PHENOMENOLOGICAL LAGRANGIANS*

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1. Introduction: A reminiscence

Julian Schwinger's ideas have strongly influenced my understanding of phenomenological Lagrangians since 1966, when I made a visit to Harvard. At that time, I was trying to construct a phenomenological Lagrangian which would allow one to obtain the predictions of current algebra for soft pion matrix elements with less work, and with more insight into possible corrections. It was necessary to arrange that the pion couplings in the Lagrangian would all be derivative interactions, to suppress the incalculable graphs in which soft pions would be emitted from internal lines of a hard-particle process. The mathematical approach I followed[1] at first was quite clumsy; I started with the old σ-model[2], in which the pion is in a chiral quartet with a 0+ isoscalar σ; then performed a space-time dependent chiral rotation which transformed $\{\pi, \sigma\}$ everywhere into $\{0, \sigma'\}$ with $\sigma' \equiv (\sigma^2 + \pi)^{1/2}$; and then re-introduced the pion field as the chiral rotation "angle". The Lagrangian obtained in this way had a complicated and unfamiliar non-linear structure, but it did have the desired property of derivative coupling, because any space-time independent part of the rotation "angle" would correspond to a symmetry of the theory, and so would not contribute to the Lagrangian.

Schwinger suggested to me that one might be able to construct a suitable phenomenological Lagrangian directly, by introducing a pion field which from the beginning would have the non-linear transformation property of chiral rotation angles, and then just obeying the dictates of chiral symmetry for such a pion field[3]. Following this suggestion, I worked out a general theory[4] of non-linear realizations of chiral $SU(2) \times SU(2)$, which was soon after generalized to arbitrary groups in elegant papers of Callan, Coleman, Wess, and Zumino[5], and has since been applied by many authors[6]. The importance of the approach suggested by Schwinger has been not only that it saves the work

involved in the transition from an ordinary linear representation like $\{\pi, \sigma\}$ to a non-linear realization, but more important, that it makes clear that the interactions of other hadrons with soft pions does not in any way depend on the chiral transformation properties of whatever fields are associated with these hadrons, but only on their isospin.

In the decade since 1967, Schwinger's ideas have evolved into what he calls "source theory"[7]. I have been pretty much out of touch with this work, mostly because of an involvement with other lines of research, but perhaps also because I found Schwinger's conceptual framework unfamiliar. Recently, several problems have led me to think again about the use of phenomenological Lagrangians, and I find that my ideas have shifted somewhat, to a point of view that seems to me to be now not too different from the point of view of source theory.

To summarize: section 2 presents an argument that phenomenological Lagrangians can be used not only to reproduce the soft pion results of current algebra, but also to *justify* these results, without any use of operator algebra. Section 3 shows how phenomenological Lagrangians can be used to calculate corrections to the leading soft pion results to any desired order in external momenta. In section 4, the renormalization group is used to elucidate the structure of these corrections. Corrections due to the finite mass of the pion are treated in section 5. Section 6 offers speculations about a possible other application of phenomenological Lagrangians.

This article is intended as a review – I doubt that any of the material presented here is entirely new. In particular, although I have not tried here to judge the extent to which the ideas described below overlap those of source theory, I would not be surprised to find that these are points which long ago appeared in Schwinger's work. In that case, I hope that he will take this paper as a little work of translation into the Vulgate, offered as a birthday present to an old friend.

2. Current algebra without current algebra

It is well known that matrix elements for soft-pion interactions can be obtained by "current algebra", that is, by a direct use of the commutation and conservation relations of the vector and axial-vector currents of chiral $SU(2) \times SU(2)$. It is also well known that the same matrix elements may also be calculated (usually more easily) from the tree graphs in an $SU(2) \times SU(2)$-invariant phenomenological Lagrangian. However, it has been widely supposed[8] that the ultimate justification of the results obtained from a phenomenological Lagrangian rests on the foundation of current algebra.

According to this method of derivation, one must first use current algebra

to show that soft-pion matrix elements are uniquely determined by the properties of the currents plus certain "smoothness" properties of the matrix elements. One then reflects that any chiral-invariant Lagrangian will have currents with the assumed properties, and that the tree graphs in such a theory will have the assumed smoothness properties. It follows that the matrix elements computed from these tree graphs must automatically reproduce the results of current algebra.

I would like to show in this section that the use of current algebra in the above line of reasoning is actually unnecessary. That is, the phenomenological Lagrangians themselves can be used to justify the calculation of soft-pion matrix elements from the tree graphs, without any use of operator algebra.

This remark is based on a "theorem", which as far as I know has never been proven, but which I cannot imagine could be wrong. The "theorem" says that although individual quantum field theories have of course a good deal of content, quantum field theory itself has no content beyond analyticity, unitarity, cluster decomposition, and symmetry. This can be put more precisely in the context of perturbation theory: if one writes down the most general possible Lagrangian, including *all* terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. As I said, this has not been proved, but any counterexamples would be of great interest, and I do not know of any.

With this "theorem", one can obtain and *justify* the results of current algebra simply by writing down the most general Lagrangian consistent with the assumed symmetry principles, and then deriving low energy theorems by a direct study of the Feynman graphs, without operator algebra. However, in order for this to be a derivation and not merely a mnemonic, it is necessary to include all possible terms in the Lagrangian, and take account of graphs of all orders in perturbation theory.

To illustrate this procedure, let us consider a theory of massless pions, governed by a chiral $SU(2) \times SU(2)$ symmetry, for which the pions serve as Goldstone bosons. For simplicity, we will "integrate out" whatever other degrees of freedom may be present – nucleons, ρ mesons, σ mesons, etc. – and consider only the pions. The Lagrangian will be $SU(2) \times SU(2)$-invariant provided it conserves isospin, and is constructed only from a chiral-covariant derivative of the pion field, which by a suitable definition of the pion field may be taken in the form[4])

$$D_\mu \pi = (\partial_\mu \pi)/(1 + \pi^2). \tag{1}$$

The most general such Lagrangian is an infinite series of operators of higher and higher dimensionality[9])

$$\mathcal{L} = -\tfrac{1}{2}g_2 D_\mu \boldsymbol{\pi} \cdot D^\mu \boldsymbol{\pi} - \tfrac{1}{4}g_4^{(1)}(D_\mu \boldsymbol{\pi} \cdot D^\mu \boldsymbol{\pi})^2$$
$$-\tfrac{1}{4}g_4^{(2)}(D_\mu \boldsymbol{\pi} \cdot D_\nu \boldsymbol{\pi})(D^\mu \boldsymbol{\pi} \cdot D^\nu \boldsymbol{\pi}) + \cdots, \tag{2}$$

where $g_d^{(n)}$ are constants of dimensionality $[\text{mass}]^{4-d}$. Since the field $\boldsymbol{\pi}$ is dimensionless, d is just equal to the number of derivatives in the interaction. As is well known, the constant g_2 is related to the pion decay amplitude $F_\pi \simeq 190$ MeV by

$$g_2 = F_\pi^2 \tag{3}$$

but it will be more convenient here to treat g_2 in parallel with the other couplings.

According to the "theorem" quoted at the beginning of this section, such a general Lagrangian has no specific dynamical content beyond the general principles of analyticity, unitarity, cluster decomposition, Lorentz invariance, and chirality, so that when it is used to calculate pionic S-matrix elements, it yields the most general matrix elements consistent with these general principles, provided that all terms of all orders in all couplings g_2, $g_4^{(1)}$, $g_4^{(2)}$, etc., are included. One does not need the methods of current algebra for justification here; the Lagrangian (2) is so general that the only conclusions that can be drawn from it are just those which follow from the general principles with which we started.

All this becomes of practical value in the calculation of matrix elements for pions of low energy. Consider the matrix element for a process involving N_e external pion lines, carrying energies proportional to some energy scale E. Such a matrix element will have dimensionality $[\text{mass}]^{D_1}$, where

$$D_1 = 4 - N_e. \tag{4}$$

The coupling constants contributing to a given term in such a matrix element will altogether have dimensionality $[\text{mass}]^{D_2}$, where

$$D_2 = \sum_d N_d(4 - d) - 2N_i - N_e. \tag{5}$$

Here N_d is the number of vertices formed from interactions with d derivatives, and N_i is the total number of internal pion lines. (The terms $-2N_i$ and $-N_e$ appear here because the pion field $\boldsymbol{\pi}$ has an unconventional normalization, with a propagator proportional to $1/g_2$ and external line "wave functions" proportional to $1/\sqrt{g_2}$. We could have used a conventionally normalized pion field $\sqrt{g_2}\boldsymbol{\pi}$, in which case the propagators and external lines would not contribute factors involving g_2, but such factors would instead be

contributed by the pion fields in the interactions. The final answer is of course the same.) Ultraviolet divergences are to be absorbed into a renormalization of the infinite number of coupling parameters, defined at renormalization points with momenta proportional to some common renormalization energy scale μ. The only quantities with non-vanishing dimensionality are the common energy scale E, the common renormalization scale μ, and the coupling constants themselves, so each term in the matrix element must take the form

$$M = E^D f(E/\mu) \tag{6}$$

with

$$D = D_1 - D_2 = 4 + \sum_d N_d(d-4) + 2N_i. \tag{7}$$

This can be conveniently re-written by using the well-known formula for the number of loops in a graph

$$N_L = N_i - \sum_d N_d + 1. \tag{8}$$

We find then

$$D = 2 + \sum_d N_d(d-2) + 2N_L. \tag{9}$$

Now suppose that the characteristic pion energy E is very small, and take the renormalization scale μ to be of order E. From (6), we see that the dominant graphs will then be those with the smallest values for the exponent D. According to eq. (9), these are just the tree graphs (i.e., $N_L = 0$) formed purely from the term in the Lagrangian with the lowest possible number d of derivatives, the $d = 2$ term

$$\mathcal{L}_2 = -\tfrac{1}{2}g_2 D_\mu \pi D^\mu \pi. \tag{10}$$

Thus without using the methods of current algebra, we arrive at the same conclusion, that matrix elements for soft pion processes may be calculated from the effective Lagrangian (10), keeping only tree graphs.

3. Corrections to soft-pion results

The real virtue of the phenomenological Lagrangian approach described in the preceding section is not that it provides an alternative derivation of a known result, but that it allows us in a systematic way to calculate corrections to this result.

For example, suppose that we want to calculate the matrix element for pion–pion scattering with Mandelstam variables s, t, u all of the same order of magnitude, and all very small. As we have seen, eq. (9) tells us that the leading term (which here is of order s) can be calculated using the $d = 2$ term (10) in the tree approximation. This gives the known matrix element[10])

$$M^{(1)}_{abcd} = 4g_2^{-1}[\delta_{ab}\delta_{cd}s + \delta_{ac}\delta_{bd}t + \delta_{ad}\delta_{bc}u]. \tag{11}$$

In the notation used here, a, b, c, d are isovector indices associated with pion lines having four-momenta P_A, P_B, P_C, P_D respectively and $s = -(P_A + P_B)^2$, $t = -(P_A - P_C)^2$, $u = -(P_A - P_D)^2$. We here set the pion masses equal to zero, so $s + t + u$ vanishes. Also, the pion–pion scattering matrix element M is normalized so that the S-matrix element is

$$S = i(2\pi)^4\delta^4(P_A + P_B - P_C - P_D)M(2\pi)^{-6}(16E_AE_BE_CE_D)^{-1/2}.$$

Now, suppose that we want also to calculate the corrections of order $s^2 \sim E^4$. Equation (9) tells us that there will arise from graphs in which there are any number of couplings (10) with $d = 2$, and either *one* vertex with $d = 4$ or *one* loop. A straight-forward calculation gives the order-s^2 corrections to M as:

$$M^{(2)}_{abcd} = \frac{\delta_{ab}\delta_{cd}}{g_2^2}\left[-\frac{1}{2\pi^2}s^2\ln(-s) - \frac{1}{12\pi^2}(u^2 - s^2 + 3t^2)\ln(-t)\right.$$

$$-\frac{1}{12\pi^2}(t^2 - s^2 + 3u^2)\ln(-u) + \frac{1}{\pi^2}(\tfrac{1}{3}s^2 + \tfrac{1}{3}t^2 + \tfrac{1}{3}u^2)\ln\Lambda^2$$

$$\left. -\tfrac{1}{2}g_4^{(1)}s^2 - \tfrac{1}{4}g_4^{(2)}(t^2 + u^2) \right] + \text{crossed terms}. \tag{12}$$

Here Λ is an ultraviolet cut-off, and "crossed terms" denotes terms given by the interchanges $b \leftrightarrow c$ and $s \leftrightarrow t$ or $b \leftrightarrow d$ and $s \leftrightarrow u$. The divergence may be eliminated by defining renormalized coupling constants

$$g_4^{(1)}(\mu) \equiv g_4^{(1)} - \frac{2}{3\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right), \tag{13}$$

$$g_4^{(2)}(\mu) \equiv g_4^{(2)} - \frac{4}{3\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right), \tag{14}$$

so that (12) becomes

$$M^{(2)}_{abcd} = \frac{\delta_{ab}\delta_{cd}}{g_2^2}\left[-\frac{1}{2\pi^2}s^2\ln\left(\frac{-s}{\mu^2}\right) - \frac{1}{12\pi^2}(u^2 - s^2 + 3t^2)\ln\left(\frac{-t}{\mu^2}\right)\right.$$

$$\left. -\frac{1}{12\pi^2}(t^2 - s^2 + 3u^2)\ln\left(\frac{-u}{\mu^2}\right) - \tfrac{1}{2}g_4^{(1)}(\mu)s^2 - \tfrac{1}{4}g_4^{(2)}(\mu)(t^2 + u^2) \right]$$

$$+ \text{crossed terms}. \tag{15}$$

It is true that (15) has a polynomial part with unknown coefficients, but it is far from an empty formula: the logarithmic terms have coefficients given by eq. (15) as definite functions of g_2. It is not surprising that chiral symmetry should have consequences of this sort, for the logarithmic branch points arise from intermediate states consisting of soft pion pairs[16]), and the matrix elements for producing and absorbing these soft pions are determined by chiral symmetry. What is noteworthy is that the coefficients of the logarithmic terms can be calculated in detail so easily, by a one-loop calculation using a suitable phenomenological Lagrangian.

4. Application of the renormalization group

Gell-Mann and Low showed long ago[11]) how renormalization group techniques could be used to gain information about the perturbation series for quantum electrodynamics. Without having to calculate Feynman graphs, they were able to show that the photon propagator $\Delta(s)$ is linear in $\log s$ in second order, linear in $\log s$ in fourth order, and quadratic in $\log s$ in sixth order, with the coefficient of $\log^2 s$ determined by the product of the coefficients of $\log s$ in second and fourth order. In much the same way, we can use renormalization group techniques to get detailed information about the perturbation series generated by a non-renormalizable phenomenological Lagrangian.

For illustration, let us consider the matrix element for an arbitrary reaction among soft pions, but now fix all scattering angles and isospin indices, so that the scattering amplitude can be written as function of the center-of-mass energy $E \equiv \sqrt{s}$ alone. According to eq. (9), terms of order E^2 arise from tree graphs involving only g_2; terms of order E^4 arise from one-loop graphs involving only g_2 plus tree graphs linear in $g_4^{(1)}(\mu)$ or $g_4^{(2)}(\mu)$; terms of order E^6 arise from two-loop graphs involving only g_2, plus one-loop graphs linear in $g_4^{(1)}$ or $g_4^{(2)}$, plus tree graphs linear in $g_6^{(n)}$ or the quadratic in the $g_4^{(n)}$; and so on. Hence the matrix element takes the form

$$M(E) = g_2^{1-N_c/2}\left\{ c_2 E^2 + g_2^{-1} E^4 \left[F\left(\frac{E}{\mu}\right) + \sum_n c_4^{(n)} g_4^{(n)}(\mu) \right] \right.$$

$$+ g_2^{-2} E^6 \left[H\left(\frac{E}{\mu}\right) + \sum_n J^{(n)}\left(\frac{E}{\mu}\right) g_4^{(n)}(\mu) + g_2 \sum_n c_6^{(n)} g_6^{(n)}(\mu) \right.$$

$$\left. \left. + \sum_{nm} C_6^{(nm)} g_4^{(n)}(\mu) g_4^{(m)}(\mu) \right] + \cdots \right\}. \tag{16}$$

Here the c_i are dimensionless functions of angle and isospin variables. The dimensionless functions F, H, J, etc., arise from loop graphs, and depend only on angle and isospin variables and on the dimensionless ratio E/μ.

The essential idea of the renormalization group method is to exploit the fact that the matrix element must be independent of the arbitrary renormalization scale μ. As applied to eq. (16), this yields the conditions

$$0 = -\frac{E}{\mu} F'\left(\frac{E}{\mu}\right) + \sum_n c_4^{(n)} \mu g_4^{(n)\prime}(\mu), \tag{17}$$

$$0 = -\frac{E}{\mu} H'\left(\frac{E}{\mu}\right) - \frac{E}{\mu} \sum_n J^{(n)\prime}\left(\frac{E}{\mu}\right) g_4^{(n)}(\mu)$$

$$+ \sum_n J^{(n)}\left(\frac{E}{\mu}\right) \mu g_4^{(n)\prime}(\mu) + g_2 \sum_n c_6^{(n)} \mu g_6^{(n)\prime}(\mu) + 2\sum_{nm} c_6^{(nm)} \mu g_4^{(n)\prime}(\mu) g_4^{(m)\prime}(\mu). \tag{18}$$

Since (17) must hold for arbitrary E and μ, both terms must be constants. For this to be true for all angles and isospins, we must then have

$$\mu g_4^{(n)\prime}(\mu) = b_4^{(n)}, \tag{19}$$

$$\frac{E}{\mu} F'\left(\frac{E}{\mu}\right) = \sum_n c_4^{(n)} b_4^{(n)} \tag{20}$$

with $b_4^{(n)}$ constants that are independent of all angle or isospin variables. Thus the terms of order E^4 can at most contain a single logarithm

$$F\left(\frac{E}{\mu}\right) = f_0 + \sum_n b_4^{(n)} c_4^{(n)} \ln\left(\frac{E}{\mu}\right). \tag{21}$$

To determine the $b_4^{(n)}$ we can compare our previous result (13), (14) with the solution of eq. (19)

$$g_4^{(n)}(\mu) = b_4^{(n)} \ln\left(\frac{\mu}{\mu_0}\right) \tag{22}$$

and find

$$b_4^{(1)} = \frac{4}{3\pi^2}, \quad b_4^{(2)} = \frac{8}{3\pi^2}. \tag{23}$$

Of course, eq. (21) holds with the same values of $b_4^{(n)}$ for all processes, not just $\pi-\pi$ scattering.

Returning now to eq. (18), we can insert (22), and find

$$0 = -\frac{E}{\mu} H'\left(\frac{E}{\mu}\right) - \frac{E}{\mu} \sum_n J^{(n)\prime}\left(\frac{E}{\mu}\right) b_4^{(n)} \ln\left(\frac{\mu}{\mu_0}\right)$$

$$+ \sum_n J^{(n)}\left(\frac{E}{\mu}\right) b_4^{(n)} + g_2 \sum_n c_6^{(n)} \mu g_6^{(n)\prime}(\mu) + 2\sum_{nm} c_6^{(nm)} b_4^{(m)} g_4^{(n)\prime}(\mu).$$

Differentiating this with respect to E then yields

$$\frac{E}{\mu}\frac{\partial}{\partial(E/\mu)}\frac{E}{\mu}\frac{\partial}{\partial(E/\mu)}H\left(\frac{E}{\mu}\right) = \sum_n b_4^{(n)}\ln\left(\frac{\mu}{\mu_0}\right)\frac{E}{\mu}\frac{\partial}{\partial(E/\mu)}\frac{E}{\mu}\frac{\partial}{\partial(E/\mu)}J^{(n)}\left(\frac{E}{\mu}\right)$$
$$+ \sum_n b_4^{(n)}\frac{E}{\mu}\frac{\partial}{\partial(E/\mu)}J^{(n)}\left(\frac{E}{\mu}\right).$$

Since this must hold for all μ, we can immediately conclude that $J^{(n)}(E/\mu)$ is linear in $\ln E/\mu$ while $H(E/\mu)$ is *quadratic* in $\ln E/\mu$:

$$H(E/\mu) = h_0 + h_1 \ln E/\mu + h_2 \ln^2 E/\mu, \tag{24}$$

$$J^{(n)}(E/\mu) = j_0^{(n)} + j_1^{(n)} \ln(E/\mu) \tag{25}$$

with coefficients of leading logarithms related by

$$h_2 = \tfrac{1}{2}\sum_n b_4^{(n)}j_1^{(n)}. \tag{26}$$

It is truly a pleasure to be able to deduce such detailed information about multi-loop graphs, without ever having to calculate any of them.

5. Symmetry breaking

In the real world, chiral symmetry is broken – rather weakly for $SU(2) \times SU(2)$, fairly strongly for $SU(3) \times SU(3)$. As a result, the "soft π" and "soft K" results of current algebra, which would be precise theorems in the limit of exact chiral symmetry, became somewhat fuzzy, depending for their interpretation on a good deal of unsystematic guesswork about the smoothness of extrapolations off the mass shell. In this section, I wish to show that phenomenological Lagrangians can serve as the basis of an approach to chiral symmetry breaking, which has at least the virtue of being entirely systematic.

Quantum chromodynamics tells us that in a world with only light u and d quark fields, the Lagrangian of the strong interactions takes the form $\mathscr{L}_0 + \mathscr{L}_1$, where $\mathscr{L}_0$ is invariant under global $SU(2) \times SU(2)$ transformations on the quark fields, and $\mathscr{L}_1$ is in some sense a small perturbation

$$\mathscr{L}_1 = m_u \bar{u}u + m_d \bar{d}d. \tag{27}$$

We may write $\mathscr{L}_1$ as the sum of third and fourth components of chiral four-vectors[15])

$$\mathscr{L}_1^{QCD} = V_3 + V_4, \tag{28}$$

$$V_3 = \tfrac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d), \tag{29}$$

$$V_4 = \tfrac{1}{2}(m_u + m_d)(\bar{u}u + \bar{d}d). \tag{30}$$

Thus the S-matrix takes the form of a sum of terms of kth order in $m_u - m_d$ and lth order in $m_u + m_d$, each term having the chiral transformation property of a traceless symmetric tensor of rank $k + l$ with k indices equal to 3 and l indices equal to 4

$$S = \sum_{kl} S^{(kl)},$$

$$S^{(kl)} \propto (m_u - m_d)^k (m_u + m_d)^l. \tag{31}$$

(Strictly speaking, this is true only with a chiral-invariant infrared cut-off.) Now, the most general phenomenological Lagrangian which gives an S-matrix of this form is itself such a sum

$$\mathscr{L}^{\mathrm{EFF}} = \sum_{k,l=0}^{\infty} \mathscr{L}^{(kl)}, \tag{32}$$

$$\mathscr{L}^{(kl)} \propto (m_u - m_d)^k (m_u + m_d)^l. \tag{33}$$

The S-matrix is therefore to be calculated with such a phenomenological Lagrangian, with $\mathscr{L}^{(kl)}$ taken as the most general function of hadronic fields and their derivatives having the chiral transformation property of a component of a traceless symmetric tensor of rank $k + l$ with k 3-indices and l 4-indices.

To see how this works in detail, let us again restrict ourselves to purely pionic processes. As is well known[12] (and shown below) the square of the pion mass is proportional to $m_u + m_d$, so $S^{(kl)}$ and $\mathscr{L}^{(kl)}$ are of order $m_\pi^{2(k+l)}$. If we calculate a pionic process near threshold, then all of the characteristic energies E are also of order m_π. Hence (9) may be modified to give the number of powers of E and/or m_π contributed by any given graph

$$\bar{D} = 2 + \sum_{d,k,l} N_{dkl}(d - 2 + 2k + 2l) + 2N_{\mathrm{L}}. \tag{34}$$

where N_{dkl} is the number of vertices with d derivatives, k powers of $m_d + m_u$, and l powers of $m_d - m_u$, and N_{L} is again the number of loops. If we regard $E \approx m_\pi$ as a small parameter, then the leading graphs are those with the smallest values of $\bar{D}$.

Now, there is no way to make a chiral scalar out of the pion field alone, with no derivatives, so there is no term in (34) with $d = k = l = 0$. There is a single chiral four-vector that can be formed with no derivatives, with components

$$V_i = (1 + \pi^2)^{-1}\pi_i, \qquad V_4 = -\tfrac{1}{2} + \frac{\pi^2}{1 + \pi^2}. \tag{35}$$

The third component is a pseudoscalar, and therefore cannot appear in the strong interaction Lagrangian. However, the fourth component is a scalar, so it yields an interaction with $d = k = 0$, $l = 1$

$$\mathcal{L}^{(01)} = -\tfrac{1}{2}m_\pi^2 g_2(1 + \pi^2)^{-1}\pi^2. \tag{36}$$

The coupling constant is fixed here by the condition that the square of the canonically normalized field $\pi' \equiv g_2^{1/2}\pi$ should have coefficient $-m_\pi^2/2$, and a constant term has been discarded. Comparison of (36) with (33) shows as already mentioned that $m_\pi^2 \propto m_u + m_d$.

According to eq. (34), the leading terms in the S-matrix are given by the sum of all tree graphs ($N_L = 0$) constructed from any number of vertices (10) with $k = l = 0$, $d = 2$ and any number of vertices (36) with $k = d = 0$, $l = 1$. For pion–pion scattering, the tree graphs consist of single vertices, formed from the terms in (10) or (36) that are quartic in the pion field. This yields precisely the formulas for the π–π scattering lengths previously derived by operator methods[10].

As before, phenomenological Lagrangians really come into their own in calculating corrections to the leading terms. Equation (34) shows that the leading corrections arise from one-loop graphs constructed from any number of vertices (10), (36), plus tree graphs constructed from any number of vertices (10), (36) and one vertex with $d + 2k + 2l = 4$. These latter vertices may be formed from functions of the pion field and its derivatives, of three different kinds:

(a) $d = 4$, $k + l = 0$. These are just the chiral scalars with four derivatives appearing in eq. (2).

(b) $d = 2$, $k + l = 1$. This is the chiral four-vector formed by multiplying the four-vector (35) with the scalar (10)

$$U_i = (1 + \pi^2)^{-3}\pi_i\partial_\mu\pi_j\partial^\mu\pi_j, \tag{37}$$

$$U_4 = -\tfrac{1}{2}(1 + \pi^2)^{-3}(1 - \pi^2)\partial_\mu\pi_j\partial^\mu\pi_j. \tag{38}$$

(c) $d = 0$, $k + l = 2$. This is the chiral tensor formed from the direct product of the four-vector (35) with itself

$$T_{ij} = (1 + \pi^2)^{-2}\pi_i\pi_j, \tag{39}$$

$$T_{i4} = -\tfrac{1}{2}(1 + \pi^2)^{-2}(1 - \pi^2)\pi_i, \tag{40}$$

$$T_{44} = \tfrac{1}{4}(1 + \pi^2)^{-2}(1 - \pi^2)^2. \tag{41}$$

The only operators here of positive parity are the chiral scalars (a), plus the components U_4, T_{ij}, and T_{44}. Hence the new terms in the effective Lagrangian which are needed to calculate the leading corrections to the tree ap-

proximation results for soft pion processes are

$$\alpha m_\pi^2 (1 + \pi^2)^{-3}(1 - \pi^2)\partial_\mu \pi \cdot \partial^\mu \pi + \beta m_\pi^4 (1 + \pi^2)^{-2}\pi_3^2$$
$$+ \gamma m_\pi^4 (1 + \pi^2)^{-2}(1 - \pi^2)^2, \tag{42}$$

where α, β, and γ are dimensionless constants of order unity, with

$$\beta/\gamma = 0((m_u - m_d)^2/(m_u + m_d)^2). \tag{43}$$

Current algebra calculations[12]) of the $K^+ - K^0$ mass difference yield the quark mass ratio $m_d/m_u = 1.8$, so the right-hand side of (43) is 0.08. It is interesting that the non-degeneracy of d and u introduces a rather large violation of isotopic spin conservation in the leading corrections to the usual soft pion results.

In calculating terms of a given order in E and/or m_π, we must include graphs with arbitrary numbers of vertices formed from the interactions (10) and (36). For the most part, the number of vertices that can actually occur are limited by the topology of the graphs; for instance for $\pi-\pi$ scattering the tree graphs have just a single quartic vertex, the one-loop graphs have two quartic vertices, and so on. However, there is never any limit on the number of times the quadratic part $-\frac{1}{2}m_\pi^2 g_2 \pi^2$ of eq. (36) can appear. To put this another and more convenient way, we must calculate all tree and loop graphs using a pion propagator $g_2^{-1}(q^2 + m_\pi^2)^{-1}$, and not expand these propagators in powers of m_π. Thus in addition to an over-all power of m_π, the matrix elements we calculate will have singularities in m_π as well as E. These singularities are of course just what is required by perturbative unitarity[13]).

There is not so much need today for refinements in the theory of pion–pion scattering. On the other hand, there is a wide variety of experimentally interesting processes where corrections to soft π or soft K theorems may be important, including $\pi N \to 2\pi N$, $K \to 2\pi$, $K \to 3\pi$, $K \to \pi\mu\nu$, $\eta \to 3\pi$, etc. The approach outlined in this section may serve as a basis for a systematic treatment of all these processes.

6. Phenomenological Lagrangians and QCD

Handy as they are, the phenomenological Lagrangians that we have been using are only phenomenological. This is brought home to us as we calculate graphs to higher and higher order in the pion energy – in each successive order, we encounter more and more unknown parameters. Beneath the phenomenological Lagrangian of the soft pions there must lie a more nearly

fundamental quantum field theory of strong interactions, which fixes all the free parameters of our phenomenological Lagrangians.

It now appears increasingly likely that this underlying theory is the renormalizable gauge theory known as quantum chromodynamics. By virtue of its asymptotic freedom, QCD predicts that the strong interactions should become weak at high energies in a certain definite way. The weakness of the interactions allows one to carry out perturbative calculations of certain quantities at high energy, and the results so far are in agreement with experiment.

However, as the energy becomes smaller, the strong interactions in QCD become stronger, and perturbation theory becomes no longer applicable. This of course is just what we want – the richness of the hadron spectrum and the absence of free quarks or gluons show clearly that perturbation theory had better *not* work at all energies. But then how do we do calculations of strong interactions at low energies?

In non-relativistic potential theory, there are well-known methods of solving the problem of calculating scattering amplitudes for potentials that are too strong to allow the use of perturbation theory. One of these methods of solution is the "quasiparticle" approach[14]. In this method, one introduces fictitious elementary particles into the theory, in rough correspondence with the bound states (or more precisely, the eigenvalues of the scattering kernel) of the theory. In order not to change the physics, one must at the same time change the potential. Since the bound states of the original theory are now introduced as elementary particles, the modified potential must not produce them also as bound states. Hence the modified potential is weaker, and can in fact be weak enough to allow the use of perturbation theory.

Following this lead, one might imagine weakening the forces of QCD by introducing some sort of infrared cut-off λ, and preserving the physical content of the theory by introducing the bound states of the theory as fictitious elementary particles. These bound states are just the ordinary hadrons, and they must be described by a chiral-invariant phenomenological Lagrangian. The parameters of the phenomenological Lagrangian would have to be functions of λ, defined by differential equations which guarantee the λ-independence of the S-matrix, with initial condition set by the requirement that the theory goes over to pure QCD in the limit $\lambda \to 0$, where there is no infrared cut-off. The hope would be that at low energy, one could continue the solution of these equations to a value of λ large enough to allow the use of perturbation theory.

It remains to be seen whether this program can be successfully carried through.

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