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On rotating charged dust in general relativity

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The problem of charged dust rotating about an axis of symmetry is considered both in Newtonian physics and in general relativity. The Newtonian problem is reduced to a single equation in the case of constant rotation, and to two coupled equations in the case of differential rotation, and some explicit cylindrically symmetric solutions are obtained. In general relativity some new cylindrically symmetric exact solutions for constant rotation are derived, and the problem of differential rotation is reduced to four coupled equations for four unknowns.

1. INTRODUCTION

In a previous paper (Islam 1977, hereafter referred to as paper I) a class was obtained of exact interior stationary solutions of the Einstein–Maxwell equations representing charged dust (i.e. pressureless matter) rotating with constant angular velocity about an axis of symmetry. This class is expressible in terms of a single axisymmetric harmonic function, i.e. solution of the flat space Laplace equation. In this paper the problem of rotating charged dust is studied further, both in Newtonian physics and in general relativity.

In § 2 we consider the problem of charged dust rotating about an axis of symmetry in Newtonian physics. Only local solutions are examined, without considering boundary conditions. For charged dust, unlike neutral dust, it appears likely that an axially symmetric (but non-cylindrically symmetric) configuration exists. We consider both rigid and differential rotation and obtain some exact solutions with cylindrical symmetry. This Newtonian analysis provides a useful background for the corresponding general relativistic problem considered in paper I and in the rest of this paper.

In § 3 we consider the Einstein–Maxwell field equations for a differentially rotating charged dust which has axial (but not necessarily cylindrical) symmetry. These are a set of coupled partial differential equations for seven unknowns of two variables. It is first shown how to reduce the number of unknowns from seven to six. The six unknowns satisfy a coupled system of seven equations. One of these equations is eliminated to obtain a system of six equations for six unknowns. This is reduced effectively to a system of four equations for four unknowns since two of the unknowns can be obtained trivially once the other four have been solved for.

The solution obtained in paper I has the property that the Lorentz force on a typical particle vanishes. Such a solution has no analogue in Newtonian physics except in the trivial case of neutral dust. However, in §2 we find Newtonian solutions with cylindrical symmetry for which the Lorentz force does not vanish. In §4 we look for the general relativistic analogue of these Newtonian solutions and find a class of cylindrically symmetric solutions expressible in terms of the solution of a single ordinary differential equation. We solve this equation for a particular value of the charge : mass ratio, thus obtaining an explicit solution. These cylindrically symmetric solutions are distinct from the solutions obtained in paper I, since here the Lorentz force does not vanish.

Although the problem of rotating charged dust is unlikely to be of astrophysical interest, I believe it to be important from the point of view of gaining insight into general relativity. This is a well-defined physical situation for which solutions exist in general relativity. This situation is rare in general relativity. The existence of non-cylindrically symmetric equilibrium solutions such as the Van Stockum (1937) and Winicour (1975) solutions for neutral dust and the solution found in paper I for charged dust suggests that a particle rotating about an axis of symmetry exerts a force parallel to the axis of symmetry (i.e. a sort of magnetic gravitation force), unlike in Newtonian gravitation. (I am grateful to W. B. Bonnor for a discussion about this.) For a conclusive proof of this one would have to show that the solutions just mentioned can be applied to finite and bounded distributions of matter. This problem is not considered here.

2. Newtonian problem

Let the number density of particles be n, and the mass and the charge of the particles be m and q respectively. There is no pressure. We envisage an axially symmetric situation in which the particles are rotating about an axis of symmetry (the z-axis) with an angular velocity which is a function of ρ (the distance from the axis) and z only. We ask if a stationary equilibrium is possible. We set the gravitational constant and the velocity of light equal to unity. The equation of motion is as follows:

$$mn(\partial/\partial t + \boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = \boldsymbol{J}\wedge\boldsymbol{B} - mn\boldsymbol{\nabla}V' + qn\boldsymbol{E}, \tag{1}$$

where $\boldsymbol{v}$ is the velocity, constrained to be given by the following (Cartesian components)

$$\boldsymbol{v} = (-\Omega y, \Omega x, 0), \quad \Omega = \Omega(\rho, z). \tag{2}$$

$\boldsymbol{J}$ is the current density given by

$$\boldsymbol{J} = qn\boldsymbol{v}, \tag{3}$$

while $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric field and the magnetic induction respectively, satisfying (in the stationary situation) the equations

$$\text{curl}\ \boldsymbol{E} = 0, \quad \text{div}\ \boldsymbol{E} = 4\pi qn, \tag{4}$$

$$\text{div}\ \boldsymbol{B} = 0, \quad \text{curl}\ \boldsymbol{B} = 4\pi\boldsymbol{J}. \tag{5}$$

V' is the Newtonian potential satisfying

$$\nabla^2 V' = 4\pi mn. \tag{6}$$

From (4) we have

$$\boldsymbol{E} = \boldsymbol{\nabla}\phi, \quad \nabla^2\phi = 4\pi qn. \tag{7}$$

Because of the symmetry of the situation, V' and ϕ are functions of ρ and z only. In Cartesian components, with $\boldsymbol{B} = (B^{(1)}, B^{(2)}, B^{(3)})$, equation (1) reads

$$m\Omega^2(-x, -y, 0) = q\Omega(xB^{(3)}, yB^{(3)}, -xB^{(1)} - yB^{(2)}) - m(V'_x, V'_y, V'_z) + q(\phi_x, \phi_y, \phi_z), \tag{8}$$

where $\phi_x = \partial\phi/\partial x$, etc. For any axially symmetric function f we have $f_x = (x/\rho)f_\rho$, $f_y = (y/\rho)f_\rho$. The second equation in (5) reads

$$(B^{(3)}_y - B^{(2)}_z, B^{(1)}_z - B^{(3)}_x, B^{(2)}_x - B^{(1)}_y) = 4\pi\Omega qn(-y, x, 0). \tag{9}$$

The current has to satisfy the equation of continuity

$$\operatorname{div} \boldsymbol{J} = q \operatorname{div}(n\boldsymbol{v}) = 0. \tag{10}$$

For $\boldsymbol{v}$ given by (2) this is satisfied identically if $n = n(\rho, z)$.

From the third component of (8) we see that $xB^{(1)} + yB^{(2)}$ must be a function of ρ and z. This implies the following:

$$B^{(1)} = xB + yC, \quad B^{(2)} = yB - xC, \tag{11}$$

where B and C are functions of ρ and z only and correspond to the radial and azimuthal components of the induction.

Consider the line integral of the induction around a circle C' given by z = constant, ρ = constant. This line integral is proportional to the current flowing through C' and must vanish identically for dust rotating around the z-axis. With the use of (11) we have

$$\oint_{C'} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{s} = -\int \rho^2 C\,\mathrm{d}\theta = 0, \tag{12}$$

where θ is the azimuthal angle. Thus we must have $C = 0$.

The first of equations (5) implies

$$\rho^2 B = b_z, \quad \rho B^{(3)} = -b_\rho, \tag{13}$$

where b is an arbitrary function of ρ and z. We now assume Ω to be constant. The last component of (8) can then be integrated to yield

$$-mV' + q\phi - q\Omega b = f(\rho), \tag{14}$$

where f is an arbitrary function of ρ only. Consistency with the first two components of (8) demands

$$f = -\tfrac{1}{2}m\Omega^2\rho^2 + f_0, \tag{15}$$

where f_0 is an arbitrary constant. The third component of (9) is satisfied identically, while the first two components yield the single equation

$$b_{\rho\rho} + b_{zz} - \rho^{-1}b_\rho = 4\pi q\Omega\rho^2 n. \tag{16}$$

Equation (16) can be converted into a Poisson equation by the substitution $b = \rho b'_\rho$ in which case

$$\nabla^2 b' \equiv b'_{\rho\rho} + b'_{zz} + \rho^{-1} b'_\rho = 4\pi q \Omega \int^\rho \rho n \,\mathrm{d}\rho. \tag{17}$$

By applying the ∇^2 operator to (14) and using (6), (7) and (16) we get a relation between b and n as follows:

$$2\pi(q^2 - m^2 - q^2\Omega^2\rho^2)\,n + m\Omega^2 = q\Omega\rho^{-1} b_\rho. \tag{18}$$

From (16) and (18) one can eliminate either n or b to get a single equation for b or n. Eliminating n, we get

$$b_{\rho\rho} + b_{zz} + \left\{-\frac{1}{\rho} - \frac{2q^2\Omega^2\rho}{q^2 - m^2 - q^2\Omega^2\rho^2}\right\} b_\rho = \frac{-2mq\Omega^3\rho^2}{q^2 - m^2 - q^2\Omega^2\rho^2}. \tag{19}$$

Thus to solve the problem one first finds a solution to equation (19), with the use of which the number density can be obtained from (18) and the magnetic induction from (13). The gravitational and electric potentials, V', ϕ can then be obtained from the Poisson equations (6) and (7). Thus, if a solution exists to (19) depending on ρ and z, then a non-cylindrically symmetric configuration can exist, unlike the case of neutral rotating dust. Whether the matter distribution can be a finite one is not clear.

As a limiting case one can set $b_z = 0$, leading to $B^{(1)} = B^{(2)} = 0$. Here an explicit solution can be obtained as follows:

$$B^{(3)} = (1/q\Omega)\,(E - m\Omega^2\rho^2)\,(a + \rho^2)^{-1}, \tag{20a}$$

$$n = a'(a + \rho^2)^{-2}, \tag{20b}$$

$$V' = (\pi m a'/a) \ln(a + \rho^2) + V'_0 \tag{20c}$$

$$\phi = (\pi q a'/a) \ln(a + \rho^2) + \phi_0, \tag{20d}$$

where

$$a \equiv \frac{m^2 - q^2}{q^2\Omega^2}, \quad a' = \frac{1}{2\pi q^2\Omega^2}\left\{\frac{m(m^2 - q^2)}{q^2} + E\right\}, \tag{20e}$$

and E, ϕ_0, V'_0 are arbitrary constants. The solution given by ($20a$–e) is well behaved everywhere (with n positive) at least for $q^2 < m^2$ and for a suitable choice of E.

We now come to differential rotation where Ω is no longer constant. The magnetic induction is again given by (13) where b satisfies (16) with Ω now a function of ρ and z. The other equations satisfied by n and b can be seen to be

$$q(\Omega_z b_\rho - \Omega_\rho b_z) = 2m\rho\Omega\Omega_z, \tag{21}$$

and

$$4\pi(m^2 - q^2 + q^2\Omega^2\rho^2)\,n = -q\Omega\{(\Omega_\rho + 2/\rho)\,b_\rho + \Omega_z b_z\} + 2m\rho\Omega\Omega_\rho. \tag{22}$$

The function n in (16) can be eliminated with the use of (22), yielding a system of two coupled equations for b and Ω. Once this system is solved, the potentials V', ϕ can be determined in principle from the Poisson equations (6) and (7). Thus if the system of coupled equations for b and Ω has a solution, then a non-cylindrically symmetric configuration is possible in this instance. Such a system is not possible if Ω is a function of ρ only (but $\Omega \neq$ constant). For then (21) implies that b is independent of z (this does not follow if Ω = constant), which in turn implies from (22) that n is a function of ρ only, leading to a cylindrically symmetric situation. Here the solution depends on one arbitrary function of ρ, for if $\Omega(\rho)$ is specified all other functions can be determined in principle.

It is easily verified that in the above solutions the Lorentz force on all particles does not vanish, for if this were the case we must necessarily have

$$V' = \tfrac{1}{2}\Omega^2\rho^2 + V_0', \tag{23}$$

where V_0' is a constant, leading, through (6), to a *constant* n. This is true only if $q = 0$, i.e. for neutral dust.

3. Field equations in general relativity

The Einstein–Maxwell interior equations for charged dust are given as follows (see paper I):

$$R_{\mu\nu} = 8\pi mn(u_\mu u_\nu - \tfrac{1}{2}g_{\mu\nu}) - 2F_\mu^{\alpha} F_{\nu\alpha} + \tfrac{1}{2}g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{24a}$$

$$F_{\mu\nu;\sigma} + F_{\nu\sigma;\mu} + F_{\sigma\mu;\nu} = 0, \tag{24b}$$

$$F^{\mu\nu}{}_{;\nu} = -4\pi J^\mu, \quad J^\mu = qnu^\mu, \tag{24c}$$

where a semicolon denotes covariant differentiation, $R_{\mu\nu}$, $F_{\mu\nu}$ are respectively the Ricci and the electromagnetic field tensors, u^μ the four-velocity, and J^μ the four-current. The symbols m, q and n have the same interpretation as in the last section. In paper I it is shown that the total energy–momentum tensor satisfies the usual conservation equation. As before, the gravitational constant and the velocity of light are set equal to unity. The equation of motion is

$$m(\mathrm{d}u^\mu/\mathrm{d}s + \Gamma^\mu_{\sigma\nu} u^\sigma u^\nu) = qF^{\mu\nu} u_\nu, \tag{25}$$

where s is the proper time along the world line of the particle. These equations reduce to (1) in the Newtonian limit.

The most general axially symmetric stationary metric can be written as follows:

$$\mathrm{d}s^2 = f\,\mathrm{d}t^2 - 2k\,\mathrm{d}\theta\,\mathrm{d}t - l\,\mathrm{d}\theta^2 - \mathrm{e}^\mu(\mathrm{d}\rho^2 + \mathrm{d}z^2), \tag{26}$$

where f, k, l and μ are all functions of ρ and z only. Setting $(x^0, x^1, x^2, x^3) = (t, \rho, z, \theta)$, the components of the four-velocity of the rotating dust are

$$u^0 = \mathrm{d}t/\mathrm{d}s = (f - 2\Omega k - \Omega^2 l)^{-\frac{1}{2}}, \quad u^1 = u^2 = 0, \quad u^3 = \mathrm{d}\theta/\mathrm{d}s = \Omega u^0, \tag{27}$$

where Ω is the angular velocity, a function of ρ and z. $F_{\mu\nu}$ is defined in the usual manner in terms of a four-potential A_μ here given by

$$(A_0, A_1, A_2, A_3) = (\phi, 0, 0, \chi), \tag{28}$$

where ϕ and χ are related to the electric and magnetic potentials respectively. Since all functions are independent of t and θ the equation of conservation of matter

$(nu^\mu)_{;\mu} = 0$ is satisfied identically, as are the t- and θ-components of (25). The other two components of (25) reduce to the following equations:†

$$\tfrac{1}{2}m(f_\rho - 2\Omega k_\rho - \Omega^2 l_\rho)(f - 2\Omega k - \Omega^2 l)^{-\frac{1}{2}} = q(\phi_\rho + \Omega\chi_\rho), \tag{29a}$$

$$\tfrac{1}{2}m(f_z - 2\Omega k_z - \Omega^2 l_z)(f - 2\Omega k - \Omega^2 l)^{-\frac{1}{2}} = q(\phi_z + \Omega\chi_z), \tag{29b}$$

with $f_\rho \equiv \partial f/\partial\rho$, etc.

Three of the field equations are as follows:

$$\begin{aligned} 2e^\mu D^{-1}R_{00} &= (D^{-1}f_\rho)_\rho + (D^{-1}f_z)_z + D^{-3}f(f_\rho l_\rho + f_z l_z + k_\rho^2 + k_z^2) \\ &= 8\pi mnD^{-1}e^\mu(f - 2\Omega k - \Omega^2 l)^{-1}\{2\Omega k(-f + \Omega k) + f(f + \Omega^2 l)\} \\ &\quad + 4D^{-3}\{(\tfrac{1}{2}fl + k^2)(\phi_\rho^2 + \phi_z^2) + kf(\phi_\rho\chi_\rho + \phi_z\chi_z) + \tfrac{1}{2}f^2(\chi_\rho^2 + \chi_z^2)\}, \end{aligned} \tag{30a}$$

$$\begin{aligned} -2e^\mu D^{-1}R_{03} &= (D^{-1}k_\rho)_\rho + (D^{-1}k_z)_z + D^{-3}k(f_\rho l_\rho + f_z l_z + k_\rho^2 + k_z^2) \\ &= 8\pi mnD^{-1}e^\mu(f - 2\Omega k - \Omega^2 l)^{-1}(fk + 2\Omega fl - \Omega^2 kl) \\ &\quad - 4D^{-3}\{\tfrac{1}{2}kl(\phi_\rho^2 + \phi_z^2) + fl(\phi_\rho\chi_\rho + \phi_z\chi_z) - \tfrac{1}{2}kf(\chi_\rho^2 + \chi_z^2)\}, \end{aligned} \tag{30b}$$

$$\begin{aligned} -2e^\mu D^{-1}R_{33} &= (D^{-1}l_\rho)_\rho + (D^{-1}l_z)_z + D^{-3}l(f_\rho l_\rho + f_z l_z + k_\rho^2 + k_z^2) \\ &= -8\pi mnD^{-1}e^\mu(f - 2\Omega k - \Omega^2 l)^{-1}(fl + 2k^2 + 2\Omega kl + \Omega^2 l^2) \\ &\quad - 4D^{-3}\{\tfrac{1}{2}l^2(\phi_\rho^2 + \phi_z^2) - kl(\phi_\rho\chi_\rho + \phi_z\chi_z) + (\tfrac{1}{2}fl + k^2)(\chi_\rho^2 + \chi_z^2)\}, \end{aligned} \tag{30c}$$

where $D^2 = fl + k^2$. An important combination of these equations, which is valid in the present case of differential rotation as well as for constant Ω considered in paper I, is the following:

$$e^\mu D^{-1}(lR_{00} - 2kR_{03} - fR_{33}) = D_{\rho\rho} + D_{zz} = 0. \tag{31}$$

It is well known how (31) can be used to transform to coordinates such that $D = \rho$, thus giving an alegebraic relation connecting f, l and k. We then have essentially seven unknown functions of ρ and z, namely μ, n, ϕ, χ, Ω, and two of f, l and k. Instead of f, k, ϕ, we shall use the functions F, K, ψ (and relabel l for convenience) as follows:

$$L = l, \quad K = k + \Omega l, \quad F = f - 2\Omega k - \Omega^2 l, \quad \psi = \phi + \Omega\chi. \tag{32}$$

When Ω is constant, as was the case in paper I, (32) amounts to transforming to a coordinate system rotating with angular velocity Ω. In the present case where Ω is a function of ρ and z, no such interpretation can be given. In terms of the new functions (29a, b) can be written

$$\tfrac{1}{2}mF^{-\frac{1}{2}}(F_\rho + 2K\Omega_\rho) = q(\psi_\rho - \Omega_\rho\chi), \tag{33a}$$

$$\tfrac{1}{2}mF^{-\frac{1}{2}}(F_z + 2K\Omega_z) = q(\psi_z - \Omega_z\chi). \tag{33b}$$

† I mention some minor errors in paper I. The first equation in (24c) and equation (25) above (in paper I these equations have numbers (3) and (10) respectively) have the wrong signs in the right-hand sides, although the explicit forms of (10) of paper I (i.e. (15a, b) of paper 1 and (29a, b) of this paper) are correct. There should be a corresponding change of sign in the right-hand side of (21a) of paper I, leading to a trivial change of sign in the charge: mass ratio (equation (27) of paper I).

Note that
$$\rho^2 = fl + k^2 = FL + K^2. \tag{34}$$

One of the main results of this paper is to show that when the field equations are written in terms of F, K and ψ (which replace f, k and ϕ) the function Ω (as opposed to Ω_ρ and Ω_z) disappears from the equations. The Ω_ρ and Ω_z (and $\Omega_{\rho\rho}$, Ω_{zz}) which remain in the equations can be eliminated with the help of (33*a*, *b*). Thus Ω and its derivatives disappear completely from the equations, reducing the number of unknowns from seven to six.

We show how to integrate (33*a*, *b*), although we shall not use this result further. Equations (33*a*, *b*) imply
$$(mF^{\frac{1}{2}} - q\psi)_\rho \, \Omega_z = (mF^{\frac{1}{2}} - q\psi)_z \, \Omega_\rho, \tag{35}$$
the solution to which is
$$\Omega = \Omega(mF^{\frac{1}{2}} - q\psi), \tag{36}$$
i.e. Ω is a function of $mF^{\frac{1}{2}} - q\psi$.

We proceed to write all the equations in terms of F, K and ψ. Equations (30*a*, *b*, *c*) can be combined to give the following three equations:
$$\begin{aligned}
&\Delta F + 4(K_\rho \Omega_\rho + K_z \Omega_z) - 2L(\Omega_\rho^2 + \Omega_z^2) + 2K\Delta\Omega + \rho^{-2}F\Sigma \\
&\quad = 8\pi mn\,\mathrm{e}^{\mu} F + 4\rho^{-2}\{(\tfrac{1}{2}LF + K^2)(\psi_\rho^2 + \psi_z^2) + KF(\psi_\rho \chi_\rho + \psi_z \chi_z) \\
&\qquad - KF\chi(\chi_\rho \Omega_\rho + \chi_z \Omega_z) - (FL + 2K^2)\chi(\psi_\rho \Omega_\rho + \psi_z \Omega_z) \\
&\qquad + (\tfrac{1}{2}LF + K^2)\chi^2(\Omega_\rho^2 + \Omega_z^2) + \tfrac{1}{2}F^2(\chi_\rho^2 + \chi_z^2)\},
\end{aligned} \tag{37a}$$
$$\begin{aligned}
&\Delta K - L\Delta\Omega - 2(L_\rho \Omega_\rho + L_z \Omega_z) + \rho^{-2}K\Sigma \\
&\quad = 8\pi mn\,\mathrm{e}^{\mu} K - 4\rho^{-2}\{\tfrac{1}{2}LK(\psi_\rho^2 + \psi_z^2) - LK\chi(\psi_\rho \Omega_\rho + \psi_z \Omega_z) \\
&\qquad + \tfrac{1}{2}LK\chi^2(\Omega_\rho^2 + \Omega_z^2) + LF(\psi_\rho \chi_\rho + \psi_z \chi_z) - LF\chi(\chi_\rho \Omega_\rho + \chi_z \Omega_z) \\
&\qquad - \tfrac{1}{2}FK(\chi_\rho^2 + \chi_z^2)\},
\end{aligned} \tag{37b}$$
$$\begin{aligned}
\Delta L + \rho^{-2}L\Sigma &= 8\pi mn\,\mathrm{e}^{\mu} F^{-1}(-FL - 2K^2) - 4\rho^{-2}\{\tfrac{1}{2}L^2(\psi_\rho^2 + \psi_z^2) \\
&\quad - LK(\psi_\rho \chi_\rho + \psi_z \chi_z) - L^2\chi(\psi_\rho \Omega_\rho + \psi_z \Omega_z) \\
&\quad + \tfrac{1}{2}L^2\chi^2(\Omega_\rho^2 + \Omega_z^2) + (\tfrac{1}{2}LF + K^2)(\chi_\rho^2 + \chi_z^2) \\
&\quad + LK\chi(\chi_\rho \Omega_\rho + \chi_z \Omega_z)\},
\end{aligned} \tag{37c}$$
where
$$\Delta \equiv \left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\right), \tag{38a}$$
and
$$\Sigma \equiv \{F_\rho L_\rho + K_\rho^2 + F_z L_z + K_z^2 + 2\Omega_\rho(KL_\rho - LK_\rho) + 2\Omega_z(KL_z - LK_z) + L^2(\Omega_\rho^2 + \Omega_z^2)\}. \tag{38b}$$

Because of (34) only two of equations (37*a*, *b*, *c*) are independent. The three other non-trivial components of (24*a*) can be written as follows:
$$\begin{aligned}
R_{11} &= -\tfrac{1}{2}(\mu_{\rho\rho} + \mu_{zz}) + \tfrac{1}{2}\rho^{-1}\mu_\rho + \tfrac{1}{2}\rho^{-2}\{F_\rho L_\rho + K_\rho^2 + 2(KL_\rho - LK_\rho)\,\Omega_\rho + L^2\Omega_\rho^2\} \\
&= 4\pi mn\,\mathrm{e}^{\mu} + \rho^{-2}\Lambda,
\end{aligned} \tag{39a}$$

$$R_{12} = \tfrac{1}{2}\rho^{-1}\mu_z + \tfrac{1}{4}\rho^{-2}\{F_\rho L_z + F_z L_\rho + 2K_\rho K_z + 2(KL_z - LK_z)\,\Omega_\rho + 2(KL_\rho - LK_\rho)\,\Omega_z + 2L^2\Omega_\rho\,\Omega_z\}$$
$$= 2\rho^{-2}\{-L(\psi_\rho - \chi\Omega_\rho)(\psi_z - \chi\Omega_z) + K(\psi_\rho\chi_z + \psi_z\chi_\rho) - K\chi(\Omega_\rho\chi_z + \Omega_z\chi_\rho) + F\chi_\rho\chi_z\}, \tag{39b}$$
$$R_{22} = -\tfrac{1}{2}(\mu_{\rho\rho} + \mu_{zz}) - \tfrac{1}{2}\rho^{-1}\mu_\rho + \tfrac{1}{2}\rho^{-2}\{F_z L_z + K_z^2 + L^2\Omega_z^2 + 2(KL_z - LK_z)\,\Omega_z\} = 4\pi mn\,\mathrm{e}^{\mu} - \rho^{-2}\Lambda, \tag{39c}$$

where

$$\Lambda \equiv L\{-(\psi_\rho - \chi\Omega_\rho)^2 + (\psi_z - \chi\Omega_z)^2\} + 2K\{\chi_\rho(\psi_\rho - \chi\Omega_\rho) - \chi_z(\psi_z - \chi\Omega_z)\} + F(\chi_\rho^2 - \chi_z^2). \tag{40}$$

The Maxwell equations (24c) reduce to the following two equations:

$$-L\Delta\psi + K\Delta\chi + L\chi\Delta\Omega - L_\rho\psi_\rho - L_z\psi_z + K_\rho\chi_\rho + K_z\chi_z + L(\chi_\rho\Omega_\rho + \chi_z\Omega_z) + \chi(L_\rho\Omega_\rho + L_z\Omega_z) = -4\pi q\rho^2\mathrm{e}^{\mu} nF^{-\frac{1}{2}}, \tag{41a}$$
$$K\Delta\psi + F\Delta\chi - K\chi\Delta\Omega + K_\rho\psi_\rho + K_z\psi_z + F_\rho\chi_\rho + F_z\chi_z - \chi(K_\rho\Omega_\rho + K_z\Omega_z) - L(\Omega_\rho\psi_\rho + \Omega_z\psi_z) + L\chi(\Omega_\rho^2 + \Omega_z^2) = 0. \tag{41b}$$

The function Ω occurs in the equations (37a, b, c), (39a, b, c) and (41a, b) only through the derivatives Ω_ρ, Ω_z, $\Omega_{\rho\rho}$ and Ω_{zz}. These can be eliminated with the use of (33a, b). We assume that this has been done. Recall that only two of (37a, b, c) are independent. The six unknown functions μ, n, ψ, χ, and two of F, K, L thus satisfy the seven coupled equations (39a, b, c), (41a, b), and two of (37a, b, c). In the appendix we shall show that one of these equations is redundant. We shall further reduce the system to one of four coupled equations for the four unknowns F, K, ψ and χ.

4. Cylindrically symmetric exact solution

In this section we look for the general relativistic analogue of the cylindrically symmetric Newtonian solutions given by (20a–d) etc. These would be distinct from the cylindrically symmetric limit of the solutions found in paper I because in the latter the Lorentz force vanishes whereas this is not the case for the solutions obtained in this section. When Ω is constant, equations (29a, b) can be integrated as follows:

$$F = (q^2/m^2)\,\psi^2, \tag{42}$$

where an arbitrary constant has been absorbed in ψ, which is undefined to within an additive arbitrary constant. In the solutions obtained in paper I we have F = constant, ψ = constant. This implies that the electromagnetic force on the particles vanish and that the gravitational force is balanced by the centrifugal force. These solutions are in general non-cylindrically symmetric. If we consider the cylindrically symmetric limit of these solutions the property of the electromagnetic force vanishing continues to hold. These solutions are thus distinct from the cylindrically symmetric solutions found in this section, for which we have $F \neq$ constant, $\psi \neq$ constant. We proceed to find these solutions. We set Ω equal to a constant, and make all functions independent of z. Equation (41 b) can be integrated to give

$$K\psi_\rho + F\chi_\rho = A\rho, \tag{43}$$

where A is an arbitrary constant. Multiplying (37 a) by K and subtracting from it F times (37 b) yields, after some reduction, an equation which can be integrated to give

$$KF_\rho - FK_\rho = \rho(4A\psi + A_0), \tag{44}$$

where A_0 is another arbitrary constant. With the use of (43) and (44), equation (48 a) can be reduced to

$$-\tfrac{1}{2}mF_{\rho\rho} - \tfrac{1}{2}m\rho^{-1}F_\rho + \tfrac{3}{4}mF^{-1}F_\rho^2 - AA_0qF^{-\frac{1}{2}} - 4mA^2 = -4\pi q^2 n\,\mathrm{e}^{\mu}F. \tag{45}$$

With the use of (43) and the relation,

$$\begin{aligned} F_\rho L_\rho + K_\rho^2 &= -\rho^2F^{-2}F_\rho^2 + 2\rho F^{-1}F_\rho + F^{-2}(KF_\rho - FK_\rho)^2 \\ &= -\rho^2F^{-2}F_\rho^2 + 2\rho F^{-1}F_\rho + \rho^2F^{-2}(4A\psi + A_0)^2, \end{aligned} \tag{46}$$

equation (37 a) can be reduced to the following equation:

$$\begin{aligned} \tfrac{1}{2}F_{\rho\rho} + \tfrac{1}{2}\rho^{-1}F_\rho + \left(-\tfrac{1}{2} - \frac{1}{4}\frac{m^2}{q^2}\right)F^{-1}F_\rho^2 + 4AA_0\frac{m}{q}F^{-\frac{1}{2}} + \tfrac{1}{2}A_0^2F^{-1} \\ -A^2 + 8\frac{m^2}{q^2}A^2 = 4\pi mn\,\mathrm{e}^{\mu}F. \end{aligned} \tag{47}$$

Eliminating $n\,\mathrm{e}^{\mu}$ from (45) and (47), we get

$$(q^2 - m^2)(F_{\rho\rho} + \rho^{-1}F_\rho - F^{-1}F_\rho^2) + 2(4m^2 - q^2)A^2 + 6qmA_0AF^{-\frac{1}{2}} + A_0^2q^2F^{-1} = 0. \tag{48}$$

We have thus reduced the problem to the single equation (48), which we have been unable to solve. Once this equation is solved the other functions can be obtained easily. The solution F = constant yields the cylindrically symmetric form of the solution obtained in paper I. We can get a distinct explicit solution if we set $A_0 = 0$, and take $q^2 = 4m^2$. (This same value of charge : mass ratio gives unusual behaviour in the earlier solution; see the end of §4 of paper I.) In this instance we get the following solution:

$$\left.\begin{aligned} &F = \xi\rho^\eta, \quad K = \frac{-4A\rho^{2-\frac{1}{2}\eta}}{\xi^{\frac{1}{2}}(4-3\eta)}, \qquad \psi = \tfrac{1}{2}\xi^{\frac{1}{2}}\rho^{\frac{1}{2}\eta}, \quad \chi = \frac{2A\rho^{2-\eta}}{\xi(4-3\eta)}, \\ &\mu = (\tfrac{3}{8}\eta^2 - \eta)\ln\rho, \quad 8\pi mn = \left(-\tfrac{1}{8}\eta^2\rho^{-2} + 2A^2\,\xi^{-1}\rho^{-\eta}\right)\mathrm{e}^{-\mu}, \end{aligned}\right\} \tag{49}$$

where ξ and η are arbitrary constants.

5. On the existence of further solutions

It is possible that non-cylindrically symmetric solutions exist in general relativity that correspond to the Newtonian solutions given by (19) and related equations for constant Ω, and by (21), (22) and related equations for differential rotation.

As mentioned earlier, these general relativistic solutions would be distinct from those obtained in paper I.

For constant Ω the Newtonian solution depends on a general solution of (19). This seems a formidable problem. It is therefore unlikely that the corresponding general relativistic problem can be reduced to a convenient set of equations. However, it may be possible to find exact solutions in general relativity corresponding to some particular explicit solution of (19). Similar remarks apply to differential rotation.

An interesting class of solutions has been obtained by Winicour (1975) for differentially rotating neutral dust. These depend on a harmonic function and an arbitrary function of one variable. There may exist a class of solutions for differentially rotating charged dust which generalizes the Winicour solutions and those found in paper I. Call these new solutions the class σ. The difficulty of the Newtonian problem of differentially rotating charged dust need not deter one from seeking the class σ, for the latter may not have any Newtonian analogue, just as the solutions found in paper I do not have a Newtonian analogue. The analysis carried out in the appendix points towards the consistency of the field equations for differential rotation, and gives one some hope that it will be possible to find the class σ of solutions.

Appendix

Here we show that one of the equations (37a, b), (39a, b, c), (41a, b) is redundant, and we reduce this system to one of four coupled equations for four unknowns.

From (39a, c) we get

$$\mu_\rho = \tfrac{1}{2}\rho^{-1}(F_z L_z + K_z^2 - F_\rho L_\rho - K_\rho^2) + \rho^{-1}(KL_z - LK_z)\,\Omega_z$$
$$- \rho^{-1}(KL_\rho - LK_\rho)\,\Omega_\rho + \tfrac{1}{2}\rho^{-1}L^2(\Omega_z^2 - \Omega_\rho^2) + 2\rho^{-1}\Lambda, \quad \text{(A 1)}$$

where Λ is defined by (40). From (39a, c) we also get

$$\mu_{\rho\rho} + \mu_{zz} = \tfrac{1}{2}\rho^{-2}\Sigma - 8\pi mn\,\mathrm{e}^{\mu}, \quad \text{(A 2)}$$

where Σ is defined by (38b). Thus (39a, c) are equivalent to (A 1) and (A 2). We show that (A 2) follows from (37a, b, c), (39b), (41a, b) and (A 1). (We recall that only two of (37a, b, c) are independent). The procedure will be to evaluate $\mu_{\rho\rho} + \mu_{zz}$ with the use of (39b) and (A 1), and show that this is the same as the right hand side of (A 2).

From (39b) and (A 1) we get the following relation after some cancellations and simplification.

$$\begin{aligned}\mu_{\rho\rho}+\mu_{zz} = [& -\tfrac{1}{2}\rho^{-1}F_\rho\Delta L-\tfrac{1}{2}\rho^{-1}L_\rho\Delta F-\rho^{-1}K_\rho\Delta K-\rho^{-1}(KL_\rho-LK_\rho)\Delta\Omega \\ & -\rho^{-1}L^2\Omega_\rho\Delta\Omega+\rho^{-1}\Omega_\rho(L\Delta K-K\Delta L) \\ & +4\rho^{-1}\chi_\rho(K\Delta\psi+F\Delta\chi-K\chi\Delta\Omega)+4\rho^{-1}\psi_\rho(K\Delta\chi-L\Delta\psi+L\chi\Delta\Omega) \\ & -4\rho^{-1}\chi\Omega_\rho(K\Delta\chi-L\Delta\psi+L\chi\Delta\Omega)-\tfrac{1}{2}\rho^{-2}\Sigma+2\rho^{-1}(K_\rho L_z-L_\rho K_z)\,\Omega_z \\ & +\rho^{-1}L\{L_\rho(\Omega_z^2-\Omega_\rho^2)-2L_z\,\Omega_\rho\,\Omega_z\}+4\rho^{-2}K(\psi_\rho\chi_\rho+\psi_z\chi_z) \\ & -2\rho^{-2}L(\psi_\rho^2+\psi_z^2)+4\rho^{-2}L\chi(\psi_\rho\,\Omega_\rho+\psi_z\,\Omega_z) \\ & -4\rho^{-2}K\chi(\chi_\rho\,\Omega_\rho+\chi_z\,\Omega_z)+2\rho^{-2}F(\chi_\rho^2+\chi_z^2)-2\rho^{-2}L\chi^2(\Omega_\rho^2+\Omega_z^2) \\ & +2\rho^{-1}L\{2\chi_\rho(\psi_\rho\,\Omega_\rho-\psi_z\,\Omega_z)+2\chi_z(\psi_z\,\Omega_\rho+\psi_\rho\,\Omega_z) \\ & +2\chi\chi_\rho(\Omega_z^2-\Omega_\rho^2)-4\chi\chi_z\,\Omega_\rho\,\Omega_z\}-4\rho^{-1}K\Omega_\rho(\chi_\rho^2+\chi_z^2) \\ & +2\rho^{-1}L_\rho\{\psi_z^2-\psi_\rho^2+2\chi(\psi_\rho\,\Omega_\rho-\psi_z\,\Omega_z)+\chi^2(\Omega_z^2-\Omega_\rho^2)\} \\ & -4\rho^{-1}L_z\{\psi_\rho\psi_z-\chi(\psi_z\,\Omega_\rho+\psi_\rho\,\Omega_z)+\chi^2\Omega_\rho\,\Omega_z\} \\ & +4\rho^{-1}K_\rho\{\psi_\rho\chi_\rho-\psi_z\chi_z+\chi(\chi_z\,\Omega_z-\chi_\rho\,\Omega_\rho)\}+4\rho^{-1}K_z\{\psi_\rho\chi_z+\psi_z\chi_\rho \\ & -\chi(\chi_z\,\Omega_\rho+\chi_\rho\,\Omega_z)\}+2\rho^{-1}F_\rho(\chi_\rho^2-\chi_z^2)+4\rho^{-1}F_z\chi_\rho\chi_z],\end{aligned} \tag{A 3}$$

where the operator Δ is defined by (38a) and Σ by (38b). Now because of (34), ΔL can be expressed in terms of ΔF and ΔK modulo first derivative terms. One thus has only the four equations (37a, b) and (41a, b) to eliminate the five expressions ΔF, ΔK, $\Delta\Omega$, $\Delta\psi$ and $\Delta\chi$. This may not have been possible in general, but in fact these expressions occur in (A 3) in just such a combination as to enable one to eliminate them completely in terms of first derivative terms with the use of (37a, b) and (41a, b). Once this is done, one can show, after a considerable amount of manipulation, that the right hand side of (A 3) reduces to the right hand side of (A 2). In the final stages one has to use the equations (33a, b). Thus we have shown (A 2) to be redundant, so that the six functions μ, n, F, K, ψ, and χ satisfy the six equations given by (39b), (41a, b), (A 1) and two of (37a, b, c), assuming that L and the derivatives of Ω have been eliminated with the use of (33a, b), (34). We proceed to reduce this to a system of four equations for four unknowns.

Elimination of μ from (39b) and (A 1) leads to the following equation:

$$\begin{aligned}[& \tfrac{1}{2}L_z\Delta F+(\tfrac{1}{2}F_z+K\Omega_z)\,\Delta L+(K_z-L\Omega_z)\,\Delta K+4(L\psi_z-K\chi_z-L\chi\Omega_z)\,\Delta\psi \\ & +4(-K\psi_z-F\chi_z+K\chi\Omega_z)\,\Delta\chi+(KL_z-LK_z-4L\chi\psi_z+L^2\Omega_z+4K\chi\chi_z \\ & +4L\chi^2\Omega_z)\,\Delta\Omega-2L_z(\psi_\rho^2-\psi_z^2)+4L_\rho\,\psi_\rho\,\psi_z+4K_z(\psi_\rho\,\chi_\rho-\psi_z\,\chi_z)-4K_\rho(\chi_\rho\,\psi_z+\chi_z\,\psi_\rho) \\ & +2F_z(\chi_\rho^2-\chi_z^2)-4F_\rho\,\chi_\rho\,\chi_z+2(K_\rho\,L_z-L_\rho\,K_z)\,\Omega_\rho+4(L\chi_z+L_z\chi)\,(\psi_\rho\,\Omega_\rho-\psi_z\,\Omega_z) \\ & +LL_z(\Omega_z^2-\Omega_\rho^2)-4(L\chi_\rho+\chi L_\rho)\,(\psi_z\,\Omega_\rho+\psi_\rho\,\Omega_z)+2LL_\rho\,\Omega_\rho\,\Omega_z+4K\Omega_z(\chi_\rho^2+\chi_z^2) \\ & +4\chi K_z(\chi_z\,\Omega_z-\chi_\rho\,\Omega_\rho)+4\chi K_\rho(\chi_z\,\Omega_\rho+\chi_\rho\,\Omega_z)+2\chi(\chi L_z+2L\chi_z)\,(\Omega_z^2-\Omega_\rho^2) \\ & +4\chi(\chi L_\rho+2L\chi_\rho)\,\Omega_\rho\,\Omega_z] = 0.\end{aligned} \tag{A 4}$$

From (39b) and (A1) we get

$$\begin{aligned}& K\Delta F-F\Delta K+(2K^2+FL)\,\Delta\Omega+4K(K_\rho\,\Omega_\rho+K_z\,\Omega_z)+2F(L_\rho\,\Omega_\rho+L_z\,\Omega_z) \\ & \quad -2LK(\Omega_\rho^2+\Omega_z^2) = 4\{K(\psi_\rho^2+\psi_z^2)+F(\psi_\rho\,\chi_\rho+\psi_z\,\chi_z)-F\chi(\chi_\rho\,\Omega_\rho+\chi_z\,\Omega_z) \\ & \quad -2K\chi(\psi_\rho\,\Omega_\rho+\psi_z\,\Omega_z)+K\chi^2(\Omega_\rho^2+\Omega_z^2)\}.\end{aligned} \tag{A 5}$$

Eliminating $n\,e^{\mu}$ from (37 a) and (41 a) we get

$$\begin{aligned}[q\{&\tfrac{1}{4}\rho^2\Delta F+\rho^2(K_\rho\Omega_\rho+K_z\Omega_z)-\tfrac{1}{2}\rho^2L(\Omega_\rho^2+\Omega_z^2)+\tfrac{1}{2}\rho^2K\Delta\Omega+\tfrac{1}{4}F\Sigma-(\tfrac{1}{2}LF+K^2)(\psi_\rho^2+\psi_z^2)\\&-KF(\psi_\rho\chi_\rho+\psi_z\chi_z)+KF\chi(\chi_\rho\Omega_\rho+\chi_z\Omega_z)+(LF+2K^2)\,\chi(\psi_\rho\Omega_\rho+\psi_z\Omega_z)\\&-(\tfrac{1}{2}LF+K^2)\,\chi^2(\Omega_\rho^2+\Omega_z^2)-\tfrac{1}{2}F^2(\chi_\rho^2+\chi_z^2)\}+\tfrac{1}{2}mF^{\frac{3}{2}}\{-L\Delta\psi+K\Delta\chi+L\chi\Delta\Omega\\&-L_\rho\psi_\rho-L_z\psi_z+K_\rho\chi_\rho+K_z\chi_z+L(\chi_\rho\Omega_\rho+\chi_z\Omega_z)+\chi(L_\rho\Omega_\rho+L_z\Omega_z)\}]=0. \qquad \text{(A 6)}\end{aligned}$$

Assuming that L and the derivatives of Ω have been eliminated with the use of (33 a, b) and (34), we see that the four equations (41 b), (A 4), (A 5) and (A 6) contain only the four unknowns F, K, ψ and χ. We have thus reduced the system to one of four equations for four unknowns. The functions n and μ can be obtained trivially once F, K, ψ, χ have been determined.

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