Effective Action and Renormalization Group Flow of Anisotropic Superconductors

Steven Weinberg[1]

Theory Group, Department of Physics, University of Texas
Austin, TX, 78712

Abstract

We calculate the effective action of a superconductor, without assuming that either the electron-electron potential or the Fermi surface obey rotational invariance. This approach leads to the same gap equation and equilibrium free energy as more conventional methods. The results are used to obtain the Gell-Mann - Low renormalization group equations for the electron-electron potential.

1. Introduction

This paper aims to demonstrate the usefulness in the theory of superconductivity of the effective action formalism of quantum field theory[2]. Although the effective action may be defined non-perturbatively, for our purposes it may be taken simply as the sum of all connected one-particle-irreducible vacuum diagrams in the presence of any background field. As we shall see, the usual assumptions of the BCS superconductivity theory allow an almost trivial calculation of the effective action, from which one may obtain all of the properties of the superconductor, including the gap field, free energy, penetration depth, and so on.

As an additional application of this formalism, we shall derive the renormalization group flow of the electron-electron interaction, a subject that has received increased attention in the last few years[2]. The renormalization group equation for the electron-electron potential is obtained here from the condition that the effective action expressed as a function of the gap field should be renormalization group invariant. From a practical point of view, the most significant difference with most earlier work is that here we make no special assumptions about the form of the Fermi surface or the rotational invariance of the electron-electron potential.[3] Just as in the rotationally in-

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[2]For a lucid review of the effective action method and references to earlier work, see S. Coleman, ref. [1].

[3]After this work was completed I learned that the renormalization group equation for non-circular Fermi surfaces in two dimensions had been derived by R. Shankar, Rev. Mod. Phys., ref. [2].

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variant case, it turns out that the renormalization group equations may be expressed as de-coupled equations for an infinite number of coupling parameters.

2. Calculation of the Effective Action

As usual in the Fermi theory of liquids, we assume that all degrees of freedom may be "integrated out" except for electrons (strictly speaking, quasi-particles with the quantum numbers of electrons) in a narrow shell around a Fermi surface, say of thickness μ in electron energy. (In the approach to be used here, this condition on electron wave numbers is implemented through the renormalization procedure, and must be verified *a posteriori*.) Also as usual, we discard all interactions that become irrelevant in the limit $\mu \to 0$. This leaves just the one-electron and two-electron terms in the Lagrangian:

$$
\begin{aligned}
L = &\sum_{s',s} \int d^3x \; \psi^\dagger(\vec{x}, s') \left[\left(-i\frac{\partial}{\partial t} + eA_0(\vec{x}, t) \right) \delta_{s',s} + E_{s',s}(-i\vec{\nabla} + e\vec{A}(\vec{x}, t)) \right] \psi(\vec{x}, s) \\
&- \frac{1}{4} \sum_{s_1',s_2',s_1,s_2} \int d^3x_1' d^3x_2' d^3x_1 d^3x_2 \; V_{s_1',s_2',s_1,s_2}(\vec{x_1}', \vec{x_2}', \vec{x_1}, \vec{x_2}) \\
&\times \psi^\dagger(\vec{x_1}', s_1') \, \psi^\dagger(\vec{x_2}', s_2') \, \psi(\vec{x_1}, s_1) \, \psi(\vec{x_2}, s_2)
\end{aligned}
\tag{1}
$$

where A^μ is an external electromagnetic vector potential, and s, s', etc. are spin indices summed over the values $\pm\frac{1}{2}$. [The coefficient of the first term is adjusted to be a simple Kronecker delta by suitable definition of the electron field operator $\psi(\vec{x}, s)$. The time argument is suppressed in all field operators.] Also, as well known, the only diagrams that are not suppressed by

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powers of μ as $\mu \to 0$ are those whose structure constrains each interacting pair of electrons to have opposite momenta (for very slowly varying external fields), so that if one electron is on the Fermi surface, then the other is also. [The Fermi surface is defined by the vanishing of any eigenvalue of the energy matrix $E_{s',s}(\vec{p})$, which is understood to include a chemical potential term. Time-reversal invariance tells us that if a momentum $\vec{p}$ is on the Fermi surface, then so is $-\vec{p}$.] In particular, the unsuppressed vacuum diagrams are those that become disconnected if we cut through an interaction vertex so as to separate the incoming from the outgoing electrons.

Before setting out to calculate these diagrams, we will introduce a pair field Ψ by a familiar trick, generally attributed to Hubbard and Stratonovich[3], which we extend here to general potentials. Add a term to the Lagrangian of the form

$$
\begin{aligned}
\Delta L = \frac{1}{4} \sum_{s_1',s_2',s_1,s_2} \int d^3x_1' d^3x_2' d^3x_1 d^3x_2 \; V_{s_1',s_2',s_1,s_2}(\vec{x_1}', \vec{x_2}', \vec{x_1}, \vec{x_2}) \\
\times \, [\Psi^\dagger_{s_1',s_2'}(\vec{x_1}', \vec{x_2}') - \psi^\dagger(\vec{x_1}', s_1') \, \psi^\dagger(\vec{x_2}', s_2')] \\
\times \, [\Psi_{s_1,s_2}(\vec{x_1}, \vec{x_2}) - \psi(\vec{x_1}, s_1) \, \psi(\vec{x_2}, s_2)]
\end{aligned}
\tag{2}
$$

and integrate over the pair field $\Psi(\vec{x_1}, \vec{x_2})$ as well as over the electron field $\psi(\vec{x})$. This clearly has no effect; the action is quadratic in Ψ, so the path integral may be evaluated by setting Ψ equal to its equilibrium value

$$
\Psi_{s_1,s_2}(\vec{x_1}, \vec{x_2}) = \psi(\vec{x_1}, s_1) \, \psi(\vec{x_2}, s_2)
$$

at which (2) vanishes. Instead of integrating over the pair field, we shall

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evaluate the effective action in the presence of a background pair field, integrating over the electron field. The term (2) has been chosen to cancel the term in the Lagrangian quartic in the electron field, leaving just quadratic terms:

$$
L + \Delta L = \sum_{s',s} \int d^3x \; \psi^\dagger(\vec{x}, s') \left[\left(-i\frac{\partial}{\partial t} + eA_0(\vec{x}, t) \right) \delta_{s',s} + E_{s',s}(-i\vec{\nabla} + e\vec{A}(\vec{x}, t)) \right] \psi(\vec{x}, s)
$$
$$
-\frac{1}{4} \sum_{s_1',s_2',s_1,s_2} \int d^3x_1' d^3x_2' d^3x_1 d^3x_2 \; V_{s_1',s_2',s_1,s_2}(\vec{x_1}', \vec{x_2}', \vec{x_1}, \vec{x_2})
$$
$$
\times \left[\psi^\dagger(\vec{x_1}', s_1') \, \psi^\dagger(\vec{x_2}', s_2') \, \Psi_{s_1,s_2}(\vec{x_1}, \vec{x_2}) + \psi(\vec{x_1}, s_1) \, \psi(\vec{x_2}, s_2) \, \Psi^\dagger_{s_1',s_2'}(\vec{x_1}', \vec{x_2}') \right]
$$
$$
+\frac{1}{4} \sum_{s_1',s_2',s_1,s_2} \int d^3x_1' d^3x_2' d^3x_1 d^3x_2 \; \Psi^\dagger_{s_1',s_2'}(\vec{x_1}', \vec{x_2}') \, V_{s_1',s_2',s_1,s_2}(\vec{x_1}', \vec{x_2}', \vec{x_1}, \vec{x_2}) \, \Psi_{s_1,s_2}(\vec{x_1}, \vec{x_2})
$$

$$(3)$$

Now, the effective action in a background pair field is given by the sum of all one-particle-irreducible vacuum diagrams — that is, all vacuum diagrams that cannot be disconnected by cutting through any one internal line[1]. On the other hand, we have already mentioned that in the limit $\mu \to 0$, we are to keep only graphs that can be disconnected by slicing through any electron-electron interaction so as to separate incoming from outgoing electrons. In using (3), this means that we are to keep only graphs that are disconnected by cutting through any internal pair field line. *Since we are to keep only graphs that both are and are not disconnected by cutting through any internal pair field line, we conclude that we must keep only graphs that have no internal pair field lines at all.* There are just two such graphs; a tree graph arising

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from the last term in (3), and a one-electron-loop graph whose value is given by the determinant of the "matrix" accompanying the terms in (3) quadratic in the electron field:

$$\Gamma[\Psi] = \frac{1}{4} \sum_{s_1',s_2',s_1,s_2} \int dt \int d^3x_1' d^3x_2' d^3x_1 d^3x_2 \; \Psi^\dagger_{s_1',s_2'}(\vec{x_1'},\vec{x_2'})$$
$$\times V_{s_1',s_2',s_1,s_2}(\vec{x_1}',\vec{x_2}',\vec{x_1},\vec{x_2}) \; \Psi_{s_1,s_2}(\vec{x_1},\vec{x_2})$$
$$-\frac{i}{2} \ln \mathrm{Det} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\dagger & -\mathcal{A}^T \end{bmatrix} + \frac{i}{2} \ln \mathrm{Det} \begin{bmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A}^T \end{bmatrix} + \Gamma[0] \qquad (4)$$

where $\mathcal{A}$ and $\mathcal{B}$ are the "matrices":

$$\mathcal{A}_{s'\vec{x}'t',s\vec{x}t} = \left[\left(-i\frac{\partial}{\partial t} + eA_0(\vec{x},t)\right)\delta_{s',s} + E_{s',s}(-i\vec{\nabla} + e\vec{A}(\vec{x},t))\right]\delta^3(\vec{x}'-\vec{x})\delta(t'-t)$$

$$(5)$$

$$\mathcal{B}_{s'\vec{x}'t',s\vec{x}t} = \Delta_{s's}(\vec{x}',\vec{x})\delta(t'-t) \qquad (6)$$

and $\Delta_{s's}(\vec{x}',\vec{x})$ is the "gap" field:

$$\Delta_{s's}(\vec{x}',\vec{x}) \equiv -\frac{1}{2}\sum_{\sigma',\sigma} \int d^3y\,d^3y'\; V_{s's\sigma'\sigma}(\vec{x}',\vec{x},\vec{y}',\vec{y})\Psi_{\sigma'\sigma}(\vec{y}'\vec{y}) \qquad (7)$$

The constant $\Gamma(0)$ represents the contribution of electrons not near the Fermi surface, for which the approximations made here are not applicable.

To avoid becoming lost in a cloud of indices, we now specialize to the case of spin-independent forces and a spin-singlet pair field, writing

$$\Psi_{\frac{1}{2}\,-\frac{1}{2}} = -\Psi_{\frac{1}{2}\,\frac{1}{2}} \equiv \Psi \qquad \Psi_{\frac{1}{2}\,\frac{1}{2}} = \Psi_{-\frac{1}{2}\,-\frac{1}{2}} = 0 \qquad (8)$$

$$V_{\frac{1}{2}\,-\frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2}} - V_{\frac{1}{2}\,-\frac{1}{2}\,-\frac{1}{2}\,\frac{1}{2}} = -V_{-\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2}} + V_{-\frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2}\,\frac{1}{2}} \equiv 2V \qquad (9)$$

$$E_{s's} = E\,\delta_{s's} \qquad (10)$$

(It is easy to extend the results here to systems with a more general spin dependence, such as liquid He^3.) It follows from (8) and (9) that

$$\Delta_{\frac{1}{2} -\frac{1}{2}} = -\Delta_{-\frac{1}{2} \frac{1}{2}} \equiv \Delta \qquad \Delta_{\frac{1}{2} \frac{1}{2}} = \Delta_{-\frac{1}{2} -\frac{1}{2}} = 0 \tag{11}$$

where

$$\Delta(\vec{x'}, \vec{x}) \equiv - \int d^3y d^3y' \, V(\vec{x'}, \vec{x}, \vec{y'}, \vec{y}) \, \Psi(\vec{y'}\vec{y}) \tag{12}$$

The effective action is now

$$\Gamma[\Psi] = \int dt \int d^3x_1' d^3x_2' d^3x_1 d^3x_2 \, \Psi^\dagger(\vec{x_1'}, \vec{x_2'}) \, V(\vec{x_1}', \vec{x_2}', \vec{x_1}, \vec{x_2}) \, \Psi(\vec{x_1}, \vec{x_2})$$
$$-i \ln \text{Det} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\dagger & -\mathcal{A}^T \end{bmatrix} + i \ln \text{Det} \begin{bmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A}^T \end{bmatrix} + \Gamma[0] \tag{13}$$

$$\mathcal{A}_{\vec{x'}t', \vec{x}t} = \left[-i\frac{\partial}{\partial t} + e\mathcal{A}_0(\vec{x}, t) + E(-i\vec{\nabla} + e\vec{A}(\vec{x}, t)) \right] \delta^3(\vec{x'} - \vec{x}) \delta(t' - t) \tag{14}$$

$$\mathcal{B}_{\vec{x'}t', \vec{x}t} = \Delta(\vec{x'}, \vec{x}) \delta(t' - t) \tag{15}$$

3. Translationally Invariant Equilibrium

First we consider the translationally invariant case, with no external electromagnetic fields. Then the pair and gap fields can be put in the form

$$\Psi(\vec{x'}, \vec{x}) = \int d^3p \, e^{i\vec{p} \cdot (\vec{x'} - \vec{x})} \Psi(\vec{p}) \tag{16}$$

$$\Delta(\vec{x'}, \vec{x}) = (2\pi)^{-3} \int d^3p \, e^{i\vec{p} \cdot (\vec{x'} - \vec{x})} \Delta(\vec{p}) \tag{17}$$

The electron-electron potential appears here in the form

$$\int d^3x_1 d^3x_2 d^3x_3 d^3x_4 \, e^{i\vec{p'} \cdot (\vec{x_1} - \vec{x_2})} \, e^{i\vec{p} \cdot (\vec{x_3} - \vec{x_4})} \, V(x_1, x_2, x_3, x_4)$$
$$\equiv \Omega_4 \, V(\vec{p'}, \vec{p}) \tag{18}$$

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where Ω_4 is the spacetime volume

$$\Omega_4 \equiv \int d^3x \int dt \tag{19}$$

The effective potential $U[\Psi]$ is defined[1] as minus the effective action per spacetime volume

$$U[\Psi] \equiv -\Gamma[\Psi]/\Omega_4$$
$$= -\int d^3p\, d^3p'\ \Psi^*(\vec{p'})V(\vec{p'},\vec{p})\Psi(\vec{p})$$
$$+\frac{i}{(2\pi)^4}\int d\omega d^3p\ \ln\left(1-\frac{|\Delta(\vec{p})|^2}{\omega^2-E^2(\vec{p})+i\epsilon}\right) + U[0] \tag{20}$$

with

$$\Delta(\vec{p}) = -\int d^3p'\ V(p,\vec{p'})\,\Psi(\vec{p'}) \tag{21}$$

(We are working here at zero temperature. For non-zero temperature, the integral over ω is of course replaced with a sum over discrete Matsubara frequencies.) Wick rotating, integrating over ω, and expressing Ψ in terms of Δ, this becomes

$$U[\Delta] = -\int d^3p\, d^3p'\ \Delta^*(\vec{p'})V^{-1}(\vec{p'},\vec{p})\Delta(\vec{p})$$
$$-\frac{1}{(2\pi)^3}\int d^3p\left[\sqrt{E^2(\vec{p})+|\Delta(\vec{p})|^2}\ -\ E(\vec{p})\right]\ + U[0] \tag{22}$$

We pause to note that the gap equation is obtained from the condition that $\Delta(\vec{p})$ be at the stationary point $\Delta_0(\vec{p})$ of $U[\Delta]$:

$$0 = \left.\frac{\delta U[\Delta]}{\delta\Delta^*(\vec{p})}\right|_{\Delta=\Delta_0}$$
$$= -\int d^3p'\ V^{-1}(\vec{p},\vec{p'})\Delta_0(\vec{p'}) - \frac{1}{2(2\pi)^3}\frac{\Delta_0(\vec{p})}{\sqrt{E^2(\vec{p})+|\Delta_0(\vec{p})|^2}}$$

or in a more familiar form

$$\Delta_0(\vec{p}) = -\frac{1}{2(2\pi)^3} \int d^3p' \frac{V(\vec{p}, \vec{p'})\, \Delta_0(\vec{p'})}{\sqrt{E^2(\vec{p'}) + |\Delta_0(\vec{p'})|^2}} \tag{23}$$

Also, the effective potential at this stationary "point" is the free energy density, which using (23) may be expressed in terms of the gap field:

$$F_{equilibrium} - F_{\Delta=0} = U[\Delta_0] - U[0]$$

$$= \frac{1}{(2\pi)^3} \int d^3p \left[\frac{|\Delta_0(\vec{p})|^2}{2\sqrt{E^2(\vec{p}) + |\Delta_0(\vec{p})|^2}} - \sqrt{E^2(\vec{p}) + |\Delta_0(\vec{p})|^2} + E(\vec{p}) \right] \tag{24}$$

This is the same as the result normally derived from the BCS ground state wave function[4]. However, it should be noted that (22) is *not* the same as the formula for the free energy as a functional of $\Delta(\vec{p})$ calculated from the BCS wave function[4]:

$$F[\Psi] = + \int d^3p\, d^3p'\, \Psi^*(\vec{p'})\, V(\vec{p'}, \vec{p})\, \Psi(\vec{p}) + \frac{1}{(2\pi)^3} \int d^3p\, E(\vec{p}) \left[1 - \sqrt{1 - 4(2\pi)^6 |\Psi(\vec{p})|^2} \right] \, . \tag{25}$$

This difference arises because the effective potential does not have the interpretation of an energy density for a composite field like $\Psi(\vec{p})$ or $\Delta(\vec{p})$, except at the stationary point of the effective potential[6]. Nevertheless, both (22) and (25) yield the same gap equation, and the same value for the equilibrium free energy.

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[4]This may be obtained e. g. from Eq. (4-64) of ref. [5], using the relations $\Psi_k = u_k v_k$ and $u_k^2 + v_k^2 = 1$ to express the coefficients u_k and v_k in terms of the pair field Ψ_k, and then converting to the notation of the present paper by replacing sums over the discrete index k with integrals over $\vec{p}$, and inserting appropriate factors of $(2\pi)^3$.

So far, the limitation of electron momenta to a thin shell around the Fermi surface has been left implicit. To make this limitation explicit, we will write the electron momenta as

$$\vec{p} = \vec{k} + \hat{n}(\vec{k})\ell \qquad d^3p = d^2k\,d\ell \qquad (26)$$

where $\vec{k}$ is on the Fermi surface (that is, $E(\vec{k}) = 0$), and $\hat{n}(\vec{k})$ is the unit vector normal to the Fermi surface at $\vec{k}$. [Here d^2k should be understood as $Jd\theta_1 d\theta_2$, where θ_1 and θ_2 are coordinates on the Fermi surface, and J is the Jacobian of the transformation from $\vec{p}$ to ℓ, θ_1, and θ_2. For a spherical Fermi surface, $\int d^2k$ just gives a factor $4\pi k_F^2$.] We will temporarily impose the condition that electron momenta are close to the Fermi surface by introducing a cut-off Λ on ℓ, chosen small enough so that $\Lambda \ll |\vec{k}|$ for all $\vec{k}$ on the Fermi surface, and so that $V(\vec{p'},\vec{p})$ and $\Delta(\vec{p})$ change negligibly as ℓ and ℓ' vary from 0 to Λ. [This cut-off will eventually be obviated by the introduction of a renormalized electron-electron potential.] With this cut-off, we may approximate

$$E(\vec{p}) = v_F(\vec{k})\ell \qquad (27)$$

where v_F is the Fermi velocity:

$$v_F(\vec{k}) = \left.\frac{\partial E(\vec{k} + \hat{n}(\vec{k})\ell)}{\partial \ell}\right|_{\ell=0} \qquad (28)$$

Eq. (22) may now be written

$$U[\Delta] = -\Lambda^2 \int d^2k\,d^2k'\,\Delta^*(\vec{k'})V^{-1}(\vec{k'},\vec{k})\Delta(\vec{k})$$
$$-\frac{1}{(2\pi)^3}\int_0^\Lambda d\ell \int d^2k\,\left[\sqrt{\ell^2 v_F^2(\vec{k}) + |\Delta(\vec{k})|^2} - \ell v_F(\vec{k})\right] + U[0] \quad (29)$$

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so that $U[\Delta]$ is now defined as a functional of the gap field on the Fermi surface only. We define a renormalized electron-electron potential at a renormalization scale μ by

$$V_\mu^{-1}(\vec{k'}, \vec{k}) \equiv -\frac{\delta^2 U[\Delta]}{\delta\Delta^*(\vec{k'})\delta\Delta(\vec{k})}\bigg|_{\Delta(\vec{k})=\Delta(\vec{k})^*=\mu} \tag{30}$$

When we express the electron-electron potential V in terms of the renormalized potential V_μ, Eq. (29) for the effective potential becomes:

$$U[\Delta] = -\int d^2k\, d^2k'\, \Delta^*(\vec{k'})V_\mu^{-1}(\vec{k'}, \vec{k})\Delta(\vec{k})$$
$$-\frac{1}{(2\pi)^3}\int_0^\Lambda d\ell \int d^2k\, \left[\sqrt{\ell^2 v_F^2(\vec{k}) + |\Delta(\vec{k})|^2} - \ell v_F(\vec{k})\right.$$
$$\left.-\frac{|\Delta(\vec{k})|^2}{2(\ell^2 v_F^2(\vec{k}) + \mu^2)^{1/2}} + \frac{\mu^2|\Delta(\vec{k})|^2}{4(\ell^2 v_F^2(\vec{k}) + \mu^2)^{3/2}}\right] + U[0] \tag{31}$$

The one-loop integral over ℓ now converges as $\ell \to \infty$, so we may remove the cut-off, and find

$$U[\Delta] = -\int d^2k\, d^2k'\, \Delta^*(\vec{k'})V_\mu^{-1}(\vec{k'}, \vec{k})\Delta(\vec{k})$$
$$+\frac{1}{2(2\pi)^3}\int d^2k\, \frac{|\Delta(\vec{k})|^2}{v_F(\vec{k})}\left[\ln\left(\frac{|\Delta(\vec{k})|}{\mu}\right) - 1\right] + U[0] \tag{32}$$

The corresponding gap equation is obtained from the condition that this expression for the effective potential be stationary:

$$\Delta_0(\vec{k}) = \frac{1}{2(2\pi)^3}\int d^2k'\, V_\mu(\vec{k}, \vec{k'})\, v_F^{-1}(\vec{k'})\Delta_0(\vec{k'})\left[\ln\left(\frac{|\Delta_0(\vec{k'})|}{\mu}\right) - \frac{1}{2}\right] \tag{33}$$

and the equilibrium free energy is

$$F_{equilibrium} = F_{\Delta=0} - \frac{1}{4(2\pi)^3}\int d^2k\, \frac{|\Delta_0(\vec{k})|^2}{v_F(\vec{k})} \tag{34}$$

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These very simple results should not be taken entirely literally, because the effective potential will always contain "irrelevant" terms of higher order in Δ arising from degrees of freedom that have been integrated out here. In particular, note that according to (32) the difference $U[\Delta] - U[0]$ vanishes for $\Delta \to 0$, is negative for sufficiently small Δ, and goes to $+\infty$ for $\Delta \to \infty$, so it must have a stationary point $\Delta_0 \neq 0$, at which the gap equation (33) is satisfied, for any renormalized electron-electron potential, whether repulsive or attractive. But this solution should not be taken seriously if it has Δ_0 so large that it is outside the range of validity of these equations. To be specific, it is easy to see from (32) that if $V_\mu(\vec{k'}, k)$ is positive (in the matrix sense) for some μ, then the minimum $\Delta_0(\vec{k})$ of the effective potential will have a scale $||\Delta_0|| \geq \sqrt{e}\mu$, where the "scale" $||\Delta||$ of an arbitrary function $\Delta(\vec{k})$ is defined by the condition

$$\int d^2k \; \frac{|\Delta(\vec{k})|^2}{v_F(\vec{k})} \; \ln\left(\frac{|\Delta(\vec{k})|}{||\Delta||}\right) \equiv 0.$$

In particular, if $V_\mu(\vec{k'}, k)$ is a positive "matrix" for μ of the order of the Debye frequency, then the solution of the gap equation (33) is physically irrelevant.

4. Renormalization Group Flow

In the Wilson approach to the renormalization group that is most familiar in condensed matter physics, we would keep the cut-off Λ finite, and derive a renormalization group equation for the Λ-dependence of the unrenormalized electron-electron potential $V(\vec{k'}, \vec{k})$ from the condition that the effective po-

tential (29) should be Λ-independent. This approach would be possible here, but it would require the introduction of "irrelevant" terms in $U[\Delta]$ of higher order in $\Delta(\vec{k})$ to keep $U[\Delta]$ Λ-independent for finite Λ. It is much simpler to apply the older approach of Gell-Mann and Low: the renormalization group equation in this approach is the condition that $U[\Delta]$ is independent of the arbitrary renormalization scale μ:

$$\mu \frac{d}{d\mu} V_\mu^{-1}(\vec{k'}, \vec{k}) = -\frac{\delta^2(\vec{k'} - \vec{k})}{2(2\pi)^3 v_F(\vec{k})} \tag{35}$$

or equivalently

$$\mu \frac{d}{d\mu} V_\mu(\vec{k'}, \vec{k}) = \frac{1}{2(2\pi)^3} \int d^2k'' \, V_\mu(\vec{k'}, \vec{k''}) \, v_F^{-1}(\vec{k''}) \, V_\mu(\vec{k''}, \vec{k}) \tag{36}$$

This of course also implies that the solution of the gap equation is independent of μ. For the special case of a spherical Fermi surface Eq. (36) is a continuous version of the discrete renormalization group equation of Benfatto and Gallavotti[2], provided we identify their constant β with $1/2(2\pi)^3 v_F$. Eq. (36) also agrees with the results of Shankar[2], with fairly obvious changes to convert his results from two to three dimensions.

These renormalization group equations can be separated into equations for the eigenvalues $\lambda_n(\mu)$ of the Hermitian kernel

$$K_\mu(\vec{k'}, \vec{k}) \equiv \frac{1}{2(2\pi)^3} \, v_F^{-1/2}(\vec{k'}) \, v_F^{-1/2}(\vec{k}) \, V_\mu(\vec{k'}, \vec{k}) \ . \tag{37}$$

From either (35) or (36), we see that the eigenvectors of K are renormalization-group invariant, while the eigenvalues are governed by the flow equations

$$\mu \frac{d}{d\mu} \lambda_n(\mu) = \lambda_n^2(\mu) \ . \tag{38}$$

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For a spherical Fermi surface these eigenvectors are just the spherical harmonics $Y_\ell^m(\hat{k})$, but we see that the decoupling of the eigenmodes of K is in fact quite general, not depending on rotational invariance.

A completely repulsive potential may be defined as one for which all eigenvectors λ_n are positive. If this is true at some starting scale μ_0 (say, the Debye frequency) then as $\mu \to 0$ the eigenvalues stay positive and become smaller, so nothing interesting happens. This conclusion may be altered by the inclusion of formally irrelevant electron-electron couplings, as in the Kohn-Luttinger effect, but is not directly affected by anisotropies. A completely or partly attractive potential is one for which all or some of the eigenvalues of K are negative. Any eigenvalue that is negative at some μ_0 becomes infinite at a finite $\mu < \mu_0$, with the eigenvalue that is largest in magnitude becoming infinite first. As already mentioned, this is the case where superconductivity actually occurs.

It is not clear what is gained by this renormalization group analysis. Within Fermi liquid theory, the effective potential arises solely from tree and one-loop graphs, so we do not need to use the renormalization group to make the sort of improvement in perturbation theory familiar in quantum electrodynamics, or to identify a weak coupling regime, as in quantum chromodynamics or the theory of critical phenomena. Of course, one can go beyond tree and one-loop graphs, but this would require that we take into account irrelevant terms in the original Lagrangian, which would involve ad-

ditional microscopic information, not just renormalization group equations.

5. Slowly Varying Electromagnetic and Goldstone Fields

Now let us return to the general case of a superconductor in a translationally-non-invariant external electromagnetic field. In the limit where this field has very small frequencies and wave numbers (smaller than the inverse correlation length), we can integrate out all degrees of freedom except those that have zero "mass," in the sense that their frequency vanishes in the limit of vanishing wave number. For a superconductor that is not close to a phase transition at which superconductivity is lost, the only such "massless" degree of freedom is the Goldstone mode, associated with the spontaneous breakdown of electromagnetic gauge invariance within the superconductor. All of the classic exact properties of superconductors (persistent currents, Meissner effect, flux quantization, and Josephson frequency) can be derived by considering only general properties of the effective action for the Goldstone mode in the presence of external electromagnetic fields[7]. But to derive values for quantities like the penetration depth in Type II superconductors, we need a detailed formula for the effective action.

The effective action for a Goldstone mode is in general obtained by setting all the fields in the effective action equal to their equilibrium values, and then subjecting them to a symmetry transformation with space-time dependent parameters equal to the Goldstone fields. In our case, where the broken

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symmetry is electromagnetic gauge invariance, this means that we must make the replacement

$$\Delta(\vec{x'}, \vec{x}, t) \rightarrow e^{-ie\phi(\vec{x'}, t)} \Delta_0(\vec{x'} - \vec{x}) e^{-ie\phi(\vec{x}, t)} \tag{39}$$

where $\phi(\vec{x})$ is the Goldstone field, and $\Delta_0(\vec{x'} - \vec{x})$ is the equilibrium gap field, given by the Fourier transform of the solution of the gap equation:

$$\Delta_0(\vec{x'} - \vec{x}) = (2\pi)^{-3} \int d^3p \ e^{i\vec{p} \cdot (\vec{x'} - \vec{x})} \Delta_0(\vec{p}) \tag{40}$$

Electromagnetic gauge invariance[5] then allows us to remove the factors $e^{-ie\phi(\vec{x'}, t')}$ and $e^{-ie\phi(\vec{x}, t)}$ in Eq. (39), by replacing the electromagnetic vector potential $A_\mu(\vec{x}, t)$ with $A_\mu(\vec{x}, t) - \partial_\mu \phi(\vec{x}, t)$. In this way, Eqs. (13)-(15) yield the effective action:

$$\Gamma[\phi, A] = \Gamma_{\Delta=0}[A] - i \ln \mathrm{Det} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\dagger & -\mathcal{A}^T \end{bmatrix} + i \ln \mathrm{Det} \begin{bmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A}^T \end{bmatrix} \tag{41}$$

where now

$$
\begin{aligned}
\mathcal{A}_{\vec{x'}t', \vec{x}t} &= \left[-i\frac{\partial}{\partial t} + eA_0(\vec{x}, t) - e\dot{\phi}(\vec{x}, t) + E(-i\vec{\nabla} + e\vec{A}(\vec{x}, t) - e\vec{\nabla}\phi(\vec{x}, t)) \right] \\
&\times \quad \delta^3(\vec{x'} - \vec{x})\delta(t' - t)
\end{aligned} \tag{42}
$$

$$\mathcal{B}_{\vec{x'}t', \vec{x}t} = \Delta_0(\vec{x'} - \vec{x})\delta(t' - t) \tag{43}$$

Quantitative properties of the superconductor such as the penetration depth can be read off from the expansion of Eq. (41) in powers of $A_0(\vec{x}, t) - \dot{\phi}(\vec{x}, t)$ and $\vec{A}(\vec{x}, t) - \vec{\nabla}\phi(\vec{x}, t)$.

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[5]The Lagrangian (1) is gauge invariant for either a local electron-electron potential, or an arbitrary electron-electron potential with a suitable dependence on the external electromagnetic fields.

Acknowledgment

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