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(The record on the title page is from an earthquake in Greece on April 5, 1965, as recorded by a long-period vertical-component seismograph at Uppsala).

Developments in Solid Earth Geophysics
4

MATHEMATICAL ASPECTS OF SEISMOLOGY

by

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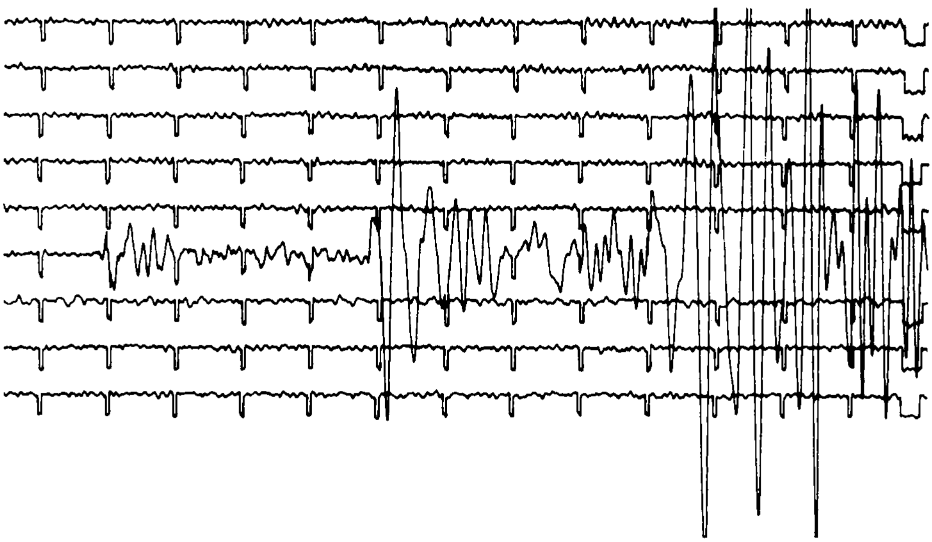
$$\rho \frac{\partial^2 s}{\partial t^2} = (\lambda + 2\mu)\ \mathrm{grad}\ \theta - 2\mu\ \mathrm{curl}\ \omega$$



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PREFACE

There is frequently a large gap between the seismological student's knowledge of mathematics and the mathematics needed to understand more advanced textbooks or original papers in the subject. As an example I may cite the fact that Bessel functions are usually not included in less advanced courses in mathematics but are fundamental to almost every branch of seismology. The purpose of the present book is to bridge this gap. It provides nearly all the mathematics needed to follow such textbooks as those of EWING, JARDETZKY and PRESS (1957) or of BREKHOVSKIKH (1960) without the need to refer continually to original works.

The book assumes the reader has a mathematics background corresponding to that usually required of seismologists, that is, equivalent to about one to two years' university studies in mathematics. It will also assume some knowledge of wave propagation phenomena and related chapters of theoretical seismology. Starting from this level of knowledge, it will give the applied mathematics necessary for advanced studies, especially of the theoretical aspects of seismology. As a consequence, about 50% of the content is applied mathematics, the remaining 50% being various applications to seismological problems. Although specifically directed to the needs in seismology, much of the book could also prove useful to other students of applied mathematics.

The book is primarily intended to facilitate studies of the more advanced parts of theoretical seismology. As a consequence, such material has not been included which is either more directly concerned with observational seismology or which can be more easily grasped even with a simple mathematical background. This is the case with the following subjects for which I give only some references to the literature listed at the end of the book:

(*1*) Map projections: see KELLAWAY (1946).
(*2*) Vector and tensor calculus: see MARGENAU and MURPHY (1943), JOOS (1956).
(*3*) Spherical trigonometry: see SMART (1936).
(*4*) Statistical handling of observational data: see LEVY and PREIDEL (1945), FISHER (1950), BLACKMAN and TUKEY (1958), ROBINSON (1967).

These topics and these references are to be taken only as examples. They could be multiplied.

The book is divided into four parts, comprising fourteen chapters altogether. Each chapter is divided into sections and sub-sections, using a decimal system. Formulas are numbered [1], [2], . . . and upwards within each section. The mathematical content has been the guide-line in arranging the material in the book. More immediate seismological applications appear at the end of the respective sections, whereas extensive problems, being applications of several mathematical methods, are collected in Part IV.

In working up the material for this book, numerous texts were consulted. As a rule, the result given here is a combination of several presentations found, and which I considered to be the best.

MARKUS BÅTH

Chapter 1

INTRODUCTION

1.1 DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

The chief differential equations of mathematical physics reduce to a few simple forms, closely related to each other even for problems which apparently diverge considerably from each other. This is naturally to be considered as a fortunate circumstance, as treatments of solutions to these equations can be made quite general.

In this section we shall only compare the equations formally without going into details about, for example, the meaning of every symbol used, etc. We start with the so-called *telegraphic equation*, valid for the propagation of a current- or tension-wave in a telegraph wire, arisen through induction in another field (e.g., the atmosphere):

$$a\frac{\partial^2\psi}{\partial t^2} + b\frac{\partial\psi}{\partial t} + c\nabla^2\psi = -ce \qquad [1]$$

∇^2 (∇ = inverted delta or nabla) is the *Laplace operator*, often also denoted by Δ; a, b, c are constants, t is time, and e denotes any sources or sinks.

Eq.[1] is quite a general equation, from which we may form a series of special cases:

(*1*) $a = 0$ gives, putting $c/b = -g^2$:

$$\frac{\partial\psi}{\partial t} - g^2\nabla^2\psi = g^2 e \qquad \text{with } e \neq 0 \qquad [2]$$

$$\frac{\partial\psi}{\partial t} - g^2\nabla^2\psi = 0 \qquad \text{with } e = 0 \qquad [3]$$

which is *the equation of heat conduction*, in [2] with sources or sinks, in [3] without.

(*2*) $b = 0$ gives, putting $c/a = -v^2$:

$$\frac{\partial^2\psi}{\partial t^2} - v^2\nabla^2\psi = v^2 e \qquad \text{with } e \neq 0 \qquad [4]$$

$$\frac{\partial^2\psi}{\partial t^2} - v^2\nabla^2\psi = 0 \qquad \text{with } e = 0 \qquad [5]$$

which is *the wave equation*, with and without sources or sinks respectively, and valid for sound waves (or generally for elastic waves), the displacement of a string or a membrane, water waves, and electromagnetic waves (v is the *phase velocity* or *wave velocity*).

A further specialization is obtained from eq.[2]–[5] by considering a steady state, i.e., one which does not change with time or for which $\partial/\partial t = 0$. We then get respectively:

$$\nabla^2\psi = -e \qquad [6]$$

which is *Poisson's equation* (with sources or sinks, e being the source density = a given point function), and:

$$\nabla^2\psi = 0 \qquad [7]$$

which is *Laplace's equation* (without sources or sinks). The same specialization of [4] and [5] is achieved by assuming an infinite velocity v.

There is one equation which cannot be derived from [1] and which is of some interest to us, namely the following:

$$\frac{\partial^2\psi}{\partial t^2} + a^2\nabla^2\nabla^2\psi = a^2 e \qquad [8]$$

and without sources or sinks:

$$\frac{\partial^2\psi}{\partial t^2} + a^2\nabla^2\nabla^2\psi = 0 \qquad [9]$$

valid for the transverse motion of a bar or a plate. It contains the *double Laplace operator* $\nabla^2\nabla^2$, which mathematically means nothing but exercising the ∇^2-operator on $\nabla^2\psi$.

Summarizing, we see that the equations given (which are the most important ones in mathematical physics) are similar in being *linear*, of the *second order* (excepting [8] and [9] which are of the fourth order), and having *constant coefficients* of the derivatives. This implies, for example, that in the wave equation [4] and [5] we assume the phase velocity v constant, independent of space and time. Having v as a function of space and time is of importance in some fields, as in quantum physics, wave mechanics and certain branches of optics. In solid earth physics, it is important when, for instance, one likes to consider v as a function of depth.

1.2 COORDINATE TRANSFORMATIONS

The special functions in applied mathematics arise in the solution of partial differential equations. The problem consists of finding those functions which satisfy a given differential equation and certain boundary conditions. The shape of these boundaries often makes it desirable to work in curvilinear coordinates q_1, q_2, q_3 instead of rectangular Cartesian coordinates x, y, z. Coordinate transformation then occurs:

$$\left.\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3) \end{aligned}\right| \qquad [1]$$

Differentiate [1]:

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial q_1}dq_1 + \frac{\partial x}{\partial q_2}dq_2 + \frac{\partial x}{\partial q_3}dq_3 \\ dy &= \frac{\partial y}{\partial q_1}dq_1 + \frac{\partial y}{\partial q_2}dq_2 + \frac{\partial y}{\partial q_3}dq_3 \\ dz &= \frac{\partial z}{\partial q_1}dq_1 + \frac{\partial z}{\partial q_2}dq_2 + \frac{\partial z}{\partial q_3}dq_3 \end{aligned}\right| \qquad [2]$$

An element of length $= dl$ is expressed by:

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2 = Q_{11}^2dq_1^2 + Q_{22}^2dq_2^2 + Q_{33}^2dq_3^2 + 2Q_{12}dq_1dq_2$$
$$+ 2Q_{13}dq_1dq_3 + 2Q_{23}dq_2dq_3$$

where:

$$Q_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}$$

and for $i = j$: [3]

$$Q_{ii} = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2$$

$i, j = 1, 2, 3$

If, in particular, $Q_{ij} = 0$ for $i \neq j$, then:

$$(dl)^2 = Q_{11}^2dq_1^2 + Q_{22}^2dq_2^2 + Q_{33}^2dq_3^2$$

and the coordinates q_i are *orthogonal curvilinear coordinates*, and we simplify the expression for $(dl)^2$:

$$(dl)^2 = Q_1^2dq_1^2 + Q_2^2dq_2^2 + Q_3^2dq_3^2$$

i.e., $Q_{ii} \to Q_i$.

By means of vector calculus, it is possible to transform various vector operations from one coordinate system to another.

Noting that $dl_i = Q_i dq_i \quad (i = 1, 2, 3)$ we have that:

$$\frac{\partial}{\partial l_i} = \frac{1}{Q_i}\frac{\partial}{\partial q_i}$$

and:

$$\operatorname{grad} f = \frac{\mathbf{i}_1}{Q_1}\frac{\partial f}{\partial q_1} + \frac{\mathbf{i}_2}{Q_2}\frac{\partial f}{\partial q_2} + \frac{\mathbf{i}_3}{Q_3}\frac{\partial f}{\partial q_3} \qquad [4]$$

($\mathbf{i}$ = unit vectors).

The transformation for div$\mathbf{v}$ is derived by using Gauss' formula. We have a volume element $dV = dl_1dl_2dl_3$ and consider the vector $\mathbf{v}(v_1, v_2, v_3)$. Gauss' formula transforms

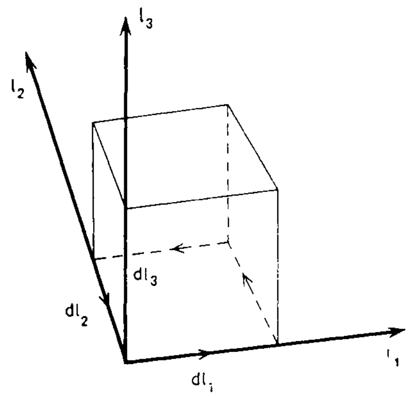


Fig.1.

a volume integral into a surface integral (S = surface):

$$\iiint_V \operatorname{div} \mathbf{v} \, dV = \iint_S \mathbf{v} \cdot d\mathbf{S} \tag{5}$$

In the present case we have the following contributions to the surface integral (see Fig.1) from one side:

$$-v_1 dl_2 dl_3; \qquad -v_2 dl_3 dl_1; \qquad -v_3 dl_1 dl_2$$

and from the opposite side:

$$v_1 dl_2 dl_3 + \frac{\partial}{\partial l_1}(v_1 dl_2 dl_3) dl_1;$$

$$v_2 dl_3 dl_1 + \frac{\partial}{\partial l_2}(v_2 dl_3 dl_1) dl_2,$$

etc.

Therefore, the right-hand side of Gauss' formula [5] becomes (net contributions):

$$\frac{1}{Q_1}\frac{\partial}{\partial q_1}(Q_2 Q_3 v_1) dl_1 dq_2 dq_3 + \frac{1}{Q_2}\frac{\partial}{\partial q_2}(Q_3 Q_1 v_2) dl_2 dq_3 dq_1$$

$$+ \frac{1}{Q_3}\frac{\partial}{\partial q_3}(Q_1 Q_2 v_3) dl_3 dq_1 dq_2$$

which is equal to divv $dl_1 dl_2 dl_3$, i.e., finally:

$$\operatorname{div}\mathbf{v} = \frac{1}{Q_1 Q_2 Q_3}\left[\frac{\partial}{\partial q_1}(Q_2 Q_3 v_1) + \frac{\partial}{\partial q_2}(Q_3 Q_1 v_2) + \frac{\partial}{\partial q_3}(Q_1 Q_2 v_3)\right] \tag{6}$$

We can now transform $\nabla^2\psi$, since:

$$\nabla^2\psi = \operatorname{div} \operatorname{grad}\psi$$

$$= \frac{1}{Q_1Q_2Q_3}\left[\frac{\partial}{\partial q_1}\left(\frac{Q_2Q_3}{Q_1}\frac{\partial\psi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{Q_3Q_1}{Q_2}\frac{\partial\psi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{Q_1Q_2}{Q_3}\frac{\partial\psi}{\partial q_3}\right)\right] \quad [7]$$

This is valid for the case when ψ is a scalar, and thus $\nabla^2\psi$ also is a scalar.

In a similar way we derive the transformation for rot**v** (or curl**v**) by using Stokes' formula, which connects a surface integral with an integral around the contour (see section 2.1.2 below):

$$\iint_S \mathrm{curl}\mathbf{v}\cdot \mathrm{d}\mathbf{S} = \int_L \mathbf{v}\cdot \mathrm{d}\mathbf{s} \quad [8]$$

In Fig.1 we need consider only one of the surfaces, say $\mathrm{d}l_1\mathrm{d}l_2$, perpendicular to the l_3-axis, i.e., we apply [8] to this surface and its contour. We get the following contributions to the right-hand side of [8], the signs being determined by the directions (positive or negative), in which we go through the contour:

$$+v_1\mathrm{d}l_1;\quad -v_2\mathrm{d}l_2;\quad v_2\mathrm{d}l_2+\frac{\partial}{\partial l_1}(v_2\mathrm{d}l_2)\mathrm{d}l_1;\quad -v_1\mathrm{d}l_1-\frac{\partial}{\partial l_2}(v_1\mathrm{d}l_1)\mathrm{d}l_2$$

The net contribution to the right-hand side of [8] is thus:

$$\frac{\partial}{\partial l_1}(v_2\mathrm{d}l_2)\mathrm{d}l_1 - \frac{\partial}{\partial l_2}(v_1\mathrm{d}l_1)\mathrm{d}l_2$$

$$= \frac{1}{Q_1}\frac{\partial}{\partial q_1}(v_2Q_2)\mathrm{d}q_2\mathrm{d}l_1 - \frac{1}{Q_2}\frac{\partial}{\partial q_2}(v_1Q_1)\mathrm{d}q_1\mathrm{d}l_2$$

$$\left[\text{using the relations } \mathrm{d}l_i = Q_i\mathrm{d}q_i;\ \frac{\partial}{\partial l_i}=\frac{1}{Q_i}\frac{\partial}{\partial q_i}\right]$$

$$= \mathrm{d}l_1\mathrm{d}l_2(\mathrm{curl}\mathbf{v})_3$$

noting that this is the corresponding component (along the l_3-axis) of [8]. We thus find that:

$$(\mathrm{curl}\mathbf{v})_3 = \frac{1}{Q_1Q_2}\frac{\partial}{\partial q_1}(v_2Q_2) - \frac{1}{Q_1Q_2}\frac{\partial}{\partial q_2}(v_1Q_1) \quad [9]$$

or completely written out in vector form:

$$\mathrm{curl}\mathbf{v} = \frac{\mathbf{i}_1}{Q_2Q_3}\left[\frac{\partial(v_3Q_3)}{\partial q_2}-\frac{\partial(v_2Q_2)}{\partial q_3}\right] + \frac{\mathbf{i}_2}{Q_3Q_1}\left[\frac{\partial(v_1Q_1)}{\partial q_3}-\frac{\partial(v_3Q_3)}{\partial q_1}\right] + \frac{\mathbf{i}_3}{Q_1Q_2}\left[\frac{\partial(v_2Q_2)}{\partial q_1}-\frac{\partial(v_1Q_1)}{\partial q_2}\right] \quad [10]$$

The transformation formulas [4], [6], [7] and [10] are valid only for *orthogonal* systems.

1.2.1 Applications to the wave equation

In seismology we have frequent use for these formulas, especially in transformation of the wave equation into a coordinate system which is suitable for a special problem. The wave equations referred to a rectangular *xyz*-system are as follows with standard notation (see, e.g., EWING, JARDETZKY and PRESS, 1957, p.6):

$$\left.\begin{aligned} \varrho \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu) \frac{\partial \theta}{\partial x} + \mu \nabla^2 u + \varrho X \\ \varrho \frac{\partial^2 v}{\partial t^2} &= (\lambda + \mu) \frac{\partial \theta}{\partial y} + \mu \nabla^2 v + \varrho Y \\ \varrho \frac{\partial^2 w}{\partial t^2} &= (\lambda + \mu) \frac{\partial \theta}{\partial z} + \mu \nabla^2 w + \varrho Z \end{aligned}\right| \qquad [11]$$

or in vector form:

$$\varrho \frac{\partial^2 \mathbf{s}}{\partial t^2} = (\lambda + 2\mu)\,\mathrm{grad\,div}\mathbf{s} + \mu(\nabla^2 \mathbf{s} - \mathrm{grad\,div}\mathbf{s}) + \varrho \mathbf{F} \qquad [12]$$

noting that $\theta = \mathrm{div}\mathbf{s}$. From vector calculus, we know that:

$$\mathrm{curl\,curl}\mathbf{s} = \mathrm{grad\,div}\mathbf{s} - \nabla^2 \mathbf{s} \qquad [13]$$

and:

$$\mathrm{curl}\mathbf{s} = 2\boldsymbol{\omega} \qquad [14]$$

For proofs, see MARGENAU and MURPHY (1943, p.149 for [13] and p.148 for [14]). [13] is easily proved by expansion in rectangular components. Note that in Margenau and Murphy's derivation of [14], s and $\boldsymbol{\omega}$ are velocities (linear and rotational, respectively) Here, they are both displacements (linear and rotational, respectively), but if we take them both per unit time, we naturally have velocities.

Apply [13] and [14] to [12], also omit the external force **F**, as is usually done in. dealing with the wave equation; then we find:

$$\varrho \frac{\partial^2 \mathbf{s}}{\partial t^2} = (\lambda + 2\mu)\,\mathrm{grad}\theta - 2\mu\,\mathrm{curl}\boldsymbol{\omega} \qquad [15]$$

(*1*) *Spherical coordinates.* This case is shown in Fig.2. We have that $(\mathrm{d}l)^2 = (\mathrm{d}R)^2 + (R\mathrm{d}\delta)^2 + (R\sin\delta\mathrm{d}\varepsilon)^2$, that is:

$$\left.\begin{aligned} Q_1 &= 1 \\ Q_2 &= R \\ Q_3 &= R\sin\delta \end{aligned}\right| \qquad [16]$$

Applying the expressions for grad: [4], and for curl: [10], we find immediately the wave equation in spherical coordinates from [15]:

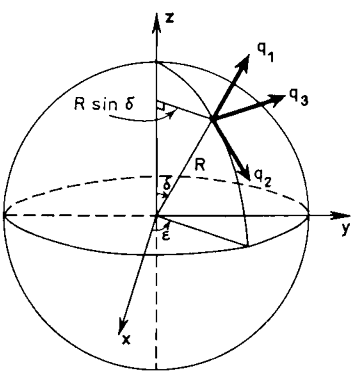
$$
\left.
\begin{aligned}
\varrho \frac{\partial^2 u_1}{\partial t^2} &= (\lambda + 2\mu) \frac{\partial \theta}{\partial R} - \frac{2\mu}{R \sin\delta} \frac{\partial}{\partial \delta} (\omega_3 \sin\delta) + \frac{2\mu}{R \sin\delta} \frac{\partial \omega_2}{\partial \varepsilon} \\
\varrho \frac{\partial^2 u_2}{\partial t^2} &= (\lambda + 2\mu) \frac{1}{R} \frac{\partial \theta}{\partial \delta} - \frac{2\mu}{R \sin\delta} \frac{\partial \omega_1}{\partial \varepsilon} + \frac{2\mu}{R} \frac{\partial}{\partial R} (R\omega_3) \\
\varrho \frac{\partial^2 u_3}{\partial t^2} &= \frac{(\lambda + 2\mu)}{R \sin\delta} \frac{\partial \theta}{\partial \varepsilon} - \frac{2\mu}{R} \frac{\partial}{\partial R} (R\omega_2) + \frac{2\mu}{R} \frac{\partial \omega_1}{\partial \delta}
\end{aligned}
\right| \quad [17]
$$

See also EWING, JARDETZKY and PRESS (1957, p.267).

From [6], [14] and [10] we deduce the following expressions for the rotational components, which could be substituted into [17] to yield equations in u_1, u_2, u_3:

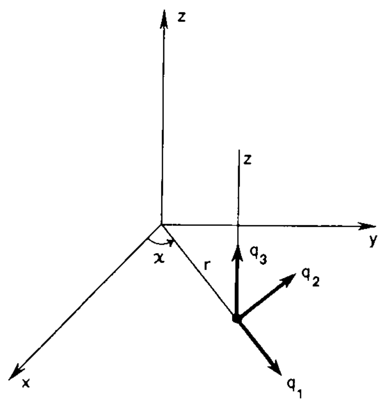
$$
\left.
\begin{aligned}
2\omega_1 &= \frac{1}{R^2 \sin\delta} \left[\frac{\partial}{\partial \delta} (u_3 R \sin\delta) - \frac{\partial}{\partial \varepsilon} (u_2 R) \right] \\
2\omega_2 &= \frac{1}{R \sin\delta} \left[\frac{\partial u_1}{\partial \varepsilon} - \frac{\partial}{\partial R} (u_3 R \sin\delta) \right] \\
2\omega_3 &= \frac{1}{R} \left[\frac{\partial}{\partial R} (u_2 R) - \frac{\partial u_1}{\partial \delta} \right] \\
\theta = \operatorname{div} s &= \frac{1}{R^2 \sin\delta} \left[\frac{\partial}{\partial R} (u_1 R^2 \sin\delta) + \frac{\partial}{\partial \delta} (u_2 R \sin\delta) + \frac{\partial}{\partial \varepsilon} (u_3 R) \right]
\end{aligned}
\right| \quad [18]
$$

In these and the following examples we are using equations containing the displacements u. If instead we introduce displacement potentials, the wave equation simplifies, and so do the coordinate transformations. See an example in section 5.3.2 below.



q_1 = R (radius)
q_2 = δ (colatitude)
q_3 = ε (azimuth)

Fig.2.



q_1 = r (radius)
q_2 = χ (azimuth)
q_3 = z (vertical)

Fig.3.

(*2*) *Cylindrical coordinates.* As above we have from Fig.3 that $(\mathrm{d}l)^2 = (\mathrm{d}r)^2 + (r\mathrm{d}\chi)^2 + (\mathrm{d}z)^2$ which gives the expressions for Q:

$$\left.\begin{aligned} Q_1 &= 1 \\ Q_2 &= r \\ Q_3 &= 1 \end{aligned}\right| \qquad [19]$$

The displacements are denoted: $u_1 = u$; $u_2 = v$; $u_3 = w$. Then, we have from [6]:

$$\theta = \operatorname{div} \mathbf{s} = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial \chi} + \frac{\partial}{\partial z}(rw)\right] = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \chi} + \frac{\partial w}{\partial z} \qquad [20]$$

and from [10]:

$$\operatorname{curl}\boldsymbol{\omega} = \frac{\mathbf{i}_1}{r}\left[\frac{\partial \omega_3}{\partial \chi} - \frac{\partial}{\partial z}(\omega_2 r)\right] + \mathbf{i}_2\left[\frac{\partial \omega_1}{\partial z} - \frac{\partial \omega_3}{\partial r}\right] + \frac{\mathbf{i}_3}{r}\left[\frac{\partial}{\partial r}(\omega_2 r) - \frac{\partial \omega_1}{\partial \chi}\right] \qquad [21]$$

Introducing [21] into [15] we get the following system of equations:

$$\left.\begin{aligned} \varrho\frac{\partial^2 u}{\partial t^2} &= (\lambda + 2\mu)\frac{\partial \theta}{\partial r} - \frac{2\mu}{r}\frac{\partial \omega_3}{\partial \chi} + 2\mu\frac{\partial \omega_2}{\partial z} \\ \varrho\frac{\partial^2 v}{\partial t^2} &= (\lambda + 2\mu)\frac{1}{r}\frac{\partial \theta}{\partial \chi} - 2\mu\frac{\partial \omega_1}{\partial z} + 2\mu\frac{\partial \omega_3}{\partial r} \\ \varrho\frac{\partial^2 w}{\partial t^2} &= (\lambda + 2\mu)\frac{\partial \theta}{\partial z} - \frac{2\mu}{r}\frac{\partial}{\partial r}(\omega_2 r) + \frac{2\mu}{r}\frac{\partial \omega_1}{\partial \chi} \end{aligned}\right| \qquad [22]$$

The components of $\boldsymbol{\omega}$ are from [14]:

$$\left.\begin{aligned} 2\omega_1 &= \frac{1}{r}\frac{\partial w}{\partial \chi} - \frac{\partial v}{\partial z} \\ 2\omega_2 &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \\ 2\omega_3 &= \frac{1}{r}\left[\frac{\partial}{\partial r}(vr) - \frac{\partial u}{\partial \chi}\right] \end{aligned}\right| \qquad [23]$$

Again, if we want the wave equation only in the displacements u we use [20] and [23] and substitute these in [22]. If we moreover assume axial symmetry (i.e., symmetry around the z-axis), we find the following system of equations:

$$\left.\begin{aligned} \varrho\frac{\partial^2 u}{\partial t^2} &= (\lambda + 2\mu)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 w}{\partial z \partial r}\right) + \mu\left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial z \partial r}\right) \\ \varrho\frac{\partial^2 w}{\partial t^2} &= (\lambda + 2\mu)\left(\frac{\partial^2 u}{\partial z \partial r} + \frac{1}{r}\frac{\partial u}{\partial z} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{\mu}{r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) \\ &\quad - \mu\left(\frac{\partial^2 u}{\partial z \partial r} - \frac{\partial^2 w}{\partial r^2}\right) \end{aligned}\right| \qquad [24]$$

In this case the problem is two-dimensional and there is no wave equation for the v-component.

See also EWING, JARDETZKY and PRESS (1957, pp.9 and 124), and LOVE (1944, p.288).

Cylindrical coordinates are frequently suitable for the wave equation, especially in layered structures, when axial symmetry is a justified assumption (the axis z being perpendicular to the layering), whereas the wave fronts are not spherical in such a case.

These examples may be sufficient to illustrate how a given equation may be expressed in any orthogonal coordinate system. The steps in this procedure are the following:

(*1*) Express $(\mathrm{d}l)^2$ in the new system, which gives the expressions for Q_1, Q_2 and Q_3.

(*2*) Use the formulas for div, grad, curl given above, as the case may be.

(*3*) Introduce these expressions into a vector form of the given equation, e.g., [12] or [15], and write down the component equations in the new coordinate system.

In the examples above, we started from [15] to get the equations in spherical and cylindrical coordinates. Instead, we could have started from [12], but then we have to observe that s is a vector and that the transformation for $\nabla^2\mathbf{s}$ does not follow [7], this one being valid only for ψ being a scalar.

We shall therefore also give the transformation formula for s being a vector. Eq.[13] can be written as follows:

$$\begin{aligned}\nabla^2\mathbf{s} &= \nabla(\nabla\cdot\mathbf{s}) - \nabla\times\nabla\times\mathbf{s}\\ &= \text{grad div}\mathbf{s} - \text{curl curl}\mathbf{s}\end{aligned} \qquad [25]$$

The component in the 1-direction of this expression is:

$$\underbrace{\frac{1}{Q_1}\frac{\partial}{\partial q_1}(\nabla\cdot\mathbf{s})}_{(\text{grad div}\mathbf{s})_1} \; \underset{-}{-} \; \underbrace{\frac{1}{Q_2}\frac{\partial}{\partial q_2}(\nabla\times\mathbf{s})_3 + \frac{1}{Q_3}\frac{\partial}{\partial q_3}(\nabla\times\mathbf{s})_2}_{(\text{curl curl}\mathbf{s})_1} \qquad [26]$$

and here we have to insert the transformations [6] for div and [10] for curl.

1.2.2 Illustration of how different functions enter into calculus

As an illustration of how different functions, appropriate to the chosen coordinate system, enter into the solutions of differential equations, let us take a simple example. Transforming the Laplace equation:

$$\nabla^2\psi = 0 \qquad [27]$$

into cylindrical polar coordinates, we get:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\chi^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad [28]$$

Separating the variables by the following substitution (which is really to be considered as an assumption):

$$\psi = R(r)\varphi(\chi)Z(z) \tag{29}$$

the Laplace equation can be written as:

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2\varphi}\frac{d^2\varphi}{d\chi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0 \tag{30}$$

This shows that if φ, Z, R satisfy the equations:

$$\left.\begin{aligned}&\frac{d^2\varphi}{d\chi^2}+n^2\varphi=0\\&\frac{d^2Z}{dz^2}-m^2Z=0\\&\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(m^2-\frac{n^2}{r^2}\right)R=0\\&m, n \text{ constants}\end{aligned}\right| \tag{31}$$

then [29] is a solution of Laplace's equation [27].

The solutions of these ordinary differential equations lead to special functions appropriate to this coordinate system: [31.1] defines the circular functions, sin and cos; [31.2] defines the exponential function; [31.3] defines the Bessel function, and this equation is called Bessel's differential equation.

1.3 THE GAMMA AND BETA FUNCTIONS

As distinct from other functions, to be treated in later chapters, the gamma and beta functions are not solutions of any differential equations, but are introduced mainly for convenience in calculus. The following set of formulas is of frequent use (m and n are any numbers, with the restrictions specified in each case).

Definition of the gamma function (or Eulerian integral of the second kind):

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}\,dx \quad \text{(converges if } n>0) \tag{1}$$

Definition of the beta function (or Eulerian integral of the first kind):

$$B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad \begin{cases} m>0\\ n>0\end{cases} \tag{2}$$

From [1] and [2] we derive the following formulas:

$$\Gamma(1)=1 \tag{3}$$

$$\Gamma(n+1)=n\Gamma(n) \tag{4}$$

$$\Gamma(n+1) \equiv \Pi(n) = n! \quad \text{(if } n \text{ is a positive integer)} \tag{5}$$

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \mathrm{d}\theta \tag{6}$$

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{7}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{8}$$

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{n!n^z}{(z+1)(z+2)\ldots(z+n)} \qquad (z > 0) \tag{9}$$

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \qquad \text{("duplication formula")} \tag{10}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z} \tag{11}$$

We give some hints for the proofs of [3]–[11]. Eq.[3] is obtained immediately from

$$\int_0^\infty e^{-x}\mathrm{d}x = [e^{-x}]_\infty^0 = 1$$

For [4] see De la Vallée Poussin (1938, p.225):

$$\int_0^\infty x^n e^{-x}\mathrm{d}x = n\int_0^\infty x^{n-1}e^{-x}\mathrm{d}x \qquad \text{(by partial integration)}$$

Thus $\Gamma(n+1) = n\Gamma(n)$. This equation can be used to define $\Gamma(n)$ in case n is a negative fraction. Eq.[5] is obtained by successive application of [4]; [6] is obtained from the definition [2] by the substitution: $x = \sin^2\theta$, $\mathrm{d}x = 2\sin\theta\cos\theta\mathrm{d}\theta$. Eq.[7] is obtained by combination of [5] and [6], applying the following formula (De la Vallée Poussin, 1938, p.226):

$$\int_0^{\pi/2} \sin^{2m-1}x \cos^{2n-1}x \,\mathrm{d}x = \frac{1}{2}\frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Eq.[8] is obtained from the formula:

$$\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

See De la Vallée Poussin (1938, pp.229–230). That is:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-\frac{1}{2}}\mathrm{d}x = 2\int_0^\infty e^{-u^2}\mathrm{d}u = \sqrt{\pi}$$

(Substitute $x = u^2$.) Eq. [9] is obtained as follows:

$$\Gamma(z+1) = \int_0^\infty e^{-x}x^z\mathrm{d}x$$

by [1] and:

$$e^{-x} = \lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^n$$

by definition of e^{-x}. We consider the integral:

$$I_n = \int_0^n \left(1 - \frac{x}{n}\right)^n x^z \mathrm{d}x$$

Then:

$$\lim_{n\to\infty} I_n = \Gamma(z+1)$$

Integrate by parts, successively:

$$I_n = \frac{1}{z+1}\int_0^n \left(1 - \frac{x}{n}\right)^{n-1} x^{z+1}\mathrm{d}x = \frac{n^z n!}{(z+1)(z+2)\ldots(z+n)}\,\frac{n}{z+n+1}$$

Then, passing to the limit $n\to\infty$ — for n infinite, $n/(z+n+1) = 1$:

$$\lim_{n\to\infty}\frac{n^z n!}{(z+1)(z+2)\ldots(z+n)} = \Gamma(z+1)$$

i.e., eq.[9], which is thus proved. Eq.[10], the duplication formula is found from:

$$\Gamma(2z) = \lim_{n\to\infty}\frac{(2n+1)^{2z}(2n+1)!}{2z(2z+1)\ldots(2z+2n+1)}$$

as seen by substituting $z \to 2z$, $n \to 2n+1$ in [9], just given. We then get:

$$\Gamma(2z) = 2^{2z}\lim_{n\to\infty}\frac{n^z n!}{2z(2z+2)\ldots(2z+2n)}\,\frac{n^z n!}{(2z+1)(2z+3)\ldots(2z+2n+1)}\cdot$$

$$\cdot\frac{(2n+1)!}{(n!)^2}$$

$$= 2^{2z}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)K$$

as $(2n+1)^{2z} = 2^{2z}n^{2z}$ and 1 may be neglected for $n\to\infty$. K is a constant, including a factor 2^n from each of the two gamma functions (in the numerators). K is determined by putting $z = 1/2$:

$$1 = 2\Gamma\left(\frac{1}{2}\right)K; \quad K = \frac{1}{2\Gamma\left(\frac{1}{2}\right)}$$

Therefore:

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

i.e., eq.[10], which should be proved.

Eq.[11] can be proved by starting from [9], written as a product:

$$\Gamma(z) = \frac{1}{z}\lim_{n\to\infty} n^z \prod_1^n \left(1+\frac{z}{n}\right)^{-1}$$

from which we immediately form its inverse:

$$\frac{1}{\Gamma(z)} = z\lim_{n\to\infty} n^{-z}\prod_1^n\left(1+\frac{z}{n}\right)$$

This is *Weierstrass' definition* of the gamma function. Multiplying the right-hand side of the last equation by:

$$\left[\lim_{n\to\infty} e^{\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)z}\right]\left[\lim_{n\to\infty}\prod_1^n e^{-\frac{z}{n}}\right] = 1$$

we get:

$$\frac{1}{\Gamma(z)} = z\left[\lim_{n\to\infty} e^{\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right)z}\right]\left[\lim_{n\to\infty}\prod_1^n\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right] = ze^{\gamma z}\prod_1^\infty\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

where $\gamma = 0.5772\ldots$ is Euler's constant. Combining the expression for $\Gamma(z)$ with the following (see DE LA VALLÉE POUSSIN, 1937, p.66):

$$\frac{\sin\pi z}{\pi z} = \prod_1^\infty\left(1-\frac{z^2}{n^2}\right)$$

we find:

$$\Gamma(z)\Gamma(-z) = -\frac{1}{z^2}\prod_1^\infty\left(1+\frac{z}{n}\right)^{-1}\left(1-\frac{z}{n}\right)^{-1} = -\frac{1}{z^2}\prod_1^\infty\left(1-\frac{z^2}{n^2}\right)^{-1} = -\frac{\pi}{z\sin\pi z}$$

In combination with [4]:

$$\Gamma(-z) = -\frac{1}{z}\Gamma(1-z)$$

this readily gives [11].

The definitions [1] and [2] often appear in slightly different forms which are equivalent to [1] and [2]. For instance, [1] can be written in the following alternative form:

$$\Gamma(n) = \int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy \qquad [12]$$

Eq.[12] is immediately obtained from [1] by the substitution $e^{-x} = y$.

In addition to the definition [2] the following formula is sometimes useful:

$$B(m, n) = \int_0^\infty \xi^{m-1}(1 + \xi)^{-m-n}d\xi \qquad [13]$$

This is obtained if in [2] we replace the integration variable x by $\xi/(1 + \xi)$:

$$x = \frac{\xi}{1+\xi}; \quad 1 - x = \frac{1}{1+\xi}$$

$$dx = \frac{d\xi}{(1+\xi)^2}$$

Then:

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^\infty \frac{\xi^{m-1}}{(1+\xi)^{m-1}} \frac{1}{(1+\xi)^{n-1}} \frac{d\xi}{(1+\xi)^2} = \int_0^\infty \xi^{m-1}(1+\xi)^{-m-n}d\xi$$

i.e., eq.[13].

The following are some related functions:

(*1*) The *error-function,* which occurs in statistics and in heat conduction problems:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}du \qquad [14]$$

See LEVY and PREIDEL (1945, pp.150–155).

(*2*) The *Fresnel integrals*, which occur in problems of electromagnetic wave motion:

$$\left.\begin{aligned} C(x) &= \int_0^x \cos\left(\frac{1}{2}\pi u^2\right)du \\ S(x) &= \int_0^x \sin\left(\frac{1}{2}\pi u^2\right)du \end{aligned}\right| \qquad [15]$$

See FRANK and VON MISES (1935, pp.850–853), and OFFICER (1958, pp.272–273). Numerical values of these and other functions can be found in JAHNKE, EMDE and LÖSCH (1960), WEAST (1964), DWIGHT (1957) and others.

Exercises on the use of beta- and gamma-functions will be found in several following chapters, especially in section 11.2.

Chapter 2

CONTOUR INTEGRATION AND CONFORMAL TRANSFORMATION

2.1 CONTOUR INTEGRATION IN THE COMPLEX PLANE

The elements of the theory of complex variables and complex functions are supposed known. Fig.4 will be enough to remind the reader of how a complex number $z = x + iy$ is represented in an *Argand diagram* and also of the definitions of *modulus* and *argument* of z. A *closed* curve (C or C') is called a *contour.* A contour is said to be *simple* if it has no multiple points (e.g., a circle or an ellipse are simple contours.) Concerning motion around a contour, we have to distinguish between (*1*) a contour which does not surround the origin (C): both $|z|$ (modulus) and $\arg z$ return to their original values; (*2*) a contour which surrounds the origin (C'): $|z|$ returns to the original value, but $\arg z$ returns to its original value $\pm 2\pi$ depending on the sense of the rotation. Counter-clockwise rotation is taken as positive, by definition.

If $z = x + iy$ and $w = u + iv$ are two complex variables, such that to every value of z, there corresponds one and only one value of w, then w is a *uniform function* of z, i.e., a function of a complex variable. For example: $w = z^2$ or $u + iv = x^2 - y^2 + 2ixy$. Here u and v are themselves real functions of the two real variables x, y: $u(x, y)$, $v(x, y)$, continuous and differentiable.

Then, if z is given a small increment $\delta z = \delta x + i\delta y$, the corresponding increment in w is $\delta u + i\delta v$ and is also small. In other words, if a point P describes a continuous curve in the z-plane between two points P and P_1, then Q describes a continuous curve

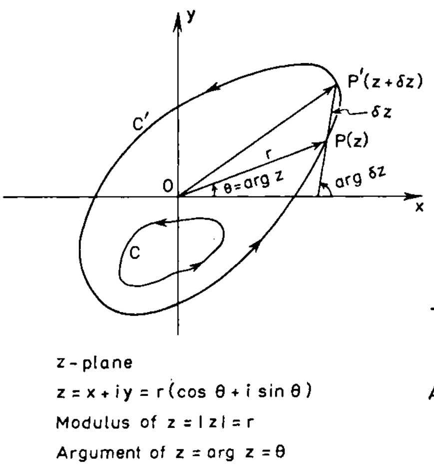


Fig.4.

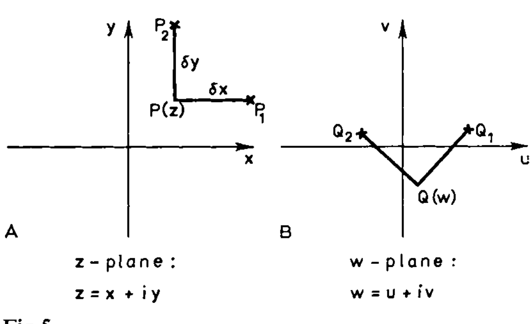


Fig.5.

in the w-plane between the corresponding points Q and Q_1 (see Fig.5):

(*1*) Consider the increment PP_1:

$$P_1\colon\ x + \delta x + iy; \qquad \delta z = \delta x$$

corresponding to QQ_1:

$$Q_1\colon\ w_1 = u(x + \delta x, y) + iv(x + \delta x, y)$$

Then the ratio of the increments is:

$$\frac{w_1 - w}{\delta z} = \frac{u(x + \delta x, y) - u(x, y)}{\delta x} + \frac{i[v(x + \delta x, y) - v(x, y)]}{\delta x}$$

and for $\delta x \to 0$:

$$\lim_{\delta x \to 0} \frac{w_1 - w}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad [1]$$

(*2*) Consider the increment PP_2:

$$P_2\colon\ x + iy + i\delta y; \qquad \delta z = i\delta y$$

corresponding to QQ_2

$$Q_2\colon\ w_2 = u(x, y + \delta y) + iv(x, y + \delta y)$$

Similarly we have:

$$\frac{w_2 - w}{\delta z} = \frac{u(x, y + \delta y) - u(x, y)}{i\delta y} + \frac{v(x, y + \delta y) - v(x, y)}{\delta y}$$

and for $\delta y \to 0$:

$$\lim_{\delta y \to 0} \frac{w_2 - w}{\delta z} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad [2]$$

In general, [1] and [2] are not equal, i.e., the ratio $\delta w/\delta z$ does not approach a unique limit as δz tends to zero, and it is not possible to extend the idea of a differential coefficient to perfectly general functions of a complex variable.

However, the two limits [1] and [2] are identical if:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad [3]$$

These are the necessary and sufficient conditions obtained by equating real and imaginary parts in [1] and [2]. They are called the *Cauchy-Riemann relations.*

Next we shall demonstrate that $\delta w/\delta z$ approaches a unique limit as δz approaches zero *by any path*, provided [3] is fulfilled.

Proof: z is given a general increment: $z + \delta z = x + \delta x + iy + i\delta y$. In consequence, w becomes:

$$\begin{aligned} w + \delta w &= u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y) \\ &= u(x, y) + iv(x, y) + u_x\delta x + u_y\delta y + iv_x\delta x + iv_y\delta y \end{aligned}$$

using Taylor's theorem, retaining terms only of the first order ($u_x = \partial u/\partial x$, etc.). Hence, as $\delta x \to 0$ and $\delta y \to 0$ independently, the limit of $\delta w/\delta z$ is:

$$\frac{(u_x + iv_x)\delta x + (u_y + iv_y)\delta y}{\delta x + i\delta y} = \frac{(u_x + iv_x)\delta x + (-v_x + iu_x)\delta y}{\delta x + i\delta y} = u_x + iv_x = v_y - iu_y$$

using conditions [3]. This means that $\delta w/\delta z$ tends to a unique limit as δz tends to zero in any manner: this limiting value is defined as the differential coefficient of w with respect to z and is denoted $\mathrm{d}w/\mathrm{d}z$. The function w is said to be *monogenic*. We shall consider only such functions.

Conjugate functions. If $u + iv = f(x + iy)$ where $f(z)$ is monogenic, then $u(x, y)$ and $v(x, y)$ are called conjugate functions. Eq.[3] gives:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

from which we get by partial differentiation:

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}$$

Both u and v satisfy the equation:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad [4]$$

i.e., Laplace equation in two dimensions.

Holomorphic functions. If within a region S of the z-plane, a function $f(z)$ is (*1*) single-valued, (*2*) finite and continuous, (*3*) monogenic, it is called *holomorphic* over the region S. The terms *regular* or *analytic* are used in the same sense.

Singularities. The function $1/(z - a)$ is holomorphic except at $z = a$. This is a *singular point* and the function has a *singularity* there.

If $f(z)$ is singular at $z = a$, but a positive integer n can be found such that $(z - a)^n \cdot f(z)$ approaches a limit, other than zero, as z approaches a, the function $f(z)$ is said to have a *pole of order n* at the point $z = a$. $1/(z - a)$ has thus a pole of order unity or a *simple pole* at $z = a$. A pole is sometimes called an *accidental singularity*.

Quite distinct is the case of an *essential singularity*, for which no n with the specified properties exists. For example $e^{1/z}$ has an essential singularity at $z = 0$:

(*1*) Approaching $z = 0$ from positive real axis: $e^{1/z} \to$ infinity.

(*2*) Approaching $z = 0$ from negative real axis: $e^{1/z} \to$ zero.

(*3*) Approaching $z = 0$ from positive imaginary axis: $e^{1/z} \to$ no definite limit.

The last point is seen from the following expression:

$$e^{1/z} = e^{1/(x+iy)} = e^{1/(iy)} = e^{-i/y} = \cos(1/y) - i\sin(1/y)$$

(along imaginary axis)

which does not approach any definite limit as $y \to 0$.

2.1.1 Curvilinear integrals

The curvilinear integral between two points A and B in the xy-plane:

$$\int_{AB} (p\mathrm{d}x + q\mathrm{d}y) \tag{5}$$

where $p(x, y)$ and $q(x, y)$ are continuous functions of x and y, is *defined* as equal to the ntegral:

$$\int_{t_0}^{t_1} \left(p \frac{\mathrm{d}x}{\mathrm{d}t} + q \frac{\mathrm{d}y}{\mathrm{d}t} \right) \mathrm{d}t$$

with the expression within brackets being a function of t, as x and y are supposed to be real functions of the parameter t, with continuous derivatives. For example $x = at^2$, $y = 2at$, then the curve is an arc of a parabola.

The definition gives:

$$\int_{AB} (p\mathrm{d}x + q\mathrm{d}y) = -\int_{BA} (p\mathrm{d}x + q\mathrm{d}y)$$

Now suppose $f(z) = u + iv$ is a function of the complex variable $z = x + iy$, then: $f(z)\mathrm{d}z = (u + iv)(\mathrm{d}x + i\mathrm{d}y) = (u\mathrm{d}x - v\mathrm{d}y) + i(v\mathrm{d}x + u\mathrm{d}y)$ and the integral:

$$\int_{AB} f(z)\mathrm{d}z$$

is *defined* as equivalent to:

$$\int_{AB} (u\mathrm{d}x - v\mathrm{d}y) + i \int_{AB} (v\mathrm{d}x + u\mathrm{d}y) \tag{6}$$

AB is the *path of integration*. If this is a closed curve, the integral is called a *contour integral*.

2.1.2 Stokes' theorem

Consider a simple contour, which is met in not more than two points by any straight line parallel to either of the axes (Fig.6). Consider the integral I over the enclosed area:

$$I = \int\int \frac{\partial p}{\partial y} \mathrm{d}x\mathrm{d}y$$

where $p(x, y)$ and $\partial p/\partial y$ are continuous at all points within and on the contour C. Integrate with respect to y:

$$I = \int_{x_1}^{x_2} [p(x, y_2) - p(x, y_1)]\mathrm{d}x$$

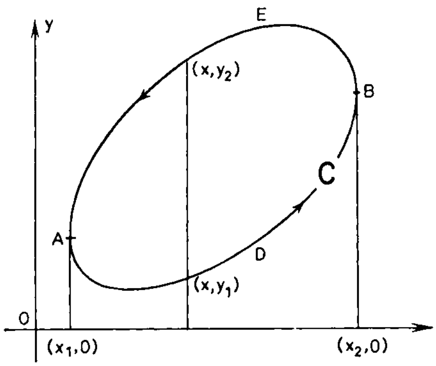


Fig.6.

Then, evaluate the following curvilinear integral in positive sense of rotation:

$$\int_C p\mathrm{d}x = \int_{ADB} p\mathrm{d}x + \int_{BEA} p\mathrm{d}x = \int_{ADB} p\mathrm{d}x - \int_{AEB} p\mathrm{d}x$$

$$= \int_{x_1}^{x_2} [p(x, y_1) - p(x, y_2)]\mathrm{d}x = -I = -\int\int \frac{\partial p}{\partial y} \mathrm{d}x\mathrm{d}y$$

Similarly, if $q(x, y)$ and $\partial q/\partial x$ are continuous at all points within and on the contour C:

$$\int_C q\mathrm{d}y = \int\int \frac{\partial q}{\partial x} \mathrm{d}x\mathrm{d}y$$

Adding the two results, we get *Stokes' theorem*:

$$\int_C (p\mathrm{d}x + q\mathrm{d}y) = \int\int \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) \mathrm{d}x\mathrm{d}y \qquad [7]$$

The same theorem is valid for more complicated contours, as seen by splitting these up into simple contours like the one considered here.

2.1.3 *Cauchy's theorem*

Let $f(z) = u + iv$ be a holomorphic function of z at all points within and on a contour C in the z-plane. Using eq.[6] and Stokes' theorem [7], we find:

$$\int_C f(z)\mathrm{d}z = \int_C (u\mathrm{d}x - v\mathrm{d}y) + i\int_C (v\mathrm{d}x + u\mathrm{d}y) = \int\int (-v_x - u_y)\mathrm{d}x\mathrm{d}y + i\int\int (u_x - v_y)\mathrm{d}x\mathrm{d}y$$

But, $f(z)$ is holomorphic, i.e., $u_x = v_y$; $u_y = -v_x$. Then the double integrals vanish, and we get *Cauchy's theorem*:

$$\int_C f(z)\mathrm{d}z = 0 \qquad [8]$$

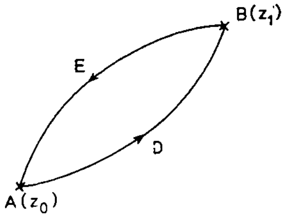


Fig.7.

As a corollary we find that:

$$\int_{z_0}^{z_1} f(z)\mathrm{d}z$$

is the same for any two paths between z_0 and z_1 provided they do not pass through or enclose any singularity of $f(z)$. This is seen by applying [8] to the contour in Fig.7:

$$\int_{ADB} f(z)\mathrm{d}z + \underbrace{\int_{BEA} f(z)\mathrm{d}z}_{= -\int_{AEB} f(z)\mathrm{d}z} = 0$$

Thus:

$$\int_{ADB} f(z)\mathrm{d}z = \int_{AEB} f(z)\mathrm{d}z$$

2.1.4 Rational functions

A function of the form $P(z)/Q(z)$ where P and Q are polynomials in z, is a rational function, by definition. Suppose $P(z)$ and $Q(z)$ are of degrees m and n respectively, and that $m \geqslant n$, then we can divide and find:

$$\frac{P(z)}{Q(z)} = F(z) + \frac{G(z)}{Q(z)}$$

$F(z)$ is of degree $m - n$ and $G(z)$ of a degree $< n$.

The polynomial $Q(z)$ may be factored:

$$Q(z) = k(z-\alpha)^a(z-\beta)^b \ldots (z-\varrho)^r$$

The only singularities of the rational function $P(z)/Q(z)$ in the finite part of the z-plane are at the points $\alpha, \beta, \ldots, \varrho$, where it has poles of orders, $a, b, \ldots, r$ respectively. Using ordinary methods of resolution into partial fractions (see, e.g., DE LA VALLÉE POUSSIN, 1938, p.106 and following) we find that:

$$\begin{aligned}\frac{P(z)}{Q(z)} = F(z) &+ A_1(z-\alpha)^{-1} + A_2(z-\alpha)^{-2} + \ldots + A_a(z-\alpha)^{-a}\\ &+ B_1(z-\beta)^{-1} + B_2(z-\beta)^{-2} + \ldots + B_b(z-\beta)^{-b}\\ &+ \ldots\\ &+ R_1(z-\varrho)^{-1} + R_2(z-\varrho)^{-2} + \ldots + R_r(z-\varrho)^{-r}\end{aligned}$$

where $F(z) = 0$ if $m < n$, but a polynomial of degree $m - n$ if $m > n$.

With centre at the point α, describe a circle C in the z-plane, whose radius R is less than the distance between α and the nearest of the points $\beta, \gamma, \ldots, \varrho$. Then, within and on this circle, the function:

$$\varphi(z) = \frac{P(z)}{Q(z)} - A_1(z-\alpha)^{-1} - A_2(z-\alpha)^{-2} - \ldots - A_a(z-\alpha)^{-a}$$

is holomorphic and the integral around the circle vanishes by Cauchy's theorem [8].

If z is any point on the circle, we have $z - \alpha = Re^{i\theta}$ and $\mathrm{d}z = iRe^{i\theta}\mathrm{d}\theta$ from which we have:

$$\int_C A_1(z-\alpha)^{-1}\mathrm{d}z = \int_0^{2\pi} iA_1\mathrm{d}\theta = 2\pi i A_1$$

Moreover, as $(z-\alpha)^s = R^s e^{i\theta s}$ we have:

$$\int_C A_s(z-\alpha)^{-s}\mathrm{d}z = \int_0^{2\pi} iA_s R^{1-s} e^{(1-s)i\theta}\mathrm{d}\theta = \frac{A_s R^{1-s}}{1-s}\left[e^{(1-s)i\theta}\right]_0^{2\pi} = 0$$

$$s = 2, 3, \ldots, a$$

noting that:

$$e^{i2\pi} = \underbrace{\cos 2\pi}_{=+1} + \underbrace{i\sin 2\pi}_{=0}$$

Therefore:

$$\int_C \varphi(z)\mathrm{d}z = \int_C \frac{P(z)}{Q(z)}\mathrm{d}z - 2\pi i A_1 = 0$$

and:

$$\int_C \frac{P(z)}{Q(z)}\mathrm{d}z = 2\pi i A_1 \qquad [9]$$

The constant A_1 is defined as the *residue* of the function at the pole α. Note that if α is a simple pole, A_1 is the limit of $(z-\alpha)[P(z)/Q(z)]$ as z tends to α:

$$A_1 = \lim_{z\to\alpha}(z-\alpha)\frac{P(z)}{Q(z)} = \frac{P(\alpha)}{Q'(\alpha)} \qquad [10]$$

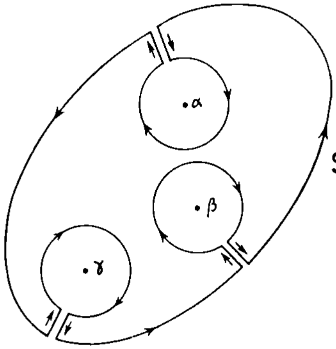


Fig.8.

the last expression for A_1 being obtained by applying l'Hospital's rule (DE LA VALLÉE POUSSIN, 1938, pp.93–96).

We generalize this to any number of poles (Fig.8) with residues A_1, B_1, . . ., L_1. The function $f(z) = \dfrac{P(z)}{Q(z)}$ is then holomorphic at all points of the region between S and the circles and therefore, with the integration path as shown in Fig.8, using Cauchy's theorem [8], we have:

$$\int f(z)\mathrm{d}z = 0$$

This integral is composed of:

$$\int_S f(z)\mathrm{d}z$$

along the surrounding contour S and the integrals around each pole, for which we already have an expression by the residue theorem. Considering the sense of rotation, we have then:

$$\int_S f(z)\mathrm{d}z = 2\pi i(A_1 + B_1 + \ldots + L_1) \tag{11}$$

i.e., the integral taken in the positive sense around any contour is equal to $2\pi i$ multiplied by the sum of the residues at the poles within the contour.

Taylor and Laurent series for rational functions. The point $z = c$ (finite) is assumed not to be a zero of $Q(z)$. With $z = c$ as centre draw a circle S with radius equal to the distance to the nearest zero of $Q(z)$. Then the function $P(z)/Q(z)$ is holomorphic at all points within (but not on) S. If z is any point within S:

$$(z-a)^{-k} = [(z-c)+(c-a)]^{-k} = (c-a)^{-k}\left[1+\frac{z-c}{c-a}\right]^{-k}$$

$$= (c-a)^{-k}\left[1+\sum_{r=1}^{\infty}A_r(z-c)^r\right]$$

where k is any one of the integers, 1, 2, 3, . . ., a and:

$$A_r = -k(-k-1)(-k-2)\ldots(-k-r+1)/r!(c-\alpha)^r$$

We have used the binomial expansion $(1+b)^n = 1 + (n/1!)b + [n(n-1)/2!]b^2 + \ldots + b^n$ which is valid, since $|(z-c)/(c-\alpha)| < 1$. Applying the same development to all other terms in the partial fraction expression for $P(z)/Q(z)$ we have:

$$\frac{P(z)}{Q(z)} = \sum_{r=0}^{\infty} C_r(z-c)^r \qquad [12]$$

called a *Taylor series.* z lies within the circle S. By putting $r = 0$ in the sum we include also the constant term.

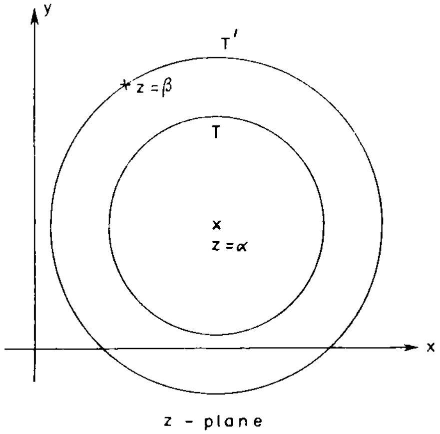


Fig.9.

Now, instead of an ordinary point c, consider one of the poles, say α. Draw two circles T and T' with their centre at $z = \alpha$, the radius of the outer circle being equal to the distance between α and the nearest of the other zeros $(\beta, \gamma, \ldots)$ of $Q(z)$—see Fig.9. Then the rational function:

$$\frac{P(z)}{Q(z)} - A_1(z-\alpha)^{-1} - A_2(z-\alpha)^{-2} - \ldots - A_a(z-\alpha)^{-a}$$

is holomorphic at all points within T' and can be expanded in a Taylor series:

$$\sum_{r=0}^{\infty} C_r(z-\alpha)^r$$

Therefore, if z lies within the annulus bounded by T and T':

$$\frac{P(z)}{Q(z)} = \sum_{s=1}^{a} A_s(z-\alpha)^{-s} + \sum_{r=0}^{\infty} C_r(z-\alpha)^r \qquad [13]$$

This is called a *Laurent series*. The terms containing the negative powers of $(z - a)$, i.e.:

$$\sum_{s=1}^{a} A_s(z - a)^{-s}$$

are said to form the *principal part* of the series.

Behaviour of a rational function at infinity. Consider the function $f(z) = P(z)/Q(z)$, substitute Z for $1/z$, and see how $f(1/Z)$ behaves as $Z \to 0$. Two cases can be distinguished:

(*1*) The degree of $P(z)$ is not greater than that of $Q(z)$; $f(1/Z)$ is finite when $Z = 0$ and can be expanded in a Taylor series:

$$f\left(\frac{1}{Z}\right) = A_0 + A_1 Z + A_2 Z^2 + \dots$$

when $|Z|$ is sufficiently small, and therefore:

$$f(z) = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots$$

when $|z|$ is sufficiently great. Since $f(z)$ is finite, when z is infinite, the point at infinity is said to be an *ordinary point* of the function.

(*2*) The degree of $P(z)$ is greater than that of $Q(z)$; $f(1/Z)$ becomes infinite when $Z = 0$, i.e., it has a pole of order p, say. We then get by Laurent's theorem [13], within an annulus with its centre at $Z = 0$, R being the outer radius in the Z-plane, i.e., $|Z| < R$:

$$f\left(\frac{1}{Z}\right) = \sum_{s=1}^{p} B_s Z^{-s} + \sum_{r=0}^{\infty} C_r Z^r$$

or:

$$f(z) = \sum_{s=1}^{p} B_s z^s + \sum_{r=0}^{\infty} C_r z^{-r} \tag{14}$$

when $|z| > R$. In this case we say that the function has a *pole of order p at infinity*, the *principal part* of the expansion there being:

$$\sum_{s=1}^{p} B_s z^s$$

Analogous results for functions in general. For any function, not necessarily rational, we have corresponding formulas, not proved here:

Taylor's theorem. $f(z)$ is assumed holomorphic at all points within a circle of radius r and centre $z = a$. Then:

$$f(z) = f(a) + (z - a)f_1(a) + \dots + (z - a)^n f_n(a)/n! + \dots \tag{15}$$

when $|z - a| < r$.

Laurent's theorem. $f(z)$ is assumed holomorphic at all points within an annulus bounded by two circles, with $z = a$ as common centre, and with radii R, r, such that r can be made as small as we please; then we have for any point z within the annulus:

$$f(z) = \sum_{n=0}^{\infty} A_n(z-a)^n + \sum_{n=1}^{\infty} B_n(z-a)^{-n} \qquad [16]$$

If B_n vanishes when n exceeds s, but B_s is not zero, the point a is a pole of order s and the residue is B_s. If an infinite number of the coefficients B are different from zero, the point a is said to be an *essential singularity.*

2.1.5 The contour integration theorem. Calculus of residues

$f(z)$ is assumed to be holomorphic at all points on a simple (closed) contour C, enclosing no singularities of $f(z)$ other than poles. Then we have by eq.[11]:

$$\int_C f(z)\mathrm{d}z = 2\pi i \sum \mathrm{Res} \qquad [17]$$

where $\sum \mathrm{Res}$ is the sum of the residues at the poles within C, the integral being taken in the positive sense around C.

Contour integration in the complex plane offers an extremely powerful method for the calculation of definite integrals. We shall illustrate this by a few typical cases.

Integration around the unit circle. Consider integrals of the type:

$$\int_0^{2\pi} \varphi(\cos\theta, \sin\theta)\mathrm{d}\theta \qquad [18]$$

where $\varphi(\cos\theta, \sin\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$. Write:

$$z = \mathrm{e}^{i\theta}; \qquad \mathrm{d}z = i\mathrm{d}\theta \mathrm{e}^{i\theta} = iz\mathrm{d}\theta; \qquad |z| = 1$$

and:

$$\left.\begin{aligned} z &= \cos\theta + i\sin\theta \\ \frac{1}{z} &= \mathrm{e}^{-i\theta} = \cos\theta - i\sin\theta \end{aligned}\right\}$$

which gives:

$$\left.\begin{aligned} \cos\theta &= \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \sin\theta &= \frac{1}{2i}\left(z - \frac{1}{z}\right) \end{aligned}\right\}$$

and so finally:

$$\int_0^{2\pi} \varphi(\cos\theta, \sin\theta)\mathrm{d}\theta = \int_C \psi(z)\mathrm{d}z = 2\pi i \sum \mathrm{Res}$$

where $\psi(z)$ is a rational function of z and C is the unit circle $|z| = 1$, and $\sum \mathrm{Res}$ is the sum of the residues of $\psi(z)$ at its poles inside C.

Evaluation of a special type of infinite integral. $Q(z)$ satisfies the following conditions:

(*1*) It is meromorphic in the upper half-plane, i.e., the only singularities are poles (no essential singularity).

(*2*) There are no poles on the real axis.

(*3*) $zQ(z) \to 0$ uniformly as $|z| \to \infty$ for $0 \leqslant \arg z \leqslant \pi$.

(*4*) $\int\limits_0^\infty Q(x)\mathrm{d}x$ and $\int\limits_{-\infty}^0 Q(x)\mathrm{d}x$ both converge. Then:

$$\int\limits_{-\infty}^{\infty} Q(x)\mathrm{d}x = 2\pi i \sum \mathrm{Res} \qquad [19]$$

where $\sum$ Res is the sum of the residues of $Q(z)$ at its poles in the upper half-plane.

Proof: Use a contour as in Fig.10. Choose R large enough to include all poles. Then, by the residue theorem [17]:

$$\int\limits_{-R}^{R} Q(x)\mathrm{d}x + \int\limits_{\Gamma} Q(z)\mathrm{d}z = 2\pi i \sum \mathrm{Res}$$

From condition (*3*), if R be large enough, $|zQ(z)| < \varepsilon$ for all points on Γ. Then:

$$\left| \int\limits_{\Gamma} Q(z)\mathrm{d}z \right| = \left| \int\limits_0^\pi Q(Re^{i\theta})Re^{i\theta}\mathrm{d}\theta \right| < \varepsilon \int\limits_0^\pi \mathrm{d}\theta = \pi\varepsilon$$

(putting $z = Re^{i\theta}$.) Let $R \to \infty$, then by condition (*4*):

$$\int\limits_{-\infty}^{\infty} Q(x)\mathrm{d}x = 2\pi i \sum \mathrm{Res}$$

which should be proved. The theorem can be extended to the case that $Q(z)$ has simple poles on the real axis.

Integrals involving many-valued functions. Riemann surface. Branch point, branch cut

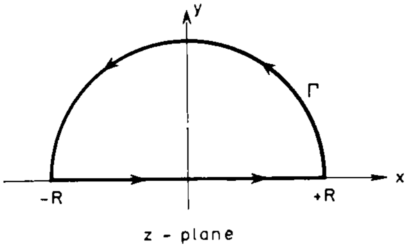


Fig.10.

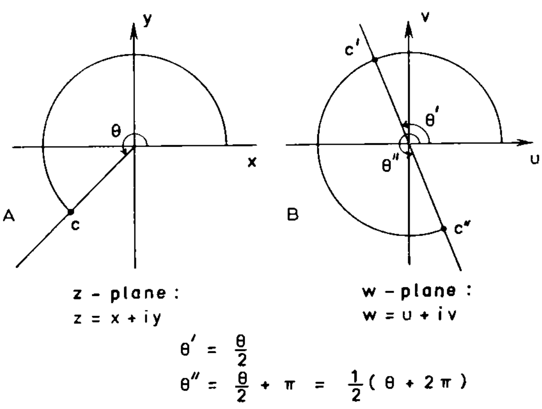


Fig.11.

and branch line. Consider the following transformation between the w- and z-planes:

$$w = z^{1/2} = \sqrt{r}\, e^{i\theta/2} \tag{20}$$

This is a *many-valued* or *multi-valued* function, as one point in the $z(x, y)$-plane is mapped in more than one point in the $w(u, v)$-plane. Thus in this case, one point c in the xy-plane is mapped in *two* points c' and c'' in the uv-plane (see Fig.11).

We shall make the mapping single-valued such that c shall correspond to c' and not to c''. Or differently expressed, we shall define the mapping such that a closed curve in the z-plane will transform into a closed curve in the w-plane, i.e., when the point has described the closed curve in the z-plane, the corresponding point in the w-plane shall also return to its original value.

As remarked already in the beginning of section 2.1, two cases can be distinguished (see Fig.4):

(*1*) the closed curve (C) in the z-plane does not encircle the origin: then the requirement of single-valuedness is fulfilled immediately.

(*2*) the closed curve (C') does encircle the origin in the z-plane: then θ changes by 2π for every closed contour around the origin, and the single-valuedness is not fulfilled.

It is necessary that the function $f(z)$ is single-valued when Cauchy's theorem [17] is applied. The single-valuedness is restored in the latter case, if we never encircle the origin. It must be avoided and this is done by having a *branch cut* from $z = 0$ to infinity and not crossing it, but instead letting the integration path follow a *branch line* around the cut (see Fig.12). The point $z = 0$ is called a *branch point*. The branch cut can be placed along any line from $z = 0$ to infinity, but it is customary to place it along the real axis.

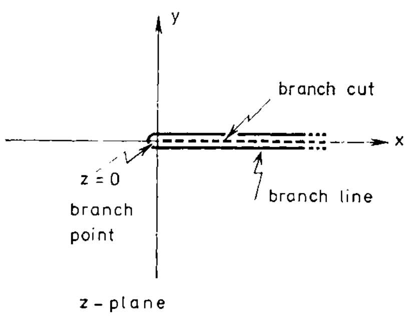


Fig.12.

The z-plane, when cut in this way, is called a *Riemann sheet* of the function $w(z)$. The contour in Fig.13 in the upper sheet corresponds to the upper half-circle in the w-plane. After passing along this contour, we make another passage, but now on the lower sheet in the z-plane, and this second passage corresponds to the lower half-circle in the w-plane. The circuits are made in alphabetical order *abcdef* and the corresponding points in the w-plane are $a'b'c'd'e'f'$. We pass from one sheet to the other at the branch cut. In this way we get a single-valued transformation from the z-plane to the w-plane. The two sheets so joined are called a *Riemann surface*.

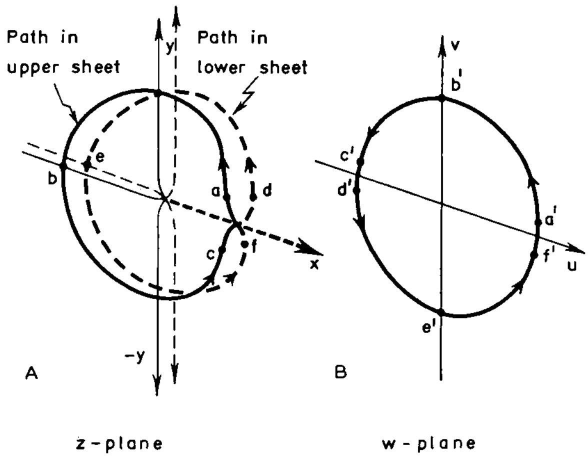


Fig.13.

Thus, a Riemann surface consists of several sheets (Riemann sheets), which meet along branch cuts. On the branch cuts, passage may be made from one sheet to another, without invalidating the single-valued transformation.

We consider a few functions from this viewpoint:

$$(1) \quad w = (z - a)^{1/2} \qquad [21]$$

(see Fig.14). The Riemann surface again consists of two sheets, but the branch point is now at $z = a$ instead of at $z = 0$.

$$(2) \quad w = (z - a)^{1/2}(z - b)^{1/2} \qquad [22]$$

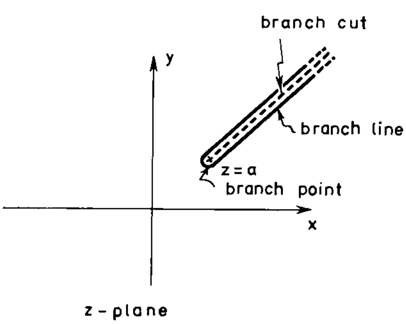


Fig.14.

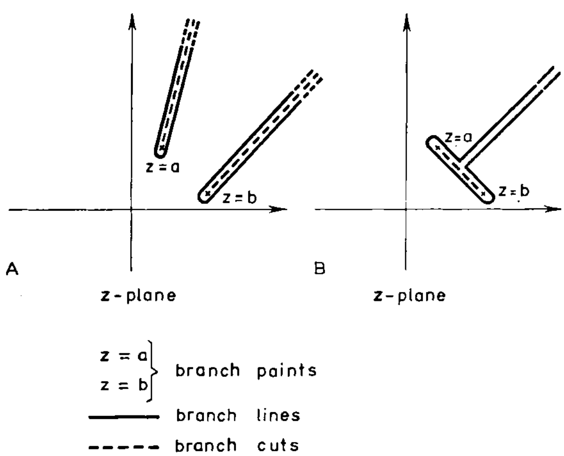


Fig.15.

(see Fig.15). There are two branch points: $z = a$ and $z = b$, and the branch cuts can be made in two ways, as illustrated in Fig.15.

(3) $\quad w = z^{1/n}$ [23]

This is a generalization of the originally considered case $w = z^{1/2}$, but now the Riemann surface consists of n sheets. This is immediately seen as with $z = re^{i\theta}$ eq.[23] becomes $w = r^{1/n}e^{i\theta/n}$, which means that one complete revolution in the z-plane corresponds only to an angle of θ/n in the w-plane. Thus, n revolutions are needed in the z-plane to perform one complete revolution in the w-plane. But, to guarantee a single-valued transformation from z to w, each one of these n revolutions has to be made on a different sheet. There are thus n sheets which in this case together make up the Riemann surface. All these sheets intersect along the branch cut.

(4) $\quad w = \log z$ [24]

Again with $z = re^{i\theta}$ [24] becomes:

$$w = \log z = \log r + i\theta = \log r + i(\theta + k \cdot 2\pi)$$

which holds for k being an integer. This follows immediately from the fact that:

$$e^{i\theta} = e^{i(\theta + k \cdot 2\pi)}$$

which is seen to be correct by applying Euler's formula to its two members. Every integer k gives a branch, so that $\log z$ is an infinitely many-valued function of z. The Riemann surface consists of infinitely many sheets, superposed on each other, with a common cut along the positive real axis in the z-plane. For each branch, there corresponds a strip of width 2π in the imaginary direction in the w-plane. There is in this case no return to the starting point.

(5) $\quad w = \sin(z^{1/2})$ is another example of a many-valued function. It is left as an exercise to the reader to study this function in the same way as above.

(6) $\quad \int\limits_0^\infty x^{a-1}Q(x)\mathrm{d}x$ [25]

where a is not an integer. x is a real variable, but the integral can easily be evaluated by replacing x by the complex variable $z = x + iy$ and integrating along a contour in the z-plane. Only, we have again to remember that z^{a-1} is a many-valued function and we have to choose a contour as shown in Fig.16. We assume that the integrand tends to zero both for $z = 0$ and $z = \infty$, and apply Cauchy's theorem [17]:

$$\int\limits_{AB} x^{a-1}Q(x)\mathrm{d}x + \underbrace{\int\limits_{\Gamma} z^{a-1}Q(z)\mathrm{d}z}_{=0} + \int\limits_{CD} z^{a-1}Q(z)\mathrm{d}z + \underbrace{\int\limits_{\gamma} z^{a-1}Q(z)\mathrm{d}z}_{=0} = 2\pi i \sum \mathrm{Res}$$

where $\sum \mathrm{Res}$ is the sum of the residues of the integrand inside the contour. We are left with the *branch line integrals*, along the upper and lower side of the real axis:

Upper side: $z = re^{i\theta} = x; \quad z^{a-1} = x^{a-1}$.

Lower side: $z = re^{i\theta} = xe^{i \cdot 2\pi}; \quad z^{a-1} = x^{a-1}e^{2\pi i(a-1)} = -x^{a-1}e^{2\pi ia}$.

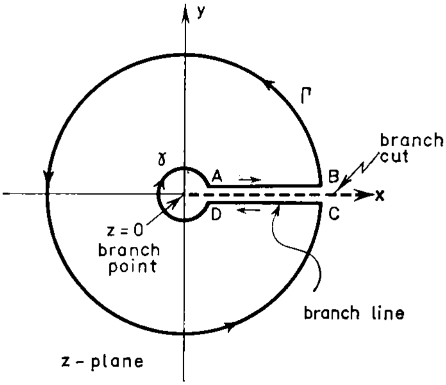


Fig.16.

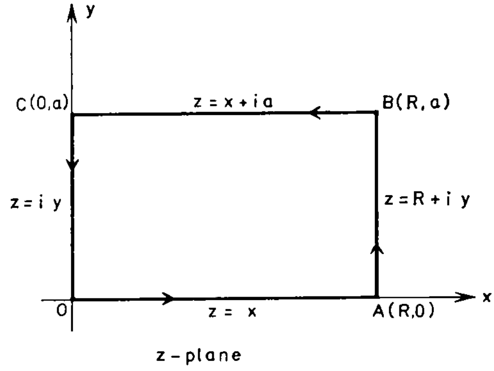


Fig.17.

Insert into the Cauchy formula above, and we arrive at the result:

$$\int_{AB} x^{a-1}Q(x)\mathrm{d}x - \int_{DC} x^{a-1}e^{2\pi ia}Q(x)\mathrm{d}x = 2\pi i\sum \mathrm{Res}$$

$$\int_0^\infty x^{a-1}Q(x)\mathrm{d}x = \frac{2\pi i\sum \mathrm{Res}}{1-e^{2\pi ia}} \tag{26}$$

Deducing integrals from known integrals. Circles and semi-circles are not better than other contours for integration. We consider now a case with a rectangular contour. Prove that:

$$\int_0^\infty e^{-x^2}\cos 2ax\ \mathrm{d}x = \frac{1}{2}\sqrt{\pi}\ e^{-a^2} \tag{27}$$

by integrating e^{-z^2} around the rectangle whose vertices are 0, R, $R + ia$, ia (Fig. 17). e^{-z^2} has no poles within or on this contour, and therefore we have by Cauchy's theorem [17]:

$$\int_0^R e^{-x^2}\mathrm{d}x + \int_0^a e^{-(R+iy)^2}i\mathrm{d}y \quad + \quad \int_R^0 e^{-(x+ia)^2}\mathrm{d}x \quad + \int_a^0 e^{y^2}i\mathrm{d}y = 0$$

Hence:

$$\int_0^R e^{-x^2}\ \mathrm{d}x + i\int_0^a e^{-R^2}e^{-2iRy+y^2}\mathrm{d}y - e^{a^2}\int_0^R e^{-x^2}(\cos 2ax - i\sin 2ax)\mathrm{d}x - i\int_0^a e^{y^2}\mathrm{d}y = 0 \tag{28}$$

Now:

$$\left| i\int_0^a e^{-R^2}e^{-2iRy+y^2}\mathrm{d}y\right| < e^{-R^2}\cdot e^{a^2}\cdot a$$

as:

$$\int_0^a \mathrm{d}y = a$$

and $|e^{-2iRy}| = 1$ and so this integral $\to 0$ as $R \to \infty$. Using the result that:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

(DE LA VALLÉE POUSSIN, 1938, pp. 229–230) we find on making $R \to \infty$ and equating *real* parts in eq.[28]:

$$\int_0^\infty e^{-x^2} \cos 2ax \, dx = \frac{\sqrt{\pi}}{2} e^{-a^2}$$

which should be proved.

Some practical rules for contour integration

(*1*) In a given integral, let the integration variable be complex. The integration is carried out along a (closed) contour in this complex plane.

(*2*) The given integral (e.g., from $-\infty$ to $+\infty$ or from 0 to $+\infty$) is made a part of the contour. Another part is frequently made up of a circular arc of infinite radius.

(*3*) Look for branch points and branch cuts, following the rules on the preceding pages. For example, if the integrand contains a factor $= (z - a)^{b/c}$, where b/c is non-integral, then $z = a$ is a branch point and the Riemann surface consists of c sheets. On the other hand, if b/c is integral, then $(z - a)^{b/c}$ is a single-valued function of z.

(*4*) Look for poles of the integrand, i.e., points at which the integrand becomes infinite.

(*5*) Evaluate the integral by Cauchy's theorem [17], remembering contributions from branch lines and from the poles.

Applications of the contour integration method to seismological problems will be given in several subsequent chapters.

2.2 CONFORMAL TRANSFORMATION

The methods of conformal transformation (or representation) are particularly useful because they frequently enable us to deduce the solution of a boundary problem for one closed region A from the solution of a corresponding boundary problem for another region B which is of a simpler type.

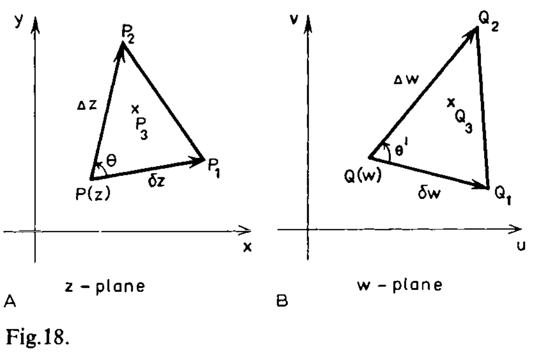


Fig.18.

Two complex variables $w = u + iv$ and $z = x + iy$ are connected by the relation $w = f(z)$ where $f(z)$ is a monogenic function of z (only such functions are considered). Then, there is a one-to-one correspondence between the z- and the w-planes (Fig.18). If PP_1 and PP_2 are sufficiently small, we shall have, because w is monogenic:

$$\frac{\delta w}{\delta z} = \frac{\Delta w}{\Delta z} \text{ or } \frac{\Delta z}{\delta z} = \frac{\Delta w}{\delta w}$$

$$\frac{PP_2}{PP_1} e^{i\theta} = \frac{QQ_2}{QQ_1} e^{i\theta'}$$

as $z = re^{i\theta}$, $z_1 = r_1 e^{i\theta_1}$, etc. Separating into real and imaginary parts, we get:

$$\frac{PP_2}{PP_1} \cos\theta = \frac{QQ_2}{QQ_1} \cos\theta'$$

and:

$$\frac{PP_2}{PP_1} \sin\theta = \frac{QQ_2}{QQ_1} \sin\theta'$$

From these two formulas it follows by squaring and adding:

$$\frac{PP_2}{PP_1} = \frac{QQ_2}{QQ_1}$$

and by dividing: $\theta = \theta'$.

Thus, two sides are proportional and the angle between them is the same, i.e., the two elementary triangles are *similar*. Linear dimensions are in the ratio $1 : |dw/dz|$ and areas in the ratio $1 : |dw/dz|^2$. The factor $|dw/dz|$ is called *magnification*.

If a point moves in the z-plane so as to trace a curve, the locus of the corresponding point in the w-plane is called the *transformed curve*. All angles are conserved as well as the sense of rotation. A closed curve C in the z-plane transforms into a closed curve D in the w-plane, but in general the curves are not similar, as the magnification varies from point to point.

Since *infinitesimal* elements of area are unaltered in shape, the transformation is said to be *conformal*. This is true for ordinary points, but not necessarily at singular points.

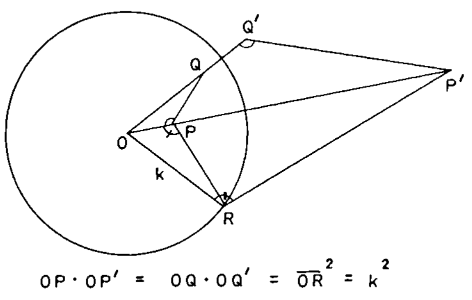


Fig.19.

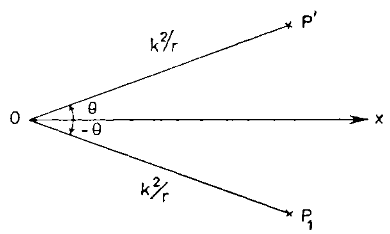


Fig.20.

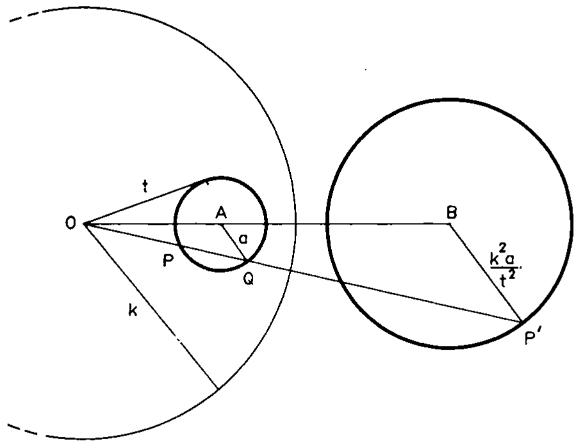


Fig.21.

Inversion with respect to a circle. In Fig.19 we have that $OP \cdot OP' = k^2$; P and P' are *inverse points* with respect to the circle, O is the *centre of inversion* and k is the *radius of inversion* (= radius of the circle). If Q, Q' are another pair of inverse points with respect to the same circle, then the triangles OPQ and $OQ'P'$ are *similar* because $OP/OQ = OQ'/OP'$ and the angle at O is common to the two triangles.

If PQ is an element of a curve, $P'Q'$ is its inverse curve element. Their tangents make equal angles with OP, but in opposite sense. This is true generally for *inverse curves.*

Reflection. In Fig.19 O = origin, P is the point $z = re^{i\theta}$ in the z-plane, then $OP' = k^2/OP$ and P' represents $(k^2/r)e^{i\theta}$ (same direction, i.e., same argument as for P). P_1 is the image (reflection) of P' in the real axis (Fig.20). Then P_1 represents:

$$\frac{k^2}{r} e^{-i\theta} = \frac{k^2}{z}$$

The inverse of a circle is also a circle, except when the centre of inversion is on the circumference, when the inverse is a straight line.

Proof: We have from Fig.21:

$$\frac{OB}{OA} = \frac{OP'}{OQ} = \frac{OP \cdot OP'}{OP \cdot OQ} = \frac{k^2}{t^2}$$

as $OP \cdot OP' = k^2$, $OP \cdot OQ = t^2$, i.e., OB/OA is constant and B a fixed point. Further:

$$\frac{BP'}{AQ} = \frac{OB}{OA} = \text{constant} = \frac{k^2}{t^2}$$

$$BP' = \frac{k^2 a}{t^2}$$

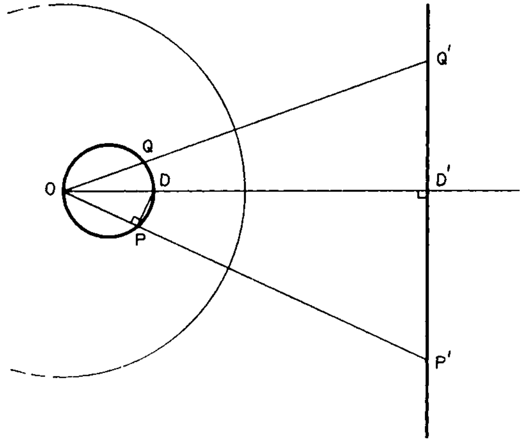


Fig.22.

i.e., the inverse of the circle with centre A and radius a is a circle with centre B and radius k^2a/t^2.

If the centre of inversion O is on the circumference, the inverse curve is a straight line $P'D'$. See Fig.22, where D, D' and P, P' are inverse points and the triangles OPD and $OD'P'$ are similar.

2.2.1 *The reciprocal transformation: the point at infinity*

We shall study the conformal transformation: $w = 1/z$. Instead of considering two different planes (w and z), they are preferably superimposed on each other, giving a one-to-one correspondence between the points z and $1/z$.

Apply inversion with respect to a unit circle, i.e., $k = 1$ (last section). If P is any given point in the z-plane, Q its transformation $1/z$, P' the inverse of P with respect to the unit circle, then Q is the image (reflection) of P' in the real axis. Symbolically we can express this as follows:

$$P(re^{i\theta}) \to Q\left(\frac{1}{r}\,e^{-i\theta}\right) \to P'\left(\frac{k^2}{r}\,e^{i\theta}\right) = P'\left(\frac{1}{r}\,e^{i\theta}\right)$$

as $k = 1$.

If P is inside the circle, Q is outside and vice-versa. Similarly, if $z = 0$, we say that its transformed point is the *point at infinity*, only one point at infinity assumed.

2.2.2 *The bilinear transformation or Möbius transformation*

This is of special interest because it is the only transformation which transforms the *whole* of the z-plane in a one-to-one manner into the whole of the w-plane. It is defined by:

$$Awz + Bw + Cz + D = 0 \qquad [1]$$

where A, B, C, D are constants (generally complex), such that $AD \neq BC$. If $AD = BC$, the relation is of no interest as it would give $z = -B/A$ or else $w = -C/A$. Solve [1] for w:

$$w = -\frac{Cz + D}{Az + B} = -\frac{C}{A} + \frac{BC - AD}{A(Az + B)}$$

which is of the form:

$$w - a = \frac{k}{z - b} \qquad [2]$$

where a, b, k are constants. Write $z - b = re^{i\theta}$ and $k = c^2e^{2i\alpha}$. Then:

$$|w - a| = \frac{c^2}{r}$$

and if $\varphi = \arg(w - a)$, then $\varphi = 2\alpha - \theta$ or $\varphi - \alpha = \alpha - \theta$. The geometrical construction of w when z is given is clarified in Fig.23. P' is the inverse to P:

$$|z' - b| = \frac{c^2}{r}$$

$$\arg(z' - b) = \arg(z - b) = \theta$$

P_1 is the image of P' in relation to the line with argument α:

$$\arg BP_1 = \arg BP' - 2(\theta - \alpha) = \theta - 2\theta + 2\alpha = 2\alpha - \theta$$

$$|BP_1| = \frac{c^2}{r} = |w - a|$$

Obviously, $Q(w)$ is the transformed point, corresponding to the given point P. The bilinear transformation is equivalent to an inversion, a reflection and a translation. Corollary: circles are transformed into circles or straight lines.

Suppose the points z_1, z_2, z_3, z_4 are transformed bilinearly into the points w_1, w_2,

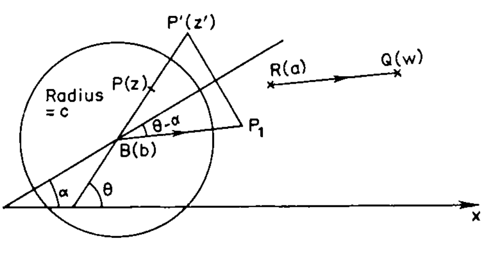


Fig.23.

w_3, w_4, all at finite distance from the origin. Then we get from the solution [2] of w:

$$w_1 - w_3 = -\frac{(BC - AD)(z_1 - z_3)}{(Az_1 + B)(Az_3 + B)}$$

etc. Hence:

$$\frac{w_1 - w_3}{w_2 - w_3} : \frac{w_1 - w_4}{w_2 - w_4} = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4}$$

or:

$$(w_1 w_2 w_3 w_4) = (z_1 z_2 z_3 z_4) \qquad [3]$$

introducing the notation for the *generalized cross-ratio* of four points. The cross-ratio is left unaltered by any bilinear transformation.

Similarly for two sets of three points:

$$(z z_1 z_2 z_3) = (w w_1 w_2 w_3) \qquad [4]$$

This transformation converts the circle passing through z_1, z_2, z_3 into the circle through w_1, w_2, w_3. Eq.[4] may be used to find the particular function which transforms one given circle into any other given circle or straight line.

Double points. The points z and w coincide if:

$$Az^2 + (B + C)z + D = 0 \qquad [5]$$

As this is an equation of the second degree, with two solutions, there are in general two distinct points which are called the *self-corresponding or double points*.

2.2.3 *The transformation $w = z + k^2/z$ with k real and positive*

We have:

$$\frac{\mathrm{d}w}{\mathrm{d}z} = 1 - \frac{k^2}{z^2}$$

which is finite at all points except $z = 0$ and not zero except at $z = \pm k$; the transformation is conformal at all other points than these. As z approaches infinity, w approaches z, and the magnification $|\mathrm{d}w/\mathrm{d}z|$ approaches unity.

Consider the transformation of the circle $|z| = c$, $c > k$ in the z-plane. At any point on this circle we have $z = c\mathrm{e}^{i\theta}$ and therefore:

$$w = u + iv = c\mathrm{e}^{i\theta} + \frac{k^2}{c}\mathrm{e}^{-i\theta} = a\cos\theta + ib\sin\theta$$

where, identifying real and imaginary parts:

$$\left.\begin{aligned} u &= a\cos\theta \\ v &= b\sin\theta \end{aligned}\right\} a = \frac{c^2 + k^2}{c}; \quad b = \frac{c^2 - k^2}{c}$$

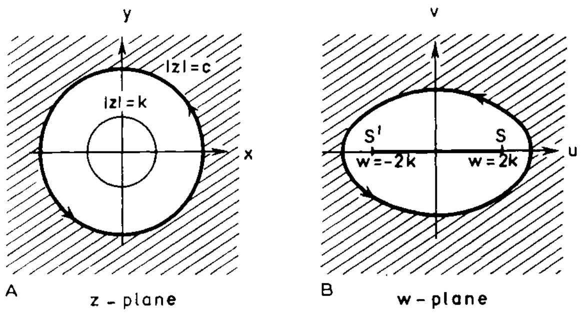


Fig.24.

As θ increases from $-\pi$ to $+\pi$, the point z describes the circle once in counter-clockwise direction and the point w moves once in the same sense around the ellipse:

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

The area outside the circle is transformed into the area outside the ellipse. See Fig.24 where corresponding areas are shaded.

The foci S and S' of the ellipse are given by:

$$w = \pm(a^2 - b^2)^{1/2} = \pm 2k$$

and the corresponding points in the z-plane are $z = \pm k$.

If, in particular, $c = k$, the major axis of the ellipse is $2a = 4k$ and the minor axis $2b$ vanishes, i.e., the ellipse degenerates into the line SS'. The area outside the circle in the z-plane becomes the area of the whole w-plane with an internal boundary SS' which may be regarded as an impassable barrier (as in hydrodynamics) or as a slit in the plane (a fault!). In either case, a point which moves in the plane must avoid crossing the barrier or slit.

As the transformation can be written:

$$z^2 - wz + k^2 = 0$$

it is clear that to any given z corresponds only one w, but to a given w corresponds in general two points z. The product of the roots being k^2, it follows that in general one of the points is inside and the other outside the circle $z = k$.

2.2.4 *The transformation* $w = \log z$

Putting $z = re^{i\theta}$ we have now that:

$$w = u + iv = \log z = \log r + i\theta$$

The annulus in the z-plane corresponds to a rectangle in the w-plane, shaded in Fig.25. The area within either boundary is represented conformally on the other. If $a \to \infty$ and $b \to 0$, the rectangle in the w-plane becomes the doubly infinite strip between the lines $v = \pm\pi$ and this corresponds to the whole of the cut z-plane.

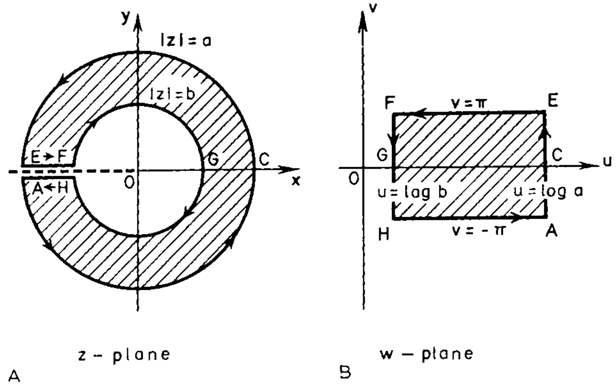


Fig.25.

2.2.5 The transformation $w = \cosh z$

Here $u + iv = \cosh x \cos y + i \sinh x \sin y$ and $u = \cosh x \cos y$; $v = \sinh x \sin y$. If x is constant, the locus of the point w is the ellipse:

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1$$

and if y is constant, the locus of w is the hyperbola:

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1$$

The two curves are confocal, the common foci being at $w = \pm 1$. A rectangular area in the z-plane is transformed into an area in the w-plane bounded by ellipses and hyperbolas (Fig.26).

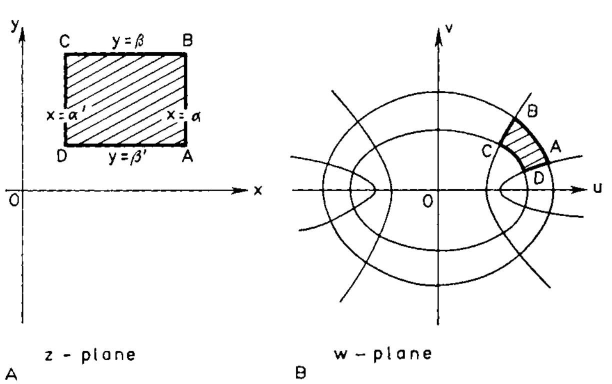


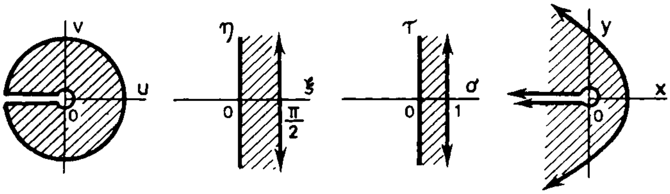
Fig.26.

2.2.6 Successive transformations. Example: $w = \tan^2(1/4\pi\sqrt{z})$

This can be considered as a combination of three successive transformations:

$$w = \tan^2 \frac{\zeta}{2} \quad , \quad \zeta = \frac{\pi}{2} t \quad , \quad t = \sqrt{z}$$

where $w = u + iv$, $\zeta = \xi + i\eta$, $t = \sigma + i\tau$, $z = x + iy$. Fig.27 illustrates these successive transformations, and it is left as an exercise to the reader to convince himself that this picture is correct.



w - plane A ζ - plane B t - plane C z - plane D

Fig.27.

2.2.7 Conformal transformation of a half-plane into a polygon (Schwarz–Christoffel transformation)

Suppose:

$$F(z) = (z - a)^{-\alpha}(z - b)^{-\beta}(z - c)^{-\gamma} \ldots (z - k)^{-\varkappa} \tag{6}$$

where $a, b, c, \ldots, k$ are n real constants arranged in ascending order and $\alpha, \beta, \gamma, \ldots, \varkappa$ are n real constants, each lying between -1 and 1. Consider how the argument of $F(z)$ varies as z moves along the real axis from $-\infty$ to $+\infty$ (see Fig.28A). When z is on the left of a, the argument of each of the numbers $z - a, z - b, z - c, \ldots, z - k$ is π. When z passes through a, the arguments are unaltered except that of the first, which *decreases* by π to 0. This means that the argument of $F(z)$ *increases* by $\alpha\pi$ when z passes a:

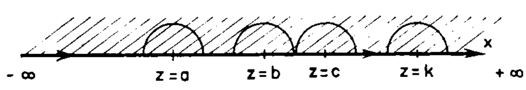
$$\Delta \arg F(z) = \Delta[-\alpha \arg(z - a)] = -\alpha(0 - \pi) = +\alpha\pi \tag{7}$$

And so on for the points $b, c, \ldots, k$. Hence the *total increase* of $\arg F(z)$ is:

$$(\alpha + \beta + \gamma + \ldots + \varkappa)\pi \tag{8}$$

Since the upper half of the z-plane is on the left of an observer moving with z, the *interior* of the polygon is the corresponding area in the w-plane (see Fig.28B). U in the w-plane corresponds to $z = \infty$. It is only at the vertices of the polygon that the transformation is not conformal; these are the only points at which $\mathrm{d}w/\mathrm{d}z$ becomes zero or infinite. The transformation can be written:

$$\frac{\mathrm{d}w}{\mathrm{d}z} = LF(z) \tag{9}$$



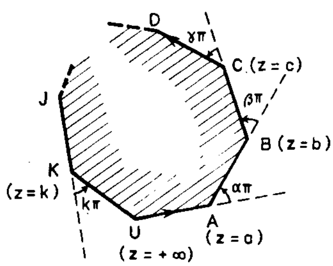


Fig.28.

(L = a complex constant) from which it follows that:

$$\arg\delta w = \arg\delta z + \arg F(z) + \arg L$$

2.2.8 Remarks on applications of conformal transformations

In applications of the conformal transformation, the problem to be solved is as follows: *given* two domains D and D' with specified boundaries, *find* the function $w = f(z)$ which will transform D into D' so that the given boundaries correspond to each other. There is no straightforward method to determine $f(z)$ in any given case. It is therefore important to know how transformations are effected by simple functions $f(z)$, such as those which have been dealt with above.

Conformal transformation has applications in the construction of maps. For example, the conformal mapping of a spherical surface on a plane is done by *Mercator's projection*. In this projection the surface of a sphere is transformed into a doubly-infinite strip between the lines with longitudes $= \pm\pi$. An unlimited number of other conformal maps may be derived from this one. For more details, see for example KELLAWAY (1946).

Chapter 3

METHODS OF STATIONARY PHASE AND OF STEEPEST DESCENT

3.1 METHOD OF STATIONARY PHASE (OR PRINCIPLE OF STATIONARY PHASE)

The method of stationary phase (due to Kelvin and Stokes) is useful in evaluating integrals of the following type:

$$u = \int_a^b \varphi(x) e^{if(x)} dx \tag{1}$$

where the circular function $e^{if(x)} = \cos f(x) + i \sin f(x)$ goes through a large number of periods, while $\varphi(x)$ changes comparatively slowly. For example, when $f(x)$ changes by 2π, then $\varphi(x)$ is supposed to vary by only a small fraction of itself. That is, the integrand approximates a constant $\varphi(x)$ multiplied by a rapidly varying function $e^{if(x)}$, which varies between $+1$ and -1. Therefore, destructive interference between the various contributions to the integral will make it vanish, *except* for those values of x for which $f(x)$ is stationary. Stationary values do not cancel each other in general—they are in general not symmetrically distributed around zero, for any function $f(x)$. At the stationary values of x, the integrand is approximately one constant, $\varphi(x)$, multiplied by another, $e^{if(x)}$.

Put $x = \alpha + \xi$, where $f(x)$ is stationary for $x = \alpha$, and ξ is small, i.e., we consider the neighbourhood of the point $x = \alpha$, the only range where we get a contribution to the integral [1]. We develop $f(x)$ in a Taylor series:

$$f(x) = f(\alpha + \xi) = f(\alpha) + \underbrace{\xi f'(\alpha)}_{=0} + \frac{\xi^2}{2!} f''(\alpha) + \frac{\xi^3}{3!} f'''(\alpha) + \ldots \tag{2}$$

$\uparrow$—— cut series here

α is supposed to be within the range of integration (Fig.29).

Extending the integration in [1] to $+\infty$ and $-\infty$, which is permitted as there are no contributions except for the neighbourhood of $x = \alpha$, we find:

$$u = \varphi(\alpha) e^{if(\alpha)} \int_{-\infty}^{\infty} e^{\frac{1}{2} i f''(\alpha) \xi^2} d\xi \tag{3}$$

To evaluate [3] we use the following formulas (see DWIGHT, 1957, p.199, eq.859.5, or JEFFREYS and JEFFREYS, 1946, p.371):

$$\begin{aligned} &\int_0^\infty \sin\left(\frac{1}{2} ax^2\right) dx = \int_0^\infty \cos\left(\frac{1}{2} ax^2\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \\ &\int_{-\infty}^{\infty} e^{\pm(\frac{1}{2}) i a x^2} dx = \int_{-\infty}^{\infty} \left[\cos\left(\frac{1}{2} ax^2\right) \pm i \sin\left(\frac{1}{2} ax^2\right)\right] dx \\ &= \sqrt{\frac{2\pi}{a}} e^{\pm i(\pi/4)} = (1 \pm i)\sqrt{\frac{\pi}{a}} \end{aligned} \tag{4}$$

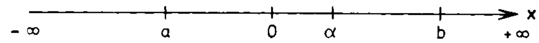


Fig.29.

remembering that $e^{\pm i(\pi/4)} = (1 \pm i)/\sqrt{2}$; [4.2] is frequently called the *complete Fresnel integral.* Applied to [3] we find:

$$u = \frac{\sqrt{\pi} \cdot \varphi(\alpha)}{\sqrt{\left|\frac{1}{2} f''(\alpha)\right|}} e^{i[f(\alpha) \pm \pi/4]} \tag{5}$$

where the upper or lower sign is to be taken in the exponent depending on whether $f''(\alpha)$ is positive or negative. If α coincides with one of the limits of integration, a or b in [1], the limits in [3] will be replaced by 0 and $+\infty$ (if α coincides with a) or by $-\infty$ and 0 (if α coincides with b):

$$\int_{x=a}^{x=b} \to \int_{\xi=a-\alpha}^{\xi=b-\alpha} \to \quad 1)\,\alpha=a \int_{\xi=0}^{\xi=\infty} \to \quad 2)\,\alpha=b \int_{\xi=-\infty}^{\xi=0}$$

Then the integral in [4.2] must be halved.

In [2] we included terms only up to the second derivative, which is correct if:

$$\frac{\xi f'''(\alpha)}{f''(\alpha)} < \varepsilon \tag{6}$$

where ε is any small number. Moreover, in [3] $\xi^2 f''(\alpha)$ is a moderate multiple of 2π (in other words, it does not vanish):

$$\xi^2 f''(\alpha) = n \cdot 2\pi \tag{7}$$

Combining [6] with [7] we have that (eliminating ξ):

$$\frac{f'''(\alpha)}{[f''(\alpha)]^{3/2}} < \varepsilon \tag{8}$$

This is an important condition which must be fulfilled in order to make [5] a valid approximation to the integral [1].

3.1.1 Waves propagating on a water surface from an initial disturbance (displacement or impulse)

The displacement η for any point x or any time t can be written as follows (cf. BULLEN, 1963, p.60):

$$\eta = \frac{1}{\pi} \int_0^\infty \varphi(k) e^{i(\sigma t - kx)} dk + \frac{1}{\pi} \int_0^\infty \varphi(k) e^{i(\sigma t + kx)} dk \tag{9}$$

Note that *dispersion* of the waves is implied in [9], as c (wave velocity) $= \sigma/k$ is generally assumed to be a function of k, the wave number. $\varphi(k)$ is called the *amplitude spectrum* of the wave and $(\sigma t \pm kx)$ its *eikonal*. This quantity, equal to some constant, is the equation for the *wave front*, i.e., the line (or surface) which connects points with the same phase. Different values of the constant correspond to different wave fronts. Eq.[9] is true for a medium of any kind, not necessarily a water surface, but we shall assume σ to be the same function of k as for free water surface waves. The two terms in [9] represent waves travelling outwards from and inwards to the initial disturbance in the direction of positive and negative x, respectively.

The integrals in [9] can be treated by the method of stationary phase. We restrict ourselves to considering outward propagation, i.e., the first integral in [9]. By comparison with [1] we then have that:

$$\begin{aligned} f(k) &= \sigma t - kx \\ f'(k) &= \frac{\mathrm{d}\sigma}{\mathrm{d}k} t - x = 0 \end{aligned} \qquad [10]$$

i.e., $f(k)$ has its stationary value for:

$$t \frac{\mathrm{d}\sigma}{\mathrm{d}k} = x \qquad [11]$$

which means that the group velocity U is:

$$U = \frac{\mathrm{d}\sigma}{\mathrm{d}k} = \frac{x}{t}$$

[11] determines k, and therefore also σ, as a function of x and t. As:

$$f''(k) = t \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2} \qquad [12]$$

we find by application of [5], retaining only the real part and extending the integration in [9] from $-\infty$ to $+\infty$:

$$\eta = \frac{\varphi(k)}{\left|\frac{\pi}{2} t(\mathrm{d}^2\sigma/\mathrm{d}k^2)\right|^{1/2}} \cos(\sigma t - kx \pm \pi/4) \qquad [13]$$

where the sign of $f''(k)$, i.e., of $\mathrm{d}^2\sigma/\mathrm{d}k^2$, defines the sign to use in [13]. Moreover. it is necessary that the condition [8] be fulfilled in order to permit us to use [5], i.e., it is necessary that:

$$\frac{t \dfrac{\mathrm{d}^3\sigma}{\mathrm{d}k^3}}{\left[t \dfrac{\mathrm{d}^2\sigma}{\mathrm{d}k^2}\right]^{3/2}} < \varepsilon \qquad [14]$$

If in [13] we replace t by x from [11]:

$$t = \frac{x}{\frac{\mathrm{d}\sigma}{\mathrm{d}k}}$$

we get:

$$\eta = \frac{\varphi(k)}{\left|\frac{\pi}{2} x \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right)^{-1}\right|^{1/2}} \cos(\sigma t - kx \pm \pi/4) \qquad [15]$$

which means that:

$$\eta \sim x^{-1/2} \qquad [16]$$

This gives the decrease of amplitude with distance, which holds everywhere except for the Airy phase (see section 3.3 below).

We now consider the special case of *water waves on deep water*, i.e.:

$$\sigma^2 = gk$$

for wave lengths small compared to depth. Also: [17]

$$\varphi(k) = 1$$

Then:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}k} = \frac{1}{2} g^{1/2} k^{-1/2}$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2} = -\frac{1}{4} g^{1/2} k^{-3/2} \qquad [18]$$

$$\frac{\mathrm{d}^3\sigma}{\mathrm{d}k^3} = \frac{3}{8} g^{1/2} k^{-5/2}$$

[18.1] combined with the condition [11] gives that:

$$k = \frac{gt^2}{4x^2}$$

and then by combination with [17.1] we get that:

$$\sigma = \frac{gt}{2x} \qquad [19]$$

and that:

$$\sigma t - kx = \frac{gt^2}{4x}$$

The solution [13] then becomes:

$$\eta = \frac{g^{1/2}t}{\pi^{1/2}x^{3/2}}\cos\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right) \qquad [20]$$

This is true if the condition [14] is fulfilled, i.e., if:

$$\sim \left(\frac{x}{gt^2}\right)^{1/2} < \varepsilon \qquad [21]$$

In other words, the approximation [20] holds only for:

$$x \ll gt^2$$

From [19.1] and the fact that $k = 2\pi/\lambda$ (λ = wave length), we find: (*1*) for given x: $\lambda \sim 1/t^2$; (*2*) for given t: $\lambda \sim x^2$. This expresses a well-known phenomenon of velocity dispersion: the longest waves travel faster and the wave lengths become shorter and shorter as we approach the rear of the wave train.

3.1.2 Seismic surface waves

Our next application will be to seismic surface waves. Here it is sufficient to refer to BULLEN (1963, pp.58–64), who deals with the same thing. For instance, if we start from our formula [13], which is true for any medium, and introduce the group velocity as follows:

$$\left.\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}k} &= \frac{x}{t} = U \\ \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2} &= \frac{\mathrm{d}U}{\mathrm{d}k} \end{aligned}\right\} \qquad [22]$$

also consider only the advancing wave, and finally change the sign of the argument of the cos-term (which does not change anything else), then we have:

$$\eta(x, t) = \frac{\varphi(k)}{\left(\dfrac{\pi}{2}\left|\dfrac{\mathrm{d}U}{\mathrm{d}k}\right| t\right)^{1/2}} \cos(kx - \sigma t \mp \pi/4) \qquad [23]$$

This agrees exactly with BULLEN's (1963, p. 61) formula (57).

Thus, at the point x, at time t, the displacement η has from [23] the form of simple harmonic waves of wave length $2\pi/k$ and a certain amplitude given in [23]. We see from [23] that at stationary values of the group velocity, i.e., for $\mathrm{d}U/\mathrm{d}k = 0$, the amplitudes will be very large. This is a common experience (Airy phase) for fundamental-mode surface waves. (Also seismic channel waves have sometimes been explained as corresponding to stationary values in higher-mode dispersion curves.) However, we must be cautious in applying [23] near stationary values of U, i.e., for $\mathrm{d}U/\mathrm{d}k = \mathrm{d}^2\sigma/\mathrm{d}k^2 = 0$, as this may not be permitted by the condition [14]. For more discussion see BULLEN

(1963); the derivation of [23] given here is probably simpler than Bullen's discussion, as he deals immediately with the dispersion problem, without first having clarified the principle of stationary phase.

3.1.3 Lateral refraction of dispersed waves[1]

This problem offers another application of the method of stationary phase. Fig.30 illustrates the case encountered in nature, when surface waves pass from one structure M

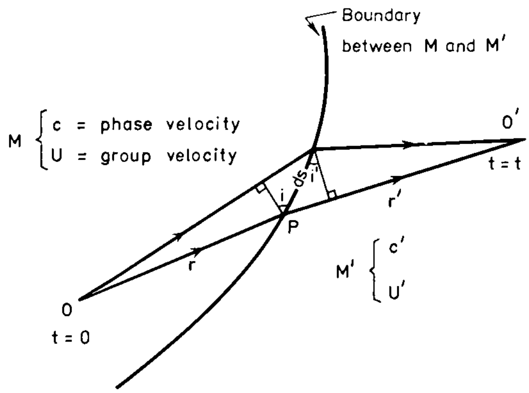


Fig.30.

to another M', e.g., in crossing the oceanic–continental boundary. A ray pencil travels in time t from O to O'. In this pencil the periods are assumed to lie in the range from $2\pi/\sigma$ to $2\pi/(\sigma + \mathrm{d}\sigma)$. By analogy with eq.[9], the displacement at O' can be written in the following form, noting that we have two integration variables, σ and s, instead of one, and that:

$$k(x - ct) = \frac{\sigma}{c}(x - ct) = \sigma\left(\frac{x}{c} - t\right) = \sigma\left(\frac{r}{c} + \frac{r'}{c'} - t\right)$$

thus:

$$\eta \sim \int\int f(\sigma)g(\theta)\mathrm{e}^{i\sigma(r/c + r'/c' - t)}\mathrm{d}\sigma\mathrm{d}s \qquad [24]$$

This is the integrated effect at O'. Here $f(\sigma)$ is analogous to $\varphi(k)$ in [9], and $g(\theta)$ is a function of propagation direction. These two functions generally vary slowly, and the only contributions to this integral will occur for stationary values of the integrand. This is a function both of s and of σ, and therefore it is stationary when the derivatives of the exponent with respect to these two variables are zero:

With respect to s:

$$\frac{1}{c}\frac{\mathrm{d}r}{\mathrm{d}s} + \frac{1}{c'}\frac{\mathrm{d}r'}{\mathrm{d}s} = 0 \qquad (\mathrm{d}t = 0 \text{ by Fermat's principle})$$

[1] BULLEN, 1963, pp.107–108.

With respect to σ:

$$r\frac{\mathrm{d}\left(\frac{\sigma}{c}\right)}{\mathrm{d}\sigma}+r'\frac{\mathrm{d}\left(\frac{\sigma}{c'}\right)}{\mathrm{d}\sigma}-t=0 \quad (c \text{ and } c' \text{ are functions of } \sigma)$$

The angle of incidence of $OP = i$, and we have that:

$$\left|\frac{\mathrm{d}r}{\mathrm{d}s}\right| = \sin i$$

$$\left|\frac{\mathrm{d}r'}{\mathrm{d}s}\right| = \sin i'$$

The first condition:

$$\frac{1}{c}\left|\frac{\mathrm{d}r}{\mathrm{d}s}\right| = \frac{1}{c'}\left|\frac{\mathrm{d}r'}{\mathrm{d}s}\right|$$

then becomes:

$$\frac{\sin i}{c} = \frac{\sin i'}{c'} \qquad [25]$$

which is the usual law of refraction, where the *phase* velocities shall be used.

The second condition can be re-written as follows:

$$\frac{\mathrm{d}\left(\frac{\sigma}{c}\right)}{\mathrm{d}\sigma} = \frac{\mathrm{d}k}{\mathrm{d}\sigma} = \frac{1}{U}$$

$$t = \frac{r}{U} + \frac{r'}{U'} \qquad [26]$$

which says that the travel time t is determined by the *group* velocities. This is to be noted in seismic applications. Also the energy of the wave motion propagates with the *group* velocity.

Lateral reflection and refraction is often referred to in seismology to explain certain features of surface-wave records. Eq.[25] suggests some interesting results, with relevance to such discussions. If a surface-wave train is incident against a boundary under a certain incidence angle $= i$, then any reflected wave train will leave the boundary under the same angle (i) for all periods involved. However, for the refracted surface waves, the angle i' will vary with the period (provided c'/c varies with the period). This would mean that the refracted waves would diverge in the second medium from every point on the boundary. This disintegration of the wave energy is part of the explanation why surface waves travelling along such a boundary are generally very weak (cf. BÅTH, 1952).

3.2 METHOD OF STEEPEST DESCENT[1]

This method (due to Debye) is closely related to the method of stationary phase, and like that method used for the approximate evaluation of complex integrals, especially in the theory of dispersion of waves and the probability theory, including statistical mechanics.

3.2.1 *Watson's lemma*

First we derive an asymptotic expansion of the integral:

$$I = \int_{-\infty}^{\infty} e^{-(1/2)a^2z^2} f(z)\mathrm{d}z \tag{1}$$

where a is real and large, $f(z)$ is analytic at $z = 0$ and bounded on the real axis. Within the convergence limits we expand $f(z)$ as follows:

$$f(z) = a_0 + a_1 z + \ldots + a_{2n-1}z^{2n-1} + R_{2n}(z) \tag{2}$$

where $z^{-2n}R_{2n}(z)$ tends to a finite limit as $z \to 0$. Then the function:

$$g(z) = \frac{[f(z) - (a_0 + a_1 z + \ldots + a_{2n-1}z^{2n-1})]}{z^{2n}} = z^{-2n}R_{2n}(z) \tag{3}$$

is bounded on the real axis; the upper bound of the modulus of $g(z)$ will be called M. Then we can write:

$$\left| \int_{-\infty}^{\infty} e^{-(1/2)a^2z^2}[f(z) - (a_0 + a_1 z + \ldots + a_{2n-1}z^{2n-1})]\, \mathrm{d}z \right| \leqslant \int_{-\infty}^{\infty} e^{-(1/2)a^2z^2} M z^{2n}\mathrm{d}z \tag{4}$$

Furthermore, from DWIGHT (1957, formula 861.7, p.201):

$$\int_{-\infty}^{\infty} e^{-(1/2)a^2z^2} z^{2n}\mathrm{d}z = \sqrt{2\pi}\,\frac{1 \cdot 3 \ldots (2n-1)}{a^{2n+1}} \tag{5}$$

Application of [5] to [4] gives:

$$\left| \int_{-\infty}^{\infty} e^{-(1/2)a^2z^2} f(z)\mathrm{d}z - \sqrt{2\pi}\left[\frac{a_0}{a} + \frac{a_2}{a^3} + \frac{1 \cdot 3}{a^5}a_4 + \ldots + \frac{1 \cdot 3 \ldots (2n-3)}{a^{2n-1}}a_{2n-2}\right] \right|$$

$$\leqslant \sqrt{2\pi}\, M \frac{1 \cdot 3 \ldots (2n-1)}{a^{2n+1}} \tag{6}$$

Multiply [6] by a^{2n} and let a tend to infinity. Writing [6] in abbreviated form as follows, after multiplication by a^{2n}:

$$|\alpha| \cdot a^{2n} \leqslant \frac{\beta}{a}$$

[1] Or: Saddle point method, German: "Sattelpunktmethode".

we see that as $a \to \infty$, [6] becomes:

$$|\alpha| \cdot \infty = 0$$

This can only be fulfilled if $\alpha = 0$, that is:

$$I \simeq \sqrt{2\pi}\left[\frac{a_0}{a} + \frac{a_2}{a^3} + \frac{1 \cdot 3}{a^5} a_4 + \ldots + \frac{1 \cdot 3 \ldots (2n-3)}{a^{2n-1}} a_{2n-2}\right] \qquad [7]$$

Frequently it is sufficiently accurate to keep only the first term in the series expansion in [7]. This is Watson's lemma. It holds also if the limits are $-B$ and $+A$ (A, B positive), as the tails give vanishing contributions.

3.2.2 *Method of steepest descent*

This method is used for approximate evaluation of integrals of the type:

$$I = \int_A^B \chi(z) e^{tf(z)} dz \qquad [8]$$

where t is large, real and positive, and $f(z)$ is analytic:

$$\left.\begin{aligned} f(z) &= \varphi(x, y) + i\psi(x, y) \\ z &= x + iy \end{aligned}\right| \qquad [9]$$

separating real and imaginary parts. The integrand, containing $\chi e^{t\varphi} \cdot e^{it\psi}$ will be large when φ is algebraically large. Compare the more simplified integral [1] in section 3.1. Therefore, the integral [8] will be large if the path AB passes through large values of φ. It will be convenient to choose the path of integration such that the largest values of φ (which make essential contributions to the integral [8]) are located as close to each other as possible. Then, an approximate value of the integral can be obtained by carrying out the integration only over the range of largest values of φ.

Differentiating [9], we get:

$$\left.\begin{aligned} f_x &= \frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz} = \varphi_x + i\psi_x \\ f_y &= \frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = i\frac{df}{dz} = \varphi_y + i\psi_y \end{aligned}\right\}$$

Identifying real and imaginary parts in the two expressions for df/dz, we find:

$$\left.\begin{aligned} \varphi_x &= \psi_y \\ \varphi_y &= -\psi_x \end{aligned}\right\} \text{Cauchy-Riemann relations (cf. section 2.1)} \qquad [10]$$

The integration path is defined in the following way:

(*1*) It *passes through* stationary (maximum) points of φ, at $z = z_0$, i.e., at points where:

$$d\varphi = \varphi_x dx + \varphi_y dy = 0 \qquad [11]$$

or where $\varphi_x = \varphi_y = 0$ for any values of dx and dy.

(*2*) It is *directed along* the negative gradient of φ, i.e., along $-\text{grad}\varphi$, which is the steepest descent; this means that the direction along the path θ is defined by:

$$\tan\theta = \frac{dy}{dx} = \frac{\varphi_y}{\varphi_x} = -\frac{\psi_x}{\psi_y}$$

by eq.[10]. Thus:

$$d\psi = \psi_x dx + \psi_y dy = 0 \tag{12}$$

and ψ is constant along the path of integration.

It should be noted that φ can never have an absolute maximum, but it can have stationary points (item *1* above). Because of eq.[10] the quantity $AC - B^2 = \varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 = -(\psi_{xy}^2 + \varphi_{xy}^2) < 0$ (see DE LA VALLÉE-POUSSIN, 1938, pp.133–135), and this means that only inflection points (stationary points) exist.

Item *1* defines the points z_0 which must be known in the evaluation of integrals of the type [8] by this method. At such a point both $d\varphi$ and $d\psi$ are $= 0$, which can happen only if $df = 0$. The sought points $z = z_0$ are obtained as solution to this equation, or $f'(z_0) = 0$.

We have to remember that the two functions φ and ψ are not completely arbitrary, but are the real and imaginary parts, respectively, of one and the same given function $f(z)$. This implies certain restrictions, expressed in the Cauchy–Riemann relations (which are essential in the whole argument).

Next, we will show that the points $z = z_0$ are "saddle points", i.e., that the curvature on the path of steepest descent is opposite to that along a path at right angles to this direction. The curvature for any direction dy/dx is:

$$d^2\varphi = \varphi_{xx}dx^2 + 2\varphi_{xy}dxdy + \varphi_{yy}dy^2 \tag{13}$$

as obtained by differentiating $d\varphi = \varphi_x dx + \varphi_y dy$, and in a direction at right angles to this one, the curvature is obtained by substituting $-dy$ for dx and dx for dy:

$$d^2\varphi_\perp = \varphi_{xx}dy^2 - 2\varphi_{xy}dxdy + \varphi_{yy}dx^2 \tag{14}$$

and therefore:

$$d^2\varphi + d^2\varphi_\perp = (\varphi_{xx} + \varphi_{yy})(dx^2 + dy^2) = 0 \tag{15}$$

because of [10]. Therefore the two curvatures are numerically equal but of opposite sign, which defines a "saddle point". The geometry of the problem is illustrated in Fig.31, in which φ is plotted as the 3rd coordinate versus x and y. Consequently, we should follow a path of steepest descent in φ to meet the requirement of concentrating large values of φ to the shortest possible segment of the integration path A to B. We shall assume that A and B are in different 'valleys" and then the path AB has to go through a saddle point —hence the alternative names "pass method" or "method of saddle points" which SOMMERFELD (1954, p.101) says are more appropriate than "method of steepest descent". In general, A and B will not themselves be on lines of steepest descents, but can be joined to them by lines within the valleys (with vanishing contributions to the integral [8]).

Near the saddle point $z = z_0$, $f(z)$ can be expanded in a series, remembering that $f'(z_0) = 0$:

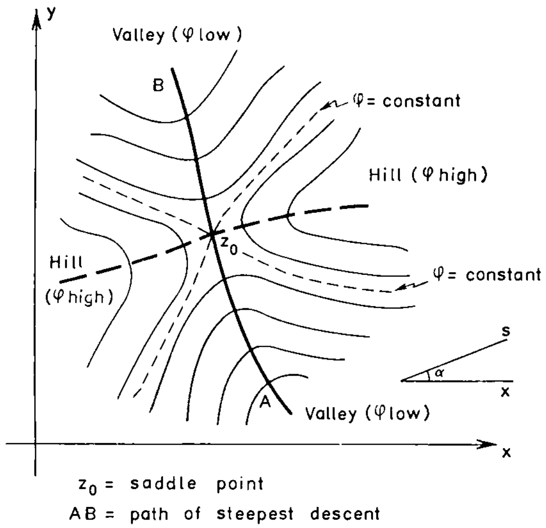


Fig.31.

$$
\left.
\begin{aligned}
f(z) &= f(z_0) + \frac{1}{2}(z - z_0)^2 f''(z_0) + \ldots \\
f(z) - f(z_0) &= -\frac{1}{2}\zeta^2
\end{aligned}
\right| \qquad [16]
$$

which also defines our new variable ζ. The path of integration (through a *maximum*) is such that the second derivative $f''(z_0)$ is real and *negative*; remember that along the path, the imaginary part of $f(z)$, i.e., ψ, is constant, and therefore its derivative zero. Then also $(z - z_0)^2 f''(z_0)$ is real and negative along the path.

By means of [16], our integral [8] can now be written:

$$
I = e^{tf(z_0)} \int \chi(z) e^{-(1/2)t\zeta^2} dz = e^{tf(z_0)} \int e^{-(1/2)t\zeta^2} \chi(z) \frac{dz}{d\zeta} d\zeta \qquad [17]
$$

The direction of the path of integration can be defined by the angle α it forms with the x-axis (Fig.31). We then have that:

$$
\left.
\begin{aligned}
z - z_0 &= re^{i\alpha} \\
\zeta^2 &= -f''(z_0) r^2 e^{2i\alpha} = r^2 |f''(z_0) e^{2i\alpha}| \\
\zeta &= \pm r |f''(z_0)|^{1/2}, \quad \text{as} \quad |e^{2i\alpha}| = 1 \\
\frac{d\zeta}{dz} &= \pm e^{-i\alpha} |f''(z_0)|^{1/2}
\end{aligned}
\right| \qquad [18]
$$

We have to choose our direction, as there are two directions α differing by π. Choosing the positive sign in [18], we find by means of Watson's lemma [7], keeping only the first term:

$$I \simeq \frac{\chi(z_0)e^{tf(z_0)}\sqrt{2\pi}\ e^{i\alpha}}{|tf''(z_0)|^{1/2}} \tag{19}$$

This is the asymptotic expression of the integral obtained by means of the method of steepest descent. Eq.[19] is a valid approximation under the same condition as in the method of stationary phase (eq.[8] in section 3.1).

The method of steepest descent can also be used in case there is a pole near the saddle point. But then the method must be modified. For details of this, the reader is referred to BREKHOVSKIKH (1960, pp.264–266).

3.2.3 *Application of the method of steepest descent*[1]

The integrals encountered in the solution of Lamb's problem are dealt with by means of contour integration in Chapter 12. The method of steepest descent offers another possibility. We consider the following integral, which is obtained in the problem of an internal compressional source and which represents the shear-wave motion, produced by the presence of a free surface:

$$\psi = -4ie^{i\omega t}\int_{-\infty}^{\infty}\frac{\zeta(2\zeta^2-k_\beta^2)}{F(\zeta)}\,e^{-\nu h-\nu' z-i\zeta x}\,d\zeta \tag{20}$$

Here we have:

$$\left.\begin{aligned} &F(\zeta) = (2\zeta^2-k_\beta^2)^2-4\zeta^2\nu\nu' \quad \text{(Rayleigh's function)}\\ &\nu^2 = \zeta^2-k_\alpha^2\ ;\quad k_\alpha = \frac{\omega}{\alpha}\\ &\nu'^2 = \zeta^2-k_\beta^2\ ;\quad k_\beta = \frac{\omega}{\beta}\end{aligned}\right| \tag{20'}$$

(h, z, x see Fig.32); α, β are compressional- and shear-wave velocities, respectively. These notations agree with EWING, JARDETZKY and PRESS' (1957) treatment of Lamb's problem, but differ from those of LAMB's original paper (1904). In EWING, JARDETZKY and PRESS (1957, p.37) formula 2-47 reads:

$$\psi = -8\int_{0}^{\infty}\frac{\zeta(2\zeta^2-k_\beta^2)}{F(\zeta)}\,e^{-\nu h-\nu' z}\sin\zeta x\,d\zeta \tag{21}$$

Extend the integration over all wave numbers ζ, from $-\infty$ to $+\infty$ (for negative ζ, the integrand is the same as for positive ζ, because of the product $\zeta\sin\zeta x$, and because other expressions in [21] are unchanged for a change of sign of ζ):

[1] EWING, JARDETZKY and PRESS, 1957, p.59.

$$\psi = -4 \int_{-\infty}^{\infty} \frac{\zeta(2\zeta^2 - k_\beta^2)}{F(\zeta)} \mathrm{e}^{-\nu h - \nu' z} \sin\zeta x \, \mathrm{d}\zeta \qquad [22]$$

Also, replace $\sin\zeta x$ by exponential functions (Re = real part of):

$$\sin\zeta x = \mathrm{Re}(i\cos\zeta x + \sin\zeta x) = \mathrm{Re}\, i(\cos\zeta x - i\sin\zeta x) = \mathrm{Re}\, i e^{-i\zeta x}$$

Finally, reintroduce the time factor (outside the integral) $\mathrm{e}^{i\omega t}$, and we arrive at eq.[20].

Comparing with [8], the exponent in [20] is $xf(\zeta)$, i.e.:

$$f(\zeta) = -\frac{\nu h}{x} - \frac{\nu' z}{x} - i\zeta = \varphi + i\psi \qquad [23]$$

The line of steepest descent is defined by $\psi(k, \tau)$ = constant, k and τ being the two coordinates.

Write:

$$\zeta = k_\alpha u \qquad [24]$$

i.e., replace the variable ζ by the new variable u. Also we put:

$$\omega = s - ic \qquad (s > 0,\ c > 0) \qquad [25]$$

i.e., take the frequency ω as a complex quantity:

$$\mathrm{e}^{i\omega t} = \mathrm{e}^{i(s-ic)t} = \underbrace{\mathrm{e}^{ist}}_{I}\,\underbrace{\mathrm{e}^{ct}}_{II}$$

I = periodic factor; II = exponential increase, from $t = 0$. In the range $0 \leqslant u \leqslant 1$, we then have:

$$-\psi = \frac{s}{\alpha}\left[u + \frac{h}{x}(1 - u^2)^{1/2} + \frac{z}{x}\left(\frac{\alpha^2}{\beta^2} - u^2\right)^{1/2}\right] \qquad [26]$$

Eq.[26] is obtained by identifying the imaginary parts on the two sides of [23], after introducing the expressions for ν, ν' from [20'], of k_α/k_β and of ζ from [24], and of ω from [25].

The saddle point ζ_0 is given by $f'(\zeta_0) = 0$, i.e.:

$$x = \frac{hu_0}{(1 - u_0^2)^{1/2}} + \frac{zu_0}{\left(\dfrac{\alpha^2}{\beta^2} - u_0^2\right)^{1/2}} \qquad [27]$$

Eq.[27] is found immediately by differentiating [23], remembering that ν and ν' are functions of ζ by [20']. We see from [27] that a saddle point exists for u_0 real. It depends on h, x and z and lies in the range $0 \leqslant u_0 \leqslant 1$. As from [24] $u_0 = \dfrac{\zeta_0}{k_\alpha}$ we also have for the saddle point that:

$$0 \leqslant \frac{\zeta_0}{k_\alpha} \leqslant 1 \text{ or } 0 \leqslant \zeta_0 \leqslant k_\alpha$$

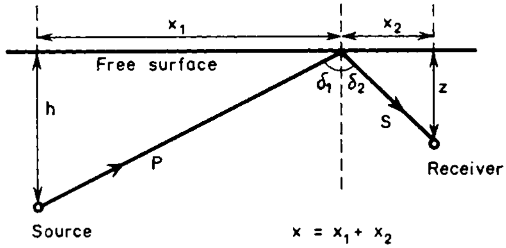


Fig.32.

i.e., the wave number corresponding to the saddle point lies on the range $0 - k_\alpha$.

The solution of the integral [20] is then obtained directly from [19]. It includes an exponential (periodic) factor of the following form, including the time factor from [20]; other factors will be part of the amplitude:

$$e^{i\omega\left[t - \frac{xu_0}{\alpha} - \frac{h}{\alpha}\left(1-u_0^2\right)^{1/2} - \frac{z}{\alpha}\left(\frac{\alpha^2}{\beta^2} - u_0^2\right)^{1/2}\right]} \tag{28}$$

The complete solution could also be written down, from [19]. This is left as an exercise to the reader.

Equations [27] and [28] permit a simple physical explanation of the obtained result: the solution corresponds to the *PS*-wave (Fig.32). This result is seen from [27] and [28] if we put $u_0 = \sin\delta_1$ (*not* $\cos\delta_1$, as in EWING, JARDETZKY and PRESS, 1957). The sine function fulfills the limitations made on u_0 (to be in the range 0–1).

We find that [27] expresses nothing but the reflection law, applied to the geometry of Fig.32:

$$x = x_1 + x_2 = h\tan\delta_1 + z\tan\delta_2$$

$$= \frac{h\sin\delta_1}{\cos\delta_1} + \frac{z\sin\delta_2}{\cos\delta_2}$$

$$= \frac{h\sin\delta_1}{\cos\delta_1} + \frac{z\dfrac{\beta}{\alpha}\sin\delta_1}{\left(1 - \dfrac{\beta^2}{\alpha^2}\sin^2\delta_1\right)^{1/2}}$$

$$= \frac{hu_0}{\left(1 - u_0^2\right)^{1/2}} + \frac{zu_0}{\left(\dfrac{\alpha^2}{\beta^2} - u_0^2\right)^{1/2}} \qquad \text{i.e., eq.[27]} \quad \left[\text{as } \frac{\sin\delta_1}{\alpha} = \frac{\sin\delta_2}{\beta}\right]$$

Similarly, [28] gives the expression for the propagation of a given, constant phase. Considering many paths between source and receiver (see Fig.32), we have to select one for which $dt = 0$ (minimum time, i.e., Fermat's principle), and then the stationary phase condition applied to [28] gives that:

$u_0 = \sin\delta_1$ varies; $\quad x, h, z, \alpha, \beta$ constant;

$$-\frac{x}{\alpha}\sin\delta_1 - \frac{h}{\alpha}\cos\delta_1 - \frac{z}{\alpha}\left(\frac{\alpha^2}{\beta^2} - \sin^2\delta_1\right)^{1/2} = \text{constant}$$

Differentiation of this expression with regard to δ_1 gives eq.[27].

The last point can also be seen in the following way: [28] can be written $e^{i\omega(t-t_1)}$ where t_1 must be the time required for the wave to travel from source to receiver (as P and later as S). The path length as P is $h\cos\delta_1 + x_1\sin\delta_1$ as seen by projecting h and x_1 on to the P-ray, and similarly the length of the S-ray is $z\cos\delta_2 + x_2\sin\delta_2$. Therefore the total travel time is:

$$t_1 = \frac{h\cos\delta_1}{\alpha} + \frac{x_1\sin\delta_1}{\alpha} + \frac{z\cos\delta_2}{\beta} + \frac{x_2\sin\delta_2}{\beta}$$

$$= \frac{h(1-u_0^2)^{1/2}}{\alpha} + \frac{xu_0}{\alpha} + \frac{z}{\alpha}\left(\frac{\alpha^2}{\beta^2} - u_0^2\right)^{1/2}$$

i.e., corresponding to [28].

3.2.4 *Comparison of the methods of steepest descent and of stationary phase*

The integral [8] studied in section 3.2.2 can also be approximated by the method of stationary phase:

$$I = \int_A^B \chi(z)e^{tf(z)}dz = \int_A^B \chi e^{t\varphi}\cdot e^{it\psi}dz \qquad [29]$$

The two methods are nearly equivalent and the main difference is shown in Table I. In both methods the integration paths pass through saddle points, but in different directions, one time keeping ψ constant, in the other keeping φ constant (see Fig.31).

Considering that essential contributions to the integral [29] are only obtained near the stationary phase ($z = z_0$), we can write the integral as follows:

TABLE I

COMPARISON OF METHODS OF STEEPEST DESCENT AND OF STATIONARY PHASE

Method	*Modulus*	*Phase*	
Steepest descent	varies	constant	along the integration path
Stationary phase	constant	varies	

$$\int_A^B \chi(z)e^{tf(z)}dz \simeq \chi(z_0)e^{tf(z_0)} \int_{-\infty}^{\infty} e^{(1/2)tf''(z_0)(z-z_0)^2}dz \tag{30}$$

where the integral is extended from $-\infty$ to $+\infty$ and evaluated by means of [4] in section 3.1:

$$\int_{-\infty}^{\infty} e^{\pm(1/2)it\frac{f''(z_0)}{i}(z-z_0)^2} dz = (1 \pm i)\left(\frac{\pi}{tf''(z_0)/i}\right)^{1/2} \tag{31}$$

Combination of [30] and [31] gives the approximate expression of the integral [29] in case of the stationary phase method:

$$I \simeq \frac{\chi(z_0)e^{tf(z_0)}\sqrt{2\pi}}{\left|\frac{tf''(z_0)}{i}\right|^{1/2}} e^{\pm i(\pi/4)} \tag{32}$$

noting that:

$$1 \pm i = \sqrt{2} \cdot e^{\pm i(\pi/4)}$$

Also as $|i| = 1$, the agreement with [19] is very good; the absolute values of the expressions [32] and [19] are the same. In both methods the integration paths pass through saddle points, but in different directions, as already mentioned. By Cauchy's theorem, deformation of paths is always permissible to some extent, and therefore we can regard the method of stationary phase as a modification of the steepest descent method and justified by it (historically, the method of stationary phase was the earlier one).

3.3 THE AIRY INTEGRAL

3.3.1 Definition

In the derivation of expressions for the Airy integral, we follow JEFFREYS and JEFFREYS (1946, pp.476–478).

Consider the integral:

$$f(z) = \frac{1}{2\pi i}\int e^{tz-t^3/3}dt \tag{1}$$

taken along any of the paths shown in Fig.33. We assume the real part of t^3 tends to $+\infty$ at the three vertices, thus making [1] converge exponentially. We have:

$$\frac{d^2}{dz^2}f(z) - zf(z) = \frac{1}{2\pi i}\int e^{tz-t^3/3}(t^2 - z)dt$$

$$= -\frac{1}{2\pi i}\int e^{tz-t^3/3}d(tz - t^3/3) = 0 \tag{2}$$

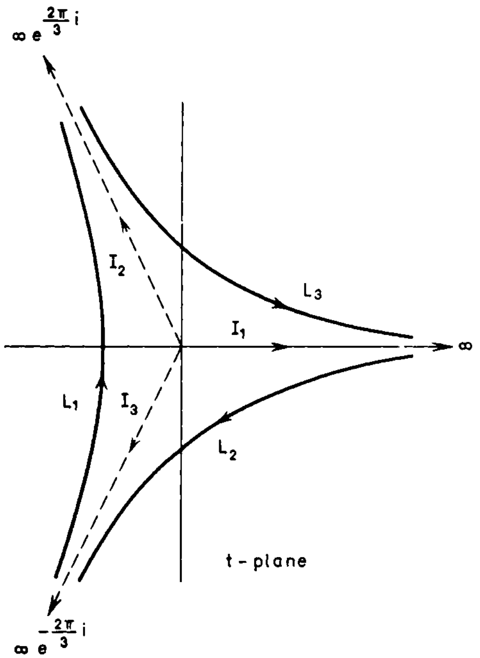


Fig.33.

as $e^{tz-t^3/3}$ tends to 0 at both limits in each case (because of the assumption concerning the real part of t^3). [2] demonstrates that the integrals [1] are solutions of the following second-order differential equation:

$$\frac{d^2Z}{dz^2} - zZ = 0 \tag{3}$$

As [3] is of the *second* order, it can have *only two* linearly independent solutions. Therefore there must be some relation between the three integrals mentioned. By Cauchy's theorem on integration in the complex plane, we see that the sum of the three integrals (Fig.33) is zero, as there is no residue:

$$\int_{L_1} + \int_{L_2} + \int_{L_3} = 0$$

We consider first the integral along L_1 (the Airy integral):

$$Ai(z) = \frac{1}{2\pi i} \int_{L_1} e^{tz-t^3/3} dt \tag{4}$$

We shall develop this integral in two ways: expressing it in a series expansion and in an asymptotic form. Put $t = is$ in [4]:

$$t^2 = -s^2; \qquad t^3 = -is^3; \qquad dt = ids$$

$$Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(sz+s^3/3)} ds = \frac{1}{\pi} \int_0^{\infty} \cos(sz + s^3/3) ds \tag{5}$$

Putting $t = is$ means considering only the imaginary part of t; then the integration is extended along the imaginary axis from $-\infty$ to $+\infty$. The positive and negative contributions to the sine term cancel, and we are left with only the cosine term. The form [5] of the integral was used by Airy.

3.3.2 Series expansion

Consider also the integrals I_1, I_2, I_3 (Fig.33):

$$I_2(z) = \frac{1}{2\pi i}\int_0^{\infty e^{2\pi i/3}} e^{tz-t^3/3}\mathrm{d}t = \frac{1}{2\pi i}\, e^{2\pi i/3}\int_0^{\infty} e^{uze^{2\pi i/3}-u^3/3}\mathrm{d}u =$$

(make substitution $t = ue^{2\pi i/3}$; $t^3 = u^3$)

$$= e^{2\pi i/3} I_1(ze^{2\pi i/3}) \qquad [6]$$

Similarly:

$$I_3(z) = e^{-2\pi i/3} I_1(ze^{-2\pi i/3})$$

Moreover:

$$\left.\begin{aligned} Ai(z) &= \int_{\infty e^{-2\pi i/3}}^{\infty e^{2\pi i/3}} = \int_{\infty e^{-2\pi i/3}}^{0} + \int_0^{\infty e^{2\pi i/3}} = -I_3(z) + I_2(z) \\ \frac{1}{2\pi i}\int_{L_2} &= -I_1(z) + I_3(z) \\ \frac{1}{2\pi i}\int_{L_3} &= -I_2(z) + I_1(z) \end{aligned}\right| \qquad [7]$$

By Cauchy's theorem we have freedom in choosing the integration path, remembering only that the end-points are the same.

By [6] and [7] all integrals studied can be expressed in terms of the integral I_1. Therefore, if we have an expression for $I_1(z)$, all the other integrals are also known. We get:

$$I_1(z) = \frac{1}{2\pi i}\int_0^{\infty} e^{tz-t^3/3}\mathrm{d}t = \frac{1}{2\pi i}\int_0^{\infty} e^{-t^3/3}\sum_{r=0}^{\infty}\frac{(tz)^r}{r!}\,\mathrm{d}t$$

using the series expansion of:

$$e^{tz} = 1 + \frac{tz}{1!} + \frac{(tz)^2}{2!} + \ldots = \sum_{r=0}^{\infty}\frac{(tz)^r}{r!}$$

Then substituting $v = t^3/3$, we find:

$$I_1(z) = \frac{1}{2\pi i}\int_0^{\infty} e^{-v}\sum_{r=0}^{\infty} 3^{-2/3}\frac{(3^{1/3}z)^r}{r!}\, v^{r/3-2/3}\mathrm{d}v$$

and using the integration rule (DWIGHT, 1957, 861.2, p.200):

$$\int_0^\infty e^{-v} v^{r/3-2/3} dv = (r/3 - 2/3)!$$

we finally get:

$$I_1(z) = \frac{1}{2\pi i} 3^{-2/3} \sum_{r=0}^{\infty} \frac{(3^{1/3}z)^r}{r!} (r/3 - 2/3)! \tag{8}$$

By [7] and [6] we then get:

$$Ai(z) = \frac{1}{2\pi i} e^{2\pi i/3} 3^{-2/3} \sum_{r=0}^{\infty} \frac{(3^{1/3}z)^r}{r!} (r/3 - 2/3)!\, e^{2\pi i r/3}$$
$$- \frac{1}{2\pi i} e^{-2\pi i/3} 3^{-2/3} \sum_{r=0}^{\infty} \frac{(3^{1/3}z)^r}{r!} (r/3 - 2/3)!\, e^{-2\pi i r/3}$$

Using Euler's formulas for the e-functions, where the only terms which are left are:

$$\cos\frac{2\pi}{3} \sin\frac{2\pi r}{3} + \sin\frac{2\pi}{3} \cos\frac{2\pi r}{3} = \sin\frac{2\pi}{3}(r + 1)$$

we find that:

$$\begin{aligned} Ai(z) &= \frac{1}{\pi} 3^{-2/3} \sum_{r=0}^{\infty} \frac{(3^{1/3}z)^r}{r!} (r/3 - 2/3)! \sin\left[\frac{2\pi}{3}(r + 1)\right] \\ &= \frac{1}{\pi} 3^{-2/3} \sin\frac{2\pi}{3} \cdot (-2/3)! \cdot \\ &\quad \cdot \left(1 + \frac{z^3}{2 \cdot 3} + \frac{z^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{z^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \ldots\right) \\ &\quad + \frac{1}{\pi} 3^{-1/3} \sin\frac{4\pi}{3} \cdot (-1/3)! \cdot \\ &\quad \cdot \left(z + \frac{z^4}{3 \cdot 4} + \frac{z^7}{3 \cdot 4 \cdot 6 \cdot 7} + \frac{z^{10}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} + \ldots\right) \end{aligned} \tag{9}$$

This expansion is valid for all values of the argument z. In particular, for $z = 0$, we get:

$$Ai(0) = \frac{1}{\pi} 3^{-2/3} \sin\frac{2\pi}{3} \cdot (-2/3)!$$

By means of the relations:

$$(2/3)!\,(-2/3)! = \frac{\frac{2\pi}{3}}{\sin\frac{2\pi}{3}}$$ (see JEFFREYS and JEFFREYS, 1946, p.435, or section 1.3)

and:

$$(2/3)! = \Gamma(5/3) = (2/3)\Gamma(2/3) \qquad \text{(see section 1.3)}$$

the expression for $Ai(0)$ can be written also in the following way:

$$Ai(0) = \frac{1}{3^{2/3}\Gamma(2/3)}$$

In [9] $Ai(z)$ has been expressed as the sum of two series: first series: $r = 0, 3, 6, 9, \ldots$; second series: $r = 1, 4, 7, 10, \ldots$ Note that $r = 2, 5, 8, \ldots$ give no contribution, as the sine term then vanishes. Both series can be shown to be convergent and separately to be solutions of [3]. We can write [9] in shorter form:

$$Ai(z) = \alpha y_1(z) - \beta y_2(z) \tag{10}$$

Using the relation:

$$z!(-z)! = \frac{\pi z}{\sin \pi z}\,; \qquad \left[(2/3)!\,(-2/3)! = \frac{\frac{2\pi}{3}}{\sin\frac{2\pi}{3}}\right]$$

we find that:

$$\left.\begin{aligned} \alpha &= 3^{-2/3}/(-1/3)! \\ \beta &= 3^{-1/3}/(-2/3)! \end{aligned}\right| \tag{11}$$

As another solution of the differential eq.[3] we take, real for real z:

$$\begin{aligned} Bi(z) &= \frac{1}{2\pi}\left(\int_{L_3} - \int_{L_2}\right) = -iI_2 - iI_3 + 2iI_1 \\ &= \frac{1}{2\pi}\, 3^{-2/3} \sum_{r=0}^{\infty} \frac{(3^{1/3}z)^r}{r!}(r/3 - 2/3)!\,[2 - e^{2(r+1)i\pi/3} - e^{-2(r+1)i\pi/3}] \\ &= \frac{1}{\pi}\, 3^{-2/3} \sum_{r=0}^{\infty} \frac{(3^{1/3}z)^r}{r!}(r/3 - 2/3)!\,[1 - \cos 2(r+1)\pi/3] \\ &= \sqrt{3}\alpha y_1 + \sqrt{3}\beta y_2 \end{aligned} \tag{12}$$

Expressions in terms of Bessel functions are also possible (see eq.[44] in section 5.3).

3.3.3 Asymptotic expressions

The series converge too slowly to be convenient for computation when $|z| > 3$. Then an asymptotic expression is preferable. This can be obtained by the *method of steepest descent*:

(1) *z real and positive*:

$$\left.\begin{aligned} f(t) &= tz - t^3/3 \\ f'(t) &= z - t^2 = 0 \\ f''(t) &= -2t \end{aligned}\right| \qquad [13]$$

and the saddle points are at $t = \pm z^{1/2}$.

(a) $t = -z^{1/2}$; $f''(t) = +2z^{1/2}$. This is therefore a minimum (passing along the real axis). Then, the path of steepest descent is perpendicular to the real axis. Such a path can be deformed into L_1 (Fig.33) without passing over a hill. Then we find by [19] in section 3.2, where in this case $\chi = 1$ and $t = 1$:

$$Ai(z) \simeq \frac{1}{2\pi i} e^{-\frac{2z^{3/2}}{3}} \frac{\sqrt{2\pi}}{(2z^{1/2})^{1/2}} i = \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2z^{3/2}}{3}} \qquad [14]$$

(b) $t = +z^{1/2}$; $f''(t) = -2z^{1/2}$. This is a maximum (2nd derivative negative), and the path of steepest descent is along the real axis from $-z^{1/2}$ to $+\infty$; but $-z^{1/2}$ is a saddle point and the paths of steepest descent away from it are to $\infty e^{\pm 2\pi i/3}$. Therefore in the present case, the paths are L_2 and L_3 (Fig.33), and the main contribution to the integral comes from the passage through $+z^{1/2}$. As:

$$Bi(z) = \frac{1}{2\pi}\left(\int_{L_3} - \int_{L_2}\right)$$

it means that we pass from the negative side ($\infty e^{\pm 2\pi i/3}$) to the positive side ($+\infty$), both along L_2 and L_3—as we have taken the reverse direction of L_2, by using the negative sign in defining $Bi(z)$. Thus we get:

$$Bi(z) \simeq 2i \frac{1}{2\pi i} e^{\frac{2z^{3/2}}{3}} \frac{\sqrt{2\pi}}{(2z^{1/2})^{1/2}} = \frac{1}{\sqrt{\pi}} z^{-1/4} e^{\frac{2z^{3/2}}{3}} \qquad [15]$$

(2) *z real and negative*, say $z = -\zeta$. The saddle points are on the imaginary axis at $\pm i\zeta^{1/2}$. At $t = +i\zeta^{1/2}$, $f''(t) = -2i\zeta^{1/2}$ and the line of steepest descent is at $3\pi/4$ to the real axis. This is seen from [18], section 3.2, which gives the direction of the path of steepest descent:

$$f(t) - f(i\zeta^{1/2}) = -(t - i\zeta^{1/2})^2 i\zeta^{1/2} = -\frac{1}{2}x^2$$

x corresponds to ζ in [18], section 3.2:

$$x = \pm\sqrt{2i\zeta^{1/2}}\,(t - i\zeta^{1/2}) = \pm e^{i\pi/4}\,|f''(t_0)|^{1/2}\,(t - i\zeta^{1/2})$$

as $i^{1/2} = e^{i\pi/4}$. Then:

$$\frac{dx}{dt} = \pm e^{i\pi/4}\,|f''(t_0)|^{1/2}; \text{ i.e., } \alpha = -\frac{\pi}{4} \to \frac{3\pi}{4}$$

Then there are two contributions to $Ai(z)$, along L_2 and along L_3:

$$Ai(-\zeta) = Ai(z) \simeq \frac{1}{2\pi i} e^{\frac{2i\zeta^{3/2}}{3}} \frac{\sqrt{2\pi}}{(2\zeta^{1/2})^{1/2}} e^{i\pi/4} + \\ + \frac{1}{2\pi i} e^{\frac{-2i\zeta^{3/2}}{3}} \frac{\sqrt{2\pi}}{(2\zeta^{1/2})^{1/2}} e^{3i\pi/4} = \frac{1}{\sqrt{\pi}} \zeta^{-1/4} \sin\left(\frac{2}{3}\zeta^{3/2} + \frac{\pi}{4}\right) \qquad [16]$$

For $Bi(z)$ the same paths are used, but again the upper one (L_3) in opposite direction, and an extra factor i is to be included:

$$Bi(-\zeta) = Bi(z) \simeq \frac{1}{\sqrt{\pi}} \zeta^{-1/4} \cos\left(\frac{2}{3}\zeta^{3/2} + \frac{\pi}{4}\right) \qquad [17]$$

Note that the integral $Ai(z)$ is defined so as to go from lower left to upper left extremum: either along L_1 or along $L_2 + L_3$. On the other hand, $Bi(z)$ is defined so as to go from left to right, along both L_2 and L_3.

The general appearance of $Ai(z)$ and $Bi(z)$ is much clearer from the asymptotic expressions, just derived, than from the series expansions:

(*1*) z positive, [14] and [15]: $Ai(z)$ decreases exponentially and $Bi(z)$ increases exponentially;

(*2*) z negative, [16] and [17]: $Ai(z)$ and $Bi(z)$ oscillate, with slowly diminishing amplitude (proportional to $\zeta^{-1/4}$) and diminishing period (the argument $(2/3\ \zeta^{3/2} + \pi/4)$ increases with increasing ζ, which corresponds to shorter and shorter waves).

We must note that the Airy function is sometimes defined in a slightly different way. For instance, in BREKHOVSKIKH (1960), the Airy function is equal to $\sqrt{\pi}$ times our Airy function.

3.3.4 Applications

Behaviour of surface waves near minimum group velocity

We go back to the discussion in section 3.1.1. Near minimum group velocity the condition [14] in section 3.1 does not hold, and therefore the formula [13] does not hold either. The reason is that in the Taylor expansion of $f(k)$, the second derivative, $f''(k)$, eq.[12] in section 3.1, now vanishes, and therefore it will be necessary to include the next (cubic) term in the development of $f(k)$. This will lead over to Airy integrals.

Consider the following integral, which is obtained from the first term in eq.[9], section 3.1, assuming that $k \to |k|$:

$$\eta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(k) e^{i(\sigma t - kx)} dk \qquad [18]$$

This means that [18] refers only to the wave propagating in the direction of increasing x.

Let index m denote values at the minimum group velocity and consider its neighbourhood: $k = k_m + k_1$. Moreover, we put phase velocity $= c$ and group velocity $= U$. Note that $U = \mathrm{d}\sigma/\mathrm{d}k$. Expand σ in a Taylor series in the neighbourhood of the minimum group velocity:

$$\sigma(k) = \sigma(k_m) + \frac{\mathrm{d}\sigma}{\mathrm{d}k} k_1 + \underbrace{\frac{1}{2!} \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2}}_{=0} k_1^2 + \frac{1}{3!} \frac{\mathrm{d}^3\sigma}{\mathrm{d}k^3} k_1^3 = k_m c_m + U_m k_1 + \frac{1}{6} U_m'' k_1^3 \qquad [19]$$

A similar expansion could be applied to the amplitude factor $\varphi(k)$, but as the minimum group velocity occurs near a maximum in this spectrum, we could approximately replace $\varphi(k)$ by its value $\varphi(k_m)$ right at the minimum group velocity and treat this as a constant in the integration. Introduce [19] into [18] and use now k_1 as integration variable ($\mathrm{d}k = \mathrm{d}k_1$ as $\mathrm{d}k_m = 0$):

$$\eta = \frac{\varphi(k_m)}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ik_m(c_m t - x)} \mathrm{e}^{ik_1(U_m t - x) + \frac{1}{6} iU_m'' t k_1^3} \, \mathrm{d}k_1 \qquad [20]$$

Make the substitution $k_1 = -i\lambda$:

$$\int_{-\infty}^{\infty} \mathrm{e}^{ik_1(U_m t - x) + \frac{1}{6} iU_m'' t k_1^3} \, \mathrm{d}k_1 = -i \int_{-i\infty}^{i\infty} \mathrm{e}^{\lambda(U_m t - x) - \frac{1}{6} U_m'' t \lambda^3} \, \mathrm{d}\lambda$$

Then, we substitute $\lambda = s \left(\frac{2}{U_m'' t}\right)^{1/3}$, which defines the new variable s, and get:

$$-i \left(\frac{2}{U_m'' t}\right)^{1/3} \int_{-i\infty}^{i\infty} \mathrm{e}^{\left[s\left(\frac{2}{U_m'' t}\right)^{1/3}(U_m t - x) - \frac{1}{3} s^3\right]} \mathrm{d}s$$

$$= \left(\frac{2}{U_m'' t}\right)^{1/3} 2\pi Ai \left[\left(\frac{2}{U_m'' t}\right)^{1/3} (U_m t - x)\right] \qquad [21]$$

Our final result is then:

$$\eta = \varphi(k_m) \left(\frac{2}{U_m'' t}\right)^{1/3} \cos\left[k_m(c_m t - x)\right] Ai \left[\left(\frac{2}{U_m'' t}\right)^{1/3} (U_m t - x)\right] \qquad [22]$$

taking only the real part of the solution.

For the minimum group velocity (the so-called *Airy phase*), $x = U_m t$. Replacing t in the amplitude factor by x from this relation, we see that for the Airy phase the amplitude decreases as $x^{-1/3}$, contrary to the case for other parts of the dispersion curve, where the amplitude decreases as $x^{-1/2}$ (see section 3.1.1.) This means that as the waves propagate, the amplitude decrease with x is slower for the Airy phase than for other waves, which means that at sufficiently large distances x the Airy phase will dominate the surface waves. This is also in good agreement with seismological observations.

Ai will reduce to a constant, when the argument is zero, eq.[9]. This happens exactly at the Airy phase: $U_m t - x = 0$.

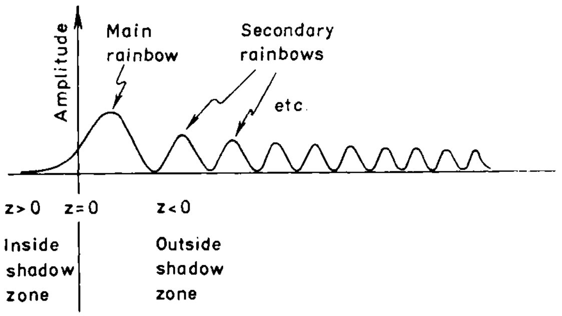


Fig.34.

Diffraction near a caustic surface

In *seismology*, we have this problem especially pronounced for diffraction at the caustic around 142–143° distance. JEFFREYS (1939) demonstrated theoretically that diffracted waves with periods equal to or less than 1 sec cannot be observed more than 3° into the shadow zone, i.e., down to about 139–140°. This was also the reason why other explanations had to be found for waves observed far into the shadow zone (see also PAYO SUBIZA and BÅTH, 1964).

In *optics*, diffraction phenomena are common. Notable examples are the diffraction phenomena in connection with the rainbow (see PERNTER and EXNER, 1922, pp.549–564 and Fig.34). This explains why the interior of a rainbow is darker than its exterior. The situation is similar for *PKP* in the earth. This is also the same diffraction problem as when the edge of an infinite opaque screen is illuminated from one side (see BAKER and COPSON, 1953).

Chapter 4

SERIES INTEGRATION

4.1 FUNDAMENTAL CONCEPTS

A function is *analytic* at a point if it is possible to expand it in a Taylor series valid in some neighbourhood of the point, in other words if the function is single-valued and possesses derivatives of all orders at the point (cf. section 2.1).

An *ordinary point* $x = x_0$ of the second-order differential equation:

$$y'' + X_1(x)y' + X_2(x)y = 0 \qquad [1]$$

is one at which the coefficients X_1, X_2 are analytic functions.

When, as is usually the case, X_1 and X_2 are polynomials of low degree, the solution is most easily found by assuming a power series of the form

$$y = \sum_{\lambda=0}^{\infty} a_\lambda (x - x_0)^\lambda \qquad [2]$$

for the solution and determining the coefficients $a_0, a_1, a_2, \ldots$ by direct substitution in [1] and equating the coefficients of the successive powers of x to zero.

Consider the simplest type of this equation: $y'' + y = 0$. Substituting:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda$$

we find:

$$\sum_{\lambda=0}^{\infty} \lambda(\lambda - 1)a_\lambda x^{\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda = 0$$

The first sum is equivalent to:

$$\sum_{\lambda=-2}^{\infty} (\lambda + 1)(\lambda + 2)a_{\lambda+2}x^\lambda$$

obtained by replacing λ by $\lambda + 2$.

Putting the sum of all coefficients of x^λ equal to zero, we have:

$$(\lambda + 1)(\lambda + 2)a_{\lambda+2} + a_\lambda = 0 \qquad [3]$$

which is called a *recurrence relation.* a_0 and a_1 are determined from boundary conditions, e.g., the values of y and y' at $x = 0$. We find the solution:

$$y = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

or, from the series expansions of cosine and sine:

$$y = a_0 \cos x + a_1 \sin x$$

$x = x_0$ is a *singular point* if X_1 and/or X_2 are not analytic at $x = x_0$ (as distinct from an ordinary point). $x = x_0$ is a *regular singular point* if X_1 and X_2 are such that eq.[1] can be written in the following form:

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0 \qquad [4]$$

and there is at least one solution of the form:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda (x - x_0)^{\varkappa+\lambda} \qquad [5]$$

If the Taylor expansions of $p(x)$ and $q(x)$ are valid for $|x - x_0| < R$, the solution [5] is valid in the same range.

Putting:

$$\left.\begin{aligned} p(x) &= \sum_{s=0}^{\infty} p_s (x - x_0)^s \\ q(x) &= \sum_{s=0}^{\infty} q_s (x - x_0)^s \end{aligned}\right| \qquad [6]$$

and substituting [5] and [6] into [4], we get:

$$\begin{aligned} &\sum_{\lambda=0}^{\infty} a_\lambda (\varkappa + \lambda)(\varkappa + \lambda - 1)(x - x_0)^{\varkappa+\lambda} \\ &+ \sum_{s=0}^{\infty} p_s (x - x_0)^s \sum_{\lambda=0}^{\infty} a_\lambda (\varkappa + \lambda)(x - x_0)^{\varkappa+\lambda} \\ &+ \sum_{s=0}^{\infty} q_s (x - x_0)^s \sum_{\lambda=0}^{\infty} a_\lambda (x - x_0)^{\varkappa+\lambda} = 0 \end{aligned} \qquad [7]$$

Equate the coefficient for $(x - x_0)^\varkappa$ to zero ($\lambda = 0$, $s = 0$):

$$a_0 \varkappa(\varkappa - 1) + p_0 a_0 \varkappa + q_0 a_0 = 0 \qquad [8]$$

Divide by a_0 ($a_0 \neq 0$ provided):

$$\varkappa^2 + (p_0 - 1)\varkappa + q_0 = 0 \qquad [9]$$

This is called the *indicial equation* (German: "determinierende Fundamentalgleichung") and determines $\varkappa$. Similarly, equating the coefficient for $(x - x_0)^{\varkappa+\lambda}$ in [7] to zero:

$$a_\lambda (\varkappa + \lambda)(\varkappa + \lambda - 1) + \sum_{s=0}^{\lambda} [p_s(\varkappa + \lambda - s) + q_s] a_{\lambda-s} = 0 \qquad [10]$$

Note that in eq.[7], in the 2nd and 3rd terms, we have also $(x - x_0)^s$, which necessitates taking a term not with index λ but with index $\lambda - s$, in the 2nd sum in [10]. Separating from the last sum, the term for $s = 0$, we have:

$$a_\lambda[(\varkappa + \lambda)(\varkappa + \lambda - 1) + p_0(\varkappa + \lambda) + q_0] + \sum_{s=1}^{\lambda} [p_s(\varkappa + \lambda - s) + q_s]a_{\lambda-s} = 0 \qquad [11]$$

This is a *recurrence relation*. The two roots $\varkappa_1, \varkappa_2$ of the indicial eq.[9] define each by [11] a series of a_λ values and the solutions:

$$\left.\begin{aligned} y_1(x) &= \sum_{\lambda=0}^{\infty} a_\lambda (x - x_0)^{\lambda+\varkappa_1} \\ y_2(x) &= \sum_{\lambda=0}^{\infty} a_\lambda' (x - x_0)^{\lambda+\varkappa_2} \end{aligned}\right| \qquad [12]$$

Three different cases arise:

(*1*) $\varkappa_1 - \varkappa_2$ is neither zero, nor an integer. The general solution is then:

$$y = y_1(x) + y_2(x) = \sum_{\lambda=0}^{\infty} a_\lambda (x - x_0)^{\lambda+\varkappa_1} + \sum_{\lambda=0}^{\infty} a_\lambda' (x - x_0)^{\lambda+\varkappa_2} \qquad [13]$$

(*2*) $\varkappa_1 = \varkappa_2$. The general solution is then:

$$\left.\begin{aligned} y &= y_1(x) + y_2(x) \\ y_1(x) &= (x - x_0)^{\varkappa_1} \sum_{\lambda=0}^{\infty} a_\lambda (x - x_0)^\lambda \\ y_2(x) &= \left(\frac{\partial y_1}{\partial \varkappa}\right)_{\varkappa=\varkappa_1} = y_1(x)\log(x - x_0) + (x - x_0)^{\varkappa_1} \sum_{\lambda=0}^{\infty} \left(\frac{\partial a_\lambda}{\partial \varkappa}\right)_{\varkappa=\varkappa_1} (x - x_0)^\lambda \end{aligned}\right| \qquad [14]$$

(*3*) $\varkappa_1 - \varkappa_2 = n$, where n is an integer. In this case all the coefficients a_λ in one of the solutions are, from some point onwards, either infinite or indeterminate. If, for instance, some of the coefficients of y become *infinite* when $\varkappa = \varkappa_2$, then we modify the form of y by replacing a_0 by $k(\varkappa - \varkappa_2)$. We then get two independent solutions by putting $\varkappa = \varkappa_2$ in the modified form of y and $\partial y/\partial \varkappa$. The result of putting $\varkappa = \varkappa_1$ in y gives only a numerical multiple of that obtained by putting $\varkappa = \varkappa_2$. An example of the case when one of the coefficients of y instead becomes *indeterminate* is given in the beginning of section 4.2.

Item *1* above must be considered the general case. For more details, the reader is referred to mathematical textbooks, e.g., PIAGGIO (1944, pp.110–117).

The integration method described is called *integration in series*. We shall study it in some detail in the following sections of this chapter. Eq.[1] is a type of differential equation which is common in applications. However, elementary methods can be used for its solution only when X_1 and X_2 are functions of special forms, but not in more general cases. Then, series integration offers a much more powerful method, as at least for certain ranges of x it is possible to express the solution in a convergent power series in x. The method is applicable to differential equations of any order, although we shall consider only second order equations. The method is often also referred to as the *Frobenius' method.*

4.1.1 Limitation in the use of the series integration method

The method of series integration cannot always be applied. For instance, one may find that all coefficients a are zero or that the exponent $\varkappa$ is infinite, or that the series does not converge. There is an important theorem, called *Fuchs' theorem*, by which one can say immediately whether the series integration method will work in any special case or not.

Fuchs' theorem: If the differential equation:

$$y'' + X_1 y' + X_2 y = 0 \qquad [15]$$

has a singular point at $x = x_0$, i.e., $X_1(x_0)$ and/or $X_2(x_0)$ are singular, then a development of the solution in a convergent power series around the point $x = x_0$ is possible if $(x - x_0)X_1(x_0)$ and $(x - x_0)^2 X_2(x_0)$ are regular. Then $x = x_0$ is a regular singular point.

An equation with no essential singularity (cf. section 2.1) in the entire complex plane is said to belong to the *Fuchsian class of differential equations*. For furthei discussion of Fuchs' theorem, I refer to textbooks, e.g., FRANK and VON MISES (1930, pp.317–322), PIAGGIO (1944, pp.211–214).

The method of series integration is of particular importance in a few types of differential equations:

Legendre: $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ [16]

Bessel: $x^2 y'' + xy' + (x^2 - n^2)y = 0$ [17]

Hermite: $y'' - 2xy' + 2ay = 0$ [18]

Laguerre: $xy'' + (1 - x)y' + ay = 0$ [19]

Gauss (hypergeometric): $(x^2 - x)y'' + [(1 + \alpha + \beta)x - \gamma]y' + \alpha\beta y = 0$ [20]

Riccati: $y' + by^2 = cx^m$ [21]

Mathieu: $y'' + (a + 16b \cos 2x)y = 0$ [22]

Quite generally, all these equations can be written in the following form:

$$X_0 y'' + X_1 y' + X_2 y = X_3 \qquad [23]$$

with the expressions for X_0, X_1, X_2, X_3 summarized in Table II.

The complete theory of series integration is a long chapter and only of more immediate mathematical interest. As the equations which we have use for are of a few well-defined types (several of them listed above), it will be sufficient to learn the solutions in these special cases, and omit the more general, mathematical discussions. The equations listed here are among the most important ones in applied mathematics, and their importance lies in the fact that solutions of partial differential equations in mathematical physics (see section 1.1) frequently lead to equations of some of these types.

TABLE II

SUMMARY OF VARIOUS DIFFERENTIAL EQUATIONS, WITH REFERENCE TO EQ.[23]

Equation	X_0	X_1	X_2	X_3
Legendre	$1 - x^2$	$-2x$	$n(n+1)$	0
Bessel	x^2	x	$x^2 - n^2$	0
Hermite	1	$-2x$	2α	0
Laguerre	x	$1 - x$	α	0
Gauss	$x^2 - x$	$(1 + \alpha + \beta)x - \gamma$	$\alpha\beta$	0
Riccati	0	1	by	cx^m
Mathieu	1	0	$a + 16b \cos 2x$	0

4.1.2 The Wronskian or the Wronskian determinant

Let y_1 and y_2 be two solutions of our linear differential eq.[1]:

$$y'' + X_1 y' + X_2 y = 0$$

i.e.:

$$\left.\begin{aligned} y_1'' + X_1 y_1' + X_2 y_1 &= 0 \\ y_2'' + X_1 y_2' + X_2 y_2 &= 0 \end{aligned}\right| \qquad [24]$$

Take eq.[24.1] $\times\, y_2$ — eq.[24.2] $\times\, y_1$, then:

$$y_1'' y_2 - y_2'' y_1 + X_1(y_1' y_2 - y_2' y_1) = 0 \qquad [25]$$

We define the *Wronskian* W of y_1, y_2:

$$W = y_1' y_2 - y_2' y_1 = \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} \qquad [26]$$

which gives after differentiation:

$$\frac{\mathrm{d}W}{\mathrm{d}x} = y_1'' y_2 - y_2'' y_1 \qquad [27]$$

Then [25] can be written as follows:

$$\frac{\mathrm{d}W}{\mathrm{d}x} + X_1 W = 0 \qquad [28]$$

which has the following solution:

$$W = Ae^{-\int X_1 dx} \tag{29}$$

with A = constant. If $A = 0$, then $W = 0$, and from [26] $y_1'y_2 = y_2'y_1$ which happens if y_1 = constant $\times$ y_2 (as seen immediately upon differentiation). For $A \neq 0$ it is possible to determine W directly from the differential eq.[1], except for the constant factor.

In conclusion, the solutions y_1 and y_2 are independent of each other only when W does not vanish.

4.2 LEGENDRE'S DIFFERENTIAL EQUATION

We begin with *Legendre's differential equation* and shall essentially follow MARGENAU and MURPHY (1943, pp.59–78). This equation reads:

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \tag{1}$$

n = constant. We assume that [1] has the following solution:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{\varkappa+\lambda} = x^\varkappa(a_0 + a_1x + a_2x^2 + \ldots) \tag{2}$$

We have to determine the coefficients a_λ in [2] such that [2] is a solution of [1]. At the moment we assume that this is possible and that [2] is a convergent series, which will permit differentiation term by term and substitution into [1]:

$$\sum_\lambda a_\lambda(\varkappa + \lambda)(\varkappa + \lambda - 1)x^{\varkappa+\lambda-2} - \sum_\lambda a_\lambda[(\varkappa + \lambda)(\varkappa + \lambda - 1)$$
$$+ 2(\varkappa + \lambda) - n(n + 1)]x^{\varkappa+\lambda} = 0 \tag{3}$$

Eq.[3] must hold for every value of x, which happens only if the coefficient of *every* power of x is identically zero. Take the coefficient of the lowest power of x, i.e., for $\lambda = 0$, i.e., the coefficient of $x^{\varkappa-2}$, and equate to zero:

$$a_0\varkappa(\varkappa - 1) = 0 \tag{4}$$

This equation is the *indicial equation* and it determines $\varkappa$:

$$\left.\begin{aligned} \varkappa &= 0 \\ \varkappa &= 1 \end{aligned}\right| \tag{5}$$

By the same principle, we have that the coefficient of $x^{\varkappa+j}$ must be zero for every integer $j \geqslant -2$. The corresponding coefficients in [3] are found from $\lambda = j + 2$ in the first sum, and $\lambda = j$ in the second sum:

$$a_{j+2}(\varkappa + j + 2)(\varkappa + j + 1) = a_j[(\varkappa + j)(\varkappa + j + 1) - n(n + 1)] \tag{6}$$

or:

$$a_{j+2} = \frac{(\varkappa + j)(\varkappa + j + 1) - n(n + 1)}{(\varkappa + j + 1)(\varkappa + j + 2)} a_j$$

which gives a *recurrence relation* between the coefficients in [2]. As [1] is of the second order, the solution should have two arbitrary coefficients; we can choose a_0 and a_1 as independent, arbitrary constants, and then all other coefficients follow from [6]. Then this case corresponds to item 3 in section 4.1, with a_0 indeterminate according to [4].

Case 1: $\varkappa = 0$. Eq.[6] becomes:

$$a_{j+2} = \frac{j(j+1) - n(n+1)}{(j+1)(j+2)} a_j \tag{7}$$

and the solution is, with a_0 and a_1 as arbitrary constants:

$$\begin{aligned} y &= \left[1 - \frac{n(n+1)}{2}x^2 - \frac{n(n+1)}{2}\cdot\frac{6-n(n+1)}{12}x^4 + \dots\right]a_0 \\ &+ \left[x + \frac{2-n(n+1)}{6}x^3 + \frac{2-n(n+1)}{6}\cdot\frac{12-n(n+1)}{20}x^5 + \dots\right]a_1 \\ &= \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \dots\right. \\ &\left. + (-1)^r\frac{n(n-2)\dots(n-2r+2)(n+1)\dots(n+2r-1)}{(2r)!}x^{2r} + \dots\right]a_0 \\ &+ \left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \dots\right. \\ &\left. + (-1)^r\frac{(n-1)(n-3)\dots(n-2r+1)(n+2)\dots(n+2r)}{(2r+1)!}x^{2r+1} + \dots\right]a_1 \end{aligned} \tag{8}$$

The general expressions for the terms in [8] can be found without difficulty, by continuing the same procedure step by step, and seeing how the general structure is.

Case 2: $\varkappa = 1$. Eq.[6] becomes now:

$$a_{j+2} = \frac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)} a_j \tag{9}$$

and the solution is:

$$\begin{aligned} y &= x\left[1 + \frac{2-n(n+1)}{6}x^2 + \frac{2-n(n+1)}{6}\cdot\frac{12-n(n+1)}{20}x^4 + \dots\right]a_0 \\ &+ x\left[x + \frac{6-n(n+1)}{12}x^3 + \frac{6-n(n+1)}{12}\cdot\frac{20-n(n+1)}{30}x^5 + \dots\right]a_1 \\ &= x\left[1 - \frac{(n-1)(n+2)}{3!}x^2 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^4 + \dots\right]a_0 \\ &+ x\left[x - \frac{(n-2)(n+3)}{12}x^3 + \frac{(n-2)(n-4)(n+3)(n+5)}{360}x^5 + \dots\right]a_1 \end{aligned} \tag{10}$$

The solutions obtained, [8] and [10], are written as follows for brevity:

$$\boxed{y = a_0 f_1(x)} + a_1 f_2(x) \qquad [8']$$

$$\boxed{y = a_0 f_3(x)} \underbrace{+ a_1 f_4(x)}_{\text{exclude}} \qquad [10']$$

We note that $f_2(x) = f_3(x)$, thus the same solution. Moreover, $f_4(x)$ is no solution at all: take, e.g., $n = 2$ and substitute $f_4(x)$ into [1]; then we find the left-hand side $= 2a_1$, which is $\neq 0$. As a consequence, we can write the significant parts of the solutions as follows:

$$y = a_0 f_1(x) \qquad [8'']$$

$$y = a_1 f_2(x) \qquad [10'']$$

Convergence test. We assumed that [2] is *convergent*, but is it convergent for all values of x? If successive terms in a series are denoted u, the convergence criterion states that:

$$\lim_{j\to\infty} \frac{|u_{j+2}|}{|u_j|} < 1 \quad \text{(De La Vallée Poussin, 1938, p.437.)} \qquad [11]$$

Applied to [8] and [10] this criterion becomes:

$$\lim_{j\to\infty} \frac{|a_{j+2}|}{|a_j|} x^2 = 1 \cdot x^2 = x^2 \qquad [12]$$

which is convergent for $x^2 < 1$. This means that the solutions [8] and [10] have significance only for $-1 < x < 1$, but for no other x-values.

Let us now find a significant solution of [1] *for* $x > 1$. We make a new assumption about the solution, instead of [2]:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{\varkappa-\lambda} \qquad [13]$$

and then proceed as above. Instead of [3] we get:

$$\sum_\lambda a_\lambda(\varkappa - \lambda)(\varkappa - \lambda - 1)x^{\varkappa-\lambda-2}$$
$$- \sum_\lambda a_\lambda[(\varkappa - \lambda)(\varkappa - \lambda + 1) - n(n+1)]x^{\varkappa-\lambda} = 0 \qquad [14]$$

The indicial equation is obtained by putting $\lambda = 0$ (coefficient of $x^\varkappa$):

$$\varkappa(\varkappa + 1) - n(n+1) = 0 \qquad [15]$$

with solutions $\varkappa = n$ and $\varkappa = -n - 1$. Depending on the value of n, we have all three cases for $\varkappa_1 - \varkappa_2$ listed in section 4.1. However, in this section we prefer to discuss the various cases from first principles. Instead of [6] we find, considering the coefficient of $x^{\varkappa-j}$:

$$a_{j-2}(\varkappa - j + 2)(\varkappa - j + 1) = a_j[(\varkappa - j)(\varkappa - j + 1) - n(n + 1)] \qquad [16]$$

($\lambda = j - 2$ in the first sum, $\lambda = j$ in the second sum) or, replacing j by $j + 2$:

$$a_{j+2} = \frac{(\varkappa - j)(\varkappa - j - 1)}{(\varkappa - j - 2)(\varkappa - j - 1) - n(n + 1)} a_j \qquad [17]$$

Case 1: $\varkappa = n$. Eq.[17] then becomes:

$$a_{j+2} = \frac{(n - j)(n - j - 1)}{(j + 2)(j - 2n + 1)} a_j \qquad [18]$$

and the solution is, choosing a_0 arbitrarily and putting $a_1 = 0$:

$$y = x^n \Big[1 - \frac{n(n-1)}{2(2n-1)} x^{-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)} x^{-4} - \dots$$
$$+ (-1)^r \frac{(n - 2r + 1)(n - 2r + 2) \dots (n - 1)n}{2r \dots 2(2n - 2r + 1) \dots (2n - 1)} x^{-2r}\Big] a_0 \qquad [19]$$

$a_1 = 0$ is a mere choice and [19] is only a *particular* solution.

Case 2: $\varkappa = -n - 1$. Eq.[17] becomes:

$$a_{j+2} = \frac{(j + n + 1)(j + n + 2)}{(j + 2)(2n + j + 3)} a_j \qquad [20]$$

and the solution, for $a_1 = 0$ again, is:

$$y = x^{-n-1} \Big[1 + \frac{(n+1)(n+2)}{2(2n+3)} x^{-2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4(2n+3)(2n+5)} x^{-4} + \dots$$
$$+ \frac{(n+1) \dots (n+2r)}{2 \cdot 4 \dots 2r(2n+3) \dots (2n+2r+1)} x^{-2r} + \dots\Big] a_0 \qquad [21]$$

The solutions [19] and [21] are independent and therefore their sum is the *general* solution of [1], convergent for $|x| > 1$ (except when n has such a value that the denominator of some coefficient vanishes, making the corresponding term infinite).

Thus, in summary: [8] and [10] are solutions for $|x| < 1$, [19] and [21] are solutions for $|x| > 1$.

The infinite series become polynomials for integer values of n. So far, n could have any (constant) value, except integral and halfintegral values. We shall first restrict ourselves to *integer* values of n. When in the recurrence relations [6] or [17] some coefficient becomes zero, all following coefficients are zero, and the series becomes a polynomial.

(*a*) *n even positive*: [8″] → [19] (by this writing we mean that these two equations become identical). With $n = 2r$, [8″] becomes:

$$y = a\left[1 - \frac{n(n+1)}{2!}x^2 + \ldots\right.$$

$$\left. + (-1)^{n/2}\frac{n(n-2)\ldots 2(n+1)\ldots(2n-1)}{n!}x^n\right] \qquad [22]$$

and [19] becomes:

$$y = ax^n\left[1 - \frac{n(n-1)}{2(2n-1)}x^{-2} + \ldots\right.$$

$$\left. + (-1)^{n/2}\frac{n!}{n(n-2)\ldots 2(n+1)\ldots(2n-1)}x^{-n}\right] \qquad [23]$$

Multiply [23] by the constant factor:

$$(-1)^{n/2}\frac{n(n-2)\ldots 2(n+1)\ldots(2n-1)}{n!}$$

and it agrees with [22].

(*b*) *n odd negative:* [8''] → [21]: By a similar argument.

(*c*) *n odd positive:* [10''] → [19]. With $n = 2r + 1$, [10''] becomes:

$$y = a\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \ldots\right.$$

$$\left. + (-1)^{(n-1)/2}\frac{(n-1)(n-3)\ldots 2(n+2)\ldots(2n-1)}{n!}x^n\right] \qquad [24]$$

and [19]:

$$y = ax^n\left[1 - \frac{n(n-1)}{2(2n-1)}x^{-2} + \ldots\right.$$

$$\left. + (-1)^{(n-1)/2}\frac{n!}{2\cdot 4\ldots(n-1)(n+2)\ldots(2n-1)}x^{-n+1}\right] \qquad [25]$$

which agree after appropriate multiplication with constant factors.

(*d*) *n even negative*: [10''] → [21]: By a similar argument.

This scheme follows immediately from the rule, that in each case we have to look for those values of n which will make the coefficients in the series expansions [8''], [10''], etc., vanish. In case of integer values of n, we have shown that the system of solutions [8''], [10''] coincides with the system [19], [21], and the last two are sufficient to give the general solution of [1] (for integer n).

Assume n to be an integer and:

(*1*) in [19] put:

$$a_0 = \frac{(2n)!}{2^n(n!)^2} = \frac{(2n-1)(2n-3)\dots 1}{n!} \qquad [26]$$

Then the solution y is denoted $P_n(x)$, and is called a *Legendre polynomial of the first kind* of degree n :

$$P_n(x) = \frac{1\cdot 3\cdot 5\dots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2}\right.$$
$$\left. + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \dots\right] \qquad [27]$$

n being a positive integer, we continue down to the constant term (a polynomial).

(*2*) Similarly, in [21] define:

$$a_0 = \frac{2^n(n!)^2}{(2n+1)!} \qquad [28]$$

and the corresponding solution [21] $y = Q_n(x)$ is called a *Legendre polynomial of the second kind:*

$$Q_n(x) = \frac{n!}{1\cdot 3\dots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \dots\right.$$
$$\left. + \frac{(n+1)\dots(n+2r)}{2\cdot 4\dots 2r(2n+3)\dots(2n+2r+1)}x^{-n-2r-1} + \dots\right] \qquad [29]$$

If n is a negative integer it follows that [29] is a polynomial.

The general solution of [1] for integer values of n is the sum of [19] or [27] *and* [21] or [29], and from above it follows that: n positive integer, [19], [27] polynomial, [21], [29] infinite series; n negative integer, [21], [29] polynomial, [19], [27] infinite series.

We then consider two cases, when n is half-integral:

(*a*) $2n$ is an odd positive integer: $2n = 2m - 1$: [19] $\rightarrow$ [21], i.e., [19] degenerates into [21]. Consider [19] for $2n = 2m - 1$, then $2n - 2m + 1 = 0$ and the denominator in the coefficient of x^{n-2m} vanishes, making the coefficient infinite, as well as all following terms in [19]. To remove these infinities we multiply the series [19] by $(m - r)$, which causes all terms of higher order than $n - 2m$ to vanish, while the others remain finite. Then the series begins with the power $x^{n-2m} = x^{-n-1}$ and this is identical with [21].

(*b*) $2n$ is an odd negative integer: [21] $\rightarrow$ [19], by analogous argument. By the method used here, we obtain only particular solutions in this and the preceding cases but no general solution.

4.2.1 Physical applications

In most applications of the Legendre equation we have $x = \cos\theta$, i.e., $-1 \leqslant x \leqslant +1$. We are only interested in functions which are finite (i.e., polynomials) for such

x-values including ± 1. This happens only for n integer. Then we have only [19] plus [21] to consider, as other solutions reduce to these. Also [21] is reduced to [19] if n is replaced by $-(n + 1)$. So we need only consider the solution [19], identical with [27] for $n \geqslant 0$. Therefore, the only solution of Legendre's equation which is of practical importance is $P_n(\cos\theta)$.

4.2.2 Relations between $P_n(x)$ and $Q_n(x)$ [1]

In [27] and [29] we defined $P_n(x)$ and $Q_n(x)$ for the case that n is an integer. Now we generalize these definitions to any value of n (i.e., not only integer values). As we cannot then use factorials in the expressions for P and Q (as factorials are defined only for integer values), we introduce instead Gauss' Π-function, which is related to the gamma function (section 1.3) and to factorials as follows:

$$\Pi(n) = \Gamma(n+1) = n! \tag{30}$$

(the last equality valid only for integer n). Then the definitions of P and Q become, corresponding to [27] and [29]:

$$P_n(x) = \frac{\Pi(2n)}{2^n\Pi(n)\Pi(n)}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \ldots\right] \tag{31}$$

agreeing with [6] in section 6.1 below, and:

$$Q_n(x) = \frac{2^n\Pi(n)\Pi(n)}{\Pi(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \ldots\right] \tag{32}$$

P and Q are linearly independent solutions of Legendre's equation. There are some (non-linear) relations between P and Q, which can be written in various forms (see section 6.2.2 below), some of which we shall now deduce.

As P_n and Q_n are solutions of Legendre's differential equation, we have:

$$\left.\begin{aligned} &(1-x^2)\frac{\mathrm{d}^2P_n}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}P_n}{\mathrm{d}x} + n(n+1)P_n = 0 \\ &(1-x^2)\frac{\mathrm{d}^2Q_n}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}Q_n}{\mathrm{d}x} + n(n+1)Q_n = 0 \end{aligned}\right| \tag{33}$$

From [33.1] Q_n — [33.2] P_n we get:

$$(x^2-1)\left(Q_n\frac{\mathrm{d}^2P_n}{\mathrm{d}x^2} - P_n\frac{\mathrm{d}^2Q_n}{\mathrm{d}x^2}\right) + 2x\left(Q_n\frac{\mathrm{d}P_n}{\mathrm{d}x} - P_n\frac{\mathrm{d}Q_n}{\mathrm{d}x}\right) = 0 \tag{34}$$

which upon integration gives:

$$(x^2-1)\left(Q_n\frac{\mathrm{d}P_n}{\mathrm{d}x} - P_n\frac{\mathrm{d}Q_n}{\mathrm{d}x}\right) = A \tag{35}$$

[1] FORSYTH (1912, pp.175–182).

A being an integration constant. Eq.[35] holds foı all values of x. We determine the value of the constant A in [35] by considering the limit of the left-hand side for $x \to \infty$. From [31] and [32], we have for $x \to \infty$, considering that $\Pi(2n + 1) = (2n + 1)\Pi(2n)$:

$$\left.\begin{aligned} Q_n \frac{\mathrm{d}P_n}{\mathrm{d}x} &\to \frac{n}{2n+1} \cdot \frac{1}{x^2} \\ P_n \frac{\mathrm{d}Q_n}{\mathrm{d}x} &\to -\frac{n+1}{2n+1} \cdot \frac{1}{x^2} \end{aligned}\right| \tag{36}$$

Introducing this into [35], still letting $x \to \infty$, we get $A = 1$, and [35] becomes:

$$Q_n \frac{\mathrm{d}P_n}{\mathrm{d}x} - P_n \frac{\mathrm{d}Q_n}{\mathrm{d}x} = \frac{1}{x^2 - 1} \tag{37}$$

or:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q_n}{P_n}\right) = \frac{1}{(1 - x^2)P_n^2} \tag{38}$$

Integrating [38] between the limits $x = \infty$ and $x = x$, remembering that for $x = \infty$ we have $Q_n(x)/P_n(x) = 0$ (see eq.[31] and [32]), we finally get:

$$\frac{Q_n}{P_n} = -\int_\infty^x \frac{\mathrm{d}x}{(x^2 - 1)P_n^2} = \int_x^\infty \frac{\mathrm{d}x}{(x^2 - 1)P_n^2} \tag{39}$$

This is one of the relations between P and Q often referred to.

Another relation (eq.[10] in section 6.2.2) is the following:

$$Q_n(x) = \frac{1}{2} P_n(x) \log \frac{x+1}{x-1} - R_n(x) \tag{40}$$

where $R_n(x)$ is a polynomial of degree $(n - 1)$ in case n is an integer. Eq.[40] can be derived from [39], which we shall now demonstrate.

The integral in [39] can be split up as follows:

$$\int_x^\infty \frac{\mathrm{d}x}{(x^2 - 1)P_n^2} = \frac{1}{2}\int_x^\infty \frac{\mathrm{d}x}{x-1} - \frac{1}{2}\int_x^\infty \frac{\mathrm{d}x}{x+1} + \int_x^\infty \frac{(1 - P_n^2)\mathrm{d}x}{(x^2 - 1)P_n^2} \tag{41}$$

Carrying out the integrations we get:

$$\int_x^\infty \frac{\mathrm{d}x}{(x^2 - 1)P_n^2} = \underbrace{-\frac{1}{2}\left[\log \frac{x+1}{x-1}\right]_x^\infty}_{= +\frac{1}{2}\log \frac{x+1}{x-1}} + \int_x^\infty \frac{(1 - P_n^2)\mathrm{d}x}{(x^2 - 1)P_n^2} \tag{42}$$

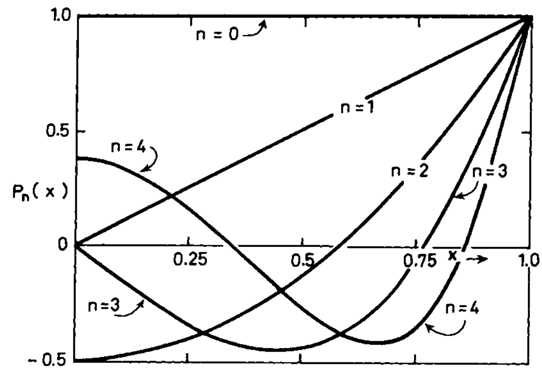


Fig.35.

$R_n(x)$ in [40] is thus:

$$R_n(x) = -P_n \int_x^\infty \frac{(1-P_n^2)\mathrm{d}x}{(x^2-1)P_n^2} \to x^n \int_x^\infty \frac{x^{2n}\mathrm{d}x}{x^{2+2n}} = x^n \int_x^\infty \frac{\mathrm{d}x}{x^2} = -x^n \left[\frac{1}{x}\right]_x^\infty = x^{n-1} \quad [43]$$

(see section 2.1.4), i.e., a polynomial in x of degree $n - 1$, which should be proved. Some of the lowest orders of $P_n(x)$ are shown in Fig.35.

In this section we have worked out the solutions of the Legendre differential equation from first principles. Instead, we could have applied the general scheme for obtaining the solutions, outlined in section 4.1. This is left as an exercise to the reader. The results should naturally be identical with those given here.

4.3 BESSEL'S DIFFERENTIAL EQUATION

As an alternative to the treatment in section 4.2, we shall here apply the general scheme of section 4.1. However, it is suggested to the reader as an exercise to work out the solutions also in the present case from first principles, just as in section 4.2. Such a treatment can be found in FORSYTH (1912, pp.182–192).

We write Bessel's differential equation as follows:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0 \quad [1]$$

or:

$$x^2y'' + xy' + (-n^2 + x^2)y = 0 \quad [2]$$

where n is in general non-integral. By identification with [4] in section 4.1, we see that $x = 0$ is a regular singular point, and that $p_0 = 1$ and $q_0 = -n^2$, as we have from [6] in 4.1 that:

$$q(x) = \sum_{s=0}^{\infty} q_s x^s = q_0 + q_1 x + q_2 x^2 + \ldots = -n^2 + x^2$$

$$\text{i.e., } q_0 = -n^2, \quad q_1 = 0, \quad q_2 = 1$$

Then from [9] in section 4.1 the indicial equation is:

$$\varkappa^2 - n^2 = 0$$

which has the roots:

$$\left.\begin{aligned} &\varkappa = \pm n \\ &\text{i.e.:} \\ &\varkappa_1 - \varkappa_2 = 2n \end{aligned}\right| \qquad [3]$$

Depending on whether n is an integer or not, we have to distinguish between the cases 1–3 of section 4.1.

(*1*) $\varkappa_1 - \varkappa_2 = 2n$; n is neither zero, nor an integer. Then, the first solution is:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+n} \qquad [4]$$

Substitute this in [2]:

$$\sum_{\lambda=0}^{\infty} [(\lambda + n)(\lambda + n - 1) + (\lambda + n) - n^2]a_\lambda x^{\lambda+n} + \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+n+2} = 0$$

which determines the coefficients a_λ. Putting the coefficient of $x^{\lambda+n} = 0$ gives:

$$a_\lambda[(\lambda + n)^2 - n^2] = -a_{\lambda-2} \qquad [5]$$

In particular, $\lambda = 1$ (note that $\lambda \geqslant 0$, according to [4], or that $a_\lambda = 0$ for $\lambda < 0$):

$$a_1[(1 + n)^2 - n^2] = 0$$

Therefore, $a_1 = 0$, and by [5], also all $a_{2\lambda+1} = 0$. From [5], replacing λ by 2λ and noting that $(2\lambda + n)^2 - n^2 = (2\lambda + 2n)2\lambda$ we have:

$$a_{2\lambda} = \frac{(-1)^\lambda a_0}{(2\lambda + 2n)(2\lambda + 2n - 2) \ldots (2n + 2) \cdot 2\lambda(2\lambda - 2) \ldots 2}$$

$$= \frac{a_0}{\lambda!(n + 1)_\lambda}\left(-\frac{1}{4}\right)^\lambda$$

obtained from the recurrence form:

$$a_{2\lambda} = \frac{-a_{2\lambda-2}}{(2\lambda + 2n) \cdot 2\lambda} \quad ; \quad \ldots \quad ; \quad a_2 = \frac{-a_0}{(2 + 2n) \cdot 2} \quad \text{for } \lambda = 1$$

and where $(n + 1)_\lambda$ stands for $(n + 1)(n + 2) \ldots (n + \lambda)$. We put:

$$a_0 = \frac{1}{2^n n!}$$

and [4] becomes:

$$y = \frac{x^n}{2^n\Gamma(n+1)} \sum_{\lambda=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^\lambda}{\lambda!(n+1)_\lambda} = J_n(x) \qquad [6]$$

In this context, eq.[6] is taken as a definition of $J_n(x)$, i.e., the *Bessel function of the first kind*, where n is not an integer. Compare eq.[27] in section 5.2 below.

Similarly, for the other root of the indicial equation:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda-n} = J_{-n}(x) \qquad [7]$$

Thus, when n is *not* an integer we may write the general solution of [1] as follows:

$$y = AJ_n(x) + BJ_{-n}(x) \qquad [8]$$

(2) $\varkappa_1 - \varkappa_2 = 2n = 0$, i.e., $n = 0$.

Put:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+\varkappa}$$

In this case:

$$a_0 = \frac{1}{2^n n!} = 1 \quad \text{for } n = 0$$

Then we get from the expression for $a_{2\lambda}$ above:

$$a_{2\lambda} = \frac{\left(-\frac{1}{4}\right)^\lambda}{\lambda!(\varkappa+1)_\lambda}$$

and:

$$y = x^\varkappa \sum_{\lambda=0}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^\lambda}{\lambda!(\varkappa+1)_\lambda}$$

In particular: $\varkappa = 0$ gives $y_0 = J_0(x)$, i.e., the first solution, by [14] in 4.1. For the second solution, we need an expression for the following derivative:

$$\frac{\partial}{\partial\varkappa}\frac{1}{(\varkappa+1)_\lambda} = \frac{\partial}{\partial\varkappa}\frac{1}{(\varkappa+1)\dots(\varkappa+\lambda)} = \frac{1}{(\varkappa+2)\dots(\varkappa+\lambda)}\frac{\partial}{\partial\varkappa}\frac{1}{\varkappa+1}$$
$$+\frac{1}{(\varkappa+1)\dots(\varkappa+\lambda)}\frac{\partial}{\partial\varkappa}\frac{1}{\varkappa+2} + \dots + \frac{1}{(\varkappa+1)\dots(\varkappa+\lambda-1)}\frac{\partial}{\partial\varkappa}\frac{1}{\varkappa+\lambda}$$

$$= \frac{\varkappa+1}{(\varkappa+1)_\lambda}\frac{\partial}{\partial\varkappa}\frac{1}{\varkappa+1} + \frac{\varkappa+2}{(\varkappa+1)_\lambda}\frac{\partial}{\partial\varkappa}\frac{1}{\varkappa+2} + \ldots + \frac{\varkappa+\lambda}{(\varkappa+1)_\lambda}\frac{\partial}{\partial\varkappa}\frac{1}{\varkappa+\lambda}$$

$$= \frac{1}{(\varkappa+1)_\lambda}\sum_{s=1}^{\lambda}(\varkappa+s)\frac{\partial}{\partial\varkappa}\frac{1}{\varkappa+s} = -\frac{1}{(\varkappa+1)_\lambda}\sum_{s=1}^{\lambda}\frac{1}{\varkappa+s}$$

Using that:

$$\frac{dA^x}{dx} = A^x \log A$$

we find:

$$\frac{\partial y}{\partial\varkappa} = y\log x - x^\varkappa \sum_{\lambda=1}^{\infty}\frac{\left(-\frac{x^2}{4}\right)^\lambda}{\lambda!(\varkappa+1)_\lambda}\sum_{s=1}^{\lambda}\frac{1}{\varkappa+s}$$

Put $\varkappa = 0$ and the second solution is $\left(\frac{\partial y}{\partial\varkappa}\right)_{\varkappa=0}$ by [14] in section 4.1, or:

$$\mathrm{Y}_0(x) = J_0(x)\log x - \sum_{\lambda=1}^{\infty}\frac{\left(-\frac{x^2}{4}\right)^\lambda}{(\lambda!)^2}\varphi(\lambda) \qquad [9]$$

where:

$$\varphi(\lambda) = \sum_{s=1}^{\lambda}\frac{1}{s}$$

$\mathrm{Y}_0(x)$ is called Neumann's Bessel function of the second kind of zero order, or simply *Neumann's function of zero order.*

Neumann's function plus any constant multiple of $J_0(x)$ is also a solution of eq.[1] with $n = 0$:

$$y'' + \frac{1}{x}y' + y = 0$$

In particular, the function:

$$Y_0(x) = \frac{2}{\pi}\left[\mathrm{Y}_0(x) - (\log 2 - \gamma)J_0(x)\right] \qquad [10]$$

is a solution, where γ is Euler's constant, defined by:

$$\gamma = \lim_{m\to\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{m} - \log m\right) = 0.5772\ldots$$

Introducing the expression [9] for Neumann's function, we obtain:

$$Y_0(x) = \frac{2}{\pi}\left(\log\frac{x}{2} + \gamma\right)J_0(x) - \frac{2}{\pi}\sum_{\lambda=1}^{\infty}\frac{\left(-\frac{x^2}{4}\right)^\lambda}{(\lambda!)^2}\varphi(\lambda) \qquad [11]$$

The function $Y_0(x)$ is called *Weber's Bessel function of the second kind of zero order.*

The complete solution is:

$$y = AJ_0(x) + BY_0(x) \qquad [12]$$

(*3*) $\varkappa_1 - \varkappa_2 = 2n$; n is an integer. By a similar argument as in (*2*), it is shown that the *complete* solution is:

$$y = AJ_n(x) + BY_n(x)$$

where:

$$Y_n(x) = \frac{2}{\pi}\left(\log\frac{x}{2} + \gamma\right)J_n(x) - \frac{1}{\pi}\sum_{\lambda=0}^{n-1}\frac{(n-\lambda-1)!}{\lambda!}\left(\frac{2}{x}\right)^{n-2\lambda}$$

$$-\frac{1}{\pi}\sum_{\lambda=0}^{\infty}\frac{(-1)^\lambda\left(\dfrac{x}{2}\right)^{n+2\lambda}}{\lambda!(\lambda+n)!}[\varphi(\lambda+n)+\varphi(\lambda)] \qquad [13]$$

This is called *Weber's Bessel function of the second kind of order n.* See JEFFREYS and JEFFREYS (1946, pp.545–546). Eq.[13] is identical with eq.[42] in JEFFREYS and JEFFREYS (1946, p.546) which is immediately seen when we consider that the relation between the so-called digamma function $F(\lambda)$ and our function $\varphi(\lambda)$ is:

$$F(\lambda) = \varphi(\lambda) - \gamma$$

so that:

$$F(\lambda+n) + F(\lambda) = \varphi(\lambda+n) + \varphi(\lambda) - 2\gamma$$

For $n = 0$ eq.[13] degenerates into [11].

Defining another related pair of Bessel functions as follows:

$$\left.\begin{aligned} H_n^{(1)}(x) &= J_n(x) + iY_n(x) \\ H_n^{(2)}(x) &= J_n(x) - iY_n(x) \end{aligned}\right| \qquad [14]$$

it is possible to write the general solution also in the following way:

$$y = AH_n^{(1)}(x) + BH_n^{(2)}(x) \qquad [15]$$

with A and B as arbitrary constants. The functions $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are called Hankel's Bessel functions of the third kind of order n or simply *Hankel functions of order n.* $H_n^{(1)}(x)$ is generally referred to as a Hankel function of the first kind, and $H_n^{(2)}(x)$ as a Hankel function of the second kind.

Adding the two equations [14] we get the following formula:

$$2J_n(x) = H_n^{(1)}(x) + H_n^{(2)}(x) \qquad [16]$$

which is used sometimes to replace a Bessel function by a pair of Hankel functions. This

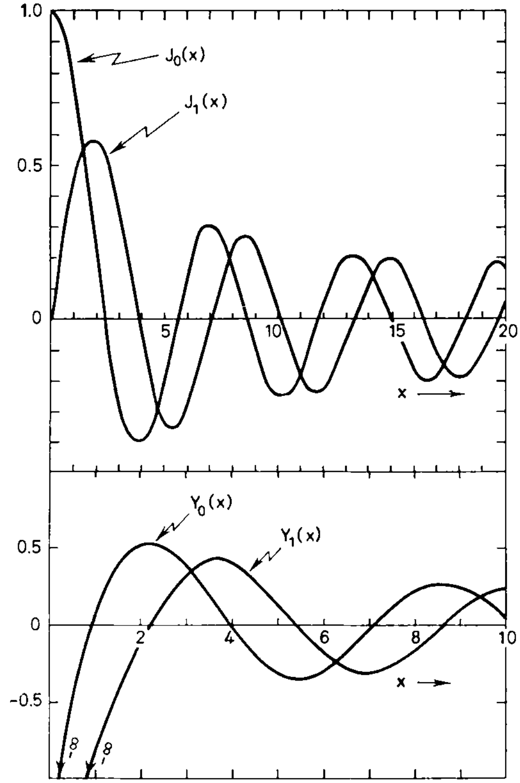


Fig.36.

may be useful for instance in an integral containing a Bessel function, where the integrations could be more easily carried out if the Bessel function is replaced according to the formula [16]; examples are given in EWING, JARDETZKY and PRESS (1957, p.98), and at many other places.

The Hankel functions bear the same relation to the Bessel functions as $e^{\pm i\nu x}$ bears to $\cos\nu x$, $\sin\nu x$ and they are used in analysis for similar reasons.

An alternative to the expression [13] for $Y_n(x)$ is the following:

$$Y_n(x) = \frac{\cos n\pi \, J_n(x) - J_{-n}(x)}{\sin n\pi} \qquad [17]$$

which can be taken as a definition of $Y_n(x)$. It can be shown that for integer orders n we get the expansion [13] by inserting the series expansions for $J_n(x)$ and $J_{-n}(x)$ from eq.[6].

This expression for $Y_n(x)$ defines always a second independent solution of the Bessel equation, for any order n.

Combining the definitions of $Y_n(x)$ in [17] and of $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ in [14], we arrive at the following formulas, by elimination of $Y_n(x)$:

$$\left.\begin{aligned} H_n^{(1)}(x) &= \frac{i}{\sin n\pi}\,[e^{-n\pi i}J_n(x) - J_{-n}(x)] \\ H_n^{(2)}(x) &= \frac{-i}{\sin n\pi}\,[e^{n\pi i}J_n(x) - J_{-n}(x)] \end{aligned}\right| \qquad [18]$$

Some of the lowest orders of $J_n(x)$ and $Y_n(x)$ are shown in Fig.36.

Finally, it should be remarked that the definitions [14] of the Hankel functions and [17] of the Weber function are valid generally, i.e., for any n, integer or not. This is then also the case with [16] and [18].

4.3.1 Application of the Wronskian to Bessel's differential equation

Consider the equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \qquad [19]$$

The Wronskian (section 4.1.2) of any two solutions y_1 and y_2 is:

$$W(y_1, y_2) = Ae^{-\int\frac{dx}{x}} = \frac{A}{x} \qquad [20]$$

In particular:

$$J_n'(x)J_{-n}(x) - J_{-n}'(x)J_n(x) = \frac{A}{x} \qquad [21]$$

where the constant A is found from the series expansions of the Bessel functions, eq.[6]:

$$A = \frac{2\sin n\pi}{\pi} \qquad [22]$$

The expression [22] is seen to be different from zero when n is neither zero, nor an integer. That is, in such cases do $J_n(x)$ and $J_{-n}(x)$ represent two linearly independent solutions of [19]. This agrees with the results above in this section.

Similarly we can demonstrate that $J_0(x)$ and $Y_0(x)$ are independent solutions for $n = 0$, and that $J_n(x)$ and $Y_n(x)$ are independent solutions for n an integer.

4.3.2 Relation between Legendre's and Bessel's differential equations

We shall derive Bessel's equation from Legendre's. We differentiate Legendre's equation:

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad [23]$$

m times, putting $z = \mathrm{d}^m y/\mathrm{d}x^m$ and using Leibniz' rule for m-time differentiation of products (FRANK and VON MISES, 1930, p.5):

$$(1 - x^2)\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} - (2m + 2)x\frac{\mathrm{d}z}{\mathrm{d}x} + [n(n + 1) - m(m + 1)]\, z = 0 \tag{24}$$

Replace the *dependent* variable z by ξ from the relation:

$$z = (1 - x^2)^{-m/2}\xi \tag{25}$$

Then [24] becomes:

$$(1 - x^2)\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}\xi}{\mathrm{d}x} + \left[n(n + 1) - \frac{m^2}{1 - x^2}\right]\xi = 0 \tag{26}$$

which is found by straight-forward differentiation of [25]. Then we replace the *independent* variable x by φ by the substitution:

$$\varphi^2 = n^2(1 - x^2) \tag{27}$$

This is done by using the following formulas:

$$\left.\begin{aligned}
&\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{\mathrm{d}\xi}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}x}\\
&\frac{\mathrm{d}\varphi}{\mathrm{d}x} = -\frac{n^2 x}{\varphi}\\
&\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} = \frac{\mathrm{d}\xi}{\mathrm{d}\varphi}\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \frac{\mathrm{d}\varphi}{\mathrm{d}x}\frac{\mathrm{d}^2\xi}{\mathrm{d}x\mathrm{d}\varphi}\\
&\frac{\mathrm{d}^2\xi}{\mathrm{d}x\mathrm{d}\varphi} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\varphi}\right) = \frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\varphi}\right)\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{\mathrm{d}^2\xi}{\mathrm{d}\varphi^2}\frac{\mathrm{d}\varphi}{\mathrm{d}x}
\end{aligned}\right| \tag{28}$$

(remember that in the last formula we differentiate a second time *with respect to x*) and then [26] transforms into:

$$\left(1 - \frac{\varphi^2}{n^2}\right)\frac{\mathrm{d}^2\xi}{\mathrm{d}\varphi^2} + \left(1 - \frac{2\varphi^2}{n^2}\right)\frac{1}{\varphi}\frac{\mathrm{d}\xi}{\mathrm{d}\varphi} + \left(1 + \frac{1}{n} - \frac{m^2}{\varphi^2}\right)\xi = 0 \tag{29}$$

Finally, in [29] let $n \to \infty$:

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\varphi^2} + \frac{1}{\varphi}\frac{\mathrm{d}\xi}{\mathrm{d}\varphi} + \left(1 - \frac{m^2}{\varphi^2}\right)\xi = 0 \tag{30}$$

which is Bessel's differential equation, thus derived from Legendre's differential equation.

4.4 HERMITE'S DIFFERENTIAL EQUATION

This equation reads:

$$y'' - 2xy' + 2ay = 0 \tag{1}$$

α = constant. We solve it by integration in series and follow exactly the procedure used in the solution of Legendre's equation in section 4.2. Thus we put:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{\varkappa+\lambda} \tag{2}$$

By differentiation of [2] and substitution into [1] we find:

$$\sum_\lambda a_\lambda(\varkappa + \lambda)(\varkappa + \lambda - 1)x^{\varkappa+\lambda-2} - 2 \sum_\lambda a_\lambda(\varkappa + \lambda - \alpha)x^{\varkappa+\lambda} = 0 \tag{3}$$

Eq.[3] gives for $\lambda = 0$ the indicial equation (i.e., power $\varkappa - 2$ of x):

$$a_0\varkappa(\varkappa - 1) = 0 \tag{4}$$

which has the roots $\varkappa = 0$ and 1. Again [3] gives for $(\varkappa + j)^{\text{th}}$ power of x the recurrence relation between the coefficients in [2]:

$$a_{j+2} = \frac{2(\varkappa + j) - 2\alpha}{(\varkappa + j + 2)(\varkappa + j + 1)} a_j \tag{5}$$

First for $\varkappa = 0$ the recurrence relation [5] becomes:

$$a_{j+2} = \frac{2j - 2\alpha}{(j + 2)(j + 1)} a_j \tag{6}$$

and the series solution is:

$$y = a_0 \left[1 - \frac{2\alpha}{2!}x^2 + \frac{2^2\alpha(\alpha - 2)}{4!}x^4 - \frac{2^3\alpha(\alpha - 2)(\alpha - 4)}{6!}x^6 + \dots \right.$$
$$\left. \dots + (-2)^r \frac{\alpha(\alpha - 2)\dots(\alpha - 2r + 2)}{(2r)!} x^{2r} + \dots \right] \tag{7}$$

Secondly, for $\varkappa = 1$, the recurrence relation [5] is:

$$a_{j+2} = \frac{2(1 + j) - 2\alpha}{(j + 3)(j + 2)} a_j \tag{8}$$

and the series solution is:

$$y = a_0 x \left[1 - \frac{2(\alpha - 1)}{3!}x^2 + \frac{2^2(\alpha - 1)(\alpha - 3)}{5!}x^4 + \dots \right.$$
$$\left. + (-2)^r \frac{(\alpha - 1)(\alpha - 3)\dots(\alpha - 2r + 1)}{(2r + 1)!} x^{2r} + \dots \right] \tag{9}$$

Quite generally, we should have expected the series to start with a_0 and with a_1 in both [7] and [9], or formally written:

$$\left.\begin{array}{ll} \varkappa = 0: & y = a_0[\ldots] + a_1[\ldots] \\ \varkappa = 1: & y = a_0[\ldots] + \underbrace{a_1[\ldots]}_{\text{no solution}} \end{array}\right| \qquad [10]$$

But, from the recurrence relations [6] and [8] we find that the second series in [10.1] is identical with the first in [10.2] and also that the second series in [10.2] is no solution of the given equation [1]; this is easily tested by direct substitution of this series in [1]. Thus, the circumstances are quite similar to Legendre's equation. And [7] and [9] together give the general solution of Hermite's equation.

So far, we have only postulated that α should be constant, But if we further restrict values of α to integers, we have two cases of great importance:

(*1*) α is an *even integer* n: [7] reduces to an even polynomial of degree n. Furthermore, we put:

$$a_0 = (-1)^{n/2}\frac{n!}{(n/2)!} \qquad [11]$$

As n is an even integer, $n/2$ is an integer, and $(n/2)!$ has sense. Then the polynomial [7] becomes:

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \ldots + \underbrace{(-1)^{n/2}\frac{n!}{(n/2)!}}_{\substack{\text{[this } (= a_0) \text{ is the} \\ \text{constant term]}}} \qquad [12]$$

The order of the terms in [12] is reversed to [7]. The correctness of [12] is easily demonstrated term by term; for instance taking $2r = n$ or $r = n/2$ in [7], we get the first term in $H_n(x)$; the next term is found by replacing $2r$ by $2r - 2 = n - 2$ in [7], or replacing r by $r - 1 = n/2 - 1$.

(*2*) α is an *odd integer* n: [9] reduces to an odd polynomial of degree n. Also, putting

$$a_0 = (-1)^{(n-1)/2}\frac{2\cdot n!}{\left(\dfrac{n-1}{2}\right)!} \qquad [13]$$

we arrive at exactly the same expression for $H_n(x)$ as in [12].

The polynomial [12] can also be written as follows in more compact form:

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}) \qquad [14]$$

That [12] and [14] are identical is easily seen by an induction proof. The identity is clear for low orders, e.g.:

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2, \qquad \text{etc.}$$

Solving for the derivative in [14], we then say that if:

$$\frac{d^n}{dx^n}(e^{-x^2}) = \frac{(2x)^n - \dfrac{n(n-1)}{1!}(2x)^{n-2} + \ldots}{(-1)^n e^{x^2}}$$

then we have also (for $n + 1$):

$$\frac{d^{n+1}}{dx^{n+1}}(e^{-x^2}) = \frac{(2x)^{n+1} - \dfrac{(n+1)n}{1!}(2x)^{n-1} + \ldots}{(-1)^{n+1} e^{x^2}}$$

This is easily shown: the derivative d/dx of the right-hand side of the last equation but one gives the right-hand side of the last equation. So if the identity of [12] with [14] holds for n, it holds also for $n + 1$. It holds for $n = 1$ (for instance); then it holds for all n, which should be proved. Fig.37 shows the Hermite polynomials of the lowest orders.

The following equation is related to Hermite's:

$$y'' + (1 - x^2 + 2a)y = 0 \qquad [15]$$

Making the substitution:

$$y = e^{-x^2/2}v \qquad [16]$$

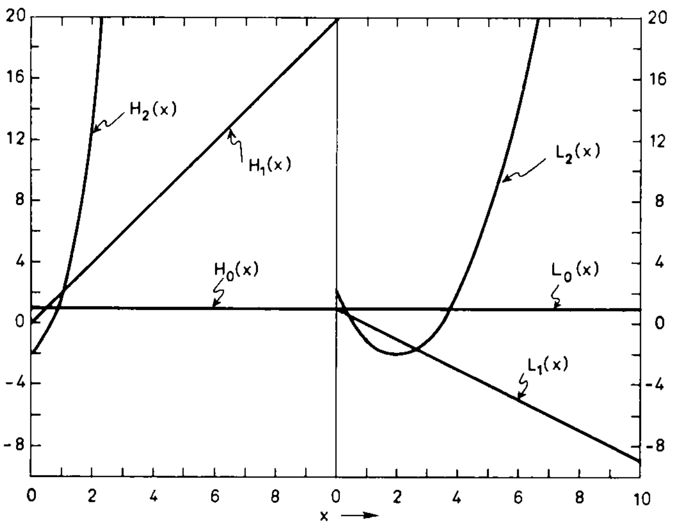


Fig.37.

and carrying out the differentiations, we find that [15] transforms into:

$$v'' - 2xv' + 2av = 0 \qquad [17]$$

which is Hermite's equation. Therefore the solution of [15] is equal to any solution of Hermite's equation, multiplied by $e^{-x^2/2}$; $y = e^{-x^2/2}H_n(x)$ is called a *Hermite function of order n* (note that Hermite function is *not* the same as Hermite polynomial).

The last example [15] illustrates a very common method in applied mathematics. Certain forms of differential equations (see Table II in section 4.1.1) are of special importance in applications and their solutions have been worked out. Therefore, if in a particular problem it is possible to transform an obtained differential equation into any of these standard forms, then we know the solutions to the differential equation, appropriate to the particular problem.

4.5 LAGUERRE'S DIFFERENTIAL EQUATION

This equation reads:

$$xy'' + (1 - x)y' + ay = 0 \qquad [1]$$

a = constant. Dividing [1] by x:

$$y'' + \frac{1-x}{x}y' + \frac{a}{x}y = 0 \qquad [2]$$

and comparing with the general type of equations we are solving by series integration (eq.[1] in section 4.1):

$$y'' + X_1y' + X_2y = 0 \qquad [3]$$

we have that $X_1 = (1 - x)/x$ and $X_2 = a/x$, which are both infinite at $x = 0$. This is a singular point. Using Fuchs' theorem (section 4.1.1), and observing that $xX_1 = 1 - x$ is finite for $x = 0$ and that $x^2X_2 = x^2a/x = ax$ is also finite for $x = 0$, we find that $x = 0$ is a nonessential singularity (i.e., $x = 0$ is a regular singular point), and the development in series about this point is therefore possible.

As before we put:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{\varkappa+\lambda} \qquad [4]$$

which inserted into [1] gives:

$$\sum_\lambda a_\lambda(\varkappa + \lambda)^2x^{\varkappa+\lambda-1} - \sum_\lambda a_\lambda(\varkappa + \lambda - a)x^{\varkappa+\lambda} = 0 \qquad [5]$$

(*1*) Consider first the lowest power of x, i.e. $x^{\varkappa-1}$, which is obtained from [5] by putting $\lambda = 0$ in the first sum (there is no contribution from second term, as λ is by definition [4] $\geqslant 0$). The indicial equation is thus:

$$a_0\varkappa^2 = 0 \qquad [6]$$

There is only one root $\varkappa = 0$, and therefore there is only one solution to [1].

(*2*) Next consider the power $x^{\varkappa+j}$, obtained by putting $\lambda = j + 1$ in the first sum of [5] and $\lambda = j$ in the second sum of [5]. This gives the recurrence relation, remembering that we have only one value of $\varkappa = 0$:

$$a_{j+1} = \frac{j - a}{(j + 1)^2} a_j \qquad [7]$$

Then we can write down our series solution [4]:

$$y = a_0 \left[1 - ax + \frac{a(a-1)}{(2!)^2} x^2 + \dots \right.$$
$$\left. \dots + (-1)^r \frac{a(a-1)\dots(a-r+1)}{(r!)^2} x + \dots \right] \qquad [8]$$

If we then restrict the discussion to positive integer values of $a = n$, the infinite series in [8] becomes a polynomial of degree n. If in addition we put $a_0 = n!$, y becomes the *Laguerre polynomial of degree n*:

$$L_n(x) = (-1)^n \left[x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2(n-1)^2}{2!} x^{n-2} + \dots + (-1)^n n! \right] \qquad [9]$$

which can also be written as:

$$L_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x}) \qquad [10]$$

That [9] corresponds to [8] is easily found, reversing the order of the terms in [8]. The constant term a_0 is given immediately. The highest power (x^n) is found by putting $r = n$ in the general term in [8], the next term (x^{n-1}) likewise by putting $r = n - 1$, etc. That [10] is identical with [9] is found by carrying out the differentiations in [10], observing that:

$$\frac{n!}{(-1)^n} = \frac{(-1)^{2n} n!}{(-1)^n} = (-1)^n n!$$

as $(-1)^{2n} = +1$. The lowest orders of the Laguerre polynomial are shown in Fig.37.

4.5.1 Associated Laguerre polynomials and functions

We shall consider two equations which are closely related to Laguerre's equation [1]:

$$(1) \quad xy'' + (k + 1 - x)y' + (a - k)y = 0 \qquad [11]$$

k positive integer. Eq.[11] is obtained by differentiating [1] k times and replacing

$d^k y/dx^k$ by y in the resulting equation. This is found by straight-forward differentiations (using Leibniz' rule). Therefore a solution of [11] for α and k positive integers is:

$$y = \frac{d^k}{dx^k} L_n(x) = L_n^k(x) \quad [12]$$

sometimes called the *associated Laguerre polynomial of degree $n - k$.*

(2) Another related equation is:

$$xy'' + 2y' + \left[n - \frac{k-1}{2} - \frac{x}{4} - \frac{k^2-1}{4x}\right] y = 0 \quad [13]$$

If we make the substitution:

$$y = e^{-x/2} x^{(k-1)/2} v(x) \quad [14]$$

we find that v satisfies the equation:

$$xv'' + (k + 1 - x)v' + (n - k)v = 0 \quad [15]$$

identical with [11]. Thus we have $v = L_n^k(x)$, and a particular solution of [13] is:

$$y = e^{-x/2} x^{(k-1)/2} L_n^k(x) \quad [16]$$

known as an *associated Laguerre function* or simply *Laguerre function.*

Note that a Laguerre *polynomial* is defined as in [9] and [10], an associated Laguerre polynomial as in [12] and a Laguerre *function* or associated Laguerre function as in [16].

Both for Hermite and Laguerre functions we have omitted recurrence relations and integral properties, as these are generally of less use in seismology. Also the presentation of Hermite and Laguerre polynomials in the form of definite integrals has been omitted, for the same reason.

4.5.2 Seismological applications of Laguerre functions

Although Hermite and Laguerre functions have their greatest importance in wave mechanics and quantum physics, they are also of significance in seismology, especially in data handling.

We define a Laguerre function as in [16], but put $k = 1$. Then [16] becomes:

$$y_{n-1} = e^{-x/2} L_n^1(x) = e^{-x/2} \frac{d}{dx} L_n(x) \quad [17]$$

Fig.38 gives an idea of how the functions [17] run. The Laguerre functions show a rapid decay with increasing x, and it is for this property that they have become important in seismology.

In spectral analyses of transient seismic pulses, the Fourier series expansion is usually used. But this method is not efficient, since we are then expanding transient phenomena in a series of steady state functions. A more efficient approach is to expand such transients in a series of functions with properties similar to the transients themselves.

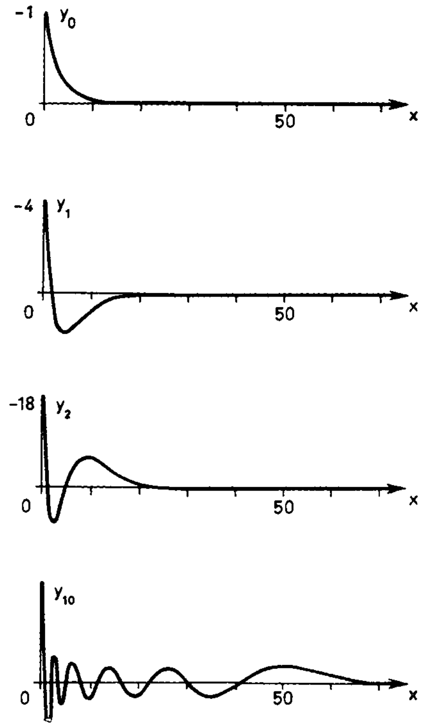


Fig.38.

Laguerre functions have this property, as seen in Fig.38, where we have to take the independent variable x to be time t or proportional to t. The advantage of using Laguerre functions in such cases over the Fourier series expansions is that the number of required terms is much less (just due to the fact that the Laguerre functions approximate the transient pulses so much better). Whereas a few hundred terms of a Fourier series may be required, the same approximation to a given signal can be achieved by only a few dozen Laguerre functions, the idea being to expand the given transient in a series of Laguerre functions:

$$f(t) = \sum_{n=0}^{m} c_n y_n(t) \qquad [18]$$

The same method has application in the transient calibration of seismometers which yields magnifications and phase responses for all frequencies.

For more details, the reader is referred to DEAN (1964).

4.6 GAUSS' (HYPERGEOMETRIC) DIFFERENTIAL EQUATION—WHITTAKER'S FUNCTIONS

Gauss' differential equation reads:

$$(x^2 - x)y'' + [(1 + \alpha + \beta)x - \gamma]y' + \alpha\beta y = 0 \qquad [1]$$

and can be solved by series integration. The parameters α, β, γ are constants and it is assumed that γ is not an integer. Writing [1] in the standard form [1] of section 4.1:

$$y'' + \underbrace{\frac{(1+\alpha+\beta)x-\gamma}{x^2-x}}_{X_1} y' + \underbrace{\frac{\alpha\beta}{x^2-x}}_{X_2} y = 0 \tag{2}$$

we see that X_1 and X_2 become singular for $x = 0$, 1 and ∞. But $(x - x_0)X_1$ and $(x - x_0)^2 X_2$ are regular in all three cases, thus by Fuchs' theorem all three singularities are non-essential.

Again assuming the power series development *around* $x = 0$:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{\varkappa+\lambda} \tag{3}$$

we find after insertion into [1]:

$$\begin{aligned}(x^2 - x) \sum_\lambda a_\lambda(\varkappa + \lambda)(\varkappa + \lambda - 1)x^{\varkappa+\lambda-2}&\\ + [(1 + \alpha + \beta)x - \gamma] \sum_\lambda a_\lambda(\varkappa + \lambda)x^{\varkappa+\lambda-1} + \alpha\beta \sum_\lambda a_\lambda x^{\varkappa+\lambda} &= 0\end{aligned} \tag{4}$$

The lowest power is $x^{\varkappa-1}$ (for $\lambda = 0$); the vanishing of its coefficient gives the indicial equation:

$$\varkappa(\varkappa - 1) + \varkappa\gamma = 0$$

or:

$$\varkappa(\varkappa - 1 + \gamma) = 0 \tag{5}$$

with the solutions $\varkappa = 0$ and $\varkappa = 1 - \gamma$.

(*I*) $\varkappa = 0$. Considering the power x^j, we get the recurrence relation:

$$a_{j+1} = \frac{(\alpha + j)(\beta + j)}{(j + 1)(j + \gamma)} a_j \tag{6}$$

and the particular solution:

$$\begin{aligned}y = a_0 \Bigg[1 &+ \frac{\alpha\beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \dots\\ &+ \frac{\alpha(\alpha+1)\dots(\alpha+r-1) \cdot \beta(\beta+1)\dots(\beta+r-1)}{r!\gamma(\gamma+1)\dots(\gamma+r-1)} x^r + \dots\Bigg]\end{aligned} \tag{7}$$

The series in brackets in [7] is called the *hypergeometric series*. It converges for $|x| < 1$, as the ratio of two consecutive terms is:

$$\lim_{j\to\infty} \frac{a_{j+1}x^{j+1}}{a_j x^j} = \lim_{j\to\infty} \underbrace{\frac{(\alpha + j)(\beta + j)}{(j + 1)(j + \gamma)}}_{= 1 \text{ for } j \to \infty} x = x < 1 \tag{8}$$

From the expression [7] we also understand the reason why at the beginning we excluded integer values of γ, as these could make some factor in the series infinite. For $\alpha = 1$ and $\beta = \gamma$ the series in [7] reduces to $1 + x + x^2 + x^3 + \ldots$, i.e., an ordinary geometric series, hence the name hypergeometric.

Using the notation:

$$(\alpha)_r = \alpha(\alpha + 1) \ldots (\alpha + r - 1) = \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)} \qquad [9]$$

(section 1.3) we can write the hypergeometric series as:

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{r!(\gamma)_r} x^r \qquad [10]$$

where the suffixes 2 and 1 denote that there are two parameters of the type α and only one of the type γ. The solution [7] is thus:

$$y = a_0 \; {}_2F_1(\alpha, \beta; \gamma; x) \qquad [11]$$

(2) $\varkappa = 1 - \gamma$. Considering the power $x^{\varkappa+j}$ in [4] we get the recurrence relation:

$$a_{j+1} = \frac{(\alpha - \gamma + j + 1)(\beta - \gamma + j + 1)}{(j + 1)(j + 2 - \gamma)} a_j \qquad [12]$$

If we make the substitutions (only a change of notation for the constants):

$$\left.\begin{aligned} \alpha' &= \alpha - \gamma + 1 \\ \beta' &= \beta - \gamma + 1 \\ \gamma' &= 2 - \gamma \end{aligned}\right| \qquad [13]$$

[12] becomes:

$$a_{j+1} = \frac{(\alpha' + j)(\beta' + j)}{(j + 1)(j + \gamma')} a_j \qquad [14]$$

that is, identical with [6]. The particular solution is thus now:

$$y = a_0 x^{1-\gamma} {}_2F_1(\alpha - \gamma + 1, \; \beta - \gamma + 1; \; 2 - \gamma; \; x) \qquad [15]$$

and the general solution of [1] is:

$$y = A \; {}_2F_1(\alpha, \beta; \gamma; x) + Bx^{1-\gamma} {}_2F_1(\alpha - \gamma + 1, \; \beta - \gamma + 1; \; 2 - \gamma; \; x) \qquad [16]$$

convergent for $|x| < 1$.

This solution has been obtained by a power series development around $x = 0$ [3]. In applications we sometimes have use for the corresponding solutions obtained by power series expansions around the other singular points, i.e., $x = 1$ and $x = \infty$. See, e.g., BREKHOVSKIKH (1960, p.177 ff.).

These solutions can be obtained by appropriate substitutions. $x = 1/z$ changes the singularities into ∞, 1, 0 (from 0, 1, ∞ in the original form). Likewise $x = 1 - z$ changes the singularities into 1, 0, ∞.

Making such substitutions in [1] and again solving in the same way, i.e., assuming a power series expansion around the value 0 for the independent variable, we see that an expansion around $z = 0$ after the substitution $x = 1/z$ is equivalent to an expansion around $x = \infty$ in the original equation. Similarly, by the substitution $x = 1 - z$, an expansion around $z = 0$ is equivalent to an expansion around $x = 1$ in the original equation. The details of the calculation are straight-forward and are left to the reader as an exercise.

The results are around the point $x = 1$:

$$y = A\,{}_2F_1(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - x)$$
$$+ B(1 - x)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1 - x) \qquad [17]$$

and around the point $x = \infty$:

$$y = Ax^{-\alpha}{}_2F_1\left(\alpha, \alpha - \gamma + 1; \alpha - \beta + 1; \frac{1}{x}\right)$$
$$+ Bx^{-\beta}{}_2F_1\left(\beta, \beta - \gamma + 1; \beta - \alpha + 1; \frac{1}{x}\right) \qquad [18]$$

Next, we shall briefly consider several important relations to which the hypergeometric series is closely related.

4.6.1 Transformation to Legendre's equation

Introduce ξ as an independent variable in [1] instead of x:

$$x = \frac{1}{2}(1 - \xi) \qquad [19]$$

Then:

$$\frac{d\xi}{dx} = -2$$

and:

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = -2\frac{dy}{d\xi}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{d\xi}\underbrace{\frac{d^2\xi}{dx^2}}_{=0} + \left(\frac{d\xi}{dx}\right)^2\frac{d^2y}{d\xi^2} = 4\frac{d^2y}{d\xi^2}$$

after which [1] becomes:

$$(1-\xi^2)\frac{d^2y}{d\xi^2}+[1+\alpha+\beta-2\gamma-(\alpha+\beta+1)\xi]\frac{dy}{d\xi}-\alpha\beta y=0 \tag{20}$$

Furthermore, select the constants such that:

$$\alpha=n+1\ ;\ \ \beta=-n\ ;\ \ \gamma=1 \tag{21}$$

after which [20] becomes:

$$(1-\xi^2)y''^2-2\xi y'+n(n+1)y=0 \tag{22}$$

i.e., *Legendre's equation.* One solution is thus:

$$y=a\ {}_2F_1\left(n+1,\ -n\ ;\ \ 1\ ;\ \frac{1-\xi}{2}\right)=aP_n(\xi) \tag{23}$$

if n is a positive integer.

4.6.2 *Tschebyscheff polynomial*

If in [20] we instead choose the constants as follows:

$$\alpha=-\beta=n\,(\text{integer})\ ;\ \ \gamma=\frac{1}{2} \tag{24}$$

[20] becomes:

$$(1-\xi^2)\frac{d^2y}{d\xi^2}-\xi\frac{dy}{d\xi}+n^2y=0 \tag{25}$$

with the general solution:

$$y(\xi)=A\ {}_2F_1\left(n,-n\ ;\ \frac{1}{2}\ ;\ \frac{1-\xi}{2}\right)$$
$$+B\left(\frac{1-\xi}{2}\right)^{1/2}{}_2F_1\left(n+\frac{1}{2},\ -n+\frac{1}{2}\ ;\ \frac{3}{2}\ ;\ \frac{1-\xi}{2}\right) \tag{26}$$

This is obtained from the general solution [16] just by making the same substitutions as we have done now. The first particular solution in [26] is called the *Tschebyscheff polynomial,* of degree n.

4.6.3 *Jacobi polynomial*

The hypergeometric series reduces to a polynomial of degree n, if $\alpha=-n$, where n is a positive integer. This polynomial is called *Jacobi polynomial*:

$$J_n(p; q; x) = {}_2F_1(-n, p + n; q; x) \tag{27}$$

where we have put:

$$\alpha = -n; \quad \beta = p + n; \quad \gamma = q$$

This polynomial satisfies the following differential equation, as seen by immediate substitution into [1]:

$$(x^2 - x)y'' + [(1 + p)x - q]y' - n(p + n)y = 0 \tag{28}$$

where we have to require that $q > 0$, to avoid any coefficient in [27] becoming infinite. Writing out the F-function in [27] we find the following expression for the Jacobi polynomial (cf. [7]):

$$J_n(p; q; x) = 1 + \sum_{\lambda=1}^{n} (-1)^\lambda \binom{n}{\lambda} \frac{(p + n)(p + n + 1) \dots (p + n + \lambda - 1)}{q(q + 1) \dots (q + \lambda - 1)} x^\lambda \tag{29}$$

where we have used the notation:

$$\binom{n}{\lambda} = \frac{n!}{\lambda!(n - \lambda)!}$$

4.6.4 Confluent hypergeometric function

In [1] we make the substitution $x = \xi/\beta$, that is:

$$\left.\begin{aligned} &\frac{d\xi}{dx} = \beta \\ &\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = \beta\frac{dy}{d\xi} \\ &\frac{d^2y}{dx^2} = \beta^2\frac{d^2y}{d\xi^2} \end{aligned}\right| \tag{30}$$

whence [1] becomes:

$$\xi\left(\frac{\xi}{\beta} - 1\right)\frac{d^2y}{d\xi^2} + \left[\left(1 + \frac{\alpha + 1}{\beta}\right)\xi - \gamma\right]\frac{dy}{d\xi} + \alpha y = 0 \tag{31}$$

with the solution:

$$y = a_0 \, {}_2F_1(\alpha, \beta; \gamma; \xi/\beta) \tag{32}$$

Then, letting $\beta \to \infty$ in [31] and [32], we get that:

$$\lim_{\beta\to\infty} {}_2F_1(\alpha, \beta; \gamma; \xi/\beta) \tag{33}$$

is a solution to:

$$\xi \frac{d^2y}{d\xi^2} + (\gamma - \xi)\frac{dy}{d\xi} - \alpha y = 0 \qquad [34]$$

[33] is easily evaluated, because:

$$\lim_{\beta\to\infty} \frac{(\beta)_r}{\beta^r} = \lim_{\beta\to\infty} \frac{\beta(\beta+1)\dots(\beta+r-1)}{\beta^r} =$$

$$= \lim_{\beta\to\infty} \frac{\beta^r \cdot 1 \cdot \left(1+\frac{1}{\beta}\right)\dots\left(1+\frac{r-1}{\beta}\right)}{\beta^r} = 1$$

and therefore [33] becomes:

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\gamma)_r} \cdot \frac{\xi^r}{r!} \qquad [35]$$

This function is called a *confluent hypergeometric function* and [34] is the *confluent hypergeometric equation.*

Solving [34] by series integration from the beginning (which is left as an exercise), we find, in perfect analogy to the solution of [1], that the general solution of [34] is:

$$y(\xi) = A\,{}_1F_1(\alpha;\gamma;\xi) + B\xi^{1-\gamma}\,{}_1F_1(\alpha-\gamma+1;2-\gamma;\xi) \qquad [36]$$

4.6.5 Whittaker's functions

In [34] we put $\xi = x$ and:

$$y(x) = x^{-\gamma/2}e^{x/2}W(x) \qquad [37]$$

We then find that the first derivative vanishes and [34] changes into:

$$\frac{d^2W}{dx^2} + \left(-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - m^2}{x^2}\right)W(x) = 0 \qquad [38]$$

where we have written:

$$k = \frac{\gamma}{2} - \alpha \quad \text{and} \quad m = \frac{1-\gamma}{2}$$

Eq.[38] is *Whittaker's differential equation* and the solutions are called *Whittaker's confluent hypergeometric functions.*

Then, with the mentioned substitutions $\gamma = 1 - 2m$; $\alpha = \frac{1}{2} - k - m$;

$W(x) = x^{\gamma/2}e^{-x/2}y(x)$ the solutions to [38] are the following two functions:

$$\left.\begin{aligned} M_{k,-m}(x) &= x^{1/2-m}e^{-x/2}\,{}_1F_1\left(\frac{1}{2}-k-m\;;\;\;1-2m\;;\;\;x\right) \\ M_{k,m}(x) &= x^{1/2+m}e^{-x/2}\,{}_1F_1\left(\frac{1}{2}-k+m\;;\;\;1+2m\;;\;\;x\right) \end{aligned}\right| \qquad [39]$$

called *Whittaker's functions*. We have assumed that γ is not an integer and therefore we have to make the same assumption for $2m$.

That [39] represents the solutions of [38] is seen by comparison with [36]:

$$[39.1]\text{: exponent of } x = \frac{\gamma}{2} = \frac{1}{2} - m\;;\quad \alpha = \frac{\gamma}{2} - k = \frac{1}{2} - m - k$$

$$[39.2]\text{: exponent of } x = \frac{\gamma}{2} + \underbrace{1-\gamma}_{\text{from [36]}} = 1 - \frac{\gamma}{2} = 1 - \frac{1}{2} + m = \frac{1}{2} + m$$

$$\alpha - \gamma + 1 = \left(\frac{1}{2} - k - m\right) - (1-2m) + 1 = \frac{1}{2} - k + m$$

$$2 - \gamma = 2 - 1 + 2m = 1 + 2m$$

Next we shall find *an integral expression for Whittaker's functions*. For the functions ${}_1F_1$ appearing in [39] we can get an expression in terms of an integral in the following way. We have the expression for ${}_1F_1$:

$${}_1F_1(\alpha;\gamma;x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!(\gamma)_r}x^r \qquad [40]$$

Compare eq.[10]. Furthermore:

$$\frac{(\alpha)_r}{(\gamma)_r} = \frac{\Gamma(\alpha+r)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma+r)} = \frac{B(\alpha+r,\gamma-\alpha)}{B(\alpha,\gamma-\alpha)} =$$

$$\text{(by[9])} \qquad \left(\text{as } B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \text{(section 1.3)}\right)$$

$$= \frac{1}{B(\alpha,\gamma-\alpha)}\int_0^1 t^{\alpha+r-1}(1-t)^{\gamma-\alpha-1}dt \qquad [41]$$

$$\left(\text{as } B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}dt \text{ (section 1.3)}\right)$$

Thus we get:

$${}_1F_1(\alpha;\gamma;x) = \frac{1}{B(\alpha,\gamma-\alpha)}\sum_{r=0}^{\infty}\frac{x^r}{r!}\int_0^1 t^{\alpha+r-1}(1-t)^{\gamma-\alpha-1}dt =$$

$$= \frac{1}{B(\alpha, \gamma - \alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1} \underbrace{\left[\sum_{r=0}^{\infty} \frac{(xt)^r}{r!}\right]}_{= e^{xt}} dt =$$

(interchanging summation and integration)

$$= \frac{1}{B(\alpha, \gamma - \alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1} e^{xt} dt \qquad [42]$$

Using the integral expression [42], we can express the functions M in [39] in integral form.

Instead of the functions M, it is customary to use a linear combination of the two functions in [39]:

$$W_{k,m}(x) = \frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2} - m - k\right)} M_{k,m}(x) + \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2} + m - k\right)} M_{k,-m}(x) \qquad [43]$$

See WHITTAKER and WATSON (1935, p.346). The advantage of using a Whittaker function as defined in [43] is that it can be used also when $2m$ is an integer. As we have seen above, [39.2] becomes infinite if $2m$ is a negative integer, and [39.1] becomes infinite if $2m$ is a positive integer. By multiplication, in [43], by $\Gamma(-2m)$ and $\Gamma(2m)$ respectively, this circumstance is eliminated, remembering that for example:

$$\Gamma(2m) = (2m-1)! = (2m-1)(2m-2)(2m-3)\ldots 1$$

The function $W_{k,m}(x)$ can also be expressed in integral form:

$$W_{k,m}(x) = \frac{e^{-x/2}x^k}{\Gamma\left(\frac{1}{2} - k + m\right)} \int_0^{\infty} t^{-k-1/2+m}\left(1 + \frac{t}{x}\right)^{k-1/2+m} e^{-t} dt \qquad [44]$$

Carrying out the differentiations and substituting in [38], we can verify that [44] is a solution of this equation. For a more complete discussion and derivation of [44], the reader is referred to WHITTAKER and WATSON (1935, pp.337–351), or to JEFFREYS and JEFFREYS (1946, pp.574–585).

An *asymptotic expansion* of $W_{k,m}(x)$ is useful, especially for large values of $|x|$. This can be derived from [44] by expanding:

$$\left(1 + \frac{t}{x}\right)^{k-1/2+m}$$

by the binomial theorem:

$$\left(1 + \frac{t}{x}\right)^{\lambda} = 1 + \frac{\lambda}{1!}\frac{t}{x} + \frac{\lambda(\lambda-1)}{2!}\left(\frac{t}{x}\right)^2 + \ldots$$

where $\lambda = m + k - \frac{1}{2}$ and then integrating term by term. Putting $a = m - k - \frac{1}{2}$ we get from [44]:

$$W_{k,m}(x) = \frac{e^{-x/2}\,x^k}{\Gamma(a+1)} \int_0^\infty t^a \left[1 + \frac{\lambda}{1!}\frac{t}{x} + \frac{\lambda(\lambda-1)}{2!}\left(\frac{t}{x}\right)^2 + \dots\right] e^{-t}\mathrm{d}t$$

Using:

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\mathrm{d}x$$

and:

$$\Gamma(n+1) = n\Gamma(n)$$

from section 1.3 and integrating term by term, we find:

$$\begin{aligned} W_{k,m}(x) &= \frac{e^{-x/2}\,x^k}{\Gamma(a+1)}\Bigg[\underbrace{\int_0^\infty e^{-t}t^a\mathrm{d}t}_{=\Gamma(a+1)} + \frac{\lambda}{x}\underbrace{\int_0^\infty e^{-t}t^{a+1}\mathrm{d}t}_{=\Gamma(a+2)\,=\,(a+1)\Gamma(a+1)} \\ &\quad + \frac{\lambda(\lambda-1)}{2!x^2}\underbrace{\int_0^\infty e^{-t}t^{a+2}\mathrm{d}t}_{=\Gamma(a+3)\,=\,(a+2)\Gamma(a+2)\,=\,(a+2)(a+1)\Gamma(a+1)} + \dots\Bigg] \\ &= e^{-x/2}\,x^k\left[1 + \frac{\lambda}{x}(a+1) + \frac{\lambda(\lambda-1)}{2!x^2}(a+2)(a+1) + \dots\right] \\ &= e^{-x/2}\,x^k\left\{1 + \frac{m^2-\left(k-\frac{1}{2}\right)^2}{x} + \frac{\left[m^2-\left(k-\frac{1}{2}\right)^2\right]\left[m^2-\left(k-\frac{3}{2}\right)^2\right]}{2!x^2} + \dots\right\} \end{aligned} \qquad [45]$$

which is the asymptotic expansion sought. We note that [45] has a finite number of terms in certain cases. Writing in general a term as:

$$m^2-\left(k-\frac{\nu}{2}\right)^2 = \left(m+k-\frac{\nu}{2}\right)\left(m-k+\frac{\nu}{2}\right)$$

$[\nu = 1, 3, 5 \dots]$

$$= \left(\underbrace{m+k-\frac{1}{2}}_{\substack{\text{positive}\\ \text{integer}\\ \text{makes}\to 0}}-\underbrace{\frac{\nu-1}{2}}_{\substack{\text{positive}\\ \text{integer}}}\right)\cdot\left(\underbrace{m-k+\frac{1}{2}}_{\substack{\text{negative}\\ \text{integer}\\ \text{makes}\to 0}}+\underbrace{\frac{\nu-1}{2}}_{\substack{\text{positive}\\ \text{integer}}}\right)$$

we find that if $m + k - \frac{1}{2}$ is a positive integer or if $m - k + \frac{1}{2}$ is a negative integer, the series [45] terminates.

The differential equation [38] is unchanged if x and k are replaced by $-x$ and $-k$ respectively. Therefore, another solution is $W_{-k,m}(-x)$. The corresponding expressions [44] and [45] can be written down immediately by making the said substitutions.

4.6.6 *Weber functions* [1]

We attempt to find the differential equation which is satisfied by:

$$w = z^{-1/2}\, W_{k,-1/4}\left(\frac{z^2}{2}\right)$$

or:

$$wz^{1/2} = W_{k,-1/4}\left(\frac{z^2}{2}\right) \qquad [46]$$

This function will satisfy Whittaker's differential equation [38], by putting $m = -1/4$ and considering $z^2/2$ as the independent variable. Inserting into [38] we find:

$$\frac{d}{zdz}\left[\frac{d(wz^{1/2})}{zdz}\right] + \left(-\frac{1}{4} + \frac{2k}{z^2} + \frac{\frac{3}{4}}{z^4}\right) wz^{1/2} = 0 \qquad [47]$$

noting that:

$$d\left(\frac{z^2}{2}\right) = zdz$$

[47] can be simplified to the following equation:

$$\frac{d^2w}{dz^2} + \left(2k - \frac{z^2}{4}\right) w = 0 \qquad [48]$$

Therefore the function:

$$D_n(z) = 2^{n/2 + 1/4} z^{-1/2}\, W_{n/2 + 1/4,\, -1/4}\left(\frac{z^2}{2}\right) \qquad [49]$$

which is of the same form as w above, will satisfy the following equation (putting $k = n/2 + 1/4$ in eq.[48]):

$$\frac{d^2 D_n(z)}{dz^2} + \left(n + \frac{1}{2} - \frac{z^2}{4}\right) D_n(z) = 0 \qquad [50]$$

This is *Weber's differential equation* and $D_n(z)$ is *Weber's function*. We have to distinguish this Weber function from Weber's Bessel function of the second kind, which was defined in eq.[11] and [13] in section 4.3.

[1] WHITTAKER and WATSON (1935, p.347).

Weber's equation is unchanged if we simultaneously replace n by $-n-1$ and z by $\pm iz$. Making these substitutions in [50], we find the equation unchanged (only a change of sign of all terms). Therefore, in addition to $D_n(z)$, also $D_{-n-1}(iz)$ and $D_{-n-1}(-iz)$ are solutions of Weber's equation.

By means of the Whittaker functions, it is possible to express the Weber functions in a series or as an integral.

Those who want more details about Weber functions are referred to WHITTAKER and WATSON (1935, pp.347–351). There are relatively few seismological applications of Weber functions. One example can be found in BREKHOVSKIKH (1960, pp.283–287), dealing with the wave field in the region close to the angle of total internal reflection.

4.6.7 Transformation into the wave equation

It is also possible to transform the hypergeometric equation into the wave equation. By means of the following substitutions:

$$\left.\begin{aligned} y &= r(z)Z(z) \\ x &= P(z) \end{aligned}\right| \qquad [51]$$

$P(z)$ being an arbitrary function, and $r(z)$ given by:

$$r(z) = r_0 x^{-\gamma/2}(1-x)^{\frac{\gamma-\alpha-\beta-1}{2}}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^{1/2} \qquad [52]$$

(r_0 = constant) equation [1] is transformed into the wave equation:

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + g(z)Z = 0 \qquad [53]$$

$g(z)$ is given by the following expression:

$$\left.\begin{aligned} g(z) = &\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\log\frac{\mathrm{d}P}{\mathrm{d}z}\right) - \frac{1}{4}\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\frac{\mathrm{d}P}{\mathrm{d}z}\right)\right]^2 \\ &-\left(\frac{\mathrm{d}}{\mathrm{d}z}\log P\right)^2\left[K_1 + K_2\frac{P}{1-P} + K_3\frac{P}{(1-P)^2}\right] \end{aligned}\right| \qquad [54]$$

where:

$$\left.\begin{aligned} 4K_1 &= \gamma(\gamma-2) \\ 4K_2 &= 1-(\alpha-\beta)^2+\gamma(\gamma-2) \\ 4K_3 &= (\alpha+\beta-\gamma)^2-1 \end{aligned}\right| \qquad [55]$$

log = natural logarithm (ln)

That [53] corresponds to the wave equation is seen if we start from the form:

$$\nabla^2\varphi + k^2(z)\varphi = 0 \qquad [56]$$

or (for two-dimensional motion):

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} + k^2(z)\varphi = 0 \qquad [57]$$

The wave number k and thus the wave velocity is assumed to be a function of z, i.e., corresponding to an inhomogeneous medium. Separating the variables by the assumption:

$$\varphi(x, z) = X(x) \cdot Z(z) \qquad [58]$$

we find the following equation for Z by substitution into [57]:

$$\frac{d^2Z}{dz^2} + [k^2(z) - b^2]Z = 0 \qquad [59]$$

i.e., the same as [53] with $g(z) = k^2(z) - b^2$ (b = constant).

The details of the transformation into [53], i.e., especially the expression [54], are left out here. The calculations are rather lengthy but quite straight-forward, and there are no essential difficulties involved. Seismological applications of this transformation can be found in BREKHOVSKIKH (1960, p.174 ff.), dealing with wave reflection from an inhomogeneous layer.

4.7 LOVE WAVES IN HETEROGENEOUS ISOTROPIC MEDIA[1]

By *heterogeneous* (or inhomogeneous) we mean that properties vary within the medium. For simplicity, it is usually assumed that variation takes place only in one direction, e.g., with depth. The term *isotropic* (cf. BULLEN, 1963, pp.19 and 30) means that the elastic behaviour is entirely independent of any particular direction, e.g., that the simple stress-strain relation holds.

Consider the coordinate system and the medium properties shown in Fig.39. As usual, μ is the modulus of rigidity, ϱ is the density, and β is the shear-wave velocity. For Love waves we then have the following displacement components:

$$u = w = 0\,; \quad v = v(x, z, t) \qquad [1]$$

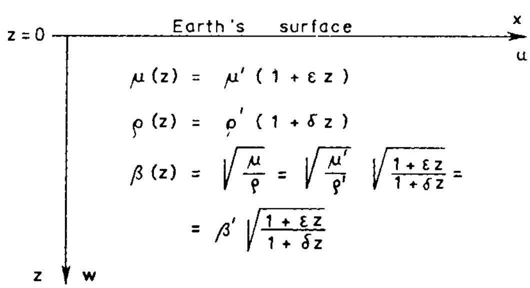


Fig.39.

[1] EWING, JARDETZKY and PRESS (1957, pp.342, 347–349).

All derivatives with respect to y vanish and the equation of motion becomes:

$$\varrho \frac{\partial^2 v}{\partial t^2} = \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{zy}}{\partial z} \tag{2}$$

neglecting body forces.

Introducing the expressions for the stress components, we get:

$$\varrho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) \tag{3}$$

As $\mu = \mu(z)$, [3] becomes:

$$\varrho \frac{\partial^2 v}{\partial t^2} = \underbrace{\mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right)}_{\nabla^2 v} + \frac{\partial \mu}{\partial z}\frac{\partial v}{\partial z} \tag{4}$$

Replacing the dependent variable v by V by the following definition:

$$v = \frac{V}{\sqrt{\mu}} \tag{5}$$

and remembering that $v = v(x, z, t)$ and $V = V(x, z, t)$, whereas $\mu = \mu(z)$, we find by straight-forward calculations that [4] transforms into:

$$\varrho \frac{\partial^2 V}{\partial t^2} = \mu \nabla^2 V + \left[\frac{1}{4\mu}\left(\frac{\partial \mu}{\partial z}\right)^2 - \frac{1}{2}\frac{\partial^2 \mu}{\partial z^2}\right] V \tag{6}$$

Assuming that:

$$V = Z(z)e^{i(\omega t - kx)} \tag{7}$$

corresponding to a wave propagating in the positive x-direction and with an amplitude which depends only on depth z, we find from [6] the following equation for Z:

$$\frac{d^2 Z}{dz^2} + \left[k_\beta^2 - k^2 + \frac{1}{4\mu^2}\left(\frac{d\mu}{dz}\right)^2 - \frac{1}{2\mu}\frac{d^2\mu}{dz^2}\right] Z = 0 \tag{8}$$

considering that:

$$k_\beta^2 = \frac{\omega^2}{\beta^2} = \frac{\omega^2 \varrho}{\mu}$$

In particular, we now consider the vertical variation of μ and ϱ as shown in Fig.39 (Meissner's case). The function assumed in Fig.39 for $\mu(z)$ leads to:

$$\frac{d\mu}{dz} = \mu'\varepsilon \quad \text{and} \quad \frac{d^2\mu}{dz^2} = 0$$

Eq.[6] is valid also with density ϱ as a function of depth z (see EWING, JARDETZKY and PRESS, 1957, p.328). The limiting velocity for infinite depth is:

$$\beta_m = \lim_{z \to \infty} \beta(z) = \beta' \left(\frac{\varepsilon}{\delta}\right)^{1/2} \tag{9}$$

If we introduce η as independent variable instead of z by means of the substitution:

$$\left.\begin{aligned} \eta &= \frac{2\gamma k}{\varepsilon}(1 + \varepsilon z) = \frac{2\gamma k}{\varepsilon}\frac{\mu}{\mu'} \\ &\text{i.e.} \\ \mathrm{d}z &= \frac{\mathrm{d}\eta}{2\gamma k} \end{aligned}\right| \tag{10}$$

with:

$$\gamma = \left(1 - \frac{c^2}{\beta_m^2}\right)^{1/2} \quad ; \quad \omega = kc$$

[8] transforms into:

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\eta^2} + \left(\frac{1}{4\eta^2} + \frac{p}{2\eta} - \frac{1}{4}\right) Z = 0 \tag{11}$$

where:

$$p = \frac{c^2 k}{\gamma \beta'^2 \varepsilon}\left(1 - \frac{\delta}{\varepsilon}\right) \tag{12}$$

The easiest way to see [12] is to start from the equation [8], and then equate the factor of Z in the transformed equation with the factor appearing in [11], this being a definition of p. It is then easily shown that this p agrees with the expression [12].

By this substitution we have obtained an equation [11], which is of the same type as Whittaker's differential equation [38] in section 4.6. In this case we have $x = \eta$, $m = 0$, $k = p/2$. Considering the statement at the end of section 4.6.5, we have the following general solution of [11]:

$$Z = AW_{p/2,0}(\eta) + BW_{-p/2,0}(-\eta) \tag{13}$$

There are two boundary conditions:

(*1*) Vanishing tangential stress at the surface:

$$p_{zy} = \mu \frac{\partial v}{\partial z} = 0 \qquad \text{for } z = 0 \tag{14}$$

(*2*) Vanishing amplitudes at great depth:

$$\lim_{\eta \to \infty} Z(\eta) = 0 \tag{15}$$

From the integral [44] in section 4.6 or from the asymptotic expansion [45] in section 4.6 we see that only the Whittaker function with positive argument fulfills the condition [15], that is we have $B = 0$, and [13] reduces to:

$$Z = AW_{p/2,0}(\eta) \tag{16}$$

The condition [14] leads to the following equation, considering [5], [7], [10], [16]:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{Z}{\sqrt{\mu}}\right) = 0$$

$$2\mu\frac{\mathrm{d}Z}{\mathrm{d}z} - Z\frac{\mathrm{d}\mu}{\mathrm{d}z} = 0$$

$$2\mu\frac{\mathrm{d}W_{p/2,0}(\eta)}{\mathrm{d}\eta} - W_{p/2,0}(\eta)\frac{\mathrm{d}\mu}{\mathrm{d}\eta} = 0$$

At $z = 0$:

$$\mu = \mu'$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}\eta} = \frac{\varepsilon\mu'}{2\gamma k} = \frac{\mu'}{\eta_0}\ ;\quad \eta_0 = \frac{2\gamma k}{\varepsilon}$$

We then find:

$$2\eta_0\left[\frac{\mathrm{d}W_{p/2,0}(\eta)}{\mathrm{d}\eta}\right]_{\eta=\eta_0} - W_{p/2,0}(\eta_0) = 0 \qquad [17]$$

Eq.[17] expresses the period equation implicitly. This is seen, as [17] is a relation between p and η_0, and from [12] we see that $p = p(c, k)$ and from the expression for η_0 we see that also $\eta_0 = \eta_0(c, k)$. Thus, [17] is an implicit relation between phase velocity c and wave number k, in other words this is the so-called period equation.

Using the integral expression [44] in section 4.6 for the Whittaker function, we can write the period equation [17] also as follows:

$$\int_0^\infty e^{-t}t^{-\frac{p+1}{2}}(t+\eta_0)^{\frac{p-3}{2}}[p-1-(t+\eta_0)]\mathrm{d}t = 0 \qquad [18]$$

This is an *exact* form of the period equation. It is found immediately upon differentiation of [44] in section 4.6, considering that the independent variable occurs both outside and inside the integral sign (for the latter the differentiation is carried out under the integral sign). An *approximate* form of the period equation can be obtained from the asymptotic expansion [45] in section 4.6 of the Whittaker function. The latter is particularly useful for large values of $|\eta|$, and then relatively simple expressions are obtained.

Before leaving this chapter, it should be remarked that another very important integration method is the integral transform method, which in fact belongs to this part of the book. This method will be dealt with in Chapter 8 below, because we need a more extensive knowledge of the Bessel functions before we read that chapter.

Chapter 5

BESSEL FUNCTIONS

5.1 ORIGIN OF BESSEL FUNCTIONS

Bessel introduced his functions in 1824 in the discussion of a problem in dynamical astronomy. Fig.40 shows the elliptical orbit of a planet P around the sun S, located in one of the foci. We define: *true anomaly* v of the planet = angle ASP (not very convenient to use); *mean anomaly* $\zeta = 2\pi \times$ area of elliptic sector ASP: area of ellipse; *eccentric anomaly* = angle u (see Fig.40).

If e = the eccentricity of the ellipse, then:

$$\zeta = u - e \sin u \tag{1}$$

This is called Kepler's equation. We give here a proof (for more details, see textbooks in spherical astronomy, e.g., SMART, 1936, pp.111–114).

Kepler's second law gives:

$$\text{area } SPA \text{ : area of ellipse } = \frac{t - \tau}{T}$$

where T = one full period; $t - \tau$ = time to pass from A to P. Thus:

$$\zeta = 2\pi \frac{t - \tau}{T} = n(t - \tau)$$

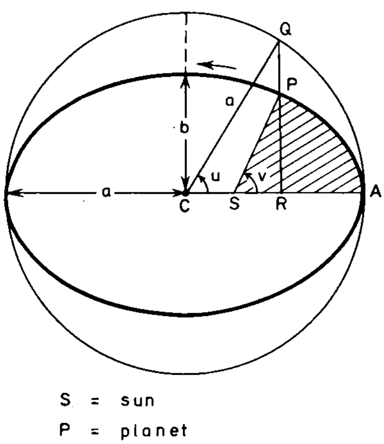


Fig.40.

where $n = 2\pi/T$. As the area of the ellipse $= \pi ab$, we have:

$$\text{area of SPA} = \pi ab \frac{t - \tau}{T} = \frac{1}{2} ab\zeta \qquad [2]$$

We can also express the same area SPA in the eccentric anomaly u, by a relatively simple geometrical argument:

$$\text{area } SPA = \text{triangle } PSR + \text{area } RPA$$

$$\text{triangle } PSR = \tfrac{1}{2} SR \cdot PR = \tfrac{1}{2} ab \sin u\,(\cos u - e)$$

Area RPA: consider strips perpendicular to the major axis, then area $RPA = b/a \times$ area $QRA = b/a$ (sector CQA — triangle QCR) $= \frac{1}{2} ab(u - \sin u \cos u)$. Then:

$$\text{area } SPA = \tfrac{1}{2} ab(u - e \sin u) \qquad [3]$$

Then by combination of [2] and [3] we find [1].

Bessel set the problem of expressing $u - \zeta$ in a sine series, i.e., to determine the coefficients c_r in the development:

$$u - \zeta = \sum_{r=1}^{\infty} c_r \sin(r\zeta) \qquad [4]$$

We multiply both sides of [4] by $\sin(s\zeta)$ and integrate from 0 to π:

$$\int_0^\pi (u - \zeta) \sin(s\zeta) \mathrm{d}\zeta = \sum_{r=1}^{\infty} c_r \int_0^\pi \sin(r\zeta) \sin(s\zeta) \mathrm{d}\zeta \qquad [5]$$

From De la Vallée Poussin (1938, p.224), we have:

$$\int_0^\pi \sin(r\zeta) \sin(s\zeta) \mathrm{d}\zeta = \frac{\pi}{2} \delta_{r,s} \qquad [6]$$

where $\delta_{r,s} = 1$ for $r = s$ and $\delta_{r,s} = 0$ for $r \neq s$ (Kronecker's delta).

Integrate the left-hand side of [5] by parts:

$$\int_0^\pi (u - \zeta) \sin(s\zeta) \mathrm{d}\zeta = \underbrace{\frac{1}{s} \left[(\zeta - u) \cos(s\zeta) \right]_0^\pi}_{= 0} + \frac{1}{s} \int_0^\pi \left(\frac{\mathrm{d}u}{\mathrm{d}\zeta} - 1 \right) \cos(s\zeta) \mathrm{d}\zeta \qquad [7]$$

because:

$$\zeta - u = 0 \quad \text{for} \quad \begin{Bmatrix} \zeta = 0 \\ \zeta = \pi \end{Bmatrix} \quad \text{according to [4]}$$

Therefore:

$$\int_0^\pi (u - \zeta) \sin(s\zeta) \mathrm{d}\zeta = \frac{1}{s} \int_0^\pi \cos(s\zeta) \mathrm{d}u \qquad [8]$$

as:

$$\int_0^\pi \cos(s\zeta)\mathrm{d}\zeta = 0$$

$r = s$ is selected, i.e., the only case with a contribution to the right-hand side of [5].

Combine eq.[8], [1], [5] and [6]:

$$\frac{1}{s}\int_0^\pi \cos(s\zeta)\mathrm{d}u = \frac{1}{s}\int_0^\pi \cos[s(u - e\sin u)]\mathrm{d}u = \frac{\pi}{2}\cdot c_s$$

(This is the right-hand side of [5] for $r = s$.) Thus we find:

$$c_s = \frac{2}{\pi s}\int_0^\pi \cos[s(u - e\sin u)]\mathrm{d}u \qquad [9]$$

We define:

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\mathrm{d}\theta \qquad [10]$$

Combine [9], [4] and [10]:

$$\begin{aligned} u - \zeta &= \frac{2}{\pi}\sum_{r=1}^{\infty}\frac{\sin(r\zeta)}{r}\int_0^\pi \cos(re\sin u - ru)\mathrm{d}u \\ &= 2\sum_{r=1}^{\infty} J_r(er)\frac{\sin(r\zeta)}{r} \end{aligned} \qquad [11]$$

The function $J_n(x)$ so defined is called *Bessel's coefficient of order n.* It should be observed that we have earlier defined $J_n(x)$ by equation [6] in section 4.3. We shall prove later (section 5.2.3) that the two definitions are identical. Instead of Bessel *coefficient* we can say Bessel *function.*

5.2. PROPERTIES OF BESSEL COEFFICIENTS

5.2.1 *An important theorem*

We shall demonstrate that $J_n(x)$ is equal to the coefficient of t^n in the expansion of:

$$\mathrm{e}^{(x/2)(t-1/t)}$$

i.e.:

$$\mathrm{e}^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \quad (n \text{ integer}) \qquad [1]$$

To prove this we need only to show that $J_n(x)$ of [1] can be expressed in the form [10] of section 5.1. We have:

$$\sum_{n=-\infty}^{\infty} (-1)^n J_{-n}(x)t^n = \sum_{n=-\infty}^{\infty} J_n(x)\left(-\frac{1}{t}\right)^n = \sum_{n=-\infty}^{\infty} J_n(x)t^n \qquad [2]$$

The second member in [2] is seen to be correct by a change of variable $t \rightarrow -1/t$ in [1]. The left-hand side of [1] is completely unchanged by this change of variable. The first member of [2] is simply obtained from the second by substituting $n \rightarrow -n$. Equating coefficients of t^n in the first and third member of [2], we get:

$$(-1)^n J_{-n}(x) = J_n(x) \tag{3}$$

Substitute $t = e^{i\theta}$ in [1]:

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta} \tag{4}$$

because:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Using [3] we write the right-hand side of [4] as follows:

$$J_0(x) + 2\sum_{m=1}^{\infty} J_{2m}(x)\cos(2m\theta) + 2i\sum_{m=0}^{\infty} J_{2m+1}(x)\sin[(2m+1)\theta] \tag{5}$$

Eq.[5] is seen from the following development:

$$\sum_{n=-\infty}^{\infty} J_n(x)[\cos(n\theta) + i\sin(n\theta)]$$

$$= J_0(x) + J_1(x)(\cos\theta + i\sin\theta) + J_2(x)(\cos 2\theta + i\sin 2\theta) + \ldots$$

$$+ \underbrace{J_{-1}(x)}_{-J_1(x)}(\cos\theta - i\sin\theta) + \underbrace{J_{-2}(x)}_{+J_2(x)}(\cos 2\theta - i\sin 2\theta) + \ldots$$

$$= J_0(x) + J_1(x)\cdot 2\,i\sin\theta + J_2(x)\cdot 2\cos 2\theta + \ldots$$

Equating real and imaginary parts in [4] we get respectively:

$$\left.\begin{aligned} \cos(x\sin\theta) &= J_0(x) + 2\sum_{m=1}^{\infty} J_{2m}(x)\cos(2m\theta) \\ \sin(x\sin\theta) &= 2\sum_{m=0}^{\infty} J_{2m+1}(x)\sin[(2m+1)\theta] \end{aligned}\right| \tag{6}$$

Multiply [6.1] by $\cos(n\theta)$ and [6.2] by $\sin(n\theta)$, integrate with respect to θ from 0 to π and use the formulas:

$$\int_0^\pi \cos(m\theta)\cos(n\theta)d\theta = \int_0^\pi \sin(m\theta)\sin(n\theta)d\theta = \frac{\pi}{2}\delta_{m,n} \tag{7}$$

We then find that:

$$\left.\begin{aligned} J_n(x) &= \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\cos(n\theta)\mathrm{d}\theta && (n = 2m,\ n \text{ is even}) \\ J_n(x) &= \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin(n\theta)\mathrm{d}\theta && (n = 2m+1,\ n \text{ is odd}) \end{aligned}\right| \qquad [8]$$

In the integration interval, sine is positive in both first and second quadrants, whereas cosine is positive in first and negative in second quadrant. Therefore in [8.1] the integral $= 0$, if n is odd, and in [8.2] the integral $= 0$, if n is even. The two formulas [8] can then be combined into one expression:

$$\begin{aligned} J_n(x) &= \frac{1}{\pi}\int_0^\pi [\cos(x\sin\theta)\cos(n\theta) + \sin(x\sin\theta)\sin(n\theta)]\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\mathrm{d}\theta \qquad [9] \end{aligned}$$

valid for all integral values of n. This is identical with [10] in section 5.1, which should be proved. In particular, for $n = 0$, we get:

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\mathrm{d}\theta \qquad [10]$$

These are *Bessel functions of the first kind.* An alternative treatment, found in some books, is to *define* $J_n(x)$ by eq.[1] and then to *prove* eq.[9].

5.2.2 Recurrence relations for the Bessel coefficients

Differentiate eq.[1]:

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \qquad [11]$$

with respect to x:

$$\frac{1}{2}\left(t - \frac{1}{t}\right) e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J'_n(x)t^n$$

Using [11], this can be written as follows:

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} [J_n(x)t^{n+1} - J_n(x)t^{n-1}] - \sum_{n=-\infty}^{\infty} J'_n(x)t^n = 0$$

Equating the coefficient for t^n to zero, we find:

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x) \qquad [12]$$

Then we differentiate [11] with respect to t instead and find:

$$\frac{x}{2}\left(1+\frac{1}{t^2}\right) e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1}$$

which is equivalent to:

$$\frac{x}{2}\sum_{n=-\infty}^{\infty} (t^n + t^{n-2})J_n(x) - \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1} = 0$$

Equating the coefficient of t^{n-1} to zero, we get:

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \qquad [13]$$

Equating coefficients to zero is justified by the fact that the relations must be valid for all t.

Add [12] and [13]:

$$xJ'_n(x) = xJ_{n-1}(x) - nJ_n(x) \qquad [14]$$

Subtract [12] from [13]:

$$xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x) \qquad [15]$$

Put $n = 0$ in [15]:

$$J'_0(x) = -J_1(x) \qquad [16]$$

Put $n = 1$ in [14]:

$$J'_1(x) = J_0(x) - \frac{1}{x} J_1(x) \qquad [17]$$

Differentiate [16] with respect to x and use [17]:

$$J''_0(x) = -J_0(x) + \frac{1}{x} J_1(x) \qquad [18]$$

Considering [16], we get from [18]:

$$J''_0(x) + \frac{1}{x} J'_0(x) + J_0(x) = 0 \qquad [19]$$

showing that $y = J_0(x)$ is a solution of the differential equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \qquad [20]$$

Similarly, we show that $y = J_n(x)$ satisfies the differential equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \qquad [21]$$

(n is integer).

Proof: Differentiate [15] with respect to x:

$$xJ''_n(x) + J'_n(x) = nJ'_n(x) - J_{n+1}(x) - xJ'_{n+1}(x) \tag{22}$$

Multiply [15] by n/x:

$$nJ'_n(x) = \frac{n^2}{x} J_n(x) - nJ_{n+1}(x)$$

Substitute $n \to n + 1$ in [14]:

$$xJ'_{n+1}(x) + (n + 1)J_{n+1}(x) = xJ_n(x)$$

By means of the last two equations [22] becomes:

$$xJ''_n(x) + J'_n(x) = \frac{n^2}{x} J_n(x) - xJ_n(x)$$

or:

$$J''_n(x) + \frac{1}{x} J'_n(x) + \left(1 - \frac{n^2}{x^2}\right) J_n(x) = 0 \tag{23}$$

i.e., eq.[21], which should be proved. This is *Bessel's equation*. This equation was studied in detail in section 4.3. Note that from the development in this section, n must be an integer in [23].

5.2.3 Series expansion for the Bessel coefficient

We write:

$$e^{(x/2)(t-1/t)} = e^{xt/2}e^{-x/2t}$$

and apply the formula $e^z = \sum_0^\infty \frac{z^n}{n!}$ (DE LA VALLÉE POUSSIN, 1938, p.455). We find:

$$e^{(x/2)(t-1/t)} = \sum_{r=0}^{\infty} \frac{(xt)^r}{2^r r!} \sum_{s=0}^{\infty} \frac{(-x)^s}{2^s t^s s!} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{r+s} \frac{t^{r-s}}{r!s!} \tag{24}$$

Putting $r - s = \pm n$, we can write [24] as follows:

$$e^{(x/2)(t-1/t)} = \sum_{n=0}^{\infty} t^n \underbrace{\sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{(n+s)!s!}}_{J_n(x)} + \sum_{n=1}^{\infty} t^{-n} \underbrace{\sum_{r=0}^{\infty} (-1)^n \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{(n+r)!r!}}_{J_{-n}(x)} \tag{25}$$

$J_n(x)$ is the coefficient of t^n in this expansion. If n is zero or a positive integer, we get:

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s} \tag{26}$$

For n a negative integer, apply eq.[3]: $(-1)^n J_{-n}(x) = J_n(x)$. We can write [26] in the following form:

$$J_n(x) = \frac{x^n}{2^n n!} \sum_{s=0}^{\infty} \frac{1}{s!(n+1)_s} \left(-\frac{x^2}{4}\right)^s \tag{27}$$

because:

$$(n+1)_s = (n+1)(n+2)\dots(n+s-1)(n+s)$$

and:

$$(n+s)! = n!\,(n+1)_s$$

or:

$$J_n(x) = \frac{x^n}{2^n n!}\, {}_0F_1\left(n+1\;;\; -\frac{x^2}{4}\right) \tag{28}$$

where ${}_0F_1$ is the sum of a hypergeometric series, as defined in eq.[10], section 4.6.

Eq.[27] is identical with [6] in section 4.3. We have thus proved that the definition [10] in section 5.1 of $J_n(x)$ is identical with the definition [6] in section 4.3, at least for integer values of n. But, because of the development in section 4.3, we can immediately generalize the definition of $J_n(x)$ to any n, integer or not. Moreover, the formulas in this chapter, which have been proved for integer values of n, could be extended to any values of n, including complex numbers, except for eq.[3] which is valid only for integer values of n.

Some properties of Bessel functions may be derived from the series expansion. Multiply [26] by x^n:

$$x^n J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2n+2s}}{s!\,(n+s)!} \left(\frac{1}{2}\right)^{n+2s}$$

Differentiate with respect to x and apply the formula:

$$\frac{2n+2s}{(n+s)!} = \frac{2}{(n-1+s)!}$$

We then get:

$$\frac{d}{dx}[x^n J_n(x)] = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2n+2s-1}}{s!(n-1+s)!} \left(\frac{1}{2}\right)^{n-1+2s} = x^n J_{n-1}(x) \tag{29}$$

applying $[26] \cdot x^n$ again. Writing [29] as follows:

$$\frac{1}{x}\frac{d}{dx}[x^n J_n(x)] = x^{n-1} J_{n-1}(x) \tag{30}$$

we see that if m is a positive integer, then:

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m [x^n J_n(x)] = x^{n-m} J_{n-m}(x) \tag{31}$$

In a similar way, we find the corresponding relations containing a factor x^{-n}:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x) \\
&\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)[x^{-n}J_n(x)] = -x^{-n-1}J_{n+1}(x) \\
&\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^m[x^{-n}J_n(x)] = (-1)^m x^{-n-m}J_{n+m}(x)
\end{aligned} \tag{32}$$

In particular for $n = 0$, $m \to n$ the last equation gives:

$$x^{-n}J_n(x) = (-1)^n\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n J_0(x) \tag{33}$$

which shows how the Bessel coefficient $J_n(x)$ may be derived from $J_0(x)$.

Behaviour of Bessel coefficients for small values of x. Since:

$$\lim_{x\to 0} {}_0F_1\left(n+1; -\frac{x^2}{4}\right) = 1$$

we have from [28]:

$$\lim_{x\to 0} x^{-n}J_n(x) = \frac{1}{2^n n!}$$

or that for small values of x, $J_n(x)$ behaves like:

$$J_n(x) \simeq \frac{x^n}{2^n n!} \tag{34}$$

5.2.4 *Integral expressions for the Bessel coefficients*

One integral expression has already been given, eq.[9]. Now we shall derive some other integral expressions.

Consider the integral:

$$I = \int_{-1}^{1}(1-t^2)^{n-1/2}\mathrm{e}^{\mathrm{i}xt}\mathrm{d}t \qquad \text{with } n > -1/2 \tag{35}$$

Develop the exponential function in a power series. Then [35] becomes:

$$I = \sum_{s=0}^{\infty}\frac{(\mathrm{i}x)^s}{s!}\int_{-1}^{1}(1-t^2)^{n-1/2}t^s\mathrm{d}t \tag{36}$$

If s is an odd integer, the integral is zero, because positive and negative values cancel in the integration interval of $+1$ to -1. If s is an even integer, $s = 2r$, then substituting $u = t^2$ (i.e., $\mathrm{d}u = 2t\mathrm{d}t$, $\mathrm{d}t = \dfrac{\mathrm{d}u}{2u^{1/2}}$; a factor 1/2 is compensated by taking only half the integration interval):

$$\int_0^1 (1-u)^{n-1/2}\,u^{r-1/2}\,\mathrm{d}u = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(r+\frac{1}{2}\right)}{\Gamma(n+r+1)} \qquad [37]$$

using the relation between beta- and gamma-functions (eq.[7] in section 1.3). Then as $i^{2r} = (-1)^r$:

$$I = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r)!} \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(r+\frac{1}{2}\right)}{\Gamma(n+r+1)} =$$

$$= \Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!\,\Gamma(n+r+1)2^{2r}} \qquad [38]$$

applying the duplication formula for the gamma-function (eq.[10] in section 1.3), multiplied by $2r$:

$$2r\cdot\Gamma\left(\frac{1}{2}\right)\Gamma(2r) = 2r\cdot 2^{2r-1}\,\Gamma(r)\,\Gamma\left(r+\frac{1}{2}\right)$$

i.e.:

$$\Gamma\left(\frac{1}{2}\right)(2r)! = 2^{2r} r!\,\Gamma\left(r+\frac{1}{2}\right)$$

Combine [38] with [26], observing that $\Gamma(n+r+1) = (n+r)!$. Then:

$$I = \Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\frac{J_n(x)}{\left(\frac{x}{2}\right)^n}$$

i.e.:

$$J_n(x) = \frac{\left(\frac{x}{2}\right)^n}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}\int_{-1}^{1}(1-t^2)^{n-1/2}e^{ixt}\mathrm{d}t \qquad [39]$$

which is equivalent to:

$$J_n(x) = \frac{x^n}{2^{n-1}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}\int_0^1 (1-t^2)^{n-1/2}\cos(xt)\mathrm{d}t \qquad [40]$$

because:

$$e^{ixt} = \cos(xt) + i\sin(xt)$$

Positive and negative sine-terms cancel and only cosines are left. Half the integration interval is taken, and compensated by multiplication with 2.

In particular, as $\Gamma(1/2) = \sqrt{\pi}$ (eq.[8] in section 1.3), we find:

$$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos(xt)}{\sqrt{(1-t^2)}}\,\mathrm{d}t \qquad [41]$$

Substitute $t = \cos\theta$ in [40], eliminating the negative sign by interchanging integration limits:

$$J_n(x) = \frac{x^n}{2^{n-1}\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}\int_0^{\pi/2} \cos(x\cos\theta)\sin^{2n}\theta\mathrm{d}\theta \qquad [42]$$

Similarly, substitute $t = \sin\theta$ in [40]:

$$J_n(x) = \frac{x^n}{2^{n-1}\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}\int_0^{\pi/2} \cos(x\sin\theta)\cos^{2n}\theta\mathrm{d}\theta \qquad [43]$$

Special case $n = 0$:

$$J_0(x) = \frac{2}{\pi}\int_0^{\pi/2}\cos(x\cos\theta)\mathrm{d}\theta = \frac{2}{\pi}\int_0^{\pi/2}\cos(x\sin\theta)\mathrm{d}\theta \qquad [44]$$

which is the same as eq.[10].

Obviously, eq.[44] could also be written in the following way:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi}\cos(x\cos\theta)\mathrm{d}\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cos(x\cos\theta)\mathrm{d}\theta + \underbrace{\frac{i}{2\pi}\int_0^{2\pi}\sin(x\cos\theta)\mathrm{d}\theta}_{=\,0\text{ (as positive and negative contributions cancel)}} =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\mathrm{d}\theta \qquad [44a]$$

We shall also derive some other useful integral expressions for $J_n(x)$. (See MARGENAU and MURPHY, 1943, pp.109–111.) We use some results from residue calculus (section 2.1). Assume that a function $f(z)$ can be expanded around a point z_0, where m is an integer:

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \ldots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots \qquad [45]$$

Assume that $z_0 = 0$. The coefficient of $1/z$ in the expansion of $f(z)$ is given by:

$$a_{-1} = \frac{1}{2\pi i}\int f(z)\mathrm{d}z \qquad [46]$$

where the integration is taken along a contour around $z = 0$, in counter-clockwise direction. Similarly, defining another function $\varphi(z)$:

$$\varphi(z) = \frac{f(z)}{z^{n+1}} = \frac{\ldots a_n z^n \ldots}{z^{n+1}} = \ldots\frac{a_n}{z}\ldots \qquad (z_0 = 0) \qquad [47]$$

we get by the same theorem the coefficient a_n of $1/z$ in the expansion of $\varphi(z)$ or, what is exactly the same, the coefficient of z^n in the expansion of $f(z)$:

$$a_n = \frac{1}{2\pi i} \int \varphi(z) \mathrm{d}z = \frac{1}{2\pi i} \int \frac{f(z)}{z^{n+1}} \mathrm{d}z \tag{48}$$

We apply [48] to equation [25]. Then we let $f(z)$ correspond to $e^{(x/2)(t-1/t)}$, and the coefficient of z^n in the expansion of $f(z)$ corresponds to the coefficient of t^n in the expansion of $e^{(x/2)(t-1/t)}$. Thus from [48]:

$$J_n(x) = \frac{1}{2\pi i} \int \frac{e^{(x/2)(t-1/t)}}{t^{n+1}} \mathrm{d}t \tag{49}$$

Eq.[49] is called *Schläfli's integral.*

Frequently, [49] appears in a slightly different form, obtainable immediately from [49] by the following substitution:

$$t = \frac{2u}{x} \tag{50}$$

which gives:

$$J_n(x) = \frac{1}{2\pi i} \left(\frac{x}{2}\right)^n \int \frac{e^{u-x^2/4u}}{u^{n+1}} \mathrm{d}u \tag{51}$$

Now [49] and [51] have been derived from [25], which was proved only for integer values of n. But [49] and [51] are valid also for non-integer values of n. This is best demonstrated by substituting [49] or [51] into the Bessel differential equation. It is found that this is identically fulfilled, for any value of n.

For non-integral n, however, we have to observe that in carrying out the contour integrations in [49] or [51], the negative real axis in the t- or u-plane is a branch cut, and we have to choose a contour as shown in Fig.41. The reason for this branch cut is the denominator, t^{n+1} or u^{n+1}, respectively. Take for example $n = -1/2$. We have $t^{1/2} = r^{1/2}e^{i\theta/2}$. At the starting-point (A): $\theta = -\pi$; $t^{1/2} = e^{-i\pi/2} = -i$ (putting $r = 1$). At the end-point (B): $\theta = \pi$; $t^{1/2} = e^{i\pi/2} = +i$.

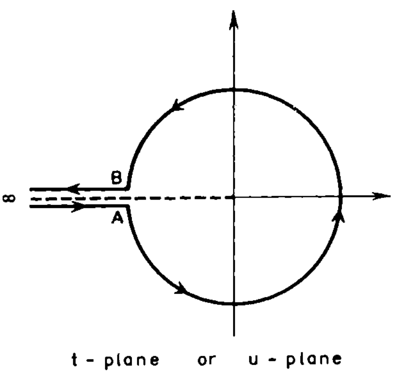


Fig.41.

We have to observe that in this case we could not use a contour similar to the one shown in Fig.16 for the reason that now we have to encircle the origin. The termini of the contour in Fig.41 are located at t or $u = -\infty$. Assuming x real and positive we see that the integrands in [49] and [51] approach zero as t or u approaches $-\infty$. This means that we have to think of the contour in Fig.41 as being closed at t or $u = -\infty$.

Still other integral expressions—for $J_0(x)$—will be demonstrated below (eq.[11] and [17] in section 5.3).

5.2.5 *The addition formula for the Bessel coefficients*

We derive an expression for $J_n(x + y)$. From eq.[1] we have:

$$e^{(1/2)(x+y)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x + y)t^n \qquad [52]$$

We write the left-hand side of [52] as a product:

$$e^{(x/2)(t-1/t)} \cdot e^{(y/2)(t-1/t))}$$

and apply eq.[1] again:

$$\sum_{n=-\infty}^{\infty} J_n(x + y)t^n = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} J_r(x)J_s(y)t^{r+s}$$

Equate the coefficients for t^n, i.e., $r + s = n$ or $s = n - r$:

$$J_n(x + y) = \sum_{r=-\infty}^{\infty} J_r(x)J_{n-r}(y)$$

We split the term on the right-hand side into three terms, so that we get Bessel coefficients of positive order only:

$$\sum_{r=-\infty}^{-1} J_r(x)J_{n-r}(y) + \sum_{r=0}^{n} J_r(x)J_{n-r}(y) + \sum_{r=n+1}^{\infty} J_r(x)J_{n-r}(y)$$

Apply eq.[3]: $(-1)^n J_{-n}(x) = J_n(x)$. Then the first term is written:

$$\sum_{r=-\infty}^{-1} (-1)^r J_{-r}(x)J_{n-r}(y) = \sum_{r=1}^{\infty} (-1)^r J_r(x)J_{n+r}(y)$$

replacing $-r$ by $+r$. Similarly, we find for the third term, replacing r by $n + r$:

$$\sum_{r=1}^{\infty} J_{n+r}(x)J_{-r}(y) = \sum_{r=1}^{\infty} (-1)^r J_{n+r}(x)J_r(y)$$

and finally:

$$J_n(x + y) = \sum_{r=0}^{n} J_r(x)J_{n-r}(y) + \sum_{r=1}^{\infty} (-1)^r[J_r(x)J_{n+r}(y) + J_{n+r}(x)J_r(y)] \qquad [53]$$

5.2.6 Integrals involving Bessel functions

Integrate eq.[29]:

$$\int_0^\alpha x^n J_{n-1}(x)\mathrm{d}x = [x^n J_n(x)]_0^\alpha$$

If $n > 0$, $x^n J_n(x) \to 0$ as $x \to 0$; therefore the lower limit $= 0$, and:

$$\int_0^\alpha x^n J_{n-1}(x)\mathrm{d}x = \alpha^n J_n(\alpha) \qquad (n > 0) \tag{54}$$

Substitute $x = \xi r$ and $\alpha = \xi a$ in [54]:

$$\int_0^a r^n J_{n-1}(\xi r)\mathrm{d}r = \frac{a^n}{\xi} J_n(\xi a) \qquad (n > 0) \tag{55}$$

A special case of importance is $n = 1$:

$$\int_0^a r J_0(\xi r)\mathrm{d}r = \frac{a}{\xi} J_1(\xi a) \tag{56}$$

Further results can be obtained from [29]:

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n J_n(x)] = x^n J_{n-1}(x)$$

Put $x = \xi r$ and $n = 1$. Then, multiplying [29] by r^2 and integrating:

$$\int_0^a r^3 J_0(\xi r)\mathrm{d}r = \int_0^a r^2 \frac{1}{\xi}\frac{\partial}{\partial r}[rJ_1(\xi r)]\mathrm{d}r$$

$$= \frac{a^3}{\xi} J_1(\xi a) - \frac{2}{\xi}\int_0^a r^2 J_1(\xi r)\mathrm{d}r$$

(by partial integration)

$$= \frac{a^3}{\xi} J_1(\xi a) - \frac{2a^2}{\xi^2} J_2(\xi a)$$

using [55]. Eq.[13] gives:

$$J_2(\xi a) = \frac{2}{\xi a} J_1(\xi a) - J_0(\xi a)$$

and finally:

$$\int_0^a r^3 J_0(\xi r)\mathrm{d}r = \frac{2a^2}{\xi^2}\left[J_0(\xi a) + \left(\frac{1}{2}a\xi - \frac{2}{a\xi}\right) J_1(\xi a)\right] \tag{57}$$

Combine with [56]:

$$\int_0^a r(a^2 - r^2)J_0(\xi r)\mathrm{d}r = \frac{4a}{\xi^3} J_1(\xi a) - \frac{2a^2}{\xi^2} J_0(\xi a) \qquad [58]$$

Several other integrals involving Bessel functions may be derived, e.g., the following:

$$\int_0^\infty J_n(ax)x^n \mathrm{e}^{-px}\mathrm{d}x = \frac{2^n\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{a^n}{(a^2+p^2)^{n+1/2}} \qquad [59]$$

$$\int_0^\infty J_n(ax)x^{n+1}\mathrm{e}^{-px}\mathrm{d}x = \frac{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{pa^n}{(a^2+p^2)^{n+3/2}} \qquad [60]$$

Eq.[59] and [60], given here without proof, are especially important. The special cases $n = 0$ and $n = 1$, which are obtained immediately, occur frequently.

Such integrals are usually proved by expanding the Bessel function in powers of ax and integrating term by term (this is justified if the series converges absolutely). See further WATSON (1944, pp.384–386).

The equations [55], [56], [58], [59] and [60] are the most important formulas in this sub-section.

5.2.7 Expansions in series composed of Bessel functions. Fourier-Bessel integrals

In Bessel's differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \left(1 - \frac{m^2}{x^2}\right)y = 0$$

we make the substitutions $x \to \lambda x$ and $x \to \mu x$ respectively:

$$\left.\begin{aligned} &\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x} + (\lambda^2x^2 - m^2)\right] J_m(\lambda x) = 0 \\ &\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x} + (\mu^2x^2 - n^2)\right] J_n(\mu x) = 0 \end{aligned}\right| \qquad [61]$$

Writing for simplicity $f(x) = J_m(\lambda x)$ and $g(x) = J_n(\mu x)$, eq.[61] become:

$$x^2f'' + xf' + (\lambda^2x^2 - m^2)f = 0$$

$$x^2g'' + xg' + (\mu^2x^2 - n^2)g = 0$$

Multiplying the first equation by g/x and the second by f/x and subtracting, we find:

$$(\lambda^2 - \mu^2)xfg + (n^2 - m^2)\frac{1}{x}fg = \frac{\mathrm{d}}{\mathrm{d}x}[x(fg' - gf')] \qquad [62]$$

We integrate this equation with respect to x from 0 to a. After reintroducing $J_m(\lambda x)$ and $J_n(\mu x)$, also remembering that $f' = \lambda J'_m(\lambda x)$ and $g' = \mu J'_n(\mu x)$, we find:

$$(\lambda^2 - \mu^2) \int_0^a x J_m(\lambda x) J_n(\mu x) \mathrm{d}x + (n^2 - m^2) \int_0^a J_m(\lambda x) J_n(\mu x) \frac{\mathrm{d}x}{x}$$

$$= a[\mu J_m(\lambda a) J'_n(\mu a) - \lambda J_n(\mu a) J'_m(\lambda a)] \qquad [63]$$

The last expression is assumed to vanish for $x = 0$. Considering eq.[12], by which we can replace the derivative of a Bessel coefficient by two other Bessel coefficients, we easily find that the expression mentioned vanishes for $x = 0$, provided the following condition is fulfilled: $n > -1$ and $m > -1$.

Put $m = n$, but still keep $\lambda \neq \mu$. Then [63] becomes:

$$\int_0^a x J_n(\lambda x) J_n(\mu x) \mathrm{d}x = \frac{a}{\lambda^2 - \mu^2} [\mu J_n(\lambda a) J'_n(\mu a) - \lambda J_n(\mu a) J'_n(\lambda a)] \qquad [64]$$

The corresponding expression for $\lambda = \mu$ is obtained by putting $\mu = \lambda + \varepsilon$ where ε is small, using Taylor's theorem and then letting ε tend to zero. The result is:

$$\int_0^a x[J_n(\lambda x)]^2 \mathrm{d}x = \frac{a^2}{2} \left\{ [J'_n(\lambda a)]^2 + \left(1 - \frac{n^2}{\lambda^2 a^2}\right) [J_n(\lambda a)]^2 \right\} \qquad [65]$$

Suppose now that λ and μ are positive roots of the transcendental equation:

$$h J_n(\lambda a) + k \lambda a J'_n(\lambda a) = 0 \qquad [66]$$

where h and k are constants. Then:

$$\int_0^a x J_n(\lambda x) J_n(\mu x) \mathrm{d}x = c_\lambda \delta_{\lambda,\mu} \qquad [67]$$

where:

$$c_\lambda = \frac{[J_n(\lambda a)]^2}{2k^2 \lambda^2} [k^2 \lambda^2 a^2 + h^2 - k^2 n^2] \qquad [68]$$

and $\delta_{\lambda,\mu}$ is Kronecker's delta. Check by substitution in [64] and [65] respectively.

Assume that an arbitrary function $f(x)$ can be expanded as follows:

$$f(x) = \sum_{i=1}^{\infty} a_i J_n(\lambda_i x) \qquad [69]$$

where the sum is taken over the positive roots of [66]. The coefficients a_i are determined as follows: multiply [69] by $x J_n(\lambda_j x)$ and integrate with respect to x from 0 to a, then:

$$\int_0^a x f(x) J_n(\lambda_j x) \mathrm{d}x = \sum_i a_i \int_0^a x J_n(\lambda_i x) J_n(\lambda_j x) \mathrm{d}x = a_j c_{\lambda_j}$$

by [67] from which it follows that:

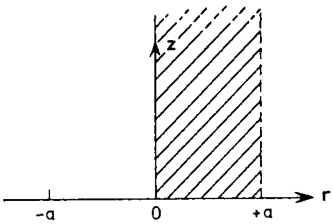


Fig.42.

$$a_j = \frac{1}{c_{\lambda_j}} \int_0^a x f(x) J_n(\lambda_j x) \mathrm{d}x \qquad [70]$$

Because of its similarity to a Fourier series, a series of the type [69] is called a *Fourier–Bessel series*. In the limit, this is a *Fourier-Bessel integral.*

In particular, if [66] reduces to the following equation:

$$J'_n(\lambda a) = 0$$

i.e., $h = 0$ and k arbitrary, then we have:

$$a_j = \frac{2\lambda_j^2}{[J_n(\lambda_j a)]^2} \frac{1}{\lambda_j^2 a^2 - n^2} \int_0^a x f(x) J_n(\lambda_j x) \mathrm{d}x \qquad [71]$$

Use of Bessel functions in potential theory. We seek a solution $\psi(r, z)$ for the half-space $a \geqslant r \geqslant 0$, $z \geqslant 0$ of the differential equation:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0 \qquad [72]$$

satisfying the following boundary conditions (Fig.42):

(1) $\psi = f(r)$ on $z = 0$

(2) $\psi \to 0$ as $z \to \infty$

(3) $\dfrac{\partial \psi}{\partial r} + \varkappa \psi = 0$ on $r = a$

(4) ψ remains finite as $r \to 0$

We have seen in section 1.2 that a function of the form $\psi = R(r)Z(z)$ is a solution of [72] provided that:

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - \lambda_i^2 Z = 0 \qquad [73]$$

and:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \lambda_i^2 R = 0 \qquad [74]$$

where λ_i is a constant. Boundary condition (*2*) requires that:

$$Z = e^{-\lambda_i z}$$

and (*4*) requires that $R = J_0(\lambda_i r)$, since the second solution $Y_0(\lambda_i r)$ would become infinite for $r = 0$. Compare eq.[11] in section 4.3 for $n = 0$ and $x = \lambda_i r$. We see that $Y_0(\lambda_i r)$ involves $\log(\lambda_i r)$, which is negative infinite for $r = 0$.

Therefore, the solution of [72] can be written:

$$\psi(r, z) = \sum_i a_i e^{-\lambda_i z} J_0(\lambda_i r) \tag{75}$$

where a_i and λ_i are constants. Boundary condition (*3*) then gives immediately by substituting [75]:

$$\lambda_i J'_0(\lambda_i a) + \varkappa J_0(\lambda_i a) = 0 \tag{76}$$

which means that the sum in [75] should be taken over the positive roots of eq.[76]. λ_i must be positive to satisfy condition (*2*). In addition, we have to find constants a_i such that condition (*1*) is fulfilled, i.e.:

$$f(r) = \sum_i a_i J_0(\lambda_i r) \tag{77}$$

By comparison of eq.[76] with eq.[66], we have in this particular case $n = 0$, $h = \varkappa$, $k = 1/a$ and therefore by eq.[68] and [70]:

$$a_i = \frac{2\lambda_i^2}{a^2(\lambda_i^2 + \varkappa^2)[J_0(\lambda_i a)]^2} \int_0^a r' f(r') J_0(\lambda_i r') \mathrm{d}r' \tag{78}$$

Inserted into [75] this gives the required solution:

$$\psi(r, z) = \frac{2}{a^2} \sum_i \frac{\lambda_i^2 e^{-\lambda_i z} J_0(\lambda_i r)}{(\lambda_i^2 + \varkappa^2)[J_0(\lambda_i a)]^2} \int_0^a r' f(r') J_0(\lambda_i r') \mathrm{d}r' \tag{79}$$

where the sum is taken over the positive roots of [76].

5.3 RELATED BESSEL FUNCTIONS

5.3.1 Hankel functions of zero order

Hankel functions are even more important than Bessel functions in applied mathematics. There are extensive treatments of such functions (especially WATSON's, 1944, book). We shall limit the discussion in this section to Hankel functions of zero order and follow FRANK and VON MISES (1930, pp.415–418).

Consider Bessel's differential equation of zero order ($n = 0$):

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + x^2 y = 0 \tag{1}$$

We shall prove that one solution of this equation is:

$$y = g(x) = e^{-ix} \int_0^\infty e^{-2ux}(u - iu^2)^{-1/2} du \qquad [2]$$

which converges for $x > 0$ and where $(u - iu^2)^{-1/2}$ only with positive real part is considered (the integration is made with respect to u from 0 to ∞). In fact, [2] can be obtained by the Laplace transform method (Chapter 8), but is here proved directly. By differentiating [2] we get (including differentiation under the integral sign):

$$g'(x) = -ig(x) - 2ug(x) = -(i + 2u)g(x)$$

and:

$$g''(x) = -(i + 2u)g'(x) = +(i + 2u)^2 g(x)$$
$$= (i^2 + 4u^2 + 4iu)g(x) = 4i(u - iu^2)g(x) - g(x)$$

Inserting into the left-hand side of [1] we get:

$$x^2 g''(x) + xg'(x) + x^2 g(x)$$
$$= [4i(u - iu^2)x^2 - x^2 - (i + 2u)x + x^2]g(x)$$
$$= [4i(u - iu^2)x^2 - (i + 2u)x]g(x)$$
$$= 4ix^2 e^{-ix} \int_0^\infty e^{-2ux}(u - iu^2)^{1/2} du$$
$$- xe^{-ix} \int_0^\infty e^{-2ux}(u - iu^2)^{-1/2}(i + 2u) du = 0 \qquad [3]$$

because the first integral can be written as follows, by partial integration:

$$\int_0^\infty e^{-2ux}(u - iu^2)^{1/2} du = \frac{1}{4x}\int_0^\infty e^{-2ux}(u - iu^2)^{-1/2}(1 - 2iu) du$$

Thus we have demonstrated that $g(x)$ is a solution of [1].

If in [2] we replace i by $-i$, i.e., form the complex conjugate $\bar{g}(x)$ of $g(x)$, we find that $\bar{g}(x)$ is also a solution of [1]. We then *define* the following two functions:

$$\left.\begin{aligned} H_0^{(1)}(x) &= \frac{2}{\pi} e^{-i\pi/4} \bar{g}(x) = \frac{2}{\pi} e^{i(x-\pi/4)} \int_0^\infty e^{-2ux}(u + iu^2)^{-1/2} du \\ H_0^{(2)}(x) &= \frac{2}{\pi} e^{i\pi/4} g(x) = \frac{2}{\pi} e^{-i(x-\pi/4)} \int_0^\infty e^{-2ux}(u - iu^2)^{-1/2} du \end{aligned}\right| \qquad [4]$$

These are the *Bessel functions of the third kind* (or *Hankel functions*) *of zero order*.

Also $J_0(x)$ and $Y_0(x)$ are solutions of [1], as already found in section 4.3. We shall demonstrate that the following relations hold between the different solutions:

$$\left.\begin{aligned} H_0^{(1)}(x) &= J_0(x) + iY_0(x) \\ H_0^{(2)}(x) &= J_0(x) - iY_0(x) \end{aligned}\right| \qquad [5]$$

i.e.:

$$J_0(x) = \mathrm{Re}\, H_0^{(1)}(x) = \frac{2}{\pi}\,\mathrm{Re}\, e^{i(x-\pi/4)} \int_0^\infty e^{-2ux}(u + iu^2)^{-1/2}\mathrm{d}u$$

$$Y_0(x) = \mathrm{Im}\, H_0^{(1)}(x) = \frac{2}{\pi}\,\mathrm{Im}\, e^{i(x-\pi/4)} \int_0^\infty e^{-2ux}(u + iu^2)^{-1/2}\mathrm{d}u \qquad [6]$$

where Re = real part of, and Im = imaginary part of. Note that in section 4.3 we took [5] as definition of the Hankel functions. Here instead we take [4] as definition and prove [5]. Naturally, the two treatments are equivalent to each other.

We limit ourselves to proving [6.1], the proof of [6.2] being analogous. We replace the variable u by another variable φ:

$$u = \cot\varphi \qquad [7]$$

Then we find that:

$$(u + iu^2)^{-1/2} = u^{-1/2}(1 + iu)^{-1/2}$$
$$= \cot^{-1/2}\varphi(1 + i\cot\varphi)^{-1/2}$$
$$= \cos^{-1/2}\varphi\,\sin\varphi(\sin\varphi + i\cos\varphi)^{-1/2}$$
$$= \cos^{-1/2}\varphi\,\sin\varphi\, e^{(i/2)(\varphi-\pi/2)} \text{ (using de Moivre's formula)} \qquad [8]$$

De Moivre's formula reads:

$$(\cos z \pm i\sin z)^n = \cos(nz) \pm i\sin(nz) = e^{\pm inz}$$

Applied to our case, we get:

$$(\sin\varphi + i\cos\varphi)^{-1/2} = \left[\cos\left(\frac{\pi}{2}-\varphi\right) + i\sin\left(\frac{\pi}{2}-\varphi\right)\right]^{-1/2}$$

$$e^{(+i)(-1/2)(\pi/2-\varphi)} = e^{(i/2)(\varphi-\pi/2)}$$

The real part of $H_0^{(1)}(x)$ becomes:

$$\frac{2}{\pi}\int_0^{\pi/2} e^{-2x\cot\varphi}\cos^{-1/2}\varphi\,\sin\varphi\,\cos\left(x + \frac{\varphi}{2} - \frac{\pi}{2}\right)\frac{\mathrm{d}\varphi}{\sin^2\varphi} \qquad [9]$$

For $x \to 0$ the integral [9] converges to:

$$\frac{2}{\pi}\int_0^{\pi/2} \cos^{-1/2}\varphi\,\sin\varphi\,\cos\left(\frac{\varphi}{2} - \frac{\pi}{2}\right)\frac{\mathrm{d}\varphi}{\sin^2\varphi} = 1 \qquad [10]$$

The evaluation of the integral in [10] can easily be made by contour integration along the unit circle (see eq.[18] in section 2.1), and is left to the reader as an exercise in such integration methods. As $J_0(0) = 1$ (see, e.g., eq.[10] in section 5.1), we have thus proved that for $x = 0$, eq.[6.1] is true, i.e., that $J_0(0)$ is the real part of $H_0^{(1)}(0)$. But as both $H_0^{(1)}(x)$ and $J_0(x)$ are solutions of [1], it is clear that our proof is valid for any value of x. In other words, as these two functions agree in one point ($x = 0$) and as they are both

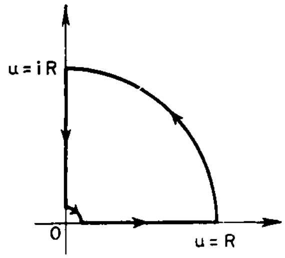


Fig.43.

solutions of the same equation [1], then they must agree for any value of x. The result permits us then from [9] to write another expression for $J_0(x)$:

$$J_0(x) = \frac{2}{\pi}\int_0^{\pi/2} e^{-2x\cot\varphi}\frac{\sin(x+\varphi/2)}{\sin\varphi\sqrt{\cos\varphi}}\,d\varphi \qquad [11]$$

After this we shall derive some other formulas for $H_0^{(1)}(x)$ and $H_0^{(2)}(x)$, simpler than [4], using contour integration. Let $f(u)$ be the integrand of [2] and perform an integration along the contour shown in Fig.43, in the complex u-plane. We have that $x > 0$ and:

$$f(u) = e^{-2ux}(u - iu^2)^{-1/2} = \frac{e^{-2ux}}{\sqrt{(u - iu^2)}} \qquad [12]$$

Putting the denominator of $f(u)$ equal to zero, we see that $f(u)$ has poles at $u = 0$ and $u = -i$, and for the contour chosen there is no pole inside the contour. Then the integral along this contour is zero:

$$\underbrace{\int_0^R f(u)du}_{\text{real axis}} + \underbrace{iR\int_0^{\pi/2} f(Re^{i\varphi})e^{i\varphi}d\varphi}_{\substack{u\to Re^{i\varphi}\\ \text{circular arc}}} - \underbrace{i\int_0^R f(iu)du}_{\substack{u\to iu\\ \text{imaginary axis}}} = 0 \qquad [13]$$

The integral along the small indentation at the origin can be immediately ignored. The second integral vanishes for R large:

$$R|f(Re^{i\varphi})| = \frac{R\cdot e^{-2Rx(\cos\varphi+i\sin\varphi)}}{\underbrace{\sqrt{[R|1 - iRe^{i\varphi}|]}}_{=R \text{ as } R\gg 1}}$$

$$\to e^{-2Rx(\cos\varphi+i\sin\varphi)} \to 0$$

as $x > 0$ and $0 \leqslant \varphi \leqslant \pi/2$. Therefore, with $R \to \infty$:

$$\int_0^\infty f(u)du = i\int_0^\infty f(iu)du = i\int_0^\infty \frac{e^{-2iux}}{\sqrt{[iu(1+u)]}}\,du = e^{i\pi/4}\int_0^\infty \frac{e^{-2iux}}{\sqrt{[u(1+u)]}}\,du \qquad [14]$$

as $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$; and, taking the square root:

$$e^{i\pi/4} = \sqrt{i}$$

Using the expression [14] for the integral, also making the substitution $2u + 1 = v$, we can write [4.2] in the following simple way:

$$H_0^{(2)}(x) = \frac{2}{\pi} e^{i(\pi/4-x)} \int_0^\infty f(u)\mathrm{d}u = \frac{2i}{\pi} \int_1^\infty \frac{e^{-ixv}}{\sqrt{(v^2-1)}} \mathrm{d}v$$

Similarly: [15]

$$H_0^{(1)}(x) = -\frac{2i}{\pi} \int_1^\infty \frac{e^{ixv}}{\sqrt{(v^2-1)}} \mathrm{d}v$$

From [15] we find immediately the following formulas:

$$H_0^{(1)}(x) = -H_0^{(2)}(-x)$$
$$H_0^{(2)}(x) = -H_0^{(1)}(-x) \qquad [16]$$

and still another expression for $J_0(x)$:

$$J_0(x) = \mathrm{Re}\, H_0^{(1)}(x) = \frac{2}{\pi} \int_1^\infty \frac{\sin(xv)}{\sqrt{(v^2-1)}} \mathrm{d}v \qquad [17]$$

As an example, use of [16] is made in EWING, JARDETZKY and PRESS (1957, p.99) and WHITE (1965, p.188). The substitution $v = \cosh u$ in [17] leads to further useful modifications of this formula.

5.3.2 Spherical Bessel functions

We start from the wave equation:

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

and transform this into spherical coordinates (see Fig.2):

$$\frac{\partial^2\psi}{\partial R^2} + \frac{2}{R}\frac{\partial\psi}{\partial R} + \frac{1}{R^2\sin\delta}\frac{\partial}{\partial\delta}\left(\sin\delta\frac{\partial\psi}{\partial\delta}\right) + \frac{1}{R^2\sin^2\delta}\frac{\partial^2\psi}{\partial\varepsilon^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \qquad [18]$$

This is found immediately upon application of eq.[7] in section 1.2. Assume a solution of [18] to be of the form (cf. BULLEN, 1963, p.255):

$$\psi = Y_{m,n}(\delta, \varepsilon)\Psi(R)\, e^{i\omega t} \qquad [19]$$

where $Y_{m,n}$ is a surface harmonic, which we shall consider in Chapter 6:

$$Y_{m,n}(\delta, \varepsilon) = \left(\frac{2n+1}{2\pi}\right)^{1/2} \left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} P_n^m(\cos\delta)\sin(m\varepsilon) \qquad [20]$$

where $m \leqslant n$ and:

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m} \qquad \text{(Ferrers' function)}$$

and:

$$P_n(x) = \frac{(2x)^n(\frac{1}{2})_n}{n!}\,{}_2F_1\left(\tfrac{1}{2}-\tfrac{1}{2}n,\ -\tfrac{1}{2}n;\ \tfrac{1}{2}-n;\ \frac{1}{x^2}\right)$$

(Legendre's polynomial of degree n; compare [27] in section 4.2.) Substitute [19] into [18] (cf. MATHEWS and WALKER, 1965, p.219):

$$\frac{d^2\Psi}{dR^2} + \frac{2}{R}\frac{d\Psi}{dR} - \frac{n(n+1)}{R^2}\Psi + \frac{\omega^2}{v^2}\Psi = 0 \qquad [21]$$

Put $\Psi = R^{-1/2}f(R)$; then [21] becomes:

$$\frac{d^2f}{dR^2} + \frac{1}{R}\frac{df}{dR} + \left[\frac{\omega^2}{v^2} - \frac{\left(n+\frac{1}{2}\right)^2}{R^2}\right]f = 0 \qquad [22]$$

Multiply [22] by v^2/ω^2 and use $\omega R/v$ as independent variable instead of R. Then the general solution of [22] is found to be:

$$f = AJ_{n+1/2}\left(\frac{\omega R}{v}\right) + BJ_{-n-1/2}\left(\frac{\omega R}{v}\right) \qquad [23]$$

and [19] becomes:

$$\psi = \frac{1}{\sqrt{R}}\,Y_{m,n}(\delta, \varepsilon)\cdot J_{\pm(n+1/2)}\left(\frac{\omega R}{v}\right)\cdot e^{i\omega t} \qquad [24]$$

$J_{\pm(n+1/2)}(x)$ are called *spherical Bessel functions.*

In particular, putting $n = \frac{1}{2}$ in eq.[6] in section 4.3, we find:

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r+1}}{r!\left(\frac{3}{2}\right)_r 2^{2r}} = \left(\frac{2}{\pi x}\right)^{1/2}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r+1}}{(2r+1)!}$$

because:

$$(2r+1)! = \Gamma(2r+2)\ ; \quad r! = \Gamma(r+1)\ ;$$

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2r+2) = 2^{2r+1}\Gamma(r+1)\Gamma\left(r+\frac{3}{2}\right)$$

(duplication formula [10] in section 1.3)

$$\left(\frac{3}{2}\right)_r = \frac{\Gamma\left(\frac{3}{2}+r\right)}{\Gamma\left(\frac{3}{2}\right)}\,; \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

Therefore:

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x \qquad [25]$$

Similarly:

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x \qquad [26]$$

using the series expansions of $\sin x$ and $\cos x$, respectively (DE LA VALLÉE POUSSIN, 1938, p.455). For other values of n, similar derivations apply, but more complicated functions arise. Numerical tables of such functions can be found in WATSON (1944) and in MATHEMATICAL TABLES PROJECT (1947).

5.3.3 Modified Bessel functions

By an argument similar to that employed in section 1.2 we can show that Laplace's equation in cylindrical coordinates:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \chi^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \qquad [27]$$

has solutions of the form:

$$\psi = \mathrm{e}^{\pm in\chi \pm imz} R(r) \qquad [28]$$

where $R(r)$ satisfies the ordinary differential equation:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \left(m^2 + \frac{n^2}{r^2}\right) R = 0 \qquad [29$$

The difference from section 1.2 is that we now assume $\mathrm{e}^{\pm imz}$ (circular) instead of e^{-mz} (exponential). Substitute in [29]:

$$r = \frac{x}{m}\,; \quad \mathrm{d}r = \frac{\mathrm{d}x}{m}\,; \quad \mathrm{d}r^2 = \frac{\mathrm{d}x^2}{m^2}$$

Then:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}R}{\mathrm{d}x} - \left(1 + \frac{n^2}{x^2}\right) R = 0 \qquad [30]$$

Just as in dealing with Bessel's equation (section 4.3), we find the following solution of [30]:

$$R = AI_n(x) + BI_{-n}(x) \qquad [31]$$

(n neither zero nor an integer) where:

$$I_n(x) = \frac{x^n}{2^n\Gamma(n+1)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^r}{r!(n+1)_r} = \frac{x^n}{2^n\Gamma(n+1)}\, {}_0F_1\left(n+1;\ \frac{1}{4}x^2\right) \qquad [32]$$

By comparison with [6] in section 4.3 we find:

$$I_n(x) = i^{-n}J_n(ix) \qquad [33]$$

If instead n is an integer, then it is shown by a similar argument as for the usua Bessel equation, that the solution is:

$$R = AI_n(x) + BK_n(x) \qquad [34]$$

where the function $K_n(x)$ is defined by:

$$K_n(x) = (-1)^{n+1}I_n(x)\left(\log\frac{x}{2} + \gamma\right) + \frac{1}{2}\sum_{r=0}^{n-1}\frac{(-1)^r(n-r-1)!}{r!}\left(\frac{x}{2}\right)^{-n+2r}$$

$$+ \frac{1}{2}(-1)^n \sum_{r=0}^{\infty} \frac{1}{r!(n+r)!}[\varphi(r) + \varphi(n+r)]\left(\frac{x}{2}\right)^{n+2r} \qquad [35]$$

The functions $I_n(x)$ and $K_n(x)$ are called *modified Bessel functions of the first and second kinds.*

The expression [35] is valid only when n is an integer. On the other hand, $K_n(x)$ is a solution of the modified Bessel equation [30], whether the order n is *an integer or not.* See WHITTAKER and WATSON (1935, p.374). That is, the function $K_n(x)$ is a solution of the differential equation, whether n is an integer or not, but the expansion [35] of $K_n(x)$ holds only for an integer n. Also in this way $K_n(x)$ corresponds to $Y_n(x)$.

There is some confusion in the definition of the function $K_n(x)$, as at least three slightly different definitions are in current use, valid for all values of n:

(*1*) WHITTAKER and WATSON (1935); we call this definition $Kw_n(x)$:

$$Kw_n(x) = \frac{\pi}{2}[I_{-n}(x) - I_n(x)]\cot n\pi \qquad [36]$$

(*2*) JEFFREYS and JEFFREYS (1946):

$$Kh_n(x) = [I_{-n}(x) - I_n(x)]\frac{1}{\sin n\pi} \qquad [37]$$

(*3*) The definition used in this book (WATSON, 1944, p.78):

$$K_n(x) = \frac{\pi}{2}Kh_n(x) = \frac{\pi}{2}[I_{-n}(x) - I_n(x)]\frac{1}{\sin n\pi} \qquad [38]$$

$Kh_n(x)$ has the advantage of being directly comparable to $Y_n(x)$, as seen from [35] and eq.[13] in section 4.3.

Just as $I_n(x)$ is defined by eq.[33], we can *define* $K_n(x)$ by the following formula:

$$K_n(x) = \frac{\pi i}{2} i^n H_n^{(1)}(ix) \qquad [39]$$

This can easily be shown to be identical with the definition [38], using the following expressions:

$$\left.\begin{array}{ll} H_n^{(1)}(ix) = J_n(ix) + iY_n(ix) & \text{(eq.[14] in section 4.3)} \\ Y_n(ix) = \dfrac{\cos n\pi \cdot J_n(ix) - J_{-n}(ix)}{\sin n\pi} & \text{(eq.[17] in section 4.3)} \\ I_{-n}(x) = i^n J_{-n}(ix) & \text{(eq.[33])} \end{array}\right| \qquad [40]$$

By means of [33] we can derive a set of recurrence formulas for $I_n(x)$ from the corresponding recurrence formulas for $J_n(x)$, derived in the sections 5.2.1–5.2.3:

$$\left.\begin{array}{l} I_{-n}(x) = I_n(x) \\ 2I_n'(x) = I_{n-1}(x) + I_{n+1}(x) \\ \dfrac{2n}{x} I_n(x) = I_{n-1}(x) - I_{n+1}(x) \\ xI_n'(x) = xI_{n-1}(x) - nI_n(x) \\ xI_n'(x) = nI_n(x) + xI_{n+1}(x) \\ I_0'(x) = I_1(x) \\ \dfrac{d}{dx}[x^n I_n(x)] = x^n I_{n-1}(x) \\ \dfrac{d}{dx}[x^{-n} I_n(x)] = x^{-n} I_{n+1}(x) \end{array}\right| \qquad [41]$$

Eq.[41.1] is valid only for integer values of n, but the other formulas in [41] are valid for any n.

The corresponding recurrence relations for $K_n(x)$ are also frequently useful. They can be derived from a combination of [41] and [38]. The following list gives the correspondence to [41], formula by formula:

$$\left.\begin{array}{l} K_{-n}(x) = K_n(x) \\ 2K_n'(x) = -K_{n-1}(x) - K_{n+1}(x) \\ -\dfrac{2n}{x} K_n(x) = K_{n-1}(x) - K_{n+1}(x) \\ xK_n'(x) = -xK_{n-1}(x) - nK_n(x) \\ xK_n'(x) = nK_n(x) - xK_{n+1}(x) \end{array}\right| \qquad [42]$$

$$
\left.
\begin{aligned}
&K_0'(x) = -K_1(x) \\
&\frac{\mathrm{d}}{\mathrm{d}x}[x^n K_n(x)] = -x^n K_{n-1}(x) \\
&\frac{\mathrm{d}}{\mathrm{d}x}[x^{-n} K_n(x)] = -x^{-n} K_{n+1}(x)
\end{aligned}
\right| \qquad [42]
$$

All the formulas [42] are valid for any value of n. A good rule, obvious from the comparison of [41] and [42], is that terms with index n have the same sign in the two sets of formulas; those with index $n - 1$ or $n + 1$ have opposite signs. This provides a "rule of thumb" in deducing immediately a recurrence relation for $K_n(x)$, when the corresponding relation for $I_n(x)$ is given.

The recurrence relations are the same for $K_n(x)$ and $Kh_n(x)$, but different from those of $Kw_n(x)$. We have seen that the recurrence relations are different for $K_n(x)$ in our definition and for $I_n(x)$. But if we define $K_n(x)$ as $Kw_n(x)$, the recurrence relations will be the same as for $I_n(x)$. Symbolically we could express this as follows:

$$[K_n(x)] = [Kh_n(x)] \neq [Kw_n(x)]$$

$$[K_n(x)] \neq [I_n(x)] = [Kw_n(x)]$$

Note that in EWING, JARDETZKY and PRESS (1957) $K_n(x)$ is not clearly defined: for instance on p.315 (6-97) refers to the definition [36] but (6-98) refers to [37] or [38].

When n is an even integer, [36] agrees with [38], but when n is an odd integer, [36] and [38] have opposite signs. This is due to the additional factor $\cos(n\pi)$ in [36], which is $= +1$ for n even, but $= -1$ for n odd. This explains the "rule of thumb" mentioned above.

Application of the Wronskian. In analogy with eq.[21] in section 4.3 the following relation can be proved for the modified Bessel functions:

$$I_n'(x)K_n(x) - I_n(x)K_n'(x) = \frac{1}{x} \qquad [43]$$

As this is different from zero (for finite values of x), it demonstrates that $I_n(x)$ and $K_n(x)$ are independent solutions of the modified Bessel equation [30]; compare section 4.1.2.

Expressions for the Airy function in Bessel functions. By comparing [9] in section 3.3 with the series expansion of Bessel functions [26] in section 5.2 we can express the Airy function in Bessel functions. Making extensive use of the relations for the gamma function:

$$(n-1)! = \Gamma(n) = (n-1)\Gamma(n-1)$$

$$n! = \Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)$$

and of [11] in section 3.3, we find the following relations, in both of which z has to be taken positive:

$$Ai(-z) = \frac{\sqrt{z}}{3}\left[J_{-1/3}\left(\frac{2z^{3/2}}{3}\right) + J_{1/3}\left(\frac{2z^{3/2}}{3}\right)\right]$$

$$Ai(z) = \frac{\sqrt{z}}{3}\left[I_{-1/3}\left(\frac{2z^{3/2}}{3}\right) - I_{1/3}\left(\frac{2z^{3/2}}{3}\right)\right] =$$

$$= \frac{\sqrt{z}}{\pi\sqrt{3}} K_{1/3}\left(\frac{2z^{3/2}}{3}\right) \qquad [44]$$

using the definition [38] of the K function.

These formulas are not given in many textbooks. The corresponding expressions by WATSON (1944, p.190) are slightly different, because of a different definition of the Airy integral. The formulas given by WATSON (1944) are transformed into those given here if we make the following substitutions in Watson's formulas:

$$t = \frac{t_1}{3^{1/3}}; \quad x = z \cdot 3^{1/3}$$

The same formulas as here have been given by BREKHOVSKIKH (1960, pp.399 and 485) except for a difference of a factor of $\sqrt{\pi}$, depending again on a different definition of the Airy integral. Brekhovskikh does not give any proofs. His formulas are unclear as he does not make a clear distinction between the J and I functions in his notation.

5.3.4 The ber and bei functions

Kelvin defined the following functions:

$$\mathrm{ber}(x) + i\,\mathrm{bei}(x) = I_0(i^{1/2}x) \qquad [45]$$

From eq.[32], we have:

$$I_n(x) = \frac{x^n}{2^n\Gamma(n+1)} \sum_{r=0}^{\infty} \frac{\left(+\frac{1}{4}x^2\right)^r}{r!(n+1)_r}$$

We find the following expressions for the ber and bei functions, considering that ber and bei are the real and imaginary parts, respectively, of $I_0(i^{1/2}x)$:

$$\left.\begin{aligned} \mathrm{ber}(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r\left(\frac{1}{4}x^2\right)^{2r}}{[(2r)!]^2} \\ \mathrm{bei}(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r\left(\frac{1}{4}x^2\right)^{2r+1}}{[(2r+1)!]^2} \end{aligned}\right| \qquad [46]$$

This is easily seen by writing the first few terms of the series development for $I_0(i^{1/2}x)$.

Two other functions are related in a similar way to the modified Bessel function of the second kind:

$$\operatorname{ker}(x) + i \operatorname{kei}(x) = K_0(i^{1/2}x) \tag{47}$$

In the same way we can write the expressions for ker and kei, taking the real and imaginary parts of $K_0(i^{1/2}x)$, defined by [38].

These new functions are of practical importance in the solution of certain differential equations, e.g., the diffusion equation, and in electrical engineering.

5.3.5 Asymptotic expansions of Bessel functions

In certain problems only a very crude approximation to the behaviour of a Bessel function is desired. In this section we shall derive asymptotic expressions for the Bessel functions, valid for large values of $|x|$, and we follow JEFFREYS and JEFFREYS (1946, pp. 543–544; 550–551).

We use the formula [49], section 5.2, and deform the integration contour of Fig.41 as shown in Fig.44. In this figure we call the expression [49] in section 5.2 along the upper path $\frac{1}{2} H_n^{(1)}(x)$ and that along the lower path $\frac{1}{2} H_n^{(2)}(x)$. This may be taken as an alternative definition of the Hankel functions. Then:

$$2J_n(x) = H_n^{(1)}(x) + H_n^{(2)}(x) \tag{48}$$

Similarly, we define $Y_n(x)$:

$$2iY_n(x) = H_n^{(1)}(x) - H_n^{(2)}(x) \qquad [49[$$

From [49], section 5.2, it then follows that:

$$H_n^{(1)}(x) = \frac{1}{\pi i} \int_{0,i}^{-\infty} e^{(x/2)(t-1/t)} \frac{dt}{t^{n+1}} \tag{50}$$

(the integration being taken from 0 over $+i$ to $-\infty$) and:

$$H_n^{(2)}(x) = \frac{1}{\pi i} \int_{-\infty,-i}^{0} e^{(x/2)(t-1/t)} \frac{dt}{t^{n+1}} \tag{51}$$

(the integration being taken from $-\infty$ over $-i$ to 0).

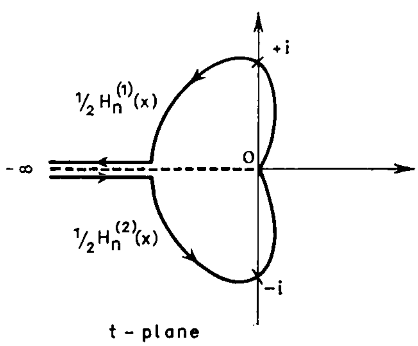


Fig.44.

The points $\pm i$ are saddle points for the integrals [50] and [51], respectively (compare section 3.2):

$$\left.\begin{aligned} & f(t) = \frac{1}{2}\left(t - \frac{1}{t}\right); \quad f'(t) = \frac{1}{2}\left(1 + \frac{1}{t^2}\right) = 0 \\ & t^2 = -1\ ; \qquad t = \pm i\ ; \\ & f''(t) = -\frac{1}{t^3}; \qquad f''(+i) = -\frac{1}{i^3} = \frac{1}{i} = -i \end{aligned}\right| \quad [52]$$

and we can evaluate approximate expressions for [50] and [51] by the method of steepest descent. Take first $H_n^{(1)}(x)$. Its integrand at $t = i$ becomes:

$$\frac{e^{(x/2)(i-1/i)}}{i^{n+1}} = e^{ix}\, e^{-(1/2)(n+1)\pi i}$$

because:

$$e^{(x/2)(i-1/i)} = e^{(x/2)(i+i)} = e^{ix}$$

and:

$$i^{-(n+1)} = (\underbrace{e^{i\,\pi/2}}_{=\,i})^{-(n+1)} = e^{-(1/2)(n+1)\,\pi i}$$

and the path of steepest descent, for x real and positive, makes an angle of $3\pi/4$ with the positive real axis (by eq.[18.4] of section 3.2). We then get the following approximate expression for [50]:

$$\begin{aligned} H_n^{(1)}(x) &\simeq \frac{1}{\pi i}\, e^{i[x-(n\pi/2)-(\pi/2)]} \left(\frac{2\pi}{x}\right)^{1/2} e^{3\pi i/4} \\ &= \left(\frac{2}{\pi x}\right)^{1/2} e^{i[x-(n\pi/2)-(\pi/4)]} \end{aligned} \quad [53]$$

Similarly, from [51]:

$$H_n^{(2)}(x) \simeq \left(\frac{2}{\pi x}\right)^{1/2} e^{-i[x-(n\pi/2)-(\pi/4)]} \quad [54]$$

Then, from [48] and [49], using [53] and [54]:

$$\left.\begin{aligned} J_n(x) &\simeq \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \\ Y_n(x) &\simeq \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \end{aligned}\right| \quad [55]$$

These are approximate expressions, valid for *large*, positive, real values of x (this is required in the method of steepest descent).

Then, the corresponding asymptotic formula for $I_n(x)$, i.e.:

$$I_n(x) \simeq \frac{e^x}{\sqrt{2\pi x}} \qquad [56]$$

can be deduced from the relation between $I_n(x)$ and $J_n(x)$, eq.[33], inserting the asymptotic expression for $J_n(ix)$ from [55.1]:

$$\begin{aligned}
I_n(x) &= i^{-n} J_n(ix) \\
&= i^{-n}\left(\frac{2}{\pi i x}\right)^{1/2} \cos\left[ix - \frac{\pi}{2}\left(n + \frac{1}{2}\right)\right] \\
&= \underbrace{i^{-(n+1/2)}}\left(\frac{2}{\pi x}\right)^{1/2} \cos\left[ix - \frac{\pi}{2}\left(n + \frac{1}{2}\right)\right] \\
&= \underbrace{e^{-(i\pi/2)(n+1/2)}} \\
&= \cos\left[\frac{\pi}{2}\left(n + \frac{1}{2}\right)\right] - i \sin\left[\frac{\pi}{2}\left(n + \frac{1}{2}\right)\right]
\end{aligned}$$

Then expand all cosines and sines, take $\cos(\pi/4)$ and $\sin(\pi/4) = 1/\sqrt{2}$ outside the brackets, and use the formulas:

$$\left.\begin{aligned}
\cos(ix) &= \cosh x = \frac{1}{2}(e^x + e^{-x}) \simeq \frac{1}{2} e^x \\
\sin(ix) &= i \sinh x = \frac{i}{2}(e^x - e^{-x}) \simeq \frac{i}{2} e^x
\end{aligned}\right\} \text{for } x \text{ large}$$

The details are left to the reader as an exercise.

In a similar way we derive asymptotic expressions for $I_{-n}(x)$ and then from [38] also for $K_n(x)$, i.e.:

$$K_n(x) \simeq \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \qquad [57]$$

The approximate expressions [56] and [57] can also be obtained as limits of the following series expansions of $I_n(x)$ and $K_n(x)$, valid for large $|x|$. They are not proved here.

$$\begin{aligned}
I_n(x) &\simeq \frac{e^x}{\sqrt{2\pi x}}\left[1 + \sum_{r=1}^{\infty} (-1)^r \frac{[4n^2 - 1^2][4n^2 - 3^2]\ldots[4n^2 - (2r-1)^2]}{r!2^{3r}x^r}\right] \\
&\to \frac{e^x}{\sqrt{2\pi x}} \qquad [58]
\end{aligned}$$

$$\begin{aligned}
K_n(x) &\simeq \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left[1 + \sum_{r=1}^{\infty} \frac{[4n^2 - 1^2][4n^2 - 3^2]\ldots[4n^2 - (2r-1)^2]}{r!2^{3r}x^r}\right] \\
&\to \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \qquad [59]
\end{aligned}$$

for large values of $|x|$. See further WHITTAKER and WATSON (1935, pp.373–374). Note that $K_n(x)$ follows the definition used in this book, i.e., eq.[38]. In case we use instead the definition [36], the asymptotic expansion for large $|x|$ becomes:

$$Kw_n(x) \simeq \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\cos(n\pi)\left[1 + \sum_{r=1}^{\infty} \frac{[4n^2-1^2][4n^2-3^2]\ldots[4n^2-(2r-1)^2]}{r!2^{3r}x^r}\right]$$

$$\to \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\cos(n\pi) \qquad [60]$$

5.4 APPLICATIONS OF BESSEL AND HANKEL FUNCTIONS

5.4.1 *Acoustic-gravity waves*[1]

We consider the vertical propagation of sound waves in an atmosphere, with the temperature decreasing linearly upwards. Put the origin $x = 0$ at the upper boundary of the atmosphere, i.e., where the temperature $T_0 = 0$ (Fig.45) and assume the following temperature law:

$$T_0 = \beta x \qquad [1]$$

The general gas law (p_0 = pressure, R = gas constant, ϱ_0 = density; index zero being used to indicate equilibrium conditions):

$$p_0 = R\varrho_0 T_0 \qquad [2]$$

gives:

$$\log\varrho_0 = \log p_0 - \log T_0 - \log R$$

$$\frac{1}{\varrho_0}\frac{d\varrho_0}{dx} = \frac{1}{p_0}\frac{dp_0}{dx} - \frac{1}{T_0}\frac{dT_0}{dx} = \frac{g\varrho_0}{p_0} - \frac{\beta}{T_0} = \frac{m}{x}$$

considering that $dp_0/dx = g\varrho_0$ (g = acceleration of gravity) and provided:

$$m = \frac{g}{R\beta} - 1 \qquad [3]$$

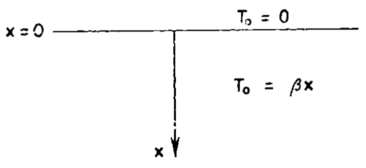


Fig.45.

[1] LAMB (1945, pp.545–547).

as seen immediately from [1] and [2]. Therefore:

$$\frac{d\varrho_0}{\varrho_0} = m\frac{dx}{x}$$

and by integration:

$$\varrho_0 \sim x^m$$

which combined with [2] and [1] gives:

$$p_0 \sim x^{m+1} \qquad [4]$$

We now specify our assumption about the temperature variation with height so that the temperature gradient is adiabatic ("convective equilibrium"), i.e.:

$$p_0 \sim \varrho_0^{\gamma}$$

where $\gamma = c_p/c_v$, c_p and c_v being the specific heats at constant pressure and at constant volume, respectively. Combination with eq.[4] gives that $x^{m+1} \sim x^{m\gamma}$, i.e., $m + 1 = m\gamma$, or:

$$m = \frac{1}{\gamma - 1}$$

From [3] we then solve β:

$$\beta = \frac{g}{R(m+1)} = \frac{(\gamma - 1)g}{\gamma R} = \beta_1 \qquad [5]$$

This is the adiabatic temperature gradient (about 1°/100 m). For temperature gradients in excess of the adiabatic value, the atmosphere becomes unstable, if the restriction to vertical motion is abandoned.

With c = the sound velocity and ξ = the vertical displacement, the equation of motion is in this case:

$$\frac{\partial^2\xi}{\partial t^2} = c^2\frac{\partial^2\xi}{\partial x^2} + \gamma g\frac{\partial\xi}{\partial x} \qquad [6]$$

where c^2 is a function of x:

$$c^2 = \frac{\gamma p_0}{\varrho_0} \underset{\text{(eq.[2])}}{=} \gamma RT_0 \underset{\text{(eq.[1])}}{=} \gamma R\beta x \qquad [7]$$

Put:

$$\tau = \int_0^x \frac{dx}{c} \underset{\text{(eq.[7])}}{=} \int_0^x \frac{dx}{\sqrt{(\gamma R\beta x)}} = \left(\frac{4x}{\gamma R\beta}\right)^{1/2} \qquad [8]$$

or:

$$x = \frac{1}{4}\gamma R\beta\tau^2$$

where τ denotes the travel time of the sound from the top of the atmosphere to the position x. By [8] we replace x as independent variable in [6] by τ and use [3]:

$$\frac{\partial^2\xi}{\partial t^2} = \frac{\partial^2\xi}{\partial\tau^2} + \frac{2m+1}{\tau}\frac{\partial\xi}{\partial\tau} \qquad [9]$$

This substitution is obvious as follows. Put $c_1 = \frac{1}{4}\gamma R\beta$ = constant. Then:

$$\frac{\partial\xi}{\partial x} = \frac{\partial\xi}{\partial(c_1\tau^2)} = \frac{1}{2c_1\tau}\frac{\partial\xi}{\partial\tau}$$

$$\frac{\partial^2\xi}{\partial x^2} = \frac{1}{2c_1\tau}\frac{\partial^2\xi}{\partial\tau^2}\frac{\mathrm{d}\tau}{\mathrm{d}x} - \frac{1}{2c_1}\frac{\partial\xi}{\partial\tau}\frac{1}{\tau^2}\frac{\mathrm{d}\tau}{\mathrm{d}x}$$

which is inserted into [6], and [9] is obtained.

We assume simple harmonic vibrations, i.e., $\xi \sim e^{i\sigma t}$ and then [9] becomes:

$$\frac{\partial^2\xi}{\partial\tau^2} + \frac{2m+1}{\tau}\frac{\partial\xi}{\partial\tau} + \sigma^2\xi = 0 \qquad [10]$$

Eq.[10] can be rewritten in the following way:

$$\frac{\mathrm{d}^2(\xi\tau^m)}{\mathrm{d}(\sigma\tau)^2} + \frac{1}{\sigma\tau}\frac{\mathrm{d}(\xi\tau^m)}{\mathrm{d}(\sigma\tau)} + \left(1 - \frac{m^2}{\sigma^2\tau^2}\right)\xi\tau^m = 0$$

which exactly agrees with Bessel's differential equation with $\xi\tau^m$ as dependent variable and $\sigma\tau$ as independent variable. As m is generally neither zero nor an integer (see eq.[3]) we have the following general solution of [10] (from eq.[8] of section 4.3):

$$\xi = \tau^{-m}[AJ_m(\sigma\tau) + BJ_{-m}(\sigma\tau)] \qquad [11]$$

From LAMB (1945, p.541), we have ($p - p_0$ being the excess pressure due to the sound wave):

$$p - p_0 = -\gamma p_0 \frac{\partial\xi}{\partial x}$$

that is:

$$p - p_0 \sim p_0 \frac{\partial\xi}{\partial x}$$

Eq.[4] and [8] give that:

$$p_0 \sim x^{m+1} \sim \tau^{2m+2}$$

$$\frac{\partial\xi}{\partial x} \sim \frac{1}{\tau}\frac{\partial\xi}{\partial\tau}$$

and therefore:

$$p - p_0 \sim \tau^{2m+1} \frac{\partial \xi}{\partial \tau} \qquad [12]$$

As this must vanish for $x = 0$ or for $\tau = 0$, we must have $B = 0$. This is seen if we differentiate [11], using the formulas [31] and [32] of section 5.2, then preferably writing the differential in the following way:

$$\frac{\mathrm{d}(\xi\sigma^{-m})}{\mathrm{d}(\sigma\tau)} = -A(\sigma\tau)^{-m}J_{m+1}(\sigma\tau) + B(\sigma\tau)^{-m}J_{-m-1}(\sigma\tau)$$

Considering that in this case $m > 0$, and considering the series expansion [6] in section 4.3 of the Bessel functions, we find that if we put $\tau = 0$, the expression [12] will vanish only if $B = 0$, because $J_{-m-1}(\sigma\tau)$ is infinite for $\tau = 0$.

Then we obtain from [11], considering two levels:

$$\frac{\xi}{\xi_1} = \left(\frac{\tau_1}{\tau}\right)^m \frac{J_m(\sigma\tau)}{J_m(\sigma\tau_1)}$$

Assume:

$$\xi_1 = \mathrm{e}^{i\sigma t} \qquad [13]$$

at a given level $\tau = \tau_1$. Then:

$$\xi = \left(\frac{\tau_1}{\tau}\right)^m \frac{J_m(\sigma\tau)}{J_m(\sigma\tau_1)} \mathrm{e}^{i\sigma t} \qquad [14]$$

This is not the equation of a propagating wave. Instead [14] represents a standing oscillation due to superposition of two wave trains of equal amplitude travelling upwards and downwards respectively. Compare BULLEN (1963, p.58). Eq.[14] becomes infinite (the case of resonance) when

$$J_m(\sigma\tau_1) = 0 \qquad [15]$$

This determines the periods $2\pi/\sigma$ of the free oscillations of the air lying above a *fixed rigid* horizontal plane for which $\tau = \tau_1$, because [15] implies that $\xi_1 = 0$.

Another problem which is closely related to the one just discussed but which needs more lengthy developments, is the *horizontal* propagation of disturbances in the atmosphere. For this case, I limit myself to some references:

(*1*) Basic theory: LAMB (1945) and PEKERIS (1948a).

(*2*) Calculation of dispersion curves: PRESS and HARKRIDER (1962) and HARKRIDER (1964).

5.4.2 Conical compressional waves [1]

We consider waves around a borehole (Fig.46), with axes as shown. We assume

[1] WHITE (1965, pp.186–189).

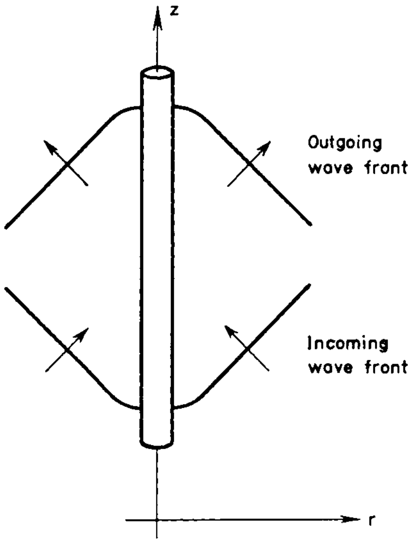


Fig.46.

circular symmetry around the borehole (axial symmetry). The wave equation for compressional waves is then (α = compressional wave velocity):

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial^2 \psi}{\partial t^2} \qquad [16]$$

(compare section 1.2.2), which can be solved by separation of the variables:

$$\psi \sim e^{Lz} e^{\Omega t} f(r) \qquad [17]$$

Substituting [17] into [16] we find the following equation in r:

$$f''(r) + \frac{1}{r} f'(r) + \left(L^2 - \frac{\Omega^2}{\alpha^2}\right) f(r) = 0 \qquad [18]$$

Putting $L = -il$ and $\Omega = i\omega$ we find the factor for $f(r)$ in [18] to be:

$$i^2 l^2 - \frac{i^2 \omega^2}{\alpha^2} = -l^2 + \frac{\omega^2}{\alpha^2} = \frac{\omega^2}{\alpha^2} - \frac{\omega^2}{c^2} = \omega^2\left(\frac{1}{\alpha^2} - \frac{1}{c^2}\right) = m^2$$

(c = wave velocity in the direction of z, i.e., along the borehole) and eq.[18] becomes:

$$f''(r) + \frac{1}{r} f'(r) + m^2 f(r) = 0 \qquad [19]$$

Changing the variable r into mr, [19] becomes:

$$f''(mr) + \frac{1}{mr} f'(mr) + f(mr) = 0 \qquad [20]$$

which is a zero-order Bessel equation. To describe travelling waves, the Bessel functions are most conveniently expressed in Hankel functions (see section 5.3.1). The solution of [16] for the case $a < |c|$ is therefore:

$$\psi = [A_1 H_0^{(1)}(mr) + A_2 H_0^{(2)}(mr)]\, e^{-ilz}\, e^{i\omega t} \qquad [21]$$
$$\text{(incident)} \qquad \text{(reflected)}$$

Note that the assumption $a < |c|$ makes m real and the argument mr of the Hankel functions real. If, on the other hand, $|c| < a$, m will become imaginary ($m = i\bar{m}$, where $\bar{m}$ is real), and the argument mr imaginary. In this case the Hankel functions can be expressed as the *modified Bessel functions* $K_0(x)$ using eq.[39] of section 5.3:

$$K_0(\bar{m}r) = \frac{i\pi}{2} H_0^{(1)}(i\bar{m}r)$$

For large values of the argument, i.e., for large values of r, the asymptotic expansions of the Hankel functions can be used (eq.[53] and [54] in section 5.3), which in this case become:

$$\left.\begin{aligned} H_0^{(1)}(mr) &\to \left(\frac{2}{\pi mr}\right)^{1/2} e^{i(mr-\pi/4)} \\ H_0^{(2)}(mr) &\to \left(\frac{2}{\pi mr}\right)^{1/2} e^{-i(mr-\pi/4)} \end{aligned}\right| \qquad [22]$$

The second term in [21] then becomes:

$$\psi \sim r^{-1/2}\, e^{-imr}\, e^{-ilz} e^{i\omega t} = r^{-1/2}\, e^{i\omega(t - mr/\omega - lz/\omega)} \qquad [23]$$

which is a compressional wave travelling in the positive z and r directions. At a given time. a certain phase has the equation:

$$mr + lz = \text{constant} \qquad [24]$$

for large r, i.e.:

$$\frac{\mathrm{d}r}{\mathrm{d}z} = -\frac{l}{m} = -\left(\frac{c^2}{a^2} - 1\right)^{-1/2} \qquad [25]$$

which defines the slope of the reflected wave front in Fig.46. Near the z-axis, i.e., for small r, the expressions [22] are not valid. The surfaces for equal phase bend in towards the z-axis, as shown in Fig.46, i.e., giving a rather blunt nose to the *conical wave*.

The first term in [21] is likewise a conical wave, but proceeding towards smaller r and greater z. Similar considerations can be made for transverse waves. The similarity between conical waves around a cylindrical borehole and plane waves near a plane boundary (see section 8.3.3) is very great, and the latter can be considered as a special case of the former.

5.4.3 *Wave reflection from an inhomogeneous half-space* [1]

The physical problem is the following. A ray is incident from a homogeneous medium (constant properties) against the boundary of another, inhomogeneous medium, i.e., one in which the properties (velocity v or refraction index n) vary with distance from the boundary. See Fig.47 (a is a positive constant). We want to find an expression for the reflection coefficient.

The wave field in the homogeneous medium consists of the incident and reflected waves:

$$\psi(x, z) = Z(z)e^{ik_0x\sin\theta_0} \tag{26}$$

where:

$$Z(z) = e^{ik_0z\cos\theta_0} + V(\theta_0)e^{-ik_0z\cos\theta_0} \tag{27}$$

k_0 is the wave number in the homogeneous medium and $V(\theta_0)$ is the reflection coefficient. The amplitude of the incident wave is assumed $= 1$.

We shall try to find an expression corresponding to [26] for the inhomogeneous medium ($z > 0$) by solving the following equation:

$$\frac{d^2Z}{dz^2} + k_0^2[n^2(z) - \sin^2\theta_0]Z = 0 \tag{28}$$

The correctness of [26] and [27] is seen immediately, considering the equation of the incident wave front:

$$x\sin\theta_0 + z\cos\theta_0 = c$$

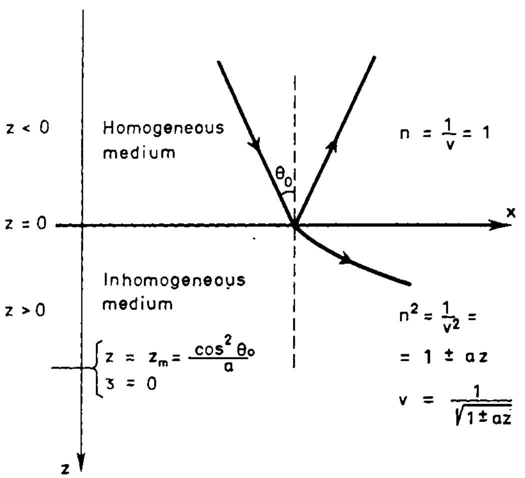


Fig.47.

[1] Brekhovskikh (1960, pp.189–193).

where c measures distance along the wave propagation. Also remember that the incident ray propagates towards positive x and positive z, and the reflected wave propagates towards positive x, but negative z. This explains the signs. A time factor $e^{i\omega t}$ is left out.

Equation [28] is obtained from [59], section 4.6, considering: (*1*) that the constant b enters into the solution of [56], section 4.6, in the form e^{ibx}, i.e., comparing with [26] and [27], we have $b = k_0 \sin\theta_0$; (*2*) that:

$$n = \frac{1}{v} = \frac{v_0}{v} = \frac{\lambda_0}{\lambda} = \frac{k}{k_0}$$

where λ = wave length and index 0 refers to the homogeneous medium and not indexed quantities to the inhomogeneous medium. Thus $k = k_0 n$ or $k^2(z) = k_0^2 n^2(z)$.

Assume now that the index of refraction n is decreasing downwards in the lower medium (i.e., the velocity is increasing downwards), which means we are choosing the minus-sign in the expression for n in Fig.47. Replace the independent variable z by ζ:

$$\zeta = \cos^2\theta_0 - az \qquad [29]$$

Then [28] transforms into:

$$\frac{d^2Z}{d\zeta^2} + \left(\frac{k_0}{a}\right)^2 \zeta Z = 0 \qquad [30]$$

Replacing the dependent variable Z by u and the independent variable ζ by w by means of the following relations:

$$\left.\begin{aligned} Z &= \frac{k_0}{a}\zeta^{1/2}u \\ \zeta &= \left(\frac{3}{2}\right)^{2/3}\left(\frac{k_0}{a}\right)^{-2/3} w^{2/3} \end{aligned}\right| \qquad [31]$$

transforms equation [30] into:

$$w^2\frac{d^2u}{dw^2} + w\frac{du}{dw} + \left(w^2 - \frac{1}{9}\right)u = 0 \qquad [32]$$

See WHITTAKER and WATSON (1935, p.205, example 6). Note, in carrying out the differentiations of Z, that this is a function both of ζ and u, that is:

$$Z = Z(\zeta, u)$$

$$\frac{dZ}{d\zeta} = \frac{\partial Z}{\partial \zeta} + \frac{\partial Z}{\partial u}\frac{du}{d\zeta} = \frac{\partial Z}{\partial \zeta} + \frac{\partial Z}{\partial u}\frac{du}{dw}\frac{dw}{d\zeta}$$

Here we insert the values of $\partial Z/\partial\zeta$, $\partial Z/\partial u$, $dw/d\zeta$, calculated from [31]. Then we can perform the second differentiation $d^2Z/d\zeta^2$ and after that substitute in [30].

Eq.[32] is a Bessel differential equation of order 1/3 and its general solution is:

$$u(w) = AH_{1/3}^{(1)}(w) + BH_{1/3}^{(2)}(w) \qquad [33]$$

We then find the following solution for Z, using [31] and including the constant factor in the constants A and B:

$$Z(w) = w^{1/3}[AH_{1/3}^{(1)}(w) + BH_{1/3}^{(2)}(w)] \tag{34}$$

for $\zeta > 0$, $z < z_m$ (see Fig.47), and:

$$Z(w) = w_1^{1/3}\,[CH_{1/3}^{(1)}(iw_1) + DH_{1/3}^{(2)}(iw_1)] \tag{35}$$

for:

$$\zeta < 0,\ z > z_m$$

$$-\zeta = \left(\frac{3}{2}\right)^{2/3}\left(\frac{k_0}{a}\right)^{-2/3} w_1^{2/3}$$

From the asymptotic expansion for large values of the argument, [54] in section 5.3, it follows that:

$$z \to \infty$$

$$w_1 \to \infty$$

$$|H_{1/3}^{(2)}(iw_1)| \sim \left(\frac{2}{\pi w_1}\right)^{1/2} e^{w_1} \to \infty \tag{36}$$

whence we have to put $D = 0$ in [35].

We have four boundary conditions for the determination of A, B, C and V. These are the continuity of vertical displacement ($\mathrm{d}Z/\mathrm{d}z$) and of pressure ($p = -\varrho\,(\partial\psi/\partial t) \sim Z$) at $z = 0$ and at $z = z_m$.

At $z = z_m$, we have $w = 0$, and in the neighbourhood of this point, the argument w is small. Therefore, we can use approximate expressions for the Hankel functions for small values of the argument. These are obtained from expressions of the Hankel functions in Bessel functions, [18] in section 4.3, using series expansions of the Bessel functions, and including only first terms (eq.[34] in 5.2). After that we express the continuity of Z and $\mathrm{d}Z/\mathrm{d}z$ at $z = z_m$, which leads to the following two relations:

$$\left.\begin{aligned} A &= B\,e^{i\pi/3} \\ C &= iB\,e^{i\pi/3} \end{aligned}\right| \tag{37}$$

More details about just this calculation can be read in BREKHOVSKIKH (1960, pp.210–212).

The conditions of continuity of Z and of $\mathrm{d}Z/\mathrm{d}z$ at $z = 0$ lead to the following two equations, considering [27] for the upper medium and [34] for the lower medium, also replacing A by the expression [37.1]:

$$\left.\begin{aligned} 1 + V &= B\left\{w^{1/3}[e^{i\pi/3}H_{1/3}^{(1)}(w) + H_{1/3}^{(2)}(w)]\right\}_{z=0} \\ ik_0\cos\theta_0(1 - V) &= B\left\{\frac{\mathrm{d}}{\mathrm{d}z}\,[w^{1/3}(e^{i\pi/3}H_{1/3}^{(1)}(w) + H_{1/3}^{(2)}(w))]\right\}_{z=0} \end{aligned}\right| \tag{38}$$

In carrying out the differentiations in eq.[38.2] we use the expressions of the

Hankel functions in Bessel functions, [18] in section 4.3, together with the rules for differentiation of Bessel functions, [30] and [32] in section 5.2. This gives:

$$\left[\frac{\mathrm{d}}{\mathrm{d}w}(w^{1/3}H^{(1)}_{1/3}(w))\right]_{z=0} = [w^{1/3}H^{(1)}_{-2/3}(w)]_{z=0} \qquad [39]$$

$$= w_0^{1/3}H^{(1)}_{-2/3}(w_0)$$

where w_0 is the value of w for $z = 0$, obtainable from [31] and [29]:

$$w_0 = -\frac{2}{3}\frac{k_0}{a}\cos^3\theta_0 \qquad [40]$$

The same formula as [39] holds for Hankel functions of the second kind, which is easily seen in the same way. Also, in carrying out the differentiations in [38] we have to notice that:

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}z}$$

$$\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_{z=0} = \frac{k_0}{a}\left(\zeta^{1/2}\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)_{z=0} = -k_0\cos\theta_0 \qquad [41]$$

again obtained by combination of [29] and [31] and finally putting $z = 0$.

Then, dividing [38.2] by [38.1] we get:

$$\frac{1-V}{1+V} = i\,\frac{\mathrm{e}^{i\pi/3}H^{(1)}_{-2/3}(w_0) + H^{(2)}_{-2/3}(w_0)}{\mathrm{e}^{i\pi/3}H^{(1)}_{1/3}(w_0) + H^{(2)}_{1/3}(w_0)} \qquad [42]$$

Replacing the Hankel functions in [42] by their expressions in Bessel functions, using the formulas already referred to, we find the following expression for the reflection coefficient V:

$$V = \frac{(J_{1/3} + J_{-1/3}) - i(J_{-2/3} - J_{2/3})}{(J_{1/3} + J_{-1/3}) + i(J_{-2/3} - J_{2/3})} \qquad [43]$$

all Bessel functions having the argument w_0. The modulus of V is unity, i.e., total reflection. V is complex, which in general implies a phase change of the reflected wave in relation to the incident wave.

Again the expressions within the parentheses in [43] can be expressed in the Airy function $Ai(-t)$ and its first derivative, using the formulas [44] in section 5.3. The formula mentioned gives:

$$Ai(-t) = \frac{\sqrt{t}}{3}[J_{-1/3}(w_0) + J_{1/3}(w_0)] \qquad [44]$$

with:

$$w_0 = \frac{2t^{3/2}}{3}$$

Differentiating [44] with respect to w_0 we find, using the formulas for differentiating Bessel functions (section 5.2.3):

$$\frac{\mathrm{d}Ai(-t)}{\mathrm{d}w_0} = \frac{\sqrt{t}}{3}\left[J_{-2/3}(w_0) - J_{2/3}(w_0)\right] \qquad [45]$$

[44] and [45] contain, on the right-hand sides within the brackets, the same expressions as enter [43] on its right-hand side. Substituting [44] and [45] into [43] we have immediately the reflection coefficient expressed in the Airy function $Ai(-t)$ and its first derivative with respect to w_0.

The other case, i.e., index of refraction increasing downwards in the second medium (or downwards decreasing velocity), can be investigated in a similar way (BREKHOVSKIKH, 1960, pp.192–193).

In concluding this chapter, let us give the following review:

(*1*) Bessel functions expressed as sums: see sections 2.3, 3.2, 3.3 and 3.4.

(*2*) Bessel functions expressed as integrals: see sections 1, 2.1, 2.4, 3.1 and 3.5.

(*3*) Integrals involving Bessel functions: see sections 2.6 and 2.7.

(*4*) Sums involving Bessel functions: see sections 2.1, 2.5 and 2.7.

Chapter 6

LEGENDRE FUNCTIONS

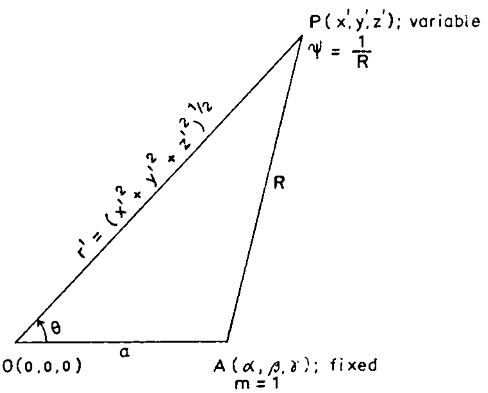
6.1 LEGENDRE POLYNOMIALS

The gravitational potential at P (Fig.48) due to a unit mass ($m = 1$) at A is:

$$\psi = \frac{1}{R} = \frac{1}{(r'^2 + a^2 - 2ar' \cos\theta)^{1/2}} =$$

$$\left.\begin{aligned} &= \frac{1}{a(1 - 2xr + r^2)^{1/2}} \qquad \text{for} \quad r = \frac{r'}{a} < 1 \\ \text{or:} \\ &= \frac{1}{r'(1 - 2xr + r^2)^{1/2}} \qquad \text{for} \quad r = \frac{a}{r'} < 1 \end{aligned}\right| \tag{1}$$

We expand $(1 - 2xr + r^2)^{-1/2}$ in ascending powers of r and denote the coefficient of r^n by $P_n(x)$. This will be shown to be a polynomial in x of degree n, called the *Legendre polynomial of degree n:*



$$R = [(x'-\alpha)^2 + (y'-\beta)^2 + (z'-\gamma)^2]^{1/2}$$
$$= (r'^2 + a^2 - 2ar' \cos\theta)^{1/2}$$
$$= (r'^2 + a^2 - 2ar' x)^{1/2}; \quad x = \cos\theta$$

Fig.48.

$$\frac{1}{(1 - 2xr + r^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)r^n \qquad [2]$$

This is a definition of $P_n(x)$. We came across $P_n(x)$ already in eq.[27] in section 4.2. We shall see later that the two definitions of $P_n(x)$ are equivalent to each other. Obviously, the discussion here has certain analogies to the discussion of Bessel functions above. Introducing [2] into [1], we see that the potential ψ, being a solution of Laplace equation (in polar coordinates), can be expressed in terms of Legendre polynomials or Legendre functions (sometimes called Legendre coefficients or Zonal harmonics).

By means of the definition [2] we shall now derive a general expression for $P_n(x)$, and also demonstiate a series of properties of $P_n(x)$.

6.1.1 *Properties of Legendre polynomials*

$P_n(x)$ is a polynomial of degree n in x, the nth Legendre polynomial

Proof. Expand the left-hand side of [2] by means of the binomial theorem:

$$(1 - 2rx + r^2)^{-1/2} = [1 - (2rx - r^2)]^{-1/2}$$

$$= 1 - \left(-\frac{1}{2}\right)(2rx - r^2) + \frac{-\frac{1}{2}\left(-\frac{1}{2} - 1\right)}{2!}(2rx - r^2)^2 - \ldots$$

$$= \underset{\text{0th term}}{1} - \underset{\text{1st term}}{\binom{-\frac{1}{2}}{1}(2rx - r^2)} + \underset{\text{2nd term}}{\binom{-\frac{1}{2}}{2}(2rx - r^2)^2} - \ldots \qquad [3]$$

From the definition [2] of $P_n(x)$ it is obvious that we need only include terms in this development up to the nth degree in r, as all following terms have higher degrees than n in r. The sum of the first n terms is a rational function of r and x, up to the degree n. (Note that r and x occur together, to the same degree, in [3]). Therefore, the coefficient of r^n, i.e., $P_n(x)$, is a rational function of degree n in x.

Next, let us find an expression for $P_n(x)$. In [3] we have used a common notation for the binomial coefficients, which quite generally can be written as follows:

$$(-1)^\nu \binom{-\frac{1}{2}}{\nu} = \frac{1 \cdot 3 \ldots (2\nu - 1)}{2 \cdot 4 \ldots 2\nu} = \frac{(2\nu)!}{2^{2\nu}(\nu!)^2} \qquad [4]$$

This can be seen immediately from the definition of the binomial coefficient. Then, the sum of the first n terms in [3] can be written as follows:

$$\sum_{\nu=0}^{n} \frac{(2\nu)!}{2^{2\nu}(\nu!)^2}(2rx - r^2)^\nu =$$

(note that for $\nu = 0$, this gives 1, the 0th term in [3])

$$= \sum_{\nu=0}^{n} \sum_{k=0}^{\nu} (-1)^k \binom{\nu}{k} \frac{(2\nu)!}{2^{2\nu}(\nu!)^2} (2rx)^{\nu-k} r^{2k} =$$

(when also $(2rx - r^2)^\nu$ is expanded by the binomial theorem)

$$= \sum_{\nu=0}^{n} \sum_{k=0}^{\nu} (-1)^k \binom{\nu}{k} \frac{(2\nu)!}{2^{2\nu}(\nu!)^2} (2x)^{\nu-k} r^{\nu+k} \tag{5}$$

(just separating r and x)

From this we get $P_n(x)$, the coefficient of r^n, as a sum of all terms for which the exponent of r is n, i.e., for which $\nu + k = n$, or $\nu = n - k$ and $\nu - k = n - 2k$:

$$P_n(x) = \frac{1}{2^n} \sum_{2k \leq n} (-1)^k \frac{(2n-2k)!}{(n-2k)!\,(n-k)!\,k!} x^{n-2k} \tag{6}$$

where the summation is extended over $k = 0, 1, 2, \ldots, n/2$. This ensures our getting a polynomial of degree n in x, as was found above to be the correct expression for $P_n(x)$. This means that if a certain n is given, the summation [6] is carried out over k, from $k = 0$ to $k = n/2$. Eq.[6] is obtained very easily, using the expression for:

$$\binom{\nu}{k} = \frac{\nu!}{k!\,(\nu-k)!}$$

It expresses the Legendre polynomial of degree n.

Eq.[6] permits us to deduce some properties of $P_n(x)$:

(*1*) The highest coefficient, i.e., the coefficient of x^n, is:

$$l_n = \frac{(2n)!}{2^n(n!)^2} \tag{7}$$

obtained by putting $k = 0$ in [6].

(*2*) Furthermore, from [6]:

$$P_n(-x) = (-1)^n P_n(x) \tag{8}$$

i.e., $P_n(x)$ is an even function of x for n even, but an odd function of x for n odd. This is immediately obvious from [6], as x and n occur in combination only in x^n, which is even or odd, if n is even or odd, respectively.

(*3*) Put $x = 0$. As we have that $0^0 = 1$, $0^1 = 0$, $0^2 = 0$, etc., we see that the only contribution we get from [6] is for the exponent $n - 2k = 0$ when $x = 0$. Also, we know that n and k are integers. Therefore:

(*a*) If n is even, then $k = n/2$ is an integer, and we get from [6]:

$$P_n(0) = (-1)^{n/2} \frac{n!}{2^n(n/2)!^2} \tag{9}$$

(*b*) If n is odd, then k cannot be an integer, $0^{n-2k} = 0$, and:

$$P_n(0) = 0 \tag{10}$$

(4) Put $x = 1$. From the original definition [2] we then have:

$$\frac{1}{1 - r} = \sum_{n=0}^{\infty} P_n(1) r^n \tag{11}$$

Expanding $1/(1 - r)$ by the binomial theorem we get:

$$\frac{1}{1 - r} = 1 + r + r^2 + r^3 + \ldots + r^n + \ldots$$

i.e., $P_n(1) = 1$; and by means of [8]:

$$P_n(-1) = (-1)^n \tag{12}$$

(5) Assigning some special values to n, we get the following expressions from [6]:

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}\right| \tag{13}$$

It is helpful in formulating such expressions to remember the rules we have derived from [6], namely, that $P_n(x)$ is a polynomial of degree n in x; the coefficient of x^n is given by [7]; $P_n(x)$ is even if n is even, i.e., such a polynomial contains only even powers of x, e.g., if $n = 6$, we have only x^6, x^4, x^2, and constant terms (or x^0); correspondingly if n is odd. Fig.35 illustrates eq.[13]. The number of maxima and minima of $P_n(x)$ increases as we go to higher degrees n.

(6) $x = \cos\theta$. From the definition [2] onwards, we made no restriction on x. But we arrived at the expression [2] by putting $x = \cos\theta$ (Fig.48). Eq.[6] then tells us that we can expand $P_n(\cos\theta)$ in powers of $\cos\theta$:

$$P_n(\cos\theta) = a \cos^n\theta + b \cos^{n-2}\theta + \ldots \tag{14}$$

Instead we can express $P_n(\cos\theta)$ in terms of cosines of multiples of θ. This could be done from [6], but more simply by starting from the original definition [2]. By Euler's formulas we have:

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i \sin\theta \\ e^{-i\theta} &= \cos\theta - i \sin \end{aligned}\right| \tag{15}$$

and thus:

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

and:

$$1 - 2r\cos\theta + r^2 = (1 - e^{i\theta}r)(1 - e^{-i\theta}r)$$

Therefore:

$$\sum_{n=0}^{\infty} P_n(\cos\theta)r^n = (1 - e^{i\theta}r)^{-1/2}(1 - e^{-i\theta}r)^{-1/2} \qquad [16]$$

Expand the two parentheses by the binomial theorem, and find the coefficient of r^n, i.e., $P_n(\cos\theta)$:

$$P_n(\cos\theta) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \frac{(2n-2k)!(2k)!}{[(n-k)!]^2(k!)^2} e^{i\theta(n-2k)} \qquad [17]$$

The method to arrive at [17] is exactly the same as used in finding [6]. By [15], we see that [17] fulfills our request, i.e., to express P_n in terms of cosines of *multiples* of θ:

$$e^{i\theta(n-2k)} = \cos[\theta(n-2k)] + i\sin[\theta(n-2k)]$$

As P_n is real, we have to take only the real part of [17] in writing down the expression for any particular P_n.

6.1.2 An orthogonal system

The polynomials P_0, P_1, P_2, P_3, . . . form an orthogonal system in the interval -1 *to* $+1$, *i.e., we have*:

$$\int_{-1}^{+1} P_m(x)P_n(x)\mathrm{d}x = 0, \qquad m \neq n; \quad m, n = 0, 1, 2 \ldots \qquad [18]$$

Proof. According to the definition [2] we have:

$$\left.\begin{aligned} \frac{1}{(1-2xu+u^2)^{1/2}} &= \sum_{m=0}^{\infty} P_m(x)u^m \\ \frac{1}{(1-2xv+v^2)^{1/2}} &= \sum_{n=0}^{\infty} P_n(x)v^n \end{aligned}\right| \qquad [19]$$

By multiplication of these two formulas and integration with respect to x we get:

$$\int_{-1}^{+1} \frac{\mathrm{d}x}{(1-2xu+u^2)^{1/2}(1-2xv+v^2)^{1/2}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{+1} P_m(x)P_n(x)u^m v^n \mathrm{d}x \qquad [20]$$

The integral on the left-hand side can be evaluated by means of a formula given by DWIGHT (1957, p.44, formula 195.01):

$$X = a + bx \ ; \quad U = f + gx \ ; \quad bg > 0$$

$$\int \frac{\mathrm{d}x}{X^{1/2}U^{1/2}} = \frac{2}{\sqrt{(bg)}} \log \left| \sqrt{(bgX)} + b\sqrt{U} \right| \qquad [21]$$

Applying [21] to the left-hand side of [20], we find that this is equal to:

$$\frac{1}{\sqrt{(uv)}} \log \frac{1 + \sqrt{(uv)}}{1 - \sqrt{(uv)}} \qquad [22]$$

As by definition, $u < 1$ and $v < 1$, i.e., $uv < 1$ and $\sqrt{(uv)} < 1$ (see introduction to section 6.1), we can apply the power series expansion of the logarithm in [22], i.e., we apply the expansion formula:

$$\log(1 + a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \dots \qquad [23]$$

which converges if $|a| < 1$ (De la Vallée Poussin, 1938, p.448). We thus find that [22] can be written as:

$$\sum_{n=0}^{\infty} \frac{2}{2n + 1} u^n v^n \qquad [24]$$

Compare this with the right-hand side of [20], where $u^m v^n$ can be put outside the integral sign. Equating coefficients of $u^m v^n$ on both sides of [20], we find that there is a contribution only for $m = n$, but that all other terms vanish. This proves eq.[18].

We can summarize the result in the following form:

$$\int_{-1}^{+1} P_m(x)P_n(x)\mathrm{d}x = \frac{2}{2n + 1} \delta_{m,n} \qquad [25]$$

where $\delta_{m,n}$ is the Kronecker delta, i.e. $= 1$ for $m = n$ and $= 0$ for $m \neq n$.

Quite generally, we say that a series of functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x), \dots$ which have the property that:

$$\int_a^b \varphi_m(x)\varphi_n(x)\mathrm{d}x = 0, \qquad m \neq n \qquad [26]$$

form an *orthogonal sequence* for the interval (a, b). If, in addition, the following relation is fulfilled for all values of n:

$$\int_a^b [\varphi_n(x)]^2\mathrm{d}x = 1 \qquad [27]$$

we say that the functions of the sequence are *normalized* and form an *orthonormal set* (orthonormal = *ortho*gonal + *normal*ized).

We thus see that the Legendre functions $P_n(x)$ form an orthogonal set, but not a normalized one. On the other hand, the set:

$$\left(n + \frac{1}{2}\right)^{1/2} P_n(x) \qquad [28]$$

forms an orthonormal set, as from [25] we then also have:

$$\int_{-1}^{+1} \left(n + \frac{1}{2}\right) P_n^2(x)\mathrm{d}x = 1 \qquad [29]$$

6.1.3 Expansion in series composed of Legendre functions

Any function $f(x)$ defined for $|x| \leqslant 1$ can be expressed in a series of Legendre functions of the form:

$$f(x) = \sum_{r=0}^{\infty} c_r P_r(x) \qquad [30]$$

Proof. The problem is to see if there are any coefficients c_r which will satisfy [30]. Assuming that the infinite series [30] converges uniformly in the range (—1, 1) to the sum $f(x)$, we may multiply each term of the series [30] by $P_n(x)$ and integrate term by term with regard to x over the range —1 to +1 (this is permitted only provided [30] is uniformly convergent):

$$\int_{-1}^{1} f(x)P_n(x)\mathrm{d}x = \sum_{r=0}^{\infty} c_r \int_{-1}^{1} P_r(x)P_n(x)\mathrm{d}x \qquad [31]$$

Applying [25] we find that the right-hand side is:

$$\sum_{r=0}^{\infty} c_r \frac{2}{2n+1} \delta_{r,n} = \frac{2c_n}{2n+1} \qquad [32]$$

which defines the coefficient c_n in the series expansion [30]:

$$c_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(\nu)P_n(\nu)\mathrm{d}\nu \qquad [33]$$

Thus, the series:

$$f(x) = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) P_n(x) \int_{-1}^{1} f(\nu)P_n(\nu)\mathrm{d}\nu \qquad [34]$$

converges uniformly to the sum $f(x)$ in the range (—1, +1), which should be proved. Eq.[34] is called the *Legendre series.*

6.1.4 Rodrigues' formula

The polynomial $P_n(x)$ is proportional to the nth derivative of $(x^2 - 1)^n$:

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n \qquad [35]$$

This is called *Rodrigues' formula.*

Proof. We start from the right-hand side of [35], excluding the constant factor for the moment:

$$\frac{d^n}{dx^n}(x^2-1)^n = \frac{d^n}{dx^n}\left[\sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\, x^{2n-2k}\right] \qquad [36]$$

where expansion of $(x^2-1)^n$ by the binomial theorem has been used. Carrying out successively the n derivations, we get:

$$\frac{d^n}{dx^n}x^{2n-2k} = \frac{(2n-2k)!}{(n-2k)!}x^{n-2k} \qquad [37]$$

Combining the right-hand side of [35] with [36] and [37] we then find:

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}\,x^{2n-2k}\right]$$

$$= \frac{1}{2^n n!}\sum_{k=0}^{n/2}(-1)^k\frac{n!}{k!\,(n-k)!}\;\frac{(2n-2k)!}{(n-2k)!}\,x^{n-2k} \qquad [38]$$

$$= P_n(x) \quad \text{by eq.[6]}$$

This proves Rodrigues' formula [35].

This formula is especially useful in evaluating definite integrals involving Legendre polynomials.

6.1.5 *Recurrence relations for Legendre polynomials*

The following is a useful collection of recurrence relations:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \qquad [39]$$

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x) \qquad [40]$$

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x) \qquad [41]$$

$$P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x) \qquad [42]$$

Proofs. [39]: Differentiate [2] with respect to r:

$$\frac{x-r}{(1-2xr+r^2)^{3/2}} = \sum_{n=0}^{\infty} nr^{n-1}P_n(x)\,; \quad |x| < 1 \qquad [39a]$$

[39a] can be written as:

$$(x-r)\sum_{n=0}^{\infty} r^nP_n(x) = (1-2xr+r^2)\sum_{n=0}^{\infty} nr^{n-1}P_n(x) \qquad [39b]$$

Equating coefficients of r^n on both sides we get:

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2nxP_n(x) + (n-1)P_{n-1}(x) \qquad [39c]$$

which immediately gives [39].

[40]: Instead, differentiate [2] with respect to x:

$$\frac{r}{(1 - 2xr + r^2)^{3/2}} = \sum_{n=0}^{\infty} r^n P_n'(x) \tag{40a}$$

[39a]/[40a] gives:

$$(x - r) \sum_{n=0}^{\infty} r^n P_n'(x) = \sum_{n=0}^{\infty} nr^n P_n(x) \tag{40b}$$

Equating coefficients of r^n gives [40]:

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x) \tag{40}$$

[41]: Differentiate [39] with respect to x:

$$(n + 1)P_{n+1}'(x) - (2n + 1)P_n(x) - (2n + 1)xP_n'(x) + nP_{n-1}'(x) = 0 \tag{41a}$$

Eliminate $P_n'(x)$ from [40] and [41a] and we get [41].

[42]: is obtained immediately from [41] minus [40].

Remark. If in [39] we replace n by $n - 1$, the formula reads:

$$P_n(x) = \frac{2n - 1}{n} xP_{n-1}(x) - \frac{n - 1}{n} P_{n-2}(x) \tag{43}$$

with $n = 2, 3, 4, \ldots$ Remembering that $P_0(x) = 1$ and $P_1(x) = x$, we find the formula [43] particularly useful for successive calculation of the expressions for $P_n(x)$. This is an alternative to the use of [6].

Likewise, we can, from [40] and [41], derive a corresponding formula for successive computation of the derivatives, $P_n'(x)$. Dividing [41] by [40] and then replacing n by $n - 1$, we find:

$$P_n'(x) = \frac{2n - 1}{n - 1} xP_{n-1}'(x) - \frac{n}{n - 1} P_{n-2}'(x) \tag{44}$$

with $n = 2, 3, 4, \ldots$ Also remember that $P_0'(x) = 0$; $P_1'(x) = 1$.

6.1.6 Schläfli's integral for $P_n(z)$:

$$P_n(z) = \frac{1}{2\pi i} \int_C \frac{(t^2 - 1)^n \mathrm{d}t}{2^n (t - z)^{n+1}} \tag{45}$$

where $z = x + iy$.

Proof. Referring to section 2.1, we know that a function $f(z)$, which is *analytic on and inside* a contour C, can for any point *a within C* be expressed in terms of an integral which depends only on the values of $f(z)$ at points *on* the contour itself:

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)\mathrm{d}z}{z - a} \tag{46}$$

Eq.[46] is obtained from [9] in section 2.1, if we replace $P(z)/Q(z)$ by $f(z)/(z - a)$. Differentiating eq.[46] n times with regard to a, we find finally that:

$$[f^{(n)}(z)]_{z=a} = \left[\frac{\mathrm{d}^n f(z)}{\mathrm{d}z^n}\right]_{z=a} = \frac{n!}{2\pi i}\int_C \frac{f(z)\mathrm{d}z}{(z-a)^{n+1}} \tag{47}$$

Combining [47] with [35], where we now write z instead of x (just to denote the general case of a complex variable) and where we identify the nth derivatives, we find Schläfli's integral:

$$P_n(z) = \frac{1}{2\pi i}\int_C \frac{(t^2-1)^n\mathrm{d}t}{2^n(t-z)^{n+1}} \tag{48}$$

where C is a contour which encircles the point z once counter-clockwise (i.e., positive circulation).

6.2 LEGENDRE FUNCTIONS

6.2.1 Legendre function of the first kind

We shall now prove that the function $y = P_n(x)$, as defined by [2] in section 6.1, is a solution of the differential equation:

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + n(n+1)y = 0 \tag{1}$$

i.e., *Legendre's differential equation for functions of degree n* (section 4.2).

Proof. There are a number of different ways to prove this. I have chosen a method which goes directly back to the recurrence relations in section 6.1.5.

Multiply [40] in section 6.1 by x:

$$x^2P_n'(x) - xP_{n-1}'(x) = xnP_n(x) \tag{2}$$

Use [42] in section 6.1 and replace $n+1$ by n:

$$P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x) \tag{3}$$

Take the difference [3] — [2]:

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - xnP_n(x) \tag{4}$$

Differentiate [4] with regard to x:

$$(1-x^2)P_n''(x) - 2xP_n'(x) = nP_{n-1}'(x) - xnP_n'(x) - nP_n(x) \tag{5}$$

On the right-hand side of [5], express $nP_{n-1}'(x)$ from [40] in section 6.1, by multiplying [40] with n. We then get our result:

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \tag{6}$$

This proves that $P_n(x)$ is *one* solution of the differential equation [1].

Eq.[6] can also be written in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)P_n'(x)] + n(n+1)P_n(x) = 0 \tag{7}$$

An alternative proof is given by WHITTAKER and WATSON (1935, p.304), who apply eq.[35], [47] and [48] in section 6.1 and arrive at a contour integral in which the integrand, corresponding to the left-hand side of [1], resumes its original value after describing C, *when n is an integer*, thus making this contour integral $= 0$:

$$(1-z^2)\frac{\mathrm{d}^2P_n(z)}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}P_n(z)}{\mathrm{d}z} + n(n+1)P_n(z)$$

$$= \frac{n+1}{2\pi i\cdot 2^n}\int_C \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}}\right]\mathrm{d}t = 0 \tag{8}$$

The proof thus far has assumed that n is an integer. But [48] in section 6.1 is still a solution of Legendre's differential equation for any n, provided that C is a contour such that the integrand assumes the same value as its original value after passing along the contour. Such contours are possible to find, and the expression [48] in section 6.1 with *any value of n* is termed the *Legendre function of degree n of the first kind.* Such a contour must enclose the singular points $t = +1$ and $t = z$ but not the point $t = -1$. This is easily seen from [8] as in going around $t = 1$ counter-clockwise the function under the integral sign resumes its original value multiplied by $e^{2\pi i(n+1)}$; similarly, going around $t = z$ counter-clockwise, the function resumes its original value multiplied by $e^{2\pi i(-n-2)}$. Going around both these points, the function under the integral sign in [8] will resume its original value *for any n*, because:

$$e^{2\pi i(n+1)}\cdot e^{2\pi i(-n-2)} = e^{-2\pi i} = 1 \tag{9}$$

On the other hand, the point $t = -1$ must not be enclosed, except for integer values of n.

We have thus proved that $P_n(x)$ is a solution of eq.[1], for any value of n. Moreover, the coefficient of x^n in the development of $P_n(x)$ given in eq.[7] of section 6.1 agrees exactly with the corresponding coefficient given in eq.[26] of section 4.2. This demonstrates that the definition [2] in section 6.1 of $P_n(x)$ is identical with the definition used in section 4.2.

This circumstance permits us to generalize the recurrence relations in section 6.1.5 to any positive n, integer or not. Quite generally, it should be emphasized that all formulas—both for Legendre and Bessel functions—are valid for integer values of n, but that caution is required in every individual case when formulas are to be applied to non-integral values of n.

6.2.2 Legendre function of the second kind

Legendre's equation [1] has in addition to $P_n(x)$ another integral, $Q_n(x)$, which is linearly independent of $P_n(x)$. $Q_n(x)$ is called the *Legendre function of degree n of the second kind* and can be expressed by the formula:

$$Q_n(x) = \frac{1}{2} P_n(x) \log \frac{x+1}{x-1} - R_n(x) \qquad [10]$$

where $R_n(x)$ is a polynomial of degree $(n-1)$. $Q_n(x)$ is infinite at $x = \pm 1$ and tends to 0 as $x \to \infty$. Eq.[10] was demonstrated in section 4.2.2.

The following formula is called *Neumann's formula* (from which [10] can be derived):

$$Q_n(x) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(t)}{x-t}\, dt \qquad \begin{cases} |x| > 1 \\ n \text{ positive integer} \end{cases} \qquad [11]$$

The general solution of Legendre's equation [1] is thus:

$$y = AP_n(x) + BQ_n(x) \qquad [12]$$

where the constants A and B have to be determined from the boundary conditions.

Even when n is an integer, $Q_n(x)$ is not a polynomial. In some problems we know that the solution [12] should be a polynomial in x. In such cases we must take the solution to be $y = AP_n(x)$.

Obviously, the Legendre functions $P_n(x)$ and $Q_n(x)$ are analogous to the Bessel functions $J_n(x)$ and $Y_n(x)$, respectively.

6.2.3 *Ferrers' associated Legendre functions*

These are defined as:

$$\left.\begin{aligned} P_n^m(x) &= (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m} \\ Q_n^m(x) &= (1-x^2)^{m/2} \frac{d^m Q_n(x)}{dx^m} \end{aligned}\right| \qquad [13]$$

m positive integer, $|x| < 1$, n unrestricted. They are of *degree n* and *order m*. P is of the *first kind*, Q of the *second kind*. They represent a more extended class of Legendre functions and satisfy a differential equation analogous to the Legendre equation [1].

We shall now derive the differential equation to which [13] are solutions. We start from Legendre's eq.[1]:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad [14]$$

We differentiate [14] m times with respect to x and write $v = d^m y/dx^m$. In carrying out the differentiations, we apply Leibniz' rule for the differentiation of a product n times:

$$(uv)^{(n)} = uv^{(n)} + \binom{n}{1} u' v^{(n-1)} + \ldots + \binom{n}{\nu} u^{(\nu)} v^{(n-\nu)} + \ldots$$

$$+ \binom{n}{n-1} u^{(n-1)} v' + u^{(n)} v \qquad [15]$$

We thus find from [14]:

$$(1-x^2)\frac{d^2v}{dx^2} - 2x(m+1)\frac{dv}{dx} + (n-m)(n+m+1)v = 0 \qquad [16]$$

Write:

$$w = (1-x^2)^{m/2}v \qquad [17]$$

by which [16] transforms into:

$$(1-x^2)\frac{d^2w}{dx^2} - 2x\frac{dw}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right] w = 0 \qquad [18]$$

Eq.[18] is *Legendre's associated differential equation* with the general solution:

$$w = AP_n^m(x) + BQ_n^m(x) \qquad [19]$$

That [18] is the sought differential equation is seen as follows. $y = P_n(x)$ is a solution of [14], as found above. Then we have from the definitions of v and w that:

$$\left.\begin{aligned} v &= \frac{d^mP_n(x)}{dx^m} \\ w &= (1-x^2)^{m/2}\frac{d^mP_n(x)}{dx^m} = P_n^m(x) \end{aligned}\right| \qquad [20]$$

Thus, w, solution of [18], is identical with $P_n^m(x)$, which is therefore a solution of [18].

In many physical applications, $|x| < 1$, e.g., $x = \cos\theta$, and then the definitions [13] are suitable. In other cases, slightly different definitions may be used, e.g., such that instead of $(1-x^2)$ one writes (x^2-1).

6.2.4 *Integral properties of the associated Legendre functions*

Corresponding to the integral [25] in section 6.1 for the Legendre functions, there are similar expressions for the associated Legendre functions:

$$\int_{-1}^{1} P_n^m(x)P_r^m(x)dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nr} \qquad [21]$$

Proof. The differential equations [18] for $P_n^m(x)$ and $P_r^m(x)$:

$$\left.\begin{aligned} &(1-x^2)\frac{d^2P_n^m(x)}{dx^2} - 2x\frac{dP_n^m(x)}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right] P_n^m(x) = 0 \\ &(1-x^2)\frac{d^2P_r^m(x)}{dx^2} - 2x\frac{dP_r^m(x)}{dx} + \left[r(r+1) - \frac{m^2}{1-x^2}\right] P_r^m(x) = 0 \end{aligned}\right| \qquad [22]$$

are multiplied by $P_r^m(x)$ and $P_n^m(x)$ respectively, and then we subtract the second from the first:

$$\frac{d}{dx}\left\{(1-x^2)\left[P_r^m(x)\frac{dP_n^m(x)}{dx} - P_n^m(x)\frac{dP_r^m(x)}{dx}\right]\right\}$$
$$+ (n-r)(n+r+1)P_r^m(x)P_n^m(x) = 0 \qquad [23]$$

Integrate between -1 and $+1$: the first term (in d/dx) then vanishes as the expression within the brackets vanishes at both $+1$ and -1. When $n \neq r$ we have that $(n-r)(n+r+1) \neq 0$; therefore, the integral over $P_n^m(x)P_r^m(x)$ must vanish.

The case with $r = n$ is proved by starting from the following formula:

$$P_n^{m+1}(x) = (1-x^2)^{1/2}\frac{dP_n^m(x)}{dx} + mx(1-x^2)^{-1/2}P_n^m(x) \qquad [24]$$

obtained from the definition [13]:

$$P_n^{m+1}(x) = (1-x^2)^{(m+1)/2}\underbrace{\frac{d^{m+1}P_n(x)}{dx^{m+1}}}_{\displaystyle = \frac{d}{dx}\underbrace{\frac{d^mP_n(x)}{dx^m}}_{\displaystyle = (1-x^2)^{-m/2}P_n^m(x) \text{ from [13]}}}$$

Squaring and integrating [24] we obtain:

$$\int_{-1}^{1}[P_n^{m+1}(x)]^2dx = \int_{-1}^{1}\left\{(1-x^2)\left[\frac{dP_n^m(x)}{dx}\right]^2 + 2mxP_n^m(x)\frac{dP_n^m(x)}{dx}\right.$$
$$\left. + \frac{m^2x^2}{1-x^2}[P_n^m(x)]^2\right\}dx =$$

(using partial integration on the first two integrals on the right side; details of this calculation are given at the end of this section)

$$= -\int_{-1}^{1}P_n^m(x)\underbrace{\frac{d}{dx}\left[(1-x^2)\frac{dP_n^m(x)}{dx}\right]}_{D}dx - m\int_{-1}^{1}[P_n^m(x)]^2dx$$
$$+ \int_{-1}^{1}\frac{m^2x^2}{1-x^2}[P_n^m(x)]^2dx \qquad [25]$$

By [22] the expression D becomes:

$$D = \frac{d}{dx}\left[(1-x^2)\frac{dP_n^m(x)}{dx}\right] = (1-x^2)\frac{d^2P_n^m(x)}{dx^2} - 2x\frac{dP_n^m(x)}{dx}$$
$$= -\left[n(n+1) - \frac{m^2}{1-x^2}\right]P_n^m(x)$$

Eq.[25] then becomes:

$$\int_{-1}^{1} [P_n^{m+1}(x)]^2 \mathrm{d}x = (n - m)(n + m + 1) \int_{-1}^{1} [P_n^m(x)]^2 \mathrm{d}x \qquad [26]$$

Repeated application of this result gives:

$$\int_{-1}^{1} [P_n^m(x)]^2 \mathrm{d}x = (n - m + 1)(n - m + 2) \cdots n \cdot$$

$$\cdot (n + m)(n + m - 1) \cdots (n + 1) \underbrace{\int_{-1}^{1} [P_n(x)]^2 \mathrm{d}x}_{= \frac{2}{2n+1} \text{ from [25] in section 6.1}} \qquad [27]$$

from which the final result follows:

$$\int_{-1}^{1} [P_n^m(x)]^2 \mathrm{d}x = \frac{2}{2n + 1} \frac{(n + m)!}{(n - m)!} \qquad [28]$$

The details of the partial integration referred to above follow. We use the partial integration formula:

$$\int u \mathrm{d}v = uv - \int v \mathrm{d}u$$

The first integral is:

$$\int_{-1}^{1} (1 - x^2) \left[\frac{\mathrm{d}P_n^m(x)}{\mathrm{d}x}\right]^2 \mathrm{d}x$$

where we put, for simplicity:

$$P \equiv P_n^m(x)$$

and identify:

$$u = (1 - x^2) \frac{\mathrm{d}P}{\mathrm{d}x}$$

$$\mathrm{d}v = \frac{\mathrm{d}P}{\mathrm{d}x} \mathrm{d}x$$

thus:

$$\mathrm{d}u = \mathrm{d}\left[(1 - x^2) \frac{\mathrm{d}P}{\mathrm{d}x}\right]$$

$$v = P$$

and:

$$\int_{-1}^{1} = \underbrace{\left[(1 - x^2) \frac{\mathrm{d}P}{\mathrm{d}x} P\right]_{-1}^{1}}_{= 0, \text{ because } 1 - x^2 \text{ is 0 at both limits}} - \int_{-1} P \frac{\mathrm{d}}{\mathrm{d}x} \left[(1 - x^2) \frac{\mathrm{d}P}{\mathrm{d}x}\right] \mathrm{d}x$$

The second integral is:

$$\int_{-1}^{1} 2mxP_n^m(x)\frac{\mathrm{d}P_n^m(x)}{\mathrm{d}x}\,\mathrm{d}x$$

where we proceed similarly:

$$u = 2mx$$

$$\mathrm{d}u = 2m\mathrm{d}x$$

$$\mathrm{d}v = P\frac{\mathrm{d}P}{\mathrm{d}x}\,\mathrm{d}x = P\mathrm{d}P$$

$$v = \frac{1}{2}P^2$$

and find:

$$\int_{-1}^{1} = \underbrace{\left[2mx\frac{1}{2}P^2\right]_{-1}^{1}} - \int_{-1}^{1} P^2 m\mathrm{d}x$$

$$= m\left[xP^2\right]_{-1}^{1} = m[P^2(+1) + P^2(-1)] = 0$$

Both P are $=0$, for $x = 1$ and $x = -1$, according to [13].

6.3 APPLICATIONS OF LEGENDRE FUNCTIONS

6.3.1 Solution of Laplace's differential equation in terms of Legendre functions. Spherical harmonics ("Kugelfunktionen")

We start from Laplace's equation in Cartesian coordinates:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad [1]$$

We are trying to find those homogeneous functions $\psi(x, y, z)$ of degree n which satisfy [1] everywhere, except possibly for the origin.

We transform [1] into spherical coordinates by means of the substitutions:

$$\left.\begin{aligned} x &= R\sin\delta\cos\varepsilon \\ y &= R\sin\delta\sin\varepsilon \\ z &= R\cos\delta \end{aligned}\right| \qquad [2]$$

(R = radius; δ = colatitude; ε = longitude; see section 1.2) after which [1] becomes

$$\frac{\partial}{\partial R}\left(R^2\frac{\partial\psi}{\partial R}\right) + \frac{1}{\sin\delta}\frac{\partial}{\partial\delta}\left(\sin\delta\frac{\partial\psi}{\partial\delta}\right) + \frac{1}{\sin^2\delta}\frac{\partial^2\psi}{\partial\varepsilon^2} = 0 \qquad [3]$$

Compare [18] in section 5.3. Eq.[3] has solutions of the form:

$$\psi = R^n e^{im\varepsilon}\Theta(\cos\delta) \qquad [4]$$

provided that Θ satisfies the following ordinary differential equation:

$$(1-u^2)\frac{d^2\Theta}{du^2} - 2u\frac{d\Theta}{du} + \left[n(n+1) - \frac{m^2}{1-u^2}\right]\Theta = 0 \qquad [5]$$

This is seen immediately by substituting [4] into [3], and writing $\cos\delta = u$.

Eq.[5] is identical with [18] in section 6.2, i.e., Legendre's associated differential equation. In particular, if we put $m = 0$ in [5], we get [1] in section 6.2, i.e., the usual Legendre's differential equation.

We see from [5] that replacing n by $-n-1$ does not change equation [5]. The only place where n enters [5] is in the term containing $n(n+1)$, and this is easily seen to be unchanged in that case. This means that besides solution [4] of degree n, there is also another solution of degree $-n-1$:

$$\psi = R^{-n-1}e^{im\varepsilon}\Theta(\cos\delta) \qquad [6]$$

The solution of [3] can be expressed as follows, in terms of Legendre functions:

$$\frac{\psi}{R^n} = e^{im\varepsilon}\Theta(\cos\delta) = A_nP_n(\cos\delta) + \sum_{m=1}^{n}(A_n^{(m)}\cos m\varepsilon + B_n^{(m)}\sin m\varepsilon)P_n^m(\cos\delta) \qquad [7]$$

ψ/R^n is called a *surface harmonic of degree n*, and ψ is called a *solid harmonic* or a *spherical harmonic of degree n*. The general solution of [3] can then be written as follows:

$$\psi(R, \delta, \varepsilon) = \sum_{n=0}^{\infty} R^n\left[A_nP_n(\cos\delta) + \sum_{m=1}^{n}(A_n^{(m)}\cos m\varepsilon + B_n^{(m)}\sin m\varepsilon)P_n^m(\cos\delta)\right] \qquad [8]$$

The integration constants are as usual determined from the boundary conditions, in the present case usually from assigned values on the surface of a given sphere.

The expression [7] is a sum of $2n+1$ terms and it represents *all* surface harmonics of degree n.

We add the following to the definitions above. On a unit sphere, $R = 1$, a surface harmonic is obviously identical with a solid harmonic (or spherical harmonic). On any sphere, $P_n(\cos\delta)$ vanishes on n parallels of latitude, as δ is colatitude and $P_n(\cos\delta) = 0$ is an equation in $\cos\delta$ of degree n, thus with n solutions in δ. It is called a *zonal harmonic*. The other terms in [7], i.e., $\cos(m\varepsilon)P_n^m(\cos\delta)$; $\sin(m\varepsilon)P_n^m(\cos\delta)$, vanish on the surface of the sphere in the following cases:

(*1*) $\cos(m\varepsilon) = 0$: this happens on $2m$ meridians. The same is true in case of $\sin(m\varepsilon) = 0$.

(*2*) The other factor $P_n^m(\cos\delta)$ is $=0$ on $n-m$ parallels of latitude. This is seen from the definition [13] in section 6.2, which gives:

$$P_n^m(\cos\delta) = (\sin\delta)^m \frac{d^mP_n(\cos\delta)}{d(\cos\delta)^m}$$

This vanishes (*1*) for $(\sin\delta)^m = 0$, i.e., at the poles; (*2*) for $d^m P_n(\cos\delta)/d(\cos\delta)^m = 0$. As $P_n(\cos\delta)$ is a polynomial of degree n in $\cos\delta$, $dP_n(\cos\delta)/d(\cos\delta)$ is a polynomial of degree $n - 1$, and so forth, and $d^m P_n(\cos\delta)/d(\cos\delta)^m$ is a polynomial of degree $n - m$, thus it has $n - m$ solutions. Therefore the case (*2*) happens on $n - m$ parallels of latitude.

Altogether, $\genfrac{}{}{0pt}{}{\cos}{\sin}(m\varepsilon)P_n^m(\cos\delta)$ vanish along the sides of right-angled quadrangles on the sphere, and are called *tesseral harmonics*. It is clear that these harmonics divide the surface of the sphere into something like a chess-board, hence the name *tesseral*. The word *tesselar* also exists, in the same meaning. In the case $n = m$, the harmonic divides the surface of the sphere into sectors, extending from pole to pole, as there are no zeroes of the function along any latitude (just like the sectors of a cut orange). Such harmonics are therefore called *sectorial*.

Let us now summarize the definitions we have introduced on harmonic functions:

(*1*) *Surface harmonic* (German: "Kugelflächenfunktion"):

$$A_n P_n(\cos\delta) + \sum_{m=1}^{n} (A_n^{(m)}\cos m\varepsilon + B_n^{(m)}\sin m\varepsilon)P_n^m(\cos\delta)$$

(*2*) *Solid harmonic* or *spherical harmonic* (German "Kugelfunktion"):

$$R^n \left[A_n P_n(\cos\delta) + \sum_{m=1}^{n} (A_n^{(m)}\cos m\varepsilon + B_n^{(m)}\sin m\varepsilon)P_n^m(\cos\delta) \right]$$

(*3*) *Zonal harmonic* (or *simple harmonic* or *Legendre harmonic*):

$$P_n(\cos\delta)$$

(*4*) *Tesseral harmonic* (specially $m = n$: *sectorial harmonic*):

$$\genfrac{}{}{0pt}{}{\cos}{\sin}(m\varepsilon)P_n^m(\cos\delta)$$

In summary we can write: Solid harmonic or Spherical harmonic $= R^n \cdot$ Surface harmonic $= R^n \cdot$ (Zonal harmonic + Tesseral harmonic).

As mentioned above, the determination of the constants in [8] can be made from the boundary values. For instance if the solution $\psi(R, \delta, \varepsilon)$ is equal to a given function $f(\delta, \varepsilon)$ on the surface of a sphere with radius $R = a$, then [8] becomes:

$$f(\delta, \varepsilon) = \sum_{n=0}^{\infty} a^n \left[A_n P_n(\cos\delta) + \sum_{m=1}^{n} (A_n^{(m)}\cos m\varepsilon + B_n^{(m)}\sin m\varepsilon)P_n^m(\cos\delta) \right] \qquad [9]$$

Assuming uniform convergence within the domain $0 \leqslant \delta \leqslant \pi$; $-\pi \leqslant \varepsilon \leqslant \pi$ the constants can be found from [9] by multiplication of this formula by $P_n^m(\cos\delta)\genfrac{}{}{0pt}{}{\cos}{\sin}(m\varepsilon)$ and carrying out integrations over the domain mentioned. This is facilitated by the formula [21] in section 6.2. For more details, see WHITTAKER and WATSON (1935, pp.393–394).

6.3.2 *Solution of the wave equation in terms of Legendre functions*

The solution of the wave equation expressed in spherical (polar) coordinates leads to associated Legendre functions. This will be discussed in more detail in Chapter 7. (See also section 5.3.2.)

6.3.3 *Some other geophysical applications of Legendre functions*

The formula [8] has obvious similarities to the expansion in a Fourier series, but in [8] the series represents a function with given values distributed over a sphere. In the case of a Fourier series, we have usually a one-dimensional representation instead, whether the variable be time or a linear coordinate. Eq.[8] can be used to represent *any* function with a given distribution over a sphere.

The analogy holds further in the following way. The first term in either case represents the mean value of the function (over the sphere, over a linear coordinate, or over the time, as the case may be). The following terms represent the (harmonic) deviations from the mean, with higher and higher frequencies. Their average value (over the sphere, etc.) is equal to zero, term by term.

Because of their close relations to the Laplace equation, the Legendre functions are of much use in *potential theory*. (See, e.g., HEISKANEN and VENING MEINESZ, 1958, p.41.)

The *topography of the earth* has been developed in spherical harmonics. See HEISKANEN and VENING MEINESZ (1958, pp.423–434). PREY (1922) developed the earth's topography in spherical harmonics up to the 16th order. Vening Meinesz took up these studies in 1950 and tried to demonstrate that this development could be explained by convection currents in the earth's interior:

1st order term: explained by 1st order current systems in an early phase of the earth's history, which not only influenced topography but also formed the core. The core development itself brought this current system to an end.

2nd–7th order terms: correspond to currents through the whole mantle. The 2nd and 3rd order terms probably came into play towards the end of the core formation.

8th–11th order terms: correspond to current systems in the upper *half* of the mantle.

12th–16th order terms: correspond to current systems in the upper *third* of the mantle.

Naturally the lower order terms correspond to the major features of the topography, whereas the higher the order is, the smaller are the details of the topography that the terms represent.

In this connection it should be mentioned that Legendre functions have a given place in the study of convection currents in the earth (this being a spherical problem). Quite generally, they are of importance in any spherical problem. Also, any distribution over a sphere can be described by means of surface harmonics, as, e.g., the earth's gravity field or its magnetic field, or even the earth's population density, etc.

Another field where spherical harmonics have a given place is the study of the earth's oscillations. (See LOVE, 1944, chapter 12; further BOLT, 1963; BULLEN, 1963, chapter 14; STONELEY, 1961.)

Chapter 7

THE WAVE EQUATION

7.1 GENERAL CONSIDERATIONS OF THE WAVE EQUATION

We consider the wave equation, which we write:

$$v^2\nabla^2\psi - \frac{\partial^2\psi}{\partial t^2} = 0 \qquad [1]$$

With regard to motions in some medium (gas, liquid or solid) we may distinguish between the following two main types:

(*1*) The medium itself is in motion. This is the usual problem encountered for the atmosphere or the oceans, but for the earth's interior it arises only in special cases, e.g., in convection currents, uplifts and depressions.

(*2*) The medium does not take part in the motion as such, but is only traversed by a wave motion. This can naturally also take place in all three media mentioned.

In both cases, three fundamental equations govern the happenings: (*A*) equation of motion; (*B*) equation of continuity; (*C*) equation of state. With regard to wave propagation through the earth's interior, it is the equation of motion, or more correctly the wave equation, which is of the greatest significance.

I am not deriving the wave equation here but I would like to summarize the conditions under which [1] is valid:

(*1*) Infinitesimal relative displacements.

(*2*) Perfectly elastic material, i.e., the stress is a homogeneous, linear function of strain and vice versa.

(*3*) Isotropic material, i.e., the elastic properties are independent of direction.

(*4*) No external forces (as gravity, friction) exist.

It is certainly of very great importance to investigate in special cases what the solutions may be if one or several of these assumptions are dropped (see further section 7.2.4). In this and the following section, we shall essentially follow MARGENAU and MURPHY (1943, pp.223–231).

7.1.1 Plane waves

Eq.[1] can be solved by the introduction of a single independent variable:

$$\xi = \alpha x + \beta y + \gamma z + vt$$

Eq.[1] then becomes:

$$\left[v^2(\alpha^2 + \beta^2 + \gamma^2) - v^2\right] \frac{d^2\psi}{d\xi^2} = 0 \qquad [2]$$

assuming α, β, γ constant and writing ∇^2 in its Cartesian form:

$$\frac{\partial \psi}{\partial x} = \frac{d\psi}{d\xi} \underbrace{\frac{\partial \xi}{\partial x}}_{=\alpha}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{d^2\psi}{d\xi^2} \frac{\partial \xi}{\partial x} \alpha = \alpha^2 \frac{d^2\psi}{d\xi^2}$$

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{d^2\psi}{d\xi^2}$$

Eq.[2] is satisfied if:

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \qquad [3]$$

Similarly $\eta = \alpha x + \beta y + \gamma z - vt$ leads to a solution $\psi(\eta)$ of [1], and the general solution can be written as follows, where the forms of f_1 and f_2 are arbitrary, remembering only that they can be differentiated twice:

$$\psi = f_1(\xi) + f_2(\eta) \qquad [4]$$

Now in [3] α, β and γ can be interpreted as direction cosines, i.e., as components of a *unit vector* **i** (this destroys generality as then α, β and γ are only real, but in [4] they may also be complex); then [4] becomes:

$$\psi = f_1(\mathbf{i} \cdot \mathbf{R} + vt) + f_2(\mathbf{i} \cdot \mathbf{R} - vt) \qquad [5]$$

[$\mathbf{i} \cdot \mathbf{R} = \alpha x + \beta y + \gamma z$, i.e., scalar multiplication of two vectors]

Physically [5] means two plane waves travelling towards $-\mathbf{i}$ and $+\mathbf{i}$, respectively, both with velocity v.

7.1.2 Spherical waves

Assume ψ to be a function of radius R and time t only (i.e., spherically symmetric radiation); then:

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{2}{R} \frac{\partial}{\partial R}$$

and [1] reduces to:

$$\frac{v^2}{R} \frac{\partial^2 (R\psi)}{\partial R^2} - \frac{\partial^2 \psi}{\partial t^2} = 0 \qquad [6]$$

Substitute $\xi = R + vt$ and $R\psi = P$. Multiply [6] by R:

$$v^2 \frac{\partial^2(R\psi)}{\partial R^2} - \frac{\partial^2(R\psi)}{\partial t^2} = 0$$

Insert P:

$$v^2 \frac{\partial^2 P}{\partial R^2} - \frac{\partial^2 P}{\partial t^2} = 0$$

and its derivatives:

$$\frac{\partial P}{\partial R} = \frac{\mathrm{d}P}{\mathrm{d}\xi} \underbrace{\frac{\partial \xi}{\partial R}}_{=1} \quad ; \quad \frac{\partial P}{\partial t} = \frac{\mathrm{d}P}{\mathrm{d}\xi} \underbrace{\frac{\partial \xi}{\partial t}}_{=v}$$

$$\frac{\partial^2 P}{\partial R^2} = \frac{\mathrm{d}^2 P}{\mathrm{d}\xi^2} \quad ; \quad \frac{\partial^2 P}{\partial t^2} = \frac{\mathrm{d}^2 P}{\mathrm{d}\xi^2} v^2$$

and [6] becomes an identity:

$$v^2 \frac{\mathrm{d}^2 P}{\mathrm{d}\xi^2} - v^2 \frac{\mathrm{d}^2 P}{\mathrm{d}\xi^2} = 0$$

This means that $P = f_1(R + vt)$ is a solution of [6]. Similarly, by another substitution $\eta = R - vt$ we find that $f_2(R - vt)$ is also a solution of [6]. The general solution is then:

$$P = f_1(R + vt) + f_2(R - vt) \qquad [7]$$

or:

$$\psi = \frac{1}{R} [f_1(R + vt) + f_2(R - vt)] \qquad [8]$$

This again means two waves, one travelling towards, the other outwards from the centre, and the factor $1/R$ is due to the geometrical spreading.

7.1.3 *Separation of variables*

We assume a solution of [1] in the form:

$$\psi = ST \qquad [9]$$

where S depends only on space coordinates, T only on time (this is the method of separation of variables for the solution of differential equations). We then have $\nabla^2\psi = \nabla^2(ST) = T\nabla^2 S$; $\partial^2\psi/\partial t^2 = S\ddot{T}$ (dots denoting time derivatives) and eq.[1] becomes:

$$v^2 \frac{\nabla^2 S}{S} = \frac{\ddot{T}}{T} \qquad [10]$$

The two sides of [10] are equal to one and the same constant, which we call $-\omega^2$. The time equation:

$$\ddot{T} + \omega^2 T = 0 \qquad [11]$$

has the solution:

$$T_\omega = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \qquad [12]$$

where ω clearly has the meaning of angular frequency. The space equation:

$$\nabla^2 S + \frac{\omega^2}{v^2} S = 0 \qquad [13]$$

is written:

$$\nabla^2 S + k^2 S = 0 \qquad [14]$$

where k is the *wave number*:

$$k = \frac{\omega}{v} = \frac{2\pi}{v\tau} = \frac{2\pi}{\lambda} \qquad [15]$$

(τ = period; λ = wave length).

The method of separation of variables can be justified only by its success, mathematically and physically, i.e., both to satisfy a given differential equation and initial and/or boundary conditions. If a separation of variables is possible and everything fulfilled, this certainly gives a solution to our problem. But is there no other solution as well? This is a mathematical question. Physically, there is generally only one solution. Therefore, the method of separation of variables does not limit the generality of the solutions, at least not when the physical application is considered.

I would like to remark that this chapter involves solutions of Laplace equation $\nabla^2 S = 0$ as special cases, namely those of the space part [14] of the wave equation when $k = 0$.

7.2 SOLUTIONS OF THE SPACE FORM OF THE WAVE EQUATION

7.2.1 One dimension[1]

In case of one dimension, eq.[14] in section 7.1 becomes:

$$\frac{d^2 S}{dx^2} + k^2 S = 0 \qquad [1]$$

with the solution:

$$S_k = a e^{ikx} + b e^{-ikx} \qquad [2]$$

[1] For example a string or a plane elastic wave.

one solution for each k. For $k = 0$, we have $S_0 = ax + b$. It is to be observed that $S = \sum_k S_k$ is *not* a solution of [1]. This is easily seen by putting S equal to a sum over two k (k_1 and k_2), perform the derivations and insert into [1]; the two terms in [1], i.e.:

$$\frac{d^2}{dx^2}\sum_k S_k \text{ and } k^2\sum_k S_k$$

can obviously not cancel each other for all values of k involved, but only for one value; in other words, [1] is not satisfied by such a solution. On the other hand $\psi = \sum_k S_k T_k$ is a solution of [1] in section 7.1, our original wave equation, as is immediately seen by substitution:

$$v^2\sum_k T_k\nabla^2 S_k - \sum_k S_k\ddot{T}_k = 0$$

(there is no k outside the summation signs to invalidate the relation). S_k = constant is the equation of a *wave front* at any given instant t.

7.2.2 *Two dimensions* [1]

(*1*) *Rectangular coordinates.* Eq.[14] in section 7.1 now reads:

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + k^2 S = 0 \qquad [3]$$

Renewed separation of variables is used:

$$S = X(x)\,Y(y) \qquad [4]$$

and [3] becomes:

$$\frac{X''}{X} + \frac{Y''}{Y} + k^2 = 0 \qquad [5]$$

because:

$$\frac{\partial S}{\partial x} = X'Y \quad ; \quad \frac{\partial^2 S}{\partial x^2} = X''Y \quad ; \quad \text{etc.}$$

Eq.[5] can be split into two equations by putting:

$$\frac{X''}{X} = -k_x^2 \quad \text{and} \quad \frac{Y''}{Y} = -k_y^2 \qquad [6]$$

subject to the condition that:

$$k_x^2 + k_y^2 = k^2 \;, \quad \text{eq.[5]} \qquad [7]$$

Solutions of [6] are:

$$X = c_1 e^{\pm ik_x x} \quad \text{and} \quad Y = c_2 e^{\pm ik_y y} \qquad [8]$$

[1] For example a membrane or vibrations of a gas in a thin layer or for cylindrical elastic waves.

and the solution of [3] then becomes:

$$S_{k_x k_y} = c_{k_x k_y} \mathrm{e}^{\pm i(k_x x \pm k_y y)} \tag{9}$$

This means that for every value of k (the wave number) we theoretically have an infinite number of solutions [9] such that k_x and k_y fulfill the condition [7]. And, as above, the corresponding solution of the wave equation is $\psi = \sum_k S_k T_k$.

Of course, a more explicit form of the solution [8] would be as follows:

$$X = a\mathrm{e}^{ik_x x} + b\mathrm{e}^{-ik_x x} \rightarrow c_1 \mathrm{e}^{\pm ik_x x}$$

the two terms representing waves propagating along negative and positive x. But in this treatment we combine the two terms in the shorter way, and then simply mean that we consider one wave (one term) at a time (but do not imply that the two waves have the same coefficient). Similarly, in the final solution [9] we may combine the + and — signs at will, i.e., all four possible combinations are equally justified, corresponding to wave propagation in the four quadrants of the xy-plane.

(2) *Polar coordinates.* In polar coordinates r and χ eq.[14] in section 7.1 becomes:

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r}\frac{\partial S}{\partial r} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \chi^2} + k^2 S = 0 \tag{10}$$

Separation of variables is used again:

$$S = P(r)\,\varphi(\chi)$$

by which [10] is transformed as follows:

$$r^2\frac{P''}{P} + r\frac{P'}{P} + \frac{\varphi''}{\varphi} + r^2k^2 = 0 \tag{11}$$

Here we put:

$$\frac{\varphi''}{\varphi} = -m^2 \tag{12}$$

after which [11] becomes:

$$r^2P'' + rP' + (k^2r^2 - m^2)P = 0 \tag{13}$$

This is a Bessel differential equation with the independent variable kr. In fact, [13] can be written as:

$$(kr)^2\frac{\mathrm{d}^2P}{\mathrm{d}(kr)^2} + kr\frac{\mathrm{d}P}{\mathrm{d}(kr)} + (k^2r^2 - m^2)P = 0 \tag{14}$$

which is seen to be identical with Bessel's differential equation:

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{15}$$

This has the solutions:

$$y = AJ_n(x) + BJ_{-n}(x) \qquad \text{when } n \text{ is not an integer} \tag{16}$$

$$y = CJ_n(x) + DY_n(x) \qquad \text{when } n \text{ is an integer} \tag{17}$$

We have to put $D = 0$, i.e., to discard the solution $Y_n(x)$, because this function contains $\log x$, which is negative infinite at $x = 0$ (see section 4.3). We denote the right-hand side of [16] or [17] by $Z_n(x)$, where $Z_n(x)$ is called the *general cylinder function* (SOMMERFELD, 1954, p.88). Applied to [13] our solution becomes:

$$S_{k,m} = Z_m(kr)e^{\pm im\chi} \tag{18}$$

or more generally:

$$S_k = \sum_m a_m Z_m(kr)\, e^{\pm im\chi} \tag{19}$$

7.2.3 *Three dimensions*

(*1*) *Rectangular coordinates.* We can immediately generalize the solution in rectangular coordinates for two dimensions and get:

$$S_{k_x k_y k_z} = c_{k_x k_y k_z}\, e^{\pm i(k_x x \pm k_y y \pm k_z z)} \tag{20}$$

under the condition that:

$$k_x^2 + k_y^2 + k_z^2 = k^2 \tag{21}$$

If k_x, k_y, k_z are taken to be real (which destroys the generality of the solution, as generally k may be real or complex), then they can be regarded as components of a vector $\mathbf{k}$, the *wave vector* (note that k_x, k_y, k_z are the wave numbers, as measured along the three axes):

$$S(\mathbf{k}) = c(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{R}} \tag{22}$$

Combining with the solution of the time part of the wave equation:

$$T_\omega \sim e^{\pm i\omega t} = e^{\pm ikvt} \quad , \quad \text{as } \omega = kv \tag{22a}$$

we find the following form for the solution of the wave equation:

$$\psi = \sum_k S_k T_k = \sum_k c(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{R} \pm kvt)} \tag{23}$$

or in integral form:

$$\psi = \int c(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{R} \pm kvt)}d\mathbf{k} \tag{24}$$

where $c(\mathbf{k})$ is simply a function of k_x, k_y, k_z and $d\mathbf{k}$ stands for the product $dk_x\, dk_y\, dk_z$. This is in fact a volume element in the k-space and strictly *not* a vector, but the way of writing [24] is used for convenience and will not cause confusion. Note in addition that k exists in [23] and [24] both in vectorial and scalar form.

Eq.[24] is a very useful form of the solution of the wave equation, and it corresponds to the *construction of a general wave* (i.e., a wave of any, three-dimensional shape) *by superposition of plane sinusoidal waves.* In section 7.3 we shall discuss in greater detail the decomposition of a spherical wave into plane waves.

Another advantage of [24] is that initial conditions can be easily introduced. For $t = 0$, we have:

$$\psi_0(x, y, z) = \int c(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{R}}d\mathbf{k} \tag{25}$$

which means that the $c(\mathbf{k})$ are given by the Fourier analysis of the assumed known function $\psi_0(x, y, z)$.

We can rewrite [24] in the following form by eliminating $c(\mathbf{k})$ between [24] and [25]. We get an expression of $c(\mathbf{k})$ from [25] by applying the *complex Fourier transform* to this equation. In the case of *one* coordinate, this transform and its inversion are (section 8.1.3):

$$\left.\begin{aligned} f(x) &= \int_{-\infty}^{\infty} c(k)e^{ikx}dk \\ c(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{-ik\xi}d\xi \end{aligned}\right| \tag{26}$$

In this case we have three space coordinates and the application of [26] then becomes:

$$\left.\begin{aligned} \psi_0(x, y, z) &= \int_{-\infty}^{\infty} c(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{R}}d\mathbf{k} \\ c(\mathbf{k}) &= \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \psi_0(x', y', z')e^{-i\mathbf{k}\cdot\mathbf{R}'}dx'dy'dz' \end{aligned}\right| \tag{27}$$

and the general solution to our problem is in this case:

$$\psi(x, y, z, t) = \frac{1}{(2\pi)^3}\iiiiiint_{-\infty}^{\infty} \psi_0(x', y', z')e^{i[\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')\pm kvt]}dx'dy'dz'dk_xdk_ydk_z \tag{28}$$

The integrations are extended from $-\infty$ to $+\infty$, both for space coordinates and for wave numbers (in physical applications, however, we deal only with positive wave numbers).

Obviously, the development in this section is equally applicable in two dimensions (section 7.2.2), where we could have considered $\mathbf{k}$ as a vector with the components k_x and k_y.

(2) *Cylindrical coordinates.* In cylindrical coordinates (see Fig.3) we have $S = S(r, z, \chi)$ and the space equation becomes:

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r}\frac{\partial S}{\partial r} + \frac{\partial^2 S}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \chi^2} + k^2 S = 0 \tag{29}$$

Separation of variables:

$$S = F(z)\,\varphi(\chi)\,P(r) \tag{30}$$

immediately yields by substitution in [29] and division by $F\varphi P$ the following three equations:

$$\left.\begin{aligned} &F'' = -\varkappa^2 F \\ &\varphi'' = -n^2\varphi \\ &r^2P'' + rP' - [(\varkappa^2 - k^2)r^2 + n^2]P = 0 \end{aligned}\right| \qquad [31]$$

In [31.3] we make the substitution: $i\sqrt{\varkappa^2 - k^2}\, r = x$, i.e.:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}P}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{\mathrm{d}P}{\mathrm{d}x}\, i\sqrt{(\varkappa^2 - k^2)}$$

$$\frac{\mathrm{d}^2P}{\mathrm{d}r^2} = \frac{\mathrm{d}^2P}{\mathrm{d}x^2}\, i^2(\varkappa^2 - k^2)$$

and [31.3] becomes:

$$x^2\frac{\mathrm{d}^2P}{\mathrm{d}x^2} + x\frac{\mathrm{d}P}{\mathrm{d}x} + (x^2 - n^2)P = 0 \qquad [32]$$

This is Bessel's differential equation, with the following solution:

$$P = Z_n(x) = Z_n[\sqrt{k^2 - \varkappa^2}\, r]$$

The solution of [29] thus becomes:

$$S_{k,\varkappa,n} = c\mathrm{e}^{\pm i(\varkappa z \pm n\chi)}Z_n[\sqrt{k^2 - \varkappa^2}\, r] \qquad [33]$$

It can be demonstrated that this solution is single-valued in χ, that is, it has only one value for any given value of χ, provided that n is any integer, positive or negative. This is seen as:

$$\mathrm{e}^{in\chi} = \mathrm{e}^{in\chi}\underbrace{\mathrm{e}^{i2\pi m}}_{=+1 \text{ for integer } m} = \mathrm{e}^{i(n\chi + 2\pi m)}$$

but also that:

$$\mathrm{e}^{in\chi} = \mathrm{e}^{in(\chi + 2\pi)}$$

Identifying the two exponents, we have that $n = m =$ integer. This means that our wave front S has only one definite value for a given value of the azimuth χ, but naturally it can vary with the azimuth. The single-valuedness is only to be taken as a limitation in the mathematically general solution, which has to be imposed for physical reasons. The solution of the wave equation becomes:

$$\psi = \sum_{k,\varkappa,n} S_{k,\varkappa,n}\mathrm{e}^{\pm ikvt} \qquad [34]$$

or more explicitly (n integer, k and $\varkappa$ varying continuously, assuming $\sqrt{(k^2 - \varkappa^2)} = \mu$ to have only real, positive values, and remembering that the cylinder function Z_n reduces to the Bessel function J_n for n integer, also putting $c = g\mu$):

$$\psi = \int \mathrm{e}^{\pm ikvt}\mathrm{d}k \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\chi} \int_0^{\infty} g_n(k,\mu)\mathrm{e}^{\pm i\sqrt{k^2-\mu^2}z} J_n(\mu r)\mu \mathrm{d}\mu \qquad [35]$$

A summation sign (and not integral) is used for n, because n assumes only integer values (it does not vary continuously as k or $\varkappa$).

In specific problems the coefficients g can be determined as follows. Suppose $t = 0$, and a disturbance is confined to the plane $z = 0$ and denoted $\psi_0(r,\chi)$. Also consider only one value of the wave length, i.e., k = constant, which eliminates integration over k. Eq.[35] then gives:

$$\psi_0(r,\chi) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\chi} \int_0^{\infty} g_n(\mu)J_n(\mu r)\mu \mathrm{d}\mu \qquad [36]$$

from which all coefficients g can be determined. Multiply both sides of [36] by $\mathrm{e}^{-in\chi}$ and integrate over χ from 0 to 2π:

$$\int_0^{\infty} g_n(\mu)J_n(\mu r)\mu \mathrm{d}\mu = \frac{1}{2\pi}\int_0^{2\pi} \psi_0(r,\chi)\mathrm{e}^{-in\chi}\mathrm{d}\chi \equiv \psi_n(r) \qquad [37]$$

for each n. This is nothing else than a *Fourier–Bessel transformation* or *Hankel transformation* (Chapter 8) of $\psi_n(r)$ and we immediately get the following equation for the determination of g:

$$g_n(\mu) = \int_0^{\infty} \psi_n(r)J_n(\mu r)r\mathrm{d}r \qquad [38]$$

(*3*) *Spherical* (*polar*) *coordinates.* In spherical (polar) coordinates R, ε, δ (see Fig.2) the space equation reads:

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial S}{\partial R}\right) + \frac{1}{R^2\sin\delta}\frac{\partial}{\partial \delta}\left(\sin\delta\frac{\partial S}{\partial \delta}\right) + \frac{1}{R^2\sin^2\delta}\frac{\partial^2 S}{\partial \varepsilon^2} + k^2S = 0 \qquad [39]$$

The substitution (separation of variables):

$$S = F(R)\theta(\delta)\varphi(\varepsilon) \qquad [40]$$

into [39] leads to:

$$\frac{\theta\varphi}{R^2}\frac{\partial}{\partial R}(R^2F') + \frac{F\varphi}{R^2\sin\delta}\frac{\partial}{\partial\delta}(\sin\delta\,\theta') + \frac{F\theta}{R^2\sin^2\delta}\varphi'' + k^2F\theta\varphi = 0$$

Multiply by $R^2/F\theta\varphi$:

$$\frac{1}{F}\frac{\mathrm{d}}{\mathrm{d}R}(R^2F') + k^2R^2 = -\frac{1}{\theta\sin\delta}\frac{\mathrm{d}}{\mathrm{d}\delta}(\sin\delta\,\theta') + \frac{m^2}{\sin^2\delta}$$

$$= \text{constant} = +n(n+1), \text{ assuming } \varphi''/\varphi = -m^2.$$

This leads to the following set of three equations, by multiplication of appropriate members by θ and by F/R^2, respectively:

$$\left.\begin{aligned}
&\varphi'' = -m^2\varphi \quad (m \text{ integer to ensure single-valuedness in } \varepsilon)\\
&\frac{1}{\sin\delta}\frac{d}{d\delta}(\sin\delta\,\theta') - \frac{m^2}{\sin^2\delta}\theta + n(n+1)\theta = 0\\
&\frac{1}{R^2}\frac{d}{dR}(R^2F') + \left[k^2 - \frac{n(n+1)}{R^2}\right]F = 0
\end{aligned}\right| \qquad [41]$$

In [41.2] we introduce $\cos\delta$ as independent variable instead of δ:

$$(1-\cos^2\delta)\frac{d^2\theta}{d(\cos\delta)^2} - 2\cos\delta\frac{d\theta}{d(\cos\delta)} + \left[n(n+1) - \frac{m^2}{1-\cos^2\delta}\right]\theta = 0$$

This is identical with eq.[18] in section 6.2, i.e., Legendre's associated differential equation. We take n integer so that the solution θ becomes a polynomial, not diverging for $\cos\delta = \pm 1$ (cf. section 6.2.3).

In [41.3] put $F = P/R$ (where P must be a function of R), and replace the variable R by τ, such that $R = \tau/k$. Then:

$$\frac{\partial}{\partial R} = k\frac{\partial}{\partial\tau}\ ; \quad \frac{\partial^2}{\partial R^2} = k^2\frac{\partial^2}{\partial\tau^2}$$

and eq.[41.3] becomes:

$$\frac{d^2P}{d\tau^2} + \left[1 - \frac{n(n+1)}{\tau^2}\right]P = 0 \qquad [42]$$

Again putting $P = \sqrt{\tau}Q$, Q being a function of τ or R, equivalent to putting $F = Q(R)\left(\frac{k}{R}\right)^{1/2}$, reduces [42] to the following form:

$$\frac{d^2Q}{d\tau^2} + \frac{1}{\tau}\frac{dQ}{d\tau} + \left[1 - \frac{\left(n+\frac{1}{2}\right)^2}{\tau^2}\right]Q = 0 \qquad [43]$$

i.e., a Bessel differential equation with the following solution:

$$Q = Z_{n+1/2}(\tau) \qquad [44]$$

From this we get:

$$F = \left(\frac{k}{R}\right)^{1/2} Z_{n+1/2}(\tau) = cR^{-1/2}Z_{n+1/2}(kR) \qquad [45]$$

The solution of the space equation is thus, with $P_n^m(\cos\delta)$ denoting associated Legendre polynomials:

$$S_k = \sum_{m,n} c_{k,m,n} P_n^m(\cos\delta)\, e^{im\varepsilon} R^{-1/2} Z_{n+1/2}(kR) \qquad [46]$$

From eq.[7] in section 6.3 we have:

$$\sum_{m=0}^{n} c_m P_n^m(\cos\delta)\, e^{im\varepsilon} = Y_n(\delta, \varepsilon) \qquad [47]$$

where Y is a surface harmonic. Then:

$$S_k = \sum_{n=0}^{\infty} c_{k,n} Y_n(\delta, \varepsilon) R^{-1/2} Z_{n+1/2}(kR) \qquad [48]$$

and the complete solution of the wave equation becomes:

$$\psi = \sum_k S_k e^{\pm ikvt} = \int dk \sum_{n=0}^{\infty} c_{k,n} Y_n(\delta, \varepsilon) R^{-1/2} Z_{n+1/2}(kR) e^{\pm ikvt} \qquad [49]$$

We shall apply this result to a simple case: the pressure wave emitted from a "breathing" sphere, assuming spherically symmetric radiation. The following two boundary conditions are valid:

(*1*) at the surface of any sphere, $R = R_0$, all points are in phase:

$p_{R=R_0} =$ constant $\cdot\, e^{-i\omega t}$, where p = pressure;

(*2*) at infinity, the wave should be an outgoing one:

$p_{R\to\infty} = f(R) e^{i(kR-\omega t)}$

Consider monochromatic radiation, i.e., only one k. In [49], there will then be no integration over k. Furthermore, at $R = R_0$, there is no functional dependence on either ε or δ; both n and m are zero. Eq.[49] becomes:

$$p = C_1 R^{-1/2} Z_{1/2}(kR) e^{-i\omega t} \qquad [50]$$

But:

$$Z_{1/2}(x) = a_1 J_{1/2}(x) + a_2 J_{-1/2}(x)$$
$$= a_1 \left(\frac{2}{\pi x}\right)^{1/2} \sin x + a_2 \left(\frac{2}{\pi x}\right)^{1/2} \cos x$$

from eq.[25] and [26] in section 5.3. Then [50] becomes:

$$p = C_2 \left[a_1 \frac{\sin(kR)}{kR} + a_2 \frac{\cos(kR)}{kR} \right] e^{-i\omega t} \qquad [51]$$

In order to satisfy the second boundary condition above, we have to put $a_1 = i$ and $a_2 = +1$, and thus the final solution is:

$$p = \frac{C_2}{kR} e^{i(kR-\omega t)} \qquad [52]$$

A more general case would be to drop the assumption of spherically symmetric radiation and to assume for the surface of the sphere instead:

$$p_{R=R_0} = \text{constant} \cdot \cos\delta \cdot e^{-i\omega t} \qquad [53]$$

when the sphere is said to emit *dipole waves*. This case, which is left as an exercise to the reader, is slightly more complicated than the previous case, but can also be solved by means of the equations we have developed.

7.2.4 *Concluding remarks*

This finishes our treatment of the wave equation in different dimensions (one, two, three) and in different coordinate systems (rectangular, cylindrical, spherical —polar). The treatment is complete and the expressions for the solutions in terms of wave functions (wave potentials) permit calculation of every quantity of interest to characterize the wave motion. Displacements are for example obtained as appropriate space derivatives of the wave potentials, etc. Formulas for this can be found in seismological textbooks.

We have dealt rather completely with the wave equation and its solutions, when it is written in the form [1] in section 7.1. Various extensions would require dropping one or several of the conditions, under which this equation is valid. These would include, e.g.:

(*1*) Finite displacements: see MURNAGHAN (1951).

(*2*) Time effects: see GUTENBERG (1959), BULLEN (1963); these are frequently so difficult that only model experiments can lead to results.

(*3*) Non-isotropic (anisotropic or aeolotropic) materials: see EWING, JARDETZKY and PRESS (1957) and BULLEN (1963).

(*4*) External forces, as gravity (see Chapter 14), friction, etc.

7.3 EXPANSION OF A SPHERICAL WAVE INTO PLANE WAVES: SOMMERFELD'S INTEGRAL

Let us assume a source of continuous spherical waves. In the atmosphere such a source could be a symmetrically pulsating sphere with a sinusoidal time variation. Then the acoustic velocity potential is:

$$\psi = \frac{V_0}{4\pi} \frac{e^{i(\omega t - kR)}}{R} \qquad [1]$$

where R = distance from the source and V_0 = "source intensity" or "source strength". $V_0 = 1$ means a "unit source". That [1] is a suitable expression for the potential can be seen by performing the operation $R^2(\partial/\partial R)$ on [1] and then letting $R \to 0$. We find:

$$\lim_{R\to 0} \left| -4\pi R^2 \frac{\partial \psi}{\partial R} \right| = | V_0 e^{i\omega t} | = V_0$$

where the left-hand side within brackets gives the volume over which the spherical waves spread out in unit time. The limit for $R \to 0$ is a measure of the source intensity $= V_0$. Instead of having $\omega t - kR$ in the exponent we could write $kR - \omega t$, as in section 7.2, as this means no other change.

Before tackling the problem of reflection and refraction of a spherical wave at a plane boundary, it is convenient to expand the spherical wave into plane waves, especially since the theory of reflection and refraction of plane waves is well-known.

We consider the spherical wave only in the form:

$$\frac{e^{-ikR}}{R} \tag{2}$$

that is, we leave out parts which do not contain R. Also if the source is at the origin:

$$R = \sqrt{(x^2 + y^2 + z^2)}$$

In the plane $z = 0$, the field of the spherical wave will have the form:

$$\frac{e^{-ikr}}{r}$$

where $r = \sqrt{(x^2 + y^2)}$. We expand this field in a double Fourier integral in the variables k_x and k_y, i.e., the wave numbers measured along the x- and y-axes, respectively:

$$\frac{e^{-ikr}}{r} = \iint_{-\infty}^{\infty} A(k_x, k_y) e^{+i(k_x x + k_y y)}\, dk_x dk_y \tag{3}$$

where $A(k_x, k_y)$ is given by:

$$(2\pi)^2 A(k_x, k_y) = \iint_{-\infty}^{\infty} \frac{e^{-ikr}}{r} e^{-i(k_x x + k_y y)}\, dx dy \tag{4}$$

Eq.[3] is called the complex Fourier transform and [4] is its inversion formula. These formulas will be explained in the next chapter, section 8.1.3.

We transform to polar coordinates r, φ_1 and use the notation:

$$k_x = q \cos\psi_1 \;;\quad k_y = q \sin\psi_1 \;;\quad q = \sqrt{(k_x^2 + k_y^2)}$$

$$x = r \cos\varphi_1 \;;\quad y = r \sin\varphi_1$$

Then $dxdy = r d\varphi_1 dr$. We obtain:

$$(2\pi)^2 A(k_x, k_y) = \int_0^{2\pi} d\varphi_1 \int_0^{\infty} e^{-ir[k + q\cos(\psi_1 - \varphi_1)]} dr \tag{5}$$

The integral over r can easily be evaluated; we introduce attenuation (only very small attenuation is sufficient), i.e., k has a negative imaginary part: $k = -ik'$, $k' > 0$, $e^{-ikr} = e^{-k'r}$ and the integration with regard to r gives:

$$\int_0^{\infty} e^{irB} dr = \left[\frac{e^{rBi}}{iB}\right]_0^{\infty} = -\frac{1}{iB} = \frac{i}{B}$$

where $B = -[k + q\cos(\psi_1 - \varphi_1)]$. That is:

$$(2\pi)^2 A(k_x, k_y) = i \int_0^{2\pi} \frac{d\varphi_1}{-[k + q\cos(\psi_1 - \varphi_1)]} = \frac{i}{k} \int_0^{2\pi} \frac{d\delta}{1 + \frac{q}{k}\cos\delta} \tag{6}$$

after substituting: $\delta = \psi_1 - \varphi_1$; $d\delta = -d\varphi_1$. Here ψ_1 is constant, $d\psi_1 = 0$, as $\tan\psi_1 = k_y/k_x$ = constant for a given direction of propagation. The integration can be carried out using the following integration formula:

$$\int_0^{2\pi} \frac{dx}{1 + a\cos x} = \frac{2\pi}{\sqrt{(1 - a^2)}}$$

if $a^2 < 1$ (DE LA VALLÉE POUSSIN, 1938, p.224). We then find from [6]:

$$A(k_x, k_y) = \frac{i}{2\pi} \frac{1}{\sqrt{(k^2 - q^2)}} = \frac{i}{2\pi\sqrt{(k^2 - k_x^2 - k_y^2)}}$$

Thus from [3]:

$$\frac{e^{-ikr}}{r} = \frac{i}{2\pi} \int\int_{-\infty}^{\infty} \frac{e^{+i(k_x x + k_y y)}}{\sqrt{(k^2 - k_x^2 - k_y^2)}} dk_x dk_y \qquad [7]$$

The formula can easily be extended to three dimensions. Formally it is sufficient for this "continuation" into space to add the term $ik_z z$ to the exponent in the integrand. In addition:

$$k^2 = k_x^2 + k_y^2 + k_z^2 \; ; \quad k_z = \sqrt{(k^2 - k_x^2 - k_y^2)}$$

Thus the three-dimensional equation becomes:

(*1*) if $z > 0$:

$$\frac{e^{-ikR}}{R} = \frac{i}{2\pi} \int\int_{-\infty}^{\infty} e^{+i(k_x x + k_y y + k_z z)} \frac{dk_x dk_y}{k_z}$$

(*2*) if $z < 0$: [8]

$$\frac{e^{-ikR}}{R} = \frac{i}{2\pi} \int\int_{-\infty}^{\infty} e^{+i(k_x x + k_y y - k_z z)} \frac{dk_x dk_y}{k_z}$$

The validity of the present "continuation" is based on the fact that the right-hand sides of the last expressions satisfy the wave equation (since it is satisfied by the integrand) and give the correct value (V_0) for the field at $z = 0$.

These formulas give the expansion of a spherical wave into plane waves. We have to combine with the time factor $e^{i\omega t}$; then we see immediately that the exponent in the integrand represents a *plane* wave, propagating in the direction given by the components k_x, k_y, k_z of the *wave vector*.

We will now transform the equations [8] into some other forms. Fig.49 gives immediately: $k_x = k \sin\delta \cos\psi_1$; $k_y = k \sin\delta \sin\psi_1$; $k_z = k \cos\delta$. The integration with respect to ψ_1 will be performed between the limits 0 and 2π. The integration with respect to δ cannot be restricted to real values of this angle. Instead we have:

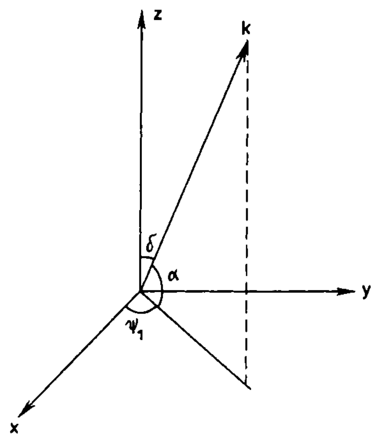


Fig.49.

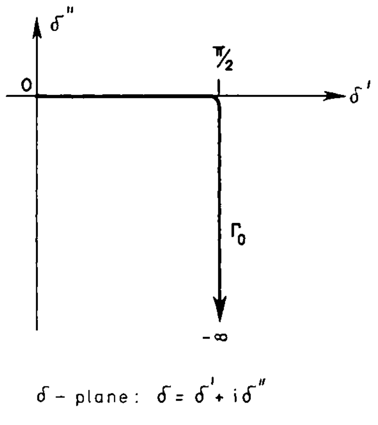


Fig.50.

$$\cos\delta = \frac{k_z}{k} \quad \begin{cases} (1) & k_x = k_y = 0;\ k_z = k;\ \delta = 0 \\ (2) & k_x \text{ or } k_y = \pm\infty;\ k_z = i\infty;\ \cos\delta = i\infty;\ \delta = \pi/2 - i\infty \end{cases}$$

remembering that:

$$\cos\left(\frac{\pi}{2} - ix\right) = \sin(ix) = i\ \sinh x = \frac{i}{2}(e^x - e^{-x})$$

This corresponds to the path of integration in the complex δ-plane which is shown in Fig.50. Furthermore, we find:

$$\frac{dk_x dk_y}{k_z} = k\sin\delta\, d\delta\, d\psi_1$$

and the formulas [8] can then be written in the following form:

$$\frac{e^{-ikR}}{R} = \frac{ik}{2\pi}\int_0^{\pi/2-i\infty}\int_0^{2\pi} e^{+i(k_x x + k_y y \pm k_z z)} \sin\delta\, d\delta\, d\psi_1 \tag{9}$$

where $+$ or $-$ shall be taken depending on whether $z > 0$ or $z < 0$, respectively, and where k_x, k_y, k_z are functions of δ, ψ_1 given above.

Thus, we see that in the expansion of a spherical wave, in addition to the usual waves in all possible directions within the limits $0 \leqslant \psi_1 \leqslant 2\pi$, $0 \leqslant \delta \leqslant \pi/2$, there will also be waves corresponding to complex values of δ. Waves of this type are called *inhomogeneous waves*. Put $\delta = \pi/2 - ia$, which means that we are on the vertical part of the contour of integration for δ (a supposed real and positive). Then we have from the formulas above: $k_x = k\cos\psi_1\cosh a$; $k_y = k\sin\psi_1\cosh a$; $k_z = ki\sinh a$. As $a \to \infty$ we obtain: $k_x \to \infty\cos\psi_1$; $k_y \to \infty\sin\psi_1$; $k_z \to i\infty$. This means that the so-called inhomogeneous waves propagate in the xy-plane ($\delta = \pi/2$) with a wave length tending to zero (wave numbers tend to infinity) and simultaneously attenuating in the z-direction (the z-factor becomes $e^{\mp\infty z}$ as k_z is imaginary).

In the expansion of a spherical wave into plane waves, it is necessary to include inhomogeneous waves in order to guarantee (*1*) the necessary singularity at $R = 0$ and (*2*) to give solutions which are bounded at all other points. At $x = y = z = 0$ a superposition of an infinite number of these waves (integral [9] above) gives an infinite value for the field. This is true for a point source. As we depart from the origin we obtain finite values, either because of attenuation (for $z \neq 0$) or because of phase interference (for $x \neq 0$ or $y \neq 0$), since all the waves have the same phase only at $x = y = 0$.

Instead of the angle of incidence δ we frequently use the angle of emergence α. We then get for $z \geqslant 0$:

$$\frac{e^{-ikR}}{R} = -\frac{ik}{2\pi} \int_{\pi/2}^{i\infty} \int_{0}^{2\pi} e^{+i(k_x x + k_y y + k_z z)} \cos\alpha \, d\alpha \, d\psi_1 \qquad [10]$$

where $k_x = k \cos\alpha \cos\psi_1$; $k_y = k \cos\alpha \sin\psi_1$; $k_z = k \sin\alpha$.

We introduce polar coordinates in the xy-plane: $x = r \cos\varphi_1$ and $y = r \sin\varphi_1$. This is the same as used above in deducing [5]. Note that ψ_1 is the corresponding angle for the wave vector k. Also introducing ν as a variable instead of α:

$$\nu = k \cos\alpha \quad ; \quad \text{i.e.,} \ \sin\alpha = \frac{i\sqrt{(\nu^2 - k^2)}}{k}$$

we can express our resulting formula (from [10] with $z \geqslant 0$) in terms of a Bessel integral. It is this solution which is usually referred to as the *Sommerfeld integral for spherical waves:*

$$\frac{e^{-ikR}}{R} = \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} e^{+ir\nu \cos(\psi_1 - \varphi_1)} \, e^{-z\sqrt{(\nu^2 - k^2)}} \frac{\nu d\nu d\psi_1}{\sqrt{(\nu^2 - k^2)}}$$

$$= \int_0^{\infty} \frac{J_0(\nu r) e^{-z\sqrt{(\nu^2 - k^2)}} \nu d\nu}{\sqrt{(\nu^2 - k^2)}} \qquad [11]$$

The introduction of the Bessel function is seen from eq.[44a] in section 5.2, replacing θ by $\psi_1 - \varphi_1$, also noting that $d\theta = d(\psi_1 - \varphi_1) = d\psi_1$ as we are integrating over all wave numbers (k), i.e., over all ψ_1, for a given direction φ_1 of the point (x, y). Eq.[11] is the same as eq.[1-41], p.13, in EWING, JARDETZKY and PRESS (1957). The development here has essentially followed BREKHOVSKIKH (1960, pp.237–244).

Reflection of spherical waves at a plane interface. By the expansion of a spherical wave into plane waves, we are back to the reflection of *plane* waves, in principle, which is relatively simple to deal with. With reference to Fig.51 we have for the total potential on the incident side, leaving out the time factor, etc.:

$$\psi = \frac{e^{-ikR}}{R} + \psi_{\text{refl}} \qquad [12]$$

Clearly, the reflected wave ψ_{refl} can be represented as a superposition of plane waves, resulting from the reflection of the plane waves, into which the original spherical wave was expanded. We have only to consider two factors:

(*1*) Upon reflection, the amplitude of each plane wave must be multiplied by the reflection coefficient $V(\alpha)$, where α is the angle of emergence.

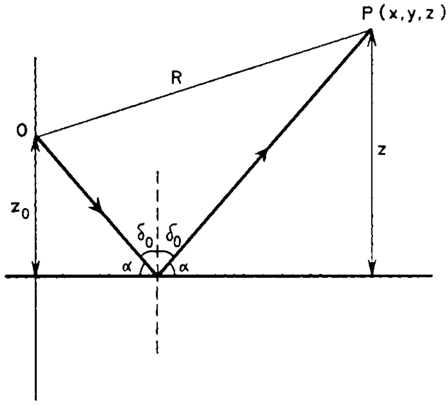


Fig.51.

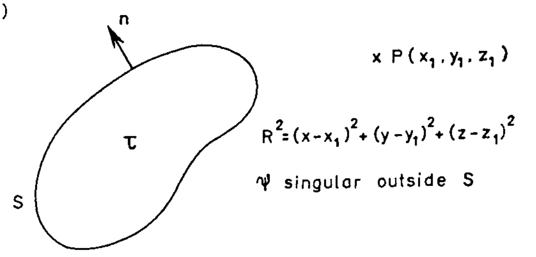


Fig.52.

(*2*) We have to take into account the phase changes as the wave travels from the source (to the boundary and then to the point of observation P; in addition we may have phase changes upon reflection, which we do not introduce here).

Using the Sommerfeld's integral, we then have for the reflected wave potential:

$$\psi'_{\text{refl}} = \int_0^\infty \frac{J_0(\nu r)\mathrm{e}^{-(z+z_0)\sqrt{(\nu^2-k^2)}}V(\alpha)\nu\mathrm{d}\nu}{\sqrt{(\nu^2-k^2)}} \qquad [13]$$

Applications of this method may be found in seismological textbooks, e.g., EWING, JARDETZKY and PRESS (1957, pp.94 ff.), etc. The usual boundary conditions may be applied to these integral expressions for the potentials, and as the entering integrands and integral limits are the same or analogous in the different potentials involved, it is possible to multiply out a factor, which has to be = 0, to fulfill the boundary conditions. These equations are then solved in the same way as in the plane-wave case.

7.4 KIRCHHOFF'S SOLUTION OF THE WAVE EQUATION

Here we shall mainly follow BAKER and COPSON (1953, pp.38–40, 42–44) and the proof starts from first principles; on the other hand, the proof given by WEBSTER (1947, pp.213–216) presupposes considerable knowledge of potential theory.

(*1*) Consider the case illustrated in Fig.52: S = a closed surface, within and on which the arbitrary function $V(x, y, z)$ and its first two derivatives are continuous, $P(x_1, y_1, z_1)$ = a point *outside* S, τ = volume enclosed by S, n = outward normal to S, R = distance from the fixed point P.

We start from *Green's theorem*, which we can write in general terms as follows (for n positive outwards):

$$\iiint_\tau (V\nabla^2 W - W\nabla^2 V)\mathrm{d}\tau = \iint_S \left(V\frac{\partial W}{\partial n} - W\frac{\partial V}{\partial n}\right)\mathrm{d}S \qquad [1]$$

Put $W = 1/R$ and note that:

$$\nabla^2 W = \nabla^2 \left(\frac{1}{R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial \frac{1}{R}}{\partial R}\right) = 0$$

from eq.[7] and [16] in section 1.2. Then eq.[1] becomes:

$$\iint_S \left(V \frac{\partial \frac{1}{R}}{\partial n} - \frac{1}{R}\frac{\partial V}{\partial n}\right) \mathrm{d}S + \iiint_\tau \frac{1}{R} \nabla^2 V \mathrm{d}\tau = 0 \qquad [2]$$

Also put:

$$V = \psi\left(x, y, z, t - \frac{R}{v}\right) = [\psi] \qquad \text{"retarded potential"} \qquad [3]$$

for outgoing spherical wave, where ψ is a solution of the wave equation:

$$\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \qquad [3a]$$

ψ is assumed to have no singularities inside or on S (the same condition as for V for [1] to be valid). By singularity, we mean a point where ψ or one of its first or second derivatives is discontinuous. ψ is the wave potential and the problem is to calculate an expression for ψ in the point P.

Before being able to apply [2] we have to form expressions for $\nabla^2 V$ and $\partial V/\partial n$. Therefore, we differentiate [3] with respect to x:

$$\frac{\partial V}{\partial x} = \left(\frac{\partial}{\partial x} - \frac{1}{v}\frac{\partial R}{\partial x}\frac{\partial}{\partial t_1}\right)\psi\left((x, y, z, \underbrace{t - \frac{R}{v}}_{= t_1}\right) = \left[\frac{\partial \psi}{\partial x}\right] - \frac{1}{v}\frac{\partial R}{\partial x}\left[\frac{\partial \psi}{\partial t}\right] \qquad [4]$$

Note that in the expression for ψ, x appears both explicit and implicit, as x is also contained in R. Derivatives where only the explicit case is contained, are denoted by square brackets; in other words, square brackets refer to "retarded" values. In the last term in [4] we have $[\partial\psi/\partial t_1]$, i.e., a derivative with respect to the "retarded" time $t_1 = t - R/v$. But as we denote retarded values by the square brackets, we will drop the subscript in t_1.

In analogy with [4] we have:

$$\frac{\partial V}{\partial n} = \left[\frac{\partial \psi}{\partial n}\right] - \frac{1}{v}\frac{\partial R}{\partial n}\left[\frac{\partial \psi}{\partial t}\right] \qquad [5]$$

This is obtained immediately from [4], the only difference being that in [4] we consider derivations with respect to the direction x, in [5] with respect to the direction n. From [4] we obtain:

(*a*) by replacing ψ by $\left[\frac{\partial \psi}{\partial x}\right]$:

$$\frac{\partial}{\partial x}\left[\frac{\partial \psi}{\partial x}\right] = \left[\frac{\partial^2 \psi}{\partial x^2}\right] - \frac{1}{v}\frac{\partial R}{\partial x}\left[\frac{\partial^2 \psi}{\partial x \partial t}\right] \qquad [6]$$

(*b*) by replacing ψ by $\left[\frac{\partial \psi}{\partial t}\right]$:

$$\frac{\partial}{\partial x}\left[\frac{\partial \psi}{\partial t}\right] = \left[\frac{\partial^2 \psi}{\partial x \partial t}\right] - \frac{1}{v}\frac{\partial R}{\partial x}\left[\frac{\partial^2 \psi}{\partial t^2}\right] \qquad [7]$$

which we will have use for below. Differentiate [4] with respect to x:

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \underbrace{\frac{\partial}{\partial x}\left[\frac{\partial \psi}{\partial x}\right]}_{\text{(apply [6])}} - \frac{1}{v}\frac{\partial R}{\partial x}\underbrace{\frac{\partial}{\partial x}\left[\frac{\partial \psi}{\partial t}\right]}_{\text{(apply [7])}} - \frac{1}{v}\frac{\partial^2 R}{\partial x^2}\left[\frac{\partial \psi}{\partial t}\right]$$

$$= \left[\frac{\partial^2 \psi}{\partial x^2}\right] - \frac{2}{v}\frac{\partial R}{\partial x}\left[\frac{\partial^2 \psi}{\partial x \partial t}\right] + \frac{1}{v^2}\left(\frac{\partial R}{\partial x}\right)^2\left[\frac{\partial^2 \psi}{\partial t^2}\right] - \frac{1}{v}\frac{\partial^2 R}{\partial x^2}\left[\frac{\partial \psi}{\partial t}\right] \qquad [8]$$

Carry out the same operations with respect to y and z and add the three formulas of type [8]:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= [\nabla^2 \psi] - \frac{2}{v}\sum \frac{\partial R}{\partial x}\left[\frac{\partial^2 \psi}{\partial x \partial t}\right] + \frac{1}{v^2}\left[\frac{\partial^2 \psi}{\partial t^2}\right]\sum\left(\frac{\partial R}{\partial x}\right)^2 - \frac{1}{v}\left[\frac{\partial \psi}{\partial t}\right]\sum\frac{\partial^2 R}{\partial x^2} =$$

(where the summation is carried out over x, y, z; using Einstein's summation convention, this can also be written as follows—two indices i mean summation over x_i)

$$= [\nabla^2 \psi] - \frac{2}{v}\frac{\partial R}{\partial x_i}\left[\frac{\partial^2 \psi}{\partial x_i \partial t}\right] + \frac{1}{v^2}\left[\frac{\partial^2 \psi}{\partial t^2}\right]\underbrace{\left(\frac{\partial R}{\partial x_i}\right)^2}_{=1} - \frac{1}{v}\left[\frac{\partial \psi}{\partial t}\right]\underbrace{\frac{\partial^2 R}{\partial x_i^2}}_{=2/R} \qquad [9]$$

The space derivatives of R are as follows:

$$R^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

$$\frac{\partial R}{\partial x} = \frac{x - x_1}{R} \quad ; \quad \sum\left(\frac{\partial R}{\partial x}\right)^2 = 1$$

$$\frac{\partial^2 R}{\partial x^2} = \frac{R^2 - (x - x_1)^2}{R^3} \quad ; \quad \sum\left(\frac{\partial^2 R}{\partial x^2}\right) = \frac{2}{R}$$

We can rewrite [9] in the following way, considering the wave equation [3a] which permits us to combine the first and third terms on the right-hand side of [9]:

$$\nabla^2 V = \frac{2}{v^2}\left[\frac{\partial^2 \psi}{\partial t^2}\right] - \frac{2}{vR}\left[\frac{\partial \psi}{\partial t}\right] - \frac{2}{v}\sum\frac{\partial R}{\partial x}\left[\frac{\partial^2 \psi}{\partial x \partial t}\right] \qquad [10]$$

Then the integrand of the volume integral of [2] can be written as follows:

$$\frac{1}{R}\nabla^2 V = \frac{2}{v^2 R}\left[\frac{\partial^2\psi}{\partial t^2}\right] - \frac{2}{vR^2}\left[\frac{\partial\psi}{\partial t}\right] - \frac{2}{v}\sum\frac{x-x_1}{R^2}\left[\frac{\partial^2\psi}{\partial x\partial t}\right] \qquad [11]$$

The following development permits us to write [11] in the form of a *divergence*

$$\frac{\partial}{\partial x}\left\{\frac{x-x_1}{R^2}\left[\frac{\partial\psi}{\partial t}\right]\right\} = \frac{\partial}{\partial x}\left(\frac{x-x_1}{R^2}\right)\left[\frac{\partial\psi}{\partial t}\right] + \frac{x-x_1}{R^2}\frac{\partial}{\partial x}\left[\frac{\partial\psi}{\partial t}\right] =$$

(by [7])

$$= \frac{1}{R^2}\left[\frac{\partial\psi}{\partial t}\right] - \frac{2(x-x_1)^2}{R^4}\left[\frac{\partial\psi}{\partial t}\right] + \frac{x-x_1}{R^2}\left[\frac{\partial^2\psi}{\partial x\partial t}\right] - \frac{(x-x_1)^2}{vR^3}\left[\frac{\partial^2\psi}{\partial t^2}\right]$$

(sum over x, y, z)

$$\sum\frac{\partial}{\partial x}\left\{\frac{x-x_1}{R^2}\left[\frac{\partial\psi}{\partial t}\right]\right\} = \frac{1}{R^2}\left[\frac{\partial\psi}{\partial t}\right] - \frac{1}{vR}\left[\frac{\partial^2\psi}{\partial t^2}\right] + \sum\frac{x-x_1}{R^2}\left[\frac{\partial^2\psi}{\partial x\partial t}\right]$$

This multiplied by $-2/v$ is equal to the right-hand side of [11], thus:

$$\frac{1}{R}\nabla^2 V = -\frac{2}{v}\sum\frac{\partial}{\partial x}\left\{\frac{x-x_1}{R^2}\left[\frac{\partial\psi}{\partial t}\right]\right\} \qquad [12]$$

That is, the integrand of the volume integral in [2] is now written in the form of a divergence:

$$\mathrm{div}\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

We can then apply the *divergence theorem* (or *Green's theorem*):

$$\iiint_\tau \mathrm{div}\mathbf{F}\,\mathrm{d}\tau = \iint_S F_n\,\mathrm{d}S \quad (n \text{ positive outwards}) \qquad [13]$$

and get:

$$\iiint_\tau \frac{1}{R}\nabla^2 V\mathrm{d}\tau = -\frac{2}{v}\iiint_\tau \sum\frac{\partial}{\partial x}\left\{\frac{1}{R}\frac{\partial R}{\partial x}\left[\frac{\partial\psi}{\partial t}\right]\right\}\mathrm{d}\tau$$

$$= -\frac{2}{v}\iint_S \frac{1}{R}\frac{\partial R}{\partial n}\left[\frac{\partial\psi}{\partial t}\right]\mathrm{d}S \qquad [14]$$

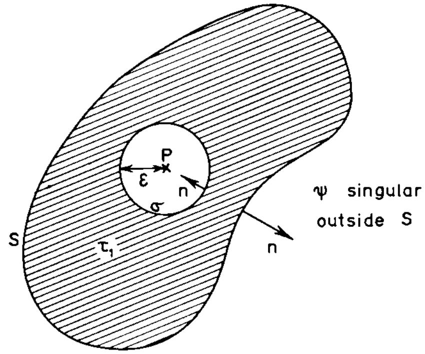


Fig.53.

Going back to eq.[2], this can now be written in the following form, taking eq.[5] for $\partial V/\partial n$ and [14] into account, also eq.[3]:

$$\iint_S \left\{ [\psi] \frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\left[\frac{\partial \psi}{\partial n}\right] - \frac{1}{vR}\frac{\partial R}{\partial n}\left[\frac{\partial \psi}{\partial t}\right] \right\} dS = 0 \qquad [15]$$

which is *Kirchhoff's formula* for the case when P and all the singularities of ψ lie *outside* S.

(*2*) Consider now the case shown in Fig.53, that is, the point $P(x_1, y_1, z_1)$ is *inside* S, but all singularities of ψ (the solution of the wave equation) are still outside S. We can again write Green's formula, eq.[2], extending the volume integral over τ_1, which is limited externally by the surface S and internally by the surface σ of a small sphere (radius ε) around P, i.e.:

$$\iint_S \left\{ V\frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial V}{\partial n} \right\} dS + \iint_\sigma \left\{ V\frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial V}{\partial n} \right\} dS$$

$$+ \iiint_{\tau_1} \frac{1}{R}\nabla^2 V d\tau = 0 \qquad [16]$$

The calculations under (*1*) above can be repeated, with the same result, namely that the volume integral can be transformed into a surface integral extended over the limiting surface, in this case $S + \sigma$:

$$\iint_S \left\{ [\psi] \frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\left[\frac{\partial \psi}{\partial n}\right] - \frac{1}{vR}\frac{\partial R}{\partial n}\left[\frac{\partial \psi}{\partial t}\right] \right\} dS$$

$$+ \iint_\sigma \left\{ [\psi] \frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\left[\frac{\partial \psi}{\partial n}\right] - \frac{1}{vR}\frac{\partial R}{\partial n}\left[\frac{\partial \psi}{\partial t}\right] \right\} dS = 0 \qquad [17]$$

Now let the radius ε tend to zero, and remembering that $(\partial/\partial n) = -(\partial/\partial R)$ on the surface σ and by writing $dS = R^2 d\Omega$ we find:

$$\iint_{\sigma} \left\{ [\psi] \frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\left[\frac{\partial \psi}{\partial n}\right] - \frac{1}{vR}\frac{\partial R}{\partial n}\left[\frac{\partial \psi}{\partial t}\right] \right\} \mathrm{d}S$$

$$= \iint_{\sigma} \left\{ - [\psi] \underbrace{\frac{\partial}{\partial R}\left(\frac{1}{R}\right)}_{= -1/R^2} + \frac{1}{R}\left[\frac{\partial \psi}{\partial R}\right] + \frac{1}{vR}\underbrace{\left(\frac{\partial R}{\partial R}\right)}_{=1}\left[\frac{\partial \psi}{\partial t}\right] \right\} \mathrm{d}S$$

$$= \iint_{\sigma} \left\{ \frac{[\psi]}{R^2} + \frac{1}{R}\left[\frac{\partial \psi}{\partial R}\right] + \frac{1}{vR}\left[\frac{\partial \psi}{\partial t}\right] \right\} R^2 \mathrm{d}\Omega$$

$$= \iint_{\sigma} \left\{ [\psi] + \underbrace{R\left[\frac{\partial \psi}{\partial R}\right] + \frac{R}{v}\left[\frac{\partial \psi}{\partial t}\right]}_{\to 0 \text{ as } R \to 0} \right\} \mathrm{d}\Omega = \iint_{\sigma} [\psi]\, \mathrm{d}\Omega = 4\pi[\psi] \qquad [18]$$

And thus [17] becomes, for $R = 0$, i.e., in the point P:

$$t - \frac{R}{v} \to t$$

$$\psi(x_1, y_1, z_1, t) = -\frac{1}{4\pi} \iint_{S} \left\{ [\psi] \frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\left[\frac{\partial \psi}{\partial n}\right] - \frac{1}{vR}\frac{\partial R}{\partial n}\left[\frac{\partial \psi}{\partial t}\right] \right\} \mathrm{d}S \qquad [19]$$

which is *Kirchhoff's formula* in the case when P is inside S and all the singularities lie outside. This case also includes [15], only that the integral vanishes for *P outside S*, i.e., the potential ψ is then $= 0$ at P.

(*3*) The third case to consider is when all the singularities of ψ lie inside the closed surface S (see Fig.54). Kirchhoff's formula, eq.[19], can be written immediately for this

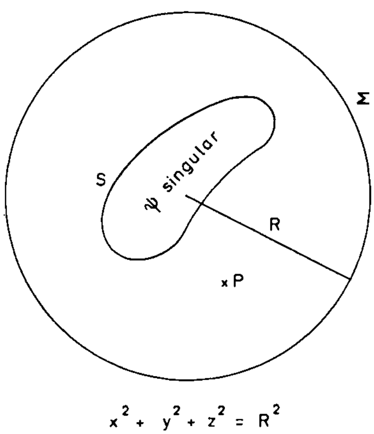


Fig.54.

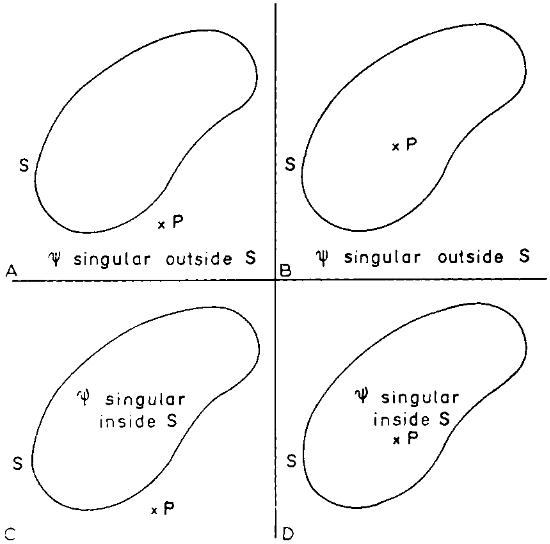


Fig.55.

case, where the surface integral shall be extended over S and Σ. If we now let $R \to \infty$, we obtain a theorem valid when ψ has no singularities outside the closed surface S, provided that the integral over Σ tends to zero. This is the case if ψ behaves like $f(vt - R)/R$ for large values of R.

In general, the different elements with which we must deal are: (*a*) a closed surface S, on which ψ and its derivatives are known, (*b*) a point P, for which the wave function is to be evaluated, and (*c*) the wave function ψ which may have singular points. With regard to the mutual distribution of these elements, there are four possibilities; see Fig.55. The cases *1–3* are those already discussed.

Case *4* can be treated in analogy with the preceding cases, by surrounding both P and singular points for ψ with small spheres, which we let tend to zero. When P or singularities of ψ are on the surface S itself, the discussion proceeds as here, only that such points are surrounded by small spheres, which at the end are made infinitely small, and one has just to calculate the limit of the surface integral in such cases.

Physically, eq.[19] states that the value of ψ at the point P at time t can be calculated from the values of ψ and its derivatives on the surface S at an earlier instant, namely at time $t - R/v$. R/v is the time that it takes for a wave to propagate from any point of S to the point P. The points on S act as *secondary sources* of the wave motion. In fact, Kirchhoff's formula is the analytical statement of *Huygens' principle:* in every wave front we can consider every point as a source of a new elementary wave system; the combination of all such elementary waves is identical with the initial single wave. This principle has played a great role in optics, but is of general validity for any wave systems.

7.4.1 *Poisson's formula*

Poisson's formula is a special case of Kirchhoff's formula, namely in the case that *the closed surface is spherical* with radius $R = vt$. Starting from [19], we have in case of a sphere:

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial R}$$

(as n is positive outwards from S)

$$dS = R^2 d\Omega$$

and:

$$\psi = -\frac{1}{4\pi}\iint_S \left\{ [\psi] \underbrace{\frac{\partial}{\partial R}\left(\frac{1}{R}\right)}_{= -\frac{1}{R^2}} - \frac{1}{R}\left[\frac{\partial \psi}{\partial R}\right] - \frac{1}{vR}\underbrace{\frac{\partial R}{\partial R}}_{=1}\left[\frac{\partial \psi}{\partial t}\right]\right\} R^2 d\Omega$$

$$= +\frac{1}{4\pi}\iint_\Omega \left\{ \underbrace{[\psi] + R\left[\frac{\partial \psi}{\partial R}\right]}_{= \left(\frac{\partial(\psi R)}{\partial R}\right)_{t_1 = t - R/v}} + \frac{R}{v}\underbrace{\left[\frac{\partial \psi}{\partial t}\right]}_{= \left(\frac{\partial \psi}{\partial t}\right)_{t_1 = t - R/v}}\right\} d\Omega$$

As the radius of the spherical surface was assumed to be $R = vt$, we find:

$$\psi = \frac{1}{4\pi}\left[\frac{\partial}{\partial(vt)} vt \iint_\Omega \psi_{t=0} d\Omega + t \iint_\Omega \left(\frac{\partial \psi}{\partial t}\right)_{t=0} d\Omega\right] \tag{20}$$

This is Poisson's solution. Note that the time variable in quantities within square brackets is $t - R/v$, in the others simply t.

7.4.2 *Helmholtz' formula*

Helmholtz' formula is another special case of Kirchhoff's formula, namely for the case when the wave potential ψ can be written in the following form:

$$\psi = we^{-i\omega t} = we^{-ikvt} \tag{21}$$

(monochromatic radiation). The retarded potential is then:

$$[\psi] = we^{ik(R-vt)} \tag{22}$$

replacing t by $t - R/v$ in [21]. Furthermore:

$$\left.\begin{aligned}\left[\frac{\partial\psi}{\partial t}\right] &= -ikv[\psi]\\ \left[\frac{\partial\psi}{\partial n}\right] &= \frac{\partial w}{\partial n}\mathrm{e}^{ik(R-vt)}\end{aligned}\right\} \qquad [23]$$

which after introduction into [19] yield:

$$\psi(x_1, y_1, z_1, t) = -\frac{1}{4\pi}\iint_S \left\{w\mathrm{e}^{ik(R-vt)}\frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial w}{\partial n}\mathrm{e}^{ik(R-vt)} + \frac{1}{vR}\frac{\partial R}{\partial n}ikv[\psi]\right\}\mathrm{d}S = -\frac{1}{4\pi}\iint_S \left\{w\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{ik(R-vt)}}{R}\right) - \underbrace{\frac{w}{R}\frac{\partial}{\partial n}\mathrm{e}^{ik(R-vt)}}_{=\frac{w}{R}\mathrm{e}^{ik(R-vt)}ik\frac{\partial R}{\partial n}} - \frac{1}{R}\frac{\partial w}{\partial n}\mathrm{e}^{ik(R-vt)} + \frac{1}{R}\frac{\partial R}{\partial n}ik[\psi]\right\}\mathrm{d}S$$

cancel

That is:

$$\psi(x_1, y_1, z_1, t) = -\frac{1}{4\pi}\iint_S \left\{w\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{ik(R-vt)}}{R}\right) - \frac{\mathrm{e}^{ik(R-vt)}}{R}\frac{\partial w}{\partial n}\right\}\mathrm{d}S \qquad [24]$$

which is Helmholtz' formula. Note again that n is positive in an outward direction from the surface S. Kirchhoff's formula is more general as it does not contain k (wave number) and is valid for any solution of the wave equation.

7.4.3 Kirchhoff's formula generalized

We can generalize Kirchhoff's formula [19] to the case, when there are also sources (or sinks) of wave motion. Returning to eq.[4] in section 1.1, the wave equation then becomes inhomogeneous and reads:

$$\frac{\partial^2\psi}{\partial t^2} - v^2\nabla^2\psi = v^2 e$$

or:

$$\nabla^2\psi + e(x, y, z, t) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \qquad [25]$$

Consider Poisson's equation:

$$\nabla^2\psi + 4\pi\varrho = 0 \tag{26}$$

where by comparison with [25] $4\pi\varrho = e$. [26] has the following solution (WEBSTER, 1947, pp.196–197):

$$\psi = \iiint_{\tau'} \frac{[\varrho]\mathrm{d}\tau'}{R} = \frac{1}{4\pi}\iiint_{\tau'} \frac{[e]}{R}\,\mathrm{d}\tau' \tag{27}$$

where $[\varrho] = \varrho(x, y, z, t - R/v)$ and the integration is extended over the volume τ' which contains sources (or sinks). It can be shown that a term like [27] has to be added to the solution of the wave equation, to cover the case [25] and Kirchhoff's equation [19] now becomes:

$$\psi(x_1, y_1, z_1, t) = \frac{1}{4\pi}\iiint_{\tau'} \frac{[e]}{R}\,\mathrm{d}\tau' - \frac{1}{4\pi}\iint_{S} \left\{ [\psi]\frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\left[\frac{\partial\psi}{\partial n}\right] - \frac{1}{vR}\frac{\partial R}{\partial n}\left[\frac{\partial\psi}{\partial t}\right]\right\}\mathrm{d}S \tag{28}$$

For a more complete treatment of this section, see WEBSTER (1947, pp.216–217).

7.5 COMMON FEATURES OF SPECIAL FUNCTIONS AND OF SPECIAL DIFFERENTIAL EQUATIONS

In our studies we have come across a number of special functions and their corresponding differential equations, as well as a number of typical differential equations of mathematical physics. In this section we shall try to see some common features among these functions and equations.

7.5.1 *Generating functions*

Formally the relation between a *generating function* $f(x, y)$ and a special function $F_n(x)$ can be written as:

$$f(x, y) = \sum_n F_n(x)y^n \tag{1}$$

A number of examples will be given.

(*1*) Legendre polynomials:

$$(1 - 2xy + y^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)y^n \tag{2}$$

In our presentation of Legendre polynomials, this was used as a *definition* of $P_n(x)$: eq.[2] in section 6.1.

(*2*) Associated Legendre polynomials:

$$\frac{(2m)!\,(1-x^2)^{m/2}y^m}{2^m m!\,(1-2xy+y^2)^{m+1/2}} = \sum_{n=m}^{\infty} P_n^m(x)y^n \qquad [3]$$

This is seen by differentiating eq.[2] m times. The left-hand side of [2] becomes:

$$\frac{d^m}{dx^m}(1-2xy+y^2)^{-1/2} = \frac{(2m)!\,y^m}{2^m m!\,(1-2xy+y^2)^{m+1/2}}$$

which is easy to derive by a proof from "$m-1$ to m" and to show it for the lowest orders of differentiation. After differentiating also the right-hand side of eq.[2] m times, we use the definition [13] in section 6.2. The summation in [3] extends from $n = m$ to infinity. This is clear, as $P_n(x)$ is a polynomial of degree n in x, and for $n < m$, i.e., for orders of differentiation exceeding the order of the polynomial, there will be no contribution to the right-hand side of [3].

(*3*) Bessel functions (of integral order):

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \qquad [4]$$

This is demonstrated in section 5.2.1.

(*4*) Hermite polynomials:

$$e^{x^2-(z-x)^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n \qquad [5]$$

(see section 4.4). Consider Hermite's differential equation:

$$y'' - 2xy' + 2ny = 0 \qquad [6]$$

and put:

$$y_n = \frac{1}{2\pi i}\int z^{-n-1}e^{x^2-(z-x)^2}\,dz \qquad [7]$$

taking a contour in the complex z-plane around a circle with its centre at the origin. Then, carrying out the following differentiations under the integral sign:

$$\left.\begin{aligned} \frac{dy_n}{dx} &= \frac{1}{2\pi i}\int 2z^{-n}e^{x^2-(z-x)^2}\,dz \\ \frac{d^2y_n}{dx^2} &= \frac{1}{2\pi i}\int 4z^{-n+1}e^{x^2-(z-x)^2}\,dz \end{aligned}\right| \qquad [8]$$

(each such differentiation being equivalent to multiplication by $2z$) and substituting in [6], we find:

$$y_n'' - 2xy_n' + 2ny_n = \frac{1}{2\pi i}\int (4z^2 - 4xz + 2n)e^{x^2-(z-x)^2}z^{-n-1}\,dz$$

$$= -\frac{2}{2\pi i}\int \frac{d}{dz}(z^{-n}e^{x^2-(z-x)^2})\,dz = 0 \quad [9]$$

The last integral is zero, because the parenthesis, being a single-valued function of z, if n is an integer, takes the same value at the initial and final points of the integration contour. Therefore, y_n in [7] is also a solution of Hermite's differential equation [6], like $H_n(x)$. Therefore, $H_n(x)$ is proportional to $y_n(x)$. The proportionality constant may be found by computing $H_n(0)$ and $y_n(0)$ for even n. We have by [11], section 4.4:

$$H_n(0) = \frac{(-1)^{n/2}n!}{(n/2)!} \quad [10]$$

and from [7], by the theorem of residues:

$$y_n(0) = \frac{1}{2\pi i}\int z^{-n-1}e^{-z^2}\,dz$$

$$= \frac{1}{2\pi i}\int \frac{1}{z^{n+1}}\left[1 - \frac{z^2}{1!} + \ldots + (-1)^{n/2}\frac{z^n}{(n/2)!} + \ldots\right]dz = \frac{(-1)^{n/2}}{(n/2)!} \quad [11]$$

the residue being the factor of $1/z$ in the integrand, as the pole is at $z = 0$. Thus, $H_n(x) = n!y_n(x)$, or:

$$H_n(x) = \frac{n!}{2\pi i}\int z^{-n-1}e^{x^2-(z-x)^2}\,dz \quad [12]$$

[12] gives an integral expression of Hermite's polynomial. It can be rewritten in the following way, starting from [7]:

$$y_n = \frac{1}{2\pi i}\int z^{-n-1}e^{x^2-(z-x)^2}\,dz$$

= the residue at the pole $z = 0$ (by the residue theorem)
= the factor of $1/z$ in the expansion of the integrand
= the factor of z^n in the expansion of $e^{x^2-(z-x)^2}$

With this interpretation of y_n, we get:

$$e^{x^2-(z-x)^2} = \sum_{n=0}^{\infty} y_n z^n = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}z^n \quad [13]$$

and this proves the formula [5].

(5) Laguerre polynomials:

$$(1-z)^{-1}e^{-xz/(1-z)} = \sum_{n=0}^{\infty}\frac{L_n(x)}{n!}z^n \quad [14]$$

(see section 4.5). The proof of this formula is exactly analogous to the proof of formula [5] above. As there we demonstrate that Laguerre's differential equation:

$$xy'' + (1-x)y' + ny = 0 \qquad [15]$$

in addition to $L_n(x)$ has a solution of the form:

$$y_n = \frac{1}{2\pi i}\int \frac{z^{-n-1}}{1-z}\, e^{-xz/(1-z)}\, dz \qquad [16]$$

and that the relation between the two solutions is $L_n(x) = n!\, y_n(x)$. An integral expression for $L_n(x)$ is thus:

$$L_n(x) = \frac{n!}{2\pi i}\int \frac{z^{-n-1}}{1-z}\, e^{-xz/(1-z)}\, dz \qquad [17]$$

and an application of the residue theorem yields [14]. The details of this proof are left as an exercise to the reader.

(6) Associated Laguerre polynomials:

$$(-1)^k(1-z)^{-1}\left(\frac{z}{1-z}\right)^k e^{-xz/(1-z)} = \sum_{n=k}^{\infty} \frac{L_n^k(x)}{n!} z^n \qquad [18]$$

(see section 4.5.1). Equation [18] is obtained by differentiating [14] k times, with respect to x:

$$\frac{d^k}{dx^k}\left(e^{-xz/(1-z)}\right) = (-1)^k e^{-xz/(1-z)}\left(\frac{z}{1-z}\right)^k$$

7.5.2 Sturm–Liouville theory

This theory provides a simple and elegant method to get a uniform representation of various differential equations, as of Legendre, Gauss, Bessel, Hermite, Tschebyscheff, Jacobi, Laguerre, Mathieu (see section 4.1.1).

A *differential operator* $D(u)$:

$$D(u) = fu'' + gu' + hu \qquad [19]$$

(f, g, h being functions of x) is said to be *self-adjoint* if $g = f'$. The differential operator in the Sturm–Liouville theory is:

$$L(u) = (pu')' - qu = pu'' + p'u' - qu \qquad [20]$$

and this is obviously self-adjoint. A second-order differential operator, which is not self-adjoint, can be made so by multiplication by $e^{\int \frac{g-f'}{f} dx}$:

$$D(u)\, e^{\int \frac{g-f'}{f} dx} = fu''\, e^{\int \frac{g-f'}{f} dx} + gu'\, e^{\int \frac{g-f'}{f} dx} + hu\, e^{\int \frac{g-f'}{f} dx} \qquad [21]$$

This is self-adjoint, which is seen by writing down the condition for this (i.e., $g = f'$) and carrying out the computation:

$$g\,e^{\int \frac{g-f'}{f}dx} = \left(f\,e^{\int \frac{g-f'}{f}dx}\right)' = f'e^{\int \frac{g-f'}{f}dx} + f\,e^{\int \frac{g-f'}{f}dx} \cdot \frac{g-f'}{f}$$

which becomes an identity. Thus all second-order differential operators can be made self-adjoint by multiplication with a factor F defined as:

$$F = e^{\int \frac{g-f'}{f}dx} \tag{22}$$

when the differential equation is written in its usual form.

Therefore all differential equations we have discussed in Chapter 4 can be written in self-adjoint form:

$$L(u) + \lambda wu = 0 \tag{23}$$

where λ = constant and $w = w(x) \geqslant 0$, and any theory for self-adjoint equations applies to all equations in Chapter 4 (see Table III).

Boundary conditions may also be written in a general form. For instance, if u and v are two solutions of a differential equation, and a and b are the boundaries, such that:

$$[vpu']_a = [vpu']_b \tag{24}$$

then the following theorem holds:

$$\int_a^b vL(u)dx = \int_a^b uL(v)dx \tag{25}$$

This is demonstrated by a straight-forward calculation:

$$\int_a^b vL(u)dx = \int v(pu')'dx - \int vqudx =$$

(introducing $L(u)$ from [20] and using partial integration)

$$= \underbrace{[vpu']_a^b} - \int v'pu'dx - \int vqudx =$$

($= 0$ by [24]) (again partial integration)

$$= \underbrace{-[v'pu]_a^b} + \int u(pv')'dx - \int vqudx =$$

($= 0$ by [24]; this is true as u and v are *any* solutions, and their positions in [24] can be interchanged)

and using the definition of $L(v)$:

$$= \int uL(v)dx$$

which should be proved. Then L is said to be *Hermitian* with respect to functions satisfying [24].

7.5.3 Deduction of one differential equation from another

As an alternative to section 7.5.2, where differential equations of the second order were reduced to the Sturm–Liouville differential equation, it is possible in some cases to reduce one equation to another (see 4.3.2, where a relation between *Legendre's and Bessel's* differential equations is demonstrated).

FORSYTH (1912, pp.201–203) similarly demonstrates a relation between *Bessel's and Riccati's* differential equations. Details will not be given here.

7.5.4 Partial differential equations of mathematical physics

The equations dealt with so far in this section are *total* differential equations of the *second order* of special types. They are total or ordinary, i.e., there is only one independent variable. They are of second order, i.e., they include derivatives up to the second order, except Riccati's equation, which is of the first order. The coefficients are generally functions of the independent variable x. The equations are *linear*, again except for Riccati's.

As distinct from these, the differential equations of mathematical physics, corresponding to certain fundamental physical phenomena, are *partial* differential equations (i.e., there is more than one independent variable) and they are usually of the *second order*. They are usually *linear*, i.e., the dependent variable exists only in its first power. The coefficients are usually *constant*.

Also the differential equations of mathematical physics are of a few types, closely related to each other (see section 1.1).

The connection between the two groups of differential equations is this: In the solution of an equation of mathematical physics (including its boundary conditions), it may be possible to reduce the given equation to one or several of the standard forms (Bessel, Legendre, etc.), and then it is an easy matter to write down the solution. The main difficulty may be just in the reduction of a problem into any of the given standard forms. The wave equation studied earlier in this chapter illustrates the general procedure.

Chapter 8

INTEGRAL TRANSFORMS

8.1 INTRODUCTION TO LAPLACE AND FOURIER TRANSFORMS

8.1.1 Definition of integral transforms

The integral transform $\bar{f}(p)$ of a function $f(x)$ is *defined* by the integral equation:

$$\bar{f}(p) = \int_a^b f(x)K(p, x)\mathrm{d}x \qquad [1]$$

$K(p, x)$ is called the *kernel* of the transform. Different transforms differ by the appearance of the function K. If the integral limits a, b are both finite, we call $\bar{f}(p)$ the *finite transform* of $f(x)$. In the following transforms, $a = 0$ or $-\infty$ and $b = +\infty$.

Laplace transform:

$$\bar{f}(p) = \int_0^\infty f(x)\mathrm{e}^{-px}\mathrm{d}x \qquad [2]$$

Fourier sine and cosine transform:

$$\bar{f}(p) = \int_0^\infty f(x) \begin{matrix}\sin\\ \cos\end{matrix}\, px\, \mathrm{d}x \qquad [3]$$

Complex Fourier transform:

$$\bar{f}(p) = \int_{-\infty}^\infty f(x)\mathrm{e}^{ipx}\mathrm{d}x \qquad [4]$$

Hankel transform (also called Fourier–Bessel transform):

$$\bar{f}(p) = \int_0^\infty f(x)xJ_n(px)\mathrm{d}x \qquad [5]$$

Mellin transform:

$$\bar{f}(p) = \int_0^\infty f(x)x^{p-1}\mathrm{d}x \qquad [6]$$

We shall be concerned mainly with the Laplace and Fourier transforms. The transforms are of great use in solving partial differential equations. Applying a transform to a partial differential equation makes it possible to temporarily exclude one of the independent variables and to leave for solution a partial differential equation in one less variable. The solution of this equation will be a function of p and the remaining variables.

This solution has to be "inverted" to recover the lost variable x. The transforms, giving $\bar{f}(p)$, have to be inverted to obtain $f(x)$. *Inversion formulas* solve the transforms above for $f(x)$, and such formulas can be obtained from Fourier's integral formula.

8.1.2 Fourier's integral formula

A function $f(x)$, of period $2\pi\lambda$, is given by the Fourier series (DE LA VALLÉE POUSSIN 1937, p.91):

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{nx}{\lambda} + b_n \sin \frac{nx}{\lambda} \right] \qquad [7]$$

The coefficients a_0, a_n, b_n are obtained by multiplying [7] successively by 1, $\cos(nx/\lambda)$, $\sin(nx/\lambda)$ and integrating with respect to x between $-\pi\lambda$ and $+\pi\lambda$. Since the circular functions are orthogonal (DE LA VALLÉE POUSSIN, 1938, p.224), i.e.:

$$\int_{-\pi\lambda}^{\pi\lambda} \begin{matrix} \sin \\ \cos \end{matrix} \frac{nx}{\lambda} \, dx = 0$$

$$\int_{-\pi\lambda}^{\pi\lambda} \begin{matrix} \sin \\ \cos \end{matrix} \frac{nx}{\lambda} \begin{matrix} \cos \\ \sin \end{matrix} \frac{mx}{\lambda} \, dx = 0$$

$$\int_{-\pi\lambda}^{\pi\lambda} \begin{matrix} \sin \\ \cos \end{matrix} \frac{nx}{\lambda} \begin{matrix} \sin \\ \cos \end{matrix} \frac{mx}{\lambda} \, dx \begin{cases} = 0 \ , & m \neq n \\ = \pi\lambda & m = n \end{cases}$$

this process yields:

$$\pi\lambda a_0 = \int_{-\pi\lambda}^{\pi\lambda} f(x')dx'$$

$$\pi\lambda a_n = \int_{-\pi\lambda}^{\pi\lambda} f(x') \cos \frac{nx'}{\lambda} \, dx'$$

$$\pi\lambda b_n = \int_{-\pi\lambda}^{\pi\lambda} f(x') \sin \frac{nx'}{\lambda} \, dx'$$

Hence, introducing the values of a_0, a_n, b_n into the series $f(x)$, we get:

$$f(x) = \frac{1}{2\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(x')dx' + \frac{1}{\pi\lambda} \sum_{n=1}^{\infty} \int_{-\pi\lambda}^{\pi\lambda} f(x') \cos \frac{n(x - x')}{\lambda} \, dx' \qquad [8]$$

Put $n/\lambda = \alpha$ and take λ very large. Then the first term will tend to zero if $\int_{-\infty}^{\infty} f(x')dx'$

converges. Moreover, consecutive values of α differ by $1/\lambda$, i.e., a small quantity. If we put $1/\lambda = \mathrm{d}\alpha$, the sum passes into an integral:

$$\pi f(x) = \int_0^\infty \mathrm{d}\alpha \int_{-\infty}^{\infty} f(x') \cos[\alpha(x - x')] \, \mathrm{d}x' \qquad [9]$$

This is *Fourier's integral formula.* For a more complete discussion of the conditions under which [9] is valid, the reader is referred to DE LA VALLÉE POUSSIN (1937, chapter 4). For instance, [9] holds for functions $f(x)$ having only a finite number of maxima, minima and ordinary discontinuities or having a bounded variation in any finite interval, in addition to the condition that $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ is absolutely convergent.

8.1.3 Inversion formulas

Inversion formula for the Laplace transform

Fourier's integral formula [9] can be written:

$$\pi f(x) = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} f(x') \cos[\alpha(x - x')] \, \mathrm{d}x' \qquad [10]$$

(integration over α has been extended over twice the original interval, which is compensated by division by 2). Now:

$$\int_{-\infty}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} f(x') \sin[\alpha(x - x')] \, \mathrm{d}x' = 0 \qquad [11]$$

as terms with positive and negative α will cancel each other (sine of a negative argument is negative). Then, Fourier's integral formula can be written:

$$2\pi f(x) = \int_{-\infty}^{\infty} \mathrm{e}^{i\alpha x}\mathrm{d}\alpha \int_{-\infty}^{\infty} f(x')\mathrm{e}^{-i\alpha x'}\mathrm{d}x' \qquad [12]$$

as:

$$\mathrm{e}^{i\alpha(x-x')} = \cos[\alpha(x - x')] + i \sin[\alpha(x - x')] \qquad \text{(Euler's formula)}$$

From the definition of the Laplace transform [2] we have:

$$\int_{\gamma-i\omega}^{\gamma+i\omega} \mathrm{e}^{xp}\bar{f}(p)\mathrm{d}p = \int_{\gamma-i\omega}^{\gamma+i\omega} \mathrm{e}^{xp}\mathrm{d}p \int_0^\infty f(x')\mathrm{e}^{-px'} \, \mathrm{d}x'$$

$$= i\mathrm{e}^{\gamma x} \int_{-\omega}^{\omega} \mathrm{e}^{ixy}\mathrm{d}y \int_0^\infty \mathrm{e}^{-iyx'}[\mathrm{e}^{-\gamma x'} f(x')] \, \mathrm{d}x' \qquad [13]$$

where we have replaced p by $p = \gamma + iy$, i.e., $\mathrm{d}p = i\,\mathrm{d}y$. Let $\omega \to \infty$; then the right-hand side of [13] becomes, because of eq.[12]:

$$= i\mathrm{e}^{\gamma x} \cdot 2\pi[\mathrm{e}^{-\gamma x} f(x)] \qquad \text{for } x > 0$$

seen immediately from [2a] as p has the same dimension as $1/x$). Eq.[2a] is called the *Carson integral equation* for $f(x)$. It will be used in Cagniard's method (section 8.4). (See also JEFFREYS and JEFFREYS, 1946, pp.429–430.)

Inversion formula for the Hankel (or Fourier–Bessel) transform

The formulas above can be extended to any number of variables, provided that:

$$\int |f(x, y, z, \ldots)| \mathrm{d}x\mathrm{d}y\mathrm{d}z \ldots$$

exists. In particular, we can write [23] and [24] for two variables x and y as follows:

$$\left.\begin{aligned} \bar{f}(s, t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\mathrm{e}^{i(sx+ty)}\mathrm{d}x\mathrm{d}y \\ 4\pi^2 f(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(s, t)\mathrm{e}^{-i(xs+yt)}\mathrm{d}s\mathrm{d}t \end{aligned}\right| \quad [25]$$

In the last expression the left-hand side should be multiplied by $(2\pi)^n$ where n is the number of variables. Compare [28] in section 7.2 and [4] in section 7.3. We replace x and y by r and θ, and we also replace s and t by p and α:

$$x = r\cos\theta \ ; \ s = p\cos\alpha$$

$$y = r\sin\theta \ ; \ t = p\sin\alpha$$

Note that the integrations in [25], extending over x and y from $-\infty$ to $+\infty$, correspond to r extending from 0 to $+\infty$ and to θ, extending from 0 to 2π (in both cases we cover the whole xy-plane once). Then [25] transforms into:

$$\left.\begin{aligned} \bar{f}(p, \alpha) &= \int_0^{\infty} r\mathrm{d}r \int_0^{2\pi} f(r, \theta)\mathrm{e}^{ipr\cos(\theta-\alpha)}\mathrm{d}\theta \\ 4\pi^2 f(r, \theta) &= \int_0^{\infty} p\mathrm{d}p \int_0^{2\pi} \bar{f}(p, \alpha)\mathrm{e}^{-ipr\cos(\theta-\alpha)}\mathrm{d}\alpha \end{aligned}\right| \quad [26]$$

Then we start from [26.1] assuming that $f(r, \theta) = f(r)\mathrm{e}^{-in\theta}$:

$$\bar{f}(p, \alpha) = \int_0^{\infty} f(r)r\mathrm{d}r \int_0^{2\pi} \mathrm{e}^{i[-n\theta+pr\cos(\theta-\alpha)]}\mathrm{d}\theta$$

$$= \int_0^{\infty} f(r)r\mathrm{d}r \cdot \mathrm{e}^{in(\pi/2-\alpha)} \int_0^{2\pi} \mathrm{e}^{i(n\varphi-pr\sin\varphi)}\mathrm{d}\varphi =$$

(introducing φ defined by $\varphi = \alpha - \theta - \pi/2$)

$$= \int_0^{\infty} f(r)r\mathrm{d}r \cdot 2\pi\mathrm{e}^{in(\pi/2-\alpha)}J_n(pr) =$$

(using eq.[10] in section 5.1)

$$= 2\pi\mathrm{e}^{in(\pi/2-\alpha)} \int_0^{\infty} f(r)rJ_n(pr)\mathrm{d}r = 2\pi\mathrm{e}^{in(\pi/2-\alpha)}\bar{f}(p) \quad [27]$$

(by the definition [5] of the Hankel transform)
Then, substituting $f(r)\,\mathrm{e}^{-in\theta}$ for $f(r, \theta)$ and the expression [27] for $\bar{f}(p, \alpha)$ in [26.2], we find:

$$2\pi f(r)\mathrm{e}^{-in\theta} = \int_0^\infty p\bar{f}(p)\mathrm{d}p \int_0^{2\pi} \mathrm{e}^{i[n(\pi/2-\alpha)-pr\cos(\theta-\alpha)]}\mathrm{d}\alpha$$

$$= \int_0^\infty p\bar{f}(p)\mathrm{d}p \cdot \mathrm{e}^{-in\theta} \int_0^{2\pi} \mathrm{e}^{i(n\varphi-pr\sin\varphi)}\mathrm{d}\varphi =$$

(introducing φ, now defined by $\varphi = \theta - \alpha + \pi/2$)

$$= \int_0^\infty p\bar{f}(p)\mathrm{d}p \cdot 2\pi\mathrm{e}^{-in\theta}J_n(pr)$$

TABLE IV

LAPLACE TRANSFORMS, EQ.[2]

$\bar{f}(p)$	$f(x)$
p^{-n-1}	$\dfrac{x^n}{\Gamma(n+1)}$, $n > -1$
$(p+\omega)^{-1}$	$\mathrm{e}^{-\omega x}$
$\dfrac{\omega}{p^2+\omega^2}$	$\sin\omega x$
$\dfrac{p}{p^2+\omega^2}$	$\cos\omega x$
$\dfrac{\omega}{p^2-\omega^2}$	$\sinh\omega x$, $p > \|\omega\|$
$\dfrac{p}{p^2-\omega^2}$	$\cosh\omega x$, $p > \|\omega\|$
$\dfrac{\mathrm{e}^{-a\sqrt{p}}}{p}$	* $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{x}}\right) = \dfrac{2}{\sqrt{\pi}} \int_{\frac{a}{2\sqrt{x}}}^\infty \mathrm{e}^{-u^2}\mathrm{d}u$, $a > 0$
$\dfrac{\mathrm{e}^{-a\sqrt{p}}}{p^{3/2}}$	** $2\sqrt{x}\ \operatorname{ierfc}\left(\dfrac{a}{2\sqrt{x}}\right)$, $a > 0$

* $\operatorname{erfc} x = 1 - \operatorname{erf} x = \dfrac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-u^2}\mathrm{d}u$

(cf. eq.[14] in section 1.3; erfc is called the complementary error function).

** $\operatorname{ierfc} x = \int_x^\infty \operatorname{erfc} u\, \mathrm{d}u$

$$\int_0^{\infty} e^{-pt} \frac{\partial T}{\partial t} dt = [e^{-pt}T]_{t=0}^{\infty} + p \int_0^{\infty} e^{-pt}T dt = p\bar{T} \tag{38}$$

Multiply [36.1] and [36.2] by the kernel, e^{-pt}, integrate from 0 to ∞, use [37] and [38], and we have the auxiliary equations:

$$\left.\begin{aligned} \varkappa \frac{d^2\bar{T}}{dx^2} &= p\bar{T} && (x > 0) \\ \bar{T} &= \int_0^{\infty} e^{-pt}T_0 dt = \frac{T_0}{p} && (x = 0) \end{aligned}\right| \tag{39}$$

The problem has thus been reduced to the solution of an *ordinary* differential equation [39.1]. The solution of [39.1] satisfying the values at $x = 0$ and $x = \infty$ is:

$$\bar{T} = \frac{T_0}{p} e^{-x\sqrt{(p/\varkappa)}}$$

Inversion to T either from the inversion formula [14] or Table IV gives:

$$T = T_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{(\varkappa t)}}\right) \tag{40}$$

the solution of our problem.

In most cases we do not find the solution of the inversion in any table, and then we have to apply the appropriate inversion formula and to solve the integral by contour integration in the complex plane. This will be discussed in section 8.2.

8.1.5 Application of Fourier transforms

Sine and cosine transforms

These are used in the same way as the Laplace transform for the solution of differential equations. The choice of sine or cosine transform is decided by the form of the boundary conditions at the lower limit of the variable selected for exclusion. Suppose T is the wanted function and we remove $\partial^2T/\partial x^2$ by a sine transform:

$$\int_0^{\infty} \sin px \frac{\partial^2 T}{\partial x^2} dx = \left[\frac{\partial T}{\partial x} \sin px\right]_{x=0}^{\infty} - p \int_0^{\infty} \cos px \frac{\partial T}{\partial x} dx \tag{41}$$

The first term on the right-hand side will vanish, if $\partial T/\partial x = 0$ for $x \to \infty$. Another partial integration, performed on the second term on the right-hand side of [41], gives:

$$\int_0^{\infty} \sin px \frac{\partial^2 T}{\partial x^2} dx = -p[T \cos px]_{x=0}^{\infty} - p^2 \int_0^{\infty} T \sin px\, dx \tag{42}$$

If also $T = 0$ for $x \to \infty$:

$$\int_0^\infty \sin px \frac{\partial^2 T}{\partial x^2} \mathrm{d}x = p\,[T]_{x=0} - p^2 \bar{T} \qquad [43]$$

$\bar{T}$ is the transform of T. Similarly, under the same assumptions, but with $\bar{T}$ now being the cosine transform:

$$\int_0^\infty \cos px \frac{\partial^2 T}{\partial x^2} \mathrm{d}x = -\left[\frac{\partial T}{\partial x}\right]_{x=0} - p^2 \bar{T} \qquad [44]$$

We conclude, that for sine transform we need to know T for $x = 0$ and for cosine transform we need to know $\partial T/\partial x$ for $x = 0$.

The heat conduction problem of section 8.1.4, solved by the sine transform
The problem to solve is the following:

$$\frac{\partial T}{\partial t} = \varkappa \frac{\partial^2 T}{\partial x^2} \qquad \begin{array}{l} x > 0 \ ; \ T = T_0 \ , \ x = 0 \ , \ t > 0 \\ t > 0 \ ; \ T = 0 \ , \ t = 0 \ , \ x > 0 \end{array} \qquad [45]$$

Use the sine transform, which is appropriate when T is given for $x = 0$. Multiply the differential equation and the initial condition by $\sin px$, integrate between 0 and ∞ with respect to x, and apply eq.[43]; we have as both T and $\partial T/\partial x$ tend to 0 as $x \to \infty$:

$$\frac{\mathrm{d}\bar{T}}{\mathrm{d}t} = \varkappa(pT_0 - p^2\bar{T}) \ , \qquad t > 0 \ ; \ \bar{T} = 0 \ , \ t = 0 \qquad [46]$$

The problem is again reduced to the solution of an *ordinary* differential equation. The solution, finite for $t > 0$ and satisfying the condition $\bar{T} = 0$ for $t = 0$, is:

$$\bar{T} = \frac{T_0}{p}(1 - \mathrm{e}^{-p^2\varkappa t}) \qquad [47]$$

The inversion formula [18] gives:

$$T = \frac{2T_0}{\pi} \int_0^\infty (1 - \mathrm{e}^{-p^2\varkappa t}) \sin xp \frac{\mathrm{d}p}{p} \qquad [48]$$

Split up the integral into its two terms. The first term is immediately evaluated, since:

$$\int_0^\infty \sin xp \frac{\mathrm{d}p}{p} = \frac{\pi}{2}$$

and we have:

$$T = T_0 \left[1 - \frac{2}{\pi} \int_0^\infty \mathrm{e}^{-p^2\varkappa t} \sin xp \frac{\mathrm{d}p}{p}\right] = T_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{(\varkappa t)}}\right) \qquad [49]$$

This is seen to be correct by applying the following formula, which can be derived from another integral formula (see DWIGHT, 1957, p.201, formula 863.3, or WEAST, 1964, p.325, formula 432; cf. section 8.2.7):

Finite Fourier transforms

We define the finite Fourier sine transform by:

$$\bar{f}(p) = \int_0^{\pi} f(x) \sin px \, dx \qquad [60]$$

p = positive integer. An upper limit $= \pi$ can usually be obtained in any particular case by substitution.

The ordinary Fourier series theory can be used to find the *inversion formula*. If $f(x)$ can be expanded in a sine series, the coefficient a_p of $\sin px$ is (derivation as in section 8.1.2):

$$a_p = \frac{2}{\pi} \int_0^{\pi} f(x) \sin px \, dx = \frac{2}{\pi} \bar{f}(p) \qquad [61]$$

using definition [60]. Hence, the inversion formula:

$$f(x) = \sum_{p=1}^{\infty} a_p \sin px = \frac{2}{\pi} \sum_{p=1}^{\infty} \bar{f}(p) \sin px \qquad [62]$$

Similarly, the finite cosine transform is defined:

$$\bar{f}(p) = \int_0^{\pi} f(x) \cos px \, dx \qquad [63]$$

p positive integer or 0, and, similarly, we find the inversion formula:

$$f(x) = \frac{1}{\pi} \bar{f}(0) + \frac{2}{\pi} \sum_{p=1}^{\infty} \bar{f}(p) \cos px \qquad [64]$$

with:

$$\bar{f}(0) = \int_0^{\pi} f(x) dx$$

i.e., $p = 0$ in [63].

Suppose, as an example, that we have a term $\partial^2 T/\partial x^2$ in a differential equation and apply the finite Fourier sine transform. By successive partial integrations, we find:

$$\int_0^{\pi} \sin px \frac{\partial^2 T}{\partial x^2} dx = \underbrace{\left[\frac{\partial T}{\partial x} \sin px\right]_{x=0}^{\pi}}_{=0} - p \int_0^{\pi} \cos px \frac{\partial T}{\partial x} dx$$

$$\left(\frac{\partial T}{\partial x} \text{ assumed finite}\right)$$

$$= -p[T \cos px]_{x=0}^{\pi} - p^2 \int_0^{\pi} T \sin px \, dx = p[T_0 - (-1)^p T_\pi] - p^2 \bar{T} \qquad [65]$$

Similarly, if the finite cosine transform is used, provided T is finite for $x = 0$:

$$\int_0^\pi \cos px \frac{\partial^2 T}{\partial x^2} \mathrm{d}x = (-1)^p \left(\frac{\partial T}{\partial x}\right)_\pi - \left(\frac{\partial T}{\partial x}\right)_0 - p^2 \bar{T} \qquad [66]$$

Use of finite sine (cosine) transforms thus requires knowledge of T (or $\partial T/\partial x$ respectively) at both extremities.

An example of the use of the finite sine transform

Consider the problem of the *steady* temperature in a long square bar when one face

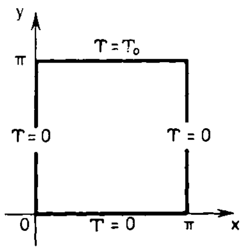


Fig.56.

is kept at constant temperature $= T_0$ and the other faces at zero temperature. Fig.56 shows the quadratic cross-section of the bar. The differential equation is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad 0 < x < \pi \ , \quad 0 < y < \pi \qquad [67]$$

with the boundary conditions:

$$\left.\begin{aligned} &T = 0 \quad \text{for } x = 0 \ , \quad x = \pi \ , \quad y = 0 \\ &T = T_0 \ \text{for } y = \pi \quad (T_0 = \text{constant}) \end{aligned}\right| \qquad [68]$$

As T is given at both ends of a *finite* interval of x, we use the *finite sine* transform:

$$\bar{T} = \int_0^\pi T \sin px \mathrm{d}x \ , \ p \text{ positive integer} \qquad [69]$$

Eq.[65] then gives, as $T = 0$ for $x = 0$ and $x = \pi$:

$$\int_0^\pi \frac{\partial^2 T}{\partial x^2} \sin px \, \mathrm{d}x = -p^2 \bar{T} \qquad [70]$$

Multiply [67] by $\sin px$ and integrate with respect to x from 0 to π, taking [70] into account:

$$\frac{\mathrm{d}^2 \bar{T}}{\mathrm{d}y^2} - p^2 \bar{T} = 0 \qquad [71]$$

Similarly, for the boundary conditions [68]:

refer mainly to Heaviside's habit of assuming that rules of calculus, found in special cases, were of general validity (frequently without proving that this was so).

8.2.1 Ordinary linear differential equations with constant coefficients

Given is the following differential equation:

$$\frac{\mathrm{d}^n x}{\mathrm{d}t^n} + a_1 \frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}} + \ldots + a_{n-1}\frac{\mathrm{d}x}{\mathrm{d}t} + a_n x = f(t) \qquad t > 0 \qquad [2]$$

with $a_1, a_2, \ldots$ constant and:

$$x = x_0\,, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = x_1\,, \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = x_2\,, \ldots, \quad \frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}} = x_{n-1}\,, \quad t = 0$$

We multiply [2] by e^{-pt} (p positive and real) and integrate:

(*1*)

$$\int_0^\infty \mathrm{e}^{-pt}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \underbrace{[\mathrm{e}^{-pt}x]_0^\infty}_{= -x_0} + p\int_0^\infty \mathrm{e}^{-pt}x\mathrm{d}t$$

[if $\lim\limits_{t\to\infty}(\mathrm{e}^{-pt}x) = 0$; this implies a certain restricting assumption on the solution $x(t)$]

(*2*)

$$\int_0^\infty \mathrm{e}^{-pt}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathrm{d}t = \underbrace{\left[\mathrm{e}^{-pt}\frac{\mathrm{d}x}{\mathrm{d}t}\right]_0^\infty}_{= -x_1} + p\int_0^\infty \mathrm{e}^{-pt}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t =$$

[if $\lim\limits_{t\to\infty}\left(\mathrm{e}^{-pt}\dfrac{\mathrm{d}x}{\mathrm{d}t}\right) = 0$; this also implies a restriction on $x(t)$]

$$= -(px_0 + x_1) + p^2\int_0^\infty \mathrm{e}^{-pt}x\mathrm{d}t$$

in every case assuming that p is so large that the integrals on the right-hand sides exist, i.e., that they are convergent. For a general term we find:

(*3*)

$$\int_0^\infty \mathrm{e}^{-pt}\frac{\mathrm{d}^r x}{\mathrm{d}t^r}\mathrm{d}t = -(p^{r-1}x_0 + p^{r-2}x_1 + \ldots + px_{r-2} + x_{r-1})$$

$$+ p^r\int_0^\infty \mathrm{e}^{-pt}x\mathrm{d}t \qquad r \leqslant n \qquad [3]$$

Carrying out the operations [3] on [2] we arrive at the auxiliary or subsidiary equation, where:

$$h(p) = p^n + a_1 p^{n-1} + \ldots + a_{n-1}p + a_n, \text{ i.e.:}$$

$$\begin{aligned} h(p) \int_0^\infty \mathrm{e}^{-pt} x \mathrm{d}t = {} & (p^{n-1}x_0 + p^{n-2}x_1 + \ldots + px_{n-2} + x_{n-1}) \\ & + a_1(p^{n-2}x_0 + p^{n-3}x_1 + \ldots + px_{n-3} + x_{n-2}) \\ & + \ldots \\ & + a_{n-2}(px_0 + x_1) \\ & + a_{n-1}x_0 \\ & + \int_0^\infty \mathrm{e}^{-pt} f(t) \mathrm{d}t \end{aligned} \qquad [4]$$

which has to be solved for x. A rule in writing down [4] is that the sum of the index of a, the exponent of p and the index of x, in each term is $= n - 1$.

If $f(t)$ has certain simple forms (e.g., zero, a constant, e^{at}, $\sin bt$, $\cos bt$, t^r with r a positive integer, or a product of any two of these forms), then the right-hand side of [4] becomes a polynomial in p, and we can divide [4] by $h(p)$:

$$\overline{x}(p) = \int_0^\infty \mathrm{e}^{-pt} x \mathrm{d}t = \frac{g(p)}{h(p)} = \sum_{r=1}^{n} \frac{g(\alpha_r)}{(p - \alpha_r)h'(\alpha_r)} \qquad [5]$$

where $g(p)$ and $h(p)$ are polynomials in p, the degree of the former being at least one less than that of the latter. The right-hand side of [5] is split into its partial fractions (assuming $h(p)$ to have n zeroes α_r, *all different*; see section 2.1.4). $\overline{x}(p)$ is the Laplace transform of $x(t)$, by definition. From [5] we get:

$$x(t) = \sum_{r=1}^{n} \frac{g(\alpha_r)}{h'(\alpha_r)} \mathrm{e}^{\alpha_r t} \qquad [6]$$

Eq.[6] is found immediately by substituting this expression into eq.[5]. This procedure of finding $x(t)$, when $\overline{x}(p)$ is given, avoids the use of the inversion formula for the Laplace transform. The method can easily be extended to the case of repeated zeroes of $h(p)$ and to complex roots.

It remains to be verified that the solution [6] is the general solution of [2] without the restrictive assumptions on $x(t)$ made in deriving [3]. Fortunately, in most applications, the restrictions are fulfilled.

The method can easily be extended to *systems* of simultaneous ordinary differential equations with constant coefficients.

8.2.2 *Some theorems for the Laplace transform*

We have seven important theorems:

(*1*) If $\bar{x}_1(p)$ and $\bar{x}_2(p)$ are the transforms of $x_1(t)$ and $x_2(t)$, then $\bar{x}_1(p) \pm \bar{x}_2(p)$ is the transform of $x_1(t) \pm x_2(t)$.

(*2*) If $\bar{x}(p)$ is the transform of $x(t)$ and $\lim\limits_{t\to\infty} e^{-pt}x(t) = 0$, then $p\bar{x}(p) - x(0)$ is the transform of dx/dt.

(*3*) If $\bar{x}(p)$ is the transform of $x(t)$ and $\lim\limits_{t\to\infty} e^{-pt} \int\limits_0^t x(\tau)d\tau = 0$, then $\dfrac{1}{p}\bar{x}(p)$ is the transform of $\int\limits_0^t x(\tau)d\tau$.

(*4*) If $\bar{x}(p)$ is the transform of $x(t)$, and $p + a > 0$, then $\bar{x}(p + a)$ is the transform of $e^{-at}x(t)$.

(*5*) If $\bar{x}(p)$ is the transform of $x(t)$, then $e^{-ap}\bar{x}(p)$, $a > 0$, is the transform of the function $X(t)$, where:

$$\left.\begin{aligned} X(t) &= 0 \qquad && 0 < t < a \\ &= x(t-a) && t > a \end{aligned}\right\}$$

(note that x is a function of $t - a$, not multiplied with $t - a$).

(*6*) If $\bar{x}_1(p)$ and $\bar{x}_2(p)$ are the transforms of $x_1(t)$ and $x_2(t)$, then $\bar{x}_1(p)\bar{x}_2(p)$ is the transform of:

$$\int\limits_0^t x_1(\tau)x_2(t-\tau)d\tau$$

and this is equal to:

$$\int\limits_0^t x_1(t-\tau)x_2(\tau)d\tau$$

(*7*) If two continuous functions $x_1(t)$ and $x_2(t)$ have the same Laplace transform $\bar{x}(p)$, then they are identically equal.

Proofs of the theorems *1–7* follow here.

(*Ad 1*) We have:

$$\bar{x}_1(p) = \int\limits_0^\infty e^{-pt}x_1(t)dt$$

$$\bar{x}_2(p) = \int\limits_0^\infty e^{-pt}x_2(t)dt$$

Then:

$$\bar{x}_1(p) \pm \bar{x}_2(p) = \int\limits_0^\infty e^{-pt}[x_1(t) \pm x_2(t)]dt$$

which should be proved.

(*Ad 2*) We start from the following expression:

$$\int_0^\infty e^{-pt}\frac{dx}{dt}dt = [e^{-pt}x(t)]_0^\infty + p\int_0^\infty e^{-pt}x(t)dt =$$

(by partial integration)

$$= -x(0) + p\bar{x}(p)$$

which should be proved.

(*Ad 3*) Again we start from the given expression and integrate by parts:

$$\int_0^\infty e^{-pt}\left[\int_0^t x(\tau)d\tau\right]dt = -\underbrace{\left[\frac{1}{p}e^{-pt}\int_0^t x(\tau)d\tau\right]_0^\infty}_{=0}$$

$$+\frac{1}{p}\int_0^\infty e^{-pt}\underbrace{\frac{d}{dt}\left[\int_0^t x(\tau)d\tau\right]}dt = \frac{1}{p}\int_0^\infty e^{-pt}x(t)dt =$$

(= x, applying the rule for differentiation with regard to the upper limit of a definite integral; see DE LA VALLÉE POUSSIN, 1938, p.214)

$$= \frac{1}{p}\bar{x}(p)$$

which should be proved.

(*Ad 4*) Again we start from the given expression:

$$\int_0^\infty e^{-pt}e^{-at}x(t)dt = \int_0^\infty e^{-(p+a)t}x(t)dt = \bar{x}(p+a)$$

(this integral exists, i.e., is convergent, if $p + a > 0$)

which should be proved.

(*Ad 5*) We start from the given expression:

$$\int_0^\infty e^{-pt}X(t)dt = \int_a^\infty e^{-pt}x(t-a)dt = \int_0^\infty e^{-p(t+a)}x(t)dt =$$

(use $t - a$ as variable instead of t)

$$= e^{-ap}\bar{x}(p)$$

which should be proved.

(*Ad 6*) Let:

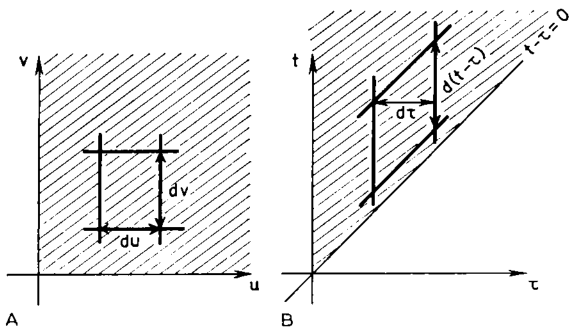


Fig.57.

$$\left.\begin{aligned}\bar{x}_1(p) &= \int_0^\infty e^{-pu}x_1(u)\mathrm{d}u \\ \bar{x}_2(p) &= \int_0^\infty e^{-pv}x_2(v)\mathrm{d}v\end{aligned}\right\} \quad (u \text{ and } v \text{ are only used as integration variables})$$

converge absolutely for $p > 0$. Then:

$$\bar{x}_1(p)\bar{x}_2(p) = \int_0^\infty e^{-pu}x_1(u)\mathrm{d}u \int_0^\infty e^{-pv}x_2(v)\mathrm{d}v$$

$$= \int_0^\infty \int_0^\infty e^{-p(u+v)}x_1(u)x_2(v)\mathrm{d}u\mathrm{d}v$$

on account of the absolute convergence. The double integral is taken over the quadrant $u > 0$ and $v > 0$ (Fig.57) and it is equal to:

$$\int_0^\infty \int_0^\infty e^{-pt}x_1(t-\tau)x_2(\tau)\mathrm{d}t\mathrm{d}\tau$$

This is seen from the substitution: $u + v = t$ and $v = \tau$, the integral now being taken over the hatched area in the (t, τ) plane $[\mathrm{d}u\mathrm{d}v \rightarrow \mathrm{d}(t-\tau)\mathrm{d}\tau \rightarrow \mathrm{d}t\mathrm{d}\tau]$. The integral becomes:

$$= \int_0^\infty e^{-pt}\left[\int_0^t x_1(t-\tau)x_2(\tau)\mathrm{d}\tau\right]\mathrm{d}t$$

$$= \int_0^\infty e^{-pt}\left[\int_0^t x_1(\tau)x_2(t-\tau)\mathrm{d}\tau\right]\mathrm{d}t$$

(instead substituting: $u + v = t$, $u = \tau$). And this is what should be proved. (*3*) is a special case of (*6*), namely for $x_1(t) = 1$.

(*7*) is proved as a special case of Lerch's theorem (not given here). Theorem (*7*) is the uniqueness theorem for Laplace transforms.

We summarize the theorems as follows:

If $\bar{x}(p)$ is the transform of $x(t)$ and:

(*2*) if $\lim_{t\to\infty} e^{-pt}x(t) = 0$, then $p\bar{x}(p) - x(0)$ is the transform of $\dfrac{dx}{dt}$

(*3*) if $\lim_{t\to\infty} e^{-pt} \int_0^t x(\tau)d\tau = 0$, then $\dfrac{1}{p}\bar{x}(p)$ is the transform of $\int_0^t x(\tau)d\tau$

(*4*) if $p + a > 0$, then $\bar{x}(p+a)$ is the transform of $e^{-at}x(t)$

(*5*) if $\begin{cases} a > 0 \\ X(t) = 0, \; 0 < t < a \\ \quad\;\; = x(t-a), \; t > a \end{cases}$ then $e^{-ap}\bar{x}(p)$ is the transform of $X(t)$

The sum (or difference) and the product of two transforms are themselves transforms given by (*1*) and (*6*), respectively. (*7*) finally is the uniqueness theorem.

A suitable exercise would be to develop similar rules for the other transforms (Fourier transform, etc.).

8.2.3 The inversion theorem for the Laplace transformation. Its evaluation by contour integration in the complex plane

The inversion theorem for the Laplace transformation is as follows.

If:

$$\bar{x}(p) = \int_0^\infty e^{-pt}x(t)dt \;, \quad \mathrm{Re}\,p > 0$$

then: [7]

$$x(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\bar{x}(p)dp$$

This offers another method to find the solution $x(t)$, i.e., an alternative method to the one used in section 8.2.1. γ is a constant greater than the real part of all the singularities of $\bar{x}(p)$. Eq.[7.2] was proved in section 8.1.3, eq.[14].

The integral in the inversion formula [7.2] is usually evaluated by application of the calculus of residues to a contour in the complex p-plane (Fig.58). Eq.[7] is integrated from A to B; this path is completed by the circle Γ of large radius R.

We shall prove the following *lemma:* If $|f(p)| < CR^{-k}$ when $p = Re^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$, $R > R_0$, and where R_0, C, k are constants and $k > 0$, then $I = \int e^{pt}f(p)dp$ taken over the arcs $BB'C$ and $AA'C$ of the circle Γ of radius R tends to zero as $R \to \infty$, provided $t > 0$.

Proof. We have:

$$p = Re^{i\theta} = R(\cos\theta + i\sin\theta)$$

$$|e^{pt}| = |e^{tR\cos\theta}|\underbrace{|e^{tiR\sin\theta}|}_{=1} = |e^{tR\cos\theta}|$$

$$dp = iRe^{i\theta}d\theta \ ; \quad |dp| = |Rd\theta|$$

On BB':

$$\theta \to \alpha = \cos^{-1}\left(\frac{\gamma}{R}\right)$$

$$|e^{pt}| = |e^{tR\cos\theta}| < e^{tR\cos\alpha} = e^{\gamma t} \quad (\alpha < \theta)$$

Thus the absolute value of the integral (I) over BB' is:

$$|I_{BB'}| < CR^{-k+1}e^{\gamma t}\int_{\alpha}^{\pi/2} d\theta = CR^{-k+1}e^{\gamma t}\sin^{-1}\left(\frac{\gamma}{R}\right)$$

Thus in the limit:

$$\lim_{R\to\infty}|I_{BB'}| = 0$$

because for $R \to \infty$, $\sin^{-1}\left(\frac{\gamma}{R}\right) \to 0$ always, but $R^{-k+1} \to 0$ only if $k > 1$. The case $0 < k < 1$ therefore needs a special investigation, using l'Hospital's rules (DE LA VALLÉE POUSSIN, 1938, pp.93–99):

$$\lim_{R\to\infty}\frac{\sin^{-1}\left(\frac{\gamma}{R}\right)}{R^{k-1}} = \lim_{R\to\infty}\frac{-\gamma}{(k-1)R^{k-2}\left(1-\frac{\gamma^2}{R^2}\right)^{1/2}R^2} = 0$$

Thus the integral over BB' vanishes as soon as $k > 0$. Similarly, we find for the integral over $B'C$:

$$|I_{B'C}| < CR^{-k+1}\int_{\pi/2}^{\pi} e^{Rt\cos\theta}d\theta = CR^{-k+1}\int_{0}^{\pi/2} e^{-Rt\sin\varphi}d\varphi <$$

(by the substitution $\theta = \pi/2 + \varphi$)

$$< CR^{-k+1}\int_{0}^{\pi/2} e^{-2Rt\varphi/\pi}d\varphi <$$

(because we always have $\sin\varphi > 2\varphi/\pi$)

$$< \frac{\pi CR^{-k}}{2t}$$

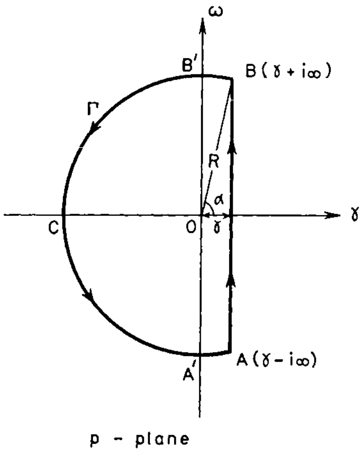


Fig.58.

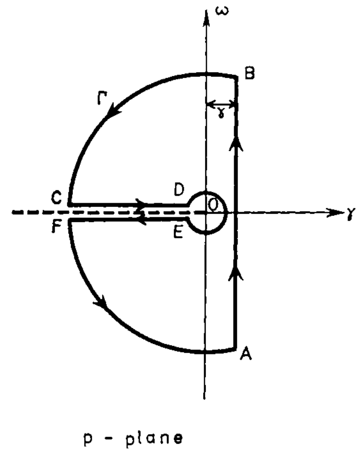


Fig.59.

(by directly carrying out the integration). Therefore:

$$\lim_{R\to\infty} |I_{B'C}| = 0$$

Similarly, we prove that the integrals over CA' and $A'A$ vanish.

This means that the integral in the inversion theorem [7.2] can be replaced by the same integral over the contour in Fig.58. This in turn can be evaluated by the rules of the residue calculus (the reason for our choice of γ, i.e., a constant greater than the real part of all singularities of $f(p)$, is to ensure that all poles are within the contour).

We can summarize the results of this section as follows:

(*a*) If $f(p)$ is a single-valued function of p, use the contour shown in Fig.58.

(*b*) If $f(p)$ has a branch point at the origin, use the contour shown in Fig.59.

8.2.4 Partial linear differential equations

The procedure is analogous to what has been developed above for ordinary differential equations. Take, for example, a quite general form of a partial differential equation:

$$\nabla^2 u + A_2(x, y, z)\frac{\partial^2 u}{\partial t^2} + A_1(x, y, z)\frac{\partial u}{\partial t} + A_0(x, y, z)u = B(x, y, z, t) \qquad [8]$$

where ∇^2 is the Laplace operator. In addition, we have both a boundary condition:

$$G(x, y, z)u + H(x, y, z)\frac{\partial u}{\partial n} = K(x, y, z, t) \qquad [9]$$

$\partial/\partial n$ being differentiation along the normal to the boundary, and initial conditions:

$$\left.\begin{aligned}&\lim_{t\to 0} u(x, y, z, t) = u_0(x, y, z)\\&\lim_{t\to 0} \frac{\partial}{\partial t} u(x, y, z, t) = u_1(x, y, z)\end{aligned}\right| \tag{10}$$

Multiply [8] by $e^{-pt}(p > 0)$ and integrate with respect to t from 0 to ∞, assuming that the integrals exist and that the Laplace operator can be placed outside the integral. We use partial integration as in section 8.2.1, which immediately gives:

$$\left.\begin{aligned}&\int_0^\infty e^{-pt} \frac{\partial u}{\partial t} \mathrm{d}t = -u_0 + p\bar{u}\\&\int_0^\infty e^{-pt} \frac{\partial^2 u}{\partial t^2} \mathrm{d}t = -(pu_0 + u_1) + p^2\bar{u}\end{aligned}\right| \tag{11}$$

From [8] we thus find the auxiliary equation:

$$\nabla^2\bar{u} + [A_2(x, y, z)p^2 + A_1(x, y, z)p + A_0(x, y, z)]\bar{u}$$
$$= A_2(x, y, z)(pu_0 + u_1) + A_1(x, y, z)u_0 + \int_0^\infty e^{-pt}B(x, y, z, t)\mathrm{d}t \tag{12}$$

and in the same way, the boundary condition [9] becomes:

$$G(x, y, z)\bar{u} + H(x, y, z) \frac{\partial \bar{u}}{\partial n} = \int_0^\infty e^{-pt}K(x, y, z, t)\mathrm{d}t \tag{13}$$

The solution is now obtained by first finding $\bar{u}$ from [12] and [13], which should generally be simpler than finding u from [8] directly; otherwise, the transformation would not have any purpose. Supposing that we have found $\bar{u}$ from [12] and [13], then we have to find $u(x, y, z, t)$ from:

$$\bar{u}(x, y, z, p) = \int_0^\infty e^{-pt}u(x, y, z, t)\mathrm{d}t \tag{14}$$

This can be done in two ways:

(*a*) From a table of Laplace transforms (Table IV).

(*b*) If (*a*) does not work, by using the Laplace inversion formula [7.2]:

$$u(x, y, z, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\bar{u}(x, y, z, p)\mathrm{d}p \tag{15}$$

Eq.[15] is solved by contour integration in the complex p-plane, as outlined in section 8.2.3.

To make the solution completely rigorous it is necessary to verify that the result obtained does satisfy the original differential equation and the initial and boundary conditions.

8.2.5 Forced vibrations of a membrane

This example incorporates the following items:

(*1*) Derivation of the equation of motion for a membrane under the action of forces (often only the equation for free vibrations is derived).

(*2*) Use of the Laplace transform to obtain the solution.

(*3*) Solution of the auxiliary equation in terms of Bessel functions.

(*4*) Application of the Cauchy theorem for integration in the complex plane to solve an integral obtained through the Laplace inversion theorem.

The problem is the following: A circular membrane of radius a is stretched by tension T, kg/cm. At $t = 0$ a uniform pressure $P_0 \sin\omega t$ is applied per unit area of the membrane. It is required to find the motion.

(*Ad 1*) Use the following notation (see FRANK and VON MISES, 1935, pp.339–340): x, y = coordinates in the plane of the membrane, z is perpendicular to it; $q(x, y)$, kg/cm² = force *per unit area* acting in the z-direction on the membrane; $u(x, y)$ = corresponding displacement of the membrane, in the z-direction, assumed small; $\mathbf{n}$ = unit vector along the normal to the deformed surface; $\mathrm{d}s(\mathrm{d}x, \mathrm{d}y, \mathrm{d}u)$ = boundary element (vector).

The tension acting on the element ds is:

$$T(\mathrm{d}s \times \mathbf{n}) \qquad [16]$$

which is easily seen, as this expression has both the right direction (i.e., perpendicular to $\mathbf{n}$ and to ds) and the right modulus ($T\mathrm{d}s$). Considering that $\mathbf{n}$ and ds are perpendicular to each other, we have by analytical geometry:

$$\cos(n, x) \underbrace{\cos(\mathrm{d}s, x)}_{=\frac{\mathrm{d}x}{\mathrm{d}s}} + \cos(n, y) \underbrace{\cos(\mathrm{d}s, y)}_{=\frac{\mathrm{d}y}{\mathrm{d}s}} + \underbrace{\cos(n, z)}_{=1} \underbrace{\cos(\mathrm{d}s, z)}_{=\frac{\mathrm{d}z}{\mathrm{d}s}} = 0 \qquad [17]$$

(for small deformations)

Differentiate $z \to u(x, y)$: $\dfrac{\mathrm{d}z}{\mathrm{d}s} = \dfrac{\partial u}{\partial x}\dfrac{\mathrm{d}x}{\mathrm{d}s} + \dfrac{\partial u}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}s}$ and insert into [17]:

$$\underbrace{\left[\cos(n, x) + \frac{\partial u}{\partial x}\right]}_{=0} \mathrm{d}x + \underbrace{\left[\cos(n, y) + \frac{\partial u}{\partial y}\right]}_{=0} \mathrm{d}y = 0 \qquad [17a]$$

As the last equation must be true for any dx, dy, the parentheses must be zero. The z-component of the vector product [16] is then as follows (using the usual rule of vector calculus for the component of a vector product):

$$(\mathrm{d}s \times \mathbf{n})_z = \underbrace{\cos(n, y)}_{\text{(i.e., projection of the unit vector on the } y\text{-axis)}}\mathrm{d}x - \cos(n, x)\mathrm{d}y = \underbrace{\frac{\partial u}{\partial x}\mathrm{d}y - \frac{\partial u}{\partial y}\mathrm{d}x}_{\text{(using [17a])}} \qquad [18]$$

The z-component of the force acting on ds is obtained by multiplication of [18] by T. By integration over the boundary s, we find the z-component of the force on the whole area:

$$\int T\left(\frac{\partial u}{\partial x}\mathrm{d}y - \frac{\partial u}{\partial y}\mathrm{d}x\right) = \int\int T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\mathrm{d}x\mathrm{d}y \qquad [19]$$

where the last integral is extended over the whole surface of the membrane. Eq.[19] is an application of Stokes' theorem (see eq.[7] in section 2.1 or MARGENAU and MURPHY, 1943, p.152):

$$\int \mathbf{V}\cdot\mathrm{d}\mathbf{s} = \int\int \nabla \times \mathbf{V}\cdot\mathrm{d}\mathbf{S}$$

where we put $\mathbf{V} = (\mathrm{d}\mathbf{s} \times \mathbf{n})/\mathrm{d}s$ and take the z-component on both sides. This must be balanced by the applied force q, so that the *equilibrium condition* becomes:

$$\int\int \left[T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + q(x, y)\right]\mathrm{d}x\mathrm{d}y = 0 \qquad [20]$$

or, as this must be valid for any dxdy:

$$T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + q(x, y) = 0 \qquad [21]$$

The *equation of motion* is then easily obtained from [21], by considering that the forces in [21] are balanced by the inertia force $\varrho(\partial^2 u/\partial t^2)$ where ϱ is the mass of the membrane *per unit area* (it is necessary to use unit area as all other terms in [21] correspond to unit area). Considering also that in this dynamical case the applied force varies with time, we use $q(x, y, t)$ instead of just $q(x, y)$. The equation of motion is then:

$$T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + q(x, y, t) - \varrho\frac{\partial^2 u}{\partial t^2} = 0 \qquad [22]$$

In [22] we change to polar coordinates (r), assuming circular symmetry of the membrane; furthermore, we divide by T and put $c^2 = T/\varrho$, and we introduce our assumed expression of the applied force: $q(x, y, t) = P_0\sin\omega t$. Then [22] becomes:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\frac{P_0\sin\omega t}{T} \qquad [23]$$

Eq.[23] is valid for $0 \leqslant r < a$ and $t > 0$ with the following conditions: boundary condition: $u = 0$ for $r = a$, $t \geqslant 0$; initial condition: $u = \partial u/\partial t = 0$ for $t = 0$, $0 \leqslant r < a$.

(*Ad 2*) The auxiliary equation to [23] is found in the usual way, i.e., by multiplication by e^{-pt} and integration over t from 0 to ∞:

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\bar{u}}{\mathrm{d}r} - \frac{p^2}{c^2}\bar{u} = -\frac{P_0\omega}{T(p^2 + \omega^2)} \qquad [24]$$

On the right-hand side we get the integral:

$$\int_0^\infty \sin\omega t\,\mathrm{e}^{-pt}\mathrm{d}t = \frac{\omega}{p^2 + \omega^2}$$

which is easily found by repeated application of partial integration. It is also given in Table IV of Laplace transforms. In [24] we write total derivatives instead of partial, as r is now the only variable (t was eliminated by the Laplace transform). The condition for [24] is that $\bar{u} = 0$ at $r = a$.

(*Ad 3*) The problem is thus reduced to solving eq.[24], which is an ordinary, linear differential equation of the second order with a right-hand member which is a constant (i.e., independent of r). From textbooks on solution of differential equations we know that the general solution of [24] is equal to the sum of a particular integral and the complementary function (see, e.g., FORSYTH, 1912, pp.65–66). This is also true when the coefficients are functions of r, i.e., not constant.

(*a*) A particular integral: Write D for $\mathrm{d}/\mathrm{d}r$; then a particular integral of [24] is:

$$\bar{u} = \frac{1}{D^2 + \frac{1}{r}D - \frac{p^2}{c^2}} \frac{-P_0\omega}{T(p^2+\omega^2)} = \frac{P_0\omega c^2}{T(p^2+\omega^2)p^2} \tag{25}$$

The particular solution can be chosen at will, but preferably the most simple form is chosen, in this case a constant.

(*b*) The complementary function is obtained by putting the right-hand side of [24] = 0:

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\bar{u}}{\mathrm{d}r} - \frac{p^2}{c^2}\bar{u} = 0 \tag{26}$$

Divide by p^2/c^2 and use $R = pr/c$ as the new variable:

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}\bar{u}}{\mathrm{d}R} - \bar{u} = 0 \tag{27}$$

This is a modified Bessel differential equation of zero order (eq.[30] in section 5.3) with the solution (A = constant):

$$\bar{u} = AJ_0(iR) = AI_0(R) = AI_0\left(\frac{pr}{c}\right) \tag{28}$$

where we have applied [33] in section 5.3. The solution $K_0(R)$ has to be discarded as it would give an infinite value of $\bar{u}$ for $r = 0$; see eq.[35] in section 5.3. The complete solution of [24] is the sum of [25] and [28]:

$$\bar{u} = \frac{P_0\omega c^2}{Tp^2(p^2+\omega^2)} + AI_0\left(\frac{pr}{c}\right) \tag{29}$$

A is determined by the boundary condition $\bar{u} = 0$ for $r = a$:

$$A = -\frac{P_0\omega c^2}{Tp^2(p^2+\omega^2)} \frac{1}{I_0\left(\frac{pa}{c}\right)} \tag{30}$$

Thus:

$$\bar{u} = \frac{P_0\omega c^2}{Tp^2(p^2+\omega^2)}\left[1 - \frac{I_0\left(\dfrac{pr}{c}\right)}{I_0\left(\dfrac{pa}{c}\right)}\right] \tag{31}$$

From [31] we then have to find $u(t)$; for the first term this can easily be done by use of a table of Laplace transforms. We have [31]: $\bar{u} = \bar{u}_1 + \bar{u}_2$, where:

$$\bar{u}_1 = \frac{P_0\omega c^2}{T}\,\frac{1}{p^2(p^2+\omega^2)} = \frac{P_0 c^2}{\omega T}\left[\underset{(I)}{\frac{1}{p^2}} - \underset{(II)}{\frac{1}{p^2+\omega^2}}\right]$$

$$\left.\begin{aligned} &(I) \quad u(t) = \frac{t}{\Gamma(2)} = t \\ &(II) \quad u(t) = \frac{1}{\omega}\sin\omega t \end{aligned}\right| \tag{32}$$

(from Table IV)

For the second term in [31], i.e., $\bar{u}_2$, we use the Laplace inversion theorem. Together we thus have for our solution $u(t)$:

$$u(t) = \frac{P_0c^2}{\omega T}\left(t - \frac{1}{\omega}\sin\omega t\right) - \frac{P_0\omega c^2}{2\pi i T}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{pt}I_0\left(\dfrac{pr}{c}\right)\mathrm{d}p}{p^2(p^2+\omega^2)I_0\left(\dfrac{pa}{c}\right)} \tag{33}$$

(*Ad 4*) The integral in [33] can be evaluated by integration in the complex plane:

	Poles:	*Residues*:
(*a*)	$p^2 = 0$ $p = 0$ (double)	$\dfrac{t}{\omega^2}$
(*b*)	$p^2 + \omega^2 = 0$ $p^2 = -\omega^2$ $p = \pm i\omega$ (simple)	$\dfrac{e^{\pm i\omega t}}{\mp 2i\omega^3}\,\dfrac{J_0\left(\dfrac{\omega r}{c}\right)}{J_0\left(\dfrac{\omega a}{c}\right)}$
(*c*)	$I_0\left(\dfrac{pa}{c}\right) = I_0(\pm i\alpha_s a)$ $= J_0(\pm\alpha_s a) = 0$ $z = \pm\alpha_s$ are roots of $J_0(az) = 0$ α_s all real and simple $p = \pm ic\alpha_s$ (simple); $s = 1, 2, \ldots \infty$	$\dfrac{e^{\pm ic\alpha_s t}J_0(r\alpha_s)}{\pm iac\alpha_s^2(\omega^2 - c^2\alpha_s^2)J_0'(a\alpha_s)}$

[34]

Note that all poles are located on the imaginary p-axis. The residues are calculated in the usual way (section 2.1). For example, in case (a) the residue is found as the coefficient of $1/p$ in the integrand, where now p is small. Expanding the integrand in [33] and neglecting small terms we get:

$$\frac{\left(1 + pt + \dfrac{p^2t^2}{2!} + \ldots\right)\left(1 + \dfrac{p^2r^2}{2^2c^2} + \ldots\right)}{p^2(p^2 + \omega^2)\left(1 + \dfrac{p^2a^2}{2^2c^2} + \ldots\right)} \rightarrow \frac{1 + pt}{p^2\omega^2}$$

$$= \frac{1}{p^2}\frac{1}{\omega^2} + \underbrace{\frac{1}{p}\frac{t}{\omega^2}}_{\text{this is the residue in case } (a)} \qquad [35]$$

The limit-values are calculated by using l'Hospital's rules (differentiation of numerator and denominator with respect to p). We also have use for the following formulas for Bessel functions:

$$I_0(z) = 1 + \frac{z^2}{2^2} + \ldots$$

(from [32] in section 5.3, which gives $I_0(0) = 1$, used in (a) above)

$$I_0(iz) = J_0(z)$$

$$I_0'(iz) = -iJ_0'(z) \qquad [36]$$

(used in (c) above).

Application of Cauchy's residue theorem to eq.[33] with the contour shown in Fig.58, as there is no branch point, then gives the final solution to our problem, where we have included only the real part (as the imaginary part has no physical significance), and where we have to remember a factor 2 in cases (b) and (c) above, as there are two poles in each of these cases. We find:

$$u(r, t) = \frac{P_0c^2}{\omega^2 T}\sin\omega t\left[\frac{J_0\left(\dfrac{\omega r}{c}\right)}{J_0\left(\dfrac{\omega a}{c}\right)} - 1\right] - \frac{2P_0\omega c}{aT}\sum_{s=1}^{\infty}\frac{\sin(c\alpha_s t)J_0(r\alpha_s)}{\alpha_s^2(\omega^2 - c^2\alpha_s^2)J_0'(a\alpha_s)} \qquad [37]$$

It has been assumed that none of the α_s is $= \omega/c$, for if this were the case, $\omega^2 - c^2\alpha_s^2 = 0$ and u in [37] would be infinite. This corresponds to resonance. Then we have *double* poles at $\pm i\omega$:

$$p = \pm ic\alpha_s = \pm ic\frac{\omega}{c} = \pm i\omega$$

Another instructive problem is offered by *the differential equation for a seismograph*. This is often solved by operational methods (see, e.g., BYERLY, 1942, chapter 8;

JEFFREYS and JEFFREYS, 1946, pp.224–227). It is left as an exercise to the reader to find these solutions but instead to use the method of the Laplace transformation. Various assumptions should be made about the variation of the ground displacement with time: (*a*) continuous harmonic (this is the usual assumption in calculation of response curves for station seismographs), (*b*) impulsive motion.

8.2.6 Radial flow of heat

This problem involves:

(*1*) Laplace transform for solving a partial differential equation with two variables.

(*2*) Solution of a modified Bessel differential equation.

(*3*) Contour integration in the complex plane.

Consider the heat flow in a long circular cylinder, radius a, with initial temperature $= 0$, and a *surface* temperature $= T_0$ for $t > 0$. The equation governing the temperature distribution $T(r, t)$, with $\varkappa$ = heat conductivity is (not derived here):

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\varkappa}\frac{\partial T}{\partial t}\,, \qquad 0 \leqslant r < a\,, \quad t > 0 \tag{38}$$

with boundary condition:

$$T = T_0 \quad \text{for} \quad r = a, \qquad t > 0 \tag{39}$$

and initial condition:

$$T = 0 \quad \text{for} \quad t = 0, \qquad 0 \leqslant r < a \tag{40}$$

(*Ad 1*) Introduce the Laplace transform:

$$\overline{T} = \int_0^\infty e^{-pt} T \, dt \tag{41}$$

Integrate by parts and use the condition [40]:

$$\int_0^\infty e^{-pt} \frac{\partial T}{\partial t}\, dt = \underbrace{[e^{-pt}T]_0^\infty}_{=0} + p \int_0^\infty T e^{-pt} dt = p\overline{T} \tag{42}$$

Multiply [38] and [39] by e^{-pt}, integrate with respect to t between 0 and ∞, and use [41] and [42]:

$$\frac{d^2\overline{T}}{dr^2} + \frac{1}{r}\frac{d\overline{T}}{dr} - q^2\overline{T} = 0, \qquad 0 \leqslant r < a \tag{43}$$

$$\overline{T} = \frac{T_0}{p} \qquad \text{for } r = a \tag{44}$$

putting $q^2 = p/\varkappa$.

(*Ad 2*) Again we have reduced the problem to the solution of an *ordinary* differential equation [43]. This is a modified Bessel equation (eq.[30] in section 5.3), and the solution is:

$$\bar{T} = AI_0(qr) + BK_0(qr) \tag{45}$$

The second term in [45] must be omitted, because $K_0(qr)$ contains a logarithmic term, which makes it infinite at $r = 0$. The solution of [43], which also satisfies [44] is:

$$\left.\begin{aligned} \bar{T} &= \frac{T_0}{p}\frac{I_0(qr)}{I_0(qa)} \\ A &= \frac{T_0}{p}\frac{1}{I_0(qa)} \end{aligned}\right| \tag{46}$$

In order to obtain the function T we have to use the inversion formula [7.2] for the Laplace transform:

$$T = \frac{T_0}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{tp}\frac{I_0(qr)}{I_0(qa)}\frac{dp}{p} \tag{47}$$

(*Ad 3*) The series for $I_0(z)$ contains only *even* powers of z (see eq.[32] in section 5.3), and therefore $I_0(qr)/I_0(qa)$, where $q^2 = p/\varkappa$, is a single-valued function of p, i.e., it has only one value for a given value of p, whether positive or negative. Therefore, in evaluating [47] by contour integration, we can choose the contour of Fig.58. The poles are found as follows: $I_0(qa) = 0$ for $p = -\varkappa\alpha_n^2$ where $\pm\alpha_n(n = 1, 2, 3, \ldots)$ are the roots of $J_0(\alpha a) = 0$. This is seen as follows. Assume $\pm\alpha_n$ to be the roots of $J_0(\alpha a)$, then:

$$J_0(\pm\alpha_n a) = 0 \;;\quad q^2 = \frac{p}{\varkappa} = -\frac{\varkappa\alpha_n^2}{\varkappa} = -\alpha_n^2$$

$$q = \pm i\alpha_n$$

$$I_0(qa) = I_0(\pm i\alpha_n a) = J_0(\pm\alpha_n a) = 0\;,$$

i.e., $I_0(qa) = 0$, which should be proved. The integral around the arc tends to zero, as the radius tends to infinity. This was proved in section 8.2.3. We have:

$$\lim_{p\to\infty}\frac{I_0(qr)}{I_0(qa)}\frac{1}{p} \to \lim_{p\to\infty}\frac{1}{p} \to 0$$

$I_0(qr)$ and $I_0(qa)$ have the same powers in p. Therefore by the residue theorem the integral in [47] is equal to $2\pi i$ times the sum of the residues at the poles of the integrand. Applying the definition of residue, we have in this case:

$$\sum_{n=1}^{\infty}\lim_{p\to-\varkappa\alpha_n^2} e^{tp}\frac{I_0(qr)}{I_0(qa)}\frac{p+\varkappa\alpha_n^2}{p} = \sum_{n=1}^{\infty} e^{-t\varkappa\alpha_n^2}J_0(\alpha_n r)\lim_{p\to-\varkappa\alpha_n^2}\frac{p+\varkappa\alpha_n^2}{I_0(qa)p}$$

because $I_0(ix) = J_0(x)$ by eq.[33] in section 5.3. According to l'Hospital's rules for calculation of limit values, we differentiate numerator and denominator with respect to p:

$$\lim_{p\to-\varkappa\alpha_n^2} \frac{p+\varkappa\alpha_n^2}{I_0(qa)\cdot p} = \lim_{p\to-\varkappa\alpha_n^2} \frac{1}{p\dfrac{\mathrm{d}I_0(qa)}{\mathrm{d}p} + \underbrace{I_0(qa)\cdot 1}_{=0}}$$

and:

$$\left[p\frac{\mathrm{d}I_0(qa)}{\mathrm{d}p}\right]_{p=-\varkappa\alpha_n^2} = \frac{1}{2}\left[q\frac{\mathrm{d}I_0(qa)}{\mathrm{d}q}\right]_{q=\pm i\alpha_n} = \frac{1}{2}\left[qa\frac{\mathrm{d}I_0(qa)}{\mathrm{d}(qa)}\right]_{q=\pm i\alpha_n}$$

$$= \frac{1}{2}[qaI_1(qa)]_{q=\pm i\alpha_n} = -\frac{1}{2}a\alpha_nJ_1(a\alpha_n)$$

because:

$$q^2 = \frac{p}{\varkappa} \qquad 2q\mathrm{d}q = \frac{\mathrm{d}p}{\varkappa}$$

$$I_0'(z) = I_1(z) \qquad I_1(iz) = iJ_1(z)$$

by eq.[33] in section 5.2. The pole $p = 0$ has residue unity:

$$\lim_{p\to0} \mathrm{e}^{tp}\frac{I_0(qr)}{I_0(qa)}\frac{p-0}{p} = \lim_{t\to0}\frac{I_0(qr)}{I_0(qa)} = 1$$

The final solution then follows from [47], with Res = residue:

$$T = \frac{T_0}{2\pi i}\cdot 2\pi i\sum\mathrm{Res} = T_0\sum\mathrm{Res} = T_0\left[1-\frac{2}{a}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-t\varkappa\alpha_n^2}J_0(\alpha_nr)}{\alpha_nJ_1(\alpha_na)}\right] \tag{48}$$

It is seen from the solution [48] that at any given point within the cylinder, i.e., for a given r, the temperature T tends to the surface temperature T_0 as t tends to infinity.

8.2.7 Heat flow in a semi-infinite solid

We return to the problem discussed in eq.[36]–[40] in section 8.1, but this time we shall apply contour integration to solve the integral, contained in the inversion formula. Using the Laplace inversion formula [7] and putting $a = x/\sqrt{\varkappa}$ we have the following inversion, noting that from section 8.1.4 $\overline{T} \sim (1/p)\mathrm{e}^{-a\sqrt{p}}$:

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{e}^{(tp-a\sqrt{p})}\frac{\mathrm{d}p}{p} \tag{49}$$

The integral has to be evaluated by contour integration in the complex p-plane, but now using the contour shown in Fig.59. The integrand has a *branch point* at the origin. The many-valuedness is avoided by a *cut* along the negative real axis as in Fig.59. The integrand is single-valued inside and on the chosen contour. Moreover, it has no poles within the contour; therefore by Cauchy's theorem, the integral around the contour vanishes, or the integral along AB (which shall be determined) is equal to the integrals along DC (note direction!) and along FE and around the small circle at the origin; the integrals along the arcs BC and FA tend to zero as the radius goes to infinity. This is seen as:

$$|\bar{f}(p)| = \left|\frac{e^{-a\sqrt{p}}}{p}\right| < |p|^{-1}$$

On DC, write $p = \alpha e^{i\pi}$ and [49] gives:

$$\frac{1}{2\pi i}\int_0^\infty e^{-\alpha t}\, e^{-ia\sqrt{\alpha}}\,\frac{d\alpha}{\alpha}$$

Similarly on FE, put $p = \alpha e^{-i\pi}$ and we get:

$$-\frac{1}{2\pi i}\int_0^\infty e^{-\alpha t}\, e^{ia\sqrt{\alpha}}\,\frac{d\alpha}{\alpha}$$

DC and FE together give (putting $\alpha = u^2$):

$$-\frac{1}{\pi}\int_0^\infty e^{-\alpha t}\sin(a\sqrt{\alpha})\,\frac{d\alpha}{\alpha} = -\frac{2}{\pi}\int_0^\infty e^{-u^2 t}\sin(au)\,\frac{du}{u}$$

This is the contribution of the branch-line integrals. The contribution from the small circle at the origin is unity:

$$p \to 0$$

$$e^{tp-a\sqrt{p}} \to 1$$

$$p = Re^{i\theta}$$

$$dp = Rie^{i\theta}d\theta = pid\theta$$

$$\frac{dp}{p} = id\theta$$

and:

$$\frac{1}{2\pi i}\int_0^{2\pi}\frac{dp}{p} = \frac{1}{2\pi i}\int_0^{2\pi} id\theta = 1$$

By Cauchy's theorem:

$$\int_{AB} + \underbrace{\int_{BC}}_{=0} + \int_{CD} + \underbrace{\int_{0}}_{-1} + \int_{EF} + \underbrace{\int_{FA}}_{=0} = 0$$

(negative rotation around O) and:

$$\int_{AB} = +1 + \int_{DC} + \int_{FE} \qquad [50]$$

Thus, the inverse is:

$$T = T_0 \left(1 - \frac{2}{\pi} \int_0^\infty e^{-u^2 t} \sin(au) \frac{du}{u}\right) \qquad [51]$$

The infinite integral in [51] can be expressed in the error function by using the following formula (see DWIGHT, 1957, p.201, formula 863.3; or WEAST, 1964, p.325, formula 432; cf. also eq.[27] in section 2.1):

$$\int_0^\infty e^{-u^2 t} \cos(au) du = \frac{1}{2} \sqrt{\left(\frac{\pi}{t}\right)}\, e^{-a^2/4t} \qquad (t > 0) \qquad [52]$$

We multiply the left-hand side of [52] by da and integrate over a from 0 to a:

$$\int_0^a da \int_0^\infty e^{-u^2 t} \cos(au) du = \int_0^\infty e^{-u^2 t} du \int_0^a \cos(au) da$$

$$= \int_0^\infty e^{-u^2 t} \sin(au) \frac{du}{u} = \qquad \frac{1}{2} \sqrt{\left(\frac{\pi}{t}\right)} \int_0^a e^{-a^2/4t} da$$

(from [52])

$$= \sqrt{\pi} \int_0^{a/2\sqrt{t}} e^{-w^2} dw \qquad = \frac{\pi}{2} \operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right)$$

$\left(\text{substituting } w = \dfrac{a}{2\sqrt{t}}\right)$ (by definition of erf)

Thus:

$$T = T_0 \left[1 - \frac{2}{\pi} \cdot \frac{\pi}{2} \operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right)\right] = T_0 \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right) \qquad [53]$$

i.e., the same result as above, expressed in eq.[40] in section 8.1.

For additional examples of the use of the Laplace transform on problems from many fields (heat conduction, hydrodynamics, electricity, diffusion, mechanics), the reader is referred to CARSLAW and JAEGER (1945).

8.3 IMPULSIVE FUNCTIONS

In classical applied mathematics it is customary to work with continuous sinusoidal phenomena. For instance, seismic waves are usually thought of as consisting of a number of such continuous waves, as for example in the usual procedure for deducing spectra and in the derivation of seismograph response. It is more rare that impulsive functions have been applied, although these often correspond better to real conditions.

8.3.1 *The Dirac delta function*

Consider the following function, which is finite only in a very limited range of the independent variable and is zero outside this range:

$$\delta_a(x) = \begin{cases} \dfrac{1}{2a} & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \qquad [1]$$

It is illustrated in Fig.60. Then it is readily shown that:

$$\int_{-\infty}^{\infty} \delta_a(x)\mathrm{d}x = \int_{-a}^{a} \delta_a(x)\mathrm{d}x = \frac{1}{2a}\int_{-a}^{a} \mathrm{d}x = 1 \qquad [2]$$

i.e., the area under the curve is unity.

Let $f(x)$ be an integrable function in the interval $(-a, a)$, then:

$$\int_{-\infty}^{\infty} f(x)\delta_a(x)\mathrm{d}x = \frac{1}{2a}\int_{-a}^{a} f(x)\mathrm{d}x = f(\theta a) \qquad (|\theta| \leqslant 1) \qquad [3]$$

where we have used the mean value theorem of integral calculus ($\frac{1}{2a}\int_{-a}^{a} f(x)\mathrm{d}x$ expresses a mean value of $f(x)$ over the interval from $-a$ to a, and this mean value is equal to some value of $f(x)$ within this range of x; see DE LA VALLÉE POUSSIN, 1938, p.208).

We define the *Dirac delta function* $\delta(x)$ as:

$$\left.\begin{aligned} &\delta(x) = \lim_{a \to 0} \delta_a(x) \\ &\text{i.e.:} \quad \delta(x) = 0 \text{ for } x \neq 0 \; ; \qquad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1 \end{aligned}\right| \qquad [4]$$

Then, the Dirac delta function is a unit spike at $x = 0$. The width of the interval, $-a$ to $+a$, where $\delta_a(x) \neq 0$ has decreased to zero. Eq.[2] becomes then an indefinite expression, but is made definite by the definition in [4]. That is, the area under the curve is still unity, which can only be attained by assuming an infinitely high pulse at $x = 0$. We have to note that this does not uniquely define the "function" $\delta(x)$. In fact, the Dirac delta function is no function at all, in the usual sense of the word function, and its properties are purely formal. Nevertheless, it is useful in applications. The delta function must be defined in combination with an ordinary function (see equations [6] and [7] below). $\delta(x)$ is a

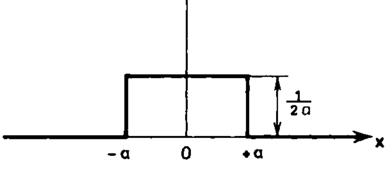


Fig.60.

generalized function or *distribution*. The notation δ has been adopted by analogy with Kronecker's delta.

The following function may be given as an example:

$$\delta(x) = \lim_{n\to\infty} \frac{\sin(2\pi nx)}{\pi x} \qquad [5]$$

It satisfies [4]:

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi nx)}{\pi x} \mathrm{d}x = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\varphi}{\varphi} \mathrm{d}\varphi = 1$$

putting $2\pi nx = \varphi$ and using DWIGHT (1957, p.198, formula 858.5). For $x = 0$:

$$\frac{2\pi n \cos(2\pi nx)}{\pi} = 2n \neq 0$$

in which $\cos(2\pi nx) = 1$. For $x \neq 0$, [5] is zero, as n runs through infinitely many values, and there are equal positive and negative contributions to the numerator.

If in [3] we let $a \to 0$, we find:

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0) \qquad [6]$$

Similarly, we find:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a) \qquad [7]$$

Both [6] and [7] are easily seen to be correct, as the only contribution is obtained for $\delta(0)$, i.e., $x = 0$ in [6] and $x = a$ in [7]. A special case of [6] is the following:

$$\int_{0}^{\infty} \mathrm{e}^{-px}\delta(x)\mathrm{d}x = 1 \qquad [8]$$

which means that the Laplace transform of the δ-function is unity. Symbolically we can write [7] as:

$$f(x)\delta(x-a) = f(a)\delta(x-a) \qquad [9]$$

remembering that this equation means that the two sides give equivalent results when used as factors in an integrand.

Using the same symbolical writing as in [9] we can prove the following relations:

$$x\delta(x) = 0 \qquad [10]$$

$$\delta(-x) = \delta(x) \qquad [11]$$

$$\delta(ax) = \frac{1}{a}\delta(x), \qquad a > 0 \qquad [12]$$

$$\delta(a^2 - x^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)], \qquad a > 0 \qquad [13]$$

Proofs:

[*10*]: In [7] put $f(x) = x$ and $a = 0$. Instead we can use the equivalent relation to [6]: $f(x)\delta(x) = f(0)\delta(x)$.

[*11*]:

$$\int_{-\infty}^{\infty} \delta(-x)\mathrm{d}x = -\int_{\infty}^{-\infty} \delta(\xi)\mathrm{d}\xi = \int_{-\infty}^{\infty} \delta(\xi)\mathrm{d}\xi$$

(put $-x = \xi$). Thus $\delta(-x) = \delta(x)$, which proves [11].

[*12*]:

$$\int_{-\infty}^{\infty} \delta(ax)\mathrm{d}x = \frac{1}{a}\int_{-\infty}^{\infty} \delta(\xi)\mathrm{d}\xi = \frac{1}{a}$$

$\left(\text{put } ax = \xi \;;\quad \mathrm{d}x = \frac{\mathrm{d}\xi}{a}\right)$. Thus $\delta(ax) = \frac{1}{a}\delta(x)$, which proves [12].

[*13*]:

$$\int_{-\infty}^{\infty} \delta(x^2 - a^2)\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\delta(\xi)\mathrm{d}\xi}{2\sqrt{(\xi + a^2)}} =$$

$\left(\text{put } x^2 - a^2 = \xi \;;\qquad 2x\mathrm{d}x = \mathrm{d}\xi \;;\qquad \mathrm{d}x = \frac{\mathrm{d}\xi}{2\sqrt{(\xi + a^2)}}\right)$

$$= \frac{1}{2a}\int_{\xi=0} \delta(\xi)\mathrm{d}\xi =$$

(only contributions from $\xi = 0$, i.e., for two points: $x = +a$ and $x = -a$)

$$= \frac{1}{2a}\int_{-\infty}^{\infty} [\delta(x - a) + \delta(x + a)]\mathrm{d}x$$

Equating the integrands in the first and last expressions gives [13].

Derivatives of $\delta(x)$

We assume that derivatives of $\delta(x)$ exist and that they can be considered as ordinary functions, including $\delta(x)$ itself. Then we can apply the rule of integration by parts:

$$\int_{-\infty}^{\infty} f(x)\delta'(x)\mathrm{d}x = [f(x)\delta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x)\mathrm{d}x = -f'(0) \qquad [14]$$

Repeating this process n times we find:

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x)\mathrm{d}x = (-1)^n f^{(n)}(0) \qquad [15]$$

The Dirac delta function can be interpreted as the derivative of the Heaviside unit function $H(x)$:

$$H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x \leqslant 0 \end{cases} \tag{16}$$

Consider the following integral:

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}H(x) = f(0) \tag{17}$$

That the right-hand side is $f(0)$ is immediately seen by the definition [16]: $H(x)$ is constant and thus $\mathrm{d}H(x) = 0$ everywhere except at $x = 0$, where there is the jump $\mathrm{d}H(x) = 1$. Therefore, there is a contribution to the integral in [17] only for $x = 0$. An integral of the type [17], more generally written as:

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}F(x)$$

is called a *Stieltjes' integral.*

Comparing [17] with [6] we find:

$$\delta(x)\mathrm{d}x = \mathrm{d}H(x) \qquad \text{or} \qquad \delta(x) = H'(x) \tag{18}$$

This gives the connection between the Dirac and the Heaviside functions. Some further properties of $H(x)$ will be studied in section 14.1.

In applications the variable x may be either time or space. Time variable: a unit instantaneous impulse at $t = 0$ (in the mechanical sense) may be regarded as due to the force $\delta(t)$; in electrical applications, an impulsive electromotive force can be written as $E_0\delta(t)$, implying a very large voltage applied for a very short time. Space variable: the $\delta(x)$-function is then useful when dealing with concentrated loads or disturbances.

8.3.2 Use of the delta function in convolution formulas [1]

We refer to the *complex Fourier transform* (eq.[4b] in section 8.1):

$$\bar{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\mathrm{d}t \tag{19}$$

and its inversion formula (eq.[24b] in section 8.1):

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\omega)\mathrm{e}^{i\omega t}\mathrm{d}\omega \tag{20}$$

We express the equivalence between these two expressions by the following symbolic writing:

$$f(t) \longleftrightarrow \bar{f}(\omega) \tag{21}$$

[1] WHITE (1965, pp.6–12).

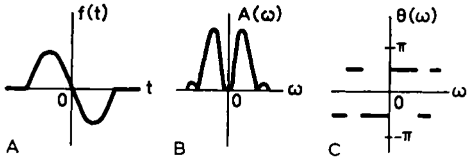


Fig.61.

We have to think of $f(t)$ and $\bar{f}(\omega)$ as representing the same phenomenon, e.g., a transient seismic signal, in one case in the time domain, in the other case in the frequency domain. $\bar{f}(\omega)$ can be expressed as follows:

$$\bar{f}(\omega) = A(\omega)\, e^{i\theta(\omega)} \qquad [22]$$

where A is amplitude and θ the phase. Fig.61 gives an example. The following properties hold for $\bar{f}(\omega)$, which are also evident from Fig.61:

$$\bar{f}(\omega) = A(\omega)e^{i\theta(\omega)}$$

$$\bar{f}(-\omega) = A(-\omega)e^{i\theta(-\omega)}$$

But from [19] we also have:

$$\bar{f}(-\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt = \text{complex conjugate of } \bar{f}(\omega)$$

in case $f(t)$ is real. Thus we get:

$$A(\omega)e^{-i\theta(\omega)} = A(-\omega)e^{i\theta(-\omega)}, \quad \text{i.e.} \quad \begin{cases} A(\omega) = A(-\omega) \\ \theta(\omega) = -\theta(-\omega) \end{cases} \qquad [23]$$

Another concept of importance is *convolution.* The convolution of two functions $f_1(t)$ and $f_2(t)$ is denoted $f_1(t) * f_2(t)$ and is defined in the following way:

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau \qquad [24]$$

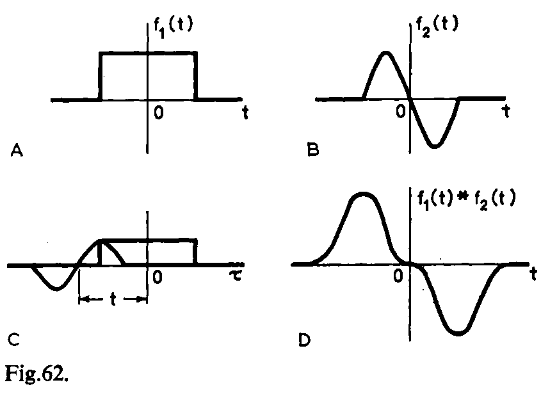


Fig.62.

Convolution is also called by a number of other names: superposition or superposition theorem (by JEFFREYS and JEFFREYS, 1946), "Faltungsintegral", Green's theorem, Duhamel's theorem, Borel's theorem and Boltzmann–Hopkinson theorem. The convolution process is illustrated in Fig.62. For each value of t, i.e., the time shift between f_1 and f_2, eq.[24] yields a point in the diagram, showing the convolved function versus t.

The convolution process is useful especially for its equivalence in terms of frequency:

$$f_1(t) * f_2(t) \longleftrightarrow \bar{f}_1(\omega)\bar{f}_2(\omega) \qquad [25]$$

where the equivalence is of the same kind as expressed in [21]. Before proceeding we shall demonstrate [25]:

$$\bar{f}_1(\omega)\bar{f}_2(\omega) = \underbrace{\int_{-\infty}^{\infty} f_1(t)e^{-i\omega t}dt}_{} \; \underbrace{\int_{-\infty}^{\infty} f_2(t)e^{-i\omega t}dt}_{} =$$

(from [19]) $(t \to \tau)$ $\qquad (t \to t - \tau;$ where $\tau =$ constant)

$$= \int_{-\infty}^{\infty} f_1(\tau)e^{-i\omega\tau}d\tau \int_{-\infty}^{\infty} f_2(t-\tau)e^{-i\omega t}e^{i\omega\tau}dt =$$

(rearrange the order of integrations)

$$= \int_{-\infty}^{\infty} e^{-i\omega t}dt \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau =$$

(use the definition [24])

$$= \int_{-\infty}^{\infty} e^{-i\omega t}dt[f_1(t) * f_2(t)]$$

and [25] is proved. The importance of [25] lies in the fact that it permits calculation of the output waveform for a given linear medium and a given input transient signal (see Fig.63). A medium is said to be *linear* when it produces only a change of phase and of amplitude. Quite generally, the "medium" can be a medium in the general sense of this word, but it can also be a discontinuity surface in the earth or it can be a seismograph.

Consider now the complex Fourier transform of the delta-function:

$$\bar{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\delta(t)dt = 1$$

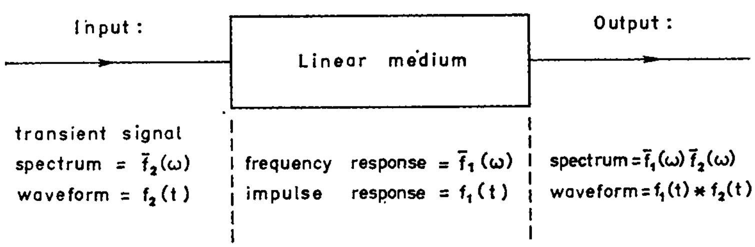


Fig.63.

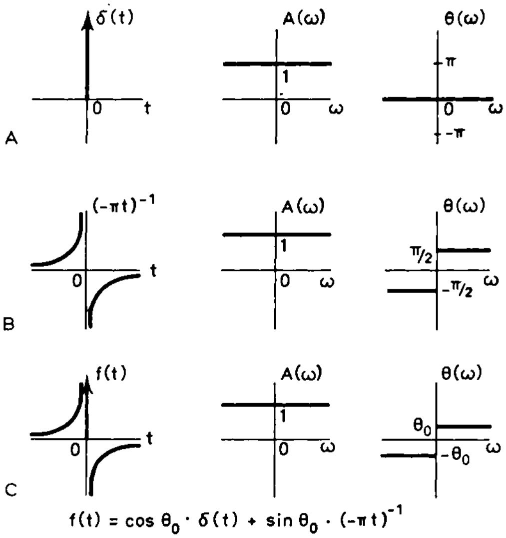


Fig.64.

(from eq.[6])

$$\bar{f}(\omega) = A(\omega)e^{i\theta(\omega)} = A(\cos\theta + i\sin\theta) = 1$$

Identify real and imaginary parts:

$$\left.\begin{aligned} A\cos\theta &= 1 \\ A\sin\theta &= 0 \end{aligned}\right\} \text{ gives: } \left.\begin{aligned} \theta &= 0 \\ A &= 1 \end{aligned}\right\} \text{ for all frequencies} \qquad [26]$$

i.e., amplitude $= 1$ and phase angle $= 0$ for all frequencies (see Fig.64). If now in Fig.63 we assume that the input waveform is the delta function, $f_2(t) \rightarrow \delta(t)$, we can obtain the impulse response $f_1(t)$ of the medium by convolving $f_1(t)$ with $f_2(t)$, i.e., from eq.[7]:

$$\int_{-\infty}^{\infty} f_1(\tau)\delta(t-\tau)\mathrm{d}\tau = f_1(t) \qquad [27]$$

If this transient response of a linear medium to a delta-function can be obtained mathematically or experimentally, the output for any input transient $f_2(t)$ can be obtained by convolution:

$$\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)\mathrm{d}\tau$$

Another function with close relations to the delta function, at least as far as the Fourier transform is concerned, is the function $(-\pi t)^{-1}$. This function and its Fourier transform are illustrated in Fig.64. The Fourier transform is derived as follows:

$$\bar{f}(\omega) = \int_{-\infty}^{\infty} \left(\frac{1}{-\pi t}\right) e^{-i\omega t}\mathrm{d}t =$$

(split up the integral)

$$= \int_{-\infty}^{0} \left(\frac{1}{-\pi t}\right) e^{-i\omega t} dt + \int_{0}^{\infty} \left(\frac{1}{-\pi t}\right) e^{-i\omega t} dt$$

By a change of variable, $t \rightarrow -\tau$, the first integral becomes:

$$- \int_{-\infty}^{0} \frac{1}{\pi\tau} e^{i\omega\tau} d\tau = + \int_{0}^{\infty} \frac{1}{\pi t} e^{i\omega t} dt$$

and:

$$\bar{f}(\omega) = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{t} (e^{i\omega t} - e^{-i\omega t}) dt =$$

(use Euler's formulas)

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{t} 2i \sin\omega t \, dt = \frac{2i}{\pi} \underbrace{\int_{0}^{\infty} \frac{\sin \omega t}{t} dt}_{\pm \frac{\pi}{2} \text{ for } \omega \gtrless 0}$$

(see DWIGHT, 1957, p.198, eq.858.5). Thus:

$$\bar{f}(\omega) = A(\omega) e^{i\theta(\omega)} = \pm i$$

and:

$$\left.\begin{array}{l} A \cos\theta = 0 \\ A \sin\theta = \pm 1 \end{array}\right\} \begin{array}{l} A = 1 \\ \theta = \pm \frac{\pi}{2} \quad \text{for } \omega \gtrless 0 \end{array}$$

The function $(-\pi t)^{-1}$ is, like the delta function, mostly used in combination with another ordinary function:

$$\int_{-\infty}^{\infty} f(t)(-\pi t)^{-1} dt = -\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) d(\log t) =$$

(partial integration)

$$= \underbrace{-\frac{1}{\pi} [f(t) \log|t|]_{-\infty}^{\infty}}_{=0} + \frac{1}{\pi} \int_{-\infty}^{\infty} \log|t| f'(t) dt \qquad [28]$$

Convolution of $f(t)$ with $(-\pi t)^{-1}$ gives a modified waveform, designated $[f(t)]_{\pi/2}$:

$$\int_{-\infty}^{\infty} f(\tau)[-\pi(t-\tau)]^{-1} d\tau = [f(t)]_{\pi/2} \qquad [29]$$

By proper combination of the delta function and the $(-\pi t)^{-1}$-function, it is possible to obtain a similar Fourier transform, i.e., with constant amplitude and a frequency which has an assigned *constant* value $= +\theta_0$ for $\omega > 0$ and $-\theta_0$ for $\omega < 0$, as demonstrated at the bottom of Fig.64. The function is given by the following formula, which is a generalization of [29]:

$$[f(t)]_{\theta_0} = \int_{-\infty}^{\infty} f(\tau)\{\cos\theta_0\,\delta(t-\tau) + \sin\theta_0\,[-\pi(t-\tau)]^{-1}\}\,d\tau$$

$$\bar{f}(\omega) = \underbrace{\int_{-\infty}^{\infty} \cos\theta_0\,\delta(t)e^{-i\omega t}dt}_{=\cos\theta_0 \text{ (from [6])}} + \underbrace{\int_{-\infty}^{\infty} \sin\theta_0\left(\frac{1}{-\pi t}\right)e^{-i\omega t}dt}_{=\pm i\sin\theta_0} \qquad [30]$$

$$\bar{f}(\omega) = A(\cos\theta + i\sin\theta) = \cos\theta_0 \pm i\sin\theta_0$$

Identify real and imaginary parts:

$$\left.\begin{aligned} A\cos\theta &= \cos\theta_0 \\ A\sin\theta &= \pm\sin\theta_0 \end{aligned}\right\} \text{thus:} \left\{\begin{aligned} A &= 1 \\ \theta &= \pm\theta_0 \qquad \text{for } \omega \gtrless 0 \end{aligned}\right.$$

For example, it can be shown that a vertically polarized shear wave (SV) of waveform $f(t)$ will give rise to a reflected waveform $[f(t)]_{\theta_0}$ near grazing incidence.

8.3.3 Seismic waves at a plane boundary [1]

In usual treatments of this problem, *continuous* harmonic wave motion is considered. The solutions obtained here are in such a form that the complex Fourier transformation can be applied and the results generalized to the reflection of *transient* waveforms.

We give first a brief recapitulation of the traditional treatment (continuous waves of one frequency) and use the following notation (Fig.65): α = longitudinal wave velocity, β = transverse wave velocity, c = phase velocity (along the boundary), ω = frequency and:

$$l = \frac{\omega}{c}$$

$$m = \omega\left(\frac{1}{\alpha^2} - \frac{1}{c^2}\right)^{1/2}$$

$$k = \omega\left(\frac{1}{\beta^2} - \frac{1}{c^2}\right)^{1/2}$$

$$c = \frac{\alpha}{\sin i_P} = \frac{\beta}{\sin i_S}$$

[1] WHITE (1965, pp.26–32).

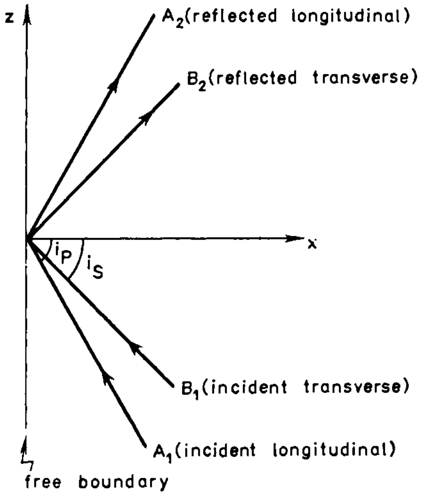


Fig.65.

Then, where $\beta < \alpha < |c|$, the solutions can be written as follows:

ongitudinal wave potential:

$$\varphi = \underset{\text{(incident)}}{(A_1 e^{imx}} + \underset{\text{(reflected)}}{A_2 e^{-imx})} e^{-ilz} e^{i\omega t}$$

transverse wave potential: [31]

$$\psi = (B_1 e^{ikx} + B_2 e^{-ikx}) e^{-ilz} e^{i\omega t}$$

For an incident longitudinal wave, i.e., $B_1 = 0$, the boundary conditions (vanishing stress at the free surface) give the following two relations:

$$\frac{A_2}{A_1} = \frac{4\left(\dfrac{c^2}{\alpha^2}-1\right)^{1/2}\left(\dfrac{c^2}{\beta^2}-1\right)^{1/2}-\left(\dfrac{c^2}{\beta^2}-2\right)^2}{4\left(\dfrac{c^2}{\alpha^2}-1\right)^{1/2}\left(\dfrac{c^2}{\beta^2}-1\right)^{1/2}+\left(\dfrac{c^2}{\beta^2}-2\right)^2} = K_1$$

$$\frac{B_2}{A_1} = \frac{4\left(\dfrac{c^2}{\alpha^2}-1\right)^{1/2}\left(\dfrac{c^2}{\beta^2}-2\right)}{4\left(\dfrac{c^2}{\alpha^2}-1\right)^{1/2}\left(\dfrac{c^2}{\beta^2}-1\right)^{1/2}+\left(\dfrac{c^2}{\beta^2}-2\right)^2} = K_2 \qquad [32]$$

Similarly, in case of an incident transverse wave (SV), i.e., $A_1 = 0$, we find:

$$\frac{A_2}{B_1} = -\frac{4\left(\dfrac{c^2}{\beta^2}-1\right)^{1/2}\left(\dfrac{c^2}{\beta^2}-2\right)}{4\left(\dfrac{c^2}{\alpha^2}-1\right)^{1/2}\left(\dfrac{c^2}{\beta^2}-1\right)^{1/2}+\left(\dfrac{c^2}{\beta^2}-2\right)^2} = K_3 \qquad [33]$$

$$\frac{B_2}{B_1} = \frac{4\left(\dfrac{c^2}{\alpha^2}-1\right)^{1/2}\left(\dfrac{c^2}{\beta^2}-1\right)^{1/2} - \left(\dfrac{c^2}{\beta^2}-2\right)^2}{4\left(\dfrac{c^2}{\alpha^2}-1\right)^{1/2}\left(\dfrac{c^2}{\beta^2}-1\right)^{1/2} + \left(\dfrac{c^2}{\beta^2}-2\right)^2} = K_4 \qquad [33]$$

This is the usual treatment given in seismological textbooks. It evidently concerns only one frequency ω, and the amplitude ratios [32] and [33] are independent of frequency, but can be expressed as functions of the phase velocity c. The *complex Fourier transform* permits us to generalize these results to transients composed of many frequencies. Suppose that A_1 (incident longitudinal) in [31] covers a band of frequencies continuously, its strength in a small frequency range $d\omega$ being $\bar{f}_1(\omega)(d\omega/2\pi)$. Integrated over all frequencies, the incoming compressional potential is then by [31]:

$$\int_{-\infty}^{\infty} \bar{f}_1(\omega)e^{imx}e^{-ilz}e^{i\omega t}\frac{d\omega}{2\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}_1(\omega)e^{i\omega\tau_1}d\omega = f_1(\tau_1) \qquad [34]$$

by the definition of the Fourier transform [20] and where:

$$\tau_1 = t + \frac{mx}{\omega} - \frac{lz}{\omega} = t + x\left(\frac{1}{\alpha^2} - \frac{1}{c^2}\right)^{1/2} - \frac{z}{c}$$

$f_1(t)$ represents the incoming compressional waveform in the time domain. From [32] we have $A_2 = K_1A_1$ for all frequencies, and hence the reflected wave strength is now $K_1\bar{f}_1(\omega)(d\omega/2\pi)$ in the frequency domain. The second term in [31.1], i.e., the reflected longitudinal wave, is now summed over all frequencies:

$$\int_{-\infty}^{\infty} K_1\bar{f}_1(\omega)e^{-imx}e^{-ilz}e^{i\omega t}\frac{d\omega}{2\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_1\bar{f}_1(\omega)e^{i\omega\tau_2}d\omega = K_1f_1(\tau_2) \qquad [35]$$

again using the Fourier transform and where:

$$\tau_2 = t - \frac{mx}{\omega} - \frac{lz}{\omega} = t - x\left(\frac{1}{\alpha^2} - \frac{1}{c^2}\right)^{1/2} - \frac{z}{c}$$

We conclude from [34] and [35] that the reflected longitudinal wave has the same waveform, i.e., the same time function $f(t)$, as the incident longitudinal wave. By similar argument, we can show that the potential of the reflected transverse wave is $K_2f_1(\tau_3)$ where:

$$\tau_3 = t - x\left(\frac{1}{\beta^2} - \frac{1}{c^2}\right)^{1/2} - \frac{z}{c}$$

and thus also has the same waveform as the incident longitudinal wave.

As an exercise, we shall now discuss the same problem in terms of the *convolution integral*. This means that we shall make use of the fact that multiplication in the frequency domain is *equivalent* to convolution in the time domain, by eq.[25]. We now have incident compressional potential (at the origin):

$$f_1(t) \longleftrightarrow \bar{f}_1(\omega) \qquad [36]$$

and Fourier transform of reflected compressional potential (at the origin):

$$\bar{f}_2(\omega) = K_1\bar{f}_1(\omega) \tag{37}$$

using [32]. Replacing A_2 by:

$$A_2 = \int_{-\infty}^{\infty} \bar{f}_2(\omega)\frac{d\omega}{2\pi} \tag{38}$$

in [31.1], the potential of the reflected longitudinal wave becomes:

$$\varphi_r(x, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_1\bar{f}_1(\omega)e^{-i(mx+lz)}e^{i\omega t}d\omega \tag{39}$$

With $\varphi_r(x, z, t) \longleftrightarrow \bar{\varphi}_r(x, z, \omega)$, comparison with [20] gives:

$$\bar{\varphi}_r(x, z, \omega) = K_1\bar{f}_1(\omega)e^{-i(mx+lz)} \tag{40}$$

Introducing the expressions for m and l, we can, from the expression [40] of the reflected potential in the frequency domain, find the corresponding potential in the time domain:

$$\bar{\varphi}_r(x, z, \omega) = K_1\bar{f}_1(\omega)e^{-i\omega\left[x\left(\frac{1}{\alpha^2}-\frac{1}{c^2}\right)^{1/2}+\frac{z}{c}\right]}$$

i.e., a product of two factors, both with known inverse transforms:

$$K_1\bar{f}_1(\omega) \longleftrightarrow K_1f_1(t)$$

$$e^{-i\omega\left[x\left(\frac{1}{\alpha^2}-\frac{1}{c^2}\right)^{1/2}+\frac{z}{c}\right]} \longleftrightarrow \delta\left[t - x\left(\frac{1}{\alpha^2}-\frac{1}{c^2}\right)^{1/2} - \frac{z}{c}\right]$$

This latter equation is seen as follows. Expressed simpler:

$$e^{-i\omega x} \longleftrightarrow \delta(t - x)$$

which is seen from the following formula:

$$\int_{-\infty}^{\infty} \delta(t - x)e^{-i\omega t}dt = e^{-i\omega x}$$

from [27]. Thus:

$$\varphi_r(x, z, t) = K_1f_1(t) * \delta\left[t - x\left(\frac{1}{\alpha^2}-\frac{1}{c^2}\right)^{1/2} - \frac{z}{c}\right] =$$

(by [25])

$$= \int_{-\infty}^{\infty} K_1f_1(\tau)\delta\left[t - x\left(\frac{1}{\alpha^2}-\frac{1}{c^2}\right)^{1/2} - \frac{z}{c} - \tau\right]d\tau =$$

(by [24])

$$= K_1 f_1 \left[t - x \left(\frac{1}{\alpha^2} - \frac{1}{c^2} \right)^{1/2} - \frac{z}{c} \right] \tag{41}$$

(by [27])

This result again shows that the reflected potential is a pulse travelling along positive x and z with the same waveform as the incident potential and multiplied by the reflection coefficient K_1. The same line of argument can be extended to the reflected transverse wave, as also to the case of an incident transverse wave (with reflected longitudinal and transverse waves).

The other two possibilities $\beta < |c| < \alpha$ and $|c| < \beta < \alpha$ can be treated similarly (see WHITE, 1965). Together this gives a unified treatment of surface and body waves at a plane boundary. A similar unified treatment of this problem is made by HASKELL (1953), applying matrix calculus (see Chapter 9).

8.3.4 Reflection of a pulse incident beyond the critical angle [1]

In classical treatments of wave behaviour at a boundary, it is generally assumed: (*1*) that the wave-front surfaces are plane, and (*2*) that the wave motion is simple harmonic (i.e., consists of only one frequency and extends infinitely in both directions).

In more modern treatments, one has tried to abandon these assumptions and to get a picture which better corresponds to reality, that is, assume: (*a*) curved wave fronts, especially spherical, which could be applied close to the wave source, and (*b*) a generalized pulse, consisting of many different frequencies.

Point (*a*) has been approached by Sommerfeld's integral expression (section 7.3). Point (*b*) has been approached in various ways (pulse functions in terms of Fourier expressions—most general—or special pulses, as the Dirac δ-function, the Heaviside function, etc.).

As a supplement to section 8.3.3 we shall now consider the *reflection of a compressional pulse beyond the critical angle at a liquid-liquid boundary*, which will include a *phase change*. The wave fronts are assumed *plane*. The waves are reflected against a medium with *higher* velocity.

The elementary theory for plane, simple harmonic waves will not be discussed here, as it does not offer any mathematical difficulties. For the case mentioned it is found that reflection is total (unchanged amplitude) and a phase change $= 2\varepsilon$ occurs, which corresponds to a time increase of $2\varepsilon/|\omega|$. The phase change ε depends only upon the properties of the media but not on frequency.

To make the following presentation more homogeneous with the rest of this book, especially with the notation used for integral transforms, we have changed some of the notation used by EWING, JARDETZKY and PRESS (1957, pp.90–92), as follows: $\hat{\varphi}_i(t) \rightarrow \overline{\varphi}_i(t)$, $g(\omega) \rightarrow \varphi_i(\omega)$, $\hat{\varphi}_r(t) \rightarrow \overline{\varphi}_r(t)$, $G(\omega) \rightarrow \varphi_r(\omega)$.

We assume the incident *pulse* to be (eq.[4a] in section 8.1):

$$\overline{\varphi}_i(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \varphi_i(\omega) e^{i\omega t} d\omega \tag{42}$$

[1] EWING, JARDETZKY and PRESS (1957, pp.90–92).

of which the complex Fourier transform is the following (eq.[24a] in section 8.1):

$$\varphi_i(\omega) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \overline{\varphi}_i(t) e^{-i\omega t} dt \tag{43}$$

Similarly, the time variation ("wave profile") in the reflected pulse can be written as the following Fourier integral:

$$\overline{\varphi}_r(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \varphi_r(\omega) e^{i\omega t} d\omega \tag{44}$$

Because of the phase change mentioned, we have:

$$\varphi_r(\omega) = \varphi_i(\omega) e^{i\omega \cdot 2\varepsilon/|\omega|} \tag{45}$$

Remember that amplitudes are the same in the reflected and incident pulses, and there is only a change of phase. We can also write this as follows:

$$\overline{\varphi}_r(t) = \overline{\varphi}_i(t) e^{i\omega \cdot 2\varepsilon/|\omega|} =$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \varphi_i(\omega) e^{i\omega t} d\omega \cdot e^{i\omega \cdot 2\varepsilon/|\omega|} =$$

(from [42], remembering that $e^{i\omega \cdot 2\varepsilon/|\omega|}$ can be placed under the integral sign, as it is independent of ω)

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \varphi_i(\omega) e^{i\omega t} d\omega \left(\cos 2\varepsilon + \frac{i\omega}{|\omega|} \sin 2\varepsilon\right) =$$

(by Euler's formula, noting that $\frac{\omega}{|\omega|}$ is omitted in the cosine term—always positive, even for negative arguments— but must be kept in the sine term)

$$= \overline{\varphi}_i(t) \cos 2\varepsilon + F(t) \sin 2\varepsilon \tag{46}$$

(remembering that ε is independent of ω) where:

$$F(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{i\omega}{|\omega|} \varphi_i(\omega) e^{i\omega t} d\omega \tag{47}$$

Eq.[47] can be rewritten, using the integration formula 858.5 in DWIGHT (1957, p.198):

$$\int_0^{\infty} \frac{\sin \omega x}{x} dx = \begin{cases} \frac{\pi}{2} \\ 0 \\ -\frac{\pi}{2} \end{cases} \text{if} \begin{cases} \omega > 0 \\ \omega = 0 \\ \omega < 0 \end{cases} \tag{48}$$

By a P preceding the integral, we mean the *principal value* of the integral, i.e., excluding singular points, where the integral diverges, as at $x = 0$ in the following integral:

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i\omega x}}{x}\,\mathrm{d}x = \mathrm{P}\left[\underbrace{\int_{-\infty}^{\infty}\frac{\cos\omega x}{x}\,\mathrm{d}x}_{\substack{(=0 \text{ as positive and negative}\\ \text{terms cancel})}} + i\int_{-\infty}^{\infty}\frac{\sin\omega x}{x}\,\mathrm{d}x\right] =$$

$$= 2i\int_{0}^{\infty}\frac{\sin\omega x}{x}\,\mathrm{d}x = \begin{cases} i\pi & \\ 0 & \text{if} \\ -i\pi & \end{cases}\begin{cases}\omega > 0\\ \omega = 0\\ \omega < 0\end{cases} \qquad [49]$$

Note here that P is needed before the first integral and before the cosine integral, but *not* before the sine integral, as:

$$\lim_{x\to 0}\frac{\sin\omega x}{x} = \omega$$

We then find that:

$$\frac{\omega}{|\omega|} = \mathrm{P}\frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i\omega x}}{x}\,\mathrm{d}x = \begin{cases} 1 & \\ 0 & \text{if} \\ -1 & \end{cases}\begin{cases}\omega > 0\\ \omega = 0\\ \omega < 0\end{cases} \qquad [50]$$

this integral being called the *Dirichlet discontinuous factor*. Eq.[47] then becomes:

$$F(t) = \frac{1}{\sqrt{(2\pi^3)}}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x}\int_{-\infty}^{\infty}\varphi_i(\omega)\mathrm{e}^{i\omega(t+x)}\mathrm{d}\omega \qquad [51]$$

$F(t)$ can be evaluated from [51] as soon as we know $\varphi_i(\omega)$ or the shape of the incident pulse. We assume the incident pulse to correspond to an explosion, i.e.:

$$\overline{\varphi}_i(t) = \begin{cases} 0 & \text{for} \quad t < 0\\ A_1\mathrm{e}^{-\sigma t} & \text{for} \quad t > 0\,, \qquad \sigma > 0\end{cases} \qquad [52]$$

Then, by equation [43]:

$$\varphi_i(\omega) = \frac{A_1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\mathrm{e}^{-\sigma t}\mathrm{e}^{-i\omega t}\mathrm{d}t =$$

(by [52] there is no contribution to this integral for $t < 0$)

$$= \frac{A_1}{\sqrt{(2\pi)}}\int_{0}^{\infty}\mathrm{e}^{-t(\sigma+i\omega)}\mathrm{d}t = \frac{A_1}{\sqrt{(2\pi)}}\,\frac{1}{\sigma + i\omega} \qquad [53]$$

Then [51] becomes:

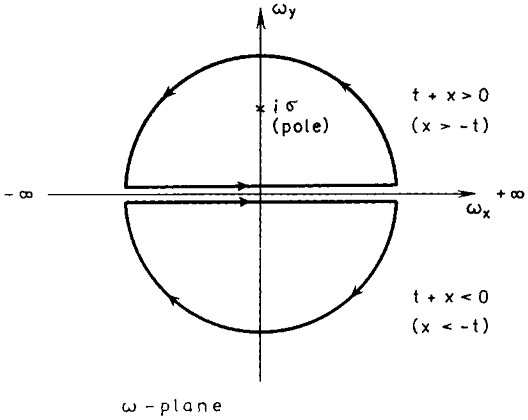


Fig.66.

$$F(t) = \frac{A_1}{2\pi^2} \mathrm{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega(t+x)}}{\sigma + i\omega} \mathrm{d}\omega \qquad [54]$$

The integral over ω is evaluated by contour integration in the complex ω-plane (see Fig.66). There is no branch point, but there is one pole of the integrand: $\sigma + i\omega = 0$ which gives $\omega = i\sigma$, located on the positive, imaginary axis. The residue Res at this pole is:

$$\mathrm{Res} = \lim_{\omega = i\sigma} (\omega - i\sigma) \frac{\mathrm{e}^{i\omega(t+x)}}{\sigma + i\omega} =$$

(evaluate by l'Hospital's rule)

$$= \frac{\mathrm{e}^{i^2\sigma(t+x)}}{i} = \frac{\mathrm{e}^{-\sigma t}\mathrm{e}^{-\sigma x}}{i} \qquad [55]$$

We consider two integration paths: lower half for $t + x < 0$, and upper half for $t + x > 0$. This is done to get vanishing contributions on the semicircular paths with infinite radius. Write the complex ω as $\omega = \omega_x + i\omega_y$. Then:

$$\mathrm{e}^{i\omega(t+x)} = \mathrm{e}^{i(\omega_x + i\omega_y)(t+x)} = \underbrace{\mathrm{e}^{i\omega_x(t+x)}}_{\text{periodic factor}} \; \underbrace{\mathrm{e}^{-\omega_y(t+x)}}_{\text{tends to zero, as the exponent } -\omega_y(t+x) \text{ tends to minus infinity}}$$

Lower half:	*Upper half*:
$\omega_y < 0$	$\omega_y > 0$
$t + x < 0$	$t + x > 0$
$-\omega_y(t + x) < 0$	$-\omega_y(t + x) < 0$

In both cases, the exponent $-\omega_y(t + x)$ tends to minus infinity, as ω_y approaches infinity, either negative (lower half) or positive (upper half). Therefore, there will be no contributions from the semicircular arcs. Also, there will be no contribution from the lower half, because of the absence of poles there. This means that the integral over ω vanishes for $x < -t$, and we need only extend the integral over x from $x = -t$ to plus infinity. From the upper half there is one contribution from the pole; see eq.[55]. As a result we can write [54] as follows, applying Cauchy's theorem:

$$F(t) = \frac{A_1}{\pi} e^{-\sigma t} \underbrace{P \int_{-t}^{\infty} \frac{e^{-\sigma x}}{x} dx}_{= -\overline{Ei}(\sigma t)} \qquad [56]$$

where $\overline{Ei}(\sigma t)$ is called the *exponential integral*. This is tabulated in WEAST (1964, pp.409–411).

Inserting [56] into [46] we have our final solution:

$$\overline{\varphi}_r(t) = \overline{\varphi}_i(t)\cos 2\varepsilon - \frac{A_1}{\pi} e^{-\sigma t}\overline{Ei}(\sigma t)\sin 2\varepsilon$$

for $t > 0$ and:

$$\overline{\varphi}_r(t) = -\frac{A_1}{\pi} e^{-\sigma t}\overline{Ei}(\sigma t)\sin 2\varepsilon \qquad [57]$$

for $t < 0$, by [52].

As expected, we find for $2\varepsilon = 0$ that $\overline{\varphi}_r(t) = \overline{\varphi}_i(t)$ and for $2\varepsilon = \pi$ that $\overline{\varphi}_r(t) = -\overline{\varphi}_i(t)$. Results for other phase changes can be calculated from [57]. They show good agreement with experiments by ARONS and YENNIE (1950).

It may be considered remarkable that [57.2] gives a reflected pulse even before the arrival of the incident pulse. This is explained by propagation of a certain amount of the incident energy through the reflecting medium at a velocity higher than that of the primary medium.

8.4 CAGNIARD'S METHOD

8.4.1 General outline of the method

This method has been given by CAGNIARD (1939). A short but very clear exposition of the method was given by DIX (1954). The book by CAGNIARD, FLINN and DIX (1962) is a translation into English of Cagniard's book of 1939, with additional explanations (see also EWING, JARDETZKY and PRESS, 1957, pp.115–121). In this presentation we shall essentially follow DIX (1954), but we have changed some of his notation so as to conform to the rest of the present book.

Physically, the method gives a solution to the problem of propagation of seismic pulses (instead of a continuous wave motion) from a spherical source in one medium. Two media in contact are also dealt with and the source is in one of these media. The problem includes first the direct waves, and secondly the waves which have been refracted and reflected at the boundary.

Mathematically, Cagniard uses the Laplace transform to solve this problem. It provides good exercise in using Laplace transforms and contour integrations in the complex plane. In more traditional work, with continuous motion, it is more appropriate to use Fourier transform.

DIX (1954) disregards the boundary and considers seismic pulse motion from a point source in an infinite medium. This simple case illustrates Cagniard's method well. The essence of this section is not to present any new mathematical tools, but rather to present a new combination of certain mathematical tools and certain seismic problems, both of which have existed for a long time.

We assume that we have a spherical pressure source in a homogeneous, perfectly elastic and isotropic body. The following steps enter into Cagniard's method:

(*a*) The displacement vector field $u(P', t)$ for time t and point P' can be written:

$$u(P', t) = \mathrm{grad}\varphi + \mathrm{curl}\psi \qquad [1]$$

φ represents the longitudinal wave and ψ the transverse wave. It is demonstrated in vector calculus that such a decomposition of a vector field is possible (see EWING, JARDETZKY and PRESS, 1957, p.8, eq.1–20′).

(*b*) The potentials φ and ψ satisfy the elastic equations of motion and the boundary conditions (at the free surface or an interface).

(*c*) The source function, i.e., the signal in the cavity, is assumed to correspond to a unit step:

$$\varphi = \frac{H\left(t - \frac{R}{v}\right)}{R} \qquad [2]$$

(for P waves) where H is the Heaviside unit function: $H(\tau) = 0$ for $\tau \leqslant 0$, and $H(\tau) = 1$ for $\tau > 0$.

(*d*) We take the Laplace transform of the whole problem, that is, take Laplace transforms of equations of motion, boundary conditions and source function using the definition [2a] in section 8.1, i.e.:

$$\left.\begin{aligned} \frac{\bar{\varphi}(r, z, p)}{p} &= \int_0^\infty \mathrm{e}^{-pt}\varphi(r, z, t)\mathrm{d}t \\ \frac{\bar{\psi}(r, z, p)}{p} &= \int_0^\infty \mathrm{e}^{-pt}\psi(r, z, t)\mathrm{d}t \end{aligned}\right| \qquad [3]$$

After application of the Laplace transform [3] the equations of motion can be solved by separation of the variables, and the solutions can then be combined so as to satisfy any boundary conditions or source conditions (e.g., item *c*).

(*e*) The next step is to recover the φ- and the ψ-response at r, z, t by an inverse transformation of [3]. Then the displacement vector is found from [1]. The results can be generalized to any source function (i.e., any input pressure).

This gives an outline of Cagniard's method.

8.4.2 Application to a source in a spherical cavity in an infinite medium

We shall now apply Cagniard's method to the simplest case: a source function according to (*c*) in a spherical cavity in an infinite medium (no boundaries). We carry through the steps (*a*)–(*d*). The linear combination of the solutions of the wave equation is made in such a way that the source function (*c*) is satisfied, whereas there are now no boundary conditions to fulfill (item *d*). When there is no interface, there will be no ψ-contribution, but only φ (i.e., direct *P* wave and no *S* waves).

The problem of *P* waves from a spherical cavity in an infinite medium has been solved by several other authors (see EWING, JARDETZKY and PRESS, 1957, p.15), which we do not need to discuss here.

The equation of motion, its Laplace transform, and the solution of the transformed equation. If we assume variations only with the radius *r* and with *z*, but no azimuthal variation, the equation for φ in cylindrical coordinates is:

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2} \qquad [4]$$

where v = longitudinal wave velocity.

Take the Laplace transform of [4]. The terms on the left-hand side do not change essentially, as shown by the following example:

$$\int_0^\infty \frac{\partial\varphi}{\partial r}\,\mathrm{e}^{-pt}\mathrm{d}t=\frac{\partial}{\partial r}\int_0^\infty \varphi\,\mathrm{e}^{-pt}\mathrm{d}t=\frac{\partial}{\partial r}\left(\frac{\overline{\varphi}}{p}\right)=\frac{1}{p}\frac{\partial\overline{\varphi}}{\partial r} \qquad [5]$$

but on the right-hand side, where *t* enters, we get:

$$\int_0^\infty \frac{\partial\varphi}{\partial t}\,\mathrm{e}^{-pt}\mathrm{d}t=\underbrace{[\varphi\mathrm{e}^{-pt}]_0^\infty}_{=0}-\int_0^\infty \varphi\mathrm{e}^{-pt}(-p)\mathrm{d}t=p\int_0^\infty \varphi\mathrm{e}^{-pt}\mathrm{d}t$$

and:

$$\int_0^\infty \frac{\partial^2\varphi}{\partial t^2}\,\mathrm{e}^{-pt}\mathrm{d}t=p^2\int_0^\infty \varphi\mathrm{e}^{-pt}\mathrm{d}t=p^2\,\frac{\overline{\varphi}}{p} \qquad [6]$$

The Laplace transform of [4] is thus:

$$\frac{\partial^2\overline{\varphi}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{\varphi}}{\partial r}+\frac{\partial^2\overline{\varphi}}{\partial z^2}-\frac{p^2}{v^2}\overline{\varphi}=0 \qquad [7]$$

which we solve by the method of separation of variables:

$$\overline{\varphi}=P(r)F(z) \qquad [8]$$

Making the substitution [8] in [7] we get the following two equations (cf. section 7.2):

$$\left.\begin{aligned} &(\lambda r)^2 \frac{d^2P}{d(\lambda r)^2} + \lambda r \frac{dP}{d(\lambda r)} + (\lambda r)^2 P = 0 \\ &\frac{d^2F}{dz^2} = \left(\lambda^2 + \frac{p^2}{v^2}\right) F \end{aligned}\right\}$$

which have the following solutions:

$$\left.\begin{aligned} &P(r) = J_0(\lambda r) \\ &F(z) = e^{\pm\left(\lambda^2 + \frac{p^2}{v^2}\right)^{1/2} z} \end{aligned}\right| \qquad [9]$$

The Bessel function Y_0 must be omitted as it is infinite all along the axis $r = 0$ (see sections 4.3 and 5.3). λ is an arbitrary parameter.

As [7] is linear, that is, we have $\overline{\varphi}$ in its first degree only, a general solution $\overline{\varphi}$ is found by adding solutions [8] with different values of the arbitrary parameter λ. Quite generally, we add or integrate all solutions by integrating over λ from 0 to ∞ (the negative values of λ need not be considered as only λ^2 enters into [7] after the substitution [8]). This gives a general expression for the left-hand side of [3.1]. The right-hand side of [3.1] can be evaluated if we introduce the expression [2] for φ:

$$\int_0^\infty e^{-pt} \frac{H\left(t - \frac{R}{v}\right)}{R} dt =$$

(substitute $t - R/v = \tau$; $dt = d\tau$)

$$= \underbrace{\int_{-R/v}}_{} ^\infty e^{-p\tau} e^{-pR/v} \frac{H(\tau)}{R} d\tau$$

$\rightarrow 0$ (as $H(\tau) = 0$ for $\tau < 0$)

$$= \frac{e^{-pR/v}}{R} \int_0^\infty e^{-p\tau} d\tau = \frac{e^{-pR/v}}{pR}$$

Thus:

$$\frac{\overline{\varphi}}{p} = \int_0^\infty e^{-pt} \frac{H\left(t - \frac{R}{v}\right)}{R} dt = \frac{e^{-pR/v}}{pR}$$

$$= \int_0^\infty G(\lambda) J_0(\lambda r) e^{-\left(\lambda^2 + \frac{p^2}{v^2}\right)^{1/2} z} d\lambda \qquad [10]$$

where p in $\overline{\varphi}/p$ is included in $G(\lambda)$ in the last member, and z is chosen positive (therefore only the minus-sign in the exponent should be used). Eq.[10] holds if:

$$G(\lambda) = \frac{\lambda}{p\left(\lambda^2 + \frac{p^2}{v^2}\right)^{1/2}} \qquad [11]$$

Proof that [11] *is the expression for $G(\lambda)$, which is needed to satisfy the initial condition, i.e.,* [2]. There are in this case no boundary conditions to satisfy. In our cylindric coordinate system, we have $R^2 = r^2 + z^2$. For $r = 0$ we can immediately show that [11] is correct by direct integration of [10], leaving out p in the denominator:

$$\underset{(J_0(0)=1)}{\int_0^\infty \frac{\lambda e^{-\left(\lambda^2+\frac{p^2}{v^2}\right)^{1/2} z}}{\left(\lambda^2+\dfrac{p^2}{v^2}\right)^{1/2}} d\lambda} = \frac{1}{2z}\int_{\lambda=0}^{\infty} \frac{e^{-\left(\lambda^2+\frac{p^2}{v^2}\right)^{1/2} z}}{\left(\lambda^2+\dfrac{p^2}{v^2}\right)^{1/2} z}\, d\underbrace{\left[\left(\lambda^2+\frac{p^2}{v^2}\right)z^2\right]}_{(\text{put } = x^2)}$$

$$= \frac{1}{z}\int_{\lambda=0}^{\infty} e^{-x}dx = \frac{1}{z}\left[-e^{-x}\right]_0^\infty = \frac{1}{z}e^{-pz/v} \qquad [12]$$

which should be proved (remembering that p was left out here, and that $R = z$, as $r = 0$). In other words, we have proved that [10] holds for $r = 0$, with the expression [11] for $G(\lambda)$.

For $r \neq 0$ we need a much more elaborate discussion to demonstrate that [11] is correct, i.e., fits to the unit step source, and we shall follow Cagniard in this treatment.

Thus, from [10] and [11], using the substitutions: $\lambda = pu$ and $1/v = S$ ("slowness"), we have to prove that:

$$\frac{e^{-pR/v}}{pR} = \int_0^\infty \frac{uJ_0(pur)e^{-p(u^2+S^2)^{1/2}z}du}{(u^2+S^2)^{1/2}} = \int_0^\infty e^{-pt}A(r, z, t)dt \qquad [13]$$

and that A will give the unit step function.

We use the following expression for the Bessel function:

$$J_0(z) = \frac{1}{\pi}\int_0^\pi \cos(z\cos\omega)d\omega = \frac{2}{\pi}\int_0^{\pi/2} \mathrm{Re}[e^{-iz\cos\omega}]\,d\omega \qquad [14]$$

(eq.[44] in section 5.2; Re = real part of). In the last step in [14] we take only half the integration interval and multiply by 2; this is permitted because of the symmetry of the integrand around the centre value $\omega = \pi/2$ (see Fig.67).

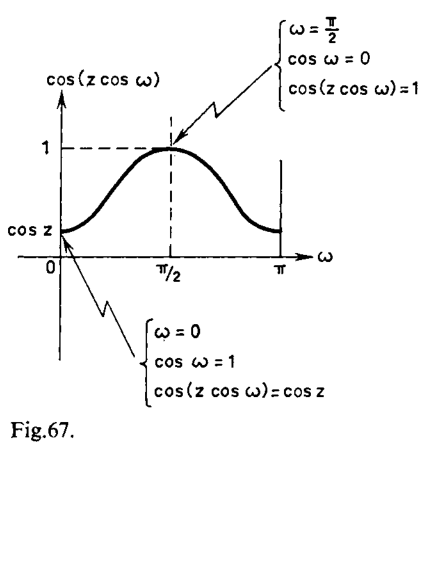


Fig.67.

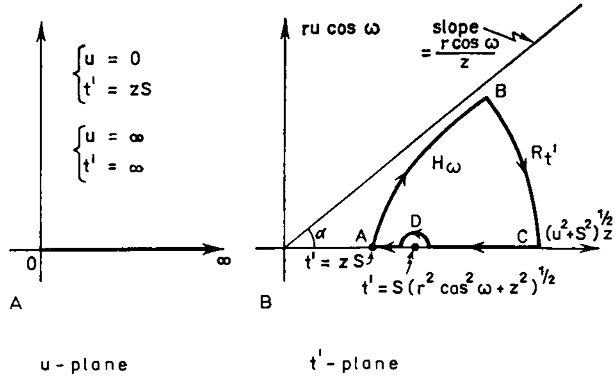


Fig.68.

In [13] put $a = +(u^2 + S^2)^{1/2}$ and use the integral expression [14] for the Bessel function:

$$\frac{2}{\pi}\int_0^\infty\int_0^{\pi/2} \mathrm{Re}[e^{-p(iur\cos\omega + az)}]\,d\omega\,\frac{u\,du}{a}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \mathrm{Re}\left[\int_0^\infty e^{-p(iur\cos\omega + az)}\,\frac{u\,du}{a}\right] d\omega \qquad [15]$$

where we have changed the integration order (see FRANK and VON MISES, 1930, p.29). Now, in order to make [15] approach the right-hand side of [13], e.g., with an integrand containing e^{-pt}, it is suitable to introduce a new variable t' (replacing u) defined as:

$$t' = iur\cos\omega + az = iur\cos\omega + (u^2 + S^2)^{1/2}z \qquad [16]$$

which gives us a factor $e^{-pt'}$ in [15]. Differentiating [16] we get:

$$\frac{\partial u}{\partial t'} = \frac{1}{ir\cos\omega + \dfrac{uz}{a}} \qquad [17]$$

and our integral [15] becomes:

$$\frac{2}{\pi}\,\mathrm{Re}\int_0^{\pi/2} d\omega \int_{H_\omega} e^{-pt'}\,\frac{u}{a}\,\frac{\partial u}{\partial t'}\,dt' \qquad [18]$$

See Fig.68: the original integration path in the u-plane, from 0 to ∞ along the real axis, is transformed into the H_ω-curve (a sort of hyperbolic curve) in the complex t'-plane. A disadvantage with the integration path H_ω is that it depends on ω; this is obvious from [16], as in passing along H_ω, we change both the real and the imaginary part of t' (and the latter contains ω). This can be avoided by making an integration only along the real axis of t' (the real part of t' does not contain ω). This is accomplished by the contour in Fig.68: $R_{t'}$ is a circular arc with great radius, and there is an indentation on the real axis of t' to avoid a pole.

The pole is obtained in this way: from [17] $\partial u/\partial t'$, which enters the integrand in [18], becomes ∞, when:

$$ir\cos\omega + \frac{uz}{(u^2+S^2)^{1/2}} = 0$$

Solving for u and t' respectively, using [16], we find the singular point:

$$\left.\begin{aligned} u &= u_0 = \frac{(\underline{+})\, iSr\cos\omega}{(r^2\cos^2\omega + z^2)^{1/2}} \\ t' &= t'_0 = S(r^2\cos^2\omega + z^2)^{1/2} \end{aligned}\right| \qquad [19]$$

The expression for t'_0 corresponds to choosing the minus-sign for u_0.

When we avoid this pole, there is no pole within the contour in the t'-plane in Fig.68, and by Cauchy's integral theorem we have:

$$\int_{H_\omega} + \underbrace{\int_{R_{t'}}}_{=0} + \int_{\infty}^{zS} = 0$$

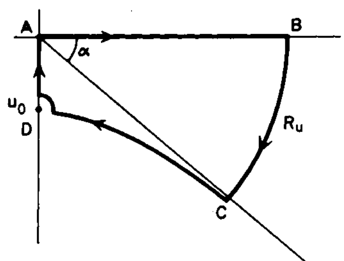
(because of the factor $e^{-pt'}$ in [18], with t' very large)
Thus:

$$\int_{H_\omega} = -\int_{\infty}^{zS} = +\int_{zS}^{\infty} \qquad [20]$$

(along real axis of t': independent of ω) and [18] becomes:

$$\frac{2}{\pi}\,\mathrm{Re}\int_0^{\pi/2} d\omega \int_{zS}^{\infty} e^{-pt'}\frac{u}{a}\frac{\partial u}{\partial t'}\, dt' \qquad [21]$$

The corresponding contour in the u-plane is shown in Fig.69; the same lettering, $ABCD$, is used as for the t'-plane. This is a good example of conformal transformation (cf. section 2.2).



u - plane

Fig.69.

We compare [21] with [13]. In order to make [21] more similar to [13], we again rearrange the order in [21]:

$$\int_{zS}^{\infty} e^{-pt'} \left[\frac{2}{\pi} \operatorname{Re} \int_{0}^{\pi/2} \frac{u}{a} \frac{\partial u}{\partial t'} \, d\omega \right] dt' \qquad [22]$$

Compare this with [13]. We see that if we define A as:

$$A(r, z, t') \begin{cases} = 0 & \text{for } t' < zS \\ = \dfrac{2}{\pi} \operatorname{Re} \displaystyle\int_{0}^{\pi/2} \frac{u}{a} \frac{\partial u}{\partial t'} \, d\omega & \text{for } t' > zS \end{cases} \qquad [23]$$

we can say that we have solved our problem, because such an A fulfills [13], i.e., it gives the unit step impulse. Eq.[23] is the *first form* of our solution.

Transformation of the expression [23] *for* $A(r, z, t')$. For $r = 0$ we immediately verify that [23] is correct. We then have:

$$\frac{\partial u}{\partial t'} = \frac{a}{uz} \quad ; \quad \frac{u}{a} \frac{\partial u}{\partial t'} = \frac{1}{z} \qquad [24]$$

and:

$$A(0, z, t')_{t' > Sz} = \frac{2}{\pi} \operatorname{Re} \underbrace{\left[\frac{u}{a} \frac{\partial u}{\partial t'}\right]}_{\text{independent of } \omega \text{ for } r = 0 \text{ by [17]}} \frac{\pi}{2} = \frac{1}{z} \qquad [25]$$

i.e., the unit step function divided by the distance from the source. This case ($r = 0$) was already verified in [12] and its consideration would be superfluous at this point.

The next problem is to consider [23] for $r \neq 0$. We again use the substitution [16], but this time replace the variable ω by the variable u, while keeping t' constant. Differentiating [16] under these conditions, we find:

$$\frac{\partial \omega}{\partial u} = \frac{ir \cos\omega + \dfrac{uz}{a}}{iur \sin\omega} \qquad [26]$$

Combining with [17] we get:

$$\frac{\partial u}{\partial t'} \frac{\partial \omega}{\partial u} = \frac{1}{iur \sin\omega} = \frac{1}{i[u^2 r^2 + (t' - az)^2]^{1/2}} \qquad [27]$$

And the integral [23] now becomes:

$$A_{t' > zS} = \frac{2}{\pi} \operatorname{Re} \int_{C_{t'}} \frac{u}{a} \frac{\partial u}{\partial t'} \frac{\partial \omega}{\partial u} \, du =$$

($C_{t'}$ is the corresponding integration path in the u-plane)

TABLE V

SCHEMATICALLY SUMMARIZED SUCCESSIVE TRANSFORMATIONS OF VARIABLES

Equation	Used variables ω	u	t'	Remarks
[15]	*	*		
[18], [21], [22]	*		*	$u \to t'$ by [16]
[23]	*			t' thrown out
[28]		*		$\omega \to u$ by [16]

$$= -\frac{2}{\pi} \operatorname{Re} i \int_{C_{t'}} \frac{u\mathrm{d}u}{a[u^2r^2 + (t' - az)^2]^{1/2}} =$$

$$= \frac{2}{\pi} \operatorname{Im} \int_{C_{t'}} \frac{u\mathrm{d}u}{a[u^2r^2 + (t' - az)^2]^{1/2}} \qquad [28]$$

using the following rule (Im = imaginary part):

$$\operatorname{Re}[i(x + iy)] = \operatorname{Re}(ix - y) = -y = -\operatorname{Im}(x + iy)$$

The successive transformations of variables are summarized in Table V.

Fig.70 illustrates the integration path $C_{t'}$ in the u-plane, from $\omega = 0$ to $\pi/2$, with t' constant. All of the properties of the figure can be derived from eq.[16]. If in [16] we put $u = x + iy$ and separate real and imaginary parts in the resulting equation, we get two equations yielding expressions for x and y:

$$x = \frac{[t'^2(z^2 - r^2\cos^2\omega) - (t'r\cos\omega + S^2z^2)(z^2 + r^2\cos^2\omega)]^{1/2}}{z^2 + r^2\cos^2\omega}$$

$$y = -\frac{t'r\cos\omega}{z^2 + r^2\cos^2\omega}$$

This is a parametric representation of the curves in Fig.70; z and r are constant. A given ω means that t' is the running parameter: we then get the hyperbola-shaped curves. On the other hand, a given t' means that ω is the running parameter: we get the heavy curve in the figure.

We also verify all properties of the curves from these expressions for x and y, e.g.:

(a) $\omega = \pi/2$ gives:

$$y = 0$$

$$x = \frac{+(t'^2 - S^2z^2)^{1/2}}{z}$$

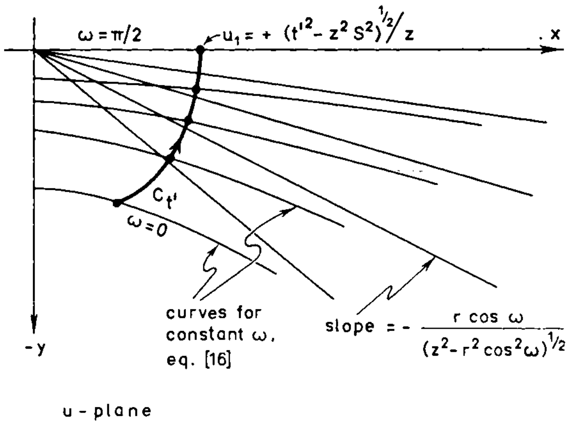


Fig.70.

(*b*) $\omega = 0$ gives:

$$x = \frac{[t'^2(z^2 - r^2) - (t'r + S^2z^2)(z^2 + r^2)]^{1/2}}{z^2 + r^2}$$

$$y = - \frac{t'r}{z^2 + r^2}$$

(*c*) the slope of the asymptotes is (note that t' increases along the ω-curves as we go away from the origin):

$$\lim_{t' \to \infty} \frac{y}{x} = - \frac{r \cos\omega}{(z^2 - r^2 \cos^2\omega)^{1/2}}$$

As we made the integration over t' along the real axis in the t'-plane (Fig.68 and eq. [22]), we should obviously select only *real* values of t' in the integration path $C_{t'}$ (Fig.70).

Considering a complex number and its conjugate: $z = x + iy$ and $\bar{z} = x - iy$ we have:

$$\text{Im}z = y = \frac{z - \bar{z}}{2i}$$

We apply this to [28]:

$$A_{t' > zS} = \frac{1}{i\pi} \left[\int_{C_{t'}} \frac{u\mathrm{d}u}{a[u^2r^2 + (t' - az)^2]^{1/2}} \right.$$

$$\left. - \int_{\bar{C}_{t'}} \frac{\bar{u}\mathrm{d}\bar{u}}{\bar{a}[\bar{u}^2r^2 + (t' - \bar{a}z)^2]^{1/2}} \right] = \frac{1}{i\pi} \int_{C_{t'} + C'_{t'}} \frac{u\mathrm{d}u}{a[u^2r^2 + (t' - az)^2]^{1/2}} \qquad [29]$$

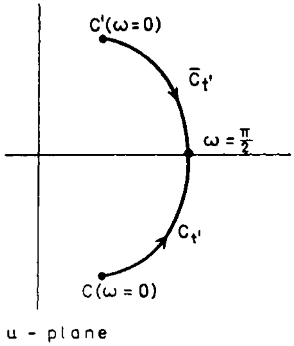


Fig.71.

$C'_{t'} = -\overline{C}_{t'}$, i.e., the integral taken along the conjugate path, but in reverse order (see Fig.71).

Evaluation of [*29*]. The integrand in [29] has the following branch points in the u-plane:

(a) At the end points C and C' of the path in [29]; $\omega = 0$ makes $\sin\omega = 0$ $(\partial u/\partial t')(\partial\omega/\partial u) = \infty$ (compare [27]). Or $u^2r^2 + (t' - az)^2 = 0$, $t' = iru + az$ for $\omega = 0$ by [16].

(b) When $a = 0$; $a = (u^2 + S^2)^{1/2} = 0$, $u^2 = -S^2$, $u = \pm iS$, i.e., at two points on the imaginary u-axis (Q and Q').

The four points C, C', Q and Q' are both *poles* and *branch points* of the integrand and the corresponding *branch cuts* are shown in Fig.72 and 73. Crossing a branch cut changes the sign of the integral.

Fig.72 gives for the integral [29]:

$$\int_{C_{t'}+C'_{t'}} = \int_{D'+D}$$

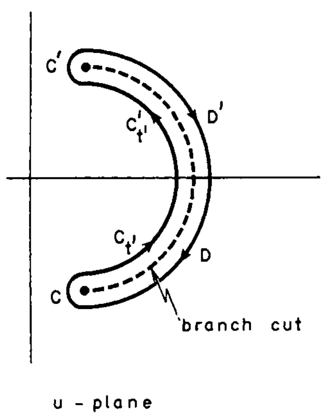


Fig.72.

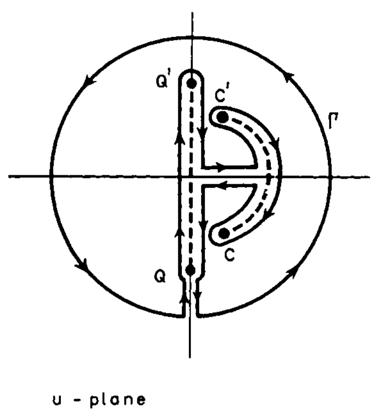


Fig.73.

They are numerically equal and the opposite direction along D, D' (change of sign) is cancelled by the fact that we are now on the opposite side of the branch cut. Therefore:

$$\int_{C_{t'}+C'_{t'}} = \frac{1}{2} \int_{C_{t'}+C'_{t'}+D'+D}$$

By comparison with Fig.8, Fig.73 gives, using Cauchy's theorem (there are no poles within the contour):

$$-\int_{CC'} - \int_{QQ'} + \int_{\Gamma} = 0$$

and where the signs ahead of the integrals are determined by the sense of rotation. In this abbreviated writing, CC' is the integral around the moon-shaped area CC', QQ' is the integral around QQ', and Γ the integral around the circle with infinite radius (Fig.73). The connections between these integrals, i.e., along the real u-axis between CC' and QQ' and along the imaginary u-axis between QQ' and Γ, give no contributions to the contour integral, as they are traversed twice in opposite directions. Neither does QQ' give any contribution, because there is a factor u in the numerator of the integrand in [29], with u having different signs in the different quadrants, together with the fact that QQ' is a branch cut. In case of the branch cut CC', there is a contribution because here u has the same sign on opposite sides of the cut.

Therefore we get:

$$\int_{CC'} = \int_{C_{t'}C'_{t'}D'D} = \int_{\Gamma} \qquad [30]$$

and:

$$A(r, z, t')_{t'>zS} = \frac{1}{2\pi i} \lim_{\Gamma\to\infty} \int_{\Gamma} \frac{u\,du}{a[u^2(r^2+z^2)+z^2S^2+t'^2-2t'az]^{1/2}} \qquad [31]$$

The integral [31] is easily evaluated by introducing polar coordinates in the u-plane, i.e., putting $u = R_1\, e^{i\theta}$ and integrating over θ from 0 to 2π. Letting $R_1 \to \infty$ we then find from [31] that:

$$A(r, z, t')_{t'>zS} = \frac{1}{(r^2+z^2)^{1/2}} \qquad [32]$$

And this is what should be proved, i.e., that A is the unit step solution of our problem.

Making the proof complete for $zS < t' < RS$ also. We have shown that A is the unit step only for $t' > zS$, but it is necessary that $t' > RS$ for this to be true; in other words, for $zS < t' < RS$, we must have $A = 0$ (see Fig.74). We cannot have any pulse before the time $t' = RS = R/v$, i.e., before the time that the pulse arrives at a distance R from the source.

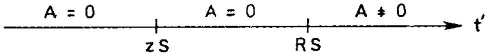


Fig.74.

To prove that $A = 0$ for $zS < t' < RS$ also, we put:

$$t' = (r^2c^2 + z^2)^{1/2}S \qquad (0 < c < 1) \tag{33}$$

which just means placing t' between the limits mentioned. Then from [16]:

$$t' = (r^2c^2 + z^2)^{1/2}S = iru\cos\omega + (u^2 + S^2)^{1/2}z \tag{34}$$

which is constant along the path $C_{t'}$, i.e., for a given constant value of t'. Solving [34] for u, we have on $C_{t'}$:

$$u = \frac{rSz(c^2 - \cos^2\omega)^{1/2}}{r^2\cos^2\omega + z^2} - i\,\frac{rS\cos\omega(r^2c^2 + z^2)^{1/2}}{r^2\cos^2\omega + z^2} \tag{35}$$

$C_{t'}$ starts at $\omega = 0$. For this ω, u becomes purely imaginary from [35]:

$$u_{\omega=0} = -i\,\frac{rS}{R^2}\left[(r^2c^2 + z^2)^{1/2} - z(1 - c^2)^{1/2}\right] \tag{36}$$

i.e., point E in Fig.75. $C_{t'}$ corresponds to some particular value of t', i.e., from [33] to some particular value of c. According to [35], we have $\cos\omega = c$ on the imaginary u-axis (on this axis, the first term of [35] must be zero and this happens only for $\cos\omega = c$).

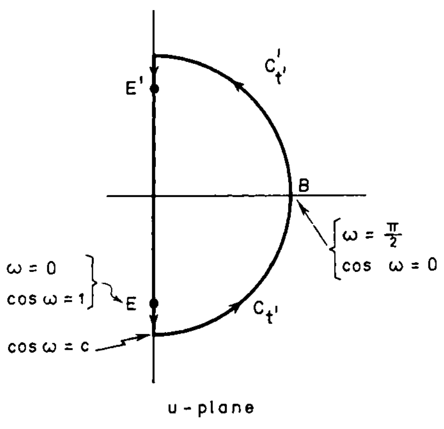


Fig.75.

The path $C_{t'}$ runs from E down the imaginary u-axis to $\cos\omega = c$ (the particular c chosen for $C_{t'}$), and after that up to B. At B we have $\omega = \pi/2$, and from [35] $u = rcS/z$.

We complete the contour by $C'_{t'}$ and $E'E$. There are no poles inside this contour; thus we have from Cauchy's integral theorem:

$$\int_{C_{t'}} + \int_{C'_{t'}} + \int_{E'E} = 0$$

or:

$$\int_{C_{t'}} + \int_{C'_{t'}} = -\int_{E'E} = +\int_{EE'}$$

and:

$$\int_{EE'} \frac{u\mathrm{d}u}{i\pi a[u^2r^2 + (t' - az)^2]^{1/2}} = 0 \qquad [37]$$

Eq.[37] is seen to be correct because positive and negative parts (above and below the origin along the imaginary u-axis) cancel each other; only the factor u in the numerator changes sign, the other factors being the same symmetrically with respect to the origin. This proves that $A = 0$ for $t' < RS$. Therefore, we can replace zS by RS in equations above, like [29], [28], [25].

8.4.3 Concluding remarks and generalization to any source function

Dɪx (1954) finishes his paper by an interesting discussion of Cagniard's method. He says, that Cagniard's method is not to use the standard Laplace transform inversion formula, but to use a series of transformations to beat the expression for the Laplace transform into the explicit Laplace transform integral, thus permitting us to extract the desired solution directly out of this integral expression. An advantage of Cagniard's method, which appears to outweigh all other methods, is that it permits exact numerical computation of examples, whereas alternate approaches usually appeal to approximations which are good only at large distances (large R). Cagniard's method is simplest at and near $r = 0$, which is just the region of maximum interest for reflection prospecting. As a conclusion of his discussion, Dɪx (1954) says that Cagniard's procedure is the simplest that he has been able to find.

In our study of Cagniard's method, we have limited ourselves to the simplest possible case: an explosive source, with a unit step as source function, in an unlimited medium. The method can be extended in two important aspects. One is to include a discontinuity surface, when reflected and refracted waves, in addition, enter into the discussion. For this, the reader is referred to Cᴀɢɴɪᴀʀᴅ (1939) or to Cᴀɢɴɪᴀʀᴅ, Fʟɪɴɴ and Dɪx (1962). The other important generalization is to consider any kind of pulse, not necessarily the unit step.

We shall briefly sketch the procedure in the last case. For this we first need the *Duhamel integrals*:

$$\left.\begin{aligned} \varphi(r, z, t) &= \int_0^t f'(t - \tau)A(r, z, \tau)\mathrm{d}\tau \\ \psi(r, z, t) &= \int_0^t f'(t - \tau)B(r, z, \tau)\mathrm{d}\tau \end{aligned}\right| \qquad [38]$$

where $f(t)$ represents the action of the source.

Proof of the Duhamel integrals [38] (Cᴀɢɴɪᴀʀᴅ, Fʟɪɴɴ and Dɪx, 1962, pp.35, 16). Let $\varphi\ (t, r, z)$ and $\psi(t, r, z)$ be the solutions of the problem. For positive real p we form their Laplace transforms:

$$\bar{\varphi}(p, r, z) = p \int_0^\infty \varphi(t, r, z) e^{-pt} dt$$

$$\bar{\psi}(p, r, z) = p \int_0^\infty \psi(t, r, z) e^{-pt} dt \quad [39]$$

and:

$$\bar{f}(p) = p \int_0^\infty f(t) e^{-pt} dt$$

$f(t)$ is the "excitation function" or "source function". *Define* $X_p(p, r, z)$ and $Y_p(p, r, z)$ as follows:

$$\bar{\varphi}(p, r, z) = X_p(p, r, z)\bar{f}(p)$$

$$\bar{\psi}(p, r, z) = Y_p(p, r, z)\bar{f}(p) \quad [40]$$

and *define* $A(t, r, z)$ and $B(t, r, z)$ by:

$$\frac{X_p(p, r, z)}{p} = \int_0^\infty e^{-p\tau} A(\tau, r, z) d\tau$$

$$\frac{Y_p(p, r, z)}{p} = \int_0^\infty e^{-p\tau} B(\tau, r, z) d\tau \quad [41]$$

The functions A, B, f, φ and ψ are thus connected by the relations:

$$\int_0^\infty e^{-pu} \varphi(u, r, z) du = p \left[\int_0^\infty e^{-p\tau} A(\tau, r, z) d\tau \right] \left[\int_0^\infty e^{-pu} f(u) du \right]$$

$$\int_0^\infty e^{-pu} \psi(u, r, z) du = p \left[\int_0^\infty e^{-p\tau} B(\tau, r, z) d\tau \right] \left[\int_0^\infty e^{-pu} f(u) du \right] \quad [42]$$

Take [42.1] and write the right-hand side in the form of a double integral; and then take ζ and $\zeta + \eta = u$ as new variables of integration (note that $f(t)$ is $= 0$ for $t \leqslant 0$, and therefore integration over f needs only to be extended from $\zeta = 0$ to $\zeta = u$, as f vanishes for higher values of ζ):

$$\int_0^\infty e^{-pu} \varphi(u) du = p \int_0^\infty \int_0^\infty e^{-p(\zeta+\eta)} A(\zeta) f(\eta) d\zeta d\eta$$

$$= p \int_0^\infty e^{-pu} du \int_0^u A(\zeta) f(u - \zeta) d\zeta \quad [43]$$

Integrate the left-hand side of [43] by parts:

$$\int_0^\infty e^{-pu} \varphi(u) du = \underbrace{\left[e^{-pu} \int_0^u \varphi(u) du \right]_0^\infty}_{\substack{= 0 \text{ at } u = \infty \\ \text{and at } u = 0}} + p \int_0^\infty e^{-pu} \left[\int_0^u \varphi(\xi) d\xi \right] du \quad [44]$$

Equating integrands on the right-hand sides of [43] and [44] we get:

$$\int_0^u \varphi(\xi)\mathrm{d}\xi = \int_0^u A(\zeta)f(u-\zeta)\mathrm{d}\zeta \qquad [45]$$

Take t as a variable instead of u and differentiate [45] with respect to t:

$$\varphi(t, r, z) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t-\tau)A(\tau, r, z)\mathrm{d}\tau = \int_0^t f'(t-\tau)A(\tau, r, z)\mathrm{d}\tau \qquad [46]$$

i.e., the Duhamel integral, which should be proved. Eq.[42.2] can be treated in the same way.

This development obviously involves a generalization to any source function $f(t)$. In our earlier discussion in this section we had $f(t) = 1$ for $t > 0$ (Heaviside unit step), and then we got $\bar{f}(p) = 1$, by carrying out the integration in the Laplace transform [39], Thus $\bar{\varphi} = X_p$, $\bar{\psi} = Y_p$ by [40], and we are back to the discussion we have covered in detail in this section.

In the more general case of *any type of source function* (not only a unit step function) Cagniard's method consists of the following operations:

(*1*) The so-called *exponential coefficients* X_p and Y_p [40] are obtained by solving the wave equation after it has undergone a Laplace transformation.

(*2*) The so-called *transmission factors* A and B are found by Carson's equations [41].

(*3*) The functions φ and ψ are determined by the *Duhamel integrals* [38].

(*4*) Displacements are then obtained from [1].

Chapter 9

MATRIX CALCULUS

9.1 INTRODUCTION

Matrix calculations are studied in the mathematics course which is supposed to be familiar to readers of this book, and are therefore omitted here. Reference is made to the following textbooks: FRANK and VON MISES (1930), BIEBERBACH and BAUER (1933), LINDELÖF and ULLRICH (1934), MARGENAU and MURPHY (1943), JEFFREYS and JEFFREYS (1946), MATHEWS and WALKER (1965), and others.

Matrices have many applications in seismology and geophysics. I would like to mention two examples: (*a*) in treatments of stress and strain: see MURNAGHAN (1951); (*b*) in deducing dispersion properties of Love and Rayleigh waves (HASKELL, 1953), or, more generally, elastic wave propagation in multi-layered media (KNOPOFF, 1964).

Especially in application (*b*), the matrix calculus has proved to offer a very powerful method to simplify otherwise very complicated calculations. For this reason I shall limit myself here to a review of the paper by HASKELL (1953).

Quite generally, matrices are useful and convenient in problems dealing with a great number of quantities of the same kind, as, e.g., stress and strain components (in general tensors of higher order), and in problems of wave propagation in multi-layered media, whether surface waves or body waves are concerned. Expressions then always become large and complicated, even with matrix calculus. However, the advantage with matrix calculus in such cases is that it provides a much better survey of the whole expression (e.g., for a large number of layers), and in numerical calculations it is more reliable, i.e., it helps one to remember better to include all terms than do traditional procedures.

The matrix method may be considered as no more than a change in notation, but the matrix notation suggests a *systematic* computational procedure, which greatly facilitates the computations. The matrix method is very useful when modern electronic computers are used for the numerical solutions.

An array of quantities or *elements:*

$$\begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \dots & \dots & \dots & \dots \\ K_{m1} & K_{m2} & \dots & K_{mn} \end{bmatrix} = [K_{ik}] \qquad [1]$$

is called a *matrix of order $m \times n$*. It has m rows and n columns. We may write it as a single symbol $= [K_{ik}]$, the first index referring to the row, the second to the column. A particular case is the square matrix (or order $n \times n$). A matrix of order $m \times 1$ has m rows and one column. Similarly, a matrix of order $1 \times n$ has one row and n columns.

9.2 HASKELL'S MATRIX METHOD FOR RAYLEIGH WAVES[1]

9.2.1 *Wave propagation in a structure with an arbitrary number of layers*

Suppose we have a layered structure, with n layers (including the lowest, semi-infinite medium); then there are in general four boundary conditions at each interior discontinuity surface (continuity of two displacement components and of two stress components), and two boundary conditions at the free surface (vanishing of two stress components), i.e., in all $4n - 2$ boundary conditions. For Love waves, there are $2n - 1$ conditions, as at each interface there is only one displacement and one stress continuity. Each boundary condition gives a homogeneous linear equation in the unknowns, of which there are as many as there are boundary conditions. This system of equations has a solution if its determinant vanishes. The determinant $= 0$ defines the dispersion equation. As seen from Table VI, two layers over a semi-infinite medium lead to a determinant of ten rows and ten columns, which involves very lengthy calculations. We shall now use the matrix method to the problem of wave propagation in a structure of an arbitrary number of layers.

TABLE VI

NUMBER OF BOUNDARY CONDITIONS FOR SURFACE WAVES IN A LAYERED STRUCTURE

Number of layers n	*Number of boundary conditions* *Rayleigh* ($4n - 2$)	*Love* ($2n - 1$)	*Remarks*
1	2	(1)	semi-infinite medium; no Love waves
2	6	3	1 layer + semi-infinite medium
3	10	5	2 layers + semi-infinite medium
4	14	7	3 layers + semi-infinite medium
6	22	11	5 layers + semi-infinite medium
8	30	15	7 layers + semi-infinite medium
10	38	19	9 layers + semi-infinite medium

We shall assume plane waves. Dispersion properties will not be affected by this assumption, since, for instance, a spherical wave can be expressed as an integral of plane waves (Sommerfeld integral; see section 7.3). With reference to Fig.76 we further assume that the layers are solid, homogeneous and isotropic and that the waves propagate in the

[1] HASKELL (1953).

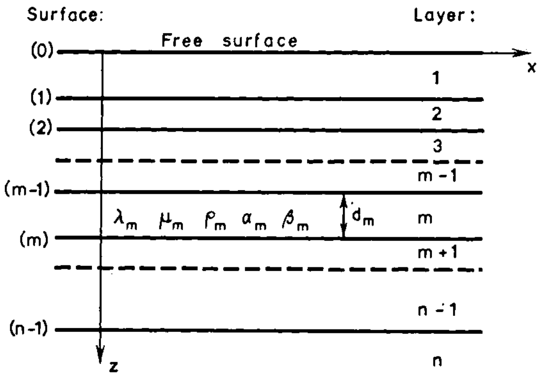


Fig.76.

direction of $+x$. We shall use the following notation (where index m refers to the layer m):

$$p = \text{angular frequency} = 2\pi/T \ (T = \text{period})$$

$$c = \text{phase velocity}$$

$$\varrho_m = \text{density}$$

$$d_m = \text{layer thickness}$$

$$\lambda_m, \mu_m = \text{Lamé parameters}$$

$$\alpha_m = \left(\frac{\lambda_m + 2\mu_m}{\varrho_m}\right)^{1/2} = P\text{-wave velocity}$$

$$\beta_m = \left(\frac{\mu_m}{\varrho_m}\right)^{1/2} = S\text{-wave velocity}$$

$$k = p/c = 2\pi/L \ (L = \text{wave length})$$

$$r_{\alpha m} = \begin{cases} +\left[\left(\dfrac{c}{\alpha_m}\right)^2 - 1\right]^{1/2} & \text{for } c > \alpha_m \\ -i\left[1 - \left(\dfrac{c}{\alpha_m}\right)^2\right]^{1/2} & \text{for } c < \alpha_m \end{cases}$$

$$r_{\beta m} = \begin{cases} +\left[\left(\dfrac{c}{\beta_m}\right)^2 - 1\right]^{1/2} & \text{for } c > \beta_m \\ -i\left[1 - \left(\dfrac{c}{\beta_m}\right)^2\right]^{1/2} & \text{for } c < \beta_m \end{cases}$$

$$u, w = \text{displacements along } x, z$$

$$\sigma = p_{zz} = \text{normal stress}$$

$\tau = p_{zx}$ = tangential stress

$$\gamma_m = 2\left(\frac{\beta_m}{c}\right)^2 \tag{1}$$

At this point, I should like to refer to the more classical treatment, which can be found in BULLEN (1963, chapter 5, pp.85–89) or in EWING, JARDETZKY and PRESS (1957).

The solution of the wave equation for the mth layer is expressed as a sum of dilatational and rotational wave solutions (see BULLEN, 1963, p.73), i.e., dilatational (P):

$$\Delta_m = \underbrace{\operatorname{div} u_i}_{= \theta \text{ (cubical dilatation)}} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \mathrm{e}^{i(pt-kx)}\,[\Delta_m' \mathrm{e}^{-ikr_{\alpha m}z} + \Delta_m'' \mathrm{e}^{ikr_{\alpha m}z}] \tag{2}$$

and rotational (S):

$$\omega_m = \underbrace{\frac{1}{2}\operatorname{curl} u_i}_{= \xi_{ij}} = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] = \mathrm{e}^{i(pt-kx)}\,[\omega_m' \mathrm{e}^{-ikr_{\beta m}z} + \omega_m'' \mathrm{e}^{ikr_{\beta m}z}] \tag{3}$$

The factor 1/2 in [3] makes the expression agree with ξ_{ij}, i.e., the rotation tensor (BULLEN, 1963, p.14). Δ_m', Δ_m'', ω_m', ω_m'' are integration constants. The relations between Δ_m and ω_m and the displacement potentials φ and ψ for P and S, respectively, used in eq.[1] in section 8.4, are seen to be $\Delta_m = \nabla^2\varphi$ and $2\omega_m = -\nabla^2\psi$, by carrying out the operations involved in the definitions [2] and [3] of Δ_m and ω_m, remembering that $\partial/\partial y = 0$. In addition, we easily find from [2] that:

$$\Delta_m = -\left(\frac{\alpha_m}{p}\right)^2\left[\frac{\partial^2\Delta_m}{\partial x^2} + \frac{\partial^2\Delta_m}{\partial z^2}\right] = \nabla^2\left[-\left(\frac{\alpha_m}{p}\right)^2\Delta_m\right]$$

i.e.:

$$\varphi = -\left(\frac{\alpha_m}{p}\right)^2\Delta_m$$

Similarly, we find that: [4]

$$\psi = +2\left(\frac{\beta_m}{p}\right)^2\omega_m$$

Consider the term in Δ_m' and Fig.77. We distinguish two cases:

(*1*) $r_{\alpha m}$ real and positive: wave front at time t_1 is $pt_1 - kx - kr_{\alpha m}z$ = constant; then $x + r_{\alpha m}z$ = constant; $\mathrm{d}x + r_{\alpha m}\mathrm{d}z = 0$; $\cot\gamma = \dfrac{A}{B} = \dfrac{\mathrm{d}x}{\mathrm{d}z} = -r_{\alpha m}$; $\cot\delta = -\cot\gamma = +r_{\alpha m}$.

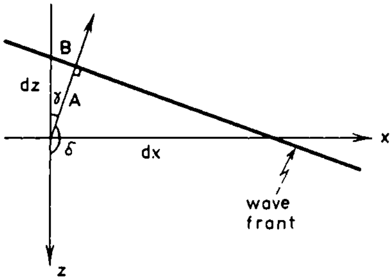


Fig.77.

The term in Δ'_m then means a plane wave propagating in a direction making an angle $= \cot^{-1} r_{\alpha m}$ with the positive z-axis, i.e., a *body wave* passing through the many-layered structure.

(*2*) $r_{\alpha m}$ imaginary and negative: $r_{\alpha m} \to -iC$ (C real constant); $-ikr_{\alpha m}z \to i^2kCz = -kCz$.

The term in Δ'_m then means a *surface wave* propagating in the $+x$-direction with an amplitude decreasing exponentially in the $+z$-direction. From the definition of $r_{\alpha m}$ there are obviously no cases other than (*1*) and (*2*).

Similar arguments can be made for the term in Δ''_m: a body wave making the same angle with $-z$ for $r_{\alpha m}$ real, and a surface wave for $r_{\alpha m}$ imaginary. Also for the solution [3].

Take the geometrical picture and let $r_{\alpha m}$ run through a series of values; if $r_{\alpha m}$ is real, there is one ray upward and one downward, making equal angles with $-z$ and $+z$, respectively. The angle between the rays decreases gradually as $r_{\alpha m}$ decreases, until $c = \alpha_m$; then $r_{\alpha m} = 0$ and the two rays coincide in one ray in the direction of $+x$ (a *P*-wave). The same is repeated at a lower value of c for the rotational solution. It is thus seen that the solutions [2] and [3] are general, including both body and surface waves, depending upon the choice of the value of c in any particular case.

9.2.2 Displacement and stress

The components of displacement and stress are as follows:

$$
\begin{aligned}
u &= -\left(\frac{\alpha_m}{p}\right)^2 \frac{\partial \Delta_m}{\partial x} - 2\left(\frac{\beta_m}{p}\right)^2 \frac{\partial \omega_m}{\partial z} \\
w &= -\left(\frac{\alpha_m}{p}\right)^2 \frac{\partial \Delta_m}{\partial z} + 2\left(\frac{\beta_m}{p}\right)^2 \frac{\partial \omega_m}{\partial x} \\
\sigma &= \varrho_m \left\{ \alpha_m^2 \Delta_m + 2\beta_m^2 \left[\left(\frac{\alpha_m}{p}\right)^2 \frac{\partial^2 \Delta_m}{\partial x^2} + 2\left(\frac{\beta_m}{p}\right)^2 \frac{\partial^2 \omega_m}{\partial x \partial z} \right] \right\} \\
\tau &= 2\varrho_m \beta_m^2 \left\{ -\left(\frac{\alpha_m}{p}\right)^2 \frac{\partial^2 \Delta_m}{\partial x \partial z} + \left(\frac{\beta_m}{p}\right)^2 \left[\frac{\partial^2 \omega_m}{\partial x^2} - \frac{\partial^2 \omega_m}{\partial z^2} \right] \right\}
\end{aligned}
\qquad [5]
$$

The expressions [5.1] and [5.2] for the displacement components are obtained from eq.[1] in section 8.4, if we take [4] into account, and then just express the x- and z-components of grad and curl. The stress components are obtained from the stress–strain relation:

$$p_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} \qquad [6]$$

(BULLEN, 1963, p.20), which applied to our case becomes:

$$\left.\begin{aligned} \sigma &= p_{zz} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z} \\ \tau &= p_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \end{aligned}\right| \qquad [7]$$

Then we use [5.1] and [5.2] to get [5.3] and [5.4].

9.2.3 *Boundary conditions and their solution*

The boundary conditions imply continuity of the four quantities [5] at each interface. Instead of considering the displacements u, w, we take $\dot{u}/c$, $\dot{w}/c$, where $\dot{u}$, $\dot{w}$ are the velocities. This can be done because c is a constant (the same for all layers) and the time derivative does not change the continuity conditions, as $\dot{\Delta}_m = ip\Delta_m$; $\omega_m = ip\omega_m$ and p (the frequency) is also the same for all layers. Therefore, we will henceforth consider the following four quantities, where the exponential functions of $ikrz$ have been expressed in trigonometric form (when r is imaginary, the trigonometric functions go over into the corresponding hyperbolic functions, sinh and cosh), and where the common factor $e^{i(pt-kx)}$ has been excluded:

$$\left.\begin{aligned} \frac{\dot{u}}{c} &= -\left(\frac{\alpha_m}{c}\right)^2 [(\Delta'_m + \Delta''_m)\cos kr_{\alpha m}z - i(\Delta'_m - \Delta''_m)\sin kr_{\alpha m}z] \\ &\quad - \gamma_m r_{\beta m}[(\omega'_m - \omega''_m)\cos kr_{\beta m}z - i(\omega'_m + \omega''_m)\sin kr_{\beta m}z] \\ \frac{\dot{w}}{c} &= -\left(\frac{\alpha_m}{c}\right)^2 r_{\alpha m}[-i(\Delta'_m + \Delta''_m)\sin kr_{\alpha m}z + (\Delta'_m - \Delta''_m)\cos kr_{\alpha m}z] \\ &\quad + \gamma_m[-i(\omega'_m - \omega''_m)\sin kr_{\beta m}z + (\omega'_m + \omega''_m)\cos kr_{\beta m}z] \\ \sigma &= -\varrho_m\alpha_m^2(\gamma_m - 1)[(\Delta'_m + \Delta''_m)\cos kr_{\alpha m}z - i(\Delta'_m - \Delta''_m)\sin kr_{\alpha m}z] \\ &\quad - \varrho_m c^2\gamma_m^2 r_{\beta m}[(\omega'_m - \omega''_m)\cos kr_{\beta m}z - i(\omega'_m + \omega''_m)\sin kr_{\beta m}z] \\ \tau &= \varrho_m\alpha_m^2\gamma_m r_{\alpha m}[-i(\Delta'_m + \Delta''_m)\sin kr_{\alpha m}z + (\Delta'_m - \Delta''_m)\cos kr_{\alpha m}z] \\ &\quad - \varrho_m c^2\gamma_m(\gamma_m - 1)[-i(\omega'_m - \omega''_m)\sin kr_{\beta m}z + (\omega'_m + \omega''_m)\cos kr_{\beta m}z] \end{aligned}\right| \qquad [8]$$

Place the origin of the z-axis in the $(m-1)$th interface (Fig.78). Then at the surface $(m-1)$ we have $z = 0$, and $\sin krz = 0$, $\cos krz = 1$. The expressions [8] can

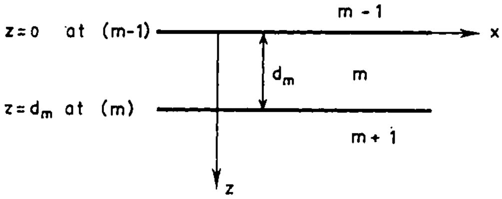


Fig.78.

now be expressed in a more compact form for the $(m-1)$th interface, including the boundary conditions at that surface, i.e., the values in the layers $m-1$ and m should be equal at the interface ($z=0$):

$$\left(\frac{\dot{u}_{m-1}}{c}, \frac{\dot{w}_{m-1}}{c}, \sigma_{m-1}, \tau_{m-1}\right) = E_m(\Delta'_m + \Delta''_m,\ \Delta'_m - \Delta''_m,\ \omega'_m - \omega''_m,\ \omega'_m + \omega''_m) \qquad \text{(4 eqs.)} \qquad [9]$$

where E_m is the following matrix, found immediately from [8]:

$$E_m = \begin{bmatrix} -\left(\frac{\alpha_m}{c}\right)^2 & 0 & -\gamma_m r_{\beta m} & 0 \\ 0 & -\left(\frac{\alpha_m}{c}\right)^2 r_{\alpha m} & 0 & \gamma_m \\ -\varrho_m \alpha_m^2(\gamma_m - 1) & 0 & -\varrho_m c^2 \gamma_m^2 r_{\beta m} & 0 \\ 0 & \varrho_m \alpha_m^2 \gamma_m r_{\alpha m} & 0 & -\varrho_m c^2 \gamma_m(\gamma_m - 1) \end{bmatrix} \qquad [10]$$

Likewise, from [8] we find the values of stress and strain at the mth interface by putting $z = d_m$ (see Fig.78):

$$\left(\frac{\dot{u}_m}{c}, \frac{\dot{w}_m}{c}, \sigma_m, \tau_m\right) = D_m(\Delta'_m + \Delta''_m,\ \Delta'_m - \Delta''_m,\ \omega'_m - \omega''_m,\ \omega'_m + \omega''_m) \qquad \text{(4 eqs.)} \qquad [11]$$

where D_m is the matrix:

$$= \begin{bmatrix} -\left(\frac{\alpha}{c}\right)^2 \cos P_m & i\left(\frac{\alpha_m}{c}\right)^2 \sin P_m & -\gamma_m r_{\beta m} \cos Q_m & i\gamma_m r_{\beta m} \sin Q_m \\ i\left(\frac{\alpha_m}{c}\right)^2 r_{\alpha m} \sin P_m & -\left(\frac{\alpha_m}{c}\right)^2 r_{\alpha m} \cos P_m & -i\gamma_m \sin Q_m & \gamma_m \cos Q_m \\ -\varrho_m \alpha_m^2(\gamma_m - 1)\cos P_m & i\varrho_m \alpha_m^2(\gamma_m - 1)\sin P_m & -\varrho_m c^2 \gamma_m^2 r_{\beta m} \cos Q_m & i\varrho_m c^2 \gamma_m^2 r_{\beta m} \sin Q_m \\ -i\varrho_m \alpha_m^2 \gamma_m r_{\alpha m} \sin P_m & \varrho_m \alpha_m^2 \gamma_m r_{\alpha m} \cos P_m & i\varrho_m c^2 \gamma_m(\gamma_m - 1)\sin Q_m & -\varrho_m c^2 \gamma_m(\gamma_m - 1)\cos Q_m \end{bmatrix} \qquad [12]$$

where $P_m = kr_{\alpha m} d_m$ and $Q_m = kr_{\beta m} d_m$.

There are four equations [9] and four equations [11], containing the four unknowns: $\Delta'_m + \Delta''_m$, $\Delta'_m - \Delta''_m$, $\omega'_m - \omega''_m$, $\omega'_m + \omega''_m$. Therefore, the unknowns can be eliminated between [9] and [11], and the result can be written symbolically in the following form:

$$\left(\frac{\dot{u}_m}{c}, \frac{\dot{w}_m}{c}, \sigma_m, \tau_m\right) = \underbrace{D_m E_m^{-1}}_{(\text{put } = a_m)} \left(\frac{\dot{u}_{m-1}}{c}, \frac{\dot{w}_{m-1}}{c}, \sigma_{m-1}, \tau_{m-1}\right) \qquad [13]$$

[13] is obtained simply by ordinary algebraic elimination of $(\Delta'_m + \Delta''_m$, $\Delta'_m - \Delta''_m$, $\omega'_m - \omega''_m$, $\omega'_m + \omega''_m)$ between [9] and [11]. In [13] E_m^{-1} is the *inverse or reciprocal matrix* to E_m and is given by:

$$E_m^{-1} = \begin{bmatrix} -2\left(\frac{\beta_m}{\alpha_m}\right)^2 & 0 & (\varrho_m\alpha_m^2)^{-1} & 0 \\ 0 & c^2(\gamma_m - 1)/\alpha_m^2 r_{\alpha m} & 0 & (\varrho_m\alpha_m^2 r_{\alpha m})^{-1} \\ (\gamma_m - 1)/\gamma_m r_{\beta m} & 0 & -(\varrho_m c^2\gamma_m r_{\beta m})^{-1} & 0 \\ 0 & 1 & 0 & (\varrho_m c^2 \gamma_m)^{-1} \end{bmatrix} \qquad [14]$$

Eq.[14] can be verified term by term from the known relation:

$$E_m E_m^{-1} = \text{unit matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [15]$$

applying the rule for multiplication of matrices (see for example BIEBERBACH and BAUER, 1933, p.93):

$$\begin{bmatrix} p_{11} & \cdots & p_{1\mu} \\ \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{n\mu} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \cdots & b_{1\mu} \\ \cdots & \cdots & \cdots \\ b_{m1} & \cdots & b_{m\mu} \end{bmatrix} \qquad [16]$$

or $\qquad P = A \cdot B$

then:

$$p_{ik} = \sum_{\sigma=1}^{\sigma=m} a_{i\sigma} b_{\sigma k}$$

The number of columns of A must be the same as the number of rows in B. In multiplying A with B, the rows of A are multiplied with the columns of B. Note that the commutative law does *not* hold for multiplication of matrices, i.e., $A \cdot B \neq B \cdot A$. The multiplication rule can be extended to any number of matrices to be multiplied.

We have written in [13] that:

$$a_m = D_m E_m^{-1} \tag{17}$$

a_m is a product of two matrices and is itself a matrix. Its elements can be found from [12] and [14] using the rule for multiplication of two matrices:

$$\begin{aligned}
(a_m)_{11} &= \gamma_m \cos P_m - (\gamma_m - 1)\cos Q_m \\
(a_m)_{12} &= i[(\gamma_m - 1) r_{\alpha m}^{-1} \sin P_m + \gamma_m r_{\beta m} \sin Q_m] \\
(a_m)_{13} &= -(\varrho_m c^2)^{-1}(\cos P_m - \cos Q_m) \\
(a_m)_{14} &= i(\varrho_m c^2)^{-1}(r_{\alpha m}^{-1} \sin P_m + r_{\beta m} \sin Q_m) \\
(a_m)_{21} &= -i[\gamma_m r_{\alpha m} \sin P_m + (\gamma_m - 1) r_{\beta m}^{-1} \sin Q_m] \\
(a_m)_{22} &= -(\gamma_m - 1)\cos P_m + \gamma_m \cos Q_m \\
(a_m)_{23} &= i(\varrho_m c^2)^{-1}(r_{\alpha m} \sin P_m + r_{\beta m}^{-1} \sin Q_m) \\
(a_m)_{24} &= (a_m)_{13} \\
(a_m)_{31} &= \varrho_m c^2 \gamma_m (\gamma_m - 1)(\cos P_m - \cos Q_m) \\
(a_m)_{32} &= i\varrho_m c^2[(\gamma_m - 1)^2 r_{\alpha m}^{-1} \sin P_m + \gamma_m^2 r_{\beta m} \sin Q_m] \\
(a_m)_{33} &= (a_m)_{22} \\
(a_m)_{34} &= (a_m)_{12} \\
(a_m)_{41} &= i\varrho_m c^2[\gamma_m^2 r_{\alpha m} \sin P_m + (\gamma_m - 1)^2 r_{\beta m}^{-1} \sin Q_m] \\
(a_m)_{42} &= (a_m)_{31} \\
(a_m)_{43} &= (a_m)_{21} \\
(a_m)_{44} &= (a_m)_{11}
\end{aligned} \tag{18}$$

If in [13] we replace m by $m - 1$ we get:

$$\left(\frac{\dot{u}_{m-1}}{c}, \frac{\dot{w}_{m-1}}{c}, \sigma_{m-1}, \tau_{m-1}\right) = a_{m-1}\left(\frac{\dot{u}_{m-2}}{c}, \frac{\dot{w}_{m-2}}{c}, \sigma_{m-2}, \tau_{m-2}\right) \tag{19}$$

Elimination of $\left(\frac{\dot{u}_{m-1}}{c}, \frac{\dot{w}_{m-1}}{c}, \sigma_{m-1}, \tau_{m-1}\right)$ between [13] and [19] gives:

$$\left(\frac{\dot{u}_m}{c}, \frac{\dot{w}_m}{c}, \sigma_m, \tau_m\right) = a_m a_{m-1}\left(\frac{\dot{u}_{m-2}}{c}, \frac{\dot{w}_{m-2}}{c}, \sigma_{m-2}, \tau_{m-2}\right) \tag{20}$$

Repeated application of this iterative procedure gives:

$$\left(\frac{\dot{u}_{n-1}}{c}, \frac{\dot{w}_{n-1}}{c}, \sigma_{n-1}, \tau_{n-1}\right) = a_{n-1} a_{n-2} \ldots a_1 \left(\frac{\dot{u}_0}{c}, \frac{\dot{w}_0}{c}, \sigma_0, \tau_0\right) \tag{21}$$

Solving [9] for the unknown constants and combining with [21], we find:

$$(\Delta_n' + \Delta_n'', \Delta_n' - \Delta_n'', \omega_n' - \omega_n'', \omega_n' + \omega_n'') = E_n^{-1} a_{n-1} a_{n-2} \ldots a_1 \left(\frac{\dot{u}_0}{c}, \frac{\dot{w}_0}{c}, \sigma_0, \tau_0\right) \tag{22}$$

Eq.[22] is quite general and is valid both for body waves and surface waves in a multi-layered medium.

Now, restricting ourselves to surface waves, we put:

$$\left.\begin{aligned} &\sigma_0 = \tau_0 = 0 \quad \text{(no stresses at the free surface; this is \textit{always} the case)} \\ &\Delta_n'' = \omega_n'' = 0 \quad \text{(to ensure that the surface waves \textit{decrease} with depth)} \end{aligned}\right| \qquad [23]$$

Also put:

$$J = E_n^{-1} a_{n-1} a_{n-2} \ldots a_1 \qquad \text{(a matrix product)} \qquad [24]$$

Then [22] simplifies to the following form:

$$(\Delta_n', \Delta_n', \omega_n', \omega_n') = J\left(\frac{\dot{u}_0}{c}, \frac{\dot{w}_0}{c}, 0, 0\right) \qquad [25]$$

or explicitly:

$$\left.\begin{aligned} \Delta_n' &= J_{11}\frac{\dot{u}_0}{c} + J_{12}\frac{\dot{w}_0}{c} \\ \Delta_n' &= J_{21}\frac{\dot{u}_0}{c} + J_{22}\frac{\dot{w}_0}{c} \\ \omega_n' &= J_{31}\frac{\dot{u}_0}{c} + J_{32}\frac{\dot{w}_0}{c} \\ \omega_n' &= J_{41}\frac{\dot{u}_0}{c} + J_{42}\frac{\dot{w}_0}{c} \end{aligned}\right| \qquad [26]$$

Eliminate Δ_n' and ω_n' by equating the right-hand sides of [26.1] and [26.2], also of [26.3] and [26.4]:

$$\frac{\dot{u}_0}{\dot{w}_0} = \frac{J_{22} - J_{12}}{J_{11} - J_{21}} = \frac{J_{42} - J_{32}}{J_{31} - J_{41}} \qquad [27]$$

The elements of the matrix J are functions of c and k; see the expressions [14] and [18]. This means that [27] implicitly expresses a relation between c and k, that is, the desired *phase-velocity dispersion function.* The advantages (at least formally) of the matrix calculation are seen from the fact that it is possible to express the dispersion equation in a simple and compact form, for any number of layers. We shall now study the solution [27] in various ways.

9.2.4 Phase difference between vertical and horizontal displacements

We put:

$$A = a_{n-1} a_{n-2} \ldots a_1$$

thus:

$$J = E_n^{-1}A \tag{28}$$

Then [27] can be written as follows:

$$-\frac{\dot{u}_0}{\dot{w}_0} = \frac{K}{L} = \frac{M}{N} \tag{29}$$

where:

$$K \sim J_{12} - J_{22}\,,\; L \sim J_{11} - J_{21}, \text{ etc.}$$

e.g.:

$$\left.\begin{aligned} K &= \gamma_n r_{\alpha n} A_{12} + (\gamma_n - 1)A_{22} - r_{\alpha n}A_{32}/\varrho_n c^2 + A_{42}/\varrho_n c^2 \\ L &= \gamma_n r_{\alpha n} A_{11} + (\gamma_n - 1)A_{21} - r_{\alpha n}A_{31}/\varrho_n c^2 + A_{41}/\varrho_n c^2 \\ M &= -(\gamma_n - 1)A_{12} + \gamma_n r_{\beta n}A_{22} + A_{32}/\varrho_n c^2 + r_{\beta n}A_{42}/\varrho_n c^2 \\ N &= -(\gamma_n - 1)A_{11} + \gamma_n r_{\beta n}A_{21} + A_{31}/\varrho_n c^2 + r_{\beta n}A_{41}/\varrho_n c^2 \end{aligned}\right| \tag{30}$$

where [14] is used (and the rule for multiplication of matrices).

Consider the elements of the matrix a_m, listed in [18]: if we denote a real quantity by *Re* and an imaginary by *Im*, this matrix can be written

$$a_m = \begin{bmatrix} Re & Im & Re & Im \\ Im & Re & Im & Re \\ Re & Im & Re & Im \\ Im & Re & Im & Re \end{bmatrix} \tag{31}$$

It starts with *Re* and then *Re* and *Im* alternate both in columns and rows. Note that $\sin P_m$ and $\sin Q_m$ and $r_{\alpha m}$ and $r_{\beta m}$, which are real or imaginary depending upon the value of c, occur only in the combinations $r_{\alpha m}^{\pm 1} \sin P_m$ and $r_{\beta m}^{\pm 1} \sin Q_m$, and therefore are always *real* (as long as c is real). Remember that sine (imaginary) is imaginary, but that cosine (imaginary) is real:

$$\underbrace{e^{-\alpha}}_{\text{real}} = e^{i^2\alpha} = e^{i(i\alpha)} = \underbrace{\cos(i\alpha)}_{\text{real}} + \overbrace{i\underbrace{\sin(i\alpha)}_{\text{imaginary}}}^{\text{real}} \tag{32}$$

The product of two matrices of the form [31] is also a matrix of the same form, i.e., with real and imaginary elements at the same places (this is easily seen from the rule of multiplication of matrices). Therefore, the same holds for A which results from repeated multiplication of matrices of the form [31]. This means that

$$\left.\begin{aligned} &A_{11},\ A_{22},\ A_{31},\ A_{42} \text{ are real} \\ &A_{12},\ A_{21},\ A_{32},\ A_{41} \text{ are imaginary} \end{aligned}\right\}$$ These are the elements entering [30], and only these are of interest.

Then consider K:

γ_n is real;

$r_{\alpha n}$ is imaginary (i.e., we consider the case $c < \alpha_n$, true for surface waves, to get vanishing amplitude for large positive z);

A_{12} is imaginary, as just found.

Together this means that K is real. Likewise, we see that N is real, while L and M are imaginary. Then [29] is of the following form:

$$\frac{\dot{u}_0}{\dot{w}_0} = iB = B \cdot e^{i\pi/2} \qquad [33]$$

that is, there is a phase difference of 90° between vertical and horizontal velocities and displacements at the free surface. This in turn implies a particle motion in the form of an ellipse with vertical and horizontal axes. But the phase difference may have either sign, i.e., the motion is not necessarily retrograde as in the simple case with Rayleigh waves on a homogeneous medium.

In the case of a medium with *dissipation*, we have to assume that k (and therefore also c) are complex:

$$k = k_r - ik_i \qquad (k_i > 0)$$

Then:

$$\begin{aligned} -ikx &= -i(k_r - ik_i)x \\ &= -ik_r x - k_i x \end{aligned} \qquad [34]$$

The complex expression for k entails a decrease of amplitude with distance, i.e., dissipation, which does not happen for real values of k. In this case $\dot{u}_0/\dot{w}_0$ would not necessarily be purely imaginary, but complex:

$$\frac{\dot{u}_0}{\dot{w}_0} = iB + C \rightarrow e^{i\alpha}$$

and:

$$\cos\alpha \sim C \qquad [35]$$

$$\sin\alpha \sim B$$

which means that other phase differences (α) than 90° between the vertical and horizontal displacements can also exist (in dissipative media). Then the otherwise vertical axis of the ellipse is sloping (forward or backward). Such observations have been made especially for explosion-generated surface waves on poorly consolidated sediments (with dissipation!).

9.2.5 *Asymptotic form for long wave lengths*

The treatment given so far is perfectly general and applicable to any number of layers and to any wave lengths. Therefore, it includes all other treatments as special cases, of which we shall consider a few.

Assume that the wave length becomes very large. Then:

$$\left.\begin{array}{ll} kd_m \to 0 & \\ P_m \to 0 & \text{(eq.[12])} \\ Q_m \to 0 & \text{(eq.[12])} \\ a_m \to \text{unit matrix} & \text{(eq.[18])} \\ J \to E_n^{-1} & \text{(eq.[24])} \end{array}\right| \qquad [36]$$

and [27] becomes, using [14]:

$$\frac{\dot{u}_0}{\dot{w}_0} = -(\gamma_n - 1)/\gamma_n r_{\alpha n} = \gamma_n r_{\beta n}/(\gamma_n - 1)$$

or:

$$(\gamma_n - 1)^2 + \gamma_n^2 r_{\alpha n} r_{\beta n} = 0 \qquad [37]$$

This is the period equation for Rayleigh waves on a semi-infinite medium (BULLEN, 1963, p.89), i.e., when the thicknesses of all superficial layers are negligible compared to the wave lengths.

9.2.6 *Asymptotic form for short wave lengths*

The matrix J in [24]:

$$J = E_n^{-1} a_{n-1} a_{n-2} \ldots a_1$$

can be written as follows, remembering that $a_n = D_n E_n^{-1}$ by eq.[17] and defining a new matrix $b_n = E_{n+1}^{-1} D_n$ (note the different order of D and E in a and b):

$$J = \overbrace{E_n^{-1} \underbrace{(D_{n-1}}_{}}^{b_{n-1}} \underbrace{E_{n-1}^{-1})}_{a_{n-1}} \ldots \quad J = E_n^{-1}(D_{n-1}E_{n-1}^{-1})(D_{n-2}E_{n-2}^{-1}) \ldots (D_1 E_1^{-1}) = b_{n-1} b_{n-2} \ldots b_1 E_1^{-1} \qquad [38]$$

which essentially is only a different grouping of the factors (by interfaces instead of by layers). For instance, a_{n-1} belongs to the layer $n-1$ (only quantities valid for this layer enter into a_{n-1}), whereas b_{n-1} belongs to the interface $n-1$, separating the layers n and $n-1$ (properties of both these layers enter the expression for b_{n-1}).

Moreover, assume $c < \beta_{n-1}$, i.e., P_{n-1} and Q_{n-1} are imaginary. Also:

$$\left.\begin{array}{l} \sin P_{n-1} \to -i \cos P_{n-1} \\ \sin Q_{n-1} \to -i \cos Q_{n-1} \end{array}\right| \qquad [39]$$

this being valid for short wave lengths, i.e. for kd_{n-1} large. Eq.[39.1] is seen in the following way:

$$\sin P_{n-1} = \sin(kr_{\alpha,n-1}d_{n-1}) = -\sin(irkd_{n-1})$$

where we put $r_{\alpha,n-1} = -ir$, r real, positive:

$$\cos P_{n-1} = \cos(kr_{\alpha,n-1}d_{n-1}) = \cos(irkd_{n-1})$$

$$\underbrace{e^{-krd_{n-1}}}_{\substack{=0 \text{ for} \\ kd_{n-1}\text{large}}} = e^{i^2krd_{n-1}} = e^{i(ikrd_{n-1})} = \cos(ikrd_{n-1}) + i\sin(ikrd_{n-1})$$

Thus:

$$\underbrace{\cos(ikrd_{n-1})}_{=\cos P_{n-1}} = \underbrace{-i\sin(ikrd_{n-1})}_{=+i\sin P_{n-1}}$$

and:

$$\sin P_{n-1} = -i\cos P_{n-1}$$

which is [39.1]. In this case the elements of the matrix:

$$b_{n-1} = E_n^{-1}D_{n-1} \qquad \text{(eq.[38])}$$

approach the following values (by multiplication of E^{-1} and D):

$$\left.\begin{aligned}
(b_{n-1})_{11} &= -(b_{n-1})_{12} = \left(\frac{\alpha_{n-1}}{\alpha_n}\right)^2 \left[\gamma_n - (\gamma_{n-1}-1)\frac{\varrho_{n-1}}{\varrho_n}\right]\cos P_{n-1} \\
(b_{n-1})_{13} &= -(b_{n-1})_{14} = \left(\frac{c}{\alpha_n}\right)^2 \gamma_{n-1}r_{\beta,n-1}\left[\gamma_n - \gamma_{n-1}\frac{\varrho_{n-1}}{\varrho_n}\right]\cos Q_{n-1} \\
(b_{n-1})_{21} &= -(b_{n-1})_{22} = \left(\frac{\alpha_{n-1}}{\alpha_n}\right)^2 \frac{r_{\alpha,n-1}}{r_{\alpha n}}\left[(\gamma_n-1) - \gamma_{n-1}\frac{\varrho_{n-1}}{\varrho_n}\right]\cos P_{n-1} \\
(b_{n-1})_{23} &= -(b_{n-1})_{24} = -\left(\frac{c}{\alpha_n}\right)^2 \frac{\gamma_{n-1}}{r_{\alpha n}}\left[(\gamma_n-1) - (\gamma_{n-1}-1)\frac{\varrho_{n-1}}{\varrho_n}\right]\cos Q_{n-1} \\
(b_{n-1})_{31} &= -(b_{n-1})_{32} = -\left(\frac{\alpha_{n-1}}{c}\right)^2 (\gamma_n r_{\beta n})^{-1}\left[(\gamma_n-1) - (\gamma_{n-1}-1)\frac{\varrho_{n-1}}{\varrho_n}\right]\cos P_{n-1} \\
(b_{n-1})_{33} &= -(b_{n-1})_{34} = -\frac{\gamma_{n-1}r_{\beta,n-1}}{\gamma_n r_{\beta n}}\left[(\gamma_n-1) - \gamma_{n-1}\frac{\varrho_{n-1}}{\varrho_n}\right]\cos Q_{n-1} \\
(b_{n-1})_{41} &= -(b_{n-1})_{42} = \left(\frac{\alpha_{n-1}}{c}\right)^2 \frac{r_{\alpha,n-1}}{\gamma_n}\left[\gamma_n - \gamma_{n-1}\frac{\varrho_{n-1}}{\varrho_n}\right]\cos P_{n-1} \\
(b_{n-1})_{43} &= -(b_{n-1})_{44} = -\frac{\gamma_{n-1}}{\gamma_n}\left[\gamma_n - (\gamma_{n-1}-1)\frac{\varrho_{n-1}}{\varrho_n}\right]\cos Q_{n-1}
\end{aligned}\right| \qquad [40]$$

Put:

$$J_{n-1} = b_{n-2}b_{n-3}\ldots b_1E_1^{-1} \qquad [41]$$

i.e., so that $J = b_{n-1}J_{n-1}$; then we can form the following expressions for the terms in [27]:

$$\begin{aligned}
J_{22} - J_{12} &= [(b_{n-1})_{11} - (b_{n-1})_{21}]\,[(J_{n-1})_{22} - (J_{n-1})_{12}] \\
&\quad + [(b_{n-1})_{13} - (b_{n-1})_{23}]\,[(J_{n-1})_{42} - (J_{n-1})_{32}] \\
J_{11} - J_{21} &= [(b_{n-1})_{11} - (b_{n-1})_{21}]\,[(J_{n-1})_{11} - (J_{n-1})_{21}] \\
&\quad + [(b_{n-1})_{13} - (b_{n-1})_{23}]\,[(J_{n-1})_{31} - (J_{n-1})_{41}] \\
J_{42} - J_{32} &= [(b_{n-1})_{31} - (b_{n-1})_{41}]\,[(J_{n-1})_{22} - (J_{n-1})_{12}] \\
&\quad + [(b_{n-1})_{33} - (b_{n-1})_{43}]\,[(J_{n-1})_{42} - (J_{n-1})_{32}] \\
J_{31} - J_{41} &= [(b_{n-1})_{31} - (b_{n-1})_{41}]\,[(J_{n-1})_{11} - (J_{n-1})_{21}] \\
&\quad + [(b_{n-1})_{33} - (b_{n-1})_{43}]\,[(J_{n-1})_{31} - (J_{n-1})_{41}]
\end{aligned} \tag{42}$$

Using the following abbreviations:

$$\begin{aligned}
K' &= (J_{n-1})_{22} - (J_{n-1})_{12} \\
L' &= (J_{n-1})_{11} - (J_{n-1})_{21} \\
M' &= (J_{n-1})_{42} - (J_{n-1})_{32} \\
N' &= (J_{n-1})_{31} - (J_{n-1})_{41} \\
R &= (b_{n-1})_{11} - (b_{n-1})_{21} \\
S &= (b_{n-1})_{13} - (b_{n-1})_{23} \\
T &= (b_{n-1})_{31} - (b_{n-1})_{41} \\
U &= (b_{n-1})_{33} - (b_{n-1})_{43}
\end{aligned} \tag{43}$$

eq.[27] can be written as follows:

$$\frac{RK' + SM'}{RL' + SN'} = \frac{TK' + UM'}{TL' + UN'} \tag{44}$$

as, e.g., $J_{22} - J_{12} = RK' + SM'$ by [42] and [43]. By cross-multiplication [44] gives:

$$(RU - ST)(K'N' - L'M') = 0 \tag{45}$$

Thus, the solution can be factored, and each factor $= 0$ represents a wave. When the expressions [40] for b_{n-1} have been introduced, the equation $RU - ST = 0$ becomes:

$$\begin{aligned}
&\varrho_n^2[(\gamma_n - 1)^2 + \gamma_n^2 r_{\alpha n} r_{\beta n}]\,[1 + r_{\alpha,n-1} r_{\beta,n-1}] - 2\varrho_n\varrho_{n-1}[(\gamma_n - 1) + \gamma_n r_{\alpha n} r_{\beta n}]\,[(\gamma_{n-1} - 1) \\
&+ \gamma_{n-1} r_{\alpha,n-1} r_{\beta,n-1}] + \varrho_n\varrho_{n-1}[r_{\alpha,n-1} r_{\beta n} + r_{\alpha n} r_{\beta,n-1}] + \varrho_{n-1}^2[(\gamma_{n-1} - 1)^2 \\
&+ \gamma_{n-1}^2 r_{\alpha,n-1} r_{\beta,n-1}]\,[1 + r_{\alpha n} r_{\beta n}] = 0
\end{aligned} \tag{46}$$

which is equivalent to Stoneley's equation for the $(n - 1)$th interface, i.e., *Stoneley waves.*

Taking instead the other factor $K'N' - L'M' = 0$, we have the same equation as [27] above but now referred to $(n - 1)$ layers instead of n. We can therefore continue the same operation on this expression, splitting it into two factors, one corresponding to Stoneley waves on the $(n-2)$th interface, and so on. By this iteration, we can split the expression [45] into factors corresponding to Stoneley waves on the successive interfaces and another factor corresponding to Rayleigh waves on the free surface.

The above argument assumes that $c < \beta_{n-1}$; in fact c must be lower than the lowest β_m in the whole structure. If in some layer $c > \beta$, K/L and M/N will not approach an asymptotic value for large k (short wave lengths) but will remain oscillatory. The equation $KN - ML = 0$ will then be satisfied by an infinite number of values of k. The corresponding roots of $KN - ML = 0$ then give the sequence of *higher modes.*

9.2.7 *The matrix a_m for a fluid layer*

The case of a fluid layer cannot be immediately obtained as a special case of the preceding formulas. For a fluid layer we have:

$$\begin{aligned}
&\beta_m = 0\,,\ \gamma_m = 0\,,\ r_{\beta m} = \infty \\
&\underset{[0\cdot\infty]}{\gamma_m r_{\beta m}} = 2\frac{\beta_m^2}{c^2}\left(\frac{c^2}{\beta_m^2} - 1\right)^{1/2} = 2\left(\frac{\beta_m^2}{c^2} - \frac{\beta_m^4}{c^4}\right)^{1/2} = 0 \text{ if } \beta_m = 0\,,\ c \neq 0
\end{aligned} \qquad [47]$$

and $E_m = 0$. E_m has only three elements different from 0: 11, 22, 31, by eq.[10]; E_m is then a singular matrix and its inverse E_m^{-1} does not exist. We then have to go back to eq.[8] and put $z = 0$:

$$\begin{aligned}
\frac{\dot{u}_{m-1}}{c} &= -\left(\frac{\alpha_m}{c}\right)^2 (\Delta_m' + \Delta_m'') \\
\frac{\dot{w}_{m-1}}{c} &= -\left(\frac{\alpha_m}{c}\right)^2 r_{\alpha m} (\Delta_m' - \Delta_m'') \\
\sigma_{m-1} &= \varrho_m \alpha_m^2 (\Delta_m' + \Delta_m'') \\
\tau_{m-1} &= 0
\end{aligned} \qquad [48]$$

The only boundary conditions left at a fluid-solid boundary are continuity of normal displacement and of normal stress, thus:

$$\begin{aligned}
&\text{from [48.3]:} \\
&\Delta_m' + \Delta_m'' = \frac{\sigma_{m-1}}{\varrho_m \alpha_m^2} \\
&\text{from [48.2]:} \\
&\Delta_m' - \Delta_m'' = -\left(\frac{c}{\alpha_m}\right)^2 r_{\alpha m}^{-1} \frac{\dot{w}_{m-1}}{c}
\end{aligned} \qquad [49]$$

Also: $\omega_m' = \omega_m'' = 0$, as there are no rotational waves in the fluid layer. The transformation:

$$(\Delta_m' + \Delta_m'', \Delta_m' - \Delta_m'', \omega_m' - \omega_m'', \omega_m' + \omega_m'') \to \left(\frac{\dot{u}_{m-1}}{c}, \frac{\dot{w}_{m-1}}{c}, \sigma_{m-1}, \tau_{m-1}\right)$$

which is the *effective* inverse of E_m (compare eq.[9]) therefore has the following matrix:

$$F_m^{-1} = \begin{bmatrix} 0 & 0 & (\varrho_m \alpha_m^2)^{-1} & 0 \\ 0 & -\left(\dfrac{c}{\alpha_m}\right)^2 r_{\alpha m}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [50]$$

The matrix D_m in this case becomes, with $\beta_m = 0$ (see eq.[12]):

$$D_m = \begin{bmatrix} -\left(\dfrac{\alpha_m}{c}\right)^2 \cos P_m & i\left(\dfrac{\alpha_m}{c}\right)^2 \sin P_m & 0 & 0 \\ i\left(\dfrac{\alpha_m}{c}\right)^2 r_{\alpha m} \sin P_m & -\left(\dfrac{\alpha_m}{c}\right)^2 r_{\alpha m} \cos P_m & 0 & 0 \\ \varrho_m \alpha_m^2 \cos P_m & -i\varrho_m \alpha_m^2 \sin P_m & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [51]$$

and finally $a_m = D_m F_m^{-1}$ (eq.[17]) now becomes by multiplication of [51] with [50]:

$$a_m = \begin{bmatrix} 0 & -ir_{\alpha m}^{-1} \sin P_m & -(\varrho_m c^2)^{-1} \cos P_m & 0 \\ 0 & \cos P_m & ir_{\alpha m}(\varrho_m c^2)^{-1} \sin P_m & 0 \\ 0 & i\varrho_m c^2 r_{\alpha m}^{-1} \sin P_m & \cos P_m & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [52]$$

Having this matrix for the fluid layer, we can carry out computations as outlined above.

Numerical computations, for example from the dispersion equations [27] or [29], have to be carried out by trial and error.

9.3 LOVE WAVES

This case is simpler, as there are only two boundary conditions at each interface: continuity of transverse displacement v and of transverse shear stress p_{zy}:

$$\begin{aligned} & u = w = 0 \\ & v_m = e^{i(pt-kx)} [v_m' e^{-ikr_{\beta m} z} + v_m'' e^{ikr_{\beta m} z}] \\ & (p_{zy})_m = \mu_m \left(\frac{\partial v}{\partial z}\right)_m = ik\mu_m r_{\beta m} e^{i(pt-kx)} [-v_m' e^{-ikr_{\beta m} z} + v_m'' e^{ikr_{\beta m} z}] \end{aligned} \qquad [1]$$

v_m' and v_m'' are integration constants.

The boundary conditions lead to the following equations for the $(m - 1)$th interface, where we put $z = 0$:

$$(\dot{v}/c)_{m-1} = ik(v'_m + v''_m)$$
$$(p_{zy})_{m-1} = ik\mu_m r_{\beta m}(v''_m - v'_m) \qquad [2]$$

At the mth interface, $z = d_m$, we get:

$$(\dot{v}/c)_m = (v'_m + v''_m)ik \cos Q_m - (v''_m - v'_m)k \sin Q_m$$
$$(p_{zy})_m = -(v'_m + v''_m)k\mu_m r_{\beta m} \sin Q_m + (v''_m - v'_m)ik\mu_m r_{\beta m} \cos Q_m \qquad [3]$$

Eliminate v'_m and v''_m between [2] and [3]:

$$(\dot{v}/c)_m = (\dot{v}/c)_{m-1} \cos Q_m + (p_{zy})_{m-1}\mu_m^{-1} r_{\beta m}^{-1} i \sin Q_m$$
$$(p_{zy})_m = (\dot{v}/c)_{m-1} i\mu_m r_{\beta m} \sin Q_m + (p_{zy})_{m-1} \cos Q_m \qquad [4]$$

Eq.[4] relates the values at $(m - 1)$ to those at m, and it is therefore equivalent to eq.[13] in the Rayleigh-wave case. The matrix a_m is obtained directly from [4]:

$$a_m = \begin{bmatrix} \cos Q_m & i\mu_m^{-1} r_{\beta m}^{-1} \sin Q_m \\ i\mu_m r_{\beta m} \sin Q_m & \cos Q_m \end{bmatrix}$$
$$= \cos Q_m \begin{bmatrix} 1 & i\mu_m^{-1} r_{\beta m}^{-1} \tan Q_m \\ i\mu_m r_{\beta m} \tan Q_m & 1 \end{bmatrix} \qquad [5]$$

Again defining the matrix product $A = a_{n-1}\, a_{n-2} \, \ldots \, a_1$ we get an iterative equation from [4], corresponding to [21] in section 9.2:

$$(\dot{v}/c)_{n-1} = A_{11}(\dot{v}/c)_0 + A_{12}(p_{zy})_0$$
$$(p_{zy})_{n-1} = A_{21}(\dot{v}/c)_0 + A_{22}(p_{zy})_0 \qquad [6]$$

Combining [6] with [2], putting $m = n$ in [2], and solving for $v'_n + v''_n$ and $v''_n - v'_n$ respectively, we find:

$$v'_n + v''_n = A_{11}(ik)^{-1}(\dot{v}/c)_0 + A_{12}(ik)^{-1}(p_{zy})_0$$
$$v''_n - v'_n = A_{21}(ik\mu_n r_{\beta n})^{-1}(\dot{v}/c)_0 + A_{22}(ik\mu_n r_{\beta n})^{-1}(p_{zy})_0 \qquad [7]$$

If $c < \beta_m$, $r_{\beta m}$ is negative imaginary, i.e. $= -ir$, where r is positive real. Then:

$$ikr_{\beta m}z = -i^2 krz = +krz$$

The conditions for the existence of free surface waves, $c < \beta_m$, $(p_{zy})_0 = 0$ and $v''_n = 0$, give, in combination with [7], the following period equation:

$$A_{21} = -\mu_n r_{\beta n} A_{11} \qquad [8]$$

In the two-layer case (i.e., one top layer over a semi-infinite medium), $A = a_1$, and [8] reduces to the well-known Love-wave dispersion equation in its simplest form (cf. BULLEN, 1963, p.92) with $n = 2$, $m = 1$:

$$\tan Q_1 = +i\frac{\mu_2 r_{\beta 2}}{\mu_1 r_{\beta 1}} \qquad [9]$$

Thus, $r_{\beta 2}$ is imaginary (to ensure decreasing amplitudes with increasing z in the lower layer), and $r_{\beta 1}$ is real (from eq.[9], to make both sides real). This means that $\beta_1 < c < \beta_2$ for Love waves, in the structure assumed. Therefore, in this case the formulas [1] in section 9.2 are:

$$\left.\begin{aligned} r_{\beta 1} &= \left[\left(\frac{c}{\beta_1}\right)^2 - 1\right]^{1/2} \\ r_{\beta 2} &= -i\left[1 - \left(\frac{c}{\beta_2}\right)^2\right]^{1/2} \end{aligned}\right| \qquad [10]$$

Introducing [10] into [9], we get exactly the same equation as eq.23 in BULLEN (1963, p.92).

Later, HASKELL (1964) developed a surface-wave theory, including various types of sources, again using layer matrices. The theory developed here (HASKELL, 1953) does not take source properties into account. Our treatment in this chapter corresponds essentially to chapter 4 in EWING, JARDETZKY and PRESS (1957), which, however, includes sources.

9.4 BODY-WAVE PROPAGATION THROUGH A MANY-LAYERED MEDIUM

As already mentioned, the matrix method of Haskell is general in the sense that it includes both body waves and surface waves. The application of this technique to body waves is therefore straight-forward and does not involve any complications. HASKELL (1960, 1962) has investigated by this method the crustal reflections of *SH* waves and of *P* and *SV* waves. The results are given in graphical form, with the surface displacement as a function of period and angle of incidence. Compared to the classical theory Haskell's treatment includes the period in the expressions (the traditional treatment assumes the wave lengths of incident waves to be very long in comparison with the thicknesses of surface layers, which in fact is frequently far from being true). See also HANNON (1964) who applied this method.

KNOPOFF (1964) has published an alternative matrix procedure for decomposition into determinants and solution, which is appropriate to machine computation. The period equation is obtained by putting a determinant = 0, and this determinant is decomposed into a product of matrices, using Laplace development by minors. Each matrix in the product is of order two only (irrespective of the number of layers) and is therefore relatively easy to calculate numerically.

Body-wave propagation in many-layered structures also occurs in interpretation of seismic field works and prospection. Also here, matrix calculation would have a place.

Chapter 10

CALCULUS OF VARIATIONS

10.1 FUNDAMENTALS OF THE CALCULUS OF VARIATIONS

We know from differential calculus that a function $y(x)$ attains a *stationary* value for $x = a$, if $y'(a) = 0$. It is maximum if $y''(a) < 0$, and minimum if $y''(a) > 0$.

In the calculus of variations the problem is related but more complicated. One significant aspect of the calculus of variations is that it can serve as a base upon which the science of mechanics is built.

10.1.1 Single independent and single dependent variable

The problem is to find such a function $y(x)$ that the integral:

$$\int_{x_1}^{x_2} f(x, y, y_x)\mathrm{d}x \qquad [1]$$

assumes a stationary value. Here $y_x = \mathrm{d}y/\mathrm{d}x$ (see Fig.79). δ is called *variation* and is used to denote small differences from the *stationary path* $y(x)$ (keeping x constant, i.e., $\delta x = 0$):

$$\delta y(x) = Y(x) - y(x)$$

$$\delta \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(Y - y) = \frac{\mathrm{d}}{\mathrm{d}x}\delta y$$

i.e., the operations δ and $\mathrm{d}/\mathrm{d}x$ are *commutative*. Furthermore: [2]

$$\delta f = f(x, Y, Y_x) - f(x, y, y_x)$$

$$= f(x, y + \delta y, y_x + \delta y_x) - f(x, y, y_x) = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y_x}\delta y_x$$

including only first order of small terms. The case $|\delta y| < \varepsilon$ is called "proximity of the 0th order", $|\delta y_x| < \varepsilon$ is called "proximity of the 1st order", etc. Here we claim "proximity of the 2nd order", which permits us to neglect higher-order terms in the development of δf.

The condition that the integral [1] shall be stationary along $y(x)$ is, then, simply that the integral shall have the same value along an adjacent path $y + \delta y$, i.e.:

$$\delta \int_{x_1}^{x_2} f\mathrm{d}x = \int_{x_1}^{x_2} \delta f\mathrm{d}x = 0 \qquad [3]$$

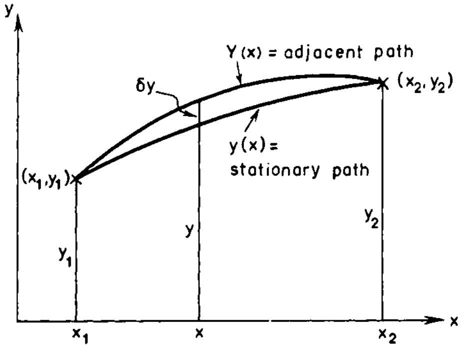


Fig.79.

which, using the two last eq.[2], becomes:

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial y_x}\frac{\mathrm{d}}{\mathrm{d}x}(\delta y)\right] \mathrm{d}x = 0 \tag{4}$$

Partial integration of the last term in [4] yields

$$\underbrace{\left[\frac{\partial f}{\partial y_x}\,\delta y\right]_{x_1}^{x_2}}_{=\,0 \text{ as } \delta y_1 = \delta y_2 = 0} - \int_{x_1}^{x_2} \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_x}\right) \delta y \mathrm{d}x \tag{5}$$

Thus, the *stationarity condition* is as follows:

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_x}\right) \delta y \mathrm{d}x = 0 \tag{6}$$

But as the integration path can be chosen arbitrarily, provided that it is still adjacent to $y(x)$, the integrand in [6] must vanish, which can be said to be our final condition:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_x} = 0 \tag{7}$$

A function y which satisfies [7] is called an *extremal function* or an *extremal.* Eq.[7] is called the *Euler equation associated with the variation problem.* We have to observe that [7] is a *necessary* condition for the function $y(x)$, but not a sufficient one. In other words, we have not shown that such a function exists. This is also true for the corresponding formulas below.

As:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x} + y_x \frac{\partial f}{\partial y} + y_{xx} \frac{\partial f}{\partial y_x} \tag{8}$$

we can also write [7] in the following form:

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f - y_x \frac{\partial f}{\partial y_x}\right) = 0 \qquad [9]$$

Hint: In [8] put everything on the left-hand side; use the expression of $\partial f/\partial y$ from [7], and we arrive at [9].

In case the curves $y(x)$ and $Y(x)$ do not meet at the end points, as in Fig.79, we have $\delta y_1 \neq 0$, $\delta y_2 \neq 0$. Then the condition [7] has to be supplemented by the following two:

$$\left[\frac{\partial f}{\partial y_x}\right]_{x_1} = 0 \ ; \quad \left[\frac{\partial f}{\partial y_x}\right]_{x_2} = 0$$

They will fix the arbitrary constants in the solution of Euler's eq.[7].

10.1.2 Several dependent variables, but one independent variable

The development in section 10.1.1 can be readily extended to this more general case. Let t (usually time) be the independent variable and $x, y, z, \ldots$ the dependent variables. The problem is then to find the functions $x(t)$, $y(t)$, $z(t), \ldots$ so that the integral:

$$\int_{t_1}^{t_2} f(t, x, y, z, \ldots, x_t, y_t, z_t, \ldots)\mathrm{d}t \qquad [10]$$

is stationary. The condition is arrived at in the same way as in section 10.1.1:

$$\int_{t_1}^{t_2} \delta f \mathrm{d}t = 0 \qquad [11]$$

Now:

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z + \ldots$$

$$+ \frac{\partial f}{\partial x_t}\delta x_t + \frac{\partial f}{\partial y_t}\delta y_t + \frac{\partial f}{\partial z_t}\delta z_t + \ldots$$

The independent variable is kept constant, as in section 10.1.1, i.e., $\delta t = 0$. Partial integration gives:

$$\int_{t_1}^{t_2} \frac{\partial f}{\partial x_t}\delta x_t \mathrm{d}t = \int_{t_1}^{t_2} \frac{\partial f}{\partial x_t}\frac{\mathrm{d}}{\mathrm{d}t}(\delta x)\mathrm{d}t = \underbrace{\left[\frac{\partial f}{\partial x_t}\delta x\right]_{t_1}^{t_2}}_{= 0 \text{ (as } \delta x = 0 \text{ at both limits)}} - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial x_t}\right)\delta x \mathrm{d}t$$

Then [11] becomes:

$$\int_{t_1}^{t_2} \left[\left(\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f}{\partial x_t} \right) \delta x + \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f}{\partial y_t} \right) \delta y + \ldots \right] \mathrm{d}t = 0$$

For arbitrary values of δx, δy, ... each integrand must be zero:

$$\left.\begin{aligned} &\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f}{\partial x_t} = 0 \\ &\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f}{\partial y_t} = 0 \\ &\ldots\ldots \end{aligned}\right| \qquad [12]$$

i.e., the same Euler equation [7] as before, but now there is one equation for each dependent variable. The functions $x, y, z, \ldots$ satisfying these differential equations, are the *extremal functions* sought.

The laws of mechanics offer an interesting application. If T = kinetic energy and V = potential energy, *Hamilton's principle* states that:

$$\int_{t_1}^{t_2} (T - V)\mathrm{d}t \qquad [13]$$

shall have a stationary value. This is an alternative method, equivalent to Newton's laws of motion. Using generalized coordinates, we write [13] as:

$$\delta \int_{t_1}^{t_2} [T(q_1, q_2, \ldots, q_n, q_{1t}, q_{2t}, \ldots, q_{nt}, t) - V(q_1, q_2, \ldots, q_n, t)]\, \mathrm{d}t = 0 \qquad [14]$$

Application of [12] leads to the famous *Lagrange's equations of motion*:

$$\frac{\partial T}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_i} = \frac{\partial V}{\partial q_i} \quad ; \quad i = 1, 2, \ldots n \qquad [15]$$

where $\dot{q}_i = (\mathrm{d}q_i/\mathrm{d}t)$. Eq.[15] is applicable also to vibrating *continuous* media, such as a string, bar, plate, membrane (see FRANK and VON MISES, 1930, p.893).

10.1.3 Several independent variables and one dependent variable

The procedure is perfectly analogous to the one applied in section 10.1.1 and 10.1.2. Put x, y, z = independent variables and $u = u(x, y, z)$ = dependent variable, to be determined in such a way that the integral:

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z, u, u_x, u_y, u_z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad [16]$$

is stationary. As above, we request that:

$$\int\int\int \delta f \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0 \qquad [17]$$

Also as above, keeping the independent variables fixed: $\delta x = \delta y = \delta z = 0$, we have:

$$\delta f = \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial u_x}\delta u_x + \frac{\partial f}{\partial u_y}\delta u_y + \frac{\partial f}{\partial u_z}\delta u_z \qquad [18]$$

Perform first an integration with respect to x, by partial integration:

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial u_x}\delta u_x \mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial f}{\partial u_x}\frac{\mathrm{d}}{\mathrm{d}x}(\delta u)\mathrm{d}x = 0 - \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u_x}\right)\delta u \mathrm{d}x \qquad [19]$$

Then [17] becomes:

$$\int\int\int \left(\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial f}{\partial u_z}\right)\delta u \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0 \qquad [20]$$

As this shall be valid for arbitrary (but adjacent) values of δu, we get:

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial f}{\partial u_z} = 0 \qquad [21]$$

i.e., the Euler equation in this case. This can be extended immediately to the case of several dependent variables, one equation for each dependent variable, as in [12].

10.1.4 Higher derivatives than the first

The problem is to find the stationary values of:

$$\int_{x_1}^{x_2} f(x, y, y_x, y_{x^2}, \ldots, y_{x^n})\mathrm{d}x \qquad [22]$$

where we have written:

$$y_{x^n} = \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$$

Euler's equation then reads:

$$f_y - \frac{\mathrm{d}}{\mathrm{d}x}f_{y_x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}f_{y_{x^2}} - \ldots + (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}f_{y_{x^n}} = 0 \qquad [23]$$

Eq.[23] is obtained by the same method as used in section 10.1.1. For the terms with derivatives higher than the first, the integration by parts has to be carried out over and over again, the same number of times as the order of the derivative. This gives the factor $(-1)^n$ in [23]. Eq. [23] assumes that all derivatives up to the order $2n$ exist and are continuous, and that at the limits (x_1 and x_2) all derivatives up to order $(n - 1)$ are given. It is suggested as an exercise to the reader to work out the details of this case, applying the same technique as has been described above.

10.1.5 Solution of Euler's differential equation in special cases

The solution of Euler's differential equation in the calculus of variations can be obtained fairly easily in special cases:

(*1*) *f does not depend on x explicitly:* $f(y, y_x)$. Consider eq.[7] and put:

$$L(y, y_x) = f(y, y_x) - y_x \frac{\partial f(y, y_x)}{\partial y_x} \tag{24}$$

Differentiate [24]:

$$\frac{dL}{dx} = f_y y_x + f_{y_x} y_{xx} - y_{xx} f_{y_x} - y_x \frac{d}{dx} f_{y_x} = y_x\left(f_y - \frac{d}{dx} f_{y_x}\right) \tag{25}$$

Because of [7]:

$$\frac{dL}{dx} = 0 \tag{26}$$

and therefore a first integral of Euler's differential equation [7] is:

$$f(y, y_x) - y_x f_{y_x}(y, y_x) = \alpha \tag{27}$$

where α = integration constant. Solving this equation with respect to y_x, we get an equation of the form:

$$\frac{dy}{dx} = y_x = \psi(y, \alpha) \quad \text{or} \quad dx = \frac{dy}{\psi(y, \alpha)} \tag{28}$$

which can be readily integrated:

$$x = \beta + \int_{y_0}^{y} \frac{dy}{\psi(y, \alpha)} \tag{29}$$

β = another integration constant. Eq.[29] defines the function $y(x)$, which was our problem to find.

(2) *The function f does not depend on y:* $f(x, y_x)$. Then [7] becomes:

$$\frac{d}{dx}\frac{\partial f}{\partial y_x} = 0$$

i.e.:

$$\frac{\partial f(x, y_x)}{\partial y_x} = \alpha \text{ (constant)}$$

Solve for y_x:

$$\left.\begin{aligned} &\frac{dy}{dx} = y_x = \varphi(x, \alpha) \\ &y = \beta + \int_{x_0}^{x} \varphi(x, \alpha) dx \end{aligned}\right| \tag{30}$$

which is the solution of Euler's equation in this case.

10.1.6 Accessory or auxiliary conditions. Lagrange's method of undetermined multipliers. Isoperimetric problems

The problem is to find the stationary value of:

$$\int f \mathrm{d}\tau$$

provided that: [31]

$$\int f_1 \mathrm{d}\tau = c_1 \; ; \quad \int f_2 \mathrm{d}\tau = c_2 \; ; \quad \ldots ; \int f_n \mathrm{d}\tau = c_n$$

The latter are the accessory conditions. All f contain the same variables, and all integration limits are the same; there may be several independent variables ($\mathrm{d}\tau = \mathrm{d}x.\ \mathrm{d}y.\ \mathrm{d}z \ldots$), and c are all constants.

The solution is found by using Lagrange's method of undetermined multipliers. Introduce n multipliers (= constant factors) $\lambda_1, \lambda_2, \ldots, \lambda_n$ so that:

$$K = f + \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n = f + \sum_i \lambda_i f_i \tag{32}$$

Then, because of [31], the integral:

$$\int K \mathrm{d}\tau \tag{33}$$

also stationary. We then have to evaluate:

$$\int \delta K \mathrm{d}\tau = 0 \tag{34}$$

which is done in the same way as in section 10.1.1. We arrive at an integral equation corresponding to [6]. The step from [6] to [7] would not be permitted in this case because of the restrictions [31], i.e., the accessory conditions, which imply that n degrees of freedom are lost. But the n multipliers λ_i are arbitrary and can be chosen at will, making up for the n lost degrees of freedom. Therefore, [7] is still valid:

$$\frac{\partial K}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial K}{\partial y_x} = 0 \tag{35}$$

The problem was to find the stationary value of $\int f \mathrm{d}\tau$, eq.[31]. As the solution K of [35] will give the stationary value of $\int K \mathrm{d}\tau$, we find from above that the stationary value of $\int f \mathrm{d}\tau$ is:

$$\begin{aligned}\int f \mathrm{d}\tau &= \int K \mathrm{d}\tau - \lambda_1 \int f_1 \mathrm{d}\tau - \ldots - \lambda_n \int f_n \mathrm{d}\tau \\ &= \int K \mathrm{d}\tau - \lambda_1 c_1 - \ldots - \lambda_n c_n \end{aligned} \tag{36}$$

which is determined, once all λ are known.

This whole class of problems is often called *isoperimetric problems*, because the problem of finding a closed plane curve of given perimeter and maximum area can be solved in this way.

As another example we consider the wave equation. Find the condition that:

$$\int\int\int (\nabla u)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int\int\int \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] \mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{37}$$

has a stationary value, with the accessory condition that:

$$\iiint u^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = c \qquad (c = \text{a constant}) \tag{38}$$

In this case:

$$\left.\begin{aligned} f &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \\ f_1 &= u^2 \end{aligned}\right| \tag{39}$$

and we have to find the stationarity condition of:

$$\iiint (f + \lambda f_1)\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{40}$$

Here we have one dependent and three independent variables, and we use [21]. We find that the condition is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \lambda u \tag{41}$$

This is the wave equation for a sinusoidal wave of *one* frequency, and where the multiplier $\lambda = -\omega^2/v^2$ (ω = frequency, v = wave velocity).

The problem of a vibrating membrane is treated in an analogous way by FRANK and VON MISES (1935, pp.357–373).

10.2 APPLICATIONS OF THE CALCULUS OF VARIATIONS

10.2.1 Sturm–Liouville equation obtained as a variational problem

In section 7.5.2 it was demonstrated that most differential equations that we have encountered (e.g., in Chapter 4) can be written in the unified form of Sturm–Liouville's equation:

$$L(u) + \lambda w u = 0 \tag{1}$$

where:

$$\left.\begin{aligned} & L(u) = (pu_x)_x - qu \\ & p = p(x)\ ; \quad q = q(x)\ ; \quad w = w(x) \geqslant 0 \\ & \lambda = \text{constant} \end{aligned}\right| \tag{2}$$

L is called the *differential operator*. There is only one independent variable x. λ is the *eigenvalue parameter*. In an equation like [1] with a parameter λ, those values of λ which lead to meaningful (i.e., not trivial) solutions are called *eigenvalues* (or *characteristic values*) and the corresponding solutions are called *eigenfunctions* (or *characteristic functions*).

Eq.[1] can be derived as a variational problem with an auxiliary condition. The integral:

$$\int (pu_x^2 + qu^2)\mathrm{d}x \tag{3}$$

should take a stationary value, with the auxiliary condition:

$$\int wu^2\mathrm{d}x = 1 \tag{4}$$

Apply the methods of section 10.1.6 and put:

$$\lambda_1 = -\lambda \ ; \quad K = f - \lambda f_1 = pu_x^2 + qu^2 - \lambda wu^2$$

The Euler equation becomes (from formula [35] in section 10.1):

$$\frac{\partial K}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial K}{\partial u_x} = 0 \tag{5}$$

Inserting the expression for K given here, we immediately arrive at the Sturm–Liouville eq.[1]. The eigenvalue λ here plays the role of a Lagrangian multiplier. Thus, solving [1] is equivalent to finding a function $u(x)$ which maximizes or minimizes [3] with the condition [4].

10.2.2 *Propagation of seismic body waves as a variational problem*

We consider first a plane problem, i.e., a plane through the ray ("plane of

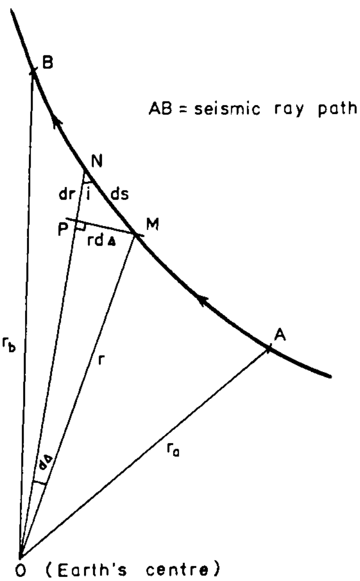


Fig.80.

propagation") and have only two coordinates: Δ and r (see Fig.80). The elementary triangle MNP gives:

$$(\mathrm{d}s)^2 = (\mathrm{d}r)^2 + r^2(\mathrm{d}\Delta)^2$$

or:

$$\mathrm{d}s = \mathrm{d}r\left[1 + r^2\left(\frac{\mathrm{d}\Delta}{\mathrm{d}r}\right)^2\right]^{1/2} \tag{6}$$

The travel time from A to B is then, if v = wave velocity:

$$t = \int_A^B \frac{\mathrm{d}s}{v} = \int_{r_a}^{r_b} \frac{\mathrm{d}r}{v}\left[1 + r^2\left(\frac{\mathrm{d}\Delta}{\mathrm{d}r}\right)^2\right]^{1/2} \tag{7}$$

Fermat's principle, which holds for all kinds of wave propagation, says that the wave propagates along such a path that the travel time t is stationary, either maximum or minimum. At this point we accept Fermat's principle without proof. However, it could be proved as follows. All contributions at the receiving point could be expressed in integral form. By the method of stationary phase (section 3.1) it can be seen that essential contributions are obtained only in case of stationary travel time. The present problem is an application of the case with one independent variable (r) and one dependent variable (Δ), see section 10.1.1. The function f corresponds in this case to the following function:

$$f\left(r, \frac{\mathrm{d}\Delta}{\mathrm{d}r}\right) = \frac{1}{v}\left[1 + r^2\left(\frac{\mathrm{d}\Delta}{\mathrm{d}r}\right)^2\right]^{1/2} \tag{8}$$

Euler's eq.[7] in 10.1 then gives:

$$\underbrace{\frac{\partial f}{\partial \Delta}}_{=0 \text{ (as } f \text{ does not contain } \Delta \text{ explicitly)}} - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\partial f}{\partial \Delta'}\right) = 0 \; ; \quad \Delta' = \frac{\mathrm{d}\Delta}{\mathrm{d}r} \tag{9}$$

We have to observe that in fact $v = v(r, \Delta)$, but disregarding horizontal inhomogeneities in the velocity field, we have $v = v(r)$. Therefore:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\partial f}{\partial \Delta'}\right) = 0$$

or:

$$\frac{\partial f}{\partial \Delta'} = \text{constant} \tag{10}$$

Use [8] to get an expression for this partial differential:

$$\frac{\partial f}{\partial \Delta'} = \frac{r}{v}\,\frac{r\,\dfrac{\mathrm{d}\Delta}{\mathrm{d}r}}{\left[1 + r^2\left(\dfrac{\mathrm{d}\Delta}{\mathrm{d}r}\right)^2\right]^{1/2}} = \text{constant} = p \tag{11}$$

From the elementary triangle MNP we have:

$$\frac{r\mathrm{d}\Delta}{\mathrm{d}r} = \tan i \qquad [12]$$

which permits us to rewrite eq.[11] as follows:

$$\frac{r}{v}\frac{\tan i}{\sqrt{(1+\tan^2 i)}} = \frac{r \sin i}{v} = p \quad \text{(ray parameter)} \qquad [13]$$

This is a well-known equation for the rays, which includes the laws of refraction and reflection. Thus, these laws follow directly from Fermat's principle as a variational problem.

All seismic waves propagate according to Fermat's principle of stationary time. Usually the travel time is a minimum, but it could also be a maximum. For example, for a longitudinal wave reflected once from the earth's outer surface, there are two possibilities: *pP* corresponding to minimum time and *PP* corresponding to maximum time. The same is true for transverse waves (*sS* and *SS*), and for mixed waves (*pS* and *PS*, *sP* and *SP*). This difference between *pP* and *PP* has been taken as an explanation for the different pulse shapes: *pP* is sharp but *PP* blunt, a rule with many exceptions.

As an exercise, it is suggested to the reader to prove that in a sphere with constant velocity throughout, *PP* has maximum travel time and *pP* minimum, using Fermat's principle. Assuming constant velocity naturally implies straight-line propagation, and is thus the simplest case. But the actual case, with velocity generally increasing downward is only a modification of this result (cf. BYERLY, 1942, pp. 202–209).

Generalized form of Fermat's principle, i.e., when v is a function of all space coordinates: $v = v(x, y, z)$. v_0 = constant is a certain reference velocity. $n = v_0/v$ = index of refraction. Then the following integral should be stationary according to Fermat's principle:

$$F = v_0 \int_A^B \mathrm{d}t = v_0 \int_A^B \frac{\mathrm{d}s}{v} = \int_A^B n\mathrm{d}s \qquad [14]$$

Use the parameter σ as independent variable and x, y, z as dependent variables:

$$\mathrm{d}s = \sqrt{(x'^2 + y'^2 + z'^2)}\mathrm{d}\sigma \qquad [15]$$

$$x' = \frac{\mathrm{d}x}{\mathrm{d}\sigma}, \quad \text{etc.}$$

and [14] becomes:

$$F = \int_A^B n(x, y, z)\sqrt{(x'^2 + y'^2 + z'^2)}\mathrm{d}\sigma = \int_A^B G(x, y, z, x', y', z')\mathrm{d}\sigma \qquad [16]$$

Euler's eq.[12] in section 10.1 give:

$$\left.\begin{aligned}\frac{\partial G}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial G}{\partial x'}\right) = 0 \\ \frac{\partial G}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial G}{\partial y'}\right) = 0\end{aligned}\right| \qquad [17]$$

$$\frac{\partial G}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial G}{\partial z'}\right) = 0 \qquad [17]$$

Eq.[17] can now be written as:

$$\begin{aligned}
&\sqrt{(x'^2 + y'^2 + z'^2)}\,\frac{\partial n}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\frac{nx'}{\sqrt{(x'^2 + y'^2 + z'^2)}}\right] = 0\\
&\sqrt{(x'^2 + y'^2 + z'^2)}\,\frac{\partial n}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\frac{ny'}{\sqrt{(x'^2 + y'^2 + z'^2)}}\right] = 0\\
&\sqrt{(x'^2 + y'^2 + z'^2)}\,\frac{\partial n}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\frac{nz'}{\sqrt{(x'^2 + y'^2 + z'^2)}}\right] = 0
\end{aligned} \qquad [18]$$

Using [15] to eliminate $\mathrm{d}\sigma$ and noting that:

$$\frac{x'}{\sqrt{(x'^2 + y'^2 + z'^2)}} = \frac{x'\mathrm{d}\sigma}{\sqrt{(x'^2 + y'^2 + z'^2)}\mathrm{d}\sigma} = \frac{\mathrm{d}x}{\mathrm{d}s}$$

we finally get:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}x}{\mathrm{d}s}\right) = \frac{\partial n}{\partial x}\ ;\ \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}y}{\mathrm{d}s}\right) = \frac{\partial n}{\partial y}\ ;\ \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}z}{\mathrm{d}s}\right) = \frac{\partial n}{\partial z} \qquad [19]$$

These are *the generalized forms of Fermat's principle* (*or of Snell's law*). It is demonstrated by OFFICER (1958) that if one starts from the assumption that rays are perpendicular to the wave front, one also arrives at [19].

It would have been more convenient above to use x_i ($i = 1, 2, 3$) instead of x, y, z and to let the summation convention hold. Then the equations read as follows:

$$\mathrm{d}s = \sqrt{(x_i'^2)}\mathrm{d}\sigma \qquad [20]$$

$$x_i' = \frac{\mathrm{d}x_i}{\mathrm{d}\sigma}$$

$$F = \int_A^B n(x_i)\sqrt{(x_i'^2)}\mathrm{d}\sigma = \int_A^B G(x_i, x_i')\mathrm{d}\sigma \qquad [21]$$

$$\frac{\partial G}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial G}{\partial x_i'}\right) = 0 \qquad [22]$$

$$\sqrt{(x_i'^2)}\,\frac{\partial n}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{nx_i'}{\sqrt{x_i'^2}}\right) = 0 \qquad [23]$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}x_i}{\mathrm{d}s}\right) = \frac{\partial n}{\partial x_i} \qquad [24]$$

Obviously, this section on Fermat's principle leads directly to a study of the *ray* properties of seismic waves. In section 11.2 we shall consider other aspects of this problem in connection with Abel's integral and its solution.

10.2.3 The variational equation of motion

We start from Hamilton's principle:

$$\delta \int_{t_1}^{t_2} (T - V)\mathrm{d}t + \int_{t_1}^{t_2} \delta W_1 \mathrm{d}t = 0 \qquad [25]$$

where T = total kinetic energy of the body concerned, V = potential energy of deformation, i.e., $\iiint W\mathrm{d}x\mathrm{d}y\mathrm{d}z$, where W = strain-energy function, W_1 = work done by the external forces when the displacement is varied. The variation in [25] is taken between fixed initial and final values (t_1 and t_2) of the time t. In [25] only the displacement is subject to variation, and it assumes constant values (no variation) at t_1 and t_2.

Eq.[25] is a more complete expression for Hamilton's principle than the one used in section 10.1.2, eq.[13], where there were no external forces, i.e., no W_1. Eq.[25] is an expression for the principle of energy conservation (1st law of thermodynamics). A good presentation of Hamilton's principle can be found in Houston (1934, pp.62–74). Above we used Hamilton's principle only for the case when the forces are conservative, i.e., they can be derived from a potential energy (Houston, 1934, p.65). But now we are considering a non-conservative system, i.e., external forces enter the problem, which cannot be expressed as derivatives of the potential energy. Assuming that these forces depend only on coordinates but not on velocities, i.e., that $\partial W_1/\partial \dot{q}_i = 0$, the Lagrangian equation reads (Houston, 1934, p.71):

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{\partial W_1}{\partial q_i} \qquad [26]$$

which corresponds to [25] above. Writing [25] as follows:

$$\delta \int L\mathrm{d}t + \int \delta W_1 \mathrm{d}t = 0 \qquad [25a]$$

$$L = T - V$$

and carrying out the variations we find [26].

Now we shall transform the terms in [25] in such a way that they are expressed in the variations of the displacements, i.e., in δu, δv, δw. The kinetic energy T is:

$$T = \iiint \frac{\varrho}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] \mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad [27]$$

which is true for *small* displacements:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial u}{\partial t} \simeq \frac{\partial u}{\partial t}$$

Taking the variation of [27] we find:

$$\delta \int |T\mathrm{d}t = \int \mathrm{d}t \iiint \varrho \left(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial v}{\partial t}\frac{\partial \delta v}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \underbrace{\left[\iiint \varrho \left(\frac{\partial u}{\partial t}\,\delta u + \frac{\partial v}{\partial t}\,\delta v + \frac{\partial w}{\partial t}\,\delta w\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z\right]_{t_1}^{t_2}}_{= 0 \text{ (as } \delta u = \delta v = \delta w = 0 \text{ at } t_1 \text{ and at } t_2)} -$$

$$- \int \mathrm{d}t \iiint \varrho \left(\frac{\partial^2 u}{\partial t^2}\,\delta u + \frac{\partial^2 v}{\partial t^2}\,\delta v + \frac{\partial^2 w}{\partial t^2}\,\delta w\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad [28]$$

where we have used partial integration on the part of the integral containing t:

$$\int \frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t}\,\mathrm{d}t = \left[\frac{\partial u}{\partial t}\,\delta u\right]_{t_1}^{t_2} - \int \delta u \frac{\partial^2 u}{\partial t^2}\,\mathrm{d}t$$

Introducing [28] into [25] we get a *variational equation of motion* (i.e., one which contains *variations* of displacements).

Considering further the variation of the potential energy in [25], we have:

$$\delta V = \iiint \delta W \mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad [29]$$

Moreover, from BULLEN (1963, p.26), using Bullen's definition of e_{ij}:

$$\delta W = \frac{\partial W}{\partial e_{xx}}\,\delta e_{xx} + \frac{\partial W}{\partial e_{yy}}\,\delta e_{yy} + \frac{\partial W}{\partial e_{zz}}\,\delta e_{zz}$$

$$+ 2\,\frac{\partial W}{\partial e_{yz}}\,\delta e_{yz} + 2\,\frac{\partial W}{\partial e_{zx}}\,\delta e_{zx} + 2\,\frac{\partial W}{\partial e_{xy}}\,\delta e_{xy} \qquad [30]$$

Partial integration of [29] gives:

$$\underset{\text{(using [30])}}{\iiint \delta W \mathrm{d}x\mathrm{d}y\mathrm{d}z} = \iiint \underbrace{\frac{\partial W}{\partial e_{xx}}\,\mathrm{d}y\mathrm{d}z}_{= u'}\ \underbrace{\frac{\partial \delta u}{\partial x}\,\mathrm{d}x}_{= \mathrm{d}v'} + \text{etc.} =$$

(partial integration:

$$\mathrm{d}u' = \frac{\partial}{\partial x}\frac{\partial W}{\partial e_{xx}}\,\mathrm{d}x \cdot \underbrace{\mathrm{d}y\mathrm{d}z}_{= \mathrm{d}S \cdot \cos(x, \nu)}$$

$$v' = \delta u)$$

$$= \iint \left\{\left[\frac{\partial W}{\partial e_{xx}}\cos(x, \nu) + \frac{\partial W}{\partial e_{xy}}\cos(y, \nu) + \frac{\partial W}{\partial e_{xz}}\cos(z, \nu)\right]\delta u + \ldots\right\}\mathrm{d}S$$

$$- \iiint \left[\left(\frac{\partial}{\partial x}\frac{\partial W}{\partial e_{xx}} + \frac{\partial}{\partial y}\frac{\partial W}{\partial e_{xy}} + \frac{\partial}{\partial z}\frac{\partial W}{\partial e_{xz}}\right)\delta u + \ldots\right] \mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad [31]$$

$\mathrm{d}S$ is a surface element corresponding to the volume element $\mathrm{d}x\mathrm{d}y\mathrm{d}z$, and ν denotes the normal direction to S.

Finally, considering the work term in [25], this can be transformed as follows:

$$\delta W_1 = \underset{\text{(body forces)}}{\iiint \varrho(X\delta u + Y\delta v + Z\delta w)\mathrm{d}x\mathrm{d}y\mathrm{d}z} + \underset{\text{(surface forces)}}{\iint (X_\nu \delta u + Y_\nu \delta v + Z_\nu \delta w)\mathrm{d}S} \qquad [32]$$

where: $X_\nu = X_x \cos(x, \nu) + X_y \cos(y, \nu) + X_z \cos(z, \nu)$.

Introducing into [25] the three expressions [28], [31] and [32], and omitting the integration over t, we arrive at an expression which may be formally written as follows:

$$\iiint [A\delta u + \ldots]\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iint [B\delta u + \ldots]\mathrm{d}S = 0 \qquad [33]$$

This can be fulfilled for all values of δu, δv, δw only if the coefficients A and B vanish:

$$\left.\begin{aligned} A &= 0 \\ B &= 0 \end{aligned}\right| \qquad [34]$$

Considering the factor of δu, we find one equation of motion:

$$\varrho \frac{\partial^2 u}{\partial t^2} = \varrho X + \frac{\partial}{\partial x}\frac{\partial W}{\partial e_{xx}} + \frac{\partial}{\partial y}\frac{\partial W}{\partial e_{xy}} + \frac{\partial}{\partial z}\frac{\partial W}{\partial e_{xz}} \qquad [35]$$

and one surface condition:

$$\frac{\partial W}{\partial e_{xx}} \cos(x, \nu) + \frac{\partial W}{\partial e_{xy}} \cos(y, \nu) + \frac{\partial W}{\partial e_{xz}} \cos(z, \nu) = X_\nu \qquad [36]$$

Similarly, we find corresponding equations for the other two variations δv and δw.

Using the summation convention we can write [35] as:

$$\varrho \frac{\partial^2 u_i}{\partial t^2} = \varrho X_i + \frac{\partial}{\partial x_j}\frac{\partial W}{\partial e_{ij}} \qquad [37]$$

and [36] as:

$$\frac{\partial W}{\partial e_{ij}} \cos(x_j, \nu) = X_{i\nu} \qquad [38]$$

The equation of motion [37] is immediately seen to agree with other more usual forms of this equation. From BULLEN (1963, p.26), we have:

$$p_{ij} = \frac{\partial W}{\partial e_{ij}}$$

and then [37] becomes:

$$\varrho \frac{\partial^2 u_i}{\partial t^2} = \varrho X_i + \frac{\partial p_{ij}}{\partial x_j}$$

i.e., identical with BULLEN (1963, eq.9, p.12). Similarly, we can rewrite [38] as:

$$p_{ij} \cos(x_j, \nu) = X_{i\nu}$$

which evidently expresses the force $X_{i\nu}$ acting on the surface with normal direction ν as the sum of the projections of the three forces p_{ij} along the normal to this surface.

Therefore, we have arrived at nothing other than well-known equations, but the purpose of this section is to demonstrate the use of Hamilton's variational principle also to continuous media.

Further applications of equations [35] and [36] may be found in LOVE (1944, pp. 167–182).

Chapter 11

INTEGRAL EQUATIONS

11.1 DEFINITIONS AND SOLUTIONS OF INTEGRAL EQUATIONS

11.1.1 Definitions

An integral equation is one which contains the unknown function under the integral sign. Such equations are of several types. An integral equation is said to be of the *first kind*, when it contains the unknown function under the integral sign only, but to be of the *second kind*, when the unknown function is contained both under and outside the integral sign. A still more general type is the following, which is called a linear integral equation of the *third kind* (linear, because it contains the unknown function $\varphi(x)$ only in its first degree; third kind, because it is more general than those of the first and second kinds):

$$g(x)\varphi(x) = f(x) + \lambda \int_a^b K(x, z)\varphi(z)\mathrm{d}z \tag{1}$$

The function $K(x, z)$ under the integral sign, is called the *kernel*, *kern*, *nucleus* or *matrix* of the equation. The kernel characterizes the integral equation. A kernel is said to be *symmetrical* when $K(x, z) = K(z, x)$, and *antisymmetrical* when $K(x, z) = -K(z, x)$. a and b may be constants or known functions of x.

The following equations are special cases of [1]:

(*1*) Fredholm's equation of the first kind: $g(x) = 0$; a, b constants:

$$f(x) = -\lambda \int_a^b K(x, z)\varphi(z)\mathrm{d}z \tag{2}$$

(*2*) Fredholm's equation of the second kind: $g(x) = 1$; a, b constants:

$$\varphi(x) = f(x) + \lambda \int_a^b K(x, z)\varphi(z)\mathrm{d}z \tag{3}$$

(*3*) Volterra's equation of the first kind: $g(x) = 0$, $a = 0$, $b = x$:

$$f(x) = -\lambda \int_0^x K(x, z)\varphi(z)\mathrm{d}z \tag{4}$$

In particular, $K(x, z) = \psi(x - z)$ gives Abel's integral equation, where ψ is a function of $(x - z)$.

(*4*) Volterra's equation of the second kind: $g(x) = 1$, $a = 0$, $b = x$:

$$\varphi(x) = f(x) + \lambda \int_0^x K(x, z)\varphi(z)\mathrm{d}z \qquad [5]$$

In particular, $K(x, z) = \psi(x - z)$ gives Poisson's integral equation.

Eq.[1] is *homogeneous* (in φ) if $f(x) = 0$. Eq.[1] is *singular*, if a or b is infinite, or if K assumes infinite values for any values of z within the range of integration, from a to b. An integral equation is *non-linear* (in the unknown function φ), if it contains φ in other powers than the first, e.g.:

$$\varphi(x) = f(x) + \lambda \int_a^b K(x, z)\varphi^n(z)\mathrm{d}z \qquad [6]$$

or:

$$\varphi(x) = f(x) + \lambda \int_a^b F[x, z, \varphi(z)]\mathrm{d}z \qquad [7]$$

11.1.2 Solutions of special types of integral equations

Integral equations can seldom be solved in finite terms, but there are extensive theories of their solution by successive approximations. Our discussion here will be far from complete, but rather limited to some examples, which will give some insight into this problem.

Solution of Fredholm's equation of the second kind. We shall demonstrate the solution of equation [3] by means of the *Liouville–Neumann series.* In eq.[3] we assume that x and z are real variables: $a \leqslant x \leqslant b$, $a \leqslant z \leqslant b$, and that $K(x, z)$ and $f(x)$ are continuous but may be complex. We assume a solution in the form of a power series in λ:

$$\varphi(x) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(x) \qquad [8]$$

This is called the Liouville–Neumann series. We substitute [8] into [3]. The resulting equation must hold for all values of λ (with λ being either an absolute constant or a parameter), and this can only happen if the coefficient of each λ-power is zero; thus:

$$\left.\begin{aligned} \varphi_0(x) &= f(x) \\ \varphi_1(x) &= \int K(x, z)\varphi_0(z)\mathrm{d}z \\ \varphi_2(x) &= \int K(x, z)\varphi_1(z)\mathrm{d}z \\ &\cdots\cdots\cdots\cdots\cdots \\ \varphi_n(x) &= \int K(x, z)\varphi_{n-1}(z)\mathrm{d}z \end{aligned}\right| \qquad [9]$$

We assumed that x and z are limited to the range $a - b$ and that $K(x, z)$ and $f(x)$ are continuous; as a consequence they have maximum values, such that:

$$|K(x, z)| \leqslant M \;;\quad |f(x)| \leqslant N \qquad [10]$$

Introduced into [9], they give:

$$|\varphi_0| \leqslant N \;;\quad |\varphi_1| \leqslant NM(b-a) \;;\quad \ldots \;;\quad |\varphi_n| \leqslant N[M(b-a)]^n \qquad [11]$$

Let us investigate the convergence of the series [8] in the usual way, i.e., the series is convergent if the ratio of successive terms is less than 1:

$$\left|\frac{\lambda^{n+1}\varphi_{n+1}(x)}{\lambda^n\varphi_n(x)}\right| < 1$$

for convergence, i.e.:

$$\left|\frac{\lambda^{n+1}N[M(b-a)]^{n+1}}{\lambda^n N[M(b-a)]^n}\right| < 1 \text{ or } |\lambda| < \frac{1}{M(b-a)} \qquad [12]$$

Convergence of [8] is guaranteed when it is fulfilled for the maximum values by [12]. For values of λ specified by [12], the series [8] represents the unique continuous solution of [3].

We shall now derive *another form of the solution of* [3]. We define the *iterated kernels* (limits of integration are a and b):

$$\left.\begin{aligned} K_1(x, z) &= K(x, z) \\ K_2(x, z) &= \int K(x, y)K(y, z)\mathrm{d}y \\ &\ldots\ldots\ldots \\ K_n(x, z) &= \int K(x, y)K_{n-1}(y, z)\mathrm{d}y \\ &= \int\int \ldots \int K(x, y_1)K(y_1, y_2) \ldots K(y_{n-1}, z)\mathrm{d}y_1\mathrm{d}y_2 \ldots \mathrm{d}y_{n-1} \end{aligned}\right| \qquad [13$$

These iterated kernels are introduced into [9]:

$$\varphi_1(x) = \int K_1(x, z)f(z)\mathrm{d}z$$

from [9.1], [9.2] and [13.1].

$$\varphi_2(x) = \int K(x, z)\varphi_1(z)\mathrm{d}z = \int K(x, y)\varphi_1(y)\mathrm{d}y =$$

(changing the integration variable z to y)

$$= \int\int K(x, y)K(y, z)f(z)\mathrm{d}y\mathrm{d}z = \int K_2(x, z)f(z)\mathrm{d}z$$

etc, or in general:

$$\varphi_n(x) = \int K_n(x, z)f(z)\mathrm{d}z \qquad [14]$$

From [10] and [13] we have:

$$|K_n(x, z)| \leqslant M^n(b-a)^{n-1} \qquad [15]$$

as in $K_n(x, z)$ in [13] there are n factors K and $(n-1)$ factors $\mathrm{d}y$; each of K is $\leqslant M$ and each of $\mathrm{d}y \leqslant b-a$. Assuming that λ still fulfills the condition [12], we can form a uniformly convergent series, called *resolvent* (German "lösender Kern"):

$$K(x, z; \lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, z) \qquad [16]$$

That [16] is uniformly convergent is seen from the ratio of successive terms:

$$\left|\frac{\lambda^{n+1}K_{n+2}(x, z)}{\lambda^n K_{n+1}(x, z)}\right| < \frac{M^{n+2}(b-a)^{n+1}}{M(b-a)M^{n+1}(b-a)^n} = 1 \tag{17}$$

Then we can write the solution of [3] also in the following form:

$$\varphi(x) = f(x) + \lambda \int K(x, z; \lambda)f(z)\mathrm{d}z \tag{18}$$

This is seen as follows:

$$\varphi_{n+1}(x) = \underset{\text{(from [14])}}{\int K_{n+1}(x, z)f(z)\mathrm{d}z} = \underset{\text{(from [9])}}{\int K(x, z)\varphi_n(z)\mathrm{d}z}$$

Multiply this equation by $\sum_{n=0}^{\infty} \lambda^n$:

$$\int \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, z)f(z)\mathrm{d}z = \int \sum_{n=0}^{\infty} \lambda^n K(x, z)\varphi_n(z)\mathrm{d}z$$

(by [16]) $\downarrow$ (by [8]) $\downarrow$

$$\int K(x, z; \lambda)f(z)\mathrm{d}z \quad = \quad \int K(x, z)\varphi(z)\mathrm{d}z = \underset{\text{(by [3])}}{\frac{1}{\lambda}[\varphi(x) - f(x)]}$$

Finally, it should be remarked that in [3] we considered $\varphi(x)$ as the unknown function, and [18] as the solution. But we could also regard [18] as the given integral equation with $f(x)$ as the unknown function and with [3] as its solution. Therefore, the functions $f(x)$ and $\varphi(x)$ have a reciprocal nature in relation to this integral equation. It is seen that the complete solution is obtained only by an iterative process, both in [8] and [18].

Volterra's equation of the second kind. The solution can be obtained also in this case by means of the Liouville-Neumann series. The calculations are closely analogous to the previous case and are not repeated here.

Volterra's equation of the first kind. With a change of notation in [4], we write this equation as:

$$g(x) = \lambda \int_0^x K(x, z)\varphi(z)\mathrm{d}z \tag{19}$$

Differentiate [19] with respect to x (note that differentiation with regard to one of the limits of the integral is made by substituting this value in the integrand; see DE LA VALLÉE-POUSSIN, 1938, p.214):

$$g'(x) = \lambda \int_0^x \frac{\partial K}{\partial x}\varphi(z)\mathrm{d}z + \lambda K(x, x)\varphi(x) \tag{20}$$

We define:

$$f(x) = \frac{g'(x)}{\lambda K(x, x)}$$

and:

$$H(x, z) = \frac{-(\partial K/\partial x)}{\lambda K(x, x)}$$

and assume that the function in the denominator $K(x, x) \neq 0$. Then [20] becomes:

$$\varphi(x) = f(x) + \lambda \int_0^x H(x, z)\varphi(z)\mathrm{d}z \qquad [21]$$

We arrive at this equation by dividing [20] by $\lambda K(x, x)$, and this division can also be made under the integral sign in [20], as the integral is over z and not over x. Eq.[21] is of the same type as [5], i.e., Volterra's equation of the second kind. It can then be solved by the methods applicable to equations of the second kind, e.g., by the Liouville–Neumann series. But this method works only when $K(x, x) \neq 0$.

An important case arises when the kernel becomes infinite at one or more points within the range of x and z, i.e., when the integral equation becomes *singular*. In such cases, the methods explained so far cannot be applied, and it is necessary to transform the equation to remove the singularity. In section 11.2 we shall discuss such a singular equation of Volterra's type of the first kind, which is important in seismology.

11.1.3 Fourier and Laplace transforms

A reciprocal nature holds between these transforms and their inversions. Take for instance the Laplace transform (section 8.1):

$$\bar{f}(p) = \int_0^\infty f(x)\mathrm{e}^{-px}\mathrm{d}x \qquad [22]$$

This is an integral equation with a kernel $= K(p, x) = \mathrm{e}^{-px}$. Comparing [22] with the general scheme of integral equations, we see that it is of the same type as Fredholm's equation of the first kind. The solution of this integral equation is its inversion:

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{f}(p)\mathrm{e}^{px}\mathrm{d}p \qquad [23]$$

Similarly, the Fourier sine and cosine transforms and the complex Fourier transform are all integral equations of the same type as Fredholm's equation of the first kind, and their kernels are respectively:

$$K(p, x) = {\sin \atop \cos} px$$

and: [24]

$$K(p, x) = \mathrm{e}^{ipx}$$

The same is true for other integral transforms (section 8.1). We may mention the Carson integral equation [2a] in section 8.1, which is also of the same type as Fredholm's equation of the first kind.

11.1.4 Relation between differential and integral equations

The physical importance of integral equations lies in the fact that most differential equations together with their boundary conditions may be reformulated in terms of a single integral equation. Such a single integral equation is often simpler to solve than a system of differential equations.

Let us consider a differential equation of the second order:

$$y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x, y) \qquad [25]$$

Repeated integration gives:

$$y'(x) = \int_0^x f[z, y(z)]\mathrm{d}z + C_1 \qquad [26]$$

$$y(x) = \int_0^x \left\{ \int_0^x f[z, y(z)]\mathrm{d}z \right\} \mathrm{d}x + C_1 x + C_2 \qquad [27]$$

That [26] and [27] are correct is seen immediately by carrying out the differentiations, remembering the rule for differentiation with regard to one of the limits of a definite integral (DE LA VALLÉE-POUSSIN, 1938, p.214). An alternative form of [27] is:

$$y(x) = \int_0^x (x - z)f[z, y(z)]\mathrm{d}z + g(x) \qquad [28]$$

$$g(x) = C_1 x + C_2$$

which is a non-linear Volterra equation of the second kind with $y(x)$ as unknown (see [5]). As $y(x)$ is the unknown also in the differential equation [25], we can say that [28] is the corresponding integral equation. That [28] agrees with [26] can be seen by carrying out the differentiations on the right-hand side of [28]: differentiating with regard to x, we first have to differentiate with regard to x under the integral sign, which gives:

$$\int_0^x f[z, y(z)]\mathrm{d}z$$

and then we have to insert the upper limit x instead of the variable z in the integrand; the contribution is then $= 0$, because of the factor $(x - z)$.

We can further transform [28] by including the two boundary conditions needed to define C_1 and C_2. We consider two sets of boundary conditions:

(*1*) For $x = 0$: $y(0) = a$ and $y'(0) = b$, a and b constants. Then [28] becomes:

$$y(x) = \int_0^x (x - z)f[z, y(z)]\mathrm{d}z + bx + a \qquad [29]$$

i.e., Volterra's equation of the second kind. The integral equation [29] includes in one single equation both the differential equation [25] and the boundary conditions (*1*).

(*2*) Other boundary conditions: $y(0) = a$, $y(1) = b$. We get $C_2 = a$, by putting $x = 0$ in [28]. $x = 1$ will define C_1:

$$b = y(1) = \int_0^1 (1 - z)f\mathrm{d}z + C_1 + a$$

thus:

$$C_1 = (b - a) - \int_0^1 (1 - z)f\mathrm{d}z \qquad [30]$$

Putting these expressions for C_1 and C_2 into [28] we get:

$$y(x) = \underbrace{a + (b - a)x}_{= h(x)} + \int_0^x (x - z)f\mathrm{d}z + x\int_0^1 (z - 1)f\mathrm{d}z$$

$$= h(x) + \int_0^x (x - z)f\mathrm{d}z + x\int_0^x (z - 1)f\mathrm{d}z + x\int_x^1 (z - 1)f\mathrm{d}z$$

$$= h(x) + \int_0^x z(x - 1)f\mathrm{d}z + \int_x^1 x(z - 1)f\mathrm{d}z \qquad [31]$$

Defining the kernel in the following way ($0 \leqslant x, z \leqslant 1$):

$$K(x, z) \begin{cases} = z(x - 1) & \text{for } x \geqslant z \\ = x(z - 1) & \text{for } x \leqslant z \end{cases} \qquad [32]$$

we can write [31] as:

$$y(x) = h(x) + \int_0^1 K(x, z)f[z, y(z)]\mathrm{d}z \qquad [33]$$

which is Fredholm's equation of the second kind, with the addition that the kernel is different in different parts of the integration interval according to [32]. Again the single integral equation [33] expresses both the differential equation [25] and the boundary conditions (*2*). The method developed in section 11.1.2 for the solution of an integral equation of this type can be applied to [33] to give $y(x)$.

A more complete and systematic treatment of integral equations and their use instead of differential equations would include the following items:

(*a*) General theory of transformation of differential equations and their boundary conditions into integral equations (use of Green's function).

(*b*) General treatment of the solution of integral equations.

Both items are beyond the scope of our discussion, and we have limited our treatment to just examples of (*a*) and (*b*).

11.2 APPLICATION TO SEISMIC RAY THEORY

Usually, seismic ray theory does not present any great mathematical difficulties. We have earlier (section 10.2.2) discussed some ray properties in connection with the calculus of variations. Here we shall discuss the determination of body-wave velocities in the earth from observed travel-time curves. We shall follow the Wiechert–Herglotz method, but give a more general solution of Abel's integral than found in BULLEN (1963, pp.119–120). We essentially follow MACELWANE and SOHON (1932, chapter 8).

11.2.1 Solution of Abel's integral equation

We start from *Volterra's equation of the first kind,* [4] in section 11.1:

$$f(h) = \lambda \int_0^h \frac{u'(y)\mathrm{d}y}{(h-y)^n} \qquad [1]$$

where the functions u and f and their derivatives are assumed continuous and $0 < n < 1$. λ = constant. The type [1] of this equation is also called Abel's integral equation.

In the following discussion we shall have use for the beta and gamma functions (see section 1.3), but first we briefly recapitulate some formulas. Beta function is defined as:

$$B(m, p) = \int_0^1 x^{m-1}(1-x)^{p-1}\mathrm{d}x \qquad \text{(eq.[2] in section 1.3)} \qquad [2]$$

$m > 0$, $p > 0$. Gamma function is defined as:

$$\Gamma(m) = \int_0^1 \left(\log \frac{1}{y}\right)^{m-1} \mathrm{d}y \qquad \text{(eq.[12] in section 1.3)} \qquad [3]$$

or:

$$\Gamma(m) = \int_0^\infty \mathrm{e}^{-x}x^{m-1}\mathrm{d}x \qquad \text{(eq.[1] in section 1.3)} \qquad [4]$$

The two functions are related by the following equation:

$$B(m, p) = \frac{\Gamma(m)\Gamma(p)}{\Gamma(m+p)} \qquad \text{(eq.[7] in section 1.3)} \qquad [5]$$

Putting $p = 1 - n$ and $z = hx$ (using z as variable instead of x), we get from [2] and [5]:

$$B(m, p) = \int_0^1 x^{m-1}(1-x)^{p-1}\mathrm{d}x = \int_0^1 \frac{x^{m-1}\mathrm{d}x}{(1-x)^n}$$

and:

$$\int_0^h \frac{z^{m-1}\mathrm{d}z}{(h-z)^n} = \frac{\Gamma(m)\Gamma(1-n)}{\Gamma(m-n+1)} h^{m-n} \qquad [6]$$

Multiply [6] by $\mathrm{d}h/(y-h)^{1-n}$ and integrate over h:

$$\int_0^y \frac{\mathrm{d}h}{(y-h)^{1-n}} \int_0^h \frac{z^{m-1}\mathrm{d}z}{(h-z)^n} = \frac{\Gamma(m)\Gamma(1-n)}{\Gamma(m-n+1)} \int_0^y \frac{h^{m-n}\mathrm{d}h}{(y-h)^{1-n}} \qquad [7]$$

The last integral in [7] can be expressed in gamma functions by a slight transformation, which makes the integral agree with the definition [2]:

$$\int_0^y \frac{h^{m-n}\mathrm{d}h}{(y-h)^{1-n}} = y^m \int_0^1 \frac{(h/y)^{m-n}\,\mathrm{d}(h/y)}{[1-(h/y)]^{1-n}} = y^m \frac{\Gamma(m-n+1)\Gamma(n)}{\Gamma(m+1)} \qquad [8]$$

(by eq.[5])

Introducing [8] into [7] and considering that $\Gamma(m+1) = m\Gamma(m)$ we find:

$$\int_0^y \frac{\mathrm{d}h}{(y-h)^{1-n}} \int_0^h \frac{z^{m-1}\mathrm{d}z}{(h-z)^n} = \Gamma(n)\Gamma(1-n)\frac{\Gamma(m)}{\Gamma(m+1)}y^m = \frac{y^m}{m}\Gamma(n)\Gamma(1-n) \qquad [9]$$

Multiply [9] by $m\varphi(m)\mathrm{d}m$ and integrate with respect to m:

$$\int_0^y \frac{\mathrm{d}h}{(y-h)^{1-n}} \int_0^h \frac{[\int \varphi(m)mz^{m-1}\mathrm{d}m]\mathrm{d}z}{(h-z)^n} = \Gamma(n)\Gamma(1-n)\int \varphi(m)y^m\mathrm{d}m \qquad [10]$$

Let our function $u(y)$ in [1] be defined as follows:

$$u(y) = \int \varphi(m)y^m\mathrm{d}m \qquad [11]$$

Thus:

$$\left.\begin{aligned} u'(y) &= \int \varphi(m)my^{m-1}\mathrm{d}m \\ u'(z) &= \int \varphi(m)mz^{m-1}\mathrm{d}m \end{aligned}\right| \qquad [12]$$

and [10] becomes:

$$\int_0^y \frac{\mathrm{d}h}{(y-h)^{1-n}} \int_0^h \frac{u'(z)\mathrm{d}z}{(h-z)^n} = \Gamma(n)\Gamma(1-n)u(y) \qquad [13]$$

We solve $u(y)$ from [13] and use the relation:

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \qquad \text{(eq.[11] in section 1.3)} \qquad [14]$$

We then find:

$$u(y) = \frac{\sin n\pi}{\pi}\int_0^y \frac{\mathrm{d}h}{(y-h)^{1-n}} \int_0^h \frac{u'(z)\mathrm{d}z}{(h-z)^n} \qquad [15]$$

Eq.[15] cannot be considered a solution of [1] before we have eliminated $u'(z)$ from the right-hand side of [15]. This is accomplished by considering [1]:

$$u(y) = \frac{\sin n\pi}{\lambda\pi}\int_0^y \frac{f(h)\mathrm{d}h}{(y-h)^{1-n}} \qquad [16]$$

This terminates our general discussion of Abel's integral equation [1] and its solution.

11.2.2 Determination of velocity distributions in the earth

We now apply the above theory to determinations of body-wave (P, S) velocity distributions in the earth from known travel-time curves. We have the following relation, corresponding to BULLEN (1963, eq.29, p.119):

$$\frac{\Delta}{p} = 2R \int_{x=\eta_p^2}^{x=\eta_0^2} \frac{\mathrm{d}\log r}{(x-p^2)^{1/2}} = 2R \int_{x=\eta_0^2}^{x=\eta_p^2} \frac{-\mathrm{d}\log r}{(x-p^2)^{1/2}} \tag{17}$$

where we express distance Δ in linear measure (not angular), R is the earth's radius, $x = \eta^2$, with $\eta = r/v$, r = any radius, v = the corresponding velocity (P or S), and p = the ray parameter. Index zero (η_0) refers to the earth's surface and index p (η_p) to the deepest point of the ray. The integration is taken along a given ray. The second part

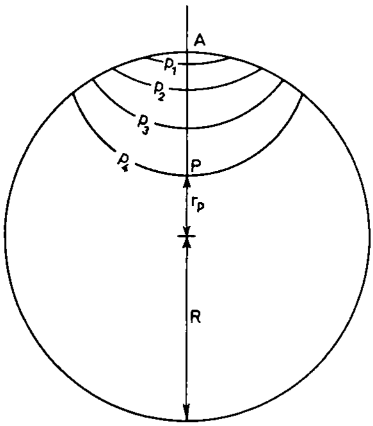


Fig.81.

of [17] with reversed sign and reversed integration path, is useful because it can be immediately identified with Abel's integral equation [1]. Identifying [1] with [17], we have:

$$\left.\begin{aligned} f(h) &= \Delta/p \\ \lambda &= 2R \\ n &= 1/2 \\ y &= \eta_0^2 - x \\ h &= \eta_0^2 - p^2 \\ u'(y)\mathrm{d}y &= -\mathrm{d}\log r \\ u(y) &= -\log r + c \end{aligned}\right| \tag{18}$$

and a direct application of the solution [16] gives, in this case:

$$-\log r = -c - \frac{1}{2\pi R} \int_{\eta_0^2}^{x} \frac{(\Delta/p)\,\mathrm{d}p^2}{(p^2 - x)^{1/2}} \tag{19}$$

The integration limits in [19] are obtained directly by comparing [16] and [18]: $h = 0$ corresponds to $p^2 = \eta_0^2$ and $h = y$ corresponds to $p^2 = x$. Note that in the general formulas, [1] is integrated over y, and [16] is integrated over h. In our case, [17], corresponding to [1], is integrated over x, i.e., along a ray, and [19] is integrated over p^2, i.e., over a family of rays (one p for each ray). Such a family of rays is shown in Fig.81, extending over a series of p-values from $A(p = \eta_0)$ to the lowest point $P(p = r/v)$. For $r = R$ there is no contribution to the integral in [19], but there is one for $r < R$. Respectively, we say that [16] is the solution of [1], and [19] is the solution of [17].

Applying [19] to $r = R$ and to $r = r_p$ we have, respectively:

$$\left.\begin{aligned} \log R &= c \\ \log r_p &= c + \frac{1}{2\pi R} \int_{\eta_0^2}^{x} \frac{(\Delta/p)\,\mathrm{d}p^2}{(p^2 - x)^{1/2}} \end{aligned}\right| \tag{20}$$

The integral in [20] is necessarily negative, as all $r < R$. Proceeding from A towards P, $\mathrm{d}p$ and $\mathrm{d}p^2$ are negative, as $p = r_p/v_p$, and going from A to P, r_p decreases and v_p increases (generally). This gives us the condition under which the method is valid:

$$\mathrm{d}p = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{v}\right)\mathrm{d}r = \underbrace{\left(\frac{1}{v} - \frac{r}{v^2}\frac{\mathrm{d}v}{\mathrm{d}r}\right)}_{>0}\underbrace{\mathrm{d}r}_{<0} < 0$$

Thus:

$$\frac{v}{r} - \frac{\mathrm{d}v}{\mathrm{d}r} > 0$$

and:

$$\frac{\mathrm{d}v}{\mathrm{d}r} < \frac{v}{r}$$

Therefore, the method is applicable to layers with velocity decreasing downwards, provided this condition is fulfilled.

Subtraction of the two eq.[20] from each other gives:

$$\log\left(\frac{R}{r_p}\right) = -\frac{1}{2\pi R} \int_{\eta_0^2}^{x} \frac{(\Delta/p)\,\mathrm{d}p^2}{(p^2 - x)^{1/2}} \tag{21}$$

thus eliminating the constant c. We then put:

$$p^2 = xQ^2 \quad \text{i.e.} \quad Q = \frac{p}{\eta_p} \tag{22}$$

introducing Q as a new variable. Eq.[21] then becomes:

$$\log\left(\frac{R}{r_p}\right) = -\frac{1}{\pi R} \int\limits_{\Delta=0}^{\Delta=\Delta_p} \frac{\Delta \mathrm{d}Q}{(Q^2-1)^{1/2}} = -\frac{1}{\pi R} \int\limits_{\Delta=0}^{\Delta=\Delta_p} \Delta \mathrm{d}q \tag{23}$$

where we have introduced $\cosh q = Q$, using the well-known formulas:

$$\frac{\mathrm{d}\cosh q}{\mathrm{d}q} = \sinh q$$

$$Q^2 - 1 = \cosh^2 q - 1 = \sinh^2 q$$

and thus:

$$\frac{\mathrm{d}\cosh q}{\sinh q} = \frac{\mathrm{d}Q}{(Q^2-1)^{1/2}} = \mathrm{d}q$$

Partial integration of [23] yields the solution of our problem:

$$\log\left(\frac{R}{r_p}\right) = \underbrace{-\frac{1}{\pi R}\Big[\Delta q\Big]_{\Delta=0}^{\Delta=\Delta_p}}_{} + \frac{1}{\pi R}\int_0^{\Delta_p} q \mathrm{d}\Delta$$

$$= 0 \quad (\text{as: } (1)\ \Delta = \Delta_p \ ; \quad Q = 1 \ ; \quad q = 0, \text{ and}$$
$$(2)\ \Delta = 0 \ ; \quad \Delta q = 0)$$

and our result is:

$$\log\left(\frac{R}{r_p}\right) = \frac{1}{\pi R}\int_0^{\Delta_p} q \mathrm{d}\Delta \tag{24}$$

Eq.[24] is the same as [30] in BULLEN (1963, p.120). In applying [24] to the earth, mechanical integration using empirically determined $q\Delta$-curves is employed. Eq. [24] then gives the radius r_p of the deepest point of the ray, and Snell's law gives the corresponding velocity. The restrictions which are imposed on this whole problem in the case of special velocity variations, e.g., inversions, do not imply any great mathematical difficulties (see BULLEN, 1963).

Chapter 12

LAMB'S PROBLEM

This chapter will be essentially based on the original paper by LAMB (1904), one of the great classical papers in seismology. Lamb's problem concerns wave propagation in a semi-infinite medium from sources located either on the surface or in the interior of the medium. Various geometrical shapes of the sources are studied, such as point sources, line sources and area sources (see also EWING, JARDETZKY and PRESS, 1957, pp.34–70).

12.1 TWO-DIMENSIONAL PROBLEM IN AN ISOTROPIC ELASTIC SOLID (AREA SOURCE, LINE SOURCE)

12.1.1 Introduction (unlimited elastic solid)

For an elastic solid, we have the following two equations of motion (in x and y coordinates):

$$\left.\begin{aligned} \varrho \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu) \frac{\partial \theta}{\partial x} + \mu \nabla^2 u \\ \varrho \frac{\partial^2 v}{\partial t^2} &= (\lambda + \mu) \frac{\partial \theta}{\partial y} + \mu \nabla^2 v \end{aligned}\right| \qquad [1]$$

where:

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad \text{(cubical dilatation)} \qquad [2]$$

Eq.[1] are satisfied by:

$$\left.\begin{aligned} u &= \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x} \end{aligned}\right| \qquad [3]$$

provided:

$$\left.\begin{aligned} \frac{\partial^2 \varphi}{\partial t^2} &= \frac{\lambda + 2\mu}{\varrho} \nabla^2 \varphi \\ \frac{\partial^2 \psi}{\partial t^2} &= \frac{\mu}{\varrho} \nabla^2 \psi \end{aligned}\right| \qquad [4]$$

(See BULLEN, 1963, p.73, or EWING, JARDETZKY and PRESS, 1957, chapter 1.) φ and ψ are the two *wave potentials* (for P and S waves, respectively). Assume simple harmonic motion with one and the same frequency in both cases:

$$\begin{aligned} &\varphi \sim e^{ipt} \\ &\psi \sim e^{ipt} \\ &\frac{\partial^2\varphi}{\partial t^2} = i^2p^2e^{ipt} = -p^2\varphi \end{aligned} \qquad [5]$$

Then eq.[4] become:

$$\begin{aligned} &(\nabla^2 + h^2)\varphi = 0 \\ &(\nabla^2 + k^2)\psi = 0 \end{aligned} \qquad [6]$$

where:

$$\begin{aligned} h^2 &= \frac{p^2\varrho}{\lambda + 2\mu} = p^2a^2 \\ k^2 &= \frac{p^2\varrho}{\mu} = p^2b^2 \end{aligned} \qquad [7]$$

a and b are the inverse velocities of P and S, and are called *wave slownesses* by Lamb (a notation introduced by Hamilton in optics).

Applying the general stress–strain relations for a perfectly elastic solid (BULLEN, 1963, pp.13, 20):

$$\begin{aligned} p_{ij} &= \lambda\theta\delta_{ij} + 2\mu e_{ij} \\ e_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \end{aligned} \qquad [8]$$

we have in our case, also taking [3] into account:

$$\begin{aligned} \frac{p_{xx}}{\mu} &= \frac{\lambda}{\mu}\,\theta + 2\frac{\partial u}{\partial x} = -k^2\varphi - 2\frac{\partial^2\varphi}{\partial y^2} + 2\frac{\partial^2\psi}{\partial x\partial y} \\ \frac{p_{yx}}{\mu} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial^2\varphi}{\partial x\partial y} - k^2\psi - 2\frac{\partial^2\psi}{\partial x^2} \\ \frac{p_{yy}}{\mu} &= \frac{\lambda}{\mu}\,\theta + 2\frac{\partial v}{\partial y} = -k^2\varphi - 2\frac{\partial^2\varphi}{\partial x^2} - 2\frac{\partial^2\psi}{\partial x\partial y} \end{aligned} \qquad [9]$$

That eq.[9] are correct can be seen as follows. Starting from the second member of [9.1], we have:

$$\frac{\lambda}{\mu}\,\theta + 2\frac{\partial u}{\partial x} = \frac{\lambda}{\mu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\frac{\partial u}{\partial x} =$$

(from [2])

$$= \left(\frac{\lambda}{\mu} + 2\right) \frac{\partial u}{\partial x} + \frac{\lambda}{\mu} \frac{\partial v}{\partial y}$$

$$= \left(\frac{\lambda}{\mu} + 2\right) \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}\right) + \frac{\lambda}{\mu} \left(\frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y}\right) =$$

(from [3])

$$= \left(\frac{\lambda}{\mu} + 2\right) \underbrace{\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right)} - 2 \frac{\partial^2 \varphi}{\partial y^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y}$$

($\nabla^2 \varphi = -h^2 \varphi$ from [6]) (this term is added and subtracted)

$$\left(\frac{\lambda}{\mu} + 2\right) h^2 = \frac{(\lambda + 2\mu) h^2}{\mu} = \frac{p^2 \varrho}{\mu} = k^2$$

(from [7])

Thus:

$$\frac{\lambda}{\mu} 0 + 2 \frac{\partial u}{\partial x} = -k^2 \varphi - 2 \frac{\partial^2 \varphi}{\partial y^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y}$$

The other two formulas in [9] are easily obtained in an analogous way.

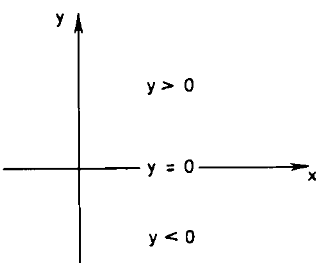


Fig. 82.

We shall consider sources localized to the plane $y = 0$ (see Fig.82). Omitting the time factor e^{ipt} we have the following solutions for $y > 0$:

$$\varphi = A e^{-\alpha y} e^{i\xi x}$$
$$\psi = B e^{-\beta y} e^{i\xi x} \qquad [10]$$

ξ (wave number in x-direction) is real and cannot be imaginary, as this would mean that the wave motion would die out or increase with increasing x, which is not the case.

α, β are either positive real (implying that the motion dies out as we go away from the source at $y = 0$) or imaginary (implying harmonic variation without amplitude decay as y increases). Putting [10] into the wave eq.[6], we find immediately:

$$\begin{aligned} \alpha^2 &= \xi^2 - h^2 \\ \beta^2 &= \xi^2 - k^2 \end{aligned} \qquad [11]$$

Likewise, for $y < 0$, we have the corresponding solutions:

$$\begin{aligned} \varphi &= A' \mathrm{e}^{\alpha y} \mathrm{e}^{i\xi x} \\ \psi &= B' \mathrm{e}^{\beta y} \mathrm{e}^{i\xi x} \end{aligned} \qquad [12]$$

Substituting the expressions for the potentials [10] into [3] and into [9] we get expressions for the displacements and the stresses, respectively. We put $y = 0$ in these expressions, and we get the displacements and the stresses in this plane:

Displacements at $y = 0$:

$$\begin{aligned} &\text{tangential} \quad & u_0 &= (i\xi A - \beta B)\mathrm{e}^{i\xi x} \\ &\text{normal} \quad & v_0 &= (-\alpha A - i\xi B)\mathrm{e}^{i\xi x} \end{aligned} \qquad [13]$$

Stresses at $y = 0$:

$$\begin{aligned} &\text{tangential} \quad & [p_{yx}]_0 &= \mu[-2i\xi\alpha A + (2\xi^2 - k^2)B]\mathrm{e}^{i\xi x} \\ &\text{normal} \quad & [p_{yy}]_0 &= \mu[(2\xi^2 - k^2)A + 2i\xi\beta B]\mathrm{e}^{i\xi x} \end{aligned} \qquad [14]$$

Corresponding expressions for the other side of the plane $y = 0$ are obtained analogously from [12] or by making the following replacements in [13] and [14]: $A \to A'$; $B \to B'$; $\alpha \to -\alpha$; $\beta \to -\beta$.

12.1.2 A periodic force acts on the plane $y = 0$ (or on a thin stratum at $y = 0$) in an unlimited elastic solid

We shall now continue our discussion by considering various kinds of forces acting on the plane $y = 0$ or the line $y = 0$, as the third coordinate does not enter our

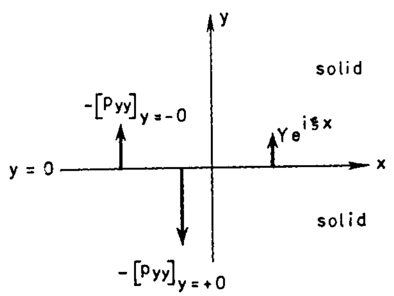


Fig.83.

present problem. First we assume that the force *per unit area* can be represented by the expression:

$$Ye^{i\xi x}e^{ipt} \tag{15}$$

As a consequence the normal stress will be discontinuous at $y = 0$; see the schematic Fig.83:

$$\begin{aligned} &-[p_{yy}]_{y=-0} + Ye^{i\xi x} = -[p_{yy}]_{y=+0} \\ \text{or:} \\ &[p_{yy}]_{y=+0} - [p_{yy}]_{y=-0} = -Ye^{i\xi x} \end{aligned} \tag{16}$$

Eq.[16] expresses the equilibrium condition for the plane $y = 0$. Fig.83 is in agreement with our sign conventions (compare BULLEN, 1963, p.8, fig.1). The material on the upper side ($y > 0$) is acted upon by a normal pressure from underneath, i.e., in amount equal to $[p_{yy}]_{y=-0}$, and with negative sign, because it acts along the inward normal of the upper half-space. The same holds for the other normal pressure acting on the lower half-space from above. Y is by definition positive upwards.

We have four conditions at the plane $y = 0$; in addition to [16] we have continuity in the tangential stress p_{yx} and in the displacements u_0 and v_0, i.e.:
stress conditions (eq.[14] and [16]):

$$\begin{aligned} &(2\xi^2 - k^2)(A - A') + 2i\xi\beta(B + B') = -\frac{Y}{\mu} \\ &-2i\xi\alpha(A + A') + (2\xi^2 - k^2)(B - B') = 0 \end{aligned} \tag{17}$$

displacement conditions (eq.[13]):

$$\begin{aligned} &i\xi(A - A') - \beta(B + B') = 0 \\ &\alpha(A + A') + i\xi(B - B') = 0 \end{aligned} \tag{18}$$

Remember that we are inside the elastic solid (no free surface). This is more general than the treatment by BULLEN (1963, pp.85–89), as he does not consider any extraneous forces. On the other hand, we have no discontinuity at $y = 0$. Eq.[17] and [18] are the four necessary and sufficient equations for calculation of the four unknowns. Using, for example, determinantal methods, we find:

$$\begin{aligned} &A = -A' = \frac{Y}{2k^2\mu} \\ &B = B' = \frac{i\xi}{\beta}\frac{Y}{2k^2\mu} \end{aligned} \tag{19}$$

Putting [19] into the potentials for $y > 0$, eq.[10], again omitting the time factor e^{ipt} we get:

$$\left.\begin{aligned}\varphi &= \frac{Y}{2k^2\mu}\, e^{-\alpha y} e^{i\xi x} \\ \psi &= \frac{Y}{2k^2\mu}\,\frac{i\xi}{\beta}\, e^{-\beta y} e^{i\xi x}\end{aligned}\right| \quad [20]$$

This in fact means that the earlier integration constants A, B in [10] have been reduced to the amplitude Y of the extraneous force.

12.1.3 A force concentrated to the line $x = 0$, $y = 0$ in an unlimited elastic solid

The force considered in section 12.1.2 has an infinite extent in the x-direction, as well as in the z-direction (this coordinate does not enter). Now we compress the force in such a way that it acts only on the line $x = 0$, $y = 0$, i.e., on the z-axis (perpendicular to our plane). This is a *line force*. Section 12.1.2 dealt with a *surface force* or *area force*.

The present case can be obtained from section 12.1.2 by superposition of an infinite number of stress distributions according to section 12.1.2 with all possible wave numbers ξ. The condition is that the distributions shall cancel for all x, except at $x = 0$. For this purpose, we put the extraneous pressure as follows:

$$Ye^{i\xi x} \to \int_{-\infty}^{\infty} \frac{Q\mathrm{d}\xi}{2\pi}\, e^{i\xi x} = \bar{f}(x) \quad [21]$$

which defines $\bar{f}(x)$. We integrate over all possible wave numbers, from $-\infty$ to $+\infty$. The principle used here is one which is common in the design of filters. Eq.[21] expresses a complex Fourier transform (cf. eq.[4] in section 8.1), and the following formula is its inversion (cf. eq.[24] in section 8.1):

$$Q = \int_{-\infty}^{\infty} \bar{f}(\eta) e^{-i\xi\eta} \mathrm{d}\eta \quad [22]$$

Then the pressure, eq.[21], can be written as a Fourier integral:

$$\bar{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \mathrm{d}\xi \int_{-\infty}^{\infty} \bar{f}(\eta) e^{-i\xi\eta} \mathrm{d}\eta \quad [23]$$

In order to obtain a concentrated normal force at $x = 0$, assume that the normal force $\bar{f}(x)$ vanishes everywhere along the x-axis except at $x = 0$, where it approaches infinity in such a way that Q in eq.[22] is finite. Then [23] reduces to [21] with Q constant.

Introducing [21] into eq.[20], these become:

$$\left.\begin{aligned}\varphi &= \frac{Q}{4\pi k^2\mu} \int_{-\infty}^{\infty} e^{-\alpha y} e^{i\xi x} \mathrm{d}\xi \\ \psi &= \frac{iQ}{4\pi k^2\mu} \int_{-\infty}^{\infty} \frac{\xi e^{-\beta y} e^{i\xi x} \mathrm{d}\xi}{\beta}\end{aligned}\right| \quad [24]$$

The exponential factors containing y, i.e., $e^{-\alpha y}$ and $e^{-\beta y}$, have to be kept under the integral signs, because α and β are functions of ξ by eq.[11].

The expressions [24] can be rewritten using results from the theory of Bessel functions (Chapter 5). From eq.[44] in section 5.2, we have:

$$J_0(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x \cos\omega)\mathrm{d}\omega = \frac{2}{\pi}\int_0^{\infty} \sin(x \cosh u)\mathrm{d}u \qquad [25]$$

The last member of [25] is obtained by the substitution $v = \cosh u$ in eq.[17] in section 5.3. In a similar way, using [15] in section 5.3, we obtain:

$$Y_0(x) = -\frac{2}{\pi}\int_0^{\infty} \cos(x\cosh u)\mathrm{d}u \qquad [26]$$

We define a function $D_0(x)$ such that:

$$D_0(x) = -Y_0(x) - iJ_0(x) = \frac{2}{\pi}\int_0^{\infty} e^{-ix\cosh u}\,\mathrm{d}u \qquad [27]$$

(by [25] and [26])

But this definition is equivalent to saying that:

$$D_0(x) = -iH_0^{(2)}(x) \qquad [28]$$

which is seen immediately from eq.[14] in section 4.3:

$$H_n^{(2)}(x) = J_n(x) - iY_n(x) \qquad [29]$$

By means of the asymptotic expansions of Bessel and Hankel functions (eq.[53]–[55] in section 5.3), we have the following expression for $D_0(x)$ valid for large values of the argument x:

$$D_0(x) \simeq -i\left(\frac{2}{\pi x}\right)^{1/2} e^{-i[x-(\pi/4)]} = \left(\frac{2}{\pi x}\right)^{1/2} e^{-i[x-(\pi/4)+(\pi/2)]} = \left(\frac{2}{\pi x}\right)^{1/2} e^{-i[x+(\pi/4)]} \qquad [30]$$

remembering that: $-i = e^{-i\pi/2}$.

It can be demonstrated that:

$$D_0(hr) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\alpha y}e^{i\xi x}\mathrm{d}\xi}{\alpha} \qquad [31]$$

with $r = (x^2 + y^2)^{1/2}$.

It is suggested as an exercise to show that [31] can be made to agree with [27], using results given by WATSON (1944, p.21, paragraph 2.21). Note also that there are similarities between [31] and eq.[11] in section 7.3. By differentiation of [31] we have:

$$\left.\begin{aligned} \frac{\partial}{\partial y} D_0(hr) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\alpha y}e^{i\xi x}\mathrm{d}\xi \\ \frac{\partial}{\partial x} D_0(kr) &= \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\xi e^{-\beta y}e^{i\xi x}\mathrm{d}\xi}{\beta} \end{aligned}\right| \qquad [32]$$

the latter from:

$$D_0(kr) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\beta y} e^{i\xi x} d\xi}{\beta}$$

(corresponding to [31]). By means of [32], our solutions [24] can be written in another form:

$$\left.\begin{aligned} \varphi &= -\frac{Q}{4k^2\mu}\frac{\partial}{\partial y} D_0(hr) \\ \psi &= \frac{Q}{4k^2\mu}\frac{\partial}{\partial x} D_0(kr) \end{aligned}\right| \qquad [33]$$

These wave potentials represent the cylindrical waves that spread out from the line source.

If instead we have used polar coordinates, which may be more appropriate to the physical conditions, i.e., if:

$$\left.\begin{aligned} x &= r\cos\chi \\ y &= r\sin\chi \end{aligned}\right| \qquad [34]$$

we would find the following expressions for the radial and the transverse displacements from [3], including the time factor e^{ipt} and considering only large distances from the origin, i.e., large r, where [30] is valid:

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\chi} &= \frac{Q}{4(\lambda+2\mu)}\left(\frac{2}{\pi hr}\right)^{1/2} e^{i[pt-hr-(\pi/4)]}\sin\chi \\ \frac{1}{r}\frac{\partial\varphi}{\partial\chi} - \frac{\partial\psi}{\partial r} &= \frac{Q}{4\mu}\left(\frac{2}{\pi kr}\right)^{1/2} e^{i[pt-hr-(\pi/4)]}\cos\chi \end{aligned}\right| \qquad [35]$$

In this case, when cylindrical waves spread out from the line source at $x = 0$, $y = 0$, there are both radial and tangential displacements, because the source is not circular-symmetric. Remember that our supposed force acts only in the direction of the y-axis. In order to have vanishing tangential displacements, it would be necessary to have a circular-symmetric force at the source.

Eq.[35] are proved in the following way. We have:

$$\frac{\partial}{\partial y} D_0(hr) = \frac{dD_0(hr)}{d(hr)}\, h \underbrace{\frac{\partial r}{\partial y}}_{=\frac{y}{r} = \sin\chi} = hD_0'(hr)\sin\chi$$

and:

$$\frac{\partial}{\partial x} D_0(kr) = kD_0'(kr)\frac{\partial r}{\partial x} = kD_0'(kr)\cos\chi$$

Introducing these expressions into [33] we get:

$$\left.\begin{aligned} \varphi &= -\frac{Q}{4k^2\mu}\, hD_0'(hr)\sin\chi \\ \psi &= \frac{Q}{4k^2\mu}\, kD_0'(kr)\cos\chi \end{aligned}\right| \qquad [36]$$

Then, the radial displacement is, using eq.[36]:

$$\frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\chi} = -\frac{Q}{4k^2\mu}\, h^2D_0''(hr)\sin\chi \underbrace{-\frac{1}{r}\frac{Q}{4k^2\mu}\, kD_0'(kr)\sin\chi}_{\to 0 \text{ (for } r \text{ large)}}$$

$$= -\frac{Q}{4(\lambda+2\mu)}\, D_0''(hr)\sin\chi =$$

$\left(\text{from [7] we have } \dfrac{h^2}{k^2} = \dfrac{\mu}{\lambda+2\mu}\right)$

$$= +\frac{Q}{4(\lambda+2\mu)}\, D_0(hr)\sin\chi \qquad [37]$$

That:

$$D_0''(hr) = -D_0(hr)$$

for r large, is seen from Bessel's differential equation (section 4.3):

$$J_0''(hr) + \underbrace{\frac{1}{hr}}_{\to 0 \text{ (for } r \text{ large)}} J_0'(hr) + J_0(hr) = 0$$

Thus:

$$J_0''(hr) = -J_0(hr)$$

Similarly we have for large r:

$$Y_0''(hr) = -Y_0(hr)$$

which proves the corresponding relation for $D_0(hr)$, considering eq.[27]. In an analogous way we have for the transverse displacement:

$$\frac{1}{r}\frac{\partial\varphi}{\partial\chi} - \frac{\partial\psi}{\partial r} = \underbrace{-\frac{1}{r}\frac{Q}{4k^2\mu}\, hD_0'(hr)\cos\chi}_{\to 0 \text{ (for } r \text{ large)}} - \frac{Q}{4k^2\mu}\, k^2D_0''(kr)\cos\chi =$$

$$= + \frac{Q}{4\mu} D_0(kr)\cos\chi \qquad [38]$$

Introducing into [37] and [38] the asymptotic expansions [30] of $D_0(hr)$ and $D_0(kr)$ for large r, we immediately have eq.[35], which should be proved.

12.1.4 A normal force acts on the surface of a semi-infinite elastic solid

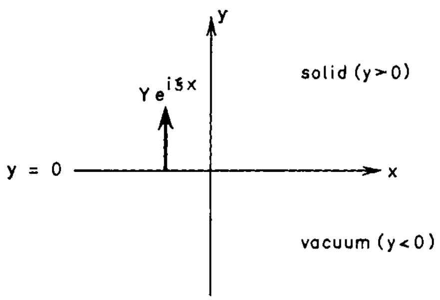


Fig.84.

On the plane $y = 0$, i.e., the surface of the elastic solid, the following forces are acting (Fig.84):

$$\left.\begin{aligned} [p_{yx}]_0 &= 0 \\ [p_{yy}]_0 &= Ye^{i\xi x} \end{aligned}\right\} \qquad [39]$$

again omitting the time factor e^{ipt}. Putting the expressions [14] into [39] we get:

$$\left.\begin{aligned} -2i\xi\alpha A + (2\xi^2 - k^2)B &= 0 \\ (2\xi^2 - k^2)A + 2i\xi\beta B &= \frac{Y}{\mu} \end{aligned}\right\} \qquad [40]$$

These correspond to the earlier equations [17], which express the condition of stress continuity at $y = 0$. Solving [40] for A and B, we find:

$$\left.\begin{aligned} A &= \frac{2\xi^2 - k^2}{F(\xi)} \frac{Y}{\mu} \\ B &= \frac{2i\xi\alpha}{F(\xi)} \frac{Y}{\mu} \end{aligned}\right\} \qquad [41]$$

where $F(\xi) = (2\xi^2 - k^2)^2 - 4\xi^2\alpha\beta$

$F(\xi)$ is called *Rayleigh's function* and $F(\xi) = 0$ is *Rayleigh's equation* (cf. BULLEN, 1963, p.89). The surface values of the displacements are obtained from [13] by introducing the expressions of A and B from [41]:

$$\left.\begin{aligned} u_0 &= \frac{i\xi(2\xi^2 - k^2 - 2\alpha\beta)\epsilon^{i\xi x}}{F(\xi)} \frac{Y}{\mu} \\ v_0 &= \frac{k^2\alpha\epsilon^{i\xi x}}{F(\xi)} \frac{Y}{\mu} \end{aligned}\right| \qquad [42]$$

12.1.5 A normal force acts on the line $x = 0$, $y = 0$ on the surface of a semi-infinite elastic solid

In Fig.84 the force now acts on the z-axis ($x = 0$, $y = 0$) in the direction of y, i.e., we have a line source on the surface of the semi-infinite elastic solid.

As in section 12.1.3 we put:

$$Y = -Q \frac{d\xi}{2\pi} \qquad [43]$$

and integrate over all wave numbers ξ from $-\infty$ to $+\infty$. The difference from section 12.1.3 is that we have a negative sign on the right-hand side of [43], which we can understand in this way: the force Y acts along the positive y, i.e., along the *inward* normal of the solid, and thus it is negative by our sign convention (Q is positive). In section 12.1.3 we did not have any free surface and thus no inward or outward normals.

The surface displacements are, in this case, immediately obtained from [42] by the substitution [43]:

$$\left.\begin{aligned} u_0 &= -\frac{iQ}{2\pi\mu} \int_{-\infty}^{\infty} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)\epsilon^{i\xi x}}{F(\xi)} d\xi \\ v_0 &= -\frac{Q}{2\pi\mu} \int_{-\infty}^{\infty} \frac{k^2\alpha e^{i\xi x} d\xi}{F(\xi)} \end{aligned}\right| \qquad [44]$$

where Q is constant and can therefore be placed outside the integral signs, for the same reason as in section 12.1.3.

12.1.6 Tangential forces acting on the surface of a semi-infinite elastic solid

By the same technique, we can easily solve the problem of tangential forces acting on the plane $y = 0$, i.e., the free surface of the solid. First, let us assume that the tangential force acts on the whole plane $y = 0$, i.e.:

$$\left.\begin{aligned} [p_{yx}]_0 &= X\epsilon^{i\xi x} \\ [p_{yy}]_0 &= 0 \end{aligned}\right| \qquad [45]$$

again omitting the time factor e^{ipt}. Proceeding as in section 12.1.4, we get:

$$\left.\begin{aligned} A &= -\frac{2i\xi\beta}{F(\xi)}\frac{X}{\mu} \\ B &= \frac{2\xi^2 - k^2}{F(\xi)}\frac{X}{\mu} \end{aligned}\right| \qquad [46]$$

By means of [46] and [13] we get the following expressions for the surface displacements:

$$\left.\begin{aligned} u_0 &= \frac{k^2\beta e^{i\xi x}}{F(\xi)}\frac{X}{\mu} \\ v_0 &= \frac{i\xi[2\alpha\beta - (2\xi^2 - k^2)]e^{i\xi x}}{F(\xi)}\frac{X}{\mu} \end{aligned}\right| \qquad [47]$$

The next case is to concentrate the tangential area force in the plane $y = 0$ into a tangential line force acting on the z-axis and in the direction of x, i.e., putting:

$$X = -\frac{P\mathrm{d}\xi}{2\pi} \qquad [48]$$

and integrating over ξ from $-\infty$ to $+\infty$. The surface displacements [47] then become

$$\left.\begin{aligned} u_0 &= -\frac{P}{2\pi\mu}\int_{-\infty}^{\infty}\frac{k^2\beta e^{i\xi x}\mathrm{d}\xi}{F(\xi)} \\ v_0 &= \frac{iP}{2\pi\mu}\int_{-\infty}^{\infty}\frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)e^{i\xi x}\mathrm{d}\xi}{F(\xi)} \end{aligned}\right| \qquad [49]$$

LAMB (1904) uses a minus sign on the right-hand side of [48]. There is no obvious reason for this. Obviously, a positive sign in [48] would do equally well.

The formulas [44] and [49] together illustrate the *principle of reciprocity* in vibration theory (see BULLEN, 1963, p.52), which is illustrated in Fig.85. That is,

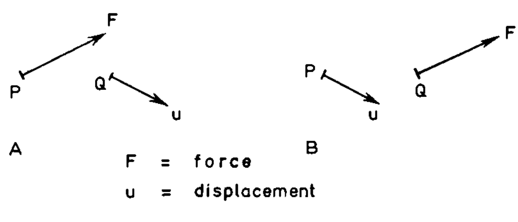


Fig.85.

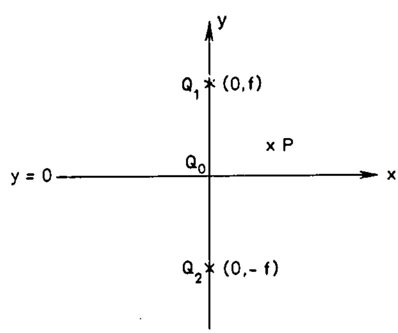


Fig.86.

application of the force F at point P of a system will produce the displacement u at point Q, and reciprocally, the force F at Q will produce the displacement u at P.

12.1.7 A case of three line sources in an unlimited elastic solid, under the condition of no stress at the plane $y = 0$

The plane $y = 0$ (Fig.86) is a "level of no stress" or a "level of no strain"; the solution of this problem shows *one* way to create such a level. Assume that line sources act along two lines, $(0, f)$ in the upper half-space and $(0, -f)$ in the lower half-space, perpendicular to the xy-plane. They are *internal sources* acting in an unlimited elastic solid. Also assume that the two sources produce only symmetrical radial motion (in the two dimensions x and y) and no transverse motion, in other words, that the velocity potentials can be assumed to have the following form:

$$\left.\begin{aligned} \varphi &= D_0(hr) + D_0(hr') \\ \psi &= 0 \end{aligned}\right| \qquad [50]$$

where:

$$r = [x^2 + (y - f)^2]^{1/2}$$

$$r' = [x^2 + (y + f)^2]^{1/2}$$

Consider a point P in the neighbourhood of the plane $y = 0$ (Fig.86). Applying formula [31] we can then write [50] in the following form:

$$\varphi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{\alpha(y-f)}e^{i\xi x}d\xi}{\alpha} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\alpha(y+f)}e^{i\xi x}d\xi}{\alpha} =$$

(from $Q_1 : y \to f - y$) (from $Q_2 : y \to f + y$)

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\cosh \alpha y}{\alpha}\, e^{-\alpha f}e^{i\xi x}d\xi \qquad [51]$$

remembering that $\cosh y = (e^y + e^{-y})/2$. Eq.[51] is the potential which is due only to the *two* line sources, so far assumed. The corresponding stresses at the plane $y = 0$ are obtained from [9]. Carrying out the differentiations of [51] and putting $y = 0$, we obtain:

$$[p_{yx}]_0 = 0$$

$$[p_{yy}]_0 = \frac{2\mu}{\pi}\int_{-\infty}^{\infty} \frac{2\xi^2 - k^2}{\alpha}\, e^{-\alpha f}e^{i\xi x}d\xi \qquad [52]$$

remembering that $\cosh y = 1$ for $y = 0$ and $\sinh y = 0$ for $y = 0$, and that $d(\cosh y)/dy = \sinh y$. Obviously, the two line sources assumed will produce no tangential stress on the plane $y = 0$, but a non-vanishing normal stress on the same plane. Our condition was that no stress at all should exist at $y = 0$. This is accomplished by introducing a third line source at $x = 0$, $y = 0$, as considered in section 12.1.4, and requesting that the corresponding normal force should be:

$$Ye^{i\xi x} = -\frac{2\mu}{\pi}\int_{-\infty}^{\infty}\frac{2\xi^2 - k^2}{\alpha}\,e^{-\alpha f}e^{i\xi x}d\xi \tag{53}$$

as seen by comparison with [39]. The combination of [52] and [53] will produce the required condition of a stress-free surface $y = 0$.

Writing out the wave potentials completely, we get the following expressions:

$$\varphi = Ae^{-\alpha y}e^{i\xi x} + \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\cosh\alpha y}{\alpha}\,e^{-\alpha f}e^{i\xi x}d\xi \tag{54a}$$

(from Q_0, eq.[10]) (from Q_1 and Q_2, eq.[51])

$$\psi = Be^{-\beta y}e^{i\xi x} \tag{54b}$$

(from Q_0, eq.[10]; no contribution from Q_1 and Q_2)

$$A = \frac{2\xi^2 - k^2}{F(\xi)}\,\frac{Y}{\mu} \quad \text{(eq.[41])}$$

$$B = \frac{2i\xi\alpha}{F(\xi)}\,\frac{Y}{\mu} \quad \text{(eq.[41])}$$

in the present case with Y as defined in eq.[53].

The displacements in the plane $y = 0$ are then obtained from [3]:

$$u_0 = \left(\frac{\partial\varphi}{\partial x}\right)_0 + \left(\frac{\partial\psi}{\partial y}\right)_0$$

$$= -\frac{2i}{\pi}\int_{-\infty}^{\infty}\xi\,\frac{(2\xi^2 - k^2)^2}{\alpha F(\xi)}\,e^{i\xi x}e^{-\alpha f}d\xi + \frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{\xi e^{-\alpha f}e^{i\xi x}d\xi}{\alpha}$$

$$+ \frac{4i}{\pi}\int_{-\infty}^{\infty}\frac{\beta\xi(2\xi^2 - k^2)}{F(\xi)}\,e^{i\xi x}e^{-\alpha f}d\xi = -\frac{4ik^2}{\pi}\int_{-\infty}^{\infty}\frac{\beta\xi e^{-\alpha f}e^{i\xi x}}{F(\xi)}\,d\xi \tag{55a}$$

$$v_0 = \left(\frac{\partial\varphi}{\partial y}\right)_0 - \left(\frac{\partial\psi}{\partial x}\right)_0 = +\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{(2\xi^2 - k^2)^2 e^{i\xi x}e^{-\alpha f}}{F(\xi)}\,d\xi$$

$$+ \frac{2i}{\pi}\int_{-\infty}^{\infty}\xi\,\frac{2i\xi}{F(\xi)}\,e^{i\xi x}(2\xi^2 - k^2)e^{-\alpha f}d\xi$$

$$= -\frac{2k^2}{\pi}\int_{-\infty}^{\infty}\frac{(2\xi^2 - k^2)e^{-\alpha f}e^{i\xi x}d\xi}{F(\xi)} \tag{55b}$$

The cases dealt with so far may serve as examples of these calculations, i.e., to find the integral solutions for displacements when source functions are given. The examples could be varied extensively. The problem we are now going to treat is the interpretation of the definite integrals, in which we have expressed the displacements.

The problem of *free Rayleigh waves*, i.e., Rayleigh waves which have left their source and are no longer under the action of any forces, follows easily from the treatment here, by putting $Y = 0$ in [40]. I am not going to discuss this theory here, as it is rather simple and can be read in seismological textbooks, e.g., EWING, JARDETZKY and PRESS (1957, pp.31–34), BREKHOVSKIKH (1960, pp.38–41), BULLEN (1963, pp.89–90).

12.1.8 Evaluation of the integral solutions for the displacements

We shall consider the case of a vertical force applied to the surface of a semi-infinite elastic solid (section 12.1.5, eq.[44]). Introducing the expression for $F(\xi)$, eq.[41], and for α and β, eq.[11], and generalizing the real variable ξ to the complex variable:

$$\zeta = \xi + i\eta$$

we see that the integrals we have to solve are the following (from eq.[44]):

(*a*) for the horizontal displacement:

$$\int_{-\infty}^{\infty} \Phi(\zeta)\mathrm{d}\zeta = \int_{-\infty}^{\infty} \frac{\zeta[(2\zeta^2 - k^2) - 2\sqrt{(\zeta^2 - h^2)}\sqrt{(\zeta^2 - k^2)}]\mathrm{e}^{i\zeta x}}{(2\zeta^2 - k^2)^2 - 4\zeta^2\sqrt{(\zeta^2 - h^2)}\sqrt{(\zeta^2 - k^2)}}\,\mathrm{d}\zeta \qquad [56]$$

(*b*) for the vertical displacement:

$$\int_{-\infty}^{\infty} \Psi(\zeta)\mathrm{d}\zeta = \int_{-\infty}^{\infty} \frac{k^2\sqrt{(\zeta^2 - h^2)}\mathrm{e}^{i\zeta x}\mathrm{d}\zeta}{(2\zeta^2 - k^2)^2 - 4\zeta^2\sqrt{(\zeta^2 - h^2)}\sqrt{(\zeta^2 - k^2)}} \qquad [57]$$

Solution is achieved by a suitable contour integration in the complex ζ-plane, and the case offers good exercise in such integration methods. This appears to be the only feasible method. Numerical integration is naturally exceedingly difficult but could possibly be made with advantage on a high-speed electronic computer.

The poles of the integrands in [56] and [57] are found as the roots of the equation:

$$F(\zeta) = (2\zeta^2 - k^2)^2 - 4\zeta^2\alpha\beta = 0 \qquad [58]$$

Solution of [58] is done in the classical treatments of free Rayleigh waves referred to at the end of section 12.1.7, and there is no need to repeat this here. There are two roots $\zeta = \pm\varkappa$, where $\varkappa > k$.

In addition, we have four branch points, due to the radicals $(\zeta^2 - h^2)^{1/2}$ and $(\zeta^2 - k^2)^{1/2}$ in [56] and [57]. The branch points are $\zeta = \pm h$ and $\zeta = \pm k$. The branch points and the poles are located on a straight line (Fig.87), whose slope γ is determined by the relation $\tan\gamma = \mathrm{Im}\,p/\mathrm{Re}\,p$. This is seen from the fact that both ζ, h and k are proportional to p ($\zeta = pc$, $h = pa$, $k = pb$; eq.[7]), and the wave slownesses c, a and b

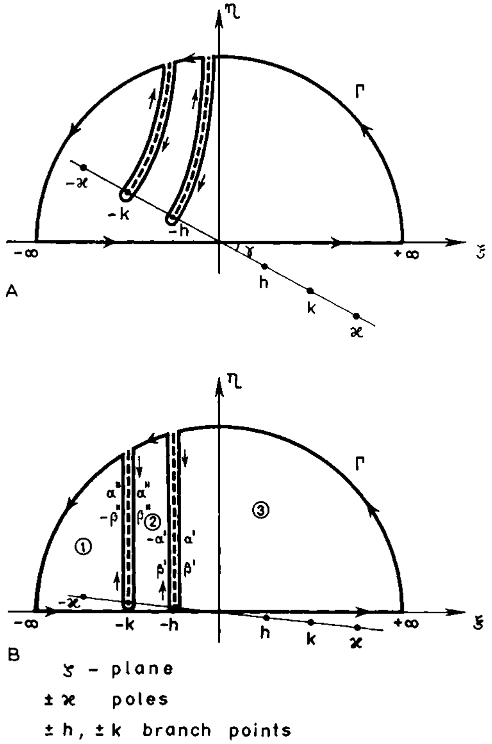


Fig.87.

are assumed real. This means that the ratios of imaginary and real parts, which determine the slope γ, are the same for $\pm\varkappa$, $\pm k$ and $\pm h$, and they are also the same as for p.

In general, the branch cuts can be chosen arbitrarily (see section 2.1.5), but in this particular case we have to request that everywhere α and $\beta \geqslant 0$, because of eq.[10] and [11]. That is, we have to require that $\mathrm{Re}(\zeta^2 - h^2)^{1/2} \geqslant 0$ and $\mathrm{Re}(\zeta^2 - k^2)^{1/2} \geqslant 0$, and on the cuts themselves these quantities have to be zero. This condition defines the cuts shown in Fig.87A.

Let us assume p complex and put $p = s - i\sigma$. Because on one of the cuts $\mathrm{Re}(\zeta^2 - k^2)^{1/2} = 0$, we can say that on this cut $(\zeta^2 - k^2)^{1/2}$ is purely imaginary, say $= iA$, where A is real. Squaring and introducing the complex expressions for ζ and $k = pb = (s - i\sigma)b$, we find:

$$\xi^2 - \eta^2 + 2i\xi\eta - (s^2 - \sigma^2 - 2is\sigma)b^2 = -A^2$$

that is, the expression on the left-hand side must be real and negative. Separating imaginary and real parts, we find:

$$\left.\begin{aligned} &\xi\eta = -s\sigma b^2 \\ &\xi^2 - \eta^2 < (s^2 - \sigma^2)b^2 \end{aligned}\right\}$$

The first of these equations defines the hyperbola-shaped cuts shown in Fig.87A, and the second equation defines the parts of the cuts which can be used. The complete contour is shown in Fig.87A.

However, we find considerable difficulty in evaluating the integrals along the hyperbola-shaped curves, and therefore it is of advantage to modify these into some other cuts for which the branch-line integrals can be more easily evaluated. Fig.87B shows the modification we shall be using here, which in fact resembles one of LAMB's (1904) solutions. The integrals along the curved branch lines in Fig.87A are equal to branch-line integrals along the straight lines perpendicular to the real axis shown in Fig.87B. This is seen immediately by Cauchy's theorem if we make a separate contour of, for example, the up-going branch at $-k$, in Fig.87A, the up-going branch at $-k$ in Fig.87B, and the infinite arc between these two branches. The curved branch-line integral is equivalent to the straight branch-line integral. But, in addition we have to shift the cuts so that they become straight lines perpendicular to the real axis. This is permitted, provided that we assume the imaginary part of p is very small. If p is real, then both the poles and the branch points are real and situated on the real axis. We can approach this situation as closely as we like, as illustrated in Fig.87B. If p is real, ζ is also real, and the cuts are defined by $\mathrm{Re}(\xi^2 - h^2)^{1/2} = 0$ and $\mathrm{Re}(\xi^2 - k^2)^{1/2} = 0$, i.e., they are the straight lines $\xi = \pm h$ and $\xi = \pm k$. This differs somewhat from the treatment by EWING, JARDETZKY and PRESS (1957, p.45), who even for a real p, allow for a complex wave velocity.

The restriction to small values of η is also suggested by the integrals [56] and [57], which have vanishing contributions for large η. For small values of η we find the following approximate expressions for the radicals (with the sign conventions as shown in Fig.87B):

$$\alpha' = -\sqrt{2h\eta}\, e^{-i\pi/4} \quad ; \quad \beta' = i\sqrt{(k^2 - h^2)}$$
$$\alpha'' = \sqrt{(k^2 - h^2)} \quad ; \quad \beta'' = -\sqrt{2k\eta}\, e^{-i\pi/4} \qquad [59]$$

These are proved as follows:

$$\alpha = \sqrt{(\zeta^2 - h^2)} \quad ; \quad \zeta = \xi + i\eta$$

α'' :

$$\zeta = -k + i\eta$$

$$\alpha'' = \sqrt{[(-k + i\eta)^2 - h^2]} = \sqrt{(k^2 + \underbrace{i^2\eta^2 - 2ik\eta}_{\text{(neglected because } \eta \text{ is small)}} - h^2)} =$$

$$= \sqrt{(k^2 - h^2)}$$

α' :

$$\zeta = -h + i\eta$$

$$\alpha' = \sqrt{[(-h + i\eta)^2 - h^2]} = \sqrt{(h^2 + i^2\eta^2 - 2ih\eta - h^2)} =$$

(η small)

$$= i\sqrt{2ih\eta} = i^{3/2}\sqrt{2h\eta} = -\sqrt{2h\eta}\, e^{-i\pi/4}$$

because:

$$i^{3/2} = \frac{i^{4/2}}{i^{1/2}} = \frac{-1}{e^{i\pi/4}}$$

The formulas [59] for β' and β'' are obtained in the same way.

We also need a relation valid for the derivative of $F(\xi)$. We have:

$$F(\xi) = (2\xi^2 - k^2)^2 - 4\xi^2(\xi^2 - h^2)^{1/2}(\xi^2 - k^2)^{1/2}$$

and thus:

$$F'(\xi) = 8\xi \left[(2\xi^2 - k^2) - (\xi^2 - h^2)^{1/2}(\xi^2 - k^2)^{1/2} - \frac{\xi^2}{2}\left(\frac{\xi^2 - k^2}{\xi^2 - h^2}\right)^{1/2} - \right.$$
$$\left. - \frac{\xi^2}{2}\left(\frac{\xi^2 - h^2}{\xi^2 - k^2}\right)^{1/2}\right]$$

which immediately gives us:

$$F'(\xi) = -F'(-\xi) \qquad [60]$$

We evaluate the integrals [56] and [57] along the contour in Fig.87B, which consists of three parts: (*1*), (*2*), (*3*). We find:

$$\int \Phi(\zeta)\mathrm{d}\zeta = \int_{(1)} \Phi(\zeta)\mathrm{d}\zeta + \int_{(2)} \Phi(\zeta)\mathrm{d}\zeta + \int_{(3)} \Phi(\zeta)\mathrm{d}\zeta = 2\pi i \,\mathrm{Res}(-\varkappa) \qquad [61]$$

assuming that branch points and poles are almost down on the real axis.

The integrals [56] and [57] tend to zero for large values of the argument ζ. Therefore, as the outer sides Γ are assumed to be at infinite distance, they give no contribution to the solutions. We thus get the following expressions:

$$\int_{(1)} \Phi(\zeta)\mathrm{d}\zeta = \int_{-\infty}^{-k} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)e^{i\xi x}\mathrm{d}\xi}{F(\xi)} +$$

(along real axis: $\zeta = \xi$)

$$+ e^{-ikx} \int_0^\infty \frac{2\zeta^2 - k^2 + 2\alpha''\beta''}{(2\zeta^2 - k^2)^2 + 4\zeta^2\alpha''\beta''} \zeta e^{-\eta x} i\mathrm{d}\eta =$$

(from the vertical path: $\zeta = -k + i\eta$; $\mathrm{d}\zeta = i\mathrm{d}\eta$)

$$= 2\pi i \,\mathrm{Res}(-\varkappa) = 2\pi i \frac{-\varkappa(2\varkappa^2 - k^2 - 2\alpha_1\beta_1)e^{-i\varkappa x}}{F'(-\varkappa)} =$$

[where $\alpha_1 = \sqrt{(\varkappa^2 - h^2)}$; $\beta_1 = \sqrt{(\varkappa^2 - k^2)}$]

$$= 2\pi i \frac{\varkappa(2\varkappa^2 - k^2 - 2\alpha_1\beta_1)e^{-i\varkappa x}}{F'(\varkappa)} \qquad [62]$$

(by eq.[60]).

$$\int\limits_{(2)} \Phi(\zeta)\mathrm{d}\zeta = \int\limits_{-k}^{-h} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)\mathrm{e}^{i\xi x}\mathrm{d}\xi}{F(\xi)} +$$

(along real axis: $\zeta = \xi$)

$$+ \mathrm{e}^{-ihx} \int\limits_0^\infty \frac{2\zeta^2 - k^2 + 2\alpha'\beta'}{(2\zeta^2 - k^2)^2 + 4\zeta^2\alpha'\beta'} \zeta \mathrm{e}^{-\eta x} i\mathrm{d}\eta -$$

(along vertical axis: $\zeta = -h + i\eta$)

$$- \mathrm{e}^{-ikx} \int\limits_0^\infty \frac{2\zeta^2 - k^2 - 2\alpha''\beta''}{(2\zeta^2 - k^2)^2 - 4\zeta^2\alpha''\beta''} \zeta \mathrm{e}^{-\eta x} i\mathrm{d}\eta \qquad [63]$$

(along vertical axis: $\zeta = -k + i\eta$; minus-sign, as the path actually goes from ∞ to 0).

$$\int\limits_{(3)} \Phi(\zeta)\mathrm{d}\zeta = \int\limits_{-h}^{\infty} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)\mathrm{e}^{i\xi x}\mathrm{d}\xi}{F(\xi)} -$$

(along real axis: $\zeta = \xi$)

$$- \mathrm{e}^{-ihx} \int\limits_0^\infty \frac{2\zeta^2 - k^2 - 2\alpha'\beta'}{(2\zeta^2 - k^2)^2 - 4\zeta^2\alpha'\beta'} \zeta \mathrm{e}^{-\eta x} i\mathrm{d}\eta \qquad [64]$$

(from vertical axis: $\zeta = -h + i\eta$)

After that, we apply [61], inserting the expressions [62], [63], [64] for the respective integrals. We introduce the notations H and K, defined as follows:

$$\left. \begin{aligned} H &= -\frac{\varkappa(2\varkappa^2 - k^2 - 2\alpha_1\beta_1)}{F'(\varkappa)} \\ K &= -\frac{k^2\alpha_1}{F'(\varkappa)} \end{aligned} \right| \qquad [65]$$

After some rearrangement we then find:

$$\int\limits_{-\infty}^{\infty} \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)\mathrm{e}^{i\xi x}\mathrm{d}\xi}{F(\xi)} = -2\pi i H \mathrm{e}^{-i\varkappa x}$$

$$+ \mathrm{e}^{-ikx} \int\limits_0^\infty \left[\frac{2\zeta^2 - k^2 - 2\alpha''\beta''}{(2\zeta^2 - k^2)^2 - 4\zeta^2\alpha''\beta''} - \frac{2\zeta^2 - k^2 + 2\alpha''\beta''}{(2\zeta^2 - k^2)^2 + 4\zeta^2\alpha''\beta''}\right] \zeta \mathrm{e}^{-\eta x} i\mathrm{d}\eta$$

$$+ \mathrm{e}^{-ihx} \int\limits_0^\infty \left[\frac{2\zeta^2 - k^2 - 2\alpha'\beta'}{(2\zeta^2 - k^2)^2 - 4\zeta^2\alpha'\beta'} - \frac{2\zeta^2 - k^2 + 2\alpha'\beta'}{(2\zeta^2 - k^2)^2 + 4\zeta^2\alpha'\beta'}\right] \zeta \mathrm{e}^{-\eta x} i\mathrm{d}\eta =$$

(or as the last two integrals can be immediately re-written)

$$= -2\pi i H \mathrm{e}^{-i\varkappa x}$$

$$+\,4ie^{-ikx}\int_0^\infty \frac{k^2(2\zeta^2-k^2)\overbrace{\alpha''\beta''}^{=\sqrt{(\zeta^2-h^2)}\sqrt{(\zeta^2-k^2)}}\zeta e^{-\eta x}\mathrm{d}\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)}+$$

(plus here because of reversed order here; $\zeta=-k+i\eta$)

$$+\,4ie^{-ihx}\int_0^\infty \frac{k^2(2\zeta^2-k^2)\overbrace{\alpha'\beta'}^{=\sqrt{(\zeta^2-h^2)}\sqrt{(\zeta^2-k^2)}}\zeta e^{-\eta x}\mathrm{d}\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)} \qquad [66]$$

$(\zeta=-h+i\eta)$

The other integral, [57], can be evaluated in the same way with the result:

$$\int_{-\infty}^{\infty}\frac{k^2\alpha e^{i\xi x}\mathrm{d}\xi}{F(\xi)}=+2\pi iKe^{-i\kappa x}$$

$$+\,8ie^{-ikx}\int_0^\infty \frac{k^2\zeta^2(\zeta^2-h^2)\beta''e^{-\eta x}\mathrm{d}\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)}$$

$$+\,2ie^{-ihx}\int_0^\infty \frac{k^2(2\zeta^2-k^2)\alpha'e^{-\eta x}\mathrm{d}\eta}{(2\zeta^2-k^2)^4+16\zeta^4(\zeta^2-h^2)(k^2-\zeta^2)} \qquad [67]$$

The only thing so far achieved is that our original integrals have been transformed into some other integrals. But the latter can be relatively easily evaluated, especially for large x (that is, at a large distance from the source) and for small η (that is, near or on the real axis ξ).

We shall use the following asymptotic expansion:

$$\int_0^\infty \eta^{1/2}f(\eta)e^{-\eta x}\mathrm{d}\eta=\frac{\Gamma(3/2)}{x^{3/2}}f(0)+\frac{\Gamma(5/2)}{x^{5/2}}\frac{f'(0)}{1!}+\frac{\Gamma(7/2)}{x^{7/2}}\frac{f''(0)}{2!}+\dots \qquad [68]$$

which is proved in the following way. Make the substitution $\eta x=\nu$; $\eta=\nu/x$. We have $x\mathrm{d}\eta=\mathrm{d}\nu$ because x is constant in the integration over η in [68]. Furthermore, by Taylor's theorem (eq.[15] in section 2.1):

$$f(\eta)=f\left(\frac{\nu}{x}\right)=f(0)+\frac{\nu}{x}f'(0)+\frac{\nu^2}{2!x^2}f''(0)+\dots$$

The integral on the left-hand side of [68] then becomes:

$$\int_0^\infty \eta^{1/2}f(\eta)e^{-\eta x}\mathrm{d}\eta=\int_0^\infty\frac{\nu^{1/2}}{x^{1/2}}f\left(\frac{\nu}{x}\right)e^{-\nu}\frac{\mathrm{d}\nu}{x}=$$

(expand the function f)

$$= \frac{1}{x^{3/2}} \int_0^\infty \nu^{1/2} e^{-\nu} d\nu \left[f(0) + \frac{\nu}{x} f'(0) + \frac{\nu^2}{2!x^2} f''(0) + \dots \right] =$$

(integrate term by term)

$$= \frac{f(0)}{x^{3/2}} \underbrace{\int_0^\infty \nu^{1/2} e^{-\nu} d\nu}_{= \Gamma(3/2)} + \frac{f'(0)}{x^{5/2}} \underbrace{\int_0^\infty \nu^{3/2} e^{-\nu} d\nu}_{= \Gamma(5/2)} + \frac{f''(0)}{2!x^{7/2}} \underbrace{\int_0^\infty \nu^{5/2} e^{-\nu} d\nu}_{= \Gamma(7/2)} + \dots$$

from which [68] follows. Here we have used the gamma function (section 1.3).

As we are considering large values of x, the first term on the right-hand side of [68] will give an adequate approximation:

$$\int_0^\infty \eta^{1/2} f(\eta) e^{-\eta x} d\eta = \frac{\sqrt{\pi}}{2} \frac{f(0)}{x^{3/2}} \tag{69}$$

The integrand in the first integral on the right-hand side of [66] can be written as follows, if we remember that in this integral $\zeta = -k + i\eta$:

$$\eta^{1/2} f(\eta) e^{-\eta x} d\eta =$$

$$\frac{-k^2[2(-k+i\eta)^2 - k^2]\sqrt{(k^2-h^2)}\sqrt{2k}\,\eta^{1/2} e^{-i\pi/4}(-k+i\eta) e^{-\eta x} d\eta}{[2(-k+i\eta)^2 - k^2]^4 + \underbrace{16(-k+i\eta)^4[(-k+i\eta)^2 - h^2][k^2 - (-k+i\eta)^2]}_{\uparrow}}$$

(this term is *exactly* $= 0$ for $\eta = 0$)

For $\eta = 0$ we then get:

$$f(0) = \frac{+k^2(2k^2 - k^2)\sqrt{(k^2 - h^2)}\sqrt{2k}\, e^{-i\pi/4} k}{(2k^2 - k^2)^4}$$

The approximate expressions [59] are used only in the numerator. Then the complete integral in question becomes:

$$4i e^{-ikx} \int_0^\infty \frac{k^2(2\zeta^2 - k^2)\alpha''\beta''\zeta e^{-\eta x} d\eta}{(2\zeta^2 - k^2)^4 + 16\zeta^4(\zeta^2 - h^2)(k^2 - \zeta^2)}$$

$$= 4i e^{-ikx} \int_0^\infty \eta^{1/2} f(\eta) e^{-\eta x} d\eta$$

$$= 2\sqrt{2\pi} \left(1 - \frac{h^2}{k^2}\right)^{1/2} \frac{i e^{-i(kx + \pi/4)}}{(kx)^{3/2}} \tag{70}$$

applying [69]. The other integrals are expressed similarly, using [69]. Then [66] becomes:

$$-2\pi i H e^{-i\varkappa x} + 2\sqrt{2\pi}\left(1-\frac{h^2}{k^2}\right)^{1/2}\frac{i e^{-i(kx+\pi/4)}}{(kx)^{3/2}}$$
$$+ 2\sqrt{2\pi}\,\frac{h^3k^2\sqrt{(k^2-h^2)}}{(k^2-2h^2)^3}\,\frac{e^{-i(hx+\pi/4)}}{(hx)^{3/2}} \qquad [71]$$

and [67] becomes:

$$+ 2\pi i K e^{-i\varkappa x} - 4\sqrt{2\pi}\left(1-\frac{h^2}{k^2}\right)\frac{i e^{-i(kx+\pi/4)}}{(kx)^{3/2}}$$
$$-\sqrt{2\pi}\,\frac{h^2k^2}{(k^2-2h^2)^2}\,\frac{i e^{-i(hx+\pi/4)}}{(hx)^{3/2}} \qquad [72]$$

Our solution of the integrals in [44] is now completed, and [44] becomes, if we use [71] and [72], respectively, and introduce the time factor e^{ipt}:

$$u_0 = -\frac{Q}{\mu} H e^{i(pt-\varkappa x)} \qquad (I)$$
$$+ \frac{Q}{\mu}\left(\frac{2}{\pi}\right)^{1/2}\left(1-\frac{h^2}{k^2}\right)^{1/2}\frac{e^{i(pt-kx-\pi/4)}}{(kx)^{3/2}} \qquad (II)$$
$$-\frac{Q}{\mu}\left(\frac{2}{\pi}\right)^{1/2}\frac{h^3k^2\sqrt{(k^2-h^2)}}{(k^2-2h^2)^3}\,\frac{i e^{i(pt-hx-\pi/4)}}{(hx)^{3/2}} \qquad (III) \qquad [73]$$

$$v_0 = -\frac{iQ}{\mu} K e^{i(pt-\varkappa x)} \qquad (I)$$
$$+ \frac{2Q}{\mu}\left(\frac{2}{\pi}\right)^{1/2}\left(1-\frac{h^2}{k^2}\right)\frac{i e^{i(pt-kx-\pi/4)}}{(kx)^{3/2}} \qquad (II)$$
$$+ \frac{Q}{2\mu}\left(\frac{2}{\pi}\right)^{1/2}\frac{h^2k^2}{(k^2-2h^2)^2}\,\frac{i e^{i(pt-hx-\pi/4)}}{(hx)^{3/2}} \qquad (III) \qquad [74]$$

The surface displacements can be written as the sum of three terms, (*I*), (*II*), (*III*). The term (*I*) is derived from the poles, (*II*) and (*III*) from the branch-line integrals.

(*I*) represents Rayleigh waves. Combining this part of u_0 and v_0 (eliminating $pt-\varkappa x$), we have the equation for the elliptic orbit:

$$\frac{u_0^2}{\left(\frac{QH}{\mu}\right)^2} + \frac{v_0^2}{\left(\frac{QK}{\mu}\right)^2} = 1 \qquad [75]$$

Some properties of the three terms are summarized in Table VII. The amplitudes of the terms (*II*) and (*III*) die out as $1/x^{3/2}$ during the wave propagation, whereas the

TABLE VII

PROPERTIES OF THE SOLUTION, EXPRESSED IN EQ.[73] - [75]

Wave property	*Term I*	*Term II*	*Term III*
Phase velocity	$p/\varkappa$, Rayleigh waves	$p/k = b^{-1}$, i.e., same as S	$p/h = a^{-1}$, i.e., same as P
Orbit at surface	elliptic, eq.[75]	elliptic, as u_0 and v_0 have a phase difference of $\pi/2$, corresponding to the factor i in v_0	linear as u_0 and v_0 are in phase
Vertical: horizontal axes at surface	$K/H = 1.47$ for $\lambda = \mu$	$2\left(1 - \frac{h^2}{k^2}\right)^{1/2}$ $= 1.63$ for $\lambda = \mu$	$\frac{k^2 - 2h^2}{2h(k^2 - h^2)^{1/2}}$ $= 0.35$ for $\lambda = \mu$

amplitude of (I) is independent of x (no frictional terms, etc., being considered). This means that at a large distance from the source, the motion consists essentially of free Rayleigh waves. Finer details can be revealed by numerical evaluation.

Thus we find a decrease as $1/x^{3/2}$ in the case of a medium with a free surface, whereas we found in eq.[35] that the decrease of the displacements is as $1/x^{1/2}$ in the case of an unlimited solid medium, in both cases with a line source. This means a more rapid decrease when we have a free surface.

TABLE VIII

COMPARISON OF NOTATIONS, USED IN LAMB'S PROBLEM

Notation in section 12.1	*Corresponding notation in* EWING–JARDETZKY–PRESS (1957)
k	k_β
h	k_α
ξ	k
η	τ
α	ν
β	ν'
a	$1/\alpha$
b	$1/\beta$
c	$1/c$
p	ω
f	G

In our treatment so far we have assumed a periodic source, acting without beginning or end. Therefore, all three waves considered, R, P and S exist simultaneously, although they propagate with different velocities. They have no beginning and no end. Therefore, the displacements u_0 and v_0 are truly the sum of the three contributions, which in this case exist simultaneously at every point x.

In order to facilitate the reading of the corresponding chapter in EWING, JARDETZKY and PRESS (1957) and to compare the present chapter more easily with the following Chapter 13, I have listed in Table VIII those notations which differ. The notation here is in almost complete agreement with the one used by LAMB (1904).

12.2 THREE-DIMENSIONAL PROBLEM IN AN ISOTROPIC ELASTIC SOLID (VOLUME SOURCE, POINT SOURCE)

12.2.1 Introduction

We start from the equations of motion in three coordinates (x, y, z):

$$\left.\begin{aligned} \varrho\frac{\partial^2 u}{\partial t^2} &= (\lambda+\mu)\frac{\partial\theta}{\partial x} + \mu\nabla^2 u \\ \varrho\frac{\partial^2 v}{\partial t^2} &= (\lambda+\mu)\frac{\partial\theta}{\partial y} + \mu\nabla^2 v \\ \varrho\frac{\partial^2 w}{\partial t^2} &= (\lambda+\mu)\frac{\partial\theta}{\partial z} + \mu\nabla^2 w \end{aligned}\right| \qquad [1]$$

where the cubical dilatation is:

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad [2]$$

We assume a simple harmonic motion, i.e., $u \sim e^{ipt}$ and $\partial^2 u/\partial t^2 = -p^2 u$. We write the displacements in the three coordinate directions as follows:

$$\left.\begin{aligned} u &= \frac{\partial\varphi}{\partial x} + u' \\ v &= \frac{\partial\varphi}{\partial y} + v' \\ w &= \frac{\partial\varphi}{\partial z} + w' \end{aligned}\right| \qquad [3]$$

which holds provided that:

$$(\nabla^2 + h^2)\varphi = 0 \qquad [4]$$

and:

$$\left.\begin{aligned}
&(\nabla^2 + k^2)u' = 0\\
&(\nabla^2 + k^2)v' = 0\\
&(\nabla^2 + k^2)w' = 0\\
&\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0
\end{aligned}\right| \qquad [5]$$

with $h^2 = p^2\varrho/(\lambda + 2\mu)$; $k^2 = p^2\varrho/\mu$ as in eq.[7] in section 12.1.

Eq.[4] and [5] are obtained by substituting the assumptions [3] into the wave equations [1]. We get the cubical dilatation:

$$\begin{aligned}
\theta &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x} + u'\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial y} + v'\right) + \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial z} + w'\right)\\
&= \nabla^2\varphi + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}
\end{aligned}$$

The first eq.[1] then becomes:

$$-\varrho p^2\left(\frac{\partial \varphi}{\partial x} + u'\right) = (\lambda + \mu)\frac{\partial}{\partial x}\left(\nabla^2\varphi + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) + \mu\nabla^2\left(\frac{\partial \varphi}{\partial x} + u'\right)$$

or, slightly rearranged:

$$\underbrace{-\varrho p^2\left(\frac{\partial \varphi}{\partial x}\right.}_{[4]} \underbrace{\left. + u'\right)}_{[5]} = \underbrace{(\lambda + 2\mu)\frac{\partial}{\partial x}\nabla^2\varphi}_{[4]} + (\lambda + \mu)\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)}_{=0} + \underbrace{\mu\nabla^2 u'}_{[5]}$$

Equating the equally marked terms gives [4] and two of the eq.[5]. The remaining equations in v' and w' in [5] are similarly obtained from the second and third eq.[1]. Of course a justified question at this point is whether there are other possibilities to fulfill the last equation above. Naturally, there are other possibilities but the one chosen here is the only one which will prove successful and make sense.

We may say that by the substitution [3] our original wave equations [1] have been transformed into new equations [5] in terms of u', v' and w' together with an "equation of continuity". A particular solution of this transformed set is:

$$\left.\begin{aligned}
u' &= \frac{\partial^2 \psi}{\partial x \partial z}\\
v' &= \frac{\partial^2 \psi}{\partial y \partial z}\\
w' &= \frac{\partial^2 \psi}{\partial z^2} + k^2\psi
\end{aligned}\right| \qquad [6]$$

provided:

$$(\nabla^2 + k^2)\psi = 0 \qquad [7]$$

Eq.[7] is seen by substituting [6] into [5]. Take for example the last equation in [5]:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

and substitute [6]:

$$\frac{\partial^3 \psi}{\partial x^2 \partial z} + \frac{\partial^3 \psi}{\partial y^2 \partial z} + \frac{\partial^3 \psi}{\partial z^3} + k^2 \frac{\partial \psi}{\partial z} = 0$$

or:

$$\frac{\partial}{\partial z}(\nabla^2 + k^2)\psi = 0$$

This holds if $(\nabla^2 + k^2)\psi$ is everywhere constant, i.e.:

$$(\nabla^2 + k^2)\psi = 0$$

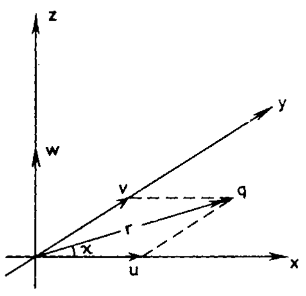
which is [7], incorporating any constant into $k^2\psi$. The same is true for the other eq.[5]:

$$(\nabla^2 + k^2)u' = (\nabla^2 + k^2)\frac{\partial^2 \psi}{\partial x \partial z} = \frac{\partial^2}{\partial x \partial z}(\nabla^2 + k^2)\psi = 0$$

(the integration order can be changed), which again holds if [7] holds, etc.

Now, φ and ψ are our two wave potentials for which [4] and [7] hold.

At this point, Lamb introduces the assumption of circular symmetry around the z-axis. It is important to remember this in the following treatment, as we are forbidden to introduce any forces which would violate this assumption; that is, any forces we introduce later must be circular-symmetric with respect to the z-axis. In other words, any force



$r = (x^2 + y^2)^{1/2}$

$q = (u^2 + v^2)^{1/2}$

Fig.88.

system we consider must have one axis of symmetry, and we orientate our coordinate system in such a way that the z-axis coincides with this axis of symmetry.

Transforming to cylindrical coordinates (Fig.88): $r = (x^2 + y^2)^{1/2}$, z, χ we have:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \underbrace{\frac{1}{r^2}\frac{\partial^2}{\partial \chi^2}} \tag{8}$$

$= 0$ (because of assumed symmetry around the z-axis)

We denote the displacement components as follows: q perpendicular to z (independent of direction in this plane); w along z. We get expressions for q and w from [3] and [6], combining the formulas for u and v into one expression (in vector form) for q:

$$\left.\begin{aligned} q &= \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z} \\ w &= \frac{\partial \varphi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi \end{aligned}\right| \tag{9}$$

The solutions in terms of our wave potentials φ and ψ of eq.[4] and [7] are now assumed to be:

$$\left.\begin{aligned} \varphi &= A e^{-\alpha z} J_0(\xi r) \\ \psi &= B e^{-\beta z} J_0(\xi r) \end{aligned}\right\} z > 0 \tag{10}$$

where $\alpha^2 = \xi^2 - h^2$; $\beta^2 = \xi^2 - k^2$.

The difference from the two-dimensional case is that we now have Bessel functions of zero order $J_0(\xi r)$ instead of the exponential factor $e^{i\xi x}$. That α and β will have the same expressions as in the two-dimensional case is immediately obvious by substituting [10] into [4] and [7]. Take for instance [4] and substitute φ from [10]; we find:

$$J_0''(\xi r) + \frac{1}{\xi r} J_0'(\xi r) + \frac{\alpha^2 + h^2}{\xi^2} J_0(\xi r) = 0$$

which is true if $\alpha^2 + h^2 = \xi^2$, as the equation then reduces to the well-known Bessel's differential equation:

$$J_0''(\xi r) + \frac{1}{\xi r} J_0'(\xi r) + J_0(\xi r) = 0$$

In the same way we prove the relation for β, using [7] and [10]. In the present case we could *not* keep the same form of the solution as in the two-dimensional case, i.e., assuming $\varphi = A e^{-\alpha z} e^{i\xi r}$, and still have the same expressions for α and β. This is verified by substituting the expression for φ into [4].

From [9] and [10] we immediately get expressions for the displacements, remembering that $J_0'(x) = -J_1(x)$:

$$\left.\begin{aligned} q &= (-\xi A e^{-\alpha z} + \xi \beta B e^{-\beta z}) J_1(\xi r) \\ w &= (-\alpha A e^{-\alpha z} + \xi^2 B e^{-\beta z}) J_0(\xi r) \end{aligned}\right| \tag{11}$$

For the stresses in the plane $z = 0$, we get by application of the fundamental stress-strain relation (eq.[8] in section 12.1):

$$\left.\begin{aligned}[p_{zr}]_0 &= \mu \left[\frac{\partial q}{\partial z} + \frac{\partial w}{\partial r}\right]_0 = \mu[2\xi\alpha A - (2\xi^2 - k^2)\xi B]J_1(\xi r)\\ [p_{zz}]_0 &= \left[\lambda\theta + 2\mu \frac{\partial w}{\partial z}\right]_0 = \mu[(2\xi^2 - k^2)A - 2\xi^2\beta B]J_0(\xi r)\end{aligned}\right| \qquad [12]$$

Corresponding expressions can be formed for $z < 0$, if [10] is replaced by:

$$\left.\begin{aligned}\varphi &= A'e^{\alpha z}J_0(\xi r)\\ \psi &= B'e^{\beta z}J_0(\xi r)\end{aligned}\right| \qquad [13]$$

12.2.2 A periodic force acts on the plane $z = 0$ in an unlimited elastic solid (area source)

The treatment is analogous to the case of two dimensions (section 12.1.2). We assume that the extraneous force $ZJ_0(\xi r)e^{ipt}$ acts per unit area on the plane $z = 0$ in the

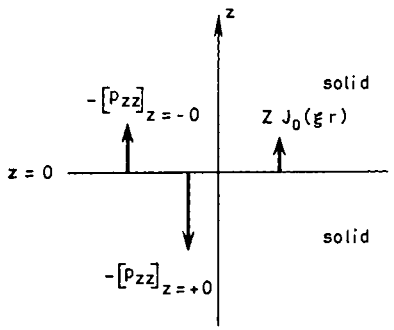


Fig.89.

direction of positive z. The acting forces are symbolically illustrated in Fig.89. We then have:

(*1*) condition for the normal stress:

$$[p_{zz}]_{z=+0} - [p_{zz}]_{z=-0} = -ZJ_0(\xi r) \qquad [14]$$

or by means of [12]:

$$(2\xi^2 - k^2)(A - A') - 2\xi^2\beta(B + B') = -\frac{Z}{\mu}$$

(*2*) continuity of tangential stress, by [12]:

$$2\alpha(A + A') - (2\xi^2 - k^2)(B - B') = 0 \qquad [15]$$

(3) continuities of displacements at $z = 0$, by [11]:

$$\left.\begin{aligned} &A - A' - \beta(B + B') = 0 \\ &\alpha(A + A') - \xi^2(B - B') = 0 \end{aligned}\right| \qquad [16]$$

The four equations [14]-[16] have the following solutions:

$$\left.\begin{aligned} &A = -A' = \frac{Z}{2k^2\mu} \\ &B = B' = \frac{Z}{2k^2\mu\beta} \end{aligned}\right| \qquad [17]$$

i.e., we get for $z > 0$ by inserting these values in [10]:

$$\left.\begin{aligned} &\varphi = \frac{Z}{2k^2\mu}\, e^{-\alpha z} J_0(\xi r) \\ &\psi = \frac{Z}{2k^2\mu}\, \frac{e^{-\beta z}}{\beta}\, J_0(\xi r) \end{aligned}\right| \qquad [18]$$

12.2.3 A periodic force concentrated to the z-axis in an unlimited elastic solid (point source)

The same method as in the two-dimensional case (section 12.1.3) can be applied here, with the difference caused by the fact that now we have the function $J_0(\xi r)$ instead of the exponential function. We put:

$$ZJ_0(\xi r) \to \int_0^\infty \frac{Q\xi}{2\pi} J_0(\xi r)\mathrm{d}\xi = \mathfrak{f}(r) \qquad [19]$$

which defines the function $\mathfrak{f}(r)$. Eq.[19] is a Hankel (or Fourier-Bessel) transformation, and its inverse is (cf.eq.[5] and [28] in section 8.1):

$$Q = \int_0^\infty \mathfrak{f}(\lambda) J_0(\xi\lambda)\lambda \mathrm{d}\lambda \qquad [20]$$

The additional factor 2π appearing in [19] is of no computational consequence. Then [19] can be written as follows:

$$\mathfrak{f}(r) = \frac{1}{2\pi} \int_0^\infty J_0(\xi r)\xi \mathrm{d}\xi \int_0^\infty \mathfrak{f}(\lambda) J_0(\xi\lambda)\lambda \mathrm{d}\lambda \qquad [21]$$

The integrations over the wave number ξ are extended from 0 to $+\infty$ instead of from $-\infty$ to $+\infty$—because the factor ξ under the integral sign would make positive and negative contributions cancel each other, as $J_0(-x) = J_0(x)$.

The infinitely many periods which are superimposed on each other are requested to fulfill the condition of a concentrated force at $z = 0$, such that Q assumes a finite (constant) value on the z-axis, and that $\bar{f}(r)$ vanishes for all values of r except on the z-axis, i.e., for $r = 0$, where it becomes infinite.

Then [18] becomes:

$$\varphi = \frac{Q}{4\pi p^2 \varrho} \int_0^\infty e^{-\alpha z} J_0(\xi r)\xi d\xi$$
$$\psi = \frac{Q}{4\pi p^2 \varrho} \int_0^\infty \frac{e^{-\beta z}}{\beta} J_0(\xi r)\xi d\xi \qquad [22]$$

($k^2\mu = p^2\varrho$). If we apply Sommerfeld's integral for spherical waves (eq.[11] in section 7.3):

$$\frac{e^{-ihR}}{R} = \int_0^\infty \frac{e^{-\alpha z} J_0(\xi r)\xi d\xi}{\alpha} \qquad [23]$$

where $R = (r^2 + z^2)^{1/2}$, [22] can also be written as follows:

$$\varphi = -\frac{Q}{4\pi p^2 \varrho} \frac{\partial}{\partial z} \frac{e^{-ihR}}{R}$$
$$\psi = \frac{Q}{4\pi p^2 \varrho} \frac{e^{-ikR}}{R} \qquad [24]$$

Substituting [24] in [9] and considering large distances R we find:

$$q = \frac{Q}{4\pi} \left(\frac{1}{\lambda + 2\mu} \frac{zr}{R^3} e^{-ihR} - \frac{1}{\mu} \frac{zr}{R^3} e^{-ikR} \right)$$
$$w = \frac{Q}{4\pi} \left(\frac{1}{\lambda + 2\mu} \frac{z^2}{R^3} e^{-ihR} + \frac{1}{\mu} \frac{r^2}{R^3} e^{-ikR} \right) \qquad [25]$$

12.2.4 A periodic force acts on the surface of a semi-infinite elastic solid (area source)

Let the semi-infinite solid occupy the half-space $z > 0$, in such a way that $z = 0$ coincides with the free surface. The treatment is similar to the two-dimensional case (section 12.1.4):

$$[p_{zz}]_0 = ZJ_0(\xi r)$$
$$[p_{zr}]_0 = 0 \qquad [26]$$

The stress conditions at $z = 0$ determine A and B. Apply [12]:

$$(2\xi^2 - k^2)A - 2\xi^2\beta B = \frac{Z}{\mu}$$
$$2\alpha A - (2\xi^2 - k^2)B = 0 \qquad [27]$$

which give:

$$A = \frac{2\xi^2 - k^2}{F(\xi)} \frac{Z}{\mu}, \qquad B = \frac{2\alpha}{F(\xi)} \frac{Z}{\mu} \tag{28}$$

where: $F(\xi) = (2\xi^2 - k^2)^2 - 4\xi^2\alpha\beta$ as in section 12.1. The surface displacements ($z = 0$) are found from [11]:

$$q_0 = -\frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)}{F(\xi)} J_1(\xi r) \frac{Z}{\mu}, \qquad w_0 = \frac{k^2\alpha}{F(\xi)} J_0(\xi r) \frac{Z}{\mu} \tag{29}$$

12.2.5 *Concentrated vertical pressure at the origin (point source)*

As in section 12.2.3, we replace Z by $-Q\xi d\xi/2\pi$ and integrate over ξ from 0 to $+\infty$. The eq.[29] then become:

$$q_0 = \frac{Q}{2\pi\mu} \int_0^\infty \frac{\xi^2(2\xi^2 - k^2 - 2\alpha\beta)}{F(\xi)} J_1(\xi r) d\xi, \qquad w_0 = -\frac{Q}{2\pi\mu} \int_0^\infty \frac{k^2\xi\alpha}{F(\xi)} J_0(\xi r) d\xi \tag{30}$$

12.2.6 *Evaluation of the integral solutions*

We use the following formulas:

$$J_0(x) = \frac{2}{\pi} \int_0^\infty \sin(x \cosh u)\, du$$

(eq.[25] in section 12.1)

$$J_1(x) = -J_0'(x) = -\frac{2}{\pi} \int_0^\infty \cos(x \cosh u) \cosh u\, du$$

(eq.[16] in section 5.2)

$$J_0(x) = -\frac{i}{\pi} \int_0^\infty (e^{ix \cosh u} - e^{-ix \cosh u})\, du$$

$$J_1(x) = -\frac{1}{\pi} \int_0^\infty (e^{ix \cosh u} + e^{-ix \cosh u}) \cosh u\, du \tag{31}$$

The last two formulas in [31] can be seen to agree with the first two, respectively, by application of Euler's formula $e^{\pm i\alpha} = \cos\alpha \pm i\sin\alpha$. Applying [31] to [30], we find:

$$\left.\begin{aligned} q_0 &= -\frac{Q}{2\pi^2\mu}\int_0^\infty \cosh u\,\mathrm{d}u \int_{-\infty}^{\infty} \frac{\xi^2(2\xi^2-k^2-2\alpha\beta)}{F(\xi)}\,e^{i\xi r\cosh u}\mathrm{d}\xi \\ w_0 &= \frac{iQ}{2\pi^2\mu}\int_0^\infty \mathrm{d}u \int_{-\infty}^{\infty} \frac{k^2\xi\alpha}{F(\xi)}\,e^{i\xi r\cosh u}\mathrm{d}\xi \end{aligned}\right| \quad [32]$$

Note that the integrations over ξ are from $-\infty$ to $+\infty$; the reason is that by [31] we get two terms for J_0 and J_1, which differ only in the sign of ξ. They can be combined into one term, and then we extend the integration interval as in [32]. Take for example the following integral:

$$\int_0^\infty \xi e^{-i\xi r\cosh u}\mathrm{d}\xi$$

and make the substitution $\xi = -\xi'$, $\mathrm{d}\xi = -\mathrm{d}\xi'$. Then the integral becomes:

$$+\int_0^{-\infty} \xi' e^{i\xi' r\cosh u}\mathrm{d}\xi' = -\int_{-\infty}^{0} \xi' e^{i\xi' r\cosh u}\mathrm{d}\xi'$$

from which the combination of the two integrals is immediately obvious.

These integrals can be evaluated by the same method as used in the two-dimensional case (section 12.1.8), but more simply we could use the results from that solution. Comparing with eq.[44] in section 12.1, which were solved before, we find that if we apply the operation $-i(\partial/\partial x)$ to [44] in section 12.1 and then replace x by $r\cosh u$ we have the integrals over ξ which appear in [32]. We start from u_0, applying $-i(\partial/\partial x)$:

$$\begin{aligned} &-i\frac{\partial}{\partial x}\int_{-\infty}^{\infty} \frac{\xi(2\xi^2-k^2-2\alpha\beta)e^{i\xi x}\mathrm{d}\xi}{F(\xi)} \\ &= -i\int_{-\infty}^{\infty} \frac{i\xi^2(2\xi^2-k^2-2\alpha\beta)e^{i\xi x}\mathrm{d}\xi}{F(\xi)} \\ &= +\int_{-\infty}^{\infty} \frac{\xi^2(2\xi^2-k^2-2\alpha\beta)e^{i\xi r\cosh u}\mathrm{d}\xi}{F(\xi)} \end{aligned} \quad [33]$$

i.e., the integral in q_0 in [32].

Thus, solutions to the integrals in [32] can be obtained by using the solutions we derived in section 12.1.8, and applying to these solutions the operation $-i(\partial/\partial x)$ and then replacing x by $r\cosh u$. We can carry out these operations on the expressions [73] and [74] in section 12.1. The calculations are rather lengthy but quite straight-forward.

The first term (Rayleigh waves) yields:

$$-i\frac{\partial}{\partial x}(e^{-i\varkappa x}) = -\varkappa\, e^{-i\varkappa x} = -\varkappa e^{-i\varkappa r\cosh u}$$

Inserted into [32.1], this gives:

$$(q_0)_{\text{Rayleigh}} = -\frac{QHi\varkappa}{\pi\mu}\underbrace{\int_0^\infty \cosh u\, e^{-i\varkappa r\cosh u} du}_{=\frac{D_1(\varkappa r)\pi}{2i}}$$

This is seen from:

$$D_0(\varkappa r) = \frac{2}{\pi}\int_0^\infty e^{-i\varkappa r\cosh u} du$$

(see [27] in section 12.1) from which follows:

$$D_1(\varkappa r) = -D_0'(\varkappa r) = \frac{2i}{\pi}\int_0^\infty e^{-i\varkappa r\cosh u}\cosh u\, du$$

Then:

$$(q_0)_{\text{Rayleigh}} = -\frac{\varkappa QH}{2\mu} D_1(\varkappa r)e^{ipt} \sim \frac{1}{r^{1/2}} \quad \text{for } r \text{ great} \qquad [34]$$

as:

$$D_1(\varkappa r) \sim \frac{1}{r^{1/2}} \quad \text{for } r \text{ great}$$

The other two terms (P and S, being contributions from the branch lines) lead to much longer expressions, but we limit our attention to the variation with r, especially for large values of r (large distance from the source). Then we have:

$$-i\frac{\partial}{\partial x}\left(\frac{e^{-ikx}}{x^{3/2}}\right) = -i\left[\underbrace{\frac{-(3/2)\,e^{-ikx}}{x^{5/2}}}_{\to 0} - \frac{ike^{-ikx}}{x^{3/2}}\right] =$$

$$= \frac{-ke^{-ikx}}{x^{3/2}} \to \frac{e^{-ikr\cosh u}}{(kr\cosh u)^{3/2}}$$

and [32] becomes:

q_0:

$$\frac{1}{(kr)^{3/2}}\int_0^\infty \frac{e^{-ikr\cosh u} du}{(\cosh u)^{1/2}} \to \frac{1}{r^{3/2}}\frac{1}{r^{1/2}} = \frac{1}{r^2} \qquad [35]$$

w_0:

$$\frac{1}{(kr)^{3/2}}\int_0^\infty \frac{e^{-ikr\cosh u}du}{(\cosh u)^{3/2}} \rightarrow \frac{1}{r^{3/2}}\frac{1}{r^{1/2}} = \frac{1}{r^2} \tag{35}$$

from the asymptotic expansion of D for large values of r (eq.[30] in section 12.1).

Thus, for an elastic medium with a free surface, the Rayleigh waves decrease as $1/r^{1/2}$ but the P and S waves as $1/r^2$ in the three-dimensional case. Therefore, at a large distance from the source, the Rayleigh waves will dominate the motion, just as in the two-dimensional case.

TABLE IX

AMPLITUDE–DISTANCE RELATIONS IN LAMB'S PROBLEM

Source	*Medium*	*Rayleigh*	*P*	*S*	*Section reference*
line	unlimited	—	$1/r^{1/2}$	$1/r^{1/2}$	12.1.3
line	half-space	r^0	$1/r^{3/2}$	$1/r^{3/2}$	12.1.5
(normal or tangential)					12.1.6
point	unlimited	—	$1/r$	$1/r$	12.2.3
point	half-space	$1/r^{1/2}$	$1/r^2$	$1/r^2$	12.2.5

As we are especially interested in the variation of *amplitudes* with distance at great distances from the source, I have summarized some of our results in Table IX. Note that these distance-variations depend on the geometry of the problems, as no assumption about damping or frictional losses has been made. I suggest that the reader see which distance-relations in Table IX are immediately obvious from simple geometrical considerations and which are not.

12.3 ARBITRARY TIME VARIATION IN THE THREE-DIMENSIONAL CASE

So far, we have been considering only harmonic variations of forces, displacements, etc., as represented by the time factor e^{ipt}. We shall now generalize the theory to arbitrary time variations, especially to impulses which have much closer relations to such phenomena as explosions and earthquakes.

In the solution of the integrals for the three-dimensional case, we arrived at the following expressions for the displacements, considering first only the Rayleigh-wave part (i.e., the contribution from poles in the integration):

$$(q_0)_{\text{Rayleigh}} = -\frac{\varkappa QH}{2\mu} D_1(\varkappa r)e^{ipt} \tag{1}$$

(same as [34] in section 12.2)

$$(w_0)_{\text{Rayleigh}} = +\frac{\varkappa QK}{2\mu} D_0(\varkappa r)e^{ipt} \qquad [1]$$

(obtained in similar way). These can be written in the following way:

$$\left.\begin{aligned}(q_0)_{\text{Rayleigh}} &= +\frac{HQ}{\pi\mu}\frac{\partial}{\partial r}\int_0^\infty e^{ip(t-cr\cosh u)}du \\ (w_0)_{\text{Rayleigh}} &= -\frac{iKQc}{\pi\mu}\frac{\partial}{\partial t}\int_0^\infty e^{ip(t-cr\cosh u)}du\end{aligned}\right\} \qquad [2]$$

These are obtained immediately, for example by starting from the integrals on the right-hand sides of [2] and using the expressions for $D_1(\varkappa r)$ and $D_0(\varkappa r)$, section 12.2.6. Note that in LAMB's (1904) notation, which I follow here, c is the "wave slowness", i.e., the inverted velocity, and thus that $\varkappa = pc$.

Eq.[2] may be said to contain a "retarded time" t':

$$t' = t - cr\cosh u \qquad [3]$$

a notation with which we are familiar from the discussion of Huygens' principle (section 7.4). Now, we replace our previous harmonic time function by an arbitrary time variation, i.e.:

$$Qe^{ipt'} \rightarrow Q(t') \qquad [4]$$

Then:

$$\frac{\partial}{\partial t} Qe^{ipt'} = ipQe^{ipt'} \rightarrow ipQ(t')$$

As the Fourier integral formula (eq.[9] in section 8.1) gives the most general expression for Q, we put:

$$Q(t') = \frac{1}{\pi}\int_0^\infty dp \int_{-\infty}^\infty Q(\lambda)\cos[p(t'-\lambda)]\, d\lambda$$

and also for a related function: [5]

$$Q^*(t') = \frac{1}{\pi}\int_0^\infty dp \int_{-\infty}^\infty Q(\lambda)\sin[p(t'-\lambda)]\, d\lambda$$

Then we have symbolically:

$$\frac{\partial Q^*}{\partial t} = pQ \qquad [6]$$

Eq.[4] and [6] permit us to write [2] in the following form:

$$
\left.\begin{aligned}
(q_0)_{\text{Rayleigh}} &= + \frac{H}{\pi\mu}\frac{\partial}{\partial r}\int_0^\infty Q(t - cr\cosh u)\mathrm{d}u \\
(w_0)_{\text{Rayleigh}} &= + \frac{Kc}{\pi\mu}\frac{\partial}{\partial t}\int_0^\infty Q^*(t - cr\cosh u)\mathrm{d}u
\end{aligned}\right| \qquad [7]
$$

LAMB (1904) discusses several different time functions for the source. We select one in which:

$$
\left.\begin{aligned}
Q(t) &= \frac{\overline{Q}}{\pi}\frac{\tau}{t^2 + \tau^2} \\
Q^*(t) &= \frac{\overline{Q}}{\pi}\frac{t}{t^2 + \tau^2}
\end{aligned}\right| \qquad [8]
$$

These satisfy [5]. τ and $\overline{Q}$ are real, positive constants. The function $Q(t)$ corresponds to the initial time variation at the source, shown in Fig.90. By substitution of [8] into [7] and evaluation for large values of r, i.e., at a large distance from the source, we finally find:

$$
\left.\begin{aligned}
(q_0)_{\text{Rayleigh}} &= - \frac{H\overline{Q}c}{4\pi\mu\tau^2}\left(\frac{2\tau}{cr}\right)^{1/2}\sin\left(\frac{\pi}{4} - \frac{3}{2}\nu\right)\cos^{3/2}\nu \\
(w_0)_{\text{Rayleigh}} &= + \frac{K\overline{Q}c}{4\pi\mu\tau^2}\left(\frac{2\tau}{cr}\right)^{1/2}\cos\left(\frac{\pi}{4} - \frac{3}{2}\nu\right)\cos^{3/2}\nu
\end{aligned}\right| \qquad [9]
$$

where $\tan\nu = (t - cr)/\tau$. Eq.[9] is shown graphically near the time $t = cr$ in Fig.91. It is obvious from [9] that the amplitudes decrease as $1/r^{1/2}$, i.e., at the same rate as for the case of a harmonic source (see Table IX).

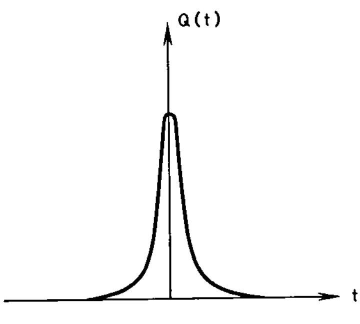


Fig.90.

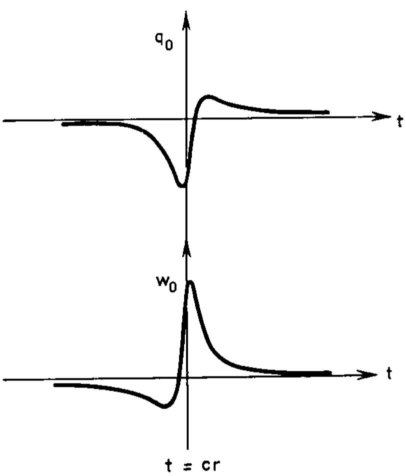


Fig.91.

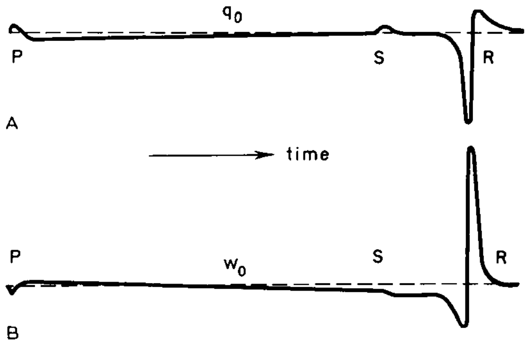


Fig.92.

We have discussed only the Rayleigh-wave part of the solution in our generalization to an arbitrary time function. The other two terms can be generalized in a similar way.

The whole record consists of only a *P*-pulse, a jerk corresponding to *S*, immediately followed by the Rayleigh-wave pulse (see Fig.92). This picture, which was constructed theoretically by LAMB (1904), is probably the first theoretical seismogram. It has only a remote resemblance to a real seismogram. Some of the reasons for the deviations may be summarized in the following points:

(*1*) The theory assumes a homogeneous elastic medium without layering and therefore with no dispersion.

(*2*) The theory assumes an infinite plane surface and takes no account of the curvature, as for the earth's surface.

(*3*) The source model may be far from real source models in earthquakes or explosions.

Still, LAMB's (1904) paper has played an enormous role in seismology.

There are several points of contact between the problems studied by LAMB (1904) and those studied by CAGNIARD (1939) (see section 8.4). We can summarize a comparison in the following points:

(*a*) One of Lamb's media is a vacuum, while Cagniard studies two media with arbitrary elastic properties.

(*b*) Lamb's source is at the surface, while Cagniard's source is in the interior of one of the media.

(*c*) Lamb studies displacements only on the surface, whereas Cagniard's solution is valid in the interior of either medium.

These remarks refer to Lamb's original paper in 1904, and not to later developments of Lamb's problem. Several famous mathematicians, notably LAPWOOD (1949), have made extensive studies of Lamb's problem.

Chapter 13

WAVE PROPAGATION IN LIQUID MEDIA

13.1 WAVE PROPAGATION IN A TWO-LAYERED LIQUID HALF-SPACE

13.1.1 Derivation of integral solutions for the displacement potentials

We assume a liquid layer overlying a liquid half-space, and we also assume a wave source located in the top layer.

The solution of this problem is of value to us for two reasons:

(*1*) It contains applications of many mathematical tools we have learnt (e.g., spherical wave fronts, contour integration, Riemann surfaces, residue calculus, Hankel functions, Fourier transform, stationary phase method).

(*2*) Once the technique for this problem has been well understood, it is a relatively easy task to apply the same technique to any model of a layered structure (i.e., with an arbitrary number of layers, liquid or solid).

In this presentation, which closely follows EWING, JARDETZKY and PRESS (1957, pp.126–144), we shall concentrate on the more difficult mathematical parts, being much shorter on easier mathematical or physical parts. We consider the situation shown in Fig.93 with two liquid layers, i.e., only longitudinal waves exist. We include a point source S in the upper layer, but assume simple harmonic waves (later, we generalize to an impulsive source).

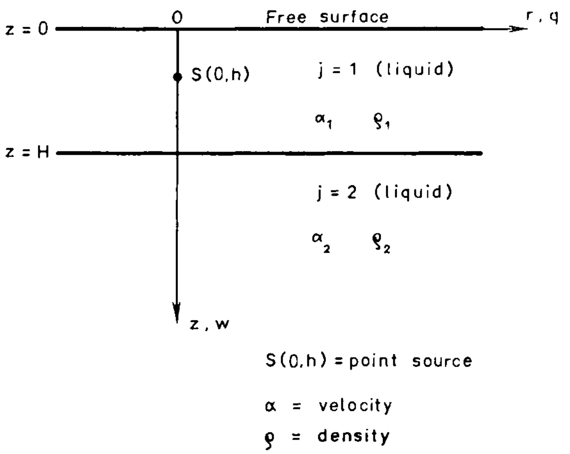


Fig.93.

In classical treatments of this and similar problems, it is assumed that there are plane, simple harmonic waves (i.e., a source at infinite distance and in continuous action). Modern developments, which better approach real situations in many cases, have improved this technique, especially in two ways: (*a*) by including a source at finite distance, i.e., no longer assuming plane wave fronts; (*b*) by allowing for any time variation of the source action, e.g., impulsive wave motion of various shapes.

In the present case we shall have no spherical symmetry, but only cylindrical symmetry around the vertical axis through S. Therefore, it is natural to use cylindrical coordinates r, z. The displacements q_j and w_j are expressed in terms of one displacement potential φ_j for P-wave motion:

$$q_j = \frac{\partial \varphi_j}{\partial r} \ ; \quad w_j = \frac{\partial \varphi_j}{\partial z} \ ; \quad j = 1,2 \tag{1}$$

The potential φ_j satisfies the wave equation:

$$\nabla^2 \varphi_j = \frac{1}{\alpha_j^2} \frac{\partial^2 \varphi_j}{\partial t^2} \ ; \quad j = 1,2 \tag{2}$$

Of course, this means that we assume our problem to fulfill all conditions under which the simple wave equation [2] is valid (see section 7.1). Further generalizations, in addition to the two mentioned above, would naturally be to drop one or several of these conditions.

For the *boundary conditions* we need an expression for the pressure p (cf. EWING, JARDETZKY and PRESS, 1957, pp.6–8, where velocity potentials were used, as distinct from displacements potentials used here). The wave equation for an ideal fluid ($\mu = 0$) without external foces ($X = Y = Z = 0$) reads, with common notation:

$$\varrho \frac{\partial^2 u}{\partial t^2} = \lambda \frac{\partial \theta}{\partial x} \tag{3}$$

Introducing the displacement potential $u = \partial\varphi/\partial x$ we get:

$$\varrho \frac{\partial}{\partial x} \frac{\partial^2 \varphi}{\partial t^2} = \lambda \frac{\partial \theta}{\partial x} \tag{4}$$

which is integrated with respect to x:

$$\varrho \frac{\partial^2 \varphi}{\partial t^2} = \lambda\theta + \underbrace{\text{constant}}_{=0} \tag{5}$$

The hydrostatic pressure is:

$$p = -\lambda\theta = -\varrho \frac{\partial^2 \varphi}{\partial t^2} = +\varrho\varphi\omega^2 \tag{6}$$

where in the last step we assume that $\varphi \sim e^{i\omega t}$, i.e., we are dealing with simple harmonic waves.

Then we can express the boundary conditions:

at $z = H$: continuity of vertical displacement

$$\frac{\partial \varphi_1}{\partial z} = \frac{\partial \varphi_2}{\partial z} \quad \text{(from [1])}$$

continuity of pressure

$$\varrho_1 \varphi_1 = \varrho_2 \varphi_2 \quad \text{(from [6])}$$

at $z = 0$: no pressure (or continuity of pressure, assuming a vacuum above layer 1)

$$\varphi_1 = 0 \quad \text{(from [6])} \qquad [7]$$

For the *displacement potentials* we assume the following expressions, omitting the time factor $e^{i\omega t}$:

$$\varphi_1 = \int_0^\infty \frac{k}{\nu_1} J_0(kr) e^{-\nu_1 |z-h|} dk +$$

(direct, spherical wave)

$$+ \int_0^\infty Q_1(k) J_0(kr) e^{-\nu_1(z-h)} dk +$$

(summation of *downward* travelling waves, which have been reflected one or more times from the boundaries)

$$+ \int_0^\infty Q_1^*(k) J_0(kr) e^{\nu_1(z-h)} dk \qquad [8]$$

(summation of corresponding *upward* travelling waves)

$$\varphi_2 = \int_0^\infty Q_2(k) J_0(kr) e^{-\nu_2(z-h)} dk \qquad [9]$$

(summation of *downward* travelling waves in the second medium).

We have to note the following. Only the direct wave (in the first medium) is spherical, and it is only for the spherical wave that we can use the expression in the first term of [8]. In other words, for spherical waves we have:

$$Q(k) = \frac{k}{\nu}$$

If a given spherical wave is reflected once, the reflected wave will still be spherical and the same expression would be applicable. However, we allow for an arbitrary number of reflections of many spherical waves, and then the resulting wave front will not in general be spherical. Hence, we have to introduce the other more general expressions, and the factors $Q(k)$ will be determined from the boundary conditions [7]. In the lower medium, eq.[9], we have to consider only *downward* propagation. This physical requirement also satisfies the mathematical request, that $\varphi_2 \to 0$ as $z \to \infty$.

The next step is quite straight-forward. The potentials are substituted into the boundary conditions [7], which gives three linear equations for solving the three unknowns Q_1, Q_1^* and Q_2. In making the substitution of [8] and [9] into [7], we can drop the integral signs and have only three relations concerning the integrands of [8] and [9]. As there are no mathematical difficulties involved I only refer to EWING, JARDETZKY and PRESS (1957, p.128). The *resulting solutions* are:

$0 \leqslant z \leqslant h$:

$$\varphi_1' = 2\int_0^\infty kJ_0(kr)\frac{\sinh\nu_1 z}{\nu_1}\,\frac{\nu_1\cosh[\nu_1(H-h)] + \delta_1\nu_2\sinh[\nu_1(H-h)]}{\nu_1\cosh\nu_1 H + \delta_1\nu_2\sinh\nu_1 H}\,\mathrm{d}k \qquad [10]$$

$h \leqslant z \leqslant H$:

$$\varphi_1'' = 2\int_0^\infty kJ_0(kr)\frac{\sinh\nu_1 h}{\nu_1}\,\frac{\nu_1\cosh[\nu_1(z-H)] - \delta_1\nu_2\sinh[\nu_1(z-H)]}{\nu_1\cosh\nu_1 H + \delta_1\nu_2\sinh\nu_1 H}\,\mathrm{d}k \qquad [11]$$

$H \leqslant z < \infty$:

$$\varphi_2 = 2\delta_1\int_0^\infty kJ_0(kr)\,\frac{\sinh\nu_1 h}{\nu_1\cosh\nu_1 H + \delta_1\nu_2\sinh\nu_1 H}\,\mathrm{e}^{-\nu_2(z-H)}\mathrm{d}k \qquad [12]$$

where $\delta_1 = \varrho_1/\varrho_2$. We note that for $z = h$, we have $\varphi_1' = \varphi_1''$. The boundary conditions are easily seen to be fulfilled by [10]–[12].

An alternative method, developed by PEKERIS (1948b), combines the three terms in [8] into one term (for $0 \leqslant z \leqslant h$) or two terms (for $h \leqslant z \leqslant H$). The terms include functions of k as factors, which are determined by the boundary conditions, and, for example, an expression for the direct wave is not explicitly written out. This method leads to results which are identical with those obtained here.

13.1.2 Evaluation of the integral solutions

Eq.[10]–[12] represent the solution of our problem, and our next task is to evaluate the integrals, contained in these formulas. In order to follow the developments in EWING, JARDETZKY and PRESS (1957, pp.128–131), we have to remember the following formulas:

$$\left.\begin{aligned}&\cosh\\&\sinh\end{aligned}\right\} x = \frac{1}{2}(\mathrm{e}^x \pm \mathrm{e}^{-x})$$
$$\cos(-ix) = \cos(ix) = \cosh x \qquad [13]$$
$$\sin(-ix) = -\sin(ix) = -i\sinh x$$

Our study of the integrals in [10]–[12] will make use of contour integration, and will lead to a representation of the wave motion by normal modes. This is most appropriate when the horizontal distance from the source is large compared to the layer thickness, so that rays propagating along different paths overlap.

Any one of the solutions [10]–[12] can be written in the following general form:

$$\varphi = 2\int_0^\infty J_0(kr)F(\nu_1,\nu_2)k\mathrm{d}k$$

$$= \int_0^\infty H_0^{(1)}(kr)F(\nu_1,\nu_2)k\mathrm{d}k + \int_0^\infty H_0^{(2)}(kr)F(\nu_1,\nu_2)k\mathrm{d}k$$

$$= \int_0^\infty G_1\mathrm{d}k + \int_0^\infty G_2\mathrm{d}k \qquad [14]$$

where we have split the Bessel function into Hankel functions of the first and second kind, respectively, a method which is frequently useful in evaluating integrals containing Bessel functions (eq.[16] in section 4.3).

We replace the integration variable k by a complex variable $\zeta = k + i\tau$ and carry out the contour integrations in the complex ζ-plane, as shown in Fig.94. In this discussion, which closely follows EWING, JARDETZKY and PRESS (1957), we assume that ω is real and that ζ is complex. This is compatible if we allow for a complex wave velocity c. On the other hand, we can regard $\zeta = k + i\tau$ as a mere substitution.

Poles. The integrands become infinite when the denominator in [10]–[12] becomes zero, i.e., when:

$$\nu_1 \cosh\nu_1 H + \delta_1\nu_2 \sinh\nu_1 H = 0 \qquad [15]$$

where $\nu_j = \pm\sqrt{(\zeta^2 - k_{\alpha_j}^2)}$ and $k_{\alpha_j} = \dfrac{\omega}{\alpha_j}$.

There is one pole P on the real axis, $k = \varkappa_n$, $\varkappa_n$ being a solution of eq.[15], which is equivalent to the following equation, considering [13]:

$$\bar{\nu}_1 \cos\bar{\nu}_1 H + i\delta_1\bar{\nu}_2 \sin\bar{\nu}_1 H = 0 \qquad [16]$$

where $\bar{\nu}_1 = -i\nu_1$ and $\bar{\nu}_2 = -i\nu_2$. Note that $\nu_1 = 0$ is *not* a singular point of the integrands in [10]–[12]. This can be seen by evaluating the integrands when passing to the limit $\nu_1 = 0$.

Branch points. Inspecting the integrands [10]–[12] we find that the radical:

$$\nu_1 = \pm\sqrt{(\zeta^2 - k_{\alpha_1}^2)}$$

will give the same value of $F(\nu_1, \nu_2)$, whether it is positive or negative. Therefore, the integrals along the corresponding branch cut (in opposite directions above and below the cut) will cancel each other. However, the other radical:

$$\nu_2 = \pm\sqrt{(\zeta^2 - k_{\alpha_2}^2)}$$

will give different values for $F(\nu_1, \nu_2)$, depending upon whether it is positive or negative. This means that the only branch point is A, where $k = k_{\alpha_2}$ on the real axis. Thus, the branch point k_{α_2} is determined only by the lower layer (half-space). It can be shown that this is generally true for any number of overlying layers. The corresponding *branch cut*

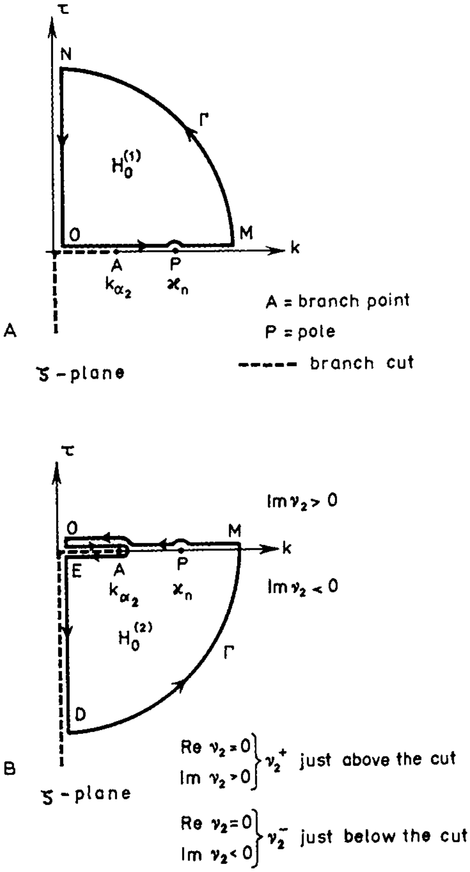


Fig. 94.

is given by the condition $\mathrm{Re}\,\nu_2 = 0$, and it runs from A to the origin O and from there to $-i\infty$. By eq.[9] we have to request that $\mathrm{Re}\,\nu_2 \geqslant 0$, in order to have φ_2 decrease with increasing z. On the cut we have $\mathrm{Re}\,\nu_2 = 0$, i.e., ν_2 is pure imaginary, say $\nu_2 = iB$, where B is real. Then we get:

$$\nu_2^2 = \zeta^2 - k_{\alpha_2}^2 = k^2 - \tau^2 + 2ik\tau - k_{\alpha_2}^2 = -B^2$$

i.e., real and negative. Separating real and imaginary parts we get: (*1*) $k\tau = 0$, which defines the cut as lying along the axes in the ζ-plane; (*2*) $k^2 - \tau^2 < k_{\alpha_2}^2$, which defines the parts of the cut we can use.

Riemann surface. In the integrands [10]–[12] there are two radicals, ν_1 and ν_2, and therefore we should expect to have a four-leaved Riemann surface. But because of the property of $F(\nu_1, \nu_2)$, explained above, there is only one branch cut to be considered, and then the Riemann surface is only two-leaved, as shown in Fig.94. In the complete circuit we have to imagine the two figures combined into one figure, such that, for example, we perform the counter-clockwise circulation first in the upper figure (starting from O

and returning to O); then we continue in the lower figure from O along the contour which is situated immediately above the cut along the real axis, following the arrows, and finally returning to O again. As $H_0^{(2)}(x)$ is the same as $H_0^{(1)}(x)$ but with $-i$ instead of i, it is natural to perform the first integral in [14] in the region of $+i$ and the latter in the region of $-i$.

Applying *Cauchy's integration theorem* to the first circuit, we get:

$$\int_0^{\infty} G_1 \mathrm{d}\zeta + \underbrace{\int_{\Gamma} G_1 \mathrm{d}\zeta}_{=0} + \int_N^{O} G_1 \mathrm{d}\zeta = 0$$

(as no poles are enclosed), that is:

$$\int_0^{\infty} G_1 \mathrm{d}\zeta = \int_O^{N} G_1 \mathrm{d}\zeta$$

For the second circuit we find:

$$\int_{OAE} G_2 \mathrm{d}\zeta + \int_E^{D} G_2 \mathrm{d}\zeta + \underbrace{\int_{\Gamma} G_2 \mathrm{d}\zeta}_{=0} + \int_{\infty}^{0} G_2 \mathrm{d}\zeta = 2\pi i \sum \operatorname{Res} G_2$$

(there is one pole P enclosed in this circuit), that is:

$$\int_0^{\infty} G_2 \mathrm{d}\zeta = \int_{OAE} G_2 \mathrm{d}\zeta + \int_E^{D} G_2 \mathrm{d}\zeta - 2\pi i \sum \operatorname{Res} G_2$$

The integrals along the infinite arcs Γ vanish, as is seen by inspection of how the integrands in [10]–[12] behave for ζ large. Inserting into [14], we find:

$$\varphi = \int_O^{N} G_1 \mathrm{d}\zeta + \int_E^{D} G_2 \mathrm{d}\zeta + \int_{OAE} G_2 \mathrm{d}\zeta - 2\pi i \sum \operatorname{Res} G_2 \qquad [17]$$

Using the following property of Hankel functions (eq.[16] in section 5.3):

$$H_0^{(1)}(\zeta r) = -H_0^{(2)}(-\zeta r) \qquad [18]$$

we can rewrite the first integral on the right-hand side of [17]:

$$\int_O^{N} G_1 \mathrm{d}\zeta = \int_0^{\infty} H_0^{(1)}(i\tau r) F(\nu_1, \nu_2^+) i\tau \cdot i\mathrm{d}\tau = -\int_0^{-\infty} H_0^{(2)}(i\tau r) F(\nu_1, \nu_2^+) i\tau \cdot i\mathrm{d}\tau$$

because on the imaginary axis ON we have $\zeta = i\tau$ and $\nu_2 = \nu_2^+$, and:

$$\int_0^{\infty} H_0^{(2)}(-x) x \mathrm{d}x = \int_0^{-\infty} H_0^{(2)}(x) x \mathrm{d}x$$

as seen by substituting $-x$ for x. Then, the first two integrals in [17] can be combined into one integral, and we get:

$$\varphi = \int_0^{-\infty} H_0^{(2)}(i\tau r)\,[F(\nu_1, \nu_2^-) - F(\nu_1, \nu_2^+)]\, i\tau \cdot i\mathrm{d}\tau$$

$$+ \int_{k_{\alpha_2}}^{0} H_0^{(2)}(kr)\,[F(\nu_1, \nu_2^-) - F(\nu_1, \nu_2^+)]k\mathrm{d}k\, -$$

(this is the branch-line integral OAE; $\zeta = k$)

$$-\,2\pi i \sum \mathrm{Res}\; G_2$$

$$= \int_L H_0^{(2)}(\zeta r)[F(\nu_1, \nu_2^-) - F(\nu_1, \nu_2^+)]\zeta \mathrm{d}\zeta\, -$$

(this is the total branch-line integral $OAED$)

$$-\,2\pi i \sum \mathrm{Res}\; G_2 \tag{19}$$

In order to evaluate Res G_2, we note that G_2 is of the following form:

$$G_2 = H_0^{(2)}(kr)\,\frac{M(k)}{N(k)} \tag{20}$$

(compare [14]), and the residue at a simple pole $k = \varkappa_n$ is given by:

$$\mathrm{Res}\; G_2 = H_0^{(2)}(\varkappa_n r)\,\frac{M(\varkappa_n)}{N'(\varkappa_n)} \quad \text{(eq.[10] in section 2.1)} \tag{21}$$

where (cf. [16]):

$$N(\varkappa_n) = \bar{\nu}_1 \cos\bar{\nu}_1 H + i\delta_1 \bar{\nu}_2 \sin\bar{\nu}_1 H = 0 \tag{22}$$

With the condition [22] we find:

$$\left.\begin{aligned} M(\varkappa_n) &= \frac{\sin\bar{\nu}_1 z \sin\bar{\nu}_1 h}{\sin\bar{\nu}_1 H}\,\varkappa_n \\ N'(\varkappa_n) &= \frac{\varkappa_n}{\bar{\nu}_1 \sin\bar{\nu}_1 H}\,[\bar{\nu}_1 H - \sin\bar{\nu}_1 H \cos\bar{\nu}_1 H - \delta_1^2 \sin^2\bar{\nu}_1 H \tan\bar{\nu}_1 H] \end{aligned}\right| \tag{23}$$

again reverting to [13]. Finally we find that the potential for the upper layer is:

$$\varphi_1' = \varphi_1'' = \int_L H_0^{(2)}(\zeta r)\,[F(\nu_1, \nu_2^-) - F(\nu_1, \nu_2^+)]\zeta \mathrm{d}\zeta$$

$$-2\pi i \sum_n \frac{H_0^{(2)}(\varkappa_n r)\bar{\nu}_1 \sin\bar{\nu}_1 h \sin\bar{\nu}_1 z}{\bar{\nu}_1 H - \sin\bar{\nu}_1 H \cos\bar{\nu}_1 H - \delta_1^2 \sin^2\bar{\nu}_1 H \tan\bar{\nu}_1 H} \tag{24}$$

the last summation being extended over all poles $\varkappa_n$. We note that this solution holds over the whole range $0 \leqslant z \leqslant H$.

Eq.[22], defining the poles, is the same as obtained when putting the determinant of the system of equations, obtained from the boundary conditions, equal to zero. It is the *period equation*, i.e., an equation relating velocity and period, thus defining the dispersion of the waves. In agreement with [10]–[12] we can also write the period equation as follows:

$$\nu_1 \cosh \nu_1 H + \delta_1 \nu_2 \sinh \nu_1 H = 0 \qquad [25]$$

Assume $\alpha_1 < \alpha_2$; then $k_{\alpha_2} < k_{\alpha_1}$. For $k > k_{\alpha_1}$ both ν_1 and ν_2 are real and positive, and there can be no root of [25]. The same happens for $0 < k < k_{\alpha_2}$. Therefore, all real roots $\varkappa_n$ of [25] must lie in the interval

$$k_{\alpha_2} \leqslant \varkappa_n \leqslant k_{\alpha_1} \qquad [26]$$

In addition, we have to convince ourselves that there are no complex roots of [25] in the first and fourth quadrants, as assumed in writing down [17]. If we write:

$$\left.\begin{aligned} H\nu_1 &= H\sqrt{(\zeta^2 - k_{\alpha_1}^2)} = p_1 + iq_1 \\ H\nu_2 &= H\sqrt{(\zeta^2 - k_{\alpha_2}^2)} = p_2 + iq_2 \end{aligned}\right| \qquad [27]$$

where $p_1 \geqslant 0$ and $p_2 \geqslant 0$ and $q_1 q_2 \geqslant 0$ (only first and fourth quadrants considered!); then [25] becomes:

$$\delta_1 \tanh(p_1 + iq_1) = -\frac{p_1 + iq_1}{p_2 + iq_2} \qquad [28]$$

Separating real and imaginary parts, we get from [28]:

$$\left.\begin{aligned} \frac{\delta_1 \tanh p_1 (1 + \tan^2 q_1)}{1 + \tanh^2 p_1 \tan^2 q_1} &= -\frac{p_1 p_2 + q_1 q_2}{p_2^2 + q_2^2} \\ \frac{\delta_1 \tan q_1 (1 - \tanh^2 p_1)}{1 + \tanh^2 p_1 \tan^2 q_1} &= \frac{p_1 q_2 - p_2 q_1}{p_2^2 + q_2^2} \end{aligned}\right| \qquad [29]$$

Eq.[29.1] cannot be fulfilled, as the left-hand side is positive and the right-hand side negative. The conclusion is that there are no poles on the Riemann surfaces used above in the integration.

Similarly, by putting $\zeta = -im$ in [25], we can show that there are no poles on the imaginary axis (that is, the equation obtained after this substitution has no real roots m).

13.1.3 Solution at large distances

Usually, branch-line integrals give contributions which vanish at large distances. The branch-line integral in [24] corresponds to the refraction arrival and diminishes as r^{-2} with distance (see EWING, JARDETZKY and PRESS, 1957, pp.93–105). Therefore, we shall restrict our attention to the residues, which, as we shall find, decrease much slower, and therefore dominate the solution at large distances.

We use the asymptotic expression of the Hankel function for large values of the argument (eq.[54] in section 5.3):

$$H_0^{(2)}(\varkappa_n r) \simeq \left(\frac{2}{\pi\varkappa_n r}\right)^{1/2} e^{i(\pi/4-\varkappa_n r)} \tag{30}$$

Furthermore, we write:

$$\begin{aligned} &\bar{\nu}_1 H = x \\ &\bar{\nu}_1 = -i\nu_1 = -i\sqrt{(k^2 - k_{\alpha_1}^2)} = -\sqrt{(k_{\alpha_1}^2 - k^2)} \\ &\bar{\nu}_1^{(n)} H = -H\sqrt{(k_{\alpha_1}^2 - \varkappa_n^2)} = x_n \end{aligned} \tag{31}$$

and:

$$V(x_n) = \frac{x_n}{x_n - \sin x_n \cos x_n - \delta_1^2 \sin^2 x_n \tan x_n} \tag{32}$$

and use the relation:

$$-i = e^{-i\pi/2} \tag{33}$$

and re-introduce the time factor $e^{i\omega t}$. Then the residue term in [24], i.e., the term including the sum on the right-hand side, becomes:

$$\varphi_{1R} = \frac{2}{H}\left(\frac{2\pi}{r}\right)^{1/2} \sum_n \frac{1}{\sqrt{\varkappa_n}} e^{i(\omega t - \varkappa r - \pi/4)} V(x_n) \sin\frac{x_n h}{H} \sin\frac{x_n z}{H} \qquad (0 \leqslant z \leqslant H) \tag{34}$$

Similar developments can be made for the second medium. It is seen from [34] that this term (from the residues) decreases slower, namely as $r^{-1/2}$, and therefore will be the dominant term at large r. Each term in [34] is called a *normal mode*, and there is one mode for each value of n. The further discussion of these results is relatively simple, involving no mathematical difficulties. The reader is therefore advised to read EWING, JARDETZKY and PRESS (1957, pp.138–141), just after this.

We can summarize the results as follows:

(*a*) Branch-line integrals correspond to refraction arrivals, which are generally unimportant at large distances.

(*b*) Residues correspond to the normal modes, which will dominate the motion at large distances.

Exceptions to this rule exist when there is constructive interference between refracted waves and those reflected from the free surface (in that case, branch-line integrals also become important at large distances; see EWING, JARDETZKY and PRESS, 1957, p.142).

13.1.4 Generalization to a pulse

In this section we have made some changes in the notation used by EWING, JARDETZKY and PRESS (1957, p.143), to make it more homogeneous with the rest of this book, especially Chapter 8, namely: $g(\omega) \to \bar{g}(\omega)$, $S(t) \to g(t)$.

Instead of harmonic wave motion, considered so far in this chapter, we generalize to an arbitrary disturbance with an initial time variation $g(t)$. We make use of the complex Fourier transform:

$$\bar{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt \quad \text{(eq.[4a] in section 8.1)} \qquad [35]$$

to obtain an expression for $g(t)$:

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{g}(\omega) e^{i\omega t} d\omega \quad \text{(eq.[24a] in section 8.1)} \qquad [36]$$

If we replace the harmonic motion by the time function $g(t)$, i.e.:

$$\underset{\text{(steady state)}}{e^{i\omega t}} \rightarrow \underset{\text{(pulse)}}{g(t)}$$

where $g(t)$ is given by [36], then eq.[34] becomes:

$$\varphi_{1R} = \frac{2}{H\sqrt{r}} \sum_n \int_{-\infty}^{\infty} \bar{g}(\omega) e^{i(\omega t - \varkappa_n r - \pi/4)} \frac{V(x_n)}{\sqrt{\varkappa_n}} \sin \frac{x_n h}{H} \sin \frac{x_n z}{H} d\omega \qquad [37]$$

$$(0 \leqslant z \leqslant H)$$

Assuming the initial disturbance to be an explosion, we can write $g(t)$ as follows:

$$g(t) = \begin{cases} e^{-\sigma t} & t > 0 \\ 0 & t < 0 \end{cases} \qquad [38]$$

with $\sigma > 0$. The corresponding complex Fourier transform is from [35]:

$$\bar{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(\sigma + i\omega)t} dt = \frac{1}{\sqrt{2\pi}(\sigma + i\omega)} \qquad [39]$$

where only $t > 0$ need be considered, because of [38]. Introducing [39] into [37], we find:

$$\varphi_{1R} = \frac{1}{H} \left(\frac{2}{\pi r}\right)^{1/2} \sum_n \int_{-\infty}^{\infty} \frac{e^{i(\omega t - \varkappa_n r - \pi/4)}}{\sqrt{\varkappa_n}(\sigma + i\omega)} V(x_n) \sin \frac{x_n h}{H} \sin \frac{x_n z}{H} d\omega \qquad [40]$$

Eq.[40] can be evaluated by the method of stationary phase (eq.[32] in section 3.2):

$$\int F(\zeta) e^{xf(\zeta)} d\zeta \simeq \frac{\sqrt{2\pi} F(\zeta_0) e^{xf(\zeta_0)}}{\sqrt{|xf''(\zeta_0)/i|}} e^{\pm i\pi/4}$$

Compared with [40] we find in our case:

$$xf(\zeta) \rightarrow i\left(\omega t - \varkappa_n r - \frac{\pi}{4}\right) = ir\left(\omega \frac{t}{r} - \varkappa_n - \frac{\pi}{4r}\right)$$

that is:

$$x \to ir$$

and:

$$f(\zeta) \to f(\omega) = \omega \frac{t}{r} - \varkappa_n - \frac{\pi}{4r}$$

Differentiating $f(\omega)$ once, we get:

$$f'(\omega) = \frac{t}{r} - \frac{d\varkappa_n}{d\omega} = 0$$

which defines the stationary phase, and differentiation twice gives:

$$f''(\omega) = -\frac{d^2\varkappa_n}{d\omega^2}$$

Introducing the group velocity U:

$$U = \frac{d\omega}{dk} = \frac{r}{t}$$

we get:

$$\frac{1}{U} = \frac{d\varkappa_n}{d\omega}$$

$$f''(\omega) = -\frac{d^2\varkappa_n}{d\omega^2} = -\frac{d}{d\omega}\left(\frac{1}{U}\right) = \frac{1}{U^2}\frac{dU}{d\omega}$$

Some more notations will be used:

$$\gamma = \frac{\omega H}{2\pi\alpha_1} \quad ; \quad c_n = \frac{\omega}{\varkappa_n} \quad ; \quad d\gamma = \frac{H}{2\pi\alpha_1} d\omega$$

With the formula for the stationary phase, we then find for large r:

$$\varphi_{1R} = \frac{2}{Hr}\sum_n U_n \left[\frac{c_n/\alpha_1}{\gamma\left|\dfrac{dU_n/\alpha_1}{d\gamma}\right|}\right]^{1/2} \frac{e^{i(\omega_0 t - \varkappa_n r - \pi/4 \pm \pi/4)}}{\sigma + i\omega_0} V(x_n)\sin\frac{x_n h}{H}\sin\frac{x_n z}{H} \qquad [41]$$

$$(0 \leqslant z \leqslant H)$$

Written in this way, the formula [41] agrees with the formula (4-93) of EWING, JARDETZKY and PRESS (1957, p.144).

It should be noticed that in [41] the potential decreases as r^{-1}, as distinct from $r^{-1/2}$ in [34]. Both formulas are valid for large r, but [34] is valid for simple harmonic motion (steady state) and [41] for a pulse. In the latter case the decrease with distance is more rapid, which is ascribed to dispersion. Note, that we have got the displacement *potentials* as functions of r, and from these expressions we can calculate the distance variation of other quantities (such as of the displacements themselves).

The development here (especially the application of the method of stationary phase) breaks down in the neighbourhood of maximum or minimum group velocity. In this case (Airy phase), a special study is needed (see section 3.3).

As a concluding remark to this section and the preceding chapter, I should like to emphasize the paramount importance that the contour integration method has in the solution of integrals encountered in wave propagation problems. This method often requires much skill and ingenuity in finding suitable integration paths and great care in evaluating the various contributions. Often, it is as difficult as it is important. The integrals which have been evaluated in this and the preceding chapter led to similar procedures, but partly for matter of illustration, we have chosen slightly different methods. In order to get further experience in these methods, it is suggested that the student work through other examples which may be found in seismological textbooks and journals.

13.2 WAVE PROPAGATION IN A LIQUID HALF-SPACE WITH VELOCITY VARYING WITH DEPTH

This section gives an example of the use of modified Bessel functions (see section 5.3). It is based on PEKERIS (1946) and EWING, JARDETZKY and PRESS (1957, pp.330–334).

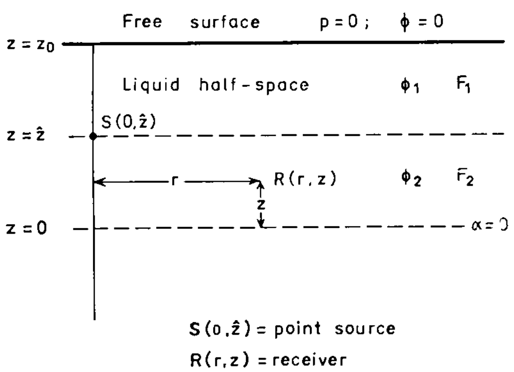


Fig.95.

Let us consider the situation illustrated in Fig.95. We assume that the sound velocity α in the liquid half-space varies with depth z, e.g., according to the law:

$$\alpha = az \qquad [1]$$

Eq.[1] means that we consider only the layer $z_0 \geqslant z \geqslant 0$. We also assume that the density variation with depth is small enough to permit us to write the wave equation in its simple form:

$$\nabla^2\varphi = \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} \qquad [2]$$

For the displacement potential φ we assume the following expression:

$$\varphi = A(k)J_0(kr)F(z) \qquad [3]$$

omitting the time factor $e^{i\omega t}$ (continuous harmonic oscillations). As the velocity varies with depth, the wave front will *not* be spherical, and we have to use the more general form for the potential φ, given in [3]. The functions $Q(k)$ in section 13.1.1 are here represented by $A(k)$. In section 13.1.1, we assumed from the start an exponential decrease of the potential with depth (or with the vertical distance from the source); here, we use instead the general function $F(z)$, which has to be determined such that [2] and the boundary conditions are fulfilled. The *exponential* decrease of φ with depth is true in media with constant properties (e.g., compare the treatment of surface waves, BULLEN, 1963, pp. 85–90). In this case, however, we have a medium in which properties vary continuously with depth. Then the rays are curved, and the wave fronts are not spherical. Their exact shape depends on the velocity–depth function. But, anyway, the depth variation of φ will be influenced.

We can write [2] also in the following form:

$$\nabla^2\varphi + k_\alpha^2\varphi = 0 \qquad [4]$$

as:

$$\varphi \sim e^{i\omega t} \quad ; \quad \frac{\partial\varphi}{\partial t} \sim i\omega\, e^{i\omega t} \quad ;$$

$$\frac{\partial^2\varphi}{\partial t^2} \sim i^2\omega^2 e^{i\omega t} = -\omega^2\varphi \quad ; \quad k_\alpha = \frac{\omega}{a}$$

We insert [3] into [4], using the expression for the Laplace operator in cylindrical coordinates, with symmetry around the z-axis (eq.[7] in section 1.2). Also considering the Bessel differential equation:

$$J_0''(x) + \frac{1}{x}J_0'(x) + J_0(x) = 0 \qquad [5]$$

we then find that $F(z)$ must fulfill the following differential equation:

$$\frac{d^2F}{dz^2} + F\left(\frac{\omega^2}{a^2z^2} - k^2\right) = 0 \qquad [6]$$

where we have put:

$$k_\alpha^2 = \frac{\omega^2}{a^2} = \frac{\omega^2}{a^2z^2} \qquad [7]$$

considering [1]. Note the difference between k_α and k. The former, k_α, is a constant at each level (for given ω, the frequency of the source). The latter, k, is to be considered as a variable, and in fact, [3] must be integrated over all values of k.

In [6] we substitute:

$$F(z) = z^{1/2}G(z) \tag{8}$$

by which [6] changes into:

$$\frac{d^2G}{dz^2} + \frac{1}{z}\frac{dG}{dz} + \left[\left(\frac{\omega^2}{a^2} - \frac{1}{4}\right)\frac{1}{z^2} - k^2\right] G = 0 \tag{9}$$

which is found immediately. Then we make the following substitution in [9]:

$$z = \frac{i\chi}{k} \tag{10}$$

i.e., we take $i\chi$ as the new variable instead of z. We also write:

$$n^2 = \frac{\omega^2}{a^2} - \frac{1}{4} \simeq \frac{\omega^2}{a^2} \tag{11}$$

Then [9] transforms into:

$$\frac{d^2G}{d(i\chi)^2} + \frac{1}{i\chi}\,\frac{dG}{d(i\chi)} - \left[1 + \frac{(in)^2}{(i\chi)^2}\right] G = 0 \tag{12}$$

which is a differential equation with Bessel functions of *order in* and *argument* $i\chi$ as solution:

$$G = A_1 I_{in}(i\chi) + B_1 K_{in}(i\chi) \tag{13}$$

Here I and K are the modified Bessel functions (section 5.3), and [8] becomes immediately:

$$F(z) = A_1 z^{1/2} I_{in}(kz) + B_1 z^{1/2} K_{in}(kz) \tag{14}$$

Here $i\chi$ in [13] has been replaced by kz from [10].

The purpose of the substitutions [8] and [10] is to transform [6] into a form, for which we know the solution, in this case a Bessel differential equation. This is a very usual procedure in order to find the solution of a differential equation.

We require that the potential φ and thus also the function $F(z)$ should decrease as we go away from the source S (see Fig.95). Above the source, z increases upward. We have:

$$\lim_{z \to \infty} K_{in}(kz) = 0 \tag{15}$$

and therefore K_{in} corresponds to a wave propagating upward. Eq.[15] is obtained from the asymptotic expression for K_{in} for large values of the argument (eq.[57] in section 5.3).

Below the source, z decreases (downward propagation) and then

$$\lim_{z \to 0} I_{in}(kz) = 0 \qquad [16]$$

Eq.[16] is found from the definition of I_{in} in usual Bessel functions (eq.[33] in section 5.3), but [16] holds only for $n > 1$. That $n > 1$ here is obvious from [11].

For the medium *below* the source, the solution [14] becomes:

$$F_2(z) = C_2 z^{1/2} I_{in}(kz) \quad \text{(condition 1)} \qquad [17]$$

In the lower medium ($\hat{z} \geqslant z \geqslant 0$) we have *only* a downgoing wave. There is no upgoing wave in this region, and there is no "reflection" from the layer $z = 0$ (where the velocity vanishes), as it would take an infinite time for a wave to reach that level.

On the other hand, in the upper layer ($z_0 \geqslant z \geqslant \hat{z}$) we have both upgoing (direct) and downgoing (reflected) waves, the former represented by $K_{in}(kz)$, the latter by $I_{in}(kz)$. So for F_1 we have to use the complete solution [14]:

$$F_1(z) = A_1 z^{1/2} I_{in}(kz) + B_1 z^{1/2} K_{in}(kz) \qquad [18]$$

At the free surface, $z = z_0$, the potential φ and therefore F vanish:

$$A_1 I_{in}(kz_0) + B_1 K_{in}(kz_0) = 0 \quad \text{(condition 2)} \qquad [19]$$

This defines A_1 expressed in B_1, and introducing this into [18], we find:

$$F_1(z) = B_1 z^{1/2} \left[K_{in}(kz) - I_{in}(kz) \frac{K_{in}(kz_0)}{I_{in}(kz_0)} \right] \qquad [20]$$

In the plane of the source S ($z = \hat{z}$) there must be continuity of the velocity components, except at the source itself. At S, the vertical velocity is in opposite directions above and below the source. Therefore, we put:

$$F_1 = F_2$$

and:

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_2}{\partial z} = -Dk \text{ at } z = \hat{z} \quad \text{(condition 3)}$$

$$(D \text{ constant}) \qquad [21]$$

The discontinuity in the vertical velocity at S becomes:

$$\frac{\partial \varphi_1}{\partial z} - \frac{\partial \varphi_2}{\partial z} = A(k) J_0(kr) \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_2}{\partial z} \right) \sim \int_0^\infty J_0(kr) k \mathrm{d}k \qquad [22]$$

$A(k)$ has vanished as the boundary conditions imply that $A(k)$ is a constant multiplying factor, depending on the source strength. [22] vanishes everywhere except at the source ($r = 0$), where it becomes infinite in such a way that its integral over the plane is finite. Therefore, [21] is in agreement with the conditions at the source.

Integrals of the type [22] are found in section 5.2.6, from which it follows that the integral vanishes (put $p = 0$, $n = 0$ in eq.[60] in section 5.2). That the integral over the plane is finite can be seen using the Fourier-Bessel transform (see EWING, JARDETZKY and PRESS, 1957, p.130).

The conditions [21] lead to the two following equations, using the expressions [20] and [17] for F_1 and F_2:

$$\left.\begin{aligned} B_1 \left[K_{in}(k\hat{z}) - I_{in}(k\hat{z}) \frac{K_{in}(kz_0)}{I_{in}(kz_0)}\right] &= C_2 I_{in}(k\hat{z}) \\ B_1 \hat{z}^{1/2} \left[K'_{in}(k\hat{z}) - I'_{in}(k\hat{z}) \frac{K_{in}(kz_0)}{I_{in}(kz_0)}\right] &= -D + C_2 \hat{z}^{1/2} I'_{in}(k\hat{z}) \end{aligned}\right\} \tag{23}$$

The latter equation in [23] is obtained if use is made of the first equation [23]. Also remember that $K'_{in}(k\hat{z})$, for instance, means the derivative with respect to kz, after which we put $z = \hat{z}$.

We solve [23] for B_1 and C_2 and obtain the following expressions in D:

$$\left.\begin{aligned} B_1 &= D\hat{z}^{-1/2} \left[\frac{I'_{in}(k\hat{z})}{I_{in}(k\hat{z})} K_{in}(k\hat{z}) - K'_{in}(k\hat{z})\right]^{-1} \\ C_2 &= D\hat{z}^{-1/2} \left[\frac{I'_{in}(k\hat{z})}{I_{in}(k\hat{z})} K_{in}(k\hat{z}) - K'_{in}(k\hat{z})\right]^{-1} \left[\frac{K_{in}(k\hat{z})}{I_{in}(k\hat{z})} - \frac{K_{in}(kz_0)}{I_{in}(kz_0)}\right] \end{aligned}\right\} \tag{24}$$

Then we make use of the following relation for the modified Bessel functions (see eq.[43] in section 5.3):

$$I'_n(x)K_n(x) - I_n(x)K'_n(x) = \frac{1}{x} \tag{25}$$

Using the series expansions of $I_n(x)$ and $K_n(x)$ (eq.[58] and [59] in section 5.3) we can show that [25] holds for any x. This is important below when we integrate over x from 0 to ∞. By means of [25] we can immediately rewrite [24] in the following form:

$$\left.\begin{aligned} B_1 &= Dk\hat{z}^{1/2} I_{in}(k\hat{z}) \\ C_2 &= Dk\hat{z}^{1/2} \left[K_{in}(k\hat{z}) - \frac{K_{in}(kz_0)}{I_{in}(kz_0)} I_{in}(k\hat{z})\right] \end{aligned}\right\} \tag{26}$$

Then we can write down the expressions for the potentials above the source (φ_1) and below the source (φ_2), using [3], [17], [20]:

$$\left.\begin{aligned} \varphi_1 &= \int_0^\infty J_0(kr) B_1(k) z^{1/2} \left[K_{in}(kz) - I_{in}(kz) \frac{K_{in}(kz_0)}{I_{in}(kz_0)}\right] \mathrm{d}k \\ \varphi_2 &= \int_0^\infty J_0(kr) C_2(k) z^{1/2} I_{in}(kz) \mathrm{d}k \end{aligned}\right\} \tag{27}$$

$B_1(k)$ and $C_2(k)$ are given by [26]. Obviously, the factor $A(k)$ in [3] is reduced to the constant D, which is a measure of the source strength. Both B_1 and C_2 are proportional to D.

In case we remove the free surface, i.e., we consider the whole space, then the expressions simplify. For $z_0 \to \infty$ we use the asymptotic expressions for $I_{in}(kz_0)$ and $K_{in}(kz_0)$, eq.[56] and [57] in section 5.3. Then we find:

$$\lim_{z_0 \to \infty} \left[\frac{K_{in}(kz_0)}{I_{in}(kz_0)} \right] = 0 \tag{28}$$

and the two potentials become:

$$\left.\begin{aligned} \varphi_1 &= D\hat{z}^{1/2} z^{1/2} \int_0^\infty J_0(kr) I_{in}(k\hat{z}) K_{in}(kz) k \mathrm{d}k \qquad (z \geqslant \hat{z}) \\ \varphi_2 &= D\hat{z}^{1/2} z^{1/2} \int_0^\infty J_0(kr) I_{in}(kz) K_{in}(k\hat{z}) k \mathrm{d}k \qquad (0 \leqslant z \leqslant \hat{z}) \end{aligned}\right| \tag{29}$$

As seen from [29] the two potentials are built up in an analogous way in this case. At $z = \hat{z}$ they are identical.

The next step is to evaluate the integral expressions for the potentials by contour integrations in the complex k-plane. For this development, the reader is referred to PEKERIS (1946).

Chapter 14

INFLUENCE OF GRAVITY ON WAVE PROPAGATION

14.1 MATHEMATICAL INTRODUCTION

We shall first prove a mathematical relation which we need in the next section (JEFFREYS and JEFFREYS, 1946, pp.359–365):

$$F(p)H(t) = \frac{1}{2\pi i} \int_L F(z) e^{tz} \frac{dz}{z} \qquad [1]$$

where $p = d/dt$ and $H(t)$ is *Heaviside's unit function* ($= 0$ for $t \leqslant 0$; $= 1$ for $t > 0$). $F(p)$ is an operator carried out on $H(t)$. The integral on the right-hand side extends along the imaginary axis from $-i\infty$ to $+i\infty$ on the positive side of this axis. This integral is called *Bromwich's integral.*

Consider the integral:

$$\int_C \frac{e^{zt}}{z^{n+1}} dz \qquad [2]$$

taken around the contour C in the (complex) z-plane in positive direction (see Fig.96). n is a positive integer and t is independent of z. Expand the exponential function:

$$\int_C \frac{e^{zt}}{z^{n+1}} dz = \int_C \frac{1 + zt + \frac{(zt)^2}{2!} + \frac{(zt)^3}{3!} + \ldots + \frac{(zt)^n}{n!} + \ldots}{z^{n+1}} dz \qquad [3]$$

from which we see that the coefficient of $1/z$ is $t^n/n!$. This is then the residue Res of the

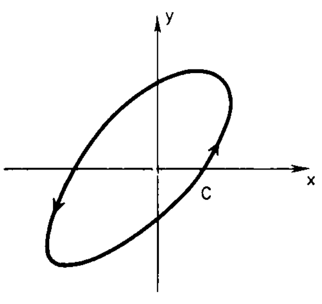


Fig.96.

integrand at the origin, $z = 0$, the only pole within C, and from the contour integration theorem (eq.[17] in section 2.1) we have:

$$\int_C \frac{e^{zt}}{z^{n+1}} dz = 2\pi i \sum \text{Res} = 2\pi i \frac{t^n}{n!} \qquad [4]$$

This formula allows an immediate generalization:

$$\frac{1}{2\pi i} \int_C \frac{e^{zt}}{z} \left(a_0 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n} \right) dz = a_0 + a_1 t + \ldots + \frac{a_n t^n}{n!} \qquad [5]$$

Define a function $F(z)$ such that:

$$F(z) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{z^n} \qquad [6]$$

which is the same as the parenthesis in the integral [5], except that n now extends to infinity. We assume $F(z)$ to be convergent for $|z| \geqslant r_1$. Then in the integral:

$$\frac{1}{2\pi i} \int_C \frac{e^{zt}}{z} F(z) dz \qquad [7]$$

corresponding to the left-hand side of [5], the integrand is a uniformly convergent series (Jeffreys and Jeffreys, 1946, pp.32–33, 324–325) and integration can be performed term by term (C is now a contour in the region where $|z| > r_1$). Hence, by immediate extension of [5]:

$$\frac{1}{2\pi i} \int_C \frac{e^{zt}}{z} F(z) dz = a_0 + \sum_{n=1}^{\infty} \frac{a_n t^n}{n!} \qquad [8]$$

The right-hand side of [8] can now be expressed in another form, using *operational calculus* (Jeffreys and Jeffreys, 1946, pp.204–205, 212–213). Define the operator Q as an *integral* over t:

$$Q f(t) = \int_0^t f(\xi) d\xi \qquad [9]$$

Putting $f(t) = 1$, we then immediately get:

$$\left.\begin{aligned} Q1 &= \int_0^t d\xi = t \\ Q^2 1 &= QQ1 = Qt = \int_0^t \xi d\xi = \frac{t^2}{2} \\ Q^3 1 &= \frac{t^3}{3!} \\ Q^n 1 &= \frac{t^n}{n!} \end{aligned}\right| \qquad [10]$$

The inverse operator is p, meaning differentiation with respect to t:

$$\left.\begin{aligned} p &= \frac{\mathrm{d}}{\mathrm{d}t} \\ Q &= p^{-1} \\ Q^n &= p^{-n} \\ Q^n 1 &= p^{-n} 1 = \frac{t^n}{n!} \end{aligned}\right| \qquad [11]$$

Then we have:

$$F(p)1 = \left(a_0 + \sum_{n=1}^{\infty} a_n p^{-n}\right) 1 = a_0 + \sum_{n=1}^{\infty} \frac{a_n t^n}{n!} \qquad [12]$$

(by the definition of F, eq.[6]) (by [11])

By combination of [12] and [8] we thus get:

$$F(p)1 = \frac{1}{2\pi i} \int_C \frac{\mathrm{e}^{zt}}{z} F(z)\mathrm{d}z \qquad [13]$$

We shall now demonstrate that the contour integral can be transformed into a line integral, using *Jordan's lemma*:

$$\int_C \frac{\mathrm{e}^{zt}}{z} F(z)\mathrm{d}z = \int_L \frac{\mathrm{e}^{zt}}{z} F(z)\mathrm{d}z \qquad [14]$$

where the path of integration is along a line L from $-i\infty$ to $+i\infty$ (on the positive side of the imaginary axis). Put:

$$\frac{F(z)}{z} = \varphi(z) \qquad [15]$$

and require $\varphi(z)$ to fulfill the following conditions (these are fulfilled by our form of this function $= F(z)/z$):

$$\left.\begin{aligned} &|z| \geqslant r_1 \qquad \mathrm{Re} z = x \leqslant c \qquad (\mathrm{Re} = \text{real part of}) \\ &|\varphi(z)| < \omega \qquad c > 0 \end{aligned}\right| \qquad [16]$$

for any arbitrarily chosen value of ω. We consider the case when $t > 0$. The contour C, equal to $ABCDA$, is shown in Fig.97 and is chosen in such a way that [16] is fulfilled. Integration then gives:

$$\left| \int_{BC} \mathrm{e}^{tz}\varphi(z)\mathrm{d}z \right| = \left| \int_c^{-X} \mathrm{e}^{t(x+iY)}\varphi(z)\mathrm{d}x \right| < \omega \left| \int_c^{-X} \mathrm{e}^{tx}\mathrm{d}x \right| = \frac{\omega}{t}(\mathrm{e}^{tc} - \mathrm{e}^{-tX}) \qquad [17]$$

because $|\mathrm{e}^{itY}| = 1$. Similarly, we get:

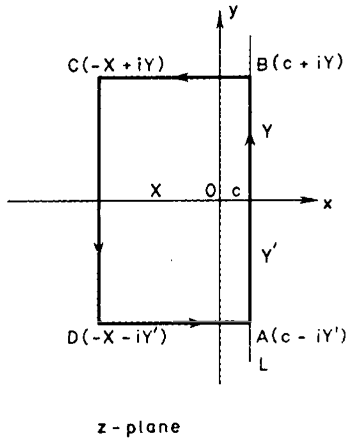


Fig.97.

$$\left| \int_{DA} e^{tz}\varphi(z)dz \right| < \frac{\omega}{t}(e^{tc} - e^{-tX}) \tag{18}$$

and:

$$\left| \int_{CD} e^{tz}\varphi(z)dz \right| = \left| \int_{Y}^{-Y'} e^{t(-X+iy)}\varphi(z)idy \right| <$$

$$< \frac{\omega}{t} e^{-tX} \underbrace{|e^{itY} - e^{-itY'}|} < \frac{2\omega}{t} e^{-tX} \tag{19}$$

(the bracket is maximum = 2 when both terms add with their maximum values). The total integral over the path $BCDA$ is thus [17] + [18] + [19], adding up the absolute values only:

$$\int_{BC} + \int_{CD} + \int_{DA} e^{tz}\varphi(z)dz < \frac{2\omega}{t} e^{ct} < \varepsilon \tag{20}$$

The latter inequality is true if ε has been chosen such that:

$$\omega < \frac{1}{2} t\varepsilon e^{-ct} \tag{21}$$

which is always possible because ω is completely arbitrary. On the other hand, the integral along AB does not vanish, as this path lies at a short distance from the origin, where the condition [16] is not fulfilled. Therefore, only the integral along L remains. The path L can be made infinitely long because of our conditions [16]. Thus, eq.[14] is proved. For $t < 0$, we could proceed in the same way, but then we have to place the rectangle $ABCD$ to the right of L. Replacing 1 in [13] by $H(t)$, we get:

$$F(p)H(t) = \frac{1}{2\pi i} \int_{L} \frac{e^{zt}}{z} F(z)dz \tag{22}$$

and this is the formula [1], which we set out to prove; cf. section 8.2.3.

In particular, put $F = 1$ in [22], and we get an expression for the Heaviside unit function:

$$H(t) = \frac{1}{2\pi i}\int_L \frac{e^{i\omega t}}{\omega}\,d\omega \tag{23}$$

where we have also replaced z by $i\omega$ to get an expression more similar to those sometimes used in seismology and where L extends parallel to the *real* axis in the ω-plane.

An explosive source can be written as follows (cf. eq.[38] in section 13.1):

$$g(t) = e^{-\sigma t} H(t) \qquad (\sigma \geqslant 0) \tag{24}$$

where $H(t)$ is the Heaviside unit function. Using the expression [23] for $H(t)$, we can write this as:

$$g(t) = \frac{1}{2\pi i}\int_L \frac{e^{i\omega t}}{\omega - i\sigma}\,d\omega = e^{-\sigma t}H(t) \tag{25}$$

We can easily see this as follows. The contour L runs from $-\infty \pm i\omega_y$ to $\infty \pm i\omega_y$ in the complex ω-plane ($\omega = \omega_x + i\omega_y$). We require the integral in [25] to vanish for large ω. Because we have:

$$e^{i\omega t} = e^{i(\omega_x + i\omega_y)t} = e^{i\omega_x t}e^{-\omega_y t}$$

this requirement is fulfilled if $\omega_y t > 0$. With regard to time t we distinguish two cases:

(*1*) $t < 0$. Then we have $\omega_y < 0$ in order to make $\omega_y t > 0$. In this case the contour L is equivalent to the infinite semicircle in the lower half-plane (see Fig.66). As this integral is zero, the integral along L is also zero.

(*2*) $t > 0$. In order to have $\omega_y t > 0$, we must have $\omega_y > 0$. This means that we now make a contour in the *upper* half-plane. There will be no contribution from the infinite semicircle, but there is a contribution from the pole $\omega = i\sigma$ located on the imaginary axis. Evaluating the residue from this pole, we find eq.[25].

In any wave propagation problem, with an initial pulse $g(t)$, we can generally write the solution corresponding to any wave, whether direct, reflected or refracted, in the following form (cylindrical coordinates supposed):

$$\frac{1}{2\pi i}\int_L \frac{f(\omega, r, z)}{\omega - i\sigma}e^{i\omega t}d\omega \tag{26}$$

where $f(\omega, r, z)e^{i\omega t}$ represents the steady-state solution. This procedure is frequently applied in seismological problems where a *pulse* source is involved (see EWING, JARDETZKY and PRESS, 1957, p.97).

14.2 BODY WAVES

In this discussion we shall essentially follow JEFFREYS (1931).

14.2.1 Deduction of the differential equation

Hitherto, we have neglected external forces in our treatment of the wave equation. In this section we shall consider the effect of gravity. The wave equation can then be written as follows with the usual notation:

$$\varrho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial \theta}{\partial x_i} + \mu \nabla^2 u_i + \varrho X_i \qquad [1]$$

where:

$$\theta = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \operatorname{div}\mathbf{u}$$

is the cubical dilatation, X_i = body force per unit mass, and $i = 1, 2, 3$. Carrying out the operation $\partial/\partial x_i$ on eq.[1] and remembering that $\partial^2/\partial x_i^2 = \nabla^2$, we get:

$$\varrho \frac{\partial^2 \theta}{\partial t^2} = (\lambda + \mu)\nabla^2\theta + \mu\nabla^2\theta + \varrho \frac{\partial X_i}{\partial x_i} = (\lambda + 2\mu)\nabla^2\theta + \varrho \frac{\partial X_i}{\partial x_i} \qquad [2]$$

The Newtonian gravitational potential V fulfills *Poisson's equation*:

$$\nabla^2 V + 4\pi\varrho = 0 \qquad [3]$$

where $V = \iiint \mathrm{d}m/r$. Furthermore, if f is the gravitational constant:

$$X_i = f \frac{\partial V}{\partial x_i} \qquad [4]$$

In the case of gravity effects on elastic waves we have to consider only *variations* of gravity within the field occupied by the elastic waves. In other words, a field of gravity which is constant throughout the whole field occupied by the elastic waves, has no effect on these. For media at rest, this gravity is compensated by supporting forces. This is clear from the derivation of [1]. The only thing of significance is when some (internal) gravitational force is introduced into the field of the elastic waves due to variations of density. Then, it is clear that no such effect will arise for *S*-waves, but only for *P*-waves (which are accompanied by dilatation and compression of the medium). We denote the disturbance of density (= deviation from its normal value) ϱ' and introduce this value in [3]. Moreover, considering the mass of a given unit volume (in the initial state, before the disturbance), we have by the principle of conservation of mass:

$$M = \varrho \cdot 1 = (\varrho + \varrho')(1 + \theta) \qquad [5]$$

and as we can neglect ϱ' compared to ϱ, we have:

$$\varrho\theta = -\varrho' \qquad [6]$$

We then get:

$$\frac{\partial X_i}{\partial x_i} = f \frac{\partial^2 V}{\partial x_i^2} = f\nabla^2 V = -4\pi f\varrho' = 4\pi f\varrho\theta \qquad [7]$$

(eq.[4]) (eq.[3]) (eq.[6])

Inserted into [2] this gives:

$$\frac{\partial^2\theta}{\partial t^2} = \frac{\lambda + 2\mu}{\varrho}\nabla^2\theta + 4\pi f\varrho\theta \qquad [8]$$

Put:

$$\frac{\lambda + 2\mu}{\varrho} = \alpha^2$$

(α = P-wave velocity) [9]

$$4\pi f\varrho = \omega^2$$

(ω = constant)

Then [8] becomes:

$$\frac{\partial^2\theta}{\partial t^2} = \alpha^2\nabla^2\theta + \omega^2\theta \qquad [10]$$

14.2.2 Solution of the differential equation in terms of cubical dilatation

We shall now restrict ourselves to considering R as the only space coordinate, i.e., R and t are the two independent variables in [10]. This will correspond to the following physical picture:

$t \leqslant 0$: no displacement for $R \geqslant a$;

$t > 0$: the sphere $R = a$ is acted upon from inside by a symmetric pressure = $AH(t)$, where $H(t)$ is the Heaviside unit function.

We use operational calculus, putting $\partial/\partial t = p$. We also write:

$$p^2 - \omega^2 = \alpha^2 q^2 \qquad [11]$$

which is taken as a definition of q. A solution of [10] is the following:

$$\theta R = C(t)e^{-qR} \qquad [12]$$

This is seen as follows; we start from:

$\frac{\partial^2\theta}{\partial t^2} - \omega^2\theta = \alpha^2\nabla^2\theta$ (which is eq.[10])

$(p^2 - \omega^2)\theta = \alpha^2\nabla^2\theta$ (using the definition of p)

$\alpha^2 q^2\theta = \alpha^2\nabla^2\theta$ (using eq.[11])

$\nabla^2\theta = q^2\theta$ (after division by α^2)

$\nabla^2\theta = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\theta}{\partial R}\right) = q^2\theta$ (considering that R is our only space coordinate; eq.[7] in section 1.2)

Put:

$$R^2 \frac{d\theta}{dR} = R \frac{d(\theta R)}{dR} - \theta R$$ (as we want to introduce θR as dependent variable)

then:

$$\frac{1}{R} \frac{d}{dR} \left[R \frac{d(\theta R)}{dR} - \theta R \right] = q^2 \theta R$$ (after introducing the last expression in the previous equation and multiplying with R)

$$\frac{d^2(\theta R)}{dR^2} = q^2 \theta R$$ (after carrying out the operation d/dR)

This equation has the following solution (immediately verified by substitution):

$$\theta R = C e^{-qR}$$

i.e., eq.[12]. The integration constant C is independent of R but depends on time t:

$$\theta R = C(t) e^{-qR} \qquad [12]$$

As q is related only to a *time* operator p, by eq.[11], we are justified to consider q as a constant in making this solution. We are considering only the space coordinate R here.

14.2.3 Solution in terms of displacement

We can express the cubical dilatation in spherical coordinates, with R as the only variable (spherical symmetry), by means of eq.[6] in section 1.2:

$$\theta = \operatorname{div}\mathbf{u} = \frac{du}{dR} + \frac{2u}{R} \qquad [13]$$

where u is the radial displacement. Multiply [13] by R^2:

$$R^2\theta = R^2 \left(\frac{du}{dR} + \frac{2u}{R} \right) = \frac{d(R^2 u)}{dR}$$

Integrate over R ($R_1 > R$):

$$\int_R^{R_1} R^2 \theta dR = \int_{R^2 u}^{0} d(R^2 u)$$

$$R^2 u = -\int_R^{\infty} R^2 \theta dR \qquad [14]$$

We assume the wave has reached R but not R_1, which means that $u = 0$ at R_1. Then we let $R_1 \to \infty$ as there will be no contiibution for larger R. This is seen as follows:

$$\lim_{R\to\infty} (R^2\theta) = \lim_{R\to\infty} (Re^{-qR}) = \lim_{R\to\infty} \frac{R}{e^{qR}} = \lim_{R\to\infty} \frac{1}{qe^{qR}} \to 0$$

(eq.[12]) (l'Hospital's rule)

We have $q > 0$, as otherwise θ would increase with increasing R by eq.[12], which is physically excluded.

Combining [14] with [12] we obtain:

$$R^2 u = -\int_R^\infty RCe^{-qR}\mathrm{d}R$$

which is solved easily by partial integration:

$$\int Re^{-qR}\mathrm{d}R = -\frac{R}{q}e^{-qR} + \frac{1}{q}\int e^{-qR}\mathrm{d}R = -\frac{R}{q}e^{-qR} - \frac{1}{q^2}e^{-qR}$$

Inserting the limits and observing that all terms are zero for $R = \infty$, we find the solution of [10] in terms of displacement u:

$$R^2 u = -\left(\frac{R}{q} + \frac{1}{q^2}\right)Ce^{-qR} \qquad [15]$$

14.2.4 Elimination of the integration constant C by introducing the pressure

We can immediately find an expression for the radial stress p_{RR} by applying the stress-strain relation:

$$p_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} \qquad [16]$$

where: $p_{ii} = \lambda\theta + 2\mu e_{ii}$; $i \to R$; $e_{RR} = \mathrm{d}u/\mathrm{d}R$ (being the only *principal extension*, i.e., the extension of unit length, in radial direction). In applications of [16] we have to note the difference between θ and e_{ii}: θ refers to the total cubical dilatation, i.e., independent of coordinates, but e_{ii} refers only to the coordinate i, the same as for p_{ii}. We thus get:

$$p_{RR} = \lambda\theta + 2\mu\frac{\mathrm{d}u}{\mathrm{d}R} = (\lambda + 2\mu)\theta - 4\mu\frac{u}{R} =$$

(only R coordinate) (eq.[13])

$$= \left[(\lambda + 2\mu) + 4\mu\left(\frac{1}{qR} + \frac{1}{q^2R^2}\right)\right]\frac{C}{R}e^{-qR} \qquad [17]$$

(eq.[12], [15])

Now, at $R = a$ we assume a radial pressure $= AH(t)$ (see section 14.2.2) and, considering the region at $R = a$ and beyond, we see that this pressure (from inside) acts along the inward normal. By the common sign definition of stress (see BULLEN, 1963, p.22), we then see that at $R = a$:

$$p_{RR} = -AH(t) \tag{18}$$

We introduce [18] and $R = a$ in [17], and eliminate C between [17] and [15]. We find:

$$R^2u = \frac{qR + 1}{(\lambda + 2\mu)q^2a^2 + 4\mu(qa + 1)} a^3Ae^{-q(R-a)}H(t) \tag{19}$$

14.2.5 Solution of [19]

As q is related to $p = \partial/\partial t$ by eq.[11], eq.[19] means that we have to carry out quite a complicated operation on $H(t)$. However, [19] can be solved using eq.[1] in section 14.1:

$$F(p)H(t) = \frac{1}{2\pi i}\int_L F(z)e^{tz}\frac{dz}{z} \tag{20}$$

where the path of integration is a line L from $-i\infty$ to $+i\infty$ on the positive side of the imaginary axis.

From [11] we have:

$$q^2 = \frac{p^2 - \omega^2}{\alpha^2}$$

$$q = \frac{\sqrt{(p^2 - \omega^2)}}{\alpha} \qquad (q > 0 \text{ by [12]})$$

Then:

$$F(p) = \frac{\left[\dfrac{\sqrt{(p^2-\omega^2)}}{\alpha}R + 1\right]e^{-\frac{\sqrt{(p^2-\omega^2)}}{\alpha}(R-a)}}{(\lambda + 2\mu)\dfrac{p^2-\omega^2}{\alpha^2}a^2 + 4\mu\left[\dfrac{\sqrt{(p^2-\omega^2)}}{\alpha}a + 1\right]} \tag{21}$$

by comparing [19] and [20]. According to [20] we replace p by the new variable z:

$$F(z) = \frac{\left[\dfrac{\sqrt{(z^2-\omega^2)}}{\alpha}R + 1\right]e^{-\frac{\sqrt{(z^2-\omega^2)}}{\alpha}(R-a)}}{(\lambda + 2\mu)\dfrac{z^2-\omega^2}{\alpha^2}a^2 + 4\mu\left[\dfrac{\sqrt{(z^2-\omega^2)}}{\alpha}a + 1\right]} \tag{22}$$

For convenience, we define a new variable $\varkappa$ such that:

$$\varkappa = \frac{\sqrt{(z^2 - \omega^2)}}{\alpha} \qquad (\omega, \alpha \text{ constants}) \tag{23}$$

Then [22] reads:

$$F(z) = F_1(\varkappa) = \frac{(\varkappa R + 1)e^{-\varkappa(R-a)}}{(\lambda + 2\mu)\varkappa^2 a^2 + 4\mu(\varkappa a + 1)} \tag{24}$$

Using the expressions [21] and [24], we apply [20] to eq.[19]:

$$R^2 u = \frac{a^3 A}{2\pi i} \int_L \frac{(\varkappa R + 1)e^{[zt-\varkappa(R-a)]}}{(\lambda + 2\mu)\varkappa^2 a^2 + 4\mu(\varkappa a + 1)} \frac{dz}{z} \tag{25}$$

Eq.[25] is then rewritten by consistently using $\varkappa$ as the new variable:

$$z^2 = \alpha^2\varkappa^2 + \omega^2$$

$$\frac{dz}{z} = \frac{\varkappa\alpha^2 d\varkappa}{\alpha^2\varkappa^2 + \omega^2}$$

$$R^2 u = \frac{a^3 A}{2\pi i} \int_L \frac{(\varkappa R + 1)e^{[t(\alpha^2\varkappa^2+\omega^2)^{1/2}-\varkappa(R-a)]}}{(\lambda + 2\mu)\varkappa^2 a^2 + 4\mu(\varkappa a + 1)} \frac{\varkappa\alpha^2 d\varkappa}{\alpha^2\varkappa^2 + \omega^2} \tag{26}$$

Eq.[26] gives the formal solution of our problem, namely, to find the displacement $u = u(R, t)$ for a spherically spreading elastic wave in the presence of gravity disturbances (these enter through ω^2, as is seen from the wave equation [10]).

14.2.6 Discussion of the solution [26]

First let us neglect gravity, i.e., we put $\omega = 0$; compare eq.[10]. Also we assume that $\lambda = \mu$, i.e., that Poisson's ratio is $= 1/4$. Eq.[26] then immediately reduces to the following:

$$R^2 u = \frac{a^3 A}{2\pi i\mu} \int_L \frac{(\varkappa R + 1)e^{\varkappa(\alpha t-R+a)}}{3\varkappa^2 a^2 + 4\varkappa a + 4} \frac{d\varkappa}{\varkappa} \tag{27}$$

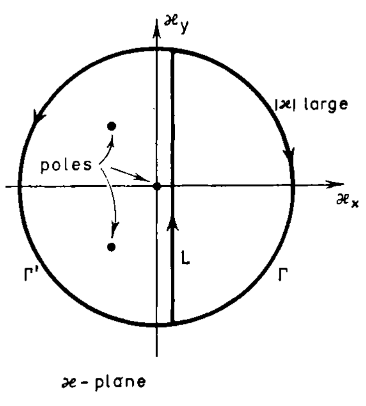


Fig.98.

Now, we have already said that the displacement u is $= 0$ for $t < (R - a)/\alpha$ or for $\alpha t < R - a$. This is clear, because $(R - a)/\alpha$ is the time required for the disturbance (i.e., the pressure pulse at the sphere of radius a) to reach a point at distance R from the origin, or at distance $R - a$ from the spherical surface, as α is the wave velocity.

We can now carry out the integration of the right-hand side of [27], by residue calculus. We choose a contour consisting of L and the infinite arc Γ' on the negative side of the imaginary axis (see Fig.98). The integral along Γ' vanishes. This is seen as follows. The exponential factor in the integrand is:

$$e^{\varkappa(\alpha t - R + a)} = e^{\varkappa c} = e^{(\varkappa_x + i\varkappa_y)c} = e^{\varkappa_x c}\, e^{i\varkappa_y c}$$

As we have contributions only for $c > 0$, the integral over Γ' will vanish if $\varkappa_x c < 0$, i.e., if $\varkappa_x < 0$, and this is the case on Γ'. The small parts of Γ' situated at the ends of L, on the positive side of the imaginary axis, give only zero contribution. There are no branch points, but we have the following poles inside the contour:

$$\left.\begin{aligned} &\varkappa = 0 \\ &\varkappa a = -\frac{2}{3}(1 \pm i\sqrt{2}) \end{aligned}\right| \qquad [28]$$

(these being the roots of: $3\varkappa^2 a^2 + 4\varkappa a + 4 = 0$). For $\alpha t > R - a$, i.e., for u different from zero, we obtain, after a simple but rather lengthy calculation, the following solution:

$$R^2 u = \frac{a^3 A}{\mu}\left\{\frac{1}{4} + \frac{1}{4a} e^{-\frac{2}{3}\frac{\alpha t - R + a}{a}} \left[\left(R - \frac{a}{2}\right)\sqrt{2}\sin\left(\frac{2\sqrt{2}}{3}\frac{\alpha t - R + a}{a}\right) - \right.\right.$$
$$\left.\left. - a\cos\left(\frac{2\sqrt{2}}{3}\frac{\alpha t - R + a}{a}\right)\right]\right\} \qquad [29]$$

Let us now study the physical side of eq.[29], which expresses the displacement u as a function of R and t: $u(R, t)$. We see that at time $t = (R - a)/\alpha$ or $\alpha t - R + a = 0$, the displacement at distance R is equal to 0. The expression for u in eq.[29] consists of a time-independent (steady) value, a sine term and a cosine term. This means that in general the displacement for $t > (R - a)/\alpha$ consists of a series of oscillations around a steady value—a lasting dislocation, depending on the assumption of $H(t)$. The sine and cosine terms have an exponential factor, which decreases with time, and this means that the oscillations die out rather rapidly.

For better illustration, let us consider two cases:

(*1*) The time variation at a point relatively near the source (Fig.99A). At the latter time indicated in the figure, the sine term is again approximately $= 0$. The ratio of the exponential factor between the two instants is: $e^{-2}/1$.

(*2*) The time variation at a point at a great distance from the source, $R \gg a$ (Fig.99B). Solving [29] for u and considering how the three terms depend upon R, we have the results as given in Table X. This means that the sine term will dominate the variation at large distances:

$$u \simeq \frac{\sqrt{2}Aa^2}{4\mu R}\sin\left(\frac{2\sqrt{2}}{3}\frac{\alpha t - R + a}{a}\right) e^{-\frac{2}{3}\frac{\alpha t - R + a}{a}} \qquad [30]$$

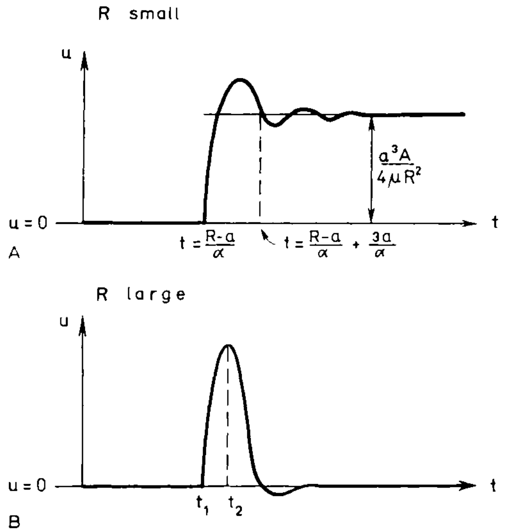


Fig.99.

The variation of u with t depends on the sine term, whereas the exponential factor causes a monotonous decrease of the amplitude of the variation.

At time t_1 the motion starts, i.e., sine = 0 and:

$$\alpha t_1 - R + a = 0 \qquad [31]$$

and at time t_2 the displacement reaches its maximum, sine = 1, and:

$$\left.\begin{aligned} &\frac{2}{3}\sqrt{2}\,\frac{\alpha t_2 - R + a}{a} = \frac{\pi}{2} \\ &\alpha t_2 - R + a = \frac{3\pi a}{4\sqrt{2}} \end{aligned}\right| \qquad [32]$$

For $t = t_1$, the exponential factor = 1, and for $t = t_2$, it is about 1/e. Taking the difference

TABLE X

PROPERTIES OF THE DISPLACEMENT u IN EQ. [29]

	Constant term	*Sine term*	*Cosine term*
Amplitude	$\sim a^3/R^2$	$\sim \frac{a^3}{a}\frac{R - a/2}{R^2}$	$\sim \frac{a^3}{a}\frac{a}{R^2}$
Variation with R	$1/R^2$	$1/R$	$1/R^2$
		(disregarding the exponential factor for the moment)	

between [32] and [31] we get:

$$t_2 - t_1 = \frac{3\pi a}{4\sqrt{2}\alpha} \tag{33}$$

which means that the time it takes for the pulse to reach its maximum, at a great distance, can be used to calculate the value of a, i.e., the source dimension. In our calculations we have assumed a spherical source $R = a$, which may be a good approximation for explosions but less good for earthquakes.

It is important to see that in both this and the preceding case, the motion is essentially only a single swing.

As an exercise, calculate the corresponding results in the case where the original disturbance, instead of being proportional to $H(t)$, has some other time function.

Now we return to the more general case, including gravity effects, i.e., to eq.[26]. The integrand in [26] has no singularity (pole) on the positive side of the imaginary axis. Then we have from Fig.98:

$$\int_L + \int_\Gamma = -2\pi i \sum \mathrm{Res} = 0$$

(negative rotation) or:

$$\int_L = -\int_\Gamma$$

or:

$$\int_{-i\infty}^{+i\infty} = -\int_\Gamma \tag{34}$$

Eq.[34] means that the integration along L in the complex $\varkappa$-plane is equivalent to an integration along the semi-circle Γ which is placed at a large distance from the origin, i.e., $|\varkappa|$ is large.

The last factor in the integrand in [26] gives, for $|\varkappa|$ large:

$$\frac{\varkappa a^2 \mathrm{d}\varkappa}{a^2\varkappa^2 + \omega^2} = \frac{a^2\varkappa^2}{a^2\varkappa^2 + \omega^2}\,\frac{\mathrm{d}\varkappa}{\varkappa} = \frac{1}{1 + \underbrace{\frac{\omega^2}{a^2\varkappa^2}}_{\to 0}}\,\frac{\mathrm{d}\varkappa}{\varkappa} \to \frac{\mathrm{d}\varkappa}{\varkappa} \tag{35}$$

and the exponential factor becomes:

$$\mathrm{e}^{t(\alpha^2\varkappa^2+\omega^2)^{1/2} - \varkappa(R-a)} = \mathrm{e}^{t(\alpha^2\varkappa^2+\omega^2)^{1/2} - \varkappa(R-a)}\,\frac{\mathrm{e}^{\varkappa\alpha t}}{\mathrm{e}^{\varkappa\alpha t}}$$

$$= \mathrm{e}^{t(\alpha^2\varkappa^2+\omega^2)^{1/2} - \varkappa\alpha t}\,\mathrm{e}^{\varkappa(\alpha t - R + a)}$$

$$\downarrow$$

$$\mathrm{e}^{t\left(\alpha\varkappa + \frac{\omega^2}{2\alpha\varkappa} + \ldots\right) - \varkappa\alpha t} = \mathrm{e}^{\frac{\omega^2 t}{2\alpha\varkappa} + \ldots} = 1 + \frac{\omega^2 t}{2\alpha\varkappa} + O(\varkappa^{-2}) + \ldots \tag{36}$$

Therefore, again assuming that $\lambda = \mu$ and including terms in $1/\varkappa$ only to their first order, i.e., neglecting $O(\varkappa^{-2})$, etc., we now have instead of [27]:

$$R^2 u = \frac{a^3 A}{2\pi i \mu} \int_L \frac{\varkappa R + 1}{3\varkappa^2 a^2 + 4\varkappa a + 4} e^{\varkappa(\alpha t - R + a)} \left(1 + \frac{\omega^2 t}{2a\varkappa}\right) \frac{d\varkappa}{\varkappa}$$

$$= \frac{a^3 A}{2\pi i \mu} \int_L \frac{\varkappa R + 1}{3\varkappa^2 a^2 + 4\varkappa a + 4} e^{\varkappa(\alpha t - R + a)} \frac{d\varkappa}{\varkappa} +$$

(this part corresponds to the solution [29])

$$+ \frac{a^3 A}{2\pi i \mu} \int_L \frac{\varkappa R + 1}{3\varkappa^2 a^2 + 4\varkappa a + 4} e^{\varkappa(\alpha t - R + a)} \frac{\omega^2 t}{2a} \frac{d\varkappa}{\varkappa^2} \qquad [37]$$

(this is the additional term, due to gravity, represented by ω^2).

The first integral in [37] is the same as [27], for which we have the solution in [29]. To solve the second integral in [37] we choose a contour consisting of L together with Γ' (see Fig.98). Again the contribution from Γ' vanishes, and we are left with the contributions from the three poles:

$$\left.\begin{aligned} &\varkappa = 0 \quad \text{(a double pole)} \\ &\varkappa = -\frac{2}{3a}(1 \pm i\sqrt{2}) \end{aligned}\right| \qquad [38]$$

The residue at the double pole at $\varkappa = 0$ is equal to the coefficient of $1/\varkappa$ in the integrand in the second term of [37]:

$$\underbrace{\frac{\varkappa R + 1}{3\varkappa^2 a^2 + 4\varkappa a + 4}}_{\to 1/4 \ (\text{for } \varkappa \to 0)} \ \underbrace{\frac{e^{\varkappa(\alpha t - R + a)}}{\varkappa^2}}_{= \frac{1}{\varkappa^2}[1 + \varkappa(at - R + a) + \ldots]} \frac{\omega^2 t}{2a} \quad \to \quad \underbrace{\frac{1}{4}(at - R + a)\frac{\omega^2 t}{2a}}_{\text{(this is the coefficient of } 1/\varkappa)}$$

The residues at the other two poles in the $\varkappa$-plane can also be calculated. The calculations are rather lengthy, and here it will be sufficient to give the order of magnitude of the contributions from these two poles. Carrying out the computations in this case, we get a formula corresponding to [29]:

$$u = \frac{a^3 A}{4\mu R^2} \left\{ 1 + \frac{\omega^2 t}{2a}(at - R + a) + \left(1 + \frac{\omega^2 at}{a}\right) e^{-\frac{2}{3}\frac{\alpha t - R + a}{a}} \left[\left(\frac{R}{a} - \frac{1}{2}\right) \right.\right.$$

$$\left.\left. \cdot \sqrt{2} \sin\left(\frac{2\sqrt{2}}{3}\frac{at - R + a}{a}\right) - \cos\left(\frac{2\sqrt{2}}{3}\frac{at - R + a}{a}\right)\right]\right\} \qquad [39]$$

where the additions to [29] consist of the terms containing ω^2.

It results that the additional terms in [39], due to gravity effects, increase with time t. However, putting in numerical values, we shall find both of them to be insignificant. In C.G.S.-units we have (approximately):

$$\left.\begin{aligned}&\frac{a}{\alpha}=1 \quad ; \quad \varrho=5 \quad ; \quad f=6.7\cdot 10^{-8}\\&\omega^2=4\pi f\varrho=4\cdot 10^{-6}\end{aligned}\right| \qquad [40]$$

The first additional term in [39] becomes:

$$\frac{\omega^2}{2}t\left(t-\frac{R-a}{\alpha}\right)=2\cdot 10^{-6}\cdot t\left(t-\frac{R-a}{\alpha}\right) \qquad [41]$$

which means that it will be small compared with unity, until t is of the order of several hundred seconds. But at a time as great as this, the displacement associated with direct P waves would be unobservable in actual earthquake conditions. Thus in practice this term is normally insignificant. For the same reason the other modification is also insignificant:

$$\frac{\omega^2 at}{\alpha}=4\cdot 10^{-6}t \qquad [42]$$

Comparing [39] with [29] we have the following two differences:

(*1*) The lasting displacement increases linearly with time in [39] but at a very slow rate; in [29] it is time-independent.

(*2*) The exponential term is also multiplied with a factor which increases linearly with time in [39] instead of being constant as in [29]; but also this rate of increase is very slow.

As in [29], the time variation is dominated by the exponential, sine and cosine terms.

It follows that gravity has little effect on the propagation of P waves. The effect on the first few swings, which constitute the noticeable portion of the disturbance when gravity is neglected, is always negligible. The steady portion is modified into a displacement that increases steadily with time, but in actual conditions this can never become great.

Our conclusion is that gravity effects are insignificant in all ordinary problems concerning seismic body waves.

BULLEN (1963, pp.75–77, 80–81) discusses gravity effects on body waves but without actually deriving the formulas.

14.3 SURFACE WAVES

For the influence of gravity on surface waves, I refer to EWING, JARDETZKY and PRESS (1957, pp.257–263). Also in this case the gravity effects are small, as evidenced by the following example. For Poisson's ratio = 1/3 and a shear-wave velocity = 4 km/sec, the velocity of Rayleigh waves is increased by about 0.2% when the wave length is about 500 km.

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