Cosmological Fluctuations Of Small Wavelength

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ABSTRACT

This paper presents a completely analytic treatment of cosmological fluctuations whose wavelength is small enough to come within the horizon well before the energy densities of matter and radiation become equal. This analysis yields a simple formula for the conventional transfer function $T(k)$ at large wave number k, which agrees very well with computer calculations of $T(k)$. It also yields an explicit formula for the microwave background multipole coefficient C_ℓ at very large ℓ.

Subject headings: cosmic microwave background — dark matter — early universe

1. Introduction

The transfer function gives the wave length dependence of the growth of perturbations in the cold dark matter density from early times to near the present. As such, it plays a central role in theoretical studies of cosmological structure formation, and it also enters in the calculation of the microwave background anisotropies of large multipole number. For general wave length the transfer function can only be calculated numerically. This paper will present a purely analytic solution of the equations governing the evolution of perturbations in the early universe in the case of small wave length,[1] which yields a simple formula for the transfer function in this case, including the numerical parameters appearing in this formula.

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[1]In speaking of small wavelengths, it is nevertheless assumed that the wavelength is large enough so that the fluctuations are far outside the horizon during the era of electron–positron annihilation, and large enough so that viscosity and heat conduction are negligible until close to the time of recombination, as is the case for all fluctuations of physical interest.

The most closely related previous work seems to be that of Hu & Sugiyama (1996). In contrast with their work, the present paper provides the justification for a crucial step in calculating fluctuations in the dark matter density (see footnote 4 below); it is entirely analytic, even in following perturbations through the era of horizon crossing and in analyzing the case of infinite wavelength (which is needed to normalize the transfer function); and explicit formulas are given for the numerical parameters in the transfer function at small wavelength and for the cosmic microwave background multipole coefficient C_ℓ at large ℓ.

2. Generalities

We consider the contents of the universe to consist of radiation plus cold dark matter plus baryons (electrons and nuclei). We include neutrinos in the radiation, neglecting the anisotropic part of their energy-momentum tensor, which makes possible a purely analytic treatment. As usual, the cold dark matter is taken to have zero pressure and only gravitational interactions. For simplicity at first we will assume local thermal equilibrium, so that the fractional changes in the baryon and radiation densities are related by $\delta\rho_B/\rho_B = 3\delta\rho_R/4\rho_R \equiv \delta_R$, which is a good approximation until late in the matter-dominated era, and we will ignore the effects of curvature and a cosmological constant, which are negligible until near the present. Later these effects and departures from equilibrium will be taken into account where they are relevant.

The evolution of compressional cosmological perturbations under these assumptions are governed by the equations:[2]

$$\frac{d}{dt}\left(a^2\psi\right) = -4\pi G\,a^2\left[\rho_D\delta_D + \left(\frac{8}{3}\rho_R + \rho_B\right)\delta_R\right]\ , \tag{1}$$

$$\dot{\delta}_D = -\psi\ , \qquad \dot{\delta}_R = -\psi + q^2 U_R\ , \tag{2}$$

$$\frac{d}{dt}\left[a^5\left(\frac{4}{3}\rho_R + \rho_B\right)U_R\right] = -\frac{4}{9}a^3\rho_R\delta_R\ . \tag{3}$$

Here q is the co-moving wave number; $a(t)$ is the Robertson–Walker scale factor; $U_R(t)$ is the radiation velocity potential; δ_D is the fractional change $\delta\rho_D/\rho_D$ in the dark matter density ρ_D; and dots indicate ordinary time derivatives. We are using a synchronous gauge, with vanishing time-time and time-space components of the metric perturbation $\delta g_{\mu\nu}$, and with the remaining gauge freedom removed by requiring that the cold dark matter velocity

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[2]These equations are a simple extension of Eqs. (15.10.50), (15.10.51), and (15.10.53) of Weinberg (1972) to the multi-fluid system considered here.

vanishes. In this gauge, all effects of gravitational perturbations for compressional modes are contained in the field $\psi(t) \equiv d(\delta g_{kk}(t)/2a^2(t))/dt$.

For general wave numbers these equations are too complicated to be solved analytically. However, for large q we can divide the evolution of the fluctuations into two *overlapping* eras, in each of which there are approximations available that allow an analytic solution.

3. Radiation Dominated Era

First, consider an era so early that ρ_D and ρ_B are much less than ρ_R, though the wavelength may be inside or outside the horizon. Here $a \propto \sqrt{t}$ and $t^2 = 3/32\pi G\rho_R$, and by eliminating U_R we obtain a pair of coupled equations for δ_R and ψ:

$$\frac{d}{dt}\left(t\,\psi\right) = -\frac{1}{t}\delta_R \,, \qquad \frac{d}{dt}\left(\sqrt{t}\frac{d\delta_R}{dt}\right) + \frac{q^2\sqrt{t}}{3a^2}\delta_R = -\frac{d}{dt}\left(\sqrt{t}\psi\right) \,. \tag{4}$$

The linear combination of the three independent solutions that grows most rapidly for small time is[3]

$$\delta_R = \frac{2N}{C^2}\left(\frac{2}{\theta}\sin\theta - \left(1 - \frac{2}{\theta^2}\right)\cos\theta - \frac{2}{\theta^2}\right) \,, \tag{5}$$

$$\psi = N\left(\frac{4}{\theta^3}\sin\theta + \frac{4}{\theta^4}\cos\theta - \frac{4}{\theta^4} - \frac{2}{\theta^2}\right) \tag{6}$$

where N is an unknown function of $\mathbf{q}$ that is presumably fixed during the era of inflation; $\theta \equiv C\sqrt{t}$; and C is the constant $C \equiv [2q\sqrt{t}/\sqrt{3}a]_{t\to0}$. Also, Eq.(2) gives

$$\delta_D = -\frac{2N}{C^2}\int_0^{C\sqrt{t}}\left(\frac{4}{\theta^3}\sin\theta + \frac{4}{\theta^4}\cos\theta - \frac{4}{\theta^4} - \frac{2}{\theta^2}\right)\theta\,d\theta \,. \tag{7}$$

Note that the fractional perturbations δ_D and δ_R are both of order ψ/q^2, justifying the neglect of the matter term in Eq. (1) when $\rho_R \gg \rho_D$.

For convenience later, it is useful to normalize the Robertson–Walker scale factor so that $a = 1$ at the time t_{EQ} when the matter density $\rho_M \equiv \rho_D + \rho_B$ and the radiation density ρ_R have a common value ρ_{EQ}. Then at early times we have $a \to (32\pi G\rho_{EQ}/3)^{1/4}\sqrt{t}$, and so $C = (q/\sqrt{3})(2\pi G\rho_{EQ}/3)^{-1/4}$. Also, q is now defined as the physical wave number q/a at $t = t_{EQ}$.

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[3]Aside from normalization, this solution is equivalent to that given for the Newtonian potential in a different gauge in Eq. (48) of Bashinsky & Bertschinger (2002).

4. Deep Inside the Horizon

Following this is an era in which the dark matter density may not be negligible, but the wavelength is well within the horizon. With the physical wave number q/a much greater than the expansion rate, there are two kinds of normal mode, that can be calculated using two different methods of approximation.

The first are the "fast" modes, for which d/dt acting on perturbations gives factors of order q/a. Inspection of Eqs. (1)–(3) shows that there is a solution with $\delta_R = O(q\psi)$, $U_R = O(\psi)$, and $\delta_D = O(\psi/q)$, so that we can neglect the term ψ on the right-hand side of Eq. (2), and even for $\rho_D > \rho_R$ we can neglect the dark matter term on the right-hand side of Eq. (1). Eliminating U_R then gives an equation for δ_R alone:

$$\frac{d}{dt}\left((1+R)a\frac{d\delta_R}{dt}\right) + \frac{q^2}{3a}\delta_R = 0 , \tag{8}$$

where $R \equiv 3\rho_B/4\rho_\gamma$. This has the well-known WKB solutions (Peebles & Yu, 1970)

$$\delta_R^\pm = (1+R)^{-1/4}\exp\left(\pm i\int\frac{q\,dt}{\sqrt{3(1+R)}a}\right) , \tag{9}$$

which would be exact for vanishing R.

Then there are "slow" modes, for which d/dt acting on perturbations gives factors of order $1/t$. Inspection of Eqs. (2)–(3) shows that in this case there is a solution with $\delta_D = O(\psi)$, $\psi \simeq q^2 U_R$. and $\delta_R = O(U_R) = O(\psi/q^2)$. It follows that even for $\rho_D < \rho_R$ we can neglect the radiation term on the right-hand side of Eq. (1), so that after eliminating the field ψ we have

$$\frac{d}{dt}\left(a^2\frac{d\delta_D}{dt}\right) = 4\pi G\, a^2\, \rho_D\delta_D . \tag{10}$$

It is convenient to convert the independent variable from t to a, using the Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\left(\rho_M + \rho_R\right) = \frac{8\pi G\rho_{\rm EQ}}{3}\left(a^{-3} + a^{-4}\right) \tag{11}$$

so that Eq. (10) reads[4]

$$a(1+a)\frac{d^2\delta_D}{da^2} + \left(1 + \frac{3a}{2}\right)\frac{d\delta_D}{da} - \frac{3}{2}(1-\beta)\,\delta_D = 0 , \tag{12}$$

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[4]Hu and Sugiyama (1996) pointed out that this equation leads to a transfer function with the asymptotic form $\ln k/k^2$, but it has not been clear why it is legitimate in deriving Eq. (12) to neglect fluctuations in the radiation energy density as a contribution to the source of the gravitational field during the radiation-dominated era. Eq. (12) was first derived by Mészáros (1974), who simply ignored fluctuations in the

where $\beta \equiv \rho_B/\rho_M = \Omega_B/\Omega_M$. The independent solutions of Eq. (12) for $\beta = 0$ were given by Mészáros (1974) and Groth & Peebles (1975):

$$f_1 = 1 + \frac{3a}{2} , \qquad f_2 = \left(1 + \frac{3a}{2} \right) \ln \left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1} \right) - 3\sqrt{1+a} . \qquad (13)$$

Hu and Sugiyama (1996) have given the solutions for general β in terms of hypergeometric functions, but the necessity of matching these solutions to those that apply after recombination leads to an extremely complicated formula for the transfer function, which obscures the dependence of the result on the baryon density. Here we will assume that β is small, though not entirely negligible, and work with solutions valid only to first order in β. The first-order solutions of Eq. (12) with the same behavior for $a \ll 1$ as the zeroth order solutions f_1 and f_2 are:

$$\delta_D^{(1,2)}(a) = f_{1,2}(a) - \frac{3\beta}{2} \int_0^a [f_1(a)f_2(b) - f_2(a)f_1(b)] \frac{f_{1,2}(b)\,db}{\sqrt{1+b}} . \qquad (14)$$

By applying Eqs. (2) and (3), we can find the fast mode solutions for $U_R^\pm$ and $\delta_D^\pm$ from Eq. (9) and the slow mode solutions for $U_R^{(1,2)}$ and $\delta_R^{(1,2)}$ from Eq. (14). These four modes a complete set of solutions of the fourth-order system of equations (1)–(3) up to the time of recombination for $q/a \gg \dot{a}/a$. The physical solution is a linear combination of these four modes, to be found by matching their behavior for $a \ll 1$ to that found in Section 3.

radiation density. Groth and Peebles (1975) neglected fluctuations in the radiation density on the grounds that the wavelength is much less than the Jeans length, which for radiation is the horizon, but the relevance of the Jeans length in an expanding universe containing both radiation and dark matter is not clear. In their Appendix B, Hu and Sugiyama (1996) neglected the contributions of perturbations in the dark matter density to the gravitational field at early times, and showed that then the contribution of perturbations in the radiation density to the gravitational field are also negligible. But this does not justify the use of equation (12). For this, it is necessary to show that perturbations in the radiation density make negligible contributions to the gravitational field when the contributions of the dark matter perturbation are *not* negligible, as is the case late in the radiation dominated era and during the cross-over from radiation to matter dominance. Liddle and Lyth (2000) on p. 107 attempted to explain the neglect of perturbations in the radiation energy density in Eq. (12) by claiming that Silk damping makes these perturbations decay away. This is incorrect. For wavelengths of physical interest Silk damping is negligible during the radiation-dominated era and through the time of radiation-matter equality. (This has been acknowledged by Liddle and Lyth in an erratum: star-www.cpes.sussex.ac.uk/ andrewl/infbook/errata.html.) The neglect of perturbations in the radiation density in Eq. (12) is explained by counting powers of $1/q$ as done here, and it applies only to the slow mode part of the solution; in the fast mode it is the perturbations in the *dark matter* density that become negligible for small wavelength.

5. Matching

Fortunately, for small wavelength there is an overlap in the two eras in which we have found solutions for δ_D, etc., satisfying *both* conditions $q/a \gg \dot{a}/a$ and $\rho_M \ll \rho_R$. In this period $C\sqrt{t} \gg 1$, and Eq. (5) gives the oscillating part of the fractional perturbation in the radiation density as $\delta_R = -(2N/C^2)\cos C\sqrt{t}$, which for $\rho_M \ll \rho_R$ fits smoothly with the linear combination of the fast solutions (9) for $q/a \gg \dot{a}/a$:

$$\delta_R^{\text{fast}} = -\frac{2N}{(1+R)^{1/4}C^2}\cos\left(\int_0^t \frac{q\,dt}{\sqrt{3(1+R)}a}\right) = -\frac{2N\sqrt{6\pi G\rho_{\text{EQ}}}}{(1+R)^{1/4}q^2}\cos\left(\int_0^t \frac{q\,dt}{\sqrt{3(1+R)}a}\right),$$
$$(15)$$

from which we also find, to leading order in $1/q$,

$$\psi^{\text{fast}} = \frac{3Na(1+R)^{1/4}(2+R)\sqrt{2\pi G\rho_{\text{EQ}}}}{q^3 t^2}\sin\left(\int_0^t \frac{q\,dt}{\sqrt{3(1+R)}a}\right), \qquad (16)$$

$$\delta_D^{\text{fast}} = \frac{3Na^2(1+R)^{3/4}(2+R)\sqrt{6\pi G\rho_{\text{EQ}}}}{q^4 t^2}\cos\left(\int_0^t \frac{q\,dt}{\sqrt{3(1+R)}a}\right). \qquad (17)$$

and

$$U_R^{\text{fast}} = \frac{2N\sqrt{2\pi G\rho_{\text{EQ}}}}{q^4 a(1+R)^{3/4}}\sin\left(\int_0^t \frac{q\,dt}{\sqrt{3(1+R)}a}\right). \qquad (18)$$

To find the coefficients in the slow modes, we note that the limit of Eq. (7) for $C\sqrt{t} \gg 1$ is

$$\delta_D \to \frac{4N}{C^2}\left(-\frac{1}{2}+\gamma+\ln C\sqrt{t}\right) = \frac{4N\sqrt{6\pi G\rho_{\text{EQ}}}}{q^2}\left(-\frac{1}{2}+\gamma+\ln\frac{aq}{\sqrt{8\pi G\rho_{\text{EQ}}}}\right), \qquad (19)$$

where $\gamma = 0.5772\ldots$ is the Euler constant. For $a \ll 1$, the solutions (14) become

$$\delta_D^{(1)} \to 1, \qquad \delta_D^{(2)} \to -\ln(a/4)-3. \qquad (20)$$

The linear combination of these solutions that fits smoothly with Eq. (19) is then

$$\begin{aligned}
\delta_D^{\text{slow}} &= \frac{4N}{C^2}\left\{\left[-\frac{7}{2}+\gamma+\ln\left(\frac{2\sqrt{3}C^2}{q}\right)\right]\delta_D^{(1)} - \delta_D^{(2)}\right\} \\
&= \frac{4N\sqrt{6\pi G\rho_{\text{EQ}}}}{q^2}\left\{\left[-\frac{7}{2}+\gamma+\ln\left(\frac{2q}{\sqrt{2\pi G\rho_{\text{EQ}}}}\right)\right]\delta_D^{(1)} - \delta_D^{(2)}\right\}.
\end{aligned} \qquad (21)$$

The slow part of the velocity potential and radiation density are given by Eqs. (2), (3), and (21) as

$$U_R^{\text{slow}} = \psi^{\text{slow}}/q^2 = -\dot{\delta}_D^{\text{slow}}/q^2 \tag{22}$$

$$\delta_R^{\text{slow}} = -3a^2 \frac{d}{dt}\left[a(1+R)U_R^{\text{slow}}\right] . \tag{23}$$

The full solution up to the time of recombination is given by $\delta_D = \delta_D^{\text{fast}} + \delta_D^{\text{slow}}$ and likewise for U_R and δ_R.

Eq. (17) shows that the fast part of δ_D is smaller than the slow part (21) by a factor of order $1/q^2 t^2$, so that for small wavelengths the full perturbation to the dark matter density is given by Eq. (21) from the time that q/a becomes much greater than $\dot{a}/a$, and even after the energy densities of matter and radiation become comparable, up to the time of recombination. But this is not true of the radiation perturbations. Comparison of Eqs. (22) and (23) with (18) and (15) shows that for large q the perturbations to the radiation velocity potential and density are dominated by the fast mode, by one and two factors of q, respectively.

6. The Transfer Function

The transfer function $T(k)$ is properly defined as the growth of the total matter density perturbation for a given present physical wave number $k \equiv q/a(t_0) = q(1 + z_{\text{EQ}})$, from early in the radiation-dominated era to late in the matter dominated era, relative to the growth that occurs in the same time interval for zero wave number. We must therefore now project the solution we have found for the density perturbations forward into the era following the time of recombination. In this era the baryonic perturbation is no longer suppressed by radiation pressure, and so it follows the same equation as the dark matter perturbation:

$$a(1+a)\frac{d^2\delta_B}{da^2} + \left(1 + \frac{3a}{2}\right)\frac{d\delta_B}{da} = a(1+a)\frac{d^2\delta_D}{da^2} + \left(1 + \frac{3a}{2}\right)\frac{d\delta_D}{da} = \frac{3}{2}\delta_M , \tag{24}$$

where $\delta_M \equiv \delta\rho_M/\rho_M = (1-\beta)\delta_D + \beta\delta_B$. This does not mean that δ_B and δ_D are equal, for they satisfy different initial conditions at recombination. But from a linear combination of these equations for δ_D and δ_B we find that

$$a(1+a)\frac{d^2\delta_M}{da^2} + \left(1 + \frac{3a}{2}\right)\frac{d\delta_M}{da} - \frac{3}{2}\delta_M = 0 , \tag{25}$$

This has the solutions (13). To find the correct linear combination of these solutions, we note that δ_B and $\dot{\delta}_B$ vanish to leading order in $1/q$ at recombination, so δ_M and and its first derivative at recombination must respectively equal $(1-\beta)\delta_D$ and its first derivative.

The total matter density perturbation after recombination is the linear combination of the solutions (13) that matches in this way with the solution (21):

$$\delta_M(a) = \frac{4N\sqrt{6\pi G\rho_{\rm EQ}}}{q^2}\Big(A_1 f_1(a) + A_2 f_2(a)\Big)\,,\tag{26}$$

where

$$A_1(q) = \left[-\frac{7}{2} + \gamma + \ln\left(\frac{2q}{\sqrt{2\pi G\rho_{\rm EQ}}}\right)\right](1 - \beta - \beta\,\mathcal{I}_{12}) - \beta\,\mathcal{I}_{22}\,,\tag{27}$$

$$A_2(q) = -1 + \beta - \beta\,\mathcal{I}_{12} - \beta\,\mathcal{I}_{11}\left[-\frac{7}{2} + \gamma + \ln\left(\frac{2q}{\sqrt{2\pi G\rho_{\rm EQ}}}\right)\right]\,,\tag{28}$$

$$\mathcal{I}_{ij} \equiv \frac{3}{2}\int_0^{a_R} \frac{f_i(a)\,f_j(a)\,da}{\sqrt{1+a}}\,.\tag{29}$$

Near the present, where $a \gg 1$, the matter density fluctuation goes to

$$\delta_M \to \frac{18 A_1 N a}{q^2}\sqrt{\frac{2\pi G\rho_{\rm EQ}}{3}}\,.\tag{30}$$

We can find the behavior of δ_M early in the matter-dominated era by taking $C\sqrt{t} \ll 1$ in Eqs. (5) and (7):

$$\delta_M \to Nt/2\,.\tag{31}$$

Eqs. (30) and (31) must be compared with the growth of δ_M for $q = 0$. In this case Eq. (2) gives $\delta_R = \delta_D = -\dot\psi$, so Eq. (1) becomes

$$\frac{d}{dt}\left(a^2\frac{d\delta_M}{dt}\right) = 4\pi G a^2\rho_{\rm EQ}\left(\frac{1}{a^3} + \frac{8}{3a^4}\right)\delta_M\tag{32}$$

The solution of Eqs. (32) and (11) that has the same behavior for $a \to 0$ as Eq. (31) is

$$\delta_M = \frac{N}{5a^2}\sqrt{\frac{3}{2\pi G\rho_{\rm EQ}}}\left(16 + 8a - 2a^2 + a^3 - 16\sqrt{1+a}\right)\tag{33}$$

This has an asymptotic behavior for $a \gg 1$:

$$\delta_M \to \frac{Na}{5}\sqrt{\frac{3}{2\pi G\rho_{\rm EQ}}}\tag{34}$$

The transfer function T then has an asymptotic behavior for large wave number given by the ratio of Eqs. (30) and (34):

$$T \to \frac{60\pi G\rho_{\rm EQ}}{q^2}A_1(q)\,,\tag{35}$$

with $A_1(q)$ given by Eq. (27). At late times the growth of δ_M may be affected by a cosmological constant or spatial curvature, but these effects are independent of wave number, and therefore cancel in the transfer function.

For $\rho_B \ll \rho_D$ we can neglect β, so Eqs. (35) and (27) give a transfer function

$$T \to \frac{60\pi G\rho_{\mathrm{EQ}}}{q^2}\left[-\frac{7}{2} + \gamma + \ln\left(\frac{2q}{\sqrt{2\pi G\rho_{\mathrm{EQ}}}}\right)\right] \;. \tag{36}$$

This can be put in more familiar terms by using the relations $\rho_{\mathrm{EQ}} = (3H_0^2\Omega_M/8\pi G)(1+z_{\mathrm{EQ}})^3$, $q = k(1 + z_{\mathrm{EQ}})$, and $1 + z_{\mathrm{EQ}} = \Omega_M/\Omega_R$, which give the transfer function in terms of the present wave number k:

$$T(k) \to \frac{45\Omega_M^2 H_0^2}{2\Omega_R k^2}\left[-\frac{7}{2} + \gamma + \ln\left(\frac{4k\sqrt{\Omega_R}}{\Omega_M H_0\sqrt{3}}\right)\right] = \frac{\ln(2.40\,Q)}{(4.07\,Q)^2} \;, \tag{37}$$

where $Q \equiv k(\mathrm{Mpc}^{-1})/\Omega_M h^2$, and in the final expression we use $\Omega_R h^2 = 4.15\times10^{-5}$. This may be compared with the BBKS numerical fit (Bardeen et al. 1986) to computer calculations of the transfer function:

$$T(k) \simeq \frac{\ln(1 + 2.34Q)}{2.34Q}\left[1 + 3.89Q + (16.1Q)^2 + (5.46Q)^3 + (6.71Q)^4\right]^{-1/4}. \tag{38}$$

This goes to $\ln(2.34Q)/(3.96\,Q)^2$ for large Q, in very good agreement with Eq. (37). Our simple calculation thus accounts not only for the form of the transfer function for large wave numbers, but also for its numerical parameters.

(Though it is not relevant to the present work, it may be noted that the BBKS formula cannot be taken seriously for small values of Q, for it has unphysical terms that are linear in Q at $Q \to 0$. Analyticity in the three-vector $\mathbf{k}$ requires that in this limit $T(k)$ should be a power series in k^2, or equivalently in Q^2.)

To assess the effect of a non-zero baryon number, we note that, to first order in $\beta = \Omega_B/\Omega_M$, the general formula (35) may be put in the form

$$T \to \frac{\ln\left(2.40\,Q\,(1 + \beta\mathcal{I}_{22})\right)}{\left[4.07\,Q\left(1 + \beta(1 + \mathcal{I}_{12})/2\right)\right]^2} \tag{39}$$

We need values for the integrals $\mathcal{I}_{ij}$ defined by Eq. (29). The upper limit on the integrals (29) is $a_R = (1 + z_{\mathrm{EQ}})/(1 + z_R)$. The redshift z_R at recombination has only a very weak dependence on cosmological parameters, and will be taken here to have the fixed value

$z_R = 1100$. The redshift z_{EQ} at matter-radiation equality is given by $1 + z_{EQ} = \Omega_M/\Omega_R$, so taking $\Omega_R h^2 = 4.15 \times 10^{-5}$, we have $a_R = 21.9\,\Omega_M h^2$. The integral $\mathcal{I}_{12}$ is given by

$$\mathcal{I}_{12}(a_R) = \frac{3}{20}\left[-22a_R - 18a_R^2 + 4\ln\left(\frac{a_R}{4}\right) + \sqrt{1 + a_R}\left(4 + 8a_R + 9a_R^2\right)\ln\left(\frac{\sqrt{1 + a_R} + 1}{\sqrt{1 + a_R} - 1}\right)\right]$$

The integral $\mathcal{I}_{22}$ is given by a lengthy expression involving Spence functions, but it converges so rapidly for likely values of a_R that for practical purposes we can use the value for a_R infinite:

$$\mathcal{I}_{22}(\infty) = 2\pi^2/5 - 3 = 0.947842 \ .$$

For instance, for $a_R = 4.38$ (corresponding to $\Omega_M h^2 = 0.2$), we have $\mathcal{I}_{22} = 0.9470$.

Eq. (39) agrees very well for large k with the numerical results of Holtzman (1989). For $\Omega_M h^2 = 0.2$ (the most plausible of the values considered by Holtzman) Eq. (39) becomes

$$T \rightarrow \frac{\ln\left(12.0\,k\,(1 + 0.947\,\beta)\right)}{\left[20.35\,k\left(1 + 1.377\,\beta\right)\right]^2} \ ,$$

with k in Mpc^{-1}. Table 1 compares the results given by this formula with the numerical results given by Holtzman (1989) for $\beta \equiv \Omega_B/\Omega_M$ equal to 0.01 and 0.1, and for various values of k. As can be seen, for these parameters the asymptotic formula (39) gives a pretty good approximation to the numerically calculated results for $k > 0.5\,\mathrm{Mpc}^{-1}$, and the numerically calculated results converge rapidly to Eq. (39) for larger values of k. But Holtzman warns that his result should not be used for $k > 3.09$ Mpc^{-1}, while the results obtained from Eq. (39) presumably become increasingly more accurate for larger values of k

We see that for any plausible value of $\Omega_M h^2$, the effect of a small baryon density on the argument of the logarithm is to replace the parameter $\Omega_M h^2$ in the definition of Q with $\Omega_M h^2(1 - .95\beta)$, while for $\Omega_M h^2$ in the range of 0.12 to 0.2, the effect of a small baryon density on the denominator of $T(k)$ is to replace the parameter $\Omega_M h^2$ in the definition of Q with $\Omega_M h^2(1 - \zeta\beta)$, with $\zeta \equiv (1 + \mathcal{I}_{22})/2$ in the range of 1.24 to 1.38. These results throw some light on a series of attempts to correct the transfer function for the effects of baryon density by re-scaling the definition of Q (usually called q) in the BBKS formula (38). Various authors attempted to correct for the baryon density by replacing $\Omega_M h^2$ in the denominator of Q with a factor $\Omega_M h^2 \exp(-2\Omega_B)$ (Peacock & Dodds 1994), or with $\Omega_M h^2 \exp(-\Omega_B - \sqrt{2h}\Omega_B/\Omega_M)$ (Sugiyama 1995) or with $\Omega_M h^2 \exp(-\Omega_B - \Omega_B/\Omega_M)$ (Liddle et al. 1996). (For a more detailed study of the effects of a finite baryon-to-dark matter ratio on the transfer function, see Eisenstein & Hu (1998).) Of course there is no reason why the baryon density should enter only in the definition of Q, and Eq. (39) shows that it does

not. But even without any detailed calculations, it is evident that these correction factors are physically impossible. The transfer function is defined with no reference to the present moment, except that it is conventionally written as a function of the present wave number k. It depends on $\Omega_M h^2$ and $\Omega_R h^2$, which enter in the formulas for k/q and $\rho_{\rm EQ}$, and it can (and does) have an additional dependence on the constant ratio of the energy densities of baryons and all matter, which is equal to Ω_B/Ω_M, but there is no way that it can depend separately on Ω_B or Ω_M or h. What we have found here is that for large wave number, the effect of a small baryon density can be crudely taken into account by replacing the parameter $\Omega_M h^2$ in the definition of Q with $\Omega_M h^2(1 - \zeta\Omega_B/\Omega_M) \simeq \Omega_M h^2 \exp(-\zeta\Omega_B/\Omega_M)$, with ζ roughly equal to unity for likely values of $\Omega_M h^2$.

7. Microwave Background Anisotropies

The fractional temperature fluctuation in a direction $\hat{n}$ takes the general form (apart from late-time effects):

$$\frac{\Delta T(\hat{n}, z)}{T} = \int p(z)\, dz \int d^3k\, e^{i(1+z)d_A(z)\hat{n}\cdot\mathbf{k}} \,\epsilon_{\mathbf{k}} \left[F(k, z) + i(\hat{n}\cdot\hat{k})G(k, z) \right] , \qquad (40)$$

where $p(z)\, dz$ is the probability that last scattering will occur between redshifts z and $z+dz$; $d_A(z)$ is the angular diameter distance to redshift z; and $\epsilon_{\mathbf{k}}$ is a primordial fluctuation amplitude, defined as proportional to $N(\mathbf{k})$, with a coefficient to be chosen below. In the synchronous gauge and hydrodynamic approximation used here, and now making the further approximation that dark matter dominates the energy density at last scattering, the form factors F and G in Eq. (40) are given by (Weinberg 2001):

$$\epsilon F = \frac{1}{3}\phi + \frac{1}{3}\delta_R , \qquad (41)$$

$$\epsilon G = -aqU_R + qt\phi/a , \qquad (42)$$

where $\phi = -4\pi G\rho_R\delta_R a^2/q^2$ is the Newtonian potential produced by dark matter density fluctuations. The first and second terms in F arise from the Sachs–Wolfe effect and intrinsic temperature fluctuations, respectively. The form factor G arises from the Doppler effect, with its first and second terms contributed by velocities produced by pressure and gravitational forces, respectively.

The conventional multipole coefficient C_ℓ is given in general by the familiar formula

$$C_\ell = 16\pi^2 \int_0^\infty \mathcal{P}(k)\, k^2\, dk \left| \int dz\, p(z)\, \left[j_\ell\!\left(kr(z)/H_0\right) F(k, z) + j_\ell'\!\left(kr(z)/H_0\right) G(k, z) \right] \right|^2 , \qquad (43)$$

where $\mathcal{P}(k)$ is the power spectral function, defined by

$$\langle \epsilon_{\mathbf{k}} \epsilon_{\mathbf{k}'} \rangle = \mathcal{P}(k)\delta^3(\mathbf{k} + \mathbf{k}') , \tag{44}$$

and

$$r(z) \equiv (1+z)d_A(z)H_0 = \int_{1/(1+z)}^{1} \frac{dx}{\sqrt{\Omega_\Lambda x^4 + \Omega_M x}} . \tag{45}$$

To fix the normalization of $\epsilon_{\mathbf{k}}$, we note that for small values of ℓ (say, $\ell < 10$) the large value of d_A makes the spherical Bessel functions in Eq. (43) oscillate rapidly except for small values of k, so C_ℓ is dominated for small ℓ by the Sachs–Wolfe term in $F(k)$, for which Eqs. (34) and (41) give the $z - independent$ small-k behavior

$$\epsilon F(k, z) \rightarrow -\frac{4\pi G\rho_R a^3 N(\mathbf{q})}{5q^2\sqrt{6\pi G\rho_{\text{EQ}}}} . \tag{46}$$

We therefore define $\epsilon_{\mathbf{k}}$ by

$$\epsilon_{\mathbf{k}} = -\frac{4\pi G\rho_R a^3}{5q^2\sqrt{6\pi G\rho_{\text{EQ}}}} N(\mathbf{q}) , \tag{47}$$

so that $F(0, z) = 1$. With this normalization, a Harrison–Zel'dovich power spectral function $\mathcal{P}(k) = Bk^{-3}$ gives $C_\ell = 8\pi^2 B\ell(\ell + 1)$ for small ℓ.

For large ℓ, the integral over k in Eq. (43) is dominated by large wave numbers. In this case, the Sachs–Wolfe term in Eq. (41) receives a contribution of order $1/k^4$ from the slow mode part (21) of δ_D and of order $1/k^6$ from the fast mode part (17). The intrinsic fluctuation term in Eq. (41) receives a contribution of order $1/k^2$ from the fast mode term (15) in δ_R, and of order $1/k^4$ from the slow mode term (23). The slow mode parts of the two terms in the Doppler form factor (42) cancel, leaving the contribution of the fast mode term (18) in U_R, which is of order $1/k^3$. We conclude from this that in the absence of dissipative effects, the temperature fluctuation is dominated for large k by the fast-mode part of the intrinsic temperature fluctuation.

But for very large k the rapidly oscillating fast mode is killed by Silk damping (i., e., photon viscosity and heat conduction) and Landau damping (cancelations due to large changes in the phase of the fast modes over the range of redshifts at which last scattering may occur). As pointed out by Hu and Sugiyama(1996), for ℓ greater than about 4,000 the dominant contribution to C_ℓ arises from the non-oscillatory terms in the perturbations. These terms, which are contributed by both the Sachs–Wolfe effect and the intrinsic temperature fluctuations, can be taken from Eqs. (83) and (84) of Weinberg (2001), with the damped terms neglected and an extra factor $T(k)$ supplied, because here we are dealing with wavelengths that come into the horizon during the radiation dominated era. This gives

$$F(k, z) \rightarrow -3R(z)\,T(k) , \qquad G(k, z) \rightarrow 0 , \tag{48}$$

so Eq. (43) becomes

$$C_\ell = 144\pi^2 \int_0^\infty \mathcal{P}(k)\, T^2(k)\, k^2\, dk \left| \int dz\, p(z)\, R(z)\, j_\ell\Big(kr(z)/H_0\Big) \right|^2 . \tag{49}$$

To do the double integral over z and k, we use an approximation of Hu and White (1996). The last-scattering probability distribution $p(z)$ is sharply peaked around a mean value $z_L \simeq 1,100$, while for sufficiently large ℓ the spherical Bessel function is even more sharply peaked at a value $\ell + 1/2$ of its argument. We therefore set z at a value where $kr(z)/H_0 = \ell + 1/2$ everywhere but in the argument of j_ℓ, and integrate over the argument of j_ℓ with k fixed, after which we set $k = (\ell + 1/2)H_0/r(z_L)$ everywhere but in the argument of $p(z)$, and integrate over that argument:

$$\begin{aligned}
C_\ell \;\to\; & 144\pi^2 H_0^3\, \mathcal{P}\left(\frac{(\ell+1/2)H_0}{r(z_L)}\right) T^2\left(\frac{(\ell+1/2)H_0}{r(z_L)}\right) R^2(z_L)\,\frac{\ell+1/2}{r^2(z_L)r'(z_L)} \\
& \times \int p^2(z)\, dz \left| \int_0^\infty j_\ell(s)\, ds \right|^2 \\
\to\; & \frac{36\pi^{5/2} H_0^3 (1+z_L)^{3/2}\sqrt{\Omega_M}}{r^2(z_L)\sigma}\, \mathcal{P}\left(\frac{\ell\, H_0}{r(z_L)}\right) T^2\left(\frac{\ell\, H_0}{r(z_L)}\right) R^2(z_L) , \tag{50}
\end{aligned}$$

where σ is defined by

$$\int p^2(z)\, dz \equiv \frac{1}{2\sqrt{\pi}\sigma} , \tag{51}$$

so that σ is the standard deviation if $p(z)$ is Gaussian. For instance, for a straight spectrum with $\mathcal{P}(k) \propto k^{-2-n_s}$, Eq. (45) gives $C_\ell \propto \ell^{-6-n_s} \ln^2 \ell$. Unfortunately the interposition of foreground objects makes it unlikely that this can be measured.

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Table 1. Values of the transfer function for $\Omega_M h^2 = 0.2$ and $\Omega_B/\Omega_M = 0.01$ or 0.1.

$k\,(\mathrm{Mpc}^{-1})$	$T(k)_{\Omega_B/\Omega_M=0.01}$[a]	$T(k)_{\Omega_B/\Omega_M=0.01}$[b]	$T(k)_{\Omega_B/\Omega_M=0.1}$[a]	$T(k)_{\Omega_B/\Omega_M=0.1}$[b]
0.1	0.161	0.0451	0.138	0.0509
0.3	0.0398	0.0337	0.0328	0.0284
0.5	0.0189	0.0169	0.0154	0.0140
1	0.00640	0.00586	0.00517	0.00480
2	0.00202	0.00187	0.00162	0.00152
3	0.000997	0.000938	0.000797	0.000762

[a]From Holtzman (1989)

[b]From Eq. (39)