The Collected Works of Professor Michio Kaku

Michio Kaku is an American theoretical physicist, futurist, and co-founder of string field theory, and is one of the most widely recognized popularizer of science in the world today. He is a professor of theoretical physics at the City College of New York and CUNY Graduate Center. Kaku has written several books about physics and has also appeared on dozens of radio and television shows and documentaries, explaining complex theories in a way everyone can understand.

The Physics of Extraterrestrial Civilizations

How advanced could they possibly be?

The late Carl Sagan once asked this question, “What does it mean for a civilization to be a million years old? We have had radio telescopes and spaceships for a few decades; our technical civilization is a few hundred years old… an advanced civilization millions of years old is as much beyond us as we are beyond a bush baby or a macaque.”

Although any conjecture about such advanced civilizations is a matter of sheer speculation, one can still use the laws of physics to place upper and lower limits on these civilizations. In particular, now that the laws of quantum field theory, general relativity, thermodynamics, etc. are fairly well-established, physics can impose broad physical bounds which constrain the parameters of these civilizations.

This question is no longer a matter of idle speculation. Soon, humanity may face an existential shock as the current list of a dozen Jupiter-sized extra-solar planets swells to hundreds of earth-sized planets, almost identical twins of our celestial homeland. This may usher in a new era in our relationship with the universe: we will never see the night sky in the same way ever again, realizing that scientists may eventually compile an encyclopedia identifying the precise co-ordinates of perhaps hundreds of earth-like planets.

Today, every few weeks brings news of a new Jupiter-sized extra-solar planet being discovered, the latest being about 15 light years away orbiting around the star Gliese 876. The most spectacular of these findings was photographed by the Hubble Space Telescope, which captured breathtaking photos of a planet 450 light years away being sling-shot into space by a double-star system.

But the best is yet to come. Early in the next decade, scientists will launch a new kind of telescope, the interferome try space telescope, which uses the interference of light beams to enhance the resolving power of telescopes.

For example, the Space Interferometry Mission (SIM), to be launched early in the next decade, consists of multiple telescopes placed along a 30 foot structure. With an unprecedented resolution approaching the physical limits of optics, the SIM is so sensitive that it almost defies belief: orbiting the earth, it can detect the motion of a lantern being waved by an astronaut on Mars!

The SIM, in turn, will pave the way for the Terrestrial Planet Finder, to be launched late in the next decade, which should identify even more earth-like planets. It will scan the brightest 1,000 stars within 50 light years of the earth and will focus on the 50 to 100 brightest planetary systems.

All this, in turn, will stimulate an active effort to determine if any of them harbor life, perhaps some with civilizations more advanced than ours.

Although it is impossible to predict the precise features of such advanced civilizations, their broad outlines can be analyzed using the laws of physics. No matter how many millions of years separate us from them, they still must obey the iron laws of physics, which are now advanced enough to explain everything from sub-atomic particles to the large-scale structure of the universe, through a staggering 43 orders of magnitude.

Physics of Type I, II, and III Civilizations

Specifically, we can rank civilizations by their energy consumption, using the following principles:

1) The laws of thermodynamics. Even an advanced civilization is bound by the laws of thermodynamics, especially the Second Law, and can hence be ranked by the energy at their disposal.

2) The laws of stable matter. Baryonic matter (e.g. based on protons and neutrons) tends to clump into three large groupings: planets, stars and galaxies. (This is a well-defined by product of stellar and galactic evolution, thermonuclear fusion, etc.) Thus, their energy will also be based on three distinct types, and this places upper limits on their rate of energy consumption.

3) The laws of planetary evolution. Any advanced civilization must grow in energy consumption faster than the frequency of life-threatening catastrophes (e.g. meteor impacts, ice ages, supernovas, etc.). If they grow any slower, they are doomed to extinction. This places mathematical lower limits on the rate of growth of these civilizations.

In a seminal paper published in 1964 in the Journal of Soviet Astronomy, Russian astrophysicist Nicolai Kardashev theorized that advanced civilizations must therefore be grouped according to three types: Type I, II, and III, which have mastered planetary, stellar and galactic forms of energy, respectively. He calculated that the energy consumption of these three types of civilization would be separated by a factor of many billions. But how long will it take to reach Type II and III status?

Shorter than most realize.

Berkeley astronomer Don Goldsmith reminds us that the earth receives about one billionth of the suns energy, and that humans utilize about one millionth of that. So we consume about one million billionth of the suns total energy. At present, our entire planetary energy production is about 10 billion billion ergs per second. But our energy growth is rising exponentially, and hence we can calculate how long it will take to rise to Type II or III status.

Goldsmith says, “Look how far we have come in energy uses once we figured out how to manipulate energy, how to get fossil fuels really going, and how to create electrical power from hydropower, and so forth; we’ve come up in energy uses in a remarkable amount in just a couple of centuries compared to billions of years our planet has been here … and this same sort of thing may apply to other civilizations.”

Physicist Freeman Dyson of the Institute for Advanced Study estimates that, within 200 years or so, we should attain Type I status. In fact, growing at a modest rate of 1% per year, Kardashev estimated that it would take only 3,200 years to reach Type II status, and 5,800 years to reach Type III status. Living in a Type I,II, or III civilization

For example, a Type I civilization is a truly planetary one, which has mastered most forms of planetary energy. Their energy output may be on the order of thousands to millions of times our current planetary output. Mark Twain once said, ”Everyone complains about the weather, but no one does anything about it.“ This may change with a Type I civilization, which has enough energy to modify the weather. They also have enough energy to alter the course of earthquakes, volcanoes, and build cities on their oceans.

Currently, our energy output qualifies us for Type 0 status. We derive our energy not from harnessing global forces, but by burning dead plants (e.g. oil and coal). But already, we can see the seeds of a Type I civilization. We see the beginning of a planetary language (English), a planetary communication system (the Internet), a planetary economy (the forging of the European Union), and even the beginnings of a planetary culture (via mass media, TV, rock music, and Hollywood films).

By definition, an advanced civilization must grow faster than the frequency of life-threatening catastrophes. Since large meteor and comet impacts take place once every few thousand years, a Type I civilization must master space travel to deflect space debris within that time frame, which should not be much of a problem. Ice ages may take place on a time scale of tens of thousands of years, so a Type I civilization must learn to modify the weather within that time frame.

Artificial and internal catastrophes must also be negotiated. But the problem of global pollution is only a mortal threat for a Type 0 civilization; a Type I civilization has lived for several millennia as a planetary civilization, necessarily achieving ecological planetary balance. Internal problems like wars do pose a serious recurring threat, but they have thousands of years in which to solve racial, national, and sectarian conflicts.

Eventually, after several thousand years, a Type I civilization will exhaust the power of a planet, and will derive their energy by consuming the entire output of their suns energy, or roughly a billion trillion trillion ergs per second.

With their energy output comparable to that of a small star, they should be visible from space. Dyson has proposed that a Type II civilization may even build a gigantic sphere around their star to more efficiently utilize its total energy output. Even if they try to conceal their existence, they must, by the Second Law of Thermodynamics, emit waste heat. From outer space, their planet may glow like a Christmas tree ornament. Dyson has even proposed looking specifically for infrared emissions (rather than radio and TV) to identify these Type II civilizations.

Perhaps the only serious threat to a Type II civilization would be a nearby supernova explosion, whose sudden eruption could scorch their planet in a withering blast of X-rays, killing all life forms. Thus, perhaps the most interesting civilization is a Type III civilization, for it is truly immortal. They have exhausted the power of a single star, and have reached for other star systems. No natural catastrophe known to science is capable of destroying a Type III civilization.

Faced with a neighboring supernova, it would have several alternatives, such as altering the evolution of dying red giant star which is about to explode, or leaving this particular star system and terraforming a nearby planetary system.

However, there are roadblocks to an emerging Type III civilization. Eventually, it bumps up against another iron law of physics, the theory of relativity. Dyson estimates that this may delay the transition to a Type III civilization by perhaps millions of years.

But even with the light barrier, there are a number of ways of expanding at near-light velocities. For example, the ultimate measure of a rockets capability is measured by something called “specific impulse” (defined as the product of the thrust and the duration, measured in units of seconds). Chemical rockets can attain specific impulses of several hundred to several thousand seconds. Ion engines can attain specific impulses of tens of thousands of seconds. But to attain near-light speed velocity, one has to achieve specific impulse of about 30 million seconds, which is far beyond our current capability, but not that of a Type III civilization. A variety of propulsion systems would be available for sub-light speed probes (such as ram-jet fusion engines, photonic engines, etc.)

How to Explore the Galaxy

Because distances between stars are so vast, and the number of unsuitable, lifeless solar systems so large, a Type III civilization would be faced with the next question: what is the mathematically most efficient way of exploring the hundreds of billions of stars in the galaxy?

In science fiction, the search for inhabitable worlds has been immortalized on TV by heroic captains boldly commanding a lone star ship, or as the murderous Borg, a Type III civilization which absorbs lower Type II civilization (such as the Federation). However, the most mathematically efficient method to explore space is far less glamorous: to send fleets of “Von Neumann probes” throughout the galaxy (named after John Von Neumann, who established the mathematical laws of self-replicating systems).

A Von Neumann probe is a robot designed to reach distant star systems and create factories which will reproduce copies themselves by the thousands. A dead moon rather than a planet makes the ideal destination for Von Neumann probes, since they can easily land and take off from these moons, and also because these moons have no erosion. These probes would live off the land, using naturally occurring deposits of iron, nickel, etc. to create the raw ingredients to build a robot factory. They would create thousands of copies of themselves, which would then scatter and search for other star systems.

Similar to a virus colonizing a body many times its size, eventually there would be a sphere of trillions of Von Neumann probes expanding in all directions, increasing at a fraction of the speed of light. In this fashion, even a galaxy 100,000 light years across may be completely analyzed within, say, a half million years.

If a Von Neumann probe only finds evidence of primitive life (such as an unstable, savage Type 0 civilization) they might simply lie dormant on the moon, silently waiting for the Type 0 civilization to evolve into a stable Type I civilization. After waiting quietly for several millennia, they may be activated when the emerging Type I civilization is advanced enough to set up a lunar colony. Physicist Paul Davies of the University of Adelaide has even raised the possibility of a Von Neumann probe resting on our own moon, left over from a previous visitation in our system aeons ago.

(If this sounds a bit familiar, that's because it was the basis of the film, 2001. Originally, Stanley Kubrick began the film with a series of scientists explaining how probes like these would be the most efficient method of exploring outer space. Unfortunately, at the last minute, Kubrick cut the opening segment from his film, and these monoliths became almost mystical entities)

New Developments

Since Kardashev gave the original ranking of civilizations, there have been many scientific developments which refine and extend his original analysis, such as recent developments in nanotechnology, biotechnology, quantum physics, etc.

For example, nanotechnology may facilitate the development of Von Neumann probes. As physicist Richard Feynman observed in his seminal essay, "There's Plenty of Room at the Bottom," there is nothing in the laws of physics which prevents building armies of molecular-sized machines. At present, scientists have already built atomic-sized curiosities, such as an atomic abacus with Buckyballs and an atomic guitar with strings about 100 atoms across.

Paul Davies speculates that a space-faring civilization could use nanotechnology to build miniature probes to explore the galaxy, perhaps no bigger than your palm. Davies says, "The tiny probes I'm talking about will be so inconspicuous that it's no surprise that we haven't come across one. It's not the sort of thing that you're going to trip over in your back yard. So if that is the way technology develops, namely, smaller, faster, cheaper and if other civilizations have gone this route, then we could be surrounded by surveillance devices."

Furthermore, the development of biotechnology has opened entirely new possibilities. These probes may act as life-forms, reproducing their genetic information, mutating and evolving at each stage of reproduction to enhance their capabilities, and may have artificial intelligence to accelerate their search.

Also, information theory modifies the original Kardashev analysis. The current SETI project only scans a few frequencies of radio and TV emissions sent by a Type 0 civilization, but perhaps not an advanced civilization. Because of the enormous static found in deep space, broadcasting on a single frequency presents a serious source of error. Instead of putting all your eggs in one basket, a more efficient system is to break up the message and smear it out over all frequencies (e.g. via Fourier like transform) and then reassemble the signal only at the other end. In this way, even if certain frequencies are disrupted by static, enough of the message will survive to accurately reassemble the message via error correction routines. However, any Type 0 civilization listening in on the message on one frequency band would only hear nonsense. In other words, our galaxy could be teeming with messages from various Type II and III civilizations, but our Type 0 radio telescopes would only hear gibberish.

Lastly, there is also the possibility that a Type II or Type III civilization might be able to reach the fabled Planck energy with their machines (10^19 billion electron volts). This is energy is a quadrillion times larger than our most powerful atom smasher. This energy, as fantastic as it may seem, is (by definition) within the range of a Type II or III civilization.

The Planck energy only occurs at the center of black holes and the instant of the Big Bang. But with recent advances in quantum gravity and superstring theory, there is renewed interest among physicists about energies so vast that quantum effects rip apart the fabric of space and time. Although it is by no means certain that quantum physics allows for stable wormholes, this raises the remote possibility that a sufficiently advanced civilizations may be able to move via holes in space, like Alice's Looking Glass. And if these civilizations can successfully navigate through stable wormholes, then attaining a specific impulse of a million seconds is no longer a problem. They merely take a short-cut through the galaxy. This would greatly cut down the transition between a Type II and Type III civilization.

Second, the ability to tear holes in space and time may come in handy one day. Astronomers, analyzing light from distant supernovas, have concluded recently that the universe may be accelerating, rather than slowing down. If this is true, there may be an anti-gravity force (perhaps Einstein's cosmological constant) which is counteracting the gravitational attraction of distant galaxies. But this also means that the universe might expand forever in a Big Chill, until temperatures approach near-absolute zero. Several papers have recently laid out what such a dismal universe may look like. It will be a pitiful sight: any civilization which survives will be desperately huddled next to the dying embers of fading neutron stars and black holes. All intelligent life must die when the universe dies.

Contemplating the death of the sun, the philosopher Bertrand Russel once wrote perhaps the most depressing paragraph in the English language: "…All the labors of the ages, all the devotion, all the inspiration, all the noonday brightness of human genius, are destined to extinction in the vast death of the solar system, and the whole temple of Mans achievement must inevitably be buried beneath the debris of a universe in ruins…"

Today, we realize that sufficiently powerful rockets may spare us from the death of our sun 5 billion years from now, when the oceans will boil and the mountains will melt. But how do we escape the death of the universe itself?

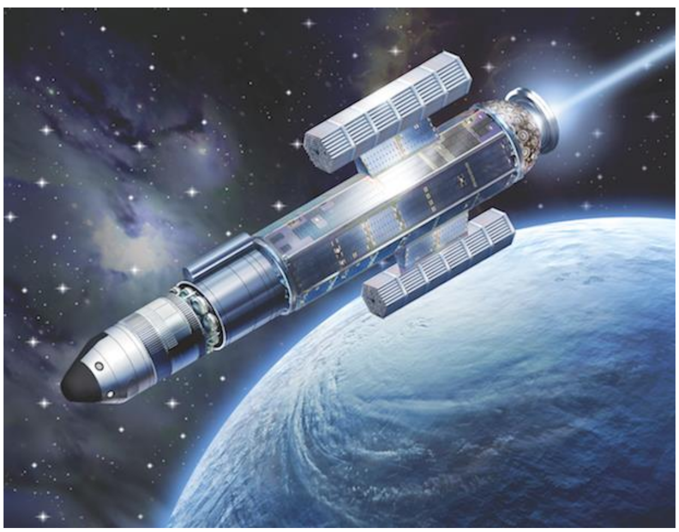
Astronomer John Barrows of the University of Sussex writes, "Suppose that we extend the classification upwards. Members of these hypothetical civilizations of Type IV, V, VI, … and so on, would be able to manipulate the structures in the universe on larger and larger scales, encompassing groups of galaxies, clusters, and superclusters of galaxies." Civilizations beyond Type III may have enough energy to escape our dying universe via holes in space.

Lastly, physicist Alan Guth of MIT, one of the originators of the inflationary universe theory, has even computed the energy necessary to create a baby universe in the laboratory (the temperature is 1,000 trillion degrees, which is within the range of these hypothetical civilizations).

Of course, until someone actually makes contact with an advanced civilization, all of this amounts to speculation tempered with the laws of physics, no more than a useful guide in our search for extra-terrestrial intelligence. But one day, many of us will gaze at the encyclopedia containing the coordinates of perhaps hundreds of earth-like planets in our sector of the galaxy. Then we will wonder, as Sagan did, what a civilization a millions years ahead of ours will look like…

The Physics of Interstellar Travel

To one day, reach the stars.



When discussing the possibility of interstellar travel, there is something called "the giggle factor." Some scientists tend to scoff at the idea of interstellar travel because of the enormous distances that separate the stars. According to Special Relativity (1905), no usable information can travel faster than light locally, and hence it would take centuries to millennia for an extra-terrestrial civilization to travel between the stars. Even the familiar stars we see at night are about 50 to 100 light years from us, and our galaxy is 100,000 light years across. The nearest galaxy is 2 million light years from us. The critics say that the universe is simply too big for interstellar travel to be practical.

Similarly, investigations into UFO's that may originate from another planet are sometimes the "third rail" of someone's scientific career. There is no funding for anyone seriously looking at unidentified objects in space, and one's reputation may suffer if one pursues an interest in these unorthodox matters. In addition, perhaps 99% of all sightings of UFO's can be dismissed as being caused by familiar phenomena, such as the planet Venus, swamp gas (which can glow in the dark under certain conditions), meteors, satellites, weather balloons, even radar echoes that bounce off mountains. (What is disturbing, to a physicist however, is the remaining 1% of these sightings, which are multiple sightings made by multiple methods of observations. Some of the most intriguing sightings have been made by seasoned pilots and passengers aboard air line flights which have also been tracked by radar and have been videotaped. Sightings like this are harder to dismiss.)

But to an astronomer, the existence of intelligent life in the universe is a compelling idea by itself, in which extra-terrestrial beings may exist on other stars who are centuries to millennia more advanced than ours. Within the Milky Way galaxy alone, there are over 100 billion stars, and there are an uncountable number of galaxies in the universe. About half of the stars we see in the heavens are double stars, probably making them unsuitable for intelligent life, but the remaining half probably have solar systems somewhat similar to ours. Although none of the over 100 extra-solar planets so far discovered in deep space resemble ours, it is inevitable, many scientists believe, that one day we will discover small, earth-like planets which have liquid water (the "universal solvent" which made possible the first DNA perhaps 3.5 billion years ago in the oceans). The discovery of earth-like planets may take place within 20 years, when NASA intends to launch the space interferometry satellite into orbit which may be sensitive enough to detect small planets orbiting other stars.

So far, we see no hard evidence of signals from extra-terrestrial civilizations from any earth-like planet. The SETI project (the search for extra-terrestrial intelligence) has yet to produce any reproducible evidence of intelligent life in the universe from such earth-like planets, but the matter still deserves serious scientific analysis. The key is to reanalyze the objection to faster-than-light travel.

A critical look at this issue must necessary embrace two new observations. First, Special Relativity itself was superceded by Einstein's own more powerful General Relativity (1915), in which faster than light travel is possible under certain rare conditions. The principal difficulty is amassing enough energy of a certain type to break the light barrier. Second, one must therefore analyze extra-terrestrial civilizations on the basis of their total energy output and the laws of thermodynamics. In this respect, one must analyze civilizations which are perhaps thousands to millions of years ahead of ours.

The first realistic attempt to analyze extra-terrestrial civilizations from the point of view of the laws of physics and the laws of thermodynamics was by Russian astrophysicist Nicolai Kardashev. He based his ranking of possible civilizations on the basis of total energy output which could be quantified and used as a guide to explore the dynamics of advanced civilizations:

Type I: this civilization harnesses the energy output of an entire planet.

Type II: this civilization harnesses the energy output of a star, and generates about 10 billion times the energy output of a Type I civilization.

Type III: this civilization harnesses the energy output of a galaxy, or about 10 billion time the energy output of a Type II civilization.

A Type I civilization would be able to manipulate truly planetary energies. They might, for example, control or modify their weather. They would have the power to manipulate planetary phenomena, such as hurricanes, which can release the energy of hundreds of hydrogen bombs. Perhaps volcanoes or even earthquakes may be altered by such a civilization.

A Type II civilization may resemble the Federation of Planets seen on the TV program Star Trek (which is capable of igniting stars and has colonized a tiny fraction of the near-by stars in the galaxy). A Type II civilization might be able to manipulate the power of solar flares.

A Type III civilization may resemble the Borg, or perhaps the Empire found in the Star Wars saga. They have colonized the galaxy itself, extracting energy from hundreds of billions of stars.

By contrast, we are a Type 0 civilization, which extracts its energy from dead plants (oil and coal). Growing at the average rate of about 3% per year, however, one may calculate that our own civilization may attain Type I status in about 100-200 years, Type II status in a few thousand years, and Type III status in about 100,000 to a million years. These time scales are insignificant when compared with the universe itself.

On this scale, one may now rank the different propulsion systems available to different types of civilizations:

Type 0

Chemical rockets

Ionic engines

Fission power

EM propulsion (rail guns)

Type I

Ram-jet fusion engines

Photonic drive

Type II

Antimatter drive

Von Neumann nano probes

Type III

Planck energy propulsion

Propulsion systems may be ranked by two quantities: their specific impulse, and final velocity of travel. Specific impulse equals thrust multiplied by the time over which the thrust acts. At present, almost all our rockets are based on chemical reactions. We see that chemical rockets have the smallest specific impulse, since they only operate for a few minutes. Their thrust may be measured in millions of pounds, but they operate for such a small duration that their specific impulse is quite small.

NASA is experimenting today with ion engines, which have a much larger specific impulse, since they can operate for months, but have an extremely low thrust. For example, an ion engine which ejects cesium ions may have the thrust of a few ounces, but in deep space they may reach great velocities over a period of time since they can operate continuously. They make up in time what they lose in thrust. Eventually, long-haul missions between planets may be conducted by ion engines.

For a Type I civilization, one can envision newer types of technologies emerging. Ram-jet fusion engines have an even larger specific impulse, operating for years by consuming the free hydrogen found in deep space. However, it may take decades before fusion power is harnessed commercially on earth, and the proton-proton fusion process of a ram-jet fusion engine may take even more time to develop, perhaps a century or more. Laser or photonic engines, because they might be propelled by laser beams inflating a gigantic sail, may have even larger specific impulses. One can envision huge laser batteries placed on the moon which generate large laser beams which then push a laser sail in outer space. This technology, which depends on operating large bases on the moon, is probably many centuries away.

For a Type II civilization, a new form of propulsion is possible: anti-matter drive. Matter-anti-matter collisions provide a 100% efficient way in which to extract energy from mater. However, anti-matter is an exotic form of matter which is extremely expensive to produce. The atom smasher at CERN, outside Geneva, is barely able to make tiny samples of anti-hydrogen gas (anti-electrons circling around anti-protons). It may take many centuries to millennia to bring down the cost so that it can be used for space flight.

Given the astronomical number of possible planets in the galaxy, a Type II civilization may try a more realistic approach than conventional rockets and use nano technology to build tiny, self-replicating robot probes which can proliferate through the galaxy in much the same way that a microscopic virus can self-replicate and colonize a human body within a week. Such a civilization might send tiny robot von Neumann probes to distant moons, where they will create large factories to reproduce millions of copies of themselves. Such a von Neumann probe need only be the size of bread-box, using sophisticated nano technology to make atomic-sized circuitry and computers. Then these copies take off to land on other distant moons and start the process all over again. Such probes may then wait on distant moons, waiting for a primitive Type 0 civilization to mature into a Type I civilization, which would then be interesting to them. (There is the small but distinct possibility that one such probe landed on our own moon billions of years ago by a passing space-faring civilization. This, in fact, is the basis of the movie 2001, perhaps the most realistic portrayal of contact with extra-terrrestrial intelligence.)

The problem, as one can see, is that none of these engines can exceed the speed of light. Hence, Type 0,I, and II civilizations probably can send probes or colonies only to within a few hundred light years of their home planet. Even with von Neumann probes, the best that a Type II civilization can achieve is to create a large sphere of billions of self-replicating probes expanding just below the speed of light. To break the light barrier, one must utilize General Relativity and the quantum theory. This requires energies which are available for very advanced Type II civilization or, more likely, a Type III civilization.

Special Relativity states that no usable information can travel locally faster than light. One may go faster than light, therefore, if one uses the possibility of globally warping space and time, i.e. General Relativity. In other words, in such a rocket, a passenger who is watching the motion of passing stars would say he is going slower than light. But once the rocket arrives at its destination and clocks are compared, it appears as if the rocket went faster than light because it warped space and time globally, either by taking a shortcut, or by stretching and contracting space.

There are at least two ways in which General Relativity may yield faster than light travel. The first is via wormholes, or multiply connected Riemann surfaces, which may give us a shortcut across space and time. One possible geometry for such a wormhole is to assemble stellar amounts of energy in a spinning ring (creating a Kerr black hole). Centrifugal force prevents the spinning ring from collapsing. Anyone passing through the ring would not be ripped apart, but would wind up on an entirely different part of the universe. This resembles the Looking Glass of Alice, with the rim of the Looking Glass being the black hole, and the mirror being the wormhole. Another method might be to tease apart a wormhole from the “quantum foam” which physicists believe makes up the fabric of space and time at the Planck length (10 to the minus 33 centimeters).

The problems with wormholes are many:

a) one version requires enormous amounts of positive energy, e.g. a black hole. Positive energy wormholes have an event horizon(s) and hence only give us a one way trip. One would need two black holes (one for the original trip, and one for the return trip) to make interstellar travel practical. Most likely only a Type III civilization would be able harness this power.

b) wormholes may be unstable, both classically or quantum mechanically. They may close up as soon as you try to enter them. Or radiation effects may soar as you entered them, killing you.

c) one version requires vast amounts of negative energy. Negative energy does exist (in the form of the Casimir effect) but huge quantities of negative energy will be beyond our technology, perhaps for millennia. The advantage of negative energy wormholes is that they do not have event horizons and hence are more easily transversable.

d) another version requires large amounts of negative matter. Unfortunately, negative matter has never been seen in nature (it would fall up, rather than down). Any negative matter on the earth would have fallen up billions of years ago, making the earth devoid of any negative matter.

The second possibility is to use large amounts of energy to continuously stretch space and time (i.e. contracting the space in front of you, and expanding the space behind you). Since only empty space is contracting or expanding, one may exceed the speed of light in this fashion. (Empty space can warp space faster than light. For example, the Big Bang expanded much faster than the speed of light.) The problem with this approach, again, is that vast amounts of energy are required, making it feasible for only a Type III civilization. Energy scales for all these proposals are on the order of the Planck energy (10 to the 19 billion electron volts, which is a quadrillion times larger than our most powerful atom smasher).

Lastly, there is the fundamental physics problem of whether "topology change" is possible within General Relativity (which would also make possible time machines, or closed time-like curves). General Relativity allows for closed time-like curves and wormholes (often called Einstein-Rosen bridges), but it unfortunately breaks down at the large energies found at the center of black holes or the instant of Creation. For these extreme energy domains, quantum effects will dominate over classical gravitational effects, and one must go to a "unified field theory" of quantum gravity.

At present, the most promising (and only) candidate for a "theory of everything", including quantum gravity, is superstring theory or M-theory. It is the only theory in which quantum forces may be combined with gravity to yield finite results. No other theory can make this claim. With only mild assumptions, one may show that the theory allows for quarks arranged in much like the configuration found in the current Standard Model of sub-atomic physics. Because the theory is defined in 10 or 11 dimensional hyperspace, it introduces a new cosmological picture: that our universe is a bubble or membrane floating in a much larger multiverse or megaverse of bubble-universes.

Unfortunately, although black hole solutions have been found in string theory, the theory is not yet developed to answer basic questions about wormholes and their stability. Within the next few years or perhaps within a decade, many physicists believe that string theory will mature to the point where it can answer these fundamental questions about space and time. The problem is well-defined. Unfortunately, even though the leading scientists on the planet are working on the theory, no one on earth is smart enough to solve the superstring equations.

Conclusion

Most scientists doubt interstellar travel because the light barrier is so difficult to break. However, to go faster than light, one must go beyond Special Relativity to General Relativity and the quantum theory. Therefore, one cannot rule out interstellar travel if an advanced civilization can attain enough energy to destabilize space and time. Perhaps only a Type III civilization can harness the Planck energy, the energy at which space and time become unstable. Various proposals have been given to exceed the light barrier (including wormholes and stretched or warped space) but all of them require energies found only in Type III galactic civilizations. On a mathematical level, ultimately, we must wait for a fully quantum mechanical theory of gravity (such as superstring theory) to answer these fundamental questions, such as whether wormholes can be created and whether they are stable enough to allow for interstellar travel.

The Physics of Time Travel

Is it real, or is it fable?

In H.G. Wells' novel, The Time Machine, our protagonist jumped into a special chair with blinking lights, spun a few dials, and found himself catapulted several hundred thousand years into the future, where England has long disappeared and is now inhabited by strange creatures called the Morlocks and Eloi. That may have made great fiction, but physicists have always scoffed at the idea of time travel, considering it to be the realm of cranks, mystics, and charlatans, and with good reason.

However, rather remarkable advances in quantum gravity are reviving the theory; it has now become fair game for theoretical physicists writing in the pages of Physical Review magazine. One stubborn problem with time travel is that it is riddled with several types of paradoxes. For example, there is the paradox of the man with no parents, i.e. what happens when you go back in time and kill your parents before you are born? Question: if your parents died before you were born, then how could you have been born to kill them in the first place?

There is also the paradox of the man with no past. For example, let's say that a young inventor is trying futilely to build a time machine in his garage. Suddenly, an elderly man appears from nowhere and gives the youth the secret of building a time machine. The young man then becomes enormously rich playing the stock market, race tracks, and sporting events because he knows the future. Then, as an old man, he decides to make his final trip back to the past and give the secret of time travel to his youthful self. Question: where did the idea of the time machine come from?

There is also the paradox of the man who is own mother (my apologies to Heinlein.) "Jane" is left at an orphanage as a foundling. When "Jane" is a teenager, she falls in love with a drifter, who abandons her but leaves her pregnant. Then disaster strikes. She almost dies giving birth to a baby girl, who is then mysteriously kidnapped. The doctors find that Jane is bleeding badly, but, oddly enough, has both sex organs. So, to save her life, the doctors convert "Jane" to "Jim."

"Jim" subsequently becomes a roaring drunk, until he meets a friendly bartender (actually a time traveler in disguise) who wisks "Jim" back way into the past. "Jim" meets a beautiful teenage girl, accidentally gets her pregnant with a baby girl. Out of guilt, he kidnaps the baby girl and drops her off at the orphanage. Later, "Jim" joins the time travelers corps, leads a distinguished life, and has one last dream: to disguise himself as a bartender to meet a certain drunk named "Jim" in the past. Question: who is "Jane's" mother, father, brother, sister, grand- father, grandmother, and grandchild?

Not surprisingly, time travel has always been considered impossible. After all, Newton believed that time was like an arrow; once fired, it soared in a straight, undeviating line. One second on the earth was one second on Mars. Clocks scattered throughout the universe beat at the same rate. Einstein gave us a much more radical picture. According to Einstein, time was more like a river, which meandered around stars and galaxies, speeding up and slowing down as it passed around massive bodies. One second on the earth was Not one second on Mars. Clocks scattered throughout the universe beat to their own distant drummer.

However, before Einstein died, he was faced with an embarrassing problem. Einstein's neighbor at Princeton, Kurt Goedel, perhaps the greatest mathematical logician of the past 500 years, found a new solution to Einstein's own equations which allowed for time travel! The "river of time" now had whirlpools in which time could wrap itself into a circle. Goedel's solution was quite ingenious: it postulated a universe filled with a rotating fluid. Anyone walking along the direction of rotation would find themselves back at the starting point, but backwards in time!

In his memoirs, Einstein wrote that he was disturbed that his equations contained solutions that allowed for time travel. But he finally concluded: the universe does not rotate, it ex- pands (i.e. as in the Big Bang theory) and hence Goedel's solution could be thrown out for "physical reasons." (Apparently, if the Big Bang was rotating, then time travel would be possible throughout the universe!)

Then in 1963, Roy Kerr, a New Zealand mathematician, found a solution of Einstein's equations for a rotating black hole, which had bizarre properties. The black hole would not collapse to a point (as previously thought) but into a spinning ring (of neutrons). The ring would be circulating so rapidly that centrifugal force would keep the ring from collapsing under gravity. The ring, in turn, acts like the Looking Glass of Alice. Anyone walking through the ring would not die, but could pass through the ring into an alternate universe. Since then, hundreds of other "wormhole" solutions have been found to Einstein's equations. These wormholes connect not only two regions of space (hence the name) but also two regions of time as well. In principle, they can be used as time machines.

Recently, attempts to add the quantum theory to gravity (and hence create a "theory of everything") have given us some insight into the paradox problem. In the quantum theory, we can have multiple states of any object. For example, an electron can exist simultaneously in different orbits (a fact which is responsible for giving us the laws of chemistry). Similarly, Schrodinger's famous cat can exist simultaneously in two possible states: dead and alive. So by going back in time and altering the past, we merely create a parallel universe. So we are changing someone ELSE's past by saving, say, Abraham Lincoln from being assassinated at the Ford Theater, but our Lincoln is still dead. In this way, the river of time forks into two separate rivers. But does this mean that we will be able to jump into H.G. Wells' machine, spin a dial, and soar several hundred thousand years into England's future? No. There are a number of difficult hurdles to overcome.

First, the main problem is one of energy. In the same way that a car needs gasoline, a time machine needs to have fabulous amounts of energy. One either has to harness the power of a star, or to find something called "exotic" matter (which falls up, rather than down) or find a source of negative energy. (Physicists once thought that negative energy was impossible. But tiny amounts of negative energy have been experimentally verified for something called the Casimir effect, i.e. the energy created by two parallel plates). All of these are exceedingly difficult to obtain in large quantities, at least for several more centuries!

Then there is the problem of stability. The Kerr black hole, for example, may be unstable if one falls through it. Similarly, quantum effects may build up and destroy the wormhole before you enter it. Unfortunately, our mathematics is not powerful enough to answer the question of stability because you need a "theory of everything" which combines both quantum forces and gravity. At present, superstring theory is the leading candidate for such a theory (in fact, it is the ONLY candidate; it really has no rivals at all). But superstring theory, which happens to be my specialty, is still to difficult to solve completely. The theory is well-defined, but no one on earth is smart enough to solve it.

Interestingly enough, Stephen Hawking once opposed the idea of time travel. He even claimed he had "empirical" evidence against it. If time travel existed, he said, then we would have been visited by tourists from the future. Since we see no tourists from the future, ergo: time travel is not possible. Because of the enormous amount of work done by theoretical physicists within the last 5 years or so, Hawking has since changed his mind, and now believes that time travel is possible (although not necessarily practical). (Furthermore, perhaps we are simply not very interesting to these tourists from the future. Anyone who can harness the power of a star would consider us to be very primitive. Imagine your friends coming across an ant hill. Would they bend down to the ants and give them trinkets, books, medicine, and power? Or would some of your friends have the strange urge to step on a few of them?)

In conclusion, don't turn someone away who knocks at your door one day and claims to be your future great-great-great grandchild. They may be right.

What to Do If You Have a Proposal for the Unified Field Theory?

…and what not to do

Due to volume of e-mail I have received (several thousand at last count) I cannot answer all requests, especially those from individuals who have a new proposal for completing Einstein's dream of a unified field theory, or a new theory of space and time.

However, I would like to give some guidelines for people who have thoughtfully pondered the question of the meaning of space-time.

1) Try to summarize the main idea or theme in a single paragraph. As Einstein once said, unless a theory has a simple underlying picture that the layman can understand, the theory is probably worthless. I will try to answer those proposals which are short and succinct, but I simply do not have time for proposals where the main idea is spread over many pages.

2) If you have a serious proposal for a new physical theory, submit it to a physics journal, just as Physical Review D or Nuclear Physics B. There, it will get the referee and serious attention that it deserves.

3) Remember that your theory will receive more credibility if your theory builds on top of previous theories, rather than making claims like "Einstein was wrong!" For example, our current understanding of the quantum theory and relativity, although incomplete, still gives us a framework for which we have not seen any experimental deviation.

Even Newtonian gravity works quite well within its domain (e.g. small velocities). Relativity is useful in its domain of velocities near the speed of light. However, even relativity breaks down for atomic distances, or gravitational fields found in the center of a black hole or the Big Bang. Similarly, the quantum theory works quite well at atomic distances, but has problems with gravity. A crude combination of the quantum theory and relativity works quite well from sub-atomic distances (10^-15 cm.) to cosmological distances (10^10 km), so your theory must improve on this!

4) Try not to use vague expressions that cannot be formulated precisely or mathematically, such as "time is quantized, " "energy is space, " or "space is twisted, " or "energy is a new dimension," etc. Instead, try to use mathematics to express your ideas. Otherwise, it's hard to understand what you are saying in a precise manner. Many referees will throw out papers which are just a collection of words, equating one mysterious concept (e.g. time) with another (e.g. light). The language of nature is mathematics (e.g. tensor calculus and Lie group theory). Try to formulate your ideas in mathematical form so that the referee has an idea of where you are coming from.

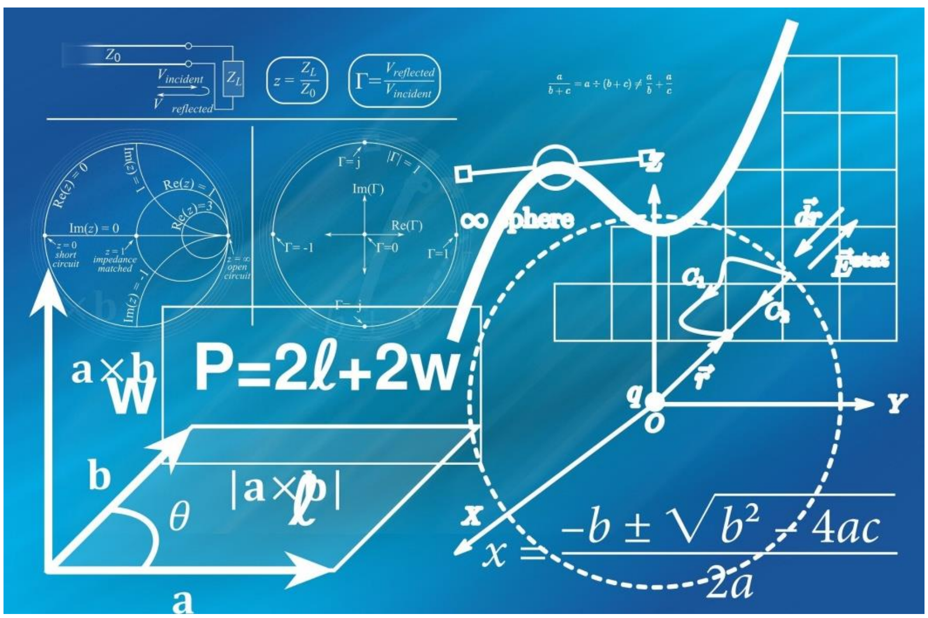
5) Once formulated mathematically, it's then relatively easy for a theoretical physicist to determine the precise nature of the theory. At the very least, your theory must contain the tensor equations of Einstein and the quantum theory of the Standard Model. If they lack these two ingredients, then your theory probably cannot describe nature as we know it. The fundamental problem facing physicists is that General Relativity and the quantum theory, when combined into a single theory, is not "renormalizable, " i.e. the theory blows up and becomes meaningless. Your proposal, therefore, has to give us a finite theory which combines these two formalisms. So far, only superstring theory can solve this key problem. Important: this means that, at the very minimum, your equations must contain the tensor equations of General Relativity and the Standard Model. If they do not include them, then your theory cannot qualify as a "theory of everything."

6) Most important, try to formulate an experiment that can test your idea. All science is based on reproducible results. No matter how outlandish your idea is, it must be accepted if it holds up experimentally. So try to think up an experiment which will distinguish your result from others. But remember, your theory has to explain the experiments that have already been done, which vindicate General Relativity and the quantum theory.

Good luck!

So You Want to Become a Physicist?

You have come to the right place



I've often been asked the question: how do you become a physicist? Let me first say that physicists, from a fairly early age, are fascinated by the universe and its fantastic wonders. We want to be part of the romantic, exciting adventure to tease apart its mysteries and understand the nature of physical reality.

That's the driving force behind our lives. We are more interested in black holes and the origin of the universe than with making tons of money and driving flashy cars. We also realize that physics forms the foundation for biology, chemistry, geology, etc. and the wealth of modern civilization. We realize that physicists pioneered the pivotal discoveries of the 20th century which revolutionized the world (e.g. the transistor, the laser, splitting the atom, TV and radio, MRI and PET scans, quantum theory and relativity, unraveling the DNA molecule was done by physicists.

But people often ask the question: do I have to be an Einstein to become a physicist? The answer is NO. Sure, physicists have to be proficient in mathematics, but the main thing is to have that curiosity and drive. One of the greatest physicists of all time, Michael Faraday, started out as a penniless, uneducated apprentice, but he was persistent and creative and then went on to revolutionize modern civilization with electric motors and dynamos. Much of the worlds gross domestic product depends on his work.

Einstein also said that behind every great theory there is a simple physical picture that even lay people can understand. In fact, he said, if a theory does not have a simple underlying picture, then the theory is probably worthless. The important thing is the physical picture; math is nothing but bookkeeping.

Steps to becoming a Physicist:

1) in high school, read popular books on physics and try to make contact with real physicists, if possible. (Role models are extremely important. If you cannot talk to a real physicist, read biographies of the giants of physics, to understand their motivation, their career path, the milestones in their career.) A role model can help you lay out a career path that is realistic and practical. The wheel has already been invented, so take advantage of a role model. Doing a science fair project is another way to plunge into the wonderful world of physics. Unfortunately, well-meaning teachers and counselors, not understanding physics, will probably give you a lot of useless advice, or may try to discourage you. Sometimes you have to ignore their advice.

Don’t get discouraged about the math, because you will have to wait until you learn calculus to understand most physics. (After all, Newton invented calculus in order to solve a physics problem: the orbit of the moon and planets in the solar system.)

Get good grades in all subjects and good SAT scores (i.e. don’t get too narrowly focused on physics) so you can be admitted to a top school, such as Harvard, Princeton, Stanford, MIT, Cal Tech. (Going to a top liberal arts college is sometimes an advantage over going to an engineering school, since it’s easier to switch majors if you have a career change.)

2) next, study four years of college. Students usually have to declare their majors in their sophomore (2nd) year in college; physics majors should begin to think about doing (a) experimental physics or (b) theoretical physics and choosing a specific field.

The standard four year curriculum:

a) first year physics, including mechanics and electricity and magnetism (caution: many universities make this course unnecessarily difficult, to weed out weaker engineers and physicists, so don’t be discouraged if you don't ace this course! Many future physicists do poorly in this first year course because it is made deliberately difficult.).

Also, take first (or second) year calculus.

b) second year physics – intermediate mechanics and EM theory.

Also, second year calculus, including differential equations and surface and volume integrals.

c) third year physics – a selection from: optics, thermodynamics, statistical mechanics, beginning atomic and nuclear theory

d) four year physics – elementary quantum mechanics

Within physics, there are many sub-disciplines you can choose from. For example, there is solid state, condensed matter, low temperature, and laser physics, which have immediate applications in electronics and optics. My own field embraces elementary particle physics as well as general relativity. Other branches include nuclear physics, astrophysics, geophysics, biophysics, etc.

Often you can apply for industrial jobs right after college. But for the higher paying jobs, it's good to get a higher degree.

3) so then there is graduate school. If your goal is to teach physics at the high school or junior college level, then obtaining a Masters degree usually involves two years of advanced course work but no original research. There is a shortage of physics teachers at the junior college and high school level.

If you want to become a research physicist or professor, you must get a Ph.D., which usually involves 4 to 5 years (sometimes more), and involves publishing original research. (This is not as daunting as it may seem, since usually this means finding a thesis advisor, who will simply assign you a research problem or include you in their experimental work.) Funding a Ph.D. is also not as hard as it seems, since a professor will usually have a grant or funding from the department to support you at a rate of about $12,000 per year or more. Compared to English or history graduate students, physics graduate students have a very cushy life.

After a Ph.D: Three sources of jobs

a) government

b) industry

c) the university

Government work may involve setting standards at the National Institute for Standards and Technology (the old National Bureau of Standards), which is important for all physics research. Government jobs pay well, but you will never become wealthy being a government physicist. But government work may also involve working in the weapons industry, which I highly discourage. (Not only for ethical reasons, but because that area is being downsized rapidly.)

Industrial work has its ebbs and flows. But lasers and semi-conductor and computer research will be the engines of the 21st century, and there will be jobs in these fields. One rewarding feature of this work is the realization that you are building the scientific architecture that will enrich all our lives. There is no job security at this level, but the pay can be quite good (especially for those in management positions – it's easier for a scientist to become a business manager than for a business major to learn science.) In fact, some of the wealthiest billionaires in the electronics industry and Silicon Valley came from physics/engineering backgrounds and then switched to management or set up their own corporation.

But I personally think a university position is the best, because then you can work on any problem you want. But jobs at the university are scarce; this may mean taking several two-year "post-doctorate" positions at various colleges before landing a teaching position as an assistant professor without tenure (tenure means you have a permanent position). Then you have 5-7 more years in which to establish a name for yourself as an assistant professor.

If you get tenure, then you have a permanent position and are promoted to associate professor and eventually full professor. The pay may average between $40,000 to $100,000, but there are also severe obstacles to this path.

In the 1960s, because of Sputnik, a tremendous number of university jobs opened up. The number of professors soared exponentially. But this could not last forever. By the mid 1970s, job expansion began inevitably to slow down, forcing many of my friends out of work. So the number of faculty positions leveled off in the 1980s.

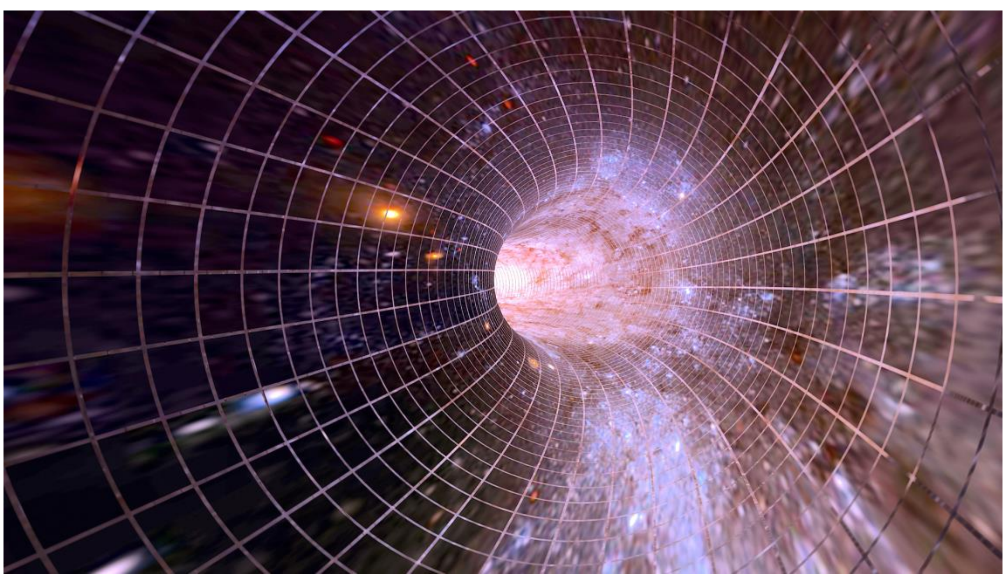
Then, many people predicted that, with the retirement of the Sputnik-generation, new jobs at the universities would open up in the 90s. Exactly the opposite took place. First, Congress passed legislation against age-discrimination, so professors could stay on as long as they like. Many physicists in their seventies decided to stay on, making it difficult to find jobs for young people. Second, after the cancellation of the SSC and the end of the Cold War, universities and government began to slowly downsize the funding for physics. As a result, the average age of a physicist increases 8 months per year, meaning that there is very little new hiring.

As I said, physicists do not become scientists for the money, so I don't want to downplay the financial problems that you may face. In fact, many superstring theorists who could not get faculty jobs went to Wall Street (where they were incorrectly called "rocket scientists"). This may mean leaving the field. However, for the diehards who wish to do physics in spite of a bad job market, you may plan to have a "fall-back" job to pay the bills (e.g. programming) while you conduct research on your own time.

But this dismal situation cannot last. Within ten years, the Sputnik-generation will finally retire, hopefully opening up new jobs for young, talented physicists. The funding for physics may never rival that of the Cold War, but physics will remain an indispensable part of creating the wealth of the 21st century. There are not many of us (about 30,000 or so are members of the American Physical Society) but we form the vanguard of the future. It also helps to join the APS and receive Physics Today magazine, which has an excellent back page which lists the various job openings around the country.

Hyperspace and a Theory of Everything

What lies beyond our 4 dimensions?



When I was a child, I used to visit the Japanese Tea Garden in San Francisco. I would spend hours fascinated by the carp, who lived in a very shallow pond just inches beneath the lily pads, just beneath my fingers, totally oblivious to the universe above them.

I would ask myself a question only a child could ask: what would it be like to be a carp? What a strange world it would be! I imagined that the pond would be an entire universe, one that is two-dimensional in space. The carp would only be able to swim forwards and backwards, and left and right. But I imagined that the concept of “up”, beyond the lily pads, would be totally alien to them. Any carp scientist daring to talk about “hyperspace”, i.e. the third dimension “above” the pond, would immediately be labelled a crank. I wondered what would happen if I could reach down and grab a carp scientist and lift it up into hyperspace. I thought what a wondrous story the scientist would tell the others! The carp would babble on about unbelievable new laws of physics: beings who could move without fins. Beings who could breathe without gills. Beings who could emit sounds without bubbles. I then wondered: how would a carp scientist know about our existence? One day it rained, and I saw the rain drops forming gentle ripples on the surface of the pond.

Then I understood.

The carp could see rippling shadows on the surface of the pond. The third dimension would be invisible to them, but vibrations in the third dimensions would be clearly visible. These ripples might even be felt by the carp, who would invent a silly concept to describe this, called “force.” They might even give these “forces” cute names, such as light and gravity. We would laugh at them, because, of course, we know there is no “force” at all, just the rippling of the water.

Today, many physicists believe that we are the carp swimming in our tiny pond, blissfully unaware of invisible, unseen uni- verses hovering just above us in hyperspace. We spend our life in three spatial dimensions, confident that what we can see with our telescopes is all there is, ignorant of the possibility of 10 dimensional hyperspace. Although these higher dimensions are invisible, their “ripples” can clearly be seen and felt. We call these ripples gravity and light. The theory of hyperspace, however, languished for many decades for lack of any physical proof or application. But the theory, once considered the province of eccentrics and mystics, is being revived for a simple reason: it may hold the key to the greatest theory of all time, the “theory of everything.”

Einstein spent the last 30 years of his life futilely chasing after this theory, the Holy Grail of physics. He wanted a theory that could explain the four fundamental forces that govern the universe: gravity, electromagnetism, and the two nuclear forces (weak and strong). It was supposed to be the crowning achievement of the last 2,000 years of science, ever since the Greeks asked what the world was made of. He was searching for an equation, perhaps no more than one-inch long, that could be placed on a T-shirt, but was so powerful it could explain every- thing from the Big Bang, exploding stars, to atoms and molecules, to the lilies of the field.

He wanted to read the mind of God. Ultimately, Einstein failed in his mission. In fact, he was shunned by many of his younger compatriots, who would taunt him with the ditty, “What God has torn asunder, no man can put together.” But perhaps Einstein is now having his revenge. For the past decade, there has been furious research on merging the four fundamental forces into a single theory, especially one that can meld general relativity (which explains gravity) with the quantum theory (which can explain the two nuclear forces and electro- magnetism).

The problem is that relativity and the quantum theory are precise opposites. General relativity is a theory of the very large: galaxies, quasars, black holes, and even the Big Bang. It is based on bending the beautiful four dimensional fabric of space and time. The quantum theory, by contrast, is a theory of the very small, i.e. the world of sub-atomic particles. It is based on discrete, tiny packets of energy called quanta. Over the past 50 years, many attempts have been tried to unite these polar opposites, and have failed. The road to the Unified Field Theory, the Theory of Everything, is littered with the corpses of failed attempts. The key to the puzzle may be hyperspace. In 1915, when Einstein said space-time was four dimensional and was warped and rippled, he showed that this bending produced a “force” called gravity. In 1921, Theodr Kaluza wrote that ripples of the fifth dimension could be viewed as light. Like the fish seeing the ripples in hyperspace moving in their world, many physicists believe that light is created by ripples in five-dimensional space-time.

But what about dimensions higher than 5?

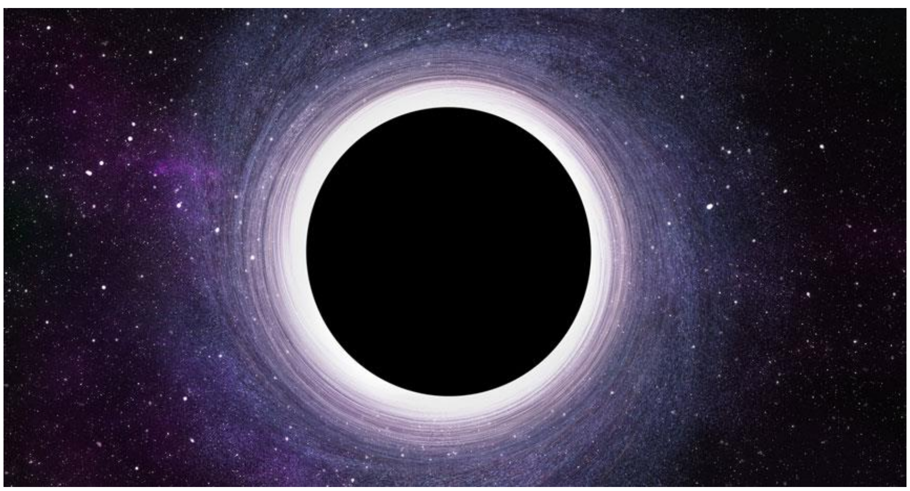
In principle, if we add more and more dimensions, we can ripple and bend them in different ways, thereby creating more forces. In 10 dimensions, in fact, we can accomodate all four fundamental forces! Actually, it’s not that simple. By naively going to 10 dimensions, we also introduce a host of esoteric mathematical inconsistencies (e.g. infinities and anomalies) that have killed all previous theories. The only theory which has survived every challenge posed to it is called superstring theory, in which this 10 dimensional universe is inhabited by tiny strings.

In fact, in one swoop, this 10 dimensional string theory gives us a simple, compelling unification of all forces. Like a violin string, these tiny strings can vibrate and create resonances or “notes”. That explains why there are so many sub- atomic particles: they are just notes on a superstring. (This seems so simple, but in the 1950s, physicists were drowning in an avalanche of sub-atomic particles. J.R. Oppenheim- er, who helped build the atomic bomb, even said, out of sheer frustration, that the Nobel Prize should go to the physicist who does NOT discover a new particle that year!) Similarly, when the string moves in space and time, it warps the space around it just as Einstein predicted. Thus, in a remarkably simple picture, we can unify gravity (as the bending of space caused by moving strings) with the other quantum forces (now viewed as vibrations of the string).

Of course, any theory with this power and majesty has a problem. This theory, because it is a theory of everything, is really a theory of Creation. Thus, to fully test the theory requires re-creating Creation! At first, this might seem hopelessly impossible. We can barely leave the earth’s puny gravity, let alone create universes in the laboratory. But there is a way out to this seemingly intractable problem. A theory of everything is also a theory of the everyday. Thus, this theory, when fully completed, will be able to explain the existence of protons, atoms, molecules, even DNA. Thus, the key is to fully solve the theory and test the theory against the known properties of the universe. At present, no one on earth is smart enough to complete the theory. The theory is perfectly well-defined, but you see, superstring theory is 21st Century physics that fell accidentally into the 20th century. It was discovered purely by accident, when two young physicists were thumbing through a mathematics book. The theory is so elegant and powerful, we were never "destined" to see it in the 20th century. The problem is that 21st century mathematics has not yet been invented yet. But since physicists are genetically predisposed to be optimists, I am confident that we will solve the theory someday soon. Perhaps a young person reading this article will be so inspired by this story that he or she will finish the theory. I can't wait!

Blackholes, Wormholes and the Tenth Dimension

Will these concepts be proven by a theory of everything?



Last June, astronomers were toasting each other with champagne glasses in laboratories around the world, savoring their latest discovery. The repaired $2 billion Hubble Space Telescope, once the laughing stock of the scientific community, had snared its most elusive prize: a black hole. But the discovery of the Holy Grail of astrophysics may also rekindle a long simmering debate within the physics community. What lies on the other side of a black hole? If someone foolishly fell into a black hole, will they be crushed by its immense gravity, as most physicists believe, or will they be propelled into a parallel universe or emerge in another time era? To solve this complex question, physicists are opening up one of the most bizarre and tantalizing chapters in modern physics. They have to navigate a minefield of potentially explosive theories, such as the possibility of "wormholes," "white holes," time machines, and even the 10th dimension! This controversy may well validate J.B.S. Haldane's wry observation that the universe is "not only queerer than we sup- pose, it is queerer than we can suppose." This delicious controversy, which delights theoretical physicists but boggles the mind of mere mortals, is the subject of my recent book, Hyperspace.

Black Holes: Collapsed Stars

A black hole, simply put, is a massive, dead star whose gravity is so intense than even light cannot escape, hence its name. By definition, it can't be seen, so NASA scientists focused instead on the tiny core of the galaxy M87, a super massive "cosmic engine" 50 million light years from earth. Astronomers then showed that the core of M87 consisted of a ferocious, swirling maelstrom of superhot hydrogen gas spinning at l.2 million miles per hour. To keep this spinning disk of gas from violently flying apart in all directions, there had to be a colossal mass concentrated at its center, weighing as much as 2 to 3 billion suns! An object with that staggering mass would be massive enough to prevent light from escaping. Ergo, a black hole.

The Einstein-Rosen Bridge

But this also revives an ongoing controversy surrounding black holes. The best description of a spinning black hole was given in 1963 by the New Zealand mathematician Roy Kerr, using Einstein's equations of gravity. But there is a quirky feature to his solution. It predicts that if one fell into a black hole, one might be sucked down a tunnel (called the "Einstein-Rosen bridge") and shot out a "white hole" in a parallel universe! Kerr showed that a spinning black hole would collapse not into a point, but to a "ring of fire." Because the ring was spinning rapidly, centrifugal forces would keep it from collapsing. Remarkably, a space probe fired directly through the ring would not be crushed into oblivion, but might actually emerge unscratched on the other side of the Einstein-Rosen bridge, in a parallel universe. This "wormhole" may connect two parallel universes, or even distant parts of the same universe.

Through the Looking Glass

The simplest way to visualize a Kerr wormhole is to think of Alice's Looking Glass. Anyone walking through the Looking Glass would be transported instantly into Wonderland, a world where animals talked in riddles and common sense wasn't so common.

The rim of the Looking Glass corresponds to the Kerr ring. Anyone walking through the Kerr ring might be transported to the other side of the universe or even the past. Like two Siamese twins joined at the hip, we now have two universes joined via the Looking Glass. Some physicists have wondered whether black holes or worm- holes might someday be used as shortcuts to another sector of our universe, or even as a time machine to the distant past (making possible the swashbuckling exploits in Star Wars). However, we caution that there are skeptics. The critics concede that hundreds of wormhole solutions have now been found to Einstein's equations, and hence they cannot be lightly dismissed as the ravings of crack pots. But they point out that wormholes might be unstable, or that intense radiation and sub-atomic forces surrounding the entrance to the wormhole would kill anyone who dared to enter. Spirited debates have erupted between physicists concerning these wormholes. Unfortunately, this controversy cannot be re-solved, because Einstein's equations break down at the center of black holes or wormholes, where radiation and sub-atomic forces might be ferocious enough to collapse the entrance. The problem is Einstein's theory only works for gravity, not the quantum forces which govern radiation and sub-atomic particles. What is needed is a theory which embraces both the quantum theory of radiation and gravity simultaneously. In a word, to solve the problem of quantum black holes, we need a "theory of everything!”

A Theory of Everything?

One of the crowning achievements of 20th century science is that all the laws of physics, at a fundamental level, can be summarized by just two formalisms: (1) Einstein’s theory of gravity, which gives us a cosmic description of the very large, i.e. galaxies, black holes and the Big Bang, and (2) the quantum theory, which gives us a microscopic description of the very small, i.e. the microcosm of sub-atomic particles and radiation. But the supreme irony, and surely one of Nature’s cosmic jokes, is that they look bewilderingly different; even the world’s greatest physicists, including Einstein and Heisenberg, have failed to unify these into one. The two theories use different mathematics and different physical principles to describe the universe in their respective domains, the cosmic and the microscopic. Fortunately, we now have a candidate for this theory. (In fact, it is the only candidate. Scores of rival proposals have all been shown to be inconsistent.) It’s called “superstring theory,” and almost effortlessly unites gravity with a theory of radiation, which is required to solve the problem of quantum wormholes. The superstring theory can explain the mysterious quantum laws of sub-atomic physics by postulating that sub-atomic particles are really just resonances or vibrations of a tiny string. The vibrations of a violin string correspond to musical notes; likewise the vibrations of a superstring correspond to the particles found in nature. The universe is then a symphony of vibrating strings. An added bonus is that, as a string moves in time, it warps the fabric of space around it, producing black holes, wormholes, and other exotic solutions of Einstein’s equations. Thus, in one stroke, the superstring theory unites both the theory of Einstein and quantum physics into one coherent, compelling picture.

A 10 Dimensional Universe

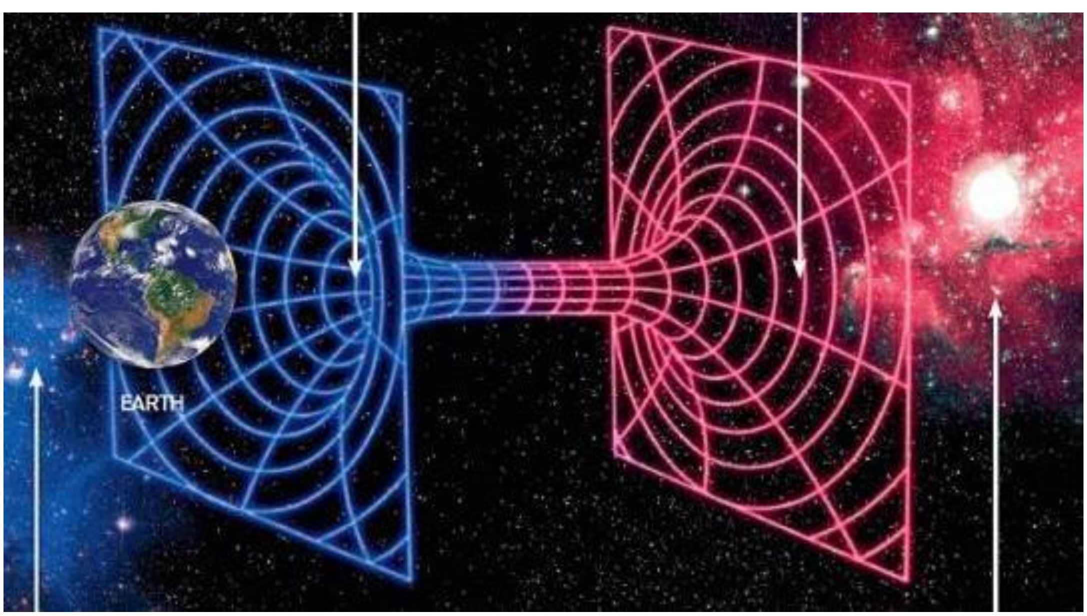
The curious feature of superstrings, however, is that they can only vibrate in 10 dimensions. This is, in fact, one of the reasons why it can unify the known forces of the universe: in 10 dimensions there is “more room” to accommodate both Einstein’s theory of gravity as well as sub-atomic physics. In some sense, previous attempts at unifying the forces of nature failed because a standard four dimensional theory is “too small” to jam all the forces into one mathematical framework. To visualize higher dimensions, consider a Japanese tea garden, where carp spend their entire lives swimming on the bottom of a shallow pond. The carp are only vaguely aware of a world beyond the surface. To a carp “scientist,” the universe only consists of two dimensions, length and width. There is no such thing as “height.” In fact, they are incapable of imagining a third dimension beyond the pond. The word “up” has no meaning for them. (Imagine their distress if we were to suddenly lift them out of their two dimensional universe into “hyperspace,” i.e. our world!) However, if it rains, then the surface of their pond becomes rippled. Although the third dimension is beyond their comprehension, they can clearly see the waves traveling on the pond’s surface. Likewise, although we earthlings cannot “see” these higher dimensions, we can see their ripples when they vibrate. According to this theory, “light” is nothing but vibrations rippling along the 5th dimension. By adding higher dimensions, we can easily accommodate more and more forces, including the nuclear forces. In a nutshell: the more dimensions we have, the more forces we can accommodate. One persistent criticism of this theory, however, is that we do not see these higher dimensions in the laboratory. At present, every event in the universe, from the tiniest sub-atomic decay to exploding galaxies, can be described by 4 numbers (length, width, depth, and time), not 10 numbers. To answer this criticism, many physicists believe (but cannot yet prove) that the universe at the instant of the Big Bang was in fact fully 10 dimensional. Only after the instant of creation did 6 of the 10 dimensions "curled up" into a ball too tiny to observe. In a real sense, this theory is really a theory of creation, when the full power of 10 dimensional space-time was manifest.

21st Century Physics

Not surprisingly, the mathematics of the 10th dimensional superstring is breathtakingly beautiful as well as brutally complex, and has sent shock waves through the mathematics community. Entirely new areas of mathematics have been opened up by this theory. Unfortunately, at present no one is smart enough to solve the problem of a quantum black hole. As Edward Witten of the Institute for Advanced Study at Princeton has claimed, "String theory is 21st century physics that fell accidentally into the 20th century." However, 21st century mathematics necessary to solve quantum black holes has not yet been discovered! However, since the stakes are so high, that hasn't stopped teams of enterprising physicists from trying to solve superstring theory. Already, over 5,000 papers have been written on the subject. As Nobel laureate Steve Weinberg said, "how can anyone expect that many of the brightest young theorists would not work on it?" Progress has been slow but steady. Last year, a significant breakthrough was announced. Several groups of physicists independently announced that string theory can completely solve the problem of a quantum black hole. (However, the calculation was so fiendishly difficult it could only be performed in two, not 10, dimensions.) So that's where we stand today. Many physicists now feel that it's only a matter of time before some enterprising physicist completely cracks this ticklish problem. The equations, although difficult, are well-defined. So until then, it's still a bit premature to buy tickets to the nearest wormhole to visit the next galaxy or hunt dinosaurs!

M-Theory: The Mother of all SuperStrings

An introduction to M-Theory



Every decade or so, a stunning breakthrough in string theory sends shock waves racing through the theoretical physics community, generating a feverish outpouring of papers and activity. This time, the Internet lines are burning up as papers keep pouring into the Los Alamos National Laboratory's computer bulletin board, the official clearing house for superstring papers. John Schwarz of Caltech, for example, has been speaking to conferences around the world proclaiming the "second superstring revolution." Edward Witten of the Institute for Advanced Study in Prince- ton gave a spell-binding 3 hour lecture describing it. The after- shocks of the breakthrough are even shaking other disciplines, like mathematics. The director of the Institute, mathematician Phillip Griffiths, says, "The excitement I sense in the people in the field and the spin-offs into my own field of mathematics … have really been quite extraordinary. I feel I've been very privileged to witness this first hand."

Cumrun Vafa at Harvard has said, "I may be biased on this one, but I think it is perhaps the most important development not only in string theory, but also in theoretical physics at least in the past two decades." What is triggering all this excitement is the discovery of something called "M-theory," a theory which may explain the origin of strings. In one dazzling stroke, this new M-theory has solved a series of long-standing puzzling mysteries about string theory which have dogged it from the beginning, leaving many theoretical physicists (myself included!) gasping for breath. M-theory, moreover, may even force string theory to change its name. Although many features of M-theory are still unknown, it does not seem to be a theory purely of strings. Michael Duff of Texas A & M is already giving speeches with the title "The theory formerly known as strings!" String theorists are careful to point out that this does not prove the final correctness of the theory. Not by any means. That may make years or decades more. But it marks a most significant breakthrough that is already reshaping the entire field.

Parable of the Lion

Einstein once said, "Nature shows us only the tail of the lion. But I do not doubt that the lion belongs to it even though he cannot at once reveal himself because of his enormous size." Einstein spent the last 30 years of his life searching for the "tail" that would lead him to the "lion," the fabled unified field theory or the "theory of everything," which would unite all the forces of the universe into a single equation. The four forces (gravity, electromagnetism, and the strong and weak nuclear forces) would be unified by an equation perhaps one inch long. Capturing the "lion" would be the greatest scientific achievement in all of physics, the crowning achievement of 2,000 years of scientific investigation, ever since the Greeks first asked themselves what the world was made of. But although Einstein was the first one to set off on this noble hunt and track the footprints left by the lion, he ultimately lost the trail and wandered off into the wilderness. Other giants of 20th century physics, like Werner Heisenberg and Wolfgang Pauli, also joined in the hunt. But all the easy ideas were tried and shown to be wrong. When Niels Bohr once heard a lecture by Pauli explaining his version of the unified field theory, Bohr stood up and said, "We in the back are all agreed that your theory is crazy. But what divides us is whether your theory is crazy enough!"

The trail leading to the unified field theory, in fact, is littered with the wreckage of failed expeditions and dreams. Today, however, physicists are following a different trail which might be "crazy enough" to lead to the lion. This new trail leads to superstring theory, which is the best (and in fact only) candidate for a theory of everything. Unlike its rivals, it has survived every blistering mathematical challenge ever hurled at it. Not surprisingly, the theory is a radical, "crazy" departure from the past, being based on tiny strings vibrating in 10 dimensional space-time. Moreover, the theory easily swallows up Einstein's theory of gravity. Witten has said, "Unlike conventional quantum field theory, string theory requires gravity. I regard this fact as one of the greatest in- sights in science ever made." But until recently, there has been a glaring weak spot: string theorists have been unable to probe all solutions of the model, failing miserably to examine what is called the "non-perturbative region," which I will describe shortly. This is vitally important, since ultimately our universe (with its wonderfully diverse collection of galaxies, stars, planets, sub- atomic particles, and even people) may lie in this "non-perturbative region." Until this region can be probed, we don't know if string theory is a theory of everything — or a theory of nothing! That's what today's excitement is all about. For the first time, using a powerful tool called "duality," physicists are now probing beyond just the tail, and finally seeing the outlines of a huge, unexpectedly beautiful lion at the other end. Not knowing what to call it, Witten has dubbed it "M-theory." In one stroke, M-theory has solved many of the embarrassing features of the theory, such as why we have 5 superstring theories. Ultimately, it may solve the nagging question of where strings come from.

"Pea Brains" and the Mother of all Strings

Einstein once asked himself if God had any choice in making the universe. Perhaps not, so it was embarrassing for string theorists to have five different self-consistent strings, all of which can unite the two fundamental theories in physics, the theory of gravity and the quantum theory.

Each of these string theories looks completely different from the others. They are based on different symmetries, with exotic names like E(8)xE(8) and O(32).

Not only this, but superstrings are in some sense not unique: there are other non-string theories which contain "super- symmetry," the key mathematical symmetry underlying superstrings. (Changing light into electrons and then into gravity is one of the rather astonishing tricks performed by supersymmetry, which is the symmetry which can exchange particles with half-integral spin, like electrons and quarks, with particles of integral spin, like photons, gravitons, and W-particles.

In 11 dimensions, in fact, there are alternate super theories based on membranes as well as point particles (called super- gravity). In lower dimensions, there is moreover a whole zoo of super theories based on membranes in different dimensions. (For example, point particles are 0-branes, strings are 1-branes, membranes are 2-branes, and so on.) For the p-dimensional case, some wag dubbed them p-branes (pronounced "pea brains"). But because p-branes are horribly difficult to work with, they were long considered just a historical curiosity, a trail that led to a dead-end. (Michael Duff, in fact, has collected a whole list of unflattering comments made by referees to his National Science Foundation grant concerning his work on p- branes. One of the more charitable comments from a referee was: "He has a skewed view of the relative importance of various concepts in modern theoretical physics.") So that was the mystery. Why should supersymmetry allow for 5 superstrings and this peculiar, motley collection of p-branes? Now we realize that strings, supergravity, and p-branes are just different aspects of the same theory. M-theory (M for "membrane" or the "mother of all strings," take your pick) unites the 5 superstrings into one theory and includes the p-branes as well. To see how this all fits together, let us update the famous parable of the blind wise men and the elephant. Think of the blind men on the trail of the lion. Hearing it race by, they chase after it and desperately grab onto its tail (a one-brane). Hanging onto the tail for dear life, they feel its one- dimensional form and loudly proclaim “It’s a string! It’s a string!”

But then one blind man goes beyond the tail and grabs onto the ear of the lion. Feeling a two-dimensional surface (a membrane), the blind man proclaims, “No, it’s really a two-brane!” Then another blind man is able to grab onto the leg of the lion. Sensing a three-dimensional solid, he shouts, “No, you’re both wrong. It’s really a three-brane!” Actually, they are all right. Just as the tail, ear, and leg are different parts of the same lion, the string and various p- branes appear to be different limits of the same theory: M-theory. Paul Townsend of Cambridge University, one of the architects of this idea, calls it “p-brane democracy,” i.e. all p- branes (including strings) are created equal. Schwarz puts a slightly different spin on this. He says, “we are in an Orwellian situation: all p-branes are equal, but some (namely strings) are more equal than others. The point is that they are the only ones on which we can base a perturbation theory.” To understand unfamiliar concepts such as duality, perturbation theory, non-perturbative solutions, it is instructive to see where these concepts first entered into physics.

Dualty

The key tool to understanding this breakthrough is something “duality.” Loosely speaking, two theories are “dual” to each other if they can be shown to be equivalent under a certain interchange. The simplest example of duality is reversing the role of electricity and magnetism in the equations discovered by James Clerk Maxwell of Cambridge University 130 years ago. These are the equations which govern light, TV, X-rays, radar, dynamos, motors, transformers, even the Internet and computers. The remarkable feature about these equations is that they remain the same if we interchange the magnetic B and electric fields E and also switch the electric charge e with the magnetic charge g of a magnetic “monopole”: E <–> B and e <–> g (In fact, the product eg is a constant.) This has important implications. Often, when a theory cannot be solved exactly, we use an approximation scheme. In first year calculus, for example, we recall that we can approximate certain functions by Taylor’s expansion. Similarly, since e^2 = 1/137 in certain units and is hence a small number, we can always approximate the theory by power expanding in e^2. So we add contributions of order e^2 + e^4 + e^6 etc. in solving for, say, the collision of two particles. Notice that each contribution is getting smaller and smaller, so we can in principle add them all up. This generalization of Taylor’s expansion is called “perturbation theory,” where we perturb the system with terms containing e^2. For example, in archery, perturbation theory is how we aim our arrows. With every motion of our arms, our bow gets closer and closer to aligning with the bull’s eye.) But now try expanding in g^2. This is much tougher; in fact, if we expand in g^2, which is large, then the sum g^2 + g^4 + g^6 etc. blows up and becomes meaningless. This is the reason why the “non-perturbative” region is so difficult to probe, since the theory simply blows up if we try to naively use perturbation theory for large coupling constant g. So at first it appears hopeless that we could ever penetrate into the non-perturbative region. (For example, if every motion of our arms got bigger and bigger, we would never be able to zero in and hit the target with the arrow.) But notice that because of duality, a theory of small e (which is easily solved) is identical to a theory of large g (which is difficult to solve). But since they are the same theory, we can use duality to solve for the non-perturbative region.

S, T, and U Dualty

The first inkling that duality might apply in string theory was discovered by K. Kikkawa and M. Yamasaki of Osaka Univ. in 1984. They showed that if you “curled up” one of the extra dimensions into a circle with radius R, the theory was the same if we curled up this dimension with radius 1/R. This is now called T- duality: R <–> 1/R When applied to various superstrings, one could reduce 5 of the string theories down to 3 (see figure). In 9 dimensions (with one dimension curled up) the Type IIa and IIb strings were identical, as were the E(8)xE(8) and O(32) strings.

Unfortunately, T duality was still a perturbative duality. The next breakthrough came when it was shown that there was a second class of dualities, called S duality, which provided a duality between the perturbative and non-perturbative regions of string theory. Another duality, called U duality, was even more powerful.

Then Nathan Seiberg and Witten brilliantly showed how another form of duality could solve for the non-perturbative region in four dimensional supersymmetric theories. However, what finally convinced many physicists of the power of this technique was the work of Paul Townsend and Edward Wit- ten. They caught everyone by surprise by showing that there was a duality between 10 dimensional Type IIa strings and 11 dimension- al supergravity! The non-perturbative region of Type IIa strings, which was previously a forbidden region, was revealed to be governed by 11 dimensional supergravity theory, with one dimension curled up. At this point, I remember that many physicists (myself included) were rubbing our eyes, not believing what we were seeing. I remember saying to myself, “But that’s impossible!”

All of a sudden, we realized that perhaps the real “home” of string theory was not 10 dimensions, but possibly 11, and that the theory wasn’t fundamentally a string theory at all! This revived tremendous interest in 11 dimensional theories and p- branes. Lurking in the 11th dimension was an entirely new theory which could reduce down to 11 dimensional supergravity as well as 10 dimensional string theory and p-brane theory.

Detractors of String Theories

To the critics, however, these mathematical developments still don’t answer the nagging question: how do you test it? Since string theory is really a theory of Creation, when all its beautiful symmetries were in their full glory, the only way to test it, the critics wail, is to re-create the Big Bang itself, which is impossible. Nobel Laureate Sheldon Glashow likes to ridicule superstring theory by comparing it with former Pres. Reagan’s Star Wars plan, i.e. they are both untestable, soak up resources, and both siphon off the best scientific brains.

Actually, most string theorists think these criticisms are silly. They believe that the critics have missed the point. The key point is this: if the theory can be solved non- perturbatively using pure mathematics, then it should reduce down at low energies to a theory of ordinary protons, electrons, atoms, and molecules, for which there is ample experimental data. If we could completely solve the theory, we should be able to extract its low energy spectrum, which should match the familiar particles we see today in the Standard Model. Thus, the problem is not building atom smashers l,000 light years in diameter; the real problem is raw brain power: of only we were clever enough, we could write down M-theory, solve it, and settle everything.

Evolving Backwards

So what would it take to actually solve the theory once and for all and end all the speculation and back-biting? There are several approaches. The first is the most direct: try to derive the Standard Model of particle interactions, with its bizarre collection of quarks, gluons, electrons, neutrinos, Higgs bosons, etc. etc. etc. (I must admit that although the Standard Model is the most successful physical theory ever proposed, it is also one of the ugliest.) This might be done by curling up 6 of the 10 dimensions, leaving us with a 4 dimensional theory that might resemble the Standard Model a bit. Then try to use duality and M- theory to probe its non-perturbative region, seeing if the symmetries break in the correct fashion, giving us the correct masses of the quarks and other particles in the Standard Model. Witten's philosophy, however, is a bit different. He feels that the key to solving string theory is to understand the under- lying principle behind the theory.

Let me explain. Einstein's theory of general relativity, for example, started from first principles. Einstein had the "happiest thought in his life" when he leaned back in his chair at the Bern patent office and realized that a person in a falling elevator would feel no gravity. Although physicists since Galileo knew this, Einstein was able to extract from this the Equivalence Principle. This deceptively simple statement (that the laws of physics are indistinguishable locally in an accelerating or a gravitating frame) led Einstein to introduce a new symmetry to physics, general co-ordinate transformations. This in turn gave birth to the action principle behind general relativity, the most beautiful and compelling theory of gravity. Only now are we trying to quantize the theory to make it compatible with the other forces. So the evolution of this theory can be summarized as: Principle -> Symmetry -> Action -> Quantum Theory According to Witten, we need to discover the analog of the Equivalence Principle for string theory. The fundamental problem has been that string theory has been evolving "backwards." As Witten says, "string theory is 21st century physics which fell into the 20th century by accident." We were never "meant" to see this theory until the next century.

Is the End in Sight?

Vafa recently added a strange twist to this when he introduced yet another mega-theory, this time a 12 dimensional theory called F-theory (F for "father") which explains the self-duality of the IIb string. (Unfortunately, this 12 dimensional theory is rather strange: it has two time co-ordinates, not one, and actually violates 12 dimensional relativity. Imagine trying to live in a world with two times! It would put an episode of Twilight Zone to shame.) So is the final theory 10, 11, or 12 dimensional?

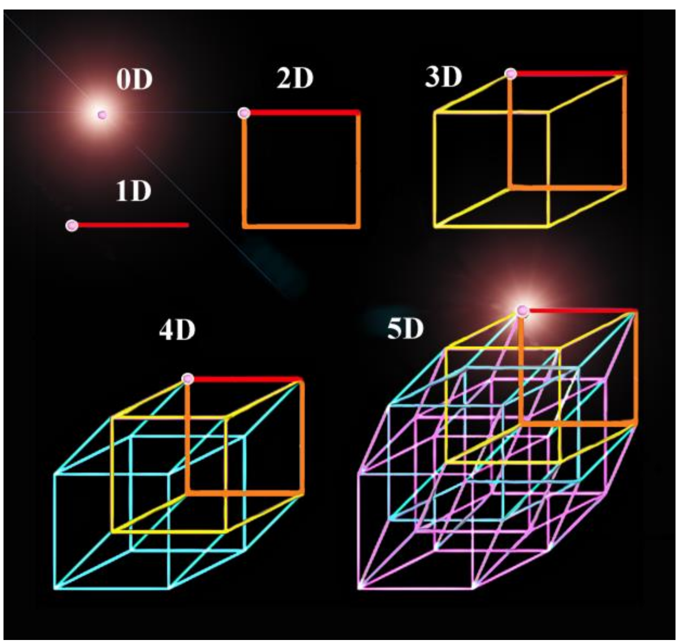
Schwarz, for one, feels that the final version of M-theory may not even have any fixed dimension. He feels that the true theory may be independent of any dimensionality of space-time, and that 11 dimensions only emerges once one tries to solve it. Townsend seems to agree, saying "the whole notion of dimensionality is an approximate one that only emerges in some semiclassical context." So does this means that the end is in sight, that we will someday soon derive the Standard Model from first principles?

I asked some of the leaders in this field to respond to this question. Although they are all enthusiastic supporters of this revolution, they are still cautious about predicting the future. Townsend believes that we are in a stage similar to the old quantum era of the Bohr atom, just before the full elucidation of quantum mechanics. He says, “We have some fruitful pictures and some rules analogous to the Bohr-Sommerfeld quantization rules, but it’s also clear that we don’t have a complete theory.”

Duff says, “Is M-theory merely a theory of supermembranes and super 5-branes requiring some (as yet unknown) non- perturbative quantization, or (as Witten believes) are the under- lying degrees of freedom of M-theory yet to be discovered? I am personally agnostic on this point.” Witten certainly believes we are on the right track, but we need a few more “revolutions” like this to finally solve the theory. “I think there are still a couple more superstring revolutions in our future, at least. If we can manage one more superstring revolution a decade, I think that we will do all right,” he says. Vafa says, “I hope this is the ‘light at the end of the tunnel’ but who knows how long the tunnel is!” Schwarz, moreover, has written about M-theory: “Whether it is based on something geometrical (like supermembranes) or some- thing completely different is still not known. In any case, finding it would be a landmark in human intellectual history.” Personally, I am optimistic. For the first time, we can see the outline of the lion, and it is magnificent. One day, we will hear it roar.

Hyperspace – A Scientific Odyssey

A look at the higher dimensions



Do higher dimensions exist? Are there unseen worlds just beyond our reach, beyond the normal laws of physics? Although higher dimensions have historically been the exclusive realm of charlatans, mystics, and science fiction writers, many serious theoretical physicists now believe that higher dimensions not only exist, but may also explain some of the deepest secrets of nature. Although we stress that there is at present no experimental evidence for higher dimensions, in principle they may solve the ultimate problem in physics: the final unification of all physical knowledge at the fundamental level.

My own fascination with higher dimensions began early in childhood. One of my happiest childhood memories was crouching next to the pond at the famed Japanese Tea Garden in San Francisco, mesmerized by the brilliantly colored carp swimming slowly beneath the water lilies. In these quiet moments, I would ask myself a silly question that a only child might ask: how would the carp in that pond view the world around them? Spending their entire lives at the bottom of the pond, the carp would believe that their "universe" consisted of the water and the lilies; they would only be dimly aware that an alien world could exist just above the surface. My world was beyond their comprehension. I was intrigued that I could sit only a few inches from the carp, yet we were separated by an immense chasm. I concluded that if there were any "scientists" among the carp, they would scoff at any fish who proposed that a parallel world could exist just above the lilies. An unseen world beyond the pond made no scientific sense. Once I imagined what would happen if I reached down and suddenly grabbed one of the carp "scientists" out of the pond. I wondered, how would this appear to the carp? The startled carp "scientist" would tell a truly amazing story, being somehow lifted out of the universe (the pond) and hurled into a mysterious nether world, another dimension with blinding lights and strange-shaped objects that no carp had ever seen before. The strangest of all was the massive creature responsible for this outrage, who did not resemble a fish in the slightest. Shockingly, it had no fins whatsoever, but nevertheless could move without them. Obviously, the familiar laws of physics no longer applied in this nether world!

The Theory of Everything

Sometimes I believe that we are like the carp living contently on the bottom of that pond; we live our lives blissfully ignorant of other worlds that might co-exist with us, laughing at any suggestion of parallel universes.

All this has changed rather dramatically in the past few years. The theory of higher dimensional space may now become the central piece in unlocking the origin of the universe. At the center of this conceptual revolution is the idea that our familiar three dimensional universe is "too small" to describe the myriad forces governing our universe. To describe our physical world, with its almost infinite variety of forms, requires entire libraries overflowing with mountains of technical journals and stacks of obscure, learned books. The ultimate goal of physics, some believe, is to have a single equation or expression from which this colossal volume of information can be derived from first principles. Today, many physicists believe that we have found the "unified field theory" which eluded Einstein for the last thirty years of his life. Although the theory of higher dimensional space has not been verified (and, we shall see, would be prohibitively expensive to prove experimentally), almost 5,000 papers, at last count, have been published in the physics literature concerning higher dimensional theories, beginning with the pioneering papers of

Theodore Kaluza and Oskar Klein in the 1920's and 30s, to the supergravity theory of the 1970s, and finally to the superstring theory of the 1980s and 90s. In fact, the superstring theory, which postulates that matter consists of tiny strings vibrating in hyperspace, predicts the precise number of dimensions of space and time: 10.

Why Can't we See the Fourth Dimension?

To understand these higher dimensions, we remember that it takes three numbers to locate every object in the universe, from the tip of your nose to the ends of the world. For example, if you want to meet some friends in Manhattan, you tell them to meet you at the building at the corner of 42nd street and 5th avenue, on the 37th floor. It takes two numbers to locate your position on a map, and one number to specify the distance above the map. It thus takes three numbers to specify the location of your lunch. (If we meet our friends at noon, then it takes four numbers to specify the space and time of the meeting.)

However, try as we may, it is impossible for our brains to visualize the fourth spatial dimension. Computers, of course, have no problem working in N dimensional space, but spatial dimensions beyond three simply cannot be conceptualized by our feeble brains. (The reason for this unfortunate accident has to do with biology, rather than physics. Human evolution put a premium on being able to visualize objects moving in three dimensions. There was a selection pressure placed on humans who could dodge lunging saber tooth tigers or hurl a spear at a charging mammoth. Since tigers do not attack us in the fourth spatial dimension, there simply was no advantage in developing a brain with the ability to visualize objects moving in four dimensions.)

Meeting a Higher Dimensional Being

To understand some of the mind-bending features of higher dimensions, imagine a two-dimensional world, called Flat land (after Edwin A. Abbott's celebrated novel) that resembles a world existing on a flat table-top. If one of the Flatlanders becomes lost, we can quickly scan all of Flatland, peering directly inside houses, buildings, and even concealed places. If one of the Flatlanders becomes sick, we can reach directly into their insides and per form surgery, without ever cutting their skin. If one of the Flatlanders is incarcerated in jail (which is a circle enclosing the Flatlander) we can simply peel the person off from Flatland into the third dimension and place the Flatlander back somewhere else. If we become more ambitious and stick our fingers and arms through Flatland, the Flatlanders would only see circles of flesh that hover around them, constantly changing shape and merging into other circles. And lastly, if we fling a Flatlander into our three dimensional world, the Flatlander can only see two dimensional cross sections of our world, i.e. a phantasmagoria of circles, squares, etc. which constantly change shape and merge (see fig. 1 and 2). Now imagine that we are "three dimensional Flatlanders" being visited by a higher dimensional being. If we became lost, a higher dimensional being could scan our entire universe all at once, peering directly into the most tightly sealed hiding places. If we became sick, a higher dimensional being could reach into our insides and perform surgery without ever cutting our skin. If we were in a maximum-security, escape-proof jail, a higher dimensional being could simply "yank" us into a higher dimension and redeposit us back somewhere else. If higher dimensional beings stick their "fingers" into our universe, they would appear to us to be blobs of flesh which float above us and constantly merge and split apart. And lastly, if we are flung into hyperspace, we would see a collection of spheres, blobs, and polyhedra which suddenly appear, constantly change shape and color, and then mysteriously disappear. Higher dimensional people, therefore, would have powers similar to a god: they could walk through walls, disappear and reappear at will, reach into the strongest steel vaults, and see through buildings. They would be omniscient and omnipotent. Not surprisingly, speculation about higher dimensions has sparked enormous literary and artistic interest over the last hundred years.

Mystics and Mathematics

Fyodor Dostoyevsky, in The Brothers Karamazov, had his protagonist Ivan Karamazov speculate on the existence of higher dimensions and non-Euclidean geometries during a discussion on the existence of God. In H. G. Wells' The Invisible Man, the source of invisibility was his ability to manipulate the fourth dimension. Oscar Wilde even refers to the fourth dimension in his play The Canterville Ghost as the homeworld for ghosts.

The fourth dimension also appears in the literary works of Marcel Proust and Joseph Conrad; it inspired some of the musical works of Alexander Scriabin, Edgar Varege, and George Antheil. It fascinated such diverse personalities as the psychologist William James, literary figure Gertrude Stein, and revolutionary socialist Vladimir Lenin. Lenin even waged a polemic on the N-th dimension with philosopher Ernst Mach in his Materialism and Empirio-Criticism. Lenin praised Mach, who "has raised the very important and useful question of a space of n-dimensions as a conceivable space," but then took him to task by insisting that the Tsar could only be overthrown in the third dimension.

Artists have been particularly interested in the fourth dimension because of the possibilities of discovering new laws of perspective. In the Middle Ages, religious art was distinctive for its deliberate lack of perspective. Serfs, peasants, and kings were depicted as if they were flat, much the way children draw people. Since God was omnipotent and could therefore see all parts of our world equally, art had to reflect His point of view, so the world was painted two-dimensionally. Renaissance art was a revolt against this flat God- centered perspective. Sweeping landscapes and realistic, three dimensional people were painted from the point of view of a person's eye, with the lines of perspective vanishing into the horizon. Renaissance art reflected the way the human eye viewed the world, from the singular point of view of the observer. In other words, Renaissance art discovered the third dimension. With the beginning of the machine age and capitalism, the artistic world revolted against the cold materialism that seemed to dominate industrial society. To the Cubists, positivism was a straitjacket that confined us to what could be measured in the laboratory, suppressing the fruits of our imagination. They asked: Why must art be clinically "realistic?" This Cubist "revolt against perspective" seized the fourth dimension because it touched the third dimension from all possible perspectives. Simply put, Cubist art embraced the fourth dimension. Picasso's paintings are a splendid example, showing a clear rejection of three dimensional perspective, with women's faces viewed simultaneously from several angles. Instead of a single point-of-view, Picasso's paintings show multiple perspectives, as if they were painted by a being from the fourth dimension, able to see all perspectives simultaneously. As art historian Linda Henderson has written, "the fourth dimension and non-Euclidean geometry emerge as among the most important themes unifying much of modern art and theory."

Unifying the Four Forces

Historically, physicists dismissed the theory of higher dimensions because they could never be measured, nor did they have any particular use. But to understand how adding higher dimensions can, in fact, simplify physical problems, consider the following example. To the ancient Egyptians, the weather was a complete mystery. What caused the seasons? Why did it get warmer as they traveled south? The weather was impossible to explain from the limited vantage point of the ancient Egyptians, to whom the earth appeared flat, like a two-dimensional plane.

But now imagine sending the Egyptians in a rocket into outer space, where they can see the earth as simple and whole in its orbit around the sun. Suddenly, the answers to these questions become obvious. From outer space, it is clear that the earth tilts about 23 degrees on its axis in its orbit around the sun. Because of this tilt, the northern hemisphere receives much less sunlight during one part of its orbit than during another part. Hence we have winter and summer. And since the equator receives more sunlight on the average than the northern or southern polar regions, it becomes warmer as we approach the equator.

In summary, the rather obscure laws of the weather are easy to understand once we view the earth from space. Thus, the solution to the problem is to go up into space, into the third dimension. Facts that were impossible to understand in a flat world suddenly become obvious when viewing a unified picture of a three dimensional earth.

The Four Fundemental Forces

Similarly, the current excitement over higher dimensions is that they may hold the key to the unification of all known forces. The culmination of 2,000 years of painstaking observation is the realization that that our universe is governed by four fundamental forces. These four forces, in turn, may be unified in higher dimensional space. Light, for example, may be viewed simply as vibrations in the fifth dimension. The other forces of nature may be viewed as vibrations in increasingly higher dimensions. At first glance, however, the four fundamental forces seem to bear no resemblance to each other. They are:

Gravity is the force which keeps our feet anchored to the spinning earth and binds the solar system and the galaxies together. Without gravity, we would be immediately flung into outer space at l,000 miles per hour. Furthermore, without gravity holding the sun together, it would explode in a catastrophic burst of energy. Electro-magnetism is the force which lights up our cities and energizes our household appliances. The electronic revolution, which has given us the light bulb, TV, the telephone, computers, radio, radar, microwaves, light bulbs, and dishwashers, is a byproduct of the electro-magnetic force.

The strong nuclear force is the force which powers the sun. Without the nuclear force, the stars would flicker out and the heavens would go dark. The nuclear force not only makes life on earth possible, it is also the devastating force unleashed by a hydrogen bomb, which can be compared to a piece of the sun brought down to earth. The weak force is the force responsible for radio active decay involving electrons. The weak force is harnessed in modern hospitals in the form of radioactive tracers used in nuclear medicine. The weak force also wrecked havoc at Chernobyl. Historically, whenever scientists unraveled the secrets of one of the four fundamental forces, this irrevocably altered the course of modern civilization, from the mastery of mechanics and Newtonian physics in the 1700s, to the harnessing of the electro-magnetism in the 1800s, and finally to the unlocking of the nuclear force in the 1900s. In some sense, some of the greatest breakthroughs in the history of science can be traced back to the gradual understanding of these four fundamental forces. Some have even claimed that the progress of the last 2,000 years of science can be understood as the successive mastery of these four fundamental forces. Given the importance of these four fundamental forces, the next question is: can they be united into one super force? Are they but the manifestations of a deeper reality? Given the fruitless search that has stumped the world's Nobel Prize winners for half a century, most physicists agree that the Theory of Everything must be a radical departure from everything that has been tried before. For example, Niels Bohr, founder of the modern atomic theory, once listened to Wolf gang Pauli's explanation of his version of the unified field theory. In frustration, Bohr finally stood up and said, "We are all agreed that your theory is absolutely crazy. But what divides us is whether your theory is crazy enough."

Today, however, after decades of false starts and frustrating dead ends, many of the world's leading physicists think that they have finally found the theory "crazy enough" to be the unified field theory. There is widespread belief (although certainly not unanimous by any means) in the world's major re search laboratories that we have at last found the Theory of Everything.

Field Theory in Higher Dimension

To see how higher dimensions helps to unify the laws of nature, physicists use the mathematical device called "field theory." For example, the magnetic field of a bar magnet resembles a spider's web which fills up all of space. To describe the magnetic field, we introduce the field, a series of numbers defined at each point in space which describes the intensity and direction of the force at that point. James Clerk Maxwell, in the last century, proved that the electro-magnetic force can be described by four numbers at each point in four dimensional space-time (labeled by A _ 1, A _ 2 , A _ 3 , A _ 4). These four numbers, in turn, obey a set of equations (called Maxwell's field equations).

For the gravitational force, Einstein showed that the field requires a total of 10 numbers at each point in four dimensions. These 10 numbers can be assembled into the array shown in fig. 3. (Since g _ 12 = g _ 21 , only 10 of the 16 numbers contained within the array are independent.) The gravitational field, in turn, obey Einstein's field equations. The key idea of Theodore Kaluza in the 1920s was to write down a five dimensional theory of gravity. In five dimensions, the gravitational field has 15 independent numbers, which can be arranged in a five dimensional array (see fig.4). Kaluza then re-defined the 5th column and row of the gravitation al field to be the electromagnetic field of Maxwell. The truly miraculous feature of this construction is that the five dimensional theory of gravity reduces down precisely to Einstein's original theory of gravity plus Maxwell's theory of light. In other words, by adding the fifth dimension, we have trivially unified light with gravity. In other words, light is now viewed as vibrations in the fifth dimension. In five dimensions, there is "enough room" to unify both gravity and light.

This trick is easily extended. For example, if we generalize the theory to N dimensions, then the N dimensional gravitational field can be split-up into the following pieces (see fig. 5). Now, out pops a generalization of the electromagnetic field, called the “Yang-Mills field,” which is known to describe the nuclear forces. The nuclear forces, therefore, may be viewed as vibrations of higher dimensional space. Simply put, by adding more dimensions, we are able to describe more forces. Similarly, by adding higher dimensions and further embellishing this approach (with something called “supersymmetry), we can explain the entire particle “zoo” that has been discovered over the past thirty years, with bizarre names like quarks, neutrinos, muons, gluons, etc. Although the mathematics required to extend the idea of Kaluza has reached truly breathtaking heights, startling even professional mathematicians, the basic idea behind unification remains surprisingly simple: the forces of nature can be viewed as vibrations in higher dimensional space.

What Happened Before the Big Bang?

One advantage to having a theory of all forces is that we may be able to resolve some of the thorniest, long-standing questions in physics, such as the origin of the universe, and the existence of “wormholes” and even time machines. The 10 dimensional superstring theory, for example, gives us a compelling explanation of the origin of the Big Bang, the cosmic explosion which took place 15 to 20 billion years ago, which sent the stars and galaxies hurling in all directions. In this theory, the universe originally started as a perfect 10 dimensional universe with nothing in it. In the beginning, the universe was completely empty. However, this 10 dimensional universe was not stable. The original 10 dimensional space-time finally “cracked” into two pieces, a four and a six dimensional universe. The universe made the “quantum leap” to another universe in which six of the 10 dimensions collapsed and curled up into a tiny ball, allowing the remaining four dimensional universe to explode outward at an enormous rate. The four dimensional universe (our world) expanded rapidly, creating the Big Bang, while the six dimensional universe wrapped itself into a tiny ball and shrunk down to infinitesimal size. This explains the origin of the Big Bang. The cur rent expansion of the universe, which we can measure with our instruments, is a rather minor aftershock of a more cataclysmic collapse: the breaking of a 10 dimensional universe into a four and six dimensional universe.

In principle, this also explains why we cannot measure the six dimensional universe, because it has shrunk down to a size much smaller than an atom. Thus, no earth-bound experiment can measure the six dimensional universe because it has curled up into a ball too small to be analyzed by even our most powerful instruments. (This will be disappointing to those who would like to visit these higher dimensions in their lifetimes. These higher dimensions are much too small to enter.)

Time Machines?

Another longstanding puzzle concerns parallel universes and time travel. According to Einstein’s theory of gravity, space-time can be visualized as a fabric which is stretched and distorted by the presence of matter and energy. The gravitational field of a black hole, for example, can be visualized as a funnel, with a dead, collapsed star at the very center (see fig. 6). Anyone unfortunate enough to get too close to the funnel inexorably falls into it and is crushed to death. One puzzle, however, is that, according to Einstein’s equations, the funnel of a black hole necessarily connects our universe with a parallel universe.

Furthermore, if the funnel connects our universe with itself, then we have a "worm hole" (see fig. 7). These anomalies did not bother Einstein because it was thought that travel through the neck of the funnel, called the "Einstein-Rosen bridge," would be impossible (since anyone falling into the black hole would be killed).

However, over the years physicists like Roy Kerr as well as Kip Thorne at the Calif. Institute of Technology have found new solutions of Einstein's equations in which the gravitational field does not become infinite at the center, i.e. in principle, a rocket ship could travel through the Einstein- Rosen bridge to an alternate universe (or a distant part of our own universe) without being ripped apart by intense gravitational fields. (This wormhole is, in fact, the mathematical representation of Alice's Looking Glass.)

Even more intriguing, these wormholes can be viewed as time machines. Since the two ends of the wormhole can connect two time eras, Thorne and his colleagues have calculated the conditions necessary to enter the wormhole in one time era and exit the other side at another time era. (Thorne is undaunted by the fact that the energy necessary to open an Einstein-Rosen bridge exceeds that of a star, and is hence beyond the reach of present-day technology. But to Thorne, this is just a small detail for the engineers of some sufficiently advanced civilization in outer space!) Thorne even gives a crude idea of what a time machine might look like when built. (Imagine, however, the chaos that could erupt if time machines were as common as cars. History books could never be written. Thousands of meddlers would constantly be going back in time to eliminate the ancestors of their enemies, to change the outcome of World War I and II, to save John Kennedy's and Abraham Lincoln's life, etc. "History" as we know it would become impossible, throwing professional historians out of work. With every turn of a time machine's dial, history would be changing like sands being blown by the wind.) Other physicists, however, like Steven Hawking, are dubious about time travel. They argue that quantum effects (such as intense radiation fields at the funnel) may close the Einstein-Rosen bridge. Hawking even advanced an experimental "proof" that time machines are not possible (i.e. if they existed, we would have been visited by tourists from the future).

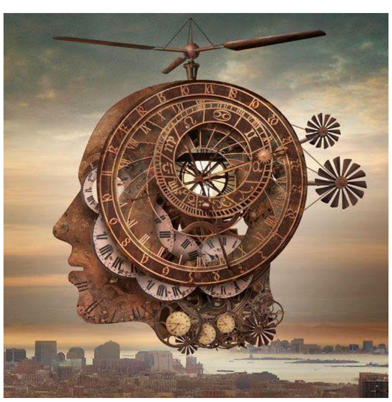
This controversy has recently generated a flurry of papers in the physics literature. The essential problem is that although Einstein's equations for gravity allow for time travel, they also break down when approaching the black hole, and quantum effects, such as radiation, take over. But to calculate if these quantum corrections are intense enough to close the Einstein-Rosen bridge, one necessarily needs a unified field theory which includes both Einstein's theory of gravity as well as the quantum theory of radiation. So there is hope that soon these questions may be answered once and for all by a unified field theory. Both sides of the controversy over time travel acknowledge that ultimately this question will be resolved by the Theory of Everything.

Recreating Creation

Although the 10 dimensional superstring theory has been called the most fascinating discovery in theoretical physics in the past decades, its critics have focused on its weakest point, that it is almost impossible to test. The energy at which the four fundamental forces merge into a single, unified force occurs at the fabulous “Planck energy,” which is a billion billion times greater than the energy found in a proton. Even if all the nations of the earth were to band together and single-mindedly build the biggest atom smasher in all history, it would still not be enough to test the theory. Because of this, some physicists have scoffed at the idea that superstring theory can even be considered a legitimate “theory.” Nobel laureate Sheldon Glashow, for example, has compared the superstring theory to the former Pres. Reagan’s Star Wars program (because it is untestable and drains the best scientific talent).The reason why the theory cannot be tested is rather simple. The Theory of Everything is necessarily a theory of Creation, that is, it must explain everything from the origin of the Big Bang down to the lilies of the field. Its full power is manifested at the instant of the Big Bang, where all its symmetries were intact. To test this theory, therefore, means recreating Creation on the earth, which is impossible with present-day technology. (This criticism applies, in fact, to any theory of Creation. The philosopher David Hume, for example, believed that a scientific theory of Creation was philosophically impossible. This was because the foundation of science depends on reproducibility, and Creation is one event which can never be reproduced in the laboratory.)

Although this is discouraging, a piece of the puzzle may be supplied by the Superconducting Supercollider (SSC), which, if built, will be the world’s largest atom smasher. The SSC (which is likely to be cancelled by Congress) is designed to accelerate protons to a staggering energy of tens of trillions of electron volts. When these sub-atomic particles slam into each other at these fantastic energies within the SSC, temperatures which have not been seen since the instant of Creation will be generated. That is why it is sometimes called a “window on Creation.” Costing /8-10 billion, the SSC consists of a ring of powerful magnets stretched out in a tube over 50 miles long. In fact, one could easily fit the Washington Beltway, which surrounds Washington D.C., inside the SSC. If and when it is built, physicists hope that the SSC will find some exotic sub-atomic particles in order to complete our present-day understanding of the four forces. However, there is also the small chance that physicists might discover “super- symmetric” particles, which may be remnants of the original superstring theory. In other words, although the superstring theory cannot be tested directly by the SSC, one hopes to find resonances from the superstring theory among the debris created by smashing protons together at energies not found since the Big Bang.

Excerpt from ‘THE FUTURE OF THE MIND’



Houdini believed that telepathy was impossible. But science is proving Houdini wrong. Telepathy is now the subject of intense research at universities around the world, where scientists have already been able to use advanced sensors to read individual words, images, and thoughts in a person's brain. This could alter the way we communicate with stroke and accident victims who are "locked in" their bodies, unable to articulate their thoughts except through blinks. But that's just the start. Telepathy might also radically change the way we interact with computers and the outside world.

Indeed, in a recent "Next 5 in 5 Forecast," which predicts five revolutionary developments in the next five years, IBM scientists claimed that we will be able to mentally communicate with computers, perhaps replacing the mouse and voice commands. This means using the power of the mind to call people on the phone, pay credit card bills, drive cars, make appointments, create beautiful symphonies and works of art, etc. The possibilities are endless, and it seems that everyone— from computer giants, educators, video game companies, and music studios to the Pentagon— is converging on this technology.

True telepathy, found in science-fiction and fantasy novels, is not possible without outside assistance. As we know, the brain is electrical. In general, anytime an electron is accelerated, it gives off electromagnetic radiation. The same holds true for electrons oscillating inside the brain, which broadcasts radio waves. But these signals are too faint to be detected by others, and even if we could perceive these radio waves, it would be difficult to make sense of them. Evolution has not given us the ability to decipher this collection of random radio signals, but computers can. Scientists have been able to get crude approximations of a person's thoughts using EEG scans. Subjects would put on a helmet with EEG sensors and concentrate on certain pictures— say, the image of a car. The EEG signals were then recorded for each image and eventually a rudimentary dictionary of thought was created, with a one- to- one correspondence between a person's thoughts and the EEG image. Then, when a person was shown a picture of another car, the computer would recognize the EEG pattern as being from a car.

The advantage of EEG sensors is that they are noninvasive and quick. You simply put a helmet containing many electrodes onto the surface of the brain and the EEG can rapidly identify signals that change every millisecond. But the problem with EEG sensors, as we have seen, is that electromagnetic waves deteriorate as they pass through the skull, and it is difficult to locate their precise source. This method can tell if you are thinking of a car or a house, but it cannot re- create an image of the car.

That is where Dr. Jack Gallant's work comes in…

VIDEOS OF THE MIND

The epicenter for much of this research is the University of California at Berkeley, where I received my own Ph.D. in theoretical physics years ago. I had the pleasure of touring the laboratory of Dr. Gallant, whose group has accomplished a feat once considered to be impossible: videotaping people's thoughts. "This is a major leap forward reconstructing internal imagery. We are opening a window into the movies in our mind," says Gallant.

When I visited his laboratory, the first thing I noticed was the team of young, eager postdoctoral and graduate students huddled in front of their computer screens, looking intently at video images that were reconstructed from someone's brain scan. Talking to Gallant's team, you feel as though you are witnessing scientific history in the making.

Gallant explained to me that first the subject lies flat on a stretcher, which is slowly inserted headfirst into a huge, state- of- the- art MRI machine, costing upward of $3 million. The subject is then shown several movie clips (such as movie trailers readily available on YouTube). To accumulate enough data, the subject has to sit motionless for hours watching these clips, a truly arduous task. I asked one of the postdocs, Dr. Shinji Nishimoto, how they found volunteers who were willing to lie still for hours on end with only fragments of video footage to occupy the time. He said the people in the room, the grad students and postdocs, volunteered to be guinea pigs for their own research.

As the subject watches the movies, the MRI machine creates a 3-D image of the blood flow within the brain. The MRI image looks like a vast collection of thirty thousand dots, or voxels. Each voxel represents a pinpoint of neural energy, and the color of the dot corresponds to the intensity of the signal and blood flow. Red dots represent points of large neural activity, while blue dots represent points of less activity. (The final image looks very much like thousands of Christmas lights in the shape of the brain. Immediately you can see that the brain is concentrating most of its mental energy in the visual cortex, which is located at the back of the brain, while watching these videos.)

Gallant's MRI machine is so powerful it can identify two to three hundred distinct regions of the brain and, on average, can take snapshots that have one hundred dots per region of the brain. (One goal for future generations of MRI technology is to provide an even sharper resolution by increasing the number of dots per region of the brain.)

At first, this 3-D collection of colored dots looks like gibberish. But after years of research, Dr. Gallant and his colleagues have developed a mathematical formula that begins to find relationships between certain features of a picture (edges, textures, intensity, etc.) and the MRI voxels. For example, if you look at a boundary, you'll notice it's a region separating lighter and darker areas, and hence the edge generates a certain pattern of voxels. By having subject after subject view such a large library of movie clips, this mathematical formula is refined, allowing the computer to analyze how all sorts of images are converted into MRI voxels. Eventually the scientists were able to ascertain a direct correlation between certain MRI patterns of voxels and features within each picture.

At this point, the subject is then shown another movie trailer. The computer analyzes the voxels generated during this viewing and re- creates a rough approximation of the original image. (The computer selects images from one hundred movie clips that most closely resemble the one that the subject just saw and then merges images to create a close approximation.) In this way, the computer is able to create a fuzzy video of the visual imagery going through your mind. Dr. Gallant's mathematical formula is so versatile that it can take a collection of MRI voxels and convert it into a picture, or it can do the reverse, taking a picture and then converting it to MRI voxels.

I had a chance to view the video created by Dr. Gallant's group, and it was very impressive. Watching it was like viewing a movie with faces, animals, street scenes, and buildings through dark glasses. Although you could not see the details within each face or animal, you could clearly identify the kind of object you were seeing.

Not only can this program decode what you are looking at, it can also decode imaginary images circulating in your head. Let's say you are asked to think of the Mona Lisa. We know from MRI scans that even though you're not viewing the painting with your eyes, the visual cortex of your brain will light up. Dr. Gallant's program then scans your brain while you are thinking of the Mona Lisa and flips through its data files of pictures, trying to find the closest match. In one experiment I saw, the computer selected a picture of the actress Salma Hayek as the closest approximation to the Mona Lisa. Of course, the average person can easily recognize hundreds of faces, but the fact that the computer analyzed an image within a person's brain and then picked out this picture from millions of random pictures at its disposal is still impressive.

The goal of this whole process is to create an accurate dictionary that allows you to rapidly match an object in the real world with the MRI pattern in your brain. In general, a detailed match is very difficult and will take years, but some categories are actually easy to read just by flipping through some photographs. Dr. Stanislas Dehaene of the Collège de France in Paris was examining MRI scans of the parietal lobe, where numbers are recognized, when one of his postdocs casually mentioned that just by quickly scanning the MRI pattern, he could tell what number the subject was looking at. In fact, certain numbers created distinctive patterns on the MRI scan. He notes, "If you take 200 voxels in this area, and look at which of them are active and which are inactive, you can construct a machine-learning device that decodes which number is being held in memory."

This leaves open the question of when we might be able to have picture quality videos of our thoughts. Unfortunately, information is lost when a person is visualizing an image. Brain scans corroborate this. When you compare the MRI scan of the brain as it is looking at a flower to an MRI scan as the brain is thinking about a flower, you immediately see that the second image has far fewer dots than the first.

So although this technology will vastly improve in the coming years, it will never be perfect. (I once read a short story in which a man meets a genie who offers to create anything that the person can imagine. The man immediately asks for a luxury car, a jet plane, and a million dollars. At first, the man is ecstatic. But when he looks at these items in detail, he sees that the car and the plane have no engines, and the image on the cash is all blurred. Everything is useless. This is because our memories are only approximations of the real thing.) But given the rapidity with which scientists are beginning to decode the MRI patterns in the brain, will we soon be able to actually read words and thoughts circulating in the mind?

READING THE MIND

In fact, in a building next to Gallant's laboratory, Dr. Brian Pasley and his colleagues are literally reading thoughts— at least in principle. One of the postdocs there, Dr. Sara Szczepanski, explained to me how they are able to identify words inside the mind.

The scientists used what is called ECOG (electrocorticogram) technology, which is a vast improvement over the jumble of signals that EEG scans produce. ECOG scans are unprecedented in accuracy and resolution, since signals are directly recorded from the brain and do not pass through the skull. The flipside is that one has to remove a portion of the skull to place a mesh, containing sixty-four electrodes in an eight-by-eight grid, directly on top of the exposed brain.

Luckily they were able to get permission to conduct experiments with ECOG scans on epileptic patients, who were suffering from debilitating seizures. The ECOG mesh was placed on the patients' brains while open- brain surgery was being performed by doctors at the nearby University of California at San Francisco.

As the patients hear various words, signals from their brains pass through the electrodes and are then recorded. Eventually a dictionary is formed, matching the word with the signals emanating from the electrodes in the brain. Later, when a word is uttered, one can see the same electrical pattern. This correspondence also means that if one is thinking of a certain word, the computer can pick up the characteristic signals and identify it. With this technology, it might be possible to have a conversation that takes place entirely telepathically. Also, stroke victims who are totally paralyzed may be able to "talk" through a voice synthesizer that recognizes the brain patterns of individual words.

Not surprisingly, BMI (brain-machine interface) has become a hot field, with groups around the country making significant breakthroughs. Similar results were obtained by scientists at the University of Utah in 2011. They placed grids, each containing sixteen electrodes, over the facial motor cortex (which controls movements of the mouth, lips, tongue, and face) and Wernicke's area, which processes information about language. The person was then asked to say ten common words, such as "yes" and "no," "hot" and "cold," "hungry" and "thirsty," "hello" and "good-bye," and "more" and "less." Using a computer to record the brain signals when these words were uttered, the scientists were able to create a rough one- to-one correspondence between spoken words and computer signals from the brain.

Later, when the patient voiced certain words, they were able to correctly identify each one with an accuracy ranging from 76 percent to 90 percent. The next step is to use grids with 121 electrodes to get better resolution. In the future, this procedure may prove useful for individuals suffering from strokes or paralyzing illnesses such as Lou Gehrig's disease, who would be able to speak using the brain-to-computer technique.

TYPING WITH THE MIND

At the Mayo Clinic in Minnesota, Dr. Jerry Shih has hooked up epileptic patients via ECOG sensors so they can learn how to type with the mind. The calibration of this device is simple. The patient is first shown a series of letters and is told to focus mentally on each symbol. A computer records the signals emanating from the brain as it scans each letter. As with the other experiments, once this one- to- one dictionary is created, it is then a simple matter for the person to merely think of the letter and for the letter to be typed on a screen, using only the power of the mind.

Dr. Shih, the leader of this project, says that the accuracy of his machine is nearly 100 percent. Dr. Shih believes that he can next create a machine to record images, not just words, that patients conceive in their minds. This could have applications for artists and architects, but the big drawback of ECOG technology, as we have mentioned, is that it requires opening up patients’ brains.

Meanwhile, EEG typewriters, because they are noninvasive, are entering the marketplace. They are not as accurate or precise as ECOG typewriters, but they have the advantage that they can be sold over the counter. Guger Technologies, based in Austria, recently demonstrated an EEG typewriter at a trade show. According to their officials, it takes only ten minutes or so for people to learn how to use this machine, and they can then type at the rate of five to ten words per minute.

SYMMETRIES AND STRING FIELD THEORY IN D=2

Michio Kaku

City College of the CUNY, N.Y., N.Y. 10031

Abstract

Prof. Wigner was well-known for his contributions to group theory and symmetries, especially the representations of the Poincaré group. Similarly, symmetry also plays a decisive role in string theory. In fact, because of these symmetries, in two dimensions the theory is, remarkably enough, exactly soluble, giving us the first non-perturbative information from string theory.

There are two ways in which to formulate 2D string theory, either as a matrix model or as a Liouville theory [1-4]. Matrix models are enormously powerful, and the emergence of a new symmetry, called $w(\infty)$, helps to explain why the model is exactly soluble. Its generators obey:

$$[Q_{jm}, Q_{j'm'}] = (jm' - j'm)Q_{j+j'-1,m+m'} \tag{1}$$

where $j = 0, 1/2, 1, 3/2, ...$ and $m = -j, ... + j$.

However, in matrix models the string degrees of freedom have been completely removed, and hence the field theoretic origin of $w(\infty)$ is rather obscure. Furthermore, the generators of $w(\infty)$ involve the so-called "discrete states," which are unusual physical states defined only at discrete momenta.

Liouville theory, by contrast, has the opposite problem: it manifestly has all the string degrees of freedom intact, but the theory is notoriously difficult to solve, even at the free level. The purpose of this paper is to show how to reformulate Liouville theory as a second quantized field theory, so that $w(\infty)$ and the discrete states emerge naturally.

To re-write Liouville theory (for $\mu = 0$) as a field theory, we will generlize the 26 dimensional non-polynomial closed string field theory, recently proposed by the author [5] and also by the Kyoto and MIT groups [6,7], which solves the long-standing problem of a covariant, closed string field theory.

Using this non-polynomial theory in two dimensions, we will show that the origin of $w(\infty)$ comes from the gauge group of the non-polynomial action. Furthermore, we will show how to reproduce the standard shifted Shapiro-Virasoro four-point amplitude from the non-polynomial Liouville field theory. This gives a cohesive, integrated description of the many seemingly divergent results found in this rapidly growing field.

Further details of this paper can be found in ref. [8].

that: $\delta A_\mu^a = f_{abc} A_\mu^b \Lambda^c + ...$ And lastly, we write down an action $\mathcal{L}$ which is an invariant under this variation. We find, for example, that the three-vertex is proportional to the structure constant of the group f_{abc}.

However, $w(\infty)$ emerges from matrix models in a highly unorthodox fashion, largely because the string degrees of freedom have been completely eliminated.

The goal of this paper is to reformulate closed string Liouville theory as a second quantized field theory, so that $w(\infty)$ emerges in the usual way. We will show that the second quantized fields transform under a certain representation of the group, that the action is an invariant of the group, and that the three-vertex is proportional to the structure constant of the group.

The action of the 26 dimensional string theory is given by the non-polynomial closed string action, first written down by the author [5] and the Kyoto and MIT groups [6,7]:

$$\mathcal{L} = \langle \Phi | Q | \Phi \rangle + \sum_{n=3}^{\infty} \alpha_n \langle \Phi^n \rangle \tag{2}$$

where $Q = Q_0(b_0 - \bar{b}_0)$, Q_0 is the usual BRST operator, and where the field $\Phi(X)$ transforms as:

$$\delta|\Phi\rangle = |Q\Lambda\rangle + \sum_{n=1}^{\infty} \beta_n |\Phi^n \Lambda\rangle \tag{3}$$

In ref. 8, we have genalized this action to describe two dimensional string theory. In this paper, we will show that:

(a) the vertices are BRST invariant

(b) the action reproduces the shifted Shapiro-Virasoro amplitude

(c) the three-vertex is proportional to the structure constant of $w(\infty)$.

2 BRST Invariance of Vertices

We first wish to show that the three-vertex is BRST invariant, i.e.

$$\sum_{i=1}^{3} Q_i |V_3\rangle = 0 \tag{4}$$

Naively, this calculation appears to be trivial, since the vertex function simply represents a delta function across three overlapping strings. Hence, we expect that the three contributions to Q cancel exactly. However, this calculation is actually rather delicate, since there are potentially anomalous contributions at the joining points. To resolve this issue, we must use point-splitting, pioneered in refs. [9,10].

We wish to make a conformal map from the multi-sheeted, three-string, world-sheet configuration in the ρ-plane to the flat, complex z-plane. Fortunately, the conformal map for the N-point function to the complex z-plane is known. The map is given by [5]:

$$\frac{d\rho(z)}{dz} = C \frac{\prod_{i=1}^{N-2} \sqrt{(z - z_i)(z - \tilde{z}_i)}}{\prod_{i=1}^{N} (z - \gamma_i)} \tag{5}$$

$$\oint_{C_1+C_2+C_3} \frac{dz}{2\pi i} c(z) \left\{ -\frac{1}{2} \left(\frac{dz'}{dz} \right) \partial X_\mu(z') \partial X^\mu(z) + \left(\frac{dz'}{dz} \right)^2 \frac{dc}{dz} b(z') + \frac{Q}{2} \partial^2 \phi(z) \right\} |V_3\rangle \tag{6}$$

where z' is infinitesimally close to z, where μ ranges over the D dimensional string modes as well as the ϕ mode, where b and c are the usual reparametrization ghosts, and C_i are infinitesimally small curves in the z-plane which encircle the joining point z_0.

The major complication to this calculation is that the Liouville ϕ field does not transform as a scalar. Instead, it transforms as:

$$\phi(\rho) \to \phi(z) + \frac{Q}{2} \log \left| \frac{dz}{d\rho} \right|^2 \tag{7}$$

This means that the energy-momentum tensor T transforms as:

$$T_{\rho\rho} \to \left(\frac{dz}{d\rho} \right)^2 T_{zz} + \left(\frac{Q}{2} \right)^2 S \tag{8}$$

where S is the Schwartzian, given by $(z'''/z') - (3/2)(z''/z')^2$. This complicates the calculation considerably, since it means that there is subtle insertion factor located at delta-function curvature singularities in the vertex function. These add non-trivial ϕ contributions to the calculation.

The final calculation is rather long [8], so we only summarize the result:

$$\left\{ -pc(z_0) \left[\frac{D}{24} - \frac{13}{12} + \frac{1}{24} + \frac{1}{8} Q^2 \right] - \frac{dc(z_0)}{dz} \left[\frac{5D}{96} - \frac{65}{48} + \frac{5}{96} + \frac{5}{32} Q^2 \right] \right\} |V_3\rangle \tag{9}$$

which cancels if:

$$D - 26 + 1 + 3Q^2 = 0 \tag{10}$$

which is precisely the consistency equation for Liouville theory in D dimensions. Thus, the vertex is BRST invariant.

3 Shifted Shapiro-Virasoro Amplitude

The next major test of the theory is whether it reproduces the shifted Shapiro-Virasoro amplitude. This calculation is highly non-trivial, since the conformal map between the multi-sheeted string-scattering Riemann sheet to the complex plane is very involved. Unlike the light cone theory, or even Witten's open string theory, the non-polynomial theory yields very complicated conformal maps.

Fortunately, for the four-point function, all conformal maps are known exactly, in terms of elliptic functions, and the calculation can be performed (see ref. 11).

For the four point function, the map in eq. (5) can be integrated exactly, giving:

$$\rho = \sum^{4} \quad gNA_i \quad \omega_i - g$$

where Π is a third elliptic function, $z = ia_i + b_i$ and $\tilde{z}_i = -ia_1 + b_1$ for complex a_i and b_i, and:

$$A_i = \frac{[(\gamma_i - b_1)^2 + a_1^2]\,[(\gamma_i - b_2)^2 + a_2^2]}{\prod_{j=1,j\neq i}^4 (\gamma_i - \gamma_j)} \tag{12}$$

$$\begin{aligned}
\omega_i &= \frac{a_1 + b_1 g_1 - \gamma_i g_1}{b_1 - a_1 g_1 - \gamma_i} \\
f_i &= \frac{1}{2}(1+\omega_i^2)^{-1/2}(k^2+\omega_i^2)^{-1/2} \\
&\quad \times \ln \frac{(k^2+\omega_i^2)^{1/2} - (1-\omega_i^2)^{1/2}\mathrm{dn}u}{(k^2+\omega_i^2)^{1/2} + (1+\omega_i^2)^{1/2}\mathrm{dn}u} \\
\phi &= \arctan\left(\frac{y - b_2 + a_1 g_1}{a_1 + g_1 b_1 - g_1 y}\right)
\end{aligned} \tag{13}$$

where: $A^2 = (b_1+b_2)^2 + (a_1+a_2)^2$, $B^2 = (b_1-b_2)^2 + (a_1-a_2)^2$, $g_1^2 = [4a_1^2 - (A-B)^2][(A+B)^2 - 4a_1^2]$, $g = 2/(A+B)$, $y_1 = b_1 - a_1 g_1$, $k'^2 = 1 - k^2 = 4AB/(A+B)^2$, $u = \mathrm{dn}^{-1}(1 - k'^2 \sin^2\phi)$

This explicit conformal map allows us to calculate the four-point amplitude. We first write the amplitude in the ρ plane, and then make a conformal map to the z-plane. Let the modular parameter be $\hat{\tau} = \tau + i\theta$, where τ is the distance between the splitting strings, and θ is the relative rotation. Let $\gamma_1 = (0, \hat{x}, 1, \infty)$. Then, with a fair amount of work, one can find the Jacobian from $\hat{\tau}$ to $\hat{x}$:

$$\frac{d\hat{\tau}}{d\hat{x}} = \frac{\pi C}{2K(k) g \hat{x}(1-\hat{x})(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)} \tag{14}$$

Then the four point amplitude can be written as:

$$\begin{aligned}
A_4 &= \langle V_3 | \frac{b_0 \bar{b}_0}{L_0 + \bar{L}_0 - 2} | V_3 \rangle \\
&= \int d\tau \int d\theta A_G \Big\langle \prod_{i=1}^4 c(\gamma_i)\bar{c}(\gamma_i) V(\gamma_i) \Big\rangle
\end{aligned} \tag{15}$$

where:

$$A_G = \int \frac{dz}{2\pi i}\frac{dz}{dw}\frac{\prod_{i<i}(\gamma_i - \gamma_j)}{\prod_{j=1}^4 (z - \gamma_j)} \tag{16}$$

$$\quad = 2\frac{g}{\pi c}\hat{x}(1-\hat{x})K(k)(\gamma_1 - \gamma_3)^3(\gamma_2 - \gamma_4)^3 \tag{17}$$

Putting everything together, we finally find:

$$A_4 = \int d^2\hat{x} \left| \hat{x}^{2p_i\cdot p_j}(1-\hat{x})^{2p_2\cdot p_3} \right|^2 \tag{18}$$

for shifted momenta p. This is precisely the shifted Shapiro-Virasoro amplitude, as expected.

4 Origin of ( )

First, notice that the field $|\Phi\rangle$ can be decomposed into a massless tachyon, the discrete states $|j,m\rangle$ (labeled by $SU(2)$ quantum numbers j,m), and BRST trivial states:

$$|\Phi\rangle = \varphi|0\rangle + \sum_{j,m} \psi_{j,m}|j,m\rangle + ... \tag{19}$$

where ... represent the BRST trivial states.

Now take the matrix element of three strings: $\langle\Phi|\langle\Phi|\langle\Phi|V_3\rangle$. When $\langle\Phi|$ is physical and on-shell, only the tachyon and discrete states survive. To compute the three-vertex function between three discrete states, we will make a conformal transformation from the three-vertex ρ complex plane to the complex z-plane. Under a complex transformation, $|j,m\rangle$ remains the same, i.e. $\Omega|j,m\rangle = |j,m\rangle$ since $|j,m\rangle$ satisfies the Virasoro conditions. Let us therefore insert $\Omega\Omega^{-1}$ inside the three-string vertex function. The $|j,m\rangle$ states remain the same, but the vertex function $|V_3\rangle_\rho$ transforms onto $|V_3\rangle_z$, which is defined on the complex z-plane

Thus, it is a simple matter to show (for the holomorphic part), that:

$$\begin{aligned} \langle j_1,m_1|\langle j_2,m_2|\langle j_3,m_3|V_3\rangle_\rho &= \langle j_1,m_1|\langle j_2,m_2|\langle j_3,m_3|V_3\rangle_z \\ &= \left\langle \Psi_{j_1,m_1}(0)\Psi_{j_2,m_2}(1)\Psi_{j_3,m_3}(\infty)\right\rangle \\ &\sim (j_1m_2 - j_2m_1)\delta_{j_3,j_1+j+2-1}\delta_{m_3,m_1+m_2} \end{aligned} \tag{20}$$

where $Q_{j,m} = \oint \frac{dz}{2\pi i}\Psi_{j,m}(z)$. That $w(\infty)$ emerges from the interaction of discrete states has been stressed in refs. [12-14]. What we have shown is that this result easily generalizes to closed string field theory.

Thus, we have shown that the three-vertex function of string field theory reproduces the structure constants of $w(\infty)$, as expected. Thus, $w(\infty)$ emerges naturally as part of the gauge invariance of the string field, as in ordinary gauge theory. (However, the geometric picture of $w(\infty)$ as area-preserving diffeomorphisms appears to be lost.)

One advantage of this formulation is that the notorious $c = 1$ barrier found in matrix models is easily broken. (However, we do not expect the model to be exactly soluble beyond this barrier.)

In summary, we have successfully formulated 2D string theory as a second quantized field theory for closed strings. Once Liouville theory is formulated as a field theory, the mysteries of 2D string theory (e.g. the field theoretic origin of $w(\infty)$, discrete states, etc.) become transparent, because everything is formulated as a standard gauge theory. See [8] for more details.

5 Acknowledgments

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Sub-critical Closed String Field Theory in D Less Than 26

Michio Kaku

Physics Dept., City College of the CUNY, N.Y., N.Y. 10031

Abstract

In this paper, we construct the second quantized action for sub-critical closed string field theory with zero cosmological constant in dimensions $2 \leq D < 26$, generalizing the non-polynomial closed string field theory action proposed by the author and the Kyoto and MIT groups for $D = 26$. The proof of gauge invariance is considerably complicated by the presence of the Liouville field ϕ and the non-polynomial nature of the action. However, we explicitly show that the polyhedral vertex functions obey BRST invariance to all orders. By point splitting methods, we calculate the anomaly contribution due to the Liouville field, and show in detail that it cancels only if $D - 26 + 1 + 3Q^2 = 0$, in both the bosonized and unbosonized polyhedral vertex functions. We also show explicitly that the four point function generated by this action reproduces the shifted Shapiro-Virasoro amplitude found from $c = 1$ matrix models and Liouville theory in two dimensions. This calculation is non-trivial because the conformal transformation from the z to the ρ plane requires rather complicated third elliptic integrals and is hence much more involved than the ones found in the usual polynomial theories.

1 Introduction

At present matrix models [1-3] give us a simple and powerful technique for constructing the S-matrix of two dimensional string theory. However, all string degrees of freedom are missing, and hence many of the successes of the theory are intuitively difficult to interpret in terms of string degrees of freedom. Features such as the discrete states [4-7] and the $w(\infty)$ algebra arise in a rather obscure fashion.

By contrast, Liouville theory [8-9] manifestly includes all string degrees of freedom, but the theory is notoriously difficult to solve, even for the free case.

In order to further develop the Liouville approach, we present the details of a second quantized field theory of closed strings defined in $2 \leq D < 26$ dimensions with $\mu = 0$. (See refs. [10-11] for work on c=1 open string field theory.)

There are several advantages to presenting a second quantized field formulation of Liouville theory:

(a) The $c = 1$ barrier, which has proved to be insurmountable for matrix models, is trivially breached for Liouville theory (although we no longer expect the model to be exactly solvable beyond $c = 1$)

(b) In principle, it should be possible to present a supersymmetric Liouville theory in field theory form, which is difficult for the matrix models approach.

(c) For $c = 1$, the rather mysterious features appearing in matrix models, which are intuitively difficult to understand, have a standard field theoretical interpretation. For example, "discrete states" arise naturally as string degrees of freedom with discrete momenta when we calculate the physical states of the theory. In other words, the $\Phi(X, b, c, \phi)$ field contains three sets of states. Symbolically, we have:

$$|\Phi(X, b, c, \phi)\rangle = |\mathrm{tachyon}\rangle + |\mathrm{discrete\,states}\rangle + |\mathrm{BRST\,trivial\,states}\rangle \qquad (1)$$

Also, the structure constants of $w(\infty)$ arise as the coefficients of the three-string vertex function, analogous to the situation in Yang-Mills theory. We see that $w(\infty)$ is just a subalgebra of the full string field theory gauge algebra. For example, if $\langle j, m|$ labels the $SU(2)$ quantum numbers of the discrete states, then we can show that the three-string vertex function $\langle \Phi^3 \rangle$, taken on discrete states, reproduces the structure constants of $w(\infty)$:

$$\begin{aligned} \langle j_1, m_1 | \langle j_2, m_2 | \langle j_3, m_3 | V_3 \rangle &\sim \left\langle \Psi_{j_1,m_1}(0) \Psi_{j_2,m_2}(1) \Psi_{j_3,m_3}(\infty) \right\rangle \\ &\sim (j_1 m_2 - j_2 m_1) \delta_{j_3, j_1+j_2-1} \delta_{m_3, m_1+m_2} \qquad (2) \end{aligned}$$

where we have made a conformal transformation from the three-string world sheet to the complex plane, and where the charges $Q_{j,m} = \oint \frac{dz}{2\pi i} \Psi_{j,m}(z)$ generate the standard $w(\infty)$ algebra:

$$[Q_{j_1,m_1}, Q_{j_2,m_2}] = (j_1 m_2 - j_2 m_1) Q_{j_1+j_2+1, m_1+m_2} \qquad (3)$$

To construct the string field theory action for non-critical strings, we first begin with the non-polynomial closed string action of the 26 dimensional string theory, first written down by the author [12] and the Kyoto and MIT groups [13,14]:

$$\mathcal{L} = \langle \Phi | Q | \Phi \rangle + \sum_{n=3}^{\infty} \alpha_n \langle \Phi^n \rangle \tag{4}$$

where $Q = Q_0(b_0 - \bar{b}_0)$, Q_0 is the usual BRST operator, and where the field Φ transforms as:

$$\delta|\Phi\rangle = |Q\Lambda\rangle + \sum_{n=1}^{\infty} \beta_n |\Phi^n \Lambda\rangle \tag{5}$$

where n labels the number of faces of the polyhedra, and there are more than one distinct polyhedra at each level. For example, there are 2 polyhedra at $N = 6$, 5 polyhedra at $N = 7$, and 14 polyhedra at $N = 8$ [12].

If we insert $\delta|\Phi\rangle$ into the action, we find that the result does not vanish, unless:

$$(-1)^n \langle \Phi || Q\Lambda \rangle + n \langle Q\Phi || \Phi^{n-1} \Lambda \rangle + \sum_{p=1}^{n-2} C_p^n \langle \Phi^{n-p} || \Phi^p \Lambda \rangle = 0 \tag{6}$$

where the double bars mean that when we join two polyhedra, the common boundary has circumference 2π. The meaning of this equation is rather simple. The first two terms on the left hand side represent the action of $\sum_i Q_i$ on the vertex function. Naively, we expect the sum of these two terms to vanish. However, naive BRST invariance is broken by the third term, which has an important interpretation. This third term consists of polyhedra with rather special parameters, i.e they are polyhedra which are at the endpoints of the modular region. Thus, these polyhedra are actually composites; they can be split in half, into two smaller polyhedra, such that the boundary of contact is 2π. This is the meaning of the double bars.

(This action also has additional quantum corrections because the measure of integration $D\Phi(X)$ is not gauge invariant. These quantum corrections can be explicitly solved in terms of a recursion relation. These corrections can be computed either by calculating these loop corrections to the measure [15], or by using the BV quantization method [16].)

If strings have equal parametrization length 2π, then we must triangulate moduli space with cylinders of equal circumference but arbitrary extension, independent of the dimension of space-time. Thus, the triangulation of moduli space on Riemann surfaces remains the same in any dimension D. Therefore, the basic structure of the action remains the same for sub-critical strings with equal parametrization length.

What is different, of course, is that the string degrees of freedom have changed drastically, and a Liouville field ϕ must be introduced. The addition of the Liouville theory complicates the proof of gauge invariance considerably, however, since this field must be inserted at curvature singularities within the vertex functions, i.e. at the corners of the polyhedra. This means that the standard proof of gauge invariance formally breaks down, and hence must be redone.

This raises a problem, since the explicit cancellation of these anomalies has only been performed for polynomial string field theory actions, not the non-polynomial one. In particular, the anomaly cancellation of the Witten field theory depends crucially on knowledge of the specific numerical value of the Neumann functions. However, the Neumann functions of the non-polynomial field theory are only defined formally. Explicit forms for them are not known. Thus, it appears that the cancellation of anomalies seems impossible.

However, we will use point splitting methods, pioneered in [17-19], which have the advantage that we can isolate those points on the world sheet where these insertion operators must be placed, and hence only need to calculate the anomaly at these insertion points. Thus, we do not need to have an explicit form for the Neumann functions; we need only certain identities which these Neumann functions obey. The great advantage of the point splitting method, therefore, is that we can show BRST invariance to all orders in polyhedra, without having to have explicit expressions for the Neumann functions. As an added check, we will calculate the anomaly in two ways, using both bosonized and unbosonized ghost variables.

Thus, we will first calculate the anomaly contribution, isolating the potential divergences coming from the insertion points and show that they sum to zero. Then we will show that our theory reproduces the standard shifted Shapiro-Virasoro amplitude.

2 BRST Invariance of Vertices

We will specify our conventions by introducing a field which combines the string variable X^i (where i labels the Lorentz index) and the Liouville field ϕ. We introduce ϕ^μ where $\mu = 0, 1, 2, ...D$ and where ϕ^D corresponds to the Liouville field, so that $\phi^\mu = \{X^i, \phi\}$.

The first quantized action is given by:

$$S = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\left\{g^{ab}\left(\partial_a X^i \partial_b X_i + \partial_a\phi\partial_b\phi\right) + Q\hat{R}\right\} \tag{7}$$

The holomorphic part of the energy-momentum tensor is therefore:

$$\begin{aligned} T^{\phi}_{zz} &= -\frac{1}{2}\left(\partial_z \phi^\mu\right)^2 - \frac{Q_\mu}{2}\left(\partial_z^2 \phi^\mu\right) \\ T^{\text{gh}}_{zz} &= \frac{1}{2}\left(\partial_z \sigma\right)^2 + \frac{3}{2}\left(\partial_z \sigma^2\right) \end{aligned} \tag{8}$$

where we have bosonized the ghost fields via $c = e^\sigma$ and $b = e^{-\sigma}$ and where $Q^\mu = (0, Q)$. Demanding that the central charge of the Virasoro algebra vanish implies that:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{9}$$

with total central charge:

$$c = D + 1 + 3Q^2 - 26 = 0 \tag{10}$$

so that $Q = 2\sqrt{2}$ for $D = 1$ (or for two dimensions if we promote ϕ to a dimension). Notice that the ghost field has a background charge of -3 and the ϕ^μ field has a background charge of $Q^\mu = (0, Q)$. This allows us to collectively place the bosonized ghost field and the ϕ^μ field together into one field. We will use the index M when referring to the collective combination of the string variable, the Liouville field, and the bosonized field. We will define

$$\begin{aligned} Q^M &= \{0, Q, -3\} \\ \phi^M &= \{X^i, \phi, \sigma\} \end{aligned} \tag{11}$$

To calculate the insertion factors in the vertex function, we must analyze the terms in the first quantized action proportional to the background charge:

$$\frac{Q^M}{8\pi}\int \sqrt{g}R\,\phi^M d^2\xi \tag{12}$$

where we have normalized the curvature on the world sheet such that $\int \sqrt{g}Rd^2\xi = 4\pi\chi$ where χ is the Euler number. In general, the curvature on the string world sheet is zero, except at isolated points where the strings join. At these interior points, the curvature is a delta function, such that $\int \sqrt{g}Rd^2\xi = -4\pi$ around these points. This means that N-point vertex functions, in general, must have insertions proportional to:

$$\prod_{j=1}^{2(N-2)}\left(e^{-Q^M\phi^M/2}\right)_j \tag{13}$$

where j labels the $2(N-2)$ sites where we have curvature singularities on the string world sheet. These insertions, in fact, are the principle complication facing us in calculating the anomalies of the various vertex functions.

The vertex is then defined as:

$$|V_N\rangle = \int B_N |V_N\rangle_0 \tag{14}$$

where B_N consist of line integrals of b operators defined over Beltrami differentials (see the Appendix for conventions for the vertex function) and $|V_N\rangle_0$ is the standard vertex function given as an over-lap condition on the string and ghost degrees of freedom which must satisfy the usual BRST condition:

$$\sum_{i=1}^{N} Q_i |V_N\rangle_0 = 0 \tag{15}$$

Notice that it is the presence of this factor B_N which prevents the vertex function from being trivially BRST invariant. The reason for this is that B_N contains line integrals of the b operators, defined over Beltrami differentials μ_k, such that:

$$T_{\mu_k} = \{Q, b_{\mu_k}\} \tag{16}$$

Whenever Q is commuted past a term in B_N, it creates an expansion or contraction of some of the modular parameters within the polyhedral vertex function. The deformation generated by T_{μ_k} is given as a total derivative in the modular parameter τ_k, i.e.

$$\int d\tau_k T_{\mu_k} \sim \int d\tau_k \frac{\partial}{\partial \tau_k} \tag{17}$$

When this deformation is integrated over the modular parameter, we find only the endpoints of the modular region. However, the endpoints of the modular region are where the polyhedra splits into two smaller polyhedra, connected by a common boundary of 2π. This, in turn, reproduces the residual terms $\langle \Phi^{n-p}||\Phi^p \Lambda\rangle$ appearing in eq. (6) which violate naive BRST invariance. Thus, the importance of this B_N term is that it gives the corrections to the naive BRST invariance equations.

Fortunately, the factor B_N remains the same even for the sub-critical case independent of the dimension of space-time. Therefore, we can ignore this term and shall concentrate instead on the properties of $|V_N\rangle_0$, which is defined as:

$$
\begin{aligned}
|V_N\rangle_0 &= \left(\prod_{j=1}^{2(N-2)} e^{-(Q^M\phi^M/2)_j}\right)\int \delta(\sum_{i=1}^{N} p_i^M + Q^M)\prod_{i=1}^{N} P_i \\
&\times \exp\left\{\sum_{r,s}^{N}\sum_{n,m=0}^{\infty}\frac{1}{2}N_{nm}^{rs}\alpha_{-n}^{Mr}\alpha_{-m}^{Ms}\right\} \\
&\times \exp\left\{\sum_{r,s}^{N}\sum_{n,m=0}^{\infty}\frac{1}{2}N_{nm}^{rs}\tilde{\alpha}_{-n}^{Mr}\tilde{\alpha}_{-m}^{Ms}\right\}\left(\prod_{i=1}^{N} d^M p_i|p_i^M\rangle\right)
\end{aligned}
\tag{18}
$$

where P_i represents the operator which rotates the string field by 2π, where j labels the insertion points, where we have deliberately dropped an uninteresting constant, and where the state vector $|p_i^M\rangle$ and the Neumann functions are defined in the Appendix.

For our calculation, we would like to commute the insertion operator directly into the vertex function. Performing the commutation, we find (for $N = 3$):

$$
\begin{aligned}
|V_3\rangle_0 &= \int \delta(p_1^M + p_2^M + p_3^M + Q^M)\prod_{i=1}^{3} P_i \exp\left\{\sum_{r,s}^{3}\sum_{n,m=0}^{\infty}\frac{1}{2}N_{nm}^{rs}\alpha_{-n}^{Mr}\alpha_{-m}^{Ns}\right. \\
&- \frac{1}{3}\frac{\bar{Q}^M}{2}\Big[\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\alpha_{-2n}^{Mr} + \sum_{r,s}^{3} N_{00}^{rs}\alpha_0^{Mr} \\
&+ \left.\sum_{r,s}^{3}\sum_{n=1}^{\infty} N_{n0}^{rs}\alpha_{-n}^{Mr} - \sum_{n=1}^{\infty}\sum_{r,s}^{3} N_{nm}^{rs}\alpha_{-n}^{Mr}\cos m\pi/2\Big]\right\}\left(\prod_{i=1}^{3} d^M p_i|p_i\rangle\right)
\end{aligned}
\tag{19}
$$

where $\bar{Q}^M = \{0, -iQ, -3\}$. The factor $1/3$ appearing before the background charge arises because we have broken up the insertion operator into three equal pieces, each defined in terms of the three different harmonic oscillators. (For simplicity, we have only presented the holomorphic part of the vertex function, and deleted $\tilde{\alpha}$ operators for convenience. It is understood that all vertex functions contain both the α and $\tilde{\alpha}$ operators.)

With these conventions, we now wish to show that the vertices of the non-polynomial theory are BRST covariant. For the three-string vertex, this means: $\sum_{i=1}^{3} Q_i|V_3\rangle_0 = 0$.

Naively, this calculation appears to be trivial, since the vertex function simply represents a delta function across three overlapping strings. Hence, we expect that the three contributions to Q cancel exactly. However, this calculation is actually rather delicate, since there are potentially anomalous contributions at the joining points.

Previous calculations of this identity were limited by the fact that they used specific information about the three-string vertex function. We would like to use a more general method which will apply for the arbitrary N-string vertex function. The most general method uses point-splitting.

We wish to construct a conformal map from the multi-sheeted, three-string world sheet configuration in the ρ-plane to the flat, complex z-plane. Fortunately, this map was constructed in [12]:

$$\frac{d\rho(z)}{dz} = C\frac{\prod_{i=1}^{N-2}\sqrt{(z-z_i)(z-\tilde{z}_i)}}{\prod_{i=1}^{N}(z-\gamma_i)} \tag{20}$$

where the N variables γ_i map to points at infinity (the external lines in the ρ plane) and the $N-2$ pair of variables $(z_i, \tilde{z}_i)$ map to the points where two strings collide, creating the ith vertex (which are interior points in the ρ plane).

The set of complex numbers $\{C, z_i, \tilde{z}_i, \gamma_i\}$ constitute an initial set of $2 + 4(N-2) + 2N = 6N - 6$ unknowns. In order to achieve the correct counting, we must impose a number of constraints. First, we must set the length of the external strings at infinity to be $\pm\pi$. In the limit where $z \to \gamma_i$, we have:

$$\lim \frac{d\rho(z)}{dz} \to \frac{\pm 1}{z-\gamma_i} \tag{21}$$

This gives us $2N$ constraints. However, by projective invariance we have the freedom of selecting three of the γ_i to be $\{0, 1, \infty\}$ Then we must subtract two, because of over-all charge conservation (taking into account that there are charges due to the Riemann cuts as well as charges located at γ_i.) Thus we have $2N + 6 - 2 = 2N + 4$ constraints.

Next, we must impose the fact that the overlap of two colliding strings at the ith vertex is given by π, such that the interaction takes place instantly in proper time τ. This is gives us:

$$\pm\, i\pi = \rho(z_i) - \rho(\tilde{z}_i) \tag{22}$$

This gives us $2(N-2)$ additional constraints, for a total of $4N$ constraints. Thus, the number of variables minus the number of constraints is given by $2N - 6$. But this is precisely the number of Koba-Nielsen variables necessary to describe N string scattering, or the number of moduli necessary to describe a Riemann surface with N punctures consisting of cylinders of equal circumference and arbitrary extension.

These moduli can be described in terms of the proper time τ separating the ith and jth vertices, as well as the angle θ separating them.

We can define

$$\hat{\tau}_{ij} = \tau_{ij} + i\theta_{ij} = \rho(z_i) - \rho(z_j) \tag{23}$$

where τ_{ij} is the proper time separating the ith and jth vertices, and θ_{ij} is the relative angle between them.

There are precisely $2N-6$ independent variables contained within the $\hat{\tau}_{ij}$, as expected. (Not all the $\hat{\tau}_{ij}$ are independent.)

In summary, we find that:

$$\begin{aligned} \{C, z_i, \tilde{z}_i, \gamma_i\} &\rightarrow 6N-6 \,\text{unknowns} \\ \{\rho'(\gamma_i), \rho(z_i) - \rho(\tilde{z}_i)\} &\rightarrow 4N \,\text{constraints} \\ \{\tau_{ij} + i\theta_{ij}\} &\rightarrow 2N-6 \,\text{moduli} \end{aligned} \tag{24}$$

The conformal map, with these constraints, describes N point scattering consisting of three-string vertices only. This is not enough to cover all of moduli space. In addition, we find a "missing region" [20]. For example, we must include the $2N-6$ moduli necessary to describe the lengths of the sides of an N sided polyhedra. The moduli describing the various polyhedra are specified by setting τ_{ij} all equal to each other. In other words, on the world sheet, the polyhedral interaction takes place instantly in τ space. Then the $2N-6$ variables necessary to describe the polyhedra can be found among the θ_{ij}.

Now that we have specified the conformal map, we can begin the calculation of the BRST invariance of the vertex functions.

First, we will find it convenient to transform the BRST operator Q into a line integral over the ρ plane. For the three-point vertex function, we have three line integrals which, for the most part, cancel each other out (because of the continuity equations across the vertex function). The only terms which do not vanish are the ones which encircle the joining points z_i and $\tilde{z}_i$.

Written as a line integral, the BRST condition becomes:

$$\begin{aligned} \sum_{i=1}^{3} Q_i |V_3\rangle_0 &= \oint_{C_1+C_2+C_3} \frac{d\rho}{2\pi} c(\rho) \\ &\times \left\{ -\frac{1}{2}(\partial_z \phi^\mu)^2 + \frac{dc}{d\rho} b(\rho) + \frac{Q}{2}(\partial_z \phi^\mu)^2 \right\} |V_3\rangle_0 \end{aligned} \tag{25}$$

where C_i are infinitesimal curves which together comprise circles which go around $\rho(z_i)$ and $\rho(\tilde{z}_i)$. Notice that this expression is, strictly speaking, divergent because they are defined at the joining point z_i, where these quantities, in general, diverge. To isolate the anomaly, we will now make a conformal transformation from the ρ plane to the z plane. When two operators are defined at the same point, as in $(\partial_z \phi^\mu)^2$, we will point split them by introducing another variable z' which is infinitesimally close to z. Then our expression becomes:

$$\begin{aligned}
\sum_{i=1}^{3} Q_i |V_3\rangle_0 &= \oint_{C_1+C_2+C_3} \frac{dz}{2\pi i} c(z) \\
&\times \left\{ -\frac{1}{2} \left(\frac{dz'}{dz} \right) \partial\phi_\mu(z') \partial\phi^\mu(z) \right. \\
&+ \left. \left(\frac{dz'}{dz} \right)^2 \frac{dc}{dz} b(z') + \frac{Q}{2} \partial^2 \phi(z) \right\} \Big| V_3 \Big\rangle_0
\end{aligned} \tag{26}$$

where z' is infinitesimally close to z, where μ ranges over the D dimensional string modes as well as the ϕ mode, where b and c are the usual reparametrization ghosts, and the C_i are now infinitesimally small curves in the z-plane which encircle the joining point, which we call z_0. In making the transition from the ρ plane to the z plane, we have made the re-definition:

$$c(\rho) = \frac{d\rho}{dz} c(z); \quad b(\rho) = \left(\frac{d\rho}{dz} \right)^{-2} b(z) \tag{27}$$

The major complication to this calculation is that the Liouville ϕ field does not transform as a scalar. Instead, it transforms as:

$$\phi(\rho) \to \phi(z) + \frac{Q}{2} \log \left| \frac{dz}{d\rho} \right| \tag{28}$$

This means that the energy-momentum tensor T transforms as:

$$\begin{aligned}
T_{\rho\rho} &\to \left(\frac{dz}{d\rho} \right)^2 T_{zz} + \left(\frac{Q}{2} \right)^2 S \\
S &= \frac{z'''}{z'} - \frac{3}{2} \left(\frac{z''}{z'} \right)^2
\end{aligned} \tag{29}$$

where S is called the Schwartzian. The form of the Schwartzian that is most crucial for our discussion will be:

$$\left(\frac{Q}{2} \right)^2 S = T^{\phi}_{zz} \left(\partial_z \phi \to \frac{Q}{2} \partial_z \log \left| \frac{dz}{d\rho} \right| \right) \tag{30}$$

This complicates the calculation considerably, since it means that there are subtle insertion factors located at delta-function curvature singularities in the vertex function. These add non-trivial ϕ contributions to the calculation.

3 Point Splitting

In order to perform this sensitive calculation, we will use the method of point splitting.

Let us examine the behavior of the various variables near the splitting point z_0 using the original conformal map in eq.(20). Near this point, we have:

$$\begin{aligned}
\frac{d\rho}{dz} &= a\sqrt{z-z_0} + b\sqrt{z-z_0}^{\,3} + ... \\
\rho(z) &= \rho(z_0) + \frac{2}{3}a(z-z_0)^{3/2}\left(1 + \frac{3}{5}\frac{b}{a}(z-z_0) + ...\right)
\end{aligned} \tag{31}$$

Now let us define $\epsilon = z - z_0$ and power expand these functions for small ϵ. For the purpose of point splitting, we introduce the variable z', which is infinitesimally close to both z and z_0, and is defined implicitly through the equation:

$$\rho(z') = \rho(z) + \frac{2}{3}a\delta \tag{32}$$

where δ is an infinitesimally small constant, which we will later set equal to zero.

We will find it convenient to introduce the following function:

$$f(\epsilon) = z' - z_0 = \epsilon\left\{1 + \sum_{n=1}^{\infty} f_n(\epsilon)\delta^n\right\} \tag{33}$$

We can easily solve for the coefficients f_n by power expanding the following equation:

$$\begin{aligned}
\rho(z') - \rho(z) &= \frac{2}{3}a\delta \\
&= \frac{2}{3}a\left(f^{3/2}(\epsilon) - \epsilon^{3/2}\right) + \frac{2b}{5}\left(f^{5/2}(\epsilon) - \epsilon^{5/2}\right) + ...
\end{aligned} \tag{34}$$

By equating the coefficients of δ, we find:

$$
\begin{aligned}
f_1 &= \frac{2}{3}\epsilon^{-3/2}\left(1 - p\epsilon\right) + ... \\
f_2 &= -\frac{1}{9}\epsilon^{-3} + ... \\
f_3 &= \frac{4}{81}\epsilon^{-9/2}\left(1 - p\epsilon\right) + ...
\end{aligned} \tag{35}
$$

where $p = b/a$.

In our calculation, we will find potentially divergent quantities, such as $1/(z' - z)$ and dz'/dz, so we will power expand all these quantities in terms of f_n in a double power expansion in ϵ and δ.

Then we easily find:

$$
\begin{aligned}
\frac{1}{z' - z} &= \frac{1}{f(\epsilon) - \epsilon} \\
&= \frac{1}{\epsilon f_1 \delta}\left(1 - \frac{f_2}{f_1}\delta + \delta^2\left(\frac{f_2^2}{f_1^2} - \frac{f_3}{f_1}\right) + ...\right) \\
\frac{1}{(z' - z)^2} &= \frac{1}{\epsilon^2 f_1^2 \delta^2}\left[1 - 2\delta\frac{f_2}{f_1}\right. \\
&+ \left.\delta^2\left(-2\frac{f_3}{f_1} + 3\frac{f_2^2}{f_1^2}\right) + ...\right]
\end{aligned} \tag{36}
$$

Also:

$$
\begin{aligned}
\frac{dz'}{dz} &= \frac{df(\epsilon)}{dz} \\
&= 1 + \sum_{n=1}^{\infty}\delta^n(\epsilon f_n)'
\end{aligned} \tag{37}
$$

We also find:

$$
\begin{aligned}
\frac{dz'}{dz}\frac{1}{(z' - z)^2} &= \epsilon^{-2} f_1^{-2}\left[(\epsilon f_2)' - 2 f_3 f_1'\right. \\
&+ \left.3 f_2^2 f_1^{-2} + (\epsilon f_1)'(-2 f_2 f_1^{-1}) + ...\right] \\
&= \epsilon^{-2}\left(\frac{5}{48} + \frac{p\epsilon}{12}\right) + ...
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
\left(\frac{dz'}{dz}\right)^2 \frac{1}{(z'-z)^2} &= \epsilon^{-2} f_1^{-2}\Big[-4(\epsilon f_1)' f_2 f_1^{-1} \\
&+ (\epsilon f_1)' + 2(\epsilon f_2)' \\
&- 2 f_3 f_1^{-1} + 3 f_2^2 f_1^{-2} + \ldots\Big] \\
&= \epsilon^{-2}\left(\frac{29}{48} + \frac{13}{12} p\epsilon\right) + \ldots
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
\left(\frac{dz'}{dz}\right)^2 \frac{1}{z'-z} &= \epsilon^{-1} f_1^{-1}\left(-f_2 f_1^{-1} + 2(\epsilon f_1)'\right) \\
&= -\frac{3}{4}\epsilon^{-1} + \ldots
\end{aligned} \tag{40}
$$

(The terms contained in ... correspond to terms which can be discarded in our approximation. For example, we will take the limit as $\epsilon \to 0$ first, and then take the limit as $\delta \to 0$. This allows us to eliminate powers of δ occurring with negative exponent. Also, there is a Riemann cut in the map in eq. (20), so we will choose the regularization scheme in ref. [19]. We can do this by altering eq. (32) slightly. We can define our point splitting by re-expressing our operators in terms of two new variables, z_1 and z_2, such that $\rho(z_1) = \rho(z) + (2/3)a\delta$ and $\rho(z_2) = \rho(z) - (2/3)a\delta$. Then operators are defined in terms of averaging over z_1 and z_2. This averaging corresponds to choosing $\rho(z') = \rho(z) + (2/3)a\delta$ and then discarding odd powers of δ.)

Now that we have defined all our regularized expressions, we can begin the process of calculating the anomaly. Let us first analyze the anomaly coming from the term $\partial_{z'}\phi(z')\partial\phi(z)$. We will commute this expression into the Neumann functions. We will then extract from this a c-number expression which represents the anomaly.

When we shove this term into the vertex function, we pick up quantities which look like $nmN^{rs}_{nm}\omega_r^n \tilde{\omega}_s^m$, where $\omega = e^{\zeta}$, where, following Mandelstam, we take ζ to be a local variable defined on the closed string, such that $\zeta = \tau + i\sigma$. ζ and ρ coincide for the closed string lying on the real axis. Fortunately, we know how to calculate this term in terms of z variables.

Let us differentiate the expression in eq. (91) in the Appendix:

$$
\frac{d}{d\zeta_s} N(\rho_r, \tilde{\rho}_s) = \delta_{rs}\left\{\frac{1}{2}\sum_{n\geq 1} \omega_r^{-n}(\tilde{\omega}_s^n + \tilde{\omega}_s^{*n}) + 1\right\}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\sum_{n,m\geq 0} nN^{rs}_{nm}\omega^n_r(\tilde{\omega}^m_s + \tilde{\omega}^{*m}_s) \\
&= \quad \frac{1}{2}\frac{dz}{d\zeta_r}\left(\frac{1}{z-\tilde{z}} + \frac{1}{z-\tilde{z}^*}\right)
\end{aligned}
\tag{41}
$$

and (by letting $\tilde{z}_s$ go to γ_s):

$$
\delta_{rs} + \sum_{n\geq 1} nN^{rs}_{n0}\omega^n_r = \frac{dz}{d\zeta_r}\frac{1}{z-\gamma_s}
\tag{42}
$$

A double differentiation leads to:

$$
\begin{aligned}
\frac{d}{d\zeta_r}\frac{d}{d\tilde{\zeta}_s}N(\rho_r,\tilde{\rho}_s) &= \delta_{rs}\frac{1}{2}\sum_{n\geq 1} n\omega^{-n}_r\tilde{\omega}^n_s + \\
&+ \frac{1}{2}\sum_{n,m\geq 1} nmN^{rs}_{nm}\omega^s_r\tilde{\omega}^m_s \\
&= \frac{1}{2}\frac{dz_r}{d\zeta_r}\frac{d\tilde{z}_s}{d\tilde{\zeta}_s}\frac{1}{(z-\tilde{z})^2}
\end{aligned}
\tag{43}
$$

We will now perform the calculation in two ways, using unbosonized ghost variables b and c, and then using the bosonized ghost variable σ.

3.1 Method I: Unbosonized Ghosts

With these identities, it is now an easy matter to calculate the action of the BRST operator on the vertex function in terms of unbosonized ghost variables b and c. Let the brackets $\langle\ \rangle$ represent the c-number expression what we obtain when we perform this commutation. Then we can show that the background-independent terms yield:

$$
\begin{aligned}
\left\langle \partial'_z\phi^\mu(z')\partial_z\phi(z)^\nu \right\rangle &= -\frac{1}{(z'-z)^2}\delta^{\mu\nu} \\
\left\langle c(z)b(z') \right\rangle &= -\frac{1}{z'-z}
\end{aligned}
\tag{44}
$$

With these expressions, we can now calculate the contribution to the anomaly due to $\partial_{z'}\phi(z')\partial_z\phi(z)$ and $\partial_z c(z)b(z')$. This calculation is simplified because the ghost insertion factor disappears in the $b-c$ formalism.

We find (dropping the background-dependent terms, for the moment):

$$
\begin{aligned}
\sum_{i=1}^{3} Q_i|V_3\rangle_0 &= \oint_{C_1+C_2+C_3} \frac{dz}{4\pi i}\Big\{c(z)\big[-\frac{dz'}{dz}\big\langle\partial_{z'}\phi^\mu(z')\partial_z\phi^\mu(z)\big\rangle \\
&+ 2\left(\frac{dz'}{dz}\right)^2\big\langle\frac{dc}{dz}(z)b(z')\big\rangle\big] - 2\frac{dc}{dz}\left(\frac{dz'}{dz}\right)^2\big\langle c(z)b(z')\big\rangle\Big\}|V_3\rangle_0 + ... \\
&= -\oint_{C_1+C_2+C_3} \frac{dz}{4\pi i}\Big\{c(z)\big[-\frac{dz'}{dz}\frac{1}{(z'-z)^2} \\
&+ 2\left(\frac{dz'}{dz}\right)^2\partial_z\frac{1}{z'-z}\big] - 2\frac{dc}{dz}\left(\frac{dz'}{dz}\right)^2\frac{1}{z'-z}\Big\}|V_3\rangle_0 + ... \\
&= -\oint_{C_1+C_2+C_3} \frac{dz}{4\pi i}\Big\{c(z)\big[-\epsilon^{-2}\left(\frac{5}{48}+\frac{p\epsilon}{12}\right) \\
&+ 2\epsilon^{-2}\left(\frac{29}{48}+\frac{13}{12}p\epsilon\right)\big] - 2\frac{dc}{dz}\left(-\frac{3}{4}\epsilon^{-1}\right)\Big\}|V_3\rangle_0 + ... \qquad (45)
\end{aligned}
$$

where ... are terms which are background-dependent. Now let us combine the three arcs C_i into one circle which goes around the joining point z_0. Integrating by parts, we find:

$$
\begin{aligned}
\sum_{i=1}^{3} Q_i|V_3\rangle_0 &= \oint_{z_0} \frac{dz}{2\pi i}\Big\{\frac{pc(z)}{z-z_0}\left[\frac{D}{24}+\frac{1}{24}-\frac{13}{24}\right] \\
&+ \frac{dc(z)}{dz}\frac{1}{z-z_0}\left[\frac{5D}{96}+\frac{5}{96}-\frac{65}{48}\right]\Big\}|V_3\rangle_0 + ... \qquad (46)
\end{aligned}
$$

The last part of the calculation is perhaps the most crucial, i.e. calculating the contribution of the term $\partial_z\phi$ to the anomaly which are background-dependent. Normally, this term does not contribute at all. However, in the presence of the insertion operator at the joining points z_i and $\tilde{z}_i$, this term does in fact contribute an important part to the anomaly.

Our task is to shove the operator $\partial_z\phi$ into the vertex function and calculate terms proportional to the background charge Q. We find:

$$
\begin{aligned}
&\partial_\rho\phi_r(\rho)|V_3\rangle_0 \\
&\qquad = (-i)^2\frac{Q}{2}\Big(\sum_{n,m\geq 0}\sum_r \omega_r^n N_{nm}^{rs}\cos(m\pi/2) \\
&\qquad -\sum_{n\geq 1}\sum_r \omega_r^n N_{n0}^{rs}\Big)|V_3\rangle_0 + ... \qquad (47)
\end{aligned}
$$

We immediately recognize the terms on the right as being functions of $1/(z-z_i)$ and $1/(z-\gamma_i)$ in eqs. (41) and (42) for the case $r \neq s$.

The contribution of the anomaly from the insertion operator is therefore given by:

$$
\begin{aligned}
\left\langle \partial_\rho \phi(\rho) \right\rangle &= -\frac{Q}{2}\left[\frac{1}{2}\sum_{i=1}^{M-2}\left(\frac{1}{z-z_i}+\frac{1}{z-\tilde{z}_i}\right)-\sum_{i=1}^{M}\frac{1}{z-\gamma_i}\right] \\
&= \frac{dz}{d\rho}\frac{Q}{2}\partial_z \log|\frac{dz}{d\rho}|
\end{aligned}
\tag{48}
$$

(As an added check on the correctness of this calculation, notice that the last step reproduces the desired transformation property of the ϕ field in eq. (28), which has an additional contribution due to the background charge Q. Thus, when we insert this term into the expression for the energy-momentum tensor, we simply reproduce the Schwartzian.)

Given this expression, we can now calculate the contribution of the background-dependent terms to the anomaly. This contribution is:

$$
\begin{aligned}
\ldots &= \oint_{z_0}\frac{dz}{2\pi i}c(z)\left[-\frac{1}{2}\left\langle \partial_z\phi\right\rangle^2+\frac{1}{2}Q\frac{d\rho}{dz}\partial_z\left(\frac{dz}{d\rho}\left\langle\partial_z\phi\right\rangle\right)\right]|V_3\rangle_0 \\
&= \oint_{z_0}\frac{dz}{2\pi i}c(z)\frac{Q^2}{4}\left(\frac{5}{8}\frac{1}{(z-z_0)^2}+\frac{1}{2}p\frac{1}{z-z_0}\right)|V_3\rangle_0
\end{aligned}
\tag{49}
$$

The last step is to put all terms together. Combining the results of eq. (46) and (49), we now easily find:

$$
\left\{pc(z_0)\left[\frac{D}{24}-\frac{13}{12}+\frac{1}{24}+\frac{1}{8}Q^2\right]+\frac{dc(z_0)}{dz}\left[\frac{5D}{96}-\frac{65}{48}+\frac{5}{96}+\frac{5}{32}Q^2\right]\right\}|V_3\rangle_0
\tag{50}
$$

which cancels if:

$$
D-26+1+3Q^2=0 \tag{51}
$$

which is precisely the consistency equation for Liouville theory in D dimensions. Thus, the vertex is BRST invariant.

3.2 Method II: Bosonized Ghosts

Next, we will show that the calculation can also be performed using the bozonized ghost variable σ. We exploit the fact that the X, ϕ, and σ field can be arranged in the same composite field ϕ^M.

When we commute $\partial_z \phi^M$ into the vertex function, we find that the σ ghost variables contribute an almost identical contribution as the ϕ variable.

Let us redo the calculation in two parts. We will calculate the background-independent terms first. This means dropping the b and c terms in eq. (45) and replacing the ϕ^μ field by ϕ^M. The calculation is straightforward, and yields:

$$\begin{aligned} \sum_{i=1}^{3} Q_i |V_3\rangle_0 &= \oint_{z_0} \frac{dz}{2\pi i} \left\{ \frac{p e^\sigma(z)}{z - z_0} \left[\frac{(D+2)}{24} \right] \right. \\ &+ \left. \frac{d e^\sigma(z)}{dz} \frac{1}{z - z_0} \left[\frac{5(D+2)}{96} \right] \right\} |V_3\rangle_0 + \ldots \end{aligned} \tag{52}$$

Next, we must calculate the background-dependent terms. We can generalize the equation which determines how the fields change when they are commuted past the insertion operators:

$$\left\langle \partial_\rho \phi^M(\rho) \right\rangle = \frac{dz}{d\rho} \frac{Q^M}{2} \partial_z \log |\frac{dz}{d\rho}| \tag{53}$$

The crucial complication is that the quadratic term in the energy-momentum tensor in eq. (8) for the ϕ field and the σ field differs by a factor of -1. This means that when we insert this expression into the BRST operator Q, we pick up an extra -1 factor, so the contribution to the anomaly from the background-dependent terms now becomes:

$$\begin{aligned} \ldots &= \oint_{z_0} \frac{dz}{2\pi i} e^{\sigma(z)} \left[\pm \frac{1}{2} \left\langle \partial_z \phi^M \right\rangle^2 + \frac{1}{2} Q^M \frac{d\rho}{dz} \partial_z \left(\frac{dz}{d\rho} \left\langle \partial_z \phi^M \right\rangle \right) \right] |V_3\rangle_0 \\ &= \oint_{z_0} \frac{dz}{2\pi i} e^\sigma(z) \frac{Q^2 - 3^2}{4} \left(\frac{5}{8} \frac{1}{(z - z_0)^2} + \frac{1}{2} p \frac{1}{z - z_0} \right) |V_3\rangle_0 \end{aligned} \tag{54}$$

where the $-(+)$ sign appears with the $\phi(\sigma)$ operator.

Now, let us put all the terms together in the calculation. We find:

$$\left\{ p e^{\sigma(z_0)} \left[\frac{D+2}{24} + \frac{1}{8}(Q^2 - 3^2) \right] + \frac{d e^{\sigma(z_0)}}{dz} \left[\frac{5(D+2)}{96} + \frac{5}{32}(Q^2 - 3^2) \right] \right\} |V_3\rangle_0 \tag{55}$$

Once again, we find that the anomaly cancels if we set:

$$D + 2 + 3(Q^2 - 3^2) = 0 \tag{56}$$

as desired. Thus, the anomaly cancels in both the bosonized and the un-bosonized expressions, although each expression is qualitatively quite dissimilar from the other. This is a check on the correctness of our calculations.

Similarly, the anomaly can be cancelled for all non-polynomial vertices. For an N-sided polyhedral vertex, we first notice that the BRST operator Q, once it is commuted past the various b operators, vanishes on the bare vertex because of the continuity equations, except at the $2(N-2)$ joining points z_i and $\tilde{z}_i$.

Second, we notice that the conformal map around each joining point in eq. (31) is virtually the same, no matter how complicated the polyhedral vertex function may be. All the messy dependence on the various constraints are buried within $\rho(z_0)$ and p. Fortunately, the dependence on these unknown factors cancels out of the calculation. This is why the calculation can be generalized to all polyhedral vertices.

Thus, the calculation of the anomaly cancellation can be performed on each of the various joining points z_i and $\tilde{z}_i$ separately. But since the calculation is basically the same for each of these joining points, we have now shown that all possible polyhedral vertex functions are all anomaly-free.

Notice that this proof does not need to know the specific value of the Neumann functions. The entire calculation just depended on knowing the derivatives of eq. (91) and how various operators behaved when commuted into the vertex function.

4 Shifted Shapiro-Virasoro Amplitude

The next major test of the theory is whether it reproduces the shifted Shapiro-Virasoro amplitude. This calculation is highly non-trivial, since the conformal map between the multi-sheeted string-scattering Riemann sheet to the complex plane is very involved. Unlike the conformal maps found in light cone theory, or even the maps found in Witten's open string theory [21-22], the non-polynomial theory yields very complicated conformal maps.

Fortunately, for the four-point function, all conformal maps are known exactly, in terms of elliptic functions, and the calculation can be performed [23-24].

For the four point function, the map in eq. (20) can be integrated exactly. We use the identity:

$$\frac{(z-z_1)(z-\tilde{z}_1)(z-z_2)(z-\tilde{z}_2)}{\prod_{i=1}^{4}(z-\gamma_i)} = 1 + \sum_{i=1}^{4} \frac{A_i}{z-\gamma_i} \tag{57}$$

where we define $z_i = ia_i + b_i$ and $\tilde{z}_i = -ia_i + b_i$ for *complex* a_i and b_i, and:

$$A_i = \frac{[(\gamma_i - b_1)^2 + a_1^2]\,[(\gamma_i - b_2)^2 + a_2^2]}{\prod_{j=1,j\neq i}^{4}(\gamma_i - \gamma_j)} \tag{58}$$

Then we can split the integral into two parts, with the result:

$$\begin{aligned}
\rho(z) &= \rho_1(z) + \rho_2(z) \\
\rho_1(z) &= \int_{y_1}^{y} \frac{N dz}{\sqrt{(z-z_1)(z-\tilde{z}_1)(z-z_2)(z-\tilde{z}_2)}} \\
\rho_2(z) &= \sum_{i=1}^{4} \int_{y_1}^{y} \frac{N A_i dz}{(z-\gamma_i)\sqrt{(z-z_1)(z-\tilde{z}_1)(z-z_2)(z-\tilde{z}_2)}}
\end{aligned} \tag{59}$$

Written in this form, we can now perform all integrals exactly, using third elliptic integrals in eq.(95) and eq. (97) in the Appendix. It is then easy to show:

$$\begin{aligned}
\rho_1(z) &= NgF(\phi, k') = Ng\mathrm{tn}^{-1}\left[\tan\phi, k'\right] \\
\rho_2(z) &= \sum_{i=1}^{4} \frac{gNA_i}{a_1 + b_1 g_1 - g_1\gamma_i}\Bigg(g_1 F(\phi, k') \\
&+ \frac{\omega_i - g_1}{1+\omega_i^2}\left[F(\phi,k') + \omega_i^2 \Pi(\phi, 1+\omega_i^2, k') + \omega_i(\omega_i^2+1) f_i\right]\Bigg)
\end{aligned} \tag{60}$$

where:

$$\begin{aligned}
\omega_i &= \frac{a_1 + b_1 g_1 - \gamma_i g_1}{b_1 - a_1 g_1 - \gamma_i} \\
f_i &= \frac{1}{2}(1+\omega_i^2)^{-1/2}(k^2+\omega_i^2)^{-1/2} \\
&\quad \times \ln\frac{(k^2+\omega_i^2)^{1/2} - (1-\omega_i^2)^{1/2}\mathrm{dn}u}{(k^2+\omega_i^2)^{1/2} + (1+\omega_i^2)^{1/2}\mathrm{dn}u} \\
\phi &= \arctan\left(\frac{y - b_1 + a_1 g_1}{a_1 + g_1 b_1 - g_1 y}\right)
\end{aligned} \tag{61}$$

and where:

$$\begin{aligned}
A^2 &= (b_1+b_2)^2 + (a_1+a_2)^2,\ B^2 = (b_1-b_2)^2 + (a_1-a_2)^2 \\
g_1^2 &= [4a_1^2 - (A-B)^2]/[(A+B)^2 - 4a_1^2],\ g = 2/(A+B) \\
y_1 &= b_1 - a_1 g_1,\ {k'}^2 = 1 - k^2 = 4AB/(A+B)^2 \\
u &= \mathrm{dn}^{-1}(1 - {k'}^2 \sin^2\phi)
\end{aligned} \tag{62}$$

After a certain amount of algebra, this expression simplifies considerably to:

$$\begin{aligned}
\rho(z) &= \sum_{i=1}^{4} \frac{gNA_i}{a_1 + b_1 g_1 - g_1\gamma_i} \frac{\omega_i - g_1}{1+\omega_i^2} \\
&\quad \times \left[\omega_i^2 \Pi(\phi, 1+\omega_i^2, k') + \omega_i(\omega_i^2+1) f_i\right]
\end{aligned} \tag{63}$$

Now that we have an explicit form for the conformal map from the flat z plane to the ρ plane, in which string scattering takes place, we must next impose the constraint that the overlap between two colliding strings is given by π. This is satisfied by imposing:

$$\begin{aligned}
\pi &= \mathrm{Im}\left[\rho(z_1) - \rho(y_1)\right] \\
&= -\frac{\pi}{2} gN \frac{(\omega_i - g_1)\omega_i}{a_1 + b_1 g_1 - g_1\gamma_i} \sum_{i=1}^{4} \frac{A_i \Lambda_0(\beta_i, k)}{\sqrt{(1+\omega_i^2)(k^2+\omega_i^2)}} \\
&= \frac{-\pi}{2} \sum_{i=1}^{4} \alpha_i \Lambda_0(\beta_i, k)
\end{aligned} \tag{64}$$

where $\alpha_i = NA_i \left[(\gamma_i - b_1)^2 + a_1^2\right]^{-1/2} \left[(\gamma_i - b_2)^2 + a_2^2\right]^{-1/2}$, where we have used eq. (100), where we have set $y = z_1$, so that $\tan\phi = i$, and where $\sin^2\beta_i = (1+\omega_i^2)^{-1}$. We have also used the fact that:

$$\Pi(\phi, 1+\omega_i^2, k') = -\frac{1}{2}\pi i \frac{\sqrt{1+\omega_i^2}}{\sqrt{k^2+\omega_i^2}} \frac{\Lambda_0(\beta_i, k) - 1}{\omega_i} \tag{65}$$

Next, we must calculate the separation between the two vertices and the relative angle of rotation between them. The proper time separating the two interactions is given by:

$$\begin{aligned}
\tau &= \mathrm{Re}\left[\rho(z_2) - \rho(z_1)\right] \\
&= g \sum_{i=1}^{4} NA_i \frac{\omega_i^2(\omega_i - g_1)\Pi(\pi/2, 1+\omega_i^2, k')}{(a_1 + b_1 g_1 - g_1\omega_i)(1+\omega_i^2)} \\
&= -K(k') \sum_{i=1}^{4} \alpha_i Z(\beta_i, k')
\end{aligned} \tag{66}$$

where we have used eq. (101) and the fact that:

$$
\begin{aligned}
\Pi(\phi, 1+\omega_i^2, k') &= \Pi(\phi_2, 1+\omega_i^2, k') - \Pi(\phi_1, 1+\omega_i^2, k') \\
\Pi(\alpha^2, k) &= -\frac{\alpha K Z(\mathrm{arcsin}\alpha^{-1}, k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}
\end{aligned}
\tag{67}
$$

and:

$$
\begin{aligned}
\mathrm{tan}\phi_1 &= i,\ \phi_1 = i\infty \\
\mathrm{tan}\phi_2 &= \frac{i}{k},\ \phi_2 = \mathrm{arcsin}\frac{1}{k'}
\end{aligned}
\tag{68}
$$

which we can show by setting $y = z_1, z_2$.

Now that we have an explicit form for τ, the next problem is to differentiate it and find the Jacobian of the transformation of τ to x.

By differentiating, we find:

$$
\begin{aligned}
d\tau &= -\sum_{i=1}^{4} \alpha_i \frac{r^2(\beta_i, k')K(k') - E(k')}{r(\beta_i, k')} d\beta_i \\
&= \frac{\pi}{2} K(k)^{-1} \sum_{i=1}^{4} \frac{\alpha_i d\beta_i}{r(\beta_i, k')} \\
&= \frac{\pi N}{2gK(k)} \sum_{i=1}^{4} \frac{d\gamma_i}{\prod_{j=1, j\neq i}(\gamma_i - \gamma_j)}
\end{aligned}
\tag{69}
$$

where $r(\theta, k') = \sqrt{1 - k'^2 \sin^2\theta}$ and where we have used eq. (102) in the Appendix. We have also used the fact that the derivative of π in eq. (64) is a constant, so:

$$
0 = \sum_{i=1}^{4} \alpha_i \frac{E(k) - k'^2 \sin^2\beta_i K(k)}{r(\beta_i, k')} d\beta_i \tag{70}
$$

This explicit conformal map allows us to calculate the four-point amplitude. We first write the amplitude in the ρ plane, and then make a conformal map to the z-plane. Let the modular parameter be $\hat{\tau} = \tau + i\theta$, where τ is the distance between the splitting strings, and θ is the relative rotation. Then, with a fair amount of work, one can find the Jacobian from $\hat{\tau}$ to $\hat{x}$.

Let us define $\hat{x}$ as:

$$
\hat{x} = \frac{(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_4)}{(\gamma_2 - \gamma_4)(\gamma_3 - \gamma_1)} \tag{71}
$$

so that:

$$d\hat{x} = \hat{x}(1-\hat{x})\frac{(\gamma_1-\gamma_3)(\gamma_2-\gamma_4)}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)(\gamma_1-\gamma_4)}d\gamma_1 \tag{72}$$

Putting everything together, we now find:

$$\frac{d\hat{\tau}}{d\hat{x}} = -\frac{\pi N}{2K(k)g\hat{x}(1-\hat{x})(\gamma_1-\gamma_3)(\gamma_2-\gamma_4)} \tag{73}$$

If we take only the tachyon component of $|\Phi\rangle$, then the four point amplitude can be written as:

$$\begin{aligned}
A_4 &= \langle V_3|\frac{b_0\bar{b}_0}{L_0+\bar{L}_0-2}|V_3\rangle \\
&= \int d\tau d\theta \left\langle V(\infty)V(1)\left(\int_C dz\frac{dz}{dw}b_{zz}\right)\left(\int_C \frac{d\bar{z}}{d\bar{w}}b_{\bar{z}\bar{z}}\right)V(\hat{x})V(0)\right\rangle \\
&= \int d^2\hat{\tau}\left|\exp\left[\sum_i\left(ip_i\cdot\phi(i)+\epsilon_i\phi(i)\right)\right)A_G\right|^2
\end{aligned} \tag{74}$$

where we must sum over all permutations so that we integrate over the entire complex plane, where b_0 defined in the ρ plane transforms into $\int_C dz(dz/dw)b_{zz}$ in the z-plane, where C is the image in the z-plane of a circle in the ρ plane which slices the intermediate closed string, where $V(z)=c(z)\tilde{c}(z)V_0(z)$, where V_0 is the tachyon vertex without ghosts, and where the ghost part A_G equals:

$$\begin{aligned}
A_G &= \int_C \frac{dz}{2\pi i}\frac{dz}{dw}\exp\left\{-\sum_{i\leq j}\langle\sigma_i\sigma_j\rangle+\sum_j\langle\sigma_j\sigma_+(z)\rangle\right\} \\
&= \int_C \frac{dz}{2\pi i}\frac{dz}{dw}\frac{\prod_{i<j}(\gamma_i-\gamma_j)}{\prod_{j=1}^4(z-\gamma_j)} && (75) \\
&= 2\frac{g}{\pi c}\hat{x}(1-\hat{x})K(k)(\gamma_1-\gamma_3)^3(\gamma_2-\gamma_4)^3 && (76)
\end{aligned}$$

(Notice that we have made a conformal transformation from the ρ world sheet to the z complex plane. In general, we pick up a determinant factor, proportional to the determinant of the Laplacian defined on the world sheet. However, after making the conformal transformation, we find that the determinant of the Laplacian on the flat z-plane reduces to a constant. Thus, we can in general ignore this determinant factor.)

Putting the Jacobian, the ghost integrand, and the string integrand together, we finally find:

$$A_4 = \int d^2\hat{x} \left| \hat{x}^{2p_1 \cdot p_2} (1-\hat{x})^{2p_2 \cdot p_3} \right|^2 \tag{77}$$

In two dimensions, we have $p_i \cdot p_j = p_i p_j - \epsilon_i \epsilon_j$ where $\epsilon_i = \sqrt{2} + \chi_i p_i$, where χ is the "chirality" of the tachyon state, so we reproduce the integral found in matrix models and Liouville theory. (The amplitude is non-zero only if the chiralities are all the same except for one external line.)

However, so far the region of integration does *not* cover the entire complex z-plane. This is because we have implicitly assumed in the constraints $\hat{\tau}_{ij} = \rho(z_i) - \rho(z_j)$ that there is no four-string interaction. However, as we have shown in [20], the complete region of integration contains a "missing region" which is precisely filled by the four string interaction. This calculation carries over, without any change, to the $D < 26$ case.

With the missing region filled by the four-string tetrahedron graph, we finally have the complete shifted Shapiro-Virasoro amplitude, as expected.

Lastly, we would like to mention the direction for possible future work. Two problems come to mind. The most glaring deficiency of this approach is that we have set the cosmological constant to zero. However, the theory becomes quite non-linear for non-zero cosmological constant, so the calculations become much more difficult.

The second problem is that we have not shown the equivalence of this approach to the Das-Jevicki action [25-6], which is the second quantized field theory of matrix models. This action is based strictly on the tachyon, so we speculate that, once we gauge away the BRST trivial states and integrate out the discrete states, our action should reduce down to the Das-Jevicki action (for $\mu = 0$). This problem is still being investigated.

5 Acknowledgments

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6 Appendix

We will find it convenient to define the holomorphic expressions for the operators as follows. (It is understood that we must double the operators in order to describe the closed string.) If we define $\phi^M = \{X^i, \phi, \sigma\}$, then:

$$\partial_z \phi^M = \sum_{n=-\infty}^{\infty} \{-i\alpha_n^i, -i\phi_n, \sigma_n\} z^{-n-1} \tag{78}$$

where:

$$[\phi_n^M, \phi_m^N] = n\delta^{MN}\delta_{n,-m} \tag{79}$$

where $\delta^{MN} = \mathrm{diag}\,\{\delta^{ij}, 1, 1\}$.

Physical states without ghost indices are defined via the conditions:

$$\begin{aligned} L_n \,|\Phi\rangle &= \bar{L}_n|\Phi\rangle = 0 \\ (L_0 - 1)|\Phi\rangle &= (\bar{L}_0 - 1)|\Phi\rangle = 0 \\ (L_0 - \bar{L}_0)|\Phi\rangle &= 0 \end{aligned} \tag{80}$$

The tachyon state is defined as:

$$|p^\mu\rangle = |p^i, \epsilon\rangle = e^{ip\cdot X + \epsilon\phi}(0)|0\rangle \tag{81}$$

where $\alpha_0^\mu|p\rangle = p^\mu|p\rangle$.

To solve for ϵ and the mass of the tachyon, we must solve the on-shell condition:

$$L_0|p,\epsilon\rangle = \bar{L}_0|p,\epsilon\rangle = \left(\frac{1}{2}p_i^2 - \frac{1}{2}\epsilon(\epsilon + Q)\right)|p,\epsilon\rangle \tag{82}$$

so that:

$$p_i^2 - \epsilon(\epsilon + Q) - 2 = 0 \tag{83}$$

To put this in more familiar mass-shell form, let us define $E = \epsilon + (1/2)Q$. Thus, the mass-shell condition can be written as:

$$p_i^2 - E^2 = -\left(\frac{1}{4}Q^2 - 1\right)^2 = -m^2 \tag{84}$$

which defines the tachyon mass. This means that the tachyon mass obeys the relation:

$$m^2 = \left(\frac{1-D}{12}\right)^2 \tag{85}$$

As a check, we find that this simply reproduces the usual relationship between the tachyon mass and dimension. So therefore the tachyon is massless in $D = 1$ (or in two dimensions, if we consider the Liouville field to be a dimension).

On the other hand, we can solve the mass-shell condition for ϵ directly, yielding:

$$\epsilon = \frac{-Q \pm \sqrt{Q^2 - 8 + 4p_i^2}}{2} \tag{86}$$

We shall be mainly interested in the case of two dimensions, or $D = 1$, so we find $Q = 2\sqrt{2}$ and:

$$\epsilon = -\sqrt{2} + \chi p \tag{87}$$

where $\chi = \pm 1$ is called the "chirality" of the tachyon state. The ground state, with arbitrary ghost number λ, can therefore be written as:

$$|p, \epsilon, \lambda\rangle = e^{ipX + \epsilon\phi + \lambda\sigma}(0)|0\rangle \tag{88}$$

where $\sigma_0|p, \lambda\rangle = \lambda|p, \lambda\rangle$. We will choose $\lambda = 1$ for the ghost vacuum.

Our tachyon state is then defined as $|p^M\rangle = |p, \epsilon, \lambda\rangle$.

In addition, we also have the $b - c$ ghost system. We define the $SL(2, R)$ vacuum in the usual way:

$$\langle 0|c_{-1}c_0c_1|0\rangle = 0 \tag{89}$$

so the ghost system has background charge -3. Then the ghost part of the tachyon field is given by $c_1\bar{c}_1|0\rangle$.

If we let $c_1|0\rangle = |-\rangle$, with ghost number $-1/2$, then the open string wave function is based on the vacua $|-\rangle$ and $c_0|-\rangle = |+\rangle$. For the closed string case, the string wave function $|\Phi\rangle$ is based on four possible vacua, so that:

$$|\Phi\rangle = \varphi_{--}|-\rangle|-\rangle + \varphi_{-+}|-\rangle|+\rangle + \varphi_{+-}|+\rangle|-\rangle + \varphi_{++}|+\rangle|+\rangle \tag{90}$$

With this ground state, we can then construct the vertex functions, once we know the Neumann functions. These can be defined via the Green's function on the string world sheet in the usual way:

$$\begin{aligned} N(\rho_r, \tilde{\rho}_s) &= -\delta_{rs}\left\{ \sum_{n\geq 1} \frac{2}{n} e^{-n|\xi_r - \tilde{\xi}_s|} \cos(n\sigma_r)\cos(n\tilde{\sigma}_s) - 2\max(\xi_r, \tilde{\xi}_s) \right\} \\ &+ 2 \sum_{n,m\geq 0} N^{rs}_{nm} e^{n\xi_r + m\tilde{\xi}_s} \cos(n\sigma_r)\cos(n\tilde{\sigma}_s) \\ &= \log|z - \tilde{z}| + \log|z - \tilde{z}^*| \end{aligned} \tag{91}$$

By taking the Fourier transform of the previous equation, one can invert the relation and find an expression for N^{rs}_{nm}:

$$N^{rs}_{nm} = \frac{1}{nm}\oint_{z_r}\frac{dz}{2\pi i}\oint_{z_s}\frac{d\tilde{z}}{2\pi i}\frac{1}{(z-\tilde{z})^2}e^{-n\rho_r(z)-m\tilde{\rho}_s(\tilde{z})}$$

$$N^{rs}_{n0} = \frac{1}{n}\oint_{z_r}\frac{dz}{2\pi i}\frac{1}{z-z_s}e^{-n\rho_r(z)} \tag{92}$$

In addition to these Neumann functions, we must also define the B_N line integrals, which are found in the calculation of any N-point tree graph and hence must appear in the vertex function as well.

We have:

$$B_N = \prod_{j=1}^{N}(b_0 - \bar{b}_0)_j \prod_{k=1}^{2N-6} b_{\mu_k} d\tau_k \tag{93}$$

where:

$$b_{\mu_k} = \int \frac{d^2\xi}{2\pi}\left(\mu_k b(z) + \text{c.c.}\right) \tag{94}$$

where τ_k are the modular parameters which specify the polyhedra, where μ_k are the $2N-6$ Beltrami differentials which correspond to the $2N-6$ quasi-conformal deformations which typify how the polyhedral vertex function changes as the moduli parameters τ_i vary. These τ_i, in turn, are functions of the angles θ_{ij}.

With these Neumann functions, we can construct the four-point scattering amplitude. However, the Jacobian from the world sheet to the complex z-plane requires elliptic integrals.

Our conventions are those of ref. [24]. First elliptic integrals are defined as:

$$\begin{aligned} F(\phi,k) &= \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \\ &= \int_0^\phi \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}} \\ &= \mathrm{sn}^{-1}(y,k) \end{aligned} \tag{95}$$

where $y = \sin\phi$ and $\phi = \mathrm{am}\ u_1$.

Second elliptic integrals are defined as:

$$\begin{aligned} E(\phi,k) &= \int_0^y \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt \\ &= \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta \end{aligned} \tag{96}$$

Third elliptic integrals are defined as:

$$
\begin{aligned}
\Pi(\phi,\alpha^2,k) &= \int_0^y \frac{dt}{(1-\alpha^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}} \\
&= \int_0^\phi \frac{d\theta}{(1-\alpha^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \\
&= \int_0^{u_1} \frac{du}{1-\alpha^2\mathrm{sn}^2u} \qquad (97)
\end{aligned}
$$

Complete first elliptic integrals are defined as:

$$
K(K) = K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\pi/2,k) \qquad (98)
$$

Complete second elliptic integrals are defined as:

$$
E(\pi/2,k) = E = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}d\theta \qquad (99)
$$

Heuman's lambda function is defined as:

$$
\Lambda_0(\phi,k) = \frac{2}{\pi}[EF(\phi,k') + KE(\phi,k') - KF(\pi,k')] \qquad (100)
$$

The Jacobi zeta function is defined as:

$$
Z(\phi,k) = E(\phi,k) - \frac{E}{K}F(\phi,k) \qquad (101)
$$

In the text, we have used the following differential equations:

$$
\begin{aligned}
\frac{d}{dk}\left[K(k')Z(\beta_i,k')\right] &= \frac{k'E(K')}{k^2}\frac{\sin\beta_i\cos\beta_i}{r(\beta_i,k')} \\
\frac{d}{dk}\Lambda_0 &= \frac{2}{\pi k}\left[E(k)-K(k)\right]\frac{\sin\beta_i\cos\beta_i}{r(\beta_i,k')} \\
\frac{d}{d\beta_i}\left[K(k')Z(\beta_i,k')\right] &= \frac{r^2(\beta_i,k')K(k')-E(k')}{r(\beta_i,k')} \\
\frac{d}{d\beta_i}\Lambda_0(\beta_i,k) &= \frac{2}{\pi r(\beta_i,k')}\left[E(k)-k^{2\prime}\sin^2\beta_iK(k)\right] \qquad (102)
\end{aligned}
$$

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HOW UNSTABLE ARE FUNDAMENTAL QUANTUM SUPERMEMBRANES?

MICHIO KAKU

Physics Dept., City College of New York
New York, N.Y. 10031, USA

ABSTRACT

1 Quantum Supermembranes

String duality, pioneered by Kikkawa and Yamasaki[1,2], represents an enormous advance in our understanding of string physics. For the first time, we can peer into the non-perturbative region of certain string theories and settle questions which have dogged the field since its very inception[3−8].

In particular, solitonic p-branes are necessary to complete our understanding of BPS-saturated states. An eleven dimensional "M-theory," in fact, must be able to incorporate both strings and solitonic membranes. Because these solitonic membranes likely have finite thickness, they are probably stable.

By contrast, fundamental quantum supermembranes are thought to have serious problems. Besides the fact that they are highly non-linear (and hence their spectrum is impossible to calculate exactly), they also have several serious physical diseases:
(a) the world volume action is not renormalizable
(b) the theory has no dilaton, so a standard KSV type perturbation theory is not possible

(c) the theory is thought to be unstable; for string-like configurations, the zero-point energy of the Hamiltonian may be zero[9].

The first problem means that an infinite number of counter-terms must be added to the world volume action to render it finite. However, perhaps these counter-terms simply represent the infinite number of background fields corresponding to excitations of the supermembrane. So having an infinite number of counter-terms is by itself not necessarily fatal.

The second problem is also not necessarily fatal if a new mechanism is found for interacting membranes, other than the standard dilaton formulation. Since we do not know how supermembranes split apart (or even if they do), it is premature to discount them on the basis of interactions.

The third problem is more serious, since it goes to the heart of whether fundamental quantum supermembranes are stable or not.

Previously, in ref. 9, this question was addressed by approximating the supermembrane action[10,11] by a SU(n) super Yang-Mills theory as $n \to \infty$. For finite n, this amounts to a convenient regulator for the theory. Although this proof is rather convincing, it depends on whether the $n \to \infty$ limit is singular or not. Perhaps there are regularization-dependent factors which enter into the picture in this delicate limit.

In this paper, we will try to address the question directly, whether the continuum theory is stable or not. By analyzing the continuum theory, we have a much more intuitive grasp of precisely where the problems may lie and where the potential infinities may occur. We will follow the basic outline of ref. 9, but adapt their calculation for our purposes.

And second, at the end of this paper, we present some rough speculations about how unstable membranes may still be made into a physical theory.

We begin with the action for the membrane, which is given by:

$$S = S_1 + S_2 \tag{1}$$

where S_1 is the usual determinant defined over a world volume:

$$S_1 = -T \int d^3\sigma \, \sqrt{-M}; \quad M_{ij} = \Pi_i^\mu \Pi_j^\nu \eta_{\mu\nu} \tag{2}$$

where:

$$\Pi_i^\mu = \partial_i X^\mu - i\bar{\theta}\Gamma^\mu \partial_i \theta \tag{3}$$

and S_2 is a Wess-Zumino term[10,11]:

$$S_2 = -T\int d^3\sigma \left[\frac{1}{2}\epsilon^{ijk}\bar{\theta}\Gamma_{\mu\nu}\partial_i\theta\left(\Pi_j^\mu\Pi_k^\nu + i\Pi_j^\mu\bar{\theta}\Gamma^\nu\partial_k\theta - \frac{1}{3}\bar{\theta}\Gamma^\mu\partial_j\theta\bar{\theta}\Gamma^\nu x\partial_k\theta\right)\right] \quad (4)$$

where $i = 1, 2, 3$ represents the three world volume indices of the membrane. Two of them, σ_1 and σ_2, represent the co-ordinates of the surface, and $\sigma_3 = \tau$ represents the time-like direction. The Greek symbols represent 11 dimensional Lorentz indices. Γ^μ are the usual Dirac matrices in 11 dimensions. X^μ is the co-ordinate of the membrane, and θ is a Majorana spinor with 32 real components.

This action is invariant under a standard reparametrization invariance:

$$\delta X^\mu(\sigma_1, \sigma_2, \sigma_3) = \epsilon^i\partial_i X^\mu(\sigma_1, \sigma_2, \sigma_3) \quad (5)$$

The Majorana spinor θ also transforms as a scalar under reparametrizations in the world volume variables.4

Under local supersymmetry, we have:

$$\delta X^\mu = \bar{\theta}\Gamma^\mu(1+\Gamma)\kappa;\ \delta\theta = (1+\Gamma)\kappa \quad (6)$$

where κ is a local parameter, and:

$$\Gamma = \frac{1}{6\sqrt{-g}}\epsilon^{ijk}\Pi_i^\mu\Pi_j^\nu\Pi_k^\rho\Gamma_{\mu\nu\rho} \quad (7)$$

and where $\Pi_i \cdot \Pi_j = g_{ij}$.

The action as it stands is intractable because of its highly coupled nature. The simplest way of simplifying and quantizing the theory is to go to the light cone gauge, where all longitudinal modes are removed. We impose:

$$\Gamma^+\theta = 0 \quad (8)$$

along with the usual bosonic constraints. A large number of terms vanishes in the light cone gauge because $\bar{\theta}\Gamma^\mu\partial_i\theta = 0$ except for $\mu = -$. In particular, the higher order coupled terms of the action disappear in this gauge.

Then the reduced equations of motion can be derived from the Hamiltonian:

$$H = \int d^2\sigma \left[\frac{1}{2}(P^I)^2 + \frac{1}{4}(\{X^I, X^J\})^2 - \frac{i}{2}\bar{\theta}\Gamma^I\{X^I, \theta\} \right] \tag{9}$$

where $I = 1, 2, ..., 9$ and:

$$\{A, B\} = \partial_1 A \, \partial_2 B - (1 \leftrightarrow 2) \tag{10}$$

and where the physical states are constrained by:

$$\{\dot{X}^I, X^I\} + \{\bar{\theta}, \theta\} = 0 \tag{11}$$

which vanishes on physical states. This constraint generates area preserving diffeomorphisms.

The problem with this Hamiltonian is that, for certain configurations of the membrane, the potential function, which is the second term in the Hamiltonian (9), vanishes. This is potentially disastrous for the theory. Let $f(\sigma_1, \sigma_2)$ represent a function of the membrane variables, and consider $X_\mu(f)$, which represents a string-like configuration. For this string-like configuration, the potential function disappears because:

$$\{X_\mu(f), X_\nu(f)\} = 0 \tag{12}$$

This means that classically, the potential function of the bosonic Hamiltonian vanishes along string-like filaments with zero area that protrude from the membrane like the quills of a porcupine. In principle, this may destabilize the Hamiltonian, allowing leakage of the wave function along these strings. In ref. 9, the potential was shown to vanish when X was approximated by fields defined in the Cartan sub-algebra of SU(n). Because the elements of the Cartan sub-algebra commute among each other, the potential term was shown to vanish.

However, it is not obvious that this means that the theory is unstable along these string-like configurations. Let us study a toy-model to understand the subtleties of the question.

As in ref. 12, let us begin with a simple quantum mechanical system in two dimensions, with the potential given by x^2y^2:

$$H_B = -\Delta + x^2y^2 \tag{13}$$

This Hamiltonian resembles the supermembrane theory because the interaction Hamiltonian vanishes along the x and y axes, so naively one may expect that the wave function can "leak" along the axes and the theory is therefore unstable. However, this is not true. Let us temporarily fix the value of x, which is defined to be large. If we move a short distance along the y axis, the potential function is a potential well for the harmonic oscillator which is quite steep for large values of x, so the leakage is quite small. For large x, the leakage is infinitesimally small. So which effect dominates?

In fact, the spectrum is actually discrete. For fixed x, the Hamiltonian obeys $H_B \geq |x|$, so the energy necessary to move the wave function to infinity is infinite. In fact,

$$H_B \geq (|x| + |y|)/2 \tag{14}$$

so the spectrum is discrete.

This toy model shows that there are subtleties with regard to the stability of even simple quantum mechanical systems. However, the theory can still become unstable if we introduce fermions and supersymmetry. The zero point energy from the fermions can cancel the $|x|$ contribution, giving us an unstable theory.

Start with the quantum mechanical system[9]:

$$H = \frac{1}{2}\{Q, Q^\dagger\} \tag{15}$$

where:

$$Q = Q^\dagger = \begin{pmatrix} -xy & i\partial_x + \partial_y \\ i\partial_x - \partial_y & xy \end{pmatrix} \tag{16}$$

The Hamiltonian reads:

$$H = \begin{pmatrix} -\Delta + x^2y^2 & x + iy \\ x - iy & -\Delta + x^2y^2 \end{pmatrix} \tag{17}$$

For fixed $|x|$, the supersymmetry of the reduced system is enough to guarantee that the energy contribution coming from the fermionic variables cancels the contribution from the bosonic variables. In fact, if we define:

$$\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{18}$$

then:

$$\xi^T H \xi = H_B - x \tag{19}$$

so the fermionic contribution cancels the x coming from the bosonic variables, and the system becomes unstable.

We can introduce normalized wave functions for this case as:

$$\Psi_t(x,y) = \chi(x-t)\pi^{-1/4}|x|^{1/4} \exp\left(-\frac{1}{2}|x|y^2\right)\xi \tag{20}$$

for the ground state. t is a parameter which will be taken to be arbitrarily large; it measures the distance we have shifted the wave function along the x direction. χ is a function which has compact support. Then we can see that:

$$\lim_{t\to\infty} = (\Psi_t, H^n \Psi_t) = \int dx\, \chi(x)^* (-\partial_x^2)^n \chi(x) \tag{21}$$

for $n = 0, 1, 2$, so we can shift the wave function as t goes to infinite without having to supply an infinite amount of energy. In fact, if E is the energy of this system, we can see that E can have any arbitrary value, corresponding to the eigenvalue of $-\partial_x^2$, where the potential vanishes. Hence, the energy spectrum is continuous.

A similar situation may happen with quantum supermembranes. Naively, the bosonic membrane theory seems to be unstable because the potential vanishes along certain string-like directions. However, the amount of energy necessary for the wave function to leak along these directions is infinite. But when we add fermions into the theory, then we must check explicitly if the fermionic contribution to the zero point energy cancels the bosonic contribution.

In ref. 9, this was studied by approximating the membrane with super Yang-Mills theory. We wish, however, to keep the continuum limit throughout, and at the very last step identify where any infinities may arise and where regularization methods may be necessary.

2 Zero Point Energy

Now let us calculate the zero point energy for the quantum supermembrane in the light cone gauge. Let us divide the original X^I membrane co-ordinate into several parts. Let x represent the co-ordinate along the string, so that:

$$x = x\left(f(\sigma_1, \sigma_2)\right) \tag{22}$$

We will let Y be the co-ordinate of the membrane which lies off the string, i.e. it cannot be written as a function of a single string variable.

In order to carry out gauge fixing, let us select out the 9th co-ordinate from I. Let the a index represent 1,2,...,8.

Now let us split the original X^I into different pieces. Not only will we split the 9th component off from the others, we will also explicitly split X^I into $x(f)$ and Y.

Then:

$$X^I = (x_9, Y_9, x_a, Y_a) \tag{23}$$

(At the end of the calculation, we will shift along the string-like configuration as some parameter $t \to \infty$, where $x_9 = x(f)$ grows like t, while x_a goes to a constant. So x_a can be dropped in relation to x, but we will keep both variables in our equations until the very last step.)

We can fix the gauge by choosing $Y_9 = 0$.

Then the Hamiltonian can be split up into several pieces:

$$H = H_1 + H_2 + H_3 + H_4 \tag{24}$$

where:

$$\begin{aligned}
H_1 &= -\frac{1}{2}\int d^2\sigma \left[\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial x_a}\right)^2\right] \\
H_2 &= -\frac{1}{2}\int d^2\sigma \left(\frac{\partial}{\partial Y_a}\right)^2 + \frac{1}{2}\int d^2\sigma d^2\bar{\sigma} d^2\sigma' \left[Y_a(\bar{\sigma}) z^T(\bar{\sigma}, \sigma') z(\sigma', \sigma) Y_a(\sigma)\right] \\
H_3 &= -\frac{i}{2}\int d^2\sigma d^2\bar{\sigma}\, \bar{\theta}(\bar{\sigma}) \left[z(\bar{\sigma}, \sigma)\gamma_9 + z_a(\bar{\sigma}, \sigma)\gamma_a\right] \theta(\sigma)
\end{aligned} \tag{25}$$

where:

$$\begin{aligned} z(\bar{\sigma},\sigma) &= \delta^2(\bar{\sigma},\sigma)\partial_{\sigma_1}x\partial_{\bar{\sigma}_2} - (1 \leftrightarrow 2) \\ z_a(\bar{\sigma},\sigma) &= \delta^2(\bar{\sigma},\sigma)\partial_{\sigma_1}x_a\partial_{\bar{\sigma}_2} - (1 \leftrightarrow 2) \end{aligned} \tag{26}$$

and the index σ is shorthand for $\{\sigma_1, \sigma_2\}$. Notice that $z(\bar{\sigma},\sigma)$ is an anti-symmetric function. Also, we have set $\Gamma_a = \gamma)a$. H_4 contains other terms in Y, which will not concern us yet.

The key factor, which will dominate our entire discussion of the zero point energy, is $z(\sigma, \bar{s}igma)$, which is the continuous matrix element which defines the diffeomorphism algebra in equation (10). In particular, we are interested in the sub-algebra of $w(\infty)$ which defines the reparametrization along the string-like filament. For elements $x(f)$, the elements of the algebra commute among each other. (In ref. 9, the counterpart of $x(f)$ are elements of the Cartan sub-algebra, which commute among each other by definition.) z is important to our discussion because the zero point energy can be defined entirely in terms of its eigenvalues.

Now consider the term H_2. We can write down an eigenfunction for H_2 as:

$$\Phi_0 = A(\det\Omega)^2 \exp\left(-\frac{1}{2}\int d^2\bar{\sigma} d^2\sigma\, Y_a(\bar{\sigma})\Omega(\bar{\sigma},\sigma)Y_a(\sigma)\right) \tag{27}$$

where Ω is yet undetermined, and A is a normalization constant, determined by:

$$1 = (\Phi_0, \Phi_0) = \int \prod_a \prod_\sigma \mathcal{D}Y_a(\sigma)\, \Phi_0^* \Phi_0 \tag{28}$$

Applying H_2 to this wave function, we find:

$$H_2\Phi_0 = 4\int d^2\sigma\, \Omega(\sigma,\sigma)\, \Phi_0 \tag{29}$$

which fixes the value of Ω to be:

$$\Omega^2(\bar{\sigma},\sigma) = \int d^2\sigma'\, z^T(\bar{\sigma},\sigma')z(\sigma',\sigma) \tag{30}$$

To find an explicit expression for the ground state energy requires that we take the trace of Ω. This is a tricky problem, since the trace may actually diverge, requiring a regularization. Let us assume that we can diagonalize the z by finding its eigenvalues. Let us introduce eigenvectors E^σ_{MN}, where $M \neq N$, as follows:

$$\int d^2\sigma\, z(\bar\sigma,\sigma)\, E^\sigma_{MN} = i\lambda_{MN} E^\sigma_{MN} \tag{31}$$

$$\int d^2\sigma\, z_a(\bar\sigma,\sigma)\, E^\sigma_{MN} = i\lambda^a_{MN} E^\sigma_{MN} \tag{32}$$

where M, N label a complete set of orthonormal functions, which can be either continuous or discrete, and λ_{MN} are the anti-symmetric eigenvalues of z. Our discussion will not depend on the explicit representation. (Since z and z_a commute, we can diagonalize them with the same eigenvectors.)

We can normalize them as follows:

$$\int d^2\sigma\, (E^\sigma_{MN})^* E^\sigma_{PQ} = \delta_{MP}\delta_{NQ} \tag{33}$$

$$\sum_{M\neq N} (E^\sigma_{MN})^* E^{\bar\sigma}_{MN} = \delta^2(\bar\sigma - \sigma) \tag{34}$$

$$(E^\sigma_{MN})^* = E^\sigma_{NM} \tag{35}$$

If we diagonalize z in terms of these eigenvalues, we find that the eigenvalue of H_2 is given by the sum of the absolute values of the eigenvalues of z:

$$\int d^2\sigma\, \Omega(\sigma,\sigma) = \sum_{M,N} |\lambda_{MN}| \tag{36}$$

$$\det \Omega = \prod_{M<N} \lambda^2_{MN} \tag{37}$$

(Because λ_{MN} is anti-symmetric, we can reduce the product over all M, N to one with $M < N$, where the precise ordering of the indices is arbitrary.) Now let us calculate the contribution of the fermionic variables to the zero point energy.

3 Fermionic Variables

The calculation of the fermionic variables is more difficult. As before our plan is to express all quantities in terms of the eigenvalues of the matrix $z(\bar{\sigma}, \sigma)$. Our calculation will resemble the path taken in ref. 9.

We now change variables to:

$$\theta(\sigma) = \sum_{M \neq N} \theta^{MN} E^{\sigma}_{MN} \tag{38}$$

The original fermionic variables are real. This means, therefore, that:

$$\theta^{MN\dagger} = \theta^{NM} \tag{39}$$

We can check that the anti-commutation relations:

$$[\theta_\alpha(\sigma), \theta_\beta(\bar{\sigma})]_+ = \delta_{\alpha,\beta}\delta^2(\sigma - \bar{\sigma}) \tag{40}$$

are transformed into:

$$\left[\theta^{MN}_\alpha, \theta^{PQ}_\beta\right]_+ = \delta_{\alpha,\beta}\delta^{MQ}\delta^{NP} \tag{41}$$

The fact that we can convert the complex θ^{MN} into its conjugate by simply reversing the lower indices is quite convenient, but it will allow us to establish creation and annihilation operators.

Then H_3 reduces to:

$$\begin{aligned} H_3 &= \frac{1}{2}\sum_{M \neq N} \theta^{NM}\left(\lambda_{MN}\gamma_9 + \lambda^a_{MN}\gamma_a\right)\theta^{MN} \\ &= \sum_{M<N} \theta^{MN\dagger}\left(\lambda_{MN}\gamma_9 + \lambda^a_{MN}\gamma_a\right)\theta^{MN} \end{aligned} \tag{42}$$

Now let us make a change in fermionic variables to eliminate the presence of γ_a in the above expression. Let us define:

$$\tilde{\theta}^{MN} = (A_{MN} + B^a_{MN}\gamma_a\gamma_9)\theta^{MN} \tag{43}$$

When we insert this expression back into the one for H_3, we demand that H_3 reduce down to a function of $\tilde{\theta}^{MN\dagger}\gamma_9\tilde{\theta}^{MN}$. We then find a system of two equations:

$$\begin{aligned} \omega_{MN}\left[A_{MN}^2 - (B_{MN}^a)^2\right] &= \lambda_{MN} \\ -2A_{MN}B_{MN}^a\omega_{MN} &= \lambda_{MN}^a \end{aligned} \tag{44}$$

whose solutions are given by:

$$\begin{aligned} A_{MN} &= \frac{1}{\sqrt{2\omega_{MN}}}\sqrt{\omega_{MN}+\lambda_{MN}} \\ B_{MN}^a &= -\frac{1}{\sqrt{2\omega_{MN}}}\frac{\lambda_{MN}^a}{\sqrt{\omega_{MN}+\lambda_{MN}}} \\ \omega_{MN} &= \sqrt{\lambda_{MN}^2 + (\lambda_{MN}^a)^2} \end{aligned} \tag{45}$$

So the expression for H_3 reduces to:

$$H_3 = \sum_{M,N} \omega_{MN}\tilde{\theta}^{MN\dagger}\gamma_9\tilde{\theta}^{MN} \tag{46}$$

Lastly, in order to eliminate the presence of γ_9, let us introduce projection operators:

$$P_\pm = \frac{1 \pm \gamma_9}{2} \tag{47}$$

so that:

$$\phi\gamma_9\theta = \phi_+\theta_+ - \phi_-\theta_- \tag{48}$$

Then H_3 becomes:

$$\begin{aligned} H_3 &= \sum_{M<N}\left(\tilde{\theta}_+^{MN\dagger}\tilde{\theta}_+^{MN} - \tilde{\theta}_-^{MN\dagger}\tilde{\theta}_-^{MN}\right) \\ &= \sum_{M<N}\left(\tilde{\theta}_+^{MN\dagger}\tilde{\theta}_+^{MN} + \tilde{\theta}_-^{MN}\tilde{\theta}_-^{MN\dagger} - 8\right) \end{aligned} \tag{49}$$

We are interested in the last constant in order to calculate the ground state energy of the system.

4 Wave Function

We can now write down the wave function. Since θ^{MN} is the conjugate to θ^{NM} for $M < N$, we can choose θ^{MN} to be annihilation operators. Let ξ_0 represent the vacuum state vector such that the annihilation operators act as follows:

$$\theta^{MN}\xi_0 = 0 \tag{50}$$

for all indices such that $M < N$. Then the ground state vector for the fermionic variables is:

$$\xi = \left[\prod_{M<N}\prod_{\alpha}^{8}\left(\tilde{\theta}_-^{MN\dagger}\right)\right]\xi_0 \tag{51}$$

In particular, this means that:

$$\begin{aligned}\tilde{\theta}_+^{MN}\xi &= 0\\ \tilde{\theta}_-^{MN\dagger}\xi &= 0\end{aligned} \tag{52}$$

With this choice, we see that:

$$H_3\xi = -8\sum_{M<N}\omega_{MN}\xi \tag{53}$$

The total wave function can now written as:

$$\Psi = \chi(x - tV, x_a)\Phi_0(x, Y_a)\xi(x, x_a) \tag{54}$$

where t becomes large as we go along the string, and V represents the asymptotic value of the string variables, which depends on the function f.

To find total energy, we now sum the contribution of H_2 and H_3:

$$(H_2 + H_3)\Psi = 8\sum_{M<N}\left(|\lambda_{MN}| - \omega_{MN}\right)\Psi \tag{55}$$

As before, let t represent a variable which measures how far we are along the string-like filament. We shall take $t \to \infty$. We make the assumption that x grows as t, while x_a approaches a constant. Then we see that ω_{MN}

asymptotically approaches $|\lambda_{MN}|$ in this limit, so that the ground state energy of H_2 and H_3 vanishes:

$$(H_2 + H_3)\,\Psi \to 0 \tag{56}$$

It is not hard to find the contribution of H_4, which is a polynomial in Y, x, x_a. We are interested in the value of the matrix element:

$$\lim_{t\to\infty} |\Psi, P\Psi| \to t^n \tag{57}$$

for some polynomial P. Since Φ_0 is just the ground state wave function for the harmonic oscillator in terms of Y, it is easy to calculate the value of $(\Phi_0, P(Y)\Phi_0)$. We find that $n = -1/2$ for every Y contained within P. For every x contained within P, we have a contribution of $n = 1$. By simply counting the number of x and Y within H_4, we see that it does not contribute to the ground state energy to the leading order, so it can be ignored.

In conclusion, we see that the principal contribution to the ground state energy comes from H_2 and H_3.

Furthermore, we see that the energy eigenvalue of the operator is continuous for the ground state, which means that the system is unstable.

We caution that there may be hidden infinities with regard to our calculation. Since the continuous matrix $z(\sigma, \bar{\sigma})$ contains derivatives, it may be possible that its eigenvalues are actually divergent. Then the cancellation of the lowest eigenvalue of $H_2 + H_3$ must be carefully regularized. However, the advantage of our discussion is that it was carried out in the continuum theory rather in super Yang-Mills theory, so we have a more intuitive understanding of where the problems may arise.

5 Discussion

Although the system is unstable, we speculate how this may still be compatible with known phenomenology. For a physical system like quantum membranes to be compatible with known physics, we have to ask:
a) why don't we see them in nature?

b) do decaying fundamental particles violate unitarity or other cherished features of quantum field theory?

To answer the first objection, we note that because the decay time of such a quantum membrane is on the order of the Planck time, it is possible that unstable membranes decay too rapidly to be detected by our instruments.

One potential flaw in this argument is that there may be different values of physical parameters, such as the mass of the membrane, for which the life-time is long. Therefore, to make a rough guess of the decay life of such a quantum membrane, we recall that the decay width of the decay of a quark-anti-quark bound state is given by:

$$\Gamma = \frac{16\pi\alpha^2}{3}\frac{|\psi(0)|^2}{M^2} \tag{58}$$

where $\phi(0)$ is the wave function of the bound state at the origin, and M is the mass of the decay product. On dimensional and kinematic grounds, we expect this formula to be roughly correct for the decay of the membrane, discarding the effect of spin, quantum numbers, etc.

We expect that $|\psi(0)|$ to be on the order of a fermi^{-3}, the rough size of the quark-anti-quark bound state. For our purposes, we assume that the membrane is on the order of the Planck length. On dimensional grounds, we therefore expect that the lifetime of the membrane to be on the order of:

$$T \sim M^2 L^{-3} \tag{59}$$

where L is the Planck length.

For relatively light-weight membranes, we find that the lifetime is much smaller than Planck times, so we will, as expected, never see these particles.

The other case is more interesting. For very massive membranes, we find that the lifetime becomes arbitrarily long, which seems to violate experiment. However, the coupling of very massive membranes, much heavier than the Planck mass, is very small, and hence barely couple to the particles we see in nature. Again, we see that unstable membranes cannot be measured in the laboratory.

In summary, light-weight membranes live too short to be detected, and massive (long-lived) membranes have vanishing coupling to the known particles.

Yet another objection that one may raise to our naive arguments is that membranes decay into other membranes, losing energy and mass with each decay, and hence the lifetime of the decaying membranes constantly changing. It is conceivable that, starting with a single membrane, the cascading sequence of daughter membranes may produce a collection of membranes with a lifetime long enough to be measured in the lab.

To estimate the effect of an infinite sequence of decaying membranes, let us analyze a simpler system: a chain of decaying objects, similar to the decay of a series of radio-nuclides.

Let N_i represent the amount of decaying material of type i. Let Ω_i represent the rate of decay of the ith material. Let ω_{ij} represent the rate at which substance i is decaying into substance j, which increases the amount of the jth substance. Then the coupled series of equations is given by:

$$\begin{aligned}
\dot{N}_1 &= -\Omega_1 N_1 \\
\dot{N}_2 &= -\Omega_2 N_2 + \omega_{12} N_1 \\
\dot{N}_3 &= -\Omega_3 N_3 + \omega_{13} N_1 + \omega_{23} N_2 \\
&\vdots \\
\dot{N}_j &= -\Omega_j N_j = \sum_{k=1}^{j-1} \omega_{kj} N_k
\end{aligned} \tag{60}$$

There is a simple solution to these coupled equations. If we use the ansatz:

$$\begin{aligned}
N_1 &= A_1 e^{-\Omega_1 t} \\
N_2 &= A_2 e^{-\Omega_2 t} + A_{12} e^{-\Omega_1 t} \\
N_3 &= A_3 e^{-\Omega_3 t} + A_{23} e^{-\Omega_2 t} + A_{13} e^{-\Omega_1 t} \\
&\vdots \\
N_j &= A_j e^{-\Omega_j t} + \sum_{i=1}^{j-1} A_{ij} e^{-\Omega_i t}
\end{aligned} \tag{61}$$

then the solution is given by:

$$
\begin{aligned}
A_{12} &= \frac{\omega_{12} A_1}{\Omega_2 - \Omega 1} \\
A_{13} &= \frac{\omega_{13} A_1}{\Omega_3 - \Omega_1} + \frac{\omega_{23}\omega_{12} A_1}{(\Omega_2 - \Omega_1)(\Omega_3 - \Omega_1)} \\
A_{23} &= \frac{\omega_{23}\omega_{12}}{(\Omega_3 - \Omega_2)(\Omega_2 - \Omega_1)} A_2
\end{aligned}
\tag{62}
$$

and so on.

The lesson we learn from this is that, even with an arbitrarily large number of decaying products, each decaying into each other, the limiting factor is the longest life-time of a single decay product. The substance with the slowest decay $e^{-\Omega_i t}$ dominates the entire series.

This gives us reasonable assurance that the infinite series of decaying membranes does not behave any worse than its longest lived membrane.

And lastly, we must ask the question of whether quantum field theory can accomodate decaying fundamental particles. Previous work by McCoy and Wu[13,14] on the field theory of decaying particles indicates that there are no fatal problems with such a theory. The pole of the two-point function corresponding to an unstable particle becomes a branch cut in this case. The principle question is whether there exists a Lehmann spectral representation of such a theory with a branch cut. In fact, two-point functions with a branch cut rather than a single pole have already been encountered in two dimensional SU(N) Yang-Mills theory in the limit $N \to \infty$, the two-dimensional SU(2) Thirring model, and two-dimensional Ising field theory.

Of course, these arguments that we have presented here are certainly not rigorous, since we do not know how membranes interact and the theory is highly non-linear. Until the interacting theory is fully calculated, we cannot know precisely whether our heuristic arguments will hold up. However, they indicate that one cannot immediately dismiss fundamental quantum membranes as a physical theory.

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A NOTE ON THE STABILITY OF QUANTUM SUPERMEMBRANES

MICHIO KAKU
Physics Dept., City College of the City University of New York
New York, N.Y. 10031, USA

We re-examine the question of the stability of quantum supermembranes. In the past, the instability of supermembranes was established by using a regulator, i.e. approximating the membrane by $SU(N)$ super Yang-Mills theory and letting $N \to \infty$. In this paper, we (a) show that the instability persists even if we directly examine the continuum theory, which then opens the door to other types of regularizations. (b) give heuristic arguments that even a theory of unstable membranes at the Planck length may still be compatible with experiment. (c) resolve a certain puzzling discrepancy between earlier works on the stability of supermembranes.

1 Quantum Supermembranes

To analyze the quantum stability of supermembranes, we start with the Hamiltonian in the light cone gauge for the supermembrane[1,2]:

$$H = \int d^2\sigma \left[\frac{1}{2}(P^I)^2 + \frac{1}{4}(\{X^I, X^J\})^2 - \frac{i}{2}\bar{\theta}\Gamma^I\{X^I, \theta\} \right] \tag{1}$$

where $I = 1, 2, ..., 9$ and $\{A, B\} = \partial_1 A\, \partial_2 B - (1 \leftrightarrow 2)$. To demonstrate the instability of this Hamiltonian without going to a $SU(N)$ super Yang-Mills theory[3], we let $f(\sigma_1, \sigma_2)$ represent a function of the membrane world-variables and consider $X_\mu(f)$, which describes a string-like configuration. The potential function in the Hamiltonian vanishes if we substitute this string-like configuration. In other words, along these string-like configurations, the wave function can "leak" out to infinity, and hence the system is unstable.

We first choose variables. Let $a = 1, 2, ...8$, and $I = 1, 2, ...9$. Then choose co-ordinates $X^I = (x_9, Y_9, x_a, Y_a)$. where we have split off the string co-ordinate by setting $x = x(f)$, while Y cannot be written as a string. We can fix the gauge by choosing $Y_9 = 0$.

Then the Hamiltonian can be split up into several pieces[4]:
$H = H_1 + H_2 + H_3 + H_4$ where:

$$H_1 = -\frac{1}{2}\int d^2\sigma \left[\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial x_a}\right)^2 \right]$$

$$
\begin{aligned}
H_2 &= -\frac{1}{2}\int d^2\sigma \left(\frac{\partial}{\partial Y_a}\right)^2 + \frac{1}{2}\int d^2\sigma d^2\bar{\sigma} d^2\sigma' \left[Y_a(\bar{\sigma}) z^T(\bar{\sigma},\sigma') z(\sigma',\sigma) Y_a(\sigma)\right] \\
H_3 &= -\frac{i}{2}\int d^2\sigma d^2\bar{\sigma}\, \bar{\theta}(\bar{\sigma}) \left[z(\bar{\sigma},\sigma)\gamma_9 + z_a(\bar{\sigma},\sigma)\gamma_a\right] \theta(\sigma) \\
z(\bar{\sigma},\sigma) &= \delta^2(\bar{\sigma},\sigma)\partial_{\sigma_1} x \partial_{\bar{\sigma}_2} - (1 \leftrightarrow 2) \\
z_a(\bar{\sigma},\sigma) &= \delta^2(\bar{\sigma},\sigma)\partial_{\sigma_1} x_a \partial_{\bar{\sigma}_2} - (1 \leftrightarrow 2) \qquad (2)
\end{aligned}
$$

and the index σ is shorthand for $\{\sigma_1, \sigma_2\}$. Notice that $z(\bar{\sigma},\sigma)$ is an anti-symmetric function. H_4 contains other terms in Y, which will vanish at the end of the calculation.

Now consider the term H_2. The eigenfunction for H_2 is:

$$
\begin{aligned}
\Phi_0 &= A(\det\Omega)^2 \exp\left(-\frac{1}{2}\int d^2\bar{\sigma} d^2\sigma\, Y_a(\bar{\sigma})\Omega(\bar{\sigma},\sigma)Y_a(\sigma)\right) \\
H_2\Phi_0 &= 4\int d^2\sigma \int d^2\sigma'\, z^T(\bar{\sigma},\sigma') z(\sigma',\sigma)\, \Phi_0 \qquad (3)
\end{aligned}
$$

Let us introduce eigenvectors E^{σ}_{MN}, where $M \neq N$, as follows:

$$
\int d^2\sigma\, z(\bar{\sigma},\sigma)\, E^{\sigma}_{MN} = i\lambda_{MN} E^{\sigma}_{MN} \qquad (4)
$$

where M, N label a complete set of orthonormal functions, which can be either continuous or discrete, and λ_{MN} are the anti-symmetric eigenvalues of z, so the eigenvalue of H_2 becomes: $\sum_{M,N} |\lambda_{MN}|$.

Now change fermionic variables to:

$$
\theta(\sigma) = \sum_{M \neq N} \theta^{MN} E^{\sigma}_{MN} \qquad (5)
$$

The original fermionic variables are real, so $\theta^{MN\dagger} = \theta^{NM}$.

Then H_3 reduces to:

$$
H_3 = \sum_{M<N} \theta^{MN\dagger} \left(\lambda_{MN}\gamma_9 + \lambda^a_{MN}\gamma_a\right) \theta^{MN} \qquad (6)
$$

After a bit of work, we find that:

$$
H_3\xi = -8\sum_{M<N} \omega_{MN}\xi \qquad (7)
$$

where: $\omega_{MN} = \{(\lambda^2_{MN} + (\lambda^a_{MN})^2\}^{1/2}$. To find total energy, we now sum the contributions of H_2 and H_3:

$$(H_2 + H_3)\Psi = 8 \sum_{M<N} (|\lambda_{MN}| - \omega_{MN}) \Psi \tag{8}$$

Then we see that ω_{MN} asymptotically approaches $|\lambda_{MN}|$ in this limit, so that the ground state energy of H_2 and H_3 vanishes. In conclusion[4], we see that the principal contributions to the ground state energy comes from H_2 and H_3, which in turn cancel if we are far from the membrane. Furthermore, we see that the energy eigenvalue of the operator is continuous for the ground state, which means that the system is unstable.

Although the system is unstable, we speculate how this may still be compatible with known phenomenology. We note that because the decay time of such a quantum membrane is on the order of the Planck time, it is possible that unstable membranes decay too rapidly to be detected by our instruments. Consider the standard decay of the quark bound state:

$$\Gamma = \frac{16\pi\alpha^2}{3} \frac{|\psi(0)|^2}{M^2} \tag{9}$$

where $\phi(0)$ is the wave function of the bound state at the origin, and M is the mass of the decay product.

We expect that $|\psi(0)|$ to be on the order of a fermi^{-3}, the rough size of the quark-anti-quark bound state. For our purposes, we assume that the membrane is on the order of the Planck length. On dimensional grounds, we therefore expect that the lifetime of the membrane to be on the order of $T \sim M^2 L^{-3}$ where L is the Planck length.

For relatively light-weight membranes, we find that the lifetime is much smaller than Planck times, so we will, as expected, never see these particles.

For very massive membranes, we find that the lifetime becomes arbitrarily long, which seems to violate experiment. However, the coupling of very massive membranes, much heavier than the Planck mass, is very small, and hence they barely couple to the particles we see in nature. Again, we see that unstable membranes cannot be measured in the laboratory.

2 Resolving a Discrepancy in the Stability of Membranes

In this section, we try to resolve a certain discrepancy with regard to the instability of supermembranes. In ref. 5, it was pointed out that it is possible to choose a gauge where the Hamiltonian becomes quadratic. Thus, it appears that the non-linearity of supermembranes, and hence their instability, is an illusion.

To resolve this puzzle, go to light cone co-ordinates: $\{X^+, X^-, X_{d-2}, X_i\}$, where $i = 1, 2, 3, ...d-3$. Our goal is to re-write everything in terms of X_i.

Choose the gauge $P^+ = p^+$; $X^+ = p^+\tau$. We still have one more gauge degree of freedom left. We choose (Δ is the volume term):

$$P_{d-2}^2 + \Delta = \partial_a X_i \partial_a X_i \tag{10}$$

This determines P_{d-2} such that the left-hand side is quartic, but the right hand side is quadratic.

Now let us solve the constraints. The $P^2 + \Delta$ constraint can be solved for P^-, which now becomes the new light cone Hamiltonian. We find:

$$P^- = \frac{1}{2p^+}\left(P_i^2 + w^{ab}\partial_a X_i \partial_b X_i\right) \tag{11}$$

Notice that the light cone Hamiltonian has now become quadratic! And lastly, X_{d-2} is fixed by the other constraints $\partial_a X_\mu P^\mu = 0$.

At this point, we now have a Hamiltonian P^- defined entirely in terms of quadratic functions, defined in terms of the canonical variables X_i, P_i. The action seems to be stable. But this contradicts all our previous results.

There is, however, a loophole to this proof. Notice that the $\dot{X}_\mu P^\mu$ term in the Lagrangian $L(X, P)$ decomposes as $\dot{X}_\mu P^\mu = p^+ P^- - \dot{X}_{d-2} P_{d-2} - \dot{X}_i P_i$.

The key point in all of this is that $\dot{X}_{d-2} P_{d-2}$ does not vanish. If we take the τ derivative of X_{d-2}, we find that this term contains non-linear functions of $\dot{X}_i$, and hence changes the canonical commutation relations between the transverse variables. Thus, the non-linearity of the theory creeps back in and is now hidden in the commutation relations. This resolves the apparent discrepancy.

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Ultra-Violet Behavior of Bosonic Quantum Membranes

Michio Kaku

Physics Dept., City College of New York
New York, N.Y. 10031, USA

ABSTRACT

We treat the action for a bosonic membrane as a sigma model, and then compute quantum corrections by integrating out higher membrane modes. As in string theory, where the equations of motion of Einstein's theory emerges by setting $\beta = 0$, we find that, with certain assumptions, we can recover the equations of motion for the background fields, i.e. $R_{\mu\nu} + (1/4)F_{\mu\alpha\beta\gamma\mu}F_{\nu}^{\alpha\beta\gamma} = 0$ and $D^{\alpha}F_{\alpha\beta\gamma} = 0$. for the membrane case. Although the membrane theory is non-renormalizable on the world volume by power counting, the investigation of the ultra-violet behavior of membranes may give us insight into the supersymmetric case, where we hope to obtain higher order M-theory corrections to 11 dimensional supergravity.

1 Introduction

At present, little is known about the action for M-theory [1,2], other than that it contains 11 dimensional supergravity in the low energy limit. Higher order corrections to 11D supergravity [3] are unknown. In this series of papers, we hope to compute these higher order corrections.

In string theory, the usual 10D supergravity action is derived by treating the original string action as a sigma model and them integrating the higher modes. By setting $\beta = 0$ via conformal invariance, we then obtain the equations of motion of 10D supergravity, plus higher order corrections to any order [4].

We would like to apply this same general technique to M-theory, treating the 11D supermembrane action [5] as a sigma model in order to compute the higher order corrections to M-theory. There are, of course, several obstacles to performing such a procedure.

First, by power counting, the membrane theory in higher dimensions is non-renormalizable on the world volume. We find that the degree of divergence for any N-point function can be made arbitrarily high by adding higher vertex corrections, thereby rending invalid the standard renormalization technique.

Second, there are problems with the quantization of supermembranes, i.e. they are quantum mechanically unstable [6]. (This instability was the original reason why many abandoned supermembrane theory soon after it was formulated. However, it may be possible to re-interpret this instability in terms of 0-branes in matrix models [7].)

Third, the precise relationship between membranes and M-theory is also not known. In particular, M-theory also contains five-branes, and perhaps higher order corrections to membranes as well.

Our philosophy, however, will be to investigate the first problem. Although the model is superficially non-renormalizable by power counting methods, it may possess enough symmetry to eliminate large classes of diverges. For example, there is no counterpart to the $\beta = 0$ equation for membranes, because there is no conformal symmetry on the world volume. However, in a later paper we will show that supersymmetry will in fact set the analogous supermembrane term to zero because of the super Bianchi identities. Thus, supersymmetry (which demands compatibility with the 11D supergravity background equations) is sufficient to render the theory one-loop renormalizable.

Our ultimate goal is to show whether or not supersymmetry is sufficient to kill the divergences of the supermembrane theory to all orders. This, in turn, would allow us to compute the higher order corrections to 11D supergravity in the M-theory action. If a recursion relation can be written for the higher order corrections, then we may be able to make statements concerning the entire theory, to all orders.

However, even if this ultimate goal is not realized, we expect to find interesting surprises. For example, we will show that, unlike the string case, one needs both one-loop and two-loop graphs in order to derive the standard equations of motion for the graviton and anti-symmetric tensor field. In the same way that the non-renormalizable four-fermion theory or the massive vector meson theory proved to be interesting laboratories for particle physics, it may turn out that supermembrane actions, even if they are inherently non-renormalizable, may be an interesting laboratory for M-theory.

2 Riemann Normal Co-ordinates

Our starting point is the bosonic membrane action:

$$L_1 = \frac{1}{2\alpha}\sqrt{\gamma}\gamma^{ij}g_{\mu\nu}\partial_i\phi^\mu\partial_j\phi^\nu \tag{1}$$

where $g_{\mu\nu}$ is the space-time metric, where Greek letters $\mu, \nu, \alpha = 0, 1, 2...10$, where γ^{ij} is the metric on the three-dimensional world volume, where Roman letters $i, j, k = 1, 2, 3$, and where ϕ^μ is the membrane co-ordinate.

To this action, we add a contribution from the anti-symmetric field:

$$L_2 = \beta\epsilon^{ijk}A_{\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \tag{2}$$

which is found in the bosonic part of the supermembrane action. The total action is then $L_T = L_1 + L_2$, with α and β being two coupling constants.

Notice that the action of this theory is gauge invariant under the transformation:

$$\delta A_{\mu\nu\lambda} = \partial_\mu\Lambda_{\nu\lambda} + ... \tag{3}$$

Notice that the action also contains a world volume metric γ^{ij}. In the usual string action, this metric can be eliminated entirely via a gauge choice and a conformal transformation. However, in the covariant membrane case, we cannot eliminate all the degrees of freedom of the non-propagating world volume metric. Instead, we will simply treat the metric γ^{ij} as a classical background field. This means that we will have to keep γ^{ij} arbitrary and quantize the theory on a classical curved world volume.

Next, we wish to power expand this action using the background field method applied to sigma models, using Riemann normal co-ordinates [6]. Let the space-time variable $\phi^\mu(\tau)$ obey a standard geodesic equation:

$$\frac{d^2\phi^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{d\phi^\rho}{d\tau}\frac{d\phi^\sigma}{d\tau} = 0 \tag{4}$$

Now expand the membrane co-ordinate ϕ^μ around a classical configuration $\phi^\mu_{\rm cl}$:

$$\phi^\mu = \phi^\mu_{\rm cl} + \pi^\mu \tag{5}$$

where π^μ is the quantum correction to the classical configuration. Now power expand π^μ in terms of ξ^μ:

$$\pi^\mu = \xi^\mu - \frac{1}{2}\Gamma^\mu_{\rho\sigma}\xi^\rho\xi^\sigma - \frac{1}{3!}\Gamma^\mu_{\rho\sigma\lambda}\xi^\rho\xi^\sigma\xi^\lambda ... \tag{6}$$

The various co-efficients in this power expansion can be laboriously computed by inserting the expression back into the geodesic equation. For example, we find that $\Gamma^\lambda_{\mu\nu}$ is the usual Christoffel symbol, and:

$$\Gamma^\mu_{\rho\sigma\lambda} = \partial_\rho\Gamma^\mu_{\sigma\lambda} - \Gamma^\alpha_{\rho\sigma}\Gamma^\mu_{\alpha\lambda} - \Gamma^\alpha_{\rho\lambda}\Gamma^\mu_{\alpha\sigma} \tag{7}$$

In general, the higher coefficients are equal to:

$$\Gamma^\nu_{\mu_1\mu_2\ldots\mu_n\alpha\beta} = D_{\mu_n}\cdots D_{\mu_1}\Gamma^\nu_{\alpha\beta} \tag{8}$$

where we take the covariant derivatives only with respect to the lower indices.

Our goal is now to power expand the Lagrangian L_1+L_2 in terms of ξ^μ, and then integrate out ξ^μ from the action. This will give us a series of potentially divergent graphs, whose structure we wish to examine.

In general, this power expansion becomes prohibitively difficult as we progress to higher and higher orders, so we will instead use the formalism introduced by Mukhi [9].

One reason why this expansion is unwieldy is because the standard Taylor expansion is non-covariant. If we have a function I and power expand it, we find:

$$I = \sum_{n=0}^{\infty} I^{(n)} \tag{9}$$

where:

$$I^{(n)} = \frac{1}{n!}\int dx_1 \xi^{\mu_1}\partial^{x_1}_{\mu_1}\int dx_2 \xi^{\mu_2}\partial^{x_2}_{\mu_2}\cdots\int dx_n \xi^{\mu_n}\partial^{x_n}_{\mu_n} I \tag{10}$$

where the co-ordinates on the three dimensional world volume are given by x_i, and where ∂^x_i is a functional derivative:

$$\partial^x_\mu = \frac{\delta}{\delta\phi^\mu(x)} \tag{11}$$

Clearly, iterating the operator:

$$\int dx\, \xi^\mu(x)\partial^x_\mu \tag{12}$$

yields non-covariant results.

Let us define instead the operator Δ:

$$\Delta = \int dx\, \xi^\mu(x) D_\mu \tag{13}$$

where D_μ is a functional covariant derivative. For example:

$$D_\mu A^\nu[\phi(y)] = \left[\partial_\mu A^\nu(\phi(x)) + \Gamma^\nu_{\mu\lambda}(\phi(x))A^\lambda(\phi(x))\right]\delta^3(x-y) \tag{14}$$

Then we can power expand the Lagrangian as follows:

$$L = \sum_{n=0}^{\infty} L^{(n)} \tag{15}$$

where:

$$L^{(n)} = \frac{1}{n!}\Delta^n L \tag{16}$$

To perform the power expansion, we derive the following identities:

$$\Delta\xi^\mu = 0$$

$$
\begin{aligned}
\Delta(\partial_i\phi^\mu) &= D_i\xi^\mu \\
\Delta(D_i\xi^\mu) &= R^\mu_{\nu\rho\sigma}\xi^\nu\xi^\rho\partial_i\phi^\sigma \\
\Delta T_{\mu_1\mu_2\ldots} &= \xi^\rho D_\rho T_{\mu_1\mu_2\ldots}
\end{aligned}
\tag{17}
$$

where $T_{\mu_1\mu_2\ldots}$ is an arbitrary tensor, and:

$$
D_i\xi^\mu = \partial_i\xi^\mu + \Gamma^\mu_{\rho\sigma}\xi^\rho\partial_i\phi^\sigma \tag{18}
$$

and:

$$
R^\mu_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma} \tag{19}
$$

Now let us power expand the original action in terms of ξ. Let us replace $\phi^\mu_{\rm cl}$ with the symbol ϕ^μ.

We find:

$$
\begin{aligned}
L_1^{(0)} &= \frac{1}{2\alpha}\sqrt{\gamma}\gamma^{ij}g_{\mu\nu}\partial_i\phi^\mu\partial_j\phi^\nu \\
L_1^{(1)} &= \frac{1}{\alpha}\sqrt{\gamma}\gamma^{ij}g_{\mu\nu}\partial_i\phi^\mu D_j\xi^\nu \\
L_1^{(2)} &= \frac{1}{2\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\nu\sigma\rho}\partial_i\phi^\mu\partial_j\phi^\sigma\xi^\nu\xi^\rho + \frac{1}{2\alpha}\sqrt{\gamma}\gamma^{ij}g_{\mu\nu}D_i\xi^\mu D_j\xi^\nu \\
L_1^{(3)} &= \frac{1}{6\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\nu\rho\sigma;\lambda}\partial_i\phi^\mu\partial_j\phi^\sigma\xi^\nu\xi^\rho\xi^\lambda + \frac{2}{3\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\nu\rho\sigma}\partial_i\phi^\mu D_j\xi^\sigma\xi^\nu\xi^\rho \\
L_1^{(4)} &= \frac{1}{24\alpha}\sqrt{\gamma}\gamma^{ij}\left[R_{\mu\nu\rho\sigma;\pi\kappa} + 4R^\lambda_{\nu\rho\mu}R_{\lambda\pi\kappa\sigma}\right]\partial_i\phi^\mu\partial_j\phi^\sigma\xi^\nu\xi^\sigma\xi^\pi\xi^\kappa \\
&+ \frac{1}{4\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\rho\sigma\nu;\pi}\partial_i\phi^\mu D_j\xi^\nu\xi^\rho\xi^\sigma\xi^\pi + \frac{1}{6\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\sigma\rho\nu}D_i\xi^\mu D_i\xi^\nu\xi^\rho\xi^\sigma \\
L_1^{(5)} &= \frac{1}{120\alpha}\sqrt{\gamma}\gamma^{ij}\left(R_{\mu\nu\alpha\beta;\gamma\delta\lambda} + 14R^\epsilon_{\alpha\beta\mu;\gamma}R_{\epsilon\delta\lambda\nu}\right)\partial_i\phi^\mu\partial_j\phi^\nu\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta\xi^\lambda \\
&+ \frac{1}{15\alpha}\sqrt{\gamma}\gamma^{ij}\left(R_{\mu\alpha\beta\nu;\gamma\delta} + 2R^\pi_{\alpha\beta\mu}R_{\pi\gamma\delta\nu}\right)\partial_i\phi^\mu D_j\xi^\nu\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta \\
&+ \frac{1}{12\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\alpha\beta\nu;\gamma}D_i\xi^\mu D_j\xi^\nu\xi^\alpha\xi^\beta\xi^\gamma \\
L_1^{(6)} &= \frac{1}{720\alpha}\sqrt{\gamma}\gamma^{ij}\left(R_{\mu\alpha\beta\nu;\gamma\delta\epsilon\lambda} + 22R_{\mu\alpha\beta\pi;\gamma\delta}R^\pi_{\epsilon\lambda\nu} + 14R^\pi_{\alpha\beta\mu;\gamma}R_{\pi\delta\epsilon\nu;\lambda}\right. \\
&+ \left.16R^\pi_{\alpha\beta\mu}R_{\pi\gamma\delta\rho}R^\rho_{\epsilon\lambda\nu}\right)\partial_i\phi^\mu\partial_i\phi^\nu\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta\xi^\epsilon\xi^\lambda \\
&+ \frac{1}{72\alpha}\sqrt{\gamma}\gamma^{ij}\left(R_{\mu\alpha\beta\nu;\gamma\delta\epsilon} + 4R^\pi_{\alpha\beta\mu;\gamma}R_{\pi\delta\epsilon\nu} + R^\pi_{\alpha\beta\nu;\gamma}R_{\pi\delta\epsilon\mu}\right)\partial_i\phi^\mu D_i\xi^\nu\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta\xi^\epsilon \\
&+ \frac{1}{40\alpha}\sqrt{\gamma}\gamma^{ij}\left(R_{\mu\alpha\beta\nu;\gamma\delta} + \frac{8}{9}R^\pi_{\alpha\beta\mu}R_{\pi\gamma\delta\nu}\right)D_i\xi^\mu D_j\xi^\nu\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta
\end{aligned}
\tag{20}
$$

The power expansion of the Lagrangian involving the anti-symmetric field is given by:

$$
\begin{aligned}
L_2^{(0)} &= \beta\sqrt{\gamma}\epsilon^{ijk}A_{\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \\
L_2^{(1)} &= 3\beta\epsilon^{ijk}A_{\mu\nu\lambda}D_i\xi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda + \beta\epsilon^{ijk}D_\rho A_{\mu\nu\lambda}\xi^\rho\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda
\end{aligned} \tag{21}
$$

Since the original action was gauge invariant in the anti-symmetric field, we wish to preserve this symmetry, so let us re-write $L_2^{(1)}$ by introducing the tensor:

$$
F_{\sigma\mu\nu\lambda} = D_\sigma A_{\mu\nu\lambda} - D_\mu A_{\nu\lambda\sigma} + D_\nu A_{\lambda\sigma\mu} - D_\lambda A_{\sigma\mu\nu} \tag{22}
$$

Because of gauge invariant, all subsequent terms will involve this covariant tensor.

Then we can write:

$$
L_2^{(1)} = \beta\epsilon^{ijk}F_{\sigma\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\sigma \tag{23}
$$

$$
L_2^{(2)} = \frac{\beta}{2}\epsilon^{ijk}\left(3F_{\sigma\mu\nu\lambda}D_i\xi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\sigma + D_\rho F_{\sigma\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\rho\xi^\sigma\right) \tag{24}
$$

$$
\begin{aligned}
L_2^{(3)} &= \frac{\beta}{6}\epsilon^{ijk}\Big(6F_{\sigma\mu\nu\lambda}D_i\xi^\mu D_j\xi^\nu\partial_k\phi^\lambda\xi^\sigma + 6D_\rho F_{\sigma\mu\nu\lambda}D_i\xi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\rho\xi^\sigma \\
&+ 3R^\mu_{\alpha\beta\gamma}\partial_i\phi^\gamma F_{\sigma\mu\nu\lambda}\xi^\alpha\xi^\beta\xi^\sigma\partial_j\phi^\nu\partial_k\phi^\lambda \\
&+ D_\pi D_\rho F_{\sigma\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\rho\xi^\sigma\xi^\pi\Big)
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
L_2^{(4)} &= \frac{\beta}{4!}\epsilon^{ijk}\Big\{6F_{\sigma\mu\nu\lambda}D_i\xi^\mu D_j\xi^\nu D_k\xi^\lambda\xi^\sigma \\
&+ 18R^\mu_{\alpha\beta\gamma}F_{\sigma\mu\nu\lambda}\partial_i\phi^\gamma D_j\xi^\nu\partial_k\phi^\lambda\xi^\alpha\xi^\beta\xi^\sigma \\
&+ 18D_\rho F_{\sigma\mu\nu\lambda}D_i\xi^\mu D_j\xi^\nu\partial_k\phi^\lambda\xi^\sigma\xi^\rho \\
&+ 9D_\pi D_\rho F_{\sigma\mu\nu\lambda}D_i\xi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\sigma\xi^\rho\xi^\pi \\
&+ 9R^\mu_{\alpha\beta\gamma}D_\rho F_{\sigma\mu\nu\lambda}\partial_i\phi^\gamma\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\alpha\xi^\beta\xi^\rho\xi^\sigma
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad D_\kappa D_\pi D_\rho F_{\sigma\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\rho\xi^\rho\xi^\pi\xi^\kappa \\
&+ \quad 3R^\mu_{\alpha\beta\gamma}F_{\sigma\mu\nu\lambda}D_i\xi^\gamma\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\alpha\xi^\beta\xi^\sigma \\
&+ \quad 3R^\mu_{\alpha\beta\gamma;\pi}\partial_i\phi^\gamma F_{\sigma\mu\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\alpha\xi^\beta\xi^\sigma\xi^\pi\Big\}
\end{aligned}
\tag{26}
$$

Now we wish to simply the action a bit. We first wish to eliminate terms linear in ξ. If we add the contribution from $L_1^{(1)}$ and $L_2^{(1)}$, we find:

$$
\begin{aligned}
L_T^{(1)} &= \frac{1}{\alpha}\sqrt{\gamma}\gamma^{ij}g_{\mu\nu}\partial_i\phi^\mu D_j\xi^\nu + \beta\epsilon^{ijk}F_{\sigma\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\sigma \\
&= \partial_i\phi^\mu\left(\frac{1}{\alpha}\sqrt{\gamma}\gamma^{ij}g_{\mu\nu}D_j\xi^\nu + \beta\epsilon^{ijk}\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\sigma F_{\sigma\mu\nu\lambda}\right) \\
&= \partial_i\phi^\mu\nabla^i\xi_\mu
\end{aligned}
\tag{27}
$$

where we define ∇ by:

$$
\nabla^i\xi_\mu = \frac{1}{\alpha}\sqrt{\gamma}\gamma^{ij}g_{\mu\nu}D_j\xi^\nu + \beta\epsilon^{ijk}\partial_j\phi^\nu\partial_k\phi^\lambda\xi^\sigma F_{\sigma\mu\nu\lambda} \tag{28}
$$

The linear term $L^{(1)}$ may be set to zero, if we impose:

$$
\nabla^i\partial_i\phi^\mu = 0 \tag{29}
$$

which then defines ϕ^μ_{cl}.

Now let us add the two contributions together from the two parts and organize them by the ξ co-ordinates.

We find:

$$
\begin{aligned}
L_T^{(2)} &= \xi^\alpha\xi^\beta L^{(2)}_{\alpha\beta} + D_i\xi^\alpha\xi^\beta L^{i(2)}_{\alpha\beta} + D_i\xi^\alpha D_j\xi^\beta L^{ij(2)}_{\alpha\beta} \\
L_T^{(3)} &= \xi^\alpha\xi^\beta\xi^\sigma L^{(3)}_{\alpha\beta\gamma} + D_i\xi^\alpha\xi^\beta\xi^\sigma L^{i(3)}_{\alpha\beta\gamma} + D_i\xi^\alpha D_j\xi^\beta\xi^\sigma L^{ij(3)}_{\alpha\beta\gamma} \\
L_T^{(4)} &= \xi^\alpha\xi^\beta\xi^\gamma\xi^\delta L^{(4)}_{\alpha\beta\gamma\delta} + D_i\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta L^{i(4)}_{\alpha\beta\gamma\delta} \\
&+ \quad D_i\xi^\alpha D_j\xi^\beta\xi^\gamma\xi^\delta L^{ij(4)}_{\alpha\beta\gamma\delta} + D_i\xi^\alpha D_j\xi^\beta D_k\xi^\gamma\xi^\delta L^{ijk(4)}_{\alpha\beta\gamma\delta}
\end{aligned}
\tag{30}
$$

where:

$$
L^{(2)}_{\alpha\beta} = \frac{1}{2\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\alpha\beta\sigma}\partial_i\phi^\mu\partial_j\phi^\sigma + \frac{\beta}{2}\epsilon^{ijk}D_\alpha F_{\beta\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda
$$

$$
\begin{aligned}
L^{i(2)}_{\alpha\beta} &= \frac{3\beta}{2}\epsilon^{ijk}F_{\beta\alpha\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda \\
L^{ij(2)}_{\alpha\beta} &= \frac{1}{2\alpha}\sqrt{\gamma}\gamma^{ij}g_{\alpha\beta}
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
L^{(3)}_{\alpha\beta\gamma} &= \frac{1}{6\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\alpha\beta\sigma;\gamma}\partial_i\phi^\mu\partial_j\phi^\sigma \\
&+ \frac{\beta}{6}\epsilon^{ijk}D_\gamma D_\beta F_{\alpha\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \\
&+ \frac{\beta}{2}\epsilon^{ijk}R^\mu_{\alpha\beta\pi}F_{\gamma\mu\nu\lambda}\partial_i\phi^\pi\partial_j\phi^\nu\partial_k\phi^\lambda \\
L^{i(3)}_{\alpha\beta\gamma} &= \frac{2}{3\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\beta\gamma\alpha}\partial_j\phi^\mu \\
&+ \beta\epsilon^{ijk}D_\beta F_{\gamma\alpha\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda \\
L^{ij(3)}_{\alpha\beta\gamma} &= \beta\epsilon^{ijk}F_{\gamma\alpha\beta\lambda}\partial_k\phi^\lambda
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
L^{(4)}_{\alpha\beta\gamma\delta} &= \frac{1}{24\alpha}\sqrt{\gamma}\gamma^{ij}\partial_i\phi^\mu\partial_j\phi^\sigma\left[R_{\mu\alpha\beta\sigma;\gamma\delta} + R^\lambda_{\alpha\beta\mu}R_{\lambda\gamma\delta\sigma}\right] \\
&+ \frac{\beta}{4!}\epsilon^{ijk}D_\delta D_\gamma D_\alpha F_{\beta\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \\
&+ \frac{9\beta}{4!}\epsilon^{ijk}R^\mu_{\alpha\beta\sigma}D_\gamma F_{\delta\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda \\
&+ \frac{3\beta}{4!}\epsilon^{ijk}R^\mu_{\alpha\beta\sigma;\delta}\partial_i\phi^\sigma F_{\gamma\mu\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda \\
L^{i(4)}_{\alpha\beta\gamma\delta} &= \frac{1}{4\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu\alpha\beta\gamma;\delta}\partial_i\phi^\mu + \frac{18\beta}{4!}\epsilon^{ijk}R^\mu_{\beta\delta\pi}F_{\delta\mu\alpha\lambda}\partial_j\phi^\pi\partial_k\phi^\lambda \\
&+ \frac{9\beta}{4!}\epsilon^{ijk}D_\delta D_\beta F_{\gamma\alpha\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda + \frac{3\beta}{4!}\epsilon^{ijk}R^\mu_{\beta\gamma\alpha}F_{\delta\mu\nu\lambda}\partial_j\phi^\nu\partial_k\phi^\lambda \\
L^{ij(4)}_{\alpha\beta\gamma\delta} &= \frac{1}{6\alpha}\sqrt{\gamma}\gamma^{ij}R_{\alpha\beta\gamma\delta} + \frac{18\beta}{4!}\epsilon^{ijk}D_\delta F_{\gamma\alpha\beta\lambda}\partial_k\phi^\lambda \\
L^{ijk(4)}_{\alpha\beta\gamma\delta} &= \frac{6\beta}{4!}\epsilon^{ijk}F_{\delta\alpha\beta\gamma}
\end{aligned}
\tag{33}
$$

Before we can begin to set up the perturbation series, we must first diagonalize the quadratic term $L^{(2)}_T$. The space-time matrix $g_{\mu\nu}$ can be eliminated in favor of the usual vierbein e^a_μ, where the Roman index a represents tangent space indices.

Now let us replace $g_{\mu\nu}$ with $e^a_\mu e^a_\nu$. With a little bit of algebra, we can move the vierbein past the derivative and prove the identity:

$$\begin{aligned} e^a_\mu D_i \xi^\mu &= e^a_\mu \left(\partial_i \xi^\mu + \partial_i \phi^\lambda \Gamma^\mu_{\lambda\nu} \xi^\nu \right) \\ &= \partial_i \xi^a + \partial_i \phi^\mu \omega^{ab}_\mu \xi^a = D_i \xi^a \end{aligned} \tag{34}$$

where $\xi^a = e^a_\mu \xi^\mu$ and where:

$$\omega^{ab}_\mu = -(\partial_\mu e^a_\nu) e^{b\nu} + e^a_\nu \Gamma^\nu_{\mu\beta} e^{b\beta} \tag{35}$$

which is self-consistent with the equation $D_\mu e^a_\nu = 0$, as desired.

In this fashion, we can now write everything with tangent space indices. We find that the only change is that we must replace Green letters $\alpha, \beta, \gamma, \delta$ with Roman letters a, b, c, d.

For the general case, we find:

$$L^{(n)}_T = \sum_{k=0}^{n-1} D_{i_1} \xi^{a_1} D_{i_2} \xi^{a_2} \cdots D_{i_k} \xi^{a_k} \cdots \xi^{a_n} L^{i_1 i_2 \cdots i_k (n)}_{a_1 \cdots a_n} \tag{36}$$

where a_i are defined in the tangent space.

3 Regularization

Now that we have power expanded the original action in terms of ξ^a, where a is the tangent space index on curved space-time, we must now integrate over the quantum field ξ^a defined on the tangent space, which will leave us with divergent terms whose structure we wish to analyze.

We will power expand around the term:

$$L^{(2)}_1 = \frac{1}{2\alpha} \sqrt{\gamma} \gamma^{ij} D_i \xi^a D_j \xi^a \tag{37}$$

where the space-time metric $g_{\mu\nu}$ has been absorbed into the vierbeins.

If we perform the integration over ξ^a, then (subject to a regularization scheme):

$$\langle T\xi^a(x)\xi^b(x')\rangle \sim \alpha\Lambda\delta^{ab} + ... \tag{38}$$

where the right hand side is linearly divergent via some large momentum scale Λ, and there are important corrections to this equation crucially dependent on the regularization scheme.

Then the first term in $L_T^{(2)}$ contributes the following term:

$$L_{ab}^{(2)}\langle T\xi^a\xi^b\rangle \tag{39}$$

which in turn yields the two equations:

$$\langle T\xi^a\xi^b\rangle \frac{1}{\alpha}\sqrt{\gamma}\gamma^{ij}R_{\mu ab\sigma}\partial_i\phi^\mu\partial_j\phi^\sigma \tag{40}$$

and:

$$\beta\langle T\xi^a\xi^b\rangle\epsilon^{ijk}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda D_a F_{b\mu\nu\lambda} \tag{41}$$

Unlike the superstring, there is no conformal symmetry by which we can set this divergent term to zero. In this paper, we will simply set the lowest order divergent term to zero by fiat. This is a weakness in this approach. This, in some sense, defines the model, i.e. the theory can only propagate on certain background fields which set the lowest order divergent term to zero.

However, for the supermembrane, we will show in a later paper that there is enough supersymmetry to allow us to set this divergent term to zero, so we have:

$$R_{\mu\nu} = 0; \quad D^a F_{abcd} = 0 \tag{42}$$

The second equation is just the equation of motion for the anti-symmetric field, as expected. However, the first equation is rather troubling, since there should be a term proportional to F^2. The fact that this term is missing means that the equations of motion are actually inconsistent. There exists no action involving $g_{\mu\nu}$ and $F_{\alpha\beta\gamma\delta}$ which yields these equations of motion. Thus, we must carefully analyze our regularization scheme and go to higher interactions. This is different from the superstring case, where the one-loop results are sufficient to yield self-consistent equations of motion. For the membrane, we find that we must go to two loops in order to obtain self-consistent results.

Now let us generalize this result to higher orders by carefully introducing a regularization scheme. There is a problem with dimensional regularization, however. If we analytically continue the integral:

$$\int \frac{d^d p}{(p^2 + m^2)} \sim \Gamma(1 - \frac{d}{2}) \tag{43}$$

we find that it is finite for $d = 3$. It diverges with a pole at $d = 4$, but is formally finite for odd dimensions. This strange result does not change for higher loops, since multiple integrals over the momenta yield factors of $\Gamma(k)$, where k is half-integral, which is again finite for $d = 3$. Furthermore, when we introduce supersymmetry, we find that dimensional regularization does not respect this symmetry, which only holds at $d = 3$ for supermembranes. Hence, dimensional regularization poses some problems. In fact, supersymmetry is so stringent, it appears that finding a suitable regularization method is problematic for any method.

We will, instead, use standard point-splitting and proper time methods, separating the points on the world volume at which the various $\xi^\mu(x)$ meet at a vertex. This, of course, will violate general covariance and supersymmetry by point-splitting. However, point-splitting methods are convenient since the divergence within a Feynman integral occur when fields are defined at the same world volume point, i.e. $x \to y$ on the world volume.

Then the two-point Green's function can be written as:

$$\langle T\xi^a(x)\xi^b(x')\rangle = -iG^{ab}(x - x') \tag{44}$$

where x and x' represents points on the three dimensional world volume, and:

$$\frac{1}{\alpha}\left[-D_i^{ab}\sqrt{\gamma}\gamma^{ij}D_j^{bc} + \sqrt{\gamma}\left(\zeta R + m^2\right)\right] G^{ac}(x, x') = \delta^3(x - x')\delta^{ab} \tag{45}$$

(Although the theory is massless, notice that we added in a small mass m^2 in order to handle infrared divergences. In non-linear sigma models of this type, it can be shown that this mass regulartor cancels against other terms in the perturbation theory. Notice that we introduce a parameter ζ which takes into account the curvature on the world volume. This term will be of interest when we introduce fermions into our formalism. However, here we can set this term ζ to zero for our case.)

The solution of this Green's function is complicated by two facts. First, the Green's function is defined over both a curved three dimensional world volume manifold and 11 dimensional space-time manifold, and hence we have to use the formalism developed for general relativity.

Second, the Green's function will in general introduce unwanted curvatures on the world volume. This is because the covariant derivative D_i contains the connection field $\partial_i \phi^\mu \omega_\mu^{ab}$. If we set this equal to A_i^{ab}, then $D_i^{ab} = \partial_i \delta^{ab} + A_i^{ab}$, which is the familiar covariant derivative in $O(D)$ gauge theory. Thus, when we invert the opertor $D_i \sqrt{\gamma} \gamma^{ij} D_j$, we encounter gauge invariant terms like the square of R_{ij}^{ab}, where

$$[D_i, D_j]^{ab} = R_{ij}^{ab} \tag{46}$$

In two dimensions, curvature terms of this sort do not contribute to the perturbation expansion to lowest order when $d = 2$ [8]. However, these terms do in fact contribute to the perturbation expansion when $d = 4$ [10]. In fact, the presence of these terms renders the quantum theory of the $d = 4$ non-linear sigma model non-renormalizable, since they introduce new counter-terms not present in the original action.

Given the potential problems with this term, let us introduce the proper time formalism [11]. Let s be the Schwinger proper time variable:

$$\tilde{G}^{ab}(x, x') = (\gamma(x))^{1/4} G^{ab}(x, x') (\gamma(x'))^{1/4} = i\alpha \int_0^\infty ds \langle x, s | x', 0 \rangle^{ab} \tag{47}$$

where $\gamma = \det \gamma^{ij}$ and we choose a positive metric on the world volume.

We impose the boundary condition:

$$\langle x, 0 | x', 0 \rangle^{ab} = \delta^{ab} \delta^3(x - x') \tag{48}$$

Now let us assume the ansatz for the Green's function:

$$\begin{aligned} \langle x, s | x', 0 \rangle &= \frac{i}{(4\pi i s)^{d/2}} \gamma(x)^{1/4} \Delta^{1/2}(x, x') \gamma(x')^{1/4} F^{ab}(x, x'; is) \\ &\times \exp\left(-\frac{\sigma(x, x')}{2is} - ism^2 \right) \end{aligned} \tag{49}$$

where $\sigma(x, x')$ is one-half the square of the distance along the geodesic between x and x', where Δ is given by:

$$\gamma(x)^{1/4}\Delta(x, x')\gamma(x')^{1/4} = -\det\left(-\sigma_{,i,j}(x, x')\right) \tag{50}$$

and:

$$F(x, x')^{ab} = \sum_{n=0}^{\infty}(is)^n a_n^{ab}(x, x') \tag{51}$$

The object of this section is to power expand the Green's function in terms of $\sigma(x, x')$. In particular, σ obeys a number of useful identities, among them:

$$\sigma_i\sigma^i = 2\sigma \tag{52}$$

where $\sigma_i = \partial_i\sigma$, and we raise and lower indices via γ^{ij}. From this identity, we can establish a large number of identities for various derivatives of σ.

When calculating Feynman diagrams, we will find that they diverge according to inverse powers of σ. Hence, we can compare the large momentum cut-off Λ to σ, i.e.

$$\Lambda \sim \sigma^{-1/2} \tag{53}$$

This new Green's function satisfies the "Schrodinger" equation:

$$-\frac{\partial}{\partial is}\langle x, s|x', 0\rangle = H\langle x, s|x', 0\rangle \tag{54}$$

where the "Hamiltonian" is given by:

$$H = -\gamma^{-1/4}D_i\gamma^{1/2}\gamma^{ij}D_j\gamma^{-1/4} + \zeta R + m^2 \tag{55}$$

If we insert the expansion of the Green's function into the defining equation for the Green's function, we are left with a constraint on the undetermined function F:

$$-\frac{\partial F}{\partial is} = \zeta RF + \frac{1}{is}\sigma^{,i}F_{,i} - \frac{1}{\Delta^{1/2}}\left(\Delta^{1/2}F\right)^{;i}_{,i} \tag{56}$$

Inserting the power expansion for F into this expression, we now have a recursive relation among the a_n coefficients appearing within F:

$$\sigma^i a_{n+1,i} + (n+1)a_{n+1} = \frac{1}{\Delta^{1/2}}\left(\Delta^{1/2}\right)_i^{;i} - \zeta R a_n \tag{57}$$

Now let us solve the a_n iteratively. The equation for a_0^{ab} give us:

$$\sigma^i D_i a_0 = 0 \tag{58}$$

The goal of this exercise is to extract out the divergent terms within the Green's function. Let us, therefore, slowly let x approach x'. Repeated differentiation of the previous formula yields:

$$\begin{aligned}
\lim_{x\to x'} D_i a_0 &= 0 \\
\lim_{x\to x'} D_i \gamma^{ij} D_j a_0 &= -\frac{1}{2} R_{ij}^{ab} \\
\lim_{x\to x'} a_{0,;i}{}^{i}{}_{;j}{}^{j} &= \frac{1}{2} \mathrm{Tr}\, R_{ij} R^{ij}
\end{aligned} \tag{59}$$

where $R_{ij}^{ab} = [D_i, D_j]^{ab}$ and:

$$\begin{aligned}
\sigma &\to \frac{1}{2}(x-x')^2 \\
\sigma_{i;j} &\to \gamma_{ij} \\
\sigma_{i;j;k;l} &\to \frac{1}{2}\left(R_{lijk} + R_{kijl}\right) \\
\Delta &\to 1
\end{aligned} \tag{60}$$

After a certain amount of algebra, we find the desired result for the coefficients a_n^{ab}:

$$\begin{aligned}
a_0 &\to \delta^{ab} \\
a_1 &\to \frac{1}{6}R - \zeta R \\
a_2 &\to \frac{1}{2}\left[(\frac{1}{6} - \zeta)R\right]^2 + \frac{1}{6}(\frac{1}{5} - \zeta)R_i^{;i} + \frac{1}{12}\mathrm{Tr}\, R_{ij}R^{ij} \\
&- \frac{1}{180}R_{ij}R^{ij} + \frac{1}{180}R_{ijkl}R^{ijkl}
\end{aligned} \tag{61}$$

Now we wish to insert these values for a_n^{ab} into the expression for the Green's function, in order to see how badly it diverges as a function of σ. In the limit as $x \to x'$, many terms drop out, and we are left with:

$$\begin{aligned}\tilde{G}(x,x')^{ab} &= i\alpha \int_0^\infty ds\, \langle x,s|x;0\rangle^{ab} \\ &= i\alpha \int_0^\infty ds\, \frac{i\sqrt{\gamma}}{(4\pi i s)^{d/2}} \exp\left(-\frac{\sigma}{2is} - ism^2\right) \sum_{n=0}^\infty (is)^n a_n^{ab} \qquad (62)\end{aligned}$$

We now use the integral:

$$\int_0^\infty dx\, x^{\nu-1} \exp\left(-\frac{i\mu}{2}[x + (\beta^2/x)]\right) = -i\pi\beta^\nu e^{i\nu\pi/2} H_{-\nu}^{(2)}(\beta\mu) \qquad (63)$$

where $H^{(2)}$ is a Bessel function of the third kind, or a Hankel function.

For the case of interest, $d = 3$, we have:

$$\tilde{G}(x,x')^{ab} = \alpha \frac{\pi\sqrt{\gamma}}{(4\pi)^{3/2}} \sum_{n=0}^\infty a_n^{ab} \left(\frac{-\sigma}{2m^2}\right)^{(1/2)(n-1/2)} H_{-n+1/2}^{(2)}\left(\sqrt{-2m^2\sigma}\right) \qquad (64)$$

We now make the definitions $\mu = 2m^2$, $\beta = (-\sigma/2m^2)^{1/2}$, $\nu = n+1-(d/2)$. To find the power expansion in σ, we use the fact that:

$$H_{-n+1/2}^{(2)} = -i(-1)^{n-1} \left(\frac{2z}{\pi}\right)^{1/2} (j_{n-1} - iy_{n-1}) \qquad (65)$$

(where we set $z = \sqrt{-2m^2\sigma}$) and the fact that:

$$\begin{aligned} j_n(z) &= z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z} \\ y_n(z) &= -z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\cos z}{z} \qquad (66)\end{aligned}$$

In particular, we find that the propagator in curved space is linearly divergent in momentum, as expected. The troublesome terms, a_1 and a_2, we see, are finite in three dimensions, and so can be dropped from our discussion (which is not the case for $d = 4$). Although they ruin the renormalization program for four dimensions, we find that they drop out in three dimensions.

For completeness, we present the entire series:

$$
\begin{aligned}
\tilde{G}(x, x')^{ab} &= \alpha i \frac{\sqrt{2\pi\gamma}}{(4\pi)^{3/2}} \left\{ \frac{e^{-iz}}{\sqrt{\sigma}} a_0^{ab} \right. \\
&+ \left. \sum_{n=1}^{\infty} (-1)^n a_n^{ab} \left(\frac{z}{-2m^2} \right)^{2n-1} \left(-\frac{1}{z}\frac{d}{dz} \right)^{n-1} \frac{e^{-iz}}{z} \right\}
\end{aligned}
\tag{67}
$$

Notice that, as $x \to x'$, we find that the integral diverges linearly with the momentum, but the troublesome curvature term involving R_{ij}^{ab} do not contribute (as they do in four dimensions). Hence, from now on, we can simply use the fact that the propagator diverges linearly with momentum.

Lastly, we can also use this formalism to compute two point functions involving derivatives. If we have two point functions like:

$$
\langle T \partial_i \xi^a(x) \partial_j \xi^b(y) \rangle = \partial_i^x \partial_j^y \langle T \xi^a(x) \xi^b(y) \rangle = \partial_i^x \partial_j^y \tilde{G}^{ab}(\sigma) \tag{68}
$$

where $\tilde{G}^{ab}(\sigma)$ is the propagator, then we can, for small distance separations, use the fact that $\sigma(x, y) \sim (1/2)(x-y)^i \gamma_{ij} (x-y)^j$, so that:

$$
\partial_i^x \partial_j^y \sigma(x, y) \sim -\gamma_{ij} \tag{69}
$$

By taking repeated derivatives of the propagator as a function of the separation σ, one can therefore contruct the contraction of an arbitrary number of ξ fields.

4 Two Loop Order

We saw earlier that the one loop result was inconsistent. An action of the form $R + F_{\mu\nu\alpha\beta}^2$ cannot have equations of motion given by $R_{\mu\nu} = 0$ and $D^\mu F_{\mu\nu\alpha\beta} = 0$ We must therefore probe the two loop result to see if we can re-establish the consistency of the model.

Consider first the case of two external lines $N = 2$.

The contraction of the term $L^{(3)}_{2,\alpha\beta}$ yields:

$$L^{i(3)}_{2,aa} \sim \epsilon^{ijk} F_{aa\nu\lambda} \partial_j \phi^\nu \partial_k \phi^\lambda = 0 \tag{70}$$

which vanishes by the anti-symmetry of the F tensor.

The most interesting two loop graph is given by the contraction of $L^{ij(3)}_{2\alpha\beta\gamma}$ with itself. This gives us the contraction:

$$\epsilon^{ijk} F_{\gamma\alpha\beta\lambda} \partial_k \phi^\lambda \langle T D_i \xi^\alpha(x) D_j \xi^\beta(x) \xi^\gamma(x) D_{\bar{i}} \xi^{\bar{\alpha}}(y) D_{\bar{j}} \xi^{\bar{\beta}}(y) \xi^{\bar{\gamma}}(y) \rangle \epsilon^{\bar{i}\bar{j}\bar{k}} F_{\bar{\gamma}\bar{\alpha}\bar{\beta}\bar{\lambda}} \partial_{\bar{k}} \phi^{\bar{\lambda}} \tag{71}$$

when $x \to y$.

If we perform the contractions over ξ, we find, with a little bit of work, the following result:

$$\beta^2 \alpha^3 \Lambda^4 \sqrt{\gamma} \gamma^{ij} \partial_i \phi^\lambda \partial_j \phi^{\bar{\lambda}} F_{cab\lambda} F^{cab}_{\bar{\lambda}} \tag{72}$$

Notice that the divergence can be absorbed into a rescaling: $\alpha = \alpha_R / \Lambda$. (In the next section, we will see that the leading divergences can in fact be absorbed by this rescaling to all orders.)

After rescaling, we find that the equation of motion of the graviton is given by:

$$R_{\mu\nu} + \frac{1}{4} F_{\mu\alpha\beta\gamma} F_\nu^{\alpha\beta\gamma} = 0 \tag{73}$$

(Unfortunately, the term proportional to $g_{\mu\nu}$ does not appear in the equations of motions, signalling a possible inconsistency. This is normally solved for the superstring case by adding an another field, the dilaton. We will see that this possible inconsistency vanishes for the supermembrane case.)

There is also a self-consistency between the equations of motion for the metric and the anti-symmetric field which must be re-established at every loop order, and hence this provides a powerful check on the correctness of any model of membranes.

Higher order graphs are easy to construct but more tedious to evaluate. We will present the contractions necessary to perform two and three loop calculations, but will not explicitly compute the graphs.

For example, the R^2 and DR two loop terms are contained in the contraction of $L^{(4)}_{1,\alpha\beta\gamma\delta}$, so we have the two loop contribution:

$$\frac{1}{24\alpha}\sqrt{\gamma}\gamma^{ij}\partial_i\phi^\mu\partial_j\phi^\sigma\left[R_{\mu\alpha\beta\sigma;\gamma\delta}+R^\lambda_{\alpha\beta\mu}R_{\lambda\gamma\delta\sigma}\right]\langle T\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta\rangle \tag{74}$$

Two loop curvature terms are also contained in the contraction of the square of $L^{(3)}_{1\alpha\beta\gamma}$. This contraction yields:

$$\begin{aligned}\langle L_1^{(3)}L_1^{(3)}\rangle &= \left(\frac{2}{3\alpha}\right)^2\sqrt{\gamma}\gamma^{ij}R_{\mu\beta\gamma\alpha}\partial_j\phi^\mu\langle TD_i\xi^\alpha(x)\xi^\beta(x)\xi^\gamma(x)\\ &\times\ D_{\bar{i}}\xi^{\bar{\alpha}}(y)\xi^{\bar{\beta}}(y)\xi^{\bar{\gamma}}(y)\rangle\sqrt{\gamma}\gamma^{\bar{i}\bar{j}}R_{\bar{\mu}\bar{\beta}\bar{\gamma}\bar{\alpha}}\partial_{\bar{j}}\phi^{\bar{\mu}}D_{\bar{i}}\xi^{\bar{\alpha}}\xi^{\bar{\beta}}\xi^{\bar{\gamma}}+\ldots\end{aligned} \tag{75}$$

These two terms give us two loop correction terms to the curvature tensor, yielding complicated combinations of R^2 and DR terms.

Lastly, we can also calculate the two and three loop contribution for the anti-symmetric field. For example, two loop corrections to the equations of motion are given by contracting $L^{(4)}_{2\alpha\beta\gamma\delta}$ with two propagators. This term is contained within:

$$\begin{aligned}\langle L_2^{(4)}\rangle &= \beta\epsilon^{ijk}\left\{\frac{1}{4!}D_\delta D_\gamma D_\alpha F_{\beta\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\right.\\ &+\left.\frac{9}{4!}R^\mu_{\alpha\beta\sigma}D_\gamma F_{\delta\mu\nu\lambda}\partial_i\phi^\mu\partial_j\phi^\nu\partial_k\phi^\lambda\right\}\langle T\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta\rangle+\ldots\end{aligned} \tag{76}$$

This gives us terms like RF and $DDDF$.

Similarly, we can also contract over the square of $L^{ij(4)}_{2\alpha\beta\gamma\delta}$, which will give us a $FDDF$ term. It is contained within:

$$\begin{aligned}\langle(L^{(4)}_{2\alpha\beta\gamma})^2\rangle &= \left(\frac{18\beta}{4!}\right)^2\epsilon^{ijk}D_\delta F_{\gamma\alpha\beta\lambda}\partial_k\phi^\lambda\langle TD_i\xi^\alpha(x)D_j\xi^\beta(x)\xi^\gamma(x)\xi^\delta(x)\\ &\times\ D_{\bar{i}}\xi^{\bar{\alpha}}(y)D_{\bar{j}}\xi^{\bar{\beta}}(y)\xi^{\bar{\gamma}}(y)\xi^{\bar{\delta}}(y)\rangle\epsilon^{\bar{i}\bar{j}\bar{k}}D_{\bar{\delta}}F_{\bar{\gamma}\bar{\alpha}\bar{\beta}\bar{\lambda}}\partial_{\bar{k}}\phi^{\bar{\lambda}}+\ldots\end{aligned} \tag{77}$$

5 Power Counting

Now let us analyze the divergence of graphs to all orders in perturbation theory. Because the coupling constant has negative dimension, we can always increase the degree of divergence of any multi-loop graph by adding more insertions. In this sense, the theory is not renormalizable. But we will see in this section how many divergences we can absorb via the coupling constant α and β.

Consider first the Lagrangian L_1 with only the metric tensor, without the anti-symmetric field. Let L be the number of loops in an arbitrarily complicated Feynman graph. Then its contribution to the over-all divergence is $3L$, due to d^3p. Let I be the number of internal lines in the graph. So its contribution is $-2L$ due to $1/p^2$. Let V_n be the number of n-point vertices in the graph. Since each n-point graph in the action has two momenta associated with it, it can contribute at most $2V_n$. Let E equal the number of external lines in the graph. Since each external line subtracts off a line which could have become an internal line, it contributes $-N$. Then the superficial divergence of any graph D is given by:

$$D = 3L - 2I + 2\sum_{n=3}^{\infty} V_n - N \tag{78}$$

Now calculate the number of momentum integrations. Each internal line contributed d^3p. Each n-point vertex contributes a momentum-conserving delta function $\delta^3(\sum p_i)$, which deletes three momentum integrations per vertex. And then there is one over-all conservation of momentum factor. The sum of these integrations, in turn, contributes an over-all $(d^3p_i)^L$ momentum integration for the loops. Thus, we have:

$$L = I - \sum_{n=3}^{\infty} V_n + 1 \tag{79}$$

Now insert the second equation into the first, and we obtain:

$$D = L - N + 2 \tag{80}$$

Notice that the degree of divergence D is just a function of the number of loops L and the number of external lines E.

Now let us see if we can re-absorb this divergence into the coupling constant α. Let Λ be the momentum cut-off for the graph. Recall that the perturbation expansion parameter is α. Then the leading divergences of the N-point amplitude A_N, symbolically speaking, diverge as:

$$A_N = \sum_{L=1}^{\infty} \alpha^{L-1} A_{N,L} \tag{81}$$

where we only compute the loop corrections.

We have just shown that $A_{N,L}$ diverges as:

$$A_{N,L} \sim \Lambda^{L-N+2} \tilde{A}_{N,L} \tag{82}$$

Now let us re-define the coupling constant as:

$$\alpha \sim \frac{\alpha_R}{\Lambda} \tag{83}$$

which we performed in the last section for the single loop.

Rescaling the graph, we now have:

$$A_N = \Lambda^{3-N} \left(\sum_{L=1} \alpha^{L-1} \tilde{A}_{N,L} \right) \tag{84}$$

Thus, the leading superficial divergence can be absorbed into α by a rescaling. The larger N, the faster the graph converges. In particular, we see that the amplitude is formally finite for $N = 3$ and beyond, but diverges still for $N = 2$. We can eliminate the $N = 2$ divergence by simply declaring that the background fields obey the standard equations of motion, thereby defining the model.

Now let us generalize the simple power counting to the general case, including the anti-symmetric field. The power counting is much worse, since we now have 3 momenta attached to each vertex function, rather than 2. The leading divergences all come from this sector.

So the degree of divergence is now given by:

$$D = 3L - 2I + 3 \sum_{n=3}^{\infty} V_n + X_N \tag{85}$$

(If some of the lines on the vertex are external lines, this reduces the degree of divergence of the graph, so we have to compensate this by adding in X_N. For example, $X_2 = -2, X_3 = -3$.)

The number of momentum integrations is given by:

$$L = I - \sum_{n=3}^{\infty} V_n + 1 \tag{86}$$

Notice that we can no longer cancel both I and $\sum V_n$ to arrive at a simple relationship involving just L and N. Thus, we need one more constraint to eliminate the vertex factors.

Let us count the number of lines in a graph. Each of the V_n vertices contributes n lines to the graph. Thus, they collectively contribute $\sum nV_n$ lines to the graph. When two of these vertices V_n are joined, they form an internal line, which is therefore counted twice. This means that the sum $\sum nV_n$ counts each internal line twice, and each external line once (since external lines are not paired off). Thus, we have:

$$\sum_{n=3}^{\infty} nV_n = 2I + N \tag{87}$$

By examining these three sets of equations, we see that, in general, it is not possible to eliminate all the V_n in a graph. Therefore, we will only concentrate on the leading divergence within a graph and ignore lower order divergences.

Let us see which vertices contribute the most to the over-all divergence. A vertex V_n contains 3 momenta. Let us say that we replace it with two small vertices V_{n_1} and V_{n+2} which are joined by an internal line. The over-all contribution from these two attached vertices is given by $3 + 3 - 2 = 4$, where the -2 comes from $1/p^2$. Thus, we can always increase the over-all divergence of a graph by replacing V_n with pairs of smaller n-point vertices. This process can be continued, until we are left with a graph with only V_4 and V_5 vertices left. Thus, the leading divergence is now given with only V_4 and V_5. If we eliminate V_4, we are left with:

$$D = 2L - \frac{1}{2}V_5 + \frac{N}{2} + X_N + 1 \tag{88}$$

(Notice that this equation depends on whether the overall number of vertex lines is even or odd. If it is even, then $V_5 = 0$.)

In this way, we can compute the over-all divergence of a graph. However, there is simple short-cut we can use. If we examine the perturbation expansion of L_1 and L_2, we see that the primary difference is that the internal vertices of L_1 contain two derivatives, while the internal vertices of L_2 contain three derivatives. Because the coupling constant $1/\alpha$ appears in front of each term in L_1, we see that each internal vertex function diverges, at most, like Λ^3. But if we let β remain a finite constant, we see that each internal vertex in L_2 diverges as Λ^3 as well. Thus, by only rescaling α but keeping β finite, we see that the divergence of the purely metric theory is identical to the theory coupled to anti-symmetric tensor fields. (β, although it is finite, will ultimately be fixed by requiring consistency in the equations of motion of the background fields).

6 Conclusion

In field theory, the study of non-renormalizable Lagrangians, such as the four-fermion model, or massive vector theories, has given us insight in deep physical processes. Likewise, the bosonic membrane action, by naive power counting, is non-renormalizable on the world volume, but may give us insight into M-theory. Although the bosonic membrane theory is ultimately probably not a consistent quantum theory, the techniques we have used here will generalize to the supermembrane case.

In this paper, we have expanded the bosonic membrane action around Riemann normal co-ordinates, treating the theory as a non-linear sigma model, and calculated the regularized propagator and higher loop graphs.

In particular, we found:

a) The standard dimensional regularization method apparently breaks down at $d = 3$, where the Gamma function no longer has a pole. Instead, we developed the proper time and point-splitting formalism in curved space for the $d = 3$ membrane action. Although we lost general covariance, this gave us an intuitive way in which to isolate all the divergences of higher graphs, since the singularities emerge when two fields touch on the world volume. It is then a simple matter to analyze complicated graphs visually and isolate their divergences.

b) The renormalization program in four dimensions for the non-linear sigma model is ruined by the presence of terms like $\mathrm{Tr}\,[R_{ij}R^{ij}]$. However, we have shown that these terms are not a problem in three dimensions.

c) We found that the single loop graph was insufficient to generate self-consistent equations of motion for the background fields. This was surprising, since setting $\beta = 0$ in the usual string formalism yields self-consistent equations of motions at the first loop level.

d) Since the formalism we have developed works for arbitrary loop level, we can calculate higher order corrections to the equations of motion. We find, at two loop level, new terms which have the form: R^2, DR, $FDDF$, FRF, etc.

e) We found that, by naive counting arguments on arbitrary loop graphs, we could absorb the leading divergences into a rescaling of the coupling constant α. By setting $\alpha \to \alpha_R/\Lambda$, where Λ is a large momentum cut-off parameter, we could absorb the leading divergences. The amplitudes then diverge as Λ^{3-N}, so the leading divergences actually vanish if $N = 3, 4, ...$ For the case $N = 2$, we set the divergence to zero, thereby yielding the equations of motions. We found that the counting of divergences remains the same if the coupling constant β for the anti-symmetric fields is finite. This doesn't mean that the action is renormalizable, of course, since we still have to analyze non-leading graphs and many other subtle problems.

One weakness of this formalism is that the equations of motion emerge only after setting one divergences to zero by hand. Hence, we have to define the theory by placing in the background fields on-shell.

In the superstring case, conformal symmetry allows us to set $\beta = 0$. However, the entire motivation of our approach is to analyze the D = 11 supermembrane, where supersymmetry is sufficient to set these lower order divergences to zero. Our ultimate goal, therefore, is to see whether supersymmetry is strong enough to control the divergences found in the supermembrane theory, and whether we can obtain the the M-theory action by expanding around higher order corrections to the standard D=11 supergravity action, and then use recursion relations to probe the entire action. The supersymmetric case will be discussed in a forthcoming paper.

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