine a function $\Psi(x)$ such that Ψ tends steadily to infinity with x and

$$\Psi(x) < \tfrac{1}{2}\phi(\sqrt{x}). \tag{2.42}$$

Then

$$\Psi(x) < \tfrac{1}{2}\phi(n) \ (\sqrt{x} \leq n \leq x).$$

---

*Since the exponent is equal to

$$\xi - \mu\left(\frac{\mu}{1\cdot 2\cdot \xi} - \frac{\mu^2}{2\cdot 3\cdot \xi^2} + \frac{\mu^3}{3\cdot 4\cdot \xi^3} - \cdots\right),$$

and

$$\frac{\mu^2}{2\cdot 3\cdot \xi^2} + \frac{\mu^3}{3\cdot 4\cdot \xi^3} + \cdots < \frac{\mu}{4\xi}$$

in virtue of (2.35)

*In this sum the values of μ are not, in general, integral: $\xi + \mu$ is integral.

In the first place

$$\phi\sqrt{\log\log n} > \Psi\sqrt{\log\log x}$$

if

$$\log\log n > \tfrac{1}{4}\log\log x,$$

which is certainly true, in virtue of (2.41), for sufficiently large values of x. Thus

$$\log\log n - \phi\sqrt{(\log\log n)} < \log\log x - \Psi\sqrt{(\log\log x)}. \tag{2.43}$$

In the second place the corresponding inequality

$$\log\log n + \phi\sqrt{(\log\log n)} > \log\log x + \Psi\sqrt{(\log\log x)} \tag{2.44}$$

is certainly true if

$$\phi\sqrt{\log\log n} > \Psi\sqrt{\log\log x} + \log 2;$$

and this also is true, in virtue of (2.41) and (2.42), for sufficiently large values of x. From (2.43), (2.44), and Theorem A, Theorem A' follows at once.

As a corollary we have

THEOREM A$''$. *The normal order of the number of prime factors of a* quadratfrei *number* is $\log\log n$.

III.

The normal order of $f(n)$.

3.1. So far we have confined our attention to numbers which have no repeated factors. When we remove this restriction, the functions $f(n)$ and $F(n)$ have to be distinguished from one another.

We shall denote by $\varpi_\nu(x)$ the number of numbers, not exceeding x, for which

$$f(n) = \nu.$$

It is obvious that

$$\varpi_\nu(x) \geq \pi_\nu(x).$$

We require an inequality for $\varpi_\nu(x)$ similar to that for $\pi_\nu(x)$ given by Lemma A.

LEMMA B. *There are absolute constants D and L such that*

$$\varpi_{\nu+1}(x) < \frac{Lx}{\log x}\frac{(\log\log x + D)^\nu}{\nu!} \tag{3.11}$$

for

$$\nu = 0, 1, 2, \ldots,\ \ x \geq 2.$$

It is plain that

$$\varpi_1(x) = \pi(x) + \pi(\sqrt{x}) + \pi(\sqrt[3]{x}) + \cdots = O\left(\frac{x}{\log x}\right).$$

The inequality is therefore true for $\nu = 0$, whatever the value of D. We shall prove by induction that it is true in general if

$$D > B + H + J, \tag{3.12}$$

where B and H have the same values as in 2.2, and

$$J = \sum_{2}^{\infty}(s+1)(2^{-s} + 3^{-s} + 5^{-s} + \cdots). \tag{3.13}$$

Consider the numbers which do not exceed x and are comprised in the table

$$\begin{array}{c} 2^a \cdot p_1^{a_1} \cdot p_2^{a_2} \cdots p_\nu^{a_\nu}, \\ 3^a \cdot p_1^{a_1} \cdot p_2^{a_2} \cdots p_\nu^{a_\nu}, \\ \dots\dots\dots\dots\dots, \\ P^a \cdot p_1^{a_1} \cdot p_2^{a_2} \cdots p_\nu^{a_\nu}, \end{array}$$

where $p_1, p_2, \ldots, p_\nu$ satisfy the same conditions as in the table of 2.2. It is plain that $P \le \sqrt{x}$ if $a = 1, P \le \sqrt[3]{x}$ if $a = 2$, and so on, so that the total number of numbers in the table does not exceed

$$\sum_{p^{a+1} \le x} \varpi_\nu\left(\frac{x}{p^a}\right).$$

If now $\omega_1, \omega_2, \ldots$ are primes and

$$\omega_1 < \omega_2 < \cdots < \omega_{\nu+1}, \quad \omega_1^{a_1}\omega_2^{a_2}\cdots\omega_{\nu+1}^{a_{\nu+1}} \le x,$$

the number $\omega_1^{a_1}\omega_2^{a_2}\cdots\omega_{\nu+1}^{a_{\nu+1}}$ will occur at least ν times in one of the tables which correspond to different values of a. We have therefore

$$\nu\varpi_{\nu+1}(x) \le \sum_{p^2 \le x} \varpi_\nu\left(\frac{x}{p}\right) + \sum_{p^3 \le x} \varpi_\nu\left(\frac{x}{p^2}\right) + \cdots. \tag{3.14}$$

Now let us suppose that (3.11) is true when $\nu - 1$ is substituted for ν. Then it is plain* that

$$\varpi_{\nu+1}(x) < \frac{Lx(\log\log x + D)^{\nu-1}}{\nu!}\left\{\sum_{p^2 \le x} \frac{1}{p\log(x/p)} + \sum_{p^3 \le x} \frac{1}{p^2\log(x/p^2)} + \cdots\right\}. \tag{3.15}$$

*See the footnote to p. 81 [footnote * on 330].

Now

$$\sum_{p^2 \le x} \frac{1}{p\log(x/p)} < \frac{\log\log x + B + H}{\log x}, \tag{3.16}$$

as we have already seen in 2.2. Also, if $p^{s+1} \le x$, we have

$$\frac{x}{p^s} \ge x^{1/(s+1)}, \quad \log\frac{x}{p^s} \ge \frac{\log x}{s+1};$$

and so

$$\sum_{p^{s+1} \le x} \frac{1}{p^s\log(x/p^s)} \le \frac{s+1}{\log x}(2^{-s} + 3^{-s} + 5^{-s} + \cdots), \tag{3.17}$$

if $s \ge 2$. Hence

$$\sum_s \sum_{p^{s+1} \le x} \frac{1}{p^s\log(x/p^s)} < \frac{\log\log x + B + H + J}{\log x} < \frac{\log\log x + D}{\log x}. \tag{3.18}$$

From (3.15) and (3.18) the truth of (3.11) follows immediately.

3.2. We can now argue with $\varpi_\nu(x)$ as we argued with $\pi_\nu(x)$ in 2.3 and the paragraphs which follow; and we may, without further preface, state the following theorems.

Theorem B. *If ϕ is a function x which tends steadily to infinity to x, then*

$$\log\log x - \phi\sqrt{(\log\log x)} < f(n) < \log\log x + \phi\sqrt{(\log\log x)}$$

for almost all numbers n less than x.

Theorem B′. *If ϕ is a function of n which tends steadily to infinity with n, then almost all numbers have between*

$$\log\log n - \phi\sqrt{(\log\log n)} \quad \textit{and} \quad \log\log n + \phi\sqrt{(\log\log n)}$$

different prime factors.

Theorem B″. *The normal order of the number of different prime factors of a number is $\log\log n$.*

IV.

The normal order of $F(n)$.

4.1. We have now to consider the corresponding theorems for $F(n)$, the number of prime factors of n when multiple factors are counted multiply. These theorems are slightly more difficult. The additional difficulty occurs, however, only in the first stage of the argument, which requires the proof of some inequality analogous to those given by Lemmas A and B. We denote by

$$\Pi_\nu(x)$$

the number of numbers, not exceeding x, for which

$$F(n) = \nu.$$

It would be natural to expect an inequality of the same form as those of Lemmas A and B, though naturally with different values of the constants. It is easy to see, however, that no such inequality can possibly be true.

For, if ν is greater than a constant multiple of $\log x$, the function

$$\frac{x}{\log x}\,\frac{(\log\log x + C)^\nu}{\nu!}$$

is — as may be seen at once by a simple approximation based upon Stirling's theorem — exceedingly small; and $\Pi_{\nu+1}(x)$, being an integer, cannot be small unless it is zero. Thus such an inequality as is suggested would shew that $F(n)$ *cannot* be of order as high as $\log x$ for any n less than x; and this is false, as we can see by taking

$$x = 2^k + 1,\ n = 2^k,\ F(n) = \frac{\log n}{\log 2}.$$

the inequality required must therefore be of a less simple character.

4.2. Lemma C. *Suppose that K and C have the same meaning as in Lemma A, and that*

$$\tfrac{9}{10} \le \lambda < 1. \tag{4.21}$$

Then

$$\prod_{\nu+1}(x) < \frac{Kx}{\log x} \tag{4.22}$$
$$\times \left\{ \frac{(\log\log x + C)^\nu}{\nu!} + \lambda\frac{(\log\log x + C)^{\nu-1}}{(\nu-1)!} + \lambda^2\frac{(\log\log x + C)^{\nu-2}}{(\nu-2)!} + \cdots \right\},$$

the series being continued to the term λ^ν.

It is plainly sufficient to prove this inequality when $\lambda = \frac{9}{10}$. In what follows we shall suppose that λ has this particular value.

We require a preliminary inequality analogous to (2.23) and (3.14). Consider the numbers which do not exceed x and are comprised in the table

$$2^a \cdot p_1 \cdot p_2 \cdots p_{\nu+1-a},$$

$$\begin{aligned}3^a \cdot p_1 \cdot p_2 \cdots p_{\nu+1-a},\\ \cdots\cdots\cdots\cdots,\\ P^a \cdot p_1 \cdot p_2 \cdots p_{\nu+1-a},\end{aligned}$$

where now $p_1, p_2, \ldots, p_{\nu+1-\alpha}$ are primes, arranged in ascending order of magnitude, but not necessarily different, and where $p_{\nu+1-\alpha} \geq 2$ in the first row, $p_{\nu+1-\alpha} \geq 3$ in the second, and so forth. Arguing as in 2.2. and 3.1, we now find

$$\nu \Pi_{\nu+1}(x) < \sum_{p^2 \leq x} \Pi_\nu\left(\frac{x}{p}\right) + \sum_{p^2 \leq x} \Pi_{\nu-1}\left(\frac{x}{p^2}\right) + \sum_{p^2 \leq x} \Pi_{\nu-2}\left(\frac{x}{p^3}\right) + \cdots. \tag{4.23}$$

This inequality differs formally from (3.14) in that the suffixes of the functions on the right-hand side sink by one from term to term.

We can now prove (4.22) by a process of induction similar in principle to that of 2.2 and 3.1. Let us suppose that the inequality is true when ν is replaced by $\nu - 1$, and let us write

$$\log\log x + C = \xi,$$

as in 2.3. Substituting in (4.23), and observing that

$$\log\log \frac{x}{p^s} + C \leq \xi,$$

we obtain

$$\begin{aligned}\Pi_{\nu+1}(x) < {}& \frac{Kx}{\nu} \sum_{p^2 \leq x} \frac{1}{p \log(x/p)} \left\{\frac{\xi^{\nu-1}}{(\nu-1)!} + \lambda \frac{\xi^{\nu-2}}{(\nu-2)!} + \cdots\right\}\\ & + \frac{Kx}{\nu} \sum_{p^3 \leq x} \frac{1}{p^2 \log(x/p^2)} \left\{\frac{\xi^{\nu-2}}{(\nu-2)!} + \lambda \frac{\xi^{\nu-3}}{(\nu-3)!} + \cdots\right\}\\ & + \frac{Kx}{\nu} \sum_{p^4 \leq x} \frac{1}{p^3 \log(x/p^3)} \left\{\frac{\xi^{\nu-3}}{(\nu-3)!} + \lambda \frac{\xi^{\nu-4}}{(\nu-4)!} + \cdots\right\} + \cdots\end{aligned} \tag{4.24}$$

Now, as we have seen already,

$$\sum_{p^2 \leq x} \frac{1}{p \log(x/p)} < \frac{\log\log x + C}{\log x}; \tag{4.251}$$

and

$$\sum_{p^{s+1} \leq x} \frac{1}{p^s \log(x/p^s)} < \frac{s+1}{\log x}(2^{-s} + 3^{-s} + 5^{-s} + \cdots) \tag{4.252}$$

if $s \geq 2$. It is moreover easy to prove that

$$(s+1)(2^{-s}+3^{-s}+5^{-s}+\cdots) < 2\lambda^s = 2(\tfrac{9}{10})^s \tag{4.2531}$$

if $s = 2$, and

$$(s+1)(2^{-s}+3^{-s}+5^{-s}+\cdots) < \lambda^s = (\tfrac{9}{10})^s \tag{4.2532}$$

if $s > 2$.[*] From (4.24) – (4.2532) it follows that

$$\begin{aligned}\Pi_{\nu+1}(x) < {} & \frac{Kx}{\nu\log x}\left\{\frac{\xi^{\nu}}{(\nu-1)!} + \lambda\frac{\xi^{\nu-1}}{(\nu-2)!} + \lambda^2\frac{\xi^{\nu-2}}{(\nu-3)!} + \cdots\right\} \\ & + \frac{2Kx}{\nu\log x}\lambda^2\left\{\frac{\xi^{\nu-2}}{(\nu-2)!} + \lambda\frac{\xi^{\nu-3}}{(\nu-3)!} + \cdots\right\} \\ & + \frac{Kx}{\nu\log x}\lambda^3\left\{\frac{\xi^{\nu-3}}{(\nu-3)!} + \lambda\frac{\xi^{\nu-4}}{(\nu-4)!} + \cdots\right\} \\ & + \cdots{}^{\dagger}.\end{aligned} \tag{4.25}$$

When we collect together the various terms on the right-hand side which involve the same powers of ξ, it will be found that the coefficient of $\xi^{\nu-p}$ is exactly

$$\frac{Kx}{\log x}\,\frac{\lambda p}{(\nu-p)!},$$

except when $p = 1$, when it is

$$\left(1-\frac{1}{\nu}\right)\frac{Kx}{\log x}\,\frac{\lambda}{(\nu-1)!}.$$

We thus obtain (4.22).

4.3. We may now argue substantially as in 2.3. We have to shew, for example, that

$$S_2 = \sum_{\nu > llx+\phi\sqrt{(llx)}} \Pi_{\nu+1}(x) = o(x); \tag{4.31}$$

and this is equivalent to proving that

$$S = \sum_{\nu > \xi+\Psi\sqrt{\xi}} \Pi_{\nu+1}(x) = o(x), \tag{4.32}$$

where Ψ is any function of x which tends to infinity with x.

---

[*] The inequalities may be verified directly for $s = 2$ and $s = 3$, and then proved to be true generally by induction

[†] The factor 2 occurs in the second line only.

We choose δ so that

$$\frac{1}{9\log(1+\frac{1}{9})} - 1 < \delta < \tfrac{1}{9},^{*} \tag{4.33}$$

and write

$$S = \sum_{\xi+\Psi\sqrt{\xi}<\nu\leq(1+\delta)\xi} \Pi_{\nu+1}(x) + \sum_{\nu>(1+\delta)\xi} \Pi_{\nu+1}(x) = S' + S'', \tag{4.34}$$

say. In S', we have

$$\Pi_{\nu+1}(x) < \frac{Kx}{\log x}\frac{\xi^\nu}{\nu!}\left\{1 + \tfrac{9}{10}\frac{\nu}{\xi} + (\tfrac{9}{10})^2\frac{\nu(\nu-1)}{\xi^2} + \cdots\right\} < \frac{K_1 x}{\log x}\frac{\xi^\nu}{\nu!}, \tag{4.351}$$

where

$$K_1 = \frac{K}{1-\frac{9}{10}(1+\delta)}; \tag{4.352}$$

and by means of this inequality we can shew, just as in 2.3, that

$$S' = o(x). \tag{4.353}$$

In discussing S'' we must use (4.22) in a different manner. We have

$$\Pi_{\nu+1}(x) < \frac{Kx}{\log x}\left(\lambda^\nu + \lambda^{\nu-1}\frac{\xi}{1!} + \lambda^{\nu-2}\frac{\xi^2}{2!} + \cdots\right) < \frac{Kx}{\log x}\lambda^\nu e^{\xi/\lambda}; \tag{4.361}$$

and so

$$S'' < \frac{Kx}{\log x}e^{\xi/\lambda}\sum_{\nu>(1+\delta)\xi}\lambda^\nu < \frac{K}{1-\lambda}\frac{x}{\log x}e^{\xi/\lambda}\lambda^{(1+\delta)\xi} = O\left\{\frac{x}{(\log x)^\eta}\right\}, \tag{4.362}$$

where

$$\eta = 1-(1/\lambda)+(1+\delta)\log(1/\lambda) = (1+\delta)\log\tfrac{10}{9} - \tfrac{1}{9} > 0, \tag{4.363}$$

by (4.33). Hence

$$S'' = o(x). \tag{4.364}$$

From (4.34), (4.353), and (4.364), it follows that $S = o(x)$.

4.4. We have therefore the following theorems.

THEOREM C. *The result of Theorem B remains true when $F(n)$ is substituted for $f(n)$.*
THEOREM C$'$, *The result of Theorem* B$'$ *remains true when the word "different" is omitted.*
THEOREM C$''$. *The normal order of the total number of prime factors of a number is* $\log\log n$.

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* $\frac{1}{9\log(1+\frac{1}{9})} - 1 < \frac{1}{1-\frac{1}{18}} - 1 < \frac{1}{9}$.

V.
The normal order of $d(n)$.

5.1. It is natural to ask whether similar theorems cannot be proved with regard to some of the other standard arithmetical functions of n, such as $d(n)$.

If
$$n = p_1^{a_1} p_2^{a_2} \cdots p_\nu^{a_\nu},$$
we have
$$d(n) = (1 + a_1)(1 + a_2) \cdots (1 + a_\nu).$$
Since
$$2 \leq 1 + a \leq 2^a$$
if $a \geq 1$, we obtain at once
$$2^\nu \leq d(n) \leq 2^{a_1 + a_2 + \cdots + a_\nu},$$
or

(5.11) $$2^{f(n)} \leq d(n) \leq 2^{F(n)}.$$

From (5.11), and Theorems B' and C', we obtain at once
THEOREM D$'$. *The inequalities*

(5.12) $$2^{\log\log n - \phi\sqrt{(\log\log n)}} < d(n) 2^{\log\log n + \phi\sqrt{(\log\log n)}},$$

where ϕ is any function of n which tends to infinity with n, are satisfied for almost all numbers n.

The inequalities (5.12) are of a much less precise type than (1.26): we cannot say that the normal order of $d(n)$ is $2^{\log\log n}$. We can however say that (to put it roughly) the normal order of $d(n)$ is *about*
$$2^{\log\log n} = (\log n)^{\log 2} = (\log n)^{0.69\ldots}.$$

It should be observed that this order is far removed from $\log n$, the *average* order f $d(n)$. The explanation of this apparent paradox is simple. The majority of numbers have about $(\log n)^{\log 2}$ divisors. But *those which have an abnormal number may have a very much larger number indeed:* the excess of the maximum order over the normal order is so great that, when we compute the average order, it is the numbers with an abnormal number of divisors which dominate the calculation. The maximum order of the number of *prime factors* is not large enough to give rise to a similar phenomenon.

Asymptotic formulæ in combinatory analysis*

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1. Introduction and summary of results

1.1 The present paper is the outcome of an attempt to apply to the principal problems of the theory of partitions the methods, depending upon the theory analytic functions, which have proved so fruitful in the theory of the distribution of primes and allied branches of the analytic theory of numbers.

The most interesting functions of the theory of partitions appear as the coefficients in the power-series which represents certain elliptic modular functions. Thus $p(n)$, the number of unrestricted partitions of n, is the coefficient of x^n in the expansion of the function

$$f(x) = 1 + \sum_1^\infty p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.\ \dagger \tag{1.11}$$

If we write

$$x = q^2 = e^{2\pi i\tau}, \tag{1.12}$$

where the imaginary part of τ is positive, we see that $f(x)$ is substantially the reciprocal of the modular function called by Tannery and Molk‡ $h(\tau)$; that, in fact,

$$h(\tau) = q^{\frac{1}{12}}q_0 = q^{\frac{1}{12}}\prod_1^\infty(1-q^{2n}) = \frac{x^{\frac{1}{24}}}{f(x)}. \tag{1.13}$$

The theory of partitions has, from the time of Euler onwards, been developed from an almost exclusively algebraical point of view. It consists of an assemblage of formal identities – many of them, it need hardly be said, of an exceedingly ingenious and beautiful character. Of *asymptotic* formluæ, one may fairly say, there are none§. So true is this, in fact, that we

*A short abstract of the contents of part of this paper appeared under the title “Une formule asymptotique pour le nombre des partitions de n, ” in the *Comptes Rendus*, January 2nd, 1917 [No. 31 of this volume].

†P. A. MacMahon, *Combinatory Analysis*, Vol. II, 1916, p. 1.

‡J. Tannery and J. Molk, *Fonctions elliptiques*, Vol. II, 1896, pp, 31 *et seq.* We shall follow the notation of this work whenever we have to quote formulæ from the theory of elliptic functions.

§We should mention one exception to this statement, to which our attention was called by Major MacMahon. The number of partitions of n *into parts none of which exceed* r is the coefficient $p_r(n)$ in the series

$$1 + \sum_1^\infty p_r(n)x^n = \frac{1}{(1-x)(1-x^2)\cdots(1-x^r)}.$$

This function has been studied in much detail, for various special values of r, by Cayley, Sylvester and Glaisher: we may refer in particular to J. J. Sylvester, “On a discovery in the theory of partitions,” *Quarterly Journal*, Vol. I, 1857, pp. 81 – 85, and “On the partition of numbers,” *ibid.*, pp. 141 – 152 (Sylvester’s have been unable to discover in the literature of the subject any allusion whatever to the question of the order of magnitude of $p(n)$.

1.2 The function $p(n)$ may, of course, be expressed in the form of an integral

$$p(n) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(x)}{x^{n+1}}\, dx, \tag{1.21}$$

by means of Cauchy's theorem, the path Γ enclosing the origin and lying entirely inside the unit circle. The idea which dominates this paper is that of obtaining asymptotic formulæ for $p(n)$ by a detailed study of the integral (1.21). This idea is an extremely obvious one; it is the idea which has dominated nine-tenths of modern research in analytic theory of numbers: and it may seem very strange that it should never have been applied to this particular problem before. Of this there are no doubt two explanations. The first is that the theory of partitions has received its most important developments, since its foundation by Euler, at the hands of a series of mathematicians whose interests have lain primarily in algebra. The second and more fundamental reason is to be found in the extreme complexity of the behavior of the generating function $f(x)$ near a point of the unit circle.

It is instructive to contrast this problem with the corresponding problems which arise for the arithmetical functions $\pi(n), \vartheta(n), \Psi(n), \mu(n), d(n), \ldots$ which have their genesis in Riemann's Zeta-function and the functions allied to it. In the latter problems we are dealing with functions defined by Dirichlet's series. The study of such functions presents difficulties far more fundamental than any which confront us in the theory of the modular functions. These difficulties, however, relate to the distribution of the zeros of the functions and their general behavior at infinity: no difficulties whatever are occasioned by the crude singularities of the functions in the finite part of the plane. The single finite singularity of $\zeta(s)$, for example, the pole at $s = 1$, is a singularity of the simplest possible character.

Works, Vol. II, pp. 86 – 89 and 90 – 99); J. W. L. Glaisher, "On the number of partitions of a number into a given number of parts", *Quarterly Journal*, Vol. XL, 1909, pp. 57 – 143; "Formulæ for partitions into given elements, derived from Sylvester's Theorem", *ibid.*, pp. 275 – 348; "Formulæ for the number of partitions of a number into the elements 1, 2, 3, ..., n upto $n = 9$", *ibid.*, Vol. XLI, 1910, pp. 94 – 112; and further references will be found in MacMahon, *loc. cit.*, pp. 59 – 71, and E. Netto, *Lehrbuch der Combinatorik*, 1901, pp. 146 – 158. Thus, for example, the coefficient of x^n in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}$$

is

$$p_3(n) = \tfrac{1}{12}(n+3)^2 - \tfrac{7}{72} + \tfrac{1}{8}(-1)^n + \tfrac{2}{9}\cos\frac{2n\pi}{3};$$

as is easily found by separating the function into partial fractions. This function may also be expressed in the forms

$$\tfrac{1}{12}(n+3)^2 + (\tfrac{1}{2}\cos\tfrac{1}{2}\pi n)^2 - (\tfrac{2}{3}\sin\tfrac{1}{3}\pi n)^2,$$
$$1 + \left[\tfrac{1}{12}n(n+6)\right], \{\tfrac{1}{12}(n+3)^2\},$$

where $[n]$ and $\{n\}$ denote the greatest integer contained in n and the integer nearest to n. These formulæ do, of course, furnish incidentally asymptotic formulæ for the functions in question. But they are, from this point of view, of a very trivial character: the interest which they possess is algebraical.

It is this pole which gives rise to the *dominant* terms in the asymptotic formulæ for the arithmetical functions associated with $\zeta(s)$. To prove such a formula rigorously is often exceedingly difficult; to determine precisely the order of the error which it involves is in many cases a problem which still defies the utmost resources of analysis. But to write down the dominant terms involves, as a rule, no difficulty more formidable than that of deforming a path of integration over a pole of the subject of integration and calculating the corresponding residue.

In the theory of partitions, on the other hand, we are dealing with functions which do not exist at all outside the unit circle. Every point of the circle is an essential singularity of the function, and no part of the contour of integration can be deformed in such a manner as to make its contribution obviously negligible. Every element of the contour requires special study; and there is no obvious method of writing down a "dominant term".

The difficulties of the problem appear then, at first sight, to be very serious. We possess, however, in the formulæ of the theory of linear transformation of the elliptic functions, an extremely powerful analytical weapon by means of which we can study the behavior of $f(x)$ near any assigned point of the unit circle*. It is to an appropriate use of these formulæ that the accuracy of our final results, an accuracy which will, we think, be found to be quite startling, is due.

1.3 It is very important, in dealing with such a problem as this, to distinguish clearly the various stages to which we can progress by arguments of progressively "deeper" and less elementary character. The earlier results are naturally (so far as the particular problem is concerned) superseded by the later. But the more elementary methods are likely to be applicable to other problems in which the more subtle analysis is impracticable.

We have attacked this particular problem by a considerable number of different methods, and cannot profess to have reached any very precise conclusions as to the possibilities of each. A detailed comparison of the results to which they lead would moreover expand this paper to a quite unreasonable length. But we have thought it worth while to include a short account of two of them. The first is quite elementary; it depends only on Euler's identity

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-x^2)^2} + \cdots \tag{1.31}$$

– an identity capable of wide generalisation – and on elementary algebraical reasoning. By these means we shew, in section 2, that

$$e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}}, \tag{1.32}$$

where A and B are positive constants, for all sufficiently large values of n.

*See G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation (II: The trigonometrical series associated with the elliptic Theta-functions)," *Acta Mathematica*, Vol. XXXVII, 1914, pp. 193 – 238, for applications of the formulæ to different but not unrelated problems.

It follows that

$$A\sqrt{n} < \log p(n) < B\sqrt{n}; \tag{1.33}$$

and the next question which arises is the question whether a constant C exists such that

$$\log p(n) \sim C\sqrt{n}. \tag{1.34}$$

We prove that this is so in section 3. Our proof is still, in a sense, "elementary." It does not appeal to the theory of analytic functions, depending only on a general arithmetic theorem concerning infinite series; but this theorem is of the difficult and delicate type which Messrs Hardy and Littlewood have called "Tauberian." The actual theorem required was proved by us in a paper recently printed in these *Proceedings*[*]. It shews that

$$C = \frac{2\pi}{\sqrt{6}}; \tag{1.35}$$

in other words that

$$p(n) = \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}(1+\epsilon)\right\}, \tag{1.36}$$

where ϵ is small when n is large. This method is one of very wide application. It may be used, for example, to prove that, if $p^{(s)}(n)$ denotes the number of partitions of n into perfect s-th powers, then

$$\log p^{(s)}(n) \sim (s+1)\left\{\frac{1}{s}\Gamma\left(1+\frac{1}{s}\right)\zeta\left(1+\frac{1}{s}\right)\right\}^{s/(s+1)} n^{1/(s+1)}.$$

It is certainly possible to obtain, by means of arguments of this general character, information about $p(n)$ more precise than that furnished by the formula (1.36). And it is equally possible to prove (1.36) by reasoning of a more elementary, though more special, character: we have a proof, for example, based on the identity

$$np(n) = \sum_{\nu=1}^{n} \sigma(\nu)p(n-\nu),$$

where $\sigma(\nu)$ is the sum of divisors of ν, and a process of induction. But we are at present unable to obtain, by any method which does not depend upon Cauchy's theorem, a result as precise as that which we state in the next paragraph, a result, that is to say, which is "vraiment asymptotique."

[*] G. H. Hardy and S. Ramanujan, "Asymptotic formulæ for the distribution of integers of various types," *Proc. London Math. Soc.*, Ser. 2, Vol. XVI, 1917, pp. 112 – 132 [No. 34 of this volume].

1.4 Our next step was to replace (1.36) by the much more precise formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}\right\}. \tag{1.41}$$

The proof of this formula appears to necessitate the use of much more powerful machinery, Cauchy's integral (1.21) and the functional relation

$$f(x) = \frac{x^{1/24}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)}\ \exp\left\{\frac{\pi^2}{6\log\ (1/x)}\right\} f(x'), \tag{1.42}$$

where

$$x' = \exp\left\{-\frac{4\pi^2}{\log\ (1/x)}\right\}. \tag{1.43}$$

This formula is a merely a statement in different notation of the relation between $h(\tau)$ and $h(T)$ where

$$T = \frac{c + d\tau}{a + b\tau}, \quad a = d = 0, b = 1, c = -1;$$

viz.

$$h(\tau) = \sqrt{\left(\frac{i}{\tau}\right)} h(T).^*$$

It is interesting to observe the correspondence between (1.41) and the results of numerical computation. Numerical data furnished to us by Major MacMahon gave the following results: we denote the right-had side of (1.41) by $\varpi(n)$.

n	$p(n)$	$\varpi(n)$	ϖ/p
10	42	48.104	1.145
20	627	692.385	1.104
50	204226	217590.499	1.065
80	15796476	16606781.567	1.051

It will be observed that the progress of ϖ/p towards its limit unity is not very rapid, and that $\varpi - p$ is always positive and appears to tend rapidly to infinity.

1.5 In order to obtain more satisfactory results it is necessary to construct some auxiliary function $F(x)$ which is regular at all points of the unit circle save $x = 1$, and has there a singularity of a type as near as possible to that of the singularity of $f(x)$. We may then hope to obtain a much more precise approximation by applying Cauchy's theorem to $f - F$ instead of to f. For although every point of the circle is a singular point of f, $x = 1$ is, to

*Tannery and Molk, *loc. cit.*, p. 265 (Table XLV, 5).

put it roughly, much the *heaviest* singularity. When $x \longrightarrow 1$ by real values, $f(x)$ tends to infinity like an exponential

$$\exp\left\{\frac{\pi^2}{6(1-x)}\right\};$$

when

$$x = re^{2p\pi i/q},$$

p and q being co-prime integers, and $r \longrightarrow 1$, $|f(x)|$ tends to infinity like an exponential

$$\exp\left\{\frac{\pi^2}{6q^2(1-r)}\right\};$$

while, if

$$x = re^{2\theta\pi i},$$

where θ is irrational, $|f(x)|$ can become infinite at most like an exponential of the type

$$\exp\left\{o\left(\frac{1}{1-r}\right)\right\}.^*$$

The function required is

$$F(x) = \frac{1}{\pi\sqrt{2}}\sum_{1}^{\infty}\Psi(n)x^n, \tag{1.51}$$

where

$$\Psi(n) = \frac{d}{dn}\left\{\frac{\cosh C\lambda_n - 1}{\lambda_n}\right\}, \tag{1.52}$$

$$C = 2\pi/\sqrt{6} = \pi\sqrt{\tfrac{2}{3}}, \quad \lambda_n = \sqrt{(n - \tfrac{1}{24})}. \tag{1.53}$$

This function may be transformed into an integral by means of a general formula given by Lindelöf[†]; and it is then easy to prove that the "principal branch" of $F(x)$ is regular all over the plane except at $x = 1$[‡]; and that

$$F(x) - \chi(x),$$

where

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[*] The statements concerning the "rational" points are corollaries of the formulæ of the transformation theory, and proofs of them are contained in the body of the paper. The proposition concerning "irrational" points may be proved by arguments similar to those used by Hardy and Littlewood in their memoir already quoted. It is not needed for our present purpose. As a matter of fact it is *generally* true that $f(x) \longrightarrow 0$ when θ is irrational, and very nearly as rapidly as $\sqrt[4]{(1-r)}$. It is in reality owing to this that our final method is so successful.

[†] E. Lindelöf, *Le calcul de résidus et ses applications à la théorie des fonctions*, (Gauthier-Villars, Collection Borel, 1905), p. 111.

[‡] We speak, of course, of the principal branch of the function, viz., that represented by the series (1.51) when x is small. The other branches are singular at the origin.

$$\chi(x) = \frac{x^{1/24}}{\sqrt{(2\pi)}} \sqrt{\left(\log \frac{1}{x}\right)} \left[\exp\left\{\frac{\pi^2}{6\log\ (1/x)}\right\} - 1\right] \tag{1.54}$$

is regular for $x = 1$. If we compare (1.42) and (1.54), and observe that $f(x')$ tends to unity with extreme rapidity when x tends to 1 along any regular path which does not touch the circle of convergence, we can see at once the very close similarity between the behaviour of f and F inside the unit circle and in the neighbourhood of $x = 1$.

It should be observed that the term -1 in (1.52) and (1.54) is-so far as our present assertions are concerned-otiose: all that we have said remains true if it is omitted; the resemblance between the singularities of f and F becomes indeed even closer. The term is inserted merely in order to facilitate some of our preliminary analysis, and will prove to be without influence on the final result.

Applying Cauchy's theorem to $f - F$, we obtain

$$p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right) + O(e^{D\sqrt{n}}), \tag{1.55}$$

where D is any number greater than

$$\tfrac{1}{2}C = \tfrac{1}{2}\pi\sqrt{(\tfrac{2}{3})}.$$

1.6 The formula (1.55) is an asymptotic formula of a type far more precise than that of (1.41). The error term is, however, of an exponential type, and may be expected ultimately to increase with very great rapidity. It was therefore with considerable surprise that we found what exceedingly good results the formula gives for fairly large values of n. For $n = 61, 62, 63$ it gives

$$1121538.972, \quad 1300121.359, \quad 1505535.606,$$

while the correct values are

$$1121505, \quad 1300156, \quad 1505499.$$

The errors

$$33.972, \quad -34.641, \quad 36.606$$

are relatively very small, and alternate in sign.

The next step is naturally to direct our attention to the singular point of $f(x)$ next in importance after that at $x = 1$, viz., that at $x = -1$; and to subtract from $f(x)$ a second auxiliary function, related to this point as $F(x)$ is to $x = 1$. No new difficulty of principle is involved, and we find that

$$p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right) + \frac{(-1)^n}{2\pi} \frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n}}{\lambda_n}\right) + O(e^{D\sqrt{n}}), \tag{1.61}$$

where D is now any number greater than $\frac{1}{3}C$. It now becomes obvious why our earlier approximation gave errors alternately of excess and of defect.

It is obvious that this process may be repeated indefinitely. The singularities next in importance are those at $x = e^{\frac{2}{3}\pi i}$ and $x = e^{\frac{4}{3}\pi i}$; the next those at $x = i$ and $x = -i$; and so on. The next two terms in the approximate formula are found to be

$$\frac{\sqrt{3}}{\pi\sqrt{2}} \cos\left(\tfrac{2}{3}n\pi - \tfrac{1}{18}\pi\right) \frac{d}{dn}\left(\frac{e^{\frac{1}{3}C\lambda_n}}{\lambda_n}\right)$$

and

$$\frac{\sqrt{2}}{\pi} \cos\left(\tfrac{1}{2}n\pi - \tfrac{1}{8}\pi\right) \frac{d}{dn}\left(\frac{e^{\frac{1}{4}C\lambda_n}}{\lambda_n}\right).$$

As we proceed further, the complexity of the calculations increases. The auxiliary function associated with the point $x = e^{2p\pi i/q}$ involves a certain $24q$-th root of unity, connected with the linear transformation which must be used in order to elucidate the behaviour of $f(x)$ near the point; and the explicit expression of this root in terms of p and q, though known, is somewhat complex. But it is plain that, by taking a sufficient number of terms, we can find a formula in which the error is

$$O(e^{C\lambda_n/\nu}),$$

where ν is a fixed but arbitrarily large integer.

1.7 A final question remains. We have still the resource of making ν a function of n, that is to say of making the number of terms in our approximate formula itself a function of n. In this way we may reasonably hope, at any rate, to find a formula in which the error is of order less than that of any exponential of the type e^{an}; of the order of a power of n, for example, or even bounded.

When, however, we proceeded to test this hypothesis by means of the numerical data most kindly provided for us by Major MacMahon, we found a correspondence between the real and the approximate values of such astonishing accuracy as to lead us to hope for even more. Taking $n = 100$, we found that the first six terms of our formula gave

$$\begin{array}{r} 190568944.783 \\ +348.872 \\ -2.598 \\ +.685 \\ +.318 \\ -.064 \\ \hline 190569291.996, \end{array}$$

while $p(100) = 190569292$; so that the error after six terms is only .004. We then proceeded to calculate $p(200)$ and found

$$\begin{array}{r} 3,972,998,993,185.896 \\ +36,282.978 \\ -87.555 \\ +5.147 \\ +1.424 \\ +0.071 \\ +0.000\ ^{*} \\ +0.043 \\ \hline 3,972,999,029,388.004, \end{array}$$

and Major MacMahon's subsequent calculations shewed that $p(200)$ is, in fact,

$$3,972,999,029,388.$$

These results suggest very forcibly that it is possible to obtain a formula for $p(n)$, which not only exhibits its order of magnitude and structure, but may be used to calculate its *exact* value for any value of n. That this is in fact so is shewn by the following theorem.

Statement of the main theorem.

THEOREM. *Suppose that*

$$\phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{C\lambda_n/q}}{\lambda_n}\right), \tag{1.71}$$

where C and λ_n are defined by the equations (1.53), *for all positive integral values of q; that p is a positive integer less than and prime to q; that $\omega_{p,q}$ is a $24q$-th root of unity, defined when p is odd by the formula*

$$\omega_{p,q} = \left(\frac{-q}{p}\right) \exp\left[-\left\{\tfrac{1}{4}(2-pq-p) + \tfrac{1}{12}\left(q-\tfrac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right], \tag{1.721}$$

and when q is odd by the formula

$$\omega_{p,q} = \left(\frac{-p}{q}\right) \exp\left[-\left\{\tfrac{1}{4}(q-1) + \tfrac{1}{12}\left(q-\tfrac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right], \tag{1.722}$$

where $\left(\frac{a}{b}\right)$ is the symbol of Legendre and Jacobi[†], *and p' is any positive integer such that $1+pp'$ is divisible by q; that*

$$A_q(n) = \textstyle\sum_{(p)} \omega_{p,q} e^{-2np\pi i/q}; \tag{1.73}$$

and that α is any positive constant, and ν the integral part of $\alpha\sqrt{n}$.
Then

$$p(n) = \textstyle\sum_1^{\nu} A_q\phi_q + O(n^{-\frac{1}{4}}), \tag{1.74}$$

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

$$\textstyle\sum_1^{\nu} A_q\phi_q. \tag{1.75}$$

---

[*] This term vanishes identically.

[†] See Tannery and Molk, *loc. cit.*, pp. 104 – 106, for a complete set of rules for the calculation of the value of $\left(\frac{a}{b}\right)$, which is, of course, always 1 or -1. When *both* p and q are odd it is indifferent which formula is adopted.

It should be observed that all the numbers A_q are real. A table of A_q from $q = 1$ to $q = 18$ is given at the end of the paper (Table II).

The proof of this main theorem is given in section 5; section 4 being devoted to a number of preliminary lemmas. The proof is naturally somewhat intricate; and we trust that we have arranged it in such a form as to be readily intelligible. In section 6 we draw attention to one or two questions which our theorem, in spite of its apparent completeness, still leaves open. In section 7 we indicate some other problems in combinatory analysis and the analytic theory of numbers to which our method may be applied; and we conclude by giving some functional and numerical tables: for the latter we are indebted to Major MacMahon and Mr. H. B. C. Darling. To Major MacMahon in particular we owe many thanks for the amount of trouble he has taken over very tedious calculations. It is certain that, without the encouragement given by the results of these calculations, we should never have attempted to prove theoretical results at all comparable in precision with those which we have enunciated.

2. Elementary proof that $e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}}$ for sufficiently large values of n.

2.1 In this section we give the elementary proof of the inequalities (1.32). We prove, in fact, rather more, viz., that positive constants H and K exist such that

(2.11) $$\frac{H}{n}e^{2\sqrt{n}} < p(n) < \frac{K}{n}e^{2\sqrt{2n}}$$

for $n \geq 1$*. We shall use in our proof only Euler's formula (1.31) and a debased form of Stirling's theorem, easily demonstrable by quite elementary methods: the proposition that

$$n!e^n/n^{n+\frac{1}{2}}$$

lies between two positive constants for all positive integral values of n.

2.2 The proof of the first of the two inequalities is slightly the simpler. It is obvious that if

$$\sum p_r(n)x^n = \frac{1}{(1-x)(1-x^2)\cdots(1-x^r)}$$

so that $p_r(n)$ is the number of partitions of n into parts not exceeding r, then

(2.21) $$p_r(n) = p_{r-1}(n) + p_{r-1}(n-r) + p_{r-1}(n-2r) + \cdots$$

We shall use this equation to prove, by induction, that

(2.22) $$p_r(n) \geq \frac{rn^{r-1}}{(r!)^2}.$$

*Somewhat inferior inequalities, of the type

$$2^{A[\sqrt{n}]} < p(n) < n^{B[\sqrt{n}]},$$

may be proved by *entirely* elementary reasoning; by reasoning, that is to say, which depends only on the arithmetical definition of $p(n)$ and on elementary finite algebra, and does not presuppose the notion of a limit or the definition of the logarithmic or exponential functions.

It is obvious that (2.22) is true for $r = 1$. Assuming it to be true for $r = s$, and using (2.21), we obtain

$$\begin{aligned} p_{s+1}(n) &\geq \frac{s}{(s!)^2}\{n^{s-1} + (n-s-1)^{s-1} + (n-2s-2)^{s-1} + \cdots\} \\ &\geq \frac{s}{(s!)^2}\left\{\frac{n^s - (n-s-1)^s}{s(s+1)} + \frac{(n-s-1)^s - (n-2s-2)^s}{s(s+1)} + \cdots\right\} \\ &= \frac{n^s}{(s+1)(s!)^2} = \frac{(s+1)n^s}{\{(s+1)!\}^2}. \end{aligned}$$

This proves (2.22). Now $p(n)$ is obviously not less than $p_r(n)$, whatever the value of r. Take $r = [\sqrt{n}]$: then

$$p(n) \geq p_{[\sqrt{n}]}(n) \geq \frac{[\sqrt{n}]}{n}\frac{n^{[\sqrt{n}]}}{\{[\sqrt{n}]!\}^2} > \frac{H}{n}e^{2\sqrt{n}},$$

by a simple application of the degenerate form of Stirling's theorem mentioned above.

2.3 The proof of the second inequality depends upon Euler's identity. If we write

$$\sum q_r(n)x^n = \frac{1}{(1-x)^2(1-x^2)^2\cdots(1-x^r)^2},$$

we have

(2.31) $$q_r(n) = q_{r-1}(n) + 2q_{r-1}(n-r) + 3q_{r-1}(n-2r) + \cdots,$$

and

(2.32) $$p(n) = q_1(n-1) + q_2(n-4) + q_3(n-9) + \cdots$$

We shall first prove by induction that

(2.33) $$q_r(n) \leq \frac{(n+r^2)^{2r-1}}{(2r-1)!(r!)^2}.$$

This is obviously true for $r = 1$. Assuming it to be true for $r = s$, and using (2.31), we obtain

$$\begin{aligned} q_{s+1}(n) \leq \frac{1}{(2s-1)!(s!)^2}\{(n+s^2)^{2s-1} &+ 2(n+s^2-s-1)^{2s-1} \\ &+ 3(n+s^2-2s-2)^{2s-1} + \cdots\}. \end{aligned}$$

Now

$$m(m-1)a^{m-2}b^2 \leq (a+b)^m - 2a^m + (a-b)^m,$$

if m is a positive integer, and a, b, and $a-b$ are positive, while if $a-b \leq 0$, and m is odd, the term $(a-b)^m$ may be omitted. In this inequality write

$$m = 2s+1, \quad a = n+s^2-ks-k \quad (k = 0, 1, 2, \ldots), \quad b = s+1,$$

and sum with respect to k. We find that

$$(2s+1)2s(s+1)^2\{(n+s^2)^{2s-1} + 2(n+s^2-s-1)^{2s-1} + \cdots\} \leq (n+s^2+s+1)^{2s+1};$$

and so

$$q_{s+1}(n) \leq \frac{(n+s^2+s+1)^{2s+1}}{(2s+1)2s(s+1)^2(2s-1)!(s!)^2} \leq \frac{\{n+(s+1)^2\}^{2s+1}}{(2s+1)!\{(s+1)!\}^2}.$$

Hence (2.33) is true generally.

It follows from (2.32) that

$$p(n) = q_1(n-1) + q_2(n-4) + \cdots \leq \sum_1^\infty \frac{n^{2r-1}}{(2r-1)!(r!)^2}.$$

But, using the degenerate form of Stirling's theorem once more, we find without difficulty that

$$\frac{1}{(2r-1)!(r!)^2} < \frac{2^{6r}K}{4r!},$$

where K is a constant. Hence

$$p(n) < 8K\sum_1^\infty \frac{(8n)^{2r-1}}{4r!} < 8K\sum_1^\infty \frac{(8n)^{\frac{1}{2}r-1}}{r!} < \frac{K}{n}e^{2\sqrt{2n}}.$$

This is the second of the inequalities (2.11).

3. Application of a Tauberian theorem to the determination of the constant C.

3.1 The value of the constant

$$C = \lim \frac{\log p(n)}{\sqrt{n}},$$

is most naturally determined by the use of the following theorem.

If $g(x) = \sum a_n x^n$ is a power-series with positive coefficients, and

$$\log g(x) \sim \frac{A}{1-x}$$

when $x \to 1$, then

$$\log s_n = \log\, (a_0 + a_1 + \cdots + a_n) \sim 2\sqrt{An}$$

when $n \to \infty$.

This theorem is a special case* of Theorem C in our paper already referred to.

Now suppose that

$$g(x) = (1-x)f(x) = \sum\{p(n) - p(n-1)\}x^n = \frac{1}{(1-x^2)(1-x^3)(1-x^4)\cdots}.$$

** Loc. cit.*, p. 129 (with $\alpha = 1$) [p. 321 of this volume].

Then

$$a_n = p(n) - p(n-1)$$

is plainly positive. And

$$\log g(x) = \sum_{2}^{\infty} \log \frac{1}{1-x^{\mu}} = \sum_{1}^{\infty} \frac{1}{\nu} \frac{x^{2\nu}}{1-x^{\nu}} \sim \frac{1}{1-x} \sum_{1}^{\infty} \frac{1}{\nu^2} = \frac{\pi^2}{6(1-x)}, \tag{3.11}$$

when $x \to 1$*. Hence

$$\log p(n) = a_0 + a_1 + \cdots + a_n \sim C\sqrt{n}, \tag{3.12}$$

where $C = 2\pi/\sqrt{6} = \pi\sqrt{\frac{2}{3}}$, as in (1.53).

3.2 There is no doubt that it is possible, by "Tauberian" arguments, to prove a good deal more about $p(n)$ than is asserted by (3.12). The functional equation satisfied by $f(x)$ shews, for example, that

$$g(x) \sim \frac{(1-x)^{3/2}}{\sqrt{2\pi}} \exp\left\{\frac{\pi^2}{6(1-x)}\right\},$$

a relation far more precise than (3.11). From this relation, and the fact that the coefficients in $g(x)$ are positive, it is certainly possible to deduce more than (3.12). But it hardly seems likely that arguments of this character will lead us to a proof of (1.41). It would be exceedingly interesting to know exactly how far they will carry us, since the method is comparatively elementary, and has a much wider range of application than the more powerful methods employed later in this paper. We must, however, reserve the discussion of this question for some future occasion.

4. Lemmas preliminary to the proof of the main theorem.

4.1 We proceed now to the proof of our main theorem. The proof is somewhat intricate, and depends on a number of subsidiary theorems which we shall state as lemmas.

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*This is a special case of a much more general theorems: see K. Knopp, "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze," *Inaugural Dissertation*, Berlin, 1907, pp. 25 *et seq.*; K. Knopp, "Über Lambertsche Reihen," *Journal für Math.*, Vol. CXLII, 1913, pp. 283 – 315; G. H. Hardy, "Theorems connected with Abel's Theorem on the continuity of power series," *Proc. London Math. Soc.*, Ser. 2, Vol. IV, 1906, pp. 247 – 265 (pp. 252, 253); G. H. Hardy, "Some theorems concerning infinite series," *Math. Ann.*, Vol. LXIV, 1907, pp. 77 – 94; G. H. Hardy, "Note on Lambert's series," *Proc. London Math. Soc.*, Ser. 2, Vol. XIII, 1913, pp. 192 – 198.

A direct proof is very easy: for

$$\nu x^{\nu-1}(1-x) < 1 - x^{\nu} < \nu(1-x),$$

$$\frac{1}{1-x}\sum \frac{x^{2\nu}}{\nu^2} < \log g(x) < \frac{1}{1-x}\sum \frac{x^{\nu+1}}{\nu^2}.$$

Lemmas concerning Farey's series.

4.21 The *Farey's series of order m* is the aggregate of irreducible rational fractions

$$p/q \quad (0 \leq p \leq q \leq m),$$

arranged in ascending order of magnitude. Thus

$$\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$$

is the Farey's series of order 7.

Lemma 4.21. *If $p/q, p'/q'$ are two successive terms of a Farey's series, then*

$$p'q - pq' = 1. \tag{4.211}$$

This is, of course, a well-known theorem, first observed by Farey and first proved by Cauchy*. Th following exceedingly simple proof is due to Hurwitz†.

The result is plainly true when $m = 1$. Let us suppose it true for $m = k$; and let $p/q, p'/q'$ be two consecutive terms in the series of order k.

Suppose now that p''/q'' is a term of the series of order $k+1$ which falls between $p/q, p'/q'$. Let

$$p''q - pq'' = \lambda > 0, \quad p'q'' - p''q' = \mu > 0.$$

Solving these equations for p'', q'' and observing that $p'q - pq' = 1$, we obtain

$$p'' = \mu p + \lambda p', \quad q'' = \mu q + \lambda q'.$$

Consider now the aggregate of fractions

$$(\mu p + \lambda p')/(\mu q + \lambda q'),$$

where λ and μ are positive integers without common factor. All of these fractions lie between p/q and p'/q'; and all are in their lowest terms, since a factor common to numerator and denominator would divide

$$\lambda = q(\mu p + \lambda p') - p(\mu q + \lambda q'),$$

and

$$\mu = p'(\mu q + \lambda q') - q'(\mu p + \lambda p').$$

Each of them first makes its appearance in the Farey's series of order $\mu q + \lambda q'$, and the *first* of them to make its appearance must be that for which $\lambda = 1, \mu = 1$. Hence

$$p'' = p + p', \quad q'' = q + q',$$

*J. Farey, "On a curious property of vulgar fraction," *Phil. Mag.*, Ser. 1, Vol. XLVII, 1816, pp. 385, 386; A. L. Cauchy, "Démonstration d'un théorème curieux sur les nombres," *Exercices de mathématiques*, Vol. I, 1826, pp. 114 – 116. Cauchy's proof was first published in the *Bulletin de la Société Philomatique* in 1816.

†A. Hurwitz, "Ueber die angenäherte Darstellung der Zahlen durch rationale Brüche," *Math. Ann.*, Vol. XLIV, 1894, pp 417-436.

$$p''q - pq'' = p'q'' - p''q' = 1.$$

The lemma is consequently proved by induction.

Lemma 4.22. *Suppose that p/q is a term of the Farey's series of order m, and $p''/q'', p'/q'$ the adjacent terms on the left and right: and let $j_{p,q}$ denote the interval*

$$\frac{p}{q} - \frac{1}{q(q+q'')}, \quad \frac{p}{q} + \frac{1}{q(q+q')}.^*$$

Then (i) *the intervals $j_{p,q}$ exactly fill up the continuum* $(0, 1)$, *and* (ii) *the length of each of the parts into which $j_{p,q}$ is divided by p/q*[†] *is greater than $1/2mq$ and less than $1/mq$.*

(i) Since

$$\frac{1}{q(q+q')} + \frac{1}{q'(q'+q)} = \frac{1}{qq'} = \frac{p'q - pq'}{qq'} = \frac{p'}{q'} - \frac{p}{q},$$

the intervals just fill up the continuum.

(ii) Since neither q nor q' exceeds m, and one at least must be less than m, we have

$$\frac{1}{q(q+q')} > \frac{1}{2mq}.$$

Also $q + q' > m$, since otherwise $(p + p')/(q + q')$ would be a term in the series between p/q and p'/q'. Hence

$$\frac{1}{q(q+q')} < \frac{1}{mq}.$$

Standard dissection of a circle.

4.22 The following mode of dissection of a circle, based upon Lemma 4.22, is of fundamental importance for our analysis.

Suppose that the circle is defined by

$$x = Re^{2\pi i\theta} \quad (0 \le \theta \le 1).$$

Construct the Farey's series of order m, and the corresponding intervals $j_{p,q}$. When these intervals are considered as intervals of variation of θ, and the two extreme intervals, which correspond to abutting arcs on the circle, are regarded as constituting a single interval $\xi_{1,1}$, the circle is divided into a number of arcs

$$\xi_{p,q},$$

where q ranges from 1 to m and p through the numbers not exceeding and prime to q[‡]. We call this dissection of the circle *the dissection* Ξ_m.

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[*] When p/q is $0/1$ or $1/1$, only the part of this interval inside $(0, 1)$ is to be taken; thus $j_{0,1}$ is $0, 1/(m+1)$ and $j_{1,1}$ is $1 - 1/(m+1), 1$.

[†] See the preceding footnote

[‡] $p = 0$ occurring with $q = 1$ only.

Lemmas from the theory of the linear transformation of the elliptic modular functions.

4.3 Lemma 4.31. *Suppose that q is a positive integer; that p is a positive integer not exceeding and prime to q; that p' is a positive integer such that $1+pp'$ is divisible by q; that $\omega_{p,q}$ is defined by the formulæ* (1.721) *or* (1.722) *; that*

$$x = \exp\left(-\frac{2\pi z}{q} + \frac{2p\pi i}{q}\right), \quad x' = \exp\left(-\frac{2\pi}{qz} + \frac{2p'\pi i}{q}\right),$$

where the real part of z is positive; and that

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

Then

$$f(x) = \omega_{p,q}\sqrt{z}\ \exp\left(\frac{\pi}{12qz} - \frac{\pi z}{12q}\right) f(x').$$

This lemma is merely a restatement in a different notation of well-known formulæ in the transformation theory.
Suppose, for example, that p is odd. If we take

$$a = p, b = -q, c = \frac{1+pp'}{q}, d = -p',$$

so that $ad - bc = 1$; and write

$$x = q^2 = e^{2\pi i\tau}, \quad x' = Q^2 = e^{2\pi iT},$$

so that

$$\tau = \frac{p}{q} + \frac{iz}{q}, \quad T = \frac{p'}{q} + \frac{i}{qz};$$

then we can easily verify that

$$T = \frac{c+d\tau}{a+b\tau}.$$

Also, in the notation of Tanner and Molk, we have

$$f(x) = \frac{q^{\frac{1}{12}}}{h(\tau)}, \quad f(x') = \frac{Q^{\frac{1}{12}}}{h(T)};$$

and the formula for the linear transformation of $h(\tau)$ is

$$h(T) = \left(\frac{b}{a}\right) \exp\left[\{\tfrac{1}{4}(a-1) - \tfrac{1}{12}[a(b-c) + bd(a^2-1)]\}\pi i\right] \sqrt{(a+b\tau)}\ h(\tau),$$

where $\sqrt{(a+b\tau)}$ has its real part positive*. A little elementary algebra will shew the equivalence of this result and ours.

The other formula for $\omega_{p,q}$ may be verified similarly, but in this case we must take

$$a=-p, b=q, c=\frac{1+pp'}{q}, d=p'.$$

We have included in the Appendix (Table I) a short table of some values of $\omega_{p,q}$, or rather of $(\log\omega_{p,q})/\pi i$.

Lemma 4.32. *The function $f(x)$ satisfies the equation*

$$f(x)=\omega_{p,q}\sqrt{\left\{\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right)\right\}}x_{p,q}^{\frac{1}{24}}\exp\left\{\frac{\pi^2}{6q^2\log(1/x_{p,q})}\right\}f(x'_{p,q}), \tag{4.321}$$

where

$$x_{p,q}=xe^{-2p\pi i/q},\quad x'_{p,q}=\exp\left\{-\frac{4\pi^2}{q^2\log(1/x_{p,q})}+\frac{2p'\pi i}{q}\right\}. \tag{4.322}$$

This is an immediate corollary from Lemma 4.31, since

$$z=\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right),\qquad e^{-\pi z/12q}=x_{p,q}^{\frac{1}{24}},$$

$$\frac{\pi}{12qz}=\frac{\pi^2}{6q^2\log(1/x_{p,q})},\qquad x'=\exp\left(-\frac{2\pi}{qz}+\frac{2p'\pi i}{q}\right)=x'_{p,q}.$$

If we observe that

$$f(x'_{p,q})=1+p(1)x'_{p,q}+\cdots,$$

we see that, if x tends to $e^{2p\pi i/q}$ along a radius vector, or indeed any regular path which does not touch the circle of convergence, the difference

$$f(x)-\omega_{p,q}\sqrt{\left\{\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right)\right\}}x_{p,q}^{\frac{1}{24}}\exp\left\{\frac{\pi^2}{6q^2\log(1/x_{p,q})}\right\}$$

tends to zero with great rapidity. It is on this fact that our analysis is based.

Lemmas concerning the auxiliary function $F_a(x)$.

4.41. The auxiliary function $F_a(x)$ is defined by the equation

$$F_a(x)=\sum_1^\infty \Psi_a(n)x^n,$$

where

$$\Psi_a(n)=\frac{d}{dn}\frac{\cosh a\lambda_n-1}{\lambda_n},$$

*Tannery and Molk, *loc. cit.*, pp. 113, 267.

$$\lambda_n = \sqrt{(n - \tfrac{1}{24})}, \quad a > 0.$$

Lemma 4.41. *Suppose that a cut is made along the segment* $(1, \infty)$ *in the plane* $f\ x$. *Then* $F_a(x)$ *is regular at all points inside the region thus defined.*

This lemma is an immediate corollary of a general theorem proved by Lindelöf on pp. 109 *et seq.* of his *Calcul des residus**.

The function

$$\Psi_a(z) = \frac{d}{dz} \frac{\cosh a\sqrt{(z - \frac{1}{24})} - 1}{\sqrt{(z - \frac{1}{24})}}$$

satisfies the conditions imposed upon it by Lindelöf, if the number which he calls α is greater than $\frac{1}{24}$; and

$$F_a(x) \int_{a - i\infty}^{a + i\infty} \frac{x^z}{1 - e^{2\pi i z}} \phi(z)\ dz, \tag{4.411}$$

if

$$x = re^{i\theta}, \quad 0 < \theta < 2\pi, \quad x^z = \exp\{z(\log r + i\theta)\}.$$

4.42. Lemma 4.42. *Suppose that* D *is the region defined by the inequalities*

$$-\pi < -\theta_0 < \theta < \theta_0 < \pi, \quad r_0 < r, \quad 0 < r_0 < 1,$$

and that $\log(1/x)$ *has its principal value, so that* $\log(1/x)$ *is one-valued, and its square root two-valued, in* D. *Further, let*

$$\chi_a(x) = \sqrt{\{\pi \log(1/x)\}} x^{\frac{1}{24}} \left[\exp\left\{ \frac{a^2}{4\log(1/x)} \right\} - 1 \right],$$

that value of the square root being chosen which is positive when $0 < x < 1$. *Then*

$$F_a(x) - \chi_a(x)$$

is regular inside D†.

We observe first that, whenθ has fixed value between 0 and 2π, the integral on the right-hand side of (4.411) is uniformly convergent for $\frac{1}{24} \le \alpha \le \alpha_0$. Hence we may take $\alpha = \frac{1}{24}$ in (4.411). We thus obtain

$$F_a(x) = ix^{\frac{1}{24}} \int_0^\infty \frac{x^{it}}{1 - e^{\frac{1}{12}\pi i - 2\pi t}} \Psi_a(\tfrac{1}{24} + it)\ dt + ix^{\frac{1}{24}} \int_0^\infty \frac{x^{-it}}{1 - e^{\frac{1}{12}\pi i + 2\pi t}} \Psi_a(\tfrac{1}{24} - it)\ dt,$$

*Lindelöf gives references to Mellin and Le Roy, who had previously established the theorem in less general forms.

†Both $F_a(x)$ and $\chi_a(x)$ are two-valued in D. The value of $F_a(x)$ contemplated is naturally that represented by the power series.

where the $\sqrt{it}$ and $\sqrt{-it}$ which occur in $\Psi_a(\frac{1}{24}+it)$ and $\Psi_a(\frac{1}{24}-it)$ are to be interpreted as $e^{(1/4)\pi i}\sqrt{t}$ and $e^{-(1/4)\pi i}\sqrt{t}$ respectively. We write this in the form

$$\begin{aligned}(4.421)\qquad F_a(x) &= X_a(x) + ix^{\frac{1}{24}}\int_0^\infty \frac{x^{it}}{e^{-\frac{1}{12}\pi i+2\pi t}-1}\Psi_a(\tfrac{1}{24}+it)\,dt \\ &\quad +ix^{\frac{1}{24}}\int_0^\infty \frac{x^{-it}}{1-e^{\frac{1}{12}\pi i+2\pi t}}\Psi_a(\tfrac{1}{24}-it)\,dt \\ &= X_a(x)+\Theta_1(x)+\Theta_2(x),\end{aligned}$$

say, where

$$X_a(x) = ix^{\frac{1}{24}}\int_0^\infty x^{it}\Psi_a(\tfrac{1}{24}+it)\,dt.$$

Now, since

$$|x^{it}| = e^{-\theta t}, \quad |x^{-it}| = e^{\theta t},$$

the functions Θ are regular throughout the angle of Lemma 4.42. And

$$X_a(x) = \frac{x^{\frac{1}{24}}}{\sqrt{i}}\int_0^\infty e^{-\lambda t}\frac{d}{dt}\left(\frac{\cosh\mu\sqrt{t}-1}{\sqrt{t}}\right)\,dt,$$

where

$$\lambda = i\log\frac{1}{x}, \quad \mu = a\sqrt{i}.$$

The form of this integral may be calculated by supposing λ and μ positive, when we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda w^2}\frac{d}{dw}\left(\frac{\cosh\mu w-1}{w}\right)\,dw &= 2\lambda\int_0^\infty e^{-\lambda w^2}(\cosh\mu w-1)\,dw \\ &= \sqrt{(\lambda\pi)}(e^{\mu^2/4\lambda}-1).\end{aligned}$$

Hence

$$(4.422)\qquad X_a(x) = \sqrt{\{\pi\log(1/x)\}}x^{\frac{1}{24}}\left[\exp\left\{\frac{a^2}{4\log(1/x)}\right\}-1\right] = \chi_a(x),$$

and the proof of the lemma is completed.

Lemmas 4.41 and 4.42 shew that $x=1$ is the sole finite singularity of the principal branch of $F_a(x)$.

4.43 Lemma 4.43 *Suppose that* $\mathbf{P}$, θ_1, *and* A *are positive constants,* θ_1 *being less than* π. *Then*

$$|F_a(x)| < K = K(\mathbf{P},\theta_1,A),$$

for

$$0\le r\le\mathbf{P}, \quad \theta_1\le\theta\le 2\pi-\theta_1, \quad 0<a\le A.$$

We use K generally to denote a positive number independent of x and of a. We may employ the formula (4.411). It is plain that

$$\left|\frac{x^z}{1-e^{2\pi iz}}\right| < Ke^{-\theta_1|\eta|},$$

$$|\Psi_a(z)| = \left|\frac{d}{dz}\left\{\frac{\cosh a\sqrt{(z-\frac{1}{24})}-1}{\sqrt{(z-\frac{1}{24})}}\right\}\right| < Ke^{K\sqrt{|\eta|}},$$

where η is the imaginary part of z. Hence

$$|F_a(x)| < K\int_{-\infty}^{\infty} e^{K\sqrt{|\eta|}-\theta_1|\eta|}\,d\eta < K.$$

4.44 Lemma 4.44. *Let c be a circle whose centre is $x=1$, and whose radius δ is less than unity. Then*

$$|F_a(x)-\chi_a(x)| < Ka^2,$$

is x lies in c and $0<a\le A, K=K(\delta,A)$ being as before independent of x and of a.
If we refer back to (4.421) and (4.422), we see that it is sufficient to prove that

$$|\Theta_1(x)| < Ka^2, \quad |\Theta_2(x)| < Ka^2;$$

and we may plainly confine ourselves to the first of these inequalities. We have

$$\Theta_1(x) = \frac{x^{\frac{1}{24}}}{\sqrt{i}}\int_0^{\infty}\frac{x^{it}}{e^{-\frac{1}{12}\pi i+2\pi t}-1}\frac{d}{dt}\left\{\frac{\cosh a\sqrt{(it)}-1}{\sqrt{(t)}}\right\}\,dt.$$

Rejecting the extraneous factor, which is plainly without importance, and integrating by parts, we obtain

$$\Theta(x) = \int_0^{\infty}\Phi(t)\frac{\cosh a\sqrt{(it)}-1}{\sqrt{(t)}}\,dt,$$

where

$$\Phi(t) = -\frac{ix^{it}\log x}{e^{-\frac{1}{12}\pi i+2\pi t}-1}+\frac{2\pi x^{it}e^{-\frac{1}{12}\pi i+2\pi t}}{(e^{-\frac{1}{12}\pi i+2\pi t}-1)^2}.$$

Now $|\theta| < \frac{1}{2}\pi$ and $|x^{it}| < Ke^{\frac{1}{2}\pi t}$. It follows that

$$|\Phi(t)| < Ke^{-\pi t};$$

and

$$|\Theta(x)| \quad < \quad K\int_0^{\infty}\frac{e^{-\pi t}}{\sqrt{t}}|\sinh^2\tfrac{1}{2}a\sqrt{(it)}|\,dt$$

$$
\begin{aligned}
&< K\int_0^\infty \frac{e^{-\pi t}}{\sqrt{t}}\{\cosh a\sqrt{(\tfrac{1}{2}t)} - \cos a\sqrt{(\tfrac{1}{2}t)}\}\ dt \\
&< K\int_0^\infty e^{-\pi w^2}\left(\cosh\frac{aw}{\sqrt{2}} - \cos\frac{aw}{\sqrt{2}}\right)\ dw \\
&= K(e^{a^2/8\pi} - e^{-a^2/8\pi}) < Ka^2.
\end{aligned}
$$

5. Proof of the main theorem.

5.1 We write

(5.11) $$F_{p,q}(x) = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}F_{C/q}(x_{p,q}),$$

where $C = \pi\sqrt{\frac{2}{3}},\ x_{p,q} = xe^{-2p\pi i/q}$; and

(5.12) $$\Phi(x) = f(x) - \sum_q\sum_p F_{p,q}(x),$$

where the summation applies to all values of p not exceeding q and prime to q, and to all values of q such that

(5.13) $$1 \le q \le \nu = [\alpha\sqrt{n}],$$

α being positive and independent of n. If then

(5.14) $$F_{p,q}(x) = \sum c_{p,q,n}x^n,$$

we have

(5.15) $$p(n) - \sum_q\sum_p c_{p,q,n} = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(x)}{x^{n+1}}\ dx,$$

where Γ is a circle whose centre is the origin and whose radius R is less than unity. We take

(5.16) $$R = 1 - \frac{\beta}{n},$$

where β also is positive and independent of n.

Our object is to shew that the integral on the right-hand side of (5.15) is of the form $O(n^{-\frac{1}{4}})$; the constant implied in the O will of course be a function of α and β. It is to be understood throughout that O's are used in this sense; $O(1)$, for instance, stands for *a function of* x, n, p, q, α, *and* β (or some only of these variables) *which is less in absolute value than a number* $K = K(\alpha, \beta)$ *independent of* x, n, p, *and* q.

We divide up the circle Γ by means of dissection Ξ_ν of 4.22, into arcs $\xi_{p,q}$ each associated with a point $Re^{2p\pi i/q}$; and we denote by $\eta_{p,q}$ the arc of Γ complementary to $\xi_{p,q}$. This being so, we have

(5.17) $$
\begin{aligned}
\int_\Gamma \frac{\Phi(x)}{x^{n+1}}\ dx &= \sum\int_{\xi_{p,q}}\frac{f(x) - F_{p,q}(x)}{x^{n+1}}\ dx - \sum\int_{\eta_{p,q}}\frac{F_{p,q}(x)}{x^{n+1}}\ dx \\
&= \sum J_{p,q} - \sum j_{p,q},
\end{aligned}
$$

say. We shall prove that each of these sums is of the form $O(n^{-\frac{1}{4}})$; and we shall begin with the second sum, which only involves the auxiliary functions F.

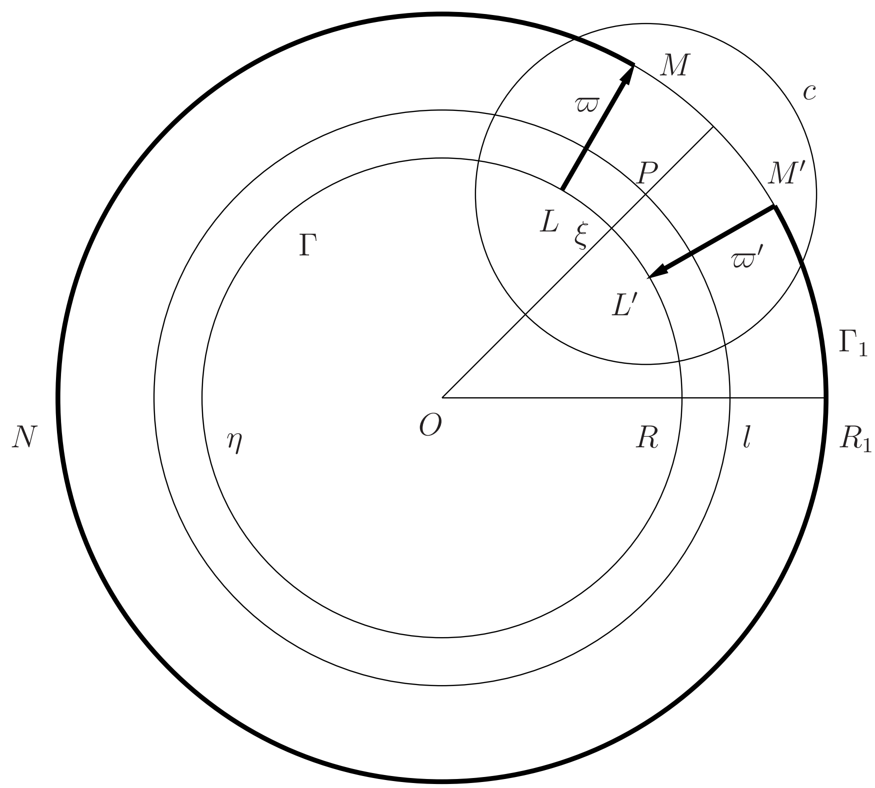
Proof that $\sum j_{p,q} = O(n^{-\frac{1}{4}})$.

5.21. We have, by Cauchy's theorem,

(5.211) $$j_{p,q} = \int_{\eta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\, dx = \int_{\zeta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\, dx,$$

where $\zeta_{p,q}$ consists of the contour $LMNM'L'$ shewn in the figure. Here L and L' are the ends of $\xi_{p,q}$, LM and $M'L'$ are radii vectors, and MNM' is part of a circle Γ_1 whose radius R_1 is greater than 1. P is the point $e^{2p\pi i/q}$; and we suppose that R_1 is small enough to ensure that all points of LM and $M'L'$ are at a distance from P less than $\frac{1}{2}$. The other circle c shewn in the figure has P as its centre and radius $\frac{1}{2}$. We denote LM by $\varpi_{p,q}$, $M'L'$ by $\varpi'_{p,q}$ and MNM' by $\gamma_{p,q}$: and we write

(5.212) $$j_{p,q} = \int_{\zeta_{p,q}} = \int_{\gamma_{p,q}} + \int_{\varpi_{p,q}} + \int_{\varpi'_{p,q}} = j^1_{p,q} + j^2_{p,q} + j^3_{p,q}.$$



The contribution of $\sum j^1_{p,q}$.

5.22. Suppose first that x lies on $\gamma_{p,q}$ and outside c. Then, in virtue of (5.11) and Lemma 4.43, we have

(5.221) $$F_{p,q}(x) = O(\sqrt{q}).$$

If on the other hand x lies on $\gamma_{p,q}$, but inside c, we have, by (5.11) and Lemma 4.44,

(5.222) $$F_{p,q}(x) - \chi_{p,q}(x) = O(q^{-\frac{3}{2}}),$$

where

$$\chi_{p,q}(x) = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}\chi_{C/q}(x_{p,q}). \tag{5.2221}$$

But, if we refer to the definition of $\chi_a(x)$ in Lemma 4.42, and observe that

$$\left|\exp\frac{a^2}{4\log(1/x)}\right| = \exp\frac{a^2\log(1/r)}{4[\{\log(1/r)\}^2+\theta^2]} < 1$$

if $x = re^{i\theta}$ and $r > 1$, we see that

$$\chi_{p,q}(x) = O(\sqrt{q}) \tag{5.223}$$

on the part of $\gamma_{p,q}$ in question. Hence (5.221) holds for all $\gamma_{p,q}$. It follows that

$$j^1_{p,q} = O(R_1^{-n}\sqrt{q}),$$

$$\sum j_{p,q} = O(R_1^{-n}\sum_q q^{\frac{3}{2}}) = O(n^{\frac{5}{4}}R_1^{-n})\ ^* \tag{5.224}$$

This sum tends to zero more rapidly than any power of n, and is therefore completely trivial.

The contributions of $\sum j^2_{p,q}$ *and* $\sum j^3_{p,q}$.

5.231. We must now consider the sums which arise from the integrals along $\varpi_{p,q}$ and $\varpi'_{p,q}$; and it is evident that we need consider in detail only the first of these two lines. We write

$$j^2_{p,q} = \int_{\varpi_{p,q}} \frac{F_{p,q}(x)-\chi_{p,q}(x)}{x^{n+1}}\,dx + \int_{\varpi_{p,q}} \frac{\chi_{p,q}(x)}{x^{n+1}}\,dx = j'_{p,q} + j''_{p,q}, \tag{5.2311}$$

say.

In the first place we have, from (5.222),

$$j'_{p,q} = O\left(q^{-\frac{3}{2}}\int_R^{R_1}\frac{dr}{r^{n+1}}\right) = O(q^{-\frac{3}{2}}n^{-1}),$$

since

$$R^{-n} = \left(1-\frac{\beta}{n}\right)^{-n} = O(1). \tag{5.2312}$$

Thus

$$\sum j'_{p,q} = O\{n^{-1}\sum_{q<O(\sqrt{n})} q^{-\frac{1}{2}}\} = O(n^{-\frac{3}{4}}). \tag{5.2313}$$

5.232. In the second place we have

$$j''_{p,q} = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}\int_{\varpi_{p,q}}\frac{\chi_{C/q}(x_{p,q})}{x^{n+1}}\,dx.$$

---

*Here, and in many passages in our subsequent argument, it is to be remembered that the number of values of p, corresponding to a given q is less than q, and that the number of values of q is of order $\sqrt{n}$. Thus we have generally

$$\sum O(q^s) = O\left(\sum_{q<O(\sqrt{n})} q^{s+1}\right) = O(n^{\frac{1}{2}s+1}).$$

It is plain that, if we substitute y for $xe^{-2p\pi i/q}$, then write x again for y, and finally substitute for $\chi_{C/q}$ its explicit expression as an elementary function, given in Lemma 4.42, we obtain

$$j''_{p,q} = O(\sqrt{q}) \int \{E(x)-1\} \sqrt{\left(\log \frac{1}{x}\right)} x^{-n-\frac{23}{24}} dx = O(\sqrt{q})J, \tag{5.2321}$$

say, where

$$E(x) = \exp\left\{\frac{\pi^2}{6q^2 \log(1/x)}\right\}, \tag{5.23211}$$

and the path of integration is now a line related to $x=1$ as $\varpi_{p,q}$ is to $x = e^{2p\pi i/q}$: the line defined by $x = re^{i\theta}$, where $R \le r \le R_1$, and θ is fixed and (by Lemma 4.22) lies between $1/2q\nu$ and $1/q\nu$.

Integrating J by parts, we find

$$\begin{aligned}(n-\tfrac{1}{24})J &= -\left[\{E(x)-1\}\sqrt{\left(\log\frac{1}{x}\right)} x^{-n+\frac{1}{24}}\right]_{r=R}^{r=R_1} \\ &\quad -\tfrac{1}{2}\int \{E(x)-1\}\left(\log\frac{1}{x}\right)^{-\frac{1}{2}} x^{-n-\frac{23}{24}} dx \\ &\quad +\frac{\pi^2}{6q^2}\int E(x)\left(\log\frac{1}{x}\right)^{-\frac{3}{2}} x^{-n-\frac{23}{24}} dx = J_1+J_2+J_3,\end{aligned} \tag{5.2322}$$

say.

5.233. In estimating J_1, J_2, and J_3, we must bear the following facts in mind.

(1) Since $|x| \ge R$, it follows from (5.2312) that $|x|^{-n} = O(1)$ throughout the range of integration.

(2) Since $1-R = \beta/n$ and $1/2q\nu < \theta < 1/q\nu$, where $\nu = [\alpha\sqrt{n}]$, we have

$$\log\left(\frac{1}{x}\right) = O\left(\frac{1}{q\sqrt{n}}\right),$$

when $r=R$, and

$$\frac{1}{\log(1/x)} = O(q\sqrt{n}),$$

throughout the range of integration.

(3) Since

$$|E(x)| = \exp\frac{\pi^2\log(1/r)}{6q^2[\{\log(1/r)\}^2+\theta^2]},$$

$E(x)$ is less than 1 in absolute value when $r>1$. And, on the part of the path for which $r<1$, it is of the form

$$\exp O\left(\frac{1}{q^2 n\theta^2}\right) = \exp O(1) = O(1).$$

It is accordingly of the form $O(1)$ throughout the range of integration.

5.234. Thus we have, first

$$J_1 = O(1)O(1)O(R_1^{-n}) + O(1)O(q^{-\frac{1}{2}}n^{-\frac{1}{4}})O(1) = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}), \tag{5.2341}$$

secondly

$$J_2 = O(1)O(q^{\frac{1}{2}}n^{\frac{1}{4}})\int_R^{R_1}\frac{dr}{r^{n+\frac{23}{24}}} = O(q^{\frac{1}{2}}n^{-\frac{3}{4}}), \tag{5.2342}$$

and thirdly

$$J_3 = O(q^{-2})O(1)O(q^{\frac{3}{2}}n^{\frac{3}{4}})\int_R^{R_1}\frac{dr}{r^{n+\frac{23}{24}}} = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}). \tag{5.2343}$$

From (5.2341), (5.2342), (5.2343), and (5.2322), we obtain

$$J = O(q^{-\frac{1}{2}}n^{-\frac{5}{4}}) + O(q^{\frac{1}{2}}n^{-\frac{7}{4}});$$

and, from (5.2321), $\qquad j''_{p,q} = O(n^{-\frac{5}{4}}) + O(qn^{-\frac{7}{4}}).$

Summing, we obtain

$$\begin{aligned} \sum j''_{p,q} &= O(n^{-\frac{5}{4}}\sum_{q<O(\sqrt{n})} q) + O(n^{-\frac{7}{4}}\sum_{q<O(\sqrt{n})} q^2) \\ &= O(n^{-\frac{1}{4}}) + O(n^{-\frac{1}{4}}) = O(n^{-\frac{1}{4}}). \end{aligned} \tag{5.2344}$$

5.235. From (5.2311), (5.2313), and (5.2344), we obtain

$$\sum j^2_{p,q} = O(n^{-\frac{1}{4}}); \tag{5.2351}$$

and in exactly the same way we can prove

$$\sum j^3_{p,q} = O(n^{-\frac{1}{4}}). \tag{5.2352}$$

And from (5.212), (5.224), (5.2351), and (5.2352), we obtain, finally,

$$\sum j_{p,q} = O(n^{-\frac{1}{4}}). \tag{5.2353}$$

Proof that $\sum J_{p,q} = O(n^{-\frac{1}{4}})$.

5.31. We turn now to the discussion of

$$\begin{aligned} J_{p,q} &= \int_{\xi_{p,q}} \frac{f(x) - F_{p,q}(x)}{x^{n+1}}\,dx \\ &= \int_{\xi_{p,q}} \frac{f(x) - X_{p,q}(x)}{x^{n+1}}\,dx - \int_{\xi_{p,q}} \frac{F_{p,q}(x) - \chi_{p,q}(x)}{x^{n+1}}\,dx \\ &\quad + \int_{\xi_{p,q}} \frac{\rho_{p,q}(x)}{x^{n+1}}\,dx \\ &= J^1_{p,q} + J^2_{p,q} + J^3_{p,q}, \end{aligned} \tag{5.311}$$

say, where

$$\rho_{p,q}(x) = \omega_{p,q}\sqrt{\left(\frac{q}{2\pi}\log\frac{1}{x_{p,q}}\right)}x_{p,q}^{\frac{1}{24}},$$

$$X_{p,q}(x) = \chi_{p,q}(x) + \rho_{p,q}(x) = \rho_{p,q}(x)E(x_{p,q}),$$

$E(x)$ being defined as in (5.23211).

Discussion of $\sum J_{p,q}^2$ *and* $\sum J_{p,q}^3$.

5.32. The discussion of these two sums is, after the analysis which precedes, a simple matter. The arc $\xi_{p,q}$ is less than a constant multiple of $1/q\sqrt{n}$; and $x^{-n} = O(1)$ on $\xi_{p,q}$. Also

$$|F_{p,q}(x) - \chi_{p,q}(x)| = O(q^{-\frac{3}{2}}),$$

by (5.222); and

(5.321) $$\sqrt{\left(\log\frac{1}{x_{p,q}}\right)} = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}),$$

since $\quad |x_{p,q}| = R = 1 - (\beta/n), \quad |amx_{p,q}| < 1/q\nu.$

Hence

$$J_{p,q}^2 = O(q^{-\frac{5}{2}}n^{-\frac{1}{2}}),$$

(5.322) $$\sum J_{p,q}^2 = O(n^{-\frac{1}{2}}\sum_{q<O(\sqrt{n})} q^{-\frac{3}{2}}) = O(n^{-\frac{1}{2}});$$

and

$$J_{p,q}^3 = O(q^{-1}n^{-\frac{3}{4}}),$$

(5.323) $$\sum J_{p,q}^3 = O(n^{-\frac{3}{4}}\sum_{q<O(\sqrt{n})} 1) = O(n^{-\frac{1}{4}}).$$

Discussion of $\sum J_{p,q}^1$.

5.33. From (4.321) and (5.2221), we have

(5.331) $$f(x) - X_{p,q}(x) = \omega_{p,q}\sqrt{\left(\frac{q}{2\pi}\log\frac{1}{x_{p,q}}\right)}x_{p,q}^{\frac{1}{24}}E(x_{p,q})\Omega(x'_{p,q}),$$

where

$$\Omega(z) = f(z) - 1 = \prod_1^\infty\left(\frac{1}{1-z^\nu}\right) - 1 = \sum_1^\infty p(\nu)z^\nu,$$

if $|z| < 1$ and

$$x'_{p,q} = \exp\left\{-\frac{4\pi^2}{q^2\log(1/x_{p,q})} + \frac{2\pi ip'}{q}\right\}.$$

Now

$$|x'_{p,q}| = \exp\left[-\frac{4\pi^2\log(1/R)}{q^2\{[\log(1/R)]^2 + \theta^2\}}\right],$$

where θ is the amplitude of $x_{p,q}$. Also

$$q^2\{[\log(1/R)]^2+\theta^2\} = O\left\{q^2\left(\frac{1}{n^2}+\frac{1}{q^2 n}\right)\right\} = O\left(\frac{1}{n}\right),$$

while $\log(1/R)$ is greater than a constant multiple of $1/n$. There is therefore a positive number δ, less than unity and independent of n and of q, such that

$$|x'_{p,q}| < \delta;$$

and we may write $\qquad \Omega(x'_{p,q}) = O(|x'_{p,q}|).$

We have therefore

$$E(x_{p,q})\Omega(x'_{p,q}) = O(|x'_{p,q}|^{-\frac{1}{24}})O(|x'_{p,q}|) = O(|x'_{p,q}|^{\frac{23}{24}}) = O(1);$$

and so, by (5.321),

$$f(x) - \chi_{p,q}(x) = O(\sqrt{q})O\left(\sqrt{\left|\log\frac{1}{x_{p,q}}\right|}\right)O(1) = O(n^{-\frac{1}{4}}).$$

And hence, as the length of $\xi_{p,q}$ is of the form $O(1/q\sqrt{n})$, we obtain

$$J^1_{p,q} = O(q^{-1}n^{-\frac{3}{4}}),$$

(5.332) $$\sum J^1_{p,q} = O(n^{-\frac{3}{4}}\sum_{q<O(\sqrt{n})} 1) = O(n^{-\frac{1}{4}}).$$

5.34. From (5.311), (5.322), (5.323), and (5.332), we obtain

(5.341) $$\sum J_{p,q} = O(n^{-\frac{1}{4}}).$$

Completion of the proof.

5.4. From (5.15), (5.17), (5.2353), and (5.341), we obtain

(5.41) $$p(n) - \sum_q \sum_p c_{p,q,n} = O(n^{-\frac{1}{4}}).$$

But

$$\sum_p c_{p,q,n} = \frac{\sqrt{q}}{\pi\sqrt{2}}A_q \frac{d}{dn}\frac{\cosh(C\lambda_n/q)-1}{\lambda_n},$$

where

$$A_q = \sum_p \omega_{p,q}e^{-2np\pi i/q}.$$

All that remains, in order to complete the proof of the theorem, is to shew that

$$\cosh(C\lambda_n/q) - 1$$

may be replaced by $\frac{1}{2}e^{C\lambda_n/q}$;

and in order to prove this it is only necessary to shew that

$$\sum_{q<O(\sqrt{n})} q^{\frac{3}{2}} \frac{d}{dn} \frac{\frac{1}{2}e^{C\lambda_n/q} - \cosh(C\lambda_n/q) + 1}{\lambda_n} = O(n^{-\frac{1}{4}}).$$

On differentiating we find that the sum is of the form

$$\sum_{q<O(\sqrt{n})} q^{\frac{3}{2}} \left\{ O\left(\frac{1}{qn}\right) + O\left(\frac{1}{n^{\frac{3}{2}}}\right) \right\} = O\left\{ \frac{1}{n} \sum_{q<O(\sqrt{n})} q^{\frac{1}{2}} \right\} = O(n^{-\frac{1}{4}}).$$

Thus the theorem is proved.

6. Additional remarks on the theorem.

6.1. The theorem which we have proved gives information about $p(n)$ which is in some ways extraordinarily exact. We are for this reason the more anxious to point out explicitly two respects in which the results of our analysis are incomplete.

6.21. We have proved that

$$p(n) = \sum A_q \phi_q + O(n^{-\frac{1}{4}}),$$

where the summation extends over the values of q specified in the theorem, for every fixed value of α; that is to say that, when α is given, a number $K = K(\alpha)$ can be found such that

$$|p(n) - \sum A_q \phi_q| < K n^{-\frac{1}{4}}$$

for every value of n. It follows that

$$p(n) = \{\sum A_q \phi_q\}, \tag{6.211}$$

where $\{x\}$ denotes the integer nearest to x, for $n \geq n_0$, where $n_0 = n_0(\alpha)$ is a certain function of α.

The question remains whether we can, by an appropriate choice of α, secure the truth of (6.211) for *all* values of n, and not merely for all sufficiently large values. Our opinion is that this is possible, and that it could be proved to be possible without any fundamental change in our analysis. Such a proof would however involve a very careful revision of our argument. It would be necessary to replace all formulæ involving O's by inequalities, containing only numbers expressed explicitly as functions of the various parameters employed. This process would certainly add very considerably to the length and the complexity of our argument. It is, as it stands, sufficient to prove what is, from our point of view, of the greatest interest; and we have not thought it worth while to elaborate it further.

6.22. The second point of incompleteness of our results is of much greater interest and importance. We have not proved either that the series

$$\sum_1^\infty A_q \phi_q$$

is convergent, or that, if it is convergent, it represents $p(n)$. Nor does it seem likely that our method is one intrinsically capable of proving these results, if they are true – a point on which we are not prepared to express any definite opinion.

It should be observed in this connection that we have not even discovered anything definite concerning the order of magnitude of A_q for large values of q. We can prove nothing better than the absolutely trivial equation $A_q = O(q)$. On the other hand we can assert that A_q is, for an infinity of values of q, effectively of an order as great as q, or indeed even that it does not tend to zero (though of course this is most unlikely).

6.3. Our formula directs us, if we wish to obtain the exact value of $p(n)$ for a large value of n, to take a number of terms of order $\sqrt{n}$. The numerical data suggest that a considerably smaller number of terms will be equally effective; and it is easy to see that this conjecture is correct.

Let us write

$$\beta = 4\pi\sqrt{\left(\tfrac{2}{3}\right)} = 4C, \quad \mu = \left[\frac{\beta\sqrt{n}}{\log n}\right],$$

and let us suppose that $\alpha < 2$. Then

$$\begin{aligned}\sum_{\mu+1}^{\nu} A_q\phi_q &= \sum_{\mu+1}^{\nu} O(q^{\frac{3}{2}})O\left(\frac{1}{qn}\right)O(e^{C\sqrt{n}/q}) = O\left(\frac{1}{n}\sum_{\mu+1}^{\nu}\sqrt{q}e^{C\sqrt{n}/q}\right)\\ &= O\left(\frac{1}{n}\int_{\mu}^{\nu}\sqrt{x}e^{C\sqrt{n}/x}\,dx\right),\end{aligned}$$

since $\sqrt{q}e^{C\sqrt{n}/q}$ decreases steadily throughout the range of summation*.

Writing $\sqrt{n}/y$ for x, we obtain

$$\begin{aligned}O\left(n^{-\frac{1}{4}}\int_{1/\alpha}^{\sqrt{n}/\mu} y^{-\frac{5}{2}}e^{Cy}\,dy\right) &= O\left\{n^{-\frac{1}{4}}\left(\frac{\sqrt{n}}{\mu}\right)^{-\frac{5}{2}}e^{C\sqrt{n}/\mu}\right\} = O\{n^{-\frac{1}{4}}(\log n)^{-\frac{5}{2}}e^{\frac{1}{4}\log n}\}\\ &= O(\log n)^{-\frac{5}{2}} = o(1).\end{aligned}$$

It follows that it is enough, when n is sufficiently large, to take

$$\left[\frac{\beta\sqrt{n}}{\log n}\right]$$

terms of the series. It is probably also *necessary* to take a number of terms of order $\sqrt{n}/(\log n)$; but it is not possible to prove this rigorously without a more exact knowledge of the properties of A_q than we possess.

6.4. We add a word on certain simple approximate formulæ for $\log p(n)$ found empirically by Major MacMahon and by ourselves. Major MacMahon found that if

*The minimum occurs when q is about equal to $2C\sqrt{n}$.

(6.41) $$\log_{10} p(n) = \sqrt{(n+4)} - a_n,$$
then a_n is approximately equal to 2 within the limits of his table of values of $p(n)$ (Table IV). This suggested to us that we should endeavour to find more accurate formulæ of the same type. The most striking that we have found is
(6.42) $$\log_{10} p(n) = \tfrac{10}{9}\{\sqrt{(n+10)} - a_n\};$$
the mode of variation of a_n is shewn in Table III.
In this connection it is interesting to observe that the function

$$13^{-\sqrt{n}} p(n)$$

(which ultimately tends to infinity with exponential rapidity) is equal to .973 for $n =$ 30000000000.

7. Further applications of the method.

7.1. We shall conclude with a few remarks concerning actual or possible applications of our method to other problems in Combinatory Analysis or the Analytic Theory of Numbers. The class of problems in which the method gives the most striking results may be defined as follows. Suppose that $q(n)$ is the coefficient of x^n in the expansion of $F(x)$, where $F(x)$ is a function of the form

(7.11) $$\frac{\{f(\pm x^a)\}^{\alpha}\{f(\pm x^{a'})\}^{\alpha'}\cdots}{\{f(\pm x^b)\}^{\beta}\{f(\pm x^{b'})\}^{\beta'}\cdots};\text{*}$$

$f(x)$ being the function considered in this paper, The a's, b's, α's, and β's being positive integers, and the number of factors in numerator and denominator being finite; and suppose that $|F(x)|$ tends exponentially to infinity when x tends in an appropriate manner to some or all the points $e^{2p\pi i/q}$. Then our method may be applied in its full power to the asymptotic study of $q(n)$, and yields results very similar to those which we have found concerning $p(n)$. Thus, if

$$F(x) = \frac{f(x)}{f(x^2)} = (1+x)(1+x^2)(1+x^3)\cdots = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots},$$

so that $q(n)$ is the number of partitions of n into odd parts, or into unequal parts†, we find that

$$\begin{aligned} q(n) \;=\; & \frac{1}{\sqrt{2}}\frac{d}{dn}J_0[i\pi\sqrt{\{\tfrac{1}{3}(n+\tfrac{1}{24})\}}] \\ & +\sqrt{2}\cos(\tfrac{2}{3}n\pi - \tfrac{1}{9}\pi)\frac{d}{dn}J_0[\tfrac{1}{3}i\pi\sqrt{\{\tfrac{1}{3}(n+\tfrac{1}{24})\}}] + \cdots. \end{aligned}$$

*Since

$$f(-x) = \frac{\{f(x^2)\}^3}{f(x)f(x^4)},$$

the arguments with a negative sign may be eliminated if this is desired.

†Cf. MacMahon, *loc. cit*, p. 11. We give at the end of the paper a table (Table V) of the values of $q(n)$ up to $n = 100$. This table was calculated by Mr. Darling.

The error after $[\alpha\sqrt{n}]$ terms is of the form $O(1)$. We are not in a position to assert that the *exact* value of $q(n)$ can always be obtained from the formula (though this is probable); but the error is certainly bounded.

If

$$F(x) = \frac{f(x^2)}{f(-x)} = \frac{f(x)f(x^4)}{\{f(x^2)\}^2} = (1+x)(1+x^3)+x^5)\cdots,$$

so that $q(n)$ is a number of partitions of n into parts which are both odd and unequal, then

$$\begin{aligned} q(n) \quad = \quad & \frac{d}{dn}J_0[i\pi\sqrt{\{\tfrac{1}{6}(n-\tfrac{1}{24})\}}] \\ & +2\cos(\tfrac{2}{3}n\pi - \tfrac{2}{9}\pi)\frac{d}{dn}J_0[\tfrac{1}{3}i\pi\sqrt{\{\tfrac{1}{6}(n-\tfrac{1}{24})\}}] + \cdots. \end{aligned}$$

The error is again bounded (and probably tends to zero).

If

$$F(x) = \frac{\{f(x)\}^2}{f(x^2)} = \frac{1}{1-2x+2x^4-2x^9+\cdots},$$

$q(n)$ has no very simple arithmetical interpretation; but the series is none the less, as the direct reciprocal of simple ϑ-function, of particular interest. In this case we find

$$q(n) = \frac{1}{4\pi}\frac{d}{dn}\frac{e^{\pi\sqrt{n}}}{\sqrt{n}} + \frac{\sqrt{3}}{2\pi}\cos(\tfrac{2}{3}n\pi - \tfrac{1}{6}\pi)\frac{d}{dn}\frac{e^{\frac{1}{3}\pi\sqrt{n}}}{\sqrt{n}} + \cdots.$$

The error here is (as in the partition problem) of order $O(n^{-\frac{1}{4}})$, and the exact value can always be found from the formula.

7.2. The method also be applied to product of form (7.11) which have (to put the matter roughly) no exponential infinities. In such cases the approximation is of much less exact character. On the other hand the problems of this character are of even greater arithmetical interest.

The standard problem of this category is that of the representation of a number as a sum of s squares, s being any positive integer odd or even*. We must reserve the application of our method to this problem for another occasion; but we can indicate the character of our main result as follows.

If $r_s(n)$ is the number of representations of n as the sum of s squares we have

$$F(x) = \sum r_s(n)x^n = (1+2x+2x^4+\cdots)^s = \frac{\{f(x^2)\}^s}{\{f(-x)\}^{2s}} = \frac{\{f(x)\}^{2s}\{f(x^4)\}^{2s}}{\{f(x^2)\}^{5s}}.$$

We find that

$$r_s(n) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}n^{\frac{1}{2}s-1}\sum\frac{c_q}{q^{\frac{1}{2}s}} + O(n^{\frac{1}{4}s}), \tag{7.21}$$

*As is well known, the arithmetical difficulties of the problem are much greater when s is odd.

where c_q is a function of q and of n of the same general type as the function A_q of this paper. The series

$$\sum \frac{c_q}{q^{\frac{1}{2}s}} \tag{7.22}$$

is absolutely convergent for sufficiently large values of s, and the summation in (7.21) may be regarded indifferently as extended over all values of q or only over a range $1 \leq q \leq \alpha\sqrt{n}$. It should be observed that the series (7.22) is quite different in form from any of the infinite series which are already known to occur in connection with this problem.

7.3. There is also a wide range of problems to which our methods are *partly* applicable. Suppose, for example, that

$$F(x) = \sum p^2(n)x^n = \frac{1}{(1-x)(1-x^4)(1-x^9)\cdots},$$

so that $p^2(n)$ is the number of partitions of n into *squares*. Then $F(x)$ is not an elliptic modular function; it possesses no general transformation theory: and the full force of our method can not be applied. We can still, however, apply some of our preliminary methods. Thus the "Tauberian" argument shews that

$$\log p^2(n) \sim 2^{-\frac{4}{3}} 3\pi^{\frac{1}{3}} \{\zeta(\tfrac{3}{2})\}^{\frac{2}{3}} n^{\frac{1}{3}}.$$

And although there is no general transformation theory, there is a formula which enables us to specify the nature of the singularity at $x = 1$. This formula is

$$\begin{aligned} \frac{1}{f(e^{-\pi z})} &= 2\sqrt{\left(\frac{\pi}{z}\right)} \exp\left\{\frac{2\pi}{\sqrt{z}}\zeta(-\tfrac{1}{2})\right\} \\ &\qquad \times \prod_1^\infty \{1 - 2e^{-2\pi\sqrt{(n/z)}} \cos 2\pi\sqrt{(n/z)} + e^{-4\pi\sqrt{(n/z)}}\}. \end{aligned}$$

By the use of this formula, in conjunction with Cauchy's theorem, it is certainly possible to obtain much more precise information about $p^2(n)$ and in particular the formula

$$p^2(n) \sim 3^{-\frac{1}{2}} (4\pi n)^{-\frac{7}{6}} \{\zeta(\tfrac{3}{2})\}^{\frac{2}{3}} e^{2^{-(4/3)} 3\pi^{(1/3)} \{\zeta(\frac{3}{2})\}^{\frac{2}{3}} n^{\frac{1}{3}}}.$$

The corresponding formula for $p^s(n)$, the number of partitions of n into perfect s-th powers, is

$$p^s(n) \sim (2\pi)^{-\frac{1}{2}(s+1)} \sqrt{\left(\frac{s}{s+1}\right)} k n^{\frac{1}{s+1} - \frac{3}{2}} e^{(s+1)kn^{1/(s+1)}},$$

where

$$k = \left\{\frac{1}{s}\Gamma\left(1 + \frac{1}{s}\right)\zeta\left(1 + \frac{1}{s}\right)\right\}^{\frac{s}{s+1}}.$$

The series (7.21) may be written in the form

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1} \sum_{p,q} \frac{\omega_{p,q}^{s}}{q^{\frac{1}{2}s}} e^{-np\pi i/q},$$

where $\omega_{p,q}$ is always one of the five numbers 0, $e^{\frac{1}{4}\pi i}$, $e^{-\frac{1}{4}\pi i}$, $-e^{\frac{1}{4}\pi i}$, $-e^{-\frac{1}{4}\pi i}$. When s is even it begins

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1} \{1^{-\frac{1}{2}s} + 2\cos(\tfrac{1}{2}n\pi - \tfrac{1}{4}s\pi)2^{-\frac{1}{2}s} + 2\cos(\tfrac{2}{3}n\pi - \tfrac{1}{2}s\pi)3^{-\frac{1}{2}s} + \cdots\}.$$

It has been proved by Ramanujan that the series gives an *exact* representation of $r_s(n)$ when $s = 4, 6, 8$; and by Hardy that this also true when $s = 3, 5, 7$. See Ramanujan, "On certain trigonometrical sums and their applications in the Theory of Numbers"; Hardy, "On the expression of a number as the sum of any number of squares, and in particular of five or seven*".

*[Ramanujan's paper referred to is No. 21 of this volume. That of Hardy was published, in the first instance, in *Proc. National Acad. of Sciences*, (Washington), Vol. IV, 1918, pp. 189 – 193, and later (in fuller form and with a slightly different title) in *Trans. American Math. Soc.*, Vol. XXI, 1920, pp. 255 – 284.]

Table I: $\omega_{p,q}$.

p	q	$\log \omega_{p,q}/\pi i$	p	q	$\log \omega_{p,q}/\pi i$	p	q	$\log \omega_{p,q}/\pi i$
1	1	0	3	11	3/22	8	15	7/18
1	2	0	4	11	3/22	11	15	-19/90
1	3	1/18	5	11	−5/22	13	15	-7/18
2	3	−1/18	6	11	5/22	14	15	-1/90
1	4	1/8	7	11	−3/22	1	16	-29/32
3	4	−1/8	8	11	−3/22	3	16	-27/32
1	5	1/5	9	11	−5/22	5	16	-5/32
2	5	0	10	11	−15/22	7	16	-3/32
3	5	0	1	12	55/72	9	16	3/32
4	5	−1/5	5	12	−1/72	11	16	5/32
1	6	5/18	7	12	1/72	13	16	27/32
5	6	−5/18	11	12	−55/72	15	16	29/32
1	7	5/14	1	13	11/13	1	17	-14/17
2	7	1/14	2	13	4/13	2	17	8/17
3	7	−1/14	3	13	1/13	3	17	5/17
4	7	1/14	4	13	−1/13	4	17	0
5	7	−1/14	5	13	0	5	17	1/17
6	7	−5/14	6	13	−4/13	6	17	5/17
1	8	7/16	7	13	4/13	7	17	1/17
3	8	1/16	8	13	0	8	17	-8/17
5	8	−1/16	9	13	1/13	9	17	8/17
7	8	−7/16	10	13	−1/13	10	17	-1/17
1	9	14/27	11	13	−4/13	11	17	-5/17
2	9	4/27	12	13	−11/13	12	17	-1/17
4	9	−4/27	1	14	13/14	13	17	0
5	9	4/27	3	14	3/14	14	17	-5/17
7	9	−4/27	5	14	3/14	15	17	-8/17
8	9	−14/27	9	14	−3/14	16	17	14/17
1	10	3/5	11	14	−3/14	1	18	-20/27
3	10	0	13	14	−13/14	5	18	2/27
7	10	0	1	15	1/90	7	18	-2/27
9	10	−3/5	2	15	7/18	11	18	2/27
1	11	15/22	4	15	19/90	13	18	-2/27
2	11	5/22	7	15	−7/18	17	18	20/27

Table II: A_q.

$$
\begin{aligned}
A_1 &= 1.\\
A_2 &= \cos n\pi.\\
A_3 &= 2\cos(\tfrac{2}{3}n\pi - \tfrac{1}{18}\pi).\\
A_4 &= 2\cos(\tfrac{1}{2}n\pi - \tfrac{1}{8}\pi).\\
A_5 &= 2\cos(\tfrac{2}{5}n\pi - \tfrac{1}{5}\pi) + 2\cos\tfrac{4}{5}n\pi.\\
A_6 &= 2\cos(\tfrac{1}{3}n\pi - \tfrac{5}{18}\pi).\\
A_7 &= 2\cos(\tfrac{2}{7}n\pi - \tfrac{5}{14}\pi) + 2\cos(\tfrac{4}{7}n\pi - \tfrac{1}{14}\pi) + 2\cos(\tfrac{6}{7}n\pi + \tfrac{1}{14}\pi).\\
A_8 &= 2\cos(\tfrac{1}{4}n\pi - \tfrac{7}{16}\pi) + 2\cos(\tfrac{3}{4}n\pi - \tfrac{1}{16}\pi).\\
A_9 &= 2\cos(\tfrac{2}{9}n\pi - \tfrac{14}{27}\pi) + 2\cos(\tfrac{4}{9}n\pi - \tfrac{4}{27}\pi) + 2\cos(\tfrac{8}{9}n\pi + \tfrac{4}{27}\pi).\\
A_{10} &= 2\cos(\tfrac{1}{5}n\pi - \tfrac{3}{5}\pi) + 2\cos\tfrac{3}{5}n\pi.\\
A_{11} &= 2\cos(\tfrac{2}{11}n\pi - \tfrac{15}{22}\pi) + 2\cos(\tfrac{4}{11}n\pi - \tfrac{5}{22}\pi) + 2\cos(\tfrac{6}{11}n\pi - \tfrac{3}{22}\pi)\\
&\qquad + 2\cos(\tfrac{8}{11}n\pi - \tfrac{3}{22}\pi) + 2\cos(\tfrac{10}{11}n\pi + \tfrac{5}{22}\pi).\\
A_{12} &= 2\cos(\tfrac{1}{6}n\pi - \tfrac{55}{72}\pi) + 2\cos(\tfrac{5}{6}n\pi + \tfrac{1}{72}\pi).\\
A_{13} &= 2\cos(\tfrac{2}{13}n\pi - \tfrac{11}{13}\pi) + 2\cos(\tfrac{4}{13}n\pi - \tfrac{4}{13}\pi) + 2\cos(\tfrac{6}{13}n\pi - \tfrac{1}{13}\pi)\\
&\qquad + 2\cos(\tfrac{8}{13}n\pi + \tfrac{1}{13}\pi) + 2\cos\tfrac{10}{13}n\pi + 2\cos(\tfrac{12}{13}n\pi + \tfrac{4}{15}\pi).\\
A_{14} &= 2\cos(\tfrac{1}{7}n\pi - \tfrac{13}{14}\pi) + 2\cos(\tfrac{3}{7}n\pi - \tfrac{3}{14}\pi) + 2\cos(\tfrac{5}{7}n\pi - \tfrac{3}{14}\pi).\\
A_{15} &= 2\cos(\tfrac{2}{15}n\pi - \tfrac{1}{90}\pi) + 2\cos(\tfrac{4}{15}n\pi - \tfrac{7}{18}\pi) + 2\cos(\tfrac{8}{15}n\pi - \tfrac{19}{90}\pi) + 2\cos(\tfrac{14}{15}n\pi + \tfrac{7}{18}\pi).\\
A_{16} &= 2\cos(\tfrac{1}{8}n\pi + \tfrac{29}{32}\pi) + 2\cos(\tfrac{3}{8}n\pi + \tfrac{27}{32}\pi) + 2\cos(\tfrac{5}{8}n\pi + \tfrac{5}{32}\pi) + 2\cos(\tfrac{7}{8}n\pi + \tfrac{3}{32}\pi).\\
A_{17} &= 2\cos(\tfrac{2}{17}n\pi + \tfrac{14}{17}\pi) + 2\cos(\tfrac{4}{17}n\pi - \tfrac{8}{17}\pi) + 2\cos(\tfrac{6}{17}n\pi - \tfrac{5}{17}\pi) + 2\cos\tfrac{8}{17}n\pi\\
&\quad + 2\cos(\tfrac{10}{17}n\pi - \tfrac{1}{17}\pi) + 2\cos(\tfrac{12}{17}n\pi - \tfrac{5}{17}\pi) + 2\cos(\tfrac{14}{17}n\pi - \tfrac{1}{17}\pi) + 2\cos(\tfrac{16}{17}n\pi + \tfrac{8}{17}\pi).\\
A_{18} &= 2\cos(\tfrac{1}{9}n\pi + \tfrac{20}{27}\pi) + 2\cos(\tfrac{5}{9}n\pi - \tfrac{2}{27}\pi) + 2\cos(\tfrac{7}{9}n\pi + \tfrac{2}{27}\pi).
\end{aligned}
$$

It may be observed that

$$
\begin{aligned}
&A_5 = 0\ (n \equiv 1,2 \pmod 5), && A_7 = 0\ (n \equiv 1,3,4 \pmod 7),\\
&A_{10} = 0\ (n \equiv 1,2 \pmod 5), && A_{11} = 0\ (n \equiv 1,2,3,5,7 \pmod{11}),\\
&A_{13} = 0\ (n \equiv 2,3,5,7,9,10 \pmod{13}), && A_{14} = 0\ (n \equiv 1,3,4 \pmod 7),\\
&A_{16} = 0\ (n \equiv 0 \pmod 2), && A_{17} = 0\ (n \equiv 1,3,4,6,7,9,13,14 \pmod{17});
\end{aligned}
$$

while $A_1, A_2, A_3, A_4, A_6, A_8, A_9, A_{12}, A_{15}$ and A_{18} never vanish.

Table III: $\log_{10} p(n) = \frac{10}{9}\{\sqrt{(n+10)} - a_n\}$.

n	a_n	n	a_n
1	3.317	10000	4.148
3	3.176	30000	4.364
10	3.011	100000	4.448
30	2.951	300000	4.267
100	3.036	1000000	3.554
300	3.237	3000000	2.072
1000	3.537	10000000	-1.188
3000	3.838	30000000	-6.796
		∞	$-\infty$

Table IV*: $p(n)$.

1	1	21	792	41	44583	61	1121505
2	2	22	1002	42	53174	62	1300156
3	3	23	1255	43	63261	63	1505499
4	5	24	1575	44	75175	64	1741630
5	7	25	1958	45	89134	65	2012558
6	11	26	2436	46	105558	66	2323520
7	15	27	3010	47	124754	67	2679689
8	22	28	3718	48	147273	68	3087735
9	30	29	4565	49	173525	69	3554345
10	42	30	5604	50	204226	70	4087968
11	56	31	6842	51	239943	71	4697205
12	77	32	8349	52	281589	72	5392783
13	101	33	10143	53	329931	73	6185689
14	135	34	12310	54	386155	74	7089500
15	176	35	14883	55	451276	75	8118264
16	231	36	17977	56	526823	76	9289091
17	297	37	21637	57	614154	77	10619863
18	385	38	26015	58	715220	78	12132164
19	490	39	31185	59	831820	79	13848650
20	627	40	37338	60	966467	80	15796476

. . . contd.

*The numbers in this table were calculated by Major MacMahon, by means of the recurrence formulæ obtained by equating the coefficients in the identity

$$(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots)\sum_0^\infty p(n)x^n = 1.$$

We have verified the table by direct calculation up to $n = 158$. Our calculation of $p(200)$ from the asymptotic formula then seemed to render further verification unnecessary.

Table IV (Contd.)

81	18004327	111	679903203	141	16670689208	171	301384802048
82	20506255	112	761002156	142	18440293320	172	330495499613
83	23338469	113	851376628	143	20390982757	173	362326859895
84	26543660	114	952050665	144	22540654445	174	397125074750
85	30167357	115	1064144451	145	24908858009	175	435157697830
86	34262962	116	1188908248	146	27517052599	176	476715857290
87	38887673	117	1327710076	147	30388671978	177	522115831195
88	44108109	118	1482074143	148	33549419497	178	571701605655
89	49995925	119	1653668665	149	37027355200	179	625846753120
90	56634173	120	1844349560	150	40853235313	180	684957390936
91	64112359	121	2056148051	151	45060624582	181	749474411781
92	72533807	122	2291320912	152	49686288421	182	819876908323
93	82010177	123	2552338241	153	54770336324	183	896684817527
94	92669720	124	2841940500	154	60356673280	184	980462880430
95	104651419	125	3163127352	155	66493182097	185	1071823774337
96	118114304	126	3519222692	156	73232243759	186	1171432692373
97	133230930	127	3913864295	157	80630964769	187	1280011042268
98	150198136	128	4351078600	158	88751778802	188	1398341745571
99	169229875	129	4835271870	159	97662728555	189	1527273599625
100	190569292	130	5371315400	160	107438159466	190	1667727404093
101	214481126	131	5964539504	161	118159068427	191	1820701100652
102	241265379	132	6620830889	162	129913904637	192	1987276856363
103	271248950	133	7346629512	163	142798995930	193	2168627105469
104	304801365	134	8149040695	164	156919475295	194	2366022741845
105	342325709	135	9035836076	165	172389800255	195	2580840212973
106	384276336	136	10015581680	166	189334822579	196	2814570987591
107	431149389	137	11097645016	167	207890420102	197	3068829878530
108	483502844	138	12292341831	168	228204732751	198	3345365983698
109	541946240	139	13610949895	169	250438925115	199	3646072432125
110	607163746	140	15065878135	170	274768617130	200	3972999029388

Table V *: $q(n)$.

n	c_n	n	c_n	n	c_n	n	c_n
1	1	26	165	51	4097	76	53250
2	1	27	192	52	4582	77	58499
3	2	28	222	53	5120	78	64234
4	2	29	256	54	5718	79	70488
5	3	30	296	55	6378	80	77312
6	4	31	340	56	7108	81	84756
7	5	32	390	57	7917	82	92864
8	6	33	448	58	8808	83	101698
9	8	34	512	59	9792	84	111322
10	10	35	585	60	10880	85	121792
11	12	36	668	61	12076	86	133184
12	15	37	760	62	13394	87	145578
13	18	38	864	63	14848	88	159046
14	22	39	982	64	16444	89	173682
15	27	40	1113	65	18200	90	189586
16	32	41	1260	66	20132	91	206848
17	38	42	1426	67	22250	92	225585
18	46	43	1610	68	24576	93	245920
19	54	44	1816	69	27130	94	267968
20	64	45	2048	70	29927	95	291874
21	76	46	2304	71	32992	96	317788
22	89	47	2590	72	36352	97	345856
23	104	48	2910	73	40026	98	376256
24	122	49	3264	74	44046	99	409174
25	142	50	3658	75	48446	100	444793

*We are indebted to Mr. Darling for this table.

On the coefficients in the expansions of certain modular functions

Proceedings of the Royal Society, A, XCV, 1919, 144 – 155

1. A very large proportion of the most interesting arithmetical functions – of the functions, for example, which occur in the theory of partitions, the theory of the divisors of numbers, or the theory of the representation of numbers by sums of squares – occur as the coefficients in the expansions of elliptic modular functions in powers of the variable $q = e^{\pi i \tau}$. All of these functions have a restricted region of existence, the unit circle $|q| = 1$ being a "natural boundary" or line of essential singularities. The most important of them, such as the functions*

$$(\omega_1/\pi)^{12}\Delta = q^2\{(1-q^2)(1-q^4)\cdots\}^{24}, \tag{1.1}$$

$$\vartheta_3(0) = 1 + 2q + 2q^4 + 2q^9 + \cdots, \tag{1.2}$$

$$12\left(\frac{\omega_1}{\pi}\right)^4 g_2 = 1 + 240\left(\frac{1^3q^2}{1-q^2} + \frac{2^3q^4}{1-q^4} + \cdots\right), \tag{1.3}$$

$$216\left(\frac{\omega_1}{\pi}\right)^6 g_3 = 1 - 504\left(\frac{1^5q^2}{1-q^2} + \frac{2^5q^4}{1-q^4} + \cdots\right), \tag{1.4}$$

are regular inside the unit circle; and many, such as the functions (1.1) and (1.2), have the additional property of having no zeros inside the circle, so that their reciprocals are also regular.

In a series of recent papers† we have applied a new method to the study of these arithmetical functions. Our aim has been to express them as series which exhibit explicitly their order of magnitude, and the genesis of their irregular variations as n increases. We find, for example, for $p(n)$, the number of unrestricted partitions of n, and for $r_s(n)$, the number of representations of n as the sum of an even number s of squares, the series

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*We follow, in general, the notation of Tannery and Molk's *Éléments de la théorie des fonctions elliptiques.* Tannery and Molk, however, write $16G$ in place of the more usual Δ.

†(1) G. H. Hardy and S. Ramanujan, "Une formule asymptotique pour le nombre des partitions de n," *Comptes Rendus*, January 2, 1917 [No. 31 of this volume]; (2) G. H. Hardy and S. Ramanujan, "Asymptotic Formulæ in Combinatory Analysis," *Proc. London Math.Soc.*, Ser. 2, Vol. XVII, 1918, pp. 75 – 115 [No. 36 of this volume]; (3) S. Ramanujan, "On Certain Trigonometrical Sums and their Applications in the Theory of Numbers," *Trans. Camb. Phil. Soc.*, Vol.XXII, 1918, pp. 259 – 276 [No. 21 of this volume]; (4) G. H. Hardy, "On the Expression of a Number as the Sum of any Number of Squares, and in particular of Five or Seven," *Proc. National Acad. of Sciences*, Vol.IV, 1918, pp. 189 – 193: [and G. H. Hardy, "On the expression of a number as the sum of any number of squares, and in particular of five," *Trans. American Math. Soc.*, Vol.XXI, 1920, pp. 255 – 284].

$$(1.5)\qquad \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right) + \frac{(-1)^n}{2\pi}\frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n/2}}{\lambda_n}\right) + \pi\sqrt{\left(\frac{3}{2}\right)}\cos\left(\frac{2}{3}n\pi - \frac{1}{18}\pi\right)\frac{d}{dn}\left(\frac{e^{C\lambda_n/3}}{\lambda_n}\right) + \cdots,$$

where $\lambda_n = \sqrt{\left(n - \frac{1}{24}\right)}$ and $C = \pi\sqrt{\left(\frac{2}{3}\right)}$, and

$$(1.6)\qquad \frac{\pi^{s/2}}{\Gamma(s/2)} n^{\frac{s}{2}-1}\{1^{\frac{-s}{2}} + 2\cos(\frac{1}{2}n\pi - \frac{1}{4}s\pi)2^{\frac{-s}{2}} + 2\cos(\frac{2}{3}n\pi - \frac{1}{2}s\pi)3^{\frac{-s}{2}} + \cdots\};$$

and our methods enable us to write down similar formulæ for a very large variety of other arithmetical functions.

The study of series such as (1.5) and (1.6) raises a number of interesting problems, some of which appear to be exceedingly difficult. The first purpose for which they are intended is that of obtaining approximations to the functions with which they are associated. Sometimes they give also an exact representation of the functions, and sometimes they do not. Thus the sum of the series (1.6) is equal to $r_s(n)$if s is 4, 6, or 8, but not in any other case. The series (1.5) enables us, by stopping after an appropriate number of terms, to find approximations to $p(n)$ of quite startling accuracy; thus six terms of the series give $p(200) = 3972999029388$, a number of 13 figures, with an error of 0.004. But we have never been able to prove that the sum of the series is $p(n)$ exactly, nor even that it is convergent.

There is one class of series, of the same general character as (1.5) or (1.6), which lends itself to comparatively simple treatment. These series arise when the generating modular function $f(q)$ of $\phi(\tau)$ satisfies an equation

$$\phi(\tau) = (a + b\tau)^n \phi\left(\frac{c + d\tau}{a + b\tau}\right),$$

where n is a positive integer, and behaves, inside the unit circle, like a rational function; that is to say, possesses no singularities but poles. The simplest examples of such functions are the reciprocals of the functions (1.3) and (1.4). The coefficients in their expansions are integral, but possess otherwise no particular arithmetical interest. The results, however, are very remarkable from the point of view of approximation; and it is in any case, well worthwhile, in view of the many arithmetical applications of this type of series, to study in detail any example in which the results can be obtained by comparatively simple analysis.

We begin by proving a general theorem (Theorem 1) concerning the expression of a modular function with poles as a series of partial fractions. This series is (as appears in Theorem 2) a "Poincaré's series": what our theorem asserts is, in effect, that the sum of a certain Poincaré's series is the only function which satisfies certain conditions. It would, no doubt, be possible to obtain this result as a corollary from propositions in the general theory of automorphic functions; but we thought it best to give an independent proof, which is interesting in itself and demands no knowledge of this theory.

2. Theorem 1. *Suppose that*

$$f(q) = f(e^{\pi i \tau}) = \phi(\tau) \tag{2.1}$$

is regular for $q = 0$, has no singularities save poles within the unit circle, and satisfies the functional equation

$$\phi(\tau) = (a + b\tau)^n \phi\left(\frac{c + d\tau}{a + b\tau}\right) = (a + b\tau)^n \phi(T), \tag{2.2}$$

n being a positive integer and, a, b, c, d any integers such that $ad - bc = 1$. Then

$$f(q) = \Sigma R, \tag{2.3}$$

where R is a residue of $f(x)/(q - x)$ at a pole of $f(x)$, if $|q| < 1$; while if $|q| > 1$ the sum of the series on the right hand side of (2.3) *is zero.*

The proof requires certain geometrical preliminaries.

3. The half-plane $\mathbf{I}(\tau) > 0$, which corresponds to the inside of the unit circle in the plane of q, is divided up, by the substitutions of the modular group, into a series of triangles whose sides are arcs of circles and whose angles are $\pi/3, \pi/3$, and 0*. One of these, which is called the *fundamental polygon* (P)†, has its vertices at the points ρ, ρ^2, and $i\infty$, where $\rho = e^{\pi i/3}$, and its sides are parts of the unit circle $|\tau| = 1$ and the lines $\mathbf{R}(\tau) = \pm\frac{1}{2}$.

Suppose that F_m is the "Farey's series" of order m, that is to say the aggregate of the rational fractions between 0 and 1, whose denominators are not greater than m, arranged in order of magnitude‡, and that h'/k' and h/k, where $0 < h'/k' < h/k < 1$, are two adjacent terms in the series. We shall consider what regions in the τ-plane correspond to P in the T-plane, when

$$T = -\frac{h' - k'\tau}{h - k\tau}, \tag{3.1}$$

$$T = \frac{h - k\tau}{h' - k'\tau}. \tag{3.2}$$

Both of these substitutions belong to the modular group, since $hk' - h'k = 1$. The points $i\infty, 1/2, -1/2$, in the T-plane correspond to $h/k, (h + 2h')/(k + 2k'), (h - 2h')/(k - 2k')$ in the τ-plane. Thus the lines $\mathbf{R}(T) = \frac{1}{2}, \mathbf{R}(T) = -\frac{1}{2}$ correspond to semicircles described on the segments

$$\left(\frac{h}{k}, \frac{h + 2h'}{k + 2k'}\right), \left(\frac{h}{k}, \frac{h - 2h'}{k - 2k'}\right)$$

respectively as diameters. Similarly the upper half of the unit circle corresponds to a semicircle on the segment

$$\left(\frac{h + h'}{k + k'}, \frac{h - h'}{k - k'}\right).$$

*It is for many purposes necessary to divide each triangle into two, whose angles are $\pi/2, \pi/3$, and 0; but this further subdivision is not required for our present purpose. For the detailed theory of the modular group, see Klien-Fricke, *Vorlesungen über die Theorie der Elliptischen Modulfunktionen*, 1890-1892.

†See Fig. 1.

‡The first and last terms are 0/1 and 1/1. A brief account of the properties of Farey's series is given in §4.2 of our paper (2)[pp. 355 – 356 of this volume].

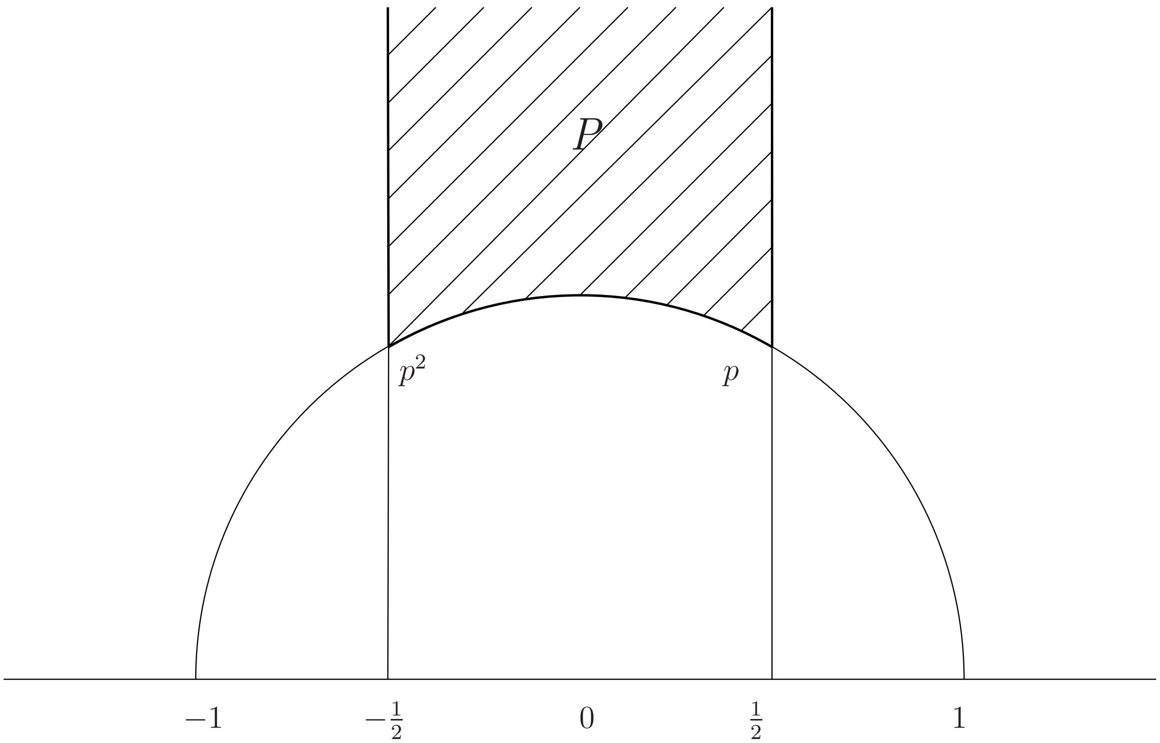

Fig 1

The polygon P corresponds to the region bounded by these three semicircles. In particular, the right hand edge of P corresponds to a circular arc stretching from h/k (where it cuts the real axis at right angles) to the point

$$\frac{h'k' + hk + \frac{1}{2}(hk' + h'k) + \frac{1}{2}i\sqrt{3}}{k^2 + kk' + k'^2} \tag{3.3}$$

corresponding to $\tau = \rho$.
Similarly we find that the substitution (3.2) correlates to P a triangle bounded by semicircles on the segments

$$\left(\frac{h'}{k'}, \frac{h' - 2h}{k' - 2k}\right), \left(\frac{h'}{k'}, \frac{h' + 2h}{k' + 2k}\right), \left(\frac{h' - h}{k' - k}, \frac{h' + h}{k' + k}\right).$$

In particular, the left hand edge of P corresponds to a circular arc from h'/k' to the point (3.3). These two arcs, meeting at the point (3.3), form a continuous path ω, connecting h/k and h'/k', every point of which corresponds, in virtue of one or other of the substitutions (3.1) and (3.2), to a point on one of the rectilinear boundaries of P*.
Performing a similar construction for every pair of adjacent fractions of F_m, we obtain a continuous path from $\tau = 0$ to $\tau = 1$. This path, and its reflexion in the imaginary axis,

*Fig. 2 illustrates the case in which $h/k = \frac{3}{5}, h'/k' = \frac{1}{2}$. These fractions are adjacent in F_5 and F_6, but not in F_7.

give a continuous path from $\tau = -1$ to $\tau = 1$, which we shall denote by Ω_m. To Ω_m corresponds a path in the q-plane, which we call H_m; H_m is a closed path, formed entirely by arcs of circles which cut the unit circle at right angles.

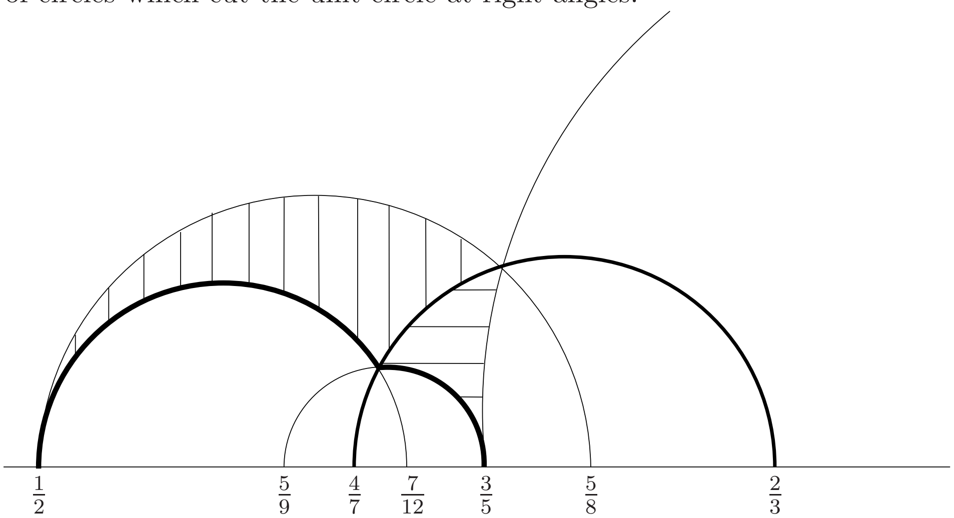

Fig 2

The region shaded horizontally corresponds to P for the substitution (3.1), that shaded vertically for the substitution (3.2). The thickest lines shew the path ω; the line of medium thickness shews the semicircle which corresponds (for either substitution) to the unit semicircle in the plane of T. The large incomplete semicircle passes through $\tau = 1$.

Since

$$\frac{h'}{k'} < \frac{h'+2h}{k'+2k}, \frac{h+2h'}{k+2k'} < \frac{h}{k},$$

the path ω from h'/k' to h/k is always passing from left to right, and its length is less than twice that of the semicircle on $(h'/k', h/k)$, i.e., than π/kk'. The total length of Ω_m is less than 2π; and, since

$$\left|\frac{dq}{d\tau}\right| = \left|\pi i e^{\pi i \tau}\right| \le \pi,$$

the length of H_m is less than $2\pi^2$. Finally, we observe that the maximum distance of Ω_m from the real axis is less than half the maximum distance between two adjacent terms of F_m, and so less than $1/2m$*. Hence Ω_m tends uniformly to the real axis, and H_m to the unit circle, when $m \longrightarrow \infty$.

4. The function $\phi(\tau)$ can have but a finite number of poles in P; we suppose, for simplicity, that none of them lie on the boundary. There is then a constant K such that $|f(q)| < K$ on the boundary of P.

*See Lemma 4.22 of our paper (2) [p. 356 of this volume].

We now consider the integral

$$\frac{1}{2\pi i}\int \frac{f(x)}{x-q}\,dx, \tag{4.1}$$

where $|q| < 1$ and the contour of integration is H_m*. By Cauchy's Theorem, the integral is equal to

$$f(q) - \Sigma R,$$

where R is a residue of $f(x)/(q-x)$ at a pole of $f(x)$ inside H_m†. To prove our theorem, then, we have merely to shew that the integral (4.1) tends to zero when $m \longrightarrow \infty$.

Let ω_1' and ω_1 be the left- and right-hand parts of ω, and ζ_1', ζ_1 and ζ the corresponding arcs of H_m. The length of ω_1 is, as we have seen, less than $\frac{1}{2}\pi/kk'$, and that of ζ_1 than $\frac{1}{2}\pi^2/kk'$. Further, we have, on ζ_1,

$$|f(x)| = |\phi(\tau)| = |h - k\tau|^n |\phi(T)| < K\left\{k\left(\frac{h}{k} - \frac{h'}{k'}\right)\right\}^n = \frac{K}{k'^n}.$$

Thus the contribution of ζ_1 to the integral is numerically less than $C/(kk'^{n+1})$, where C is independent of m; and the whole integral (4.1) is numerically less than

$$2C\Sigma\frac{1}{kk'}\left(\frac{1}{k^n} + \frac{1}{k'^n}\right), \tag{4.2}$$

where the summation extends to all pairs of adjacent terms of F_m.

When ν is fixed and $m > \nu$, the number of terms of F_m whose denominators are less than ν is a function of ν only, say $N(\nu)$. If h/k is one of these, and h'/k' is adjacent to it, $k + k' > m$‡, and so $k' > m - \nu$. Thus the terms of (4.2) in which either k or k' is less than ν contribute less than $8CN(\nu)/(m-\nu)$. The remaining terms contribute less than

$$\frac{4C}{\nu^n}\sum\frac{1}{kk'} = \frac{4C}{\nu^n}.$$

Hence the sum (4.2) is less than

$$\frac{8CN(\nu)}{m-\nu} + \frac{4C}{\nu^n},$$

and it is plain that, by choice of first ν and then m, this may be made as small as we please. Thus (4.1) tends to zero and the theorem is proved. It should be observed that ΣR must, for the present at any rate, be interpreted as meaning the limit of the sum of terms corresponding to poles inside H_m; we have not established the absolute convergence of the series.

We supposed that no pole of $\phi(\tau)$ lies on the boundary of P. This restriction, however, is in no way essential; if it is not satisfied, we have only to select our "fundamental polygon" somewhat differently. The theorem is consequently true independently of any such restriction.

*Strictly speaking, $f(x)$ is not defined at the points where H_m meets the unit circle, and we should integrate round a path just inside H_m and proceed to the limit. The point is trivial, as $f(x)$, in virtue of the functional equation, tends to zero when we approach a cusp of H_m from inside.

†We suppose m large enough to ensure that $x = q$ lies inside H_m.

‡See our paper (2), *loc. cit.*, [p. 356]

So far we have supposed $|q| < 1$. It is plain that, if $|q| > 1$, the same reasoning proves that

$$\Sigma R = 0. \tag{4.3}$$

5. Suppose in particular that $\phi(\tau)$ has one pole only, and that a simple pole at $\tau = \alpha$, with residue A. The complete system of poles is then given by

$$\tau = \mathbf{a} = \frac{c + d\alpha}{a + b\alpha} \quad (ad - bc = 1), \tag{5.1}$$

If a and b are fixed, and (c, d) is one pair of solutions of $ad - bc = 1$, the complete system of solutions is $(c + ma, d + mb)$, where m is an integer. To each pair (a, b) correspond an infinity of poles in the plane of τ; but these poles correspond to two different poles only in the plane of q, viz,

$$q = \pm\mathbf{q} = \pm e^{\pi i \mathbf{a}}, \tag{5.2}$$

the positive and negative signs corresponding to even and odd values of m respectively. It is to be observed, moreover, that different pairs (a, b) may give rise to the same pole $\mathbf{q}$. The residue of $\phi(\tau)$ for $\tau = \mathbf{a}$ is , in virtue of the functional equation (2.2),

$$\frac{A}{(a + b\alpha)^{n+2}};$$

and the residue of $f(q)$ for $q = \mathbf{q}$ is

$$\frac{A}{(a + b\alpha)^{n+2}} \left(\frac{dq}{d\tau}\right)_{\tau = \mathrm{a}} = \frac{\pi i A \mathbf{q}}{(a + b\alpha)^{n+2}}.$$

Thus the sum of the terms of our series which correspond to the poles (5.2) is

$$\frac{\pi i A}{(a + b\alpha)^{n+2}} \left(\frac{\mathbf{q}}{q - \mathbf{q}} - \frac{\mathbf{q}}{q + \mathbf{q}}\right) = \frac{2\pi i A}{(a + b\alpha)^{n+2}} \frac{\mathbf{q^2}}{q^2 - \mathbf{q}^2}.$$

We thus obtain:

Theorem 2. *If $\phi(\tau)$ has one pole only in P, viz., a simple pole at $\tau = \alpha$ with residue A, and $|q| < 1$, then*

$$f(q) = 2\pi i A \sum \frac{1}{(a + b\alpha)^{n+2}} \frac{\mathbf{q^2}}{q^2 - \mathbf{q}^2}, \tag{5.3}$$

where

$$\mathbf{q} = \exp\left(\frac{c + d\alpha}{a + b\alpha}\right) \pi i;$$

c, d being any pair of solutions of $ad - bc = 1$, and the summation extending over all pairs a, b, which give rise to distinct values of $\mathbf{q}$. *If $|q| > 1$, the sum of the series on the right-hand side of* (5.3) *is zero.*

If $\phi(\tau)$ has several poles in P, $f(q)$, of course, will be the sum of a number of series such as (5.3). Incidentally, we may observe that it now appears that the series in question are absolutely convergent.

6. As an example, we select the function

$$f(q) = \frac{\pi^6}{216\omega_1^6 g_3} = \frac{1}{1 - 504\sum_1^\infty \frac{r^5 q^{2r}}{1-q^{2r}}} = \sum_0^\infty p_n x^n, \tag{6.1}$$

say, where $x = q^2$. It is evident that p_n is always an integer; the values of the first 13 coefficients are

$p_0 = 1$,
$p_1 = 504$,
$p_2 = 270648$,
$p_3 = 144912096$,
$p_4 = 77599626552$,
$p_5 = 41553943041744$,
$p_6 = 22251789971649504$,
$p_7 = 11915647845248387520$,
$p_8 = 6380729991419236488504$,
$p_9 = 3416827666558895485479576$,
$p_{10} = 1829682703808504464920468048$,
$p_{11} = 979779820147442370107345764512$,
$p_{12} = 524663917940510191509934144603104$;

so that p_{12} is a number of 33 figures.

By means of the formulæ*

$$g_3 = \frac{8}{27}(e_1 - e_3)^2(1 + k^2)(1 - \frac{1}{2}k^2)(1 - 2k^2),$$

$$e_1 - e_3 = \left(\frac{\pi}{2\omega_1}\right)^2 \{\vartheta_3(0)\}^4, \frac{2K}{\pi} = \{\vartheta_3(0)\}^2,$$

we find that

$$\frac{1}{f(q)} = \left(\frac{2K}{\pi}\right)^6 (1 + k^2)(1 - \frac{1}{2}k^2)(1 - 2k^2).$$

The value of n is 6. The poles of $f(q)$ correspond to the value of τ which make $K = k^2$ equal to $-1, 2$ or $\frac{1}{2}$. It is easily verified† that these values are given by the general formula

$$\tau = \frac{c + di}{a + bi} \quad (ad - bc = 1),$$

so that

$$\mathbf{q} = \exp\left(\frac{c + di}{a + bi}\pi i\right) = \exp\left(\frac{ac + bd}{a^2 + b^2}\pi i - \frac{\pi}{a^2 + b^2}\right). \tag{6.2}$$

*All the formulæ which we quote are given in Tannery and Molk's Tables; see in particular Tables XXXVI (3), LXXI (3), XCVI, CX (3).

†A full account of the problem of finding τ when κ if given will be found in Tannery and Molk, *loc. cit.*, Vol. III, ch. 7 ("On donne k^2 ou g_2, g_3; trouver τ ou ω_1, ω_3").

The value of α is i*. In order to determine A we observe that

$$-504\frac{d}{dq}\left(\frac{1^5q^2}{1-q^2}+\frac{2^5q^4}{1-q^4}+\cdots\right)=-\frac{1008}{q}\left\{\frac{1^6q^2}{(1-q^2)^2}+\frac{2^6q^4}{(1-q^4)^2}+\cdots\right\}.$$

The series in curly brackets is the function called by Ramanujan† $\Phi_{1,6}$ and‡

$$1008\Phi_{1,6}=Q^2-PR,$$

where

$$P=\frac{12\eta_1\omega_1}{\pi^2},\quad Q=12g_2\left(\frac{\omega_1}{\pi}\right)^4,\quad R=216g_3\left(\frac{\omega_1}{\pi}\right)^6.$$

Here $R=0$, so that

$$1008\Phi_{1,6}=Q^2=1+480\Phi_{0,7}{}^{\S}=1+480\left(\frac{1^7q^2}{1-q^2}+\frac{2^7q^4}{1-q^4}+\cdots\right).$$

Hence we find that

$$A=i/\pi C,\quad 2\pi iA=-2/C,$$

where

(6.3) $$C=1+480\left(\frac{1^7}{e^{2\pi}-1}+\frac{2^7}{e^{4\pi}-1}+\cdots\right).$$

Another expression for C is

(6.4) $$C=144\left(\frac{K_0}{\pi}\right)^8,$$

where

(6.41) $$K_0=\int_0^{\pi/2}\frac{d\theta}{\sqrt{(1-\frac{1}{2}\sin^2\theta)}}=\frac{\{\Gamma(1/4)\}^2}{4\sqrt{\pi}}.$$

We have still to consider more closely the values of a and b, over which the summation is effected. Let us fix k, and suppose that (a,b) is a pair of positive solutions of the equation $a^2+b^2=k$. This pair gives rise to a system of eight solutions, viz.,

$$(\pm a,\pm b),(\pm b,\pm a).$$

But it is obvious that, if we change the signs of both a and b, we do not affect the aggregate of values of $\mathbf{a}$. Thus we need only consider the four pairs

$$(a,b),(a,-b),(b,a),(b,-a).$$

*It will be observed that in this case α is on the boundary of P; see the concluding remarks of §4. As it happens, $\tau=i$ lies on that edge of P (the circular edge) which was not used in the construction of H_m, so that our analysis is applicable as it stands.

†S. Ramanujan, "On Certain Arithmetical Functions," *Trans. Camb. Phil. Soc.*, Vol. XXII, pp. 159 – 184 (p. 163) [No. 18 of this volume, p. 179].

‡Ramanujan, *loc. cit.*, p. 164 [p. 181].

§Ramanujan, *loc. cit.*, p. 163 [p. 180].

If a or b is zero, or if $a = b$, these four pairs reduce to two.
It is easily verified that, if (a, b) leads to the pair of poles

$$q = \pm\mathbf{q} = \pm \exp\left(\frac{ac+bd}{a^2+b^2}\pi i - \frac{\pi}{a^2+b^2}\right),$$

then $(a, -b)$ and (b, a) each lead to $q = \pm\bar{\mathbf{q}}$, where $\bar{\mathbf{q}}$ is the conjugate of $\mathbf{q}$. Thus, in general (a, b) and the solutions derived from it lead to four distinct poles, viz., $\pm\mathbf{q}$ and $\pm\bar{\mathbf{q}}$. These four reduce to two in two cases, when $\mathbf{q}$ is real, so that $\mathbf{q} = \bar{\mathbf{q}}$, and when $\mathbf{q}$ is purely imaginary, so that $\mathbf{q} = -\bar{\mathbf{q}}$. It is easy to see that the first case can occur only when $k = 1$, and the second when $k = 2$*.
If $k = 1$ we take $a = 1, b = 0, c = 0, d = 1$; and $\mathbf{q} = \bar{\mathbf{q}} = e^{-\pi}$. If $k = 2$ we take $a = 1, b = 1, c = 0, d = 1$; and $\mathbf{q} = -\bar{\mathbf{q}} = ie^{-\pi/2}$. The corresponding terms in our series are

$$\frac{1}{1-q^2e^{2\pi}}, \frac{1}{2^4(1+qe^{\pi})}.$$

If $k > 2$, and is a sum of two coprime squares a^2 and b^2, it gives rise to terms

$$\frac{1}{(a+bi)^8}\frac{1}{1-(q/\mathbf{q})^2} + \frac{1}{(a-bi)^8}\frac{1}{1-(q/\bar{\mathbf{q}})^{\mathbf{2}}}.$$

There is, of course, a similar pair of terms corresponding to every other distinct representation of k as a sum of coprime squares. Thus finally we obtain the following result:

Theorem 3. *If*

$$f(q) = \frac{\pi^6}{216\omega_1^6 g_3} = \frac{1}{\left(1 - 504\sum_1^\infty \frac{r^5q^{2r}}{1-q^{2r}}\right)} = \sum_0^\infty p_n q^{2n},$$

and $|q| < 1$, *then*

$$\frac{1}{2}Cf(q) = \frac{1}{1-q^2e^{2\pi}} + \frac{1}{2^4(1+q^2e^{\pi})} + \sum\left\{\frac{1}{(a+bi)^8}\frac{1}{1-(q/\mathbf{q})^2} + \frac{1}{(a-bi)^8}\frac{1}{1-(q/\bar{\mathbf{q}})^{\mathbf{2}}}\right\}; \tag{6.5}$$

*When a and b are given, we can always choose c and d so that $|ac+bd| \le \frac{1}{2}(a^2+b^2)$. If $\mathbf{q}$ is real, we have $ad - bc = 1$ and $ac + bd = 0$ simultaneously; whence

$$(a^2+b^2)(c^2+d^2) = 1.$$

If $\mathbf{q}$ is purely imaginary, we have

$$ad - bc = 1, 2|ac+bd| = a^2+b^2,$$

whence

$$(c^2+d^2)^2 = (|ac+bd| - c^2 - d^2)^2 + 1.$$

This is possible only if $c^2 + d^2 = 1$ and $|ac + bd| = 1$, whence $a^2 + b^2 = 2$.

where

$$C = 1 + 480\left(\frac{1^7}{e^{2\pi}-1} + \frac{2^7}{e^{4\pi}-1} + \cdots\right) = \frac{9\pi^4}{16\{\Gamma(3/4)\}^{16}},$$

$$\mathbf{q} = \exp\left(\frac{c+di}{a+bi}\pi i\right) = \exp\left(\frac{ac+bd}{a^2+b^2}\pi i - \frac{\pi}{a^2+b^2}\right),$$

and $\bar{\mathbf{q}}$ *is the conjugate of* $\mathbf{q}$. *The summation applies to every pair of coprime positive numbers* a *and* b, *such that* $k = a^2 + b^2 \geq 5$, *such pairs, however, only being counted as distinct if they correspond to independent representations of* k *as a sum of squares. If* $|q| > 1$, *then the sum of the series on the right-hand side of* (6.5) *is zero.*

7. It follows that

$$\frac{1}{2}Cp_n = e^{2n\pi} + \frac{(-1)^n}{2^4}e^{n\pi} + \sum\left\{\frac{1}{(a+bi)^8}\mathbf{q}^{-2n} + \frac{1}{(a-bi)^8}\bar{\mathbf{q}}^{-2n}\right\} = \sum_{(\lambda)}\frac{c_\lambda(n)}{\lambda^4}e^{2n\pi/\lambda}, \tag{7.1}$$

say. Here λ is the sum of two coprime squares, so that

$$\lambda = 2^{a_2}5^{a_5}13^{a_{13}}17^{a_{17}}\cdots,$$

where a_2 is 0 or 1 and 5, 13, 17, ... are the primes of the form $4k+1$; and the first few values of $c_\lambda(n)$ are

$$c_1(n) = 1, c_2(n) = (-1)^n, c_5(n) = 2\cos\left(\tfrac{4}{5}n\pi + 8\arctan 2\right),$$

$$c_{10}(n) = 2\cos\left(\tfrac{3}{5}n\pi - 8\arctan 2\right), c_{13}(n) = 2\cos\left(\tfrac{10}{13}n\pi + 8\arctan 5\right).$$

The approximations to the coefficients given by the formula (7.1) are exceedingly remarkable. Dividing by $\frac{1}{2}C$, and taking $n = 0, 1, 2, 3, 6$, and 12, we find the following results:

$$\begin{array}{r} (0)\quad 0.944 \\ +0.059 \\ -0.003 \\ \hline p_0 = 1.000 \end{array} \qquad \begin{array}{r} (1)\quad 505.361 \\ -1.365 \\ +0.004 \\ \hline p_1 = 504.000 \end{array} \qquad \begin{array}{r} (2)\quad 270616.406 \\ +31.585 \\ +0.009 \\ \hline p_2 = 270648.000 \end{array}$$

$$\begin{array}{r} (3)\quad 144912827.002 \\ -730.900 \\ -0.101 \\ -0.001 \\ \hline p_3 = 144912096.000 \end{array} \qquad \begin{array}{r} (6)\quad 22251789962592450.237 \\ +9057051.688 \\ +2.081 \\ -0.006 \\ \hline p_6 = 22251789971649504.000 \end{array}$$

$$\begin{array}{r} (12)\qquad 524663917940510190119197271938395.329 \\ +1390736872662028.140 \\ +2680.418 \\ +0.130 \\ -0.014 \\ -0.003 \\ \hline p_{12} = 524663917940510191509934144603104.000 \end{array}$$

An alternative expression for C is

$$C = 96^2 e^{-8\pi/3}\{(1 - e^{-4\pi})(1 - e^{-8\pi})\cdots\}^{16},$$

by means of which C may be calculated with great accuracy*. To five places we have $2/C = 0.94373$, which is very nearly equal to $352/373 = 0.94370$.
It is easy to see directly that p_n lies between the coefficients of x^n in the expansions of

$$\frac{1}{(1 - 535x)(1 + 31x)}, \quad \frac{1 - 7.5x}{(1 - 535.5x)(1 + 24x)},$$

and so that

$$\frac{(535)^{n+1} - (-31)^{n+1}}{566} \le p_n \le \frac{352(535.5)^n + 21(-24)^n}{373}.$$

The function

$$\Omega(x) = \sum_{(\lambda)} \frac{c_\lambda(x)}{\lambda^4} e^{2x\pi/\lambda}$$

has very remarkable properties. It is an integral function of x, whose maximum modulus is less than a constant multiple of $e^{2\pi|x|}$. It is equal to p_n, an integer, when $x = n$, a positive integer; and to zero when $x = -n$. But we must reserve the discussion of these peculiarities for some other occasion.

*Gauss, *Werke*, Vol. III, pp. 418 – 419, gives the values of various powers of $e^{-\pi}$ to a large number of figures.