Black Holes, Hawking Radiation, and the Firewall (for CS229)

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Abstract

Here I give a friendly presentation of the the black hole information problem and the firewall paradox for computer science people who don't know physics (but would like to). Most of the notes are just requisite physics background. There are six sections. 1: Special Relativity. 2: General Relativity. 3: Quantum Field Theory. 4. Statistical Mechanics 5: Hawking Radiation. 6: The Information Paradox.

Contents

1 Special Relativity

1.1 Causality and light cones

There are four dimensions: the three spatial dimensions and time. Every "event" that happens takes place at a coordinate labelled by

$$(t, x, y, z).$$

However, it is difficult to picture things in four dimensions, so usually when we draw pictures we just throw away the two extra spatial dimensions, labelling points by

$$(t, x).$$

With this simplification, we can picture all points on the 2D plane.

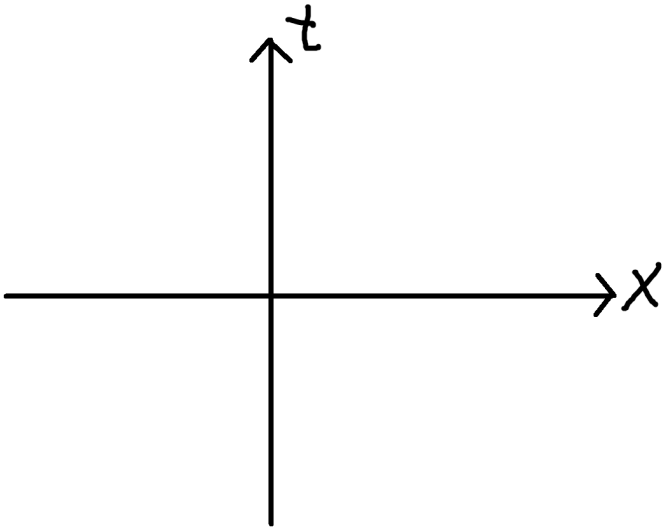


Figure 1: space-time as a 2D plane.

If something moves with a velocity v, its "worldine" will just be given by

$$x = vt. \tag{1}$$

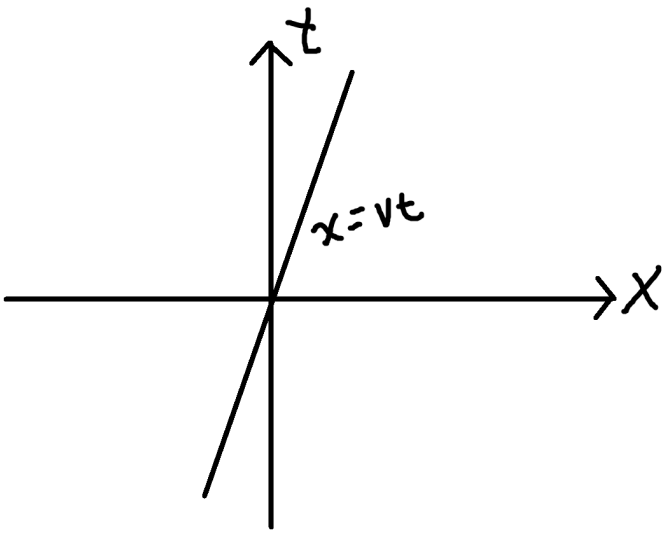


Figure 2: The worldline of something moving with velocity v.

A photon travels with velocity c. Physicists love to work in units where $c = 1$. For example, the x axis could be measured in light years and the t axis could be measured in years. In these units, the worldline of a light particle always moves at a 45° angle. (This is a very important point!)

Because nothing can travel faster than light, a particle is always constrained to move within its "lightcone."

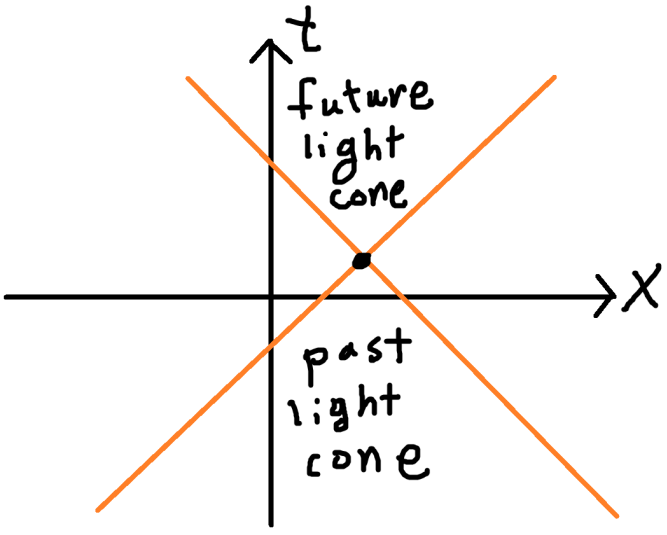


Figure 3: lightcone.

The "past light cone" consists of all of the space-time points that can send a message that point. The "future light cone" consists of all of the space-time points that can receive a message from that point.

1.2 Space-time interval

In special relativity, time passes slower for things that are moving. If your friend were to pass you by in a very fast spaceship, you would see their watch tick slower, their heartbeat thump slower, and their mind process information slower.

If your friend is moving with velocity v, you will see their time pass slower by a factor of

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2}$$

For small v, $\gamma \approx 1$. As v approaches c, γ shoots to infinity.

Let's say your friend starts at a point (t_1, x_1) and moves at a constant velocity to a point (t_2, x_2) at a constant velocity v.

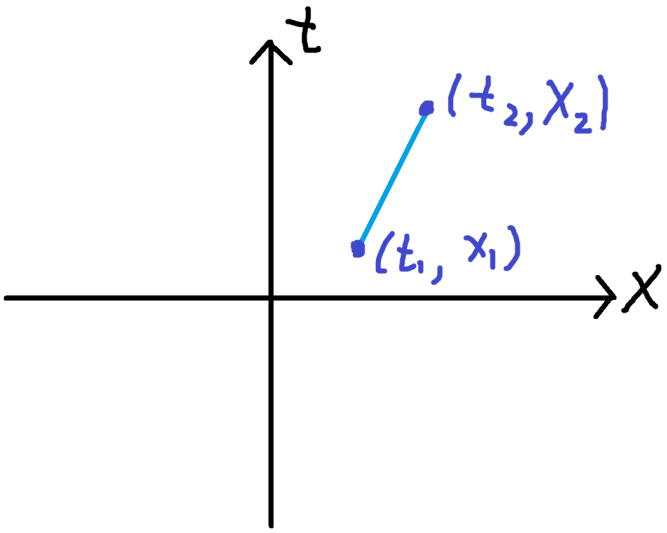


Figure 4: A straight line between (t_1, x_1) and (t_2, x_2).

Define

$$\Delta t = t_2 - t_1$$
$$\Delta x = x_2 - x_1.$$

From your perspective, your friend has moved forward in time by Δt. However, because time passes slower for your friend, their watch will have only ticked forward the amount

$$\Delta \tau \equiv \frac{\Delta t}{\gamma}. \tag{3}$$

Here, s is the so-called "proper time" that your friend experiences along their journey from (t_1, x_1) to (t_2, x_2).

Everybody will agree on what $\Delta\tau$ is. Sure, people using different coordinate systems will not agree on the exact values of t_1, x_1, t_2, x_2, or v. However, they *will* all agree on the value of $\Delta\tau$. This is because $\Delta\tau$ is a physical quantity! We can just look at our friend's watch and see how much it ticked along its journey!

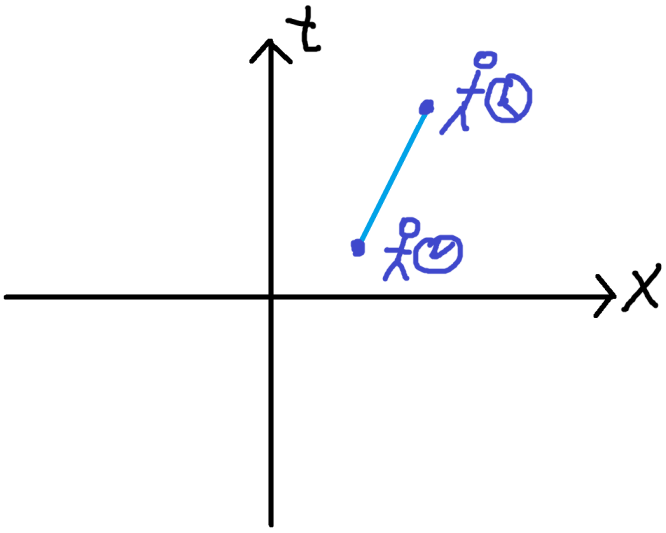


Figure 5: The time elapsed on your friend's watch during their journey is the invariant "proper time" of that space-time interval.

Usually, people like to write this in a different way, using $v = \frac{\Delta x}{\Delta t}$.

$$\begin{aligned}(\Delta\tau)^2 &= \frac{(\Delta t)^2}{\gamma^2} \\ &= (\Delta t)^2(1 - \tfrac{v^2}{c^2}) \\ &= (\Delta t)^2 - \tfrac{1}{c^2}(\Delta x)^2\end{aligned}$$

This is very suggestive. It looks a lot like the expression

$$(\Delta x)^2 + (\Delta y)^2$$

which gives an invariant notion of distance on the 2 dimensional plane. By analogy, we will rename the proper time $\Delta\tau$ the "invariant space-time interval" between two points. It gives the "distance" between two space-time points.

Note that if we choose two points for which $\Delta\tau = 0$, then those points can only be connected by something traveling at the speed of light. So points with a space-time distance $\Delta\tau = 0$ are 45° away from each other on a space-time diagram.

1.3 Penrose Diagrams

Penrose diagrams are used by physicists to study the "causal structure of space-time," i.e., which points can affect (and be affected by) other points. One difficult thing about our space-time diagrams is that t and x range from $-\infty$ to ∞. Therefore, it would be nice to reparameterize them so that they have a finite range. This will allow us to look at all of space-time on a finite piece of paper.

Doing this will severely distort our diagram and the distances between points. However, we don't really care about the *exact* distances between points. The fonly thing we care about preserving is 45° angles. We are happy to distort everything else.

To recap, a Penrose diagram is just a reparameterization of our usual space-time diagram that

1. is "finite," i.e. "compactified," i.e. can be drawn on a page
2. distorts distances but perserves 45° angles
3. lets us easily see how all space-time points are causally related.

So let's reparameterize! Define new coordinates u and v by

$$u \pm v = \arctan(t \pm x). \tag{4}$$

As promised, $u, v \in (-\frac{\pi}{2}, \frac{\pi}{2})$. So now let's draw our Penrose diagram!

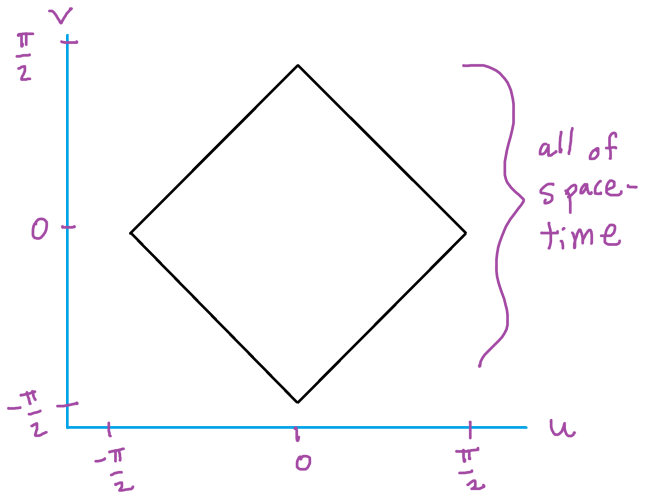


Figure 6: The Penrose diagram for flat space.

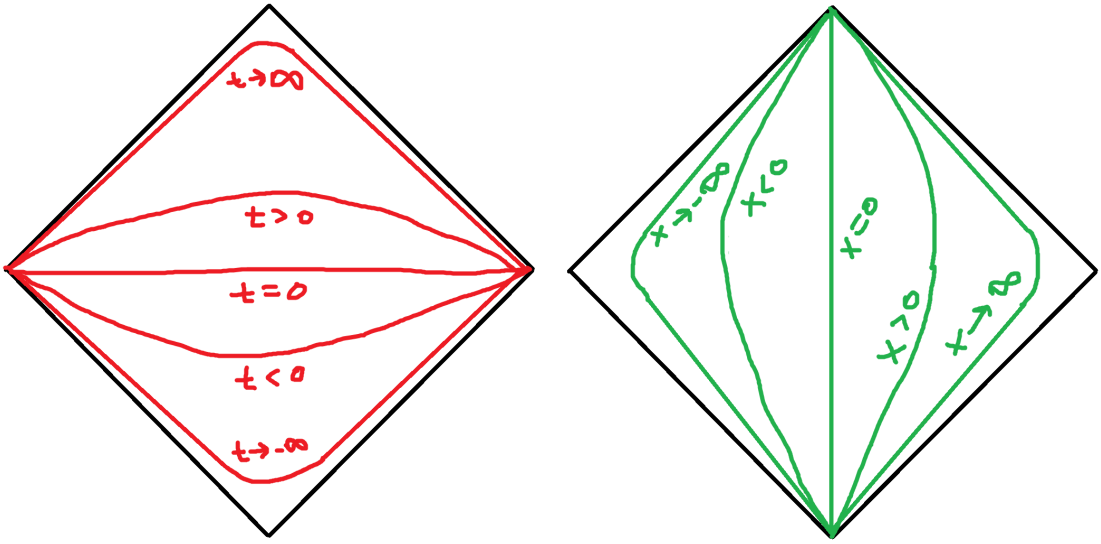


Figure 7: Lines of constant t and constant x.

Let's talk about a few features of the diagram. The bottom corner is the "distant past." All particles moving slower than c will emerge from there. Likewise, he top corner is the "distant future," where all particles moving slower than c will end up. Even though each is just one point in our picture, they really represent an infinite number of points.

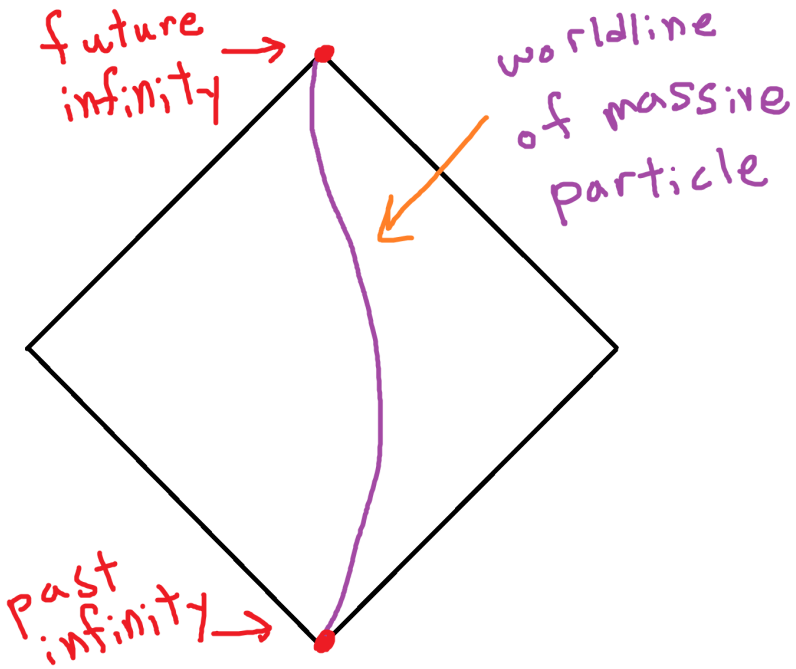


Figure 8: The worldline of a massive particle.

The right corner and left corner are two points called "spacelike in-

fintiy." Nothing physical ever comes out of those points.

The diagonal edges are called "lightlike infinty." Photons emerge from one diagonal, travel at a 45° angle, and end up at another diagonal.

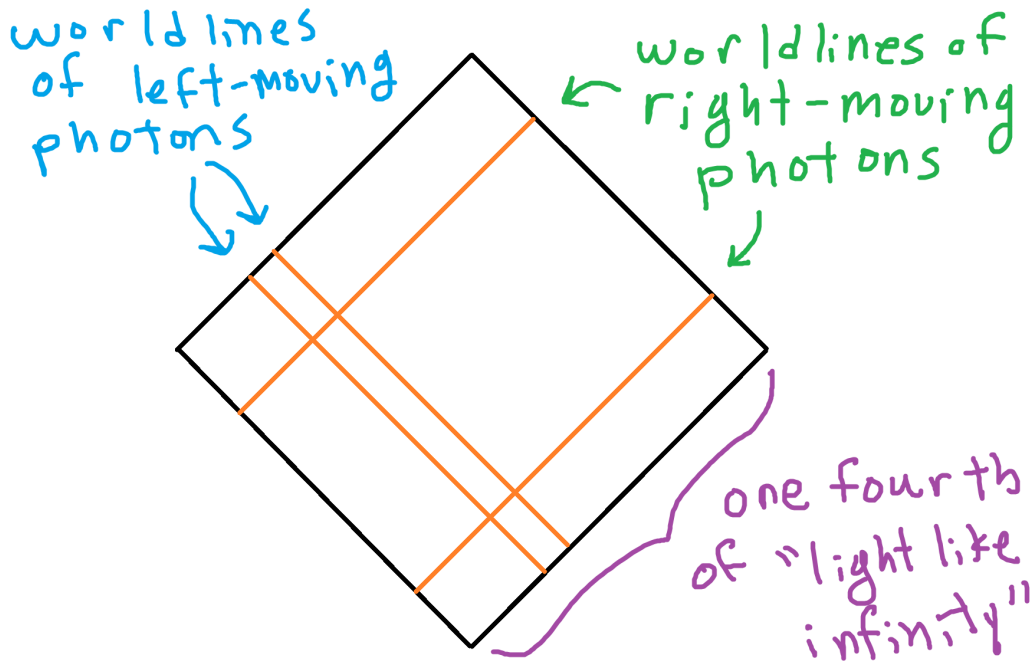


Figure 9: Worldlines of photons on our Penrose diagram.

From this point forward, we "set" $c = 1$ in all of our equations to keep things simple.

2 General Relativity

2.1 The metric

Space-time is actually curved, much like the surface of the Earth. However, locally, the Earth doesn't look very curved. While it is not clear how to measure large distances on a curved surface, there is no trouble measuring distances on a tiny scale where things are basically flat.

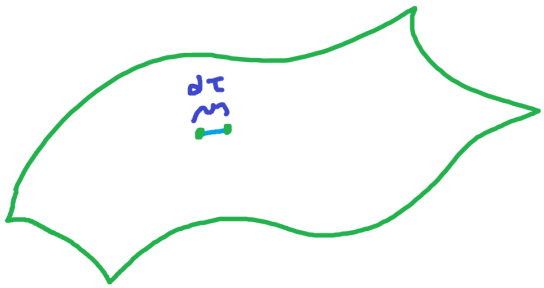


Figure 10: A curved surface is flat on tiny scales. Here, the distance, A.K.A proper time, between nearby points is labelled $d\tau$.

Say you have two points which are very close together on a curved space-time, and an observer travels between the two at a constant velocity. Say the two points are separated by the infinitesimal interval

$$dx^\mu = (dt, dx, dy, dz)$$

where $\mu = 0, 1, 2, 3$.

In general we can write the proper time $d\tau$ elapsed on the observer's watch by

$$d\tau^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu. \tag{5}$$

for some 16 numbers $g_{\mu\nu}$.

Eq 5 might be puzzling to you, but it shouldn't be. If anything, its just a definition for $g_{\mu\nu}$. If two nearby points have a tiny space-time distance $d\tau$, then $d\tau^2$ necessarily has to be expressible in the above form for two close points. There are no terms linear in dx^μ because they would not match the dimensionality of our tiny ds^2 (they would be "too big"). There are no terms of order $(dx^\mu)^3$ because those are too small for our consideration. Therefore, Eq. 5, by just being all possible quadratic

combinations of dx^μ, is the most general possible form for a distance we could have. I should note that Eq 5 could be written as

$$d\tau^2 = a^T M a$$

where the vector $a = dx^\mu$ and the 4×4 matrix $M = g_{\mu\nu}$.

In general relativity, $g_{\mu\nu}$ is called the "metric." It varies from point to point. People always define it to be symmetric, i.e. $g_{\mu\nu} = g_{\nu\mu}$, without loss of generality.

The only difference between special relativity and general relativity is that in special relativity we only think about the flat metric

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{6}$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{7}$$

However, in general relativity, we are interested in *dynamical* metrics which vary from point to point.

I should mention one further thing. Just because Eq. 5 says $d\tau^2 =$ (something), that doesn't mean that $d\tau^2$ is the square of some quantity "$d\tau$." This is because the metric $g_{\mu\nu}$ is not positive definite. We can see that for two nearby points that are contained within each other's light-cones, $d\tau^2 > 0$. However, if they are outside of each other's lightcones, then $d\tau^2 < 0$, meaning $d\tau^2$ is not the square of some $d\tau$. If $d\tau^2 = 0$, the the points are on the "rim" of each other's light cones.

While the metric gives us an infinitesimal notion of distance, we have to integrate it in order to figure out a macroscopic notion of distance. Say you have a path in space-time. The total "length' of that path $\Delta\tau$ is just the integral of $d\tau$ along the path.

$$\Delta\tau = \int d\tau = \int \sqrt{\sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu} \tag{8}$$

If an observer travels along that path, then s will be the proper time they experience from the start of the path to the end of the path. Remember that the proper time is still a physical quantity that all observers can agree on. Its just how much time elapses on the observer's watch.

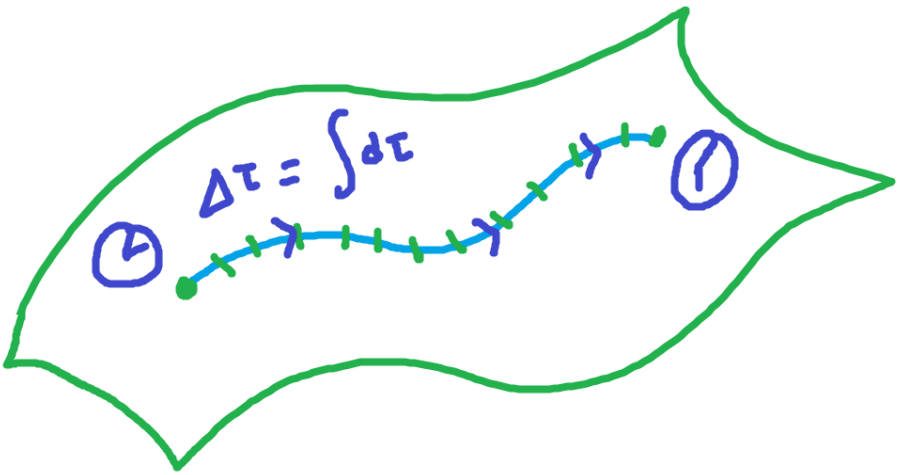


Figure 11: The proper time Δt along a path in space time gives the elapsed proper time for a clock which follows that path.

2.2 Geodesics

Let's think about 2D flat space-time again. Imagine all the paths that start at (t_1, x_1) and end at (t_2, x_2). If we integrate $d\tau$ along this path, we will get the proper time experienced by an observer travelling along that path.

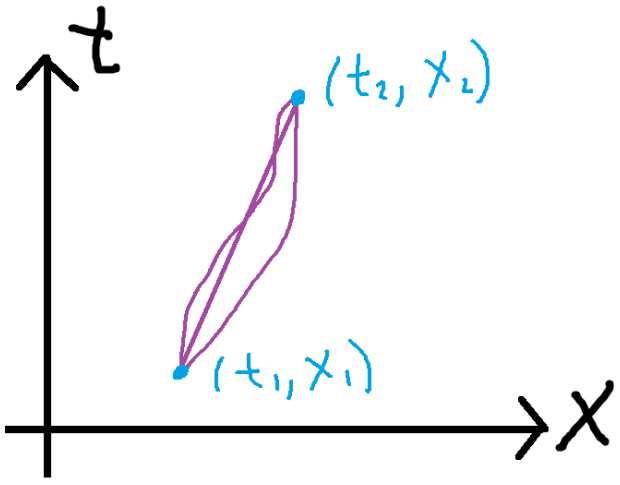


Figure 12: Each path from (t_1, x_1) to (t_2, x_2) has a different proper time $\Delta\tau = \int d\tau$.

Remember that when things travel faster, time passes slower. The more wiggly a path is, the faster that observer is travelling on average,

and the *less proper time passes for them.* The observer travelling on the straight path experiences the most proper time of all.

Newton taught us that things move in straight lines if not acted on by external forces. There is another way to understand this fact: things move on paths that maximize their proper time when not acted on by an external force.

This remains to be true in general relativity. Things like to move on paths that maximize

$$\Delta\tau = \int d\tau.$$

Such paths are called "geodesics." It takes an external force to make things deviate from geodesics. Ignoring air resistance, a sky diver falling to the earth is moving along a geodesic. However you, sitting in your chair, are not moving along a geodesic because your chair is pushing up on your bottom, providing an external force.

2.3 Einstein's field equations

> Space-time tells matter how to move; matter tells space-time how to curve.
>
> — John Wheeler

Einstein's field equation tells you what the metric of space-time is in the presence of matter. This is the equation that has made Einstein truly immortal in the world of physics. It took him almost 10 years to come up with it and almost died in the process.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{9}$$

Here, G is Newton's gravitational constant and c is the speed of light. $g_{\mu\nu}$ is the metric. $R_{\mu\nu}$ is something called the "Ricci curvature tensor." R is called the "scalar curvature." Both $R_{\mu\nu}$ and R depend on $g_{\mu\nu}$ and its derivatives in a very complicated way. $T_{\mu\nu}$ is something called the "stress energy tensor."

I will not explain all of the details, but hope to give you a heuristic picture. First off, notice the free indicies μ and ν. Einstein's equation is actually 16 equations, one for each choice of μ of ν from 0 to 3. However, because it is actually symmetric under the interchange of μ and ν, it is only 10 independent equations. They are extremely non-linear partial differtial equations.

The stress energy tensor $T_{\mu\nu}$ can be thought of as a shorthand for the energy density in space. Wherever there is stuff, there is a non-zero $T_{\mu\nu}$. The exact form of $T_{\mu\nu}$ depends on what the "stuff" actually is.

More specifically, the different components of $T_{\mu\nu}$ correspond to different physical quantities.

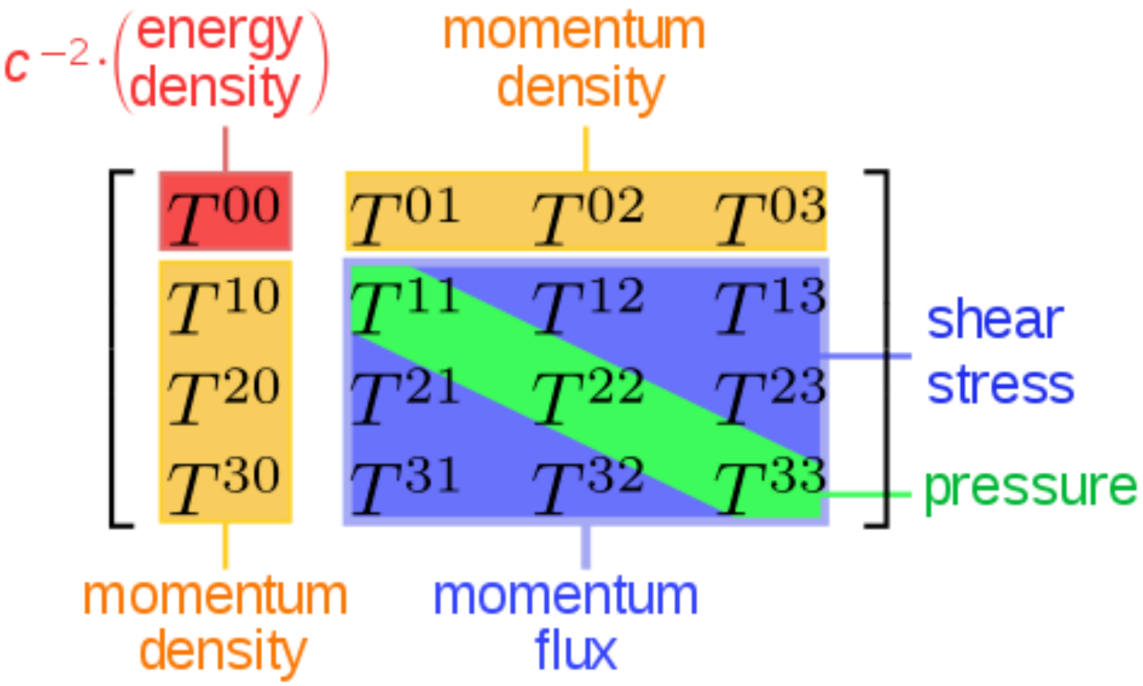


Figure 13: Components of the $T_{\mu\nu}$, taken from Wikipedia.

Roughly, Einstein's equation can be understood as

$$\text{something depending on curvature} \approx G \times \text{stuff density}. \tag{10}$$

This is what Wheeler meant by "matter tells space-time how to curve."

Take the sun, for example. The sun is very massive, and therefore space-time is very curved in the sun. Because the sun distorts space-time, its radius is actually a few kilometers bigger than you would naively expect from flat space. At the location of the sun there is an appreciable $T_{\mu\nu}$, and likewise a lot of curvature.

Once you get away from the physical location of the sun into the vacuum of space, $T_{\mu\nu} = 0$ and the curvature gradually dies off. This curvature is what causes the Earth to orbit the sun. Locally, the Earth is travelling in a straight line in space-time. But because space-time is curved, the Earth's path appears to be curved as well. This is what Wheeler meant by "space-time tells matter how to move."

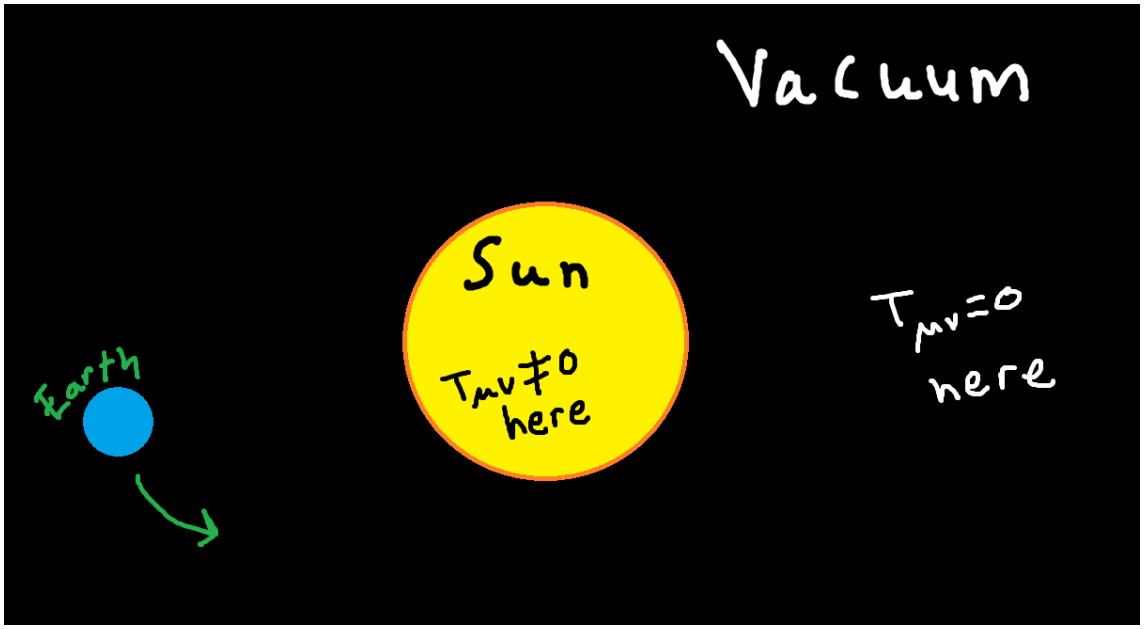


Figure 14: $T_{\mu\nu}$ is large where there is stuff, and 0 in the vacuum of space.

Notice, however, that the Earth itself also has some matter density, so it curves space-time as well. The thing is that it curves space-time a lot less than the sun does. If we want to solve for the motion of the Earth, we pretend is doesn't have any mass and just moves in the fixed "background" metric created by the sun. However, this is only an approximation.

2.4 The Schwarzschild metric

What if T_{00} is infinite at one point (a delta function) and 0 everywhere else? What will the metric be then? We have to solve the Einstein field equations to figure this out. (This is just a messy PDEs problem, but its not *so* messy. For reference, it took me about 5 hours to do it while following along with a book.) Thankfully, the answer is very pretty. Setting $c = 1$,

$$d\tau^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2GM}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{11}$$

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{2GM}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{2GM}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{12}$$

Here we are using the spherical coordinates

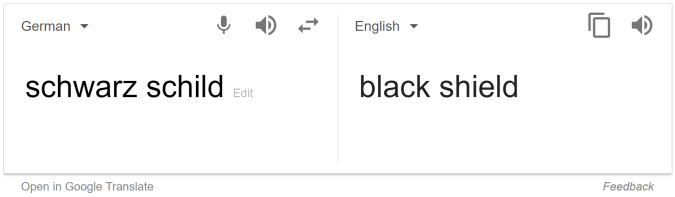
$$(t, r, \theta, \phi)$$

where r is the radial coordinate and θ and ϕ are the "polar" and "azimuthal" angles on the surface of a sphere, respectively.

This is the first metric that was found using Einstein's equations. It was derived by a German man named Karl Schwarzschild. He wanted to figure out what the metric was for a non-rotating spherically symmetric gravitating body of mass M, like the sun. Outside of the radius of the sun, the Schwarzschild metric does give the correct form for the metric there. Inside the sun, the metric needs to be modified and becomes more complicated

Interestingly, the metric "blows up" at the origin $r = 0$. Karl Schwarzschild just assumed that this wasn't physical. Because a real planet or star would need the metric to be modified inside of its volume, this singularity would not exist in those cases. He assumed that the singularity would not be able to form in real life under any circumstances. Einstein himself was disturbed by the singularity, and made a number of flawed arguments for why they can't exist. We know now that he wasn't right, and that these singularities really do form in real life inside of what we call "black holes."

In one of the amazing coincidences of history, "Schwarz" means "black" in German while "schild" means "shield." It appears that Karl Schwarzschild was always destined to discover black holes, even if he himself didn't know that.



2.5 Black Holes

Let's see if we can get an intuitive feel for black holes just by looking at the Schwarzschild metric. First, note that there is an interesting length

$$r_s = 2GM. \tag{13}$$

This is the "Schwarzschild radius." As I'm sure you heard, anything that enters the Schwarzschild radius, A.K.A. the "event horizon," cannot ever escape. Why is that?

Note that at $r = r_s$, the dt component of the metric becomes 0 and the dr component becomes infinite. This particular singularity isn't "real." It's a "coordinate singularity." There are other coordinates we could use, like the Kruskal–Szekeres coordinates that do not have this unattractive feature. We will ignore this.

The more important thing to note is that the dt and dr components flip signs as r dips below $2GM$. This is very significant. Remember that the flat space metric is

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{14}$$

The only thing that distinguishes time and space is a sign in the metric! This sign flips once you cross the event horizon.

Here is why this is important. Say that a massive particle moves a tiny bit to a nearby space-time point which is separated from the original point by $d\tau$. If the particle is moving slower than c, then $d\tau^2 > 0$. However, inside of a black hole, as per Eq. 11, we can see that when $d\tau^2 > 0$, the particle must either be travelling into the center of the black hole or away from it. This is just because 11 is of the form

$$d\tau^2 = \begin{cases} (+)dt^2 + (-)dr^2 + (-)d\theta^2 + (-)d\phi^2 & \text{if } r > 2GM \\ (-)dt^2 + (+)dr^2 + (-)d\theta^2 + (-)d\phi^2 & \text{if } r < 2GM \end{cases}$$

were $(+)$ denotes a positive quantity and $(-)$ denotes a negative quantity. In order that $d\tau^2 > 0$, we much have $dt^2 > 0$ outside of the event horizon but $dr^2 > 0$ inside the horizon, so dr cannot be 0.

Furthermore, if the particle started outside of the even horizon and then went in, travelling with $dr < 0$ along its path, then by continuity it has no choice but to keep travelling inside with $dr < 0$ until it hits the singularity

The reason that a particle cannot "turn around" and leave the black hole is the exact same reason why you cannot "turn around" and go back in time. If you think about it, there is a similar "horizon" between you and your childhood. You can never go back. If you wanted to go back in time, at some point you would have to travel faster than the speed of light (faster than 45°).

The r coordinate becomes "time-like" behind the event horizon.

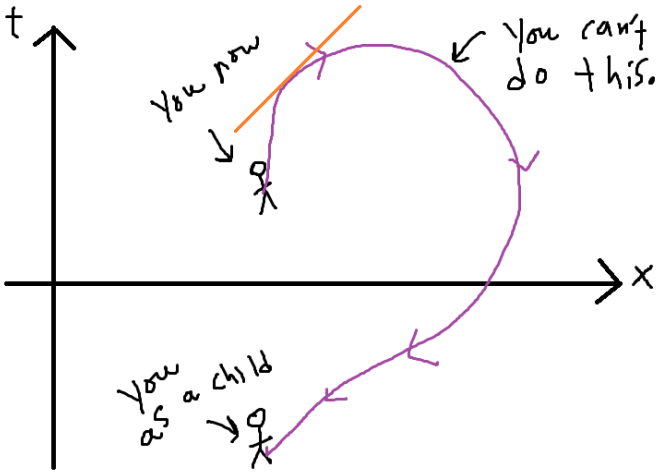


Figure 15: Going back in time requires going faster than c, which is impossible.

Outside of a black hole, we are forced to continue aging and die, t ever increasing. Inside of a black hole, we would be forced to hit the singularity and die, r ever decreasing. Death is always gently guiding us into the future.

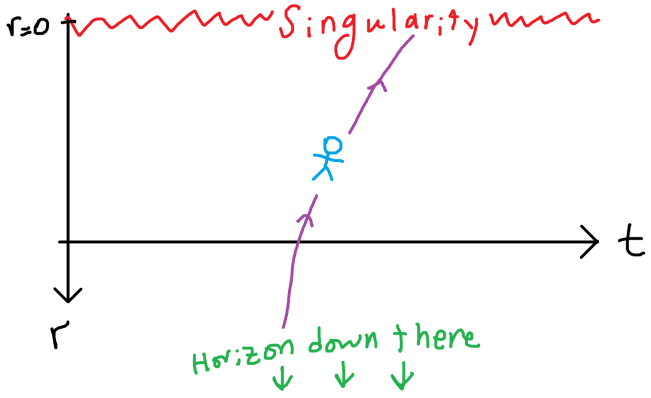


Figure 16: Once you have passed the event horizon of a black hole, r and t "flip," so now going into the future means going further into the black hole until you hit the singularity.

2.6 Penrose Diagram for a Black Hole

If we get rid of the angular θ and ϕ coordinates, our Schwarzschild space-time only has two coordinates (t, r). Once again, we can cook up new coordinates that allow us to draw a Penrose diagram. Here is the result.

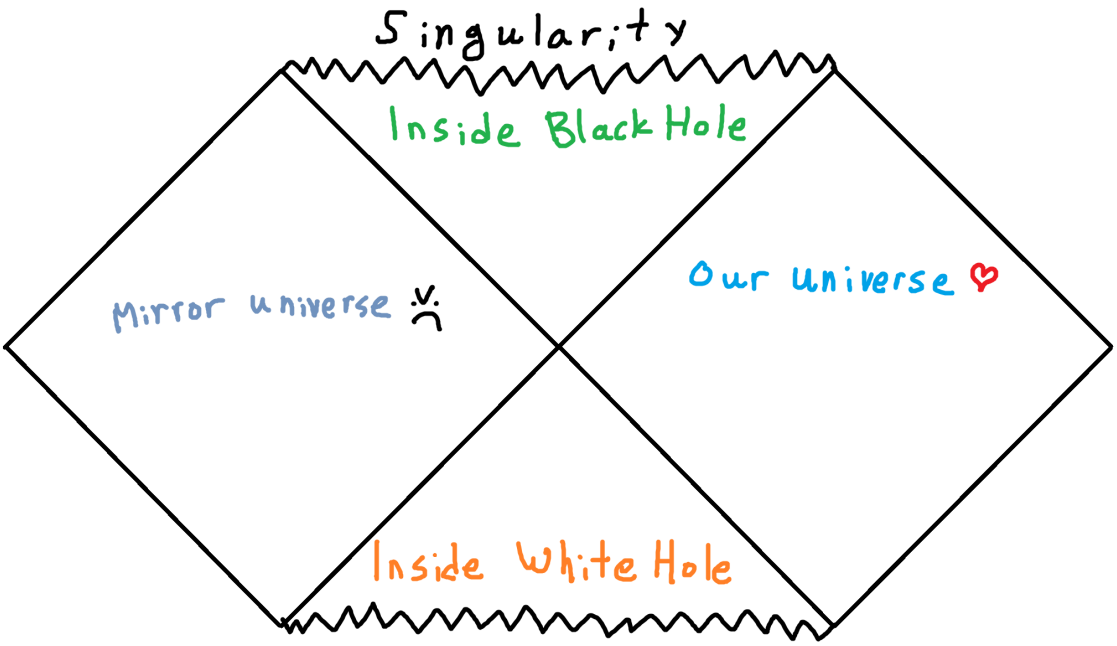


Figure 17: Penrose diagram of maximally extended space-time with Schwarzschild metric.

There is a lot to unpack here. Let's start with the right hand diamond. This is space-time outside of the black hole, where everyone is safe. The upper triangle is the interior of the black hole. Because the boundary is a 45° angle, once you enter you cannot leave. This is the event horizon. The jagged line up top is the singularity that you are destined to hit once you enter the black hole. From the perspective of people outside the black hole, it takes an infinite amount of time for something enter the black hole. It only enters at $t = +\infty$

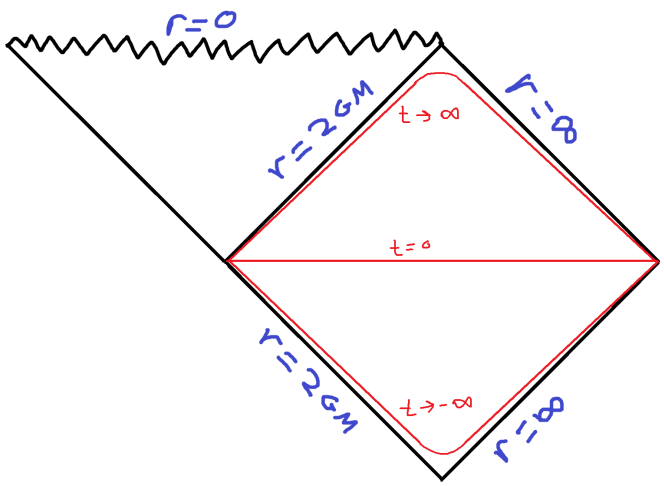


Figure 18: Penrose diagram of black hole with some lines of constant r and t labelled.

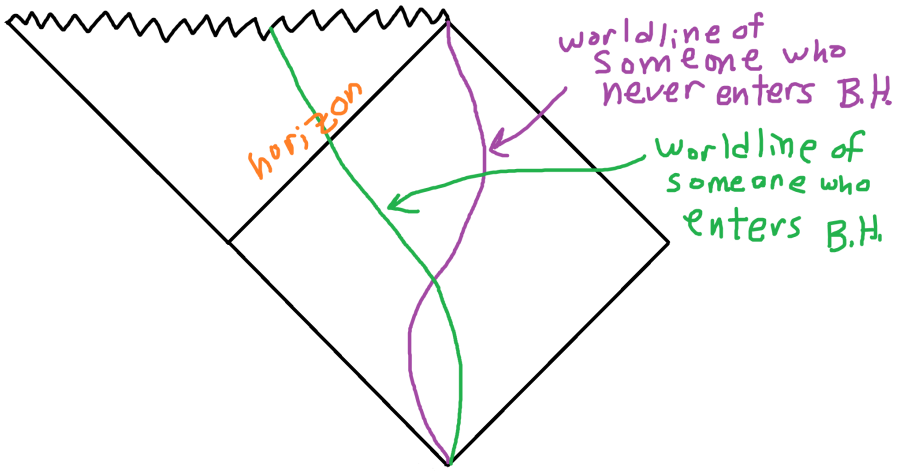


Figure 19: Two worldlines in this space-time, one which enters the black hole and one which does not.

I'm sure you noticed that there are two other parts to the diagram. The bottom triangle is the interior of the "white hole" and the left hand diamond is another universe! This other universe is invisible to the Schwarzschild coordinates, and only appears once the coordinates are "maximally extended."

First let's look at the white hole. There's actually nothing too crazy about it. If something inside the black hole is moving away from the singularity (with $dr > 0$) it has no choice but to keep doing so until it leaves the event horizon. So the stuff that starts in the bottom triangle is the stuff that comes out of the black hole. (In this context, however, we call it the white hole). It enters our universe at $t = -\infty$. It is impossible for someone on the outside to enter the white hole. If they try, they will only enter the black hole instead. This is because the can't go faster than 45°!

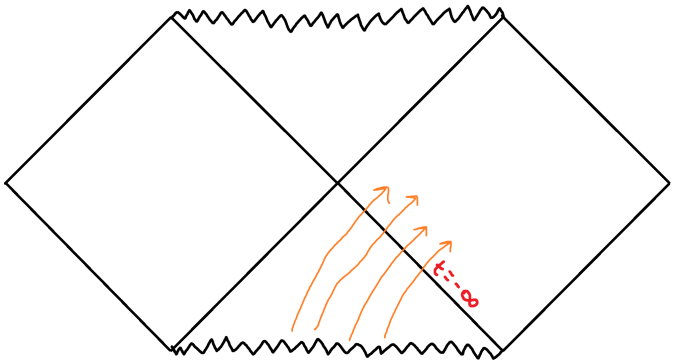


Figure 20: Stuff can come out of the white hole and enter our universe at $t = -\infty$.

Okay, now what the hell is up with this other universe? Its exactly the same as our universe, but different. Note that two people in the different universes can both enter the black hole and meet inside. However, they are both doomed to hit the singularity soon after. The two universes have no way to communicate outside of the black hole.

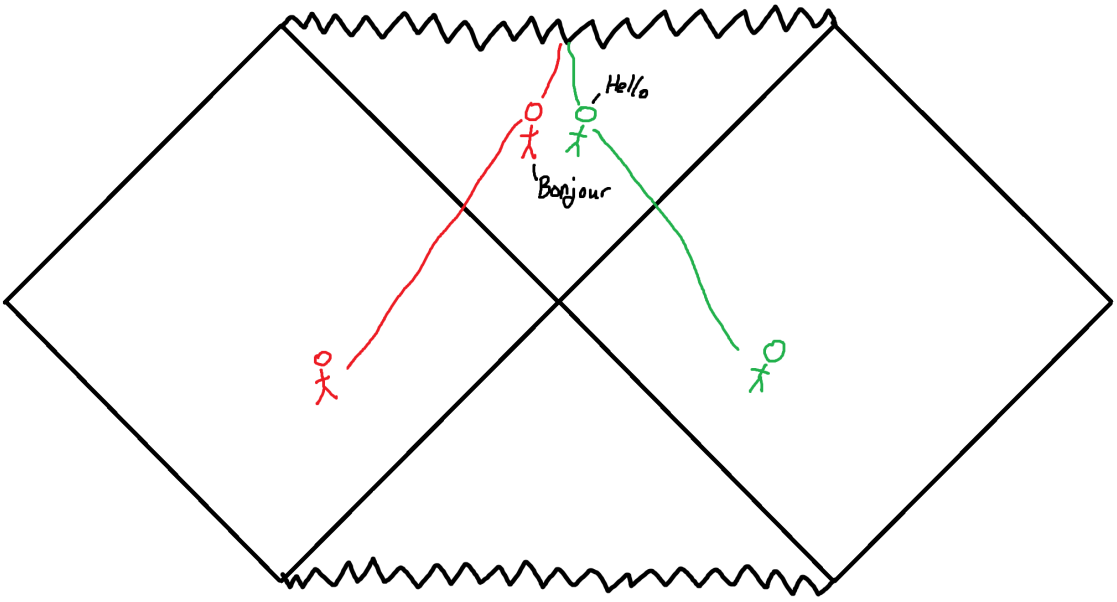


Figure 21: People from parallel universes can meet inside the black hole.

But wait! Hold the phone! Black holes exist in real life, right? Is there a mirror universe on the other side of every black hole????

No. The Schwarzschild metric describes an "eternal black hole" that has been there since the beginning of time and will be there until the end of time. Real black holes are not like this. They form when stars collapse. It is more complicated to figure out what the metric is if you want to take stellar collapse into account, but it can be done. I will not write the metric, but I will draw the Penrose diagram.

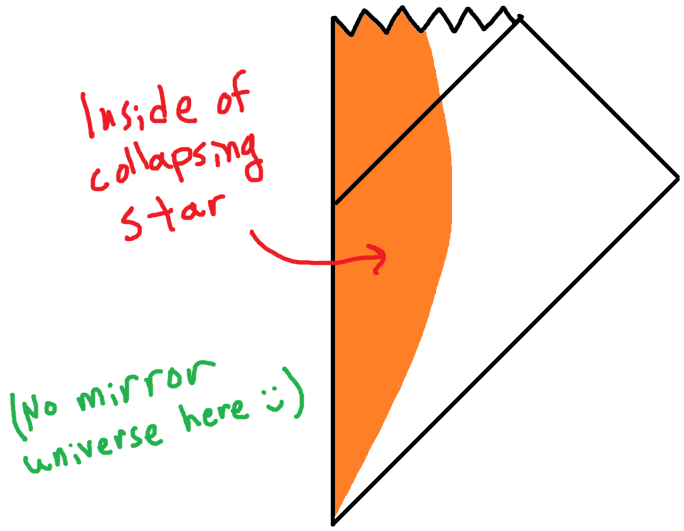


Figure 22: A Penrose diagram for a black hole that forms via stellar collapse.

Because the black hole forms at a some finite time, there is no white hole in our Penrose diagram. Likewise, there is no mirror universe.

Its interesting to turn the Penrose diagram upside down, which is another valid solution to Einstein's equations. This depicts a universe in which a white hole has existed since the beginning of the universe. It keeps spewing out material, getting smaller and smaller, until it disappears at some finite time. No one can enter the white hole. If they try, they will only see it spew material faster and faster as they get closer. The white hole will dissolve right before their eyes. That is why they can't enter it.

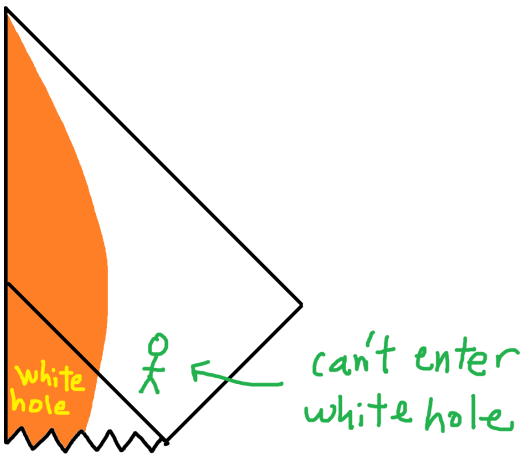


Figure 23: The Penrose diagram for a white hole that exists for some finite time.

2.7 Black Hole Evaporation

I have not mentioned anything about quantum field theory yet, but I will give you a spoiler: black holes evaporate. This was discovered by Stephen Hawking in 1975. They radiate energy in the form of very low energy particles until they do not exist any more. This is a unique feature of what happens to black holes when you take quantum field theory into account, and is very surprising. Having said that, this process is extremely slow. A black hole with the mass of our sun would take 10^{67} years to evaporate. Let's take a look at the Penrose diagram for a black hole which forms via stellar collapse and then evaporates.

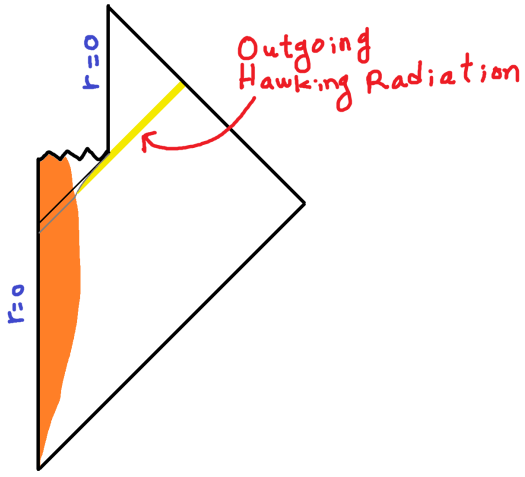


Figure 24: The Penrose diagram for a black hole which forms via stellar collapse and then evaporates.

3 Quantum Field Theory

While reading this section, forget I told you anything about general relativity. This section only applies to flat Minkowski space and has nothing to do with black holes.

3.1 Quantum Mechanics

Quantum mechanics is very simple. You only need two things. A Hilbert space and Hamiltonian. Once you specify those two things, you are done!

A Hilbert space $\mathcal{H}$ is just a complex vector space. States are elements of the Hilbert space.

$$|\psi\rangle \in \mathcal{H}. \tag{15}$$

Our Hilbert space also has a positive definite Hermitian inner product.

$$\langle\psi|\psi\rangle > 0 \text{ if } |\psi\rangle \neq 0. \tag{16}$$

A Hamiltonian $\hat{H}$ is just a linear map

$$\hat{H} = \mathcal{H} \to \mathcal{H} \tag{17}$$

that is self adjoint.

$$\hat{H}^\dagger = \hat{H} \tag{18}$$

States evolve in time according to the Schrödinger equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = -\frac{i}{\hbar}\hat{H}|\psi\rangle\,. \tag{19}$$

Therefore states evolve in time according to

$$U(t)|\psi\rangle \equiv \exp\Big(-\tfrac{i}{\hbar}t\hat{H}\Big)|\psi\rangle\,. \tag{20}$$

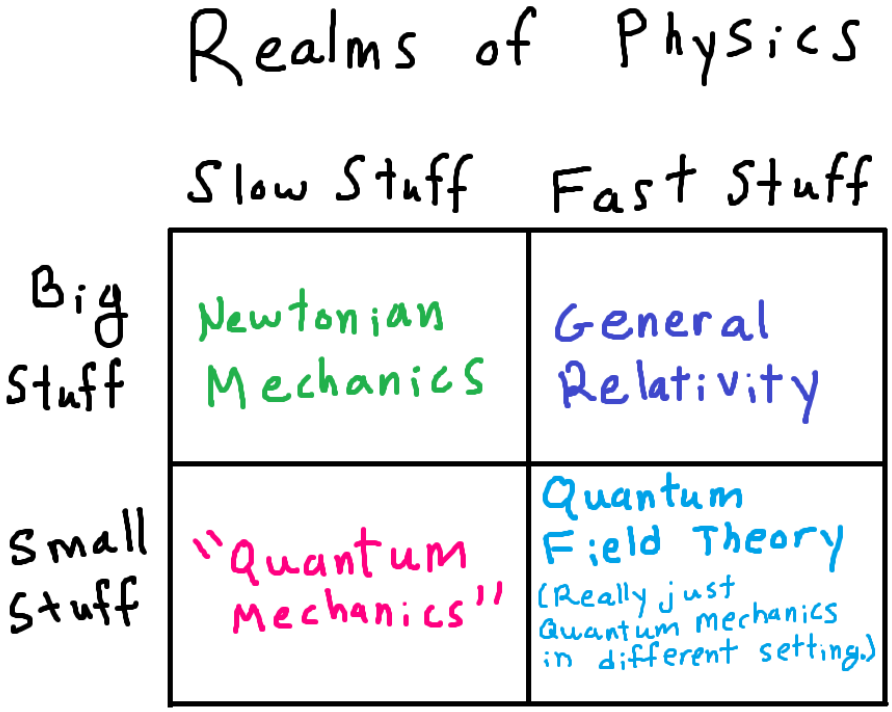
Because $\hat{H}$ is self adjoint, $U(t)$ is unitary.

$$U(t)^\dagger = U(t)^{-1} \tag{21}$$

(Sometimes the Hamiltonian itself depends on time, i.e. $\hat{H} = \hat{H}(t)$. In these cases the situation isn't so simple.)

I really want to drive this point into your head. Once you have a Hilbert space $\mathcal{H}$ and a Hamiltonian $\hat{H}$, you are DONE!

3.2 Quantum Field Theory vs Quantum Mechanics



Different areas of physics need different theories to describe them. People usually use the term "quantum mechanics" to describe things that are small but moving very slow. This is the domain of chemistry. However, once things travel very fast, there is enough energy to make new particles and destroy old ones. This is the domain of quantum field theory.

However, mathematically speaking, quantum field theory is just *subset* of quantum mechanics. States in quantum field theory live in a Hilbert space and evolve according to a Hamiltonian just like in quantum mechanics.

I am going to be extremely ambitious here and literally just tell you what this Hilbert space and Hamiltonian *actually is* for a very simple quantum field theory. However, I will not describe to you the Hilbert space of the actual quantum fields we see in real life like the photon field, the electron field, etc. Actual particles have a confusing property called "spin" which I don't want to get into. I will instead tell you about the quantum field theory of a fictitious "spin 0" particle that could theoretically exist in real life but doesn't appear to. Furthermore, this particle will not "interact" with any other particle, making its analysis particularly simple.

3.3 The Hilbert Space of QFT: Wavefunctionals

A classical field is a function $\phi(\mathbf{x})$ from space into $\mathbb{R}$.

$$\phi : \mathbb{R}^3 \to \mathbb{R}. \tag{22}$$

We denote the space of smooth functions on $\mathbb{R}^3$ by $C^\infty(\mathbb{R}^3)$.

$$\phi \in C^\infty(\mathbb{R}^3). \tag{23}$$

Each particular ϕ is called a "classical field configuration." Each value $\phi(\mathbf{x})$ for some particular $\mathbf{x}$ is called a "field variable."

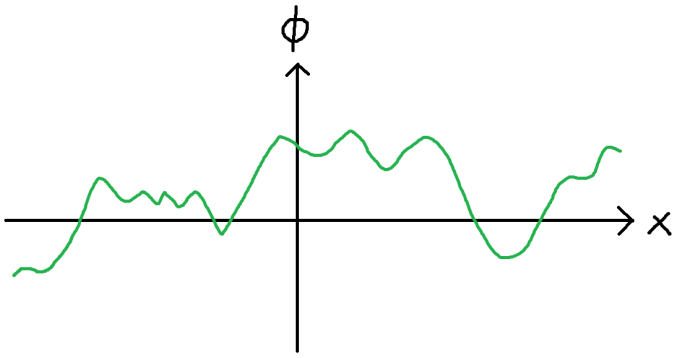


Figure 25: A classical field assigns a real number to each point in space. I have suppressed the three spatial dimensions into just one, x, for simplicity.

Now I'm going to tell you what a quantum field state is. Are you ready? A quantum field state is a *functional* from classical fields to complex numbers.

$$\Psi : C^\infty(\mathbb{R}^3) \to \mathbb{C} \tag{24}$$

$$\mathcal{H}_{\text{QFT}} = \text{all such wave functionals} \tag{25}$$

These are called "wave functionals." Let's say you have two wave functionals Ψ and Φ. The inner product is this infinite dimensional integral, which integrates over all possible classical field configurations:

$$\langle \Psi | \Phi \rangle = \int \prod_{\mathbf{x} \in \mathbb{R}^3} d\phi(\mathbf{x}) \Big(\Psi[\phi]^* \Phi[\phi] \Big). \tag{26}$$

Obviously, the product over $\mathbf{x} \in \mathbb{R}^3$ isn't mathematically well defined. However, there's a whole branch of physics called "lattice field theory" where people discretize space into lattices in order to compute things

on a super computer. Furthermore, physicists have many reasons to believe that if we had a full theory of quantum gravity, we would realize that quantum field theory as we know it does break down at very tiny Planck-length sized distances. Most likely it would not be anything as crude as a literal lattice, but something must be going on at really small lengths. Anyway, because we don't have a theory of quantum gravity, this is the best we can do for now.

The physical interpretation is that if $|\Psi[\phi]|^2$ is very big for a particular ϕ, then the quantum field is very likely to "be" in the classical field configuration ϕ.

Note that we have a basis of wave functionals given by

$$\Psi_{\phi_0}[\phi] \propto \begin{cases} 1 & \text{if } \phi = \phi_0 \\ 0 & \text{if } \phi \neq \phi_0 \end{cases} \tag{27}$$

for all $\phi_0 \in C^\infty(\mathbb{R}^3)$. We can write them as

$$|\Psi_{\phi_0}\rangle$$

(You should think of these as the i in $|i\rangle$. Each classical field ϕ_0 labels a "coordinate" of the QFT Hilbert space.) All other wave functionals can be written as a linear combination of these wave functionals with complex coefficients. However, this basis of the Hilbert space is physically useless. You would never ever see a quantum field state like these in real life. (The reason is that they have infinite energy.) I will tell you about a more useful basis for quantum field states a bit later.

3.4 Two Observables

An observable $\hat{\mathcal{O}}$ is a linear map

$$\hat{\mathcal{O}} : \mathcal{H} \to \mathcal{H} \tag{28}$$

that is self adjoint

$$\hat{\mathcal{O}}^\dagger = \hat{\mathcal{O}}. \tag{29}$$

Because it is self adjoint, all of its eigenvalues must be real numbers. An eigenstate $|\psi\rangle$ of $\hat{\mathcal{O}}$ that satisfies

$$\hat{\mathcal{O}}\,|\psi\rangle = \lambda\,|\psi\rangle \tag{30}$$

for some $\lambda \in \mathbb{R}$ has the interpretation of having definite value λ under the measurement corresponding to $\hat{\mathcal{O}}$.

There are two important sets of observables I have to tell you about for these wave functionals. They are called

$$\hat{\phi}(\mathbf{x}) \quad \text{and} \quad \hat{\pi}(\mathbf{x}). \tag{31}$$

There are an infinite number of them, one for each $\mathbf{x}$. You should think of the measurements $\hat{\phi}(\mathbf{x})$ and $\hat{\pi}(\mathbf{x})$ as measurements occurring at the point $\mathbf{x}$ in space. They are linear operators

$$\hat{\phi}(\mathbf{x}) : \mathcal{H}_{\text{QFT}} \to \mathcal{H}_{\text{QFT}} \quad \text{and} \quad \hat{\pi}(\mathbf{x}) : \mathcal{H}_{\text{QFT}} \to \mathcal{H}_{\text{QFT}}$$

which are defined as follows.

$$\left(\hat{\phi}(\mathbf{x})\Psi\right)[\phi] \equiv \phi(\mathbf{x})\Psi[\phi] \tag{32}$$

$$\left(\hat{\pi}(\mathbf{x})\Psi\right)[\phi] \equiv \frac{\hbar}{i}\frac{\delta}{\delta\phi(\mathbf{x})}\Psi[\phi] \tag{33}$$

(Use the hats to help you! $\hat{\phi}(\mathbf{x})$ is an operator acting on wave functionals, while ϕ is the classical field configuration at which we are evaluating our wave-functional. $\phi(\mathbf{x})$ is just the value of that input ϕ at $\mathbf{x}$.)

First let's talk about $\hat{\phi}(\mathbf{x})$. It is the observable that "measures the value of the field at $\mathbf{x}$." For example, the expected field value at $\mathbf{x}$ would be

$$\langle\Psi|\,\hat{\phi}(\mathbf{x})\,|\Psi\rangle = \int \prod_{\mathbf{x}'\in\mathbb{R}^3} d\phi(\mathbf{x}')|\Psi[\phi]|^2\phi(\mathbf{x}).$$

Note that our previously defined Ψ_{ϕ_0} are eigenstates of this operator. For any $\phi_0 \in C^\infty(\mathbb{R}^3)$, we have

$$\hat{\phi}(\mathbf{x})\,|\Psi_{\phi_0}\rangle = \phi_0(\mathbf{x})\,|\Psi_{\phi_0}\rangle\,. \tag{34}$$

The physical interpretation of $\hat{\pi}(\mathbf{x})$ is a bit obscure. First off, if you don't know,

$$\frac{\delta}{\delta\phi(\mathbf{x})} \tag{35}$$

is called the "functional derivative." It is defined by

$$\frac{\delta\phi(\mathbf{x})}{\delta\phi(\mathbf{y})} \equiv \delta^3(\mathbf{x}-\mathbf{y}). \tag{36}$$

($\delta^3(\mathbf{x}-\mathbf{y})$ is the three dimensional Dirac delta function. It satisfied $\int d^3x f(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{y}) = f(\mathbf{y})$ for any $f : \mathbb{R}^3 \to \mathbb{C}$, where $d^3x = dxdydz$

is the three dimensional volume measure.) This is just the infinite dimensional version of the partial derivative in multivariable calculus.

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}. \tag{37}$$

Basically, $\hat{\pi}(\mathbf{x})$ measures the rate of change of the wave functional with respect to one particular field variable $\phi(\mathbf{x})$. (The i is there to make it self-adjoint.) I don't want to get bogged down in its physical interpretation.

3.5 The Hamiltonian

Okay, I've now told you about the Hilbert space, the inner product, and a few select observables. Now I'm going to tell you what the Hamiltonian is and then I'll be done!

$$\hat{H} = \frac{1}{2}\int d^3x \Big(\hat{\pi}^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2\Big) \tag{38}$$

Done! (Here m is just a real number.)

(Now you might be wondering where I got this Hamiltonian from. The beautiful thing is, I do not have to tell you! I am just telling you the *laws*. Nobody truly knows where the laws of physics come from. The best we can hope for is to know them, and then derive their consequences. Now obviously I am being a bit cheeky, and there are many desirable things about this Hamiltonian. But you shouldn't worry about that at this stage.)

I used some notation above that I have not defined. I am integrating over all of space, so really I should have written $\hat{\pi}(\mathbf{x})$ and $\hat{\phi}(\mathbf{x})$ but I suppressed that dependence for aesthetics. Furthermore, that gradient term needs to be written out explicitly.

$$(\nabla\hat{\phi})^2 = \big(\partial_x\hat{\phi}\big)^2 + \big(\partial_y\hat{\phi}\big)^2 + \big(\partial_z\hat{\phi}\big)^2$$

where

$$\partial_x\hat{\phi}(x,y,z) = \lim_{\Delta x\to 0}\frac{\hat{\phi}(x+\Delta x,y,z) - \hat{\phi}(x,y,z)}{\Delta x}.$$

Let's get an intuitive understanding for this Hamiltonian by looking at it term by term.

The

$$\hat{\pi}(\mathbf{x})^2$$

term means that a wavefunctional has a lot of energy if it changes quickly when a particular field variable is varied.

For the other two terms, let's imagine that our fields are well approximated by the state Ψ_{ϕ_0}, i.e. it is one of those basis states we talked about previously. This means it is "close" to being a "classical" field.

$$|\Psi\rangle \approx |\Psi_{\phi_0}\rangle \,. \tag{39}$$

Then the

$$(\boldsymbol{\nabla}\hat{\phi})^2$$

term means that a wavefunctional has a lot of energy if ϕ_0 has a big gradient. Similarly, the

$$m^2\hat{\phi}^2$$

term means the wave functional has a lot of energy if ϕ_0 is non-zero in a lot of places.

3.6 The Ground State

I will now tell you what the lowest energy eigenstate of this Hamiltonian is. It is

$$\Psi_0[\phi] \propto \exp\left(-\frac{1}{2\hbar}\int d^3k\sqrt{\mathbf{k}^2+m^2}|\phi_{\mathbf{k}}|^2\right) \tag{40}$$

where

$$\phi_{\mathbf{k}} \equiv \int \frac{d^3x}{(2\pi)^{3/2}}\phi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{41}$$

are the Fourier components (or "modes") of the classical field configuration $\phi(\mathbf{x})$. Because ϕ is real, $\phi^*_{\mathbf{k}} = \phi_{-\mathbf{k}}$. Note d^3k is the three-dimensional volume measure over $\mathbf{k}$-space, and $\mathbf{k}^2 = |\mathbf{k}|^2$. The bigger $|\mathbf{k}|$ is, the higher the "frequency" of the Fourier mode is.

Let's try and understand this wave functional qualitatively. It takes its largest value when $\phi(\mathbf{x}) = 0$. The larger the Fourier components of the classical field, the smaller Ψ_0 is. Therefore the wave functional outputs very tiny number for classical fields that are far from 0. Furthermore, because of the $\sqrt{\mathbf{k}^2+m^2}$ term, the high frequency Fourier components are penalized more heavily that the low frequency Fourier components. Therefore, the wave functional Ψ_0 is very small for big and jittery classical fields, and very large for small and gradually varying classical fields.

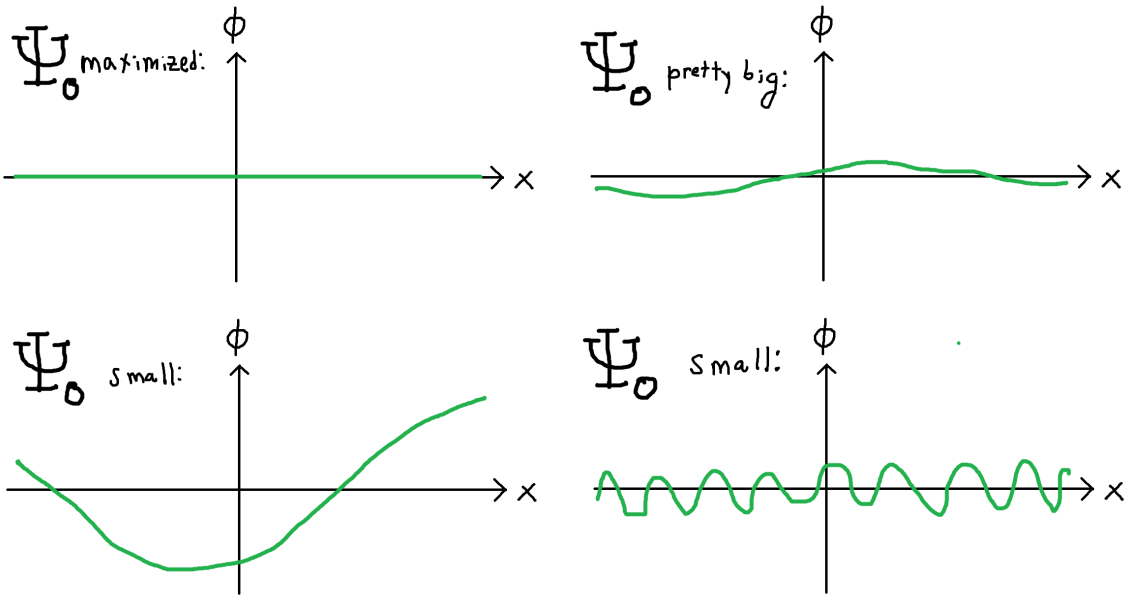


Figure 26: Some sample classical field configurations and the relative size of Ψ_0 when evaluated at each one. The upper-left field maximizes Ψ_0 because it is 0. The upper-right field is pretty close to 0, so Ψ_0 is still pretty big. The lower-left field makes Ψ_0 small because it contains large field values. The lower-right field makes Ψ_0 because its frequency $|\mathbf{k}|$ is large even though the Fourier-coefficient is not that large.

First, let's recall what we mean by ground state. Because $|\Psi_0\rangle$ is an energy eigenstate,

$$\hat{H}\,|\Psi_0\rangle = E_0\,|\Psi_0\rangle \tag{42}$$

for some energy E_0. However, any other energy eigenstate will necessarily have an eigenvalue that is bigger than E_0.

Intuitively speaking, why is $|\Psi_0\rangle$ the ground state? It's because it negotiates all of the competing interests of the terms in the Hamiltonian to minimize it's eigenvalue. Recall that there are three terms in the Hamiltonian from Eq. 38. Let's go through all three terms how see how Ψ_0 tries to minimize each one.

1. The $\hat{\pi}^2$ term doesn't want the functional to vary too quickly as the classical field input is changed. This is minimized because Ψ_0 varies like a Gaussian in terms of the Fourier components $\phi_{\mathbf{k}}$.

2. The $(\nabla\hat{\phi})^2$ term is minimized when likely classical field configurations have small gradients. This is minimized because of the $\sqrt{\mathbf{k}^2+m^2}$ factor, which penalizes high-gradient jittery fields more harshly than small-gradient gradually varying fields.

3. The $m^2\phi^2$ term wants likely classical field configurations to have field values $\phi(\mathbf{x})$ close to 0. This is minimized by making Ψ_0 peak around the classical field configuration $\phi(\mathbf{x}) = 0$.

Now that we have some appreciation for the ground state, I want to rewrite it in a suggestive way:

$$\begin{aligned}\Psi_0[\phi] &\propto \exp\left(-\frac{1}{2\hbar}\int d^3k \sqrt{\mathbf{k}^2+m^2}|\phi_{\mathbf{k}}|^2\right) \\ &\propto \prod_{\mathbf{k}\in\mathbb{R}^3} \exp\left(-\frac{1}{2\hbar}\sqrt{\mathbf{k}^2+m^2}|\phi_{\mathbf{k}}|^2\right).\end{aligned}$$

We can see that $|\Psi_0\rangle$ "factorizes" nicely when written in terms of the Fourier components $\phi_{\mathbf{k}}$ of the classical field input.

3.7 Particles

You ask, "Alright, great, I can see what a quantum field is. But what does this have to do with *particles*?"

Great question. These wave functionals seem to have nothing to do with particles. However, the particle states are hiding in these wave functionals, somehow. It turns out that we can make wave functionals that describe a state with a certain number of particles possessing some specified momenta $\hbar\mathbf{k}$. Here is how you do it:

Let's say that for each $\mathbf{k}$, there are $n_{\mathbf{k}}$ particles present with momenta $\hbar\mathbf{k}$. Schematically, the wavefunctionals corresponding to these states are

$$\Psi[\phi] \propto \prod_{\mathbf{k}\in\mathbb{R}^3} F_{n_{\mathbf{k}}}(\phi_{\mathbf{k}}, \phi_{-\mathbf{k}}) \exp\left(-\frac{1}{2\hbar}\sqrt{\mathbf{k}^2+m^2}|\phi_{\mathbf{k}}|^2\right). \tag{43}$$

for some set of functions $F_{n_{\mathbf{k}}}$.

However, people never really work in terms of these functions $F_{n_{\mathbf{k}}}$, whatever they are. More commonly, states are written in terms of "occupation number" notation. We would the write state in Eq. 43 as

$$|\Psi\rangle = |n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3}, \ldots\rangle\,. \tag{44}$$

These states are definite energy states because they are eigenstates of the Hamiltonian.

$$\hat{H}\,|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3}, \ldots\rangle = \left(E_0 + \sum_{\mathbf{k}\in\mathbb{R}^2} n_{\mathbf{k}}\sqrt{\hbar^2\mathbf{k}^2+m^2}\right)|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3}, \ldots\rangle \tag{45}$$

(Remember that E_0 is the energy of the ground state $|\Psi_0\rangle$.) If you ever took a class in special relativity, you would have learned that the energy E of a particle with momentum $\vec{p}$ and mass m is equal to

$$E^2 = p^2c^2 + m^2c^4. \tag{46}$$

That is exactly where that comes from! (Remember we set $c = 1$.) This is exactly the energy for a collection of particles with mass m and momentum $\hbar\mathbf{k}$! The ground state is just the state when all $n_{\mathbf{k}} = 0$.

Not every state can be written in the form of Eq. 44. However, every state *can* be written in terms of a *linear combination* of states of that form. Therefore, we now have two different ways to understand the Hilbert space of quantum field theory. On one hand, we can think of them as wave functionals. On the other hand, we can think of them in terms of particle occupation numbers. These are really two different bases for the same Hilbert space.

There's something I need to point out. These particle states I've written are completely "delocalized" over all of space. These particles do not exist at any particular location. They are infinite plane waves spread out over the whole universe. This is because they are energy (and momentum) eigenstates, meaning they have a well-defined energy. If we wanted to "localize" these particles, we could make a linear combination of particles of slightly different momenta in order to make a Gaussain wave packet. This Gaussian wave packet would not have a perfectly well defined energy or momentum, though. There would be some uncertainty because it is a superposition of energy eigenstates.

So if we momentarily call $|\mathbf{k}\rangle$ to be the state containing just one particle with momentum $\mathbf{k}$, then particle state which is a wavepacket of momentum $\mathbf{k}_0$ and frequency width σ could be written as

$$|\mathbf{k}_0\rangle_{\text{Gaussian}} \propto \int d^3k \exp\left(-\tfrac{(\mathbf{k}-\mathbf{k}_0)^2}{2\sigma^2}\right) |\mathbf{k}\rangle \,.$$

I have included a picture of a wavepacket in the image below. However, don't forget that our QFT "wavepacket" is really a complicated wave functional, and does not have any interpretation as a classical field.

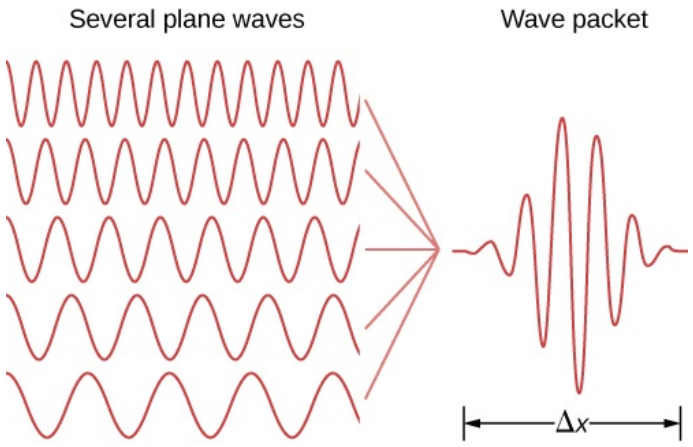


Figure 27: A localized wave packet is the sum of completely delocalized definite-frequency waves. Note that you can't localize a wave packet into a volume that isn't at least a few times as big as its wavelength.

There's four final things you might be wondering about particles. Firstly, where are the "anti-particles" you've heard so much about? The answer is that there are no anti-particles in the quantum field I've described here. This is because the classical field configurations are functions $\phi : \mathbb{R}^3 \to \mathbb{R}$. If our classical fields were functions $\phi : \mathbb{R}^3 \to \mathbb{C}$, then we would find that there are two types of particles, one of which we would call "anti particles." Secondly, I should say that the particles I've described are bosons. That means we can have as many particles as we want with some momentum $\hbar\mathbf{k}$. In other words, our occupation numbers can be any positive integer. A fermionic field is different. Fermionic fields can only have occupation numbers of 0 or 1, so they are rather "digital" in that sense. Fermionic quantum field states therefore do not have the nice wavefunctional interpretation that bosonic quantum fields have. Thirdly, the particle we've constructed here has no spin, i.e. it is a "spin 0" particle. The sorts of particles we're most used to, like electrons and photons, are not of this type. They have spin $\frac{1}{2}$ and spin 1, respectively. Fourthly, where are the Feynman diagrams you've probably heard so much about? Feynman diagrams are useful for describing particle *interactions*. For example, an electron can emit or absorb a photon, so we say the electron field interacts with the photon field. I have only told you here about non-interacting particles, which is perfectly sufficient for our purposes. Feynman diagrams are often used to compute "scattering amplitudes." For example, say I send in two electron wave packets into each other with some momenta and relative angles, wait a while, and then observe two electrons wave packets leaving with new momenta at different relative angles. Physicists use Feynman diagrams as a tool in

order to calculate what the probability of such an event is.

3.8 Entanglement properties of the ground state

We have now looked out our Hilbert space $\mathcal{H}_{\text{QFT}}$ in two different bases: the wavefunctional basis and the particle basis. Both have their strengths and weaknesses. However, I would like to bring up something interesting. Thinking in terms of the wavefunctional basis, we can see that $\mathcal{H}_{\text{QFT}}$ can be decomposed into a tensor product of Hilbert spaces, one for each position $\mathbf{x}$ in space.

$$\mathcal{H}_{\text{QFT}} = \bigotimes_{\mathbf{x}\in\mathbb{R}^3} \mathcal{H}_{\mathbf{x}} \tag{47}$$

(Once again, we might imagine that our tensor product is not truly taken over all of $\mathbb{R}^3$, but perhaps over a lattice of Planck-length spacing, for all we know.) Each local Hilbert space $\mathcal{H}_{\mathbf{x}}$ is given by all normalizable functions from $\mathbb{R} \to \mathbb{C}$. Following mathematicians, we might call such functions $L^2(\mathbb{R})$.

$$\mathcal{H}_{\mathbf{x}} = L^2(\mathbb{R})$$

Fixing $\mathbf{x}$, each state in $\mathcal{H}_{\mathbf{x}}$ simply assigns a complex number to each possible classical value of $\phi(\mathbf{x})$. Once we tensor together all $\mathcal{H}_{\mathbf{x}}$, we recover our space of field wave functionals. The question I now ask you is: what are the position-space entanglement properties of the ground state?

Let's back up a bit and remind ourselves what the ground state again. We wrote it in terms of the Fourier components:

$$\Psi_0[\phi] \propto \exp\left(-\frac{1}{2\hbar}\int d^3k \sqrt{\mathbf{k}^2+m^2}|\phi_{\mathbf{k}}|^2\right)$$

$$\phi_{\mathbf{k}} \equiv \int \frac{d^3x}{(2\pi)^{3/2}}\phi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$$

We can plug in the bottom expression into the top expression to express $\Psi_0[\phi]$ in terms of the position space classical field $\phi(\mathbf{x})$.

$$\begin{aligned}\Psi_0[\phi] &\propto \exp\left(-\frac{1}{2\hbar}\iiint d^3k \frac{d^3x d^3y}{(2\pi)^3} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\sqrt{\mathbf{k}^2+m^2}\phi(\mathbf{x})\phi(\mathbf{y})\right) \\ &\propto \exp\left(-\frac{1}{2\hbar}\iint \frac{d^3x d^3y}{(2\pi)^3} f(|\mathbf{x}-\mathbf{y}|)\phi(\mathbf{x})\phi(\mathbf{y})\right)\end{aligned}$$

One could in principle perform the k integral to compute $f(|\mathbf{x}-\mathbf{y}|)$, although I won't do that here. (There's actually a bit of funny business you have to do, introducing a "regulator" to make the integral converge.) The important thing to note is that the values of the field variables $\phi(\mathbf{x})$ and $\phi(\mathbf{y})$ are entangled together by $f(|\mathbf{x}-\mathbf{y}|)$, and the wave functional Ψ_0 does not factorize nicely in position space the way it did in Fourier space. The bigger $f(|\mathbf{x}-\mathbf{y}|)$ is, the larger the entanglement between $\mathcal{H}_\mathbf{x}$ and $\mathcal{H}_\mathbf{y}$ is. We can see that in the ground state, the value of the field at one point is quite entangled with the field at other points. Indeed, there is a lot of short-range entanglement all throughout the universe. However, it turns out that $f(|\mathbf{x}-\mathbf{y}|)$ becomes very small at large distances. Therefore, nearby field variables are highly entangled, while distant field variables are not very entangled.

This is not such a mysterious property. If your quantum field is in the ground state, and you measure the value of the field at some $\mathbf{x}$ to be $\phi(\mathbf{x})$ then all this means is that nearby field values are likely to also be close to $\phi(\mathbf{x})$. This is just because the ground state wave functional is biggest for classical fields that vary slowly in space.

You might wonder if this entanglement somehow violates causality. Long story short, it doesn't. This entanglement can't be used to send information faster than light. (However, it does have some unintuitive consequences, such as the Reeh–Schlieder theorem.)

Let me wrap this up by saying what this has to do with the Firewall paradox. Remember, in this section we have only discussed QFT in flat space! However, while the space-time at the horizon of a black hole is curved, it isn't curved *that* much. Locally, it looks pretty flat. Therefore, one would expect for quantum fields in the vicinity of the horizon to behave much like they would in flat space. This means that low energy quantum field states will still have a strong amount of short-range entanglement because short-range entanglement lowers the energy of the state. (This is because of the $(\nabla\hat{\phi})^2$ term in the Hamiltonian.) However, the Firewall paradox uses the existence of this entanglement across the horizon to make a contradiction. One resolution to the contradiction is to say that there's absolutely no entanglement across the horizon whatsoever. This would mean that there is an infinite energy density at the horizon, contradicting the assumption that nothing particularly special happens there.

4 Statistical Mechanics

4.1 Entropy

Statistical Mechanics is a branch of physics that pervades all other branches. Statistical mechanics is relevant to Newtonian mechanics, relativity, quantum mechanics, and quantum field theory.

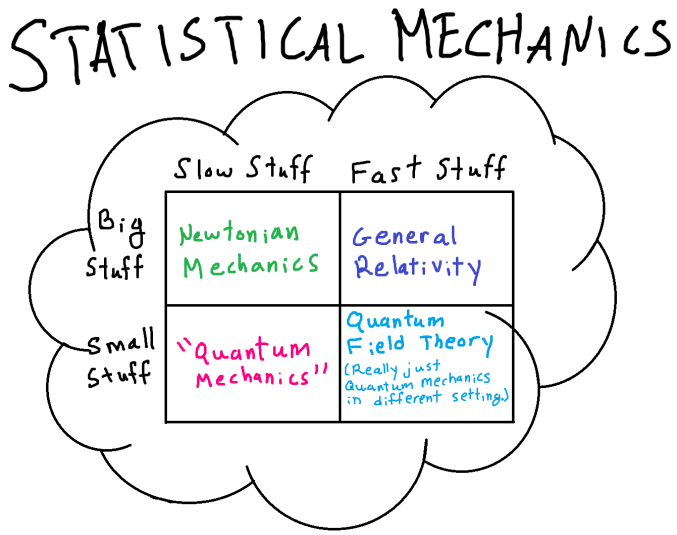


Figure 28: Statistical mechanics applies to all realms of physics.

Its exact incarnation is a little different in each quadrant, but the basic details are identical.

The most important quantity in statistical mechanics is called "entropy," which we label by S. People sometimes say that entropy is a measure of the "disorder" of a system, but I don't think this a good way to think about it. But before we define entropy, we need to discuss two different notions of state: "microstates" and "macrostates."

In physics, we like to describe the real world as mathematical objects. In classical physics, states are points in a "phase space." Say for example you had N particles moving around in 3 dimensions. It would take $6N$ real numbers to specify the physical state of this system at a given instant: 3 numbers for each particle's position and 3 numbers for each particle's momentum. The phase space for this system would therefore just be $\mathbb{R}^{6N}$.

$$(x_1, y_1, z_1, p_{x1}, p_{y1}, p_{z1}, \ldots x_N, y_N, z_N, p_{xN}, p_{yN}, p_{zN}) \in \mathbb{R}^{6N}$$

(In quantum mechanics, states are vectors in a Hilbert space $\mathcal{H}$ instead of points in a phase space. We'll return to the quantum case a bit later.)

A "microstate" is a state of the above form. It contains absolutely all the physical information that an omniscent observer could know. If you were to know the exact microstate of a system and knew all of the laws of physics, you could in principle deduce what the microstate will be at all future times and what the microstate was at all past times.

However, practically speaking, we can never know the true microstate of a system. For example, you could never know the positions and momenta of every damn particle in a box of gas. The only things we can actually measure are macroscopic variables such as internal energy, volume, and particle number (U, V, N). A "macrostate" is just a set of microstates. For examples, the "macrostate" of a box of gas labelled by (U, V, N) would be the set of all microstates with energy U, volume V, and particle number N. The idea is that if you know what macrostate your system is in, you know that your system is equally likely to truly be in any of the microstates it contains.

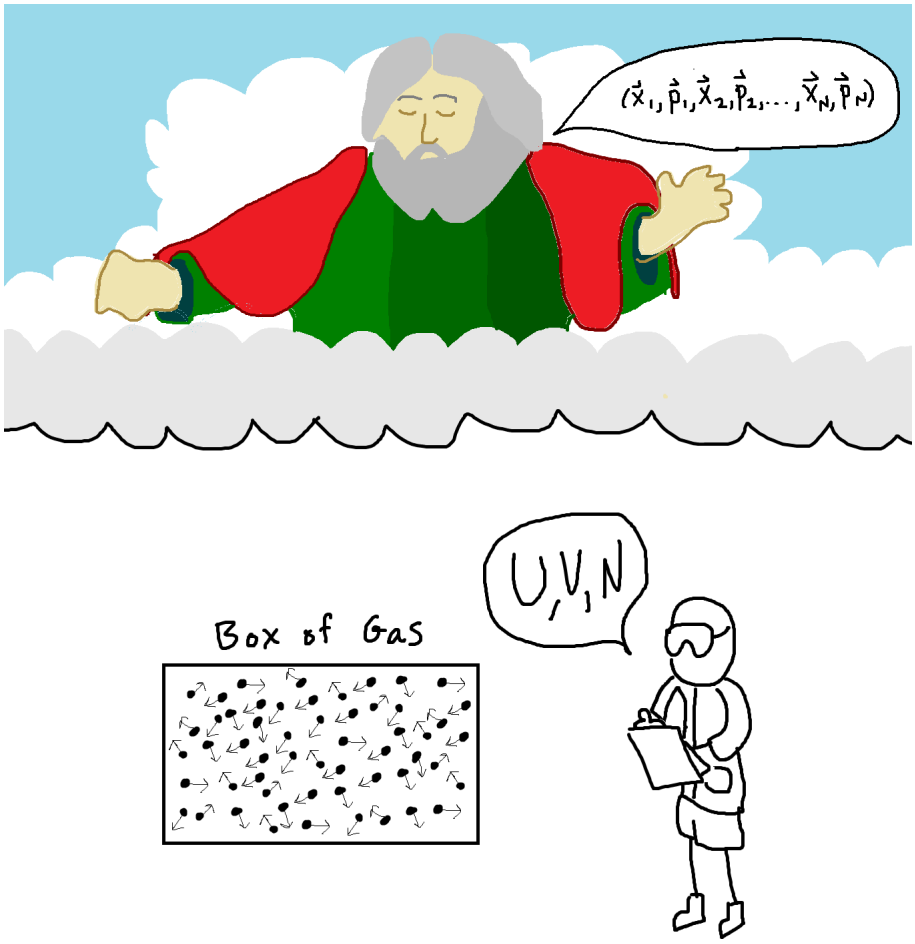


Figure 29: You may know the macrostate, but only God knows the microstate.

I am now ready to define what entropy is. Entropy is a quantity asso-

ciated with a macrostate. If a macrostate is just a set of Ω microstates, then the entropy S of the system is

$$S \equiv k \log \Omega. \tag{48}$$

Here, k is Boltzmann's constant. It is a physical constant with units of energy / temperature.

$$k \equiv 1.38065 \times 10^{-23} \frac{\text{Joules}}{\text{Kelvin}} \tag{49}$$

The only reason that we need k to define S is because the human race defined units of temperature before they defined entropy. (We'll see how temperature factors into any of this soon.) Otherwise, we probably would have set $k = 1$ and temperature would have the same units as energy.

You might be wondering how we actually count Ω. As you probably noticed, the phase space $\mathbb{R}^{6N}$ is not discrete. In that situation, we integrate over a phase space volume with the measure

$$d^3x_1 d^3p_1 \ldots d^3x_N d^3p_N.$$

However, this isn't completely satisfactory because position and momentum are dimensionful quantities while Ω should be a dimensionless number. We should therefore divide by a constant with units of position times momentum. Notice, however, that because S only depends on $\log \Omega$, any constant rescaling of Ω will only alter S by a constant and will therefore never affect the change in entropy ΔS of some process. So while we have to divide by a constant, whichever constant we divide by doesn't affect the physics.

Anyway, even though we are free to choose whatever dimensionful constant we want, the "best" is actually Planck's constant h! Therefore, for a classical macrostate that occupies a phase space volume Vol,

$$\Omega = \frac{1}{N!} \frac{1}{h^{3N}} \int_{\text{Vol}} \prod_{i=1}^{N} d^3x_i d^3p_i. \tag{50}$$

(The prefactor $1/N!$ is necessary if all N particles are indistinguishable. It is the cause of some philosophical consternation but I don't want to get into any of that.)

Let me now explain why I think saying entropy is "disorder" is not such a good idea. Different observers might describe reality with different macrostates. For example, say your room is very messy and disorganized. This isn't a problem for you, because you spend a lot of time

in there and know where everything is. Therefore, the macrostate you use to describe your room contains very few microstates and has a small entropy. However, according to your mother who has not studied your room very carefully, the entropy of your room is very large. The point is that while everyone might agree your room is messy, the entropy of your room really depends on how little you know about it.

4.2 Temperature and Equilibrium

Let's say we label our macrostates by their total internal energy U and some other macroscopic variables like V and N. (Obviously, these other macroscopic variables V and N can be replaced by different quantities in different situations, but let's just stick with this for now.) Our entropy S depends on all of these variables.

$$S = S(U, V, N) \tag{51}$$

The temperature T of the (U, V, N) macrostate is then be defined to be

$$\frac{1}{T} \equiv \left.\frac{\partial S}{\partial U}\right|_{V,N}. \tag{52}$$

The partial derivative above means that we just differentiate $S(U, V, N)$ with respect to U while keeping V and N fixed.

If your system has a high temperature and you add a bit of energy dU to it, then the entropy S will not change much. If your system has a small temperature and you add a bit of energy, the entropy will increase a lot.

Next, say you have two systems A and B which are free to trade energy back and forth.

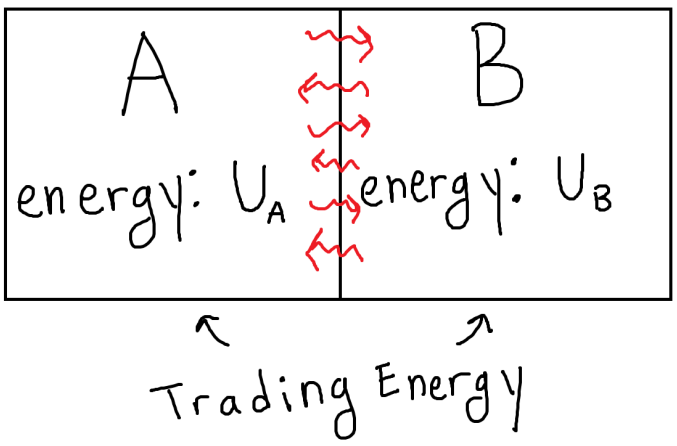


Figure 30: Two systems A and B trading energy. $U_A + U_B$ is fixed.

Say system A could be in one of Ω_A possible microstates and system B could be in Ω_B possible microstates. Therefore, the total AB system could be in $\Omega_A \Omega_B$ possible microstates. Therefore, the entropy S_{AB} of both systems combined is just the sum of entropies of both sub-systems.

$$S_{AB} = k \log(\Omega_A \Omega_B) = k \log \Omega_A + k \log \Omega_B = S_A + S_B \qquad (53)$$

The crucial realization of statistical mechanics is that, all else being equal, a system is most likely to find itself in a macrostate corresponding to the largest number of microstates. This is the so-called "Second law of thermodynamics": for all practical intents and purposes, the entropy of a closed system always increases over time. It is not really a physical "law" in the regular sense, it is more like a profound realization.

Therefore, the entropy S_{AB} of our joint AB system will increase as time goes on until it reaches its maximum possible value. In other words, A and B trade energy in a seemingly random fashion that increases S_{AB} on average. When S_{AB} is finally maximized, we say that our systems are in "thermal equilibrium."

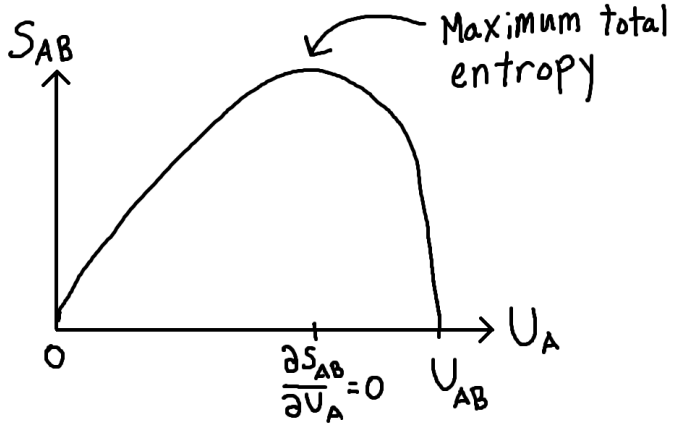


Figure 31: S_{AB} is maximized when U_A has some particular value. (It should be noted that there will actually be tiny random "thermal" fluctuations around this maximum.)

Let's say that the internal energy of system A is U_A and the internal energy of system B is U_B. Crucially, note that the total energy of combined system

$$U_{AB} = U_A + U_B$$

is constant over time! This is because energy of the total system is conserved. Therefore,

$$dU_A = -dU_B.$$

Now, the combined system will maximize its entropy when U_A and U_B have some particular values. Knowing the value of U_A is enough though, because $U_B = U_{AB} - U_A$. Therefore, entropy is maximized when

$$0 = \frac{\partial S_{AB}}{\partial U_A}. \tag{54}$$

However, we can rewrite this as

$$\begin{aligned} 0 &= \frac{\partial S_{AB}}{\partial U_A} \\ &= \frac{\partial S_A}{\partial U_A} + \frac{\partial S_B}{\partial U_A} \\ &= \frac{\partial S_A}{\partial U_A} - \frac{\partial S_B}{\partial U_B} \\ &= \frac{1}{T_A} - \frac{1}{T_B}. \end{aligned}$$

Therefore, our two systems are in equilibrium if they have the same temperature!

$$T_A = T_B \tag{55}$$

If there are other macroscopic variables we are using to define our macrostates, like volume V or particle number N, then there will be other quantities that must be equal in equibrium, assuming our two systems compete for volume or trade particles back and forth. In these cases, we define the quantities P and μ to be

$$\frac{P}{T} \equiv \left.\frac{\partial S}{\partial V}\right|_{U,N} \qquad\qquad \frac{\mu}{T} \equiv -\left.\frac{\partial S}{\partial N}\right|_{U,V}. \tag{56}$$

P is called "pressure" and μ is called "chemical potential." In equilibrium, we would also have

$$P_A = P_B \qquad\qquad \mu_A = \mu_B. \tag{57}$$

(You might object that pressure has another definition, namely force divided by area. It would be incumbent on us to check that this definition matches that definition in the relevant situation where both definitions have meaning. Thankfully it does.)

4.3 The Partition Function

Figure 32: If you want to do statistical mechanics, you really should know about the partition function.

Explicitly calculating Ω for a given macrostate is usually very hard. Practically speaking, it can only be done for simple systems you understand very well. However, physicists have developed an extremely powerful way of doing statistical mechanics even for complicated systems. It turns out that there is a function of temperature called the "partition function" that contains all the information you'd care to know about your macrostate when you are working in the "thermodynamic limit." This function is denoted $Z(T)$. Once you compute $Z(T)$ (which is usually much easier than computing Ω) it is a simple matter to extract the relevant physics.

Before defining the partition function, I would like to talk a bit about heat baths. Say you have some system $\mathcal{S}$ in a very large environment $\mathcal{E}$. Say you can measure the macroscopic variables of $\mathcal{S}$, including its energy E at any given moment. (We use E here to denotes energy instead of U when talking about the partition function.) The question I ask is: if the total system has a temperature T, what's the probability that $\mathcal{S}$ has some particular energy E?

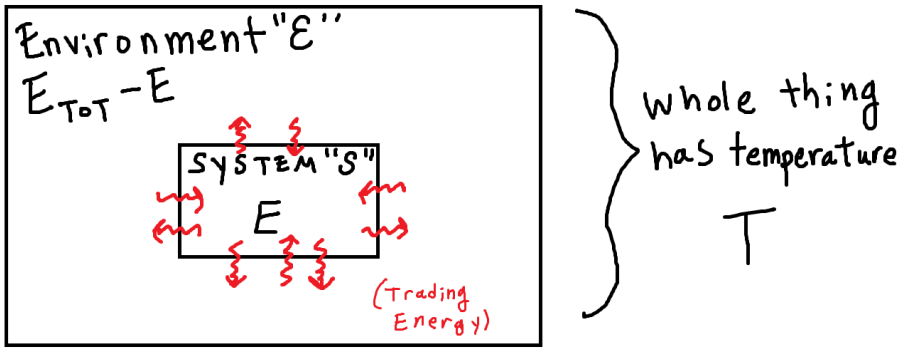


Figure 33: A large environment $\mathcal{E}$ and system $\mathcal{S}$ have a fixed total energy E_{tot}. $\mathcal{E}$ is called a "heat bath" because it is very big. The combined system has a temperature T.

We should be picturing that $\mathcal{S}$ and $\mathcal{E}$ are evolving in some complicated way we can't understand. However, their total energy

$$E_{\text{tot}} = E + E_{\mathcal{E}} \tag{58}$$

is conserved. We now define

$$\begin{aligned} \Omega_{\mathcal{S}}(E) &\equiv \text{num. microstates of } \mathcal{S} \text{ with energy } E \\ \Omega_{\mathcal{E}}(E_{\mathcal{E}}) &\equiv \text{num. microstates of } \mathcal{E} \text{ with energy } E_{\mathcal{E}}. \end{aligned} \tag{59}$$

Therefore, the probability that $\mathcal{S}$ has some energy E is proportional to the number of microstates where $\mathcal{S}$ has energy E and $\mathcal{E}$ has energy $E_{\text{tot}} - E$.

$$\text{Prob}(E) \propto \Omega_{\mathcal{S}}(E)\Omega_{\mathcal{E}}(E_{\text{tot}} - E) \tag{60}$$

Here is the important part. Say that our heat bath has a lot of energy: $E_{\text{tot}} \gg E$. As far as the heat bath is concerned, E is a very small amount of energy. Therefore,

$$\begin{aligned} \Omega_{\mathcal{E}}(E_{\text{tot}} - E) &= \exp\left(\frac{1}{k} S_{\mathcal{E}}(E_{\text{tot}} - E)\right) \\ &\approx \exp\left(\frac{1}{k} S_{\mathcal{E}}(E_{\text{tot}}) - \frac{E}{kT}\right) \end{aligned}$$

by Taylor expanding $S_{\mathcal{E}}$ in E and using the definition of temperature. We now have

$$\text{Prob}(E) \propto \Omega_{\mathcal{S}}(E) \exp\left(-\frac{E}{kT}\right).$$

$\Omega_{\mathcal{S}}(E)$ is sometimes called the "degeneracy" of E. In any case, we can easily see what the ratio of $\text{Prob}(E_1)$ and $\text{Prob}(E_2)$ must be.

$$\frac{\text{Prob}(E_1)}{\text{Prob}(E_2)} = \frac{\Omega_{\mathcal{S}}(E_1)e^{-E_1/kT}}{\Omega_{\mathcal{S}}(E_2)e^{-E_2/kT}}$$

Furthermore, we can use the fact that all probabilities must sum to 1 in order to calculate the absolute probability. We define

$$\begin{aligned} Z(T) &\equiv \sum_E \Omega_{\mathcal{S}}(E)e^{-E/kT} \\ &= \sum_s e^{-E_s/kT} \end{aligned} \tag{61}$$

where $\sum_s$ is a sum over all states of $\mathcal{S}$. Finally, we have

$$\text{Prob}(E) = \frac{\Omega_{\mathcal{S}}(E)e^{-E/kT}}{Z(T)} \tag{62}$$

However, more than being a mere proportionality factor, $Z(T)$ takes on a life of its own, so it is given the special name of the "partition function." Interestingly, $Z(T)$ is a function that depends on T and not E. It is not a function that has anything to do with a particular macrostate. Rather, it is a function that has to with *every* microstate at some temperature. Oftentimes, we also define

$$\beta \equiv \frac{1}{kT}$$

and write

$$Z(\beta) = \sum_s e^{-\beta E_s}. \tag{63}$$

The partition function $Z(\beta)$ has many amazing properties. For one, it can be used to write an endless number of clever identities. Here is one. Say you want to compute the expected energy $\langle E \rangle$ your system has at temperature T.

$$\begin{aligned} \langle E \rangle &= \sum_s E_s \text{Prob}(E_s) \\ &= \frac{\sum_s E_s e^{-\beta E_s}}{Z(\beta)} \\ &= -\frac{1}{Z}\frac{\partial}{\partial \beta} Z \\ &= -\frac{\partial}{\partial \beta} \log Z \end{aligned}$$

This expresses the expected energy $\langle E \rangle$ as a function of temperature. (We could also calculate $\langle E^n \rangle$ for any n if we wanted to.)

Where the partition function really shines is in the "thermodynamic limit." Usually, people define the thermodynamic limit as

$$N \to \infty \quad \text{(thermodynamic limit)} \tag{64}$$

where N is the number of particles. However, sometimes you might be interested in more abstract systems like a spin chain (the so-called "Ising model") or something else. There are no "particles" in such a system, however there is still something you would justifiably call the thermodynamic limit. This would be when the number of sites in your spin chain becomes very large. So N should really just be thought of as the number of variables you need to specify a microstate. When someone is "working in the thermodynamic limit," it just means that they are considering very "big" systems.

Of course, in real life N is never infinite. However, I think we can all agree that 10^{23} is close enough to infinity for all practical purposes. Whenever an equation is true "in the thermodynamic limit," you can imagine that there are extra terms of order $\frac{1}{N}$ unwritten in your equation and laugh at them.

What is special about the thermodynamic limit is that $\Omega_{\mathcal{S}}$ becomes, like, really big...

$$\Omega_{\mathcal{S}} = (\text{something})^N$$

Furthermore, the entropy and energy will scale with N

$$S_{\mathcal{S}} = N S_1 \qquad\qquad E = N E_1$$

In the above equation, S_1 and E_1 can be thought of as the average amount of entropy per particle.

Therefore, we can rewrite

$$\begin{aligned} \text{Prob}(E) &\propto \Omega_{\mathcal{S}}(E) \exp\big(-\tfrac{1}{kT} E\big) \\ &= \exp\big(\tfrac{1}{k} S_{\mathcal{S}} - \tfrac{1}{kT} E\big) \\ &= \exp\big(N\big(\tfrac{1}{k} S_1 - \tfrac{1}{kT} E_1\big)\big). \end{aligned}$$

The thing to really gawk at in the above equation is that the probability that $\mathcal{S}$ has some energy E is given by

$$\text{Prob}(E) \propto e^{N(\ldots)}.$$

I want you to appreciate how insanely big $e^{N(\ldots)}$ is in the thermodynamic limit. Furthermore, if there is even a miniscule change in $(\ldots)$,

Prob(E) will change radically. Therefore, Prob(E) will be extremely concentrated at some particular energy, and deviating slightly from that maximum will cause Prob(E) to plummit.

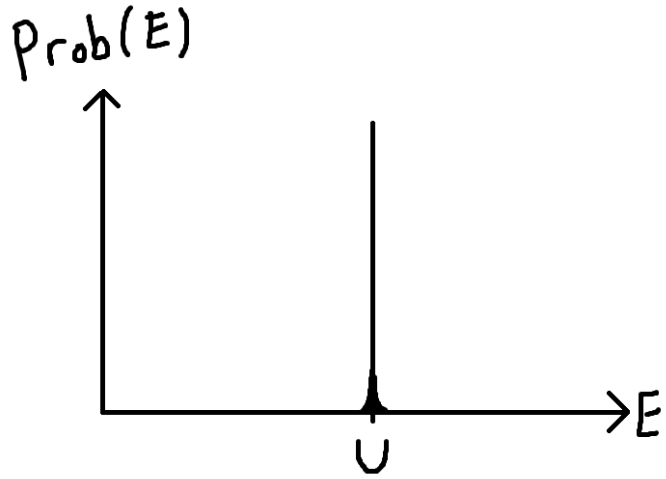


Figure 34: In the thermodynamic limit, the system $\mathcal{S}$ will have a well defined energy.

We can therefore see that if the energy U maximizes Prob(E), we will essentially have

$$\text{Prob}(E) \approx \begin{cases} 1 & \text{if } E = U \\ 0 & \text{if } E \neq U \end{cases}.$$

Let's now think back to our previously derived equation

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z(\beta).$$

Recall that $\langle E \rangle$ is the expected energy of $\mathcal{S}$ when it is coupled to a heat bath at some temperature. The beauty is that in the thermodynamic limit where our system $\mathcal{S}$ becomes very large, we don't even have to think about the heat bath anymore! Our system $\mathcal{S}$ is basically just in the macrostate where all microstates with energy U are equally likely. Therefore,

$$\langle E \rangle = U \qquad \text{(thermodynamic limit)}$$

and

$$U = -\frac{\partial}{\partial \beta} \log Z(\beta) \tag{65}$$

is an *exact* equation in the thermodynamic limit.

Let's just appreciate this for a second. Our original definition of $S(U)$ was

$$S(U) = k \log(\Omega(U))$$

and our original definition of temperature was

$$\frac{1}{T} = \frac{\partial S}{\partial U}.$$

In other words, T is a function of U. However, we totally reversed logic when we coupled our system to a larger environment. We no longer knew what the exact energy of our system was. I am now telling you that instead of calculating T as a function of U, when N is large we are actually able to calculate U as a function of T! Therefore, instead of having to calculate $\Omega(U)$, we can just calculate $Z(T)$ instead.

I should stress, however, that $Z(T)$ is still a perfectly worthwhile thing to calculate even when your system $\mathcal{S}$ isn't "big." It will still give you the exact average energy $\langle E \rangle$ when your system is in equilibrium with a bigger environment at some temperature. What's special about the thermodynamic limit is that you no longer have to imagine the heat bath is there in order to interpret your results, because any "average quantity" will basically just be an actual, sharply defined, "quantity." In short,

$$Z(\beta) = \Omega(U) e^{-\beta U} \qquad \text{(thermodynamic limit)} \tag{66}$$

It's worth mentioning that the other contributions to $Z(\beta)$ will also be absolute huge; they just won't be as stupendously huge as the term due to U.

Okay, enough adulation for the partition function. Let's do something with it again. Using the above equation there is a very easy way to figure out what $S_{\mathcal{S}}(U)$ is in terms of $Z(\beta)$.

$$\begin{aligned} S_{\mathcal{S}}(U) &= k \log \Omega_{\mathcal{S}}(U) \\ &= k \log\big(Z e^{\beta U}\big) \quad \text{(thermodynamic limit)} \\ &= k \log Z + k\beta U \\ &= k\big(1 - \beta \frac{\partial}{\partial \beta}\big) \log Z \end{aligned}$$

(Gah. Another amazing identity, all thanks to the partition function.)

This game that we played, coupling our system $\mathcal{S}$ to a heat bath so we could calculate U as a function of T instead of T as a function of U, can be replicated with other quantities like the chemical potential μ (defined in Eq. 57). We could now imagine that $\mathcal{S}$ is trading particles

with a larger environment. Our partition function would then be a function of μ in addition to T.

$$Z = Z(\mu, T)$$

In the thermodynamic limit, we could once again use our old tricks to find N in terms of μ.

4.4 Free energy

Now that we're on an unstoppable victory march of introductory statistical mechanics, I think I should define a quantity closely related to the partition function: the "free energy" F.

$$F \equiv U - TS \tag{67}$$

(This is also called the "Helmholtz Free Energy.") F is defined for any system with some well defined internal energy U and entropy S when present in a larger environment which has temperature T. Crucially, *the system does not need to be in thermal equilibrium with the environment.* In other words, free energy is a quantity associated with some system which may or may not be in equilibrium with an environment at temperature T.

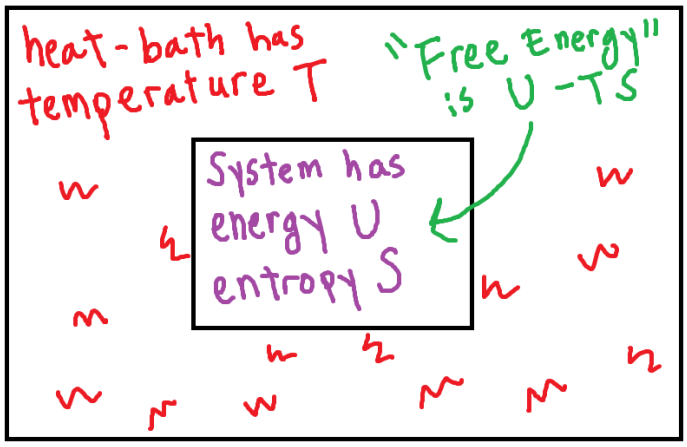


Figure 35: A system with internal energy U and entropy S in a heat bath at temperature T has free energy $F = U - TS$.

Okay. So why did we define this quantity F? The hint is in the name "free energy." Over time, the system will equilibriate with the environment in order to maximize the entropy of the whole world. While doing so, the energy U of the system will change. So if we cleverly leave our system in a larger environment, under the right circumstances we

can let the second law of thermodynamics to do all the hard work, transferring energy into our system at no cost to us! I should warn you that ΔF is actually not equal to the change in internal energy ΔU that occurs during this equilibriation. This is apparent just from its definition. (Although it does turn out that F is equal to the "useful work" you can extract from such a system.)

The reason I'm telling you about F is because it is a useful quantity for determining what will happen to a system at temperature T. Namely, in the thermodynamic limit, the system will *minimize* F by equilibriating with the environment.

Recall Eq. 66 (reproduced below).

$$Z(\beta) = \Omega(U)e^{-\beta U} \qquad \text{(thermodynamic limit)}$$

If our system $\mathcal{S}$ is in equilibrium with the heat bath, then

$$\begin{aligned} Z(\beta) &= \exp\left(\tfrac{1}{k}S - \beta U\right) \quad \text{(at equilibrium in thermodynamic limit)} \\ &= \exp(-\beta F). \end{aligned}$$

First off, we just derived another amazing identity of the partition function. More importantly, recall that U, as written in Eq. 66, is defined to be the energy that maximizes $\Omega(U)e^{-\beta U}$, A.K.A. the energy that maximizes the entropy of the world. Because we know that the entropy of the world always wants to be maximized, we can clearly see that F wants to be minimized, as claimed.

Therefore, F is a very useful quantity! It always wants to be minimized at equilibrium. It can therefore be used to detect interesting phenomena, such as phase transitions.

4.5 Phase Transitions

Let's back up a bit and think about a picture we drew, Fig. 34. It's a very suggestive picture that begs a very interesting question. What if, at some critical temperature T_c, a new peak grows and overtakes our first peak?

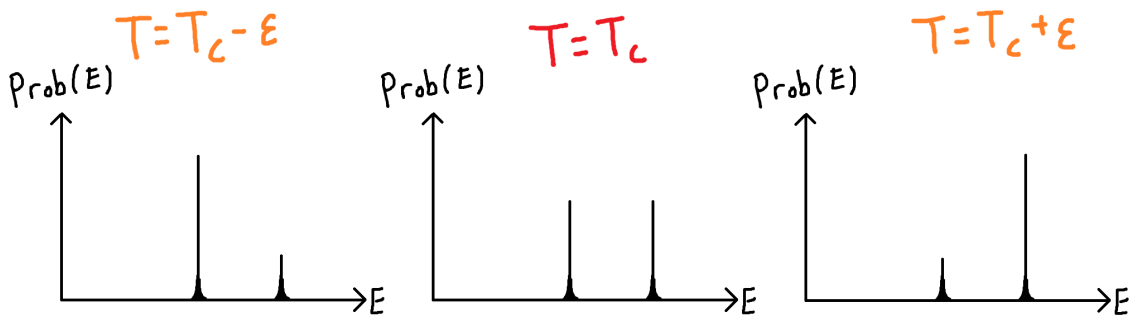


Figure 36: A phase transition, right below the critical temperature T_c, at T_c, and right above T_c.

This can indeed happen, and is in fact what a physicist would call a "first order phase transition." We can see that will be a discontinuity in the first derivative of $Z(T)$ at T_c. You might be wondering how this is possible, given the fact that from its definition, Z is clearly an analytic function as it is a sum of analytic functions. The thing to remember is that we are using the thermodynamic limit, and the sum of an infinite number of analytic functions may not be analytic.

Because there is a discontinuity in the first derivative of $Z(\beta)$, there will be a discontinuity in $E = -\frac{\partial}{\partial\beta}\log Z$. This is just the "latent heat" you learned about in high school. In real life systems, it takes some time for enough energy to be transferred into a system to overcome the latent heat energy barrier. This is why it takes so long for a pot of water to boil or a block of ice to melt. Furthermore, during these lengthy phase transitions, the pot of water or block of ice will actually be at a constant temperature, the "critical temperature" (100°C and 0°C respectively). Once the phase transition is complete, the temperature can start changing again.

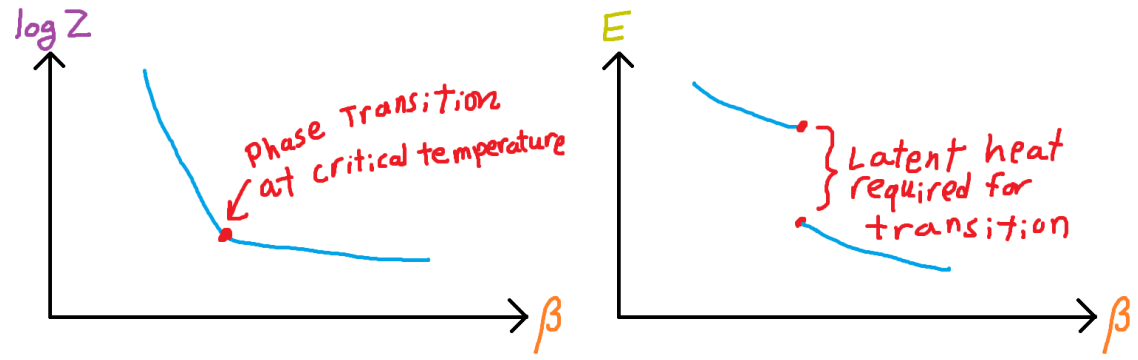


Figure 37: A discontinuity in the first derivative of Z corresponds to a first order phase transition. This means that you must put a finite amount of energy into the system called "latent heat" at the phase transition before the temperature of the system will rise again.

4.6 Example: Box of Gas

For concreteness, I will compute the partition function for an ideal gas. By ideal, I mean that the particles do not interact with each other.

Let N be the number of particles in the box and m be the mass of each particle. Suppose the particles exist in a box of volume V. The positions and momenta of the particles at $\vec{x}_i$ and $\vec{p}_i$ for $i = 1 \dots N$. The energy is given by the sum of kinetic energies of all particles.

$$E = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m}. \tag{68}$$

Therefore,

$$\begin{aligned} Z(\beta) &= \sum_s e^{-\beta E_s} \\ &= \frac{1}{N!}\frac{1}{h^{3N}} \int \prod_{i=1}^{N} d^3x_i d^3p_i \exp\left(-\beta \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m}\right) \\ &= \frac{1}{N!}\frac{V^N}{h^{3N}} \prod_{i=1}^{N} \int d^3p_i \exp\left(-\beta \frac{\vec{p}_i^2}{2m}\right) \\ &= \frac{1}{N!}\frac{V^N}{h^{3N}} \left(\frac{2m\pi}{\beta}\right)^{3N/2} \end{aligned}$$

If N is large, the thermodynamic limit is satisfied. Therefore,

$$\begin{aligned} U &= -\frac{\partial}{\partial\beta} \log Z \\ &= -\frac{3}{2} N \frac{\partial}{\partial\beta} \log\left(N!^{\frac{-2}{3N}} \left(\frac{V}{h^3}\right)^{\frac{2}{3}} \frac{2m\pi}{\beta}\right) \\ &= \frac{3}{2}\frac{N}{\beta} \\ &= \frac{3}{2} NkT. \end{aligned}$$

You could add interactions between the particles by adding some potential energy between V each pair of particles (unrelated to the volume V).

$$E = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m} + \frac{1}{2} \sum_{i,j} V(|\vec{x}_i - \vec{x}_j|) \tag{69}$$

The form of $V(r)$ might look something like this.

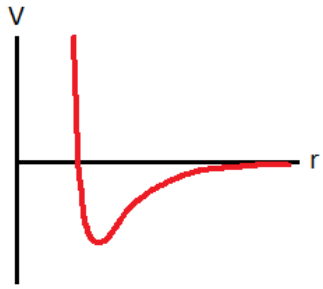


Figure 38: An example for an interaction potential V between particles as a function of distance r.

The calculation of $Z(\beta)$ then becomes more difficult, although you could approximate it pretty well using something called the "cluster decomposition." This partition function would then exhibit a phase transition at a critical temperature between a gas phase and a liquid phase. It is an interesting exercise to try to pin down for yourself where all the new states are coming from at the critical temperature which make $Z(\beta)$ discontinuous. (Hint: condensation.)

Obviously, the attractions real life particles experience cannot be written in terms of such a simple central potential $V(r)$. It's just a simplified model. For example, there should be some angular dependence to the potential energy as well which is responsible for the chemical structures we see in nature. If we wanted to model the liquid-to-solid transition, we'd have to take that into account.

4.7 Shannon Entropy

So far, we have been imagining that that all microstates in a macrostate are equally likely to be the "true" microstate. However, what if you assign a different probability p_s to each microstate s? What is the entropy then?

There is a more general notion of entropy in computer science called "Shannon entropy." It is given by

$$S = -\sum_s p_s \log p_s. \tag{70}$$

It turns out that entropy is maximized when all the probabilities p_s are equal to each other. Say there are Ω states and each $p_s = \Omega^{-1}$. Then

$$S = \log \Omega \tag{71}$$

matching the physicist's definition (up to the Boltzmann constant).

One tiny technicality when dealing with the Shannon entropy is interpreting the value of

$$0 \log 0.$$

It is a bit troublesome because $\log 0 = -\infty$. However, it turns out that the correct value to assign the above quantity is

$$0 \log 0 \equiv 0.$$

This isn't too crazy though, because

$$\lim_{x \to 0} x \log x = 0.$$

4.8 Quantum Mechanics, Density Matrices

So far I have only told you about statistical mechanics in the context of classical mechanics. Now it's time to talk about quantum mechanics.

There is something very interesting about quantum mechanics: states can be in superpositions. Because of this, even if you know the exact quantum state your system is in, you can still only predict the probabilities that any observable (such as energy) will have a particular value when measured. Therefore, there are *two* notions of uncertainty in quantum statistical mechanics:

1. Fundemental quantum uncertainty

2. Uncertainty due to the fact that you may not know the exact quantum state your system is in anyway. (This is sometimes called "classical uncertainty.")

It would be nice if we could capture these two different notions of uncertainty in one unified mathematical object. This object is called the "density matrix."

Say the quantum states for your system live in a Hilbert space $\mathcal{H}$. A density matrix ρ is an operator

$$\rho : \mathcal{H} \to \mathcal{H}. \tag{72}$$

Each density matrix is meant to represent a so-called "classical superposition" of quantum states.

For example, say that you are a physics PhD student working in a lab and studying some quantum system. Say your lab mate has prepared the system in one of two states $|\psi_1\rangle$ or $|\psi_2\rangle$, but unprofessionally forgot

which one it is in. This would be an example of a "classical superposition" of quantum states. Usually, we think of classical superpositions as having a thermodynamical nature, but that doesn't have to be the case.

Anyway, say that your lab mate thinks there's a 50% chance the system could be in either state. The density matrix corresponding to this classical superposition would be

$$\rho = \frac{1}{2} |\psi_1\rangle \langle\psi_1| + \frac{1}{2} |\psi_2\rangle \langle\psi_2| .$$

More generally, if you have a set of N quantum states $|\psi_i\rangle$ each with a classical probability p_i, then the corresponding density matrix would be

$$\rho = \sum_{i=1}^{N} p_i |\psi_i\rangle \langle\psi_i| . \tag{73}$$

This is useful to define because it allows us to extract expectation values of observables $\hat{\mathcal{O}}$ in a classical superposition. But before I prove that, I'll have to explain a very important operation: "tracing."

Say you have quantum state $|\psi\rangle$ and you want to calculate the expectation value of $\hat{\mathcal{O}}$. This is just equal to

$$\langle\hat{\mathcal{O}}\rangle = \langle\psi| \hat{\mathcal{O}} |\psi\rangle . \tag{74}$$

Now, say we have an orthonormal basis $|\phi_s\rangle \in \mathcal{H}$. We then have

$$1 = \sum_s |\phi_s\rangle \langle\phi_s| . \tag{75}$$

Therefore, inserting the identity, we have

$$\begin{aligned} \langle\hat{\mathcal{O}}\rangle &= \langle\psi| \hat{\mathcal{O}} |\psi\rangle \\ &= \sum_s \langle\psi| \hat{\mathcal{O}} |\phi_s\rangle \langle\phi_s|\psi\rangle \\ &= \sum_s \langle\phi_s|\psi\rangle \langle\psi| \hat{\mathcal{O}} |\phi_s\rangle . \end{aligned}$$

This motivates us to define something called the "trace operation" for any operator $\mathcal{H} \to \mathcal{H}$. While we are using an orthonormal basis of $\mathcal{H}$ to define it, it is actually independent of which basis you choose.

$$\mathrm{Tr}(\ldots) \equiv \sum_s \langle\phi_s| \ldots |\phi_s\rangle \tag{76}$$

We can therefore see that for our state $|\psi\rangle$,

$$\langle\hat{\mathcal{O}}\rangle = \mathrm{Tr}\Big(|\psi\rangle\langle\psi|\,\hat{\mathcal{O}}\Big). \tag{77}$$

Returning to our classical superposition and density matrix ρ, we are now ready to see how to compute the expectation values.

$$\begin{aligned}\langle\hat{\mathcal{O}}\rangle &= \sum_i p_i \langle\psi_i|\,\hat{\mathcal{O}}\,|\psi_i\rangle \\ &= \sum_i p_i \,\mathrm{Tr}\Big(|\psi_i\rangle\langle\psi_i|\,\hat{\mathcal{O}}\Big) \\ &= \mathrm{Tr}\Big(\rho\hat{\mathcal{O}}\Big)\end{aligned}$$

So I have now proved my claim that we can use density matrices to extract expectation values of observables.

Now that I have told you about these density matrices, I should introduce some terminology. A density matrix that is of the form

$$\rho = |\psi\rangle\langle\psi|$$

for some $|\psi\rangle$ is said to represent a "pure state," because you know with 100% certainty which quantum state your system is in. Note that for a pure state,

$$\rho^2 = \rho \quad \text{(for pure state)}.$$

It turns out that the above condition is a necessary and sufficient condition for determining if a density matrix represents a pure state.

If a density matrix is instead a combination of different states in a classical superposition, it is said to represent a "mixed state." This is sort of bad terminology, because a mixed state is not a "state" in the Hilbert space $\hat{\mathcal{H}}$, but whatever.

4.9 Example: Two state system

Consider the simplest Hilbert space, representing a two state system.

$$\mathcal{H} = \mathbb{C}^2$$

Let us investigate the difference between a quantum superposition and a classical super position. An orthonormal basis for this Hilbert space is given by

$$|0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \qquad\qquad |1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

Say you have a classical superposition of these two states where you have a 50% probability that your state is in either state. Then

$$\begin{aligned}\rho_{\text{Mixed}} &= \frac{1}{2}\left|0\right\rangle\left\langle 0\right| + \frac{1}{2}\left|1\right\rangle\left\langle 1\right| \\ &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.\end{aligned}$$

Let's compare this to the pure state of the quantum super position

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle.$$

The density matrix would be

$$\begin{aligned}\rho_{\text{Pure}} &= \Big(\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle\Big)\Big(\frac{1}{\sqrt{2}}\left\langle 0\right| + \frac{1}{\sqrt{2}}\left\langle 1\right|\Big) \\ &= \frac{1}{2}\big(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right|\big) \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\end{aligned}$$

The pure state density matrix is different from the mixed state because of the non-zero off diagonal terms. These are sometimes called "interference terms." The reason is that states in a quantum superposition can "interfere" with each other, while states in a classical superposition can't.

Let's now look at the expectation value of the following operators for both density matrices.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

They are given by

$$\left\langle\sigma_z\right\rangle_{\text{Mixed}} = \text{Tr}\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = 0$$

$$\left\langle\sigma_z\right\rangle_{\text{Pure}} = \text{Tr}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = 0$$

$$\left\langle\sigma_x\right\rangle_{\text{Mixed}} = \text{Tr}\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = 0$$

$$\left\langle\sigma_x\right\rangle_{\text{Pure}} = \text{Tr}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = 1$$

So we can see that a measurement given by σ_z cannot distinguish between ρ_{Mixed} and ρ_{Pure}, while a measurement given by σ_x can distinguish between them! There really is a difference between classical super positions and quantum superpositions, but you can only see this difference if you exploit the off-diagonal terms!

4.10 Entropy of Mixed States

In quantum mechanics, pure states are microstates and mixed states are the macrostates. We can define the entropy of a mixed state drawing inspiration from the definition of Shannon entropy.

$$S = -k \operatorname{Tr}(\rho \log \rho) \tag{78}$$

This is called the Von Neumann Entropy. If ρ represents a classical superposition of orthonormal states $|\psi_i\rangle$, each with some probability p_i, then the above definition exactly matches the definition of Shannon entropy.

One thing should be explained, though. How do you take the logarithm of a matrix? This is actually pretty easy. Just diagonalize the matrix and take the log of the diagonal entries. Thankfully, density matrices can always be diagonalized (they are manifestly self-adjoint and therefore diagonalizable by the spectral theorem) so you don't have to do anything more complicated.

4.11 Classicality from environmental entanglement

Say you have two quantum systems A and B with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. If you combine the two systems, states will live in the Hilbert space

$$\mathcal{H}_A \otimes \mathcal{H}_B.$$

Say that $|\phi_i\rangle_A \in \mathcal{H}_A$ comprise a basis for the state space of $\mathcal{H}_A$ and $|\phi_j\rangle_B \in \mathcal{H}_B$ comprise a basis for the state space $\mathcal{H}_B$. All states in $\mathcal{H}_A \otimes \mathcal{H}_B$ will be of the form

$$|\Psi\rangle = \sum_{i,j} c_{ij} |\phi_i\rangle_A |\phi_j\rangle_B$$

for some $c_{ij} \in \mathbb{C}$.

States are said to be "entangled" if they can **not** be written as

$$|\psi\rangle_A |\psi\rangle_B$$

for some $|\psi\rangle_A \in \mathcal{H}_A$ and $|\psi\rangle_B \in \mathcal{H}_B$.

So, for example, if $\mathcal{H}_A = \mathbb{C}^2$ and $\mathcal{H}_B = \mathbb{C}^2$, then the state

$$|0\rangle \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{i}{\sqrt{2}} |1\rangle \right)$$

would not be entangled, while the state

$$\frac{1}{\sqrt{2}} \Big(|0\rangle |0\rangle + |1\rangle |1\rangle \Big)$$

would be entangled.

Let's say a state starts out unentangled. How would it then become entangled over time? Well, say the two systems A and B have Hamiltonians $\hat{H}_A$ and $\hat{H}_B$. If we want the systems to interact weakly, i.e. "trade energy," we'll also need to add an interaction term to the Hamiltonian.

$$\hat{H} = \hat{H}_A \otimes \hat{H}_B + \hat{H}_{\text{int}}.$$

It doesn't actually matter what the interaction term is or if it is very small. All that matters is that if we really want them to interact, its important that the interaction term is there at all. Once we add an interaction term, we will generically see that states which start out unentangled become heavily entangled over time as A and B interact.

Say for example you had a system $\mathcal{S}$ described by a Hilbert space $\mathcal{H}_\mathcal{S}$ coupled to a large environment $\mathcal{E}$ described by a Hilbert space $\mathcal{H}_\mathcal{E}$. Now, maybe you are an experimentalist and you are really interested in studying the quantum dynamics of $\mathcal{S}$. You then face a very big problem: $\mathcal{E}$. Air molecules in your laboratory will be constantly bumping up against your system, for example. This is just intuitively what I mean by having some non-zero $\hat{H}_{\text{int}}$. The issue is that, if you really want to study $\mathcal{S}$, you desperately *don't* want it to entangle with the environment, because you have no control over the environment! This is why people who study quantum systems are always building these big complicated vacuum chambers and cooling their system down to fractions of a degree above absolute zero: they want to prevent entanglement with the environment so they can study $\mathcal{S}$ in peace!

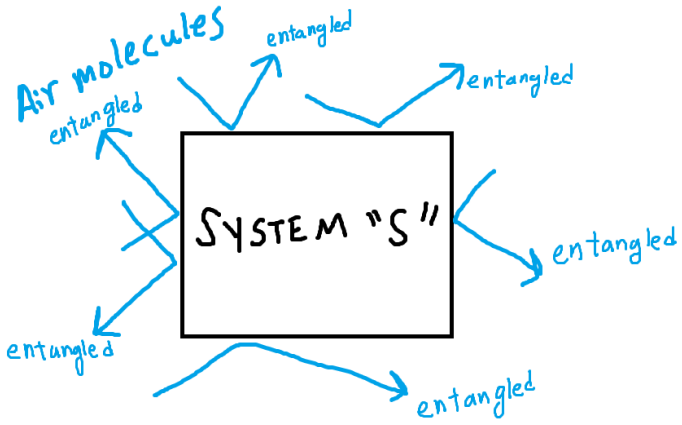


Figure 39: Air molecules bumping up against a quantum system $\mathcal{S}$ will entangle with it.

Notice that the experimentalist will not have access to the observables in the environment. Associated with $\mathcal{H}_\mathcal{S}$ is a set of observables $\hat{\mathcal{O}}_\mathcal{S}$. If you tensor these observables together with the identity,

$$\hat{\mathcal{O}}_\mathcal{S} \otimes 1_\mathcal{E}$$

you now have an observable which only measures quantities in the $\mathcal{H}_\mathcal{S}$ subsector of the full Hilbert space. The thing is that entanglement within the environment gets in the way of measuring $\hat{\mathcal{O}}_\mathcal{S} \otimes 1_\mathcal{E}$ in the way the experimenter would like.

Say, for example, $\mathcal{H}_\mathcal{S} = \mathbb{C}^2$ and $\mathcal{H}_\mathcal{E} = \mathbb{C}^N$ for some very big N. Any state in $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$ will be of the form

$$c_0 \left|0\right\rangle \left|\psi_0\right\rangle + c_1 \left|1\right\rangle \left|\psi_1\right\rangle \tag{79}$$

for some $c_0, c_1 \in \mathbb{C}$ and $\left|\psi_0\right\rangle, \left|\psi_1\right\rangle \in \mathcal{H}$. The expectation value for our observable is

$$\begin{aligned}\langle \hat{\mathcal{O}}_\mathcal{S} \otimes 1_\mathcal{E} \rangle =& \Big(c_0^* \left\langle 0\right| \left\langle \psi_0\right| + c_1^* \left\langle 1\right| \left\langle \psi_1\right| \Big) \hat{\mathcal{O}}_\mathcal{S} \otimes 1_\mathcal{E} \Big(c_0 \left|0\right\rangle \left|\psi_0\right\rangle + c_1 \left|1\right\rangle \left|\psi_1\right\rangle \Big) \\ =& |c_0|^2 \left\langle 0\right| \hat{\mathcal{O}}_\mathcal{S} \left|0\right\rangle + |c_1|^2 \left\langle 1\right| \hat{\mathcal{O}}_\mathcal{S} \left|1\right\rangle + 2\,\mathrm{Re}\left(c_0^* c_1 \left\langle 0\right| \hat{\mathcal{O}}_\mathcal{S} \left|1\right\rangle \left\langle \psi_0 | \psi_1\right\rangle\right)\end{aligned}$$

The thing is that, if the environment $\mathcal{E}$ is very big, then any two random given vectors $\left|\psi_0\right\rangle, \left|\psi_1\right\rangle \in \mathcal{H}_\mathcal{E}$ will generically have almost no overlap.

$$\langle \psi_0 | \psi_1 \rangle \approx e^{-N}$$

(This is just a fact about random vectors in high dimensional vector spaces.) Therefore, the expectation value of this observable will be

$$\langle \hat{\mathcal{O}}_\mathcal{S} \otimes 1_\mathcal{E} \rangle \approx |c_0|^2 \left\langle 0\right| \hat{\mathcal{O}}_\mathcal{S} \left|0\right\rangle + |c_1|^2 \left\langle 1\right| \hat{\mathcal{O}}_\mathcal{S} \left|1\right\rangle .$$

Because there is no cross term between $|0\rangle$ and $|1\rangle$, we can see that when we measure our observable, our system $\mathcal{S}$ seems to be in a classical superposition, A.K.A a mixed state!

This can be formalized by what is called a "partial trace." Say that $|\phi_i\rangle_{\mathcal{E}}$ comprises an orthonormal basis of $\mathcal{H}_{\mathcal{E}}$. Say we have some density matrix ρ representating a state in the full Hilbert space. We can "trace over the $\mathcal{E}$ degrees of freedom" to recieve a density matrix in the $\mathcal{S}$ Hilbert space.

$$\rho_{\mathcal{S}} \equiv \mathrm{Tr}_{\mathcal{E}}(\rho) \equiv \sum_i {}_{\mathcal{E}}\langle\phi_i|\,\rho\,|\phi_i\rangle_{\mathcal{E}}\,. \tag{80}$$

You be wondering why anyone would want to take this partial trace. Well, I would say that if you can't perform the $\mathcal{E}$ degrees of freedom, why are you describing them? It turns out that the partially traced density matrix gives us the expectation values for any observables in $\mathcal{S}$. Once we compute $\rho_{\mathcal{S}}$, by tracing over $\mathcal{E}$, we can then calculate the expectation value of any observable $\hat{\mathcal{O}}_{\mathcal{S}}$ by just calculating the trace over $\mathcal{S}$ of $\rho_{\mathcal{S}}\hat{\mathcal{O}}_{\mathcal{S}}$:

$$\mathrm{Tr}\Big(\rho\hat{\mathcal{O}}_{\mathcal{S}} \otimes 1_{\mathcal{E}}\Big) = \mathrm{Tr}_{\mathcal{S}}(\rho_{\mathcal{S}}\hat{\mathcal{O}}_{\mathcal{S}}).$$

Even though the whole world is in some particular state in $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$, when you only perform measurements on one part of it, that part might as well only be in a mixed state for all you know! Entanglement looks like a mixed state when you only look at one part of a Hilbert space. Furthermore, when the environment is very large, the off diagonal "interference terms" in the density matrix are usually very close to zero, meaning the state looks very mixed.

This is the idea of "entanglement entropy." If you have an entangled state, then trace out over the states in one part of the Hilbert space, you will recieve a mixed density matrix. That density matrix will have some Von Neumann entropy, and in this context we would call it "entanglement entropy." The more entanglement entropy your state has, the more entangled it is! And, as we can see, when you can only look at one tiny part of a state when it is heavily entangled, it appears to be in a classical superposition instead of a quantum superposition!

The process by which quantum states in real life become entangled with the surrounding environment is called "decoherence." It is one of the most visciously efficient processes in all of physics, and is the reason why it took the human race so long to discover quantum mechanics. It's very ironic that entanglement, a quintessentially quantum phenomenon, when taken to dramatic extremes, hides quantum mechanics from view

entirely!

I would like to point out an important difference between a classical macrostate and a quantum mixed state. In classical mechanics, the subtle perturbing effects of the environment on the system make it difficult to keep track of the exact microstate a system is in. However, in principle you can always just re-measure your system very precisely and figure out what the microstate is all over again. This isn't the case in quantum mechanics when your system becomes entangled with the environment. The problem is that once your system entangles with the environment, that entanglement is almost certainly never going to undo itself. In fact, it's just going to spread from the air molecules in your laboratory to the surrounding building, then the whole univeristy, then the state, the country, the planet, the solar system, the galaxy, and then the universe! And unless you "undo" all of that entanglement, the show's over! You'd just have to start from scratch and prepare your system in a pure state all over again.

4.12 The Quantum Partition Function

The quantum analog of the partition function is very straightforward. The partition function is defined to be

$$\begin{aligned} Z(T) &\equiv \mathrm{Tr}\exp\Big(-\hat{H}/kT\Big) \\ &= \sum_s e^{-\beta E_s}. \end{aligned} \tag{81}$$

Obviously, this is just the same $Z(T)$ that we saw in classical mechanics! They are really not different at all. However, there is something very interesting in the above expression. The operator

$$\exp\Big(-\hat{H}/kT\Big)$$

looks an awful lot like the time evolution operator

$$\exp\Big(-i\hat{H}t/\hbar\Big)$$

if we just replace

$$-\frac{i}{\hbar}t \longrightarrow -\beta.$$

It seems as though β is, in some sense, an "imaginary time." Rotating the time variable into the imaginary direction is called a "Wick Rotation,"

and is one of the most simple, mysterious, and powerful tricks in the working physicist's toolbelt. There's a whole beautiful story here with the path integral, but I won't get into it.

Anyway, a mixed state is said to be "thermal" if it is of the form

$$\begin{aligned} \rho_{\text{Thermal}} &= \frac{1}{Z(T)} \sum_s e^{-E_s/kT} \left|E_s\right\rangle \left\langle E_s\right| \\ &= \frac{1}{Z(\beta)} e^{-\beta \hat{H}} \end{aligned} \tag{82}$$

for some temperature T where $|E_s\rangle$ are the energy eigenstates with eigenvalues E_s. If you let your system equilibriate with an environment at some temperature T, and then trace out by the environmental degrees of freedom, you will find your system in the thermal mixed state.

5 Hawking Radiation

5.1 Quantum Field Theory in Curved Space-time

When you have some space-time manifold in general relativity, you can slice it up into a bunch of "space-like" surfaces that represent a choice of instances in time. These are called "time slices." All the normal vectors of the surface must be time-like.

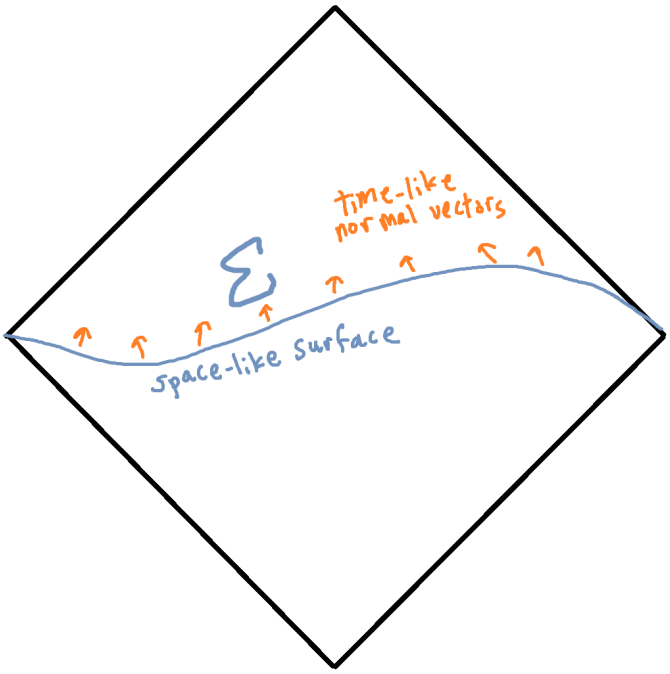


Figure 40: A "timeslice" is a "space-like" surface, meaning its normal vectors are time-like.

Once you make these time slices, you can formulate a quantum field theory in the space-time. A quantum field state on a time slice Σ is just a wave functional

$$\Psi : C^\infty(\Sigma) \to \mathbb{C}.$$

(Of course, once again this is just the case for a spin-0 boson, and will be more complicated for different types of quantum fields, such as the ones we actually see in nature.) Therefore, we have a different Hilbert space for each time slice Σ.

This might seem a bit weird to you. Usually we think about all states as living in one Hilbert space, and the state evolves in time according to the Schrödinger equation. Here, we have a different Hilbert space for each time slice and the Schrödinger equation evolves a state from one Hilbert space into a state in a different Hilbert space. This is just a

convenient way of talking about quantum fields in curved space-time, and is nothing "new" exactly. We are not really modifying quantum mechanics in any way, we're just using new language.

5.2 Hawking Radiation

In 1974, Stephen Hawking considered what would happen to a quantum field if star collapsed into a black hole [4]. If the quantum field started out in its ground state (with no particles present) before the star collapsed, Hawking discovered that after the collapse, the black hole would begin emitting particles that we now call "Hawking Radiation."

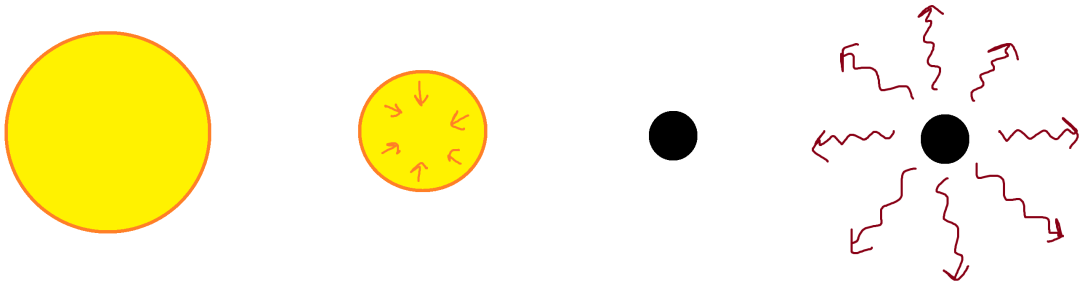


Figure 41: A star collapses, becomes a black hole, then immediately starts emitting Hawking radiation.

The reason for this is very subtle, and difficult to explain in words. Perhaps one day I will be able to explain why the black hole emits Hawking radiation in a way that is both intuitive and correct, but as of now I cannot, so I won't. I will say, however, that the emission of Hawking radiation crucially relies on the fact that different observers have different notions of what they would justifiably call a particle. While there were initially no particles in the quantum field before the black hole formed, the curvature caused by black hole messes up the definition of what a "particle is," and so all of a sudden particles start appearing out of nowhere. You shouldn't necessarily think of the particles as coming off of the horizon of the black hole, even though the formation of the horizon is crucial for Hawking radiation to be emitted. Near the horizon, the "definition of what a particle is" is a very fuzzy thing. However, once you get far enough away from the black hole, you would be justified in claiming that it is emitting particles. Capisce?

Now, in real life, for a black hole that has approximately the mass of a star, this Hawking radiation will be extremely low-energy, perhaps even as low-energy as Jeb. In fact, the bigger the black hole, the lower the energy of the radiation. The Hawking radiation from any actually existing black hole is far too weak to have been detected experimentally.

5.3 The shrinking black hole

However, Hawking didn't stop there. The black hole is emitting particles, and those particles must come from somewhere. Furthermore, Einstein's theory of general relativity tells us that energy has some effect on space-time, given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

However, there is an issue. What is $T_{\mu\nu}$ for the quantum field? In quantum mechanics, a state can be a superposition of states with different energies. However, there is only one space-time manifold, not a superposition of multiple space-time manifolds! So what do we do?

The answer? We don't know what to do! This is one to view the problem of quantum gravity. We're okay with states living on time-slices in a curved manifold. No issues there! But when we want to study how the quantum field then affects the manifold its living on, we have no idea what to do.

In other words, we have a perfectly good theory of classical gravity. But we don't know what the "Hilbert space" of gravity is! There are many proposals for what quantum gravity could be, but there are no proposals that are all of the following:

1. Consistent
2. Complete
3. Predictive
4. Applicable to the universe we actually live in
5. Confirmed by experiment.

In fact, maybe there is no "Hilbert space for gravity," and in order to figure out the correct theory of quantum gravity we will have to finally supersede quantum mechanics. But there are currently no proposals that do this. For example, the notion of a Hilbert space remains intact in both string theory and loop quantum gravity.

But certainly we don't need to know the complete theory of quantum gravity in order to figure out what happens to our black hole, right? For example, all of the particles in the earth and the sun are quantum in nature, and yet we have no trouble describing the motion of Earth's orbit. So even though we don't have a complete theory of quantum gravity, we can still analyze certain situations, right?

Indeed. While the stress energy tensor for a quantum field does not have a definite value, we can still define the *expectation value* for the stress energy tensor, $\langle \hat{T}_{\mu\nu} \rangle$. We could then guess that the effect of the quantum field on space time is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} \langle \hat{T}_{\mu\nu} \rangle.$$

This is the so called "semi-classical approximation" which Hawking used to figure out how the radiation affects the black hole. This has caused much consternation since.

You might argue that a black hole is a very extreme object because of its singularity. Presumably, one would need a theory of quantum gravity to properly describe what goes on in the singularity of a black hole where space-time is infinitely curved. So then why are we using the semi-classical approximation in a situation where it does not apply?

The answer is that, yes, we are not yet able to describe the singularity of the black hole. However, we are not trying to. We are only trying to describe what is going on at the horizon, where space time is not particularly curved at all. So our use of the semi-classical approximation ought to be perfectly justified.

Anyway, because energetic particles are leaving the black hole, Hawking realized that, assuming the semi-classical approximation is reasonable, the black hole itself will actually *shrink*, which would never happen classically!

Because of this, the black hole will shrink more and more as time goes on, emitting higher energy radiation as it does so. Therefore, as it gets smaller it also shrinks faster. The Hawking radiation would eventually become very energetic and detectable by astronomers here on Earth. Presumably, in its final moments it would explode like a firecracker in the night sky. (The semi-classical approximation would certainly not apply as the black hole poofs out of existence.)

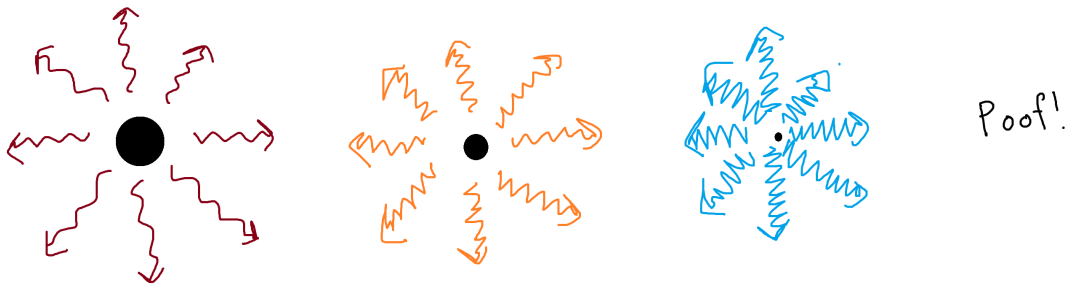


Figure 42: The black hole evaporating, emitting increasingly high energy Hawking radiation, shrinking, and then eventually disappearing.

However, we have never actually seen this happen. The black holes that we know about are simply too big and would be shrinking too slowly. A stellar mass black hole would take 10^{67} years to evaporate in this way.

But maybe much smaller black holes formed due to density fluctuations in the early universe instead of from stellar collapse. Perhaps these black holes would just be finishing up their evaporation process now and would be emitting Hawking radiation energetic enough to detect. While plausible, this has never been observed. These would be called "primordial black holes" but remain hypothetical.

5.4 Hawking Radiation is thermal

But Hawking didn't stop there [5]! You see, people outside of the black will not be able to measure the quantum field degrees of freedom inside the black hole. They will only be able to perform measurements on a subsector of the quantum field Hilbert space corresponding to the field variables outside of the event horizon. As far as an outside observer would know, the quantum field state would be in a mixed state and not a pure state.

So, Hawking went and traced over the field degrees of freedom that were hidden behind the event horizon, and found something surprising: the mixed state was thermal! It was as though the black hole of mass M had a temperature

$$T = \frac{\hbar c^3}{8\pi k G M}. \tag{83}$$

Identifying the energy of the black hole with Mc^2, you can use the definition of temperature ($\frac{1}{T} = \frac{\partial S}{\partial Mc^2}$) to deduce that the black hole also had an entropy

$$S = \frac{4\pi k G M^2}{\hbar c}. \tag{84}$$

However, at this point we can only understand the temperature and the entropy of black holes in terms of analogies. In other words, the black holes radiates "as if" is has a temperature and "as if" it has an entropy. However, remember that we defined entropy to be the log of the number of microstates a system could be in. Notice that this "entropy" was not derived with any notion of a microstate. It's just that the black hole behaves "as if" it had microstates.

5.5 Partner Modes

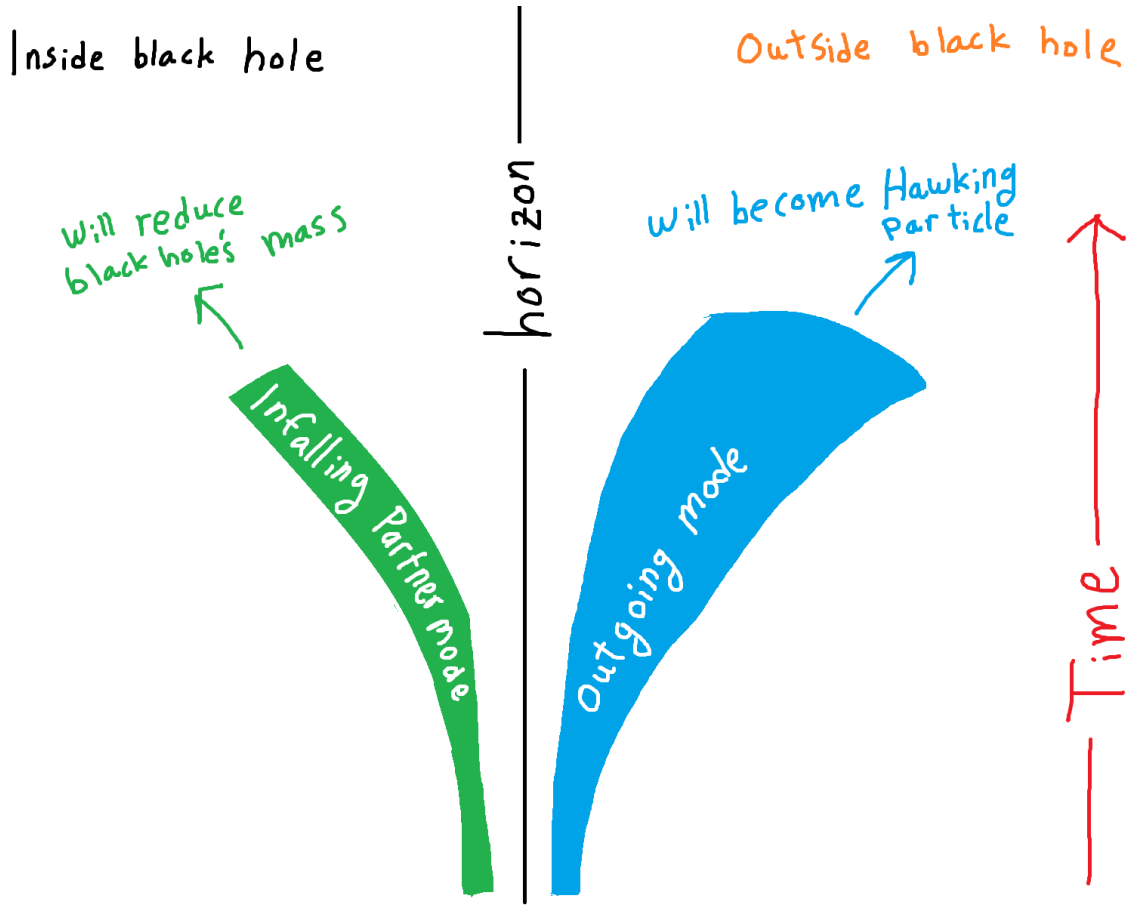


Figure 43: A cartoon of the Hawking partner modes. The shaded region shows the width of the Gaussian wavepackets. The outgoing mode redshifts and spreads.

Even though the causes of Hawking radiation are subtle, it would still be nice to have some sort of physical picture in our heads so we can think about it properly. As I have said before, even though the Hawking radiation really does come out in the form of particles, they are not really particle when they are near the horizon. Instead, we should call them "modes." If you want to just mentally replace the word "mode" with particle from here on out, be my guest, but realize that there are more subtle issues involved. For example, I shortly will use the term "mode occupation number." This should be understood to be similar to the "particle occupation number" we discussed previously.

Anyway, surrounding the horizon are pairs of modes. One is an "outgoing mode" which will go on to become a Hawking particle. The other is the "infalling partner mode," which you can think of as having negative energy. It will go on to fall into the black hole and reduce its

mass. This is drawn in Fig. 43. Note that the outgoing mode starts out near the horizon with a small wavelength and high energy, but its wavelength gets redshifted as it escapes the gravitational pull of the black hole.

The crucial thing about these two modes is that they are heavily entangled. By that, I mean that if the outgoing mode has some occupation number then the infalling mode must also have the same occupation number. (Speaking fuzzily, for every particle that comes out, one partner particle must fall in.) So if we think about the Hilbert space of a single mode (assuming we are talking about approximately localize wavepackets) we can imagine states are given by linear combinations of states of the form

$$|n\rangle_{\mathbf{k}}$$

where the integer n is the occupation number of the $\mathbf{k}$ mode. The Hilbert space of the partner modes is given by

$$\mathcal{H}_{\text{partners}} = \mathcal{H}_{\text{infalling}} \otimes \mathcal{H}_{\text{outgoing}}. \tag{85}$$

Hawking's discovery was that the modes were entangled sort of like

$$\sum_n f(n) \, |n\rangle_{\mathbf{k},\text{in}} \, |n\rangle_{\mathbf{k},\text{out}} \, . \tag{86}$$

Hopefully you can see what I mean by the modes being entangled.

To reiterate, when we trace out by the infalling mode, the density matrix of the outgoing mode looks thermal. Therefore, the outside observer will not be able to see superpositions between different occupation number states in the outgoing mode. This is just another way of saying that

$$\frac{1}{\sqrt{2}} \, |0\rangle \, |0\rangle + \frac{1}{\sqrt{2}} \, |1\rangle \, |1\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}} \, |0\rangle + \frac{1}{\sqrt{2}} \, |1\rangle$$

are different. It's just now our Hilbert space is spanned by mode occupation numbers instead of 0 and 1.

6 The Information Paradox

6.1 What should the entropy of a black hole be?

Pretend that you didn't know black holes radiate Hawking radiation. What would you guess the entropy of the black hole to be, based on the theory of general relativity?

An outside observer can measure a small number of quantities which characterize the black hole. (This is assuming the black hole has finished its collapsing process and has settled down into a stable configuration.) There's obviously the mass of the black hole, which is its most important quantity. Interestingly, if the star was spinning before it collapsed, the black hole will also have some angular momentum and its equator will bulge out a bit. So the black hole is also characterized by an angular momentum vector. Furthermore, if the star had some net electric charge, the black hole will also have a charge.

However, if an outside observer knows these quantities, they will know everything about the black hole. So we should expect for the entropy of a black hole to be 0.

But maybe that's not quite fair. After all, the contents of the star should somehow be contained in the singularity, hidden behind the horizon. Interestingly, *all* of the specific details of the star from before the collapse do not have any effect on the properties of the resulting black hole. The only stuff that matters it the *total* mass, *total* angular momentum, etc. That leaves an infinite number of possible stars that could all have produced the same black hole. So actually, we should expect the entropy of a black hole to be ∞.

However, instead of being 0 or ∞, it seems as though the actual "entropy" of a black hole is an average of the two: finite, but stupendously large. Here are some numerical estimates taken from [3]. The entropy of the universe (minus all the black holes) mostly comes from cosmic microwave background radiation, and is about 10^{87} (setting $k = 1$). Meanwhile, the entropy of a solar mass black hole is 10^{78}. The entropy of our sun, as it is now, is a much smaller 10^{60}. The entropy of the supermassive black hole in the center of our galaxy is 10^{88}, larger than the rest of the universe combined (minus black holes). The entropy of any of the largest known supermassive black holes would be 10^{96}.

There is a simple "argument" which suggests that black holes are the most efficient information storage devices in the universe: if you wanted to store a lot of information in a region smaller than a black hole horizon, it would probably have to be so dense that it would just be a black hole

anyway, as it would be contained inside its own Schwarzschild horizon.

6.2 The Area Law

Most things we're used to, like a box of gas, have an entropy that scales linearly with its volume. However, black holes are not like most things. The surface area of a black hole is just

$$A = 4\pi R^2$$

where R is its Schwarzschild radius. Using it, we can rewrite the entropy of the black hole as

$$S = \frac{kc^3}{4\hbar G} A.$$

Interestingly, the black hole's entropy scales with its *area*, not its *volume*. This is a profound and mysterious fact which many people spend a lot of time thinking about.

Sometimes, physicists like to define something called the "Planck length"

$$l_p \equiv \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}\text{m}.$$

The Planck length has no known physical significance to physics, although it is widely assumed that one would need a quantum theory of gravity to describe physics on this length scale. This is because there's only one way to combine the fundamental constants G, c, and $\hbar$ into a quantity with dimensions of length. The entropy of the black hole can be rewritten as

$$S = \frac{kA}{4l_p^2}.$$

So it seems as though the entropy of the black hole is (one fourth times) the number of Planck-length-sized squares it would take to tile the horizon area. (Perhaps the microstates of the black hole are "stored" on the horizon?)

Using "natural units" where $k = c = \hbar = G = 1$, we can write this as

$$S = \frac{A}{4}$$

which is very pretty.

Interestingly, physicists realized that the area of a black hole acted much like an entropy before they knew about Hawking radiation. For example, the way in which a black hole's area could only increase (according to classical general relativity) seemed reminiscent of the second

law of thermodynamics. Moreover, when two black holes merge, the area of the final black hole will always exceed the sum of the areas of the two original black holes.

6.3 Non unitary time evolution?

Let's assume that Hawking's semi-classical approximation was justified and consider what happens as a black hole emits radiation which appears to be in a mixed state from the outside. (It should be noted that the state only *looks* mixed because the degrees of freedom on the outside are entangled with the degrees of freedom on the inside.) Once the black hole disappears, however, it takes that entanglement with it! Therefore, the process of black hole evaporation, when combined with the disappearance of the black hole, seem to evolve a pure state into a mixed state, something which is impossible via unitary time evolution! Remember that pure states only become mixed states whenever we decide to perform a partial trace; they never become mixed because of Schrödinger's equation. But Hawking argued that black hole evaporation was unlike anything we had seen before: he said that the information of what went into the black hole disappears along with the black hole, and all that's left over is a bunch of crummy uninformative radiation. (He also pointed out that this evaporation would violate many known laws of physics such as conservation of baryon number and lepton number. While the star was composed mostly of protons, neutrons, and electrons, the Hawking radiation will be comprised mostly of photons.)

If the process of black hole evaporation is truly "non-unitary," it would be a first for physics. It would mean that once the black hole disappears, the information of what went into it is gone for good. Nobody living in the post-black-hole universe could figure out exactly what went into the black hole, even if they knew all there was to know about the radiation.

6.4 No. Unitary time evolution!

Look, we don't have a theory of quantum gravity, okay? We'd really like to know what it is, but we don't. So what should we do to remedy this? One possibility is to look for currently known physical principles that we have reason to believe should still hold even in the deeper theory.

For example, Einstein noted that somebody freefalling in a windowless elevator would have no way to tell that they weren't really in a windowless space ship floating in outer space. Einstein called this "the

principle of equivalence" and used it to help him figure out his theory of general relativity. In other words, general relativity, which is the more fundamental theory of gravity, left behind a "clue" in the less fundamental theory of Newtonian gravity. If you correctly identify physical principles which should hold in the more fundamental theory, you can use them to figure out what that more fundamental theory actually is.

Physicists now believe that "conservation of information" is one of those principles, on par with the principle of equivalence. Because information is never truly lost in any known physical process, and because it sounds appropriately profound, it might useful to adopt the attitude that information is *never* lost, and see where that takes us.

In that spirit, many physicists disagree with Hawking's original claim that information is truly lost in the black hole. They don't know exactly *why* Hawking was wrong, but they think that if they *assume* Hawking is wrong, it will help them figure out something about quantum gravity. (And I think that does make some sense.)

But then what is the paradox in the "information paradox?" Well, there is no paradox in the literal sense of the word. See, a paradox is when you derive a contradiction. But the thing we derive, that information is lost in the black hole, is only a "contradiction" if we assume that information is never lost to an outside observer. (And if we're being honest, seeing as we do not yet have a theory of quantum gravity, we don't yet know for sure if that's false.) In other words, it's only a "paradox" if we assume it's a paradox, and that's not much of a paradox at all.

But so what. Who cares. These are just words. Even if it's not a "paradox" in the dictionary sense of the word, its still something to think about nonetheless.

To summarize, most physicists believe that the process of black hole evaporation should truly be unitary. If they knew *how* it was unitary, there would no longer be a "paradox."

There's one possible resolution I'd like to discuss briefly. What if the black hole never "poofs" away in the final stage of evolution, but some quantum gravitational effect we do not yet understand stabilizes it instead, allowing for some Planck-sized object to stick around? Such an object would be called a "remnant." The so called "remnant solution" to the information paradox is not a very popular one. People don't like the idea of a very tiny, low-mass object holding an absurdly large amount of information and being entangled with a very large number other of particles. It seems much more reasonable to people that the information of what went into the black hole is being released via the radiation in a

way too subtle for us to currently understand.

6.5 Black Hole Complementarity

"Radical conservatism" is a phrase that has become quite popular in the physics community. A "radical conservative" is someone that tries to modify as few laws of physics as possible (that's the conservative part) and through their dogmatic refusal to modify these laws and go wherever their reasoning leads (that's the radical part) is able to derive amazing things.

What happens if we adopt a radically conservative attitude with regards to unitary evaporation? What crazy consequences can we derive?

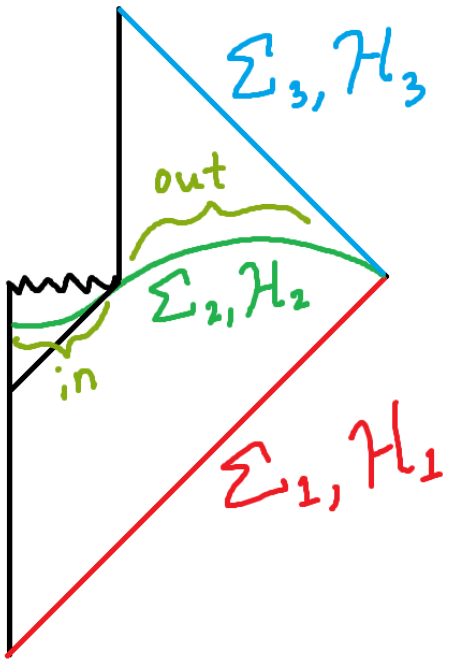


Figure 44: The Penrose diagram containing a black hole which evaporates, with some time-slices drawn. Σ_1 is the time slice in the infinite past and Σ_3 is the time slice in the infinite future. Σ_2 passes through the point where the black hole poofs out of existence, dividing the slice into two halves.

In Fig 44 above, I have drawn the Penrose diagram containing a universe with an evaporating black hole. I have drawn three time slices, Σ_1, Σ_2, and Σ_3. Each time slice comes equipped with a quantum field Hilbert space $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$, as discussed. Note that Σ_2 is split into an "in" half and an "out" half. We may therefore write

$$\mathcal{H}_2 = \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}. \tag{87}$$

Furthermore, let

$$U_{ji} : \mathcal{H}_i \to \mathcal{H}_j$$

be the unitary time evolution operator that evolves a state in $\mathcal{H}_i$ to a state in $\mathcal{H}_j$. Note that

$$U_{ij} = U_{ji}^{-1}.$$

Crucially, the Hamiltonian for our quantum field is *local*. That means that the degrees of freedom on the "in" half of Σ_2 can't make it out to Σ_3. However, it turns out this entire picture is incompatible with unitary time evolution. Why?

Well, consider the unitary operator

$$U_{23}U_{31}.$$

This evolves an initial state on Σ_1 to a state on Σ_3, and then de-evolves it backwards to a state on Σ_2. Say we have some initial state

$$|\psi_1\rangle \in \mathcal{H}_1$$

and act on it with $U_{23}U_{31}$. We will call the result $|\psi_2\rangle$:

$$|\psi_2\rangle \equiv U_{23}U_{31} |\psi_1\rangle \in \mathcal{H}_2.$$

However, if we want an outside observer to be able to reconstruct what went into the black hole, the the density matrix corresponding to $|\psi_2\rangle$ must be pure once we trace out by the "in" degrees of freedom. That is,

$$\mathrm{Tr}_{\mathrm{in}}(|\psi_2\rangle \langle\psi_2|)$$

must be pure. This is only possible if

$$|\psi_2\rangle = |\psi_{\mathrm{in}}\rangle |\psi_{\mathrm{out}}\rangle$$

for some

$$|\psi_{\mathrm{in}}\rangle \in \mathcal{H}_{\mathrm{in}} \qquad\qquad |\psi_{\mathrm{out}}\rangle \in \mathcal{H}_{\mathrm{out}}.$$

Therefore, inverting our unitary operator, we can now write

$$|\psi_1\rangle = U_{13}U_{32} |\psi_{\mathrm{in}}\rangle |\psi_{\mathrm{out}}\rangle .$$

Here comes the key step. If the Hamiltonian is local, and only the "out" part of a state can go off to affect the state on Σ_3, then if we replace $|\psi_{\mathrm{in}}\rangle$ with some other state, then the above equation should still hold. In other words, we should have both equations

$$|\psi_1\rangle = U_{13}U_{32} |\psi_{\mathrm{in}}\rangle |\psi_{\mathrm{out}}\rangle$$
$$|\psi_1\rangle = U_{13}U_{32} |\psi'_{\mathrm{in}}\rangle |\psi_{\mathrm{out}}\rangle$$

for any two distinct states

$$|\psi_{\text{in}}\rangle\,, |\psi'_{\text{in}}\rangle \in \mathcal{H}_{\text{in}}.$$

However, subtracting one of those equations from the other, we see that

$$0 = U_{13}U_{32}\big(\,|\psi_{\text{in}}\rangle - |\psi'_{\text{in}}\rangle\,\big)\,|\psi_{\text{out}}\rangle\,.$$

This is a contradiction because unitary operators must be invertible! (Some of you might recognize that we have emulated the proof of the "no cloning" theorem of quantum mechanics. Here, however, we have proven something more like a "no destruction" theorem, seeing as $\mathcal{H}_{\text{in}}$ crashes into the singularity and is destroyed.)

So wait, what gives? When we assumed that time evolution was unitary, we derived a contradiction. What is the resolution to this contradiction?

One possible resolution is to postulate that the inside of the black hole *does not exist*.

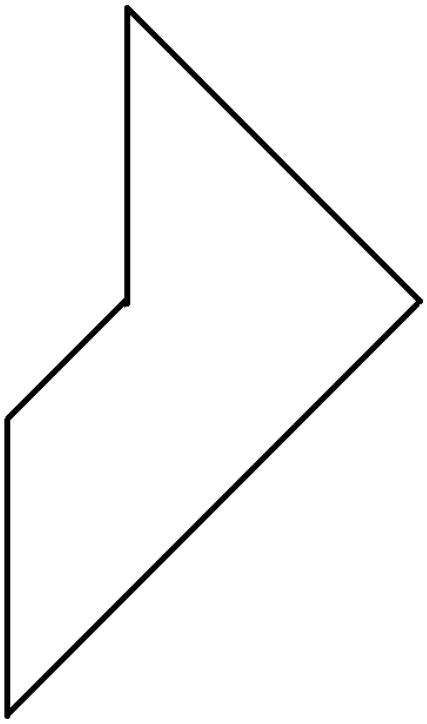


Figure 45: Maybe there is no space-time beyond the horizon a black hole.

However, that doesn't seem very conservative. According to Einstein's theory of relativity, anyone should be able to jump into a black hole and see the inside for themselves. Locally speaking, there is nothing particularly special about the horizon. Sticking to our dogma of

"radical conservatism" we should still allow for people to jump into the black hole and see things the way Einstein's theory would predict they would see it. The crucial realization is that, for the person who jumped into the black hole, the *outside* universe may as well not exist.

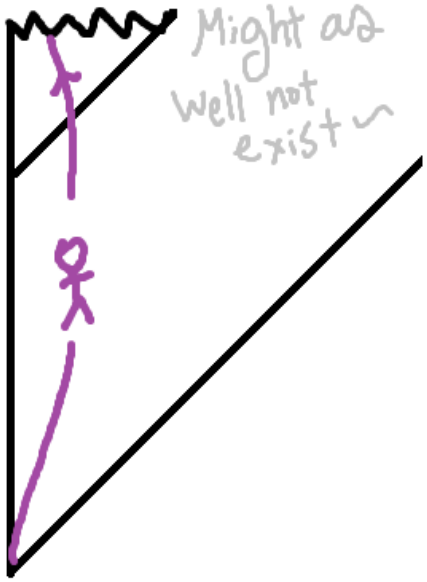


Figure 46: Maybe someone who jumps into a black hole relinquishes the right to describe what goes on outside of it.

The most radically conservative conclusion we could make is that somebody on the outside doesn't believe the interior of the black hole exists, somebody on the inside doesn't believe the exterior exists, and that they are *both right*. This hypothesis, formulated in the early 1990's, has been given the name of "Black Hole Complementarity." The word "complementarity" comes from the fact that two observers give different yet complementary views of the world. Very spiritual.

The two biggest advances in physics, namely the development of relativity theory and quantum theory, have taught us strange things about the nature of "observation." Namely, it seems as though we are not entitled to ascribe reality to things which are unmeasurable. Black Hole Complementarity (BHC) fits right into that philosophy.

But wait. Let's say I remain safe and warm on the outside of the black hole while somebody else jumps in. If I watch them as they enter the black hole, what will I see happen to them?

Leonard Susskind suggested that, according to someone on the outside, the infalling observer never makes it past the horizon. Susskind hypothesized that there is something called a "stretched horizon," which

the region of space that is contained within one Planck length of the horizon.

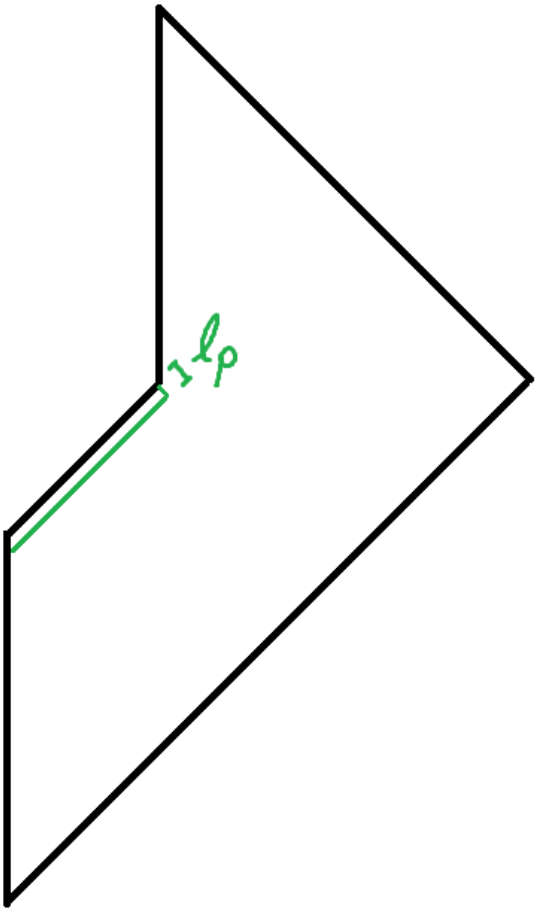


Figure 47: The "stretched horizon" is the region that is within one Planck length l_p of the horizon.

First, as the infalling observer nears the horizon, the outside observer will see them drape themselves over the horizon like a table cloth. (This is actually a prediction of general relativity.) In the limit that the infalling observer is much less massive than the black hole, they will never actually enter the black hole but only asymptotically approach the horizon. However, if the infalling observer has some finite mass, their own gravitational field will distort the horizon a bit to allow the observer to enter it at some very large yet finite time.

Susskind proposed that something different happens. Instead of entering the black hole at some finite time, the infalling observer will instead be stopped at the stretched horizon, which is quite hot when you get up close. At this point they will be smeared all over the horizon like cream cheese on a bagel. Then, the Hawking radiation coming off of the horizon will hit the observer on its way out, carrying the information

about them which has been plastered on the horizon.

So the outside observer, who is free to collect this radiation, should be able to reconstruct all the information about the person who went in. Of course, that person will have burned up at the stretched horizon and will be dead. From the infalling observer's perspective, however, they were able to pass peacefully through the black hole and sail on to the singularity. So from their perspective they live, while from the outside it looks like they died. However, no contradiction can be reached, because nobody has access to both realities.

Having said this, in order that we can't derive a contradiction, it must take some time for the infalling observer to "thermalize" (equilibriate) on the horizon. Otherwise, the outside observer could see the infalling observer die and then rocket themselves straight into the black hole themselves to meet the alive person once again before they hit the horizon, thus producing a contradiction.

Somehow, according to the BHC worldview, the information outside the horizon is redundant with the information inside the horizon. Perhaps the two observers are simply viewing the same Hilbert space through different bases.

6.6 The Firewall Paradox

People were finally growing content with the BHC paradigm when in 2012, four authors with the combined initials of "AMPS" published a paper [2] titled "Black Holes: Complementarity or Firewalls?" Unlike the "information paradox," the firewall paradox is a proper paradox. The AMPS paper claimed to show that BHC is self-contradictory. Now, as is always the case with these things, people have since claimed to have found countless unstated assumptions that AMPS made, and have attempted to save BHC by considering what happens when these assumptions are removed. Having said that, it should be noted that the Firewall paradox is definitely much more robust than most other "paradoxes" of similar ilk and has still not be conclusively refuted.

In order to understand the Firewall paradox, I need to introduce a term called the "Page time." Named after Don Page, the "Page time" refers to the time when the black hole has emitted enough of its energy in the form of Hawking radiation that its entropy has (approximately) halved. Now the question is, what's so special about the Page time?

Imagine we have watched a black hole form and begin emitting Hawking radiation. Say we start collecting this radiation. At the beginning of this process, most of the information of what went into the

black hole remains near the black hole (perhaps in the stretched horizon). Therefore, the radiation we collect at early times will still remain heavily entangled with the degrees of freedom near the black hole, and as such the state will look mixed to us because we cannot yet observe all the complicated entanglement.

Furthermore, as we continue collect radiation, generically speaking the radiation will still be heavily entangled with those near-horizon degrees of freedom.

However, once we hit the Page time, something special happens. The entanglement entropy of the outgoing radiation finally starts *decreasing*, as we are finally able to start seeing entanglements between all this seemingly random radiation we have painstakingly collected. Don Page proposed the following graph of what he entanglement entropy of the outgoing radiation should look like. It is fittingly called the "Page curve."

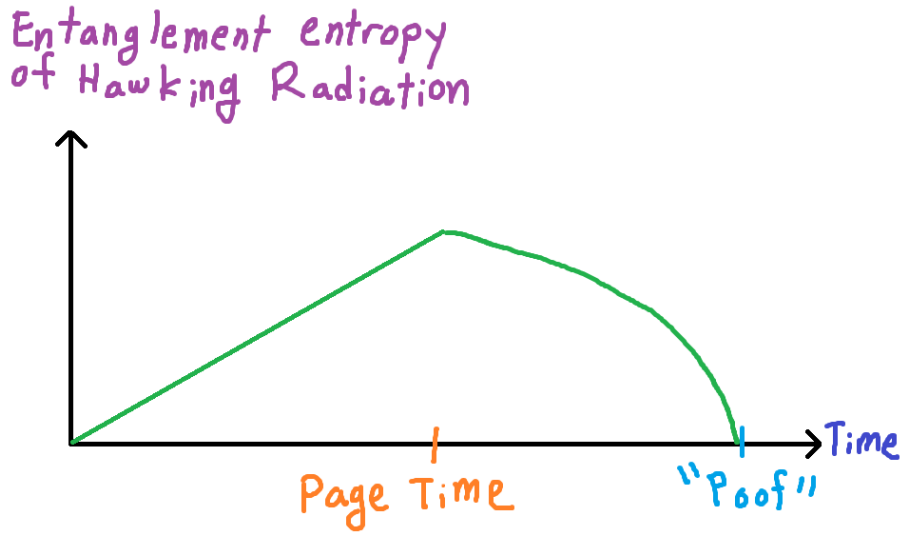


Figure 48: The Page curve

Some people like to say that if one could calculate the Page curve from first principles, the information paradox would be solved.

The Page curve starts by increasing linearly until the Page time. Let me explain the intuition behind the shape of this graph. As more and more information leaves the black hole in the form of Hawking radiation, we are "tracing out" fewer and fewer of the near-horizon degrees of freedom. The dimension of our density matrix grows bigger and bigger, and because the outgoing radiation is still so entangled with the near-horizon degrees of freedom, the density matrix will still have off diagonal terms which are essentially zero. Recall that if you tensor together a Hilbert space of dimension n with a Hilbert space of dimension m, the resulting Hilbert space has dimension $n \times m$. Therefore, once the black hole's

entropy has reduced by half, the dimension of the Hilbert space we are tracing out finally becomes smaller than the dimension of the Hilbert space we are not tracing out. The off-diagonal terms spring into our density matrix, growing in size and number as the black hole continues to shrink. Finally, once the black hole is gone, we can easily see that all the resulting radiation is in a pure state.

Let me now dumb down the thought experiment conducted in the AMPS paper. (I will try to keep the relevant details but not reproduce the technical justifications for why this thought experiment should work, and to be honest I do not understand all of them.) Say an observer, commonly named Alice, collects all the Hawking radiation coming out of a black hole and waits for the Page time to come and go. At maybe about 1.5 times the Page time, Alice is now able to see significant entanglement in all the radiation she has collected. Alice then dives into the black hole, and sees an outgoing Hawking mode escaping.

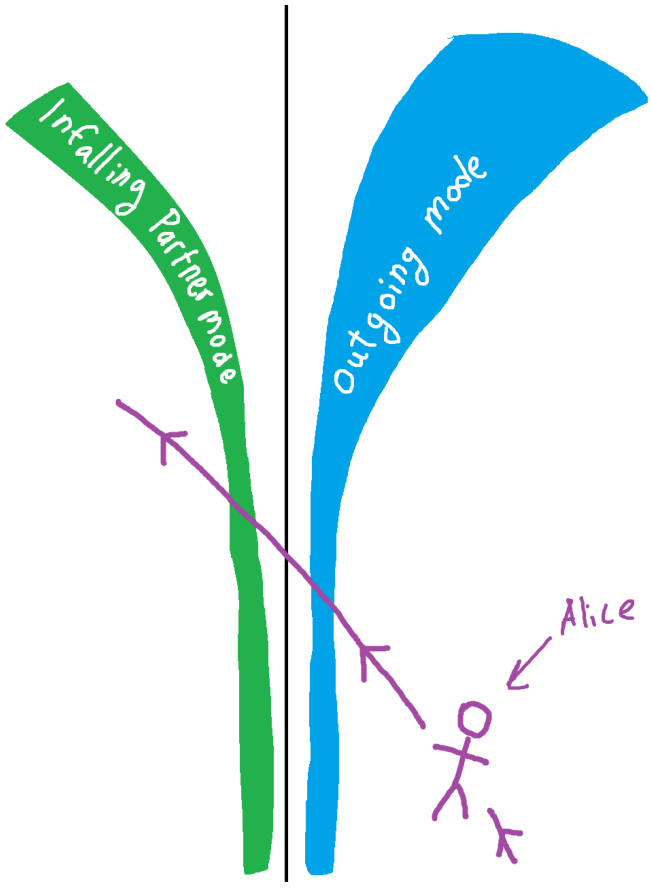


Figure 49: Alice diving into the black hole after the Page time to see the outgoing mode emerge, just like in Fig. 43.

However, the outgoing mode must be closely entangled with an infalling partner mode. This is the "short range entanglement" I've men-

tioned before. (Here I am using the so-called "no drama" postulate, which is really just the equivalence principle. Alice ought to still be able to use regular old quantum field theory just fine as she passes through the horizon. As I explained previously, a quantum field which is not highly entangled on short distances will have a very large energy density, thus violating the "no drama" postulate.) The contradiction is that the outgoing mode cannot be entangled *both* with all the radiation Alice has already collected and *also* with the nearby infalling mode.

Why not? Well, it has to do with something called the "strong subadditivity of entanglement entropy." Say you tensor together three Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_C$.

$$\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$$

If you have a density matrix representing a (possibly mixed) state in $\mathcal{H}_{ABC}$.

$$\rho_{ABC} : \mathcal{H}_{ABC} \to \mathcal{H}_{ABC}$$

you can perform a partial trace over either $\mathcal{H}_C$ or $\mathcal{H}_A$ to get the density matrices ρ_{AB} and ρ_{BC}.

$$\rho_{AB} \equiv \mathrm{Tr}_C(\rho_{ABC}) \qquad\qquad \rho_{BC} \equiv \mathrm{Tr}_A(\rho_{ABC})$$

Likewise, you can also calculate the density matrix that comes from tracing over both A and C or both B and C.

$$\rho_B \equiv \mathrm{Tr}_{AC}(\rho_{ABC}) \qquad\qquad \rho_A \equiv \mathrm{Tr}_{BC}(\rho_{ABC})$$

You can then calculate the entanglement entropies for each density matrix.

$$\begin{aligned} S_{AC} &\equiv -\mathrm{Tr}_{AC}(\rho_{AC} \log \rho_{AC}) \\ S_{AB} &\equiv -\mathrm{Tr}_{AB}(\rho_{AB} \log \rho_{AB}) \\ S_A &\equiv -\mathrm{Tr}_A(\rho_A \log \rho_A) \\ S_B &\equiv -\mathrm{Tr}_B(\rho_B \log \rho_B) \\ S_{ABC} &\equiv -\mathrm{Tr}(\rho_{ABC} \log \rho_{ABC}) \end{aligned}$$

Finally, the statement of the "strong sub-additivity" of entanglement entropy is

$$S_{AB} + S_{BC} \geq S_B + S_{ABC}. \tag{88}$$

It turns out that the above inequality always holds.

Now, to the particular case at hand,

$A =$ all the Hawking radiation that came out before Alice jumped in
$B =$ the next outgoing mode leaving the horizon
$C =$ the infalling partner mode on the other side of the horizon

We will use all of the assumptions of BHC to modify Eq. 88 until we reach a contradiction. (Note that S_{ABC} is *not* zero because ρ_{ABC} is *not* pure. There are still other degrees of freedom, namely the rest of the Hawking radiation, that don't belong to A, B, or C.)

The first fact we will use is the "no drama" principle. This says that, while crossing the event horizon, Alice should be able to describe her surroundings using regular old quantum field theory, just like Hawking said you could. This means that she shouldn't have to know about A to describe B and C, because according to Hawking's original calculation, B and C really shouldn't be entangled with A! Because A should be completely unentangled with BC, we have

$$S_{BC} = 0 \quad \text{and} \quad S_A = S_{ABC}.$$

Using the two equations above, Eq. 88 then becomes

$$S_{AB} \geq S_B + S_A. \tag{89}$$

The second fact we will use is that, because Alice is conducting this experiment after the Page time, the emission of the B mode will decrease the entanglement entropy.

$$S_A > S_{AB}$$

Therefore, we can modify Eq. 89 once again:

$$S_A > S_B + S_A. \tag{90}$$

Finally, just like in Hawking's original calculation, we know that the reduced density matrix ρ_B must be thermal. Therefore,

$$S_B > 0$$

giving us a contradiction.

Morally speaking, the above argument shows that BHC wants "too much." If all the information comes out of the black hole, then the outgoing mode must be highly entangled with all the radiation that already came out once the Page time has passed. But if we ascribe

to the "no drama" principle, then Alice shouldn't need to know about all that old radiation to describe what's happening near the horizon. The relevant degrees of freedom should be right in front of her, just like Hawking thought originally.

(Another way people like to explain this paradox is to evoke something called the "monogamy of entanglement," saying that the outgoing mode can't both be entangled with near-horizon degrees of freedom and all the outgoing radiation.)

Now I'm sure there's a question on your mind. Where does any "Firewall" come into this? Well, one suggestion that the AMPS paper makes to resolving the paradox is to say that the outgoing Hawking mode *isn't* entangled with any near-horizon degrees of freedom in the way QFT predicts. In other words, they suggest ditching the no-drama principle. As I discussed earlier in the notes, breaking entanglement on short distances in quantum field theory means that the energy density becomes extremely high, due to the gradient term in the Hamiltonian. This would be the so-called "Firewall." Perhaps it means that spacetime ends at the Horizon, and that you really can't enter a black hole after all.

One final thing I should mention is that Alice doesn't actually have to cross the horizon in order to figure out if the outgoing mode and the infalling partner mode are entangled. It is enough for her to conduct repeated measurements on multiple different outgoing modes. For example, say you could conduct measurements on many spins, with the knowledge that they were all prepared the same way. You may start by conducting measurements using the observable σ_z. If all the measurements come out to be $+1$, then you can be pretty sure that they were all in the $+1$ eigenstate of σ_z. However, if half are $+1$ and the other half are -1, then you don't yet know if your states are in mixed state or just in a superposition of σ_z eigenstates. You could then conduct measurements with σ_x and σ_y on the remaining spins to figure out if your states really were mixed they whole time. Going back to Alice, she could try to detect superpositions between the different $|n\rangle_{\mathbf{k},\text{out}}$ states for many different modes $\mathbf{k}$. If there are no such superpositions, she would deduce that the outgoing modes really are entangled with their infalling partner modes without ever entering the black hole.

6.7 Harlow Hayden

I will now very briefly introduce one proposed resolution to the Firewall Paradox. I think a very nice discussion of this is given in Lecture

6 of [1]. The question we must ask is: why should Alice be allowed to describe everything she sees using density matrices, anyway? Certainly, in order to actually reach a contradiction, there first must be some measurement she could conduct which could actually show that the outgoing mode B really is entangled with all the old radiation A. But how can she perform this measurement anyway?

In order to do this, she would have to first "distill the q-bits" in A which are entangled with B. But doing that is not so easy. In fact, it turns out that is a very difficult computation for a quantum computer to do. It would probably take a quantum circuit of exponential size to do, and by the time Alice finished, the black hole would have already evaporated. That is, the problem is likely to be intractable. It takes exponential time to distill the bit, but only polynomial time for the black hole go away. More specifically, Harlow and Hayden showed that if Alice is able to distill the entanglement in time, then $SZK \subseteq BQP$. Apparently, computer scientists have many reasons to believe that that is not the case.

This would be a pretty weird resolution to the Firewall paradox. What happens if Alice just gets, like, really lucky and finishes her distillation in time to jump in? (I should mention that not enough is known about the Harlow Hayden resolution to know if such luck is really possible. However, it also cannot yet be ruled out.) Would the firewall exist in that case? Computer scientists are fine with resolutions like Harlow and Hayden's, because they don't really care about the case where you're just super lucky. It's of no concern to them. But physicists are not used to the laws of physics being altered so dramatically by luck, even if the luck required is exponentially extreme. Can a whole region of space-time really go away just like that?

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