THE ELECTRICAL PRINCIPLES OF TELECOMMUNICATIONS

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Preface

The book is intended to cater for the requirements of more than one class of reader. It is of primary importance to students who are preparing for the Principles B and C examinations of the CGLI No. 271 Course. Secondly, there are those technicians who, although employed in the field of telecommunications, need a book to which they can refer in an attempt to keep abreast of a developing subject. Thirdly, the authors have in mind all those actively engaged in the teaching of electrical principles with a light current bias, who require a supply of examination-type examples in their everyday work. This diversity of readership has led to a somewhat unusual layout of the complete text in a single volume. Care has been taken to offer guidance to the student by printing B or C at the beginning and end of topics that are appropriate to a particular syllabus. This should not interfere with continuity in the book for other readers or in any way detract from its overall value. Instead it should be regarded as a means of gaining two advantages:

1. It has allowed the authors scope to explain certain topics without the restriction of artificial boundaries created by the need for two syllabuses.
2. It avoids the repetition that would inevitably arise from publishing two books.

When it is felt necessary to continue beyond the Principles C requirements, the opportunity has been taken of pursuing some topics a stage further in order to bring them to a more logical conclusion. The section of work on field effect transistors to the exclusion of valves was considered to be desirable.

As far as possible, the standard of mathematics in the various proofs and derivations has been kept within that taught in the third and fourth years of the CGLI No. 271 Course. The circuit diagrams are in accordance with B.S. 3939 and SI units have been used throughout. Altogether there are 216 examples included in the text, of which 45 per cent have worked solutions.

The authors would like to place on record their grateful appreciation for the valuable contribution made by the City and Guilds of London Institute in allowing past examinations questions to be used. It ought to be stated

that the authors accept responsibility for the solutions and every effort has been made to ensure their accuracy. Finally, the three typists – Mrs J. Lowe, Mrs M. Miller and Miss K. Gibbons – deserve a special word of commendation for the cheerful and efficient way in which they applied themselves to the task of producing the script.

Wigan and District Mining and Technical College R.L.
D.N.

Contents

1 Electrostatics

CAPACITANCE AND THE ELECTRIC FIELD

(B) Capacitance is the property by which the elements in an electric circuit store an electric charge. A capacitor is a circuit element or piece of apparatus designed to have a capacitance of definite or controllable value, and consists essentially of two electric conductors separated from each other by an insulating material called the dielectric. The simplest form of capacitor comprises two flat metal plates mounted on insulating supports, facing each other and parallel, and with a small gap intervening. Imagine the space surrounding the plates to be completely evacuated. If the plates are connected to a battery, a current of electricity flows for a brief instant, thus causing

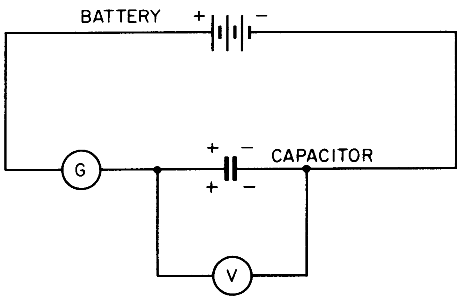


figure 1.1

the passage of a charge of Q coulombs of electricity round the circuit. The passage of this charge may be observed by a suitable galvanometer (a ballistic galvanometer) connected in the circuit, the galvanometer giving a 'kick' or 'throw' when the battery is connected. The magnitude of this throw is proportional to the quantity Q coulombs passing through the galvanometer to charge the capacitor. The passage of the charge Q causes the potential difference between the two plates to become equal to the potential difference of the battery, one plate thus becoming positively charged and the other negatively charged, as shown in figure 1.1.

It is important to note that the charge Q flowing from the positive terminal of the battery through the connecting wire into the capacitor is equal in magnitude to the charge Q flowing from the capacitor through the connecting wire to the negative terminal of the battery, the current in all parts of the circuit at any instant being the same. The capacitor becomes charged as the result of a transfer of Q coulombs from the negative to the positive plate.

Connect a voltmeter across the capacitor as shown in figure 1.1 and assume that the voltmeter takes no current. The voltmeter now reads the potential difference between the plates, and this is equal to the potential difference of the battery. The space between the plates of the capacitor is in a state of electric stress and an electric field is said to exist between the plates.

Suppose now the two connecting wires to be disconnected from the battery, the two ends being kept separated. The voltmeter reading will be seen to be initially unchanged, but may subsequently decrease slowly due to leakage resistance between the plates. With perfect insulating supports no decrease of voltage would occur for a considerable time, and the capacitor would therefore retain its charge.

If the ends of the two connecting wires are now joined together, the galvanometer will again be observed to deflect. This deflection will be of similar magnitude to that observed while charging but will be in the opposite direction. At the same time the capacitor becomes discharged and the voltmeter reading falls to zero. The similarity in magnitude of the two galvanometer deflections indicates that all the charge put into the capacitor comes out on discharge, leakage effects due to imperfect insulation being neglected. If a small filament lamp is included in the circuit during the discharge, it will be seen to light for an instant, thus indicating that energy was stored in the charged capacitor.

If the above charge-discharge procedure is repeated for several different values of battery p.d., it will be found that the charge (Q coulombs) is always proportional to the applied p.d. (V volts). This constant ratio of charge to p.d. is termed the capacitance of the capacitor. The unit of capacitance is the Farad (symbol F). A capacitor of capacitance 1 farad takes a charge of 1 coulomb when a p.d. of 1 volt is applied. Thus

$$\frac{\text{Charge (coulombs)}}{\text{Applied p.d. (volts)}} = \text{Capacitance (farads)}$$

$$\frac{Q}{V} = C$$

and

$$Q = CV \tag{1.1}$$

In practice, the farad (F) is an inconveniently large unit, and the capacitance is commonly expressed in microfarads (μF) or in picofarads (pF) where

$$1\ \mu\text{F} = 10^{-6}\ \text{F}$$

and

$$1\ \text{pF} = 10^{-12}\ \text{F}$$

Example 1.1 A 20 μF capacitor is connected to a d.c. supply of 200 V. Calculate the charge taken by the capacitor.

$$Q = CV$$
$$= 20 \times 10^{-6}\,(\mathrm{F}) \times 200\,(\mathrm{V})$$
$$= 0{\cdot}004 \text{ coulomb or } 0{\cdot}004\ \mathrm{C}$$

Permittivity

If the space between the parallel plates of the capacitor in figure 1.1 is now filled with an insulating material, it will be found that the capacitance has increased. With air and other gases, the increase is very small indeed, and such capacitors are assumed to have the same capacitance as they would have if the space between the plates were evacuated. If, however, the dielectric introduced is a solid or liquid insulating material, the capacitance is increased by a factor that depends upon the insulating material used. This factor is termed the *relative permittivity* of the dielectric. Thus, if a sheet of mica completely fills the space between the plates, the capacitance is about 6 times greater than when that space is either evacuated or filled with air. The relative permittivities of some common dielectric materials are given in table 1.1.

table 1.1

Dielectric	Air	Mineral oil	Glass	Mica	Paraffin wax
Relative permittivity	1·0006	2·2	5–10	3–7	2·5–4

Suppose the p.d. applied to the capacitor be now steadily increased, thus increasing the electric stress in the dielectric between the plates. A p.d. will eventually be reached at which the insulation will break down, as a result of a disruptive discharge passing through it. Such breakdown of a dielectric can lead to a short-circuit on the supply, and the p.d. at which a capacitor normally operates must, therefore, be kept well below the value at which breakdown occurs. However, the electric stress in the dielectric is not solely dependent upon the p.d. between the plates, but depends also upon the distance between the plates. Thus, for a parallel plate capacitor the electric stress is given by the p.d. across the plates divided by the distance between the plates, and this stress is constant at all points in the dielectric. The electric stress at any given point is also referred to as the *electric field strength* or *electric force* at that point. For the parallel plate capacitor, let the p.d. between the plates be V volts and the distance between them be d metres. Then, electric field strength or electric force $E = V/d$ volts/metre acting in the direction from the positive to the negative charge. In general, the field strength at any point in an electric field is given by the potential gradient at that point. Thus, at any point in the field,

$$E = \frac{dV}{dx} \text{ volts/metre} \tag{1.2}$$

An *electric field* may be defined as a region of space in which an electric charge experiences a force. The direction in which the field acts at any point is defined as the direction in which a small positive charge would tend to move if placed in the field at that point. A line of force is the imaginary line that would be followed by a small positive charge if free to move under the action of the force due to the electric field. The whole space between the electrodes of a capacitor is penetrated by imaginary lines of force. These lines of force originate at the electrode that is positively charged, and terminate at the electrode that is negatively charged. Lines of force cannot intersect, because the line of force defines the direction of the electric force, and this cannot

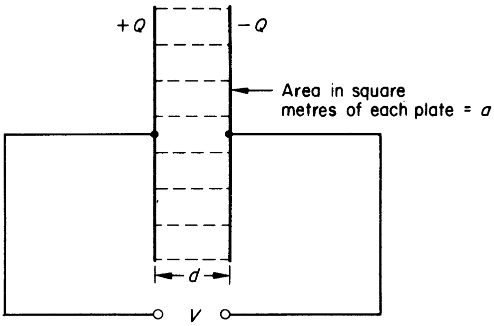


figure 1.2

have two directions at any point. Consider now a parallel plate capacitor that has plate dimensions which are large compared to the distance between them, so that the lines of force pass straight across the gap with negligible fringing at the edges of the plates, as shown in figure 1.2. Assume also that the plates are *in vacuo.*

Let the charge on the plates be $+Q$ and $-Q$ coulombs. The electric field between the plates is termed an *electric flux*, the electric flux originating at the positive plate and terminating at the negative plate. The unit of electric flux is the Coulomb (symbol C), and one coulomb of flux is defined as originating at a positive charge of one coulomb and terminating at a negative charge of one coulomb. Thus, in figure 1.2, the total flux between the plates is Q coulombs, and it is uniformly distributed. Because the cross-sectional area of the electric field is a square metres, the flux density at any point in the field is given by

$$D = \frac{Q}{a} \text{ coulombs/metre}^2 \tag{1.3}$$

The electric flux density D at any point in the field is directly proportional to the electric force E at that point, and the ratio of electric flux density to electric force is thus a constant. This constant is termed the *permittivity of free space,* and it is represented by the symbol ϵ_0.

$$\frac{D}{E} = \epsilon_0 \tag{1.4}$$

CALCULATION OF VALUES

As

$$\frac{D}{E} = \frac{Q}{a} \div \frac{V}{d} = \frac{Q}{V} \times \frac{d}{a} = C\frac{d}{a}$$

$$\epsilon_0 = C\frac{d}{a}$$

and

$$C = \epsilon_0 \frac{a}{d} \tag{1.5}$$

Capacitance

Clearly, the capacitance of a parallel plate capacitor may be calculated if the value of ϵ_0 is known. ϵ_0 may be determined experimentally by charging a capacitor of known dimensions to a known potential difference V, and then discharging it through a ballistic galvanometer in order to measure the charge Q. Then, as both Q and V are known, C may be calculated from $C = Q/V$; and, as a and d are known, ϵ_0 may be calculated from $\epsilon_0 = C \times d/a$. From the results of careful experiments, ϵ_0 is found to have the value $10^7/4\pi c^2$ (where c is the velocity of light in metres per second), so that $\epsilon_0 = 8{\cdot}85 \times 10^{-12}$ farads per metre.

If the dielectric between the plates of the capacitor is a solid or liquid dielectric of relative permittivity ϵ_r, the capacitance is then given by

$$C = \epsilon_r \epsilon_0 \frac{a}{d} \text{ farads} \tag{1.6}$$

Practical capacitors as used in electronic circuits often consist of two long thin strips of metal foil and two equally long thin strips of waxed paper wound spirally together and soaked in hot paraffin wax, thus forming in effect two metal surfaces of large area separated by a thin layer of solid dielectric. Another type of capacitor consists of two sets of metal vanes, one set being fixed, the other set being arranged so that its vanes can be moved into and out of the space between the fixed vanes without touching them, thus forming a variable capacitor. This type of variable capacitor is commonly used in tuning a radio receiver.

Example 1.2 A 1 μF capacitor is spirally wound using two strips of 10 cm wide metal foil separated by waxed paper dielectric 20 μm thick. The dielectric has a relative permittivity of 3·0. Calculate the total length of metal foil required.

As can be seen in figure 1.3 *both* surfaces of the metal foil are effective in forming the capacitor (apart from a short length of one strip on the outer surface of the assembly, the effect of which is neglected in the solution). Let l metres be the total length of strip required; then, length of each strip $= l/2$ m. Width of each strip $= 0{\cdot}1$ m.

$\therefore$ cross-sectional area of the electric field $= 2 \times l/2 \times 0{\cdot}1 = 0{\cdot}1l$ m^2.

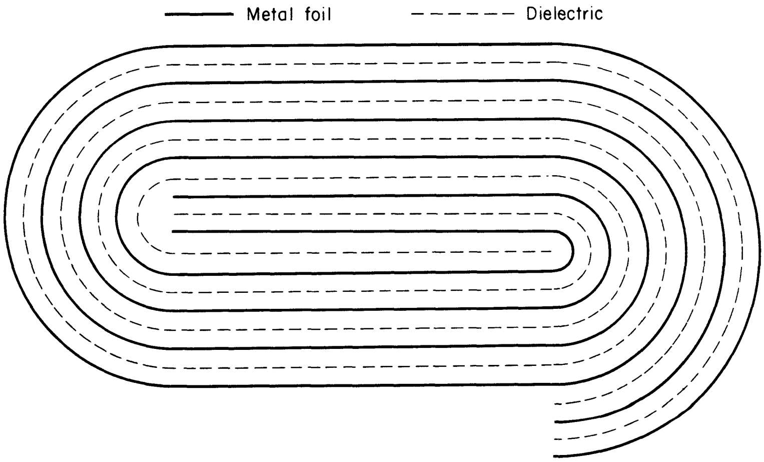


figure 1.3

From equation 1.6

$$C = \frac{\epsilon_r \epsilon_0 a}{d}$$

$$1 \times 10^{-6} = \frac{3{\cdot}0 \times 8{\cdot}85 \times 10^{-12} \times 0{\cdot}1l}{0{\cdot}02 \times 10^{-3}}$$

$$l = \frac{1 \times 10^{-6} \times 0{\cdot}02 \times 10^{-3}}{3{\cdot}0 \times 8{\cdot}85 \times 10^{-12} \times 0{\cdot}1}$$

$$= \frac{200}{26{\cdot}55} = 7{\cdot}53 \text{ m}$$

Example 1.3 A multi-plate capacitor consists of 9 parallel plates separated by sheets of mica 0·2 mm thick and of relative permittivity 6·0. The area of one side of each plate is 400 cm^2 and alternate plates are connected together. Calculate the capacitance.

Figure 1.4 shows the arrangement, from which it is seen that there are 8 sheets of dielectric through which the electric field passes (that is, one less than the number of plates). The cross-sectional area of the electric field passing through each dielectric is equal to the area of one side of each plate.

$\therefore$ Total cross-sectional area of the electric field

$$= 8 \times 400 \times 10^{-4}$$
$$= 0{\cdot}32\ \text{m}^2$$

$$\therefore \qquad C = \frac{\epsilon_r \epsilon_0 a}{d}$$
$$= \frac{6{\cdot}0 \times 8{\cdot}85 \times 10^{-12} \times 0{\cdot}32}{0{\cdot}2 \times 10^{-3}}$$
$$= 0{\cdot}085 \times 10^{-6}\ \text{F}$$
$$= 0{\cdot}085\ \mu\text{F}$$

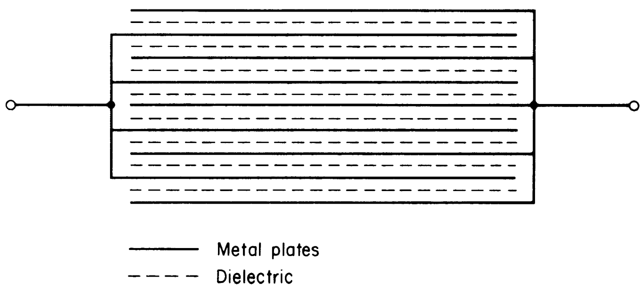


figure 1.4

Energy stored in a charged capacitor

Let a capacitor of C farads be charged at a constant rate of I amperes for T seconds, as shown in figure 1.5. At any time t in the charging process, the charge on the capacitor is $q = It$ coulombs, and the potential difference is

$$v = \frac{q}{C} = \frac{It}{C} \text{ volts}$$

The p.d. is thus proportional to the time t and is represented by the straight line OA, the final value of the p.d. being given by $V = IT/C$ volts.

Average p.d. across C during charging period = $\frac{1}{2}V$ volts.

Average power supplied to $C = \frac{1}{2}VI$ watts.

Therefore energy supplied to C = average power $\times$ time

$$= \tfrac{1}{2}VIT \text{ joules}$$

$$= \tfrac{1}{2}VQ \text{ joules}$$

$$= \tfrac{1}{2}VCV \text{ joules}$$

$$= \tfrac{1}{2}CV^2 \text{ joules}$$

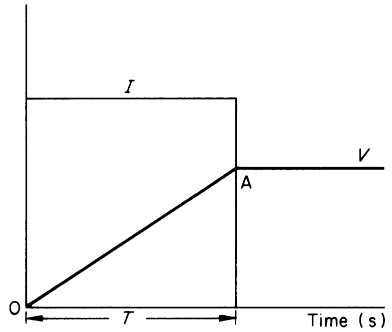


figure 1.5

This energy is stored in the electric field

$\therefore$ Energy stored $= \tfrac{1}{2}CV^2$ joules. (1.7)

The ability of a capacitor to store energy has many practical applications. It should be noted, however, that the amount of energy which can be stored is very small indeed. Capacitors, for example, cannot be used to store electric energy in connection with electricity supplies, or for driving electric motors. A 100 μF capacitor when charged to a p.d. of 200 V stores $\tfrac{1}{2} \times 100 \times 10^{-6} \times 200^2 = 2$ joules of energy. This energy is sufficient to light a 40 W lamp for only 0·05 second.

Capacitors in parallel

Let three capacitors be connected in parallel to a d.c. supply of V volts as shown in figure 1.6.

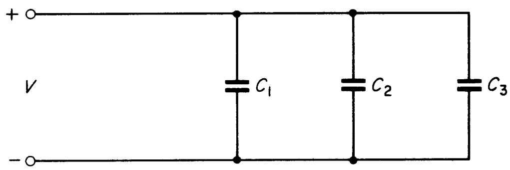


figure 1.6

The charge on each individual capacitor is given by

$$Q_1 = C_1 V$$
$$Q_2 = C_2 V$$
$$Q_3 = C_3 V$$

The resultant capacitance C is the capacitance of a single capacitor in which the same total charge Q is produced by the same supply voltage. That is

$$Q = Q_1 + Q_2 + Q_3$$

$$\therefore \qquad CV = C_1 V + C_2 V + C_3 V$$

$$= V(C_1 + C_2 + C_3)$$

$$\therefore \qquad C = C_1 + C_2 + C_3 \tag{1.8}$$

Extending the argument to *any* number of capacitors connected in parallel, it will be seen that *the resultant capacitance is always equal to the sum of the individual capacitances.*

Capacitors in series

Let three capacitors be connected in series to a d.c. supply of V volts as shown in figure 1.7.

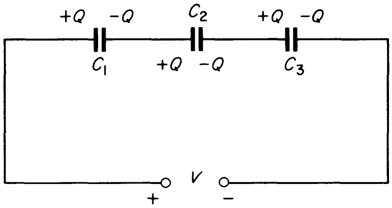


figure 1.7

The key point to note is that the current in all parts of the circuit during the charging period is the same. Thus, the charge of $+Q$ coulombs transferred from the positive supply terminal to the first plate of C_1 is equal to the charge transferred from the second plate of C_1 to the first plate of C_2. Similarly, Q coulombs are transferred from the second plate of C_2 to the first plate of C_3, and Q coulombs are transferred from the second plate of C_3 to the negative supply terminal. Each capacitor thus receives the same charge Q, this also being the charge taken from the supply by the capacitor assembly.

Let the p.d.'s across the capacitors when charged be V_1, V_2, and V_3 respectively. Then

$$C_1 V_1 = Q \qquad C_2 V_2 = Q \qquad C_3 V_3 = Q$$

$$\therefore \qquad V_1 = \frac{Q}{C_1} \qquad V_2 = \frac{Q}{C_2} \qquad V_3 = \frac{Q}{C_3}$$

As the capacitors are in series, the sum of these voltages must equal the supply voltage V

$$\therefore \qquad \frac{Q}{C_1}+\frac{Q}{C_2}+\frac{Q}{C_3}=V$$

The resultant capacitance C is the capacitance of a single capacitor in which the same charge Q is produced by the same supply voltage. Thus,

$$V=\frac{Q}{C}$$

$$\therefore \qquad \frac{Q}{C}=\frac{Q}{C_1}+\frac{Q}{C_2}+\frac{Q}{C_3}$$

$$\therefore \qquad \frac{1}{C}=\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3} \qquad (1.9)$$

Extending the argument to *any* number of capacitors connected in series, it will be seen that *the reciprocal of the resultant capacitance is always equal to the sum of the reciprocals of the individual capacitances.*

Example 1.4 Two capacitors C_1 and C_2, of capacitance 2 μF and 4 μF respectively, are connected in parallel, and this combination is then connected in series with a 3 μF capacitor C_3 to a d.c. supply of 1200 V (as shown in figure 1.8). Calculate the total capacitance of the arrangement; the charge on each capacitor; the p.d. across each capacitor; and the energy stored in each capacitor.

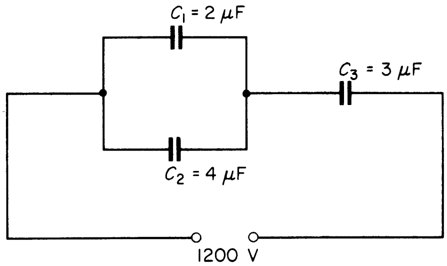


figure 1.8

C_1 and C_2 are equivalent to a single capacitor of $2 + 4 = 6$ μF. The total capacitance is then given by

$$\frac{1}{C}=\frac{1}{6}+\frac{1}{3}=\frac{3}{6}$$

$$\therefore \qquad C = 2\ \mu\text{F}$$

With 1200 V applied, the total charge Q supplied is given by

$$Q = CV = 2 \times 10^{-6} \times 1200 = 0{\cdot}0024 \text{ coulomb}$$

This is also the charge on C_3, and on the capacitance of 6 μF due to C_1 and C_2 in parallel. This charge divides between C_1 and C_2 in proportion to the capacitance.

$$\therefore \quad \text{Charge on } C_1 = \frac{2}{6} \times 0{\cdot}0024 = 0{\cdot}0008 \text{ C}$$

$$\text{Charge on } C_2 = \frac{4}{6} \times 0{\cdot}0024 = 0{\cdot}0016 \text{ C}$$

and

$$\text{Charge on } C_3 = 0{\cdot}0024 \text{ C}$$

$$\text{P.D. across } C_1 \text{ and } C_2. = \frac{0{\cdot}0024}{6 \times 10^{-6}} = 400 \text{ V}$$

$$\text{P.D. across } C_3 = \frac{0{\cdot}0024}{3 \times 10^{-6}} = 800 \text{ V}$$

$$\text{Energy stored in } C_1 = \tfrac{1}{2} \times 2 \times 10^{-6} \times 400^2 = 0{\cdot}16 \text{ J}$$

$$\text{Energy stored in } C_2 = \tfrac{1}{2} \times 4 \times 10^{-6} \times 400^2 = 0{\cdot}32 \text{ J}$$

$$\text{Energy stored in } C_3 = \tfrac{1}{2} \times 3 \times 10^{-6} \times 800^2 = 0{\cdot}96 \text{ J}$$

Summing up these stored energy values,

total energy stored = 1·44 J

This agrees with the value obtained by using the total capacitance and the total applied voltage, that is

$$\text{total energy stored} = \tfrac{1}{2} \times 2 \times 10^{-6} \times 1200^2 = 1{\cdot}44 \text{ J}$$

CHARGE AND DISCHARGE OF A CAPACITOR THROUGH A RESISTOR

Consider first a capacitor connected to a battery and assume the connecting wires and battery to have negligible resistance. The p.d. across the capacitor rises immediately to equal that of the battery, a large current flowing for a very short time. But if a resistance is connected in series with the capacitor, a smaller current (which gradually falls to zero) flows for a much longer time. For the same capacitor and the same supply voltage, the total charge displaced from one plate to the other remains the same whatever the circuit resistance. The time taken for this total charge displacement to take place, however, and hence for the capacitor voltage to rise to equal that of the supply voltage, depends upon the circuit resistance.

Time constant

Let a capacitor of capacitance C farads be connected in series with a resistance of R ohms to a supply of V volts, as shown in figure 1.9.

At the instant of closing the switch S, the capacitor is uncharged: both plates are at the same potential and the capacitor voltage v_C is zero. The p.d. v_R across the resistor is thus equal to the supply voltage V and the current flowing is $I = V/R$ amperes. As the current flows and charge is displaced from one capacitor plate to the other, the capacitor voltage v_C increases (as shown in figure 1.10). This voltage acts to oppose the supply voltage V, with the result that the voltage v_R across the resistance, and the current, both fall. Ultimately, the capacitor voltage v_C rises to equal the supply voltage V, and the current then reaches zero.

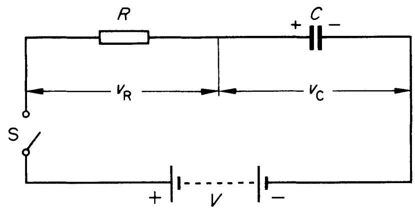


figure 1.9

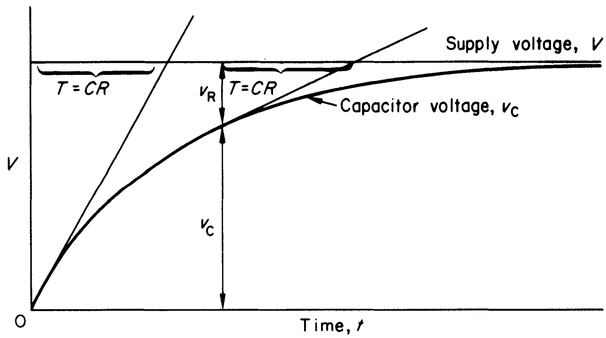


figure 1.10

The final charge on the capacitor is $Q = CV$ coulombs. If the initial current $I = V/R$ continued to flow, the time taken for the charge to reach its final value would be $T = CV/(V/R) = CR$ seconds. This time T, the time taken for the capacitor to become fully charged if the charging current continued at its initial value, is called the *time constant* of the circuit.

$$\text{Time constant } T = CR \text{ seconds} \tag{1.10}$$

where C is the capacitance in farads and R is the resistance in ohms.

If the initial current I continued to flow, the capacitor voltage would increase at its initial rate, this rate being given by the slope of the capacitor voltage/time curve at the origin. A line drawn from the origin 0 to cut the supply voltage V line at a time $T = CR$ seconds is thus a tangent to the v_C/t curve.

The charge curves

Consider next what happens when the p.d. across the capacitor has reached some value v_C. The p.d. across the resistor is then $v_R = V - v_C$ and the current is $i = (V - v_C)/R$. The charge on C is $q = Cv_C$ and the charge which remains to be displaced for C to become fully charged is $C(V - v_C)$. If the charging current i at this instant continued to flow, the time required for this would be $C(V - v_C) \div (V - v_C)/R = CR$ seconds. This is the time constant T, which is, therefore, also given by the time taken for the capacitor to become fully charged if the charging current continued unchanged at *any* given instant. The p.d. across C would then continue to increase at the same rate, so that a line drawn from the v_C/t curve to cut the supply voltage V line CR seconds later must also be a tangent to the v_C/t curve. Consequently, by drawing lines at the correct slope at a succession of different times, the charging curve v_C/t may be constructed. This has been done in figure 1.11.

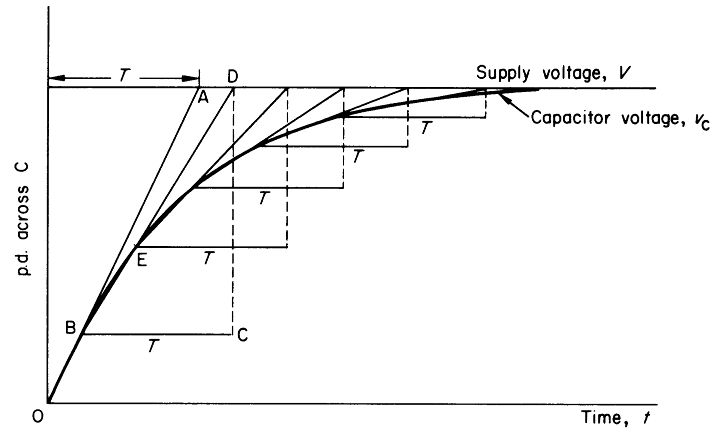


figure 1.11

Starting at O, the line OA is drawn to cut the V line at $T = CR$ seconds later. On OA, from a point B near the origin, draw BC = T seconds and erect perpendicular CD. Join BD. On BD, choose another point E near B, and repeat this procedure. A series of lines is thus drawn cutting the V line T seconds after the instant considered. Next, a curve is drawn such that these lines are all tangents to it, and this curve then portrays capacitor p.d. plotted against time. The curve of charging current against time may be drawn in a similar manner. At any instant, as the charging current is given by $i = (V - v_C)/R$, it is proportional to $(V - v_C)$, and the charging current curve is thus the inverse of that of capacitor p.d. The graphical construction is shown in figure 1.12.

Naturally, the curves graphically derived above for the variation of capacitor p.d. and charging current can be expressed in the form of mathematical equations.

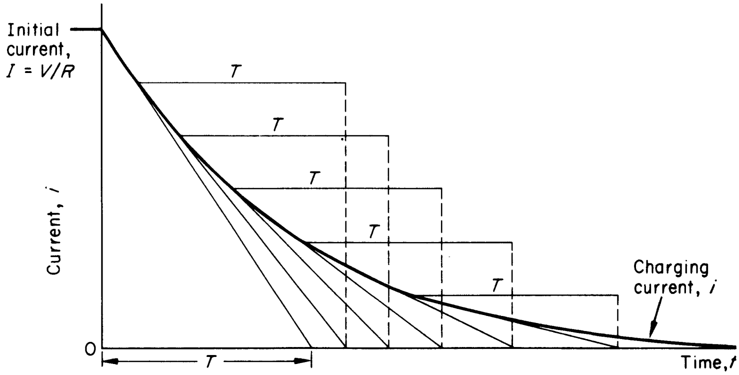


figure 1.12

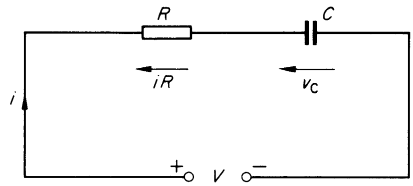


figure 1.13

As the voltage V across the circuit is at all times equal to the sum of the voltages across R and C; and as at any time t during the charging process the charge on the capacitor is q coulombs, and the current i is the rate of change of q, it follows that

$$V = iR + v_C \tag{1.11}$$

$$= R\frac{dq}{dt} + v_C$$

$$= CR\frac{dv_C}{dt} + v_C$$

$$\frac{1}{CR}(V - v_C) = \frac{dv_C}{dt}$$

$$dt = CR\frac{1}{V - v_C}\,dv_C$$

$$t = CR\int\frac{1}{V - v_C}\,dv_C$$

$$= -CR\ln(V - v_C) + \text{a constant}$$

(ln = Napierian logarithm, or $\log_e$)

When $t = 0$, $v_C = 0$

$$\therefore \qquad 0 = -CR \ln V + \text{a constant}$$

and the value of the constant is thus $CR \ln V$

$$\therefore \qquad t = -CR \ln (V - v_C) + CR \ln V$$

$$= CR \ln \left(\frac{V}{V - v_C} \right)$$

$$\therefore \qquad \frac{t}{CR} = \ln \left(\frac{V}{V - v_C} \right)$$

$$\exp (t/CR) = \frac{V}{V - v_C}$$

$$\exp (-t/CR) = \frac{V - v_C}{V} = 1 - \frac{v_C}{V}$$

$$\frac{v_C}{V} = 1 - \exp (-t/CR)$$

$$\therefore \qquad v_C = V\,[1 - \exp (-t/CR)] \qquad (1.12)$$

The capacitor p.d. thus increases exponentially, as shown in figure 1.14. To assist in sketching such curves, it is useful to determine the times taken for v_C to reach 0·1, 0·9 and 0·99 of its final value V. Substituting $0{\cdot}1\,V$ for v_C:

$$0{\cdot}1\,V = V[1 - \exp (-t/CR)]$$

$$\exp (-t/CR) = 1 - 0{\cdot}1 = 0{\cdot}9$$

$$\exp (t/CR) = 1{\cdot}111$$

$$\frac{t}{CR} = \ln 1{\cdot}11 = 0{\cdot}105$$

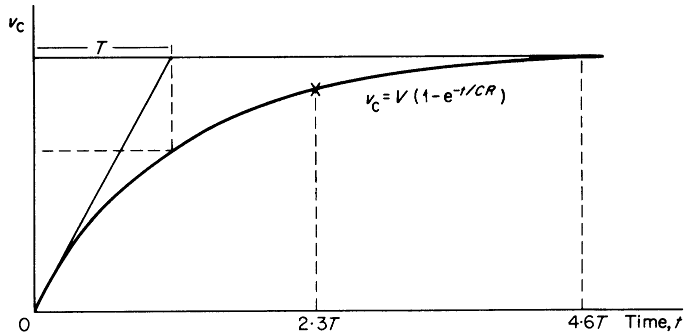


figure 1.14

$$\therefore \qquad t = 0{\cdot}105CR$$

$$t = 0{\cdot}105T$$

Thus in a time equal to $0{\cdot}105T$ the capacitor voltage v_C reaches 0·1 of its final value V. Similarly, the times required for v_C to reach $0{\cdot}9V$ and $0{\cdot}99V$ are approximately $2{\cdot}3T$ and $4{\cdot}6T$ respectively.

If the initial rate of increase of v_C was maintained, v_C would reach V in a time $T = CR$ seconds. Substituting this value for t in equation 1.12

$$v_C = V[1-\exp(-1)]$$

$$= V\left(1-\frac{1}{2{\cdot}718}\right)$$

$$\therefore \qquad v_C = 0{\cdot}632V \qquad (1.13)$$

In a time equal to the time constant T, the capacitor p.d. thus attains 0·632 (63·2 per cent) of its final steady value V.

The expression for the charging current may be obtained from equation 1.12

$$v_C = V[1-\exp(-t/CR)]$$

$$Cv_C = CV[1-\exp(-t/CR)]$$

$$\therefore \qquad q = Q[1-\exp(-t/CR)]$$

where Q is the final charge on the capacitor.

$$\therefore \qquad \frac{dq}{dt} = \frac{Q}{CR}\exp(-t/CR)$$

$$= \frac{V}{R}\exp(-t/CR)$$

$$\therefore \qquad i = I\exp(-t/CR) \qquad (1.14)$$

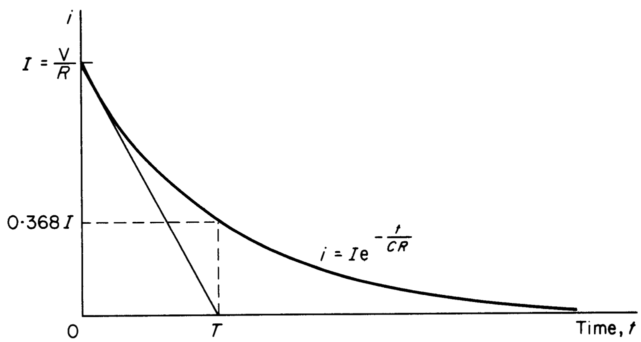


figure 1.15

The charging current thus decreases exponentially, as shown in figure 1.15.

If the initial rate of decrease of current was maintained, the charging current would fall to zero in a time $T = CR$ seconds. Substituting this value for t in equation 1.14

$$i = I \exp(-1) = I \frac{1}{2{\cdot}718}$$

$$\therefore \qquad i = 0{\cdot}368I \tag{1.15}$$

In a time equal to the time constant T, the charging current thus falls to 0·368 (36·8 per cent) of its initial value I.

The discharge curves

Referring again to figure 1.13, suppose the capacitor to have been charged to a p.d. of v_0 volts, due to a charge of q_0 coulombs, and that the supply voltage V is then removed and the two ends of the connecting wire joined together so that the capacitor discharges through the resistance R. Since $V = 0$ equation 1.11 becomes

$$0 = iR + v_C$$

$$= R\frac{dq}{dt} + v_C$$

$$= CR\frac{dv_C}{dt} + v_C$$

$$-\frac{v_C}{CR} = \frac{dv_C}{dt}$$

$$dt = -\frac{CR}{v_C}\,dv_C$$

$$t = -CR\int \frac{1}{v_C}\,dv_C$$

$$= -CR \ln v_C + \text{a constant}$$

When $t = 0$, $v_C = v_0$

$$\therefore \qquad 0 = -CR \ln v_0 + \text{a constant}$$

and the value of the constant is thus $+CR \ln v_0$

$$\therefore \quad t = -CR \ln v_C + CR \ln v_0$$

$$= CR \ln \frac{v_0}{v_C}$$

$$\frac{t}{CR} = \ln \frac{v_0}{v_C}$$

$$\exp(t/CR) = \frac{v_0}{v_C}$$

$$\therefore \quad v_C = v_0 \exp(-t/CR) \tag{1.16}$$

To obtain the expression for the discharge current:

$$Cv_C = Cv_0 \exp(-t/CR)$$

$$q = q_0 \exp(-t/CR)$$

$$\frac{dq}{dt} = -\frac{q_0}{CR} \exp(-t/CR)$$

$$= -\frac{v_0}{R} \exp(-t/CR)$$

$$\therefore \quad i = -i_0 \exp(-t/CR) \tag{1.17}$$

where i_0 is the initial value of the discharge current, the negative sign indicating that the current on discharge flows in the reverse direction to that of the charging current.

Thus the capacitor p.d. and the current on discharge both decrease exponentially, as shown in figure 1.16, and in a time $T = CR$ seconds, both decrease to 0·368 of their initial value.

The curves for capacitor p.d. and current on discharge may also be obtained graphically in a manner similar to that used for the charging curves.

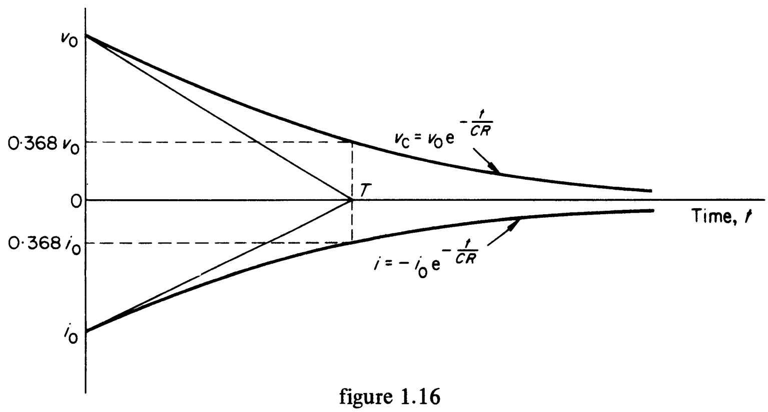


figure 1.16

Example 1.5 A 20 μF capacitor is charged to a p.d. of 200 V and then discharged through a resistor of 50 000 Ω. Derive the curve representing the discharge current.

$$\text{Time constant } T = CR = 20 \times 10^{-6}\,(\text{F}) \times 50\,000\,(\Omega)$$

$$= 1 \text{ s}$$

Initial value of discharge current

$$i_0 = \frac{200}{50\,000} = 0{\cdot}004 \text{ A}$$

$$= 4 \text{ mA}$$

The curve representing the discharge current is derived in figure 1.17, and is drawn taking the direction of this current as positive.

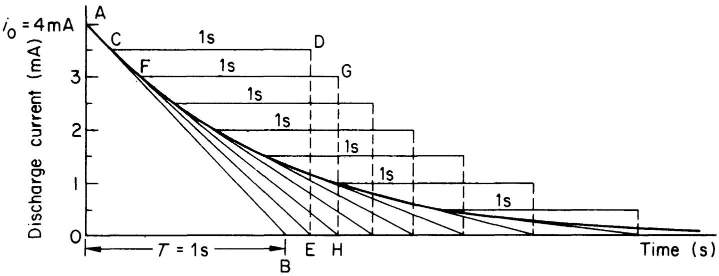


figure 1.17

To a suitable scale, mark off OA = i_0 = 4 mA, and OB = T = 1 s. Join AB. From a point C on AB corresponding to 3·5A, draw CD = 1 s, and draw DE vertically. Join CE. From a point F on CE corresponding to 3A, draw FG = 1 s, and draw GH vertically. Join FH. Repeat this construction for current intervals of 0·5A to obtain a series of lines AB, CE, etc, and draw a curve to which these lines are tangents. This curve represents the variation of discharge current with time.

Example 1.6 A capacitor C is being charged through a series resistor R from a constant voltage source V. Write down an expression for

(a) the voltage across the capacitance and
(b) the current in the resistance t seconds after switching on, the capacitor being initially uncharged.

The circuit of figure 1.18 is used to control another device such that, when the p.d. across AB reaches 63·2 volts, the capacitance C is instantly discharged, the applied voltage V then being allowed to charge it through R as before. Sketch a current/time curve for the current in R throughout this sequence.

(CGLI Principles B, 1964)

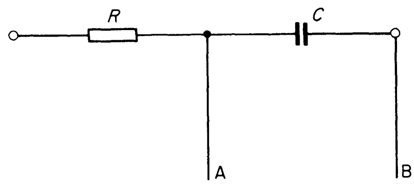


figure 1.18

Assuming that $V = 100$ volts, $C = 2\ \mu\text{F}$, $R = 100\,000\ \Omega$, find (c) the initial current and (d) the charging current at the instant before the capacitor is discharged.

(a) From equation 1.12, $v_C = V[1 - \exp(-t/CR)]$.

(b) From equation 1.14, the current in the resistance t seconds after switching on is given by

$$i = \frac{V}{R} \exp(-t/CR)$$

(c) Initial current

$$I = \frac{V}{R} = \frac{100}{100\,000} = 0{\cdot}001 \text{ A}$$
$$= 1 \text{ mA}$$

(d) The capacitor is discharged when the p.d. across it reaches 63·2 V, the supply voltage being 100 V. Hence, discharge takes place at a time equal to the time constant $T = CR$ seconds. From equation 1.15, the charging current at the instant before discharge is given by

$$i = 0{\cdot}368I$$
$$= 0{\cdot}368 \times 0{\cdot}001 \text{ A}$$
$$= 0{\cdot}368 \text{ mA}$$

The time constant $T = 2 \times 10^{-6} \times 100\,000 = 0{\cdot}2$ s.

Figure 1.19 shows the current plotted against time.

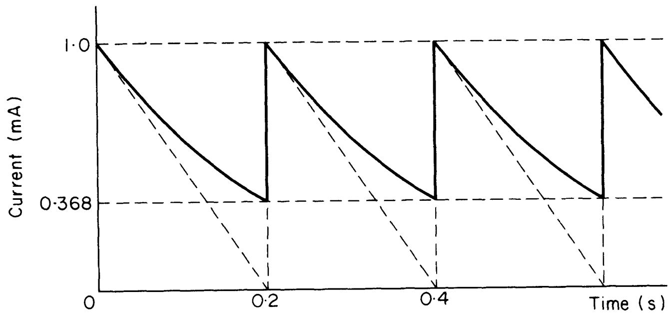


figure 1.19

Example 1.7 A 150 V battery can be switched across a circuit consisting of a 20 μF capacitor in series with a 100 kΩ resistor. Give an expression for the subsequent current/time relationship. Calculate (a) the time constant, (b) the initial current, and (c) the final current.

When the capacitor is fully charged the battery is disconnected. A 10 μF capacitor is then joined in parallel with the 20 μF capacitor, no charge being lost in the process. The combination is now discharged through the resistor. How long from the commencement of the discharge will the voltage take to fall to 37 V?

(CGLI Principles B, June 1967)

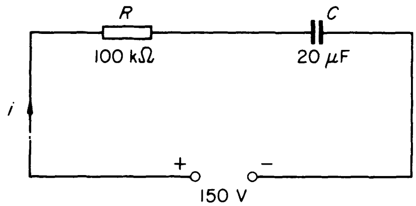


figure 1.20

(a) Time constant

$$T = CR = 20 \times 10^{-6} \times 100\,000$$

$$= 2\,\mathrm{s}$$

(b) Initial current

$$I = \frac{V}{R} = \frac{150}{100\,000} = 0{\cdot}0015\ \mathrm{A}$$

(c) Final current, when C is fully charged, is zero.

The current i is given by

$$i = \frac{V}{R}\,[1 - \exp(-t/CR)]$$

$$= 0{\cdot}0015\,[1 - \exp(-t/2)]$$

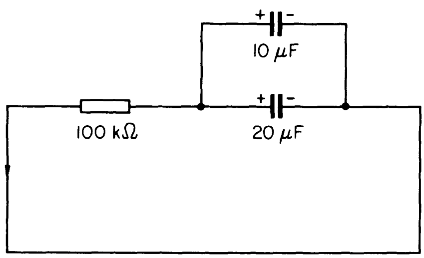


figure 1.21

Charge Q on the 20 μF capacitor when fully charged

$$= 20 \times 10^{-6} \text{ (F)} \times 150 \text{ (V)}$$

$$= 3000 \times 10^{-6} \text{ C}$$

Capacitance of 10 μF and 20 μF capacitors in parallel = 30 μF.
Since no charge is lost

$$3000 \times 10^{-6} = 30 \times 10^{-6} \times V$$

where V is the p.d. across the parallel combination.

$$\therefore \qquad V = \frac{3000 \times 10^{-6}}{30 \times 10^{-6}} = 100 \text{ V}$$

To find the time taken for the p.d. to fall to 37 V,

(From equation 1.16):

$$37 = 100 \exp(-t/CR)$$

(where $CR = 30 \times 10^{-6} \times 100\,000 = 3$ s)

$$\therefore \qquad 37 = 100 \exp(-t/3)$$

$$\exp(+t/3) = \frac{100}{37} = 2{\cdot}703$$

$$t/3 = \ln 2{\cdot}703 = 0{\cdot}9944$$

$$\therefore \qquad t = 2{\cdot}983 \text{ s}$$

More easily, the solution is obtained by assuming 37 V to be sufficiently near 36·8 V for the required time to be given by the time constant of the circuit. Then,

$$t = CR = 3 \text{ s} \qquad \text{(B)}$$

(C) PROPERTIES OF CAPACITORS

An ideal capacitor has the property of capacitance only. Under a.c. conditions, the capacitor current is then in exact quadrature with the p.d. across the capacitor. No practical capacitor, however, is perfect. For capacitors with air or other gas as dielectric, the departure of the capacitor from the ideal is very small indeed, and such capacitors are regarded as being loss-free. Capacitors having solid or liquid insulation material as the dielectric, however, have a significant power loss when working under a.c. conditions, the main causes of imperfection being dielectric leakage and dielectric absorption.

(1) Dielectric leakage. No dielectric material has an infinitely high insulation resistance, hence a continuous leakage current occurs, accompanied by a power loss.

(2) Dielectric absorption. If a capacitor is suddenly connected to a direct voltage, it is observed that a small charging current continues to flow for some time after the main charge has taken place, and then gradually falls to zero, as shown in figure 1.22.

The charging process has two phases, the first of a very short duration when the initial charge takes place, followed by a second phase of much longer duration when the capacitor continues to absorb charge. Similarly, on discharge, a small current continues to flow for some time after the main discharge has taken place. When an absorptive capacitor is charged and then discharged, if the discharging connections are quickly removed, it will be found that the p.d. between the plates gradually rises again, as though the

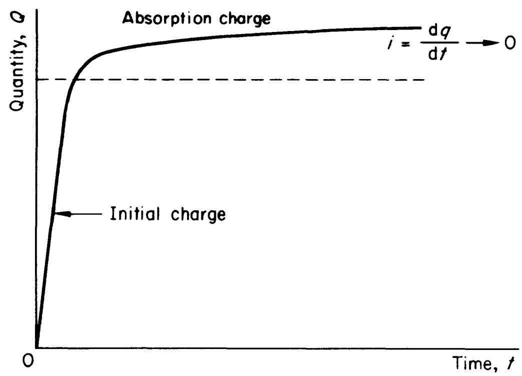


figure 1.22

capacitor were charging itself. This is known as the 'residual' effect, and arises because the molecular changes that take place in the dielectric when the capacitor is charged or discharged cannot take place instantaneously. The effect, termed *dielectric hysteresis*, is similar to that of hysteresis in a ferro-magnetic material and, under a.c. conditions, leads to a power loss. At very high frequencies the effect of absorption is negligible, because the time available for absorption to occur is very short. At lower frequencies, however, the effect of absorption leads to an increase in the effective capacitance and a power loss in the dielectric.

Equivalent circuits

A practical capacitor is thus seen to have power losses from the effects of both dielectric leakage and dielectric absorption, though these losses are negligible in a capacitor having air or gas as the dielectric. Hence, under a.c. conditions, the current must have a component in phase with the p.d. across the capacitor, and thus cannot be in exact quadrature with the p.d. The power loss can be regarded as occurring in an imaginary equivalent resistance, and the imperfect capacitor is then represented by an equivalent circuit comprising either a low resistance r_S in *series* with a capacitance C_S, or a high resistance R_P in *parallel* with a capacitance C_P. The values of r_S and C_S (or of R_P and C_P) are so chosen that the current taken by these equivalent circuits at the stated frequency is the same, and at the same phase angle, as the current taken by the actual capacitor.

Figure 1.23 shows the equivalent series circuit together with the phasor diagram for the circuit, the capacitor current I being taken as reference phasor. The small voltage drop Ir_S is in phase with the current, and the voltage drop $I \times 1/\omega C_S$ lags the current by 90°. The phasor sum of these two voltage drops equals the supply voltage V. The current I is seen to lead the voltage

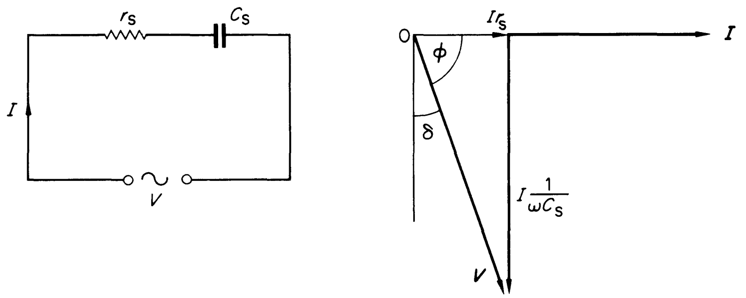


figure 1.23

V by an angle ϕ, somewhat less than 90°. The angle $(90 - \phi) = \delta$ is known as the loss angle, and is a useful criterion for indicating the quality of the capacitor. In capacitors with mica as the dielectric, the loss angle is very small; but with paraffin wax the loss angle becomes relatively large, though not exceeding 0·03 radians (about 2 degrees). The loss angle may be expressed in terms of the components of the equivalent circuit. From the phasor diagram

$$\tan\delta = \frac{Ir_S}{I/\omega C_S} = r_S\omega C_S$$

As δ is a very small angle, $\tan\delta$ is very nearly equal to δ (in radians) and also very nearly equal to the power factor ($\cos\phi$).

$$\therefore \qquad \delta \simeq \cos\phi \simeq \tan\delta = r_S\omega C_S \tag{1.18}$$

Another way of expressing the quality of a capacitor is in terms of its Q-factor: defined in terms of the ratio of the volt-amperes taken by the capacitor to the power dissipated in watts. A perfect capacitor thus has a Q-factor of infinity.

In general

$$Q = \frac{VI}{W}$$

or

$$Q = \frac{1}{\cos\phi} \tag{1.19}$$

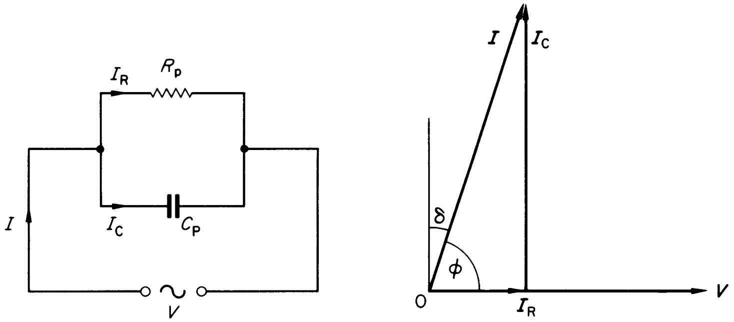


figure 1.24

Figure 1.24 shows the equivalent parallel circuit together with the phasor diagram, the supply voltage V being taken as reference phasor. Again, the current I leads the voltage V by an angle ϕ, somewhat less than 90°, and $(90° - \phi) = \delta$. From the phasor diagram

$$\tan\delta = \frac{I_R}{I_C} = \frac{V}{R_p V\omega C_p} = \frac{1}{R_p\omega C_p}$$

and

$$\delta \simeq \cos\phi \simeq \tan\delta = \frac{1}{R_p\omega C_p} \tag{1.20}$$

Example 1.8 A sinusoidal voltage of 25 V RMS at a frequency of 5 kHz is applied to a 1·59 μF capacitor having a loss-angle of 4×10^{-4} rad. Calculate for the capacitor (a) its Q-factor, (b) its equivalent parallel resistance, (c) its equivalent series resistance, (d) the power dissipated.

(a) From equation 1.18

$$\cos\phi \simeq \delta = 0{\cdot}0004$$

$$\therefore \qquad Q\text{-factor} = \frac{1}{\cos\phi} = 2500$$

(b) From equation 1.20

$$\delta = \frac{1}{R_p\omega C_p}$$

$$0{\cdot}0004 = \frac{1}{R_p \times 2\pi \times 5000 \times 1{\cdot}59 \times 10^{-6}}$$

$$\therefore \qquad R_p = \frac{1}{0{\cdot}0004 \times 10 \times 5000 \times 10^{-6}}$$

$$= 50\,000\ \Omega$$

(c) From equation 1.18

$$\delta = r_s \omega C_s$$

$$0{\cdot}0004 = r_s \times 2\pi \times 5000 \times 1{\cdot}59 \times 10^{-6}$$

$$\therefore \qquad r_s = \frac{0{\cdot}0004}{2\pi \times 5000 \times 1{\cdot}59 \times 10^{-6}}$$

$$= 0{\cdot}008\ \Omega$$

(d) P.D. across R_p in the equivalent parallel circuit

$$= 25\ \text{V RMS}$$

$$\therefore \quad \text{Power dissipated} = \frac{V^2}{R_p} = \frac{25^2}{50\,000} = 0{\cdot}0125\ \text{W}$$

Alternatively, using the equivalent series circuit:

$$X_s = \frac{10^6}{2\pi \times 5000 \times 1{\cdot}59} = 20\ \Omega$$

$$Z = (0{\cdot}008^2 + 20^2)^{1/2} = 20\ \Omega$$

$$I = \frac{V}{Z} = \frac{25}{20} = 1{\cdot}25\ \text{A}$$

$$\therefore \quad \text{Power dissipated} = I^2 r_s = 1{\cdot}25^2 \times 0{\cdot}008 = 0{\cdot}0125\ \text{W}$$

ELECTRIC FIELD CONFIGURATIONS

Capacitance has been seen to be a property of those elements of an electric circuit which consist of two electric conductors separated from each other by an insulating material. This chapter has so far concentrated on parallel plate capacitors, but a length of cable or a pair of parallel wires will also possess the property of capacitance and, if the conductors are at different potentials, an electric field will exist between them. When sketching the electric field between the conductors, it is useful to imagine the lines of force being under tension so that they tend to take the shortest path, but are often prevented from so doing by lateral repulsive pressure from adjacent lines. This may result in lines being forced to follow a curved path (for example, it is this lateral pressure that leads to the fringing of the field at the edges of a parallel plate capacitor, as shown in figure 1.25). Lines of force leaving a positively charged conductor or entering a negatively charged conductor must enter or leave at right-angles to the surface of the conductor, for the surface of a conductor must be at equipotential (that is, all points must be at the same potential), and thus cannot have a component of electric force along its surface. The field configurations for a parallel plate capacitor, a co-axial cable, and an air spaced 2-wire line are sketched in figure 1.25.

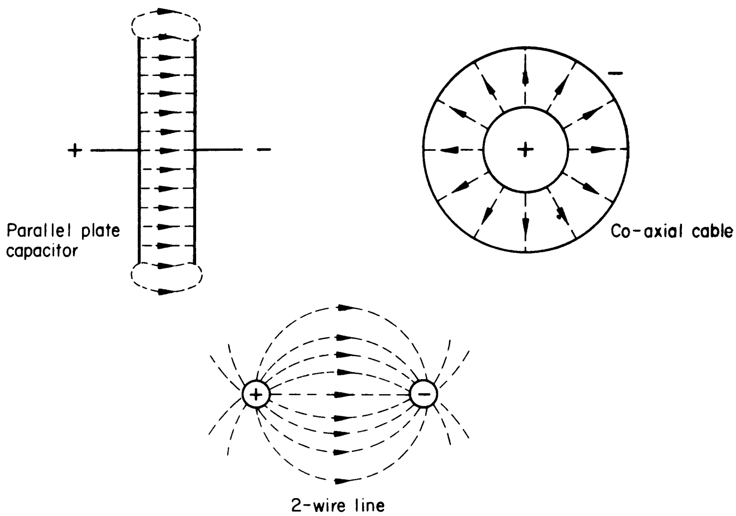


figure 1.25 (C)

MISCELLANEOUS EXAMPLES

(B) ***Example 1.9*** Capacitors of 6 μF and 3 μF are connected in series and charged from a constant-voltage 100 V source. Determine:

(a) The voltage across each capacitor
(b) The charge on each capacitor
(c) The total energy stored in them

The individual capacitors are now each charged to 100 V and connected in series aiding. What would happen if a 200 V supply of the same polarity were connected across the combination?

[(a) 33·3 V; 66·7 V (b) 200 μC (c) 0·01 J. Nothing.]

(CGLI Principles B, 1964)

Example 1.10 State the relationship between the voltage across a capacitor, the charge it carries, and its capacitance.

Derive an expression for the energy stored by a charged capacitor.

A parallel-plate capacitor having two plates each of effective area 200 cm^2, spaced 4 mm apart, is immersed in a tank of oil of relative permittivity 5. Calculate its capacitance.

An identical capacitor, but with air, not oil, as the dielectric, is connected in parallel with the first, the two being then charged in parallel from a 500 V battery. What is the energy stored in the combination?

The battery is then disconnected and the oil drained from the tank. Calculate the final potential difference across the capacitors.

[221 pF; 33·15 μJ; 1500 V]

(CGLI Principles B, 1965)

Example 1.11 An uncharged capacitor (C) in series with a resistor (R) is connected to a constant voltage supply (V). Why does the voltage across the capacitor rise slowly?

Write down an expression for the voltage across the capacitor after time (t) seconds. Sketch a curve of voltage against time.

What is the meaning of time constant for the circuit?

A capacitor of 10 μF is charged through a series resistor from a 100 V battery. What value of resistor will give a time constant of 0·1 second?

For this circuit calculate the current flowing:

(a) at the instant of switching on
(b) after a time equal to the time constant
(c) after a long period of time

[(a) 10 mA (b) 3·68 mA (c) 0]

(CGLI Principles B, 1966)

Example 1.12 When a capacitor (C) in series with a resistor (R) is connected to a constant voltage source (V) the p.d. across the capacitor rises slowly. Explain the reason for this. Give expressions for (a) the time constant of the circuit, (b) the p.d. across the capacitor, (c) the charging current, in relation to the circuit constants and time from switching on the voltage.

A circuit consisting of 50 μF in series with a resistor is to have a time constant of 2 seconds. What value of resistor will be needed? If the circuit is connected to a 100 V d.c. supply, calculate the charging current (i) initially, (ii) 2 seconds after switching on.

[(a) $T = CR$ (b) $v_C = V[1 - \exp(-t/CR)]$ (c) $V[\exp(-t/CR)]/R$
$R = 40$ kΩ (i) 2·5 mA (ii) 0·92 mA]

(CGLI Principles B, 1968) (B)

(C) ***Example 1.13*** Draw a phasor diagram to show the phase relationship between the applied voltage and the current in a capacitor and hence explain the meaning of the term loss angle. State the main reasons for power loss in a capacitor with a solid dielectric.

An alternating voltage of 1 V r.m.s. at a frequency of 1 kHz is applied to a capacitor whose reactance is 1000 Ω. Calculate (a) the capacitance, (b) the current in the circuit.

If the loss angle of the capacitor is 10^{-4} radians, calculate (c) the power dissipated, (d) the equivalent series resistance.

[(a) 0·159 μF (b) 1 mA (c) 0·1 μW (d) 0·1 Ω]

(CGLI Principles C, 1966)

Example 1.14 Sketch diagrams showing the electric field configurations in (i) a parallel plate capacitor, (ii) a length of air-spaced 2-wire line. Label the sketches to show the polarity of the applied e.m.f. and the direction of the electric flux.

At a frequency of 2 kHz a capacitor can be represented by a reactance of 500 Ω in parallel with a resistance of 2 MΩ. Calculate for this capacitor (a) the capacitance, (b) the power factor, (c) the Q-factor, (d) the power dissipated when a p.d. of 3 V r.m.s. at 2 kHz exists across the capacitor.

[(a) 0·159 μF (b) 0·00025 (c) 4000 (d) 4·5 μW]

(CGLI Principles C, 1969)

Example 1.15 Why is it sometimes convenient to represent the loss resistance of a capacitor as a series resistance whilst at other times it is represented as a parallel resistance?

A capacitor of 0·4 μF with a power factor of $2{\cdot}5 \times 10^{-4}$ and a capacitor of 0·6 μF with a power factor of 5×10^{-4} are connected:

(i) in parallel
(ii) in series

Calculate the capacitance and power factor of each combination.

[(i) 1·0 μF; 4×10^{-4} (ii) 0·24 μF; $3{\cdot}5 \times 10^{-4}$]

(CGLI Principles C, 1971) (C)

2 Electromagnetism and the transformer

ELECTROMAGNETISM

(B) When an electric current flows in a conductor, magnetic forces are produced in the area around the conductor, a magnetic field being established because of the current in the conductor. The direction in which the magnetic field acts at any point may be defined as the direction in which a small isolated north pole, imagined to be removed from a permanent magnet, would tend to move if placed in the field at that point. A line of force is then the imaginary line that would be followed by the pole if free to move under the action of the force due to the magnetic field. An isolated north pole cannot, of course, have any physical existence, unlike an isolated positive charge of electricity, and the concept of the direction of a magnetic field being defined in terms of the force on an isolated north pole is simply to emphasise the similarities between the electric field discussed in chapter 1, and the magnetic field now being dealt with. More practically, magnetic fields may be mapped out using iron filings, which do not, however, give the direction in which the magnetic field is defined as acting. If, though, a small compass needle, which may be regarded as an iron filing capable of rotating about a pivot, is placed in the field, the direction in which the needle points is defined as the direction in which the field acts. This direction is dependent upon the direction of the current in the conductor. The use of iron filings to map out the field round a current-carrying conductor shows that magnetic flux lines surrounding the conductor take the form of concentric circles, and that the intensity of the magnetic field decreases as the distance from the conductor increases. This effect is shown in figure 2.1, where the small circle and cross represent the cross-section of a long straight conductor, the cross indicating that the direction of current flow is away from the observer. The direction in which the lines of force act, as indicated by a small compass needle, is also shown, and is seen to be clockwise.

The direction in which the field acts may be determined using Maxwell's screw rule. If a right-handed screw is turned so as to travel along the conductor in the direction of the current, the corresponding turning movement gives the

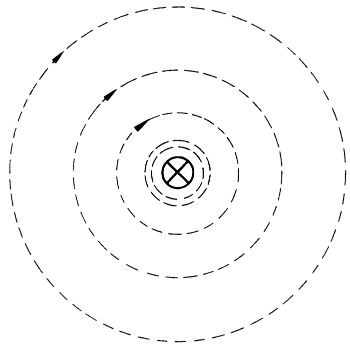


figure 2.1

direction of the lines of force. The lines of force are continuous, thus forming closed loops, and link with the current which is responsible for their production, the current-carrying conductor itself being part of a closed loop. This effect of magnetic lines of force and current linking with each other is illustrated in figure 2.2.

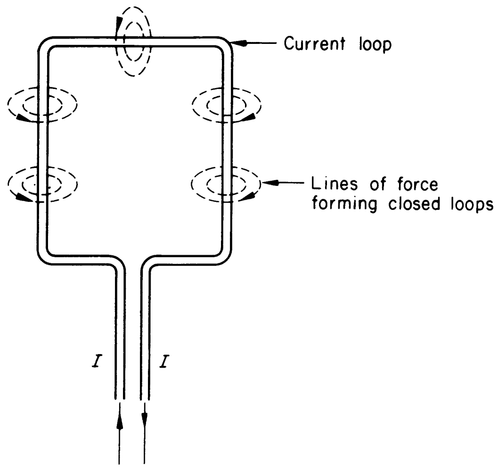


figure 2.2

The strength of the magnetic field produced is dependent upon the current I amperes flowing in the loop. Suppose, now, the loop to consist of N turns of wire. The current embraced by the magnetic field, termed a magnetic flux, is then IN amperes, where I is the current in the coil and is obviously the same in each turn. This product of current and turns, or the ampere-turns of the coil, is responsible for the production of the magnetic flux, and is termed the *magneto-motive force*, or the MMF, of the coil. The unit of magneto-motive force is thus the ampere-turn (symbol At).

Force between two long parallel conductors

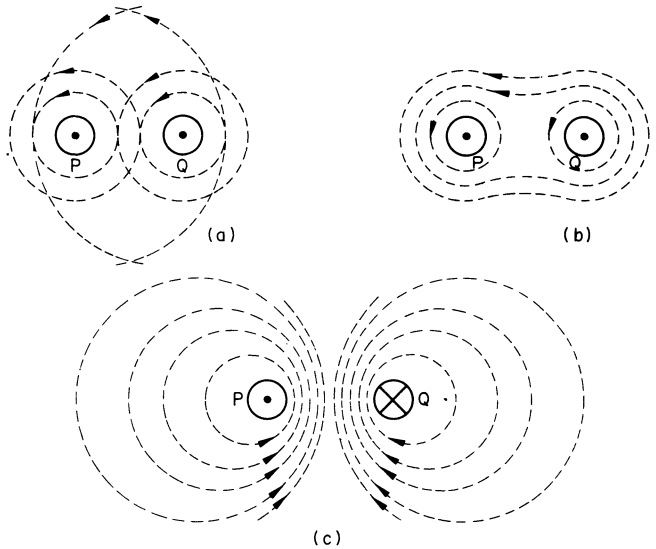


figure 2.3

Let P and Q in figure 2.3a be the cross-section of two parallel conductors carrying equal currents towards an observer, as indicated by the dots within the conductor. Each conductor tends to produce its own magnetic field shown as concentric lines of force. Between the conductors the two fields tend to neutralise each other, but in the space to the left of P and the right of Q the two fields assist each other. The resultant field distribution is shown in figure 2.3b. Since the lines of force are in tension, tending to follow the shortest possible path, the two conductors tend to move towards each other, resulting in a force of attraction between them.

If the two conductors carry currents in opposite directions, as shown in figure 2.3c, the two fields assist each other in the space between the conductors, resulting in a lateral pressure between the lines of magnetic force. The circular lines of force are no longer symmetrical about each conductor, but are displaced as shown, resulting in a strengthened field between the conductors, and a weakened field to the left of P and the right of Q. The lateral repulsive pressure between the lines of force in the space between the two conductors leads to a force of repulsion between them.

Definition of the ampere

The lateral force between two long parallel current carrying conductors is directly proportional to the product of the currents in the two conductors. The unit of current, the *ampere*, is defined as *that current which, flowing in* *two infinitely long conductors, situated* in vacuo *and separated 1 metre between centres, causes each conductor to have a force acting upon it of* 2×10^{-7} *newtons per metre length of conductor.*

Force on a current-carrying conductor in a magnetic field

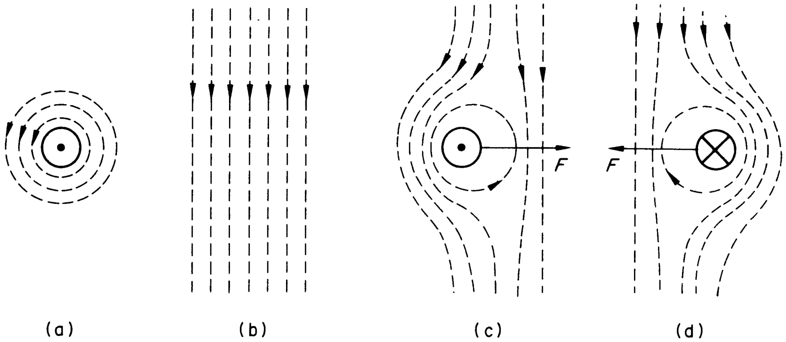


figure 2.4

In figure 2.4, the cross-section of a conductor carrying current towards an observer is shown at (a), together with its magnetic field. Also shown at (b) is another magnetic field considered to be uniform, and acting vertically downwards.

If the conductor is placed in the field, it is seen that on the left side the field produced by the current is in the same direction and therefore strengthens the main field, while on the right side it opposes and therefore weakens the main field. The resultant field produced is shown at (c). Since the lines of force are under tension and try to shorten, there is a force on the conductor tending to move it to the right, or to the weakest part of the field. Reversing the direction of the current reverses the direction of the resultant force, as shown at (d). The conductor, the direction of the main field and the direction of the force are mutually at right-angles. The magnitude of the force is proportional to the length of the conductor, the current in the conductor, and the density of the magnetic field.

Let F = force on the conductor, in newtons
I = current in amperes
l = length of conductor in metres
B = density of the magnetic field, that is, the flux density.

Then

$$F \propto BIl$$

The unit of flux density is such that a conductor carrying 1 ampere at right-angles to a uniform field has a force of 1 newton per metre acting upon it. This unit is termed the *tesla* (symbol T). Hence, in a field of density B teslas,

$$\text{Force on conductor} = BIl \text{ newtons} \tag{2.1}$$

For a uniform magnetic field of density B teslas having a cross-sectional area a square metres, the total flux Φ is measured in *webers* (symbol Wb), and is given by

$$\Phi\,(\text{Wb}) = B\,(\text{T}) \times a\,(\text{m}^2)$$

from which

$$B = \frac{\Phi}{a} \tag{2.2}$$

The magnetic circuit

A group of lines of magnetic flux which follow a similar path and form a closed loop is termed a magnetic circuit. In general, the magnetic field produced by a current carrying coil consists of an indefinite number of magnetic circuits in parallel, the MMF of the coil producing the magnetic flux in each circuit in a manner analogous to an electro-motive force producing currents in a number of parallel electric circuits. Consider for example the magnetic field produced by a coil of wire wound as a solenoid, as in figure 2.5.

The magnetic field produced is similar to that of a bar magnet. The group of flux lines represented by the dotted line A forms a magnetic circuit, as do the group of flux lines represented by the dotted line B. Flux is established in each of these circuits by the MMF of the coil, and these circuits are in

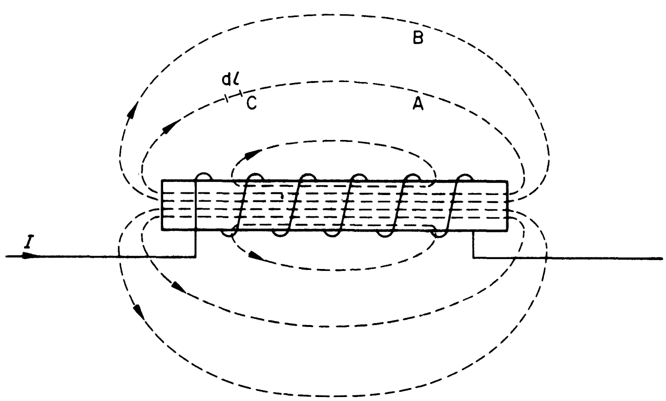


figure 2.5

parallel. The whole field can be divided up into an indefinite number of such parallel circuits. To maintain the flux at any point in a magnetic circuit, say point C on flux line A, a 'magnetic potential gradient' is required, this being referred to as the ***magnetising force*** or ***magnetic field strength*** at the point. The magnetising force H is given by the ampere-turns per metre required to maintain the flux at any point. The flux density around the magnetic circuit represented by dotted line A is not constant. The ampere-turns required to maintain the flux over an elemental length dl is given by $H\,dl$. The total number of ampere-turns is the summation of the quantity $H\,dl$ for all the elemental lengths round the magnetic circuit. This is equal to those ampere-turns of the coil which are completely linked by this magnetic circuit.

Permeability of free space

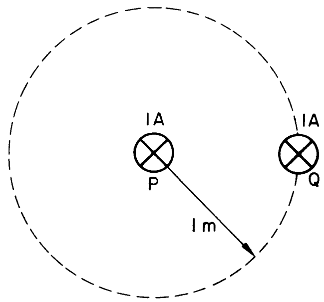


figure 2.6

Figure 2.6 shows the cross-section of a long conductor P situated *in vacuo* and carrying a current of 1 A, the return conductor being situated at a distance large enough for the current in it to have negligible effect on the magnetic field in the vicinity of P. The lines of force around P thus take the form of concentric circles. Consider one of these lines of force as shown, at a radius of 1 m from the centre of the conductor, representing a group of flux lines of density B teslas, which form a magnetic circuit of length 2π metres. The MMF producing the flux in this magnetic circuit is 1 At, and the magnetising force at all points on this circuit, that is the magnetising force at a radius of 1 m from the centre of conductor P, is thus $1/2\pi$ ampere-turns per metre.

Now the flux density B *in vacuo* is directly proportional to the magnetising force H, that is

$$B \propto H$$

or

$$B = \mu_0 H$$

where μ_0 is a constant termed the *permeability of free space.*

Thus, for the case considered,

$$B = \mu_0 \times \frac{1}{2\pi} \text{ teslas}$$

Assume now a second long conductor Q as shown, also carrying 1 A, parallel to P and 1 m from it. The force on this conductor, per metre length, is, from equation 2.1, given by

$$F = B\,(\mathrm{T}) \times 1\,(\mathrm{A}) \times 1\,(\mathrm{m})$$

$$= \mu_0 \times \frac{1}{2\pi} \text{ newtons}$$

But from the definition of the ampere, the force per metre on Q is 2×10^{-7} newtons.

$$\therefore \qquad \mu_0 \times \frac{1}{2\pi} = 2 \times 10^{-7}$$

and

$$\mu_0 = 4\pi \times 10^{-7}$$

The ratio $B/H = \mu_0$ will be seen later to have the units Henrys per metre (H/m), thus

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m} \tag{2.3}$$

If the conductors considered above are situated in air or other non-magnetic material instead of being *in vacuo*, the ratio B/H is almost exactly the same. Thus, for non-magnetic materials,

$$\frac{B}{H} = \mu_0 = 4\pi \times 10^{-7}$$

and

$$B = \mu_0 H \tag{2.4}$$

The force between two long parallel conductors d metres between centres in air may now be determined. Figure 2.7 shows two such conductors P and Q, carrying currents I_1 and I_2 respectively.

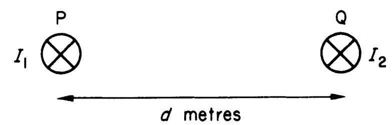


figure 2.7

Magnetising force H at Q due to I_1 in P $= \dfrac{I_1}{2\pi d}$ At/m.

$\therefore$ Flux density B at Q $= \dfrac{\mu_0 I_1}{2\pi d}$ T

$\therefore$ Force on Q (from $F = BIl$) per metre length

$$= \frac{\mu_0 I_1}{2\pi d} I_2$$

$$= \frac{4\pi \times 10^{-7} \times I_1 I_2}{2\pi d}$$

$$= 2 \times 10^{-7} \frac{I_1 I_2}{d} \text{ newtons} \qquad (2.5)$$

Similarly, force on P per metre length

$$= 2 \times 10^{-7} \frac{I_1 I_2}{d} \text{ newtons}$$

Example 2.1 Calculate the magnetising force and the flux density at a distance of 5 cm in air from the centre of a long straight conductor which carries a current of 1000 A.

Two long parallel conductors situated 5 cm between centres in air each carry a current of 1000 A. Calculate the force on each conductor.

$$\text{Magnetising force } H = \frac{I}{2\pi d} = \frac{1000 \text{ (A)}}{2\pi \times 0{\cdot}05 \text{ (m)}}$$

$$= \frac{10\,000}{\pi} \text{ At/m}$$

$$\text{Flux density } B = \mu_0 H = 4\pi \times 10^{-7} \times \frac{10\,000}{\pi}$$

$$= 0{\cdot}004 \text{ T}$$

Force on each conductor per metre length

$$= 2 \times 10^{-7} \frac{I_1 I_2}{d}$$

$$= 2 \times 10^{-7} \frac{1000 \times 1000}{0{\cdot}05}$$

$$= 4 \text{ N}$$

The toroid

In most engineering apparatus, the magnetic flux flows largely through iron, and follows certain definite paths which are easily measurable. The simplest form of magnetic circuit consists of a ring carrying a uniformly wound magnetising winding (a toroid) as shown in figure 2.8.

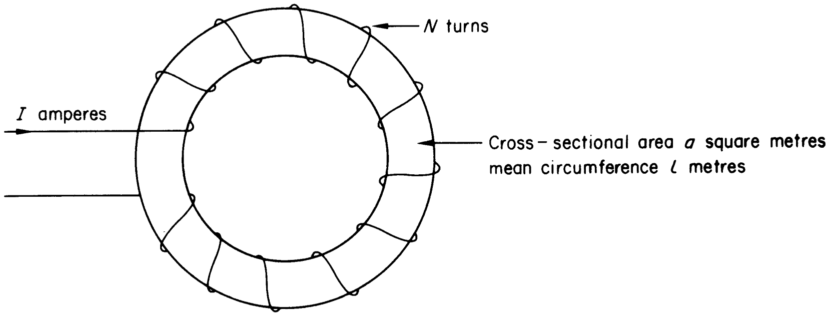


figure 2.8

As the coil is uniformly wound, the MMF is distributed evenly round the magnetic path, and the magnetic flux is confined wholly to the ring. If the mean diameter of the ring is large compared with its radial thickness, the flux may be regarded as evenly distributed throughout its sectional area. At all points within the ring, the magnetising force is given by

$$H = \frac{IN}{l} \text{ At/m}$$

Consider, first, the ring to be made of non-magnetic material. The flux density B within the ring is then given by $B = \mu_0 H$.

Relative permeability

If the non-magnetic material is now replaced by a ferro-magnetic material, the flux produced by the same MMF is increased by a factor that depends upon the material used. This factor is termed the *relative permeability* (symbol μ_r) of the material. With certain nickel-iron alloys, μ_r may be as high as 200 000, though for most magnetic materials in common use the values of μ_r are very much smaller.

The flux density in the ring is thus given by

$$B = \mu_r \mu_0 H \tag{2.6}$$

The product $\mu_r \mu_0$ is termed the *absolute permeability.*

The calculations involved in a magnetic circuit are similar to those involved in an electric circuit. Figure 2.9a shows a magnetic circuit consisting of an iron of relative permeability μ_r, wound with a coil producing an MMF of IN ampere-turns. Figure 2.9b shows an electric circuit consisting of a length of resistance wire of resistivity ρ ohm-metres connected to a battery of EMF E volts.

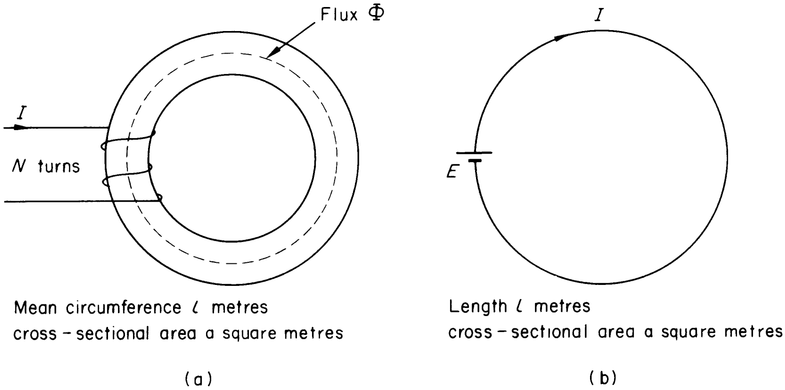


figure 2.9

For the electric circuit

$$\text{Current } I = \frac{E}{R} \qquad \text{or} \qquad E = I \times R$$

where R is the resistance of the circuit and is given by $R = \rho l/a$ ohms.

For the magnetic circuit

$$\begin{aligned} \text{Flux } \Phi &= Ba \\ &= \mu_r \mu_0 H a \\ &= \mu_r \mu_0 a \frac{IN}{l} \\ &= \frac{IN}{l/\mu_r \mu_0 a} \end{aligned}$$

$$\therefore \qquad \Phi = \frac{\text{MMF}}{S} \qquad \text{or} \qquad \text{MMF} = \Phi \times S \tag{2.7}$$

where S is the *reluctance* of the circuit, and is given by

$$S = \frac{l}{\mu_r \mu_0 a} \tag{2.8}$$

For non-magnetic materials $\mu_r = 1$ and thus $S = l/\mu_0 a$.

Composite magnetic circuits

Suppose now that the iron is not continuous, but consists of a number of separate pieces of lengths and cross-sectional areas l_1 and a_1, l_2 and a_2, l_3 and a_3, etc., butting up against each other to form a magnetic circuit of length Σl. The magnetic circuit is then analogous to an electric circuit consisting of a number of resistances in series, and is treated in a similar manner. The reluctance of each piece of iron is calculated separately as $S = l/\mu_r\mu_0 a$, and the ampere-turns required for each piece is separately obtained as the product of this reluctance and the flux passing through the piece. The total MMF for the complete magnetic circuit is then given by

$$\text{MMF} = \Phi_1 S_1 + \Phi_2 S_2 + \Phi_3 S_3 + \ldots \tag{2.9}$$

which reduces to

$$\text{MMF} = \Phi(S_1 + S_2 + S_3 + \ldots) \tag{2.10}$$

for the simple series circuit, where the flux passing through all the pieces is the same.

If there are air gaps in the magnetic circuit, by design or by the pieces of iron making bad contact with each other, there are additional reluctance terms of the form $S = g/\mu_0 a$ where g is the length of an air gap. A given length of air gap requires μ_r times the ampere-turns for the same length of iron, and since μ_r is large, such air gaps are important.

Example 2.2 The electromagnet of figure 2.10 has a U-shaped yoke made of iron of relative permeability 1600; has a uniform cross-sectional area of 5 cm^2, and a mean length of 25 cm. The armature is made of iron of relative permeability 1500; has a cross-sectional area of 4 cm^2, and a mean effective

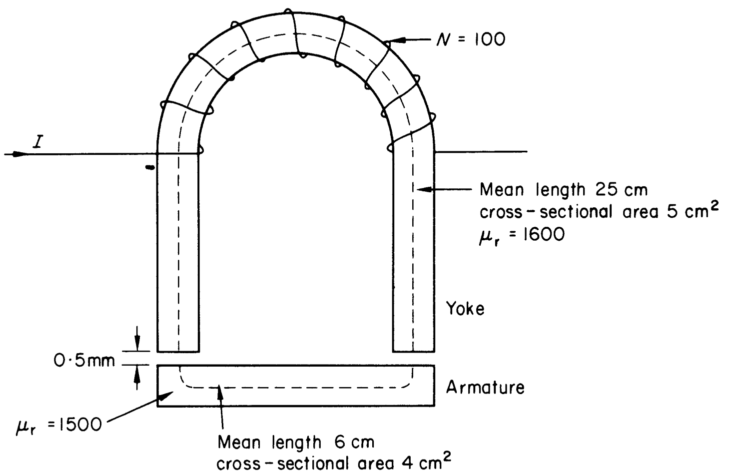


figure 2.10

length of 6 cm. There are air gaps of length 0·5 mm between each pole face and armature and, owing to fringing, each air gap has an effective area of 6 cm^2. A magnetising coil of 100 turns is wound on the yoke. Calculate the current required to produce a flux of 0·5 mWb in each air gap.

$$\text{Reluctance of yoke} = \frac{l}{\mu_r\mu_0 a} = \frac{0{\cdot}25}{1600 \times 4\pi \times 10^{-7} \times 5 \times 10^{-4}}$$

$$= 248\,600$$

Ampere-turns for yoke

$$= \Phi \times S = 0{\cdot}5 \times 10^{-3} \times 248\,600$$

$$= 124{\cdot}3$$

Reluctance of armature

$$= \frac{l}{\mu_r\mu_0 a} = \frac{0{\cdot}06}{1500 \times 4\pi \times 10^{-7} \times 4 \times 10^{-4}}$$

$$= 79\,580$$

Ampere-turns for armature

$$= 0{\cdot}5 \times 10^{-3} \times 79\,580$$

$$= 39{\cdot}8$$

Reluctance of two air gaps in series

$$= \frac{l}{\mu_0 a} = \frac{2 \times 0{\cdot}5 \times 10^{-3}}{4\pi \times 10^{-7} \times 6 \times 10^{-4}}$$

$$= 1\,326\,000$$

Ampere-turns for air gaps

$$= 0{\cdot}5 \times 10^{-3} \times 1\,326\,000$$

$$= 663$$

$$\text{Total MMF} = 663 + 39{\cdot}8 + 124{\cdot}3$$

$$= 827$$

$$\therefore \qquad I = \frac{827}{100} = 8{\cdot}27 \text{ amperes}$$

Example 2.3 An inductor has a core built of stampings of the shape shown in figure 2.11, the coil being on the centre limb. There is a 1 mm air gap in the centre limb which has a cross-sectional area of 4 cm^2. All the other paths in the core have a cross-sectional area of 2 cm^2. The mean path lengths of the magnetic flux in each portion of the core are as shown. If the relative permeability of the iron is 800, find the current needed in a coil of 500 turns to produce a total flux in the air gap of 0·8 mWb. Neglect fringing.

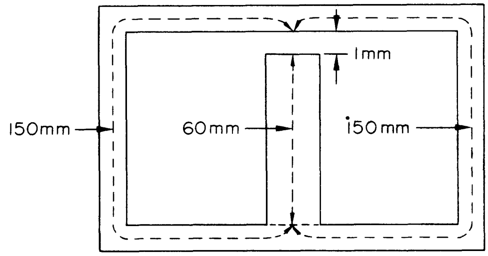
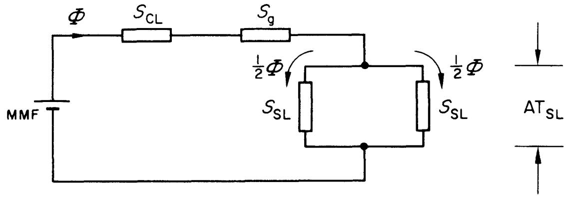


figure 2.11

The two identical side limbs are in parallel, hence half the flux passes through each side limb. The ampere-turns which produce the flux in one side limb also produce the flux in the other side limb, hence only one side limb need be considered when calculating the required ampere-turns. This is clearly seen in the equivalent circuit in an electrical form shown in figure 2.12.



S_{CL} = Reluctance of centre limb
S_{SL} = Reluctance of side limbs
S_g = Reluctance of air gap

figure 2.12

The total MMF required is then given by

$$\text{MMF} = \Phi S_{CL} + \Phi S_g + \tfrac{1}{2}\Phi S_{SL}$$

From the dimensions given:

Reluctance of centre limb

$$S_{CL} = \frac{l}{\mu_r \mu_0 a} = \frac{0{\cdot}06}{800 \times 4\pi \times 10^{-7} \times 4 \times 10^{-4}}$$

$$= \frac{3 \times 10^7}{64\pi}$$

Reluctance of air gap

$$S_g = \frac{l}{\mu_0 a} = \frac{0{\cdot}001}{4\pi \times 10^{-7} \times 4 \times 10^{-4}}$$

$$= \frac{40 \times 10^7}{64\pi}$$

Reluctance of each side limb

$$S_{SL} = \frac{l}{\mu_r \mu_0 a} = \frac{0{\cdot}15}{800 \times 4\pi \times 10^{-7} \times 2 \times 10^{-4}}$$

$$= \frac{15 \times 10^7}{64\pi}$$

$\therefore$ MMF of magnetising coil

$$= 0{\cdot}8 \times 10^{-3} \times \frac{10^7}{64\pi}\left(3 + 40 + \frac{15}{2}\right)$$

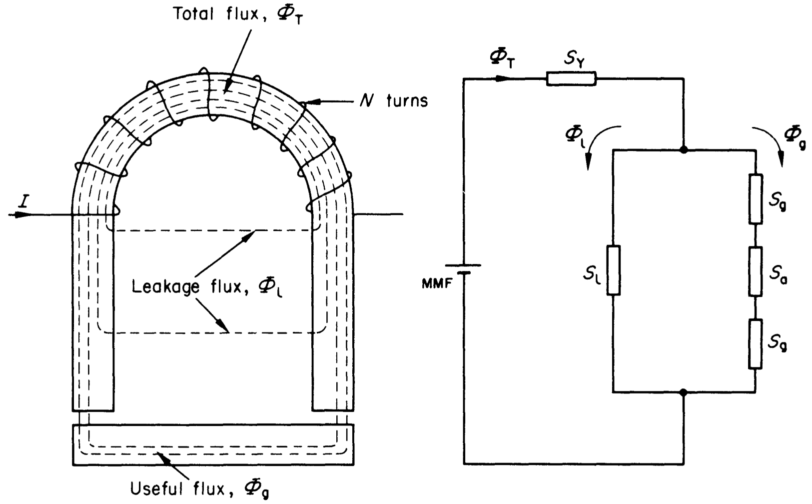
$$= \frac{8000 \times 50{\cdot}5}{64\pi} = 2010 \text{ At}$$

$$\therefore \qquad \text{Current } I = \frac{2010}{500} = 4{\cdot}02 \text{ A}$$

The electric circuit analogy of the magnetic circuit is not as perfect as would appear from the foregoing, though the method used for the calculations is still quite valid. In an electric circuit, the conductivity of the wire is about 10^{12} times that of the insulation surrounding the wire. As a result, the wire is of relatively small cross-sectional area, and the current is confined entirely to the wire, with negligible leakage. In a magnetic circuit, the permeability of the iron is only a few hundred times that of the air or other non-magnetic material surrounding it. As a result, the iron parts of the circuit are of relatively large cross-sectional area; but, even so, considerable leakage of magnetic flux may occur, particularly if the leakage paths available have a large cross-sectional area. Figure 2.13a shows a U-shaped yoke carrying a magnetising winding, and an armature across the poles with air gaps between the pole faces and the armature. In such a magnetic circuit, the object is to produce a flux in these air gaps (that is between yoke and armature) so as to exert as much attractive force on the armature as possible.

Any flux which does not traverse the path between yoke and armature serves no useful purpose and is a leakage flux. The total flux is the sum of the useful flux and the leakage flux. For simple calculations, it is assumed that the total flux Φ_T linked by the magnetising winding is confined to the iron path until it reaches the air gap, where it divides into the gap flux Φ_g and the leakage flux Φ_l. Figure 2.13b shows the equivalent circuit in an electrical form, where it is seen that the leakage path is in parallel with the useful path. The number of ampere-turns $\Phi_g(S_g + S_a + S_g)$ required to establish the useful flux also establishes the leakage flux, hence the ampere-turns $\Phi_l S_l$ required for the leakage path need not be separately calculated. The total MMF required is then given by

$$\text{MMF} = \Phi_T S_Y + \Phi_g(2S_g + S_a) \text{ At}$$



(a) (b)

S_Y = Reluctance of yoke
S_g = Reluctance of air gap
S_a = Reluctance of armature
S_l = Reluctance of leakage path

figure 2.13

The ratio Φ_g/Φ_T or Useful flux/Total flux is termed the *leakage coefficient*, the value of which normally lies between 1·0 and 1·4.

A second reason for the electric circuit analogy not being perfect concerns the relative permeability of the iron parts of the magnetic circuit. In an electric circuit the conductivity of the wire is independent of the current density. But the relative permeability of the iron varies with the flux density. Thus the flux density is not proportional to the magnetising force. Curves of B against H must be obtained experimentally, by measuring a series of values of B for corresponding values of H, to obtain a *normal magnetisation curve.* The shape of the B/H curve depends upon the quality of the magnetic material and upon such details as heat treatment and rolling. Magnetisation curves typical of materials used in electrical engineering are given in figure 2.14a, and figure 2.14b gives the relationship between the relative permeability and the flux density for the same materials.

In practical magnetic circuit calculations it is usual to determine the value of B for each part of the circuit and then to refer to the magnetisation curve for the particular material to obtain the corresponding value of H.

Example 2.4 The electromagnet of figure 2.15 has a U-shaped yoke made of wrought iron; has a uniform cross-sectional area of 5 cm^2, and a mean length of 25 cm. The armature, also of wrought iron, has a cross-sectional

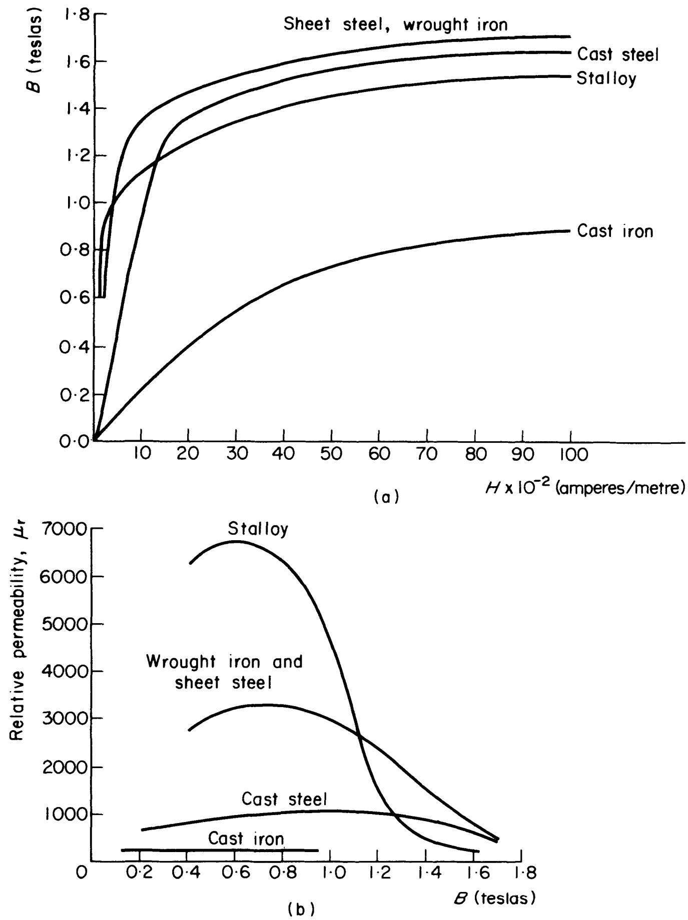


figure 2.14

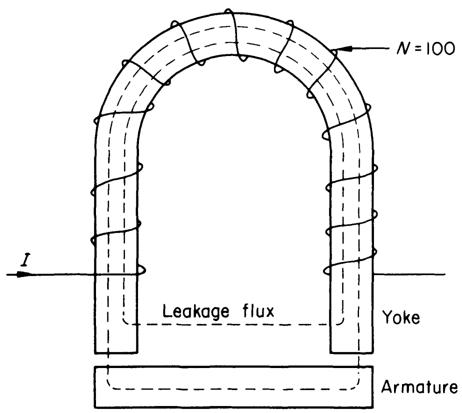


figure 2.15

area of 4 cm^2 and a mean effective length of 6 cm. There are air gaps of length 0·5 mm between each pole face and the armature; and, owing to fringing, each air gap has an effective area of 6 cm^2. A magnetising coil of 100 turns is wound on the yoke. Using the magnetisation curve for wrought iron in figure 2.14a, and assuming a leakage coefficient of 1·2, calculate the current required to produce a flux of 0·5 mWb in each air gap.

Reluctance of air gap

$$= \frac{l}{\mu_0 a}$$

∴ Reluctance of two air gaps in series

$$= \frac{2 \times 0{\cdot}0005}{4\pi \times 10^{-7} \times 6 \times 10^{-4}} = 1\,326\,000$$

∴ Ampere-turns for air gaps

$$= 0{\cdot}5 \times 10^{-3} \times 1\,326\,000 = 663$$

$$\text{Flux in armature} = 0{\cdot}5 \text{ mWb}$$

$$B \text{ in armature} = \frac{0{\cdot}5 \times 10^{-3}}{4 \times 10^{-4}}$$

$$= 1{\cdot}25 \text{ T}$$

From magnetisation curve,

$$H = 650$$

∴ Ampere turns for armature $= H \times l$

$$= 650 \times 0{\cdot}06$$

$$= 39$$

$$\text{Flux in yoke} = 1{\cdot}2 \times 0{\cdot}5$$

$$= 0{\cdot}6 \text{ mWb}$$

$$B \text{ in yoke} = \frac{0{\cdot}6 \times 10^{-3}}{5 \times 10^{-4}}$$

$$= 1{\cdot}2 \text{ T}$$

From magnetisation curve,

$$H = 550$$

∴ Ampere turns for yoke $= H \times l$

$$= 550 \times 0{\cdot}25$$

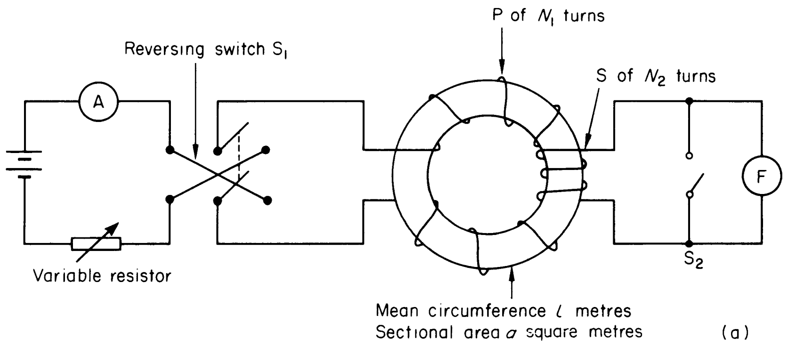
$$= 138$$

$$\text{Total MMF} = 663 + 39 + 138$$
$$= 840 \text{ At}$$
$$I = \frac{840}{100} = 8{\cdot}4 \text{ A}$$

Experimental determination of the normal magnetisation curve

The normal B/H curve for a sample of iron may be obtained using the apparatus shown in figure 2.16a. The sample takes the form of a closed ring that has a mean diameter which is large compared to its radial thickness, and that carries a uniformly wound magnetising winding P of N_1 turns (so that magnetic leakage is negligible). A secondary winding S of N_2 turns is connected to the terminals of a flux-meter, which is an instrument calibrated to read the *change* in flux linkage of the coil to which it is connected. Thus, if the flux linking coil S changes from Φ_1 to Φ_2 webers, the change in flux linkages is $N_2(\Phi_1 - \Phi_2)$ weber-turns, and the fluxmeter deflection indicates that value.

Switch S_2 is first closed, and the sample is then demagnetised by adjusting the primary current to give a value of H corresponding to OM in figure 2.16b (where the sample is magnetically saturated) and then reducing this current



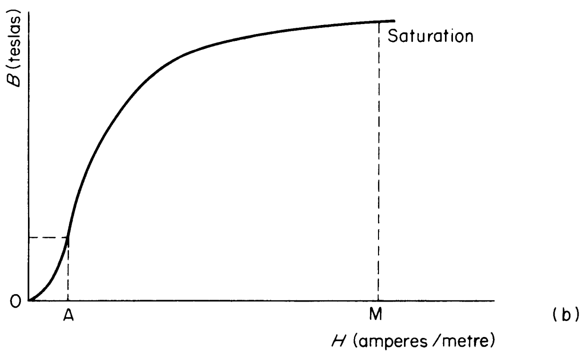


figure 2.16

to zero, while rapidly reversing S_1 alternately to right and left. The current is then adjusted to give a value of $H_1 = IN/l$ (corresponding to OA) and S_1 is reversed several times to bring the sample to a steady magnetic state. S_2 is then opened, and S_1 is reversed, the deflection of the fluxmeter being noted. Let Φ_1 be the flux in the ring corresponding to H_1. Then, since H_1 is reversed, change in flux $= 2\Phi_1$.

Change in flux linkages

$$= 2\Phi_1 N_2 = \text{fluxmeter deflection}$$

$$\therefore \qquad \Phi_1 = \frac{\text{fluxmeter deflection}}{2N_2}$$

and

$$B_1 = \frac{\Phi_1}{a}$$

Corresponding values of H and B are thus obtained. The test is successively repeated at ascending values of H until the point of magnetic saturation is reached. From the data obtained, the normal magnetisation curve can then be plotted.

Electromagnetic induction

Whenever there is a *change* in the amount of magnetic flux linking with a coil, there is an electromotive force induced in the coil. This EMF produces a current if the coil forms part of a closed electrical circuit. It is important to note that the EMF is induced only when the magnetic flux linking the coil is *changing*. The changing flux can be produced either statically or dynamically. For the *statically* induced EMF, figure 2.17 shows two coils A and B wound on an iron ring which forms a magnetic circuit.

A current I flowing in coil A produces a magnetic flux Φ which links with coil B. Let coil B have N turns and suppose that a high resistance voltmeter V

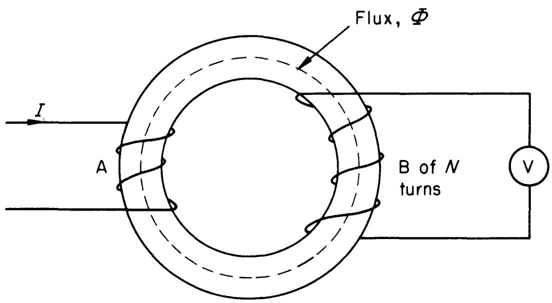


figure 2.17

is connected across the two ends of this coil to measure the induced EMF. Now suppose the current I in coil A to be reduced in such a way that the flux Φ linking coil B is reduced at a constant rate. It will be observed that while the flux is changing the voltmeter indicates a constant value of induced EMF, but that when the flux is constant the induced EMF is zero. By varying the rate at which the current in coil A is reduced, the rate of change of the flux Φ linking coil B may be varied, and it will be found that the value of the induced EMF is directly proportional to the rate of change of the flux. Finally, suppose the number of turns N of coil B to be varied. It is found that the induced EMF in coil B for a given rate of change of flux is directly proportional to the number of turns. The product of the number of turns on the coil and the flux linking the coil is termed the flux-linkages ΦN. The results obtained above are summarised in *Faraday's law of electromagnetic induction,* which states that 'the EMF induced in a circuit by a linked magnetic field is proportic nal to the rate of change of flux-linkages with time'. The induced EMF is thus given by

$$e \propto \frac{\mathrm{d}(N\Phi)}{\mathrm{d}t}$$

In engineering calculations, N is usually constant, and the induced EMF is then given by

$$e \propto N\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

This expression leads to an alternative definition of the unit of magnetic flux, the weber: an EMF of 1 volt is induced in a coil of 1 turn when the flux linking the coil is changing at the rate of 1 weber per second. Hence

$$e \text{ (volts)} = N \text{ (turns)} \times \frac{\mathrm{d}\Phi}{\mathrm{d}t} \text{ (webers/second)}$$

If the experiment just described is now repeated, but increasing the current in coil A to cause the flux linking coil B to increase instead of decrease, it will be found that the induced EMF is in the opposite direction to that formerly obtained. *Lenz's law* states that 'the direction of an induced EMF is such that the EMF tends to oppose the change that produces it'. Again consider an iron ring wound with two coils A and B, as shown in figure 2.18.

The current in coil A produces a flux Φ in the ring. By applying Maxwell's screw rule, Φ is seen to act anti-clockwise round the ring. Suppose this flux to be decreasing; then, by Lenz's law, the induced EMF in coil B must be such that it tends to prevent such decrease. If the circuit of coil B is closed through resistance R, a current flows in coil B, thereby producing a flux that tends to prevent the flux in the ring from decreasing. The flux produced by coil B must therefore act anti-clockwise round the ring; and, by applying

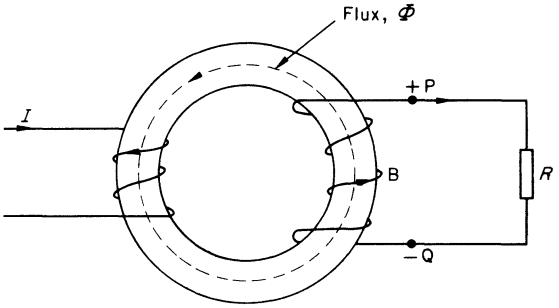


figure 2.18

the screw rule to coil B, it will be apparent that the current in coil B is as shown, and that terminal P at the top of coil B will be of positive polarity.

Faraday's law and Lenz's law may be combined into a single equation for the induced EMF

$$e = -N\frac{d\Phi}{dt} \tag{2.11}$$

the negative sign indicating the direction of the induced EMF as specified by Lenz's law.

Statically induced EMFs by mutual induction as just described form the basis of operation of the *transformer.*

Referring again to figure 2.18, it can be seen that the flux produced by the current in coil A must also link with coil A. Thus, the changing current in coil A also causes an EMF to be induced in coil A. This is known as the *EMF of self induction* and its value is given by $e = N(d\Phi/dt)$ (N now being the number of turns on coil A). The direction of this induced EMF is such that the EMF tends to oppose the change in current; thus, if the current is decreasing, the EMF of self-induction is in the *same* direction as the current flow, thereby tending to maintain the current.

Example 2.5 A magnetic flux of 0·4 mWb completely links with a coil of 600 turns. If the flux is reversed in 0·1 s, determine the average value of the EMF induced in the coil.

The flux is first reduced to zero and then increased to its full value in the reverse direction, giving a total change of flux of 0·8 mWb. This change occurs in 0·1 s, giving an average rate of change of

$$\frac{0{\cdot}8}{0{\cdot}1} = 8 \text{ mWb/s} = 0{\cdot}008 \text{ Wb/s}$$

$$\therefore \quad \text{Induced EMF} = N\frac{d\Phi}{dt} = 600 \times 0{\cdot}008$$

$$= 4{\cdot}8 \text{ V}$$

For the *dynamically* induced EMF, the EMF is induced in a conductor because of that conductor's motion in a magnetic field.

In figure 2.19 a magnetic field, considered to be uniform and acting vertically downwards, is shown at (a), (b) and (c). Also shown at (a) is the cross-section of a straight conductor moving at uniform velocity downwards and thus parallel to the lines of force of the magnetic field. The conductor does not cut the lines of force, and a voltmeter connected across its ends

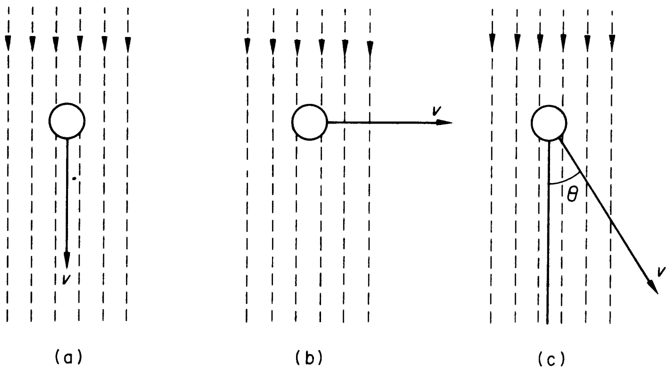


figure 2.19

would show that no EMF is induced in it. At (b) the cross-section of the conductor is shown moving at uniform velocity through the field and in a direction perpendicular to its own length and to the lines of force. The conductor now cuts the lines of force, and a voltmeter connected across its ends would show that an EMF is induced in it. By varying the active length l of the conductor in the field, the flux density B of the field, and the velocity v of the conductor in the field, it has been established that the induced EMF e is directly proportional to each of these quantities. That is,

$$e \propto Blv$$

Again, this expression leads to an alternative definition of the unit of magnetic flux density, the tesla: a magnetic field has a flux density of 1 T when an EMF of 1 V is induced in a straight conductor 1 m long moving at a velocity of 1 m/s in a direction perpendicular to the lines of force of the field. Hence

$$e \text{ (volts)} = B \text{ (teslas)} \times l \text{ (metres)} \times v \text{ (metres/second)}$$

or

$$e = Blv \qquad (2.12)$$

If the conductor is moving at an angle θ to the direction of the field, as in figure 2.19c, the magnitude of the induced EMF is proportional to that component of velocity which is perpendicular to the direction of the field.

Hence

$$e = Blv \sin\theta \tag{2.13}$$

The direction of the induced EMF is obtained using Lenz's law. If the conductor forms part of a closed circuit, a current flows in it in such a direction as to produce an opposing force to that causing motion of the conductor. The field produced by the current causes a strengthening of the resultant field ahead of the conductor and a weakening of the field behind the conductor, thus producing the opposing force as shown in figure 2.20.

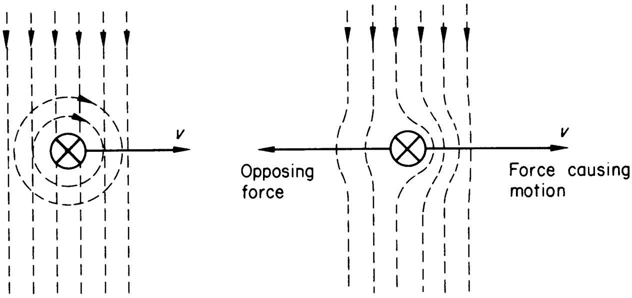


figure 2.20

The direction of the induced EMF may also be determined using Fleming's Right Hand Rule. The thumb, first finger, and second finger of the right hand are held mutually at right angles. The thumb is pointed in the direction of motion of the conductor relative to the field, and the first finger in the direction of the magnetic field. The second finger then points in the direction of the induced current (for example, in figure 2.20, the induced current flows into the paper; that is, perpendicular to the plane of the paper).

The two views of an induced EMF outlined above, in terms of a change of flux-linkages and in terms of flux-cutting, are of course mutually compatible. Let a conductor of active length l metres be situated at P in figure 2.21a, at right-angles to a uniform magnetic field of B teslas, and let the conductor be joined by straight wires to a resistance R outside the field as shown in the sectional view in figure 2.21b. Assume the conductor to be moved from P to Q, a distance of d metres, at uniform velocity v metres/second.

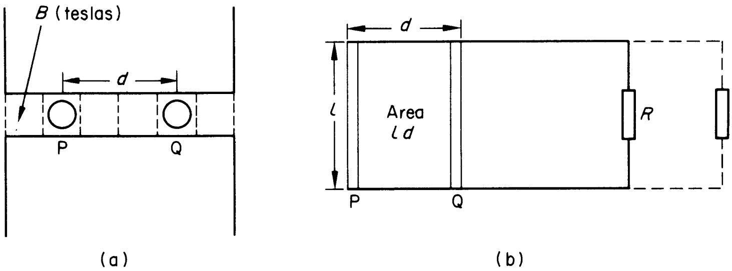


figure 2.21

The change of flux linking the circuit is given by B (T) × ld (m^2) = Bld webers. The time taken for this change is d/v seconds. The number of turns N in the circuit is 1.

$$\therefore \quad \text{Induced EMF } e = N\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

$$e = 1 \times \frac{Bld}{d/v} = Blv \text{ volts}$$

This is the value of the EMF obtained in terms of flux cutting. Figure 2.21 also illustrates the principle of the generator, which is used as an energy converting device for converting energy in a mechanical form into energy in an electric form. When a force of F newtons acts upon a conductor, the mechanical energy W_M joules expended on the conductor in moving a distance d metres from P to Q in time t seconds is given by

$$\begin{aligned} W_M \text{ (J)} &= F \text{ (N)} \times d \text{ (m)} \\ &= BIl \times d \\ &= BIl \times vt \\ &= Blv \times It \\ &= EIt \\ &= W_E \text{ joules} \end{aligned}$$

which is the electric energy expended. This energy is in turn dissipated as heat that arises as a result of the resistance of the circuit.

Example 2.6 A conductor of length 40 cm moves at right-angles to its length at 20 m/s in a uniform magnetic field of density 1·2 T. Calculate the EMF induced in the conductor when the direction of motion is (1) perpendicular to the field, (2) inclined at 30° to the direction of the field.

(1) $$\begin{aligned} e &= Blv = 1{\cdot}2 \times 0{\cdot}4 \times 20 \\ &= 9{\cdot}6 \text{ V} \end{aligned}$$

(2) $$\begin{aligned} e &= Blv \sin\theta = 9{\cdot}6 \sin 30^\circ \\ &= 4{\cdot}8 \text{ V} \end{aligned}$$

Hysteresis

If the magnetising force H applied to a specimen of iron that has been completely demagnetised is increased from zero to some maximum value represented by OM in figure 2.22, then the relationship between B and H is represented by curve OA. If H is then gradually reduced to zero, it will be observed that the B/H curve for the decreasing values lies above that for the increasing values, so that when H is again zero, B still has some definite value. The iron has in fact retained some magnetism. The density of the flux remaining in the iron is termed the *remanent flux density,* and it is represented in the figure by OR. This effect of values of B lagging behind those of H is known as *hysteresis.* To reduce B to zero, H must be increased in the reverse direction to a value represented in figure 2.22 by OC. This value of H is termed the *coercive force.*

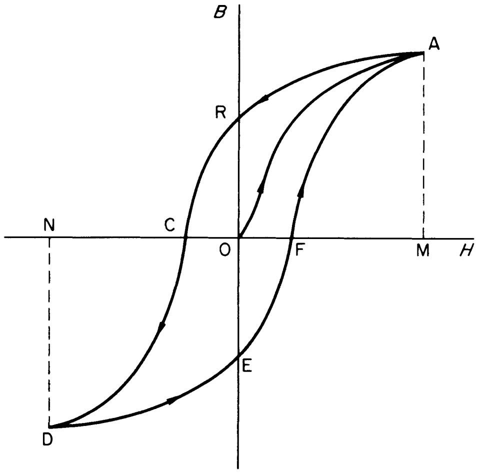


figure 2.22

If H is now increased in the reverse direction to the same value (represented by ON) as in the initial direction, the curve follows the path CD such that the flux density ND is equal in value to that represented by MA in the original direction. If H is again reversed and brought back to the value OM in the original direction, the curve DEFA is obtained, similar to the curve ARCD. The complete closed figure is termed a *hysteresis loop.* To take the iron through the complete series of magnetic states represented by the loop, an alternating magnetising force has been applied, of peak-to-peak value represented by NM. If this same alternating force is maintained, the iron continues to follow the same series of magnetic states. For any given sample of iron, a number of hysteresis loops may be obtained, each for a different maximum value of the magnetising force. These loops lie within one another, as shown in figure 2.23. The apexes of the various loops lie on the normal magnetisation curve, and the largest loop corresponds to magnetic saturation.

The value of the remanent flux density OQ obtained when the material has been taken to saturation is termed the *remanence,* and the corresponding value OP of the coercive force is termed the *coercivity.* Values of coercivity vary greatly for different magnetic materials. A high value of coercivity is desirable

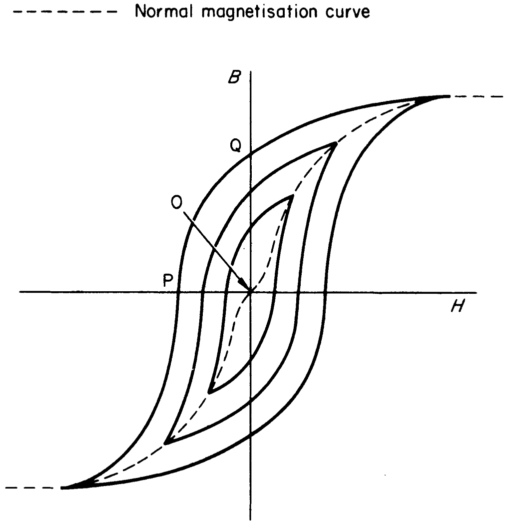


figure 2.23

for permanent magnet materials, a typical hysteresis loop for which is shown in figure 2.24a. This loop, in terms of units of $B \times H$, encloses a large area. For materials subject to alternating magnetising forces (as are used in building a power transformer) a low coercivity is desirable, and a typical loop is shown in figure 2.24b. It will be seen that the hysteresis loop for such materials encloses only a small area. Hysteresis causes an energy loss which is dissipated as heat, and this energy loss is proportional to the area of the hysteresis loop, in terms of units of $B \times H$.

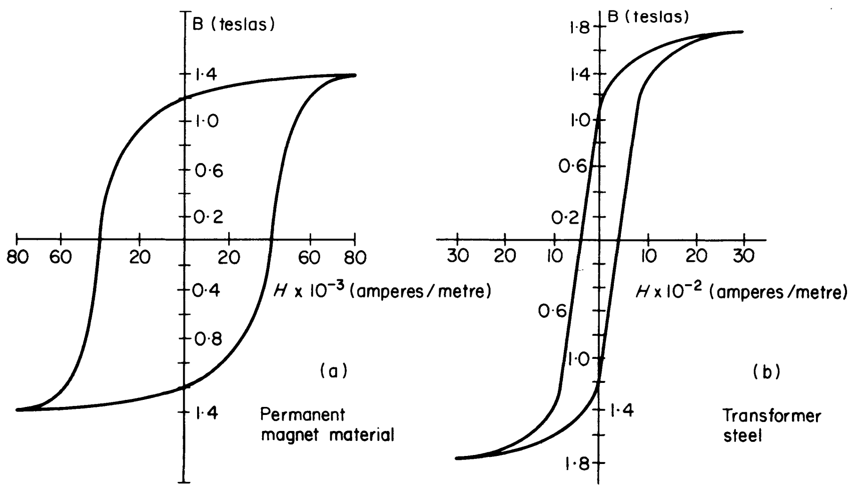


figure 2.24

Magnetism associated with iron, and hysteresis loss, may be explained in terms of the *domain* theory. As the electrons in an atom are always in motion, they constitute in effect an electric current, which is inevitably accompanied by a magnetic field. In a piece of magnetic material, the atoms are arranged in groups containing many atoms, each group occupying a minute volume known as a domain. Each domain constitutes a saturated permanent magnet, and when the piece of iron as a whole is unmagnetised, the domains are arranged in a random manner. On the application of a weak external magnetising force, the domains begin to line up with the field, thus increasing the available flux density. As the magnetising force is increased, the domains come more and more into line, until they eventually reach that complete alignment which corresponds to magnetic saturation of the piece of material.

If the external magnetising force is alternating, energy is expended in realigning the magnetic domains. The realignment tends to lag behind the varying magnetising force, thus producing the hysteresis effect. The energy expended in one complete cycle is proportional to the area of the hysteresis loop, this energy being dissipated as heat. For a given material the power loss due to hysteresis is thus also proportional to the frequency.

Eddy currents

Whenever the intensity of a magnetic field is changing, an EMF is induced in any circuit with which the flux links, and if the circuit is complete, a current will flow. Figure 2.25a shows an alternating flux passing through a solid iron core.

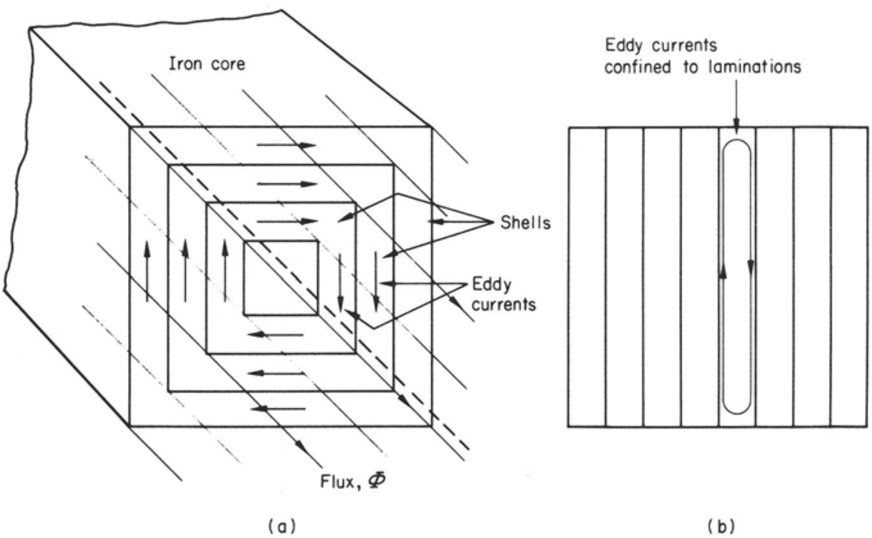


figure 2.25

The core may be imagined to be divided up into concentric shells forming closed circuits, each circuit linking with some proportion of the alternating flux. The resultant induced alternating EMF causes alternating currents, known as eddy currents, to circulate in the shells. Because the current path has resistance, however, an energy loss occurs and this is dissipated as heat. The induced EMF and eddy currents are proportional to the rate of change of the flux, which is proportional to the frequency. As energy loss is proportional to the square of the current, it is thus also proportional to the square of the frequency.

Eddy current loss is minimised by restricting the free path available for current flow. At low frequencies (50 Hz and audio frequencies) the iron core is assembled from thin sheets of iron (called laminations) of good magnetic quality and high resistivity, each sheet being coated with a layer of insulating material (for example, varnish) to increase the electrical resistance between adjacent sheets. The core is designed so that the direction of the alternating flux is along the plane of the laminations. (Figure 2.25b shows the cross-section of such a core, the flux direction being perpendicular to the plane of the paper.) The insulating layers now prevent the flow of current from sheet to sheet; the induced EMF in each sheet is low, because the eddy current path links with only a small flux; and the available current path is of much increased resistance; so that eddy currents are therefore reduced to a minimum.

As the frequency of the alternating flux is increased the thickness of the laminations must be further reduced to ensure that induced EMFs are kept sufficiently low. When it is impracticable to make laminations any thinner, core losses can be minimised by using iron-dust cores. The iron dust is mixed with a binding material and compressed in a mould which gives it a finished appearance comparable with a piece of solid iron. Eddy current paths are now restricted to the size of a dust particle. Such cores are used at radio frequencies.

Inductance

When referring to figure 2.18, it was observed that the changing current in coil A caused an EMF to be induced within that coil, this EMF of self induction being given by

$$e = -N\frac{\mathrm{d}\Phi}{\mathrm{d}t} \text{ volts}$$

This property of a circuit element by which a changing current in that element causes an EMF to be induced within it is known as *self-inductance,* or, more simply, *inductance* (symbol L). Circuit elements are frequently designed to possess inductance, the element then taking the form of a coil wound so as to link with a magnetic circuit. The induced EMF is proportional to the *rate of change* of current, and, in accordance with Lenz's law, is in such a direction as to tend to prevent the change in current. The unit of inductance is the *Henry* (symbol H), which is the inductance of a circuit in which an EMF of 1 V is induced for each *ampere per second* rate of change of current. The induced EMF is proportional to the inductance and to the rate of change of current; thus

$$e\,(\mathrm{V}) = -L\,(\mathrm{H}) \times \frac{\mathrm{d}i}{\mathrm{d}t}\;(\mathrm{A/s})$$

or

$$= -L\frac{\mathrm{d}i}{\mathrm{d}t} \tag{2.14}$$

the negative sign satisfying Lenz's law.

Consider a coil of inductance L henrys connected to a supply V volts as shown in figure 2.26a, and assume the coil to have no resistance. To satisfy Kirchhoff's law for the circuit, the induced EMF must both equal and oppose the applied voltage, that is

$$V = -e$$

$$V = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

and

$$\frac{\mathrm{d}i}{\mathrm{d}t}\,(\mathrm{A/s}) = \frac{V\,(\text{volts})}{L\,(\mathrm{H})}$$

The current in the coil thus increases at a constant rate as shown in figure 2.26b.

It may be noted that, for a purely resistive circuit connected to a d.c. supply, current $I = V/R$, and for a purely inductive connected to a d.c. supply, rate of change of current $\mathrm{d}i/\mathrm{d}t = V/L$.

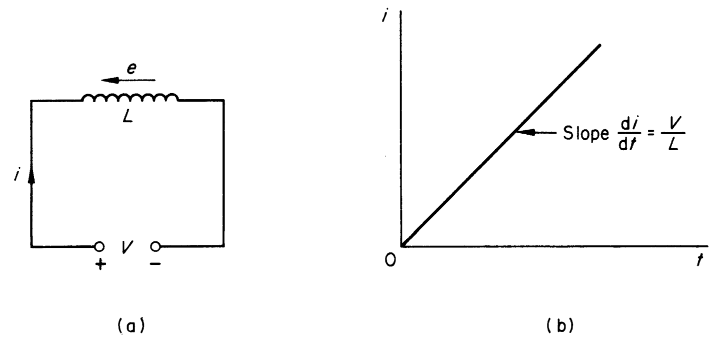


figure 2.26

Equating the two expressions for the induced EMF,

$$L\frac{di}{dt} = N\frac{d\Phi}{dt}$$

$$\therefore \qquad L = N\frac{d\Phi}{di}$$

$$= \frac{\text{Change of flux-linkages (Wb-turns)}}{\text{Change of current (A)}}$$

For a coil with a non-magnetic core, the flux is directly proportional to the current, hence $d\Phi/di$ is constant and the inductance is constant, being given by

$$L\ (\mathrm{H}) = \text{Flux linkages in weber-turns per ampere} \tag{2.15}$$

For a coil in which the magnetic circuit lies wholly or partially in iron, the flux is not proportional to the current, the relationship between the two depending upon the quality of the iron, and upon any air gaps in the magnetic circuit. Air gaps are frequently included in the magnetic circuit of an inductor so as to make the value of the inductance much less dependent upon that of the current. When iron is present $d\Phi/di$ is not constant, hence the inductance is not constant, but varies with the current and with the amount by which the current is caused to change.

Example 2.7 A coil of 200 turns is wound on an iron ring. When the current is increased from 3 A to 3·8 A the flux is increased from 0·3 mWb to 0·36 mWb. Calculate (a) the inductance of the coil over this range. Calculate also the inductance when the current is alternating, having peak values of (b) 3 A and (c) 3·8 A.

(a)
$$L = \frac{\text{Change of flux-linkages}}{\text{Change of current}}$$

$$= \frac{200(0{\cdot}36 - 0{\cdot}3)10^{-3}}{3{\cdot}8 - 3{\cdot}0}$$

$$= \frac{200 \times 0{\cdot}06 \times 10^{-3}}{0{\cdot}8}$$

$$= 0{\cdot}015\ \mathrm{H}$$

(b) When the current is alternating, both current and flux reverse, giving changes of 6 A and 0·6 mWb respectively.

Hence, inductance of coil

$$= \frac{200 \times 0{\cdot}6 \times 10^{-3}}{6}$$

$$= 0{\cdot}02 \text{ H}$$

(c) Changes of current and flux are now 7·6 A and 0·72 mWb respectively. Hence

$$L = \frac{200 \times 0{\cdot}72 \times 10^{-3}}{7{\cdot}6}$$

$$= 0{\cdot}0189 \text{ H}$$

For an iron cored coil, the conditions for which the inductance has been determined must thus be specified.

Inductance of a coil with a core of constant permeability

Let a coil of N turns be uniformly wound over a ring of constant permeability $\mu = \mu_r\mu_0$, the ring having a mean circumference of l metres and a cross-sectional area of a square metres. Let the current in the coil be I amperes.

Then flux in ring $= Ba = \mu_r\mu_0 Ha$ (from equation 2.6)

$$= \mu_r\mu_0 \frac{IN}{l} a$$

$$\text{Flux linkages} = \mu_r\mu_0 \frac{IN^2}{l} a$$

and inductance L = Flux linkages/ampere

$$= \mu_r\mu_0 N^2 \frac{a}{l} \tag{2.16}$$

∴ The inductance of the coil is proportional to (number of turns)^2. This expression enables the units of μ_0 to be determined. Rearranging equation 2.16,

$$\mu_0 = \frac{Ll}{\mu_r N^2 a}$$

$$= \frac{1}{\mu_r N^2} \times L\ (\text{H}) \times \frac{l\ (\text{m})}{a\ (\text{m}^2)}$$

$\dfrac{1}{\mu_r N^2}$ being dimensionless; the units of μ_0 are henrys per metre.

Energy stored in the magnetic field of an inductance

Let the current in a coil having a constant inductance of L henrys increase at a uniform rate from zero to I amperes in t seconds, as shown in figure 2.27.

While the current is growing, the voltage required to neutralise the EMF of self-induction is given by $V = L\,(\mathrm{d}i/\mathrm{d}t) = L\,(I/t)$ volts, and the average value of the current is $I/2$ amperes.

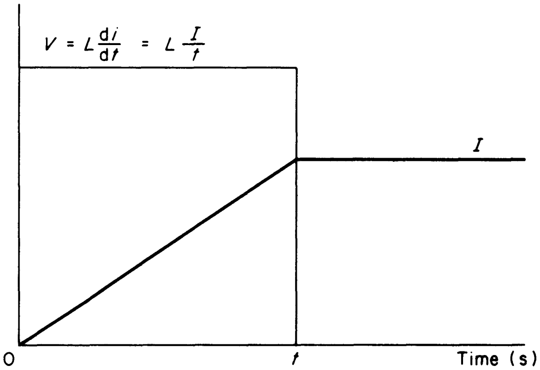


figure 2.27

Average power supplied

$$= V\frac{I}{2}$$

$$= L\frac{I}{t}\frac{I}{2} \text{ watts}$$

$$\text{Energy supplied} = \text{Average power} \times \text{time}$$

$$= L\frac{I^2}{2t}t$$

$$= \tfrac{1}{2}LI^2 \text{ joules} \qquad (2.17)$$

This energy is stored in the magnetic field associated with the inductance.

Example 2.8 A 2 H inductor carrying a direct current of 4 A produces a magnetic flux of 10 mWb. Calculate (a) the number of turns on the coil, (b) the energy stored in the magnetic field.

(a) $$L = \frac{N\Phi}{I}$$

$$\text{No. of turns } N = \frac{LI}{\Phi} = \frac{2 \times 4}{10 \times 10^{-3}} = 800 \text{ turns}$$

(b) Energy stored $= \tfrac{1}{2}LI^2 = \tfrac{1}{2} \times 2 \times 4^2 = 16$ J

Growth and decay of current in an inductive circuit

Consider a coil of inductance L henrys and resistance R ohms to be connected to a d.c. supply of V volts as shown in figure 2.28.

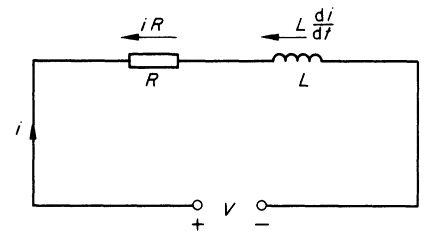


figure 2.28

When first connected to the supply, the current is zero, and must grow to its final steady value $I = V/R$. As the current is growing it sets up an increasing magnetic field thereby causing an EMF of self-inductance that opposes the growth of the current, and is proportional to the rate of growth of the current at the instant considered. The supply voltage may therefore be considered as having two components: one overcoming the resistance and maintaining the current at the value that it has attained, while the other, by overcoming the EMF of self-inductance, causes the current to grow still further. As the current i increases, the component overcoming the resistance increases, until finally the whole of the supply voltage is required to maintain the current, and no further increase can take place. At any time during the growth period, when the current has some value i,

$$V = iR + L\frac{di}{dt} \tag{2.18}$$

Time constant

The growth of the current with time is shown in figure 2.29. When the current is zero at time $t = 0$, its rate of change $di/dt = V/L$. If the current

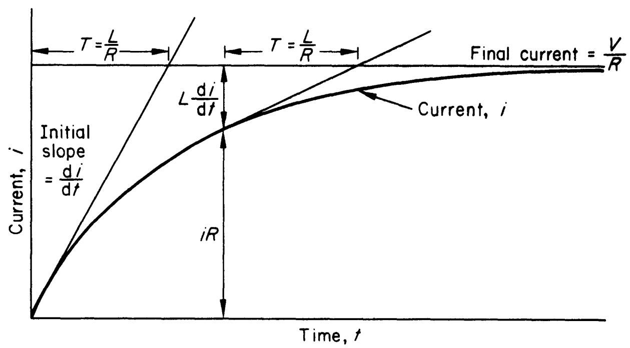


figure 2.29

continued to increase at this initial rate, it would reach its final steady value $I = V/R$ in time T, where $T = (V/R) \div (V/L) = L\,(\mathrm{H})/R\,(\Omega)$ seconds. This time is called the *time constant* of the circuit.

$$\text{Time constant } T = \frac{L}{R} \text{ seconds} \tag{2.19}$$

A line drawn from the origin O to cut the final current line I at a time $T = L/R$ seconds is a tangent to the i/t curve.

Consider next what happens when the current reaches some value i, the p.d. across the resistance then being iR. The component of V overcoming the EMF of self-inductance is

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = V - iR$$

from which

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{V - iR}{L} = \frac{1}{L}(V - iR)$$

The difference between the final current I and the current at the instant considered is

$$I - i = \frac{V}{R} - i = \frac{1}{R}(V - iR).$$

If the rate of increase of current retained the value at this instant, the current would reach its final value in a time given by

$$t = \frac{(1/R)(V - iR)}{(1/L)(V - iR)} = \frac{L}{R} \text{ seconds}$$

This is equal to the time constant T. The time constant T is thus also given by the time taken for the current to reach its final steady value if its rate of increase continued unchanged at the instant considered. A line drawn from the i/t curve to cut the final current I line L/R seconds later is thus also a tangent to the i/t curve.

By drawing lines at the correct slope at a succession of different times, the current curve i/t may be constructed, by following precisely the same procedure as that described for the growth of capacitor voltage in figure 1.11. The curve is shown in figure 2.30.

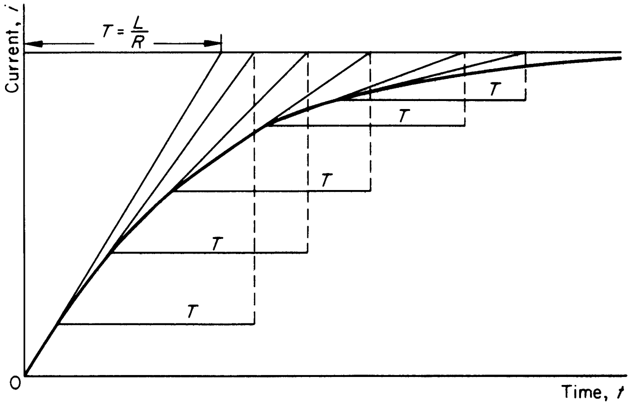


figure 2.30

The expression for the current may be obtained by solving equation 2.18 for the current

$$V = iR + L\frac{di}{dt}$$

$$\frac{di}{dt} = \frac{V - iR}{L}$$

$$dt = \frac{L}{V - iR}\, di$$

$$t = L\int \frac{1}{V - iR}\, di$$

$$= -\frac{L}{R}\ln (V - iR) + \text{a constant}$$

(Note: $\ln = \log_e$)

when $t = 0, i = 0$

$$0 = -\frac{L}{R}\ln V + \text{a constant}$$

and the value of the constant is thus $L/R \ln V$

$$\therefore \qquad t = -\frac{L}{R}\ln(V-iR) + \frac{L}{R}\ln V$$

$$= \frac{L}{R}\ln\left(\frac{V}{V-iR}\right)$$

$$\frac{Rt}{L} = \ln\left(\frac{V}{V-iR}\right)$$

$$\exp\left(\frac{Rt}{L}\right) = \frac{V}{V-iR}$$

$$\exp\left(-\frac{Rt}{L}\right) = \frac{V-iR}{V} = 1 - \frac{iR}{V}$$

$$\frac{iR}{V} = 1 - \exp\left(-\frac{Rt}{L}\right)$$

$$i = \frac{V}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right] \qquad (2.20)$$

The current thus increases exponentially as shown in figure 2.31.

If the initial rate of increase were maintained, the current would reach its final value in a time $T = L/R$ seconds. Substituting this value for t in equation 2.20

$$i = \frac{V}{R}\,(1 - e^{-1}) = 0{\cdot}632\,\frac{V}{R}$$

$$= 0{\cdot}632I \qquad (2.21)$$

In a time equal to the time constant, the current thus rises to 0·632 (63·2 per cent) of its final value.

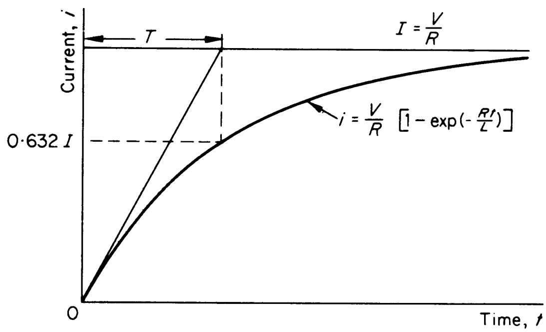


figure 2.31

The energy stored in the magnetic field of a large inductor used in a d.c. circuit may be appreciably large, and if the circuit is broken by a switch, this energy is partially dissipated as heat in an arc produced at the switch contacts. A resistor is frequently connected across the inductor when the supply is switched off, so that the energy is dissipated slowly as the current decays. The decay of the current, when the inductance is constant, is then exponential, the relationship between current and time being given by

$$i = I \exp\left(-\frac{Rt}{L}\right) \tag{2.22}$$

where I is the initial value of the current.

Example 2.9 A 20 H relay coil having a resistance of 150 Ω is connected in series with a 250 Ω resistor across an 80 V battery. The relay operates at 20 mA. Give an expression for the current/time relation in this circuit. Calculate (a) the time constant, (b) the initial current at switch-on, (c) the final current, (d) the time from switching on until the relay operates. Sketch the current/time curve showing these values.

(a) Time constant $T = \dfrac{L}{R} = \dfrac{20}{400} = 0{\cdot}05\ \text{s}$

(b) Initial current at switch-on is zero.

(c) Final current $= \dfrac{V}{R} = \dfrac{80}{400} = 0{\cdot}2\ \text{A}$

The current/time relationship is thus

$$\begin{aligned} i &= \frac{V}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right] \\ &= 0{\cdot}2\left[1 - \exp\left(-\frac{t}{0{\cdot}05}\right)\right] \\ &= 0{\cdot}2\,[1 - \exp(-20t)] \end{aligned}$$

This assumes that the inductance remains constant at 20 H after the relay operates.

(d) The relay operates when

$$\begin{aligned} i &= 0{\cdot}02\ \text{A} \\ 0{\cdot}02 &= 0{\cdot}2\,[1 - \exp(-20t)] \\ \exp(-20t) &= \frac{0{\cdot}2 - 0{\cdot}02}{0{\cdot}2} \end{aligned}$$

$$\exp(20t) = \frac{1}{0{\cdot}9} = 1{\cdot}111$$

$$20t = \ln 1{\cdot}111 = 0{\cdot}105$$

$$t = 0{\cdot}00525 \text{ s}$$

$$= 5{\cdot}25 \text{ ms}$$

Sketch of current/time curve:

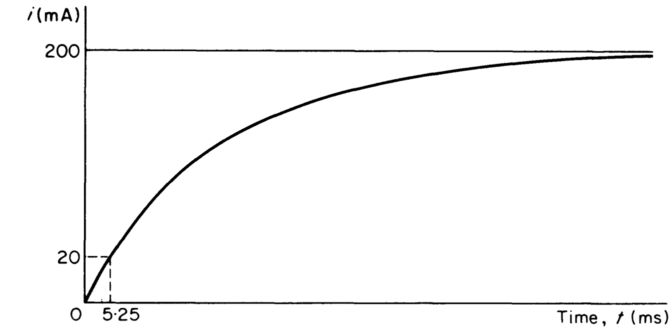


figure 2.32

Example 2.10 A relay coil has a resistance of 180 Ω and an inductance of 6 H. At time $t = 0$, a d.c. supply of 50 V is applied to the coil, and the relay operates at time $t = 5$ ms. Calculate the current then flowing in the coil, and the rate of change of current at the instant before the relay operates. Calculate also the rate of change of current at the instant after the relay operates, if the inductance increases to 12 H. Sketch the current/time curve.

Before the relay operates:

$$\text{Time constant } T = \frac{L}{R} = \frac{6}{180} = \frac{1}{30} \text{ s}$$

The equation relating i and t is

$$i = \frac{50}{180}\,[1 - \exp(-30t)]$$

$$= 0{\cdot}278\,[1 - \exp(-30 \times 0{\cdot}005)]$$

$$= 0{\cdot}278\,[1 - \exp(-0{\cdot}15)]$$

$$= 0{\cdot}278 \times 0{\cdot}1393$$

$$= 0{\cdot}0387 \text{ A}$$

Current when relay operates

$$= 38{\cdot}7\,\text{mA}$$

$$\text{Final current } I = \frac{V}{R} = 278\,\text{mA}$$

Rate of change of current at the instant before relay operates

$$= \frac{278 - 38{\cdot}7}{T} = \frac{239{\cdot}3}{1/30}\ \text{mA/s}$$

$$= 7{\cdot}18\,\text{A/s}$$

After the relay operates

$$T = \frac{12}{180} = \frac{1}{15}\ \text{s}$$

Rate of change of current at the instant after the relay operates

$$= \frac{278 - 38{\cdot}7}{1/15}\ \text{mA/s} = 3{\cdot}59\ \text{A/s}$$

After the relay operates, the equation relating i and t is

$$i = 0{\cdot}278\,[1 - \exp(-15t)]$$

The current/time curve is shown in figure 2.33. The current follows the first exponential curve until the relay operates, and then follows the second exponential curve. While the relay is operating, the sudden change in flux produces an induced EMF which temporarily reduces the current, leading to the transient effect shown. The transient soon dies out, the second exponential curve then being followed. The solution given above ignores the effect of the transient.

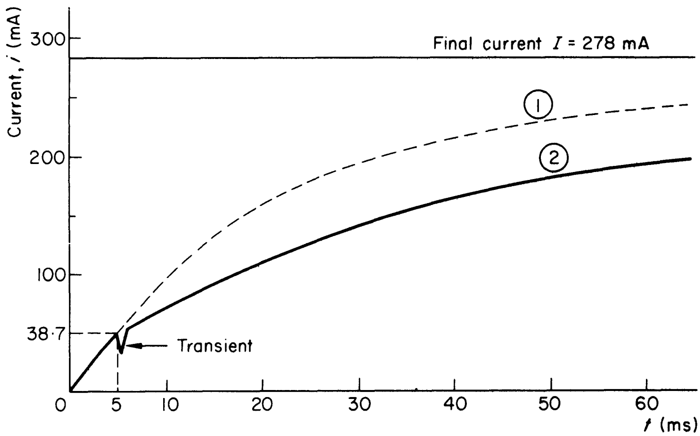


figure 2.33

(B)

Properties of coils

An ideal coil has the property of inductance only, and under a.c. conditions the coil current is then in exact quadrature with the p.d. across the coil. However, practical coils are far from perfect in this respect, and the electric properties vary according to the conditions of use, being particularly affected by current and frequency. All coils have some resistance, and must, therefore, have a power loss in overcoming that resistance. If the coil also has an iron core, then hysteresis and eddy current losses will lead to a further power loss. For convenience, this loss also can be represented as occurring in an equivalent resistance. At any *one* frequency the coil may be represented by either an equivalent series circuit or an equivalent parallel circuit, as shown in figure 2.34a and 2.34b.

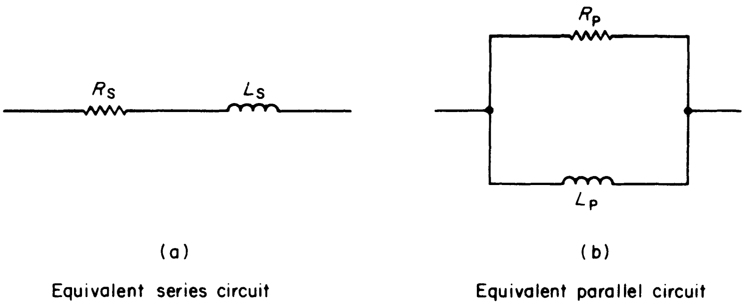


figure 2.34

R_S and R_P are such that the power losses in these equivalent components represent the power loss due to all causes in the actual coil.

In addition to resistance imperfection, the coil may have imperfection due to self-capacitance. Adjacent turns of a coil are separated by the insulation, which is a dielectric, and are at different potentials, so the capacitance effect is distributed throughout the winding, as shown in figure 2.35a. For most purposes, however, it can be represented by a single capacitance connected across the coil terminals as shown in figure 2.35b.

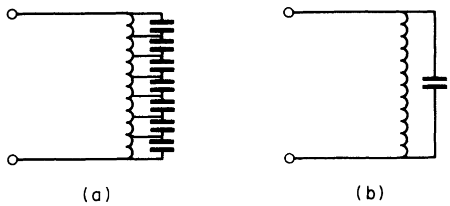


figure 2.35

The complete equivalent circuit for a coil, using the equivalent series representation, thus takes the form shown in figure 2.36a.

Values of C, R_S and L_S at any given frequency may be separately measured. Alternatively, the complete impedance of the circuit as a whole may be measured, as $R_e + j\omega L_e$ as shown in figure 2.36b. R_e and L_e are the effective resistance and inductance of the coil, including the self-capacitance effect.

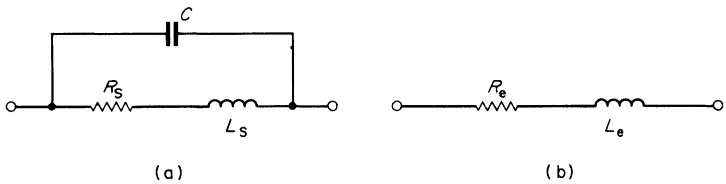


figure 2.36

(C)

[Readers may find it advisable to study the review of basic alternating current theory at the beginning of chapter 3 before proceeding to the next section of this chapter.]

(B) THE TRANSFORMER

The transformer consists essentially of two separate coils linked by a common magnetic circuit. With some small transformers used in electronics the magnetic circuit lies entirely in air, resulting in considerable leakage flux. Power transformers, however, and transformers used at audio frequencies, have magnetic circuits lying in iron arranged so that magnetic leakage is as small as possible. Ideally, the whole of the flux produced by one coil links completely with the other coil. The reluctance of the magnetic circuit is kept as low as possible by the use of high quality iron, thus minimising the ampere-turns required to establish the magnetic flux. Again, ideally, negligible ampere-turns are required for this purpose. Iron losses produced by the alternating magnetic flux are minimised by laminating the core, and by the use of iron with small hysteresis loss. In the ideal transformer these losses are negligible. The theory of the actual transformer assumes an ideal transformer with no power loss, no magnetising current and no voltage drops due to resistance and magnetic leakage, and external circuit elements are then added to the circuit of the ideal transformer to represent departures of the actual transformer from the ideal.

The usefulness of the transformer lies in the fact that it enables energy to be transferred from one circuit to another without any conductive connection, and that during the transfer the energy can be changed from one voltage level to another.

Voltage and current relationships for the ideal transformer

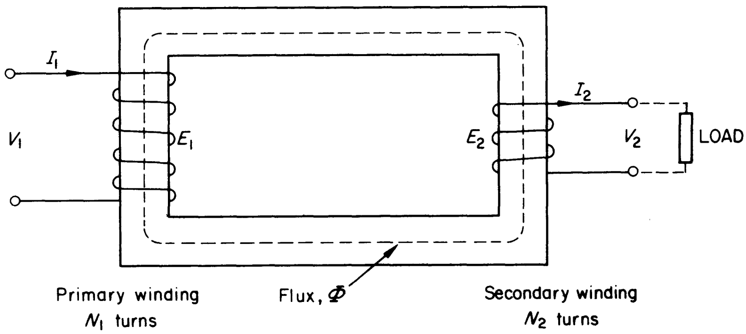


figure 2.37

Figure 2.37 shows a primary winding of N_1 turns and a secondary winding of N_2 turns linked by a common flux Φ, the primary winding being connected to an a.c. supply of V_1 volts at a frequency of f hertz. Consider first the secondary winding to be open-circuited. The primary winding acts simply as a highly inductive coil, the primary current (negligible for the ideal transformer) establishing the flux Φ in the magnetic circuit. This flux is alternating and at any instant its value is given by $\Phi = \Phi_M \cos \omega t$. The alternating flux induces an EMF of self-inductance in the primary winding, its value at any instant being given by

$$e_1 = -N_1 \frac{d\Phi}{dt} \qquad \text{(from equation 2.11)}$$

$$= -N_1 \frac{d(\Phi_M \cos \omega t)}{dt}$$

$$= +N_1 \Phi_M \omega \sin \omega t$$

The maximum value of this occurs when $\sin \omega t = 1$,

then $$E_{1_M} = N_1 \Phi_M \omega$$

and RMS value $$E_1 = \frac{\omega}{\sqrt{2}} \Phi_M N_1$$

$$= \frac{2\pi f}{\sqrt{2}} \Phi_M N_1$$

$\therefore$ $$E_1 = 4{\cdot}44 f \Phi_M N_1 \text{ volts} \qquad (2.23)$$

The EMF per turn is thus $4{\cdot}44f\Phi_M$ volts for both primary and secondary windings, since the flux also links with the secondary. The induced EMF in the secondary winding of N_2 turns is thus

$$E_2 = 4{\cdot}44f\Phi_M N_2 \text{ volts} \tag{2.24}$$

Evidently

$$\frac{E_2}{E_1} = \frac{N_2}{N_1} \tag{2.25}$$

This EMF, or turns ratio, is called the 'ratio of transformation'.

Phase relationships

Figure 2.38a shows the waveforms representing the flux Φ, the induced EMFs, and the secondary and primary terminal voltages.

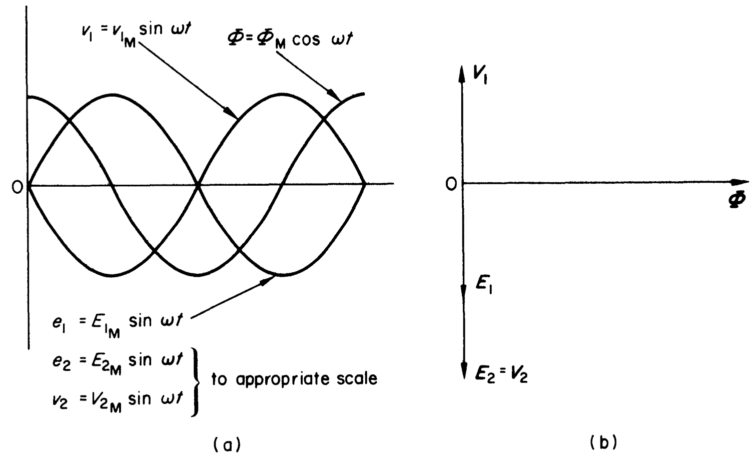


figure 2.38

The induced EMFs are zero when the flux is maximum, for the rate of change of flux is then zero. The induced EMFs are maximum when the flux is zero, because the rate of change of flux is then maximum. In accordance with Lenz's law, the induced EMFs are positive maximum when the flux is zero and changing from positive to negative, the induced EMFs thus tending to prevent the change in flux. The induced EMFs, e_1 and e_2 are thus in phase with each other, both lagging the flux by 90°; e_2 provides the secondary output voltage v_2; v_1 opposes e_1; therefore v_1 and e_1 are in phase opposition. To satisfy Kirchhoff's law, $v_1 = -e_1$, that is, the whole of the applied voltage v_1 is absorbed in neutralising the EMF induced in the primary by the main flux. Figure 2.38b shows the corresponding phasor diagram for a transformation ratio of greater than 1, drawn in terms of RMS values, the flux Φ being the reference phasor.

The effect of current in the secondary winding

If a load impedance is now connected across the secondary terminals, current flows in the winding in the direction of the induced EMF, and a field is thus set up in the core in opposition to that set up by the primary winding, the ampere-turns of the secondary tending to demagnetise the core. But from the EMF equation, the flux is proportional to the applied voltage V_1. As V_1 is constant, Φ_M must also be constant. The primary current thus increases to produce primary ampere-turns equal and opposite to the demagnetising secondary ampere-turns. In the ideal transformer, negligible ampere-turns are required to produce the flux.

Thus,

$$I_1N_1 + I_2N_2 = 0$$

$$I_1N_1 = -I_2N_2$$

and

$$\frac{I_1}{I_2} = -\frac{N_2}{N_1} \tag{2.26}$$

It is thus seen that

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} = -\frac{I_1}{I_2}$$

Also $V_1I_1 = -V_2I_2$; that is, while the primary is *absorbing* power from the supply, the secondary is *delivering* equal power to the load.

The phasor diagram for the ideal transformer is shown in figure 2.39, drawn for a load at lagging power factor.

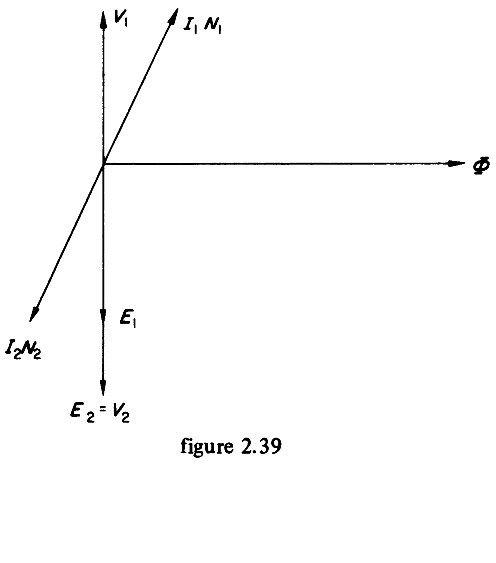


figure 2.39

Example 2.11 An ideal transformer having a primary winding of 2000 turns has a 25 Ω resistive load across its secondary. A 100 V 50 Hz input produces 75 V across the load. Calculate (a) the number of turns on the secondary winding, (b) the equivalent input resistance of the transformer on load, (c) the current in each winding on load.

(CGLI Principles B, 1967)

(a) The transformer is ideal, and $V_1 = E_1$, $V_2 = E_2$

$$\therefore \qquad \frac{V_1}{V_2} = \frac{N_1}{N_2}$$

$$N_2 = N_1 \frac{V_2}{V_1} = 2000 \times \frac{75}{100} = 1500 \text{ turns}$$

(b) Let R_1 be the equivalent input resistance, then input power $= I_1^2 R_1$. Output power $= I_2^2 R_2$ where R_2 is the load resistance.

$$\therefore \qquad I_1^2 R_1 = I_2^2 R_2$$

$$R_1 = R_2 \left(\frac{I_2}{I_1}\right)^2$$

$$= R_2 \left(\frac{N_1}{N_2}\right)^2$$

$$= 25 \times \left(\frac{2000}{1500}\right)^2$$

$$= 44{\cdot}44\ \Omega$$

(c) Current in primary

$$= \frac{100}{44{\cdot}44} = 2{\cdot}25\ \text{A}$$

$$\text{Load current} = \text{secondary current} = \frac{75}{25}$$

$$= 3\ \text{A}$$

(B)

(C) The actual transformer

The actual transformer departs from the ideal in several respects, of which the most important (as far as this treatment is concerned) relates to the primary current which flows when the secondary is open circuited. Since the reluctance of the magnetic circuit is not zero, a primary MMF is required to establish the flux, a small no-load current thus flowing in the primary winding when the secondary is open circuited. The alternating flux in the iron core also leads to power losses due to hysteresis and eddy currents. The no-load current thus has two components: (1) a magnetising component $I_{0\text{m}}$ which establishes the flux and is in phase with the flux and therefore lags the primary applied voltage by 90°; and (2) an active or power component $I_{0\text{a}}$ which provides for the iron losses and is in phase with the primary applied voltage. In figure 2.40 an equivalent circuit taking these components into account is shown at (a), with the corresponding phasor diagram shown at (b). The phase angle ϕ_0 of the no-load current is such that $\tan \phi_0 = I_{0\text{m}}/I_{0\text{a}}$. ϕ_0 is usually somewhat less than 90°.

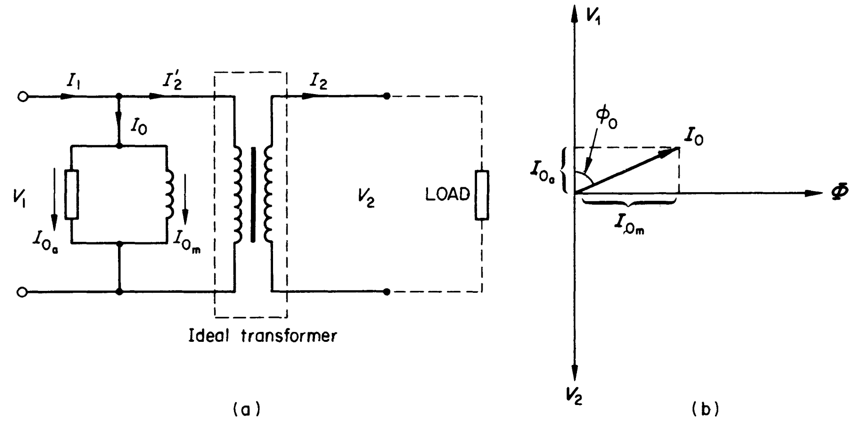


figure 2.40

Consider now a load impedance to be connected across the secondary terminals, so that the secondary current is I_2. The secondary ampere-turns I_2N_2 give rise to an *increase* in primary current, such that the *increase* in primary ampere-turns balances the secondary ampere-turns. Let the increase in primary current be I_2', then

$$I_2'N_1 = -I_2N_2$$

and

$$I_2' = -I_2\frac{N_2}{N_1} \tag{2.27}$$

I_2' is termed the secondary current referred to the primary, and is in phase opposition to I_2. The total primary current is then given by the phasor sum of I_2' and I_0, as shown in the phasor diagram of figure 2.41.

The phasor diagram is drawn for a load of lagging power factor, I_2 lagging V_2 by an angle ϕ_2; $I_2' = I_2(N_2/N_1)$ is shown in phase opposition to I_2; I_1 is given by the phasor sum of I_2' and I_0, and lags the applied voltage by an angle ϕ_1.

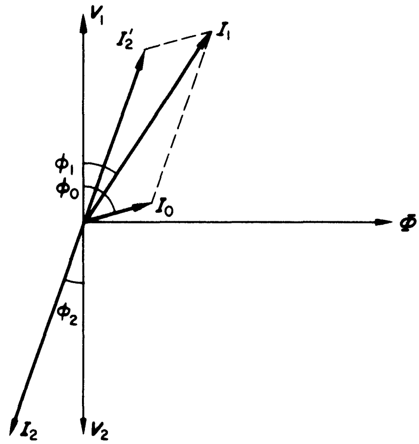


figure 2.41

Example 2.12 An iron-cored transformer has a primary to secondary turns ratio of 1: 2 and the no-load primary inductance is 0·5 H. In all other respects the transformer may be considered ideal. Calculate:

(a) the inductance of the secondary winding with the primary on open circuit;

(b) the overall inductance of the two windings connected (i) in series aiding, (ii) in series opposing;

(c) the magnetising current when a voltage of 1 V RMS at a frequency of 800 Hz is applied to the primary winding

(d) the total primary current when, under the conditions of (c), a resistance of 10 kΩ is connected to the secondary winding.

(CGLI Principles C, 1967)

(a) Using equation 2.16,

$$L = \mu_r \mu_0 N^2 \frac{a}{l}$$

As the transformer is ideal in all other respects, the inductance of any coil on its core is proportional to (number of turns)^2.

Let number of turns in primary $= N$

Then number of turns in secondary $= 2N$

$$\therefore \qquad \text{Inductance of secondary} = 0{\cdot}5 \times \left(\frac{2N}{N}\right)^2$$

$$= 2{\cdot}0\,\text{H}$$

(b) (i) Series aiding, number of effective turns $= 3N$

$\therefore$ $$\text{Overall inductance} = 0{\cdot}5 \times \left(\frac{3N}{N}\right)^2 = 4{\cdot}5\ \text{H}$$

(ii) Series, opposing, number of effective turns $= 2N - N$

$$= N$$

$\therefore$ $$\text{Overall inductance} = 0{\cdot}5\ \text{H}$$

(c) Reactance X_L of primary $= 2\pi \times 800 \times 0{\cdot}5$

$$= 2514\ \Omega$$

$$\text{Magnetising current} = \frac{1}{2520} = 0{\cdot}398\ \text{mA}$$

$$= 0{\cdot}4\ \text{mA (approximately)}$$

(d) $$V_2 = 2\ \text{volts}$$

$$I_2 = \frac{2}{10\,000} = 0{\cdot}2\ \text{mA}$$

$$I_2' = 0{\cdot}2 \times \frac{2}{1} = 0{\cdot}4\ \text{mA}$$

As the load is resistive, I_2 is in phase with V_2, and I_2' is in phase with V_1; I_0 lags V_1 by 90°; and these conditions are shown in the phasor diagram of figure 2.43.

$$I_1 = (0{\cdot}4^2 + 0{\cdot}4^2)^{1/2}$$

$$= \sqrt{2} \times 0{\cdot}4$$

$$= 0{\cdot}566\ \text{mA}$$

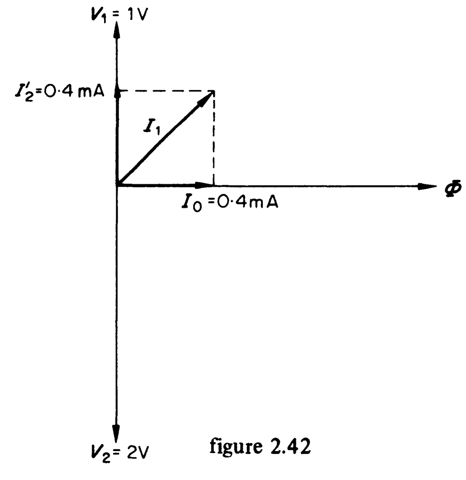


figure 2.42

(B) MISCELLANEOUS EXAMPLES

Example 2.13 The coil of a moving coil loudspeaker has a mean diameter of 5 cm and is wound with 1000 turns. It is situated in a radial magnetic field of 0·4 T. Calculate the force on the coil when the current in it is 20 mA.

[0·753 N]

Example 2.14 A conductor 8 cm long is moving at a uniform speed of 5 m/s at right-angles to its length and to a uniform magnetic field. The EMF induced in the conductor is 0·2 V. Calculate the density of the field. If the conductor forms part of a closed circuit of total resistance 0·05 Ω, calculate the force on the conductor.

[0·5 T; 0·16 N]

Example 2.15 Explain the meaning of relative permeability for a magnetic material.

Describe an experiment to determine the relative permeability for an iron specimen at various flux densities.

A stalloy core has relative permeabilities of 6000 at a flux density of 0·6 T and 2000 at 1·2 T; the μ_r/B characteristic is a straight line between these points. Calculate the current needed in a coil of 1000 turns on a core of this material to produce a flux density of 0·9 T. The mean flux path is 0·2 m long.

[0·275 A]

(CGLI Principles B)

Example 2.16 Define the term 'relative permeability' of a magnetic material. Explain briefly why iron has a high value of relative permeability.

The core of an electromagnet is made of an iron rod 1 cm diameter, bent into a circle of mean diameter 10 cm, a radial air gap of 1 mm being left between the ends of the rod.

Calculate the direct current needed in a coil of 2000 turns uniformly spaced around the core to produce a magnetic flux of 0·2 mWb in the air gap.

Assume that the relative permeability of the iron is 150, that the magnetic leakage is 1·2 and that the air gap is parallel.

[3·56 A]

(CGLI Principles B)

Example 2.17 Explain briefly under what conditions it is advantageous to use in a magnetic circuit:

(a) a laminated iron core
(b) a granulated iron (iron dust) core

A magnet core consists of a stack of 20 circular ring laminations each 0·5 mm thick with outer diameter 5·5 cm and inner diameter 3·5 cm. The relative permeability of the iron is 2000. A radial air gap of 2 mm is cut in this core. Calculate the direct current that will be required in a coil of 1000 turns uniformly distributed around the core to produce a magnetic flux of 0·3 mWb in the air gap. Assume that magnetic leakage is negligible. The permeability of free space is $4\pi \times 10^{-7}$ H/m.

[4·95 A]

(CGLI Principles B)

Example 2.18 Describe an experiment to determine the magnetisation (B/H) curve for a ring-shaped specimen of iron. Sketch a curve to show a typical result.

Define 'relative permeability' and show how it can be deduced from the results of your experiment.

What deductions on the hysteresis loss in the iron can be made from the B/H loop?

(CGLI Principles B)

Example 2.19 What is eddy current loss and under what conditions does it occur?

Explain with diagrams why, by constructing a magnetic core of laminations, the eddy current loss in it can be reduced. Show in your diagrams the relative directions of the magnetic flux and the induced EMF relative to the laminations.

What property should be possessed by the iron from which laminations for a low-frequency transformer core are made?

(CGLI Principles B)

Example 2.20 Explain what is understood by the term EMF of self-induction.

The current in a circuit consisting of an inductor of 0·1 H which has a resistance of 50 Ω is increasing uniformly from zero at a rate of 5 A/s. Draw to scale graphs of this current and of the voltage across the circuit for the first 0·01 of a second. If the coil is suddenly short-circuited at the end of this period, calculate the total energy that will have to be dissipated.

[0·125 mJ]

(CGLI Principles B)

Example 2.21 The coil of an electromagnetic relay has an inductance of 5 H when the relay armature is not operated. It increases to 20 H when the armature is operated. Explain why there is an increase in inductance. The relay coil circuit has a d.c. resistance of 240 Ω and is energised from a 50 V battery supply. The armature operates with 20 mA in the coil. Sketch a curve showing how the current rises. Give expressions for the current/time relations for the two portions of the curve. What are the two time constants?

[$i = 0{\cdot}209[1 - \exp(-48t)]; i = 0{\cdot}209[1 - \exp(-12t)]$; 20·9 ms; 83·3 ms]

(CGLI Principles B)

Example 2.22 A constant voltage, V, is maintained across an inductance of L henrys in series with a resistance of R ohms. Write down an expression for the current that flows in the circuit t seconds after switching on. On what factors does the rate of rise of current depend? What is the initial rate of rise of current?

A relay coil of resistance 200 Ω and inductance 8 H is connected in series with a 100 Ω resistor and 60 V battery. The relay operates when the current in its coil is 31·6 mA. How long does it take to operate?

The operation of the relay armature increases the inductance of the coil to 20 H. Sketch the current/time curve from the moment of switch-on, showing the effects of this increase in inductance.

How much energy is stored in the magnetic field when the current has reached a constant value?

[4·6 ms; 0·4 J]

(CGLI Principles B, 1963) (B)

(C) ***Example 2.23*** Explain why the inductance of the primary winding of a transformer should be made as high as possible.

A transformer, which may be assumed ideal in all other respects, has a primary inductance of 2 H. Its primary to secondary turns ratio is 2·5 to 1.

A p.d. of 5 V RMS at a frequency of $10^4/2\pi$ Hz is maintained across the primary winding. Calculate the magnitudes of the primary and secondary currents:

(a) when the secondary terminals are on open-circuit
(b) when the secondary terminals are connected through a 3·2 kΩ resistance.

[(a) 0·25 mA; (b) 0·354 mA]

(CGLI Principles C)

Example 2.24 When is a transformer said to be ideal? Explain why the magnetising current of a transformer increases as the frequency of a constant applied voltage is reduced.

An ideal transformer has a turns-ratio of 1:2. What is the impedance measured at the primary when a resistance of 2512 Ω is connected to the secondary? How is this value modified at a frequency of 200 Hz if the inductance of the primary winding is 0·5 H?

[628 Ω; 446 Ω]

(CGLI Principles C, 1964)

Example 2.25 Discuss the factors which cause power loss in an iron-cored transformer.

An EMF of 1·5 V RMS at a frequency of $1000/\pi$ Hz is applied to the primary of a step-up transformer with a turns-ratio of 2:1. When the secondary terminals are on open circuit the primary current is 0·5 mA and lags the applied voltage by 90°. Calculate the primary inductance.

When a resistor is connected to the secondary terminals, the magnitude of the primary current rises to 1·3 mA. Determine the value of the resistor.

[1·5 H; 5 Ω]

(CGLI Principles C, 1970) (C)

3 Alternating-current circuit theory

(B) BASIC CONCEPTS

This chapter opens with a brief review of the basic principles of alternating currents before applying those principles to circuit theory. An alternating voltage of sinusoidal waveform as normally used can be expressed in the form:

$$v = V_M \sin \omega t$$

where v is the value of the voltage at time t, V_M is the maximum or peak value, ω is the angular frequency in radians per second, and is equal to $2\pi f$ where f is the frequency in *Hertz* (symbol Hz).

1 Hertz = 1 cycle per second. The time taken for the voltage wave to complete 1 cycle is termed the periodic time T. Evidently $T = 1/f$.

This voltage wave is illustrated in figure 3.1.

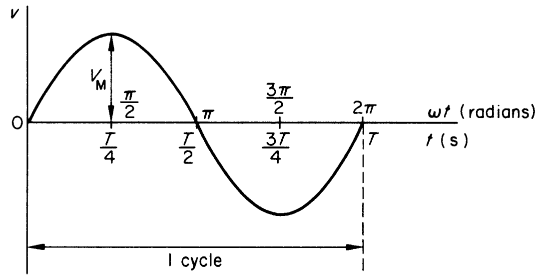


figure 3.1

Phasor representation

Two or more voltages of the same frequency may be added together by drawing the sine waves representing the voltages, and adding these. It is more convenient, however, to draw the *phasors* which represent the voltages and add these. A phasor is a rotating vector of length equal to the peak value of the voltage it represents, and is considered to rotate anti-clockwise at ω rad/s, the projection on the vertical axis being equal to the instantaneous value v. By convention, phasor diagrams are drawn for the instant $t = 0$. Since the value of $\sin \omega t = 0$ when $t = 0$, the phasor representing the quantity $V_M \sin \omega t$ is drawn horizontal. The value of $V_M \sin(\omega t - \theta)$ when $t = 0$ is $V_M \sin(-\theta)$, hence the phasor representing this voltage makes an angle $-\theta$ with the horizontal axis. Figure 3.2a shows the sine waves representing two voltages:

$$v_1 = V_{1_M} \sin \omega t$$

and

$$v_2 = V_{2_M} \sin(\omega t - \theta)$$

Thus v_2 lags v_1 by an angle θ. The resultant voltage v_R is obtained by adding instantaneous values of v_1 and v_2 together. Figure 3.2b shows the phasors representing v_1 and v_2, together with the resultant voltage v_R obtained by adding the phasors together, this addition being made by observing the normal rules for addition of vectors by graphical means.

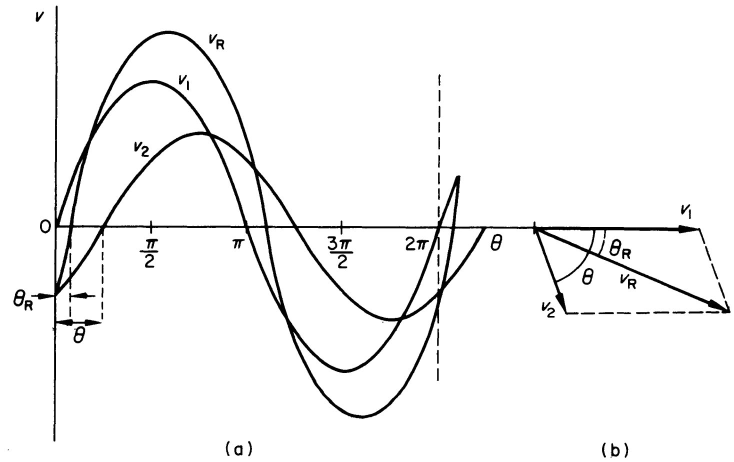


figure 3.2

Using the phasor diagram, the resultant voltage v_R may be readily determined and expressed in the form $v_R = V_{R_M} \sin(\omega t - \theta_R)$.

The principles that have been outlined apply in precisely the same way to waveforms of sinusoidal currents, which can be expressed in the form

$$i = I_M \sin(\omega t \pm \theta)$$

RMS values

The value used to denote an alternating current (or voltage) is defined in terms of the average power developed in a pure resistance R through which the current is flowing, and is termed the *root-mean-square*, or the *effective*, value of the current. If i is the current at any instant, the power at any instant is given by i^2R. This same power could be developed by a steady current I, which is thus the effective value of the alternating current. If $i = I_M \sin \omega t$, then

$$i^2 = I_M^2 \sin^2 \omega t$$

$$= I_M^2 \times \frac{1}{2}(1 - \cos 2\omega t)$$

$$= \frac{I_M^2}{2} - \frac{I_M^2}{2} \cos 2\omega t$$

The mean value of the second term is zero over a complete cycle, hence the mean square value of a sinusoidal current is half the square of its peak value.

$$I^2 = \frac{I_M^2}{2}$$

and

$$I = \frac{I_M}{\sqrt{2}} = \frac{I_M}{1{\cdot}414} = 0{\cdot}707 I_M \tag{3.1}$$

Similarly, the RMS or effective value of a sinusoidal voltage is given by

$$V = \frac{V_M}{\sqrt{2}}$$

Terms denoting RMS values are indicated by capital letters without suffixes.

The phasor diagrams discussed above were drawn with the phasors of length equal to the *peak* values of the voltages or currents they represent. The shape of the diagrams and the phase relationships between the various quantities remain unaffected if the phasors are drawn to represent RMS values instead of peak values. This is more convenient and is usually done.

The RMS value is the value indicated by most instruments. Some instruments, however, are actuated by the *average* value of a full-wave rectified alternating current. The average value of a full wave rectified sinusoidal current is given by

$$I_{AV} = \frac{2I_M}{\pi} = 0{\cdot}637 I_M \tag{3.2}$$

For an alternating current of any waveform, the ratio (RMS value/average value) is termed the *form factor.*

For a sinusoidal waveform

$$\text{Form factor} = \frac{I_M \times \pi}{\sqrt{(2)} \times 2I_M} = \frac{0{\cdot}707}{0{\cdot}637} = 1{\cdot}11 \tag{3.3}$$

Circuit possessing resistance only

Let a voltage $v = V_M \sin \omega t$ be applied to a pure resistance, as shown in figure 3.3a.
The instantaneous value of the current is given by

$$i = \frac{v}{R} = \frac{V_M}{R} \sin \omega t = I_M \sin \omega t$$

The current is thus in phase with the voltage, and is shown so in the waveform diagram of figure 3.3b, the corresponding phasor diagram, drawn with RMS values, being shown in figure 3.3c.

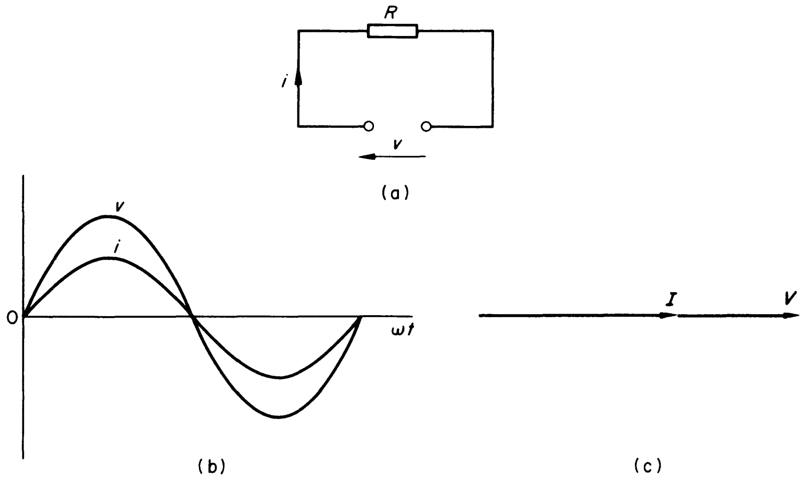


figure 3.3

Now

$$\frac{V_M}{R} = I_M$$

Dividing by $\sqrt{2}$ to obtain the RMS values:

$$I = \frac{V}{R} \tag{3.4}$$

Circuit possessing inductance only

Let a sinusoidal voltage v be applied to a coil having an inductance L henrys and negligible resistance, as shown in figure 3.4a. Let the current flowing in the coil be given by $i = I_M \sin \omega t$ as shown in the waveform diagram of figure 3.4b.

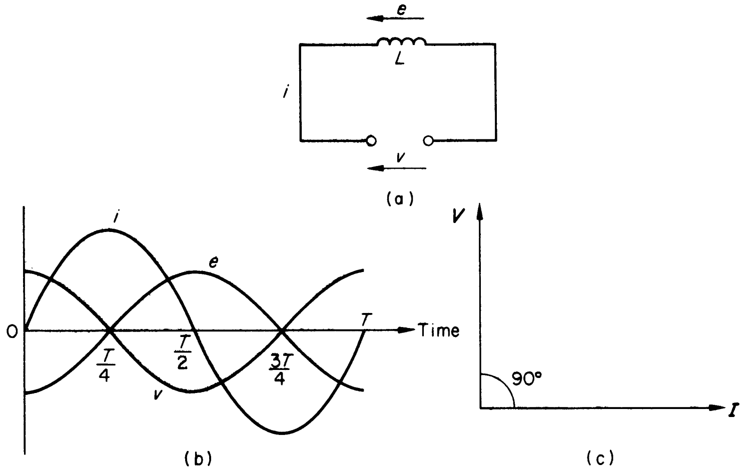


figure 3.4

At any instant t, the current is changing and hence induces in the coil an EMF e whose value is proportional to the rate of change of current at the instant considered. Thus, at time $t = 0$, the current is zero, but its rate of change is maximum, hence the induced EMF is maximum. Further, by Lenz's law, the induced EMF is a negative maximum, tending to prevent the current increasing in the positive direction. At time $t = T/4$, the current is maximum, but its rate of change is zero, hence the induced EMF is zero. Extending this argument to the remainder of the cycle, it is seen that the induced EMF is represented by a sine wave lagging behind the current wave by $\pi/2$ rad, or 90°. This induced EMF opposes the applied voltage at all instants, hence the applied voltage is represented by a sine wave leading the current by $\pi/2$ rad, or 90°. In a circuit possessing inductance only, the current thus lags behind the applied voltage by 90°. This result is illustrated in the phasor diagram of figure 3.4c.

Reactance associated with inductance

The induced EMF is given by

$$e = -L\frac{\mathrm{d}i}{\mathrm{d}t}$$

$$= -L\frac{\mathrm{d}(I_\mathrm{M} \sin \omega t)}{\mathrm{d}t}$$

$$= -LI_\mathrm{M}\omega \cos \omega t$$

the induced EMF thus lagging the current by 90°.

The maximum value of e occurs when $\cos \omega t = 1$, thus

$$E_M = -LI_M\omega$$

Dividing by $\sqrt{2}$ to obtain the RMS values

$$E = -LI\omega$$

The applied voltage V opposes E

$$\therefore \quad V = LI\omega$$

and

$$I = \frac{V}{\omega L} \tag{3.5}$$

The quantity ωL is termed the *inductive reactance* (symbol X_L) of the circuit.

$$X_L = \omega L = 2\pi f L \tag{3.6}$$

The expression $I = V/\omega L$ is of the same form as $I = V/R$ for the resistive circuit, thus reactance is measured in ohms. The inductive reactance is proportional to the frequency, and, for a given voltage, the current is inversely proportional to the frequency (as shown in figure 3.5).

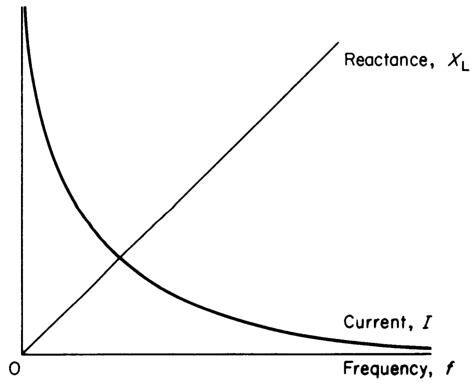


figure 3.5

Circuit possessing capacitance only

Let a voltage $v = V_M \sin \omega t$ be applied to a capacitor of capacitance C farads as shown in figure 3.6a. The charge on the capacitor at any instant is then given by $q = CV_M \sin \omega t$ and is thus proportional to the applied voltage, as shown in the waveform diagram of figure 3.6b. As the voltage changes in the positive direction, the charge on the capacitor changes in the positive direction, and current i flows in the positive direction. At time $t = 0$, the charge is zero, but its rate of change in the positive direction is a maximum, hence the current is a positive maximum. At time $t = T/4$, the charge is a positive maximum, but its rate of change is zero, hence the current is zero. Again extending the argument to the remainder of the cycle, it is seen that the current is represented by a sine wave leading the voltage wave by $\pi/2$ rad or 90°. This result is illustrated in the phasor diagram of figure 3.6c.

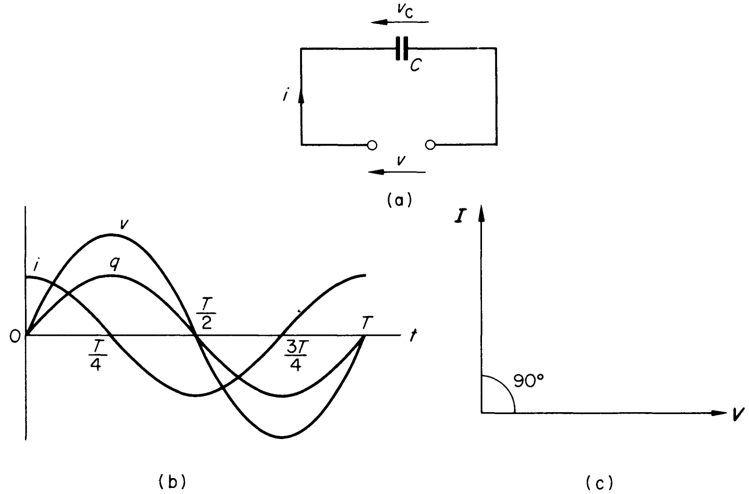


figure 3.6

Reactance associated with capacitance

The current i is given by

$$i = \frac{dq}{dt}$$

$$= \frac{d(Cv)}{dt}$$

$$= C\frac{d(V_M \sin \omega t)}{dt}$$

$$= CV_M\omega \cos \omega t$$

the current thus leading the voltage by 90°. The maximum value of i occurs when $\cos \omega t = 1$.

$$I_M = CV_M\omega$$

Dividing by $\surd 2$ to obtain the RMS values:

$$I = CV\omega$$

$$\therefore \qquad \frac{V}{I} = \frac{1}{\omega C} \qquad (3.7)$$

The quantity $1/\omega C$ is termed the *capacitive reactance* (symbol X_C) of the circuit, and is measured in ohms.

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} \qquad (3.8)$$

The capacitive reactance is inversely proportional to the frequency, and for a given voltage the current is directly proportional to the frequency, as shown in figure 3.7.

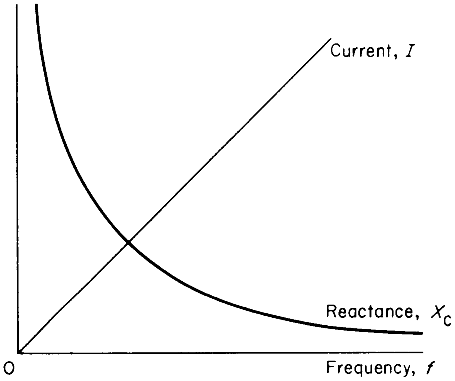


figure 3.7

Example 3.1 Write down the frequency, the RMS and the peak values of a voltage wave expressed as: $v = 14{\cdot}1 \sin 1000\pi t$.

Write down expressions for the current flowing when this voltage is applied across:

(a) a 5 Ω resistor,
(b) a 1 mH inductor of negligible resistance, and
(c) a 150 μF capacitor.

Sketch the waveforms of these currents showing clearly:

(a) the phase relationship of each current to the applied voltage,
(b) the peak value of each current.

(CGLI Principles B, 1966)

The general expression for a sinusoidal voltage v of frequency f Hz with peak value V_M at any instant t is given by

$$v = V_M \sin \omega t$$
$$= V_M \sin 2\pi f t$$

But, $v = 14{\cdot}1 \sin 1000\pi t$

$\therefore$ Peak value $V_M = 14{\cdot}1\,\text{V}$

and $2\pi f = 1000\pi$, so that $f = 500$ Hz.

Also, RMS value $V = \dfrac{14{\cdot}1}{\sqrt{2}} = 10\,\text{V}$

(a) Voltage applied to 5 Ω resistor:

$$i = \frac{V_M}{R} \sin 2\pi f t$$

$$= \frac{14{\cdot}1}{5} \sin 1000\pi t$$

$$= 2{\cdot}82 \sin 1000\pi t$$

(b) Voltage applied to 1 mH inductor:

$$X_L = 2\pi f L = 1000\pi \times 0{\cdot}001 = \pi\,\Omega$$

$$i = \frac{V_M}{X_L} \sin\left(2\pi f t - \frac{\pi}{2}\right)$$

$$= \frac{14{\cdot}1}{\pi} \sin\left(1000\pi t - \frac{\pi}{2}\right)$$

$$= 4{\cdot}48 \sin\left(1000\pi t - \frac{\pi}{2}\right)$$

(c) Voltage applied to 150 μF capacitor:

$$X_C = \frac{1}{2\pi f C} = \frac{10^6}{1000\pi \times 150} = 2{\cdot}12\,\Omega$$

$$i = \frac{V_M}{X_C} \sin\left(2\pi f t + \frac{\pi}{2}\right)$$

$$= \frac{14{\cdot}1}{2{\cdot}12} \sin\left(1000\pi t + \frac{\pi}{2}\right)$$

$$= 6{\cdot}65 \sin\left(1000\pi t + \frac{\pi}{2}\right)$$

The required waveforms are shown in figure 3.8.

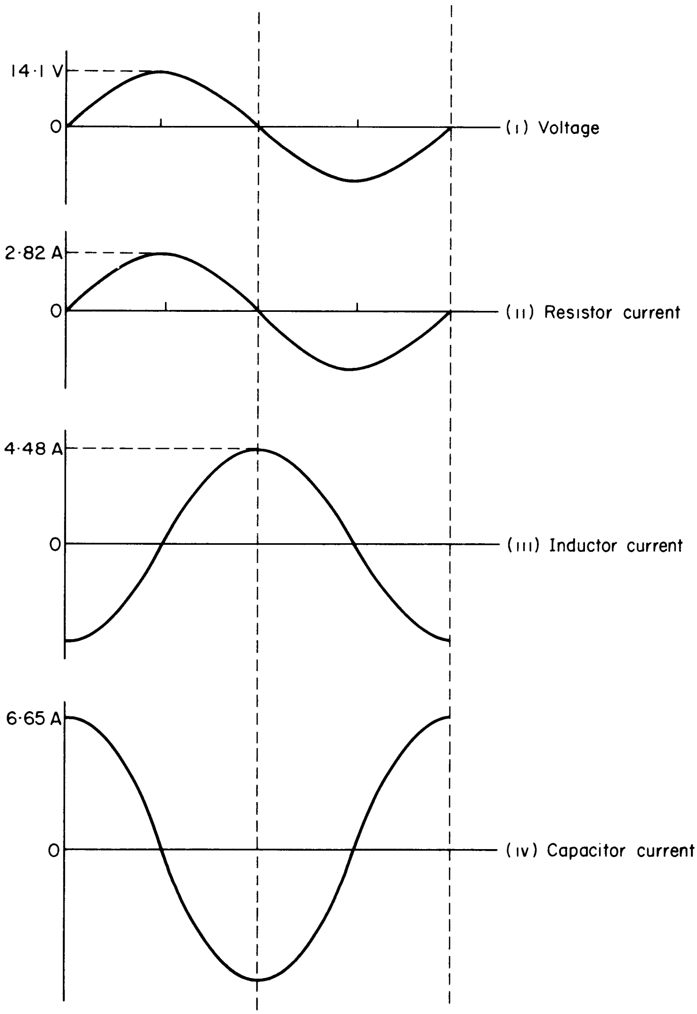


figure 3.8

Power

(i) Resistance only in circuit.

The voltage v and current i are in phase. The power at any instant is given by $v \times i$, and in figure 3.9 corresponding values of v and i have been multiplied together to obtain the waveform of the power. The power wave is seen to be a sine wave varying at twice the frequency of the supply voltage, but about a mean value of $(V_M I_M)/2$. This is also shown mathematically as follows:

$$v = V_M \sin \omega t$$

$$i = I_M \sin \omega t$$

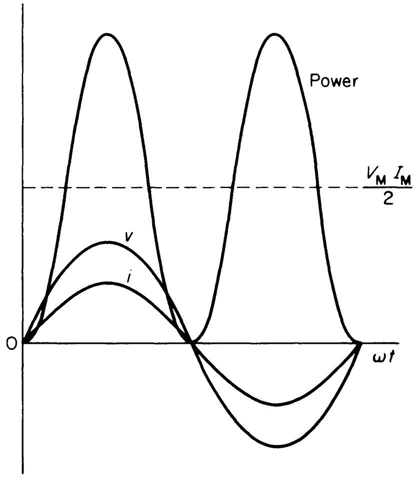


figure 3.9

Instantaneous power $= vi$

$$= V_M I_M \sin^2 \omega t$$

$$= \frac{V_M I_M}{2} (1 - \cos 2\omega t)$$

$$= \frac{V_M I_M}{2} - \frac{V_M I_M}{2} \cos 2\omega t$$

The mean value of this over a whole number of cycles is $(V_M I_M)/2$, because the mean value of the second term is zero. The average power is thus given by

$$P = \frac{V_M I_M}{2} = \frac{V_M}{\sqrt{2}} \frac{I_M}{\sqrt{2}} = VI = I^2 R = \frac{V^2}{R} \tag{3.9}$$

where V and I are RMS values, the current being in phase with the voltage.

(ii) Inductance only in circuit.

The voltage v and current i are now 90° out of phase, v leading by 90°. In figure 3.10 corresponding values of v and i have been multiplied together to obtain the waveform of the power.

Let

$$v = V_M \sin\left(\omega t + \frac{\pi}{2}\right) = V_M \cos \omega t$$

and

$$i = I_M \sin \omega t$$

$$vi = V_M I_M \cos \omega t \sin \omega t$$

$$= \frac{V_M I_M}{2} \sin 2\omega t$$

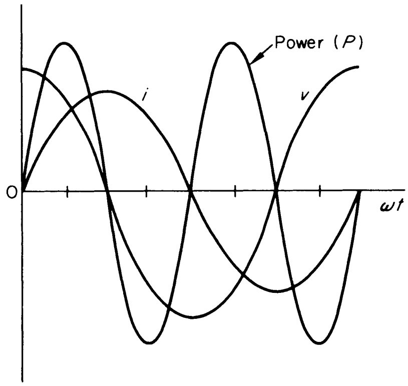


figure 3.10

The power wave is seen to be a sine wave varying at twice the supply frequency but about a mean value of zero. Hence the mean power taken from the supply over a whole number of cycles is zero. Power is taken during the first quarter-cycle while the current is increasing, the energy supplied during this period being stored in the magnetic field. During the next quarter-cycle the current decreases and the energy is returned to the supply

(iii) Capacitance only in circuit.

The voltage v and current i are again 90° out of phase and the mean power taken from the supply is zero. The energy stored in the electric field while the voltage is increasing during one quarter-cycle is returned to the supply when the voltage decreases during the next quarter-cycle.

Series circuits

Most a.c. circuits contain resistance, inductance and capacitance combined in a variety of ways, the simplest and most common combination being that of resistance and inductance in series. The resistance and inductance need not be physically separate quantities. Thus a coil of wire possesses both resistance and inductance which cannot be physically separated. For calculation purposes, however, the effects of these quantities may be separately assessed, and the overall effect obtained by a correct combination of the separate effects.

Resistance and inductance

In figure 3.11 the voltage $V_R = IR$ is in phase with the current, and the voltage $V_L = IX_L$ leads the current by 90°. The phasor diagram is drawn in figure 3.12a, the current I being taken as the reference phasor. The applied voltage V is given by the phasor sum of V_R and V_L.

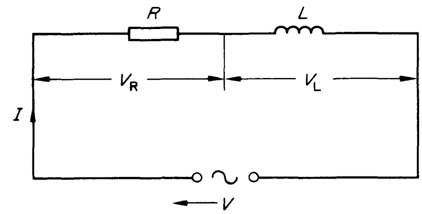


figure 3.11

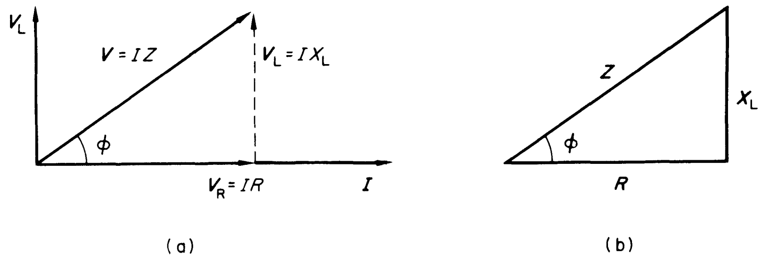


figure 3.12

Since the phasors IR and IX_L are at right-angles to each other, the applied voltage V is given by

$$V = [(IR)^2 + (IX_L)^2]^{1/2}$$
$$= I(R^2 + X_L^2)^{1/2}$$

and

$$\frac{V}{I} = \sqrt{(R^2 + X_L^2)} \tag{3.10}$$

The quantity $\sqrt{(R^2 + X_L^2)}$ is termed the *impedance* Z of the circuit and is measured in ohms. The relationship between R, X_L and Z is shown in figure 3.12b, and is termed the impedance triangle. This triangle is similar to the voltage triangle of V_R, V_L and V in figure 3.12(a).

$$Z = \sqrt{(R^2 + X_L^2)} = [R^2 + (\omega L)^2]^{1/2} \tag{3.11}$$

From the phasor diagram of figure 3.12a the current is seen to lag the applied voltage by an angle ϕ, where

$$\tan \phi = \frac{V_L}{V_R} = \frac{X_L}{R} \tag{3.12}$$

that is, the tangent of the phase angle is given by the ratio of reactance to resistance. In a circuit containing resistance and inductance in series, the current is thus given by

$$I = \frac{V}{Z} = \frac{V}{\sqrt{(R^2 + X_L^2)}} \tag{3.13}$$

and lags the applied voltage by an angle ϕ where $\tan\phi = X_L/R$.

Power factor

The supply voltage has two components, V_L across the inductance and V_R across the resistance. The mean power absorbed in the inductance is zero, hence power is absorbed in the resistance only, this power being given by the product of the RMS voltage across, and the RMS current in, the resistance. Thus:

$$P = V_R I = V\cos\phi I = VI\cos\phi \tag{3.14}$$

The product of the supply voltage V and the current I, that is VI, is sometimes termed the 'apparent power' or the 'volt-amperes'; and $\cos\phi$ is termed the *power factor (p.f.)*, because it is the factor by which the apparent power or volt-amperes must be multiplied to give the true power in watts. For sinusoidal voltages and currents, the power factor is given by

P.F. = cosine of angle of phase difference between current and applied voltage,

or

$$\text{P.F.} = \frac{\text{watts}}{\text{volt-amperes}}$$

For a circuit possessing resistance and inductance, the current lags the applied voltage, and the power factor is referred to as lagging.

Resistance and capacitance

Figure 3.13 shows the circuit diagram, phasor diagram and impedance triangle for this combination, which is similar to that of R and L in series.

$$Z = \sqrt{(R^2 + X_C^2)} = \left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]^{1/2} \tag{3.15}$$

The current *leads* the applied voltage by an angle ϕ where $\tan\phi = X_C/R$. The p.f. is again given by

$$\begin{aligned}\text{P.F.} &= \cos\phi \\ &= \frac{\text{watts}}{\text{volt-amperes}}\end{aligned}$$

and this power factor is referred to as leading.

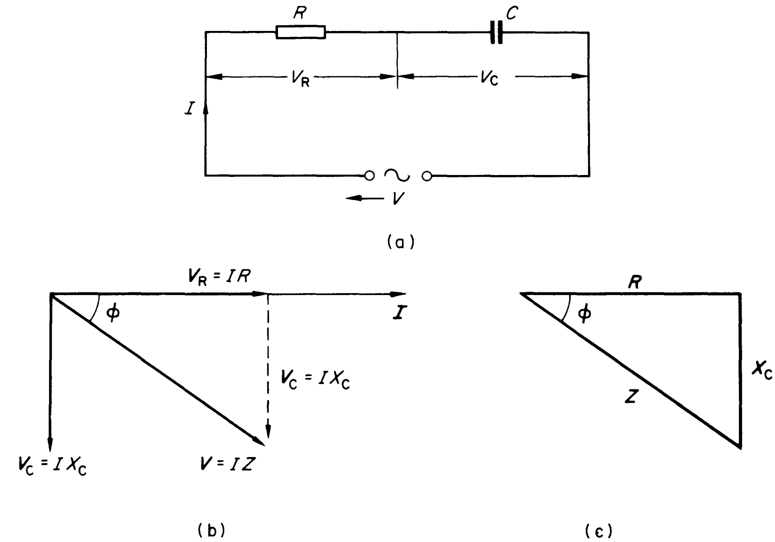


figure 3.13

Resistance, inductance and capacitance

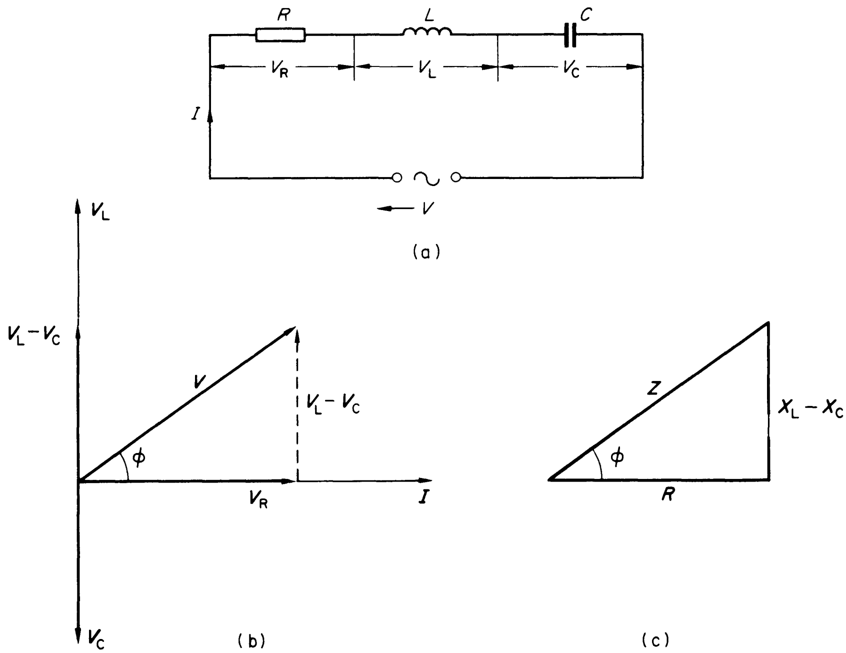


figure 3.14

Figure 3.14 shows the circuit diagram at (a) and the phasor diagram at (b). V_L leads the current by 90° and V_C lags the current by 90°. The case shown assumes $V_L > V_C$, that is $X_L > X_C$. The applied voltage V is given by the phasor sum of V_L, V_C and V_R. This is most conveniently obtained by finding first the phasor sum of V_L and V_C, that is $V_L - V_C$ (arithmetically) and then finding the phasor sum of $(V_L - V_C)$ and V_R in the normal way. From the phasor diagram it is seen that

$$V = [V_R^2 + (V_L - V_C)^2]^{1/2}$$

$$= [(IR)^2 + (IX_L - IX_C)^2]^{1/2}$$

$$= I[R^2 + (X_L - X_C)^2]^{1/2}$$

$$\frac{V}{I} = [R^2 + (X_L - X_C)^2]^{1/2}$$

$$= \sqrt{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]} \qquad (3.16)$$

The quantity $\sqrt{[R^2 + (X_L - X_C)^2]}$ is the impedance of the circuit, and is measured in ohms. The impedance triangle is shown at (c). When $X_L > X_C$ the circuit is predominantly inductive and the current lags the applied voltage. When $X_L < X_C$ the circuit is predominantly capacitive and the current leads the applied voltage. The phase angle ϕ is given by $\tan\phi = (X_L - X_C)/R$. Thus, when $\tan\phi$ is positive the current lags the voltage, and when $\tan\phi$ is negative the current leads the voltage. In a circuit containing resistance, inductance and capacitance in series, the current is given by

$$I = \frac{V}{Z} = \frac{V}{\sqrt{[R^2 + (X_L - X_C)^2]}} \qquad (3.17)$$

Example 3.2 A circuit consists of a coil of inductance 0·1 H and effective resistance 20 Ω connected in series with a capacitor of 50 μF to a 100 V 50 Hz supply. Calculate the current in the circuit and the p.d.'s across the coil and capacitor. Sketch the phasor diagram.

$$X_L = 2\pi \times 50 \times 0{\cdot}1 = 31{\cdot}4\ \Omega$$

$$X_C = \frac{10^6}{2\pi \times 50 \times 50} = 63{\cdot}7\ \Omega$$

$$X_L - X_C = -32{\cdot}3\ \Omega$$

$$Z = \sqrt{[20^2 + (-32{\cdot}3)^2]} = 38\ \Omega$$

$I = 100/38 = 2{\cdot}63$ A and leads the applied voltage by an angle

$$\tan^{-1}\frac{32{\cdot}3}{20} = 58{\cdot}2°$$

P.D. across capacitor $= IX_C = 2{\cdot}63 \times 63{\cdot}7 = 168$ V

Impedance of coil $= \sqrt{(20^2 + 31{\cdot}4^2)} = 37{\cdot}2\ \Omega$

P.D. across coil $= 2{\cdot}63 \times 37{\cdot}2 = 97{\cdot}6$ V.

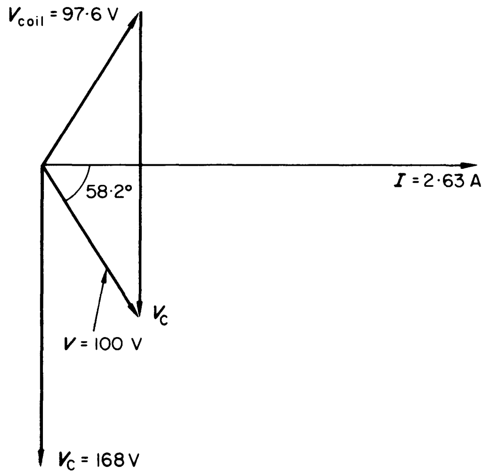


figure 3.15

Series or voltage resonance

An important characteristic of the series $R-L-C$ circuit is that the total reactance of the circuit can be zero even though the reactances of the coil and capacitor are not zero and indeed may each be quite large. The circuit reactance is given by the arithmetic difference between coil and capacitor reactances. Since the reactance of the coil increases with an increase of frequency while that of the capacitor decreases, there must be some frequency at which these two reactances are equal. This frequency is termed the *resonant frequency* f_0. At resonance,

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0^2 = \frac{1}{\sqrt{(LC)}}$$

and

$$f_0 = \frac{1}{2\pi\sqrt{(LC)}} \qquad (3.18)$$

At the resonant frequency, the impedance of the circuit is due entirely to its resistance and the current is given by $I = V/R$. If the resistance is small compared with the individual reactances, then the p.d.s across the latter, namely IX_L and IX_C, may be many times the supply voltage. The circuit may thus be used for voltage magnification. The voltage developed across the coil (or capacitor) at resonance divided by the supply voltage is termed the *Q factor* of the circuit. The smaller the resistance of the circuit the greater is the Q factor.

$$Q = \frac{V_L}{V} = \frac{I\omega_0 L}{IR} = \frac{\omega_0 L}{R} \tag{3.19}$$

Since at resonance $1/\omega_0 C = \omega_0 L$ then $Q = 1/\omega_0 CR$.

In these expressions the resistance R is the effective resistance of the circuit at the resonant frequency. This resistance increases as the frequency increases, largely because of the tendency for the current to flow in the area near the surface of the conductor, leaving the centre of the conductor carrying little current (the skin effect). In practical cases the Q factor of a resonant circuit at high frequency is limited to about 200.

Below the resonant frequency $1/\omega C$ is larger than ωL and the circuit impedance increases. Above the resonant frequency ωL is larger than $1/\omega C$ and again the circuit impedance increases. Curves of current against frequency thus take the form shown in figure 3.16.

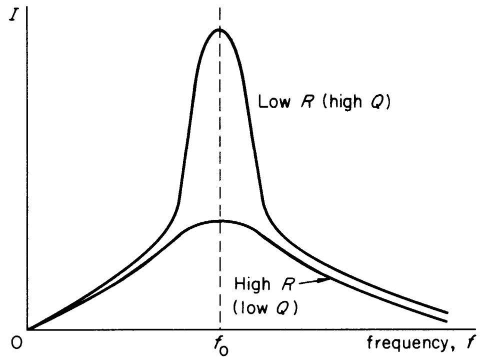


figure 3.16

At the resonant frequency the maximum current is much larger for circuits of low resistance than for those of high resistance. The current/frequency curve is thus much sharper for high-Q circuits than for those with low Q.

Example 3.3 Give expressions for the reactances of an inductor L henrys and a capacitor C farads at a frequency f hertz. Sketch the reactance/frequency curves for these two components, starting at zero frequency.

An air-cored inductor takes 40 mA when a 10 V 50 kHz supply is connected across it. A capacitor, C farads, is connected in series with the inductor and when the frequency of the supply is varied the current through the circuit rises to a maximum of 1 A. If the supply frequency is then 100 kHz calculate:

(a) the resistance of the inductor;
(b) the inductance;
(c) the capacitance; and
(d) the Q factor at resonance.

(CGLI Principles B, 1970)

(a) $$\text{Coil resistance} = \frac{V}{I} \text{ at resonance}$$

$$= \frac{10}{1} = 10\,\Omega$$

(b) $$\text{At 50 kHz, } Z = \frac{10}{0{\cdot}04} = 250\,\Omega$$

but

$$Z = \sqrt{(R^2 + X_L^2)}$$

$\therefore$

$$250 = \sqrt{[10^2 + (2\pi \times 50\,000L)^2]}$$

$$250^2 = 10^2 + (10^5\pi L)^2$$

$$L^2 = \frac{62\,400}{10^{10} \times \pi^2}$$

$$= 6340 \times 10^{-10}$$

$$L = 79{\cdot}6 \times 10^{-5}\,\text{H} = 796\,\mu\text{H}$$

This assumes that the resistance of the coil is constant.

(c) $$\omega_0 L = \frac{1}{\omega_0 C}$$

$\therefore$

$$C = \frac{1}{\omega_0^2 L} = \frac{1}{4\pi^2 \times 10^{10} \times 796 \times 10^{-6}}\,\text{F}$$

$$= 0{\cdot}00318\,\mu\text{F}$$

(d) $$Q = \frac{\omega_0 L}{R} = \frac{2\pi \times 100\,000 \times 796 \times 10^{-6}}{10} = 50$$

The first part of the question has been covered in the text.

Parallel circuits

When a circuit consists of several parallel branches, the magnitude and phase of the current in each branch depends upon the electrical elements in the branch, while the total current is the phasor sum of the branch currents. The voltage applied to the circuit is common to all branches, and when drawing the phasor diagram this voltage is therefore taken as the reference phasor and is drawn horizontal. Consider the circuit shown in figure 3.17a.

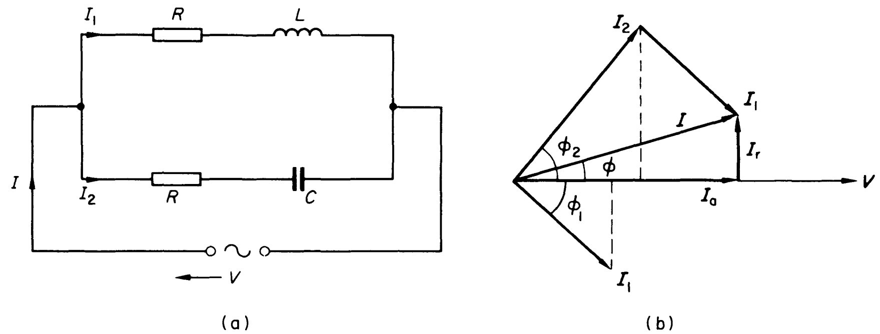


figure 3.17

The current I_1 and its phase angle ϕ_1 are first calculated and drawn as shown, I_1 lagging V by an angle ϕ_1 because the circuit is inductive. Similarly, I_2 and ϕ_2 are calculated and drawn with I_2 leading the voltage because this branch is capacitive. When these currents are drawn to a suitable current scale, the total current I and its phase angle ϕ may be determined by phasor addition as shown in figure 3.17b.

Alternatively, the branch currents may be resolved into active components (that is, $I_1 \cos \phi_1$ and $I_2 \cos \phi_2$) in phase with the voltage, and reactive components (that is, $I_1 \sin \phi_1$ and $I_2 \sin \phi_2$) in quadrature (at 90°) with the voltage. Active components are in phase with each other and are added arithmetically to obtain the total active component I_a. Reactive components are also added arithmetically, taking sign into account, to obtain the total reactive component I_r. The total current is then given by

$$I = \sqrt{(I_a^2 + I_r^2)} \tag{3.20}$$

and its phase angle is given by

$$\tan \phi = \frac{I_r}{I_a} \tag{3.21}$$

The method may obviously be used for any number of branches.

Example 3.4 Determine by means of phasors drawn to scale, or otherwise, the magnitude of the current flowing and its phase relative to the supply voltage when each of the circuits shown in figure 3.18 is connected separately across a 100 V, 100 kHz supply.

Find the total current taken and its phase relative to the supply voltage when the two circuits are connected in parallel.

(CGLI Principles B, 1968)

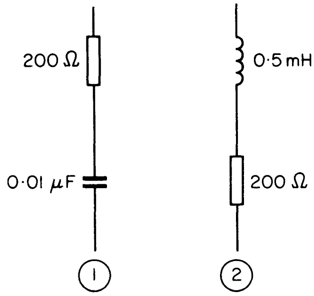


figure 3.18

Circuit 1:

$$X_C = \frac{10^6}{2\pi \times 100 \times 10^3 \times 0{\cdot}01} = 159\ \Omega$$

$$Z_1 = \sqrt{(200^2 + 159^2)} = 256\ \Omega$$

$$I_1 = \frac{100}{256} = 0{\cdot}391\ \text{A}$$

$$\tan\phi_1 = \frac{159}{200} = 0{\cdot}795$$

$$\therefore \qquad \phi_1 = 38^\circ\ 30'\ \text{leading}$$

Circuit 2:

$$X_L = 2\pi \times 100 \times 10^3 \times 0{\cdot}5 \times 10^{-3} = 314\ \Omega$$

$$Z_2 = \sqrt{(200^2 + 314^2)} = 372\ \Omega$$

$$I_2 = \frac{100}{372} = 0{\cdot}269\ \text{A}$$

$$\tan\phi_2 = \frac{314}{200} = 1{\cdot}57$$

$$\therefore \qquad \phi_2 = 57^\circ\ 30'\ \text{lagging}$$

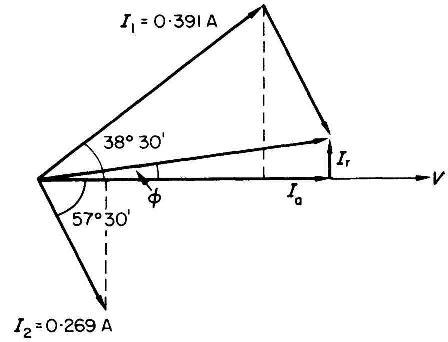


figure 3.19

Figure 3.19 (not to scale) shows I_1 and I_2 with their correct phase relationships to the voltage. Then

total active component

$$\begin{aligned} I_a &= 0{\cdot}391 \cos 38^\circ\, 30' + 0{\cdot}269 \cos 57^\circ\, 30' \\ &= 0{\cdot}391 \times 0{\cdot}783 + 0{\cdot}269 \times 0{\cdot}537 \\ &= 0{\cdot}306 + 0{\cdot}145 \\ &= 0{\cdot}451 \end{aligned}$$

total reactive component

$$\begin{aligned} I_r &= 0{\cdot}391 \sin 38^\circ\, 30' - 0{\cdot}269 \sin 57^\circ\, 30' \\ &= 0{\cdot}391 \times 0{\cdot}623 - 0{\cdot}269 \times 0{\cdot}843 \\ &= 0{\cdot}243 - 0{\cdot}227 \\ &= 0{\cdot}016 \end{aligned}$$

total current $\qquad I = \sqrt{(0{\cdot}451^2 + 0{\cdot}016^2)} = 0{\cdot}4513$ A

$$\tan \phi = \frac{0{\cdot}016}{0{\cdot}451} = 0{\cdot}0354$$

$\therefore \qquad \phi = 2^\circ$ leading

Admittance, conductance and susceptance

In a d.c. circuit comprising a number of resistances in parallel, the joint resistance of the circuit is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$$

or

$$G = G_1 + G_2 + G_3$$

where G is the conductance.

Similarly, for an a.c. circuit comprising a number of impedances in parallel, the joint impedance of the circuit is given by

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \ldots$$

or

$$Y = Y_1 + Y_2 + Y_3 + \ldots$$

where Y is the admittance, the summation being phasorial, not arithmetic.

Consider first a single impedance comprising R and X in series, as shown in figure 3.20a.

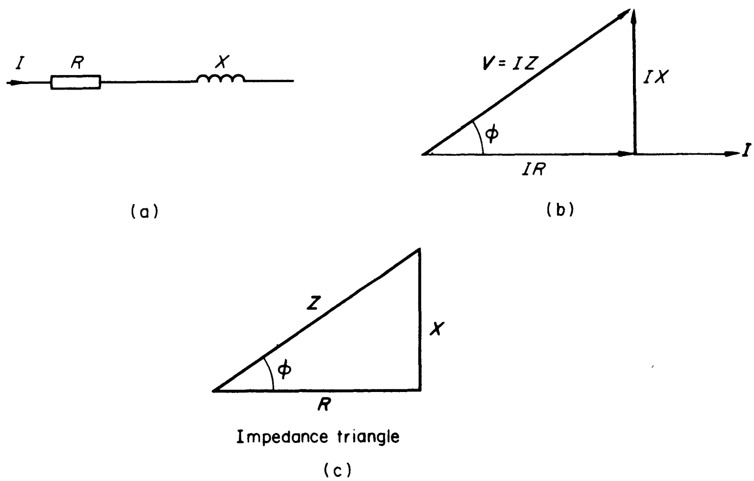


figure 3.20

The phasor diagram showing the current and voltages is shown at (b), and the impedance triangle at (c).

The circuit may be replaced by an equivalent parallel circuit having a total *admittance* Y, which has two components: G the *conductance* and B the *susceptance*, as shown in figure 3.21a.

The phasor diagram is shown at (b) and since the circuit comprises two components in parallel, the voltage V is taken as the reference phasor. The current through G is given by $I_G = V \times G$ and is in phase with V. The current through B is given by $I_B = V \times B$ and lags the voltage by 90°. The total current I is the phasor sum of I_G and I_B, and is equal to the current I in the original impedance, lagging the voltage by the same angle ϕ. The admittance triangle of figure 3.21c is thus similar to the impedance triangle of

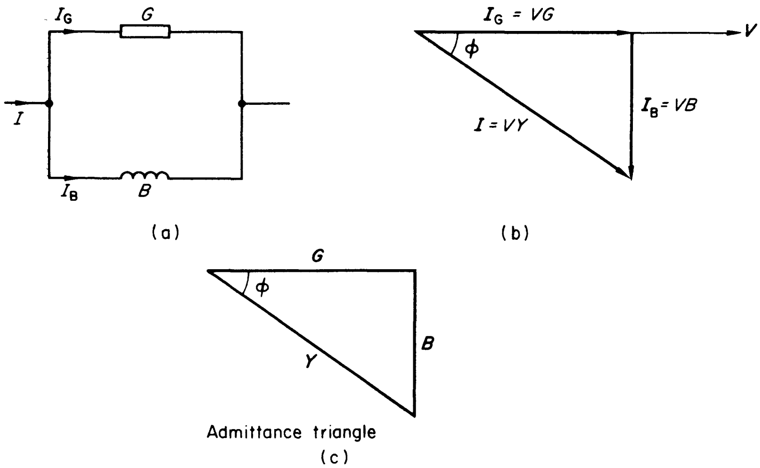


figure 3.21

figure 3.20c, except that it is inverted, a positive reactance X being replaced by a negative susceptance B. As the triangles are similar

$$\frac{B}{G} = \frac{X}{R} \tag{i}$$

and

$$G^2 + B^2 = Y^2 = \frac{1}{Z^2} = \frac{1}{R^2 + X^2} \tag{ii}$$

From (i)

$$B = G\frac{X}{R}$$

Substituting in (ii)

$$G^2 + G^2\frac{X^2}{R^2} = \frac{1}{R^2 + X^2}$$

$$G^2(R^2 + X^2) = \frac{R^2}{R^2 + X^2}$$

$$G^2 = \frac{R^2}{(R^2 + X^2)^2}$$

$$\therefore \qquad G = \frac{R}{Z^2} \tag{3.22}$$

Substituting in (i)

$$B = \frac{X}{Z^2} \tag{3.23}$$

Example 3.5 The circuit shown in figure 3.22 is connected to a 100 V, 100 kHz supply. Calculate the total current taken and its phase angle relative to the supply voltage.

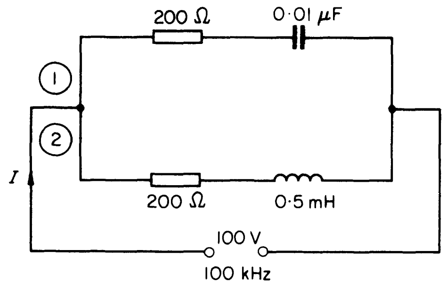


figure 3.22

This is the problem already solved in example 3.4 by phasor addition of the branch currents. Using the admittance method, each branch impedance is first converted into an equivalent parallel circuit with conductance G and susceptance B, as shown in figure 3.23.

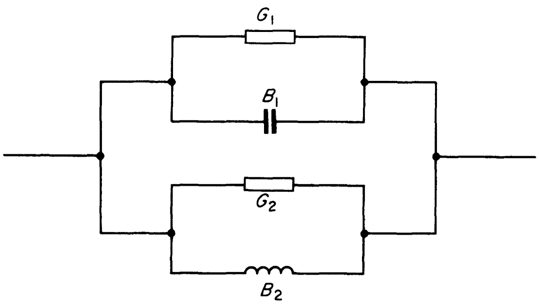


figure 3.23

The total conductance G is then given by the arithmetic sum of the branch conductances, and the total susceptance B is given by the arithmetic sum of the branch susceptances, taking sign into account.

From example 3.4

$$Z_1 = 256\,\Omega \text{ and } Z_2 = 372\,\Omega$$

$$G_1 = R_1/Z_1^2 = 200/256^2 = 0{\cdot}00306 \text{ S (siemens)}$$

$$G_2 = R_2/Z_2^2 = 200/372^2 = 0{\cdot}00145 \text{ S}$$

$$G = G_1 + G_2 = 0{\cdot}00451 \text{ S}$$

$$B_1 = X_1/Z_2^2 = 159/256^2 = 0{\cdot}00243 \text{ S}$$

$$B_2 = X_2/Z_2^2 = -314/372^2 = -0{\cdot}00227 \text{ S}$$

$$B = B_1 + B_2 = 0{\cdot}00016 \text{ S}$$

$$\text{Total admittance } Y = \sqrt{(G^2 + B^2)}$$

$$= \sqrt{(0{\cdot}00451^2 + 0{\cdot}00016^2)}$$

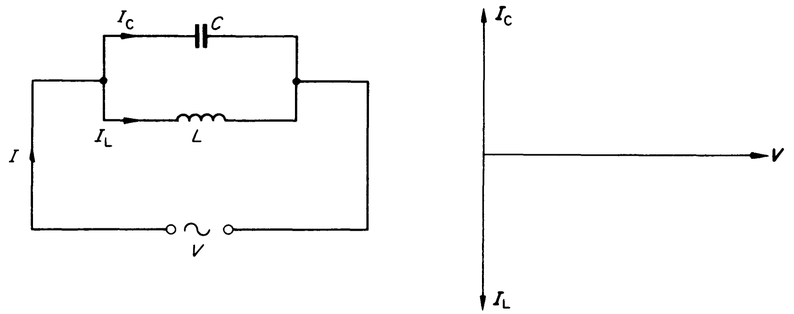
$$= 0{\cdot}004514 \text{ S}$$

$$\text{Current } I = VY = 100 \times 0{\cdot}00451 = 0{\cdot}451 \text{ A}$$

$$\tan\phi = \frac{B}{G} = \frac{0{\cdot}00016}{0{\cdot}00451} = 0{\cdot}0354$$

$$\therefore \quad \phi = 2^\circ \text{ leading}$$

Parallel or current resonance



(a) (b)

figure 3.24

Consider first an inductance L, of negligible resistance, connected in parallel with a perfect capacitor C. The reactance of the inductor increases with an increase of frequency while that of the capacitor decreases. At some frequency f_0 these reactances are equal, hence the currents I_L and I_C are equal, but opposite in phase, as shown in the phasor diagram of figure 3.24b.

This frequency is termed the *resonant frequency* f_0. At this resonant frequency the current taken from the supply is zero, even though each of the branch currents I_C and I_L may be quite large. These branch currents are maintained by the interchange of energy between the magnetic field of the inductor and the electric field of the capacitor. At this frequency of resonance,

$$X_L = X_C$$

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0 = \frac{1}{\sqrt{(LC)}}$$

and

$$f_0 = \frac{1}{2\pi\sqrt{(LC)}} \tag{3.24}$$

This expression is the same as that for the series resonant circuit. In practice, however, the coil has some resistance, and at the resonant frequency a current flows in the main circuit from the supply, this current being of sufficient magnitude to supply the power losses due to the coil resistance. The resonant frequency f_0 also differs slightly from that obtained by assuming the coil resistance to be zero. Resonance is defined as occurring when the small current I taken from the supply is in phase with the supply voltage V, as shown in the phasor diagram of figure 3.25.

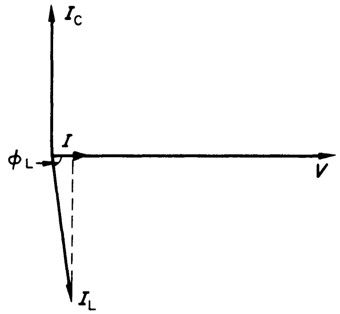


figure 3.25

In the phasor diagram I_L lags V by an angle ϕ_L, approaching but somewhat less than 90°.

$$I_L = \frac{V}{Z_L} = \frac{V}{\sqrt{[R^2 + (\omega_0 L)^2]}}$$

(where R is the effective resistance of the coil at the resonant frequency).

$$I_C = \frac{V}{1/\omega_0 C} = V\omega_0 C$$

But, at resonance, $I_C = I_L \sin\phi_L$, and as $\sin\phi_L = X_L/Z_L$

$$\therefore \qquad \left(\frac{V}{\sqrt{[R^2+(\omega_0 L)^2]}}\right)\left(\frac{\omega_0 L}{\sqrt{[R^2+(\omega_0 L)^2]}}\right) = V\omega_0 C$$

$$\therefore \qquad \frac{L}{R^2+(\omega_0 L)^2} = C$$

$$R^2+(\omega_0 L)^2 = \frac{L}{C}$$

$$(\omega_0 L)^2 = \frac{L}{C} - R^2$$

$$\therefore \qquad \omega_0^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$\omega_0 = \sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)}$$

and

$$f_0 = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)} \qquad (3.25)$$

Putting $R = 0$, equation 3.24 is obtained. Also when $R = 0$, the current taken from the supply at the resonant frequency is zero. An increase in coil

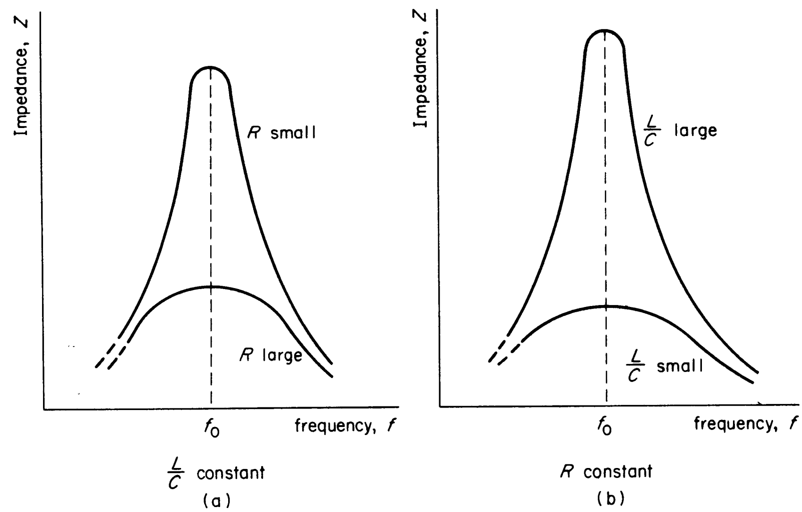


figure 3.26

resistance is seen to cause an *increase* in the supply current. Thus, as the coil resistance increases, the impedance of the circuit at the resonant frequency decreases. Figure 3.26a shows how the impedance of the circuit varies with frequency, for different values of coil resistance. The curves are seen to be sharper for low values of coil resistance than for high values.

The ratio L/C also affects the shape of the impedance/frequency curves; for example, for a given value of coil resistance, an increase in the ratio L/C causes the impedance/frequency curve to become sharper (that is, increases the selectivity). This is shown in figure 3.26b.

The impedance of the circuit at resonance is given by $Z = V/I$; but reference to figure 3.25 shows that

$$I = \frac{I_C}{\tan \phi_L}$$

$$\therefore \qquad Z = \frac{V}{I_C} \tan \phi_L$$

$$= X_C \frac{X_L}{R}$$

$$= \left(\frac{1}{2\pi f_0 C}\right)\left(\frac{2\pi f_0 L}{R}\right)$$

$$= \frac{L}{CR}$$

Thus, at resonance, the parallel combination of a coil and a capacitance is equivalent to a non-inductive resistance of L/CR ohms. This quantity is called the *dynamic impedance* Z_D of the circuit. Thus

$$Z_D = \frac{L}{CR} \tag{3.26}$$

The lower the coil resistance and the higher the ratio L/C, the higher is the dynamic impedance.

Q-factor

The current circulating in the $L + C$ combination at resonance may be many times the current taken from the supply. The current in the capacitor at resonance, divided by the supply current, is termed the Q-factor of the circuit, and is thus a measure of current magnification. Thus

$$Q = \frac{I_C}{I} = \frac{I_L \sin \phi_L}{I} = \frac{\sin \phi_L}{\cos \phi_L} = \tan \phi_L = \frac{\omega_0 L}{R} \tag{3.27}$$

Example 3.6 One branch of a parallel circuit consists of a coil of inductance 0·2 H and effective resistance 10 Ω, and the other is a 40 μF capacitor. Calculate:

(i) the resonant frequency of the circuit,
(ii) the dynamic impedance, and
(iii) the Q-factor.

(i)
$$f_0 = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)}$$
$$= \frac{1}{2\pi}\sqrt{\left(\frac{10^6}{0{\cdot}2 \times 40} - \frac{10^2}{0{\cdot}2^2}\right)}$$
$$= \frac{1}{2\pi}\sqrt{\left(\frac{10^6}{8} - \frac{10^6}{400}\right)}$$
$$= \frac{1000}{2\pi}\sqrt{\left(\frac{50-1}{400}\right)}$$
$$= \frac{7000}{2\pi \times 20} = 55{\cdot}7 \text{ Hz}$$

(ii)
$$Z_D = \frac{L}{CR} = \frac{0{\cdot}2}{40 \times 10^{-6} \times 10}$$
$$= 500\ \Omega$$

(iii)
$$Q = \frac{\omega_0 L}{R} = \frac{7000 \times 0{\cdot}2}{20 \times 10}$$
$$= 7{\cdot}0$$

In the above example, it is seen that the term R^2/L^2 is small compared with $1/LC$. In a practical tuned circuit, the effective resistance of the coil is many times the d.c. resistance. It is, however, kept as low as possible and is usually small compared with the reactance of either branch. Thus, in high frequency tuned circuits, R^2/L^2 is quite negligible compared with $1/LC$, and the resonant frequency is given by:

$$f_0 = \frac{1}{2\pi\sqrt{(LC)}}$$

Example 3.7 A coil of inductance 0·2 mH and effective resistance 10 Ω at the operating frequency is tuned to resonance at 1 MHz. Determine the value of capacitance required. Determine also the dynamic impedance and the Q-factor of the circuit. If the circuit is now shunted by a non-inductive resistance of 157 kΩ, determine the dynamic impedance of the combined circuit.

$$f_0 = \frac{1}{2\pi\sqrt{(LC)}}$$

$$\therefore \qquad C = \frac{1}{4\pi^2 f_0^2 L}$$

$$C = \frac{1}{4\pi^2 \times 10^{12} \times 0{\cdot}2 \times 10^{-3}} = 127 \times 10^{-12}\ \text{F}$$

$$= 127\ \text{pF}$$

$$Z_\text{D} = \frac{L}{CR} = \frac{0{\cdot}2 \times 10^{-3}}{127 \times 10^{-12} \times 10} = 157\ \text{k}\Omega$$

$$Q = \frac{\omega_0 L}{R} = \frac{2\pi \times 10^6 \times 0{\cdot}2 \times 10^{-3}}{10} = 126$$

The circuit may be considered as consisting of inductance, capacitance and resistance in parallel as shown in figure 3.27, the resistance being equal to $Z_\text{D} = L/CR$. A 157 kΩ resistance connected in parallel with the circuit thus reduces the dynamic impedance to $157/2 = 78{\cdot}5$ kΩ.

Further work on parallel tuned circuits will be found in chapter 8 where circuits of this type are used as the collector load of tuned RF amplifiers.

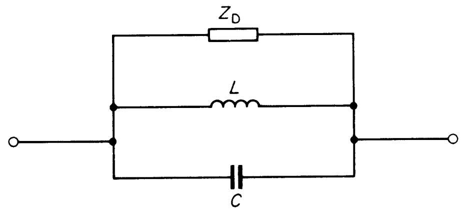


figure 3.27

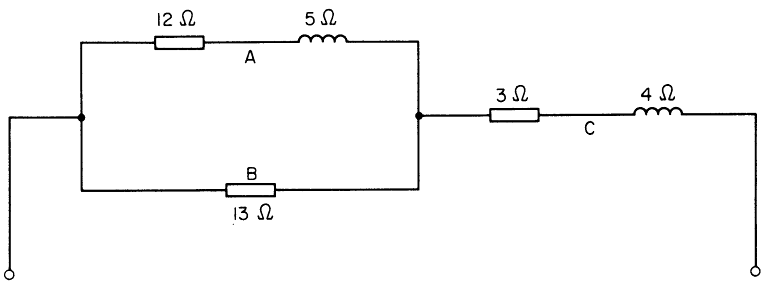


figure 3.28

MORE ADVANCED THEORY

Example 3.8 A series–parallel circuit has the component values shown in figure 3.28 when the supply frequency is 50 Hz.

Calculate:

(a) the total impedance and the phase angle of the complete circuit
(b) the current taken from a 10 V, 50 Hz supply.

(a) A circuit such as A has impedance and admittance triangles as shown in figure 3.29.

From the admittance triangle,

$$\text{conductance } G = Y \cos\theta$$

where admittance

$$Y = \frac{1}{Z}$$

and

$$\theta = \text{phase angle}$$

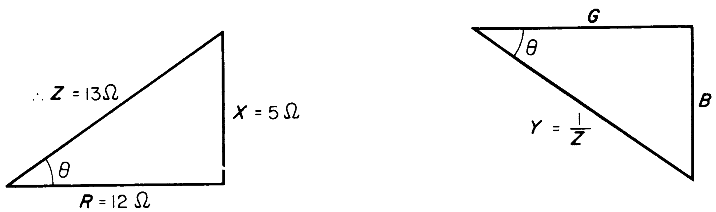


figure 3.29

∴

$$G = \frac{1}{Z} \times \frac{R}{Z} = \frac{R}{Z^2}$$

that is,

$$G_A = \frac{12}{169}\ \text{S}$$

also, susceptance

$$B = Y \sin\theta$$

∴

$$B = \frac{1}{Z} \times \frac{X}{Z} = \frac{X}{Z^2}$$

that is,

$$B_A = \frac{5}{169}\ \text{S}$$

Furthermore, the phase angle of a circuit = $\tan^{-1} B/G$

$$\therefore \qquad \theta_A = \tan^{-1} \frac{5}{12}$$

It can be seen that $G_B = (1/13)$ S and its phase angle is zero. The total admittance of the parallel circuit is given by

$$Y_{(A+B)} = \sqrt{\left[\left(\frac{12}{169} + \frac{1}{13}\right)^2 + \left(\frac{5}{169}\right)^2\right]} = \frac{1}{169}\sqrt{650} = 0{\cdot}1509 \text{ S}$$

Since the phase angle is

$$\tan^{-1} \frac{\text{Susceptance}}{\text{Conductance}}$$

∴ Phase angle of A and B combined

$$= \tan^{-1} \frac{5}{169} \times \frac{169}{25} = \tan^{-1} 0{\cdot}2$$

$$\therefore \qquad \text{Phase angle} = 11{\cdot}3^\circ$$

Thus the combination of A and B in parallel can be regarded as equivalent to a resistance $[(1/0{\cdot}1509) \cos 11{\cdot}3^\circ]$ Ω in series with an inductive reactance of $[(1/0{\cdot}1509) \sin 11{\cdot}3^\circ]$ Ω.

∴ Resistance of the complete circuit

$$= 6{\cdot}63 \times 0{\cdot}9806 + 3 = 9{\cdot}5\ \Omega$$

Reactance of the complete circuit

$$= 6{\cdot}63 \times 0{\cdot}1959 + 4 = 5{\cdot}3\ \Omega$$

$$\therefore \quad \text{Total impedance} = \sqrt{(9{\cdot}5^2 + 5{\cdot}3^2)} = 10{\cdot}9\ \Omega$$

Phase angle of the complete circuit

$$= \tan^{-1} \frac{5{\cdot}3}{9{\cdot}5} = 29{\cdot}2^\circ$$

(b) Current taken from a 10 V, 50 Hz supply

$$= \frac{10}{10{\cdot}9} = 0{\cdot}917 \text{ A}$$

In a circuit containing inductance, this current will be lagging behind the supply voltage.

A good deal of the labour involved in the above calculation could have been avoided if the various impedances had been treated as *complex quantities.*

This technique requires a working knowledge of the *Operator* j, the main properties of which will now be explained.

'Operator j' and complex numbers

Consider the quadratic equation

$$x^2 - 4x + 13 = 0$$

By the formula for solving quadratic equations

$$x = \frac{4 \pm \sqrt{(4^2 - 52)}}{2} = 2 \pm \sqrt{-9}$$

$\sqrt{-9}$ is not a real number but an imaginary quantity. It can, however, be given a value if the symbol j is introduced as an OPERATOR to take care of the negative sign under the square root.

Thus,

$$x = 2 \pm \sqrt{-9} = 2 \pm \mathrm{j}\sqrt{9} = 2 \pm \mathrm{j}3$$

Roots of quadratic equations obtained in this form are said to be 'complex' numbers because they comprise real and imaginary parts. It should also be noted that as j is equivalent to $\sqrt{-1}$, then $\mathrm{j}^2 = -1$.

Argand diagram

Complex numbers may be plotted on a graph in which the conventional X and Y axes are replaced by the REAL and IMAGINARY axes respectively

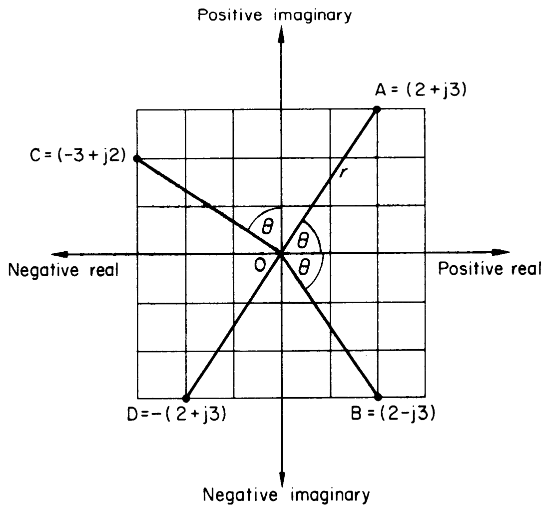


figure 3.30

as shown in figure 3.30. It should be stated that $2 + j3$ can mean either the point A or, as is much more relevant in a.c. circuit calculations, it can represent the phasor ***OA*** in magnitude and phase with respect to the positive REAL axis. For example, it could be an impedance comprising a 2 Ω resistance in series with an inductive reactance of 3 Ω. Alternatively, it could indicate that a current of magnitude OA leads a voltage by the angle θ, as shown in figure 3.30.

When the phasor ***OA*** is multiplied by j, it becomes $-3 + j2$ and takes up a position OC on the Argand diagram. It is not difficult to show that OA and OC are the same length and that the angle between them is 90°. Therefore j can be regarded as an 'operator' which has the effect of advancing a phasor by 90°. If the process were repeated, the phasor rotates in an anti-clockwise direction through another right-angle and then assumes the position OD in anti-phase with OA.

Cartesian and polar forms

In figure 3.30 let the length of OA (called the *modulus*) and the angle (called the *argument*) which it makes with the positive real axis be r and θ respectively. By the Theorem of Pythagoras,

$$r = \sqrt{[(\text{real term})^2 + (\text{imaginary term})^2]} = \sqrt{13} \text{ units}$$

and

$$\tan\theta = \frac{\text{imaginary term}}{\text{real term}} = 1{\cdot}5$$

This leads to another way of expressing complex numbers. In place of the cartesian form $(2 + j3)$, ***OA*** may equally be written in its polar form as $\sqrt{13}\,\angle\tan^{-1} 1{\cdot}5$. It is the second of these two forms which is the more convenient when electric problems require multiplication and division. Before applying any of these results to a network, there are certain rules of manipulation which should be thoroughly understood by the reader.

Addition and subtraction

These operations can only be performed while the complex numbers are in their cartesian form. The rule is to treat them as algebraic quantities by combining 'like' terms separately; for example

1. $(2-j3)+(1+j2)=(2+1)+j(-3+2)=3-j1$
2. $(2+j3)-(1-j2)=(2-1)+j(3+2)=1+j5$

Multiplication

Once again, complex numbers in cartesian form are treated as algebraic quantities and then 'like' terms collected, as follows:

$$(2 + j3)(1 - j2) = 2(1 - j2) + j3(1 - j2) = 2 - j4 + j3 + 6$$
$$= 8 - j1$$

The principle involved when complex numbers in their polar form are being multiplied is illustrated in figure 3.31.

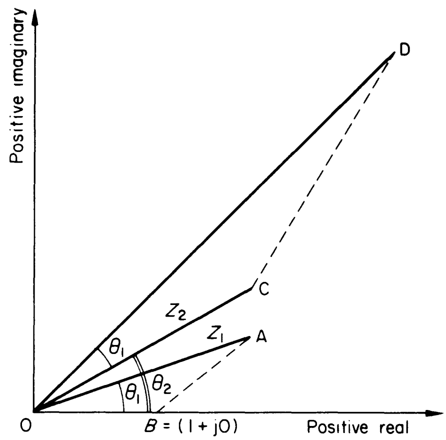


figure 3.31

The two complex numbers Z_1 and Z_2 are to be multiplied together. By construction, A is joined to B = (1 + j0) and then a triangle OCD, similar to AOB, is superimposed on OC as shown.

From the diagram,

$$\frac{OD}{OA} = \frac{OC}{OB}$$

$$\therefore \qquad OD = \frac{OA \times OC}{OB}$$

But OB is 1 unit in length so that OD has a length which is numerically equal to the modulus of $Z_1 \times Z_2$. Furthermore, the angle between OD and the positive real axis is $(\theta_1 + \theta_2)$. Mathematically, this important result may be written as

$$Z_1\underline{/\theta_1} \times Z_2\underline{/\theta_2} = Z_1Z_2\underline{/\theta_1 + \theta_2} \tag{3.28}$$

If the two complex numbers Z_1 and Z_2 had been equal, the answer would be of the form

$$(Z_1\underline{/\theta})^2 = Z_1^2\underline{/2\theta}$$

Division

Using the same Argand diagram as in figure 3.30 it can be verified that

$$\frac{Z_2\underline{/\theta_2}}{Z_1\underline{/\theta_1}} = \frac{Z_2}{Z_1}\underline{/\theta_2 - \theta_1} \qquad (3.29)$$

Thus the two main rules when dealing with complex numbers in polar form may be stated quite simply as:

(a) moduli obey the rules of Arithmetic;
(b) arguments obey the rules of Indices.

If the numbers are already given in cartesian form, it is still possible to perform the operation of division without the necessity of converting them into polar form first. The method is referred to as *rationalisation*, in which the quotient is multiplied throughout by the *conjugate* of the denominator. This is the complex number that differs from the denominator only in the sign of its imaginary term; for example

$$\frac{8 - \mathrm{j}1}{1 - \mathrm{j}2} = \frac{(8 - \mathrm{j}1)(1 + \mathrm{j}2)}{(1 - \mathrm{j}2)(1 + \mathrm{j}2)} = \frac{(8 + 2) + \mathrm{j}(-1 + 16)}{(1 + 4) + \mathrm{j}(-2 + 2)}$$

$$= \frac{10 + \mathrm{j}15}{5} = 2 + \mathrm{j}3$$

Equating

As complex numbers represent phasor quantities, they can be equated only if they are identical in both magnitude and phase. For example:

Let $\qquad Z_1 = a + \mathrm{j}b$

and $\qquad Z_2 = c + \mathrm{j}d$

If $\qquad Z_1 = Z_2$

$$a + \mathrm{j}b = c + \mathrm{j}d$$

$$\therefore \qquad a - c = -\mathrm{j}(b - d).$$

Squaring throughout,

$$(a - c)^2 = -(b - d)^2$$

Terms such as $(a-c)^2$ and $(b-d)^2$ must either be positive or zero since they are the squares of real numbers. The only reasonable conclusion is that each side of the last equation is zero because of the negative sign in front of $(b-d)^2$. Hence $a=c$ and $b=d$ which shows that Z_1 and Z_2 are equal only if both their real terms are equal, and their imaginary terms are equal. The usefulness of these results is demonstrated in many of the examples given in chapter 10 on a.c. bridges and elsewhere.

Example 3.9 Evaluate the following:

(a) $\dfrac{3+\mathrm{j}2}{(1-\mathrm{j})^2}$ (b) $\sqrt{\left[\dfrac{(4+\mathrm{j}3)(4-\mathrm{j}3)}{5-\mathrm{j}12}\right]}$

(a)
$$\frac{3+\mathrm{j}2}{(1-\mathrm{j})^2}=\frac{3+\mathrm{j}2}{1-\mathrm{j}2+\mathrm{j}^2}=\frac{3+\mathrm{j}2}{0-\mathrm{j}2}=\frac{(3+\mathrm{j}2)(0+\mathrm{j}2)}{(0-\mathrm{j}2)(0+\mathrm{j}2)}$$
$$=\frac{-4+\mathrm{j}6}{4}=-1+\mathrm{j}1{\cdot}5$$

(b)
$$\sqrt{\left[\frac{(4+\mathrm{j}3)(4-\mathrm{j}3)}{5-\mathrm{j}12}\right]}$$

Note: 4 + j3 lies in the first quadrant whereas 4 − j3 and 5 − j12 are both in the fourth quadrant of an Argand diagram where the argument has a tangent that is negative.

Therefore, the given expression, converted to polar form, becomes

$$\sqrt{\left[\frac{5\ \tan^{-1}(0{\cdot}75)\times 5\ \tan^{-1}(-0{\cdot}75)}{13\underline{/\tan^{-1}(-2{\cdot}4)}}\right]}$$
$$=\left[\frac{25\underline{/0^\circ}}{13\underline{/-67{\cdot}4^\circ}}\right]=\frac{5}{\sqrt{13}}\underline{/\frac{67{\cdot}4^\circ}{2}}=1{\cdot}386\underline{/33{\cdot}7^\circ}$$

Figure 3.32 shows this answer on an Argand diagram, from which a way of converting back again to cartesians may be understood.

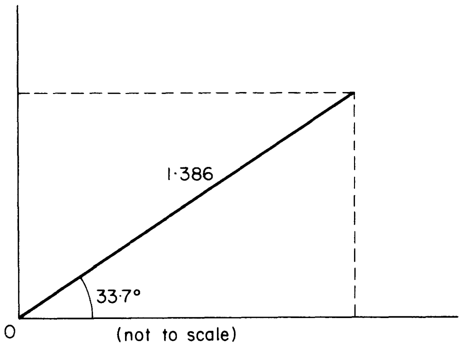


figure 3.32

Hence

$$1{\cdot}386\underline{/33{\cdot}7^\circ} = 1{\cdot}386\,(\cos 33{\cdot}7^\circ + \mathrm{j}\sin 33{\cdot}7^\circ)$$
$$= 1{\cdot}386\,(0{\cdot}832 + \mathrm{j}0{\cdot}555)$$
$$= 1{\cdot}152 + \mathrm{j}0{\cdot}77$$

Now that the basic principles of complex numbers have been explored, a selection of typical a.c. problems will be investigated starting with another look at example 3.8. The circuit of figure 3.28 can be visualised as a number of 'lumped' impedances as in figure 3.33.

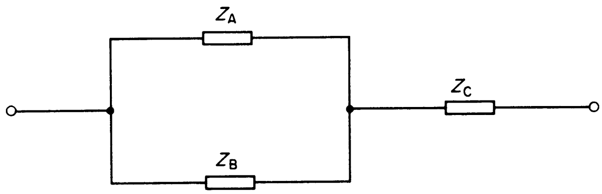


figure 3.33

If this had been a d.c. circuit with resistive elements, the overall resistance could have been written on sight as:

$$\text{Total resistance, } R = \frac{R_A R_B}{R_A + R_B} + R_C$$

In exactly the same way, the overall impedance can be evaluated, provided that the individual impedances are treated as complex numbers, that is

$$\text{Total impedance } Z = \frac{Z_A Z_B}{Z_A + Z_B} + Z_C$$

Substituting the values given,

$$Z = \frac{(12 + \mathrm{j}5)(13 + \mathrm{j}0)}{(12 + \mathrm{j}5) + (13 + \mathrm{j}0)} + 3 + \mathrm{j}4$$
$$= \frac{(12 + \mathrm{j}5)\,13}{25 + \mathrm{j}5} + 3 + \mathrm{j}4$$
$$= \frac{(156 + \mathrm{j}65)(25 - \mathrm{j}5)}{25^2 + 5^2} + 3 + \mathrm{j}4$$
$$= 6{\cdot}5 + \mathrm{j}1{\cdot}3 + 3 + \mathrm{j}4 = 9{\cdot}5 + \mathrm{j}5{\cdot}3$$

∴ Total impedance,

$$Z = 10{\cdot}9\underline{/29{\cdot}2^\circ}\ \Omega$$

which agrees with the answer previously obtained.

Example 3.10 An alternating EMF of 100 V, 50 Hz is applied across points AB of the circuit in figure 3.34.

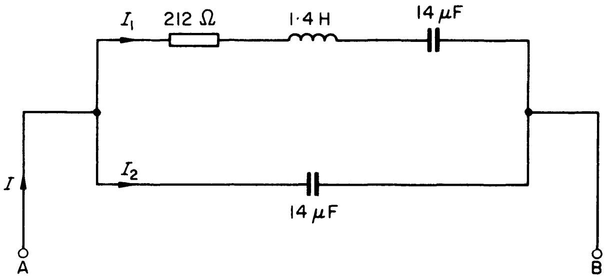


figure 3.34

Determine the current flowing in the resistor, stating your answer in polar form relative to the supply voltage. What is the magnitude of the total current taken from the supply? Sketch a phasor diagram to illustrate these two answers.

Impedance of upper branch

$$Z_1 = R + \mathrm{j}(X_L - X_C)$$

where $\quad X_L = 2\pi \times 50 \times 1{\cdot}4 = 440\ \Omega$

and

$$X_C = \frac{10^6}{2\pi \times 50 \times 14} = 228\ \Omega$$

∴ $\quad Z_1 = (212 + \mathrm{j}212)\ \Omega$ or $300\underline{/45^\circ}\ \Omega$

and current I_1 in the upper branch

$$= \frac{100\underline{/0^\circ}}{300\underline{/45^\circ}} = 0{\cdot}33\underline{/-45^\circ}\ \mathrm{A}$$

This current in the resistor therefore has a value of 0·33 A and lags behind the supply voltage by 45°, the reactance of the upper branch being predominantly inductive at 50 Hz.

Current in the lower branch,

$$I_2 = \frac{V}{-jX_C} = \frac{100\angle 0^\circ}{228\angle -90^\circ}$$

$$= 0{\cdot}44\angle 90^\circ \text{ or } j0{\cdot}44 \text{ A}$$

that is, I_2 leads the supply voltage by 90°.

Total current, I from the supply

$$= 0{\cdot}33\angle -45^\circ + j0{\cdot}44$$

$$= 0{\cdot}33\,(\cos 45^\circ - j \sin 45^\circ) + j0{\cdot}44$$

$$= 0{\cdot}234 - j0{\cdot}234 + j0{\cdot}44$$

$$= 0{\cdot}234 + j0{\cdot}206$$

The magnitude (or modulus) of this current, written as $|I|$ is given by:

$$|I| = \sqrt{(0{\cdot}234^2 + 0{\cdot}206^2)} = 0{\cdot}311 \text{ A}$$

The various phasors are illustrated on figure 3.35.

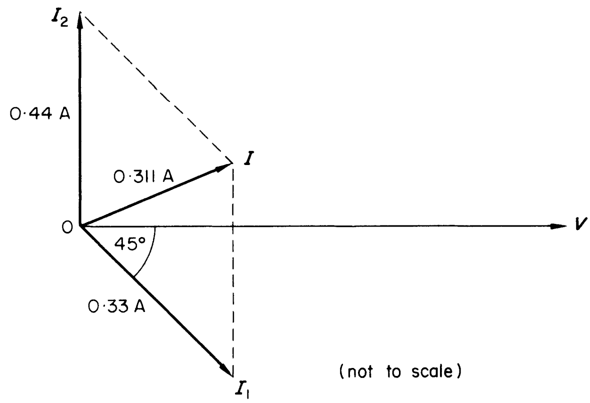


figure 3.35

Calculating power

The instantaneous power in any circuit is the product of the instantaneous values of current and voltage. One would suppose, therefore, that the power in an a.c. circuit also could be found by multiplying the current and voltage when both were expressed as complex numbers. This, however, is not correct. Suppose the RMS values of voltage and current are $V\angle\theta$ and $I\angle\phi$ with respect to the same reference phasor. Experience shows that the actual power in a circuit is calculated by multiplying the product of V and I by the cosine of the phase angle between them; that is

$$P = VI\cos(\theta - \phi)$$

where $\cos(\theta - \phi)$ is the 'power factor' of the circuit. Similarly, the reactive volt-amperes are given by $VI\sin(\theta - \phi)$. To arrive at these same results when using complex numbers, however, it is first essential to change the sign of the j term (or the argument) of either V or I. Applying this method to the example considered, $V\underline{/\theta} \times I\underline{/\phi}$ becomes

$$\begin{aligned} V\underline{/\theta} \times I\underline{/-\phi} &= VI\underline{/\theta - \phi} \\ &= VI\cos(\theta - \phi) + \mathrm{j}VI\sin(\theta - \phi) \\ &= (\text{Power}) + \mathrm{j}(\text{Reactive volt-amperes}) \end{aligned}$$

Example 3.11 The circuit shown in figure 3.36 is connected across a 200 V, 50 Hz supply. Calculate the power and reactive volt-amperes in branch A, which is a coil of 5 Ω resistance and 0·1 H inductance.

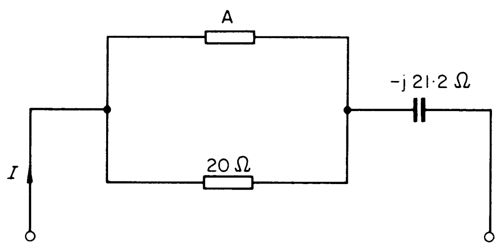


figure 3.36

Impedance of coil A $= R + \mathrm{j}X$

where $X = 2\pi \times 50 \times 0{\cdot}1 = 31{\cdot}4\ \Omega$

$\therefore \quad R + \mathrm{j}X = (5 + \mathrm{j}31{\cdot}4)\ \Omega$

Total impedance of circuit

$$\begin{aligned} &= \frac{(5 + \mathrm{j}31{\cdot}4)\,20}{20 + 5 + \mathrm{j}31{\cdot}4} - \mathrm{j}21{\cdot}2 \\ &= \frac{(100 + \mathrm{j}628)(25 - \mathrm{j}31{\cdot}4)}{25^2 + 31{\cdot}4^2} - \mathrm{j}21{\cdot}2 \\ &= 13{\cdot}8 + \mathrm{j}7{\cdot}8 - \mathrm{j}21{\cdot}2 \\ &= (13{\cdot}8 - \mathrm{j}13{\cdot}4)\ \Omega \end{aligned}$$

In polar form,

$$\text{impedance} = 19{\cdot}4\angle{-44{\cdot}2^\circ}\ \Omega$$

which is sometimes written as

$$19{\cdot}4\overline{\backslash 44{\cdot}2^\circ}\ \Omega$$

$$\therefore \qquad \text{Current, } I = \frac{200\angle 0^\circ}{19{\cdot}4\angle{-44{\cdot}2^\circ}}\ \text{A} = 10{\cdot}3\angle 44{\cdot}2^\circ\ \text{A}$$

This current produces a voltage drop across the two parallel branches, which have an effective impedance of $13{\cdot}8 + \text{j}7{\cdot}8 = 15{\cdot}8\angle 29{\cdot}5^\circ\ \Omega$

Voltage drop across coil A

$$= 10{\cdot}3\angle 44{\cdot}2^\circ \times 15{\cdot}8\angle 29{\cdot}5^\circ$$

$$= 163\angle 73{\cdot}7^\circ\text{V}$$

$\therefore$ Current in the coil A

$$= \frac{163\angle 73{\cdot}7^\circ}{\text{Impedance of coil A}}$$

Now the coil impedance

$$= 5 + \text{j}31{\cdot}4 = 31{\cdot}8\angle 80{\cdot}9^\circ \text{ in polar form}$$

$\therefore$ Current in the coil A

$$= \frac{163\angle 73{\cdot}7^\circ}{31{\cdot}8\angle 80{\cdot}9^\circ}$$

$$= 5{\cdot}13\angle{-7{\cdot}2^\circ}\ \text{A}$$

Changing the sign of the phase angle of current

$$163\angle 73{\cdot}7^\circ \times 5{\cdot}13\angle 7{\cdot}2^\circ = 835\angle 80{\cdot}9^\circ = 835\,(\cos 80{\cdot}9^\circ + \text{j}\sin 80{\cdot}9^\circ)$$

$$= 132\,\text{W} + 824 \text{ reactive VA}$$

Note: If the question had merely asked for the power in the coil, it would have been permissible to calculate this from I_A^2R where I_A is the RMS value of coil current. The principle here is that only the resistive part of the coil absorbs power. Hence, power in coil A $= 5{\cdot}13^2 \times 5 = 132$ W as before.

There are a.c. circuits in which the power factor should be improved – possibly to unity – so that more efficient use is made of the available voltage and current rating of power supply equipment. Complex numbers provide a convenient way of solving such problems, as an alternative to the method based on a phasor diagram. The next example illustrates the use of the ‘Operator j’ for this purpose.

Example 3.12 Impedances of $(40 + j50)\,\Omega$ and $(80 + j100)\,\Omega$ are connected in parallel across a 50 Hz supply. Find the admittance of each branch and of the complete circuit. Calculate the value of capacitance that must be connected in parallel with this combination if the resultant power factor is to be unity.

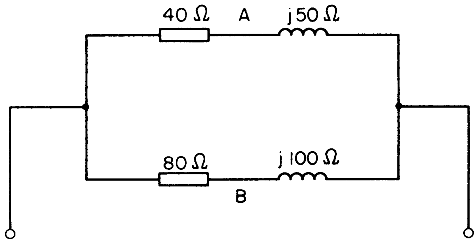


figure 3.37

$$Y_A = \frac{1}{40 + j50} = \frac{40 - j50}{40^2 + 50^2} = (0{\cdot}00975 - j0{\cdot}0122)\text{ S}$$

∴ Admittance of A,

$$|Y_A| = \sqrt{(0{\cdot}00975^2 + 0{\cdot}0122^2)} = 0{\cdot}0156\text{ S}$$

Similarly for branch B,

$$Y_B = \frac{1}{80 + j100} = (0{\cdot}00488 - j0{\cdot}0061)\text{ S}$$

$$\therefore \qquad |Y_B| = 0{\cdot}00782\text{ S}$$

Hence, the total admittance of the circuit,

$$Y = Y_A + Y_B = (0{\cdot}01463 - j0{\cdot}0183)\text{ S}$$

$$\therefore \qquad |Y| = 0{\cdot}0234\text{ S}$$

If the power factor is to be increased to unity, it means that the complete circuit must draw a current that is in phase with the supply voltage. It follows, therefore, that the reactive term ($-j0{\cdot}0183$) in the total admittance must be eliminated. This can be achieved, as the question suggests, by adding in parallel with the circuit a capacitor of which the susceptance is $+j0{\cdot}0183$; that is

$$\frac{1}{-j/\omega C} = j\omega C = j0{\cdot}0183$$

$$\therefore \qquad C = \frac{0{\cdot}0183 \times 10^6}{2\pi \times 50} = 58{\cdot}2\,\mu\text{F}$$

Network theorems

Kirchhoff's laws

Any chapter on Circuit Theory must contain a reference to the laws of Kirchhoff because they are the fundamentals from which many of the other circuit theorems are derived. The two laws can be stated as follows:

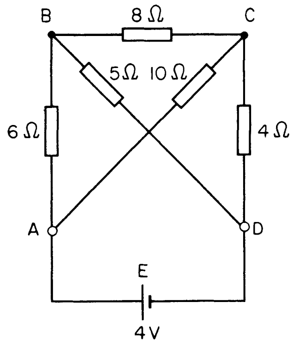
1. At any point in a circuit, the instantaneous (or phasor) sum of the currents is zero.
2. In any closed network, the instantaneous (or phasor) sum of the potential differences across the circuit elements is equal to the algebraic (or geometrical) sum of the electromotive forces.

These laws can be applied equally to d.c. and a.c. circuits, the words in parentheses referring to the latter. Worked examples of both types will now be shown.

Example 3.13 The three resistors AB, BC, CD of a network forming three sides of a square are 6 Ω, 8 Ω and 4 Ω respectively. Resistors AC of 10 Ω and BD of 5 Ω are also connected in the circuit. If a p.d. of 4 V is maintained between terminals A and D, calculate the p.d. between the mid-points of AC and BD.

The network and its equivalent circuit may be drawn as in figure 3.38.

The first step is to label the equivalent circuit with arrows to indicate current in the various branches. Care should be taken to keep the number of unknowns to a minimum and thereby reduce the amount of algebra required



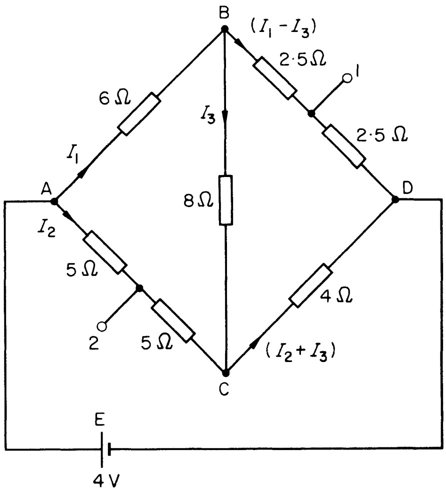


figure 3.38

for their solution. Network equations are then produced from the different closed loops, simplifying where possible, as follows:

Network ABCA,

$$6I_1 + 8I_3 - 10I_2 = 0$$

Dividing by 2

$$\therefore \qquad 3I_1 - 5I_2 + 4I_3 = 0 \qquad \text{(i)}$$

Network BDCB,

$$5(I_1 - I_3) - 4(I_2 + I_3) - 8I_3 = 0$$

$$\therefore \qquad 5I_1 - 4I_2 - 17I_3 = 0 \qquad \text{(ii)}$$

Network ABDEA,

$$6I_1 + 5(I_1 - I_3) = 4$$

$$\therefore \qquad 11I_1 - 5I_3 = 4 \qquad \text{(iii)}$$

4(i)–5(ii) will eliminate I_2 between these two equations, leaving

$$-13I_1 + 101I_3 = 0 \qquad \text{(iv)}$$

Furthermore, 13(iii) + 11(iv) will eliminate I_1 from these two equations, so that

$$1046I_3 = 52$$

$$\therefore \qquad I_3 = \frac{52}{1046} = 0{\cdot}0497 \text{ A}$$

Substituting for I_3 in (iii)

$$11I_1 = 4 + 5 \times 0{\cdot}0497$$

$$\therefore \qquad I_1 = 0{\cdot}386 \text{ A}$$

Substituting for I_1 and I_3 in (i)

$$3 \times 0{\cdot}386 + 4 \times 0{\cdot}0497 = 5I_2$$

$$\therefore \qquad I_2 = 0{\cdot}2714 \text{ A}$$

Hence, the p.d. between A and terminal 2

$$= 5I_2 = 1{\cdot}357 \text{ V}$$

$$\text{Potential of } 2 = 4 - 1{\cdot}357 = 2{\cdot}643 \text{ V}$$

Similarly, the p.d. between A and terminal 1

$$= 6I_1 + 2{\cdot}5(I_1 - I_3)$$
$$= 8{\cdot}5I_1 - 2{\cdot}5I_3$$
$$= 3{\cdot}28 - 0{\cdot}124 = 3{\cdot}156 \text{ V}$$

$$\text{Potential of } 1 = 4 - 3{\cdot}156 = 0{\cdot}844 \text{ V}$$

∴ P.D. between terminals 1 and 2

$$= 2{\cdot}643 - 0{\cdot}844 = 1{\cdot}8 \text{ V approximately.}$$

Example 3.14 A circuit containing R, L and C is connected as shown in figure 3.39. The two signal sources each have an EMF of 1·0 V but v_1 lags 90° behind v_2. Calculate the current supplied by v_2 and state the answer in polar form.

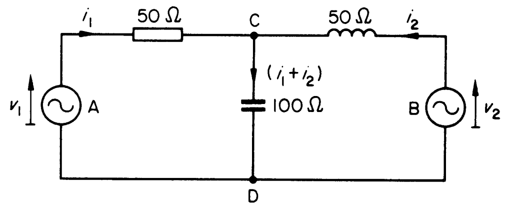


figure 3.39

Using v_2 as the reference phasor, the network equations are as follows:

Network ACD,

$$0 - \text{j}1 = 50i_1 - \text{j}100(i_1 + i_2)$$

$$\therefore \quad -\text{j}1 = (50 - \text{j}100)\,i_1 - \text{j}100i_2 \qquad \text{(i)}$$

Network BCD,

$$1 + \text{j}0 = \text{j}50i_2 - \text{j}100(i_1 + i_2)$$

$$\therefore \quad 1 = -\text{j}100i_1 - \text{j}50i_2 \qquad \text{(ii)}$$

2(ii)–(i) leaves

$$2 + \text{j}1 = (-50 - \text{j}100)\,i_1$$

$$\therefore \quad i_1 = -\frac{2+\text{j}1}{50(1+\text{j}2)} = -\frac{(2+\text{j}1)(1-\text{j}2)}{50(1^2+2^2)}$$

$$= -\frac{(4-\text{j}3)}{250}$$

$$\therefore \quad i_1 = -0{\cdot}016 + \text{j}0{\cdot}012$$

Substituting for i_1 in (ii)

$$1 = \mathrm{j}1{\cdot}6 + 1{\cdot}2 - \mathrm{j}50i_2$$

$$\therefore \qquad i_2 = \frac{0{\cdot}2 + \mathrm{j}1{\cdot}6}{\mathrm{j}50} = \frac{1{\cdot}61\angle 82{\cdot}9^\circ}{50\angle 90^\circ} = 32{\cdot}2\angle -7{\cdot}1^\circ \text{ mA}$$

Thus the signal source B supplies a current of 32·2 mA which lags by 7·1° behind v_2. (B)

(C) *Superposition theorem*

Networks having more than one source of EMF can be solved by applying the Superposition theorem. With this method, each EMF is considered separately and the effects are subsequently superimposed. The theorem can be stated as follows:

> In any network containing a number of sources of EMF, the resultant current in any branch is the algebraic (or phasor) sum of the currents that would be produced by each EMF acting alone, when all other sources are replaced by their respective internal resistances (or impedances).

Two applications of this theorem will now be demonstrated.

Example 3.15 Calculate the current flowing in the 15 Ω resistor of figure 3.40 given that the internal resistances of V_1 and V_2 are 5 Ω and 12 Ω respectively.

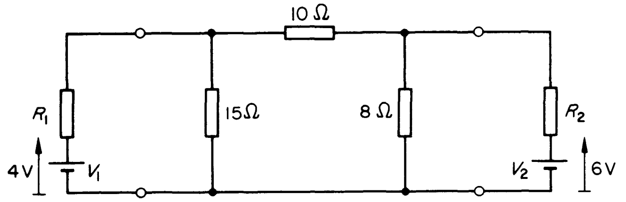


figure 3.40

With the 6 V battery replaced by R_2 = 12 Ω, an equivalent circuit can be drawn as in figure 3.41a.

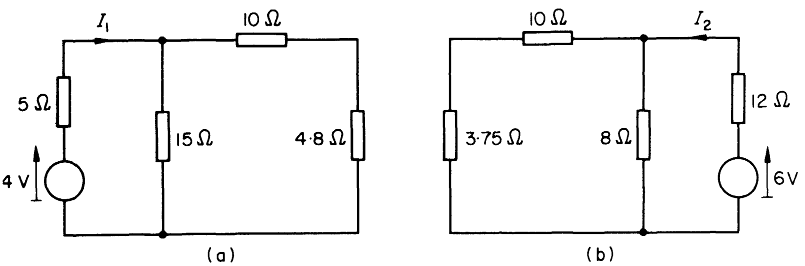


figure 3.41

Total circuit resistance

$$= 5 + \frac{15 \times 14{\cdot}8}{29{\cdot}8} = (5 + 7{\cdot}45)\,\Omega$$

$$\therefore \qquad I_1 = \frac{4}{12{\cdot}45}\ \text{A}$$

Voltage drop in 5 Ω resistance

$$= 1{\cdot}606\ \text{V}$$

∴ P.D. across 15 Ω resistor

$$= 4 - 1{\cdot}606 = 2{\cdot}394\ \text{V}$$

Hence current due to V_1 in 15 Ω resistor

$$= \frac{2{\cdot}394}{15} = 0{\cdot}16\ \text{A}$$

This current would be flowing downwards in the diagram. Similarly with the 4 V battery replaced by $R_1 = 5\ \Omega$ the equivalent circuit would appear as figure 3.41b.

Total circuit resistance

$$= 12 + \frac{8 \times 13{\cdot}75}{21{\cdot}75} = (12 + 5{\cdot}06)\,\Omega$$

$$\therefore \qquad I_2 = \frac{6}{17{\cdot}06} = 0{\cdot}352\ \text{A}$$

Voltage drop in 12 Ω resistance

$$= 4{\cdot}22\ \text{V}$$

∴ P.D. across 8 Ω resistor

$$= 6 - 4{\cdot}22 = 1{\cdot}78\ \text{V}$$

By proportion, p.d. across 3·75 Ω resistance

$$= \frac{1{\cdot}78 \times 3{\cdot}75}{13{\cdot}75} = 0{\cdot}485\ \text{V}$$

Hence current due to V_2 in 15 Ω resistor

$$= \frac{0{\cdot}485}{15} = 0{\cdot}0324\ \text{A}$$

This current also flows downwards in figure 3.41b and can be superimposed on the 0·16 A already calculated to give a total of 0·1924 A. The reader may like to use the same procedure to verify that the current in the 8 Ω resistor is 0·32 A. Both answers obtained in this circuit can also be checked by Kirchhoff's laws.

Example 3.16 A voltage of complex waveform is applied to a circuit containing a coil of resistance 12 Ω and inductance of 20 mH. Given that the voltage has an equation:

$$v = 150 \sin \omega t + 25 \sin (3\omega t + \pi/3)$$

calculate the power dissipated in the coil. The fundamental frequency is 50 Hz.

In accordance with the superposition theorem, the complex waveform of voltage can be treated as two distinct sources of EMF. One of these has a fundamental frequency of 50 Hz whilst the other is a third harmonic.

Consider the fundamental having $f = 50$ Hz.

Total impedance of the coil

$$= 12 + j314 \times 0{\cdot}02$$

$$= (12 + j6{\cdot}28)\ \Omega$$

$\therefore$ Fundamental component of current,

$$I_1 = \frac{150}{\sqrt{2}(12 + j6{\cdot}28)}$$

As it is only the resistive part of the coil impedance which dissipates power, this amount of power can be calculated from $|I_1|^2R$ where $|I_1|$ is the RMS value of current.

$$|I_1| = \frac{150}{\sqrt{[2(12^2 + 6{\cdot}28^2)]}} = 7{\cdot}83 \text{ A}$$

Fundamental power $= 7{\cdot}83^2 \times 12 = 736$ W

Similarly for the third harmonic having $f = 150$ Hz

Total impedance of the coil

$$= 12 + j942 \times 0{\cdot}02 = (12 + j18{\cdot}84)\ \Omega$$

$\therefore$ Third harmonic component of current,

$$I_3 = \frac{25}{\sqrt{2}(12 + j18{\cdot}84)}$$

Modulus of third harmonic,

$$|I_3| = \frac{25}{\sqrt{[2(12^2 + 18{\cdot}84^2)]}} = 0{\cdot}791 \text{ A}$$

Third harmonic power

$$= 0{\cdot}791^2 \times 12 = 7{\cdot}5 \text{ W}$$

∴ Total power dissipated in coil

$$= 743{\cdot}5 \text{ W}$$

Note that the resultant current responsible for the total power of 743·5 W in the coil is not (7·83 + 0·791) A. These two components cannot be added in this way because they are of different frequencies. If they must be combined, it is by means of the equation

$$I = \sqrt{(I_1^2 + I_3^2)}$$

$$\therefore \quad I = \sqrt{(7{\cdot}83^2 + 0{\cdot}791^2)} = 7{\cdot}871 \text{ A}$$

Hence the total power in the resistive part of the coil is $7{\cdot}871^2 \times 12 = 743{\cdot}5$ W as already calculated. The same technique for finding the RMS value of a complex wave is used in chapter 4 for dealing with an amplitude-modulated wave.

Thévenin's theorem

This is a very important theorem in the field of telecommunications and a good deal of circuit analysis is based upon it. The underlying principle of the theorem can be stated as follows:

> In any system of resistors (or impedances), the current in any particular branch is that which would result if an EMF, equal to the p.d. that would appear across the gap were that branch opened, were introduced into the branch when all other EMFs were replaced by their respective internal resistances (or impedances).

Examples of amplifier performance in chapter 5 and elsewhere have been solved using Thévenin's theorem. In this section, to illustrate the procedure, two typical problems have been chosen.

Example 3.17 (adapted from example 3.13)

From the circuit of figure 3.42, calculate the current in the 8 Ω resistor by means of Thévenin's theorem.

The steps in this method are as follows:

1. Imagine the branch BD open-circuited and calculate the p.d. across the gap in the circuit due to the 4 V battery.

$$\text{Current via ABC} = \frac{4}{11} \text{ A}$$

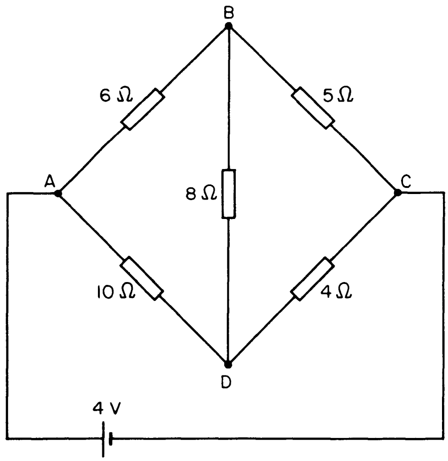


figure 3.42

$$\therefore \quad \text{Potential of B} = 4 - \left(6 \times \frac{4}{11}\right) = +1{\cdot}82\ \text{V}$$

$$\text{Current via ADC} = \frac{4}{14}\ \text{A}$$

$$\therefore \quad \text{Potential of D} = 4 - \left(10 \times \frac{4}{14}\right) = +1{\cdot}14\ \text{V}$$

Hence p.d. between B and D = 1·82 − 1·14 = 0·68 V

2. Replace the 4 V battery with its internal resistance (zero in this case) and inject a source of 0·68 V so as to drive current from the more positive terminal B to D. It is advisable to draw an equivalent circuit, as in figure 3.43, at this stage.

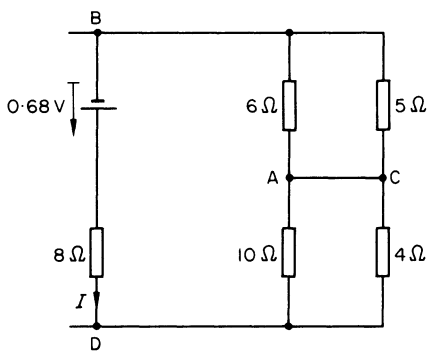


figure 3.43

Total resistance across effective battery terminals

$$= 8 + \frac{6 \times 5}{11} + \frac{10 \times 4}{14} = 13{\cdot}59\ \Omega$$

$$\therefore \qquad I = \frac{0{\cdot}68}{13{\cdot}59} = 0{\cdot}05 \text{ A}$$

as already calculated for I_3 in example 3.13.

Example 3.18 A π-section is formed of two 300 Ω resistors as the shunt components and a 2 μF capacitor as the series element. An oscillator of 600 Ω internal resistance is adjusted to give an EMF of 9 V at an angular frequency of 10^3 radians per second. Calculate the voltage developed across one of the 300 Ω resistors if the oscillator is connected across the other.

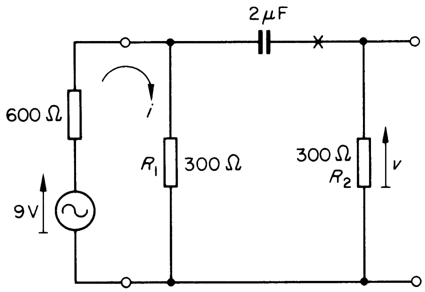


figure 3.44

The input of figure 3.44 can be reduced to a simple series circuit by means of Thévenin's theorem. Consider the circuit opened at the point X. Thus the voltage drop across the gap is that due to current *i* flowing in R_1.

By proportion, voltage drop in $R_1 = 9 \times 300/(300 + 600) = 3$ V.

This EMF is now injected in the gap X and the signal voltage replaced by its internal resistance of 600 Ω as shown in the simplified equivalent circuit of figure 3.45.

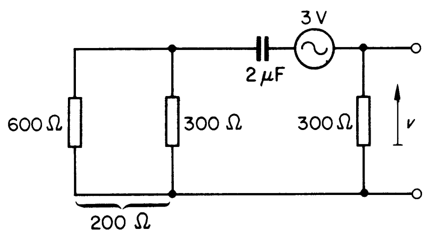


figure 3.45

Impedance seen by 3 V generator

$$= 300 + \frac{600 \times 300}{600 + 300} - jX_C$$

∴ Current supplied by the equivalent Thévenin generator is

$$\frac{3}{500 - jX_C}$$

where

$$X_C = \frac{10^6}{10^3 \times 2} = 500\,\Omega$$

∴ Voltage v developed across R_2

$$= \frac{3 \times 300}{500 - j500}$$

that is

$$|v| = \frac{900}{\sqrt{(500^2 + 500^2)}} = 1{\cdot}272\,\text{V}$$

Norton's theorem

It is sometimes preferable to regard the generator as a current source, rather than as a source of EMF. The internal impedance of the generator is then connected in parallel with the other circuit impedances across the equivalent generator terminals. Norton's theorem for such an arrangement can be stated as follows:

> Any two-terminal configuration of sources and impedances can be replaced by a current generator in parallel with a single impedance Z. The current from the generator is the short-circuit current in the original system, while Z is the effective impedance of the network between its two terminals when all sources are replaced by their respective internal impedances.

As with Thévenin's theorem, two examples will now be solved to show the application of Norton's theorem.

Example 3.19 Two batteries are connected in parallel as shown in figure 3.46. By Norton's theorem, find the current supplied to a load of 10·8 Ω connected to terminals AB. Check the answer by means of the Superposition theorem.

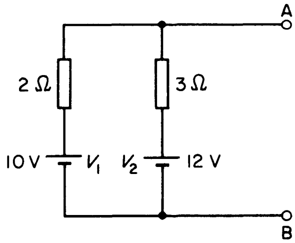


figure 3.46

(a) By Norton's theorem.

With AB short-circuited, the total current flowing from A to B is

$$I_{SC} = \frac{10}{2} + \frac{12}{3} = 9\ \text{A}$$

Impedance Z between A and B when the batteries are replaced by their internal resistances is:

$$Z = \frac{2 \times 3}{2 + 3} = 1{\cdot}2\ \Omega$$

Therefore, an equivalent circuit may be drawn in terms of a current generator as shown in figure 3.47.

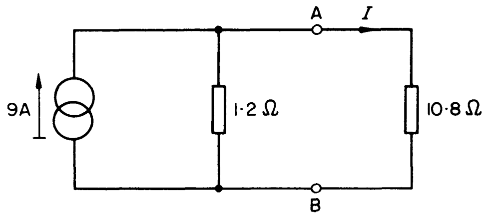


figure 3.47

By inverse proportions, the current I in the load is

$$I = 9 \times \frac{1{\cdot}2}{1{\cdot}2 + 10{\cdot}8} = 0{\cdot}9\ \text{A}$$

(b) By Superposition theorem.

Current supplied by V_1

$$= \frac{10}{2 + \dfrac{3 \times 10{\cdot}8}{13{\cdot}8}} = 2{\cdot}3\ \text{A}$$

Of this current, the portion which flows in the 10·8 Ω load is

$$2{\cdot}3 \times \frac{3}{3 + 10{\cdot}8} = 0{\cdot}5 \text{ A}$$

Similarly, current supplied by V_2 is

$$\frac{12}{3 + \dfrac{2 \times 10{\cdot}8}{12{\cdot}8}} = 2{\cdot}56 \text{ A}$$

Current flowing in the load due to V_2 is

$$2{\cdot}56 \times \frac{2}{2 + 10{\cdot}8} = 0{\cdot}4 \text{ A}$$

Superimposing these two components, current in the load is 0·9 A, as before.

Example 3.20 Evaluate the equivalent Norton generator for the circuit of figure 3.48. Hence, find the current which flows in a load of $(180 + j140)\,\Omega$ that is connected between terminals AB, given that the internal impedance of the oscillator is $(100 + j0)\,\Omega$ and the frequency of operation is $5000/2\pi$ Hz.

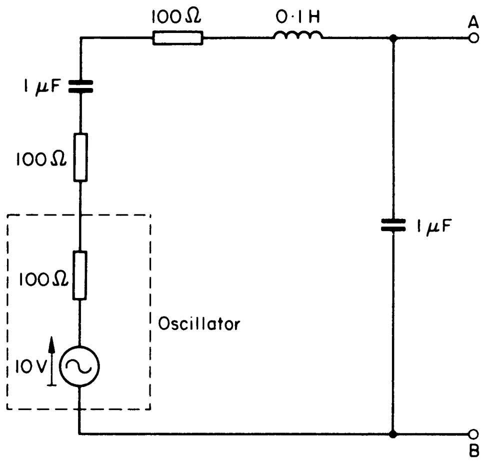


figure 3.48

Reactance of 1 μF capacitor at a frequency of $5000/2\pi$ Hz is $10^6/(5000 \times 1) = 200\,\Omega$. Reactance of 0·1 H inductance at the same frequency is $5000 \times 0{\cdot}1 = 500\,\Omega$.

Applying Norton's theorem to the given circuit, current flowing in a short-circuit between A and B is given by

$$I_{SC} = \frac{10}{300 + j(500 - 200)}$$

$$= \frac{1}{30(1 + j1)} = \frac{1 - j1}{60} \text{ (after rationalising)}$$

Impedance between A and B when oscillator is replaced by its internal resistance of $100\,\Omega$ is made up of a capacitive reactance of $-j200\,\Omega$ in parallel with $(300 + j300)\,\Omega$, that is

$$Z_{AB} = \frac{-j200(300 + j300)}{300 + j100} = \frac{-j200(3 + j3)(3 - j1)}{10}$$

$$= -j20(12 + j6) = (120 - j240)\,\Omega$$

A simplified equivalent circuit can now be drawn with the load connected as in figure 3.49.

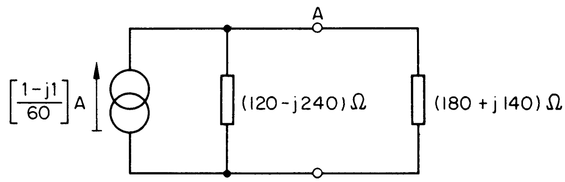


figure 3.49

By inverse proportions, current I flowing in the load can be calculated.

$$I = \frac{1 - j1}{60} \times \frac{120(1 - j2)}{(120 - j240) + (180 + j140)}$$

$$= \frac{2(1 - j1)(1 - j2)}{100(3 - j1)}$$

As it is only the size of the current that is required, this may conveniently be found from the modulus of the above expression without the laborious step of rationalising.

$$|I| = \frac{2|(-1 - j3)|}{100|(3 - j1)|} = \frac{1}{50}\frac{\sqrt{10}}{\sqrt{10}} = 0{\cdot}02 \text{ A}$$

Thus, the current in the load is 20 mA.

The same answer may also be found using Thévenin's theorem, the amount of work involved in the calculation being roughly equal. It is fair to say that the choice between Thévenin's and Norton's theorem for problems of this type is a matter of personal opinion. From the student's point of view, there is no doubt that examples ought to be solved by both methods, so that a proper judgment can be made as to the relative merits of the two theorems.

Maximum power-transfer theorem

In electrical engineering generally, it is the constant aim of designers and users alike to obtain the maximum power output from a certain equipment or circuit. This is particularly true in telecommunications where many circuits comprise devices connected in series and these are required to handle very small input signals (for example, from space satellites). For maximum transfer of power it is necessary that each component is 'matched' to its neighbours. To understand this statement, consider the signal source of internal resistance R_S, feeding a variable load resistance R_L, as shown in figure 3.50

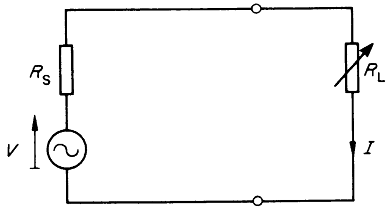


figure 3.50

Current in the load,

$$I = \frac{V}{R_S + R_L}$$

Power in the load,

$$P = I^2 R_L = \frac{V^2 R_L}{(R_S + R_L)^2}$$

When R_L is zero or infinity, no power can be absorbed in the load because either the voltage or current is zero. It is logical, therefore, that at some intermediate setting of R_L the power transferred to it reaches a maximum. This unique setting can be found by differentiating P with respect to R_L and equating to zero, as follows

$$\frac{dP}{dR_L} = V^2 \left[\frac{(R_S + R_L)^2 - R_L \times 2(R_S + R_L)}{(R_S + R_L)^4} \right]$$

$$= 0 \text{ for maximum value of } P$$

This expression will only be zero if

$$(R_S + R_L)^2 = 2R_L(R_S + R_L)$$

that is, if $\quad R_L = R_S$

Thus the condition for maximum power to be transferred from the source to the load is that their resistances should be equal. It can be shown that in general for maximum power transfer, the load impedance $Z_L \underline{/\theta}$ should be the conjugate of the source impedance $Z_S \underline{/\phi}$,
that is

$$Z_L \underline{/\theta} = Z_S \underline{/-\phi} \tag{3.30}$$

Example 3.21 Derive an expression for the current flowing in the resistance R in the circuit of figure 3.51. Hence, calculate the value of R that gives maximum power dissipated in it. What is the value of power dissipated in R under these conditions?

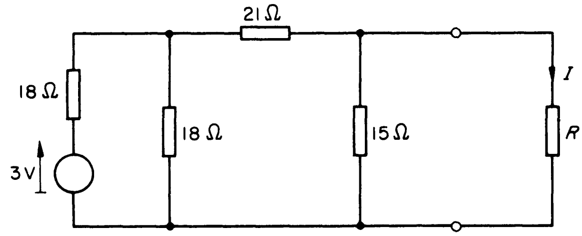


figure 3.51

Applying Thévenin's theorem, the circuit can be simplified in stages as shown in figure 3.52.

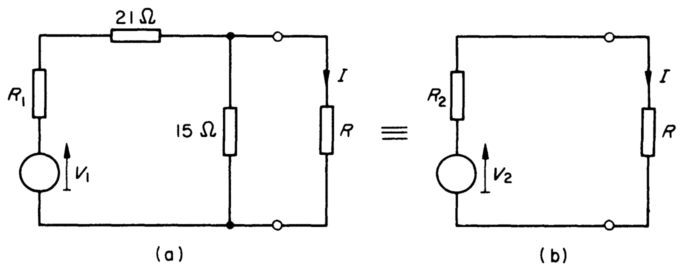


figure 3.52

$$R_1 = \frac{18 \times 18}{36} = 9\ \Omega \qquad R_2 = \frac{15 \times 30}{45} = 10\ \Omega$$

$$V_1 = 3 \times \frac{18}{36} = 1{\cdot}5\ \text{V} \qquad V_2 = 1{\cdot}5 \times \frac{15}{45} = 0{\cdot}5\ \text{V}$$

From figure 3.51b, current in R

$$= \frac{0{\cdot}5}{10 + R}\ \text{A}$$

By inspection, the condition for maximum power in R will be satisfied when $R = 10\ \Omega$.

With this value of R,

$$I = \frac{0{\cdot}5}{20} = 25\ \text{mA}$$

$\therefore$ Power dissipated in R

$$= \left(\frac{25}{10^3}\right)^2 \times 10 = 6{\cdot}25\ \text{mW}$$

Example 3.22 Two coils of 600 Ω reactance and negligible resistance are connected in series across the terminals of a 24 V signal generator. A variable resistance R connected in parallel with one of the coils is adjusted until the signal generator is delivering maximum power to the circuit. Calculate the required value of R and the power in the circuit under these conditions.

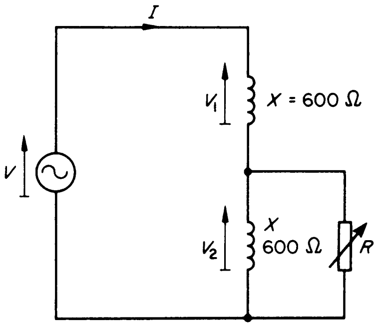


figure 3.53

Total circuit impedance

$$= jX + \frac{jRX}{R + jX} = \frac{j2RX - X^2}{R + jX}$$

$$\therefore \qquad I = \frac{V(R + jX)}{j2RX - X^2}$$

Hence, voltage drop across R

$$= \frac{V(R + jX)}{j2RX - X^2} \times \frac{jRX}{R + jX}$$

$$\therefore \qquad V_2 = \frac{jVR}{j2R - X}$$

Modulus of this p.d.

$$|V_2| = \frac{VR}{\sqrt{(4R^2 + X^2)}}$$

Now, the power P delivered by the signal generator is that dissipated in R only.

$$P = \frac{|V_2|^2}{R} = \frac{V^2R}{4R^2 + X^2}$$

It has been shown that for a maximum,

$$\frac{dP}{dR} = V^2\left[\frac{(4R^2 + X^2) - R \times 8R}{(4R^2 + X^2)^2}\right] = 0$$

$$\therefore \qquad 4R^2 + X^2 = 8R^2$$

and the condition for maximum power is

$$R = \frac{X}{2}$$

Substituting for X, $R = 300\ \Omega$.

As $$P = \frac{V^2R}{4R^2 + X^2}$$

for this circuit,

$$P_{\text{max}} = \frac{24^2 \times 300}{4 \times 300^2 + 600^2} = 0{\cdot}24\ \text{W}$$

Impedance matching

It may be necessary to increase or decrease the effective load presented to a source in order to transfer the maximum power. This is achieved with very little power loss by using a *matching transformer* between the source and the load. Consider a transformer with resistance R connected to its secondary terminals as shown in figure 3.54.

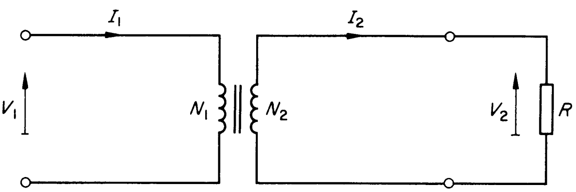


figure 3.54

If the turns ratio $N_2/N_1 = \mathrm{n}$, therefore $I_2 = I_1/\mathrm{n}$ in the ideal case, as already shown in chapter 2. An equivalent circuit is required in which R is referred to the primary. The two circuits will only be equivalent if the power absorbed in the resistance remains unchanged.

From figure 3.54 and figure 3.55, $I_2^2R = I_1^2R'$

$$\therefore \qquad R' = \left(\frac{I_2}{I_1}\right)^2 R = \frac{R}{\mathrm{n}^2} \tag{3.31}$$

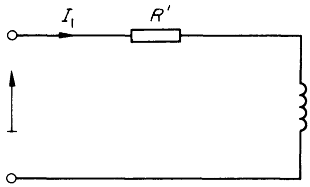


figure 3.55

Thus a step-down transformer, in which n is less than unity, will reflect an increased value of resistance into the primary circuit. This technique is often used in power amplifiers (see chapter 7) where the relatively high output resistance of a valve or transistor has to be matched to a load of low impedance such as a loudspeaker.

Example 3.23 A loudspeaker, having a resistance of 20 Ω, is to be supplied from an amplifier whose output stage can be represented as an EMF of 100 V RMS in series with an internal resistance of 2 kΩ. Find the power output from the loudspeaker if:

(a) it is connected directly to the amplifier, and
(b) a 15 : 1 step-down transformer is used for matching purposes.

What is the correct transformation ratio for maximum power in the loudspeaker and how much power is delivered under these conditions?

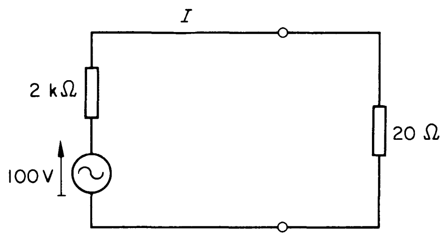


figure 3.56

(a) *Directly connected.*

$$I = \frac{100}{2{\cdot}02}\ \text{mA}$$

$$\therefore \qquad \text{Output power} = \left(\frac{100}{2{\cdot}02 \times 10^3}\right)^2 \times 20 = 49\ \text{mW}$$

(b) *Transformer coupled.* Resistance of loudspeaker referred to primary

$$= 15^2 \times 20 = 4{\cdot}5\ \text{k}\Omega$$

$$\therefore \qquad \text{Output power} = \left(\frac{100}{6{\cdot}5 \times 10^3}\right)^2 \times 4{\cdot}5 \times 10^3 = 1{\cdot}065\ \text{W}$$

For maximum power transfer, reflected resistance = 2 kΩ

$$\therefore \qquad n = \sqrt{\left(\frac{20}{2 \times 10^3}\right)} = \frac{1}{10}$$

(that is, a step-down transformation ratio of 10 : 1). Under these conditions

$$\text{Output power} = \left(\frac{100}{4 \times 10^3}\right)^2 \times 2 \times 10^3 = 1{\cdot}25\ \text{W}$$

(*Note*. More examples of this type will be dealt with, as appropriate, in the section on power amplifiers in chapter 7.)

Example 3.24 A source of open-circuit EMF of 4 V at 159 Hz has an internal impedance equivalent to a 600 Ω resistor in series with a 1·25 μF capacitor. Calculate:

(a) the components of a two-element series circuit which will take maximum power from the source, and
(b) the value of this power.

Determine also the components of a two-element parallel circuit which will also take maximum power from the source.

In what value of resistance connected to the source will maximum power be developed and what is that power?

(CGLI Principles C, 1971)

(a)

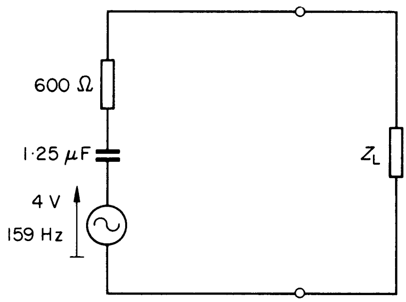


figure 3.57

$$X_C = \frac{10^6}{2\pi \times 159 \times 1{\cdot}25}$$

$$= 800\ \Omega$$

It has already been stated that for maximum power to be transferred to the load, its impedance should be the conjugate of the source impedance.

Since

$$Z_S = (600 - j800)\ \Omega$$

$$\therefore \qquad Z_L = (600 + j800)\ \Omega$$

This represents a resistance of 600 Ω in series with a reactance of 800 Ω due to a coil of 0·8 H inductance.

(b) Power absorbed in Z_L is given by I^2R where I is the modulus of the current in the circuit and R is the real part of Z_L.

Since

$$I = \frac{4}{(600 - j800) + (600 + j800)} = \frac{1}{300}\ \text{A}$$

(This is the maximum value of current possible in this circuit, corresponding to series resonance.)

$$\therefore \qquad \text{Power in load} = \frac{1}{300^2} \times 600 = \frac{1}{150}\ \text{W}$$

Admittance of the load

$$= \frac{1}{600 + j800} = \frac{1}{200(3 + j4)}$$

which, after rationalising, may be written as

$$G - jB = \frac{200(3 - j4)}{4 \times 10^4(3^2 + 4^2)} = \frac{6 - j8}{10^4}$$

Equating real terms

$$G = \frac{1}{R} = \frac{6}{10^4}$$

$$\therefore \qquad R = 1{\cdot}67\ \text{k}\Omega$$

Equating imaginary terms

$$B = \frac{1}{\omega L} = \frac{8}{10^4}$$

$$\therefore \qquad L = 1{\cdot}25\ \text{H}$$

Therefore, by definition, a resistance of 1·67 kΩ in parallel with an inductance of 1·25 H is equivalent to the original series circuit because it takes the same power from the source.

The condition for maximum power to be transferred to a load such as a pure resistance, whose phase angle cannot be varied, is that the moduli of source and load impedance must be equal.

Hence, resistance of load = $|Z_S| = 1\ \text{k}\Omega$

As

$$I = \frac{4}{(600 - \text{j}800) + 10^3}$$

$$= \frac{4}{800(2 - \text{j}1)} = \frac{1}{200(2 - \text{j}1)}$$

$$\therefore \qquad |I| = \frac{1}{200\sqrt{5}}$$

Power absorbed in load

$$= \left(\frac{1}{200\sqrt{5}}\right)^2 \times 10^3 = \frac{1}{200}\ \text{W}$$

compared with the absolute maximum of (1/150) W when there is complete freedom to adjust both the magnitude and phase angle of the load.

Example 3.25 A moving-coil microphone generates an EMF of 100 μV for a certain level of sound, and its internal resistance is 50 Ω. It is coupled to an amplifier whose input resistance is 950 Ω. Calculate the power delivered to the amplifier when the microphone is connected:

(a) directly to the amplifier input terminals, and
(b) through a 1: 5 step-up transformer.

Hence, state the insertion gain in dB achieved by using the transformer.

What would be the turns ratio of the transformer that would deliver maximum power to this amplifier?

(a) Directly coupled (see figure 3.58).

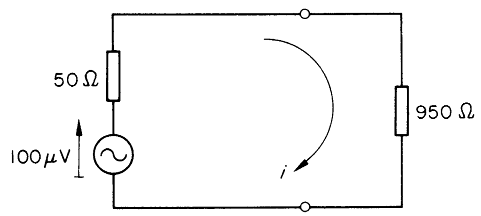


figure 3.58

Current i in the circuit

$$= \frac{100}{10^6} \times \frac{1}{10^3} = 0{\cdot}1\,\mu\text{A}$$

$\therefore$ power input to amplifier

$$= \left(\frac{0{\cdot}1}{10^6}\right)^2 \times 950 = \frac{9{\cdot}5}{10^{12}}\,\text{W}$$

(b) Transformer coupled (see figure 3.59).

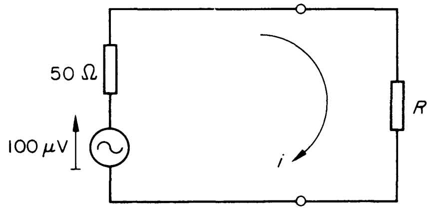


figure 3.59

$$R = \frac{950}{5^2} = 38\,\Omega$$

referred to primary.

$$\text{Circulating current } i = \frac{100}{10^6} \times \frac{1}{88} = 1{\cdot}136\,\mu\text{A}$$

$\therefore$ power input to amplifier

$$= \frac{1{\cdot}136^2}{10^{12}} \times 38 = \frac{49{\cdot}1}{10^{12}}\,\text{W}$$

$$\text{Hence, insertion gain} = 10 \log_{10} \frac{49{\cdot}1}{9{\cdot}5} \text{ decibel} = 7{\cdot}13\text{ dB}$$

For maximum power transfer,

$$n = \sqrt{\left(\frac{950}{50}\right)} = 4{\cdot}36$$

that is, the transformer required for matching the microphone to the amplifier has a step-up ratio of 1: 4·36. (C)

MISCELLANEOUS EXAMPLES

(B) ***Example 3.26*** Use phasors drawn to scale to find the voltage across a circuit consisting of a coil of self-inductance 0·2 H and resistance 50 Ω connected in series with a capacitor of 10 μF, the circuit carrying a current of 2 A at 100 Hz.

Determine the voltage across the terminals of the inductor.

What is the phase angle between the current flowing and the voltage across the whole circuit?

[120·3 V; 33·7°]

(CGLI Principles B, 1966)

N.B. Verify the answers by means of complex numbers.

Example 3.27 What is meant by the self-inductance of a coil? State the factors on which this depends.

A coil dissipates 150 W when a battery maintains 30 V across its terminals. The coil dissipates the same power when an a.c. supply of 60 V at 50 Hz replaces the battery. Although the power dissipation is the same in each circuit, why are the a.c. and d.c. voltages different?

Neglecting any core losses, calculate:

(a) the resistance of the coil
(b) the current taken in each circuit
(c) the phase angle between the voltage and current in the a.c. circuit
(d) the inductance

[(a) $R = 6\,\Omega$; (b) 5 A; (c) 60°; (d) 0·0331 H]

(CGLI Principles B, 1966)

Example 3.28 Give expressions for the reactances of an inductor of L henrys and a capacitor of C farads at a frequency f Hz.

Sketch the reactance/frequency curves for these two components, starting at zero frequency.

An air-cored inductor takes 40 mA when a 10 V, 50 kHz supply is connected across it. A capacitor of C farads is connected in series with the inductor, and when the frequency of the supply is varied the current through the circuit rises to a maximum of 1·0 A. If the supply frequency is then 100 kHz, calculate:

(a) the resistance of the inductor
(b) the inductance
(c) the capacitance
(d) the Q-factor at resonance

[(a) 10 Ω; (b) 796 μH; (c) 0·00317 μF; (d) 50]

(CGLI Principles B, 1970)

Example 3.29 Explain the meaning of (a) reactance, (b) impedance.

A 1 H inductor having a resistance of 50 Ω is connected in series with a 250 Ω resistor. A 20 V, 50 Hz supply is connected across this series circuit.

Draw a phasor diagram to represent the voltages and current in the circuit. Determine the voltage across the 250 Ω resistor.

[11·5 V]

(CGLI Principles B, 1965)

Example 3.30 What is meant by the Q-factor of a tuned circuit?

Derive an expression for the resonant frequency and impedance at resonance of a circuit consisting of a coil of inductance L and resistance R, connected in parallel with a capacitor C. If $L = 0{\cdot}05$ H, $R = 5$ Ω and $C = 0{\cdot}1$ μF, calculate the frequency of resonance. Find for this circuit and at this frequency (a) the Q-factor, (b) the impedance.

[2260 Hz; 141·4; 100 kΩ]

(CGLI Principles B, 1967)

Example 3.31 An inductor of 0·32 H having a resistance of 100 Ω is connected across a 100 V, 50 Hz supply. Determine by means of a phasor diagram the magnitude and phase of the current.

A capacitor is to be connected to the inductor in order to reduce the phase angle of the current from the source to 30°. By means of a phasor diagram, determine the value of capacitance needed and show how you would connect it to the circuit.

[0·707 A; I lags 45° behind V; 75·3 μF]

(CGLI Principles B, 1964)

Example 3.32 The two circuits shown in figure 3.60 have the same value of inductance L and are each supplied with current at 1 kHz. By means of phasors, determine the value of L for which the phase angle between supply voltage and current is the same in each circuit. Calculate this phase angle.

The two circuits are now connected in series by joining B and C. A 200 V, 1 kHz supply is connected across AD. Using phasors, or otherwise, determine the current taken from the supply.

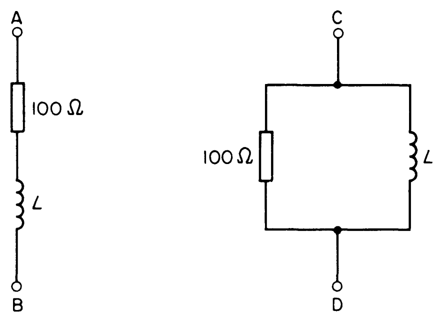


figure 3.60

[15·9 mH; 45°; 0·944 A]

(CGLI Principles B, 1963)

Example 3.33 Explain the meaning of 'power factor'. A 2 kHz, 50 V RMS supply is delivering power to equipment consisting of a 300 Ω resistor in series with a capacitor of $(2{\cdot}5/4\pi)\,\mu$F. Draw up a report of an experiment to measure the power delivered to the equipment and the power factor. In your report explain, with a circuit, the method used. Give details of the equipment needed and calculate the answers you should obtain.

[3 W; 0·6; I leading V]

(CGLI Principles B, 1970)

(*Note.* The answers given may be obtained by use of complex numbers.)
A circuit for measuring the power factor is given in figure 3.61 with brief notes of explanation.

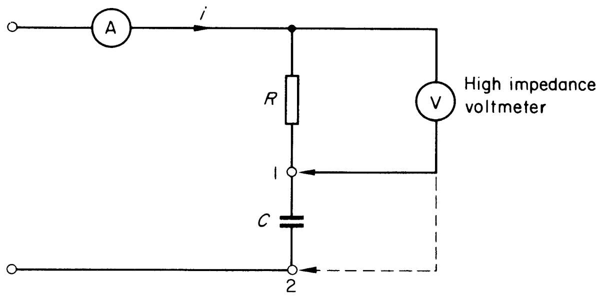


figure 3.61

Total power $= i^2R$ watts.
With voltmeter moving-contact in position 1, let voltmeter reading be V_1.
∴ Power in circuit = power in $R = iV_1$.
With moving-contact in position 2, let voltmeter reading be V_2.
∴ Apparent power in circuit $= iV_2$.
By definition, power factor

$$= \frac{\text{true power}}{\text{apparent power}} = \frac{V_1}{V_2}$$ (B)

(C) ***Example 3.34*** A coil has an inductance of 200 μH and a Q-factor of 100 at a frequency of 400 kHz. It is connected in series with a loss-free capacitor across a constant voltage source of 10 mV RMS at 400 kHz. Calculate, for this circuit at resonance:

(a) the value of the capacitance
(b) the current flowing
(c) the power dissipated
(d) the voltage across the capacitor
(e) the current when a resistance of 5 Ω is connected in series with the circuit.

[(a) 0·00079 μF; (b) 1·99 mA; (c) 19·9 μW; (d) 1 V; (e) 1 mA]

(CGLI Principles C, 1966)

Example 3.35 A coil of inductance 800 μH is connected in parallel with an 800 pF loss-free capacitor. The circuit is driven from a constant current (high impedance) source at the resonant frequency. If the driving current is 100 μA and the voltage developed across the circuit is 20 V, sketch the phasor diagram. Calculate:

(a) the frequency
(b) the current through the capacitor
(c) the Q-factor of the coil
(d) the voltage when a resistance of 200 kΩ is connected across the circuit.

[200 kHz; 20 mA; 200; 10 V]

(CGLI Principles C, 1965)

Example 3.36 An a.c. supply of variable frequency is connected to terminals A and B in figure 3.62. At a certain frequency the voltage across L_2 is 10 V and the current in L_1 is 20 mA. Draw the phasor diagram relating currents and voltages in the circuit for this condition and hence determine the supply frequency. Deduce from the phasor diagram the total current taken from the supply at this frequency. Verify your answer by solving the problem with the aid of complex numbers.

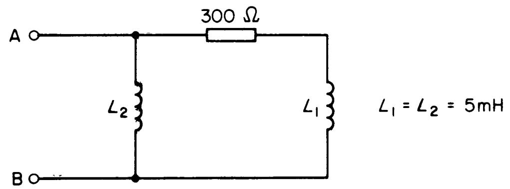


figure 3.62

[12·7 kHz; 42·7 mA]

Example 3.37 The admittance of a circuit is found to be (0·06 + j0·08) S at a frequency of 7·96 kHz. Calculate the resistance and capacitance in the circuit if these two components are connected:

(a) in parallel
(b) in series

[16·67 Ω, 1·6 μF; 6 Ω, 2·5 μF]

Example 3.38 A resistor of 20 Ω and a capacitor of unknown value are connected in parallel across a 100 V, 50 Hz supply. The current taken from the supply is measured as 6 A. The frequency is later adjusted until the current has fallen to 5·5 A. Calculate the new value of frequency.

[35 Hz]

Example 3.39 Calculate the value of capacitance, to be connected across the supply terminals of figure 3.63, which improves the power factor to unity. The frequency of the supply is 25 Hz.

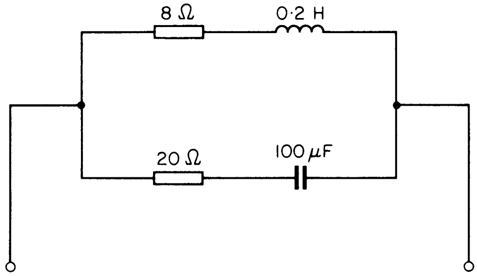


figure 3.63

[99·2 μF]

Example 3.40 The four arms of a bridge network ABCD have resistance values as follows: AB = R_1; BC = 80 Ω, CD = R_2, DA = 10 Ω. Across AC (A positive) is connected a 2 V battery of negligible internal resistance whilst a milliammeter of 10 Ω internal resistance is connected between B and D as a detector. If the current supplied from the battery is 100 mA and the meter shows 20 mA flowing from B to D, calculate:

(a) the values of R_1 and R_2
(b) the current in every part of the circuit

[10 Ω; 17·5 Ω; I_{AB} = 40 mA, etc.]

Example 3.41 State Thévenin's theorem and Norton's theorem. Deduce the Thévenin and Norton equivalent circuits for terminals AB in the network shown in figure 3.64.

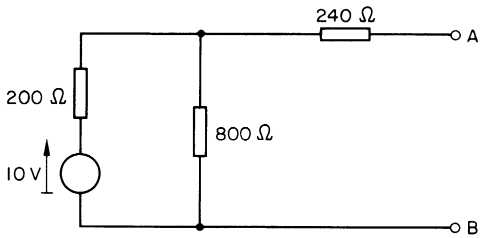


figure 3.64

A 100 Ω load resistor is connected across AB. Use the Thévenin's equivalent circuit to calculate the load current and the Norton equivalent circuit to calculate the load voltage, and compute the load power.

[16 mA; 1·6 V; 25·6 mW]

(CGLI Principles C, 1969)

Example 3.42 A resistance of 100 Ω, a capacitance of 2 μF and an inductance of 0·02 H are connected in series across an alternator of negligible internal impedance. If the EMF of the alternator is $1{\cdot}0 \sin 5000t + 0{\cdot}5 \sin 10\,000t$ calculate the RMS values of applied voltage and circuit current.

What power is being dissipated in the circuit and what is the power factor?

[0·79 V; 7·32 mA; 5·36 mW; 0·927] (CGLI Principles III, 1959)

Example 3.43 An alternating voltage of $1{\cdot}0 \sin 500t + 0{\cdot}5 \sin 1500t$ is applied across a capacitor which can be represented by a 0·5 μF capacitor shunted by a resistance of 4 kΩ. Determine:

(a) the RMS value of applied voltage
(b) an expression for the instantaneous value of the current supplied by the source
(c) the RMS value of this current
(d) the energy dissipated in the circuit during 2π milliseconds
(e) the power factor of the circuit

[0·79 V; $0{\cdot}354 \sin(500t + \pi/4) + 0{\cdot}396 \sin(1500t + 1{\cdot}25)$ mA; 0·375 mA; 0·982 microjoule; 0·527] (CGLI Principles C, 1961)

Example 3.44 A source of EMF 2 V and frequency $10^5/2\pi$ Hz has an internal impedance that may be represented by a 400 Ω resistance in series with 3 mH inductance. The source is connected to terminals PQ of the network shown in figure 3.65.

Calculate the value of the component to produce the reactance X and the turns ratio 'n' which together cause maximum power to be developed in the 4 Ω resistor. What is the value of this power?

What value of turns ratio 'n' would be required for maximum load power if the reactance X were omitted?

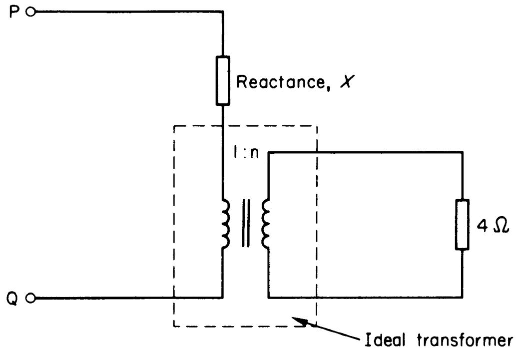


figure 3.65

[1/30 μF, 10 : 1 step-down; 2·5 mW, 11·2 : 1 step-down] (CGLI Principles C, 1969)

Example 3.45 Show how the insertion of a transformer can increase the transfer of power from a source to a load. What is the meaning of 'insertion gain'?

Calculate the insertion gain when a step-up transformer with a turns ratio of 1: 3 is connected between a source of 200 Ω and a load resistance of 1·8 kΩ. Express the result in dB. Does this value of turns ratio produce the maximum power transfer?

[4·44 dB; yes]

(CGLI Principles C, 1964)

Example 3.46 A moving-coil microphone has an internal resistance of 10 Ω and generates an EMF of 1 V RMS. It is connected via a step-up transformer of turns ratio 1: n to a studio cable, which has an input resistance of 100 Ω. Show that the power delivered to the cable is given by the expression

$$P = \frac{1}{\left(n + \dfrac{10}{n}\right)^2}\ \mu\text{W}$$

Sketch the graph showing the variation of this power as n is increased from 0 to 10, and find the value of n for maximum power transfer.

[$n = \sqrt{10}$]

(CGLI Principles C, 1966)

4 Amplitude and frequency modulation

(C) THE AMPLITUDE-MODULATED (AM) WAVE

The transmission of audio frequencies (for example, speech and music) over a distance is usually performed by operating on one of the characteristics of a radio frequency wave. This wave, known as the carrier, will originally be sinusoidal with constant amplitude and frequency. It is these two characteristics which are affected by the modulation processes, as will be described in the chapter.

Consider a carrier of amplitude A and frequency f' which is being modulated by an audio signal whose amplitude and frequency are B and f'' respectively. The effect is shown graphically in figure 4.1. As the name suggests, the amplitude is modulated between the limits $\pm(A + B)$ and $\pm(A - B)$ at a frequency of f''. The modulated carrier still appears to be a waveform of radio frequency but its amplitude is varying from one instant to the next in sympathy with the audio signal. When analysed mathematically, some very important results emerge.

Amplitude at any instant $= A + B \sin 2\pi f''t$. Treating the modulated carrier as a voltage wave, its equation can be written as

$$v = (A + B \sin 2\pi f''t) \sin 2\pi f't$$

$$= A\left(1 + \frac{B}{A} \sin 2\pi f''t\right) \sin 2\pi f't \qquad (4.1)$$

The ratio B/A which occurs in the above expression is referred to as the modulation factor or depth of modulation, M. Substituting $a = 2\pi f'$ and $b = 2\pi f''$ equation 4.1 becomes:

$$v = A(1 + M \sin bt) \sin at \qquad (4.2)$$

(It should be noted that the waveform described by equation 4.2 no longer contains a simple term of frequency f''.)

Expanding equation 4.2,

$$v = A \sin at + AM \sin at \sin bt$$

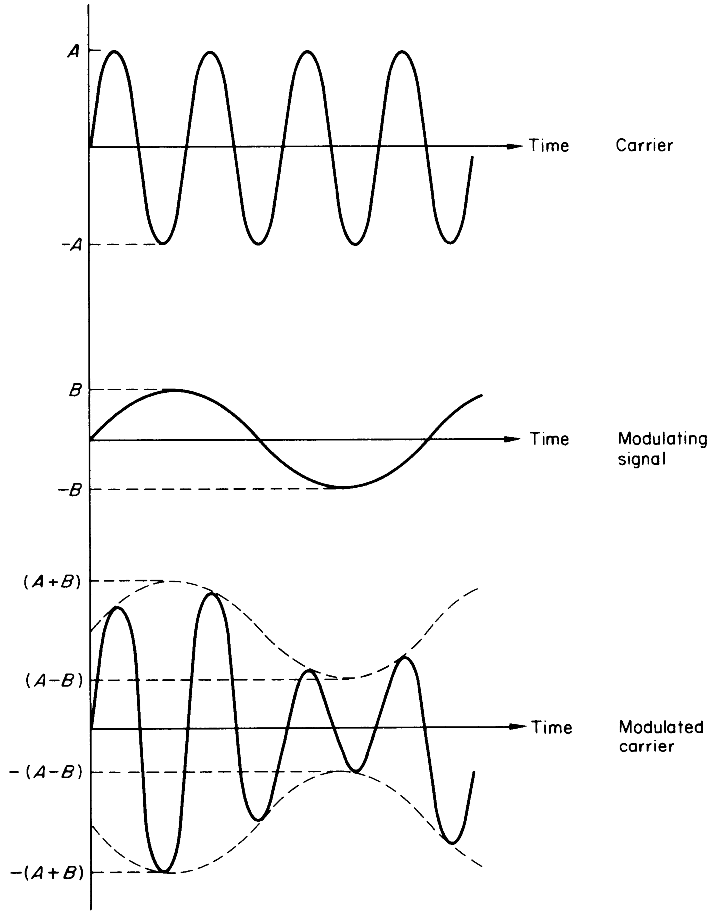


figure 4.1

From the expression

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

subtract

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

leaving

$$\cos(x + y) - \cos(x - y) = -2 \sin x \sin y$$

from which

$$\sin x \sin y = ½[\cos(x - y) - \cos(x + y)]$$

Applying this result to the amplitude modulated wave

$$v = A \sin at + \frac{AM}{2}\,[\cos(a-b)t - \cos(a+b)t]$$

Thus the waveform now consists of three components:

$A \sin 2\pi f't$	original carrier
$\frac{AM}{2} \cos 2\pi(f' - f'')t$	lower sidewave
$\frac{AM}{2} \cos 2\pi(f' + f'')t$	upper sidewave

When the appropriate values of f' and f'' are substituted, it will be found that the modulating signal is effectively raised into the RF band. It has been replaced by components having frequencies symmetrically displaced from f'. Only in this way can it be transmitted and eventually have information extracted from it at the receiving end. In practice, the modulating signal has not a single frequency but is usually a complex wave containing many frequencies within the audio ranges, so that two sidewaves appear for each of these frequencies. The net result is an amplitude-modulated wave that can be resolved into a carrier together with lower and upper sidebands. Figure 4.2 shows their relative positions on a frequency scale.

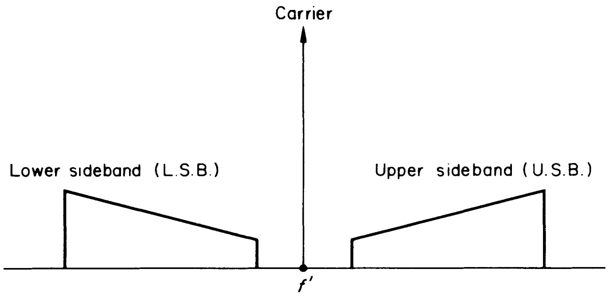


figure 4.2

Example 4.1 Three sinusoidal signals are represented by the expressions

$$v_1 = 1{\cdot}0 \sin 2\pi \times 10^5 t$$

$$v_2 = 0{\cdot}5 \cos 15\pi \times 10^4 t$$

$$v_3 = -\,0{\cdot}5 \cos 25\pi \times 10^4 t$$

By plotting them to a common time base, show that when added together they represent an AM wave. State the carrier frequency, the modulating frequency and the modulation factor (depth of modulation).

Since

$$v_1 = 1{\cdot}0 \sin 2\pi \times 10^5 t = 1{\cdot}0 \sin 2\pi f't$$

∴ Carrier frequency

$$f' = 100\,\text{kHz}$$

Similarly, frequency of v_2

$$= 75\text{ kHz} = (100 - 25)\text{ kHz}$$

and frequency of v_3 $= 125\text{ kHz} = (100 + 25)\text{ kHz}$

that is, v_2 and v_3 are the sidewaves due to a modulating frequency of 25 kHz. This can be verified from figure 4.3 by noting that the envelope shows the modulating signal has a periodic time of 40 μs.

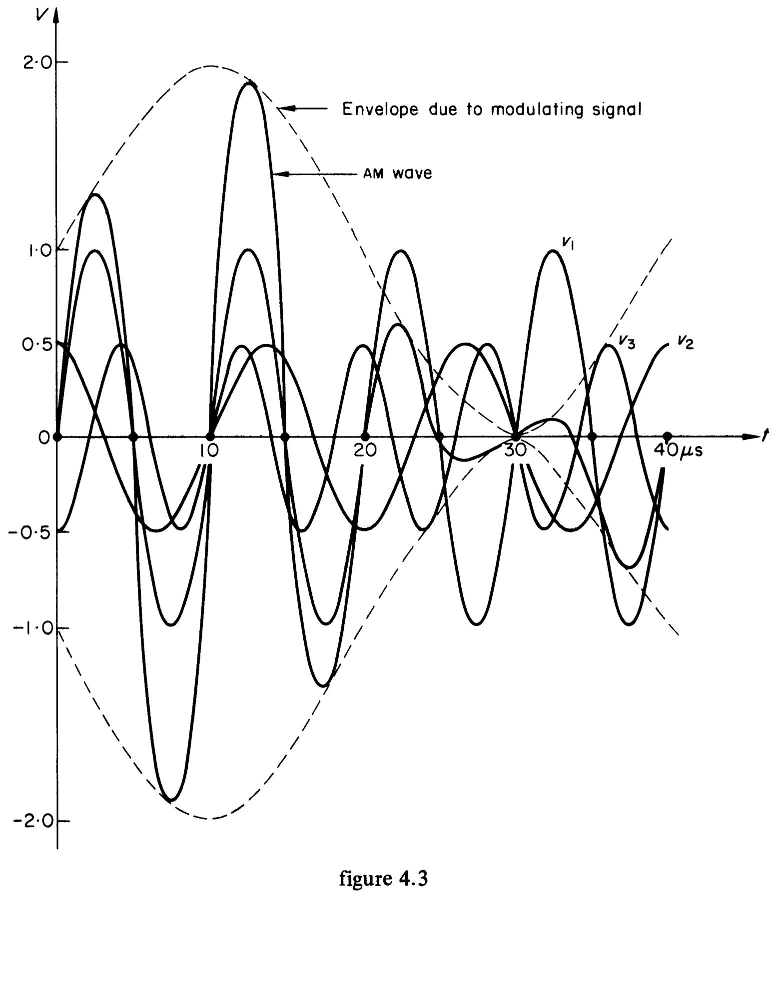


figure 4.3

Previous theory shows that the amplitude of the sidewaves is $AM/2$.

$\therefore$ Depth of modulation, M

$$= \frac{2 \times \text{amplitude of sidewaves}}{\text{Amplitude of carrier, } A}$$

From data

$$M = \frac{2 \times 0{\cdot}5}{1{\cdot}0} = 1$$

This represents 100 per cent modulation, which can again be confirmed from the graph of figure 4.3.

Amplitude modulation

Assume that a carrier and modulating signal are fed simultaneously into the same non-linear device (that is, a diode, valve, or transistor, etc.) in which the graph relating current and voltage has some curvature. Such a device can be represented by an equation of the form

$$i = av + bv^2 + cv^3 + \ldots$$

The system is shown schematically in figure 4.4.

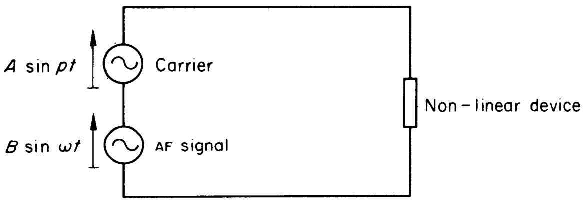


figure 4.4

From the diagram, the resultant EMF is given by

$$v = A \sin pt + B \sin \omega t$$

Hence

$$i = \mathrm{a}(A \sin pt + B \sin \omega t) + \mathrm{b}(A \sin pt + B \sin \omega t)^2 + \ldots$$

Terms beyond those quoted above may be neglected in a practical modulator because of the smallness of the coefficient c. Removing some of the brackets,

$$v = \mathrm{a}A \sin pt + \mathrm{a}B \sin \omega t + \mathrm{b}(A^2 \sin^2 pt + 2AB \sin pt \sin \omega t + B^2 \sin^2 \omega t) + \ldots$$

The remaining bracket now contains a term, $2\mathrm{b}AB \sin pt \sin \omega t$ which can be written as $\mathrm{b}AB[\cos (p - \omega) t - \cos (p + \omega) t]$ as already shown. Furthermore, the 'squared' terms can be dealt with as follows:

As

$$\sin x \sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)]$$

therefore when $x = y$

$$\sin^2 x = \tfrac{1}{2}(\cos 0 - \cos 2x)$$

$$= \tfrac{1}{2}(1 - \cos 2x)$$

Applying this result

$$\mathrm{b}A^2 \sin^2 pt = \frac{\mathrm{b}A^2}{2}(1 - \cos 2pt)$$

$$= \frac{\mathrm{b}A^2}{2} - \frac{\mathrm{b}A^2}{2}\cos 4\pi f' t$$

These terms can be recognised as d.c. (independent of time) and the second harmonic of the carrier (frequency being $2f'$). Regrouping all terms on the right-hand side of the main equation, the reader can show that

$$i = \begin{cases} \dfrac{\mathrm{b}}{2}(A^2 + B^2) & \text{d.c. components} \\ + \mathrm{a}B \sin \omega t & \text{audio signal} \\ + \mathrm{a}A \sin pt & \text{carrier} \\ + \mathrm{b}AB[\cos(p-\omega)t - \cos(p+\omega)t] & \text{sidewaves} \\ - \dfrac{\mathrm{b}}{2}(A^2 \cos 2pt + B^2 \cos 2\omega t) & \text{second harmonics} \end{cases}$$

By connecting a linear device (that is, a resistance or preferably a tuned circuit) in the path of this current, a voltage can be developed which contains the required modulation products. It was explained in chapter 3 that the frequency response of a parallel resonant circuit is as shown in figure 3.26. Clearly a circuit of this type will give a voltage output in which the various terms are treated according to their frequency. When tuned to the carrier frequency f', output will be a maximum over a limited range about f' depending on the selectivity of the resonant circuit. Thus, the circuit will reject theoretical components that are far removed from f' but will accept those near to f' (for example, carrier and pair of sidewaves). Hence, the final output from the modulator will contain only the three components that make up an amplitude modulated wave.

Example 4.2 A sinusoidal RF signal of 5 V (peak) at a frequency of 1 MHz and an AF signal of 2 V (peak) at a frequency of 1 kHz are applied in series to the grid of a triode. The anode characteristic at the operating point is given by the expression

$$I_a = (5 + v + 0{\cdot}05v^2)\ \text{mA}$$

where v = a.c. component of grid voltage.

Find the magnitude and frequency of the various components of the anode current and hence show that the triode functions as an amplitude modulator. Estimate the depth of modulation of the resultant AM wave.

Let the RF signal be $5 \sin pt$ where $p = 2\pi \times 10^6$ rad/s.
Let the AF signal be $2 \sin \omega t$ where $\omega = 2\pi \times 10^3$ rad/s.

$$\therefore \quad v = 5 + 5 \sin pt + 2 \sin \omega t + 0{\cdot}05(5 \sin pt + 2 \sin \omega t)^2$$

$$= 5 + 5 \sin pt + 2 \sin \omega t + 0{\cdot}05(25 \sin 2pt + 20 \sin pt \sin \omega t + 4 \sin^2 \omega t)$$

$$= 5 + 5 \sin pt + 2 \sin \omega t + \frac{1{\cdot}25}{2}(1 - \cos 2pt) + \frac{1}{2}[\cos(p-\omega)t - \cos(p+\omega)t] + \frac{0{\cdot}2}{2}(1 - \cos 2\omega t)$$

Thus the various modulation products can be tabulated as shown in table 4.1:

table 4.1

	Magnitude, mA (peak)	Frequency, kHz	
$5 + 0{\cdot}625 + 0{\cdot}1$	5·725	0	d.c.
$5 \sin pt$*	5·0	1000	Carrier
$2 \sin \omega t$	2·0	1	AF
$0{\cdot}625 \cos 2pt$	0·625	2000	2nd harmonic of carrier
$0{\cdot}1 \cos 2\omega t$	0·1	2	2nd harmonic of AF
$0{\cdot}5 \cos(p - \omega)t$*	0·5	999	Lower side frequency
$0{\cdot}5 \cos(p + \omega)t$*	0·5	1001	Upper side frequency

Note. Only the three components marked * constitute the AM wave. It can be seen, however, that the triode being a non-linear device acts as an amplitude modulator.

By definition, depth of modulation M

$$= \frac{2 \times \text{amplitude of sidewaves}}{\text{Amplitude of carrier}}$$

$$\therefore \quad M = \frac{2 \times 0{\cdot}5}{5} = 0{\cdot}2$$

that is, carrier is modulated to a depth of 20 per cent. It is worth commenting that an ideal modulator supplied with carrier and AF signals of amplitude

5 V and 2 V respectively would have produced a depth of modulation of 40 per cent. In the triode of example 4.2 the curvature of the mutual characteristic, revealed by the coefficient 0·05, results in an AM wave that has only half the maximum possible modulation factor.

Power in an AM wave

The power dissipated in any a.c. circuit containing resistance is proportional to the square of the RMS voltage or current; that is

$$P = \frac{V^2}{R} \quad \text{or} \quad I^2R$$

As RMS value of a sine wave is 0·707 times the amplitude,

$\therefore$ power $\propto$ (amplitude)^2

It has already been shown that an AM wave can be resolved into a carrier and two sidewaves, all of which are sinusoidal. Therefore these three components contain the total power transmitted as an AM wave. For example, if P_C = power in the carrier and M = modulation factor,

$\therefore$ power in each sidewave

$$P_S = \frac{M^2}{2} P_C$$

$$= \frac{M^2 P_C}{4}$$

Hence, total power in AM wave

$$= P_C + 2\frac{M^2 P_C}{4}$$

$$= P_C\left(1 + \frac{M^2}{2}\right) \tag{4.3}$$

Example 4.3 A transmitter radiates 9 kW of power when the carrier is unmodulated. Calculate the total power radiated if the carrier is then modulated by two sine waves of different frequencies, the modulation factors being 0·5 and 0·4 respectively.

Equation 4.3 can be extended to include any number of modulating signals; that is

$$\text{total power} = P_C\left(1 + \frac{M_1^2}{2} + \frac{M_2^2}{2} + \dots\right)$$

$$\therefore \qquad \text{total power} = 9\left(1 + \frac{0{\cdot}25}{2} + \frac{0{\cdot}16}{2}\right)$$

$$= 9 \times 1{\cdot}205 = 10{\cdot}845 \text{ kW}$$

RMS value of AM wave

In the previous example it was stated that the total power in an AM wave is

$$P = P_C\left(1 + \frac{M_1^2}{2} + \frac{M_2^2}{2} + \ldots\right)$$

As $P = V^2/R$ and $P_C = V_C^2/R$, where V and V_C are the RMS values of the AM wave and carrier respectively,

$$\therefore \qquad V^2 = V_C^2\left(1 + \frac{M_1^2}{2} + \frac{M_2^2}{2} + \ldots\right)$$

that is,

$$V = V_C\sqrt{\left(1 + \frac{M_1^2}{2} + \frac{M_2^2}{2} + \ldots\right)} \tag{4.4}$$

Example 4.4 An amplitude modulated voltage is represented by the expression

$$e = 5[1 + 0{\cdot}6 \cos(6280t)] \sin(2\pi \times 10^4 t) \text{ volts}$$

State:

(a) the modulation depth
(b) the modulating frequencv
(c) the period of the carrier wave
(d) the peak instantaneous value of the modulated wave.

Expand the expression and calculate the RMS voltage of the lower side frequency component.

The modulated wave is applied across a resistance of $1000\,\Omega$. What is the power dissipated?

(CGLI Principles C, 1966)

The given expression is similar in form to equation 4.2, the only exception being that $\cos(6280t)$ takes the place of $\sin bt$. It ought to be realised that

$$\sin(b + 90^\circ) = \sin b \cos 90^\circ + \cos b \sin 90^\circ = \cos b$$

that is, substituting a cosine term for a sine term merely adjusts the phase of the modulating signal by 90°. Therefore, by comparison with equation 4.2,

(a) $M = 0{\cdot}6$
(b) Modulating frequency $= 6280/2\pi$ Hz $= 1$ kHz

(c) Period of carrier wave = $1/10^4\,\text{s} = 100\,\mu\text{s}$

(d) Peak instantaneous value of AM wave

$$\begin{aligned} &= A + B \\ &= A + 0{\cdot}6A \qquad (\text{as } B/A = 0{\cdot}6) \\ &= 1{\cdot}6A \\ &= 8{\cdot}0\,\text{V} \qquad (\text{as } A = 5\,\text{V}) \end{aligned}$$

Expanding the expression

$$\begin{aligned} e &= 5\sin(2\pi \times 10^4 t) + 3\sin(2\pi \times 10^4 t)\cos(2\pi \times 10^3 t) \\ &= 5\sin(2\pi \times 10^4 t) + 1{\cdot}5[\sin(2\pi \times 1{\cdot}1 \times 10^4 t) \\ &\qquad + \sin(2\pi \times 0{\cdot}9 \times 10^4 t)] \end{aligned}$$

It can now be seen that the amplitude of the sidewaves is 1·5 V, so that RMS value of lower side frequency component = $1{\cdot}5/\sqrt{2} = 1{\cdot}062$ V.

When the AM wave is applied to a 1000 Ω resistor, the power dissipated is due to the three theoretical components, that is, carrier and two sidewaves. As amplitude of carrier is unchanged at 5 V, therefore power P_C relative to carrier component is

$$P_C = \left(\frac{5}{\sqrt{2}}\right)^2 \times \frac{1}{10^3}\,W = 12{\cdot}5\text{ mW}$$

From equation 4.3

$$\text{Total power} = P_C\left(1 + \frac{M^2}{2}\right) = 12{\cdot}5\left(1 + \frac{0{\cdot}6^2}{2}\right) = 14{\cdot}75\text{ mW}$$

Demodulation

This is the process of recovering information at audio-frequencies from the AM wave. A simple demodulator, which can be treated graphically, is shown in figure 4.5.

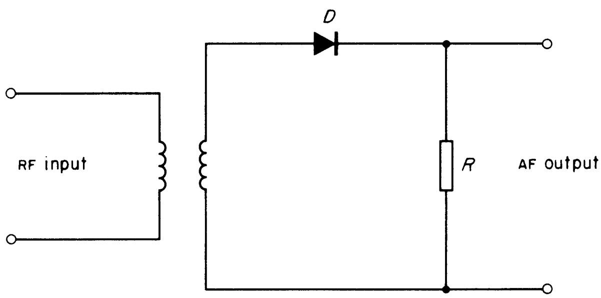


figure 4.5

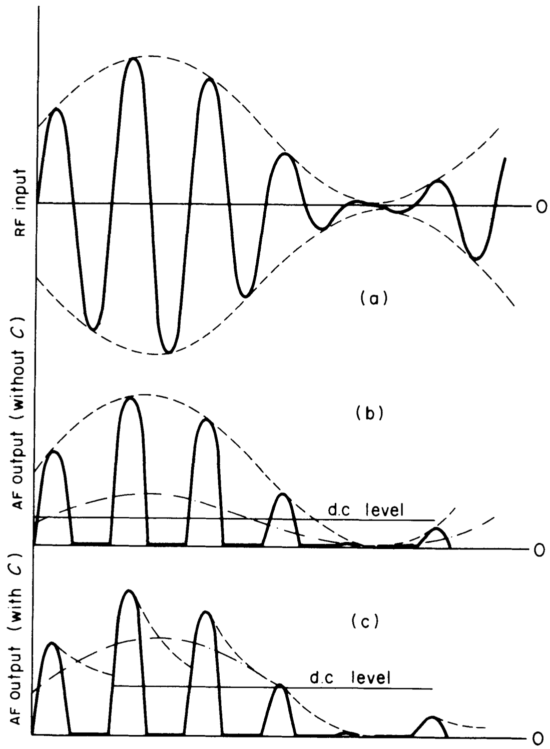


figure 4.6

Input and output waveforms for this circuit are given in the graph of figure 4.6a and 4.6b. During positive half-cycles of incoming signal, diode D conducts. The resultant output voltage developed across R is therefore a series of positive pulses that are subsequently passed on to an AF amplifier. *RC* coupling removes the d.c. level (see chapter 6) and as the amplifier cannot respond to the carrier frequency, the final output will approximate to the chain line in figure 4.6b. This curve follows the mean value of half-cycles of a sine wave; that is, peak value $\div \pi$ when averaged over complete cycles of carrier.

By connecting a capacitor C across the resistance R the output is increased in amplitude. As the carrier wave rises, C is charged with a short time constant via the low forward resistance of D. Thereafter as the carrier wave falls, C discharges much more slowly via R. Thus C and R combine to act as a smoothing circuit which maintains the average output at a higher level than before, with a ripple at carrier frequency superimposed upon it. Again the d.c. level is suppressed by *RC* coupling (see chapter 6) and the ripple is removed because the bandwidth of the AF amplifier is insufficient to accommodate such a high frequency. See figure 4.6c.

Instead of using the unidirectional properties of a diode as the demodulator, any of the non-linear devices mentioned earlier in the chapter can be utilised. Suppose its characteristic at the operating point is described by the equation

$$i = \mathrm{a} + \mathrm{b}v + \mathrm{c}v^2 + \ldots$$

and

$$v = A(1 + M \sin \omega t) \sin pt$$

Substituting for v

$$i = \mathrm{a} + \mathrm{b}A(1 + M \sin \omega t) \sin pt + \mathrm{c}A^2(1 + M \sin \omega t)^2 \sin^2 pt + \ldots$$

The squared term contains

$$\mathrm{c}A^2 2M \sin \omega t \sin^2 pt$$
$$= \mathrm{c}A^2 2M \sin \omega t \tfrac{1}{2}(1 - \cos 2pt)$$

which contains

$$\mathrm{c}A^2 2M \sin \omega t \tfrac{1}{2}$$
$$= \mathrm{c}A^2 M \sin \omega t$$

which is the required audio-frequency. Thus, the non-linear device is capable of extracting a term proportional to the original modulating signal. Unfortunately, the same squared term also has

$$\mathrm{c}A^2M^2 \sin^2 \omega t \ \sin^2 pt$$
$$= \mathrm{c}A^2M^2 [\tfrac{1}{2}(1 - \cos 2\omega t)] [\tfrac{1}{2}(1 - \cos 2pt)]$$

which contains

$$- \mathrm{c}A^2M^2 \tfrac{1}{2} \cos 2\omega t \tfrac{1}{2}$$
$$= -\frac{\mathrm{c}A^2M^2}{4} \cos 2\omega t$$

This is the second harmonic of AF and is very difficult to separate from the fundamental. It has already been seen that in order to inject the maximum power into the sidebands of an AM wave, the modulation factor M should be as large as practicable. As $M = 1$, percentage second harmonic distortion is given by

$$\frac{\mathrm{c}A^2M^2}{4} \times \frac{100}{\mathrm{c}A^2M} = 25 \text{ per cent}$$ (see page 248)

Obviously this amount of distortion is too high and M will have to be reduced until an acceptable compromise is reached. A more effective solution, however, is to employ single sideband transmission, the principles of which will be described after the next example.

Example 4.5 A carrier wave of frequency 100 kHz and amplitude 2 V has been modulated to a depth of 60 per cent at a frequency of 1 kHz. It is applied to the base of a transistor for which the relationship between collector current and base voltage is given by

$$i = 10 + 2{\cdot}5v + 0{\cdot}15v^2$$

where i = collector current in mA and v = base-emitter voltage in V.

Calculate the amplitude of the 1 kHz and 2 kHz components in the resultant collector current waveform.

From data

$$v = 2(1 + 0{\cdot}6 \sin 2\pi \times 10^3t) \sin 2\pi \times 10^5t$$

$$\therefore \quad i = 10 + 5(1 + 0{\cdot}6 \sin 2\pi \times 10^3t) \sin 2\pi \times 10^5t + 0{\cdot}6(1 + 0{\cdot}6 \sin 2\pi \times 10^3t)^2 \sin^2 2\pi \times 10^5t$$

By comparison with previous results, $c = 0{\cdot}15, A = 2, M = 0{\cdot}6$.

$\therefore$ Amplitude of 1 kHz component

$$= cA^2M = 0{\cdot}15 \times 4 \times 0{\cdot}6$$
$$= 0{\cdot}36 \text{ mA}$$

Furthermore, amplitude of 2 kHz component

$$= \frac{cA^2M^2}{4} = \frac{0{\cdot}15 \times 4 \times 0{\cdot}36}{4}$$
$$= 0{\cdot}054 \text{ mA}$$

Single sideband transmission

It should be realised that no information can be extracted from the carrier component, even though a high proportion of the power in the AM wave is required for its transmission. In the interest of economy, therefore, it would be an advantage if the carrier could be suppressed altogether at the transmitter and subsequently replaced by an oscillator at the receiver. By this method, much less power would need to be radiated from the aerial of the transmitter without losing any of the effects of amplitude modulation. Indeed the process can be carried a stage further by suppressing one of the sidebands also. The information originally used to modulate the carrier is still available. Not only is the power requirement reduced to a minimum but also the bandwidth occupied by the signal has been halved. Hence twice as many channels can be accommodated in the same frequency range. The technique whereby single sideband transmission is achieved (balanced modulators or filters) is outside the scope of this book. Reference should be made to one of the standard textbooks on communications equipment.

FREQUENCY MODULATION (FM)

It may be concluded from the chapter so far that amplitude modulation happens automatically whenever sine waves of different frequency are applied simultaneously to a non-linear device. By comparison, frequency modulation is more difficult to achieve in practice. There are, however, certain advantages to be gained by this technique, one of them being the improvement in signal-to-noise ratio. The noise (any unwanted signal) appearing at the input of a receiver usually contains a component which affects the amplitude of the required signal. It originates from such equipment as electric machines, ignition systems, etc., in which a spark is being generated. The resultant radiation is a complex wave that occupies a large bandwidth, and it is therefore impossible to remove it simply by filters. One way of dealing with it is to suppress it at the source. Alternatively, much of this impulsive noise which is eventually picked up by the receiver can be eliminated by using an FM system. In principle, because the FM wave has constant amplitude, the effect of any increase arising from noise can be dealt with in the receiver. The method is to pass the signal to an amplifying stage that is designed to saturate and hence to give an output of constant amplitude which is relatively free from noise.

A second advantage of FM compared with AM is its relative freedom from interference. This is an important factor in portable transmitter-receivers as used by the Armed Forces and the Police. Consider an AM receiver with two sinusoidal signals of different frequency applied simultaneously to its input terminals. The first signal, of amplitude A, is regarded as the 'wanted' signal, while the other, of amplitude B, is 'unwanted' but has been accepted by the tuned circuit. In general, the phasor sum of the two voltages is shown as C in figure 4.7.

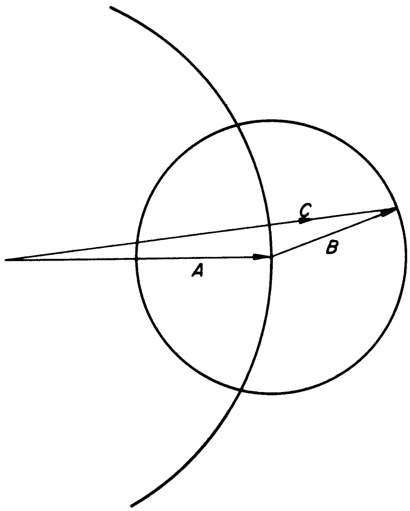


figure 4.7

As the phasors rotate with their own angular frequency, B will either advance or be retarded relative to A. Therefore if A is imagined to be stationary, the resultant C will vary in a cyclic manner between the limits $A \pm B$. It will be impossible to distinguish this combined input from a genuine AM wave. Interference will be serious, even for a fairly small amplitude B received from an adjacent station.

The effects of interference in a corresponding FM system will be discussed when the concept of *phase deviation* has been dealt with later in the present chapter.

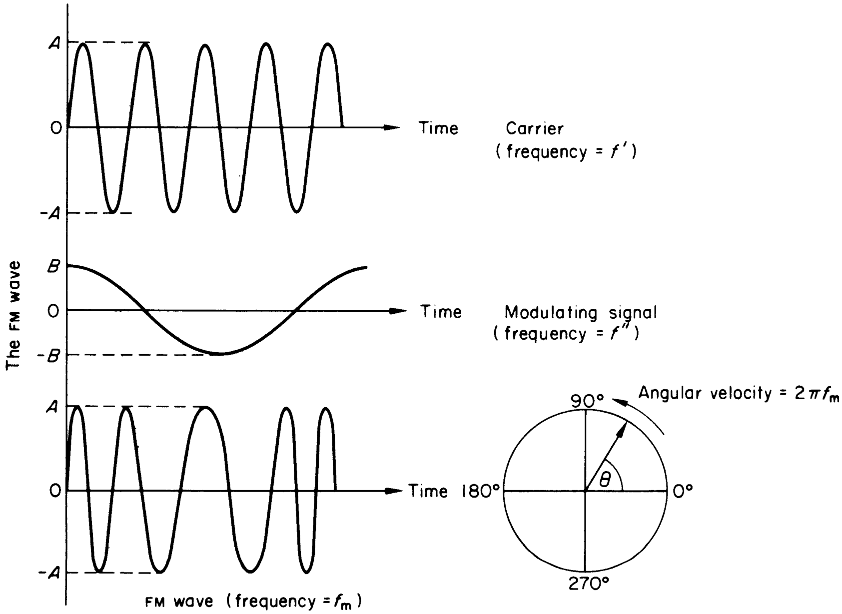


figure 4.8

If a phasor rotates with constant angular velocity ω the angle swept out in a time t seconds is ωt radian where $\omega = 2\pi f$. In an AM wave, however, ω is not constant but is continuously varying in response to the frequency modulation. The angle generated by a phasor of constant length A, corresponding to the amplitude of the carrier wave, must now be found by integration. In figure 4.8

$$\theta = 2\pi \int_0^t f_m \, dt$$

where f_m = instantaneous frequency of FM wave

$$= f'(1 + k \cos bt) \tag{4.5}$$

In this equation, the coefficient k, known as the degree of modulation, is proportional to the amplitude B of the modulating signal which is shown as a cosine wave for convenience.

Hence,

$$\theta = 2\pi f' \int_0^t (1 + \mathrm{k} \cos bt)\, \mathrm{d}t$$

$$= 2\pi f' \left[t + \frac{k}{b} \sin bt \right]_0^t$$

$$= 2\pi f' t + \frac{2\pi f' \mathrm{k}}{b} \sin bt$$

where $f'\mathrm{k}$ is called the frequency deviation

$$\therefore \qquad \theta = at + \frac{a\mathrm{k}}{b} \sin bt \tag{4.6}$$

Modulation index (or deviation ratio), m, is defined as the ratio to the modulating frequency of the amount by which the carrier frequency is deviated from its original value. Combining this definition with equation 4.6

$$m = \frac{kf'}{f''} = \frac{ak}{b}$$

$$\therefore \qquad \theta = at + m \sin bt \tag{4.7}$$

The FM wave is still of the form

$$v = A \sin \theta$$

where θ has the value shown in equation 4.7.

$$\therefore \qquad v = A \sin (at + m \sin bt) \tag{4.8}$$

An extreme case of modulation is that produced by a square wave on a sinusoidal carrier. The results are illustrated in figure 4.9 where they can also be compared with the corresponding AM wave.

Phase deviation

In figure 4.8 the frequency of the FM wave is shown rising and falling in sympathy with the amplitude of the modulating signal. Any increase in frequency must cause the phase of the FM wave to lead that of the original carrier. Consequently at the end of each positive half cycle of audio signal, the FM wave will have had its phase advanced by the maximum amount relative to the carrier. This angle of phase difference is referred to as the peak phase deviation, and its value can be calculated from equation 4.7.

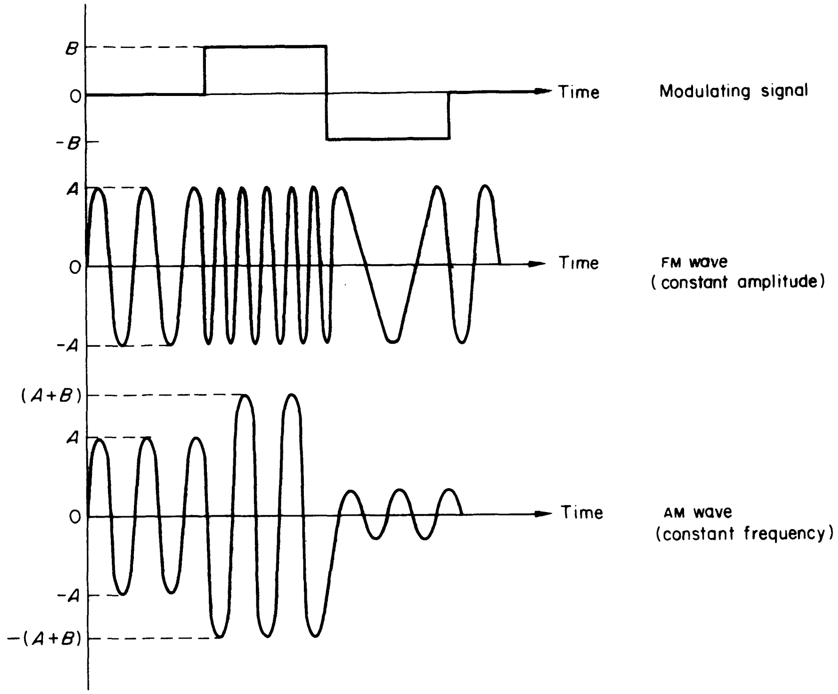


figure 4.9

In general, $\theta = (at + m \sin bt)$ radians. The maximum and minimum values which θ can have occur when $\sin bt = \pm 1$, that is,

$$\theta_{\text{max}} = (at + m) \text{ radians}$$

$$\theta_{\text{min}} = (at - m) \text{ radians}$$

$\therefore$ Peak phase deviation in radians is numerically equal to the modulation index, m. For example, if a frequency deviation of 75 kHz is produced by a given amplitude of 15 kHz modulating signal,

Modulation index,

$$m = \frac{75}{15} = 5$$

$\therefore$ Peak phase deviation = 5 radians

If the two signals A and B of figure 4.7 were applied simultaneously to the input of an FM receiver, variations in amplitude of the resultant voltage C could be removed by the limiter stage. The phase deviation of C relative to A, however, must still be taken into account, and this is shown in figure 4.10.

It is clear from figure 4.10 that with the given amplitudes of $\boldsymbol{A}$ and $\boldsymbol{B}$, the phase of $\boldsymbol{C}$ deviates at most by $\Delta\theta = \pm \frac{1}{2}$ radian approximately with respect to $\boldsymbol{A}$. Compared with the normal operating values of peak phase deviation, any interference from $\boldsymbol{B}$ is much less severe than in the AM receiver.

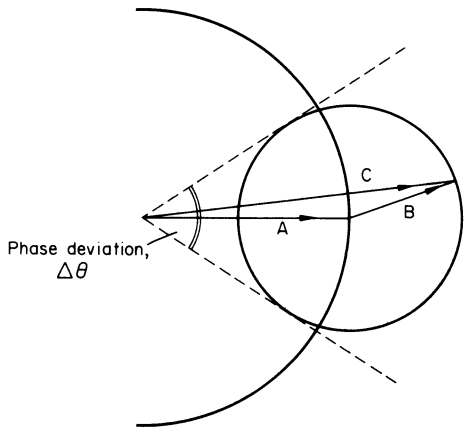


figure 4.10

The nature of the FM wave

The nature of any FM wave can now be investigated by expansion of equation 4.8, using $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

$$v = A\,[\sin at \cos(m \sin bt) + \cos at \sin(m \sin bt)]$$

From Maclaurin's theorem

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Applying these results to the previous equation

$$v = A\left[\sin at\left(1 - \frac{m^2 \sin^2 bt}{2!} + \frac{m^4 \sin^4 bt}{4!} - \ldots\right)\right.$$

$$\left. + \cos at\left(m \sin bt - \frac{m^3 \sin^3 bt}{3!} + \ldots\right)\right]$$

Neglecting terms above second order

$$v = A\left[\sin at\left(1 - \frac{m^2 \sin^2 bt}{2!}\right) + \cos at(m \sin bt)\right]$$

$$= A\left\{\sin at\left[1 - \frac{m^2}{2!} \times \frac{1}{2}(1 - \cos 2bt)\right] + \cos at(m \sin bt)\right\}$$

$$= A\left[\sin at\left(1 - \frac{m^2}{4}\right) + m \cos at \sin bt + \frac{m^2}{4} \sin at \cos 2bt\right]$$

From the expression

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

subtract

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

leaving

$$\sin(x + y) - \sin(x - y) = 2 \cos x \sin y$$

from which

$$\cos x \sin y = \tfrac{1}{2}[\sin(x + y) - \sin(x - y)]$$

Similarly, by adding the same two expressions,

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$$

$$\therefore \qquad \sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$

Applying this result to the FM wave

$$v = A\left(1 - \frac{m^2}{4}\right)\sin at + \frac{mA}{2}[\sin(a + b)t - \sin(a - b)t] + \frac{m^2 A}{8}[\sin(a + 2b)t + \sin(a - 2b)t]$$

that is, FM wave = carrier + 2 pairs of sidewaves. Furthermore, if third-order terms had been included in the expansion, another pair of sidewaves would have appeared. It will be appreciated, therefore, that the FM wave theoretically comprises an infinite number of these sidewaves. In practice, however, the series of coefficients converges, particularly if the modulation index m is small. For all practical purposes the minimum bandwidth required for the transmission of an FM wave may be taken as

$$\text{B.W.} = 2\,(\text{Peak frequency deviation} + \text{Modulating frequency})$$
$$= 2f''(m+1) \tag{4.9}$$

Example 4.6 A 50 MHz carrier wave has its frequency modulated by a 10 kHz sinusoidal signal which produces a frequency deviation of 4 kHz. Calculate the deviation ratio. Show that to a first approximation, the frequency spectrum of the FM wave consists of the carrier and two pairs of sidewaves. Given that the unmodulated wave has an amplitude of 100 mV, calculate the amplitude of the components of the resultant FM wave. Estimate the bandwidth needed to transmit this wave.

$$\text{As deviation ratio} = \frac{\text{Frequency deviation}}{\text{Modulating frequency}} \quad \therefore m = \frac{4}{10} = 0{\cdot}4$$

It has already been verified that with a small value of m, the FM wave can be resolved into a carrier with two pairs of sidewaves. Substituting $A = 100$ mV and $m = 0{\cdot}4$

$\therefore$ Amplitude of carrier component

$$= 100\left(1 - \frac{0{\cdot}4^2}{4}\right)$$

$$= 96 \text{ mV}$$

Amplitude of first pair of sidewaves

$$= \frac{0{\cdot}4 \times 100}{2} = 20 \text{ mV}$$

Amplitude of second pair of sidewaves

$$= \frac{0{\cdot}4^2 \times 100}{8} = 2 \text{ mV}$$

Thus the amplitude of the carrier has been slightly reduced to allow power at the various side frequencies to be transmitted in the FM wave.

Minimum bandwidth $= 2 \times 10(0{\cdot}4 + 1)$ kHz $= 28$ kHz.

MISCELLANEOUS EXAMPLES

Example 4.7 Explain the meaning of the term 'amplitude-modulated carrier wave'. Is it true that the frequency bandwidth needed to transmit an AM wave is wider than that of the original modulating signal? Give reasons for your answer. Why is carrier transmission used in telecommunications?

(CGLI Principles B, 1965)

Example 4.8 A carrier $E \sin \omega t$ is amplitude modulated to a depth of about 50 per cent by a signal of

(a) sine waveform at 1 kHz
(b) square waveform with a period of 1 ms

Carefully sketch the modulated waveforms, showing for each the periods of the carrier and of the signal. Explain briefly what is meant by 'side frequencies' and 'sidebands', and illustrate these meanings by reference to the two modulated carriers.

(CGLI Principles C, 1964)

Note: A square wave may be represented as a fundamental plus a series of odd harmonics

$$v = \frac{V_{max}}{\pi}\left(\sin \phi t + \frac{1}{3}\sin 3\phi t + \frac{1}{5}\sin 5\phi t + \ldots\right).$$

Example 4.9 Briefly describe an experiment whereby the depth of modulation in an AM wave can be determined on a CRO.

When a carrier is modulated by a sinusoidal audio signal, the maximum and minimum amplitudes of the resultant wave are found to be 3 V and 2 V respectively. Calculate the depth of modulation and the amplitude of the theoretical sidewaves.

[0·2; 0·25 V]

Note: With the AM wave applied to the Y-amplifier while the modulating signal provides the X-deflection instead of the internal time-base generator, the trace on the screen of a CRO is as shown in figure 4.11.
The trace will be a stationary sine wave, bounded by a trapezium if the AM wave is free from distortion. The width of it is a measure of the amplitude of the modulating signal and can be varied at will by means of the X-gain

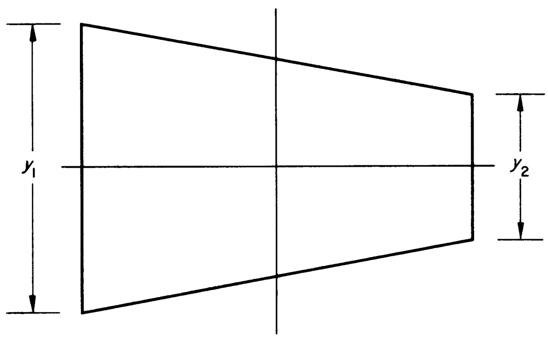


figure 4.11

of the CRO. But the vertical heights y_1 and y_2 are used to estimate modulation depth; for example

$$y_1 = \text{maximum peak-to-peak amplitude}$$

$$y_2 = \text{minimum peak-to-peak amplitude of AM wave}$$

∴ Amplitude of carrier, A

$$= \frac{\text{Average of } y_1 \text{ and } y_2}{2}$$

$$= \frac{y_1 + y_2}{4}$$

Peak-to-peak value of modulating signal

$$= \frac{\text{Difference between } y_1 \text{ and } y_2}{2}$$

∴ Amplitude of modulating signal, B

$$= \frac{y_1 - y_2}{4}$$

By definition, $M = \frac{B}{A}$ ∴ Depth of modulation $= \frac{y_1 - y_2}{y_1 + y_2}$

Example 4.10 If the radiated power from the aerial of a transmitter increases from 10 kW to 11·25 kW when the carrier is sinusoidally modulated, calculate from first principles the modulation factor.

[0·5]

Example 4.11 Verify that a device whose current/voltage characteristic may be written as: $I = \mathrm{a}V + \mathrm{b}V^2$ can be used to demodulate an AM wave. The mutual characteristic of a triode is known to be

$$I = (8 + 2v_g + 0{\cdot}05v_g^2) \text{ mA}$$

where v_g = grid signal in volts.

A high-frequency carrier of 3 V amplitude, which has been modulated to a depth of 60 per cent by an audio signal, is connected between grid and cathode of this valve. Calculate the amplitude of the carrier and fundamental components of anode current.

[6 mA; 0·27 mA]

Example 4.12 A 2 MHz sinusoidal carrier is frequency-modulated by a 1 V peak, 2 kHz sinusoidal signal using a linear modulator. The modulated carrier frequency varies between a maximum instantaneous frequency of 2000 kHz and a minimum instantaneous frequency of 1980 kHz. Calculate the peak phase deviation.

Using the same modulator and same carrier, to what values will the maximum and minimum instantaneous frequencies and peak phase deviations change if the signal amplitude and frequency are changed to

(a) 2 V peak at 2 kHz
(b) 0·5 V peak at 1 kHz
(c) 2 V peak at 100 Hz?

Estimate the minimum practical bandwidth of a circuit required to pass all three modulated waves.

[10 rad 2040–1960 kHz; 20 rad 2010–1990 kHz; 10 rad 2040–1960 kHz; 400 rad 84 kHz]

(CGLI Principles C, 1969)

Example 4.13 A 3 MHz sinusoidal carrier is frequency-modulated by a 2 V peak, 3 kHz sinusoidal signal using a linear modulator. The modulated carrier frequency varies between maximum and minimum instantaneous values of 3010 and 2990 kHz respectively.

Calculate

(a) the sensitivity of the modulator in kHz per volt
(b) the peak phase deviation

Using the same modulator and the same carrier, to what values will the peak frequency and the peak phase deviation change if the signal amplitude and frequency are changed to

(c) 1 V peak at 2 kHz
(d) 10 V peak at 0·2 kHz?

Estimate the minimum practical bandwidth of a circuit required to pass all three modulated waves.

[5 kHz/V, 10/3 rad 3005–2995 kHz; 2·5 rad 3100–2900 kHz; 250 rad 100·4 kHz]

(CGLI Principles C, 1971)

Example 4.14 A VHF signal is modulated by a 16 kHz signal which causes the frequency to deviate by ± 3·2 kHz. Find from the equation of the FM wave

(a) the amplitude of the first two pairs of sidewaves in terms of the unmodulated carrier V_{max}
(b) the percentage of the total power transmitted which is concentrated in the first pair of sidewaves

[0·1 V_{max}; 0·005 V_{max}; 2%] (C)

5 Basic transistor circuit theory

(B) HYBRID PARAMETERS AND EQUIVALENT CIRCUITS

Consider a network that has two input and two output terminals labelled with currents and voltages as shown in figure 5.1.

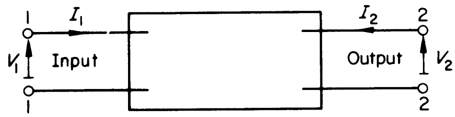


figure 5.1

The behaviour of any circuit or device that can be represented in this way is perfectly described by the relationships that exist between the various currents and voltages. There are four possible combinations that are of particular interest in the transistor, and these are illustrated in figure 5.2.

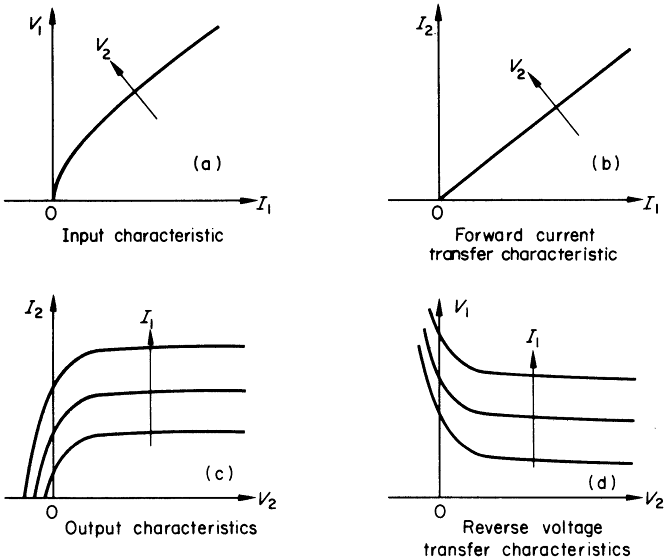


figure 5.2

To obtain the graph of figure 5.2a, the d.c. voltage V_2 across the output terminals is maintained constant at some convenient value whilst the input current I_1 is varied by means of series resistance. At each setting of I_1 the corresponding input voltage V_1 is recorded and the results plotted as shown. In figure 5.2c it is the input current I_1 which is set at a convenient value whilst corresponding readings of V_2 and I_2 are recorded. Similarly, all other graphs in the set can be obtained experimentally. The slope of every graph at an operating point Q is very significant, each being a measure of one of the transistor hybrid parameters (h-parameters); for example

$\dfrac{dV_1}{dI_1}$ = slope of input characteristic with V_2 constant, is denoted by h_i, and is stated in ohms.

$\dfrac{dI_2}{dI_1}$ = slope of forward current transfer characteristic with V_2 constant, and is denoted by h_f.

V_2 can only be regarded as 'constant' if supplied from a source of zero internal resistance. In other words, the condition required during the measurement of h_i and h_f is that the output terminals are short-circuited to alternating signals which would otherwise cause V_2 to vary.

$\dfrac{dI_2}{dV_2}$ = slope of output characteristic with I_1 constant, is denoted by h_o, and is stated in siemens.

$\dfrac{dV_1}{dV_2}$ = slope of reverse voltage transfer characteristic with I_1 constant, and is denoted by h_r.

Conversely, during measurement of h_o and h_r the requirement that I_1 should remain 'constant' is only satisfied if the input terminals are open-circuited to alternating current.

In subsequent operation as an amplifier, it is essential that the emitter-base junction of the transistor has forward bias whilst the collector-base junction has reverse bias. Therefore, the input circuit invariably includes a low resistance whereas the output circuit has relatively high resistance. Consequently, it is not difficult to simulate the necessary short-circuit condition for measurement of h_i and h_f. Similarly, the required open-circuit termination for measurement of h_o and h_r is readily arranged.

All these graphs obtained for a transistor have considerable curvature with the exception of figure 5.2b. For this reason, the location of Q (the operating point) should be clearly stated or else a range of values for the slope will emerge for the same transistor.

Example 5.1 Calculate the parameters h_i, h_f and h_o for the transistor whose input and output characteristics are given in figure 5.3.

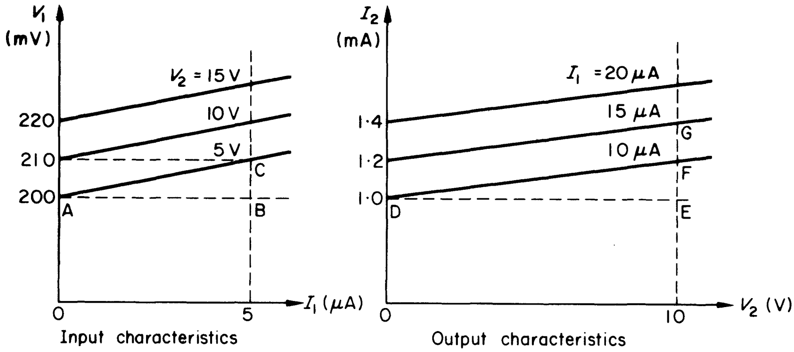


figure 5.3

From the input characteristics

$$h_i = \left.\frac{\delta V_1}{\delta I_1}\right|_{V_2 \text{ constant}} = \text{slope of graphs, } \frac{BC}{AB}$$

$$\therefore \qquad h_i = \frac{10}{10^3} \times \frac{10^6}{5} = 2 \text{ k}\Omega$$

From the output characteristics

$$h_o = \left.\frac{\delta I_2}{\delta V_2}\right|_{I_1 \text{ constant}} = \text{slope of graphs, } \frac{EF}{DE}$$

$$\therefore \qquad h_o = \frac{0{\cdot}2}{10^3} \times \frac{1}{10} = 20 \ \mu\text{S}$$

Also,

$$h_f = \left.\frac{\delta I_2}{\delta I_1}\right|_{V_2 \text{ constant}}$$

Let V_2 be constant at 10 V. Therefore, for a change of 5 μA in I_1, the current I_2 increases from F to G, that is, by 0·2 mA.

$$\therefore \qquad h_f = \frac{0{\cdot}2}{10^3} \times \frac{10^6}{5} = 40$$

Amplification

Assume that a transistor (N–P–N type) is to be used as a common-emitter amplifier.

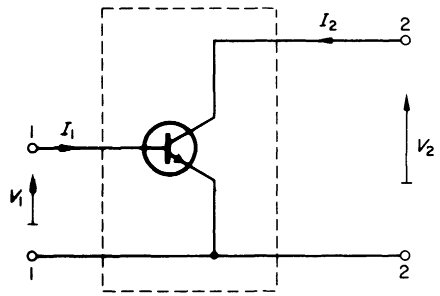


figure 5.4

Figure 5.4 shows the transistor with base bias voltage V_1 and collector supply voltage V_2 connected. I_1 and I_2 represent the no-signal (quiescent) values of current flowing in the transistor under these conditions. Any increase of V_1 will produce a corresponding increase in I_1 according to the graph of figure 5.2a. It should be noted that the base-emitter junction included in the input circuit is always biased in its forward direction. Therefore, it is reasonable that the input circuit should have a relatively low resistance, typically 1 kΩ. Furthermore, by transistor action, any increase in I_1 is transferred in amplified form to the collector circuit. Variations in I_2 due to this effect have already been indicated on figure 5.2b. Combining these two facts, it is now possible to derive a simple equivalent circuit that can be used to evaluate the performance of a transistor when connected as an amplifier. Firstly, it must be accepted that the transistor is correctly biased to some appropriate point on the characteristics; that is, is operating in its active region. Secondly, we can ignore the quiescent values of voltage and current, replacing them with changes that occur as a result of injecting an alternating signal v_i into the circuit.

The amount of current i_1 flowing into the base will be governed by the slope of the input characteristic. Thus, it is the series resistance component h_i that is primarily responsible for limiting signal current into the transistor. It is found that base current controls a much larger current in the collector. The transistor behaves like a current generator, having amplified the input current by a factor h_f which is the slope of the forward current transfer characteristic. Figure 5.5 shows the simplified equivalent circuit in terms of

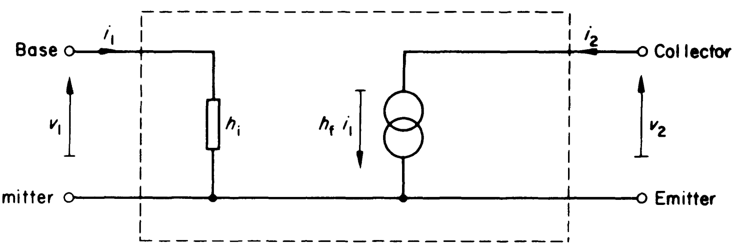


figure 5.5

two h-parameters. The other parameters, h_o and h_r have been omitted until they can be dealt with later in the chapter. By inspection:

from the input circuit,

$$v_1 = h_i i_1 \tag{i}$$

and from the output circuit

$$i_2 = h_f i_1 \tag{ii}$$

With a load R_L connected across the output terminals, an output voltage is developed as shown in figure 5.6.

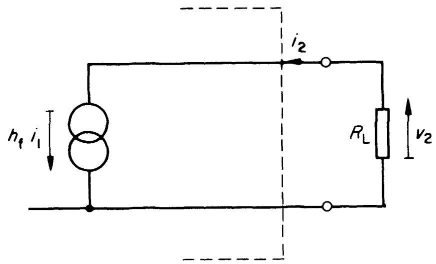


figure 5.6

As i_2 is instantaneously flowing upwards through R_L, the given output voltage v_2 must be stated as negative, that is

$$v_2 = -i_2 R_L \tag{iii}$$

By manipulation of these three basic equations, various properties of the transistor as an amplifier can be deduced.

Current gain A_i

From the block diagram of figure 5.1 it can be seen that I_1 flowing into the network produces I_2 also flowing into the output terminal. If this polarity is in fact found to be so in any actual circuit, it must indicate phase reversal of current within the network. The reader can verify that of all the three possible connections for a transistor, only the common-emitter configuration as shown in figure 5.4 provides 180° phase shift of current between input and output circuits. It is advisable to adopt a mathematical system which yields a negative answer whenever such phase reversal occurs. This can be achieved by defining current gain as

$$A_i = -\frac{i_2}{i_1} \tag{5.1}$$

$$\therefore \qquad A_i = -h_f \tag{5.2}$$

Input impedance Z_{in}

Practical amplifiers usually contain a number of stages connected in cascade (that is, the output of one stage is fed into the input of the next). It was shown in chapter 3, that (in order to eliminate losses) each stage should ideally be matched to its neighbours. It is, therefore, important to know the input impedance of any given amplifier. In this simplified treatment, where the transistor is assumed to be operating at audio-frequencies, internal capacities may be neglected and the input impedance is purely resistive and given by

$$Z_{in} = \frac{v_1}{i_1} \tag{5.3}$$

From equation (i)

$$Z_{in} = h_i \tag{5.4}$$

Voltage gain A_v

From equations (i) and (iii)

$$\frac{v_2}{v_1} = -\frac{i_2 R_L}{h_i i_1}$$

But $i_2 = h_f i_1$ from equation (ii)

$$\therefore \qquad A_v = -\frac{h_f R_L}{h_i} \tag{5.5}$$

An alternative method of calculating the voltage gain can be found if the right-hand side of equation 5.5 is combined with equations 5.2 and 5.4. It then becomes

$$A_v = \frac{A_i R_L}{Z_{in}} \tag{5.6}$$

This result could have been obtained by treating the amplifier as a 'black box', as follows:

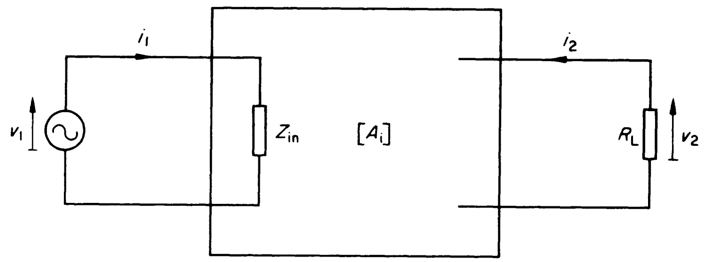


figure 5.7

From figure 5.7

$$A_v = \frac{v_2}{v_1}$$

where $v_2 = -i_2 R_L$ and $v_1 = i_1 Z_{in}$

$$\therefore \qquad A_v = \left(-\frac{i_2}{i_1}\right)\frac{R_L}{Z_{in}} = A_i \frac{R_L}{Z_{in}} \quad \text{as before}$$

It may well be that the internal resistance of the signal source producing v_1 in figure 5.7 has to be taken into account. In this case, the input circuit is modified as shown in figure 5.8.

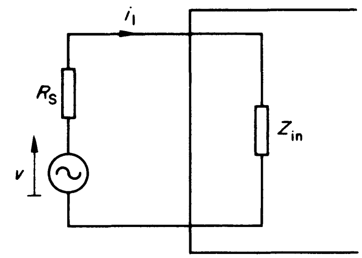


figure 5.8

From figure 5.8,

$$i_1 = \frac{v}{R_s + Z_{in}} \quad \text{or} \quad v = i_1(R_s + Z_{in})$$

With the output circuit unchanged, the overall voltage gain is given by

$$\frac{v_2}{v} = \left(-\frac{i_2}{i_1}\right)\left|\frac{R_L}{(R_s + Z_{in})}\right|$$

that is, overall voltage gain

$$= A_i \frac{R_L}{(R_s + Z_{in})} \qquad (5.7)$$

Output impedance, Z_{out}

It will be seen from figure 5.6 that across the output terminals there is the load R_L shunted by the output resistance of the transistor. The latter has generally such a high value that the output impedance of the amplifier may be taken as R_L without introducing serious error. By a more rigorous treatment later in the chapter, it will be shown that the output admittance of the transistor is modified by a term depending on the internal resistance of the signal source.

The equations derived for A_i, Z_{in} and A_v can now be used to investigate the performance of a common-emitter amplifier by substituting the values given for h_{ie} and h_{fe} as in the following example. (*Note:* The letter e now appearing in the suffix shows that these h-parameters apply to the transistor in its common-emitter configuration.)

Example 5.2 The transistor used in a common-emitter stage has h-parameters

$$h_{ie} = 1{\cdot}45\ \mathrm{k\Omega},\ h_{fe} = 41$$

Estimate current gain, voltage gain and the input impedance when the load resistance is (a) 1 kΩ, and (b) 5 kΩ.

(a) From equation 5.2

$$\text{Current gain, } A_i = -h_{fe} = -41$$

From equation 5.4

$$\text{Input impedance, } Z_{in} = h_{ie} = 1{\cdot}45\ \mathrm{k\Omega}$$

From equation 5.6

$$\text{Voltage gain, } A_v = A_i \frac{R_L}{Z_{in}} = -\frac{41 \times 1}{1{\cdot}45} = -28{\cdot}3$$

(b) Using the same method with the load increased to 5 kΩ, current gain, and input impedance, are unaffected. There is, however, a proportionate increase in voltage gain because of the new value of R_L; that is

$$A_v = -\frac{41 \times 5}{1{\cdot}45} = -141 \text{ approximately}$$

(*Note:* The negative signs in the answers for A_i and A_v indicate that both current and voltage have suffered phase reversal in passing through the transistor.) (B)

Analytical approach to *h*-parameters

(C) The simplified treatment of the transistor so far described in this chapter is found to be rather limited in its scope. It cannot, for example, adequately predict the performance or be used for the design of common-base or common-collector amplifiers. The main reason is that no attempt has yet been made to include the effects of either output impedance or internal feedback in the transistor. If appropriate terms are added to the equivalent circuit of figure 5.5, equations can be derived which are perfectly general in the sense that they apply to all configurations.

A fraction of the current from the equivalent current generator $h_f i_1$ never finds its way into the load. From the output characteristics of figure 5.2c it can be seen that the transistor has a relatively high output resistance, $1/h_o$, resulting mainly from the reverse biased collector-base junction. This term provides a parallel path across the generator terminals and therefore shunts part of the output current away from the load, with resultant fall in current gain. A modified circuit for the transistor can be drawn as shown in figure 5.9. Applying Kirchhoff's first law to point A

$$i_2 = h_f i_1 + h_o v_2 \tag{5.8}$$

which is the more accurate form of equation (ii) already quoted. Substituting for v_2 from equation (iii)

$$i_2 = h_f i_1 - h_o i_2 R_L$$

$$\therefore \quad i_2(1 + h_o R_L) = h_f i_1$$

Hence

$$A_i = -\frac{i_2}{i_1} = -\frac{h_f}{1 + h_o R_L} \tag{5.9}$$

which is the more accurate form of equation 5.2.

Experiments show that as the terminal voltage V_2 is varied, there are corresponding small changes in V_1. The relationship between these quantities is illustrated by the general shape of the graphs in figure 5.2d. The cause of this phenomenon is internal feedback within the device. To make allowance for this, a small reverse voltage $h_r v_2$ is included in the equivalent circuit of figure 5.9. It is the reverse voltage feedback factor h_r which, perhaps more than any other single term, is responsible for the differences that exist between the properties of transistors and valves.

Applying Kirchhoff's second law to the input circuit

$$v_1 = h_i i_1 + h_r v_2 \tag{5.10}$$

which is the more complete form of equation (i).

Substituting

$$v_2 = -i_2 R_L = A_i i_1 R_L$$

$$\therefore \quad v_1 = i_1(h_i + h_r A_i R_L)$$

Hence

$$Z_{in} = \frac{v_1}{i_1} = h_i + h_r A_i R_L \tag{5.11}$$

It should be noted that Z_{in} is no longer a single term h_i but is modified due to the presence of h_r. The input impedance of the amplifier now depends to some extent on the load R_L connected across its output terminals. Substituting from equation 5.9 in equation 5.11

$$Z_{in} = h_i - \frac{h_f h_r R_L}{1 + h_o R_L}$$

Dividing the last term throughout by R_L

$$Z_{in} = h_i - \frac{h_f h_r}{G_L + h_o} \tag{5.12}$$

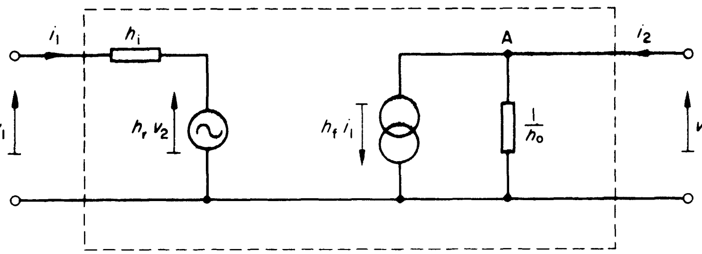


figure 5.9

With a signal source of internal resistance R_S connected to the input terminals, the output impedance of the amplifier is also modified by the term h_r. This will now be investigated with the aid of figure 5.10.

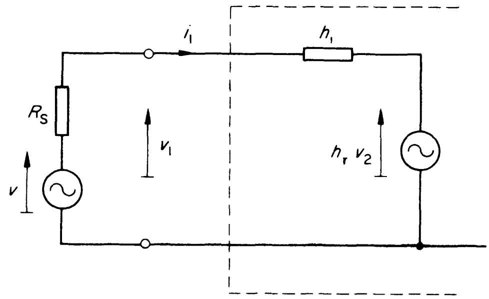


figure 5.10

If the output impedance were to be measured, the procedure would be to inject a voltage v_2 between the output terminals and measure the resultant current i_2. But care must be taken to replace the signal source v by its own internal resistance R_S during the test. Under these conditions

$$v_1 = -i_1 R_S$$

Substituting for v_1 in equation 5.10

$$-i_1 R_S = h_i i_1 + h_r v_2$$

$$\therefore \qquad i_1 = -\frac{h_r v_2}{h_i + R_S}$$

Substituting for i_1 in equation 5.8

$$i_2 = -\frac{h_f h_r v_2}{h_i + R_S} + h_o v_2$$

$$= \left(h_o - \frac{h_f h_r}{h_i + R_S}\right) v_2$$

∴ Output admittance,

$$Y_{out} = \frac{i_2}{v_2} = h_o - \frac{h_f h_r}{h_i + R_S} \qquad (5.13)$$

Note that the form of this equation is similar to that of equation 5.12. If the output impedance Z_{out} is required, it is advisable to substitute the known h-parameters in equation 5.13 and then invert the answer (see example 5.3).

Comparison of the three configurations of amplifier

The equations thus derived for the amplifier can be applied equally to all three configurations. It is only necessary to insert the appropriate h-parameters as shown. At the same time it is instructive to compare the individual characteristics of the different amplifiers that all use the same transistor.

Example 5.3 A transistor is to be used, as an amplifier, in all its three configurations. The h-parameters and recommended load resistances are shown in table 5.1.

table 5.1

	C.B.	C.E.	C.C.
h_i	17 Ω	1·45 kΩ	1·5 kΩ
h_f	−0·98	41	−42
h_o	1·6 μS	42 μS	42 μS
h_r	8×10^{-4}	$7{\cdot}6 \times 10^{-4}$	1
R_L	200 kΩ	5 kΩ	1 kΩ

In each case the amplifier is supplied from a signal generator of 600 Ω internal resistance. Calculate and compare:

(a) current gain
(b) input impedance
(c) voltage gain
(d) output impedance

(a) Current gain

From equation 5.9

$$A_i = -\frac{h_{fb}}{1 + h_{ob}R_L} \quad \text{(in common base)}$$

$$= \frac{0{\cdot}98}{1 + \frac{1{\cdot}6}{10^6} \times 2 \times 10^5}$$

$$= \frac{0{\cdot}98}{1{\cdot}32} = 0{\cdot}74$$

Current gain is therefore less than unity and output current from the collector is in phase with the input current to the emitter.

In common-emitter

$$A_i = -\frac{h_{fe}}{1 + h_{oe}R_L}$$

$$= -\frac{41}{1 + \frac{42}{10^6} \times 5 \times 10^3} = -33{\cdot}9$$

There is therefore a reasonable amount of current gain in the common-emitter amplifier although the phase of current has been shifted through 180° by the transistor.

In common-collector

$$A_i = -\frac{h_{fc}}{1 + h_{oc}R_L}$$

$$= \frac{42}{1 + \frac{42}{10^6} \times 10^3}$$

$$= \frac{42}{1{\cdot}042} = 40{\cdot}3$$

Again there is current gain, with output and input currents in phase.

(b) Input impedance

From equation 5.11

$$Z_{in} = h_{ib} + A_i h_{rb} R_L \quad \text{(in common base)}$$

$$= 17 + 0{\cdot}74 \times \frac{8}{10^4} \times 2 \times 10^5$$

$$= 17 + 118 = 135\ \Omega$$

This is a very low input impedance, typical of the common-base circuit. It would have been even lower but for the resistance reflected through the transistor from the load due to the internal feedback factor h_{rb}.

In common-emitter

$$Z_{in} = h_{ie} + A_i h_{re} R_L$$

$$= 1450 - 33{\cdot}9 \times \frac{7{\cdot}6}{10^4} \times 5 \times 10^3$$

$$= 1450 - 129 = 1321\ \Omega$$

The input impedance is found to be rather less (9 per cent) than the approximation used in example 5.2a.

In common-collector

$$Z_{in} = h_{ic} + A_i h_{rc} R_L$$

$$= (1{\cdot}5 + 40{\cdot}3 \times 1 \times 1)\ \text{k}\Omega$$

$$= 41{\cdot}8\ \text{k}\Omega$$

Thus the common-collector amplifier is seen to have the highest input impedance of all three configurations.

(c) Voltage gain

From equation 5.7

$$A_v = A_i \frac{R_L}{(R_S + Z_{in})}$$

In common base

$$A_v = \frac{0{\cdot}74 \times 2 \times 10^5}{600 + 135} = 201$$

It is clear that the common-base amplifier provides voltage gain in spite of its lack of current gain.

In common-emitter

$$A_v = -\frac{33{\cdot}9 \times 5 \times 10^3}{600 + 1321} = -88{\cdot}2$$

It follows that, if the phase of current is reversed by the common-emitter amplifier, the same must also be true of voltage.

In common-collector

$$A_v = \frac{40{\cdot}3 \times 1}{0{\cdot}6 + 41{\cdot}8} = 0{\cdot}95$$

Note that there is a voltage gain of less than unity in this type of amplifier. The reader may like to verify that the phase of the output voltage is the same as that at the input; that is, the output terminal (emitter) follows the input (base) in potential – hence the name *emitter follower* given to this stage.

(d) Output impedance

In common base

$$Y_{\text{out}} = h_{\text{ob}} - \frac{h_{\text{fb}}h_{\text{rb}}}{h_{\text{ib}} + R_{\text{S}}} \quad \text{(from equation 5.13)}$$

$$= \frac{1{\cdot}6}{10^6} + \frac{0{\cdot}98 \times 8}{(17 + 600)\,10^4}$$

$$= \frac{1}{10^6}\left[1{\cdot}6 + \frac{0{\cdot}98 \times 8}{6{\cdot}17}\right]$$

$$= \frac{2{\cdot}87}{10^6}$$

$$\therefore \text{Output impedance} = \frac{10^3}{2{\cdot}87}\ \text{k}\Omega = 348\ \text{k}\Omega$$

If the collector load were taken into account, the stage would still have a high effective output impedance of 127 kΩ (that is, 200 kΩ in parallel with 348 kΩ).

In common-emitter

$$Y_{\text{out}} = \frac{42}{10^6} - \frac{41 \times 7{\cdot}6}{(1450 + 600)\,10^4}$$

$$= \frac{1}{10^6}(42 - 15{\cdot}2)$$

$$= \frac{26{\cdot}8}{10^6}$$

$$\therefore \text{Output impedance} = \frac{10^3}{26{\cdot}8}\ \text{k}\Omega = 37{\cdot}3\ \text{k}\Omega$$

Combining with the load of 5 kΩ, the total output impedance of the common-emitter stage is 4·4 kΩ. This result is much less than that of the common-base configuration. Because it is nearer to its own input impedance, it is possible to connect common-emitter stages in cascade without serious losses due to impedance mismatch.

In common-collector

$$Y_{out} = \frac{42}{10^6} + \frac{42 \times 1}{1500 + 600}$$

$$= \frac{1}{10^6}(42 + 2 \times 10^4)$$

To a close approximation

$$Y_{out} = \frac{2 \times 10^4}{10^6}$$

$$\therefore \text{Output impedance} = \frac{10^2}{2} = 50\ \Omega$$

The effective output impedance is then found to be only 47·6 Ω. It should now be apparent that the common-collector stage, with its very high input impedance and low output impedance, is suitable for matching purposes. It is used as a buffer stage between a high impedance source and a low impedance load. At the same time, the phase of the signal is preserved.

All the results of this example are listed for convenience in table 5.2, so that the reader may form a much clearer idea of the characteristics produced in amplifiers using the same transistor but in each of its configurations.

table 5.2

	C.B.	C.E.	C.C.
A_i	0·74	−33·9	40·3
Z_{in}	135 Ω	1·32 kΩ	41·8 kΩ
A_v	201	−88·2	0·95
Z_{out} (omitting R_L)	348 kΩ	37·3 kΩ	50 Ω

STABILITY OF THE OPERATING POINT

In the present chapter it has been assumed that each transistor is operating under the correct bias conditions. Furthermore, having located the operating point on the appropriate part of the transistor characteristics, it is implied that the amount of bias remains constant throughout. In a practical circuit this is rarely true, because changes occur which cause the operating point to shift thereby altering the values of the *h*-parameters. It is possible, however, to design the circuit so that these changes are minimised. After explaining one of the principal sources of instability in the transistor, a series of circuits will be analysed by a unified method in order to calculate their degree of stability.

Leakage current

In figure 5.11 the transistor, shown in its common-base mode, is assumed to have the correct d.c. voltages applied. As a result of reverse bias, a leakage current that is dependent on junction temperature flows out of the collector. Indeed it can be shown that in a germanium transistor this leakage current doubles for every 9°C rise in temperature.

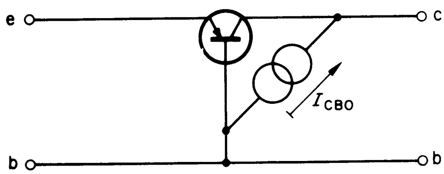


figure 5.11

Leakage current is designated I_{CBO} in figure 5.11. It may be interpreted as 'Current from the collector (I_C) in common-base (B) with the emitter, being the third terminal, open-circuited (O)'. When emitter current flows, leakage current is superimposed upon it so that, in general, collector current is given by

$$I_C = aI_E + I_{CBO} \tag{5.14}$$

(For all practical purposes $a = -h_{fb}$.)

With constant emitter current, equation 5.14 can be differentiated with respect to I_{CBO}

$$\frac{dI_C}{dI_{CBO}} = 0 + 1$$

Stability factor

The finite change in collector current, written as δI_C, for a given change in leakage current δI_{CBO} is known as the stability factor S; which, for the common-base configuration, is seen to have a value of unity. This is its minimum value and means that the common-base amplifier has the greatest thermal stability.

In the common-emitter configuration, the effects of temperature are much more pronounced.

As always, the collector-base junction has reverse bias, causing I_{CBO} to flow across it. This current can only be supplied from the internal base of the transistor shown in figure 5.12 because the base terminal is not connected. By transistor action, I_B controls a current flow of βI_B in the collector circuit. (Normally the current transfer factor β may be equated to h_{fe} without

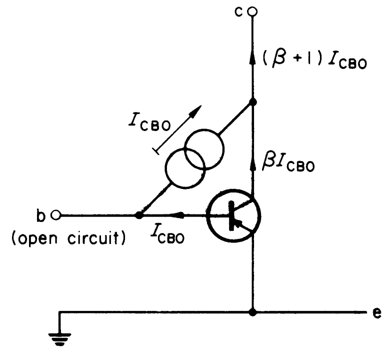


figure 5.12

introducing serious error.) As a result, the total current flowing out of the collector terminal is

$$I_C = (\beta + 1)I_{CBO}$$

This is referred to as I_{CEO}, which by convention means 'Collector current (I_C) in common-emitter (E) with the base open-circuited (O)'. When current actually flows into the base from some external circuit, the resultant collector current again consists of two components

$$I_C = \beta I_B + I_{CEO}$$

$$\therefore \quad I_C = \beta I_B + (\beta + 1)I_{CBO} \tag{5.15}$$

Differentiating with respect to I_{CBO} and maintaining I_B constant

$$S = (\beta + 1)$$

which is the maximum value it can have, showing that the basic common-emitter amplifier has poor temperature stability. Consider a germanium transistor in which I_{CBO} is 5 μA at 25°C. This will increase to 40 μA at 52°C but is still an insignificant term compared with a nominal operating current of 1 mA. At low levels of current, β is probably only about 50 per cent of the value quoted when operating normally as an amplifier. Even so, I_{CEO} may well be 125 μA at 25°C, increasing to 1 mA at 52°C. In the actual common-emitter amplifier, this level of leakage current cannot be tolerated. The presence of I_{CEO} increases the power dissipated at the collector junction whose temperature must rise. This causes a further increase in I_{CEO} so that a regenerative feedback loop exists which can cause the temperature to reach a point where the transistor is rapidly destroyed. This is the phenomenon known as *thermal runaway*. It can be avoided by a combination of circuit design and the provision of adequate energy transfer through a heat sink.

Evaluation of stability factor

A method for evaluating S is illustrated in the following circuits. In this treatment, the voltage drop V_{BE} across the emitter-base junction (0·2 V for a germanium transistor) is neglected throughout. It is permissible to do this because only the stability factor S, relating changes in I_C and I_{CBO}, is being investigated. It should be appreciated, however, that V_{BE} is a very significant term in governing collector current. They are related by another stability factor, which is outside the scope of this book, but is given by

$$S_V = \frac{\delta I_C}{\delta V_{BE}}$$

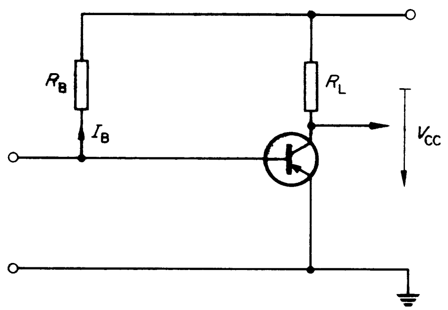


figure 5.13

Applying Kirchhoff's second law to figure 5.13

$$V_{CC} = I_B R_B$$

Multiplying by β throughout

$$\therefore \qquad \beta V_{CC} = \beta I_B R_B$$

From equation 5.15

$$\beta I_B = I_C - (\beta + 1) I_{CBO}$$

$$\therefore \qquad \beta V_{CC} = [I_C - (\beta + 1) I_{CBO}] R_B$$

Differentiating with respect to I_{CBO}, all other terms being regarded as constant,

$$0 = \left[\frac{dI_C}{dI_{CBO}} - (\beta + 1)\right] R_B$$

$$\therefore \qquad S = (\beta + 1) \qquad (5.16)$$

Thus the circuit of figure 5.13 is unstabilised and the transistor is liable to self-destruction by thermal runaway.

By taking one end of the bias resistor R_B off the supply and reconnecting it to the collector instead, thermal stability is improved.

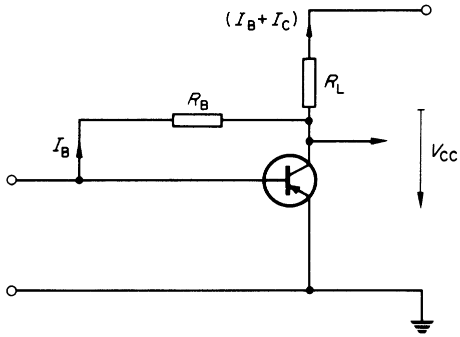


figure 5.14

Any increase in I_{CBO} due to temperature rise will cause a corresponding increase in I_C as already shown in the equation

$$I_C = \beta I_B + (\beta + 1) I_{CBO}$$

But the increased voltage drop in R_L causes less voltage to be available across the potential divider comprising R_B and the emitter-base junction. Therefore V_{BE} and consequently I_B are both reduced accordingly so as to partially offset the original change in I_C.

From the circuit of figure 5.14

$$V_{CC} = I_B R_B + (I_B + I_C) R_L$$
$$= I_B(R_B + R_L) + I_C R_L$$

Multiplying throughout by β

$$\beta V_{CC} = \beta I_B(R_B + R_L) + \beta I_C R_L$$

Substituting for βI_B as before

$$\beta V_{CC} = [I_C - (\beta + 1) I_{CBO}](R_B + R_L) + \beta I_C R_L$$
$$= I_C [R_B + (\beta + 1) R_L] - I_{CBO}(\beta + 1)(R_B + R_L)$$

Differentiating with respect to I_{CBO}, all other terms treated as constants,

$$0 = S [R_B + (\beta + 1) R_L] - (\beta + 1)(R_B + R_L)$$

$$\therefore \qquad S = \frac{(\beta + 1)(R_B + R_L)}{R_B + (\beta + 1) R_L} \qquad (5.17)$$

If it is remembered that $\beta = a/(1-a)$

$$\therefore \qquad \beta + 1 = \frac{a + 1 - a}{1 - a} = \frac{1}{1 - a}$$

Hence, dividing equation 5.17 throughout by $\beta + 1$, it may be written as

$$S = \frac{R_B + R_L}{(1-a)R_B + R_L} \tag{5.18}$$

Example 5.4 A common-emitter stage of the type shown in figure 5.14 has a collector load R_L of 2·7 kΩ. The supply voltage is 6 V and, when conducting normally, V_{BE} may be taken as 0·3 V. If the transistor has $\beta = 49$ and the operating value of emitter current is to be 1 mA, calculate a suitable bias resistor R_B, and hence determine the stability factor S. Comment on the result.

Neglecting leakage current

$$I_E = I_B + I_C$$
$$= (\beta + 1) I_B$$

$$\therefore \qquad I_B = \frac{I_E}{\beta + 1} = 20\ \mu A$$

Applying Kirchhoff's second law to figure 5.14 and including V_{BE}

$$V_{CC} - V_{BE} = I_B R_B + (I_B + I_C) R_L$$
$$= I_B [R_B + (\beta + 1) R_L]$$

Substituting the values given

$$6{\cdot}0 - 0{\cdot}3 = 0{\cdot}02 [R_B + 50 \times 2{\cdot}7]$$

$$\therefore \qquad R_B = \frac{5{\cdot}7}{0{\cdot}02} - 135 = 150\ k\Omega$$

Alternatively, it can be seen from the circuit diagram that the voltage across R_B is 6 V less V_{BE} and volt drop in R_L. As the current in R_B is 20 μA,

$$\therefore \qquad R_B = \frac{6 - 0{\cdot}3 - 2{\cdot}7}{20}\ M\Omega = 150\ k\Omega$$

From equation 5.18

$$S = \frac{R_B + R_L}{(1-a)R_B + R_L}$$

where

$$a = \frac{\beta}{\beta + 1} = \frac{49}{50} \qquad \therefore a = 0{\cdot}98$$

Hence

$$S = \frac{152 \cdot 7}{0 \cdot 02 \times 150 + 2 \cdot 7} = \frac{152 \cdot 7}{5 \cdot 7} = 26 \cdot 8$$

which is quite an improvement on $S = 50$ for the unstabilised C.E. stage using the same transistor. It should be noted, however, that S has been reduced by introducing negative feedback into the circuit via R_B. This has the effect of reducing the gain of the amplifier unless it is decoupled as shown in figure 5.15.

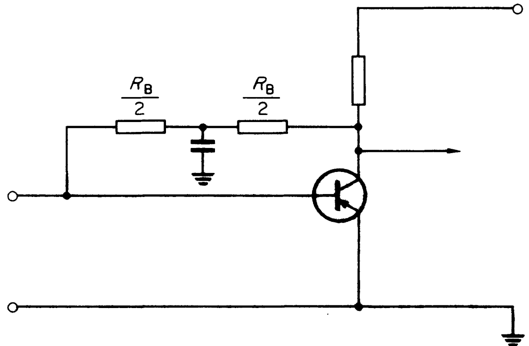


figure 5.15

By connecting a capacitor between the mid-point of R_B and earth, the value of gain is preserved, at the same time stabilising the operating point against changes in temperature with $S = 26 \cdot 8$.

Three-resistor bias

The third type of bias arrangement to be analysed comprises three resistors connected as shown in figure 5.16. It is an attempt to provide a base potential that is less dependent on I_B than in the two previous auto-bias circuits.

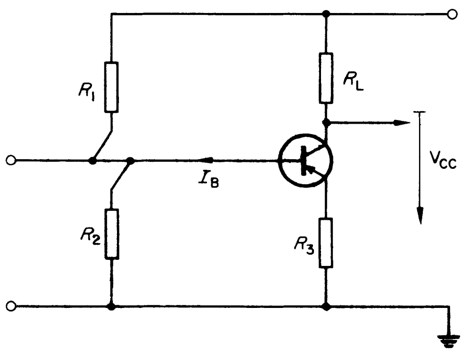


figure 5.16

When dealing with this bias network, the input is first simplified by means of Thévenin's theorem. Imagine a gap opened in the base connecting lead. With zero current flowing in the base-emitter junction and R_3, one side of the gap will be at earth potential. At the other side, the potential is determined by R_1 and R_2 across the supply voltage V_{CC}.

$\therefore$ Voltage across the gap, V'

$$V' = \frac{R_2}{R_1 + R_2} \times V_{CC}$$

I_B can now be regarded as being due to V', with V_{CC} replaced by its own internal resistance. This being a constant voltage source, the internal resistance is zero and results in R_1 and R_2 being effectively connected in parallel; that is, I_B appears to come from a voltage supply V' through $R' = R_1R_2/(R_1 + R_2)$. Thus the simplified equivalent circuit of figure 5.17 can be drawn for analysing by the method already explained.

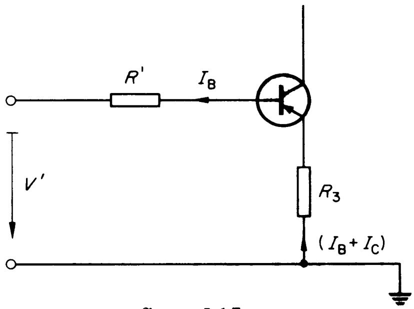


figure 5.17

By Kirchhoff's second law

$$V' = (I_B + I_C)R_3 + I_BR'$$
$$= I_B(R_3 + R') + I_CR_3$$

Multiplying by β throughout

$$\beta V' = \beta I_B(R_3 + R') + \beta I_C R_3$$

Substituting for βI_B

$$\beta V' = [I_C - (\beta + 1)I_{CBO}](R_3 + R') + \beta I_C R_3$$
$$= I_C[(\beta + 1)R_3 + R'] - I_{CBO}(\beta + 1)(R_3 + R')$$

Differentiating with respect to I_{CBO} and treating all other terms as constants

$$0 = S[(\beta + 1)R_3 + R'] - (\beta + 1)(R_3 + R')$$

$$\therefore \qquad S = \frac{(\beta + 1)(R_3 + R')}{(\beta + 1)R_3 + R'} \qquad (5.19)$$

which can also be written as

$$S = \frac{R_3 + R'}{R_3 + (1 - a)R'} \tag{5.20}$$

For good thermal stability, R' should be small compared with R_3. But there are factors which limit the choice of suitable resistors:

(a) If R_3 is too large, it deprives the transistor of operating voltage V_{CE}
(b) If R' is too small, two possibilities arise:

(i) R' tends to short-circuit the incoming signal with consequent loss of gain. Any loss of gain due to negative feedback derived from R_3 can be restored by decoupling this resistor by a large capacitor.
(ii) R' is small because one (or both) of its component parts is also small. This leads to excessive current drain from the supply voltage V_{CC}. Bleeder current via R_1 and R_2 is usually limited to $10 I_B$ in a practical circuit, as shown in the following example.

Example 5.5 A C.E. amplifier stage with 3-resistor bias as in figure 5.16 has $R_L = 2{\cdot}7\,\text{k}\Omega$, $V_{CC} = 6\,\text{V}$, $\beta = 49$, $I_E = 1\,\text{mA}$ and $R_3 = 1\,\text{k}\Omega$. Calculate suitable values for R_1 and R_2 assuming that the current through R_2 is ten times the operating current in the base and $V_{BE} = -0{\cdot}3\,\text{V}$. Hence estimate the stability factor S.

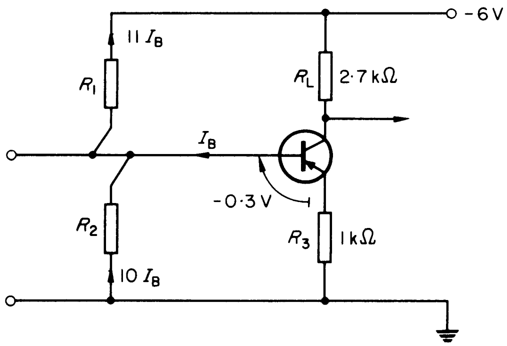


figure 5.18

As

$$I_E = (\beta + 1) I_B = 1\,\text{mA}$$

$\therefore$ $$I_B = 20\,\mu\text{A}$$

$\therefore$ $$\text{Current in } R_2 = 10 I_B = 200\,\mu\text{A}$$

From figure 5.18 voltage drop across $R_2 = I_E R_3 + V_{BE}$

$$= 1{\cdot}0 + 0{\cdot}3 = 1{\cdot}3\text{ V}$$

$$\therefore \qquad R_2 = \frac{1{\cdot}3}{0{\cdot}2} = 6{\cdot}5\text{ k}\Omega$$

Voltage drop across $R_1 = 6 - 1{\cdot}3 = 4{\cdot}7$ V whilst the current through it is 220 μA.

$$\therefore \qquad R_1 = \frac{4{\cdot}7}{0{\cdot}22} = 21{\cdot}4\text{ k}\Omega$$

Using the nearest preferred values, $R_1 = 22\text{ k}\Omega$ and $R_2 = 6{\cdot}8\text{ k}\Omega$

Hence

$$S = \frac{R_3 + R'}{R_3 + (1 - a)R'} \qquad \text{(from equation 5.20)}$$

where

$$R' = \frac{22 \times 6{\cdot}8}{28{\cdot}8} = 5{\cdot}2\text{ k}\Omega$$

and

$$a = \frac{\beta}{\beta + 1} = 0{\cdot}98$$

$$\therefore \qquad S = \frac{1 + 5{\cdot}2}{1 + 0{\cdot}02 \times 5{\cdot}2} = \frac{6{\cdot}2}{1{\cdot}104}$$

$\therefore$ Stability factor, $S = 5{\cdot}61$

It is clear from the last example that this amplifier has much more thermal stability than that of example 5.4 which uses the same transistor. It remains to be seen how this improvement has been achieved at the expense of a reduction of input impedance and current gain. Example 5.6 is included to illustrate these particular features of a practical amplifier circuit.

Example 5.6 A transistor has the following h-parameters:

$$h_{ie} = 1{\cdot}1\text{ k}\Omega, \qquad h_{fe} = 99,$$

$$h_{oe} = 139\ \mu\text{S and } h_{re} = 3{\cdot}75 \times 10^{-4}$$

It is connected as shown in figure 5.19, the capacitors having negligible reactance at the frequency being considered. Calculate the current flowing in R_5 and the stability factor S for the amplifier.

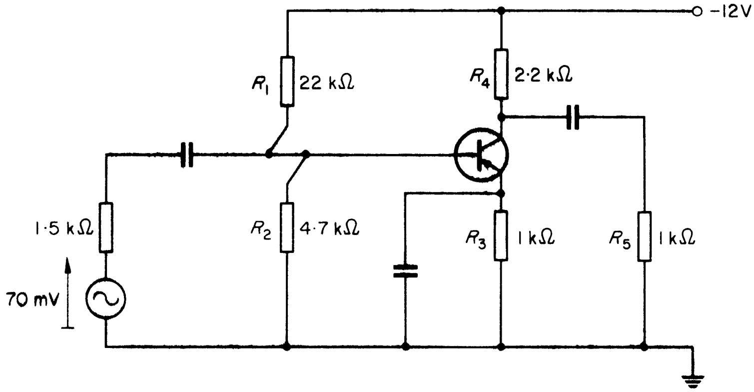


figure 5.19

The actual input circuit can be simplified by application of Thévenin's theorem, as follows:

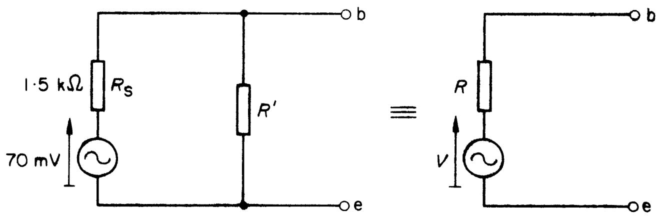


figure 5.20

In figure 5.20

$$R' = \frac{R_1 R_2}{R_1 + R_2} = 3{\cdot}87 \text{ k}\Omega$$

which shunts the amplifier input, thereby reducing Z_{in}

$$R = \frac{R' R_S}{R' + R_S} = 1{\cdot}08 \text{ k}\Omega$$

and the equivalent Thévenin generator

$$V = 70 \times \frac{3{\cdot}87}{1{\cdot}5 + 3{\cdot}87} = 50{\cdot}5 \text{ mV}$$

The output circuit can also be reduced to a single series resistance:

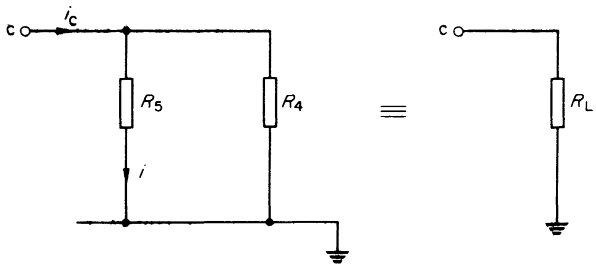


figure 5.21

In figure 5.21

$$R_L = \frac{R_4 R_5}{R_4 + R_5} = 688\ \Omega$$

From equation 5.9

$$A_i = -\frac{h_{fe}}{1 + h_{oe}R_L}$$

$$= -\frac{99}{1 + \dfrac{139 \times 688}{10^6}} = -90{\cdot}36$$

From equation 5.11 the input impedance of the transistor is given by

$$Z_{in} = h_{ie} + A_i h_{re} R_L = 1100 - 90{\cdot}36 \times \frac{3{\cdot}75}{10^4} \times 688$$

$$= 1077\ \Omega$$

$$\therefore\ \text{Current into base} = \frac{V}{R + Z_{in}} = \frac{50{\cdot}5}{1{\cdot}08 + 1{\cdot}077} = 23{\cdot}4\ \mu\text{A}$$

Hence, current in the collector circuit, $i_c = 90{\cdot}36 \times 23{\cdot}4 = 2{\cdot}12$ mA. Now this current divides at the collector terminal according to the relative values of R_4 and R_5, that is

$$\text{Current in } R_5 = i_c \times \frac{R_4}{R_4 + R_5} = 1{\cdot}45\ \text{mA}$$

If the same amplifier had been biased by a single resistor R_B between base and the negative supply rail, such that R_B was at least $10Z_{in}$, the resultant signal current into the base would have been $i_b = 70/(R_S + Z_{in}) = 27{\cdot}2\ \mu$A rather than the $23{\cdot}4\ \mu$A just calculated. Thus the overall current gain of the amplifier is less than it could have been because of the resistors R_1 and R_2 shunting some signal current away from the transistor.

FIELD EFFECT TRANSISTORS

Before leaving the subject of transistor parameters and equivalent circuits, some mention must be made of the field effect transistor (FET). There are two main types, both of which rely for their action on the control of current flow by means of an electric field. The advantages to be gained using this principle can be summarised as follows:

(a) As in the valve, they use only one kind of carrier and are therefore described as *unipolar.* It has been seen in the present chapter that the conventional transistor is *bipolar* since both majority and minority carriers play a part in conduction. It is the minority carriers, generated as a by-product of heat and possibly radiation, that give rise to leakage currents and consequent instability.
(b) The FET is noted for an input impedance that may be as high as $10^6\,\text{M}\Omega$.

In the present state of the art there are difficulties in manufacture that will no doubt be overcome eventually. There seems to be a relatively large spread in FET parameters, and internal capacitance limits the effective bandwidth.

Construction and basic principles

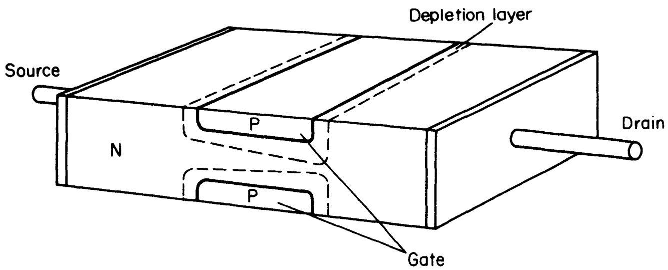


figure 5.22

A narrow channel of N-type semi-conductor is almost completely surrounded by a P–N junction. The two internally connected sections of the gate G correspond to the base of a transistor. The source S and drain D can be compared with emitter and collector respectively. As with a normal transistor, there are certain essential bias conditions that must be satisfied. These are shown in the basic amplifier circuit of figure 5.23.

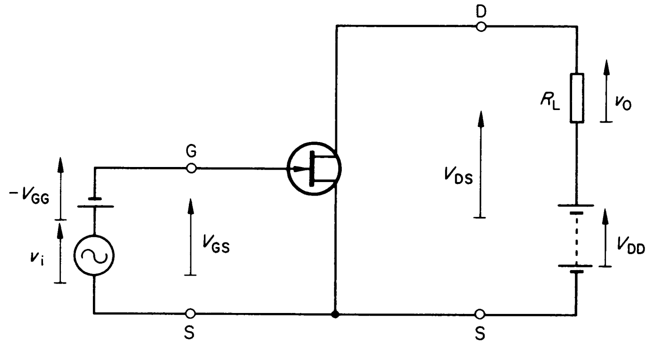


figure 5.23

With the gate and source short-circuited, the channel conducts equally in both directions. Its resistance (typically 100 Ω) depends on its dimensions and the resistivity of the semi-conductor used in its construction. But when the gate is biased negatively with respect to the channel, depletion layers increase in width. Due to the relatively high impurity content of the gate, these depletion layers spread mainly into the channel as shown schematically in figure 5.22. In so doing, they cause these areas of the channel to revert to intrinsic semi-conductor and thus restrict the number of majority carriers (electrons in this case) that are available for transporting charge through the channel. Furthermore, because of the voltage drop along the length of the channel, reverse bias is greatest at the drain end. This is the point at which the depletion layers have their maximum width. A reverse bias voltage V_{GS} of more than about 5 V can so effectively close the path for carriers to the drain that current is almost 'pinched-off'. The value of V_{GS} required to do this is referred to as the *pinch-off* voltage and it is found that the channel resistance under these conditions approaches 10^3 MΩ. It should be appreciated that current through the channel can never be completely cut off because it is the same current that originally contributed to the narrowing of the gap through which it flows. Up to the pinch-off voltage, the channel behaves like a resistance whose ohmic value is a function of V_{GS}; that is, it is a voltage dependent resistor (VDR). When used as an amplifier, however, the FET is operated in the pinch-off region of the characteristics shown in figure 5.24. The negative bias is somewhat less than 5 V but the channel, having the drain positive with respect to the gate, is working under saturation conditions. Hence the current supplied to the load appears to come from a high impedance source, giving the FET a family of characteristics similar to those of a pentode.

It was not until 1966 that field effect transistors were manufactured on a commercial scale. The problem that had to be solved was the production of a thin, lightly-doped channel between two heavily-doped regions of the opposite conductivity type. It was soon realised that the construction shown

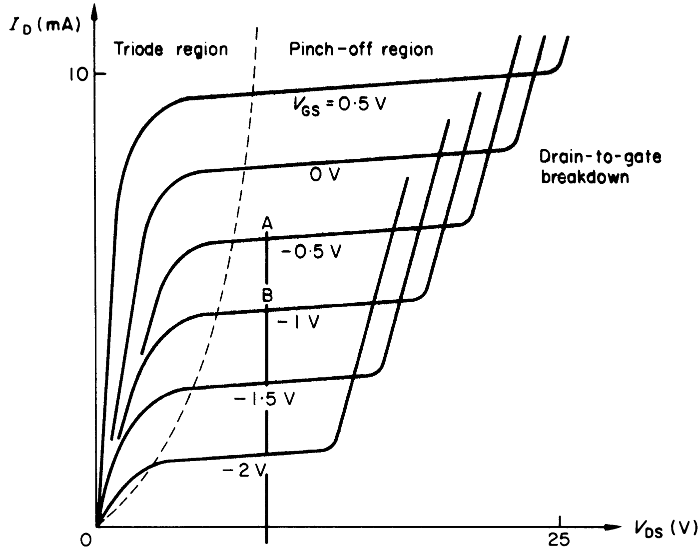


figure 5.24

in figure 5.22, in which an impurity has to be diffused from both sides of the original chip, was not very feasible. As an alternative, the single-sided geometry of figure 5.25 is much more practicable.

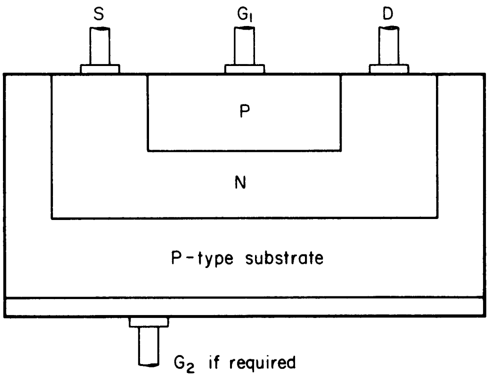


figure 5.25

The P-type substrate and the gate G_1 have low resistivity compared with the N-type channel, causing the depletion layers to spread mainly into the latter as already explained.

Equivalent circuit

This can be deduced from the drain characteristics of figure 5.24 by a method not unlike that used for the junction transistor considered earlier in this chapter. Imagine an ordinate erected at some convenient value of V_{DS} in the pinch-off region as shown. The rate of change of drain current with respect to V_{GS} is known as the mutual conductance, g_m (or transconductance) of the FET. Written mathematically, this may be stated as

$$g_m = \left.\frac{\delta I_D}{\delta V_{GS}}\right|_{V_{DS}\ \text{constant}} \tag{5.21}$$

From points A and B on figure 5.24, the reader can estimate that the transconductance of this particular FET is about 2·3 mA/V. Since g_m is a measure of the a.c. component of current flowing out of the drain due to the input signal voltage v_{gs}, the FET may be represented as a current generator $g_m v_{gs}$. Moreover, the drain characteristics also indicate that the device has a high output resistance that must be shown between the drain-source terminals, in parallel with the external load resistance as in figure 5.26.

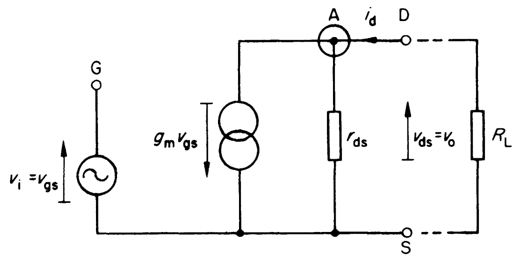


figure 5.26

The fundamental equation for an FET can readily be obtained from figure 5.26 by applying Kirchhoff's first law to point A. Since the layout of this circuit is identical to that for a pentode, it is not surprising to find that calculations involving the FET as an amplifier are also similar. When terminated with a load R_L, the effective resistance across the 'Norton equivalent generator' is $r_{ds}R_L/(r_{ds} + R_L)$.

$$\therefore \qquad v_o = -g_m v_{gs} \frac{r_{ds}R_L}{r_{ds} + R_L}$$

$$\text{Voltage gain, } \frac{v_o}{v_{gs}} = -\frac{g_m r_{ds} R_L}{r_{ds} + R_L}$$

If the product $g_m r_{ds}$ is written as the voltage amplification factor μ

$$A_v = -\frac{\mu R_L}{r_{ds} + R_L} \tag{5.22}$$

as for a triode.

Furthermore, if r_{ds} is sufficiently high compared with R_L that r_{ds} can be neglected, equation 5.22 becomes

$$A_v = -\frac{\mu R_L}{r_{ds}} = -g_m R_L \tag{5.23}$$

as for a pentode.

Example 5.7 The FET used as the first stage of a voltage amplifier has the following parameters: Transconductance, $g_m = 2$ mA/V; Drain resistance, $r_{ds} = 100$ kΩ. Calculate the voltage gain when the load resistance connected in the drain circuit is

(a) 50 kΩ
(b) 5 kΩ

(a) From equation 5.22

$$A_v = -\frac{\mu R_L}{r_{ds} + R_L}$$

where $\mu = 2 \times 100 = 200$

$$\therefore \qquad |A_v| = \frac{200 \times 50}{150} = 66{\cdot}7$$

(b) When $R_L = 5$ kΩ, which is very much less than r_{ds}, the approximation of equation 5.23 can be used without serious error, that is

$$|A_v| = g_m R_L = 10$$

THE MOS TRANSISTOR

The *metal-oxide-semiconductor* transistor is another type in which current flow is controlled by an applied electric field. Some authorities consider that the MOST offers greater advantages than the FET in certain applications. One of these advantages arises from the fact that it is constructed without any P–N junctions across which current has to flow. The alternative principle is illustrated in figure 5.27.

It will be noted that the gate G is insulated from the substrate by a layer of silicon dioxide, SiO_2 which gives the MOST an input resistance exceeding 10^4 MΩ. These two components behave as a parallel-plate capacitor with

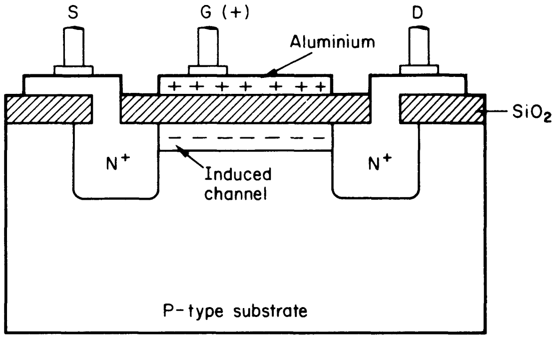


figure 5.27

SiO_2 as the dielectric. If the substrate is earthed while a positive potential is applied to the gate, negative charges are induced in a channel as shown in figure 5.27. In a region of P-type conductivity, these negative charges must be regarded as minority carriers whose density depends on the gate potential. By this method, a channel of variable conductivity is provided in the material of the substrate along which current can flow from source to drain.

The basic principle of the MOST which has just been described briefly has been extended in recent years. It now includes a *depletion MOST* in which there is diffused an N-type channel between source and drain. In subsequent operation, the gate is made negative so as to induce positive minority carriers in the channel and thereby control drain current. For full description of this and other related MOS transistors, the reader should refer to a transistor textbook or a manufacturer's application report. (C)

MISCELLANEOUS EXAMPLES

(B) ***Example 5.8*** Sketch a test circuit for obtaining the collector-current/collector-voltage characteristics of a transistor. Give a brief description, with typical resulting curves, of an experiment determining these characteristics for a transistor connected in the following configurations:

(a) common emitter, and (b) common base.

Sketch a simple transistor amplifier circuit for either (a) or (b), showing the polarities of the battery connections.

(CGLI Principles B, 1966)

(Note: The apparatus would be connected as shown in figure 5.28.)

Particular care should be taken with the following points of technique when performing the experiment:

1. The ammeter recording the output current I_2 should be connected on the 'transistor side' of the voltmeter. If this precaution is not observed, the

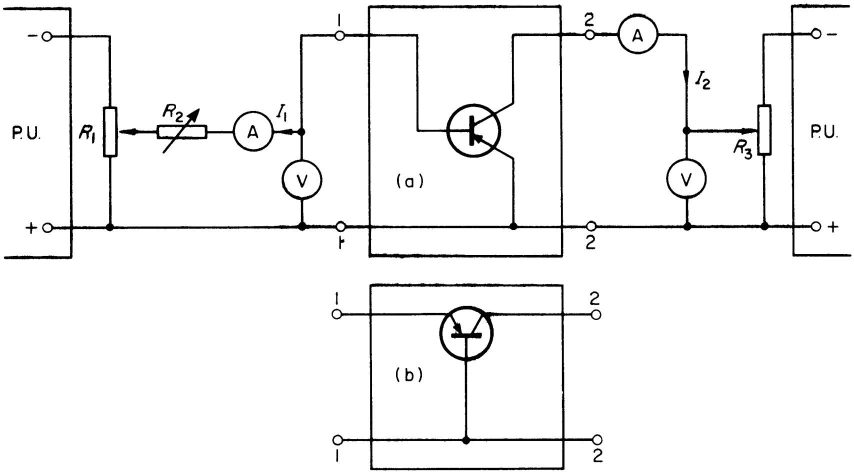


figure 5.28

current reading will include that taken by the voltmeter and the output characteristics will appear steeper than they should be; that is, the internal resistance of the voltmeter will be shunting the output parameter h_o, which is very high in the common base configuration.

2. Conversely, on the left of figure 5.28, the ammeter recording the input current I_1 should be connected on the 'supply side' of the voltmeter. In this way the voltmeter will indicate the voltage across the transistor input terminals, not including the internal resistance of the ammeter in series with the input parameter h_i, which is quite small in the common base configuration.

3. The variable resistance R_2 is used to simulate a *constant current* source supplying the input current. It artificially increases the internal resistance of the source and thereby prevents too much interference at the input terminals due to adjustments in the output circuit. R_2 should therefore be maintained as high as is practicable throughout the experiment.

4. The polarity of the input power unit must be reversed when the common-base connection (b) is introduced into the circuit.

Example 5.9 A simple single-stage amplifier using one transistor is shown in figure 5.29. Explain the function of each component and give typical values.

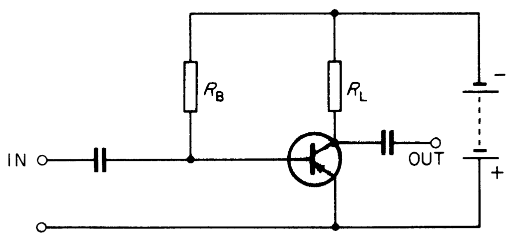


figure 5.29

Describe an experiment to determine the gain of this amplifier when it is connected to a resistor as output. Include details of the circuit and test equipment that would be used.

(CGLI Principles B, 1970)

Note: It is advisable to use voltmeters with very high internal resistance (digital voltmeters if possible) when making measurements in this circuit. Also, a cathode ray oscilloscope is essential to observe the input and output waveforms. False voltmeter readings are obtained if the waveforms are distorted by clipping either or both half-cycles of signal voltage. (B)

(C) ***Example 5.10*** Define the h-parameters of a transistor. Show how these can be used to determine the current gain of a common-base amplifier when a load resistance of R_L is connected in its collector circuit. Comment on the current gain obtained from a transistor operating in its common-base mode.

Example 5.11 The h-parameters of a transistor, connected as a common-emitter stage with a 1 kΩ collector load, are given as:

$$h_{ie} = 1{\cdot}1\ \text{k}\Omega, \quad h_{fe} = 50, \quad h_{oe} = 25\ \mu\text{S}$$

If h_{re} can be neglected, calculate the current and voltage gains of the stage.

[−48·8; −44·3]

Example 5.12 If the transistor amplifier of example 5.11 is supplied with an input of 20 mV from a signal source whose internal resistance is 600 Ω, estimate the voltage output developed across the collector load.

[0·574 V]

Example 5.13 Two transistors have the h-parameters given in table 5.3:

table 5.3

	h_{fe}	h_{oe}
TR_1	98	140 μS
TR_2	49	40 μS

Calculate the value of load resistance R_L that will cause these transistors to have the same current gain when connected in their C.E. configuration. What is the resultant value of current gain?

[16·7 kΩ; −29·37]

Example 5.14 Due to the spread of characteristics, a certain transistor type is found to have h_{ie} with a range of values from 0·4 kΩ to 1·5 kΩ. Similarly, h_{fe} can vary between 30 and 75. If h_{oe} is constant at 200 μS and this type of transistor is used as a common-emitter amplifier with a collector load of 2·2 kΩ, calculate the maximum and minimum values of voltage gain.

[−286·5; −30·6]

Example 5.15 A transistor is connected in its common-base configuration and biased to the required operating point. The results obtained when tested under small-signal conditions are shown in figure 5.30. Draw a clearly labelled equivalent circuit diagram and add the calculated values of the transistor *h*-parameters.

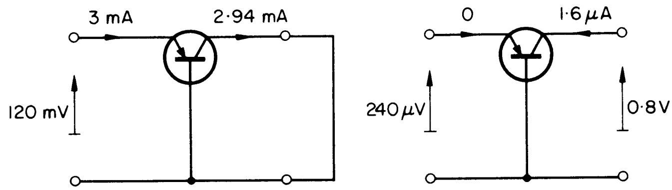


figure 5.30

[40 Ω; −0·98; 2 μS; 3 X 10^{-4}]

Example 5.16 Figure 5.31 shows a small-signal equivalent circuit of a transistor with the values of the hybrid parameters appropriate to the common-emitter connection. Assuming that the bias conditions remain unchanged throughout, calculate:

(a) the input resistance and the current gain with the collector load on short circuit
(b) the voltage gain and the power gain with the collector load of resistance 50 kΩ.

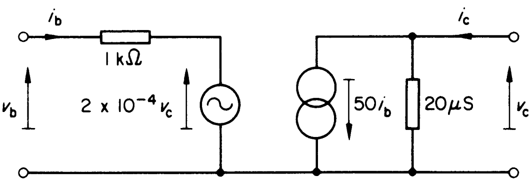


figure 5.31

[1 kΩ, −50; −1667, 41·7 X 10^3]

(CGLI Principles C, 1967)

Example 5.17 In the circuit of figure 5.32, I_{CBO} = 5 μA and a = 0·975. Find I_{CEO} and use it to calculate the emitter current if V_{BE} = − 0·3 V. Also calculate the p.d. across the transistor V_{CE} and the stability factor S for this circuit.

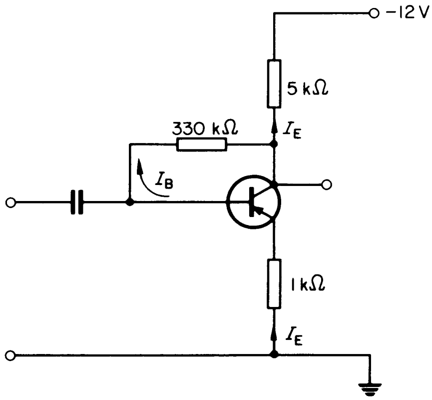


figure 5.32

[0·2 mA; 0·936 mA; −6·39 V; 23·6]

Note:

From the circuit diagram,

$$1I_E + 330I_B + 5I_E = 12 - 0{\cdot}3$$

$$\therefore \quad 330I_B + 6I_E = 11{\cdot}7 \qquad \text{(i)}$$

where

$$I_B = I_E - I_C$$

$$= I_E - (\beta I_B + I_{CEO})$$

$$\therefore \quad (\beta + 1)I_B = I_E - I_{CEO}$$

and

$$I_B = \frac{I_E - I_{CEO}}{\beta + 1} = \frac{I_E - 0{\cdot}2}{40}$$

Substituting for I_B in (i)

$$330\left(\frac{I_E - 0{\cdot}2}{40}\right) + 6I_E = 11{\cdot}7$$

$$\therefore \quad I_E = \frac{11{\cdot}7 + 1{\cdot}65}{8{\cdot}25 + 6} = 0{\cdot}936 \text{ mA}$$

Example 5.18 The transistor, connected as shown in figure 5.33, has the following particulars:

$$a = 0{\cdot}98, \ I_{CBO} = 10\,\mu\text{A}, \ V_{BE} = 0{\cdot}2\,\text{V}$$

Resistance values in the circuit are

$$R_B = 220\,\text{k}\Omega, \ R_E = 470\,\Omega, \ R_L = 3{\cdot}3\,\text{k}\Omega$$

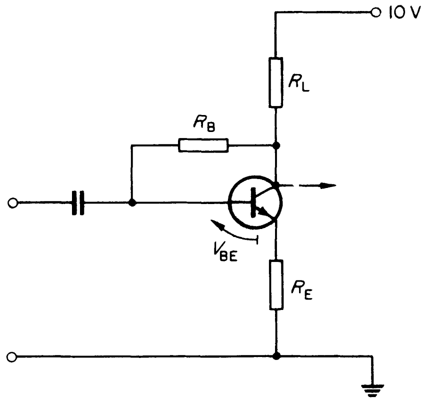


figure 5.33

Determine I_C and V_{CE} at the operating point of the transistor and also find from first principles the stability factor S.

[1·45 mA; 4·6 V; 27·4]

Example 5.19 Calculate the output voltage of the C.E. stage shown in figure 5.34 assuming that all capacitors have negligible reactance at the operating frequency.

$$h_{ie} = 1{\cdot}3\ \text{k}\Omega,\ h_{fe} = 49,\ h_{oe} = 50\ \mu\text{S},\ h_{re} = 10^{-3}$$

Also estimate the change in collector current if I_{CBO} increases by 5 μA due to a rise in temperature.

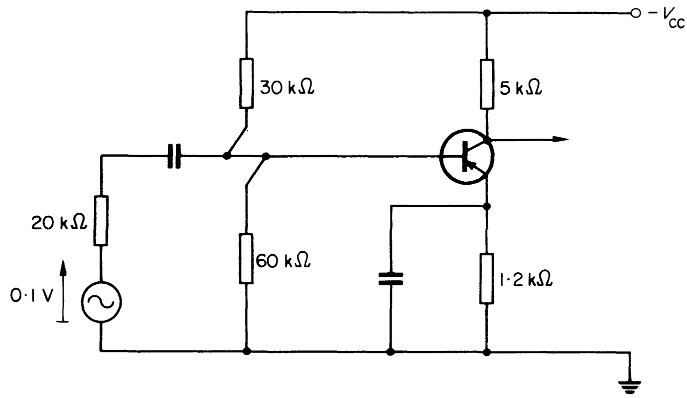


figure 5.34

[880 mV; 66·2 μA]

Example 5.20 In the C.E. configuration shown in figure 5.35, the currents and voltages are related by the equations

$$v_1 = 10^3 i_1 + 10^{-3} v_2 \qquad \text{(i)}$$

$$i_2 = 80 i_1 + 2 \times 10^{-5} v_2 \qquad \text{(ii)}$$

Define each of the four h-parameters and state their values, with appropriate units, for the given transistor. If the signal current flowing into the base is $100\,\mu$A and a load of $1\,\text{k}\Omega$ is connected between collector and emitter, calculate the output current.

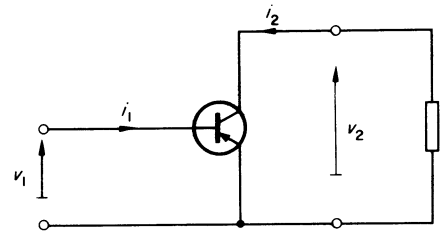


figure 5.35

[$1\,\text{k}\Omega$, 80, $20\,\mu$S, 10^{-3}; 7·85 mA]

(CGLI Principles C, 1966)

Example 5.21 Why is bias stabilisation used in common emitter transistor amplifiers? Figure 5.36 shows the circuit of a transistor with bias stabilisation. Identify the components of the bias circuit and explain the stabilising action.

At the bias point, the transistor has $h_{ie} = 2\,\text{k}\Omega$, $h_{fe} = 50$ and $h_{oe} = 5 \times 10^{-5}$ siemens, h_{re} being negligible. Calculate the voltage gain of the stage, assuming the bias components and all capacitors to have negligible effect on the signal.

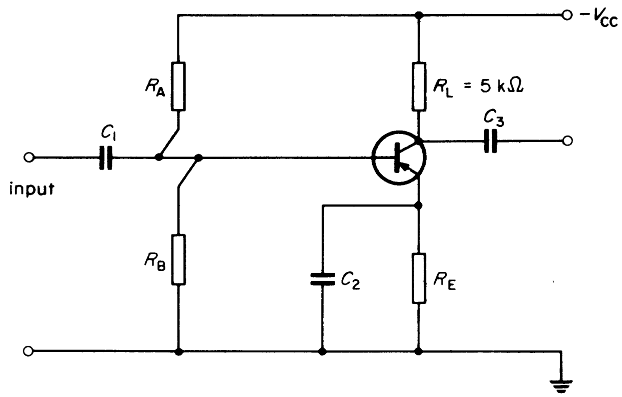


figure 5.36

[-100]

(CGLI Principles C, 1969)

Example 5.22 Explain what is meant by the term 'thermal runaway'. What is the major difference in behaviour in this respect between the two circuits of figure 5.37?

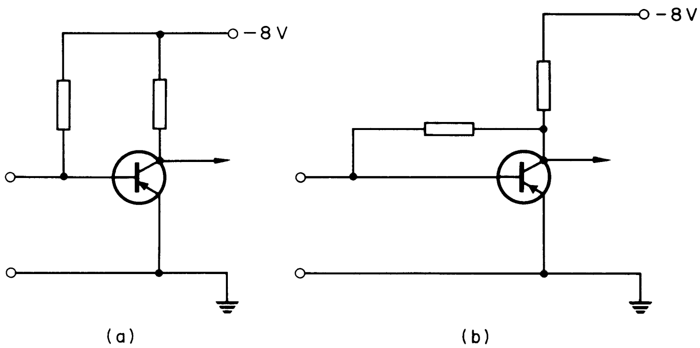


figure 5.37

Define the term 'stability factor, S' and derive an expression for it in circuit (b). If in such a circuit $\beta = 25$ and the operating values of I_C and V_{CE} are 1 mA and −4 V respectively, calculate the value of S. What precautions must be taken to avoid loss of gain due to negative feedback at the signal frequency?

[Neglecting V_{BE}, $S = 13{\cdot}5$] (C)

6 Transistor multi-stage and wideband amplifiers

(C) When the gain of a single transistor is not sufficiently high to handle very weak signals, two or more stages may be connected in cascade. The choice of configuration to be used in an amplifier of this type is important for the following reasons:

1. The common-base mode has a low input and high output impedance. The mismatch between stages is so great that no advantage is gained by connecting them in series. Indeed it can be shown that the overall voltage gain is approximately equal to that of the output stage only. For example, from the previous chapter, voltage gain of first stage

$$A'_{\mathrm{v}} = A'_{\mathrm{i}} \frac{R_{\mathrm{L}}}{Z'_{\mathrm{in}}}$$

where A'_{i} and Z'_{in} are the current gain and input impedance respectively of the first stage. For the common-base amplifier, $Z_{\mathrm{out}} \triangleq R_{\mathrm{L}}$ and this would preferably approximate to the input impedance of the second stage for maximum power transfer. Furthermore, identical stages would have equal input impedances (considered to be real at audio-frequencies).

$$\therefore \qquad R_{\mathrm{L}} \triangleq Z_{\mathrm{in}} \text{ of second stage} \triangleq Z'_{\mathrm{in}}$$

As current gain in this configuration is less than unity, it follows that the voltage gain of the first stage makes no contribution to the amplifier as a whole.

2. The common-collector mode has a voltage gain of less than unity, as already explained in chapter 5. Hence it is not possible, without a transformer, to increase voltage gain by cascading. Again the stages would be incompatible from the point of view of impedance matching.

3. The common-emitter mode suffers from none of the defects outlined and is therefore invariably used for multi-stage amplifiers. To isolate the stages so that signals may pass freely between them without disturbing any bias potentials, resistance-capacitance coupling (first developed for valve circuits) can be used.

RESISTANCE–CAPACITANCE COUPLED AMPLIFIERS

Consider the RC-coupled common-emitter amplifier shown in figure 6.1.

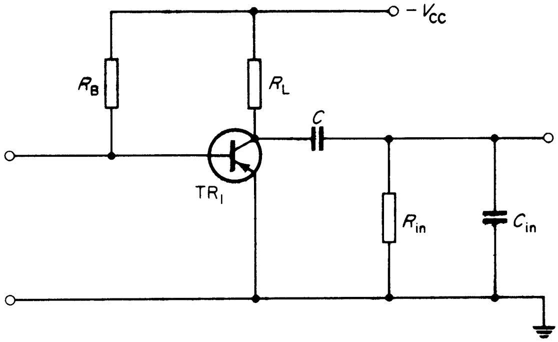


figure 6.1

An equivalent circuit can be drawn for the purpose of analysis, neglecting the bias resistor R_B and the two parameters h_{oe} and h_{re}. The components R_{in} and C_{in} are to simulate the effective input circuit of the second stage. In the following treatment, the equivalent circuit of figure 6.2 will be studied within three different ranges of frequency.

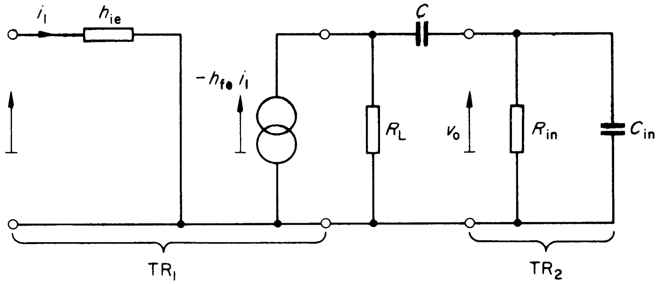


figure 6.2

Low-frequency response

At low frequencies, C_{in} can be neglected.

By Thévenin's theorem, the output circuit of TR_1 can be reduced to an equivalent voltage generator in series with R_L as shown in figure 6.3.

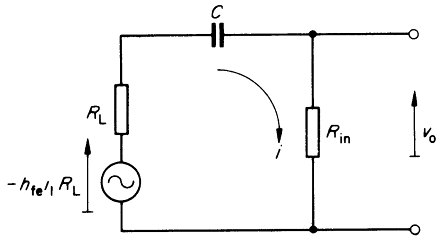


figure 6.3

From figure 6.3, the circulating current,

$$i = -\frac{h_{fe} i_1 R_L}{(R_L + R_{in}) - j/\omega C}$$

and

$$v_o = iR_{in}$$

$$= -\frac{h_{fe} i_1 R_L R_{in}}{(R_L + R_{in}) - j/\omega C}$$

where

$$i_1 = \frac{v_i}{h_{ie}} \qquad \text{(from figure 6.2)}$$

Hence, voltage gain,

$$A_v = \frac{v_o}{v_i} = -\frac{h_{fe} R_L R_{in}}{h_{ie}[(R_L + R_{in}) - j/\omega C]} \tag{6.1}$$

It should be remembered that the negative sign in front of equation 6.1 indicates phase reversal of the signal as it passes from the base of TR_1 to that of TR_2. But the presence of a negative j-term in the denominator means that, after rationalisation, there will be a positive j-term in the numerator. Thus, the phase of the signal will be advanced by the coupling capacitor so that the net result is a phase shift of less than 180°: that is, v_o lags behind v_i by angle ϕ which is less than 180° at LF.

Medium-frequency response

At medium frequencies, both C and C_{in} are negligible. Omitting the reactive term from equation 6.1, voltage gain at MF is given by

$$A_v = -\frac{h_{fe} R_L R_{in}}{h_{ie}(R_L + R_{in})} \tag{6.2}$$

There are several points to note about equation 6.2. Firstly, it represents the maximum voltage gain of the amplifier, as calculated between the bases of the two transistors. Secondly, the phase shift ϕ which occurs between these two points is 180° and the gain remains at its maximum so long as the reactive terms are negligible. Thirdly, it will be seen that when $(R_L + R_{in})$ = the modulus of $1/\omega C$ in equation 6.1, then

$$\text{LF voltage gain} = -\frac{h_{fe} R_L R_{in}}{h_{ie}(R_L + R_{in})(1 - j1)}$$

that is, at LF

$$|A_v| = \frac{h_{fe} R_L R_{in}}{h_{ie}(R_L + R_{in})\sqrt{2}}$$

∴ Low frequency gain

$$= \frac{A_V \text{ at MF}}{\sqrt{2}} = 0{\cdot}707 \text{ mid-frequency gain.}$$

Let the frequency at which this happens be f_1, then

$$R_L + R_{in} = \frac{1}{2\pi f_1 C}$$

$$f_1 = \frac{1}{2\pi C(R_L + R_{in})} \tag{6.3}$$

f_1 is referred to as the *lower cut-off frequency*, whose real significance will be explained later in this chapter. For the moment it is sufficient to realise that the gain is 70 per cent of its maximum. The phase angle of the equivalent circuit is 45° at f_1 due to the reactance of the coupling capacitor being numerically equal to the series resistance ($R_L + R_{in}$).

High-frequency response

At high frequencies, only C is negligible. The equivalent circuit must now be re-drawn as in figure 6.4.

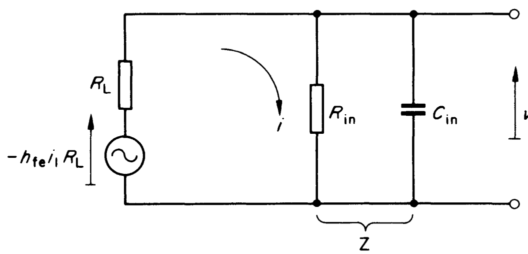


figure 6.4

From the diagram,

$$Z = \frac{R_{in}}{1 + j\omega C_{in}R_{in}}$$

also

$$i = -\frac{h_{fe}i_1R_L}{R_L + Z}$$

and

$$v_0 = -\frac{h_{fe}i_1R_LZ}{R_L + Z}$$

Substituting v_i/h_{ie} for i_1 and $R_{in}/(1+j\omega C_{in}R_{in})$ for Z

$$v_0 = -\frac{h_{fe}v_i}{h_{ie}}\left[\frac{R_LR_{in}}{(1+j\omega C_{in}R_{in})\left(R_L + \dfrac{R_{in}}{1+j\omega C_{in}R_{in}}\right)}\right]$$

∴ HF voltage gain,

$$\frac{v_0}{v_i} = -\frac{h_{fe}}{h_{ie}}\left[\frac{R_LR_{in}}{(R_L+R_{in}) + j\omega C_{in}R_{in}R_L}\right] \tag{6.4}$$

The positive j-term in the denominator has the effect of retarding the phase of v_0 relative to v_i with the result that the angle of lag ϕ at HF is somewhat greater than 180°. Furthermore, the equivalent circuit can be modified yet again by Thévenin's theorem to become the series circuit of figure 6.5.

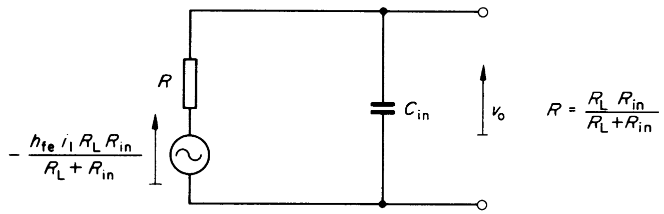


figure 6.5

It can be readily verified that the EMF of the equivalent Thévenin generator is the same as the output voltage at MF. When the frequency is increased to the point where the modulus of $1/\omega C_{in} = R$, the voltage gain falls once more to 70 per cent of its maximum. Let the *upper cut-off frequency* be f_2.

$$\therefore \qquad \frac{1}{2\pi f_2 C_{in}} = R$$

so that,

$$f_2 = \frac{1}{2\pi C_{in}R} \tag{6.5}$$

Thus f_2 is the frequency at which the reactance due to internal and stray capacity in the circuit is numerically equal to the parallel resistance of the equivalent circuit. Again the phase shift in the amplifier departs by 45° from its mid-frequency value of 180°.

Overall frequency response

Figure 6.6 shows a typical frequency response for an RC-coupled amplifier. The loss of gain at LF is caused by the coupling capacitor C which can be regarded as a series component. At HF it is the internal and stray capacitance, shunting the input of the second transistor, which is responsible for a fall in gain. The frequency range over which the voltage (or current) gain exceeds 70 per cent of its maximum is referred to as the amplifier *bandwidth* and more will be read about this later in the chapter.

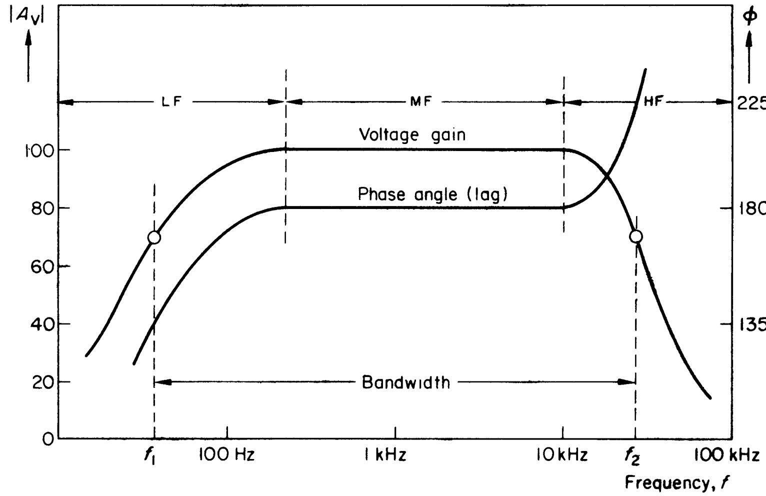


figure 6.6

Example 6.1 The second stage of an *RC*-coupled transistor amplifier has an input impedance of 1 kΩ at audio frequencies. Neglecting all other forms of LF attenuation, calculate the minimum size of coupling capacitor in order that the gain at 20 Hz shall be 70 per cent of its mid-frequency value. The load resistor of the first stage is 2·2 kΩ.

From equation 6.3

$$f_1 = \frac{1}{2\pi C(R_L + R_{in})}$$

Frequency f_1 at which gain falls to 70 per cent of its maximum is 20 Hz

$$\therefore \qquad C = \frac{10^6}{2\pi \times 20(2{\cdot}2 + 1{\cdot}0)10^3}\ \mu\text{F}$$

so that minimum coupling capacitor = 2·5 μF approximately.

Example 6.2 An amplifier uses a transistor having $h_{fe} = 150$, $h_{ie} = 1{\cdot}5$ kΩ and $R_L = 5$ kΩ, coupled to a second transistor via a 5 μF capacitor. The input circuit of the second stage may be represented by a resistance of

1·2 kΩ shunted by 1600 pF. Calculate:

(a) the mid-band voltage gain
(b) the lower cut-off frequency, f_0
(c) the higher cut-off frequency, f_2.

(a) From equation 6.2

$$A_v = -\frac{h_{fe}R_L R_{in}}{h_{ie}(R_L + R_{in})}$$

It is instructive to compare this expression with equation 5.6 in which

$$A_v = A_i \frac{R_L}{Z_{in}}$$

In the RC-coupled amplifier, the effective collector load of the first stage is $R = R_L R_{in}/(R_L + R_{in})$ from figure 6.3 with C omitted. Also, because h_{oe} and h_{re} are not given, equation 6.2 makes use of the approximations $A_i = -h_{fe}$ (the short-circuit current gain), and $Z_{in} = h_{ie}$. Thus, equations 5.6 and 6.2 are of the same form.

Substituting the values given

$$A_v = -\frac{150 \times 5 \times 1{\cdot}2}{1{\cdot}5(5 + 1{\cdot}2)} = -96{\cdot}8$$

so that, the maximum (mid-band) voltage gain is 96·8.

(b) From equation 6.3

$$f_1 = \frac{10^6}{2\pi \times 5(5 + 1{\cdot}2)10^3}\ \text{Hz} = 5{\cdot}13\ \text{Hz}$$

(c) From equation 6.5

$$f_2 = \frac{1}{2\pi C_{in} R}$$

where

$$R = \frac{R_L R_{in}}{R_L + R_{in}} = 0{\cdot}968\ \text{k}\Omega$$

$$\therefore \qquad f_2 = \frac{10^{12}}{2\pi \times 1600 \times 0{\cdot}968 \times 10^3}\ \text{Hz} = 102{\cdot}7\ \text{kHz}$$

Example 6.3 With the amplifier of example 6.2, estimate the gain at frequencies of 15·9 Hz and 79·6 kHz. Express the answers in polar form.

From equation 6.1, the LF gain is given by

$$A_v = -\frac{h_{fe}R_L R_{in}}{h_{ie}[(R_L + R_{in}) - jX_C]}$$

$$= -\frac{150 \times 5 \times 1{\cdot}2}{1{\cdot}5[(5 + 1{\cdot}2) - jX_C]}$$

where X_C must be stated in kΩ. At $f = 15{\cdot}9$ Hz,

$$X_C = \frac{10^3}{2\pi \times 15{\cdot}9 \times 5}\ \text{k}\Omega = 2\ \text{k}\Omega$$

$$\therefore \qquad A_v = -\frac{600}{6{\cdot}2 - j2} = \frac{600\angle 180^\circ}{6{\cdot}5\angle -18^\circ} = 92\angle 198^\circ$$

In other words, the amplifier has a voltage gain of 92 when the frequency is 15·9 Hz. The voltage at the base of TR_2 lags 162° behind the original input signal.

From equation 6.4

HF gain

$$= -\frac{h_{fe}}{h_{ie}}\left[\frac{R_L R_{in}}{(R_L + R_{in}) + j\omega C_{in} R_L R_{in}}\right]$$

$$= -\frac{150}{1{\cdot}5 \times 10^3}\left[\frac{5 \times 1{\cdot}2 \times 10^6}{(5 + 1{\cdot}2)10^3 + j2\pi \times 79{\cdot}6 \times 10^3 \times \frac{1600}{10^{12}} \times 5 \times 1{\cdot}2 \times 10^6}\right]$$

$$= -100\left(\frac{6}{6{\cdot}2 + j4{\cdot}8}\right) = \frac{600\angle 180^\circ}{7{\cdot}84\angle 38^\circ} = 76{\cdot}6\angle 142^\circ$$

Thus, the amplifier voltage gain at 79·6 kHz is 76·6 with v_o leading v_i by 142°. (C)

THE DECIBEL NOTATION

(B) Modern telecommunication equipment often contains several units in cascade, each making a contribution to the overall gain of the system. It would be convenient if some simple method were available whereby the total effect (power gain, etc.) could be stated in terms of its component parts. Consider an equipment comprising three networks connected as shown in figure 6.7.

If the power measured at various points in the circuit is as labelled, then

$$\text{Power gain of } N_1 = \frac{P_2}{P_1} = A_1$$

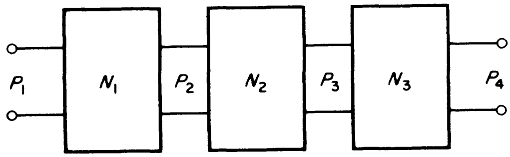


figure 6.7

Power gain of $N_2 = \dfrac{P_3}{P_2} = A_2$ etc.

Hence the overall power gain,

$$\frac{P_4}{P_1} = \frac{P_4}{P_3} \times \frac{P_3}{P_2} \times \frac{P_2}{P_1}$$

$$\therefore \qquad A = A_3 \times A_2 \times A_1$$

Power ratios in logarithmic units

The labour involved in calculating the overall gain of a system containing many such items can be tedious, but the work can be simplified by expressing power ratios such as P_2/P_1 in terms of a logarithmic unit called the *bel* (after A. G. Bell, who first patented the telephone).

For example,

$$\text{Power gain } A_1 = \log_{10}\frac{P_2}{P_1} \text{ bel, where } P_1 \text{ is the reference level.}$$

Unfortunately, the bel was found to be too large for most practical purposes, so that it has now been superseded by the *decibel* (dB).

$$\text{Thus, power gain } A_1 = 10 \log_{10}\frac{P_2}{P_1} \text{ dB} \tag{6.6}$$

It will be seen from equation 6.6 that if P_2 is less than P_1 the logarithm will be a negative quantity, indicating that network N_1 is producing attenuation. The sign of a power ratio must therefore always be taken into account when assessing the overall performance of the system. For example, in figure 6.7 let $A_1 = 0{\cdot}631, A_2 = 131{\cdot}8$ and $A_3 = 19{\cdot}95$. (*Note:* $\log_{10} 0{\cdot}631 = \bar{1}{\cdot}8 = -0{\cdot}2$.) Converting these to decibels the individual power gains are $A_1 = -2$ dB, $A_2 = 21{\cdot}2$ dB, $A_3 = 13$ dB.

$\therefore$ Total power gain $= -2 + 21{\cdot}2 + 13 = 32{\cdot}2$ dB

that is,

$$10 \log_{10}\frac{P_4}{P_1} = 32{\cdot}2$$

$$\therefore \qquad P_4 = P_1 \times \text{antilog } 3{\cdot}22 = 1660P_1$$

Power levels

In its present form the decibel can be used to express power ratios only. However, by adopting a particular reference level, all other amounts of power can be related to it. The British standard of reference corresponds to a power of one milliwatt in a 600 Ω resistor. By this notation,

(a) $$1\ \text{W} \equiv 10 \log_{10} \frac{1}{10^{-3}} = 30\ \text{dB above 1 mW}$$

(b) $$2\ \mu\text{W} \equiv 10 \log_{10} \frac{2 \times 10^{-6}}{10^{-3}} = 10 \times \bar{3}{\cdot}3010 = 27\ \text{dB below 1 mW}$$

Alternatively, '30 dB with respect to 1 mW' may be abbreviated to 30 dBm. Similarly, a power of 2 μW, being '27 dB below 1 mW' can be stated as − 27 dBm.

Example 6.4 In a multi-stage amplifier, the gains of its component parts are given as 11·2 dB, 7·67 dB, − 5·7 dB and 17 dB respectively. If the power supplied to the input terminals corresponds to 10 dBm, calculate the output power.

$$\text{Total gain} = 11{\cdot}2 + 7{\cdot}67 - 5{\cdot}7 + 17$$

$$\therefore \quad 10 \log_{10} A = 30{\cdot}17\ \text{dB}$$

$$A = \text{anti-log}\ 3{\cdot}017 = 1040$$

$$\therefore \quad \text{Output power}, P_0 = 1040 P_i$$

where input power,

$$P_i = 10\ \text{dBm}$$

Now 10 dBm means that the power into the amplifier is 10 dB above 1 mW, so that

$$10 \log_{10} \frac{P_i}{10^{-3}} = 10$$

$$\therefore \quad \frac{P_i}{10^{-3}} = 10 \text{ and } P_i = 10\ \text{mW}$$

$$\therefore \quad \text{Output power} = \frac{1040 \times 10}{10^3} = 10{\cdot}4\ \text{W}$$

Current and voltage gain

The powers P_1, P_2, etc. may well have been calculated from measurement of current or voltage. For example, P_1 could be evaluated from $I_1^2 R_1$ where R_1 is the real part of the input impedance of network N_1 in figure 6.7. If $P_2 = I_2^2 R_2$ where R_2 is the real part of the output impedance, and $R_1 = R_2$

$$\therefore \qquad \frac{P_2}{P_1} = \left(\frac{I_2}{I_1}\right)^2$$

Expressed in decibels,

$$\text{power gain} = 10 \log_{10}\left(\frac{I_2}{I_1}\right)^2 = 20 \log_{10}\frac{I_2}{I_1}\,\text{dB}$$

that is, power gain is $20 \log_{10}$ (current gain) dB.

Alternatively, in terms of voltage,

$$P_1 = \frac{V_1^2}{R_1} \quad \text{and} \quad P_2 = \frac{V_2^2}{R_2}$$

where V_1 and V_2 are the input and output voltages respectively.

$$\therefore \qquad \frac{P_2}{P_1} = \frac{V_2^2}{R_2} \times \frac{R_1}{V_1^2}$$

$$\text{If } R_1 = R_2 \quad \text{then} \quad \frac{P_2}{P_1} = \left(\frac{V_2}{V_1}\right)^2$$

that is, power gain is $20 \log_{10}$ (voltage gain) dB.

Combining the last two results, the number of decibels, N

$$= 20 \log_{10} A_i = 20 \log_{10} A_v$$

With constant usage over the years, it has been found convenient to regard the quantity N as the decibel current or voltage gain of an amplifier which has not necessarily its input and output impedances matched. Thus the current gain in dB is taken as

$$N = 20 \log_{10} A_i \text{ dB} \tag{6.7}$$

Similarly, the voltage gain in dB is calculated from

$$N = 20 \log_{10} A_v \text{ dB} \tag{6.8}$$

Example 6.5 In a public address system, a power of 5 W is to be supplied to a loudspeaker whose speech coil has a resistance of 8 Ω and negligible inductance. If the input resistance of the amplifier is 10 kΩ and the input signal is 100 mV, calculate the power and voltage gains required from the amplifier in dB.

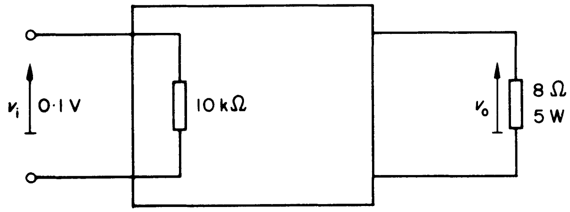


figure 6.8

From figure 6.8,

$$\text{input power } P_i = \frac{0{\cdot}1^2}{10^4} = 1\ \mu\text{W}$$

Since output power,

$$P_0 = 5\ \text{W}$$

$$\therefore \qquad \text{Power gain} = 10\log_{10}\frac{5}{10^{-6}}\ \text{dB}$$

$$= 10 \times 6{\cdot}6990 = 67\ \text{dB}$$

From the output circuit

$$v_0 = \sqrt{(5 \times 8)} = 2\sqrt{10}\ \text{V}$$

$$\therefore \qquad \text{Voltage gain} = 20\log_{10}\frac{2\sqrt{10}}{0{\cdot}1} = 20 \times 1{\cdot}8010 = 36\ \text{dB}$$ (B)

Bandwidth of amplifier

(C) We can now have a second look at the frequency response of a typical RC-coupled amplifier, as described in the early part of this chapter. It was then stated that at the cut-off frequencies f_1 and f_2, the voltage gain of the amplifier had fallen to 70 per cent of its maximum mid-frequency value. This may have seemed somewhat arbitrary at the time, but now that the decibel notation has been introduced a more detailed study of amplifier performance at f_1 and f_2 is possible.

$$\left|\frac{v_0}{(v_0)_{\max}}\right| = \frac{1}{\sqrt{2}} = 0{\cdot}707 \text{ at } f_1 \text{ and } f_2$$

$$\therefore \qquad \text{Relative gain} = 20\log_{10} 0{\cdot}707 = 20 \times \bar{1}{\cdot}85 = -3\ \text{dB}$$

Thus f_1 and f_2 are sometimes referred to as the *lower 3 dB point* and the *upper 3 dB point,* respectively. From previous theory, $P_0/(P_0)_{\max} = [v_0/(v_0)_{\max}]^2 = (1/\sqrt{2})^2 = 0{\cdot}5$, if the input and output resistances are assumed equal. Hence, relative power gain $= 10\log_{10} 0{\cdot}5 = 10 \times \bar{1}{\cdot}7 = -3$ dB.

At the same frequencies f_1 and f_2, the power transferred to the load has therefore fallen to 50 per cent of its maximum value. It is for this reason that f_1 and f_2 are called *half-power points* and are used to define the useful bandwidth of the amplifier.

Example 6.6 An amplifier uses a transistor having $h_{ie} = 1{\cdot}5\ \text{k}\Omega$ and $h_{fe} = 155$. Its load resistor R_L is $5\ \text{k}\Omega$ and it is coupled to a second stage via a capacitor of $1\ \mu\text{F}$. The input circuit of the second stage may be represented as a resistance of $1{\cdot}2\ \text{k}\Omega$ shunted by 4000 pF. Calculate:

(a) the mid-band voltage gain in dB
(b) the two 3 dB frequencies.

(a) From equation 6.2

mid-band voltage gain

$$= -\frac{h_{fe}R_LR_{in}}{h_{ie}(R_L + R_{in})}$$

Omitting the negative sign, because we are only concerned with the magnitude of this gain when converting to decibels,

∴ mid-band voltage gain

$$= \frac{155 \times 5 \times 1{\cdot}2}{1{\cdot}5(5 + 1{\cdot}2)} = 100$$

that is, maximum voltage gain $= 20 \log_{10} 100 = 40$ dB.

(b) From equation 6.3

$$f_1 = \frac{1}{2\pi C(R_L + R_{in})}$$

$$= \frac{10^6}{2\pi(5 + 1{\cdot}2)10^3} = 25{\cdot}7\ \text{Hz}$$

From equation 6.5

$$f_2 = \frac{1}{2\pi C_{in}R}$$

where

$$R = \frac{R_LR_{in}}{R_L + R_{in}} = \frac{5 \times 1{\cdot}2}{5 + 1{\cdot}2} = 0{\cdot}968\ \text{k}\Omega$$

∴

$$f_2 = \frac{10^{12}}{2\pi \times 4000 \times 0{\cdot}968 \times 10^3}\ \text{Hz} = 41{\cdot}1\ \text{kHz}$$

BANDWIDTH REQUIREMENTS

The main types of signal to be received and amplified can be subdivided into speech, music, and video signals.

Speech

In conversation, vowel sounds are generated by breathing air out of the lungs and through the vocal chords at the back of the throat. These consist of two mucous membranes, controlled by elastic ligaments under various degrees of tension. In the male voice, the average range of fundamental frequencies produced in speech is 100 Hz to 500 Hz whereas the corresponding range of the female voice is 200 Hz to 1 kHz. Consonants, however, are produced in many different ways depending on the language or even the dialect being used. For example, the sound of 'th' is formed by a partial opening between the tongue and the teeth. A jet of air under the edge of the top teeth is responsible for the higher frequencies needed to emit such a sound. The letter 's', due to air forced between the teeth, calls for frequencies up to 10 kHz. The whole range of these and similar effects associated with clear speech would require a band of frequencies to be transmitted extending from 100 Hz to 10 kHz. But such an extravagant system would need relatively expensive microphones and receivers. As it happens, the human ear can adapt itself to the signals reproduced in a much restricted bandwidth without losing the sense of the words. The fact that a letter 's' begins to sound like 'z' when the bandwidth is halved is not likely to cause too much confusion in the mind of the listener. There is, therefore, nothing to be gained by providing a channel that could accommodate all the frequencies produced by the human voice. If relatively few of them are necessary, then these can be economically handled by commercial grades of microphones and receivers. In practice it is found that for telephone circuits, satisfactory transmission of speech is achieved within the bandwidth 300 Hz to 3·4 kHz. Another advantage to be gained by restricting the bandwidth is that it limits the pick-up of unwanted signals (classified as Noise), the power of which is directly proportional to bandwidth.

Music

It is the frequency response of the human ear which is the deciding factor in determining the bandwidth required for the transmission and reception of music. The audible range for children is approximately 30 Hz to 18 kHz but sensitivity at the upper end of the spectrum is seriously impaired in older people. The bandwidth just quoted is sufficient to contain almost all the frequencies generated by musical instruments. These all operate on the principle of vibrations: for example, strings (violin and guitar); column of air (flute and trumpet); reeds (oboe and some organ pipes); and metal bars (chimes and xylophone). If the highest frequencies, which are the harmonics responsible for the characteristic tones of the instrument, are attenuated by the amplifier or loudspeaker, this will not necessarily be obvious to the listener, as a certain amount of automatic adjustment occurs. Nevertheless, this effort can become rather tiring after a time as the listener tries to make up for the deficiencies in the system. Taking into account the practical difficulties of reproducing a realistic impression of the original performance, a compromise is found by adopting a bandwidth from 50 Hz to 8·5 kHz (or 15 kHz for high quality circuits).

In this section bandwidths have been specified by quoting the lower and upper limits of frequency, f_1 and f_2. Another method is sometimes used in which the number of octaves to be transmitted is stated. The word *octave* is borrowed from the definition that notes are an octave apart if one is eight notes either above or below the other on the musical scale. In the technical sense, f_2 is one octave above f_1 if $f_2 = 2f_1$. For example, if $f_1 = 50$ Hz and $f_2 = 100$ Hz (which are the frequencies of hum associated with half-wave and full-wave rectifier circuits having inadequate smoothing) then f_2 is said to be one octave above f_1. In general, $f_2 = 2^n f_1$, where n = number of octaves.

Taking logs to base 10

$$\log_{10} f_2 = n \log_{10} 2 + \log_{10} f_1$$

$$\therefore \qquad n = 3{\cdot}32 \log_{10} \frac{f_2}{f_1} \text{ octaves} \tag{6.9}$$

Example 6.7 A certain transmission requires a bandwidth extending from 50 Hz to 12·8 kHz. Calculate the number of octaves required.

From equation 6.9

$$n = 3{\cdot}32 \log_{10} \frac{12{\cdot}8 \times 10^3}{50}$$

$$= 3{\cdot}32 \times 2{\cdot}408$$

$$= 8 \text{ octaves}$$

Video signals

In order to understand the bandwidth requirements for video signals, it is important that the nature of such signals should first be investigated. The height of a TV picture (or 'field') is composed of either 405 or 625 lines. Each of these lines can be considered as comprising small square elements, the brilliance of which is varied by means of the signal applied to the grid of the cathode-ray tube. As the width of the picture is designed to be 4/3 times the height (*aspect ratio* is 4/3), the number of elements in each line is $4n/3$ where n is the total number of lines in the field. Therefore, the total number of elements in a complete picture is $4n^2/3$.

It is essential that the electro-luminescent surface of the screen can respond to the high frequency signals controlling the density of electrons arriving on it. Thus the screen is required to have *short persistence*: that is, the brightness of an element falling to 1 per cent of its maximum in less than 10 μs. In spite of this, the whole picture should not appear to flicker and must therefore be scanned by the electron beam at not less than 50 times every second. At this frequency the human eye is quite capable of retaining successive images and thus creating the impression of a continuous picture. If N is the number of times the picture is scanned per second,

Fundamental signal frequency

$$= \frac{1}{2}\left(\frac{4n^2}{3} N\right)$$

$$= \frac{2n^2N}{3} \text{ hertz} \qquad (6.10)$$

In its simplest form, an alternating signal of constant amplitude would reproduce a dotted line across the screen with each cycle producing two elements as shown in figure 6.9.

For example, for a 405 line system, $f = {}^1/_3 \times 2 \times 405^2 \times 50 = 5{\cdot}5$ MHz. It should be borne in mind that a sinusoidal signal of this frequency will only cause the brilliance to vary gradually from one element to the next. Consequently, the definition or sharpness of the picture will be rather poor. Ideally the waveform should be square but this would necessitate an even greater bandwidth than already suggested. By a simple modification, however, it is possible to halve the frequency without serious loss of picture quality. The method is to scan all the odd-numbered lines in the field in 4 milliseconds

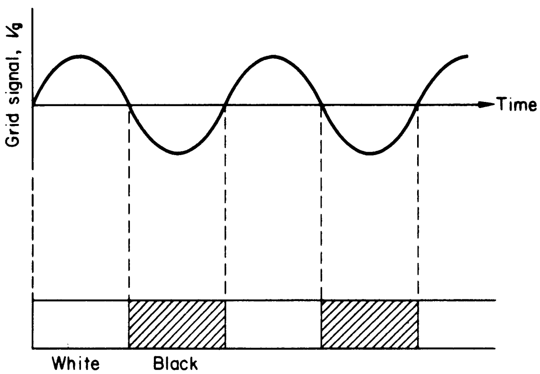


figure 6.9

before returning to the top left-hand corner and repeating the process for the even-numbered lines (*interlaced scanning*). The observer is deceived into thinking that the whole picture is being scanned with a frequency of 50 Hz whereas in fact it is the two almost coincident halves which are being displayed 25 times in every second. With N effectively reduced by a factor of 2, the net result, from equation 6.10, is that the signal frequency becomes 2·75 MHz. An acceptable picture is generated in this way and the problems of transmission and amplification are simplified. The corresponding video frequency for a 625 line system can be calculated as 6·5 MHz. For practical purposes, a frequency of about 5·5 MHz is found to be satisfactory.

Having established suitable frequency limits, the video signal can be used to modulate a carrier as explained for audio signals in chapter 4. But, of course, the carrier must have a frequency many times greater than 5·5 MHz for a 625 line system. Channel 21, for example, has a vision carrier of 471·25 MHz. Furthermore, it would be advantageous to employ single sideband transmission. For technical reasons at the receiving end, however, only a portion of one sideband is suppressed in the so-called *vestigial sideband system.* Typical bandwidths required for 405 and 625 lines are shown schematically in figure 6.10.

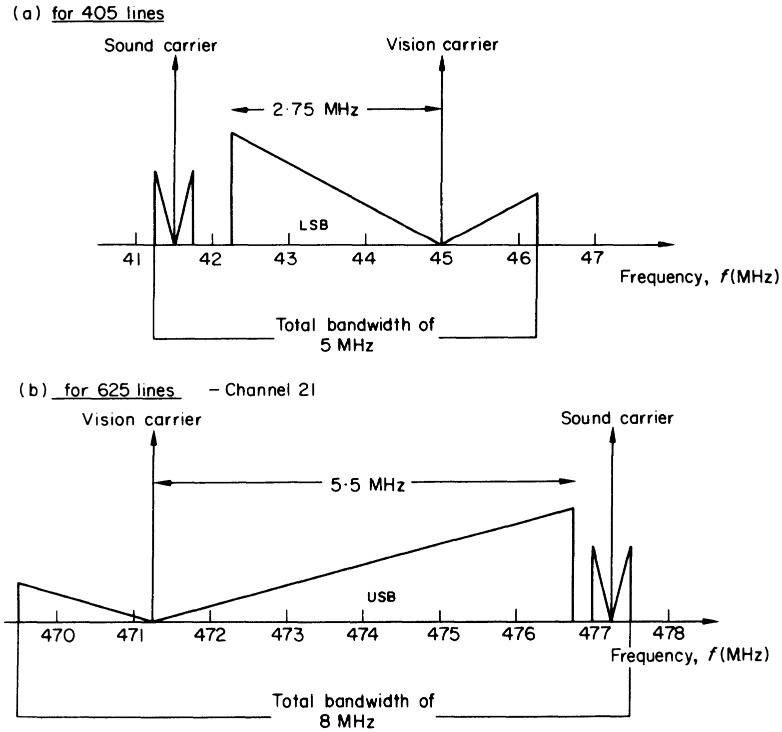


figure 6.10

MISCELLANEOUS EXAMPLES

Example 6.8 The two transistors used in a resistance-capacitance coupled amplifier each have $h_{ie} = 1 \cdot 1$ kΩ and $h_{fe} = 50$. The collector load of the first stage is a $2 \cdot 7$ kΩ resistor and the capacitor coupling the two stages has a value of $2 \cdot 5$ μF. The input impedance of the second transistor may be regarded as h_{ie} shunted by a capacitance of 2100 pF. Calculate the maximum voltage gain that can occur from the base of the first transistor to the base of the second transistor. Also, estimate the frequencies between which the voltage gain falls from its maximum value by not more than 30 per cent.

[$-35 \cdot 5$; 16·75 Hz, 97 kHz]

Example 6.9

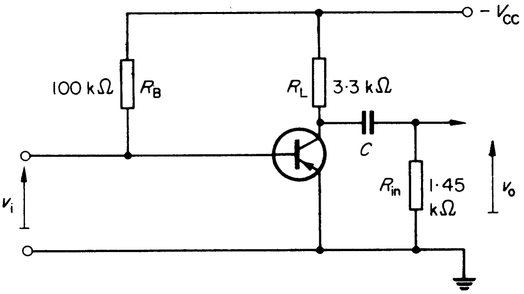


figure 6.11

The transistor shown in figure 6.11 has $h_{fe} = 99$ and $h_{ie} = 1 \cdot 1$ kΩ.

Find:

(a) the maximum value of the ratio v_o/v_i
(b) the value of C which gives the ratio $v_o/v_i = 30$ at a frequency of $50/\pi$ Hz.

With the calculated value of C, show that v_o/v_i is given by $82 \cdot 3 \angle 155 \cdot 5°$ when the frequency is increased to 100 Hz.

[$90 \cdot 6 \angle 180°$; 0·737 μF] (C)

(B) ***Example 6.10*** A common-base amplifier has a current gain of $-1 \cdot 94$ dB when the collector load is a resistance R_L. Calculate the current flowing from the collector into R_L if the emitter current is 2 mA.

[1·6 mA]

Example 6.11 If the amplifier of example 6.10 has a voltage gain of 400, calculate the power gain in dB. Also find the power in the load R_L if the power supplied to the transistor is 20 μW.

[25·05 dB; 6·4 mW]

Example 6.12 A signal generator incorporates a voltmeter that registers the EMF of the source. Between the points across which the voltmeter is connected and the output terminals there is a control that can be set at readings of 20, 40 and 60 dB of attenuation. What is the output voltage of the signal generator at each of these settings if the voltmeter is reading 2·5 V throughout?

With the control set at 40 dB, what must the voltmeter read if the signal generator is to deliver 15 mV at the output terminals?

[250 mV, 25 mV, 2·5 mV; 1·5 V]

Example 6.13 A signal generator of internal resistance 600 Ω is matched to the input impedance of a common-emitter amplifier. The power supplied to the amplifier corresponds to − 37 dBm and the current gain of the amplifier is 40 dB when operating with a load resistance of 1·2 kΩ. Calculate the input voltage and the voltage gain in dB under these conditions.

[10·95 mV; 46 dB] (B)

(C) ***Example 6.14*** Define the bandwidth of an amplifier in terms of the decibel. The transistor shown in figure 6.12 has $h_{ie} = 1$ kΩ and $h_{fe} = 100$. Estimate the maximum signal voltage developed across the collector load resistance, stating any approximations made.

Neglecting any causes of low frequency attenuation, other than the coupling capacitor C, calculate the frequency at the lower 3 dB point.

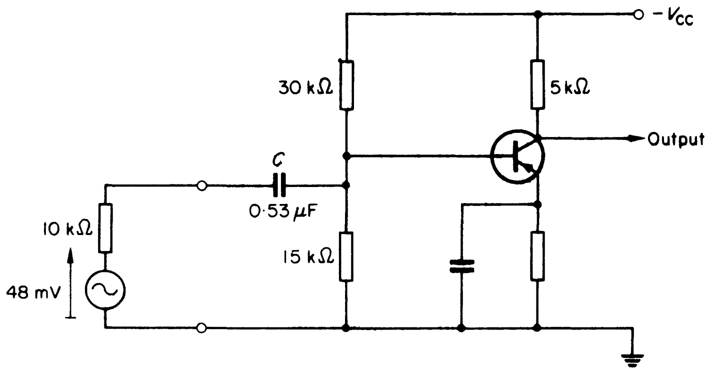


figure 6.12

[2 V; 50 Hz]

Example 6.15 What average range of frequency is

(a) produced by the human voice
(b) detected by the human ear?

Describe briefly how vowel sounds and consonants are produced and give the approximate ranges in frequency which occur in them.

Thence, or otherwise, discuss the effects on the transmission of speech which would be caused by eliminating all frequencies (i) above 1·5 kHz, (ii) below 1·5 kHz.

(CGLI Principles C, 1961)

Example 6.16 What range of frequencies are required for the transmission of:

(a) speech on a telephone circuit
(b) music on a programme circuit?

Why do these requirements differ?

A signal in the frequency range 100 Hz to 5 kHz amplitude-modulates the carrier of a radio transmitter which radiates at 4 MHz. A super-heterodyne receiver with an intermediate frequency of 500 kHz receives this transmission. Draw diagrams to show the frequency bands in use:

(a) before modulation
(b) on transmission
(c) just prior to demodulation.

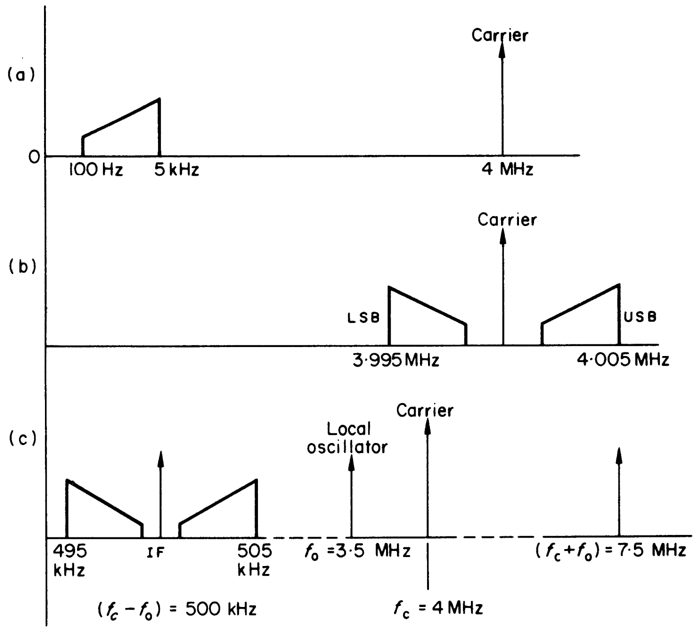


figure 6.13

Briefly indicate why each frequency translation is necessary.

(CGLI Principles C, 1964)

Note: The answers to the graphical part of the question are given in figure 6.13. (C)

7 Application of load lines to transistor amplifiers

(B) Imagine a power transistor connected in its common-emitter configuration as shown in the circuit of figure 5.28. By manipulation of the variable resistors R_1 and R_2, current into the base may be set to any convenient value. If now the resistor R_3 is varied over its maximum range to control the positive collector potential, corresponding values of V_{CE} and I_C can be recorded. When plotted on a graph, these figures represent a static output characteristic for this transistor. By repeating the experiment for different settings of base current, a whole family of curves can be plotted as shown in figure 7.1.

It should be noted that this set of curves has been obtained under static conditions; that is, the quantity V_{CE} is repeatedly adjusted to a particular voltage while collector current I_C is being measured, for a range of values

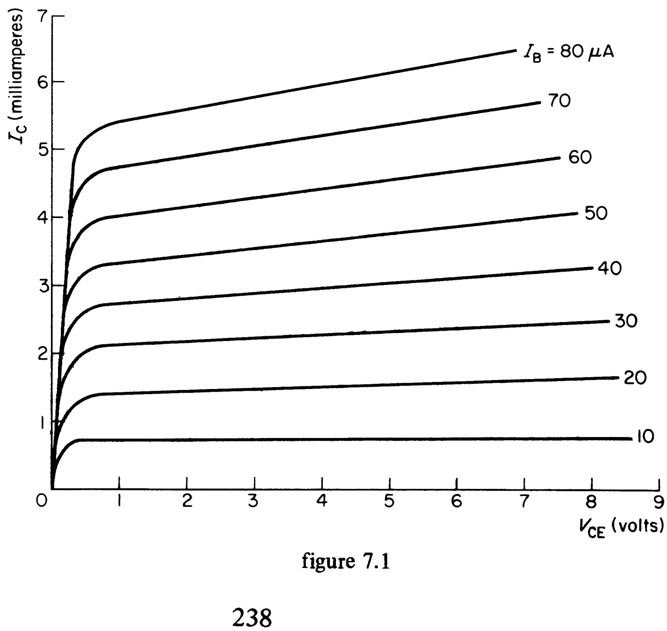


figure 7.1

of I_B. This rather artificial state of affairs cannot happen in a practical amplifier circuit. The effect of a change in I_B as a result of the incoming signal is to modulate the flow of majority carriers through the collector load. Therefore, not only does I_C fluctuate in sympathy with I_B but also V_{CE} must change due to the voltage drop in the collector load. In fact, with the common-emitter configuration, the voltage output between collector and earth is in anti-phase with the input signal between base and earth. Nevertheless, static characteristics do help to predict the performance of the transistor as an amplifier when a given value of load resistance is connected.

CONSTRUCTION OF LOAD LINES

Consider a load resistance R_L connected in the basic common-emitter amplifier shown in figure 7.2.

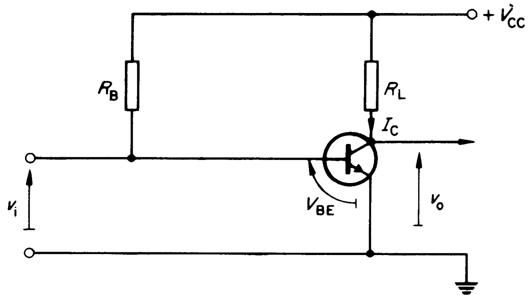


figure 7.2

It will be seen that only a single supply is used to provide the essential bias for the two junctions of the transistor, that is

base-emitter junction ... ... forward bias
base-collector junction ... ... reverse bias.

This is achieved by means of the bias resistor R_B (typically 100 kΩ) which combines with the base-emitter junction to form a potential divider between V_{CC} and earth. In this way, the low forward resistance of the junction derives a voltage drop V_{BE}, that is, of the correct polarity and has a value of 0·3 V or 0·6 V depending on whether the transistor is germanium or silicon, respectively. At all times, the p.d. between collector and emitter, V_{CE}, is given by the equation

$$V_{CE} = V_{CC} - I_C R_L$$

$$\therefore \qquad I_C = -\frac{V_{CE}}{R_L} + \frac{V_{CC}}{R_L} \qquad (7.1)$$

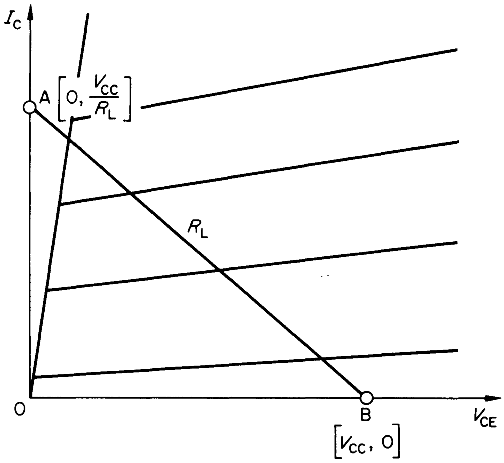


figure 7.3

which, when superimposed on the axes of figure 7.1, is a straight line having a downward slope of $1/R_L$, as in figure 7.3.

Substituting $V_{CE} = 0$ in equation 7.1, makes $I_C = V_{CC}/R_L$; so that the co-ordinates of point A are 0 and V_{CC}/R_L. Substituting $I_C = 0$ in equation 7.1, makes $V_{CE} = V_{CC}$, so that the co-ordinates of point B are V_{CC} and 0. Thus, the line joining AB represents the load resistor R_L with the known value of supply voltage, V_{CC}. Having constructed the load line in this way, figure 7.3 can then be used to calculate the gain of the amplifier as shown in the following example.

Example 7.1 The output characteristics of a transistor in its common-emitter mode are given in figure 7.4. Superimpose a load line for $R_L = 2\,\text{k}\Omega$ assuming that the supply voltage is 6 V. If the quiescent (no-signal) base current is 40 μA and an alternating input signal of 20 μA (peak) is applied to the base, estimate the current gain of the amplifier.

Given that the input resistance of the amplifier under these conditions is 1 kΩ, what is the value of voltage gain?

The load line has already been added to figure 7.4 by the method just explained and the quiescent point Q has been inserted at the intersection of the load line with the characteristic for $I_B = 40\,\mu$A. Any transistor amplifier in which base and collector currents are flowing under no-signal conditions is said to be a 'Class A' amplifier. For example, in this circuit the bias arrangement ensures that $I_B = 40\,\mu$A and $I_C = 2$ mA so long as the supply voltage of 6 V remains switched on and no other signal is connected to the base. Points A and B mark the limits of base current as the input signal varies over its maximum range. It will be seen that the corresponding excursion of I_C produces a total change (peak-to-peak) of 1·65 mA.

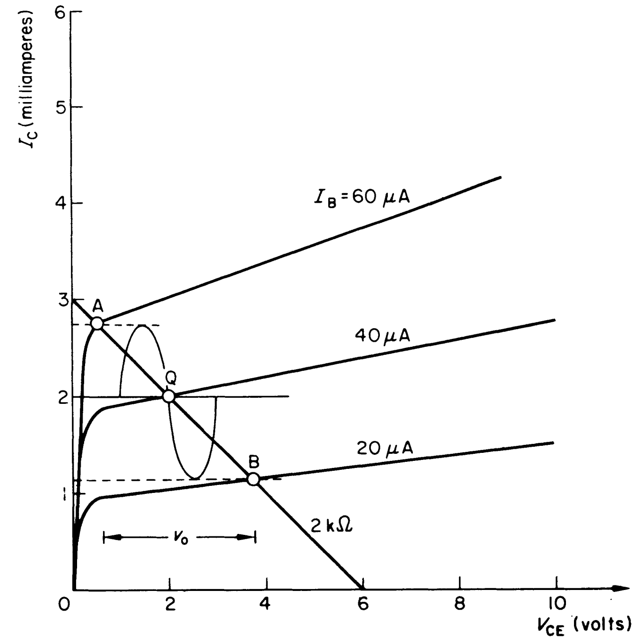


figure 7.4

Working in peak-to-peak values in μA

$$A_i = \frac{1650}{40} = 41{\cdot}25$$

Furthermore, an input signal of $20\ \mu$A would require a driving voltage of 20 mV if Z_{in} is 1 kΩ. From figure 7.4 the total output voltage swing v_o is estimated to be 3·3 V when measured on the horizontal scale directly below A and B.

$$\therefore \qquad A_v = \frac{v_o}{v_i} = \frac{3{\cdot}3}{40 \times 10^{-3}} = 82{\cdot}5$$

This agrees with the answer obtained by substituting in equation 5.6, which states

$$A_v = \frac{A_i R_L}{Z_{in}}$$

$$= \frac{41{\cdot}25 \times 2}{1} = 82{\cdot}5$$

Occasionally, the values specified for a transistor amplifier are those of V_{CE} and I_C (or I_B) at the operating point. This type of problem then requires a slightly modified construction for the load line, because the supply voltage is not stated. The method is illustrated in the next example.

Example 7.2 The output characteristics of a PNP transistor may be assumed linear between the points quoted in the following table:

table 7.1

I_B (mA)	I_C (A) $V_{CE} = -2$ V	I_C (A) $V_{CE} = -16$ V
-10	$-0{\cdot}67$	$-0{\cdot}7$
-30	$-1{\cdot}45$	$-1{\cdot}63$
-50	$-2{\cdot}32$	$-2{\cdot}65$

The transistor is to be used as a common-emitter amplifier with a collector load resistance of 5 Ω. At the operating point, $V_{CE} = -6$ V and $I_C = -1{\cdot}5$ A. Construct the appropriate load line and from it estimate:

(a) the supply voltage required for the amplifier
(b) the current gain under these conditions

Figure 7.5 shows the output characteristics plotted to a base of $-V_{CE}$. To fix the slope of the load line, a convenient value of $-V_{CE}$ is chosen (say, 10 V) and the construction already described is completed; thus producing the dotted line of figure 7.5. This line must now be moved to a new position

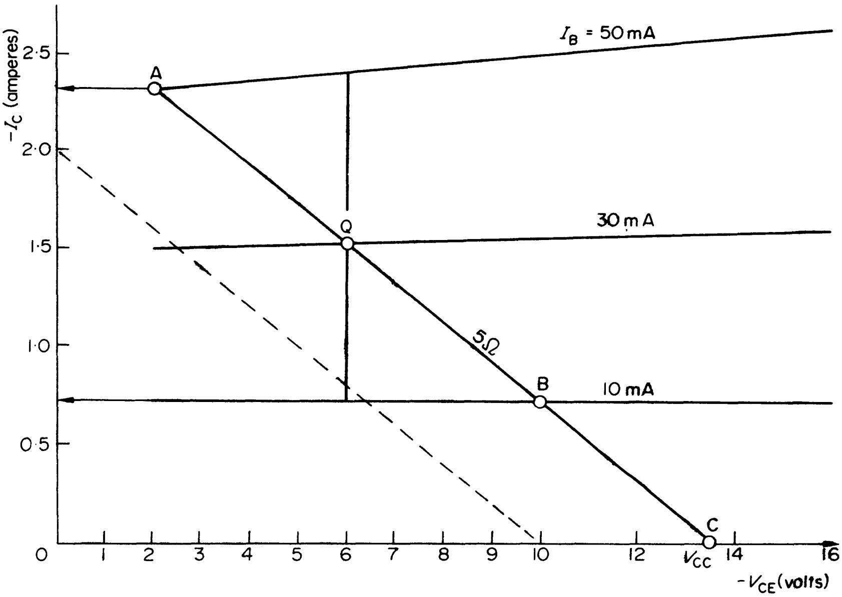


figure 7.5

parallel to its old position and such that it passes through the point Q where $V_{CE} = -6$ V and $I_C = -1{\cdot}5$ A.

(a) Extending this straight line downwards until it meets the horizontal axis determines V_{CC} at the point of intersection, C; so that, in the present example, supply voltage required $= -13{\cdot}5$ V.
(b) To find the current gain of the amplifier, a suitable peak value of input current (say, 20 mA) is chosen. As the signal varies over the complete range, the collector current is driven between points A and B. Total change of collector current $= (2{\cdot}32 - 0{\cdot}7)$ A $= 1{\cdot}62$ A.

$$\therefore \qquad \text{Current gain} = \frac{1{\cdot}62}{40 \times 10^{-3}} = 40{\cdot}5$$

Yet another variation of the technique of drawing load lines will be developed later in this chapter in connection with transformer-coupled loads. It should also be noted that a vertical line through the operating point Q on figure 7.5 can be used to find h_{fe} (or β) for this particular transistor. Such a vertical line represents an R_L of zero value (that is, a short-circuit), and it will be remembered that the definition of h_{fe} requires such a short-circuit termination at the output. Thus

$$h_{fe} = \frac{I_C}{I_B} = \frac{2{\cdot}43 - 0{\cdot}68}{40 \times 10^{-3}}$$

$$= 43{\cdot}75 \text{ at the point Q.} \qquad \text{(B)}$$

(C) LARGE-SIGNAL TRANSISTOR AMPLIFIERS

The transistor used in the output stage of an amplifier is required to deliver power, probably to a load of low resistance such as a loudspeaker, a servo-motor, or a relay. This application of transistors raises certain problems that will now be explained in some detail.

Limitations on power output

In the first place the power output from a transistor is limited by at least five factors, and these will now be considered in turn.

1. Maximum collector voltage $(V_{CE})_{max}$ is the high value of reverse bias which starts the processes of breakdown. It produces a combination of zener and avalanche effects; or, possibly, short-circuit of the base region. The collector-base depletion layer is widened to such an extent that it may contact the other junction. The net result leads to the upward sweep apparent in the characteristics of figure 7.6.
2. Maximum collector current $(I_C)_{max}$ is restricted by a reduction in current amplification between base and collector as current density increases.

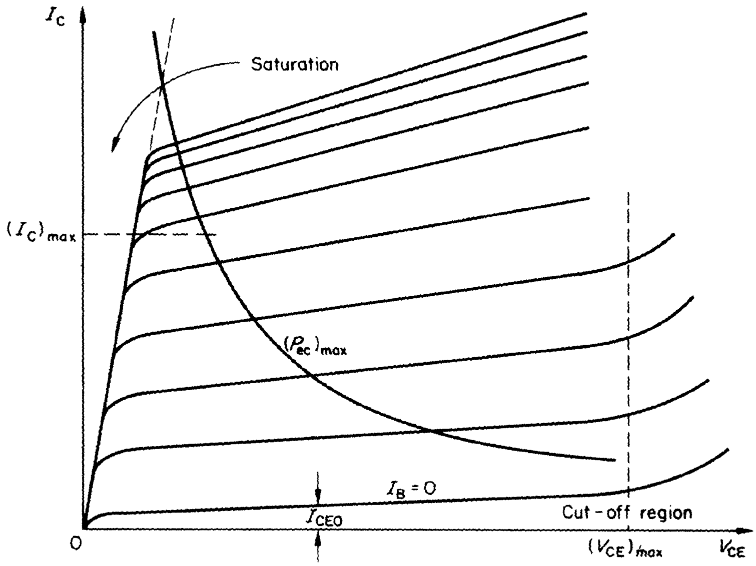


figure 7.6

This causes the output characteristics to come closer together with consequent curvature in the I_C/I_B graph (as will be seen later in figure 7.8c).

3. It is interesting to see what happens to the power drawn by a transistor from the d.c. supply. Under no-signal conditions, all this power (equal to the product of the co-ordinates V_{CE} and I_C at the quiescent point) is absorbed in the transistor. But when a signal is applied between base and emitter, some of the power is delivered as useful output to the load. Consequently, power dissipated at the collector falls to a value given by the difference between 'd.c. power' input and 'a.c. power' output. Collector conversion efficiency is defined as the ratio

$$\text{efficiency} = \frac{\text{power output at signal frequencies}}{\text{power input from d.c. supply}} \tag{7.2}$$

For a Class A amplifier, it can be shown that this efficiency cannot exceed 50 per cent. (See figure 7.7.)

In the ideal case, where the signal causes current and voltage swings to occupy the entire length of the load line, maximum signal power output

$$= \frac{V_q}{\sqrt{2}} \times \frac{I_q}{\sqrt{2}} = \frac{V_q I_q}{2}$$

As the corresponding d.c. input power is $V_q I_q$, it is clear that the maximum

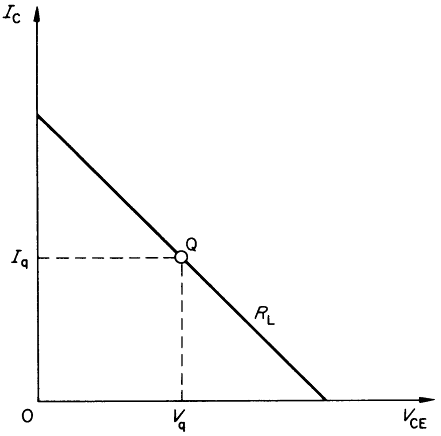


figure 7.7

conversion efficiency is 50 per cent. In actual fact, taking saturation, cut-off, and other circuit losses into account, a more realistic figure for efficiency is closer to 25 per cent.

Whenever power is dissipated at the collector junction of a transistor, the transistor temperature must increase above its surroundings. If this temperature is not to rise excessively, heat must be removed at a suitable rate, and the maximum rate achievable for any given transistor will depend mainly on the method of mounting that transistor. Acceptable temperature maxima are taken as 75°C for germanium transistors and 150°C for transistors made of silicon.

At every point on the curve for $(P_{ec})_{max}$ shown in figure 7.6, the product of the co-ordinates is equal to the maximum power that the transistor can safely dissipate as heat under steady state conditions. (The construction of this curve is further explained in example 7.4.) It is worth mentioning, however, that the steady state or continuous maximum rating of the transistor given by $(P_{ec})_{max}$ may be exceeded under certain carefully controlled conditions. An important example of this in telecommunication occurs when the transistor is used in pulse circuits, in which junction temperature depends not only on the height of the pulse but also on the pulse width in relation to the periodic time. For example, if the pulse occupies only a tenth of the periodic time, the *duty cycle* is said to be 10 per cent, and pulse power in excess of $(P_{ec})_{max}$ may safely be delivered.

4. Saturation prevents the transistor operating to the left of the 'knee' of the output characteristics shown in figure 7.6. Any attempt to drive it so hard into conduction will only result in clipping one half-cycle of the output waveform.

5. Below the characteristic for $I_B = 0$ (see figure 7.6), the transistor is cut off and its signal output is obviously zero. The only current then flowing in the collector circuit is the leakage component I_{CEO}.

Effect on *h*-parameters

A second major problem arising out of the large-signal operation of a transistor is the effect on its *h*-parameters. As the load invariably has a small resistance, the parameters h_{oe} and h_{re} lose their significance. The other two, h_{ie} and h_{fe} must be modified to satisfy the new operating conditions. It should be stressed from the outset that no form of equivalent circuit can be used for the purpose of analysis, and a graphical method is the only alternative.

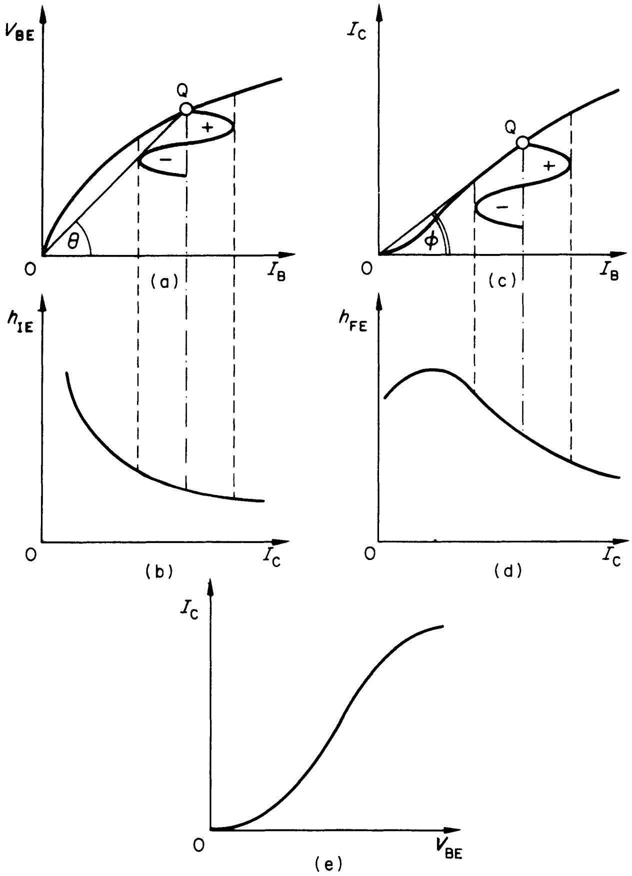


figure 7.8

On the input characteristic of figure 7.8a, the gradient of the line joining Q to the origin is the d.c. or *large-signal value* of h_{ie}, and is denoted by h_{IE}. From this

$$\tan\theta = h_{IE} = \left.\frac{V_{BE}}{I_B}\right|_{V_{CE}\ \text{constant}}$$

Similarly, from the current transfer characteristic of figure 7.8c, the gradient $\tan\phi$ is the d.c. or *large-signal value* of h_{fe}; hence

$$h_{FE} = \frac{I_C}{I_B}\bigg|_{V_{CE}\ \text{constant}}$$

Both parameters h_{IE} and h_{FE} are plotted to a base of I_C in figures 7.8b and 7.8d, from which it will be noted that their values depend considerably upon the choice of quiescent point Q. Whenever a transistor is operating over a large part of these characteristics, non-linearities must be a source of distortion. For example, the input characteristic (being exponential near the origin) causes negative half-cycles of base current to be opposed by a relatively high input impedance. The graph of h_{FE}, however, shows that these same negative half-cycles receive the greater amplification. Thus the two effects occurring simultaneously tend to cancel each other, thereby producing the reasonably linear transfer characteristic of figure 7.8e.

Harmonic distortion

Consider a sinusoidal input signal applied to the base of a common-emitter stage, as shown on the transfer characteristic of figure 7.9. The non-linearity of the graph has been exaggerated to illustrate the effect on the output current, i_o.

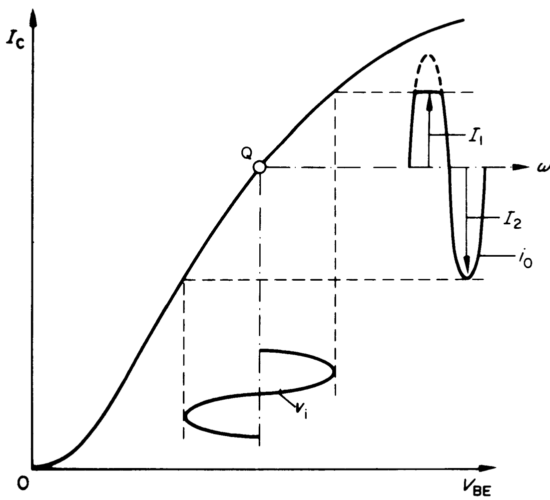


figure 7.9

With the operating point Q as a false origin, the output current i_o can be expressed as a polynomial function of the input voltage v_i. This is simply a power series in which the coefficients of the various terms take account of the gradient and non-linearity of the graph, as follows

$$i_o = av_i + bv_i^2 + cv_i^3 + \ldots$$

Substituting $\quad v_i = V \sin \omega t$

$$i_o = aV \sin \omega t + bV^2 \sin^2 \omega t + \ldots$$

$$= aV \sin \omega t + \frac{bV^2}{2}(1 - \cos 2\omega t) + \ldots$$

$$\therefore \quad i_o = \underset{\text{(d.c.)}}{\frac{bV^2}{2}} + \underset{\text{(fundamental)}}{aV \sin \omega t} - \underset{\text{(second harmonic)}}{\frac{bV^2}{2} \cos 2\omega t} + \ldots \tag{7.3}$$

If the curvature of the graph is small, third and higher harmonics are insignificant and are therefore neglected in this proof.

Substituting $\omega t = \pi/2$ radians in equation 7.3

$$I_1 = \frac{bV^2}{2} + aV + \frac{bV^2}{2} \tag{i}$$

Similarly, substituting $\omega t = 3\pi/2$ radians

$$-I_2 = \frac{bV^2}{2} - aV + \frac{bV^2}{2} \tag{ii}$$

Now the amplitude of the second harmonic term in equation 7.3 is $bV^2/2$. This may be found by dividing by 4 the sum of equations (i) and (ii).

$$\frac{bV^2}{2} = \frac{I_1 - I_2}{4}$$

Furthermore, the amplitude of the fundamental can be evaluated by dividing by 2 the difference between equations (i) and (ii).

$$aV = \frac{I_1 + I_2}{2}$$

By definition, the percentage of second harmonic distortion present in the output current waveform is given by

$$\frac{\text{Amplitude of second harmonic}}{\text{Amplitude of fundamental}} \times 100 \tag{7.4}$$

$\therefore$ Second harmonic distortion

$$= \frac{I_1 - I_2}{4} \times \frac{2}{I_1 + I_2} \times 100$$

$$= \frac{1}{2}\left(\frac{I_1 - I_2}{I_1 + I_2}\right) \times 100 \text{ per cent} \tag{7.5}$$

Note that for a transistor power amplifier, it is the half-cycle of high collector current which is more likely to suffer distortion. This means that I_2 is greater than I_1; and so, in equation 7.5, the numerator is merely taken as the difference between the peak values of the two half-cycles. For example, if the amount of second harmonic distortion in the output current waveform of a given amplifier is 5 per cent (which is an acceptable maximum),

$$5 = \frac{1}{2}\left(\frac{I_2 - I_1}{I_1 + I_2}\right) \times 100$$

$$I_1 + I_2 = 10(I_2 - I_1)$$

$\therefore 11I_1 = 9I_2$, showing that the peak values are in the ratio 9 : 11.

Example 7.3 A load line is drawn on the output characteristics of a power transistor, through an operating point at which the quiescent collector current is 33·5 mA. In response to the input signal applied between base and emitter, the maximum and minimum values of collector current are found to be 49 mA and 14 mA, respectively. Calculate the percentage second harmonic distortion contained in the output.

Peak value of positive half-cycle $I_1 = 49 - 33{\cdot}5 = 15{\cdot}5$ mA

Peak value of negative half-cycle $I_2 = 33{\cdot}5 - 14 = 19{\cdot}5$ mA

From equation 7.5

Percentage second harmonic distortion

$$= \frac{1}{2}\left(\frac{19{\cdot}5 - 15{\cdot}5}{19{\cdot}5 + 15{\cdot}5}\right) \times 100$$

$$= \frac{50 \times 4}{35} = 5{\cdot}7 \text{ per cent}$$

It can be shown that any waveform that becomes asymmetrical as a result of non-linear transistor characteristics will contain even harmonics. Of these, the most significant is the second harmonic, as already discussed. In electrical engineering generally, however, waveforms produced by rotating machines and transformers are symmetrical. Any departure from a pure sine wave in one half-cycle is repeated in the next, thereby indicating the presence of third and other odd harmonics.

The transformer-coupled load

For maximum power to be transferred to a given load, it is often necessary to include a step-down matching transformer; and, in such an amplifier, it is usually assumed that the resistance of the primary winding is zero. Therefore, under no-signal conditions, the value of V_{CE} is the same as supply voltage V_{CC}.

In other words, the d.c. load line is vertical, and the operating point Q is immediately above V_{CC}. But the a.c. load line through Q has a gradient given by the reciprocal of the load resistance referred to the primary, as shown in example 7.4.

Example 7.4 The output characteristics for a germanium transistor are a series of straight lines between the points quoted in table 7.2:

table 7.2

	I_C (mA)	
I_B (μA)	$V_{CE} = 0{\cdot}4$ V	$V_{CE} = 10$ V
20	0·9	1·5
60	2·8	4·2
100	4·8	6·6

Construct these characteristics and superimpose the load line for the AF amplifier shown in figure 7.10. Also add a collector dissipation curve for 20 mW.

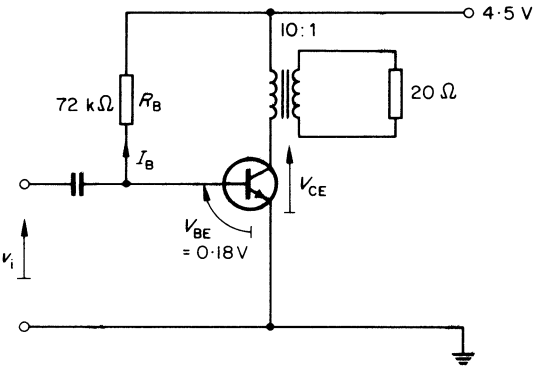


figure 7.10

Estimate the signal output power and collector dissipation when the input signal v_i causes the base current to vary by $\pm 40\,\mu$A. The output characteristics have been plotted in figure 7.11. Applying Kirchhoff's second law to the base-emitter circuit

$$4{\cdot}5 - 0{\cdot}18 = I_B R_B$$

$$\therefore \qquad I_B = \frac{4{\cdot}32}{72}\ \text{mA} = 60\ \mu\text{A}$$

From the output circuit, resistance of the load referred to the primary

$$= 10^2 \times 20 = 2\ \text{k}\Omega$$

This can be represented by a line joining 6 V and 3 mA on the two axes of the graph, but this line must then be transferred (parallel to itself) to where it passes through the point Q. At this point it is known that $I_B = 60\,\mu A$; and, assuming no voltage drop in the transformer primary, $V_{CE} = V_{CC} = 4{\cdot}5$ V. Given that the excursion of base current is $\pm 40\,\mu A$ due to the input signal, the operative portion of the load line is from A to B.

Since $(P_{ec})_{max} = 20$ mW, corresponding points on this dissipation curve are as shown in table 7.3.

table 7.3

V_{CE} (V)	2·5	4	5	6	8	10
I_C (mA)	8	5	4	3·3	2·5	2

This hyperbola has been added to the graph of figure 7.11.

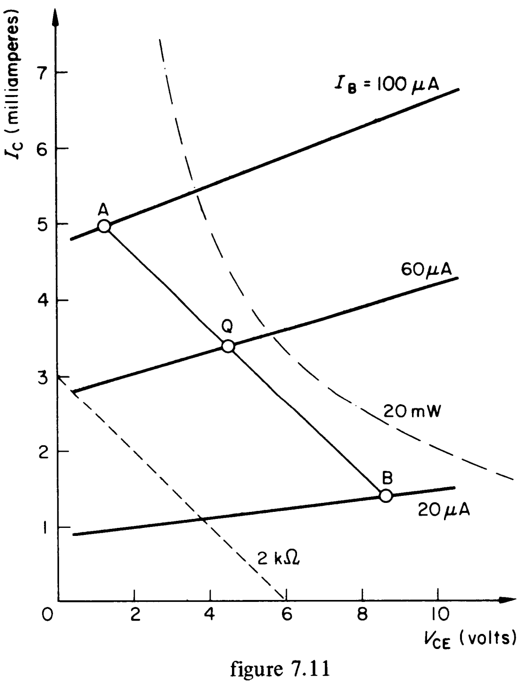


figure 7.11

Assuming sinusoidal variations of output current and voltage

$$\text{Output power} = \frac{\delta V_{CE}}{2\sqrt{2}} \times \frac{\delta I_C}{2\sqrt{2}} \text{ mW}$$

where δV_{CE} and δI_C are peak-to-peak values.

From the graph

$$\delta V_{CE} = 8{\cdot}6 - 1{\cdot}2 = 7{\cdot}4 \text{ V}$$

$$\delta I_C = 5{\cdot}0 - 1{\cdot}4 = 3{\cdot}6 \text{ mA}$$

$$\therefore \quad \text{Output power} = \frac{7{\cdot}4 \times 3{\cdot}6}{8} = 3{\cdot}33 \text{ mW}$$

Collector dissipation on load

$$= \text{Power from d.c. supply} - \text{output power}$$

$$= (4{\cdot}5 \times 3{\cdot}4) - 3{\cdot}33$$

$$= 15{\cdot}3 - 3{\cdot}33 \triangleq 12 \text{ mW}$$

It can also be seen that the collector conversion efficiency is

$$\frac{3{\cdot}33}{15{\cdot}3} \times 100 = 21{\cdot}7 \text{ per cent}$$

This figure would have been higher if the operating point Q and the load line had been nearer to the maximum dissipation curve. Ideally, Q should be situated at the point where the collector dissipation curve touches the centre of the load line. In practice, however, the d.c. term in equation 7.3 causes Q to drift upwards and thus there is a danger that the safe limit of collector dissipation may be exceeded with damaging results.

There may be a lack of any definite information regarding the location of the a.c. load line other than it must pass through a specified operating point. In these circumstances, as the load line is slewed about Q, a range of output powers can be evaluated for a given size of input signal. It is good practice to aim for maximum output with minimum distortion. A useful compromise can be reached by drawing the load line so that it intersects the characteristic corresponding to the maximum instantaneous base current just below the 'knee' as shown in example 7.5.

Example 7.5 Plot the characteristics of an NPN power transistor as given in table 7.4.

table 7.4

Collector emitter voltage V_{CE}	Collector current I_C (mA) for base current I_B (mA)							
	I_B	0	0·4	0·8	1·2	1·6	2·0	2·4
2	I_C	0·8	17	36	54	54	54	54
4		1·0	17·5	37	54	74	95	114
10		1·5	18	38	56	76	96·5	120
20		2·0	20	39	58	78	100	124
40		3·0	24	43	64	84	107	132

The transistor is to be used as the output stage of an AF amplifier with its collector connected in series with the primary of an output transformer to a 20 V d.c. supply. It is so mounted that the maximum permitted collector dissipation is 1·2 W.

(a) Plot the dissipation curve on the characteristics, select a suitable bias point, and draw an a.c. load line to give maximum output power at moderate distortion

(b) Estimate, for a sinusoidal input signal,
  (i) the maximum output power
  (ii) the collector dissipation when the transistor delivers this power
  (iii) the no-signal collector dissipation

(CGLI Principles C, 1971)

The output characteristics are plotted in figure 7.12. A collector dissipation curve for 1·2 W has been added by plotting the points in table 7.5:

table 7.5

V_{CE} (V)	10	15	20	25	30	40
I_C (mA)	120	80	60	48	40	30

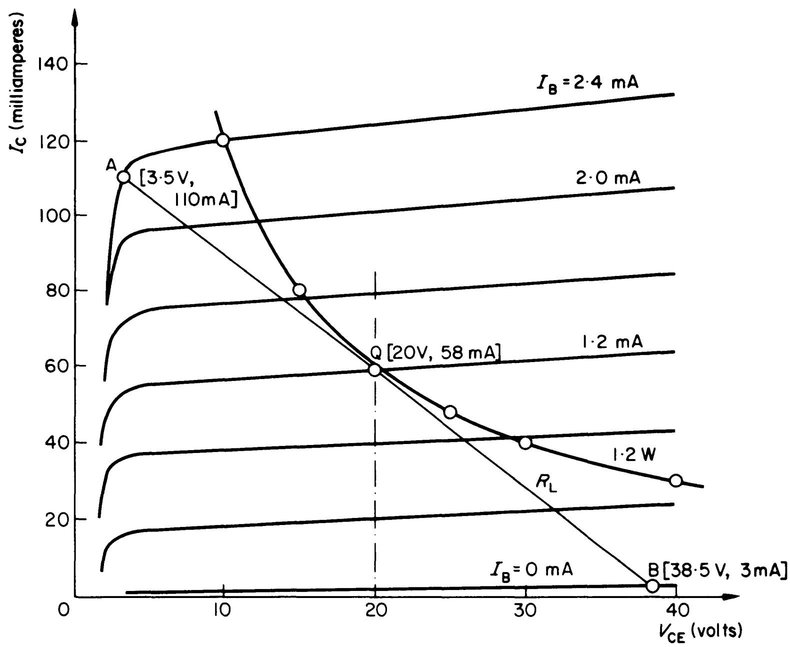


figure 7.12

Neglecting any resistance in the primary winding of the output transformer, the d.c. load line is drawn vertically through $V_{CC} = 20$ V. To allow maximum swing of base current due to the incoming signal, Q is chosen to be at the point where the d.c. load line cuts the $I_B = 1{\cdot}2$ mA characteristic. The a.c. load line is drawn through Q to cut the curve for $I_B = 2{\cdot}4$ mA at a point A, and is then extended in the other direction until it intersects the curve for $I_B = 0$ at B. (For convenience in the later calculations, the co-ordinates of A, B and Q are labelled on the graph.)

(i) Maximum output power

$$= \frac{\delta V_{CE} \times \delta I_C}{8} = \frac{35 \times 107}{8} = 468 \text{ mW}$$

(iii) No-signal collector dissipation

$$= 20 \times 58 \text{mW} = 1{\cdot}16 \text{W}$$

(ii) Combining (i) and (iii), collector dissipation with full output

$$= 1160 - 468$$

$$= 692 \text{ mW}$$

Further investigation into this question reveals that the two half-cycles of output current have amplitudes of 52 mA and 55 mA.

∴ Percentage second harmonic distortion

$$= \frac{1}{2}\left(\frac{55-52}{55+52}\right) \times 100 = 1{\cdot}4 \text{ per cent}$$

Also, the collector conversion efficiency

$$= \frac{468}{1160} \times 100 = 40{\cdot}3 \text{ per cent}$$ (C)

MISCELLANEOUS EXAMPLES

(B) ***Example 7.6*** A silicon transistor has the output characteristics shown in figure 7.1. It is connected as a power amplifier as shown in figure 7.13. The transformer has a step-down ratio of 5 : 1 and supplies a load resistor R_2 of 48 Ω. Calculate the resistance of the effective collector load and construct a load line for this resistance, assuming that the collector current at the operating point is 2·9 mA. (A copy of the characteristics must be re-drawn on graph paper for this purpose.) Also calculate:

(a) a suitable value for the bias resistor R_B assuming that $V_{BE} = 0{\cdot}5$ V
(b) the current gain of the amplifier for an input signal to the base of peak value ± 30 μA

[1·2 kΩ; 100 kΩ; 68·3]

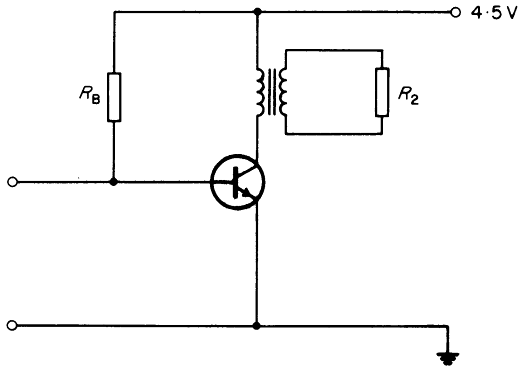


figure 7.13

Example 7.7 The amplifier of figure 7.13 is re-designed so that a 1·2 kΩ resistor is connected directly into the collector circuit. If the operating point is to remain in the same position as in the previous example, what would be the new value of supply voltage required? (B)

[8 V]

(C) *Example 7.8* Re-construct on graph paper the transistor output characteristics shown in figure 7.1 for I_B having values of 10 μA, 40 μA and 70 μA. Add a load line for 1·2 kΩ given that the supply voltage is 8 V. If the quiescent base current is 40 μA, varying by ± 30 μA due to the incoming signal, estimate:

(i) the power dissipated at the collector
(ii) the collector conversion efficiency
(iii) the percentage second harmonic distortion in the output waveform, under these conditions.

[10·5 mW; 19·6 per cent; 1·22 per cent]

Example 7.9 A transistor supplies power of 850 mW to a resistive load of 4 kΩ. The collector current at the quiescent point is 31 mA, increasing to 32 mA when delivering the above output. Determine the percentage second harmonic distortion in the output, assuming this to be the only harmonic present.

[4·86 per cent]

Example 7.10 A silicon transistor in the common-emitter mode is used as a power amplifier with an ideal 16 : 1 step-down transformer in its collector circuit. The operating point is determined by a resistor connected between the base and the supply voltage of 15 V. Under no-signal conditions, I_B = 300 μA. Construct the output characteristics from table 7.6, and hence calculate:

(a) the value of a suitable bias resistor if $V_{BE} = 0{\cdot}6$ V
(b) the power supplied to a load of 3·5 Ω resistance connected across the transformer secondary when the sinusoidal input current has a peak value of 200 μA
(c) the conversion efficiency of the amplifier when driven as in (b)

table 7.6

	I_C (mA)		
V_{CE} (V)	$I_B = 100\,\mu$A	300 μA	500 μA
5	5·5	17	30·5
25	6	19	34

[48 kΩ; 77·5 mW; 28·7 per cent]

Example 7.11 A power amplifier operating in Class A, with a load of 5 Ω referred to the primary of the coupling transformer, has a quiescent collector current of 2·7 A. During one half-cycle it is driven to a point on its output characteristics where $V_{CE} = 0{\cdot}5$ V and $I_C = 4{\cdot}9$ A. During alternate half-cycles I_C falls to a minimum value of 0·1 A. Estimate:

(a) the output power at fundamental frequency
(b) the percentage harmonic distortion of the output current waveform
(c) the supply voltage
(d) the maximum power dissipated at the collector, assuming that it occurs at the mid-point of the load line

[14·4 W; 4·16 per cent; 11·5 V; 31·25 W]

Example 7.12 The figures given in the transistor data in table 7.7 refer to the transistor used in a transformer-coupled Class A amplifier.

table 7.7

	I_C (mA)			
V_{CE} (V)	$I_B = 0$	0·2 mA	0·3 mA	0·5 mA
2·5	0·5	11·3	17·0	29·5
5·0	0·5	12·0	18·0	31·0
30·0	0·5	13·6	20·5	34·5

Plot these characteristics and add a collector dissipation curve for 250 mW. If the supply voltage is 16 V and the load resistor is 75 Ω connected across the secondary of a 3·5 : 1 step-down transformer, construct the optimum a.c. load line and use it to calculate the maximum power output under these conditions.

[100 mW] (C)

8 Transistors at high frequency and tuned amplifiers

(C) When a semi-conductor junction is formed, diffusion of carriers causes a depletion layer to appear on either side of the junction. The width of this layer depends on the impurity content of the semi-conductor. Thus, there is a layer of relatively low conductivity intrinsic material sandwiched between regions of impurity (extrinsic) semiconductor. These form a tiny parallel-plate capacitor across the junction, with the depletion layer acting as the dielectric. Forward bias on the emitter-base junction has the effect of reducing the width of the depletion layer and thereby increasing its capacitance. Conversely, the collector-base junction has a relatively small capacitance because, although of larger area, the separation of its 'plates' has been increased by reverse bias. The so-called *depletion layer capacitors* are not the only reactive elements in a transistor. A further source of capacitance is the base region itself. Due to the lack of potential gradient across the base, carriers tend to be stored in it and to cross it only by mutual repulsion (that is, they move from areas of high to low concentration). This process of migration across the base involves a delay that can be represented on an equivalent circuit as a *diffusion capacitance.*

EFFECT OF HF ON *h*-PARAMETERS

The overall effect of these internal capacitors is that all the *h*-parameters become complex functions of frequency, which limits their usefulness in small-signal audio-frequency circuit analysis. Experiments show that in the common-base configuration, when the short-circuit current from the collector is compared with emitter current, the current transfer factor h_{fb} varies in magnitude with frequency as shown in figure 8.1.

It can be seen that $|h_{fb}|$ maintains its low frequency value until at HF (typically 50 MHz), it begins to diminish. The frequency at which it has fallen 3 dB below its maximum is known as the *cut-off frequency*, f_{hfb}.

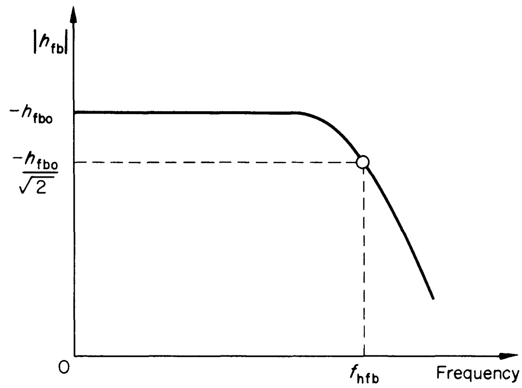


figure 8.1

Beyond this point, $|h_{fb}|$ rolls off at the rate of 20 dB per decade; that is, multiplying the frequency by a factor of 10 reduces the current transferred to the collector by 20 dB. It may be proved that such a high frequency response can be represented by an equation of the form

$$h_{fb} = \frac{h_{fbo}}{1 + \dfrac{f}{jf_{hfb}}} \tag{8.1}$$

Hence

$$|h_{fb}| = \frac{|h_{fbo}|}{\sqrt{\left[1 + \left(\dfrac{f}{f_{hfb}}\right)^2\right]}} \tag{8.2}$$

For example, when $f = f_{hfb}$, $h_{fb} = (h_{fbo})/\sqrt{2}$ as shown in figure 8.1. Equation 8.1 can be adapted to the common-emitter configuration if it is remembered that the current transfer factor h_{fe} is equal to $-h_{fb}/(1 + h_{fb})$.

$$\therefore \qquad h_{fe} = \frac{-\dfrac{h_{fbo}}{1 + j(f/f_{hfb})}}{1 + \dfrac{h_{fbo}}{1 + j(f/f_{hfb})}}$$

Multiplying throughout by $[1 + j(f/f_{hfb})]$

$$h_{fe} = \frac{-h_{fbo}}{1 + h_{fbo} + j(f/f_{hfb})}$$

Dividing throughout by $(1 + h_{\text{fbo}})$

$$h_{\text{fe}} = \frac{h_{\text{feo}}}{1 + \text{j}\dfrac{f}{f_{\text{hfb}}(1 + h_{\text{fbo}})}}$$

where

$$h_{\text{feo}} = \frac{-h_{\text{fbo}}}{1 + h_{\text{fbo}}} \text{ is the LF value of } h_{\text{fe}}.$$

If it is accepted that $1/(1 - h_{\text{fbo}})$ is approximately the same as $(h_{\text{fbo}})/(1 - h_{\text{fbo}})$, the previous result may be written as

$$h_{\text{fe}} = \frac{h_{\text{feo}}}{1 + \text{j}\dfrac{f . h_{\text{feo}}}{f_{\text{hfb}}}}$$

In exactly the same way as for equation 8.2, the modulus of h_{fe} will also fall by 3 dB below its maximum at a frequency f_{hfe} when $(f . h_{\text{feo}})/(f_{\text{hfb}}) = 1$.

Hence

$$f_{\text{hfe}} = \frac{f_{\text{hfb}}}{h_{\text{feo}}} \tag{8.3}$$

and

$$h_{\text{fe}} = \frac{h_{\text{feo}}}{1 + \text{j}\dfrac{f}{f_{\text{hfe}}}} \tag{8.4}$$

A basic circuit (omitting d.c. supplies) that may be used to verify equation 8.4 is shown in figure 8.2.

As the input impedance of the transistor is negligible when compared with the 1 MΩ resistor, current into the base of the transistor is given by

$$I_{\text{b}} = \frac{v_{\text{i}}}{10^6}$$

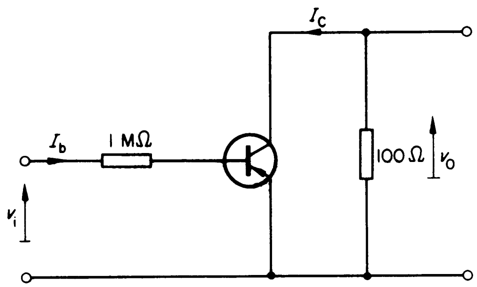


figure 8.2

The collector should ideally be directly connected to the emitter; but, in order to develop a voltage drop v_0, a 100 Ω resistor is included in the circuit, this being small compared with the transistor output impedance.

$$\therefore \text{ collector current } I_c = \frac{v_0}{100}$$

Combining the last two equations

$$h_{fe} = \frac{I_c}{I_b} = \frac{10^4 v_0}{v_i}$$

If a phase-sensitive voltmeter were used for the measurement of these voltages, it could be confirmed that at HF the transistor ceases to be a purely resistive device and it acquires a phase angle, which amounts to 45° at f_{hfe}. This can also be confirmed mathematically from equation 8.4.

Example 8.1 A transistor has a short-circuit current transfer factor of 0·975 at LF which falls to 0·69 at 40 MHz. Calculate the corresponding value of $|h_{fb}|$ at 60 MHz and the frequency at which $|h_{fb}|$ has fallen to 0·762.

From the data, $h_{fbo} = -0{\cdot}975$ and $f_{hfb} = 40$ MHz.

Substituting in equation 8.2

$$|h_{fb}| = \frac{0{\cdot}975}{\sqrt{\left[1 + \left(\frac{60}{40}\right)^2\right]}} = 0{\cdot}541$$

Similarly

$$0{\cdot}762 = \frac{0{\cdot}975}{\sqrt{\left[1 + \left(\frac{f}{40}\right)^2\right]}}$$

Squaring etc.

$$1 + \left(\frac{f}{40}\right)^2 = \left(\frac{0{\cdot}975}{0{\cdot}762}\right)^2 = 1{\cdot}64$$

$$\therefore \qquad f = 40\sqrt{(0{\cdot}64)} = 32 \text{ MHz}$$

Example 8.2 Tests on a transistor in its common-base configuration, to measure the short-circuit current 'gain' at various frequencies, yielded the results shown in table 8.1

table 8.1

f (MHz)	0·001	0·5	1·0	2·0	3·0	4·0	5·0
$\lvert h_{fb}\rvert$	0·985	0·835	0·695	0·422	0·32	0·236	0·191

Verify that these results show $|h_{fb}|$ to be related to f by equation 8.2. Also calculate the corresponding values of short-circuit current gain when the same transistor is tested in its common-emitter configuration at frequencies of 1 kHz, 0·5 MHz and 1 MHz. Comment on the results of these calculations.

If

$$h_{fb} = \frac{h_{fbo}}{\sqrt{\left[1 + \left(\frac{f}{f_{hfb}}\right)^2\right]}}$$

$$\therefore \qquad \left(\frac{h_{fbo}}{h_{fb}}\right)^2 = 1 + \left(\frac{f}{f_{hfb}}\right)^2 \text{ after squaring.}$$

Dividing throughout by $(h_{fbo})^2$

$$\frac{1}{h_{fb}{}^2} = \frac{1}{(h_{fbo}f_{hfb})^2} f^2 + \frac{1}{(h_{fbo})^2}$$

This expression is of the form

$$y = mx + c, \text{ which is linear.}$$

The only satisfactory way of confirming that the experimental results obey equation 8.2 is to see if the graph obtained by plotting $1/(h_{fb})^2$ to a base of f^2 is a straight line.

By calculation, a new set of results can now be drawn up as in table 8.2.

table 8.2

f^2	0·00001	0·25	1·0	4·0	9·0	16·0	25·0
$1/(h_{fb})^2$	1·03	1·43	2·07	5·6	10·8	18·0	27·5

These values are plotted in figure 8.3 from which it can be seen that the graph is linear, thus verifying that the results satisfy equation 8.2.
From the original data, $|h_{fbo}| = 0·985$ and falls by 3 dB at a frequency of 1 MHz. Hence $f_{hfb} = 1$ MHz and the corresponding values of h_{fe} can now be calculated.

As

$$h_{feo} = \frac{|h_{fbo}|}{1 - |h_{fbo}|}$$

$$\therefore \qquad h_{feo} = \frac{0{\cdot}985}{1 - 0{\cdot}985} = 65{\cdot}6$$

From equation 8.3

$$f_{hfe} = \frac{f_{hfb}}{h_{feo}} = 15{\cdot}2 \text{ kHz}$$

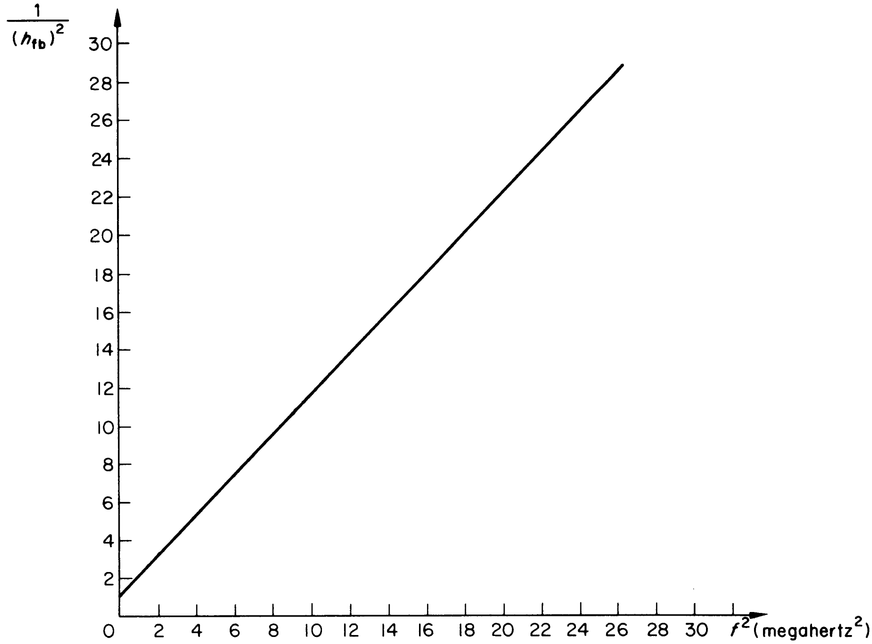


figure 8.3

From equation 8.4

$$|h_{fe}| = \frac{h_{feo}}{\sqrt{\left[1 + \left(\frac{f}{f_{hfe}}\right)^2\right]}}$$

$\therefore$ At a frequency of 0·5 MHz or 500 kHz

$$|h_{fe}| = \frac{65{\cdot}6}{\sqrt{\left[1 + \left(\frac{500}{15{\cdot}2}\right)^2\right]}}$$

As $(500/15{\cdot}2)^2$ is very much greater than 1

$$\therefore \qquad |h_{fe}| = 65{\cdot}6 \times \frac{15{\cdot}2}{500} = 2{\cdot}0$$

Similarly at a frequency of 1 MHz

$$|h_{fe}| = 65{\cdot}6 \times \frac{15{\cdot}2}{1000} = 1{\cdot}0$$

Two very significant observations can be made from these calculations: firstly, the cut-off frequency f_{hfe} occurs much earlier in the common-emitter mode, which means that the common-base method of connection makes much better use of its frequency capability than does the common-emitter mode; secondly, at a particular frequency (1 MHz in this example), the transistor ceases to be an amplifier, as its short-circuit gain in the common-emitter mode falls to unity. The particular frequency at which this latter effect happens is known as the *transition frequency* and is denoted by f_T. Substituting these values in equation 8.4 it will be seen that

$$1 = \frac{h_{feo}}{\sqrt{\left[1 + \left(\frac{f_T}{f_{hfe}}\right)^2\right]}}$$

Making the same approximation as before

$$1 = \frac{h_{feo} f_{hfe}}{f_T}$$

$$\therefore \qquad f_T = h_{feo} f_{hfe}$$

which is referred to as the *short-circuit current gain – bandwidth product* because it contains the maximum short-circuit current gain h_{feo} and the operative bandwidth under these conditions, f_{hfe}. Furthermore $h_{feo} f_{hfe}$ is equal to f_{hfb} as shown in equation 8.3. Therefore, for all practical purposes,

$$f_T = f_{hfb} \tag{8.5}$$

TUNED RF AMPLIFIERS

In the field of Telecommunications there is a need not only for wide-band amplifiers of the type considered in chapter 6 but also for amplifiers that are highly selective and can be tuned to a particular frequency. In this way, signals of the required wavelength can be accepted by the amplifier while all others are rejected. Tuned amplifiers rely basically on the use of a parallel resonant circuit as the collector load; and it is this circuit that will now be examined beyond the point reached in chapter 3, so that the operation of tuned amplifiers may be understood.

Impedance

Consider the impedance of the circuit shown in figure 8.4 tuned to resonate at f_0.

$$Z = \frac{-\frac{j}{\omega C}(R + j\omega L)}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{\frac{L}{C} - j\frac{R}{\omega C}}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

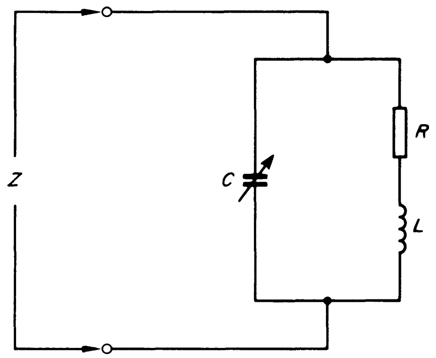


figure 8.4

In a circuit of high Q-factor

$$\frac{\omega_0 L}{R} \gg 1$$

Multiplying throughout by C

$$\therefore \qquad \frac{\omega_0 LC}{CR} \gg 1$$

Regrouping terms

$$\frac{L}{C} \gg \frac{R}{\omega_0 C}$$

Even at frequencies slightly removed from resonance

$$\frac{L}{C} > \frac{R}{\omega C}$$

so that the expression for Z becomes

$$Z = \frac{\dfrac{L}{C}}{R + \mathrm{j}\left(\omega L - \dfrac{1}{\omega C}\right)}$$

Dividing throughout by R

$$Z = \frac{\dfrac{L}{CR}}{1 + \mathrm{j}\left(\dfrac{\omega L}{R} - \dfrac{1}{\omega CR}\right)}$$

$$= \frac{Z_D}{1 + \mathrm{j}\left(\dfrac{\omega L}{R} - \dfrac{1}{\omega CR}\right)} \qquad (8.6)$$

where Z_D is the *dynamic impedance* of the tuned circuit. Let δ be the fractional deviation away from resonance, that is

$$\delta = \frac{\omega - \omega_0}{\omega_0}$$

or

$$\omega = \omega_0(1 + \delta)$$

Substituting for ω in equation 8.6

$$Z = \frac{Z_D}{1 + j\left[\dfrac{\omega_0 L}{R}(1 + \delta) - \dfrac{1}{\omega_0 CR(1 + \delta)}\right]}$$

$$= \frac{Z_D}{1 + jQ\left[(1 + \delta) - \dfrac{1}{(1 + \delta)}\right]}$$

$$= \frac{Z_D}{1 + jQ\left[\dfrac{1 + 2\delta + \delta^2 - 1}{(1 + \delta)}\right]}$$

$$= \frac{Z_D}{1 + jQ\delta\left(\dfrac{2 + \delta}{1 + \delta}\right)}$$

For minimal frequency deviation,

$$\left(\frac{2 + \delta}{1 + \delta}\right) \triangleq 2$$

To a close approximation

$$Z = \frac{Z_D}{1 + j2Q\delta}$$

Hence

$$|Z| = \frac{Z_D}{\sqrt{(1 + 4Q^2\delta^2)}} \tag{8.7}$$

Gain and bandwidth

It will now be shown that the frequency response of a transistor amplifier that uses this tuned circuit for its collector load follows the same law as that expressed in equation 8.7. Consider the basic transistor equations

$$v_1 = h_i i_1 + h_r v_2 \qquad \text{(i)}$$

$$i_2 = h_f i_1 + h_o v_2 \qquad \text{(ii)}$$

Dividing (ii) by (i), having neglected h_o and h_r

$$\frac{i_2}{v_1} \simeq \frac{h_f}{h_i}$$

For the common-emitter amplifier being considered, this expression may be written as

$$\frac{\delta I_2}{\delta V_1} \simeq \frac{h_{fe}}{h_{ie}}$$

Now the rate of change of collector current I_2 with respect to base-emitter voltage V_1 is known as the *transconductance* g_m of the transistor (already mentioned in connection with the FET in chapter 5). Therefore, to a first approximation,

$$g_m = \frac{h_{fe}}{h_{ie}}$$

which makes it possible to draw a simplified equivalent circuit for the tuned RF amplifier as in figure 8.5.

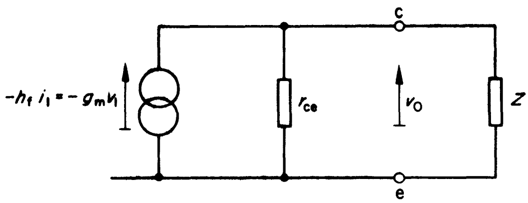


figure 8.5

Temporarily neglecting the output resistance of the transistor r_{ce} which appears between collector and emitter, voltage gain is given by

$$\frac{v_o}{v_1} = -\frac{g_m v_1 Z}{v_1} = -g_m Z \qquad (8.8)$$

Hence, for a constant value of g_m, changes in load impedance with frequency produce proportional changes in amplifier gain. Combining equations 8.7 and 8.8, voltage gain is given by

$$|A_v| = \frac{A_{vo}}{\sqrt{(1 + 4Q^2\delta^2)}} \qquad (8.9)$$

where A_{vo} = voltage gain at resonance corresponding to the dynamic impedance Z_D.

Furthermore, at the 3 dB points (which set a limit to the useful bandwidth of the amplifier)

$$|A_v| = \frac{A_{v0}}{\sqrt{2}}$$

Hence

$$1 + 4Q^2\delta^2 = 2, \text{ so that } \delta = \pm\frac{1}{2Q}$$

$$\therefore \qquad \frac{\omega_2 - \omega_0}{\omega_0} = +\frac{1}{2Q}, \qquad \frac{\omega_1 - \omega_0}{\omega_0} = -\frac{1}{2Q}$$

Subtracting one equation from the other

$$\frac{(f_2 - f_1)}{f_0} = \frac{1}{Q}$$

$\therefore$ Bandwidth

$$(f_2 - f_1) = \frac{f_0}{Q} \tag{8.10}$$

Thus, the gain of the amplifier varies with frequency as shown in the graph of figure 8.6.

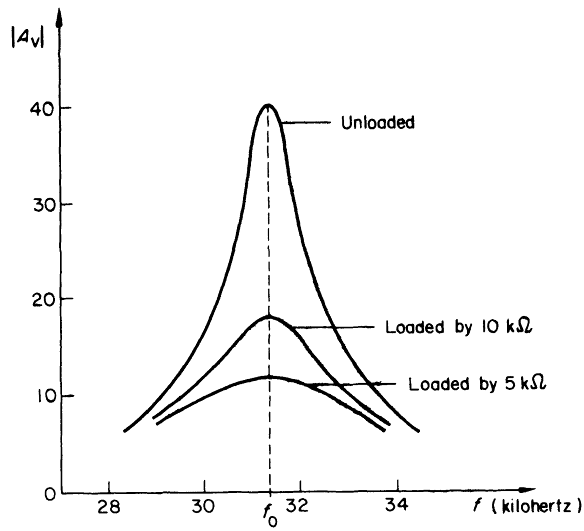


figure 8.6

Example 8.3 A tuned-collector amplifier uses a transistor with $g_m = 50$ mA/V and a very high value of r_{ce}. The coil of the tuned circuit has an inductance of 150 μH and $Q = 20$. It is tuned to resonate at 796 kHz by a parallel capacitor. Calculate:

(a) the gain at resonance f_0
(b) the gain at frequencies of $(f_0 \pm 10)$ kHz
(c) its effective bandwidth.

(a) As

$$Z_D = \frac{L}{CR} = \frac{\omega_0 L}{\omega_0 CR}$$

$$Z_D = Q\omega_0 L$$

Substituting the values given

$$Z_D = \frac{20 \times 2\pi \times 796 \times 150}{10^6} \text{ k}\Omega$$

$$= 15 \text{ k}\Omega$$

$$\therefore \qquad |A_{v0}| = g_m Z_D = 750$$

(b) From equation 8.9

$$A_v = \frac{A_{v0}}{\sqrt{(1 + 4Q^2\delta^2)}} \quad \text{where} \quad \delta = \pm\frac{10}{796}$$

$$= \frac{750}{\sqrt{\left[1 + 4 \times 20^2 \times \left(\frac{10}{796}\right)^2\right]}}$$

$A_v = 670$ at frequencies 10 kHz away from resonance.

(c) From equation 8.10

$$\text{Bandwidth} = \frac{f_0}{Q} = \frac{796}{20} = 39{\cdot}8 \text{ kHz}$$

This may be expressed as

$$\text{Bandwidth} = (796 \pm 19{\cdot}9) \text{ kHz}$$

Effect of loading

In most practical amplifier circuits, however, the value of output resistance r_{ce} compared with Z_D is not so high that it can be regarded as negligible. Also this type of amplifier is probably the first of a number of stages in a receiver. Therefore, its tuned circuit must be loaded by the input impedance of the next stage, thus modifying the overall frequency response. Assuming the input impedance over the operating frequency range of the subsequent stage to be purely resistive and represented by R, Z_D will be loaded as shown in figure 8.7.

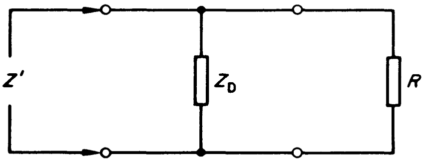


figure 8.7

Effective load on the collector at resonance

$$Z' = \frac{RZ_D}{R + Z_D} < Z_D$$

As

$$Z_D = \frac{L}{CR} = \frac{\omega_0 L}{\omega_0 CR}$$

this may be written as either $Q\omega_0 L$ or $Q/\omega_0 C$.

In both expressions it should be noted that the resonant frequency (and hence ω_0) is hardly affected by loading. Therefore, it can be deduced that the *loaded Q-factor* of the circuit and the effective load presented to the transistor are proportional. This being so, the loading of the tuned amplifier by another stage not only reduces the gain at resonance but also affects the bandwidth. In other words there is both a loss of sensitivity and a reduction in selectivity. To allow for this in calculations, the term Q that appears in equations 8.9 and 8.10 must be replaced by its loaded value Q', as shown in the next example.

Example 8.4 Calculate the mutual conductance g_m for the transistor required in a single-tuned RF amplifier operating at 465 kHz if its voltage gain at resonance is to be 78·6 dB, neglecting any loading effect. The tuned circuit consists of 200 pF in parallel with an inductor of $Q = 100$.

What are the new values of maximum gain in dB and bandwidth when the amplifier supplies another stage whose input resistance is 40 kΩ?

Using the alternative expression for dynamic impedance

$$Z_D = \frac{Q}{\omega_0 C} = 171\ \text{k}\Omega$$

$A_{v0} = 78{\cdot}6$ dB, which is equivalent to a gain of 8511

$$\therefore \qquad g_m Z_D = 8511$$

that is,

$$g_m = \frac{8511}{171} = 49{\cdot}8\ \text{mA/V}$$

When loaded with 40 kΩ

$$Z' = \frac{171 \times 40}{211} = 32{\cdot}4\,\text{k}\Omega$$

∴ New value of gain at resonance,

$$A'_{\text{vo}} = 49{\cdot}8 \times 32{\cdot}4$$

$$= 1614$$

$$\therefore \quad A'_{\text{vo}} = 20 \log_{10} 1614 = 64{\cdot}16\,\text{dB}$$

The loaded Q-factor, Q', being proportional to impedance is given by

$$Q' = \frac{100 \times 32{\cdot}4}{171} = 18{\cdot}95$$

$$\text{New bandwidth } B' = \frac{f_0}{Q'} = \frac{465}{18{\cdot}95} = 24{\cdot}5\,\text{kHz}$$

which may be written as (465 ± 12·25) kHz.

MISCELLANEOUS EXAMPLES

Example 8.5 A transistor connected in the common-base mode has a short-circuit current amplification factor of 0·985 at LF which falls to 0·695 at 10 MHz. Determine $|h_{\text{fb}}|$ at 8 MHz and h_{fe} at 152 kHz.

[0·77; 46·5]

Example 8.6 A tuned-collector amplifier operates at resonance. For the transistor, $g_{\text{m}} = 20$ mA/V and r_{ce} may be neglected. The collector load consists of a 45 μH inductor of 30 Ω resistance tuned by a parallel capacitor of 500 pF. Calculate the dynamic impedance of the amplifier load and hence find the stage gain at resonance.

[3 kΩ; 60]

Example 8.7 A single-stage, common-emitter, tuned transistor amplifier has an effective load resistance at resonance of 25 kΩ. The transistor hybrid parameters are $h_{\text{ie}} = 2$ kΩ, $h_{\text{fe}} = 55$ and both h_{re} and h_{oe} are negligible. The signal source connected between base and emitter has an open-circuit EMF of 3 mV and resistive internal impedance of 500 Ω. Calculate:

(a) the signal voltage across the amplifier input terminals
(b) the input signal current
(c) the signal power in the collector load.

(Assume the effects of biasing and coupling components on the signal to be negligible.)

Sketch a typical practical circuit diagram for the amplifier.

(CGLI Principles C, 1971)

Note: When the output to the next stage is taken from the point where the collector is connected to the 'lower' end of L, the tuned circuit is shunted by R_{in} and the frequency response is affected in the manner already explained. By connecting the collector and output to a tapping on L, however, this loading effect is considerably reduced. The choke now acts as an auto-transformer having a step-down ratio. For example, if the tapping is made at a point a quarter of the way from the 'upper' end of L as shown in figure 8.8, the turns ratio is effectively 4 : 1. Consequently, resistance reflected across the tuned circuit is increased to 16 times R_{in}.

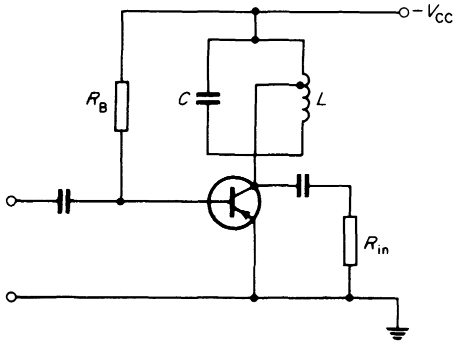


figure 8.8

[2·4 mV; 1·2 μA; 109 μW]

Example 8.8 An RF amplifier uses a transistor ($r_{ce} = 30\,\text{k}\Omega, g_m = 50\,\text{mA/V}$). In its collector circuit is a 20 μH coil of Q-factor 50 tuned to parallel resonance at 1·592 MHz. The output voltage across the tuned circuit is fed to a second stage having an input resistance of 30 kΩ. Calculate the effective bandwidth of the amplifier and its gain at resonance. Also find the range of frequency over which the gain exceeds 240.

[53 kHz; 300; 29·8 kHz] (C)

9 Feedback in amplifiers and oscillators

NEGATIVE FEEDBACK

In the amplifiers described in the four previous chapters, no attempt was made to connect any part of the output signal back again to the input terminals via an external circuit. Such amplifiers tend in practice, however, to exhibit a variable, and therefore unreliable, performance. The amplifiers used in modern communications systems, analogue computers and measuring equipment must be extremely stable and free from distortion, and these requirements can be satisfied only by the application of a controlled amount of negative feedback (NFB): that is, by taking a fraction of the output signal and feeding it back to the input terminals in opposition to the signal already there.

The principles of feedback

Consider the phasor diagram of figure 9.1 which shows the input and output voltage of a typical *RC*-coupled amplifier as already examined in chapter 6.

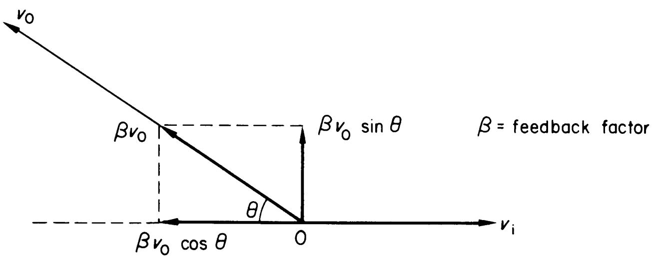


figure 9.1

The voltages v_i and v_o are shown at some particular frequency within the bandwidth of the amplifier. If a portion β of the output is fed back to the input through a resistive network, the feedback signal βv_o is in phase with the voltage from which it is derived. Resolving it into two components at right-angles, it can be seen that one of them is in anti-phase with v_i. Thus, when connected in series with v_i, the net result is that the input to the amplifier has apparently fallen below its original value. Clearly, this must also reduce the overall amplification (as will be analysed in more detail shortly).

Benefits of negative feedback

However, the benefits to be achieved by means of negative feedback far outweigh this loss of gain. They can be summarised as follows:

(a) the gain of the amplifier is less affected by the changes that inevitably occur in the system
(b) the frequency response is made more linear and the bandwidth is extended
(c) the amount of phase shift which occurs when signals of differing frequencies pass through the amplifier is reduced
(d) harmonic distortion (as explained in chapter 7) is reduced, thus providing an output of greater purity
(e) the internal impedances of the amplifier can be modified, within limits, for the purposes of maximum power transfer.

Only the first item in this list will be examined in detail, the others being outside the scope of this book. Before doing so, however, it is as well to investigate just two of the several ways by which negative feedback can be applied. These are illustrated in the following diagrams.

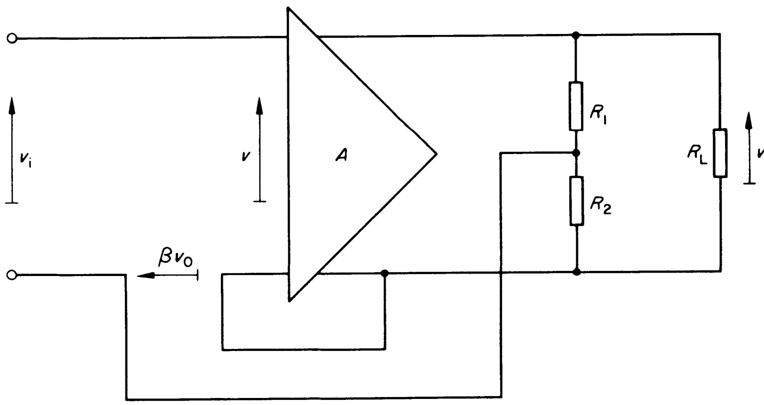


figure 9.2

Figure 9.2 shows the feedback signal βv_o, derived from the output voltage v_o, injected in series with the input signal v_i, and therefore described as *series voltage feedback*. The feedback factor β is given by

$$\beta = \frac{R_2}{R_1 + R_2} \tag{9.1}$$

It is important that $(R_1 + R_2)$ should be very much greater than the existing load resistor R_L so that the shunting effect imposed is negligible.

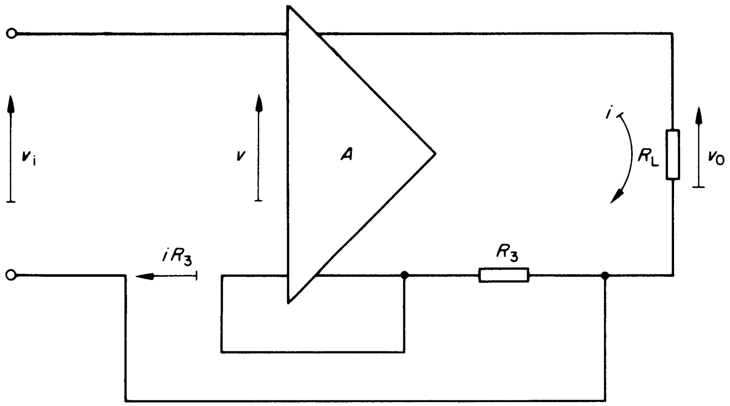


figure 9.3

In figure 9.3, the feedback signal iR_3 is a function of the current in the load, and is therefore described as *series current feedback.* If R_3 is chosen so as to be very much less than R_L, then the feedback factor (which is the voltage fed back divided by the output voltage) is approximately given by

$$\beta = \frac{R_3}{R_L} \tag{9.2}$$

Both circuits have similar effects on amplifier performance, except that the output impedance of the amplifier is decreased in figure 9.2 and increased in figure 9.3; that is, *voltage* NFB gives the amplifier the characteristics of a 'constant' voltage source (with its low output impedance), whereas the amplifier with *current* NFB behaves as a 'constant' current source.

A practical way of applying NFB is illustrated in figure 9.4.

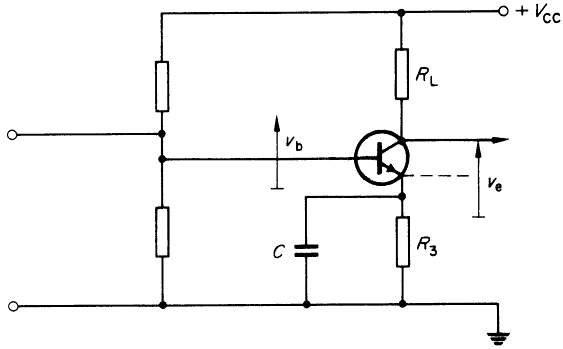


figure 9.4

In figure 9.4, if the emitter decoupling were omitted (or became an open-circuit as a result of a fault developing in the capacitor), R_3 of the 3-resistor bias network would introduce series current NFB. In the positive half-cycle, when the base potential v_b rises in response to the incoming signal, the emitter potential v_e also rises. This makes v_{be} received by the transistor seem less than it would be otherwise, as a result of the voltage drop in the undecoupled resistor R_3. Thus, the negative feedback in series with the input voltage is really a function of output current.

If the feedback circuit contains reactance, it becomes frequency sensitive and can be used in an audio amplifier to provide tone control.

Effect on gain

Consider the feedback amplifier shown in figure 9.5.

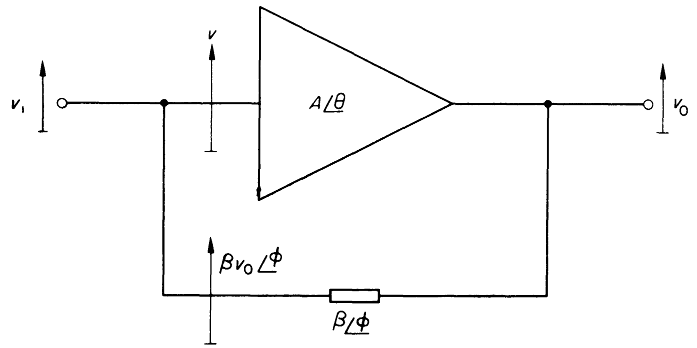


figure 9.5

In the diagram, $A\angle\theta$ is the open-loop gain of the amplifier expressed in magnitude and phase. Similarly, the term $\beta\angle\phi$ in the feedback loop is to account for any possible attenuation and phase shift that may occur in transferring the signal from output to input. At the amplifier output, the voltage must always be $A\angle\theta$ times the signal appearing at its input terminals.

$$\therefore \quad v_o = A\angle\theta\, v = A\angle\theta\,(v_i + \beta\angle\phi\, v_o)$$

$$= Av_i\angle\theta + \beta A v_o\angle\theta + \phi$$

Hence

$$v_o(1-\beta A\angle\theta + \phi) = Av_i\angle\theta$$

Thus, the effective gain of the amplifier with feedback may be calculated from

$$A' = \frac{v_o}{v_i} = \frac{A\angle\theta}{1-\beta A\angle\theta + \phi} \tag{9.3}$$

The general equation 9.3 can now be used to distinguish between positive and negative feedback. For example, if the gain v_o/v_i increases as a result of feedback being applied, the voltage fed back to the input must have a component in phase with the original input v_i, and the feedback is described as *positive.* Conversely, a fall in gain resulting from feedback is an indication that the feedback voltage has a component in series opposition to v_i, and the feedback is described as *negative.* The mathematical discrimination between positive and negative feedback may be stated as follows:

If $\left|\dfrac{A'}{A}\right| > 1$, feedback is positive

If $\left|\dfrac{A'}{A}\right| < 1$, feedback is negative.

Example 9.1 At two particular frequencies, the open-loop gains of an amplifier are $2000\angle 180^\circ$ and $1000\sqrt{2}\angle 135^\circ$ respectively. Calculate the corresponding gains of the amplifier when 0·2 per cent of the output is fed back in series with the input through a resistive network.

Let

$$A_1 = 2000\angle 180^\circ$$

Then

$$A_1' = \frac{2000\angle 180^\circ}{1 - 4\angle 180^\circ} \qquad \text{(from equation 9.3)}$$

$$= \frac{2000\angle 180^\circ}{1 - 4\,(\cos 180^\circ + \mathrm{j}\sin 180^\circ)}$$

$$= \frac{2000\angle 180^\circ}{1 + 4 - \mathrm{j}0}$$

$$\therefore \qquad A_1' = \frac{2000\angle 180^\circ}{5\angle 0^\circ} = 400\angle 180^\circ$$

Similarly, let $A_2 = 1000\sqrt{2}\angle 135^\circ$. From a knowledge of amplifier frequency response, this value will be recognised as occurring at the upper 3 dB point.

$$A_2' = \frac{1000\sqrt{2}\angle 135^\circ}{1 - 2\sqrt{2}\angle 135^\circ}$$

$$= \frac{1000\sqrt{2}\angle 135^\circ}{1 + 2 - \mathrm{j}2}$$

The reader may care to simplify the above expression by means of complex numbers and thereby show that the gain with feedback at this frequency is

$392\angle 168{\cdot}7°$. It is instructive to note from these answers that:

(a) the gain has fallen below its open-loop value at both frequencies, showing that feedback is negative
(b) the fall in gain of the feedback amplifier at the higher frequency is no longer 3 dB below that at mid-band. Neither has the phase shift through the amplifier deviated by 45° from its mid-frequency value of 180°.

It may be concluded, therefore, that another effect of negative feedback is to increase the operating bandwidth of an amplifier.

Stability of gain

Assuming that the phase shift through a single-stage amplifier is 180° over most of its operating frequency range, and that the feedback circuit is resistive, equation 9.3 can be written as

$$A' = \frac{A\angle 180°}{1-\beta A\angle 180°}$$

$$\therefore \qquad |A'| = \frac{A}{1+\beta A} \tag{9.4}$$

which, being less than unity, shows that the feedback is always negative.

Differentiating with respect to A

$$\frac{dA'}{dA} = \frac{(1+\beta A) - A\beta}{(1+\beta A)^2}$$

$$= \frac{1}{(1+\beta A)^2}$$

Hence

$$\frac{dA'}{dA} = \frac{A}{(1+\beta A)} \times \frac{1}{A(1+\beta A)}$$

$$= \frac{A'}{A(1+\beta A)}$$

Thus, the fractional change in the gain of the amplifier with negative feedback may be expressed in terms of the original change as follows

$$\frac{dA'}{A'} = \frac{dA}{A} \times \frac{1}{(1+\beta A)} \tag{9.5}$$

If the open-loop gain A changes for any reason, the effect on A' is therefore reduced by a factor of $(1 + \beta A)$. Furthermore, if βA is designed to be much greater than unity, equation 9.4 shows that A' tends towards $1/\beta$. Under these conditions, the overall gain of the feedback amplifier, although much reduced, is no longer dependent on any term such as transistor parameters or component values which would normally affect the open-loop gain A.

Example 9.2 The gain of an amplifier is initially 2000 before 2 per cent of its output is applied as negative feedback. Calculate the percentage reduction in overall gain if the open-loop gain falls by 40 per cent because of a faulty component.

Initially,

$$A = 2000$$

$$\therefore \qquad A' = \frac{2000}{1 + 0{\cdot}02 \times 2000} = 48{\cdot}78$$

Finally, when A has fallen to 1200

$$A' = \frac{1200}{1 + 0{\cdot}02 \times 1200} = 48{\cdot}0$$

$$\therefore \qquad \text{Change in } A' = \frac{0{\cdot}78}{48{\cdot}78} \times 100 = 1{\cdot}6 \text{ per cent}$$

Thus the gain has been stabilised within a small percentage.

Example 9.3 A 3-stage amplifier with NFB is required to have an overall gain of 100. As the result of a fall in supply voltage, the open-loop gain of each stage is reduced by 10 per cent, but feedback is such that the overall gain drops by only 1 per cent. Calculate the value of the feedback factor required to produce this result. Also find the open-loop gain of the complete amplifier.

Let the gain per stage without NFB $= A$.

$\therefore$ Open-loop gain of complete amplifier is initially A^3.

With NFB

$$100 = \frac{A^3}{1 + A^3}$$

from which

$$\beta = \frac{A^3 - 100}{100\beta A^3} \qquad \text{(i)}$$

Finally, when overall gain has fallen by 1 per cent,

$$99 = \frac{(0{\cdot}9A)^3}{1 + \beta(0{\cdot}9A)^3}$$

Hence

$$\beta = \frac{(0{\cdot}9A)^3 - 99}{99(0{\cdot}9A)^3} \tag{ii}$$

Equating (i) and (ii)

$(A^3 - 100)\,99 \times 0{\cdot}729A^3 = (0{\cdot}729A^3 - 99)\,100A^3$

Regrouping terms

$$0{\cdot}729A^3(100 - 99) = 99 \times 100(1 - 0{\cdot}729)$$

$\therefore$ Open-loop gain of amplifier

$$= \frac{9900 \times 0{\cdot}271}{0{\cdot}729}$$

$$= 3680$$

Substituting for A^3 in equation (i)

$$\beta = \frac{3680 - 100}{100 \times 3680}$$

$$= \frac{358}{368} \times \frac{1}{100} \quad \text{or} \quad 0{\cdot}973 \text{ per cent}$$

THE NYQUIST CRITERION

It was seen in chapter 6 that the gain of an amplifier varies not only in magnitude but also in phase, as the frequency changes. This was illustrated in the graphs of figure 6.6, where gain and phase shift were plotted separately to a common base of frequency. There is, however, another method of presenting the same information, the results of which can then be adapted to include amplifiers with feedback. In an amplifier of this type, the signal v at its input terminals, is given by

$$v = v_i + v_{fb}$$

where v_{fb} = feedback voltage.

Therefore in general, the applied signal v_i is the phasor difference between v and $\beta v_0\angle\phi$ as shown in figure 9.5. Let v be represented by the unit phasor OP on the Argand diagram of figure 9.6.

As βA is usually greater than unity in practice, the feedback voltage derived from a resistive network when the amplifier is operating at mid-band frequencies may be shown as OQ where the coordinates of Q are $-\beta A$, j0. But as the frequency changes, reactive elements in the amplifier circuit cause the locus of Q to follow a 'spiral', starting at O when the frequency is zero. If, at a particular frequency, $v_{fb} = OQ_2$, it may readily be deduced that Q_2P

represents the applied signal v_i in magnitude and phase relative to v. As point Q_2 is outside the unit circle, constructed on P as its centre, $v_i > v$ confirming that under these conditions the feedback is negative. At points such as Q_1 and Q_3 where the locus of Q enters the unit circle, v_i and v are equal in magnitude showing that feedback is on the verge of becoming positive.

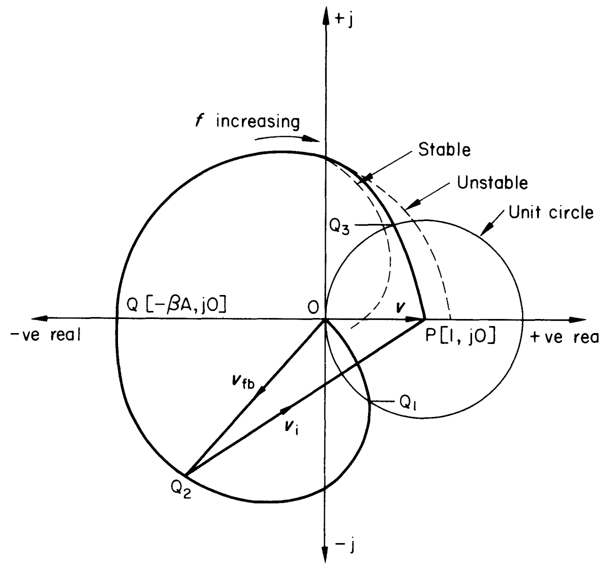


figure 9.6

This does not necessarily mean that oscillations are imminent, but the locus of Q having moved into the circle will undoubtedly cause amplifier gain to rise. If, at HF, the locus of Q eventually cuts the positive real axis, then the feedback voltage will be in phase with v. Self-oscillations will occur if this point of intersection is beyond P [1, j0] and the amplifier becomes unstable. The Nyquist criterion states that the amplifier remains stable so long as the locus of Q does not enclose the point P in figure 9.6.

Example 9.4 The voltage gain of a given amplifier at a certain frequency is $250\angle 55°$. Determine the overall gain with feedback if the feedback factor is adjusted to the following values:

(a) $\beta = 0{\cdot}004\angle 35°$
(b) $\beta = 0{\cdot}004\angle 0°$
(c) $\beta = 0{\cdot}004\angle -55°$

Comment briefly on the results obtained from these calculations.

(a) From equation 9.3,

$$A' = \frac{250\,\underline{/55^\circ}}{1 - 0{\cdot}004 \times 250\,\underline{/55^\circ + 35^\circ}}$$

$$= \frac{250\,\underline{/55^\circ}}{1 - 1\,\underline{/90^\circ}}$$

$$= \frac{250\,\underline{/55^\circ}}{1 - 1(\cos 90^\circ + \mathrm{j}\sin 90^\circ)}$$

$$= \frac{250\,\underline{/55^\circ}}{1 - (0 + \mathrm{j}1)} = \frac{250\,\underline{/55^\circ}}{1 - \mathrm{j}1}$$

∴ Gain with feedback

$$A' = \frac{250\,\underline{/55^\circ}}{\sqrt{2}\,\underline{/-45^\circ}} = 177\,\underline{/100^\circ}$$

As $\left|\dfrac{A'}{A}\right| < 1$ ∴ feedback is negative

(b) When $\beta = 0{\cdot}004\,\underline{/0^\circ}$

∴ $$A' = \frac{250\,\underline{/55^\circ}}{1 - 1\,\underline{/55^\circ}}$$

$$= \frac{250\,\underline{/55^\circ}}{1 - 1(0{\cdot}574 + \mathrm{j}0{\cdot}819)}$$

$$= \frac{250\,\underline{/55^\circ}}{0{\cdot}922\,\underline{/-62{\cdot}5^\circ}}$$

∴ $$A' = 271\,\underline{/117{\cdot}5^\circ}$$

This time $\left|\dfrac{A'}{A}\right| > 1$ showing that the feedback is positive.

(c) $$A' = \frac{250\,\underline{/55^\circ}}{1 - 1\,\underline{/0^\circ}} = \frac{250\,\underline{/55^\circ}}{1 - 1(\cos 0^\circ + \mathrm{j}\sin 0^\circ)} = \infty$$

There, the amplifier is unstable and begins to oscillate.

The analytical approach to the problem of instability in feedback amplifiers is to investigate further into equation 9.3. It was seen in the last example that it is quite possible for the expression $\beta A\,\underline{/\theta + \phi}$ to equal

$1\angle 0^\circ$. Whenever this happens, the amplifier must break into oscillation. There are two requirements to be fulfilled simultaneously:

1. $$\beta A = 1 \tag{9.6}$$

2. $$\angle \theta + \phi = 0^\circ \tag{9.7}$$

Equation 9.6 states that the amplifier built into the oscillator circuit must be capable of supplying the losses in the system so as to maintain the input signal at its original level. On the other hand, equation 9.7 shows that it is not sufficient merely to feed back the correct size of signal, but that it is also important that the signal fed back must be in phase with the signal that caused it in the first place.

(C) TRANSISTOR SINUSOIDAL OSCILLATORS

There are many different transistor oscillators, a few representative types of which will now be described. It should be noted that each consists of two basic components:

(a) an amplifier to provide the gain necessary to offset attenuation in the feedback loop
(b) a frequency-defining element that is designed to satisfy the phase requirement.

The parallel tuned circuit, consisting of a high-Q coil and a variable capacitor, has certain attractive advantages when used as the frequency-determining network. Its construction is simple and circuit losses are small. Furthermore, it can discriminate against unwanted harmonics and thereby produce an output waveform that is reasonably sinusoidal. If necessary, the capacitor may be arranged to have a negative temperature coefficient (that is, one in which capacitance falls as the temperature rises) to eliminate any frequency drift caused by the inherent positive temperature coefficient of the inductance. It has already been explained in chapter 3 that if the operating value of Q can be preserved at a high value, the resonant frequency calculated from

$$f_0 = \frac{1}{2\pi\sqrt{(LC)}} \tag{9.8}$$

is fairly accurate even though this formula is strictly true only for a series resonant circuit. Indeed, it can be verified experimentally that the frequency of an oscillator using a parallel LC circuit is not far removed from that given by the previous equation.

Tuned-collector oscillator

Figure 9.7 shows the circuit based on an amplifier with three-resistor bias.

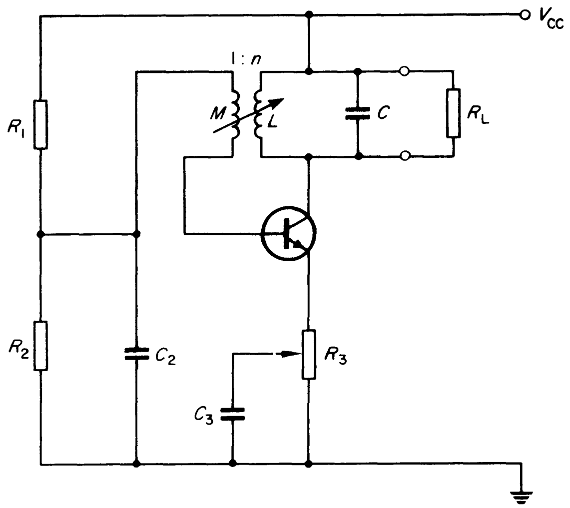


figure 9.7

The mere operation of switching-on the supply produces a transient in the collector load from which the parallel-tuned circuit selects the fundamental component. By mutual coupling, a signal is then injected into the base, whence it returns to the collector circuit (after amplification) with the correct polarity to reinforce the original change. In this way, the amplitude of oscillations soon builds up to a value that is more than sufficient to supply the resistive losses in the system. It will be seen later that the oscillator has the ability to regulate its own output automatically and thereby to reach a steady state.

Certain other features of this interesting circuit require explanation before it is analysed in some detail. The emitter resistor R_3 is shown partly decoupled by C_3 thereby introducing an adjustable amount of current NFB, so that not only are the gain and amplitude of output waveform stabilised, but also any harmonic content is simultaneously reduced. Voltage is fed back from the collector circuit into the input by means of mutual inductance M. The value chosen for capacitor C_2 ensures that it presents a low reactance at the operating frequency and thereby connects the feedback voltage between base and emitter without interfering with the d.c. bias conditions. If there is any possibility of detuning the LC circuit by stray capacitance either in the transistor or from the next stage, the collector may be connected to a tapping point on the coil. (See note on tapped coils in the tuned amplifier at the end of chapter 8.)

Neglecting h_{oe} and h_{re}, a simplified equivalent circuit of the oscillator can be drawn as in figure 9.8.

As for all other oscillators using a parallel-tuned circuit as the frequency-defining element, the frequency of oscillation may be calculated with reasonable accuracy from equation 9.8.

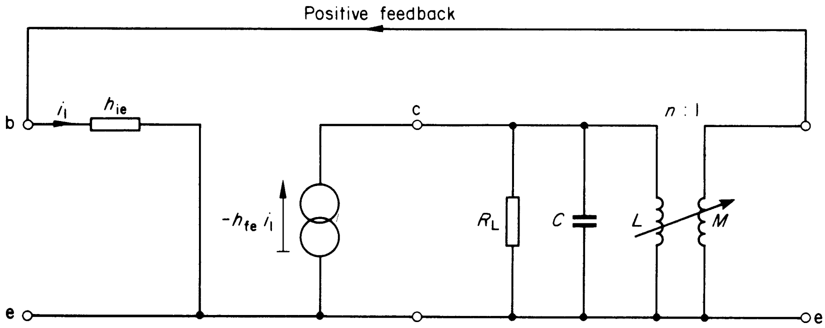


figure 9.8

As the reflected resistance across the primary of the air-cored transformer is n^2h_{ie}

$\therefore$ Total conductance across current generator

$$= \frac{1}{R_L} + \frac{1}{n^2h_{ie}}$$

Voltage across transformer primary

$$= -\frac{h_{fe}\,i_1}{\dfrac{1}{R_L} + \dfrac{1}{n^2h_{ie}}}$$

Hence, the current flowing in h_{ie} due to transformer action is

$$\frac{1}{nh_{ie}}\left(\frac{h_{fe}\,i_1}{\dfrac{1}{R_L} + \dfrac{1}{n^2h_{ie}}}\right)$$

(Note that the negative sign has been omitted from the last equation.) By design, the feedback connections must be made in such a way that phase reversal in the transistor is neutralised by a further 180° of phase shift between primary and secondary of the transformer. Thus the phase requirement of equation 9.7 is fulfilled. The other condition for oscillation, as stated in equation 9.6, is satisfied only if the current fed back into the base is numerically equal to i_1

$$\therefore \qquad \frac{h_{fe}\,i_1}{nh_{ie}G_L + 1/n} = i_1$$

Hence

$$h_{fe} = nh_{ie}G_L + 1/n \tag{9.9}$$

where G_L is the conductance of the load and the last term $1/n$ may be negligible.

This is called the 'maintenance equation' of the oscillator because it gives the minimum value of current gain h_{fe} that is necessary for continuous oscillation. It may well be that h_{fe} is somewhat larger than that indicated by equation 9.9 so as to make certain that oscillations start when the supply is first switched on. Subsequently, due to a combination of the curvature of transistor characteristics, auto-bias and saturation effects, the operating point automatically settles down to a steady state where equation 9.6 is exactly balanced. If the loaded Q-factor can be kept at a high value, it allows any change of phase due to stray capacitance in the system to be offset by minimal self-adjustment of f_0. It is also an advantage to select the largest practicable value for the transformer step-down ratio n. The reflected resistance term n^2h_{ie} is then so high that the majority of the oscillatory power is developed in the load R_L rather than in h_{ie}.

Example 9.5 A tuned-collector oscillator is to be constructed in which the air-cored transformer has a step-down ratio of 20 : 1. If the load resistance is 2·8 kΩ and h_{ie} of the transistor is 1·4 kΩ, calculate the minimum value of h_{fe} required.

From equation 9.9

$$(h_{fe})_{min} = 20 \times \frac{1{\cdot}4}{2{\cdot}8} + \frac{1}{20}$$

$$= 10{\cdot}05$$

It is obvious that the $1/n$ term is insignificant, particularly as the transistor chosen would probably have an h_{fe} of about 20.

The Colpitts oscillator

As with the previous oscillator, an amplifier having 3-resistor bias is taken for the basic circuit, as shown in figure 9.9.

Feedback from output to input is provided by a parallel resonant circuit comprising L, C_1 and C_2. The capacitor C_4 blocks the d.c. path from collector to base but is chosen to have negligible reactance at the oscillator frequency. The capacitance which causes L to resonate and thereby determines the oscillator frequency is due to C_1 and C_2 in series. It can be proved that in order to satisfy the phase requirement for oscillation, the pure reactances connected to the emitter must be of the same kind. In the Colpitts oscillator they are both capacitances, having a common connection indirectly with the emitter via the low reactance of C_3 in figure 9.9. Experience shows that C_2 should be much greater than C_1 as quoted in the next example.

When the supply is switched on, a transient once more energises the tuned circuit. Normally resistive losses in the circuit would damp the oscillations so that they decay exponentially to zero, but instead they are maintained by the

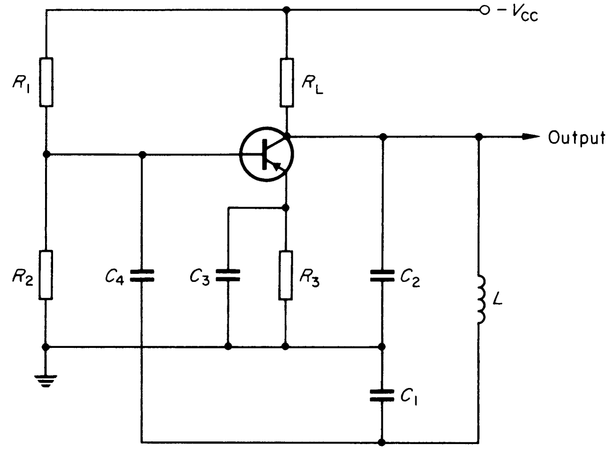


figure 9.9

transistor which converts power from the d.c. supply into oscillatory power at the operating frequency.

The feedback network is re-drawn in figure 9.10 together with its phase diagram at the oscillator frequency.

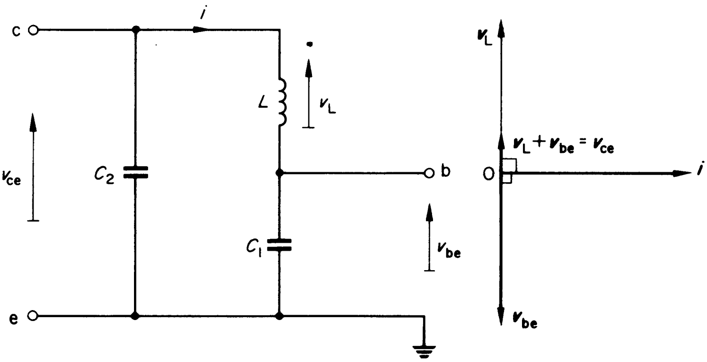


figure 9.10

It will be seen that so long as X_L exceeds the reactance of C_1, the two voltages v_{be} and v_{ce} are in anti-phase. This means that when the collector potential is going through a positive half-cycle, the base potential is being driven negatively. The resultant phase reversal through the transistor reinforces the initial change at the collector by positive feedback and oscillations can be maintained.

Example 9.6 In the Colpitts oscillator of figure 9.9, $C_1 = 100\,\text{pF}$, $C_2 = 0{\cdot}01\,\mu\text{F}$ and $L = 10\,\text{mH}$. Estimate the frequency of oscillation if the circuit components whose values are not specified are to be neglected.

Since the frequency-determining element in this case consists of L tuned by C_1 and C_2 in series,

$\therefore$ Effective capacitance

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

$$\triangleq C_1$$

because C_2 is so much greater (100 times) than C_1.

Frequency of oscillation, from equation 9.8, is given by

$$f_0 = \frac{1}{2\pi\surd(LC)} = \frac{10^6}{2\pi\surd(0{\cdot}01 \times 100)} \text{ Hz}$$

$$= 159 \text{ kHz}$$

The Hartley oscillator

This is the dual of the Colpitts oscillator just described in that the feedback circuit has the inductance and capacitive components interchanged. This requires the coil to be divided into two portions as shown in figure 9.11.

A positive feedback path from collector to base is now provided by the components $L_1 L_2$ and the tuning capacitor C. By a method similar to that used for the Colpitts oscillator, with $1/j\omega C_1$ and $1/j\omega C_2$ replaced by $j\omega L_1$ and $j\omega L_2$ respectively, it can be verified that the frequency of oscillation is given by:

$$f_0 = \frac{1}{2\pi\surd[(L_1 + L_2)C]} \tag{9.10}$$

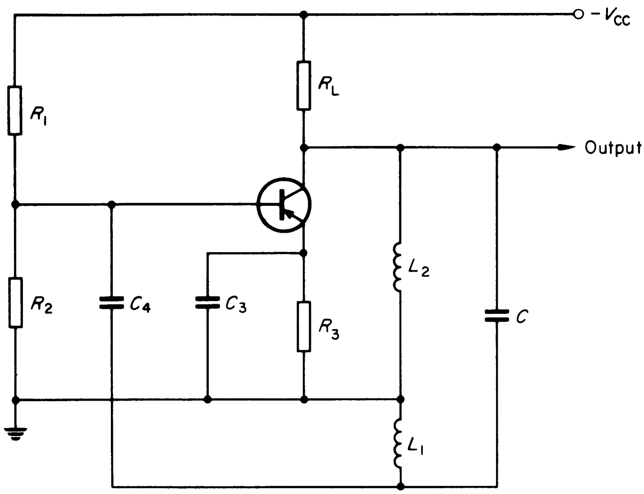


figure 9.11

that is, C tuned to resonance by L_1 and L_2 in series. This result, of course, assumes that the two parts of the coil are magnetically shielded from each other so as to eliminate any mutual inductance. As always, the two reactances connected to the emitter are of the same sign. (C)

The Wien bridge oscillator

In recent years there has been a movement away from the LC circuit as a frequency-defining element in oscillators. The inclusion of a coil has frequently presented problems in design, and, with the advent of integrated circuits, inductance is a quantity not readily available in practice. There are, however, certain oscillators that avoid the use of inductance altogether and instead employ RC circuits to fulfil the phase requirements. The Wien bridge oscillator is a case in point, and is often manufactured as a signal generator having a frequency range that is typically 15 Hz–1 MHz. The basic circuit is shown in figure 9.12.

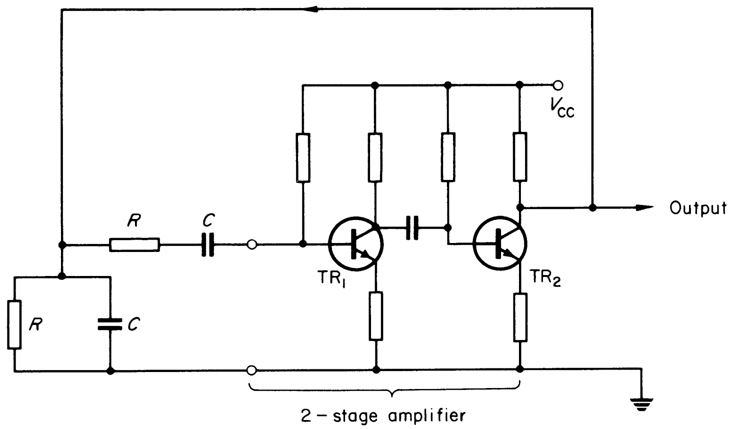


figure 9.12

Current output from the collector of TR_2 is fed back to the base of TR_1 through a series/parallel arrangement of resistance and capacitance. As overall current gain is required to have only a relatively small value, the emitter resistors are not decoupled. The resultant current NFB is therefore a means of ensuring that the output waveform is sinusoidal. The essential connections may be reduced to the block diagram of figure 9.13.

The impedance Z_1 comprises a resistor R and capacitor C in series, whereas in Z_2 there is a resistor R in parallel with capacitor C. Thus Z_1 and Z_2 combine to form an attenuator by which only a fraction of the feedback current i arrives at the amplifier input. Neglecting the input impedance of the

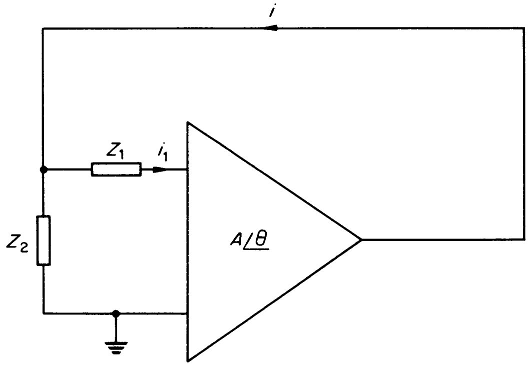


figure 9.13

amplifier, on the assumption that it is small compared with Z_1, the current i_1 can be calculated in terms of i.

From figure 9.13

$$i_1 = i\,\frac{Z_2}{Z_1 + Z_2} \qquad \text{by inverse proportions}$$

where

$$Z_1 = R + \frac{1}{\mathrm{j}\omega C}$$

and

$$Z_2 = \frac{R}{1 + \mathrm{j}\omega CR}$$

$$\therefore \qquad i_1 = \left(\frac{\dfrac{R}{1 + \mathrm{j}\omega CR}}{R + \dfrac{1}{\mathrm{j}\omega C} + \dfrac{R}{1 + \mathrm{j}\omega CR}} \right) i \qquad \text{(i)}$$

For continuous oscillation, $A\underline{/\theta}\, i_1 = i$; therefore, amplifier current gain, $A\underline{/\theta} = i/i_1$.

From equation (i)

$$\frac{i}{i_1} = \left(R + \frac{1}{\mathrm{j}\omega C} + \frac{R}{1 + \mathrm{j}\omega CR}\right)\left(\frac{1 + \mathrm{j}\omega CR}{R}\right)$$

$$= 1 + \mathrm{j}\omega CR + \frac{1 + \mathrm{j}\omega CR}{\mathrm{j}\omega CR} + 1$$

that is

$$A\underline{/\theta} = 3 + \mathrm{j}\omega CR - \frac{\mathrm{j}}{\omega CR} \qquad \text{(ii)}$$

With a two-stage amplifier, the operative value of θ must be zero so that equation (ii) becomes

$$A\underline{/0^\circ} = A + \mathrm{j}0 = 3 + \mathrm{j}\omega CR - \frac{\mathrm{j}}{\omega CR}$$

Equating real terms,

$$A = 3 \tag{9.11}$$

which shows the minimum value of amplifier current gain necessary to maintain continuous oscillation.

Equating imaginary terms,

$$0 = \omega CR - \frac{1}{\omega CR}$$

$$\therefore \qquad \omega = \frac{1}{CR} \quad \text{and} \quad f_0 = \frac{1}{2\pi CR} \tag{9.12}$$

Note that with the Wien bridge oscillator, the frequency is inversely proportional to C instead of to $\sqrt{C}$ as it was in the previous types of oscillator described. In fact, when the Wien bridge oscillator is used as a signal generator both resistance and capacitance are made variable. The method employed in practice is to select the capacitors in pairs by switching and then to make use of twin-ganged resistors that are continuously variable. In this way, capacitance is used to change the range of frequency, and resistance provides a means of fine adjustment within a particular range.

Example 9.7 In a Wien bridge oscillator, the resistors are variable between 100 Ω and 10 kΩ. Determine the values of capacitance which will enable the oscillator to be used between the limits of 100 Hz and 100 kHz in two non-overlapping ranges.

From equation 9.12

$$f_0 = \frac{1}{2\pi CR}$$

$$\therefore \qquad C = \frac{1}{2\pi f_0 R}$$

At the minimum setting of frequency, R would be at its maximum value of 10 kΩ.

$$\therefore \qquad C_1 = \frac{10^6}{2\pi \times 100 \times 10^4}\ \mu\mathrm{F} = 0{\cdot}159\ \mu\mathrm{F}$$

When R is adjusted to 100 Ω

$$f_0 = \frac{10^6}{2\pi \times 0{\cdot}159 \times 100} = 10\,\text{kHz}$$

Thus the first range of frequency, using two capacitors of 0·159 μF each, extends from 100 Hz to 10 kHz. Taking the latter figure as the minimum setting for the upper frequency range, corresponding to the maximum value of R, the other value of C can now be calculated.

$$C_2 = \frac{10^6}{2\pi \times 10^4 \times 10^4} = 0{\cdot}00159\ \mu\text{F}$$

With R reduced to its minimum value of 100 Ω

$$f_0 = \frac{10^6}{2\pi \times 0{\cdot}00159 \times 100} = 1\,\text{MHz}$$

as required.

RELAXATION OSCILLATORS

Each of the oscillator circuits so far described is designed to give a sinusoidal output. Indeed, it is one of the criteria for these oscillators that their waveform should contain no more than the minimum amount of harmonic distortion in spite of changes in operating conditions. There is, however, a need for another group of oscillators whose output is far from sinusoidal.

The saw-tooth generator

The time-base generator (mentioned in chapter 11 in connection with the cathode ray oscilloscope) with its 'saw-tooth' output waveform, is one of this group of *relaxation oscillators.* The basic principle of its operation is that a capacitor is charged from a d.c. supply until an electronic switch relaxes the condition in order to allow the capacitor to discharge partially.

Consider the simple neon time-base generator shown in figure 9.14.

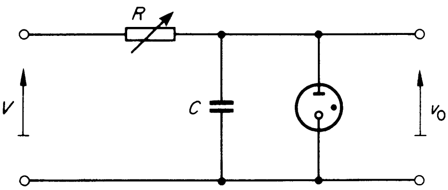


figure 9.14

When the supply is first switched on, the p.d. across C builds up to the striking voltage V_s of the neon in the tube. At this point, the resistance of the ionised gas is so small that the capacitor is virtually short-circuited and therefore discharges rapidly. But, in so doing, the p.d. between its plates tends to fall below V_m which is the voltage required to maintain ionisation in the neon tube. Thus the charging process can recommence as soon as the short-circuit across C is removed. The cycle then repeats itself at a frequency that can be estimated. It is generally assumed that negligible time is taken for the partial discharge that causes flyback.

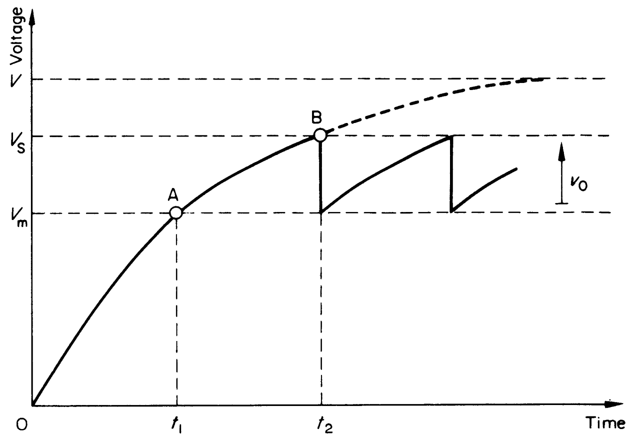


figure 9.15

Example 9.8 A time-base circuit of the type shown in figure 9.14 is connected to a 300 V supply. The capacitor of 0·05 μF is charged via a resistor which can be varied from 22 kΩ to 220 kΩ. If the striking voltage and maintaining (or 'extinguishing') voltage for the neon tube are 150 V and 120 V respectively, find from first principles the range of repetition frequencies available from the time-base generator.

In chapter 1, equation 1.13 stated that

$$v_C = V[1 - \exp(-t/CR)]$$

This may be translated into the terms used on the graph of figure 9.15 so that it becomes: Instantaneous voltage across C

$$= V[1 - \exp(-t/CR)]$$

At the point A

$$V_m = V[1 - \exp(-t_1/CR)]$$

$$\therefore \quad V - V_m = V \exp(-t_1/CR) \qquad \text{(i)}$$

Similarly, at the point B

$$V_S = V[1 - \exp(-t_2/CR)]$$

$$\therefore \qquad V - V_S = V\exp(-t_2/CR) \qquad \text{(ii)}$$

Dividing (i) by (ii)

$$\frac{V - V_m}{V - V_S} = \exp\,[(t_2 - t_1)/CR]$$

Taking natural logarithms (that is, to base e)

$$\ln\left(\frac{V - V_m}{V - V_S}\right) = \frac{t_2 - t_1}{CR}$$

But $t_2 - t_1$ is the periodic time T of the sawtooth waveform

$$\therefore \qquad T = CR\,.\ln\left(\frac{V - V_m}{V - V_S}\right) \text{ second} \qquad (9.13)$$

from which the repetition frequency $f = 1/T$.

Substituting the values given when $R = 22\ \text{k}\Omega$

$$T = \frac{0{\cdot}05}{10^6} \times 22 \times 10^3 \ln\left(\frac{300 - 120}{300 - 150}\right) \text{ second}$$

$$= \frac{1{\cdot}1}{10^3} \ln\frac{6}{5} = 0{\cdot}2 \text{ millisecond}$$

$\therefore$ Maximum repetition frequency, $f_1 = 5$ kHz.

With the variable resistor at its maximum value, the rate of increase of voltage across C is reduced. Consequently the repetition frequency is also reduced in the same proportion; that is

Minimum repetition frequency, $f_2 = 500$ Hz.

The astable multivibrator

One other important member of the relaxation oscillator family is the *astable* (or *free-running*) *multivibrator* now to be described. Imagine two transistors connected back-to-back as shown in figure 9.16.

At the instant of switching on, both transistors try to conduct; but, because there will inevitably be inequalities in the two circuits, suppose that TR_1 carries more collector current than TR_2. The fall in collector potential of TR_1 is transmitted by C_1 to the base of TR_2 which is consequently driven towards cut-off. In so doing, the collector of TR_2 rises up to V_{CC} at a rate governed by the time constant C_2R_2 taking the base of TR_1 with it, thus

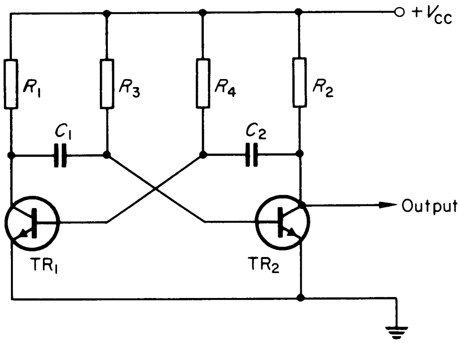


figure 9.16

reinforcing the original change in TR_1 by positive feedback. A condition is rapidly achieved in which TR_1 is ON (saturated) and TR_2 is OFF. The graph of base and collector potentials, plotted to a base of Time in figure 9.17 illustrates the changes that subsequently occur (with the relevant time constants) as TR_1 and TR_2 conduct alternately.

TR_2 will remain in the OFF state only so long as its base is held at a negative potential with respect to earth by means of the charge stored in C_1. But whenever TR_1 is ON there is a leakage path for this charge round the loop which includes C_1 and R_3. Therefore, as the base potential of TR_2 is allowed to rise through zero, TR_2 begins to conduct, and its own collector potential falls because of the voltage drop in R_2. From this time onwards, the cycle of operations already described for TR_1 is repeated, with the result that a relatively square-wave output is generated at one of the collectors. When either transistor is switched from one state to the other, the changes in collector potential are so rapid that the charge in the capacitor connected to it is temporarily unchanged. Alternatively, the leading edge of the square waveform at the collector may be regarded as such a high-frequency component that the reactance of the coupling capacitors is negligible. For both these reasons, the opposite base also receives the leading edge of the pulse whose magnitude should theoretically be $\pm V_{CC}$. It should be noted, however, that a forward biased emitter-base junction has a low resistance compared with R_3 and R_4, so that, in the actual circuit, the base potential can never rise much above earth. This process is known as *diode clamping* and results in the very small positive spike shown in the graph of base potential in figure 9.17.

The half-period T can be calculated with reasonable accuracy if the following assumptions are made:

1. Changes in collector potential are transmitted without attenuation to the opposite base.

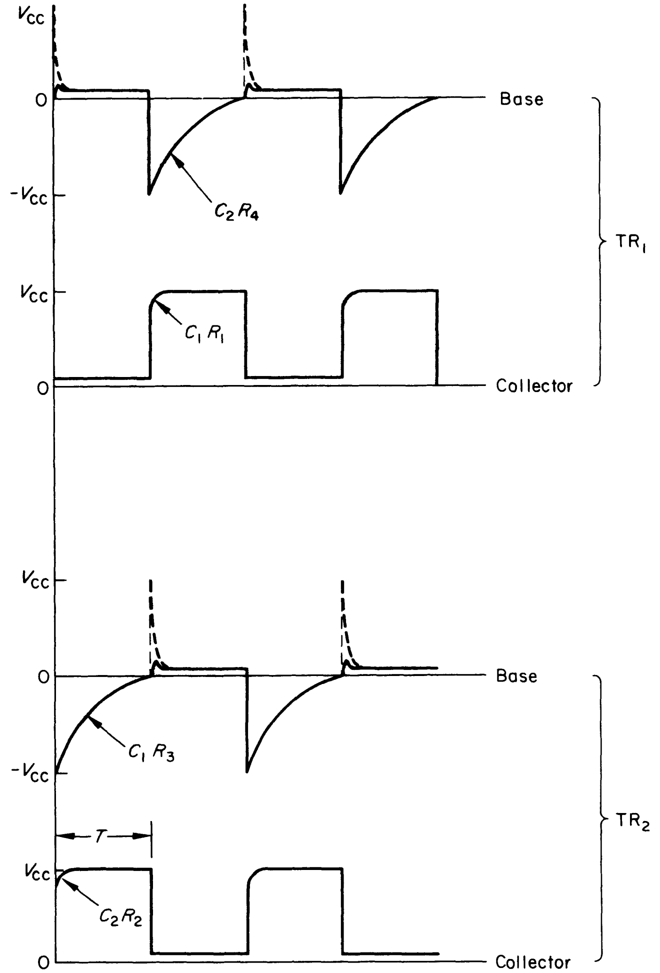


figure 9.17

2. Either transistor is turned ON as soon as its base potential reaches zero.
3. No allowance is made for internal voltage drop in the transistors; that is, a base, having been driven down to $-V_{CC}$, recovers towards $+V_{CC}$.

Consider the base of TR_2, its potential changing with time as shown in figure 9.18. Instantaneous potential at the point A, when expressed with reference to the false origin O′, can be written in the form

$$V_{CC} = 2V_{CC}[1 - \exp(-T/C_1R_3)]$$

$$\therefore \quad \exp(T/C_1R_3) = 2$$

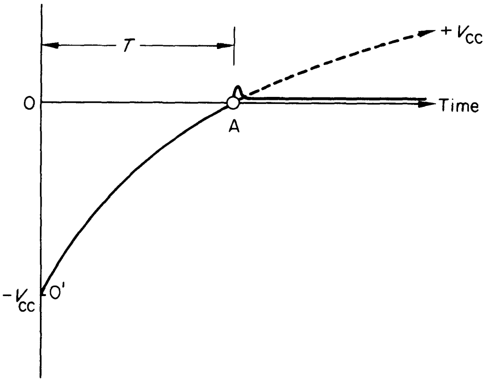


figure 9.18

Taking natural logarithms

$$\frac{T}{C_1R_3} = \ln 2 = 0{\cdot}693$$

$$\therefore \qquad T = 0{\cdot}7C_1R_3 \text{ approximately} \tag{9.14}$$

Thus, the repetition frequency of the multivibrator can be estimated from

$$f = \frac{1}{0{\cdot}7(C_1R_3 + C_2R_4)} \tag{9.15}$$

Example 9.9 The free-running multivibrator of figure 9.16 is required to give a symmetrical output having a periodic time of 1 millisecond.
If $R_3 = R_4 = 36$ kΩ, calculate the size of coupling capacitors required.

As the half-period $T = 0{\cdot}5$ ms, then from equation 9.14

$$C_1 = \frac{T}{0{\cdot}7R_3} = \frac{0{\cdot}5 \times 10^6}{10^3 \times 0{\cdot}7 \times 36 \times 10^3}\ \mu\text{F}$$

$$= 0{\cdot}01985\ \mu\text{F}$$

For a symmetrical output, the two coupling capacitors would be $0{\cdot}02\ \mu$F each.

MISCELLANEOUS EXAMPLES

Example 9.10 By means of block diagrams, illustrate the difference between *series current* and *series voltage* negative feedback. For each type, give the equation for the feedback factor.

In the circuits of figure 9.19, calculate R given that the feedback factor is to be 2 per cent.

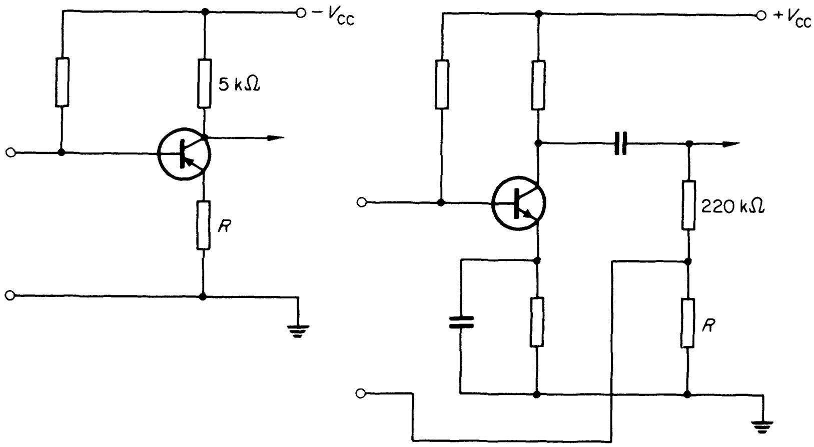


figure 9.19

[100 Ω; 4·5 kΩ]

Example 9.11 An amplifier has an open-loop gain of 20 000 and 0·02 of its output voltage is fed back in series opposition to the input signal. If changes in the amplifier cause the open-loop gain to fall by 25 per cent, find the limits of the gain in the feedback amplifier.

[49·875; 49·834]

Example 9.12 The voltage gain of an amplifier is to be reduced from 33·6 dB to 23 dB by the application of negative feedback. Calculate the percentage of feedback required.

[5%]

Example 9.13 Discuss the advantages and disadvantages of applying negative feedback to a transistor amplifier.

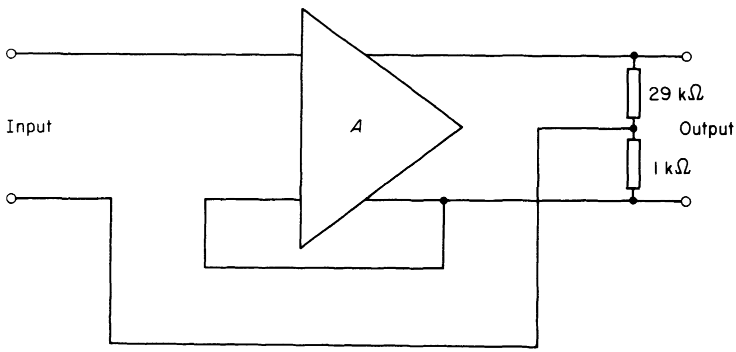


figure 9.20

An amplifier, of open-loop gain 2000 and output resistance 15 kΩ, has a phase shift of 180° between its input and output terminals in the operating range of frequency. A feedback circuit is then added as shown in figure 9.20. Comment on the component values used in the feedback network and calculate the resultant amplifier gain.

Note: It must be assumed that the output resistance quoted is the collector load.

[$30\underline{/91{\cdot}5^\circ}$]

(C) ***Example 9.14*** Figure 9.21 shows a tuned circuit that is to be used in conjunction with a suitably biased transistor to produce sinusoidal oscillations.

To which transistor electrodes should points X, Y and Z be effectively connected (either directly or through suitable capacitors of negligible reactance)?

Sketch a typical practical circuit for the oscillator and briefly explain its operation.

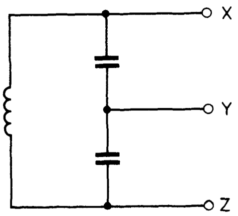


figure 9.21

(CGLI Principles C, 1969)

Example 9.15 Sketch a typical circuit diagram for a Hartley-type oscillator using a transistor.

Explain the operation of the circuit stating quite clearly the purpose of each component in the circuit.

(CGLI Principles C, 1971)

Example 9.16 The coil in the tuned circuit of a Hartley oscillator has a tapping point connected indirectly to the emitter in such a way that the two portions have self-inductances of 190 μH and 2·2 mH. If the mutual inductance between them is 55 μH and the tuning capacitor is adjusted to 2400 pF, calculate the oscillator frequency. (C)

[65 kHz]

Example 9.17 Derive an expression for the frequency of oscillation of a Wien bridge oscillator, stating any assumptions made. What value of capacitors would be required for a frequency of 15·9 kHz if the resistors were 100 kΩ each?

[100 pF]

Example 9.18 The RC coupling components in an astable multivibrator are as follows:

$$R_3 = 2R_4 = 22\ \mathrm{k}\Omega$$

$$C_1 = C_2 = 0{\cdot}5\ \mu\mathrm{F}$$

These components occupy positions as in figure 9.16. Show that the mark/space ratio of the output waveform is 2 : 1 and estimate its repetition frequency.

[86·5 Hz]

10 Alternating-current bridges

(B) A.C. bridges form the most convenient method of measuring effective resistance, self-inductance, mutual inductance, capacitance, and other associated quantities (such as the Q-factor of coils and loss angles of capacitors) at frequencies up to the lower radio-frequency range. A great many a.c. bridges are in use, dependent upon the measurements to be made and the equipment available for measuring, but they all have a common origin in the d.c. Wheatstone bridge. A.C. bridges, however, are more complicated to use and greater care is necessary to avoid errors, mainly because two quantities are usually required (for example, both the resistance and the inductance of a coil). Two variable standard components are therefore required in the bridge network, because two conditions must be satisfied simultaneously in order to achieve balance. It is an advantage to choose an a.c. bridge in which the two balance conditions are independent of each other, because then a true balance of the bridge can be rapidly obtained by alternate adjustments of the two variable standard components. The bridges dealt with in this chapter are of this nature.

Other complications in a.c. bridge operation arise from stray electric and stray magnetic fields, which cause stray currents to flow in certain arms of the bridge and thereby adversely affect the true balance of the bridge. Because of this, the use of a.c. bridges in their simplest forms is restricted to the frequency range up to a few kilohertz; but, within this range, care in the layout of bridge components enables good accuracy to be attained, if good quality standard components are used. For higher frequencies, additional measuring techniques and the use of screened leads and components are necessary, but a discussion of these is beyond the range of this chapter.

BRIDGES FOR MEASUREMENT OF INDUCTANCE AT LOW FREQUENCY

Consider first the normal Wheatstone bridge network of resistances P, Q, R and S, connected to an alternating source, and with the d.c. galvanometer replaced by an a.c. null detector D (such as a vibration galvanometer or a pair of headphones). The circuit is shown in figure 10.1a.

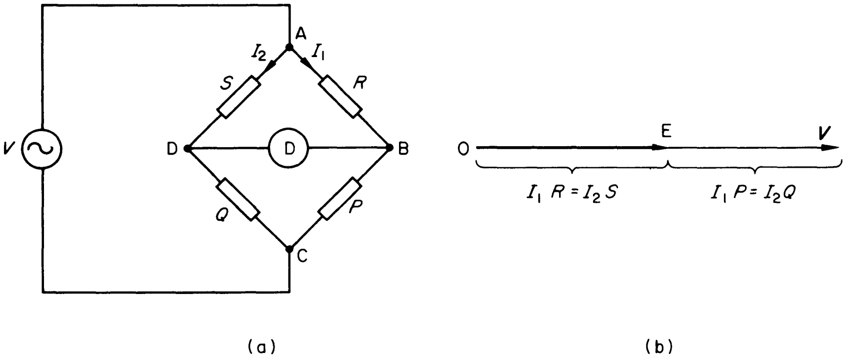


figure 10.1

When the bridge is balanced, no current flows through the detector. The voltage drops I_1R and I_2S are equal in magnitude and phase, as are the voltage drops I_1P and I_2Q. The phasor diagram for this condition is shown in figure 10.1b, where the supply voltage V is taken as reference phasor V. The sum of the voltage drops I_1R and I_1P equal the supply voltage V. The sum of the voltage drops I_2S and I_2Q also equal the supply voltage V. Note that the point E on the phasor diagram represents the potential of the two points B and D on the bridge, because the potentials of these two points must be brought to coincidence for a null reading on the detector and hence for balance of the bridge.

At balance

$$I_1R = I_2S$$

$$I_1P = I_2Q$$

from which, by division

$$\frac{R}{P} = \frac{S}{Q} \tag{10.1}$$

For balance, the potentials of the points B and D must be equal at all instants, so the introduction of inductance or capacitance into one arm of the bridge necessitates the introduction of inductance or capacitance into other arms of the network. The particular bridge network chosen depends upon the magnitudes of the inductances or capacitances, the time constants of the impedances, and the equipment available for making a measurement.

Maxwell's inductance bridge

This bridge is suitable for the measurement of inductances of medium magnitude, the unknown inductance L being measured in terms of a standard variable inductance L_S. The bridge circuit is shown in figure 10.2a.

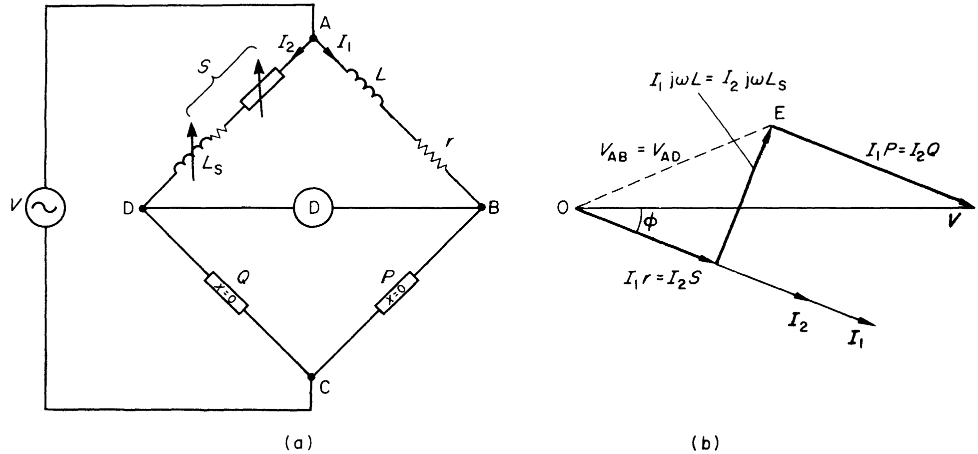


figure 10.2

P and Q are resistances used to form the ratio arms. The resistance S consists of the standard variable resistance plus the known resistance of the standard inductance. To obtain balance, it may be necessary to connect the resistance box in series with L instead of with L_S. The bridge is balanced by successive adjustments of L_S and of the resistance box, the bridge quickly converging to balance because the two adjustments are independent of each other. Then, at balance

$$I_1 Z_{AB} = I_2 Z_{AD}$$

and

$$I_1 Z_{BC} = I_2 Z_{DC}$$

which, by division, gives

$$\frac{Z_{AB}}{Z_{BC}} = \frac{Z_{AD}}{Z_{DC}}$$

Now the impedances Z_{AB}, Z_{AD}, Z_{BC}, and Z_{DC} are complex impedances, and must be written as such; that is

$$Z_{AB} = r + j\omega L$$

$$Z_{AD} = S + j\omega L_S$$

$$Z_{BC} = P$$

$$Z_{DC} = Q$$

$$\therefore \quad \frac{r + j\omega L}{P} = \frac{S + j\omega L_S}{Q}$$

$$\therefore \quad r + j\omega L = \frac{P}{Q}(S + j\omega L_S)$$

This means that the phasor quantity $r + j\omega L$ is equal to another phasor quantity $(P/Q)S + (P/Q)(j\omega L_S)$. For these two phasor quantities to be represented by the same phasor, clearly the real parts of the two phasor quantities must be equal, and the imaginary parts of the two phasor quantities must be equal. Hence, equating real terms

$$r = S\frac{P}{Q} \tag{10.2}$$

and equating imaginary terms

$$\omega L = \omega L_S \frac{P}{Q}$$

$$\therefore \qquad L = L_S \frac{P}{Q} \tag{10.3}$$

These balance equations for r and L both involve the ratio P/Q, this bridge being an example of a class of bridges known as *ratio bridges.*

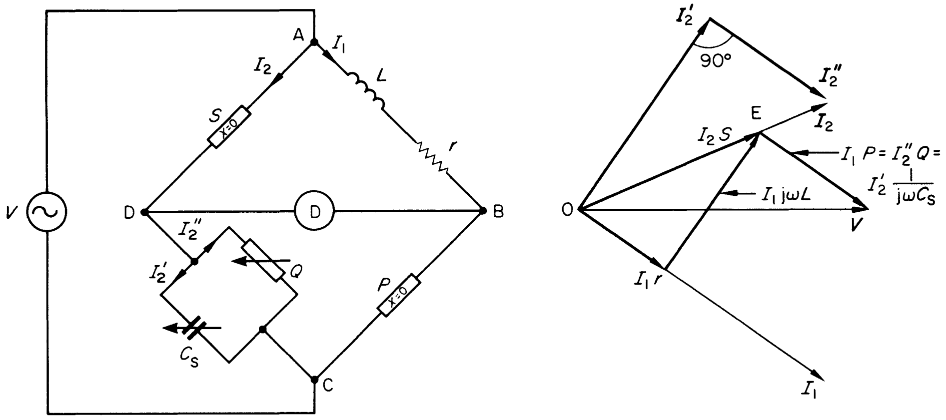
The balance equations are also independent of ω. Thus, Maxwell's inductance bridge will operate satisfactorily even if the supply frequency varies during the time that a measurement is being made.

The phasor diagram (not to scale) for the bridge is shown in figure 10.2b. Again, the supply voltage V is taken as the reference phasor V. The current I_1 lags V by the angle ϕ and produces a voltage drop I_1r in phase with the current, and a voltage drop $I_1j\omega L$ leading the current by 90°. The voltage drop I_1P is in phase with I_1 and is thus drawn parallel to I_1 so as to make the three voltage drops I_1r, $I_1j\omega L$ and I_1P sum to V. The potential of the point B on the bridge circuit is thus fixed on the phasor diagram at E. This also represents the potential of the point D on the bridge circuit. Hence, the phasor representing I_1P must also represent I_2Q. The current I_2 is thus in phase with I_1 though not necessarily equal to it. Finally, as the voltage drops I_2S and $I_2j\omega L_S$ are in quadrature and sum to point E on the phasor diagram, they must be represented by the same phasors as I_1r and $I_1j\omega L$ respectively.

Maxwell's LC bridge

A variable standard capacitor is perhaps a much more common component in a laboratory than a variable standard inductor, and may be used to measure the equivalent resistance and inductance of a coil with the bridge network shown in figure 10.3a.

The bridge is balanced by successive adjustments of the variable resistance Q and the variable capacitance C_S, the two adjustments again being independent of each other. C_S may conveniently consist of a variable decade capacitor in parallel with a variable air capacitor to give a fine adjustment of the capacitance. Z_{DC} consists of Q and C_S in parallel.



(a) (b)

figure 10.3

$$Z_{DC} = \frac{Q\dfrac{1}{j\omega C_s}}{Q + \dfrac{1}{j\omega C_s}}$$

$$= \frac{Q}{1 + j\omega C_s Q}$$

$\therefore$ At balance

$$\frac{S(1 + j\omega C_s Q)}{Q} = \frac{r + j\omega L}{P}$$

$$PS + PS\,j\omega C_s Q = Qr + Q\,j\omega L$$

Equating real terms

$$PS = Qr$$

$$\therefore \qquad r = \frac{PS}{Q} \tag{10.4}$$

Equating imaginary terms

$$PS\omega C_s Q = Q\omega L$$

$$\therefore \qquad L = PSC_s \tag{10.5}$$

These balance equations for r and L are both seen to involve the product PS of the resistance arms, this bridge being an example of a class of bridges known as *product bridges.*

The balance equations again are independent of ω, and the bridge can be used satisfactorily at frequencies up to about 3 kHz, at which point stray capacitances commence to introduce errors.

A useful application of the bridge is for the measurement of iron losses in an iron-cored inductor. The resistance r measured on the bridge is the effective resistance of the inductor, which consists of the ohmic resistance r_W of the winding plus an equivalent resistance r_i in which the iron losses are represented. First, r_W can be measured by d.c. methods, and then r is measured on the bridge. During this measurement an electronic voltmeter can be connected across P (it will have negligible effect on the bridge balance) to measure the current I_1 in the inductor. The total power loss in the inductor is then given by $I_1^2 r$ and the iron loss is given by $w_i = I_1^2(r - r_W)$.

The phasor diagram for the balanced bridge is shown in figure 10.3b. Again, the supply voltage V is taken as reference phasor V, and the voltage drops $I_1 r$, $I_1 \mathrm{j}\omega L$ and $I_1 P$ are drawn as for the Maxwell inductance bridge. Point E on the phasor diagram then represents the potentials of points B and D on the bridge network. Hence, the voltage drop $I_2 S$ is given by OE. The current I_2 is in phase with this voltage drop and is shown thus. This current I_2 divides at the junction of C_S and Q: I_2' flowing in C_S, and I_2'' in Q. The voltage drops in C_S, Q and P are all equal in both magnitude and phase, because B and D are both at the same potential and C is common to all three. Hence, the phasor representing $I_1 P$ also represents the voltage drops $I_2'' Q$ and $I_2'(1/\mathrm{j}\omega C_S)$. The currents I_2' and I_2'' can now be drawn on the phasor diagram, these currents summing to I_2, and being mutually in quadrature. From the tip of the I_2 phasor, the line of I_2'' is drawn as shown, in phase with and therefore parallel to the voltage drop $I_2'' Q$. From the origin O, a line is drawn, at 90° to the voltage drop $I_2'(1/\mathrm{j}\omega C_S)$ and thus perpendicular to the line of I_2''. The intersection of these two lines fixes the lengths of the phasors representing I_2' and I_2'' and $I_2' + I_2'' = I_2$ (phasorially). This completes the phasor diagram.

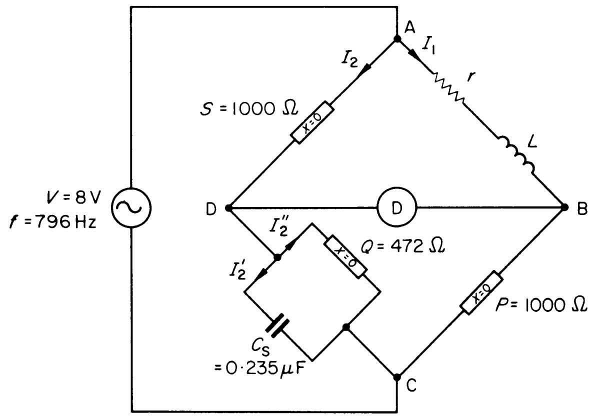


figure 10.4

Example 10.1 In a balanced Maxwell LC bridge network, AB is an unknown inductor of inductance L and effective series resistance r; BC and DA are non-inductive resistances of 1000 Ω, and DC consists of a resistance Q of 472 Ω in parallel with a capacitance C_S of 0·235 μF. A potential difference of 8 V at a frequency of 796 Hz is connected between the points A and C. Draw to scale a phasor diagram showing the currents and potential differences in the bridge, and from it determine the values of r and L. Obtain also the balance equations of the bridge in terms of r and L; and, hence, verify the values obtained from the phasor diagram.

Figure 10.4 shows the given arrangement. To draw the phasor diagram to scale, the complex value of the current I_2 is required, from which the voltage drop V_{AD} is calculated.

$$\omega = 2\pi f = 2\pi \times 796 = 5000$$

$$Z_{DC} = \frac{Q}{1 + j\omega C_S Q}$$

$$= \frac{472}{1 + j5000 \times 0{\cdot}235 \times 10^{-6} \times 472}$$

$$= \frac{472}{1 + j0{\cdot}554} = \frac{472(1 - j0{\cdot}554)}{1^2 + 0{\cdot}554^2}$$

$$= \frac{472(1 - j0{\cdot}554)}{1{\cdot}307} = 361 - j200\ \Omega$$

$$Z_{ADC} = 1361 - j200\ \Omega$$

$$\therefore \qquad I_2 = \frac{8 + j0}{1361 - j200} = \frac{8(1361 + j200)}{1361^2 + 200^2}$$

$$= \frac{8(1361 + j200)}{1{\cdot}893 \times 10^6} = (5{\cdot}75 + j0{\cdot}845)10^{-3}\ \text{A}$$

$$\therefore \qquad V_{AD} = I_2 S = 5{\cdot}75 + j0{\cdot}845\ \text{V}$$

Figure 10.5 shows the phasor diagram drawn to scales of 1 cm = 1 V and 0·5 cm = 1 mA. The supply voltage phasor V is first drawn, and the two components of V_{AD} are then measured off as shown, to fix the point E representing the potentials of points B and D on the bridge. The phasor I_1P is then drawn and measured, from which $I_1P = 2{\cdot}4$ V. As $P = 1000\ \Omega$, $I_1 = 2{\cdot}4$ mA. This is set off as shown, by drawing I_1 from the origin O parallel to the phasor I_1P. From E, a perpendicular to I_1 is drawn, meeting I_1 produced in F. Then OF $= I_1 r$ is measured, giving $I_1 r = 5{\cdot}02$ V.

$$\therefore \qquad r = \frac{I_1 r}{I_1} = \frac{5{\cdot}02}{2{\cdot}4 \times 10^{-3}} = 2092\ \Omega$$

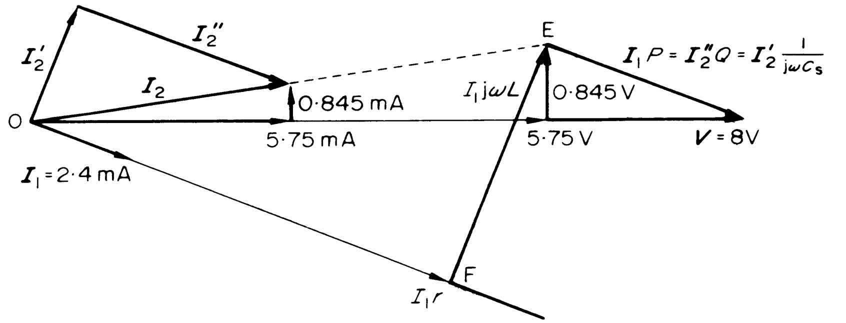


figure 10.5

EF $= I_1\omega L$ is then measured, giving $I_1\omega L = 2{\cdot}88$ V

$$\therefore \qquad \omega L = \frac{2{\cdot}88}{2{\cdot}4 \times 10^{-3}} = 1200\ \Omega$$

$$\therefore \qquad L = \frac{1200}{5000} = 0{\cdot}24\ \text{H}$$

The currents I_2', I_2'' and I_2 are drawn to the given scale on the phasor diagram by following the procedure already described.

The balance equations for the bridge have been previously obtained, giving

$$r = \frac{PS}{Q} \qquad \text{(equation 10.4)}$$

and

$$L = PSC_s \qquad \text{(equation 10.5)}$$

Substituting given values,

$$r = \frac{1000 \times 1000}{472} = 2119\ \Omega$$

$$L = 1000 \times 1000 \times 0{\cdot}235 \times 10^{-6} = 0{\cdot}235\ \text{H}$$

The results are seen to be in close agreement. Phasor diagrams to scale, of course, are not normally used in a.c. bridge calculations, because results are far more easily obtained by deriving the balance equations. To understand the operation of bridges, however, the phasor diagrams (not necessarily to scale) for all bridges studied should be drawn.

Hay's bridge

This bridge is most usefully used for the measurement of inductors of high value and with large time constants (that is, having relatively small resistance).

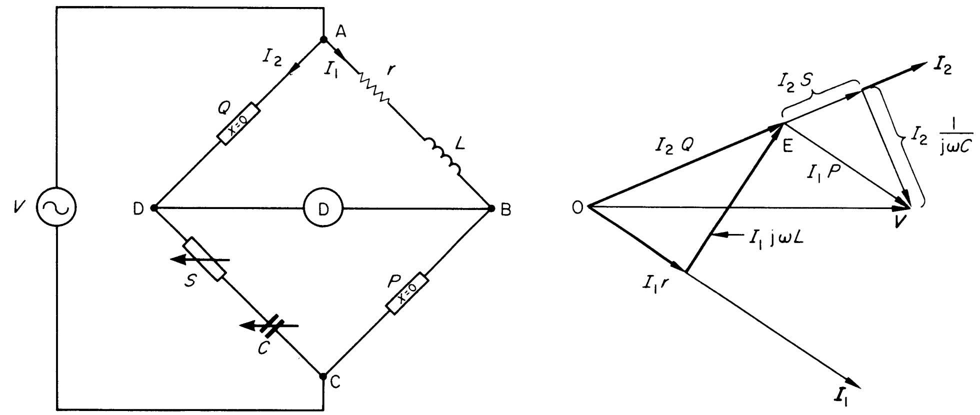


figure 10.6

Figure 10.6(a) shows the circuit diagram for the bridge, in which P and Q are fixed non-inductive resistances, and S and C are the variable components to bring the bridge to balance. At balance

$$\frac{r + j\omega L}{P} = \frac{Q}{S + \dfrac{1}{j\omega C}}$$

$$r + j\omega L = \frac{PQ\,j\omega C}{1 + j\omega CS}$$

$$= \frac{PQ\,j\omega C(1 - j\omega CS)}{1 + \omega^2C^2S^2}$$

$$= \frac{PQ\,j\omega C + \omega^2C^2SPQ}{1 + \omega^2C^2S^2}$$

Equating real parts,

$$r = \frac{\omega^2C^2PQS}{1 + \omega^2C^2S^2} \tag{10.6}$$

Equating imaginary parts,

$$L = \frac{CPQ}{1 + \omega^2C^2S^2} \tag{10.7}$$

From these equations, it is seen that the balance equations involve ω^2, and it would appear that the bridge is badly frequency dependent. The bridge, however, is used for inductors of large time constants. By combining equations 10.6 and 10.7, it can be seen that $\omega L/r = 1/\omega CS$. If $\omega L/r > 10$, then

$\omega CS < 0{\cdot}1$ and $\omega^2C^2S^2 < 0{\cdot}01$; so that, as far as the L balance is concerned, the bridge is only slightly dependent on frequency. Again, both equations involve both the variables C and S, a condition which would appear to make the bridge difficult to balance. For coils with large time constants, however, and where $\omega^2C^2S^2 \ll 1$, this is not a difficulty, the balance for L being then largely independent of that for r, and involving to any appreciable extent only one variable component, C.

The phasor diagram for the bridge is shown in figure 10.6b. The supply voltage V is again the reference phasor V, and the voltage drops I_1r, $I_1 \mathrm{j}\omega L$ and I_1P are drawn as in the previous diagrams. The point E again gives the potential of points B and D on the bridge, and OE gives the voltage drop I_2Q. This fixes the phase of the current I_2. Now the voltage drop $V_{BC} = I_1P$. This is also equal to V_{DC}, but V_{DC} has two components: I_2S in phase with I_2 and $I_2(1/\mathrm{j}\omega C)$ lagging I_2 by 90°. Both these voltage drops sum, phasorially, to I_1P and are shown thus on the phasor diagram.

Hay's bridge is useful for measuring the Q-factor of a coil, without necessarily measuring L and r separately

$$Q = \frac{\omega L}{r}$$

$$= \frac{1}{\omega CS} \quad \text{(as shown on page 308)} \qquad (10.8)$$

The Q-factor of the coil is thus given in terms of ω, C and S when the bridge is balanced.

Modified Hay's bridge

Hay's bridge is also useful, when suitably modified, for the measurement of incremental inductance, an important example being the inductance of a filter

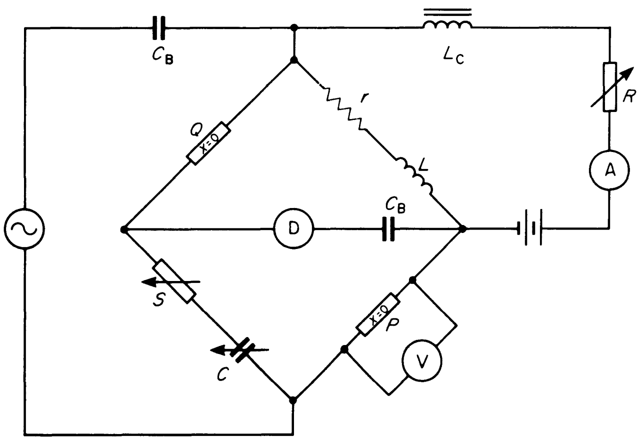


figure 10.7

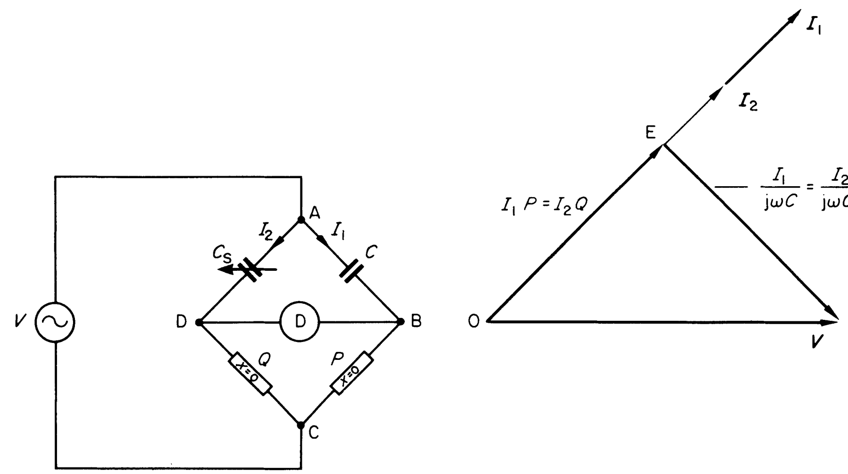
choke (as used in rectifier smoothing circuits) to ripple superimposed on direct current. The value of the inductance then depends upon both the magnitude of the direct current and the magnitude of the alternating ripple component. The modification required is shown in figure 10.7. The capacitors C_B prevent direct current flowing in the a.c. source and in the detector. L_C is an inductor large enough to prevent alternating current flowing in the circuit that comprises the d.c. source and the bridge resistance P.

The inductance L and effective resistance r of the filter choke is required for known values of direct and alternating currents. The direct current in the choke is measured by the ammeter A. The valve voltmeter connected across P enables the alternating current to be measured, because P is known.

BRIDGES FOR MEASUREMENT OF CAPACITANCE AT LOW FREQUENCY

These bridges are used for the measurement of capacitance and associated quantities, such as equivalent series and equivalent shunt resistance, dielectric loss and loss angle, and power factor. These quantities have already been discussed and defined in chapter 1. The capacitance referred to here need not necessarily be an actual capacitor, but may be an insulator, or a length of cable where the insulation of the cable is the dielectric whose properties are to be measured.

De Sauty's capacitance bridge



(a) (b)

figure 10.8

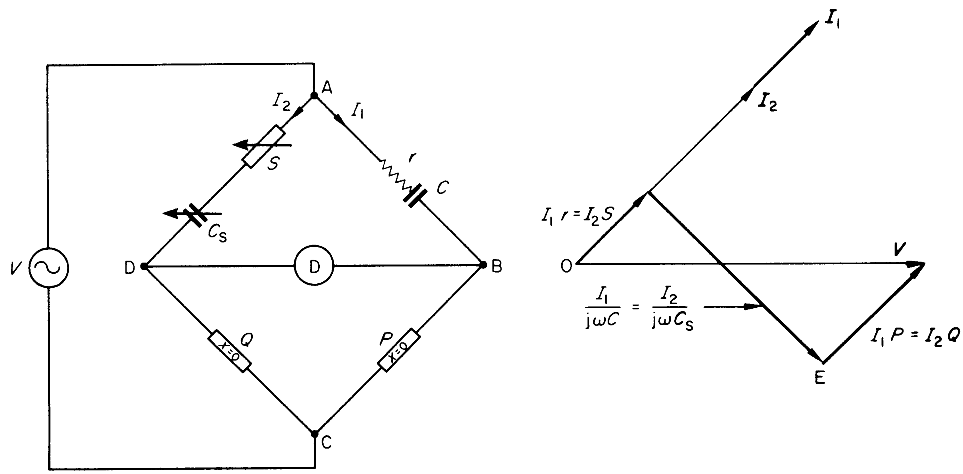
In its simplest form, this bridge is used to compare an unknown perfect capacitor C with a variable standard perfect capacitor C_S. P and Q in figure 10.8a form ratio arms.

At balance

$$\frac{1}{j\omega C_S Q} = \frac{1}{j\omega CP}$$

$$\therefore \qquad C = C_S \frac{Q}{P} \qquad (10.9)$$

If C is an imperfect capacitor, represented by its equivalent series circuit of C and r in series, then a variable standard resistance S must be inserted in series with C_S. Figure 10.9a shows the modified arrangement, and figure 10.9b the phasor diagram for the bridge at balance.



(a) (b)

figure 10.9

At balance

$$\frac{r + \dfrac{1}{j\omega C}}{P} = \frac{S + \dfrac{1}{j\omega C_S}}{Q}$$

$$Qr + \frac{Q}{j\omega C} = PS + \frac{P}{j\omega C_S}$$

Equating real terms,

$$Qr = PS$$

$$\therefore \qquad r = S \frac{P}{Q} \qquad (10.10)$$

Equating imaginary terms,

$$\frac{Q}{\omega C}=\frac{P}{\omega C_S}$$

$$\therefore \qquad C=C_S\frac{Q}{P} \tag{10.11}$$

Note that the effect of the ratio arms is inverted when calculating C.

A disadvantage of the De Sauty bridge is that the standard resistor S must be very small if the unknown capacitor is a low-loss type having a small equivalent series resistance. If a low resistance standard is not available, a high resistance standard may be connected in parallel with C_S. The balance conditions will then be different and will give the equivalent parallel resistance of the unknown if it is assumed to be represented by its equivalent parallel circuit.

Schering's bridge

This bridge avoids the need for having an inconveniently low series (or high parallel) resistance with C_S in the previous bridge by putting a variable standard capacitor C_1 in parallel with the resistance Q. C_S is then usually a fixed standard capacitor, and Q is arranged to be variable, the advantage being that Q is a variable standard of medium range. The arrangement is shown in figure 10.10a, and the phasor diagram at balance in figure 10.10b.

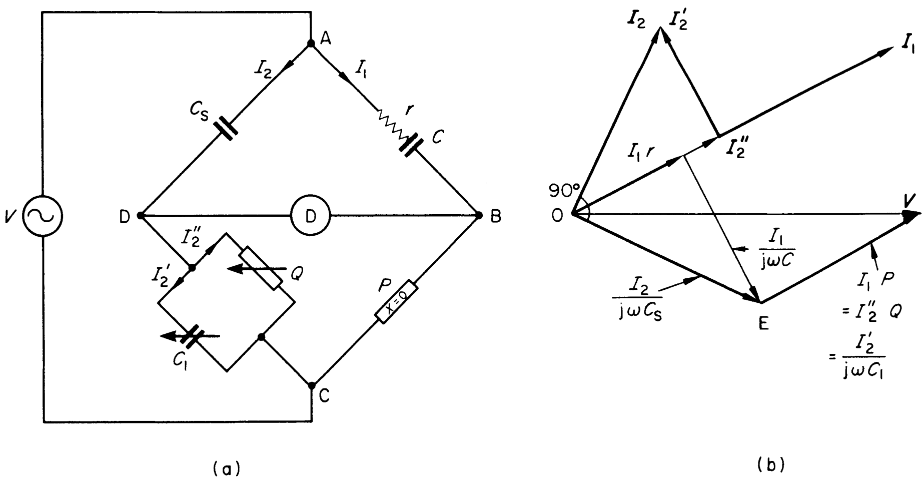


figure 10.10

At balance

$$\frac{r + \dfrac{1}{j\omega C}}{P} = \frac{1 + j\omega C_1 Q}{j\omega C_S Q}$$

$$j\omega C_S Q r + Q\frac{C_S}{C} = P + j\omega C_1 QP$$

Equating real parts

$$C = C_S\frac{Q}{P} \tag{10.12}$$

Equating imaginary parts

$$\omega C_S Q r = \omega C_1 QP$$

$$\therefore \qquad r = P\frac{C_1}{C_S} \tag{10.13}$$

The bridge is extensively used for the measurement of $\tan\delta$ of the unknown capacitor, where δ is the loss angle, and

$$\begin{aligned} \tan\delta &= \omega r C \\ &= \omega P\frac{C_1}{C_S}\cdot C_S\frac{Q}{P} \\ &= \omega C_1 Q \end{aligned} \tag{10.14}$$

Tan δ can thus be determined directly in terms of the supply frequency and the values of the standard components C_1 and Q at balance. Since δ is small, $\omega C_1 Q$ is very nearly the loss angle (radians) and the power factor.

The Schering bridge is also much used for measurements at high voltage, when the unknown capacitor C may be a short length of cable or an insulator. The low voltage value of $\tan\delta$ does not apply at high voltage, hence $\tan\delta$ must be measured at the voltage at which the cable or insulator is to be used. The standard capacitor C_S is a high-voltage capacitor with compressed gas as dielectric, and is loss free, with relatively low capacitance. The values of the impedances in arms BC and DC are chosen to be relatively low, while the impedances in arms AB and AD are very high in comparison. Thus, most of the supply voltage exists across arms AB and AD, while the voltages across arms BC and DC are quite small. These latter are the arms that contain the variable components, which, when point C is earthed, can be operated in safety. In addition, spark gaps are connected across arms BC and DC. These gaps are set to break down at a voltage of about 100 V, and give protection in the event of a breakdown of one of the high-voltage capacitors.

Example 10.2 A Schering bridge used for a test on a capacitor at 50 Hz is set up as follows: AB is the capacitor under test, of series resistance r and capacitance C; BC is a non-reactive resistor P of 500 Ω; CD is a non-reactive resistor Q in parallel with a capacitor C_1; and DA is a standard air capacitor C_s of 50 pF. Balance is obtained with $C_1 = 0{\cdot}224\,\mu$F and $Q = 340$ Ω. Determine the equivalent series resistance r, the equivalent series capacitance C and the loss angle of the test capacitor.

The equations for the required values have been derived on page 312.

$$r = P\frac{C_1}{C_s} = 500\frac{0{\cdot}224 \times 10^{-6}}{50 \times 10^{-12}}$$

$$= 2{\cdot}24\ \mathrm{M}\Omega$$

$$C = C_s\frac{Q}{P} = 50 \times 10^{-12}\frac{340}{500}$$

$$= 34\ \mathrm{pF}$$

$$\text{Loss angle} \simeq \tan\delta = \omega C_1 Q$$

$$= 2\pi \times 50 \times 0{\cdot}224 \times 10^{-6} \times 340$$

$$= 0{\cdot}024\ \text{radian}$$

$$= 1{\cdot}38°$$

(B)

(C) TRANSFORMER RATIO-ARM BRIDGES

The effects of stray electric fields limit the frequency above which the previous bridges can be satisfactorily operated in their simplest forms. For higher frequencies, additional measuring techniques are necessary, and leads and components must be screened. An alternative method of reducing these stray capacitance effects is available in the transformer ratio-arm bridge, in which the resistance ratio arms of the conventional four-arm bridge are replaced by a pair of inductively coupled windings wound together on a common core.

Simple TRA bridge

The arrangement in its simplest form is shown in figure 10.11. P and Q are two identical windings wound together on a toroidal core of high permeability. The currents flowing in these are such that the magnetising effect of P opposes that of Q. When the bridge is balanced, points B and D are at the same potential, so that equal currents flow in P and Q and thus produce no magnetisation of the core. Consequently, apart from a very small effect due to the resistances of the windings, there is no potential difference between C and B or between C and D. The unknown impedance Z is of the same type and magnitude as the standard impedance Z_S, and the whole of the supply voltage V appears across each of these impedances, thereby giving greater sensitivity than would be obtained using the same supply voltage and a conventional bridge. Any stray capacitance between D and C and between B and C is ineffective because at balance these points are at the same potential. Stray capacitance between A and C is also ineffective because it is merely connected across the supply.

Higher sensitivity TRA bridge

To increase sensitivity, the detector may be connected to a third winding on the core, as shown in figure 10.12. This figure also illustrates a useful feature of the bridge whereby a single impedance in a mesh-connected system of impedances may be measured without disconnecting it from the system.

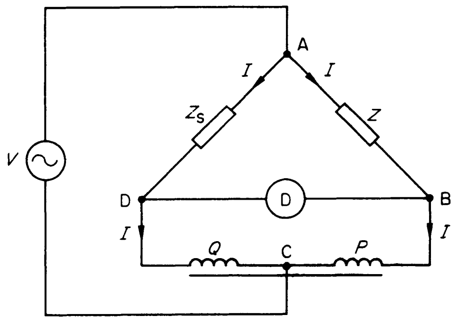


figure 10.11

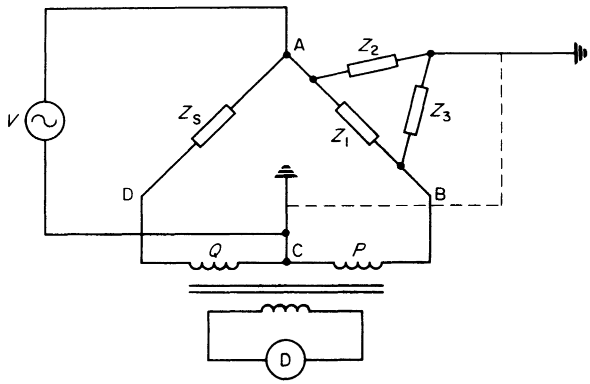


figure 10.12

Suppose it is required to measure Z_1 to the exclusion of the effects of Z_2 and Z_3. The junction of Z_2 and Z_3 is connected to earth, and point C on the bridge is also earthed. (Alternatively, these two points may be directly connected together.) Z_2 is then directly across the supply and does not affect the balance.

Z_3 is connected across winding P, which has no voltage across it at balance. Thus no current flows in Z_3, which also does not affect the balance. Z_1 alone is measured. An example of one such measurement is the determination of grid-to-anode capacitance of a triode valve in the presence of grid-to-cathode and anode-to-cathode capacitances.

Wider range TRA bridge

The range of the bridge can be extended by the use of adjustable ratio arms, shown in figure 10.13.

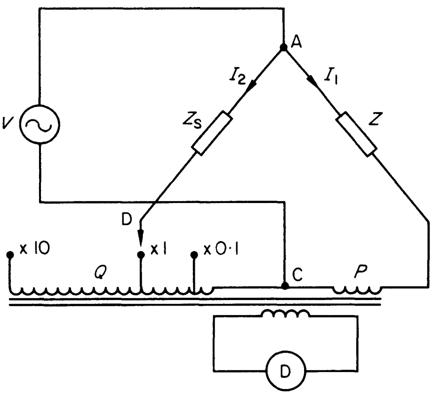


figure 10.13

At balance, zero detector deflection means that there is no flux in the core, and the ampere turns of P and Q are equal. Let the number of turns on P and Q be T_1 and T_2 respectively. Then, $I_1T_1 = I_2T_2$. Also, at balance, $I_1Z = I_2Z_S$. Combining these two equations

$$\frac{Z}{Z_S} = \frac{T_1}{T_2}$$

$$Z = \frac{T_1}{T_2} Z_S \tag{10.15}$$

Admittance bridge

In commercial transformer ratio-arm bridges, the standard impedance Z_S usually takes the form of a standard variable capacitance in parallel with a standard variable resistance, the unknown impedance Z being measured in the form of its equivalent parallel circuit. Such bridges are frequently referred to as admittance bridges, because they measure the equivalent parallel components of the unknown impedance. To measure inductance on such a

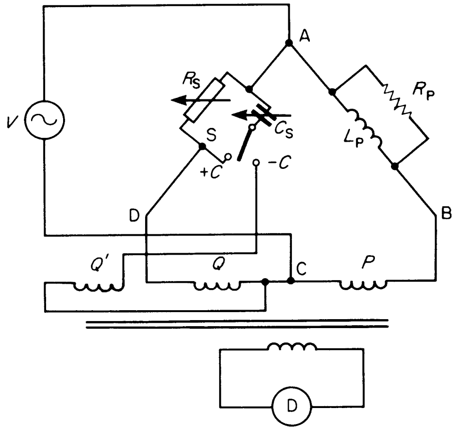


figure 10.14

bridge, the transformer winding connection to the standard capacitor must be reversed, as in figure 10.14, which shows the principle of the method. Q and Q' have equal turns and are wound on the core in the same sense. For measurements of capacitance, the switch S is in the +C position. For inductance measurement the switch S is in the −C position. As the substitution of equivalent parallel capacitance C_P by an equivalent parallel inductance L_P reverses the phase of the quadrature component of current in P, balance can be obtained only by reversing the quadrature component of current in Q. This is easily achieved by reversing the magnetic effect of this component by passing it through Q' in the opposite sense. The unknown equivalent parallel inductance L_P is thus measured in terms of 'negative capacitance'. Its value is then determined from the capacitance reading C_S at balance, using the expression $L = 1/\omega^2 C$ where C is in farads and ω is the angular frequency of the bridge supply.

Example 10.3

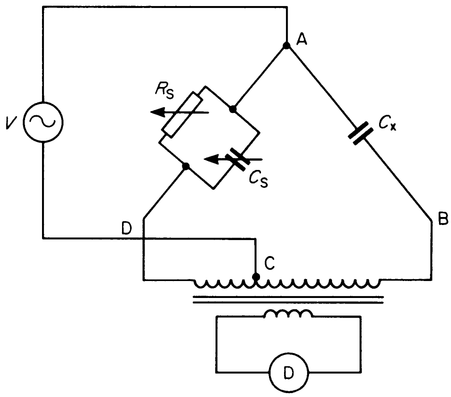


figure 10.15

In the transformer ratio-arm bridge shown in figure 10.15, the winding DB has 500 turns, and is tapped at C such that DC has 100 turns and CB has 400 turns. An unknown capacitor C_X is connected in arm AB, and balance is obtained with $C_S = 0{\cdot}0226\,\mu\text{F}$ and $R_S = 0{\cdot}44\,\text{M}\Omega$. Calculate the capacitance of C_X and its shunt loss resistance R_X.

From equation 10.15,

$$Z_{AB} = \frac{T_1}{T_2} Z_S$$

where $T_1 = 400$ and $T_2 = 100$.

Working in admittances

$$Y_{AB} = \frac{1}{Z_{AB}} \quad \text{and} \quad Y_S = \frac{1}{Z_S}$$

$$\therefore \qquad Y_{AB} = \frac{T_2}{T_1} Y_S$$

where

$$Y_{AB} = \frac{1}{R_X} + j\omega C_X \quad \text{and} \quad Y_S = \frac{1}{R_S} + j\omega C_S$$

$$\therefore \qquad \frac{1}{R_X} + j\omega C_X = \frac{100}{400}\left(\frac{1}{R_S} + j\omega C_S\right)$$

Equating real terms

$$\frac{1}{R_X} = \frac{1}{4R_S}$$

$$\therefore \qquad R_X = 4R_S = 1{\cdot}76\ \text{M}\Omega$$

Equating imaginary terms

$$\omega C_X = \frac{\omega C_S}{4}$$

$$\therefore \qquad C_X = \frac{0{\cdot}0226}{4}$$

$$= 0{\cdot}00565\ \mu\text{F} \qquad \text{(C)}$$

(B) APPARATUS REQUIRED FOR BRIDGE CIRCUITS

The apparatus required for the construction and use of an a.c. bridge network is seen from the preceding sections to consist of suitable standards with which to construct the branches, of means of supplying the bridge circuit, and of a.c. null detectors to indicate when a balance has been attained.

The standards required consist of fixed and variable standards of resistance, inductance and capacitance. Their construction requires that they be as free from residuals as possible. Resistance boxes, for example, must be non-inductively and non-capacitively constructed particularly for use at frequencies exceeding 500 Hz. For higher frequency work, resistance and capacitance standards should have conducting shields to make stray capacitances definite and constant so that their effect can be eliminated by suitable measuring techniques. The accuracy attainable in a bridge measurement will, of course, depend upon the accuracy of the standard components used. As any measurement may depend upon the values of several standards used, the errors associated with such standards are cumulative, and good quality standards are therefore essential if an acceptable accuracy of measurement is to be achieved.

The choice of a suitable source of supply depends upon the frequency to be used for the measurement. For mains frequency, step-down transformers or alternators may be used. For higher frequencies, an oscillator is suitable. If the bridge balance conditions involve frequency, then the supply frequency must be stable and precisely known, and there must be no harmonics in the supply.

The null detector chosen depends upon supply frequency. For the mains frequency of 50 Hz, a vibration galvanometer is ideal. It may be mechanically tuned to resonance at the supply frequency, at which it will then be highly sensitive, and the effects of any harmonics in the supply will be eliminated. For higher frequencies, a sensitive valve voltmeter, or a cathode-ray oscilloscope, may be used. Many commercial bridges use an a.c. amplifier with a *magic eye* tuning indicator as the detector. Finally, headphones at audio-frequency, typically around 1000 Hz, may also be used.

MISCELLANEOUS EXAMPLES

Example 10.4 Using the components shown in figure 10.16, describe a bridge suitable for measuring an unknown capacitance C_X at audio-frequency. What limitations must be set on the type of capacitor that can be measured with this circuit?

What type of detector would be suitable? Give the balance conditions for the bridge. Calculate the value of C_X when $P = Q = 100\,\Omega$, $C_1 = 0{\cdot}1\,\mu$F and a balance is obtained when $C = 0{\cdot}025\,\mu$F.

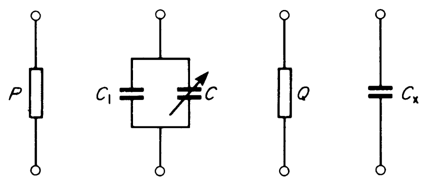


figure 10.16

How should the circuit of this bridge be altered to enable it to measure low-grade capacitors equivalent to a capacitance in series with a resistance?

[$C_X = 0{\cdot}125\,\mu F$]

(CGLI Principles B, 1966)

Example 10.5 An a.c. bridge consists of an arm AB of unknown inductance L and resistance r; arms BC and DA are non-inductive resistances each of 100 Ω; and arm CD is a resistance R_4 in parallel with a capacitor C. Balance is obtained with $R_4 = 140\,\Omega$ and $C = 1{\cdot}34\,\mu F$. The supply to the bridge, connected to A and C, is 6 V, 796 Hz. Draw a phasor diagram to scale, showing currents and voltages in the bridge and use the diagram to determine the values of L and r. Check the results algebraically, by determining the balance equations for the bridge.

[0·0134 H; 71·4 Ω]

Example 10.6 The four arms of a Hay alternating current bridge are arranged as follows: AB contains an iron-cored coil of unknown inductance L and equivalent series resistance r; BC is a non-reactive resistor P of 1000 Ω; CD is a non-reactive variable resistor R in series with a standard variable capacitor C; DA is a non-reactive resistor Q of 2000 Ω. The bridge is supplied to points A and C from a source of frequency 500 Hz. Derive the conditions of balance of the bridge. If balance is obtained with $R = 54{\cdot}6\,\Omega$ and $C = 0{\cdot}32\,\mu F$, calculate the values of L and r.

Draw a circuit diagram of the bridge showing the modifications required for the measurement of the inductance of the coil when carrying direct current, and explain how both the direct and alternating components of the coil current may be measured.

[0·638 H; 110 Ω] **(B)**

(C) ***Example 10.7*** Discuss the advantages of the transformer ratio-arm bridge compared with the conventional four-arm bridge.

A transformer ratio-arm bridge consists of a low-resistance coil BD of 400 turns wound on a ring of high permeability magnetic alloy, a tapping being made at a point C such that BC has 80 turns. Arm AB consists of a standard variable capacitor C_S in series with a standard variable resistor R_S and arm AD consists of a capacitor under test. An a.c. null detector is connected to a secondary winding on the same ring, and a supply at a frequency of 796 Hz is connected to points A and C. Balance occurs when $C_S = 0{\cdot}16\,\mu F$ and $R_S = 6{\cdot}4\,\Omega$. Calculate the capacitance and the shunt loss resistance of the capacitor under test.

[0·04 μF; 0·976 MΩ]

Example 10.8 Explain the operating principle of the 3-winding transformer bridge (the admittance bridge).

A simple 3-winding transformer bridge has identical windings. An unknown impedance, having the form of a series resonant circuit in parallel with a resistor, is connected across the appropriate terminals. At 1 kHz, balance is obtained with a 500 Ω resistor in parallel with a 0·159 μF capacitor across the 'standard' or 'measuring' terminals of the bridge. At 2 kHz, balance is obtained with 500 Ω across the 'standard' terminals and a 0·0795 μF capacitor in parallel with the unknown.

Calculate the admittance of the unknown at each frequency.

Deduce an equivalent circuit for the unknown assuming it to be formed from ideal components.

[At 1 kHz, $Y_X = 0{\cdot}002 + \mathrm{j}0{\cdot}001$ S. At 2 kHz, $Y_X = 0{\cdot}002 - \mathrm{j}0{\cdot}002$ S]

(CGLI Principles C, June 1970) (C)

11 Measurements and measuring instruments

(B) This chapter is concerned largely with measurements made at audio and radio frequencies, and with instruments designed to impose minimal loading on to any circuit under test.

TYPES OF INSTRUMENTS

Unknown voltages are measured at these frequencies by using a moving-coil instrument to obtain a deflection that is proportional to the RMS, mean, or peak value of the alternating voltage being measured. The moving coil instrument being essentially a d.c. instrument, the alternating quantity is thus arranged to produce a proportional direct quantity for operation of the instrument.

Rectifier instruments

These are used for the measurement of audio-frequency alternating currents or voltages; the alternating quantity being full-wave rectified by means of small bridge-connected metal rectifiers to produce a direct current through the moving-coil instrument. The circuit for a voltmeter is shown in figure 11.1a.

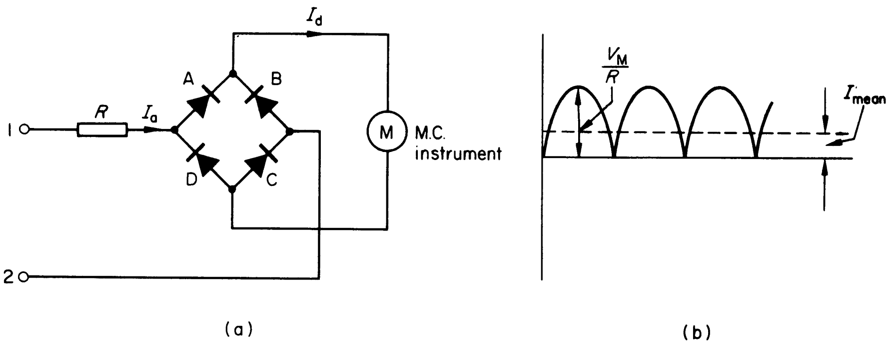


figure 11.1

Assume the rectifiers to be perfect, and consider the voltage being measured to be sinusoidal and of peak value V_M. When terminal 1 is positive, rectifiers A and C conduct with zero resistance, the peak value of the current being given by $I_M = V_M/R$. When terminal 2 is positive, rectifiers B and D conduct with zero resistance, the peak value of the current again being given by $I_M = V_M/R$. The current I_d in the moving-coil instrument is in the same direction for both half-cycles of the supply voltage, as shown in figure 11.1b. The mean value of I_d is $0{\cdot}637 I_M$ and is proportional to $0{\cdot}637 V_M$; so the deflection of the instrument is proportional to the full wave rectified average value of the voltage being measured. For many purposes, however, it is not the mean value but the RMS value that is required. For a sine wave, the ratio RMS/Mean = 1·11, and the scale of the instrument is marked to read the RMS value by multiplying each point on the scale by a factor of 1·11. The readings of this instrument will no longer be accurate if the unknown voltage is non-sinusoidal; hence, some care is required. However, the mean value of an unknown voltage can be obtained by dividing the instrument reading by 1·11; and, if the form factor of this voltage is known, the RMS value also may be determined by multiplying this mean value by the known form factor.

Rectifiers used are normally of the copper-oxide type and the current/voltage characteristics are not linear, particularly for low values of current density where the resistance tends to increase. Also, as the resistance in the reverse direction is not infinite, there is a small reverse current. Consequently, the effective resistance of the instrument circuit varies with the value of the current flowing, and the scale is therefore not uniform. In practice, the high multiplier resistance R swamps the non-linear rectifier resistance so that scales are linear for instrument ranges exceeding about 50 V.

The non-linear nature of the rectifier characteristics introduces difficulties in the design of rectifier-type instruments for use as ammeters. The rectifier-type ammeter consists of a bridge-connected rectifier and moving-coil instrument connected across the secondary winding of a current transformer. The secondary current, which flows in the instrument circuit, is then proportional to the primary current being measured and is independent of the resistance of the secondary circuit, provided this is not too high.

Rectifier instruments using metal rectifiers are accurate to within 2 per cent of full-scale deflection over a frequency range of 20 to 4000 Hz, and are usable at reduced accuracy up to 50 kHz. The accuracy is limited by the shunting capacitance across the rectifier plates. Germanium and silicon rectifiers are being increasingly used because of their much lower capacitance, and extend the use of rectifier instruments into the higher radio frequency range.

Thermo-couple instruments

These instruments measure the true RMS value of voltage or current, irrespective of waveform, over a frequency range from zero to 100 MHz, the upper frequency being limited by stray capacitance between the thermo-couple input terminals. The current to be measured (or a current proportional to the voltage being measured) passes through a small heater element that is in thermal contact with a thermo-couple comprising two wires of dissimilar metals welded together at one end, the cold ends of the wires being connected to a sensitive moving coil instrument as shown in figure 11.2.

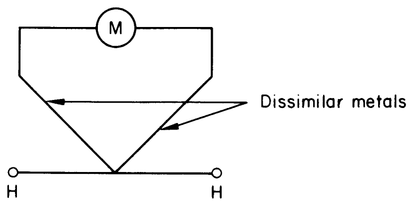


figure 11.2

Typical junction materials are copper/eureka or iron/eureka. The thermo-electric EMF generated depends upon the materials, but is in the order of 50 μV per centigrade degree temperature difference between hot and cold junctions. The magnitude is proportional to the square of the heater (measured) current, and the deflecting torque on the instrument is proportional to the mean square value of the measured current. The instrument is thus scaled to read the true RMS value, independent of waveform. Plug-in thermo-couples are frequently available to give many ranges for a single moving coil instrument. The thermo-couples are of either the vacuum type or the convection type. In the former, the assembly is mounted in a small evacuated glass bulb, thermal contact between the heater element and the junction being by means of a vitreous insulating bead. This type is used for currents ranging from 1 mA to 1 A full scale. In the convection type, the assembly is mounted in air inside a protective enclosure. There is no physical contact between junction and heater, thermal contact being due to air convection. This type is used for full-scale currents up to several amperes. For voltage measurement, a sensitive thermo-couple instrument is connected in series with a high non-inductive resistance. The chief disadvantage of thermo-couple instruments is the low overload capacity, even momentary overloads of 100 per cent being sufficient in some cases to damage the thermo-couples.

ELECTRONIC VOLTMETERS

In the instruments considered above, the whole of the power required to operate the instrument is drawn from the source being measured. This imposes a load on the source which leads to an alteration in the measured quantity. Where this alteration could be significant, it is attractive to obtain the operating power from some source external to the quantity being measured. This is the chief advantage of electronic voltmeters, and enables such instruments to have an input resistance that in some designs can be as high as several megohms per volt. There is a large range of electronic voltmeters now available. Electronic voltmeters have been largely based upon vacuum tube valves, the very high grid circuit resistance of the triode valve being an attractive feature. In addition, solid-state devices are now being used, with germanium and silicon diodes replacing the diode valve, and with the transistor and field effect transistor replacing the triode valve.

Diode voltmeters

In the simplest type of *diode voltmeter* shown in figure 11.3a, although the diode performs the same function as in the bridge-rectifier type already described, it can provide only half-wave rectification, so that the mean value indicated by the meter reading is $V_M/\pi = 0{\cdot}318\,V_M$.

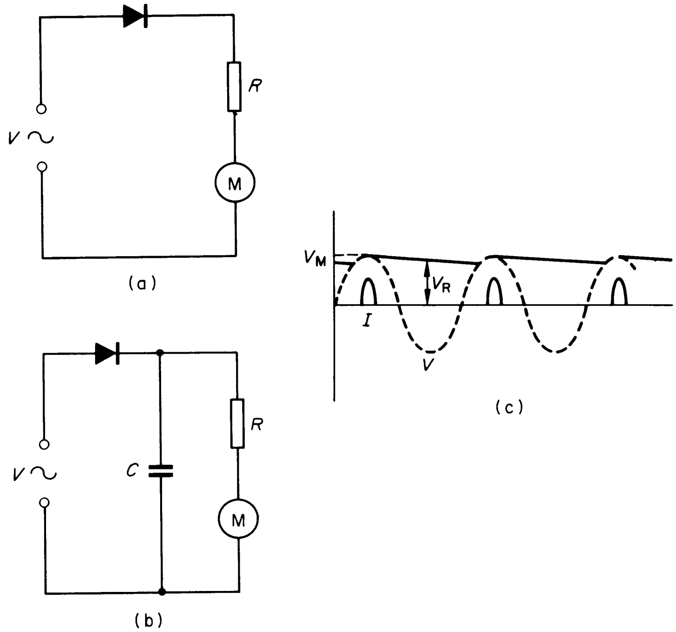


figure 11.3

If a large capacitor is connected in parallel with the instrument as shown in figure 11.3b, the meter reading now indicates the peak value of the voltage being measured, but may be scaled to indicate the RMS value. The diode conducts during the positive half-cycle, C being charged up to the peak value V_M. If there were no leakage, C would remain charged to V_M indefinitely. In practice some discharge takes place through R (this resistance being necessary in order that the rectified voltage V_R can follow changes in the measured voltage V). V_R thus falls slightly between successive positive peaks as shown in figure 11.3c, discharge taking place through R until the instantaneous value of the following positive going wave is greater than the charge voltage on the capacitor. The diode then conducts, C being recharged to the peak voltage due to the flow of charging current I. Provided that the time constant CR is long compared with the periodic time of the applied voltage wave, the capacitor voltage is almost always equal to the peak value of the applied voltage. The current through R is thus very nearly V_M/R and the power dissipated in R is given approximately by $P = V_M^2/R = 2V^2/R$ where V is the RMS value of the voltage being measured. This power is supplied by the capacitor while it discharges through R.

The input power to the circuit from the supply is given by $P = VI$ where I is the RMS value of the supply current. This input power recharges the capacitor.

$$VI = \frac{2V^2}{R} \quad \text{and} \quad \frac{V}{I} = \frac{R}{2}$$

Hence, the input resistance R_i to the circuit is given by

$$R_i = \frac{R}{2} \tag{11.1}$$

The circuit shown in figure 11.3b requires a d.c. path through the source being measured. This is not always possible or desirable. Figure 11.4 shows an alternative connection that does not require a conductive source, as the capacitor blocks any d.c. component applied by the voltage source.

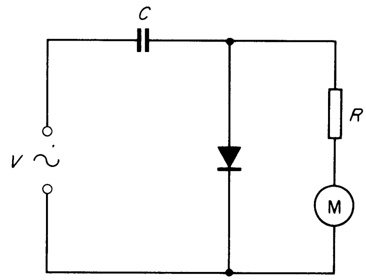


figure 11.4

In this circuit the resistance R is continuously connected across V and absorbs power even when the diode is missing. The input resistance to the circuit is now given by R in parallel with $R/2$, thus

$$R_i = \frac{R}{3} \tag{11.2}$$

A high input resistance is desirable but in the circuits shown is not always possible. If the meter M is to be a reasonably robust instrument requiring an appreciable operating current, then R cannot be too high. If the meter M is omitted, however, the d.c. voltage developed across R is available for operating a d.c. valve voltmeter of a type to be described, and a high input resistance may be retained. The circuits employed produce some amplification of small d.c. voltages but stability rather than voltage gain is the most important factor, and this feature is provided in the *balanced bridge type*, the principle of which is illustrated below.

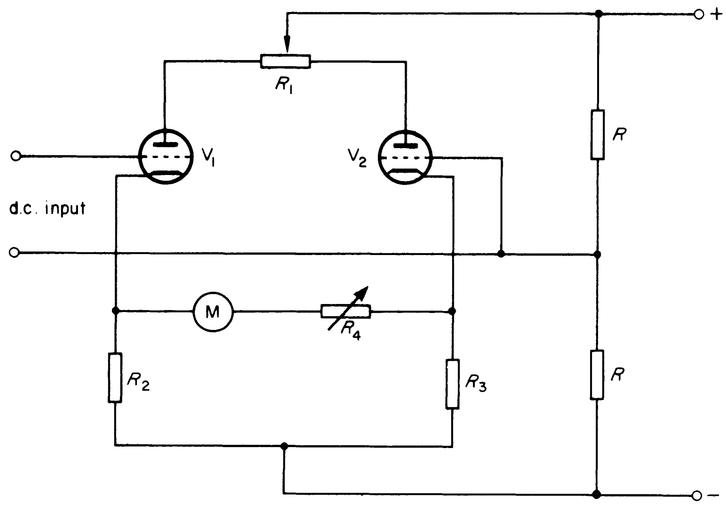


figure 11.5

Balanced-bridge voltmeters

Figure 11.5 shows the cathode-loaded form commonly used. R_2 and R_3 are equal resistors which, with triodes V_1 and V_2, form a bridge circuit. V_1 and V_2 are preferably in the form of a double triode, thus eliminating drift due to changes in anode and heater voltages since both halves of the circuit vary in the same manner. The meter M is a moving coil instrument actuated by the bridge out-of-balance current. The resistor R_4 provides a range change facility. With the d.c. input at zero the two triodes are operating at the same grid potential and R_1 is adjusted to balance the bridge, so that M reads zero. When an unknown d.c. voltage is applied such that the grid of V_1 becomes more positive its resistance effectively decreases and the bridge becomes unbalanced, the deflection of M being a measure of the unknown voltage. If the unknown voltage is such that the grid of V_1 becomes more negative its resistance effectively increases and the unbalance of the bridge tends to deflect M in the opposite direction. However, a change-over switch is provided to reverse this condition so that M deflects only in the forward direction, and the voltmeter can thus measure unknown voltages of either polarity.

A potential divider of several megohms connected between the unknown d.c. voltage and the voltmeter input terminals enables the voltmeter to measure a wide range of d.c. voltages. This is shown in figure 11.6, which also shows the triodes replaced by field effect transistors, with the advantages of increased portability and lower operating voltage.

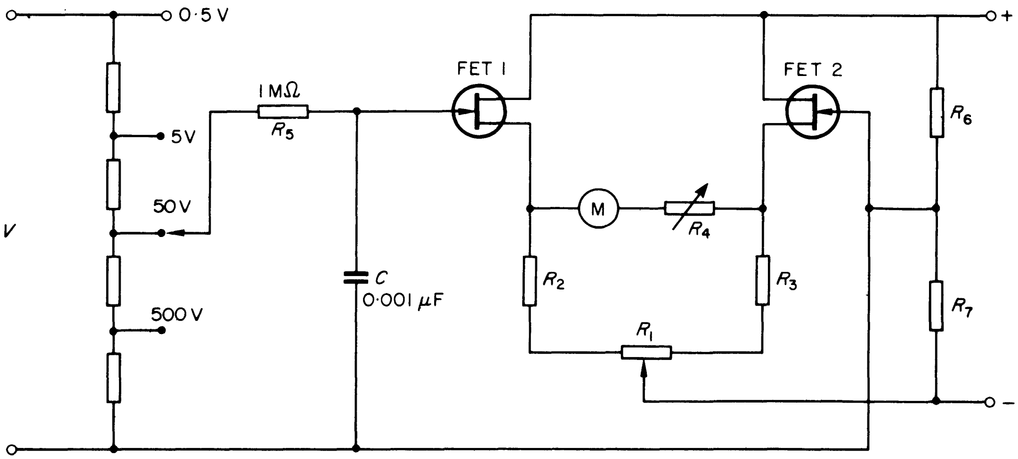


figure 11.6

The operation of the instrument is identical with that using triode valves. Application of an unknown voltage to the gate of FET_1 changes its effective resistance thus unbalancing the bridge and causing the meter to deflect. R_5 and C form a filter to remove any stray fluctuations picked up by the test leads.

Rectifier-amplifier voltmeters

The d.c. electronic voltmeter may be preceded by a diode rectifier for the measurement of alternating voltages, to form a *rectifier-amplifier* type. The circuit of figure 11.4 is commonly used for the rectifier section, the meter M being omitted and the voltage developed across R being taken for amplification. The measured voltage is then proportional to the mean value of the voltage across R, the instrument being scaled to read the RMS value on the assumption that the voltage being measured is of sinusoidal waveform. The voltage developed across R contains an alternating component in addition to the mean direct component required for measurement, hence a filter R_1C_1 is included between the diode circuit and the d.c. amplifier, as shown in figure 11.7, this filter removing the alternating component.

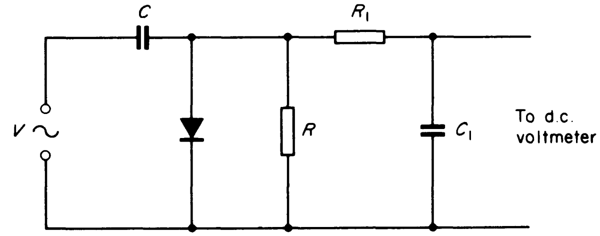


figure 11.7

The rectifier-amplifier type is commonly used for the measurement of alternating voltages at frequencies up to 1 GHz. At these high frequencies it is essential that the capacitance between input leads to the voltmeter should be minimised, so the rectifier section is mounted in a probe to bring it very close to the point at which the voltage is to be measured. As the rectifier section output is d.c. the screened cable connecting it to the d.c. voltmeter section may then be of any convenient length.

Amplifier-rectifier voltmeters

As the alternating voltage to be measured becomes smaller, rectifiers become less efficient and increasingly non-linear, and it becomes desirable to amplify the voltage to be measured before rectifying. This leads to the *amplifier-rectifier* type of instrument, in which the main problem is to design the amplifier so that it has a constant gain over the whole of the desired frequency range. To stabilise the gain and make it constant, negative feedback is used in the amplifier, and this also has the effect of increasing the bandwidth of the amplifier to the required level. The output of the amplifier may be taken to a rectifier type instrument of a type already described.

It is usually desirable to provide for several different voltage ranges, a voltage divider or attenuator as in figure 11.6 being required to enable the voltage level input to the amplifier to be kept the same on all voltage ranges. It is, however, difficult to cover a large number of voltage ranges and still retain a high input resistance, hence a pre-amplifier in the form of a cathode follower circuit in valve voltmeters (or a source follower in transistorised voltmeters) may precede the variable attenuator. Figure 11.8 shows the voltmeter in block diagram form.

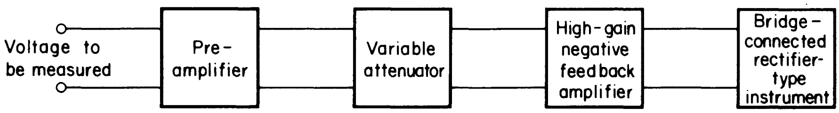


figure 11.8

RESONANCE METHODS

Owing to the difficulty of constructing a.c. bridges for use at high frequencies, resonance methods are often used for the measurement of the parameters of a component or circuit. These methods are not capable of giving the same high accuracy as a well constructed bridge, but for many purposes at high frequencies errors of 2 per cent or more are acceptable, particularly if the measurement is carried out in a simple manner. Resonance methods based on a tuned circuit are used to measure such quantities as self-inductance, self-capacitance and equivalent resistance of coils, Q-factor of coils, and capacitance and loss resistance of capacitors. They are commonly used over the frequency range 20 kHz to 100 MHz, and may be classified into two main types, according to the method used for injecting a voltage at the desired frequency into the circuit under test. The first of these types, injection by mutual coupling, is usually used where the measuring equipment is assembled from separate components. The second type, using resistance injection, is used in the commercial instrument called the Q meter.

Injection by mutual coupling

The EMF in the circuit to be tested is obtained by magnetic coupling between the coil L of the circuit and another coil connected to the output terminals of an oscillator operating over the desired frequency range. The coupling must be loose in order that the frequency and magnitude of the induced EMF should be independent of changes in the test circuit. The principle is illustrated in figure 11.9.

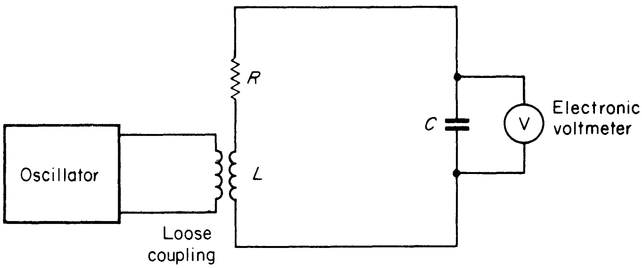


figure 11.9

The circuit is tuned so as to obtain a maximum voltmeter reading, either by varying the frequency f or capacitance C. It may be noted that the circuit is not tuned to resonance for this condition, resonance being defined as the condition for maximum current when $\omega L = 1/\omega C$. But for methods to which these techniques apply, and for coils for which the Q-factor exceeds about 10, the difference between f (or C) for resonance and f (or C) for maximum voltage is negligible. The voltmeter chosen must have a high input resistance so as to have negligible effect on the circuit.

To measure the properties of a coil, the capacitor C consists of two calibrated air capacitors, one variable up to about 1000 pF, the other variable over a small range to measure small changes in C. Several different methods are available, some of which are outlined below.

Measurement of inductance and self-capacitance

The variable capacitor is first set to a value C_1 towards the minimum on its scale and the oscillator frequency is varied to obtain maximum voltage. Let this frequency be f_1, then

$$\omega_1 L = \frac{1}{\omega_1 C_T} \quad \text{and} \quad L = \frac{1}{\omega_1^2 C_T}$$

where C_T is the total capacitance of the circuit. This includes the accurately known capacitance C_1 together with the input capacitance C_v of the valve voltmeter, the self-capacitance C_s of the coil, and other stray capacitance which may be considered negligible if the circuit is properly arranged. Let the sum of C_v, C_s and other strays equal C_a, then

$$L = \frac{1}{\omega_1^2(C_1 + C_a)}$$

where C_1 and ω_1 are known.

The variable capacitor is next set to a higher value C_2 and maximum voltage obtained at a frequency f_2. Then,

$$L = \frac{1}{\omega_2^2(C_2 + C_a)}$$

where C_2 and ω_2 are known.

The two equations obtained each contain the two unknowns, L and C_a, hence these two values can be calculated. For increased accuracy, however, it is preferable to obtain a series of corresponding values of f and C at maximum voltage. For each of these corresponding values,

$$L = \frac{1}{\omega^2(C + C_a)} \tag{11.3}$$

$$= \frac{1}{4\pi^2 f^2 (C + C_a)}$$

and

$$\frac{1}{f^2} = 4\pi^2 L(C + C_a) \tag{11.4}$$

A graph of $1/f^2$ plotted against C is thus a straight line, irregular errors in the readings tending to average out if the straight line that most nearly passes through all the plotted points is drawn. Figure 11.10 shows the form of graph obtained.

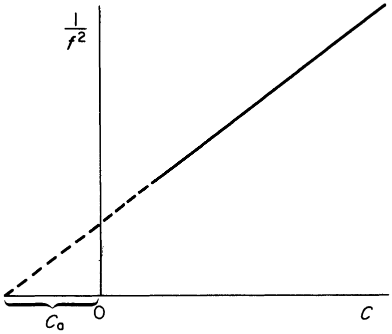


figure 11.10

C_a is then the negative reading of the intercept on the C axis, and the slope of the graph is equal to $4\pi^2L$. Alternatively, C_a and L can be calculated by reading off corresponding values of C and $1/f^2$ at two different points on the graph, and solving the two simultaneous equations obtained.

If all other stray capacitances are neglected, $C_a = C_s + C_v$. Hence the self-capacitance of the coil may be calculated if the valve voltmeter capacitance C_v is known.

Measurement of *Q*-factor and effective resistance

The reactance variation method is commonly used. In figure 11.11, the coil under test has inductance L and effective resistance R and is loosely coupled to the output coil of the oscillator.

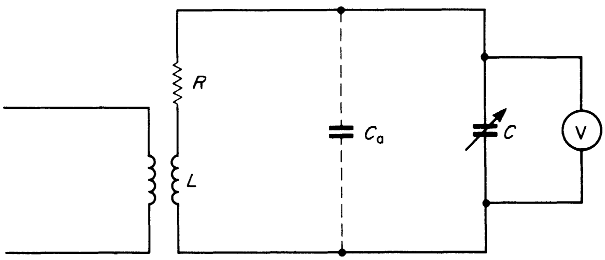


figure 11.11

With the oscillator set to the desired frequency, the circuit capacitance is varied to obtain resonance at a value C_0 as indicated by the valve voltmeter showing a maximum deflection V_0. The circuit capacitance C_0 is the sum of the calibrated capacitor C and the stray capacitances C_a.

Having obtained resonance, the capacitance C is then varied until the voltmeter reading is $V_0/\sqrt{2}$. Two values of C, say C_1 and C_2, are possible, as indicated in figure 11.12.

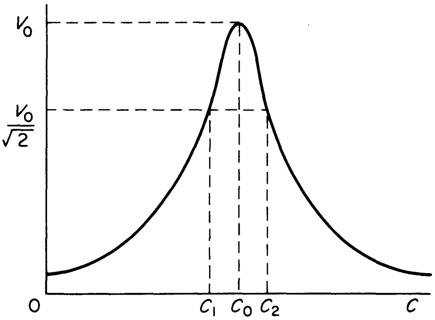


figure 11.12

Then

$$R = \frac{C_2 - C_1}{2\omega C_0^2} \tag{11.5}$$

and

$$Q = \frac{2C_0}{C_2 - C_1} \tag{11.6}$$

The derivation of equations 11.5 and 11.6, though not difficult, is somewhat lengthy, and it is not considered necessary to devote space to the proofs. The expressions for R and Q are simple and give good results provided $\Delta C = C_2 - C_1$ can be accurately determined. Hence, the need for an interpolating capacitor variable over a small range to measure small changes in capacitance.

As an alternative to varying the capacitance to obtain the half power points of $V = V_0/\sqrt{2}$, the frequency may be varied, and the current in the oscillator coil kept constant.

Let f_0 = frequency at which the voltmeter reading is a maximum of V_0 and f_1 and f_2 = frequencies at which the voltmeter reading is $V_0/\sqrt{2}$, as indicated in figure 11.13.

Then

$$R = \frac{\omega_2 - \omega_1}{\omega_0^2 C} \tag{11.7}$$

and

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} \tag{11.8}$$

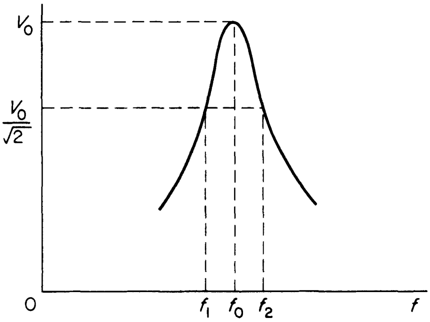


figure 11.13

Example 11.1 In an experiment, series resonance is obtained at various frequencies between a coil of unknown inductance and a variable capacitor, the results being given in table 11.1 below. The circuit is known to possess stray capacitance, which is assumed to be of constant value, in parallel with the variable capacitor. By plotting a suitable graph, or otherwise, deduce the values of the stray capacitance and the coil inductance.

table 11.1

Resonant frequency (kHz)	100	150	200	250	300
Capacitor setting (pF)	940	385	190	100	51

With the frequency constant at 200 kHz and with constant EMF induced in the coil the maximum reading V_0 of the capacitor voltage is noted. It is found that the reading is $V_0/\sqrt{2}$ when the capacitor setting is either 184·1 pF or 195·9 pF. If all circuit loss may be assumed to be in the coil, determine the effective resistance of the coil and its Q-factor.

The values in table 11.1 are used to obtain corresponding values of $1/f^2$ against C, as in table 11.2.

table 11.2

f(Hz) ($\times 10^6$)	0·1	0·15	0·2	0·25	0·3
f^2 ($\times 10^{12}$)	0·01	0·0225	0·04	0·0625	0·09
$1/f^2$ ($\times 10^{-12}$)	100	44·44	25	16	11·11
C (F) ($\times 10^{-12}$)	940	385	190	100	51

From equation 11.4, $1/f^2 = 4\pi^2 L(C + C_a)$.

The graph of $1/f^2$ against C is plotted in figure 11.14. The plotted points are all seen to lie on a straight line. Choosing two corresponding values of $1/f^2$ and C

$$100 \times 10^{-12} = 4\pi^2 L(940 + C_a)\,10^{-12} \qquad \text{(i)}$$

$$16 \times 10^{-12} = 4\pi^2 L(100 + C_a)\,10^{-12} \qquad \text{(ii)}$$

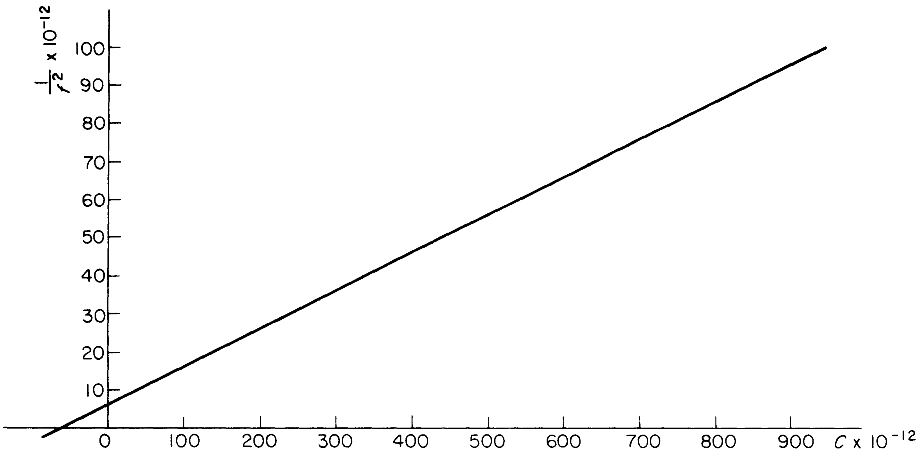


figure 11.14

Dividing (i) by (ii)

$$\frac{100}{16} = \frac{940 + C_a}{100 + C_a}$$

$$10\,000 + 100C_a = 15\,040 + 16C_a$$

$$84C_a = 5040$$

$$C_a = 60\text{ pF}$$

$\therefore$ stray capacitance = 60 pF

Substitute in equation (i)

$$100 \times 10^{-12} = 4\pi^2 L(940 + 60)\,10^{-12}$$

$$L = \frac{0{\cdot}1}{4\pi^2} = 2{\cdot}54\text{ mH}$$

From equation 11.5

$$R = \frac{C_2 - C_1}{2\omega C_0^2}$$

C_0 is the total circuit capacitance when the voltage across the capacitance is a maximum, that is

$$C_0 = 190 + 60 = 250\text{ pF}$$

$$\therefore \qquad R = \frac{(195{\cdot}9 - 184{\cdot}1)\,10^{-12}}{2 \times 2\pi \times 200 \times 10^3 \times (250 \times 10^{-12})^2}$$

$$= \frac{118}{0{\cdot}5\pi} = 75{\cdot}1\ \Omega$$

From equation 11.6

$$Q = \frac{2C_0}{C_2 - C_1}$$

$$\therefore \qquad Q = \frac{2 \times 250 \times 10^{-12}}{11{\cdot}8 \times 10^{-12}} = 42{\cdot}4$$

The Q-meter

An important instrument, primarily designed for the direct and rapid measurement of the Q-factor of coils, is the Q-meter. The instrument can also be used for the measurements outlined above. The circuit diagram showing the principal features is given in figure 11.15.

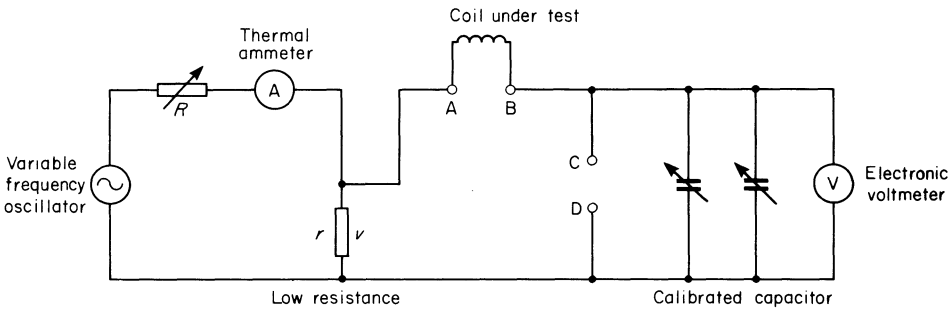


figure 11.15

A known RF voltage is provided by passing a standard current I, monitored by the thermal ammeter A through a low resistance r. This constant voltage is injected across the test circuit. The coil to be tested is connected across the inductor terminals AB, and is tuned to resonance at the desired frequency by the low-loss calibrated variable capacitor forming part of the instrument. The input capacitance of the voltmeter is taken into account in the calibration of the variable capacitor. Since Q is given by the maximum voltage indicated on the voltmeter, at resonance, divided by the injected voltage, which is constant, the voltmeter is scaled to read Q directly.

A typical value for the low-resistance r is $0{\cdot}04\ \Omega$, and with a current I of $0{\cdot}5$ A the injected voltage is $0{\cdot}02$ V. The value of r is sufficiently small to be negligible compared with the resistance of the test circuit, and the current flowing in r is practically constant at the value I passing through the thermal ammeter. In some Q-meters, the electronic voltmeter is first connected across r, and the injected voltage is thus measured and adjusted to a standard value.

A capacitor to be tested is connected to the capacitor terminals CD in parallel with the calibrated capacitor and the resonant circuit is completed by a low-loss standard inductor. Inductors suitable for use with the instrument are available, covering typically a range from 0·1 μH to 50 mH. (B)

(C) THE CATHODE-RAY OSCILLOSCOPE

This versatile instrument produces a spot of light which can be caused to move in strict accordance with a voltage under examination, even if the voltage is changing at an extremely rapid rate of several hundred megahertz. Additionally, the oscilloscope provides facilities for moving the spot in two directions simultaneously, making it possible to examine in detail the waveform of an oscillation, and to carry out measurements on the magnitude, form, phase and frequency of the waveform, and of its relationship to other quantities. The voltage under examination may be obtained in a variety of ways, enabling the oscilloscope to investigate conditions not only in electrical circuits, but also in many other devices such as mechanical structures in which the condition to be investigated is made to produce, by means of a transducer, an analogous electric signal.
The basis of the oscilloscope is the cathode-ray tube, the fundamental features of which comprise (i) the electron gun, including focusing devices, (ii) the deflecting systems and (iii) the fluorescent screen. Figure 11.16 shows the principle features, and it will be seen that the electrode system is enclosed in an evacuated glass bulb.

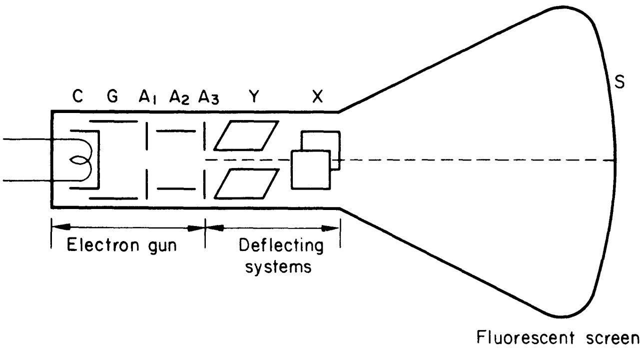


figure 11.16

The electron beam

The electron gun produces a stream of electrons emerging at high velocity, these electrons being focused so that they impinge on the screen over a very small area. The screen is coated on the inside with a fine powder that fluoresces under the impact of the electrons, thus forming a spot of light. The brilliance of the light spot depends upon the number per second of the impinging electrons, that is, upon the beam current. This is controlled by the potential on the control grid G. Electrons are emitted by thermionic emission from the cathode C, and are attracted by the positive potential on the first anode A_1 which is in the form of a disc with a small hole at its centre. Some of these electrons pass through the hole to form the beam current. The grid G is in the form of a cylinder, and carries a variable negative bias with respect to the cathode. Variation of this negative potential varies the number of electrons passing through the anode hole, and forms the *brilliance* control of the oscilloscope. Electrons that pass through the hole in A_1 would tend to diverge and thus form a diffuse beam of large cross-section. This is because electrons are emitted from the cathode in varying directions and thus tend to form a diverging beam, and also because electrons in the beam tend to repel each other. To produce a small spot of light on the screen it is thus necessary to focus this divergent beam. This can be done in a variety of ways; in the modern oscilloscope electrostatic focusing is the rule, and this requires two or more anodes at different electrical potentials. The most common arrangement takes the form shown in figure 11.17. Anodes A_1 and A_3 are discs with small central holes, are connected together, and are at a positive potential of the order of 2000 V with respect to the cathode. Anode A_2 is a cylinder at a variable positive potential less than that of the other two anodes. A_2 is thus negative with respect to A_1 and A_3. In figure 11.17a is shown the electric field distribution between the three anodes in the form of lines of force *acting on an electron* in the field.

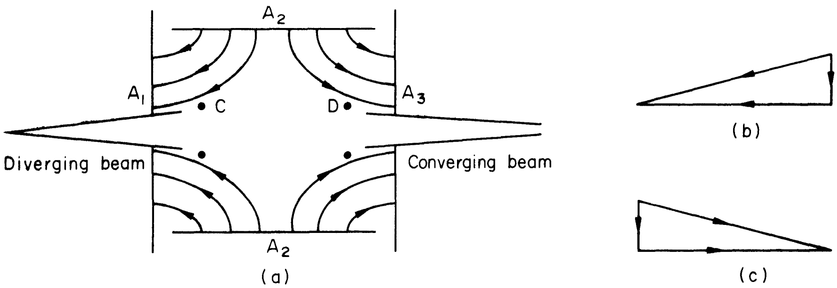


figure 11.17

Focusing

Consider any point C in the left-hand half of the electric field. The direction of the line of force at C may be resolved into two components as shown in figure 11.17b, one component parallel to the axis and the other directed towards the axis. Electrons entering the field through the hole in A_1 and which are diverging from the axis as shown, come under the influence of these forces, and the first component tends to slow the electrons down, while the second deflects them towards the axis. The electrons then enter the right-hand half of the field, the direction of the lines of force again having two components as shown in figure 11.17c. The electrons are thus accelerated and further deflected towards the axis, so that when they emerge through the hole in anode A_3 they form a convergent beam. The amount of convergence produced depends upon the velocity of the electrons and upon the relative potentials of the three anodes. In the arrangement shown, variation of the potential of anode A_2 is arranged to focus the beam to a small spot on the screen. This forms the *focus* control.

Deflection

To enable a voltage waveform to be examined, the spot of light produced on the screen must be deflected in accordance with the variation of the voltage. The electron beam emerging from the gun assembly thus enters the deflecting system, which consists of two pairs of plates termed respectively the Y and X plates. Consider first the Y plates. Two plates are arranged one on each side of the beam as shown in figure 11.18a. When a voltage is applied across these plates the beam is attracted towards the more positive plate and repelled from the more negative one, so that it changes its course, the spot of light on the screen thus changing its position in the vertical direction. The deflection of the spot is directly proportional to the voltage applied between the plates. If the voltage between the plates is continually varying, the light spot on the screen follows the variations instantaneously and exactly. For a given voltage applied to the plates, the actual movement of the spot depends upon several factors. The deflecting force depends upon the voltage applied to the plates, and upon the distance between the plates; that is, upon the field strength in volts per metre. To enable a large deflecting force to be produced and at the same time avoid the electron beam hitting the plates and disappearing, the plates are frequently given a diverging shape, as shown in figure 11.18b.
A second factor governing the deflection of the spot is the length of the tube. For a given angular deflection of the beam, the longer the tube the greater is the deflection of the spot. Finally, the deflection depends upon the velocity of the electrons in the beam, which in turn depends upon the voltage on the final anode. A given deflecting voltage produces a greater spot deflection in a tube that has a low anode voltage than in a tube that has a high anode voltage, because in the former case the electrons are travelling slower and are thus in the deflecting field for a longer period. For a given tube, however, operating at a given final anode voltage, the spot deflection is proportional to the voltage applied to the deflecting plates. The sensitivity is usually stated in terms of the voltage required to cause 1 cm deflection on the screen. Full screen deflection is obtained with peak-to-peak voltages varying between 20 V and 200 V.

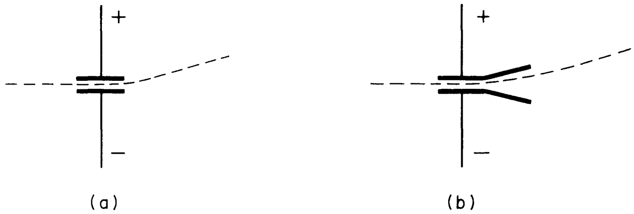


figure 11.18

Time base

An alternating voltage applied to the Y plates causes the spot on the screen to trace a straight vertical line on the screen, of length proportional to the peak-to-peak value of the alternating voltage. To examine the variation of such a voltage with time, another voltage representing time is applied to the X plates, this voltage varying linearly with time. Such a voltage causes the spot to move horizontally from left to right across the screen at a uniform rate, and the circuit used to produce this voltage is termed a *time base*. Additionally, when the spot has completed its traverse across the screen, it must return almost instantaneously and commence another traverse, this process being continually repeated. The voltage applied to the X plates must therefore have the *sawtooth* waveform shown in figure 11.19.

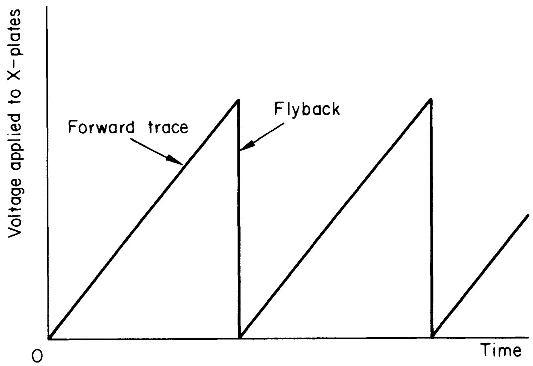


figure 11.19

If the frequency of the time base is equal to the frequency of a continuously repeated voltage wave applied to the Y plates, the spot will move over the same path during each traverse of the screen, thus producing a stationary picture of the voltage waveform. Figure 11.20 illustrates this, where it is seen that in order to produce one complete wave of the sinusoidal voltage shown, the time-base frequency must be precisely the same as the frequency of the voltage applied to the Y plates.

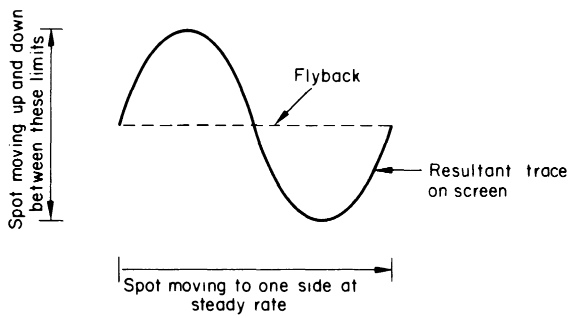


figure 11.20

To produce n complete waves on the screen, the time-base frequency must be precisely $1/n$ times that of the voltage applied to the Y plates.

Example 11.2 A cathode-ray tube has a 'Y' sensitivity of 60 V/cm and an 'X' sensitivity of 65 V/cm. A 50 V RMS 10 kHz sinusoidal supply is applied to the Y plates and a 260 V peak-to-peak variable frequency saw-tooth voltage is applied to the X plates. The frequency of the saw-tooth is varied until a stationary display of five cycles appears on the screen. Deduce the total height and width of the display and the rate of rise of the saw-tooth voltage.

(CGLI Principles B, 1971)

Figures 11.21a and 11.21b show the waveforms of the voltages applied to the Y and X plates respectively.

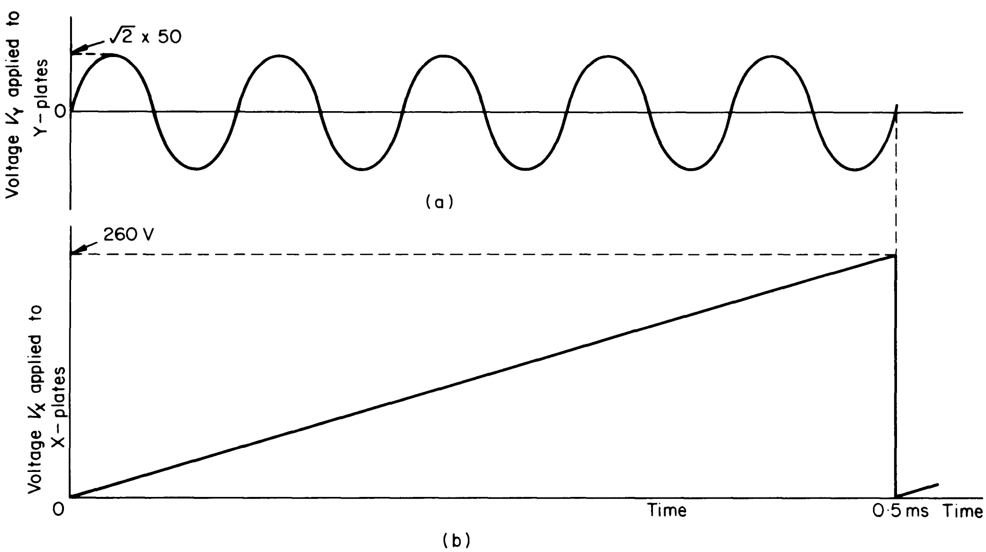


figure 11.21

Peak-to-peak value of voltage applied to Y plates

$$= 2\sqrt{2} \times 50 = 141{\cdot}4\ \text{V}$$

'Y' sensitivity = 60 V/cm (given)

Total height of display

$$= \frac{141{\cdot}4}{60} = 2{\cdot}36\ \text{cm}$$

Periodic time T of the 10 kHz waveform

$$= \frac{1}{10\,000} = 0{\cdot}1\ \text{ms}$$

Time for 5 cycles = 0·5 ms

The voltage applied to the X plates (that is, the time-base voltage) must grow in a linear manner for 0·5 ms and then be reduced to zero in negligible time, this cycle then repeating.

Peak-to-peak value of voltage applied to X plates

$$= 260\ \text{V (given)}$$

'X' sensitivity = 65 V/cm

∴ Total width of display

$$= \frac{260}{65} = 4\ \text{cm}$$

Rate of rise of saw-tooth voltage

$$= \frac{260}{0{\cdot}5 \times 10^{-3}}$$

$$= 520\,000\ \text{V/s}$$

Synchronisation

The time-base voltage is obtained by charging a capacitor at constant current, the capacitor voltage thus increasing at a constant rate. This voltage is applied to the X plates, thus causing the spot to traverse the screen from left to right at constant rate. After the required time, an electronic switch connected across the capacitor is closed. This discharges the capacitor, hence the spot returns to the left-hand side of the screen. At the same time, the electronic switch is opened and the cycle of operations is repeated. It is difficult to ensure that, unaided, the time of transit across the screen is exactly the same as, or an exact multiple of, the time of one wave of the voltage applied to the Y plates. In order to ensure this, the time base must be synchronised with the voltage under examination so that successive traces may be superimposed on one another. Time bases may be *free running*, in which case synchronisation is achieved by feeding a proportion of the voltage under examination into the circuit producing the saw-tooth waveform. This results in reducing the rate of increase of the sawtooth voltage if the time base speed is too fast, or (conversely) increasing the rate of increase of the saw-tooth voltage if the time-base speed is too slow. The modern tendency, however, is to use a *triggered* time base in which the observed waveform is used to control the repetition frequency. The synchronising system is often elaborate, but the principle is illustrated in the circuit of figure 11.22a. The voltage to be observed is connected to the Y input of the oscilloscope. In addition to supplying the Y plates, the same signal is taken to a pulse shaping circuit, arranged to produce a series of rectangular pulses of the same frequency as the waveform under examination. These rectangular pulses are then taken to the input terminals of the sawtooth generator, in which capacitor C_2 is charged at constant current through transistor TR_2. TR_1 acts as a switch, with TR_2 and R_4 as its emitter load. The base current in TR_2 is controlled by RV_1, the limits being restricted by R_2 and R_3 so that TR_2 operates only on the constant current part of its characteristic.

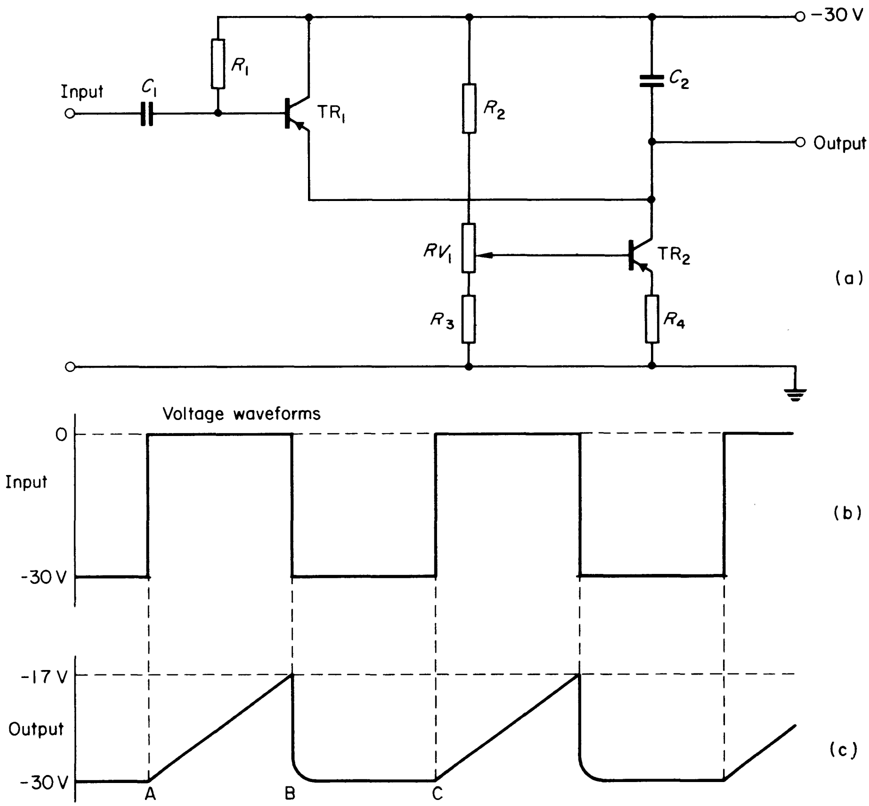


figure 11.22

The operation of the circuit may be followed by reference to figures 11.22b and 11.22c, which show the voltage waveforms at the input and output terminals respectively. Up to point A, TR_1 conducts with a small voltage V_{CE}, conduction taking place through TR_1, TR_2 and R_4 in series. C_2 charges up to this small voltage. At point A the fast rising input pulse turns off TR_1, but TR_2 continues to conduct and allows C_2 to charge so that the voltage across it rises towards 30 V. This allows the p.d. between the output terminals to become less negative as shown in figure 11.22c. The charging current through C_2 is constant and the output voltage must thus rise linearly. At point B the negative-going input pulse switches TR_1 on; and, in so doing, provides a low resistance discharge path for C_2. Thus the circuit is restored to its original state, ready to be triggered by the next positive-going pulse at point C. The length of the sweep produced can be controlled by means of C_2 and RV_1.

Power supplies

The basic circuit used to supply the voltages to the various electrodes in the tube, and to the deflector plates, is shown in figure 11.23.

It is usual to connect the final anode to earth, hence the heated cathode is operated at a high negative potential with respect to earth. In the circuit shown, anodes A_1 and A_3 are connected to earth, and anode A_2, which is

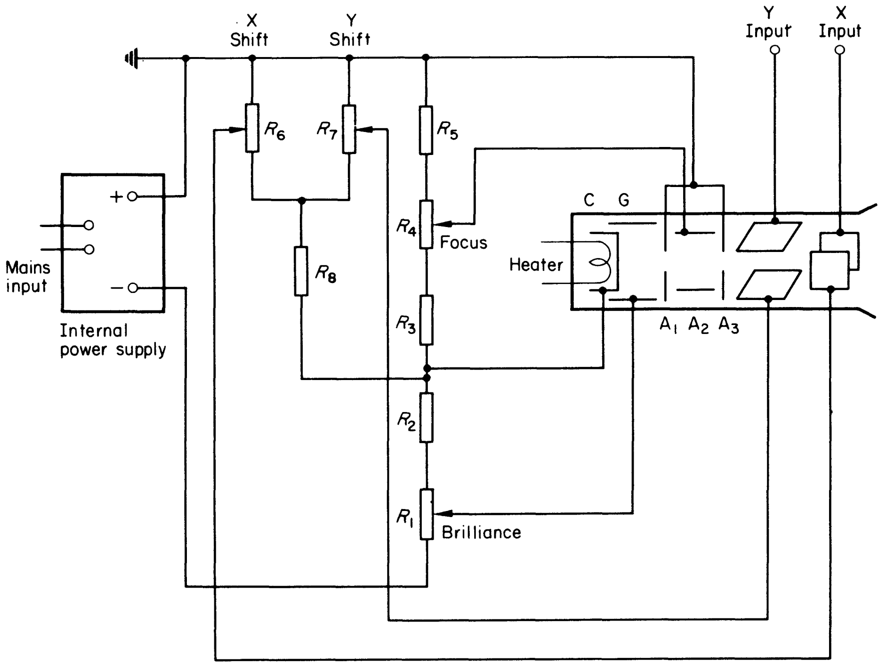


figure 11.23

negative with respect to A_1 and A_3, is connected to potentiometer R_4 which forms the *focus* control. Potentiometer R_1 supplies the grid G with a variable voltage negative with respect to the cathode C, and forms the *brilliance* control.

Shift facilities

It is desirable to be able to move the trace on the screen to any desired position, and this can be done by operation of X and Y shift controls. One plate of each pair of deflecting plates is connected to a potentiometer: in figure 11.23, X plate to R_6 and a Y plate to R_7. The spot on the screen takes up a permanent deflection according to where the potentiometers are set. This deflection makes no difference to the deflection produced by varying voltages applied to the other plates of each pair. The effect of the *shift* potentiometers is simply to move the trace on the screen in the vertical (by means of R_7) or horizontal (by means of R_6) directions.

Amplifiers and attenuators

If the voltage under test is applied directly to the Y plates, it may be either too low to produce adequate deflection, or so high that the deflection is greater than can be fully displayed. To enable widely different input voltages to be displayed, both input amplifiers and attenuators are required, and these must provide a strictly linear amplification or attenuation. To enable a given oscilloscope to respond accurately to a wide range of voltages and frequencies, such amplifiers and attenuators are often available as plug-in units. X amplifiers also are desirable, because it may be required to connect external signals to the X input.

The screen

Various types of screen are available in cathode-ray tubes, some being especially suitable for visual observation and some for photographic work. The colour of the spot on the screen is determined by the material of the screen. For visual work, zinc silicate is commonly used, giving a green light. For photographic work a blue colour is preferred, obtainable with cadmium tungstate. The time taken for the fluorescence of the screen to disappear after the impinging electrons in the beam have moved away may be from a few microseconds up to several minutes. Short persistence is desirable for rapidly recurring waveforms, and long persistence for transient observations.

The potential of the fluorescent screen in normal operation becomes slightly positive due to secondary emission. A graphite coating on the inside of the tube provides a return path for the electrons and thus completes the circuit for the beam current. (C)

MISCELLANEOUS EXAMPLES

Example 11.3 Explain the principle of the thermocouple ammeter. Why can this type of meter be especially useful for high-frequency measurements?

(CGLI Principles B, 1965)

Example 11.4 Describe the principle of operation of a meter suitable for measuring current of a few milliamperes at audio frequencies. State the factors that (a) limit the sensitivity of the instrument and (b) reduce its accuracy as the frequency increases.

(CGLI Principles B, 1966)

Example 11.5 Why is the input impedance of a valve voltmeter high? How can a valve voltmeter be used to measure RF current?

A valve voltmeter is used with the circuit of figure 11.24 to find the effective resistance (r) of an inductor of 0·001 H at 100 kHz. The valve voltmeter reads 5 V when connected across AB and 43·9 V when across BC. Find values of

(a) the 100 kHz current in the circuit
(b) the effective series resistance (r) of the inductor.

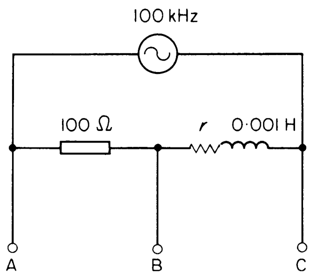


figure 11.24

[(a) 50 mA; (b) 613 Ω]

(CGLI Principles B, 1968)

Example 11.6 Discuss the applications and limitations of resonance methods of measuring impedance.

A coil and a calibrated capacitor are connected in parallel with each other and with a valve-voltmeter having an input capacitance of 10 pF. An EMF is is induced in the coil from a calibrated oscillator. At each of a series of frequencies the capacitor is adjusted until the maximum reading is shown by the voltmeter, the readings being:

table 11.3

f (kHz)	130	140	160	200	300	400
C (pF)	670	572	429	260	93	36

Determine the inductance of the coil and its self-capacitance, deriving any formulae used.

[2·12 mH; 30 pF]

Example 11.7 A circuit consisting of a coil and a variable low-loss capacitor connected in series is tuned to resonance using a Q-meter. If the variable capacitor is then set to 350 pF, the frequency is 500 kHz, and the Q-factor is indicated as 90, calculate the effective inductance and resistance of the coil. Neglect the self-capacitance of the coil and other stray capacitance.

[0·289 μH; 10·1 Ω]

Example 11.8 With the aid of a suitable sketch or sketches explain the operation of a cathode-ray tube having an electrostatic gun and deflection system. How is the tube used to display a repetitive voltage waveform?

A cathode-ray tube is operating and a well-defined display of a waveform appears on the screen. Explain how the display will change if the effective final anode voltage of the tube is reduced by a small but significant amount.

(CGLI Principles C, 1970)

Example 11.9 A cathode-ray tube has X and Y sensitivities each of 50 V/cm. A waveform displayed on the tube has a peak-to-peak vertical amplitude of 5 cm, and horizontally two full cycles occupy 4·8 cm. The time-base voltage increases at the rate of 60 V/ms. Calculate the RMS voltage and the frequency of the waveform applied to the Y-plates.

[88·2 V; 500 Hz]

12 Electrical machines

(B) An electrical machine functions essentially as an energy convertor, converting energy either from an electrical form into a mechanical form when operating as a motor, or from a mechanical form into an electrical form when operating as a generator. The basic principles applied in this energy conversion process have been outlined in chapter 2, where equation 2.1 gives the force $F = BIl$ newtons acting on a current-carrying conductor, and equation 2.13 gives the EMF $e = Blv \sin\theta$ volts induced in such a conductor. The generation of an EMF is maintained only as long as a magnetic flux continues to be cut by conductors.

The simple alternator

Consider first a single rectangular loop of wire of length l metres and breadth b metres rotating with uniform angular velocity ω radians per second about an axis perpendicular to the direction of a uniform magnetic field of density B teslas, as shown in figure 12.1a.

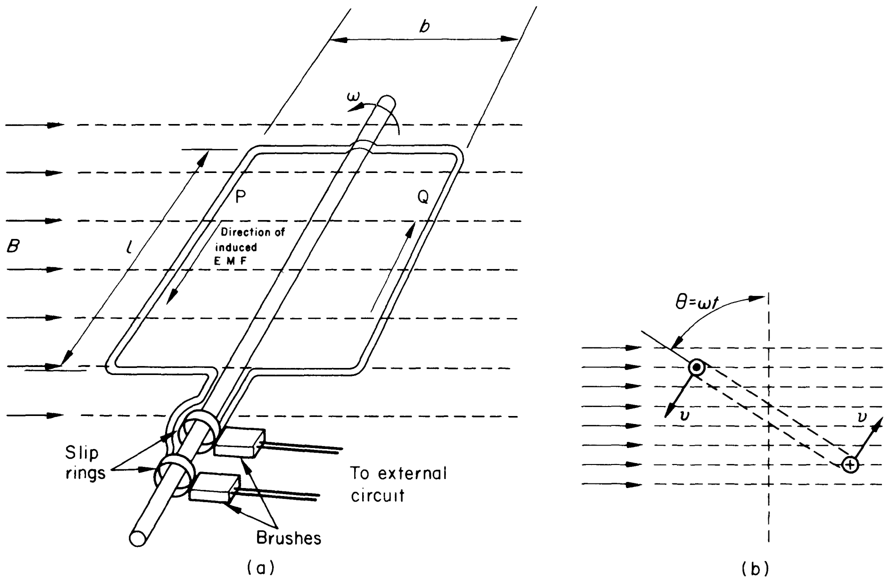


figure 12.1

At time $t = 0$ let the plane of the loop be perpendicular to the direction of the field, so that at this instant the sides of the loop are moving in a direction parallel to the field with velocity $v = (b/2)\,\omega$ metres/second. The induced EMF is then zero. After any time t, the plane of the loop will have moved through an angle $\theta = \omega t$ radians, as shown in figure 12.1b. Sides P and Q then have components of velocity perpendicular to the direction of the field, and, from equation 2.13, the induced EMF in each side is given by

$$e = Blv \sin\theta$$

$$= Bl\frac{b}{2}\omega \sin\omega t \quad \text{volts}$$

These induced EMFs are in opposite directions in each conductor, and are thus additive round the loop, hence the total EMF induced in the loop is given by

$$e = 2Blv \sin\theta$$

$$= Blb\omega \sin\omega t$$

$$= Ba\omega \sin\omega t \quad \text{volts}$$

where $a = lb$, that is, the area enclosed by the loop.

If the single loop is replaced by a coil having n turns, an EMF of equal value is induced in each turn, and the resultant EMF induced in the coil is given by

$$e = Ban\omega \sin\omega t \quad \text{volts} \tag{12.1}$$

The maximum value of this EMF occurs when $\sin\omega t = 1$, the plane of the coil then being parallel to the direction of the magnetic field. Evidently the maximum value of the EMF is given by

$$E_{\mathrm{M}} = Ban\omega \quad \text{volts} \tag{12.2}$$

and the EMF at any instant is given by

$$e = E_{\mathrm{M}} \sin\omega t \quad \text{volts} \tag{12.3}$$

A curve of e may be plotted against $\theta = \omega t$ for one complete revolution of the coil, starting when $\theta = 0$ at time $t = 0$, the induced EMF then being zero. The EMF first increases to a positive maximum when $\theta = \pi/2$ radians and $\sin\theta = 1$, then decreases to zero when $\theta = \pi$ radians and $\sin\theta = 0$. The induced EMF then reverses in direction and increases to an equal negative maximum when $\theta = 3\pi/2$ radians and $\sin\theta = -1$, and finally decreases to zero when $\theta = 2\pi$ radians and $\sin\theta = 0$. This corresponds to one complete revolution of the coil, during which the EMF has passed through one complete cycle of a sinusoidal waveform, which is repeated for each successive revolution. The curve of figure 12.2 is thus obtained. The frequency of the induced EMF is equal to the speed of the coil in revolutions per second.

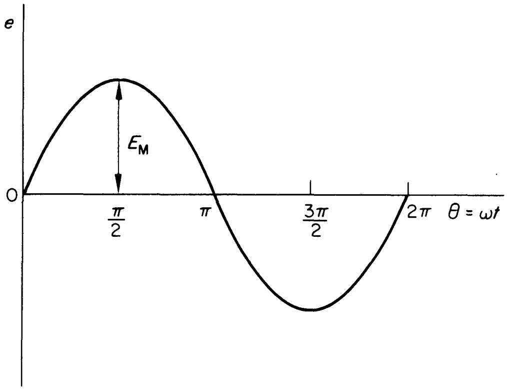


figure 12.2

Slip rings

If the ends of the coil are each connected to two metal rings, termed *slip rings,* mounted on but insulated from the spindle, as shown in figure 12.1a, continuous connection may be made with a stationary external circuit by means of two conducting spring-loaded contacts or *brushes,* made of carbon, which bear on the slip rings.

Example 12.1 A thin rectangular coil of length l metres and breadth b metres with n turns rotates at an angular velocity of ω rad/s in a uniform parallel magnetic field about an axis symmetrically situated in its plane. Derive an expression for the instantaneous EMF induced in the coil.

If $n = 500$, $l = 0{\cdot}05$ m, $b = 0{\cdot}02$ m, $\omega = 500$ rad/s, $B = 0{\cdot}06$ T and the coil has a resistance of 10 Ω, calculate

(a) the RMS value of the induced EMF
(b) the power delivered into a 50 Ω resistor connected across the output of the coil.

(CGLI Principles B, 1970)

From equation 12.1

$$e = Ban\omega \sin \omega t \quad \text{volts}$$

where $a = lb$.

Substituting given values

$$e = 0{\cdot}06 \times 0{\cdot}05 \times 0{\cdot}02 \times 500 \times 500 \sin \omega t$$

$$= 15 \sin \omega t \quad \text{volts}$$

$$E_M = 15 \text{ V}$$

∴ (a) RMS value

$$E = \frac{15}{\sqrt{2}} = 10{\cdot}6\,\text{V}$$

(b) The EMF generated acts in a circuit consisting of a coil of resistance 10 Ω in series with a load of resistance 50 Ω.

$$\text{Load current} = \frac{\text{RMS voltage}}{\text{circuit resistance}}$$

$$= \frac{10{\cdot}6}{60} = 0{\cdot}177\,\text{A}$$

Power delivered to load

$$= 0{\cdot}177^2 \times 50$$

$$= 1{\cdot}57\,\text{W}$$

THE D.C. GENERATOR

In d.c. machines the field system is stationary and the armature, carrying the active conductors, rotates. Figure 12.3a shows an armature consisting of a single loop of wire wound in slots on a laminated iron cylinder rotating between the poles of an electromagnet, with a uniform air gap in which a magnetic field is produced.

Figure 12.3b shows a sectional view of the arrangement, with the armature rotating in an anti-clockwise direction. During the first half revolution, conductor A cuts the flux leaving the N pole, and the EMF induced in the conductor when it is in a position such as A′ is towards an observer, as indicated by the dot. Conductor B meanwhile cuts the flux entering the S pole, and the EMF induced in it when the conductor is in a position such as B′ is away from an observer, as indicated by the cross. The EMFs induced in the two conductors are thus in the same direction round the loop formed by the two conductors. While the conductors are cutting the flux, the induced EMFs are almost constant, because the field in the air gap is assumed to be uniform. During the first half revolution, therefore, the EMF induced in the loop rises rapidly from zero when the conductors are midway between the poles (as at A and B) to a maximum value when the conductors are cutting the magnetic flux, and then falls rapidly to zero when the conductors are again midway between the poles. During the second half-revolution, the EMF induced in the coil is similar to that induced during the first half-revolution, but is in the reverse direction because conductor A now cuts the flux entering the S pole while conductor B cuts the flux leaving the N pole. The EMF induced in the loop is thus an alternating EMF with a flat-topped waveform, as shown in figure 12.4.

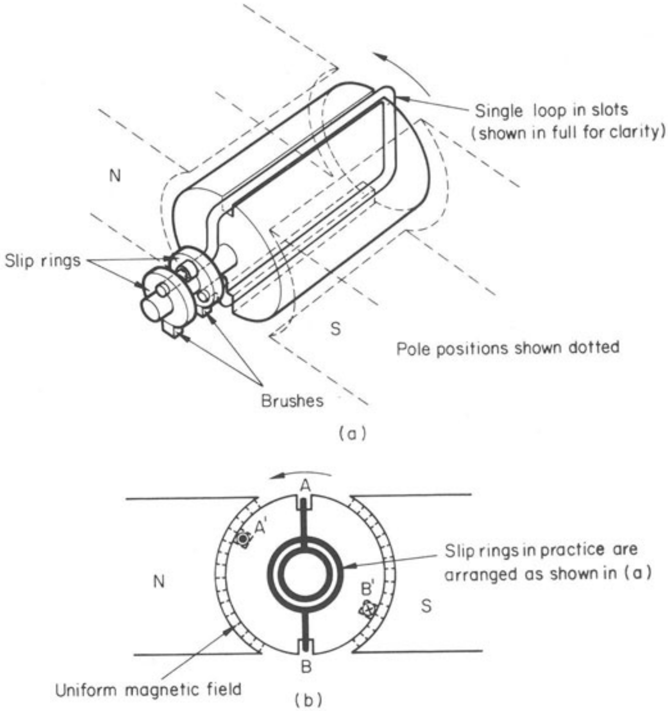


figure 12.3

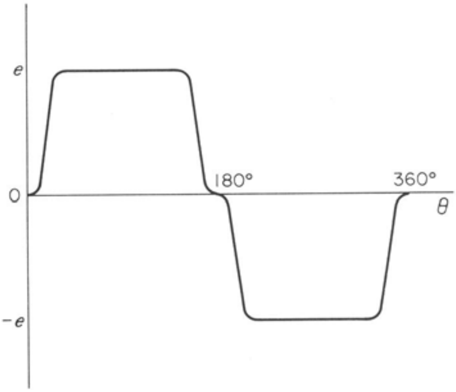


figure 12.4

Commutator

Suppose now that a metal ring is divided into two equal segments by cutting it along a diameter, and that the two segments S_A and S_B are mounted on the shaft of the armature, but insulated from the shaft and from each other. Let the two ends A and B of the coil be connected to segments S_A and S_B

respectively, and let fixed brushes b_1 and b_2 bear on the outside of each segment. Figure 12.5 shows the arrangement, but the brushes are shown inside the segments for greater clearness.

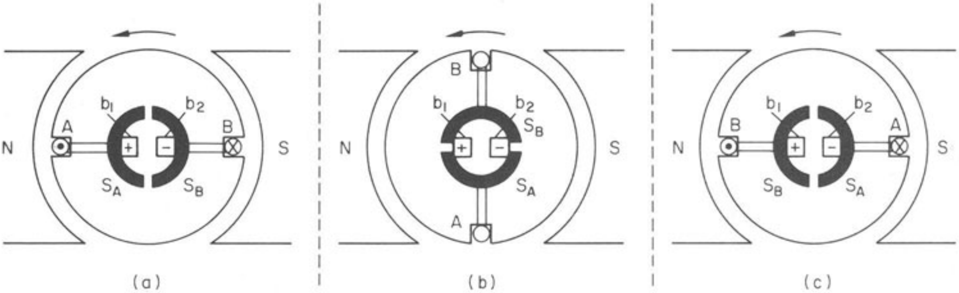


figure 12.5

In figure 12.5a, the EMF in A is induced in a direction towards segment S_A, and the brush b_1 is therefore positive. The EMF in B is induced away from segment S_B and the brush b_2 is therefore negative. In figure 12.5b, both conductors are midway between poles; thus, neither is cutting magnetic flux, and the EMF induced in the coil is zero. In figure 12.5c, the EMF in A is now in a direction away from segment S_A, but this segment now makes contact with brush b_2 which is therefore still negative. Similarly, the EMF in B is now towards segment S_B, and this segment is now making contact with brush b_1 which is therefore still positive. Brush b_1 is always connected to a conductor that is cutting the flux from a N pole. Similarly brush b_2 is always connected to a conductor that is cutting the flux entering a S pole. Although the EMF induced in the coil is alternating, the EMF between the brushes is always in the same direction. Thus, the two segments act as a mechanical switch, and the complete arrangement is termed a *commutator.*

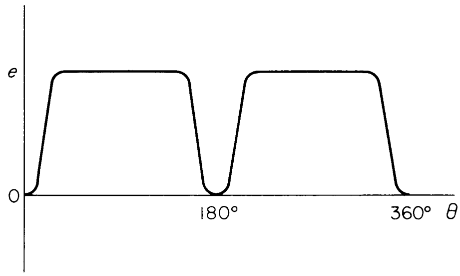


figure 12.6

The EMF between the brushes for one complete revolution of the armature is shown in figure 12.6; and, though varying in magnitude, it is always in the same direction. Twice in each revolution the brushes join together both commutator segments, thus short-circuiting the coil. This occurs, however, at those instants when no EMF is being induced in the coil.

Armature

The simple single coil and two-part commutator is not a practicable arrangement, since the EMF, and therefore the current flowing in a load resistance connected to the brushes, is far from constant. If, however, a number of coils are used, distributed in slots over the armature periphery, with a corresponding increase in the number of commutator segments, then the EMF and current can be made reasonably steady. The actual way in which the armature is wound is beyond the scope of this chapter, but the principle is illustrated in figure 12.7a which shows an armature with the conductors distributed in slots round the periphery, and with the armature rotating in the magnetic field produced by a pair of poles. All the conductors under the influence of the N pole have EMFs induced out towards an observer, as shown by the dots, while conductors under the influence of the S pole have EMFs induced away from an observer, as shown by the crosses. Conductors (such as A and A′) which are one *pole pitch* apart comprise one coil, the two ends of which are connected to adjacent commutator segments. Thus, the whole winding is made up of a series of coils, each connected to adjacent commutator segments, so that the winding itself forms a complete closed circuit as shown in figure 12.7b.

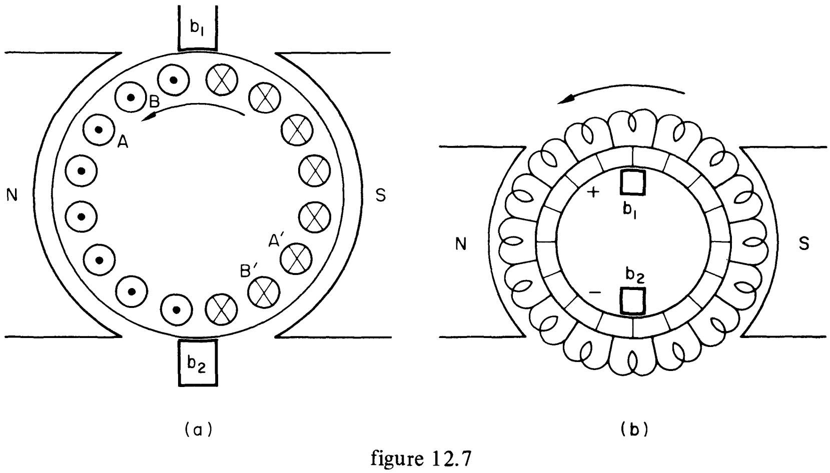


figure 12.7

The fixed brushes b_1 and b_2 are positioned so that they make contact with, and periodically short-circuit, coils at a time when no EMF is being induced in them. They are thus positioned midway between the poles, on the *magnetic neutral axis*, as shown in figure 12.7a. Between the brushes, however, lie a number of coils connected in series, and forming two equal parallel circuits. The EMF in each coil has the waveform shown in figure 12.4; but, because of the commutator action, the EMF between the brushes has the waveform shown in figure 12.6. The total EMF between the brushes is obtained by adding several such waveforms together. Figure 12.8 shows the result of adding together the EMFs of two coils AA′ and BB′, the resultant EMF now never falling to zero. With several coils in series, the fluctuation in the EMF between the brushes is still further reduced as shown in figure 12.8. Some small fluctuation, termed *commutator ripple*, is still present, but in a practical machine the large number of coils on the armature ensures that for all practical purposes the output EMF between the brushes is constant in value.

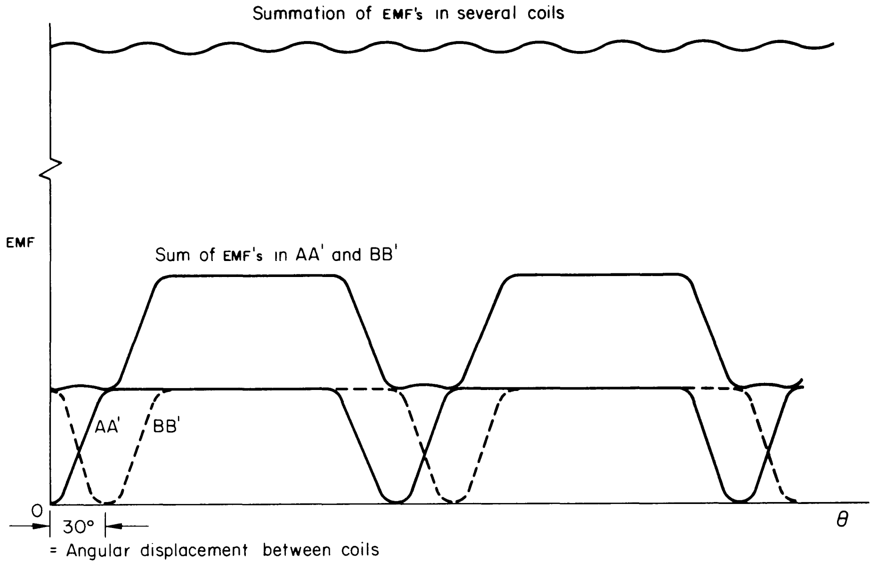


figure 12.8

Types of armature winding

With two pole machines, the arrangement of the winding is such that there are always two paths in parallel through the winding. If Z is the total number of conductors on the armature, the number of conductors connected in series between the brushes is then $Z/2$. However, a number of pairs of poles arranged alternately N and S may be used, with the two conductors that comprise one coil being arranged one pole-pitch apart. The coils may then be connected together in either of two basic ways: firstly, they may be connected so as to form two paths in parallel, irrespective of the number of poles, the arrangement then being known as a *wave* winding; secondly, they may be connected to form as many paths in parallel through the armature as there are poles on the machine, the arrangement then being termed a *lap* winding. The generated EMF between the output terminals of a multi-pole machine with a wave-wound armature is clearly greater than that for the same machine with a lap-wound armature, because there are more conductors connected in series. Nevertheless, as the conductors on the two armatures are all rated to carry equal currents, the total current output for the lap-wound armature is greater than that for the wave-wound armature, because there are more conductors connected in parallel. As a result, the power outputs of the two armatures are identical.

The EMF equation

Let Z = the total number of conductors on the armature
p = number of *pairs* of poles
Φ = flux in webers entering or leaving each pole
a = number of *pairs* of paths in parallel through the armature ($a = 1$ for a wave winding; $a = p$ for a lap winding)
n = speed of armature in revolutions per second.

Flux cut by each conductor in one revolution = $2p\Phi$ webers.
Flux cut by each conductor per second = $2p\Phi n$ webers.
Average induced EMF in each conductor = $2p\Phi n$ volts.
Number of conductors in series = $Z/2a$.
Total EMF between brushes = $2p\Phi n \times Z/2a$.

$$E = \Phi Z n \frac{p}{a} \quad \text{volts} \tag{12.4}$$

Example 12.2 A 4-pole d.c. generator has an armature with 480 conductors which is driven at 800 rev/min. The useful flux per pole is 25 mWb. Each conductor is capable of carrying 60 A without overheating. Calculate the generated EMF and the total armature current if the winding is connected (a) wave and (b) lap. Calculate also the electrical power generated in each case.

(a) Wave ($a = 1$ and $P/a = 2$)

$$E = \Phi Z n \frac{p}{a}$$

$$= 25 \times 10^{-3} \times 480 \times \frac{800}{60} \times 2$$

$$= 320\,\text{V}$$

Total armature current I_a

$$= \text{current per conductor} \times \text{paths in parallel}$$

$$= 60 \times 2$$

$$= 120\,\text{A}$$

$$\text{Power generated} = E \times Ia$$
$$= 320 \times 120$$
$$= 38{\cdot}4 \text{ kW}$$

(b) Lap ($a = p$ and $p/a = 1$)

$$E = \Phi Z n \frac{p}{a}$$
$$= 25 \times 10^{-3} \times 480 \times \frac{800}{60} \times 1$$
$$= 160 \text{ V}$$
$$I_a = 60 \times 4 = 240 \text{ A}$$
$$\text{Power generated} = E \times Ia$$
$$= 160 \times 240$$
$$= 38{\cdot}4 \text{ kW}$$

Methods of excitation

The field coils of a d.c. machine are connected in series and together comprise the field circuit. The armature winding is connected via the brushes to the terminals of the machine. Figure 12.9 shows the usual way of representing these two windings.

There are several ways of supplying the field current, and the characteristics of the machine are largely determined by the method used. The field circuit may be supplied from some entirely separate d.c. source, this being referred to

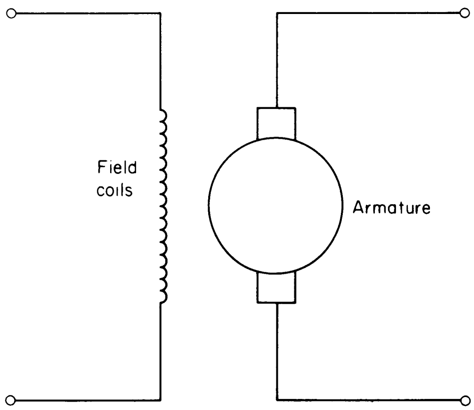


figure 12.9

as *separate excitation*. It is, however, an advantage if the generator supplies its own field current, an arrangement that is referred to as *self-excitation*. This may be achieved in three ways: the field circuit may be connected direct to the armature terminals and in parallel with the external load circuit, this being referred to as *shunt-excitation*; the field circuit may be connected in series with the external load circuit so that the load current flows through the field coils, this being termed *series-excitation*; or the machine may have both shunt field and series field circuits, this being termed *compound-excitation*.

Characteristic curves

The EMF of the d.c. generator varies with the speed and with the field current, and the terminal voltage varies with the current flowing in the load. Curves showing the way in which these quantities vary are known as *characteristic curves* for the machine. The relationships between the EMF and the field current, and the EMF and the speed, are obtained with the circuit of figure 12.10a, in which the field circuit is excited from a separate and adjustable d.c. supply, and where the unloaded generator can be driven at variable speed. The voltmeter V measures the armature terminal voltage which, since the machine is unloaded, is equal to the induced EMF.

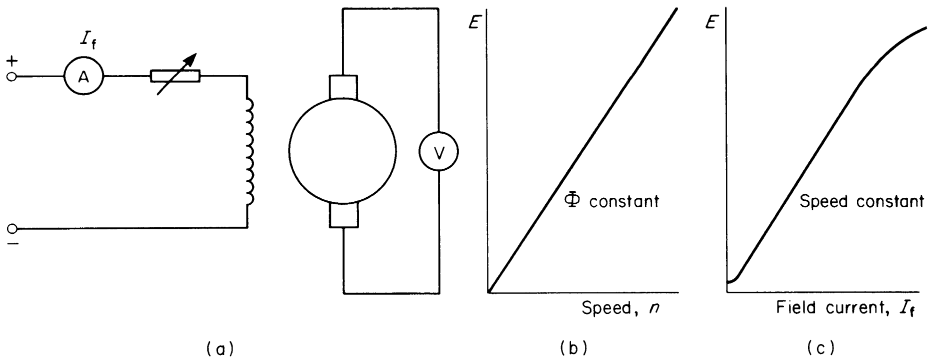


figure 12.10

From equation 12.4

$$E = \Phi Z n \frac{p}{a} \quad \text{volts}$$

and, as $Z(p/a)$ is a constant

$$E = k\Phi n$$

The relationship between the generated EMF and the speed is therefore obtained by keeping the field current I_f, and therefore the flux Φ, constant and taking a number of readings of corresponding values of generated EMF and speed. These readings when plotted are found to lie on a straight line through the origin as shown in figure 12.10b, in accordance with the EMF equation.

If the speed is now kept constant and the field current I_f is gradually increased from zero, the curve of figure 12.10c is obtained. The EMF at first increases linearly with the field current, but the rate of increase of the EMF soon becomes less as the field current is increased because of the gradual magnetic saturation of the iron parts of the magnetic circuit. The curve is similar in shape to the magnetisation curve for iron, and the curve of E against I_f is known as the *magnetisation curve* for the machine, or alternatively as the *open circuit characteristic* (o.c.c.).

If, on reaching a given maximum value, the field current is gradually reduced, it is found that the descending curve lies slightly above that obtained with ascending values of field current. This is the result of hysteresis in the iron. When the field current is reduced to zero, a small EMF is generated because of the residual flux. This small EMF is important in the operation of a self-excited generator.

Load characteristics of the separately-excited generator

Figure 12.11a shows the circuit used to determine the way in which the terminal voltage varies with the load current. As the field current is supplied from an entirely separate source, it remains entirely unaffected by the load current, and thus the flux and the generated EMF remain practically constant for all values of load current.

The terminal voltage V, however, falls in a linear manner as the load current I_L increases, as shown in figure 12.11b, the terminal voltage V being less than the generated EMF E by the volts drop $I_a r_a$ in the armature. Hence

$$V = E - I_a r_a \qquad (12.5)$$

where $I_a = I_L$.

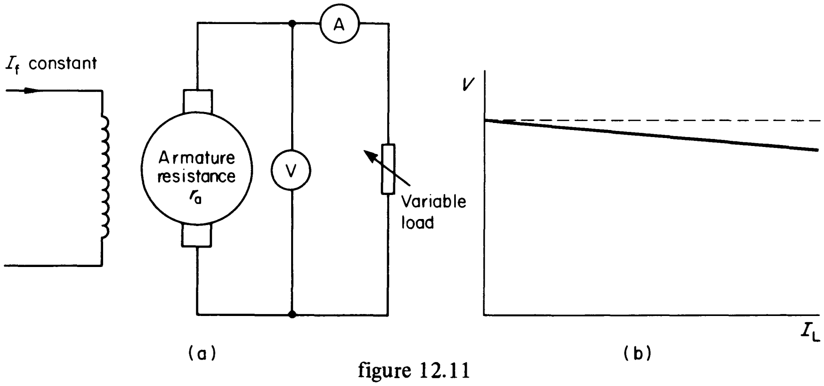


figure 12.11

The shunt-excited generator

The field circuit in the shunt-excited generator is connected directly to the armature terminals and in parallel with the load as shown in figure 12.12.

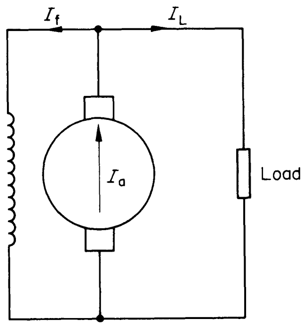


figure 12.12

The total armature current is given by the sum of the field and load currents. As the field coils contain a large number of turns of relatively thin wire, the field circuit has a high resistance so that the field current is small. The product of this small current and the large number of turns provides the necessary MMF to establish the magnetic flux. The self-excitation property depends upon the small residual flux, which is present after the field system has once been magnetised. When the armature is rotated, the small EMF generated because of the residual flux produces a small field current which, provided that it is in the correct direction, causes an increase in the flux. This in turn leads to an increase in the EMF and in the field current, the process thus being cumulative. If the open-circuit characteristic was a straight line, the conditions would be unstable and the EMF and field current would increase indefinitely. But because the curve bends over as the iron in the magnetic circuit saturates, the rate of increase of the EMF eventually becomes less than the rate of increase of the field current, so that a stable point is reached, the position of which depends upon the resistance of the field circuit. The building-up process may be followed by reference to figure 12.13, where the straight line OP represents the resistance of the field circuit, based upon the fact that the voltage applied to the field circuit divided by the current in the field circuit is constant and equal to the resistance of that circuit.

Assume the generator EMF to be building up, so that when the field current is OX, the EMF is XZ. Of this, XY volts are required to maintain the current against the resistance of the circuit, leaving YZ volts available to increase the current against the EMF of self-inductance [$L(\mathrm{d}i/\mathrm{d}t)$] produced by the increase. Hence, the EMF rises until the point P is reached, where the resistance line intersects the open-circuit characteristic. At this point, no

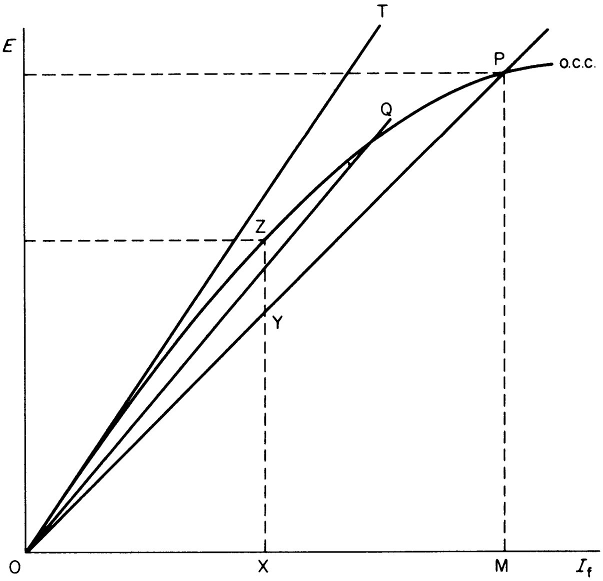


figure 12.13

further increase is possible, and the field current becomes constant at OM, which produces an EMF MP just sufficient to maintain the current against the resistance of the field circuit.

Critical resistance

If a variable resistance is included in the field circuit, different resistance lines can be obtained. If the field circuit resistance is increased, a resistance line of greater slope is obtained, cutting the o.c.c. curve at Q, thus reducing the generated EMF. If the field circuit resistance continues to be increased, the resistance line OT is obtained, tangential to the o.c.c. curve. This resistance is termed the *critical resistance*, above which the generator fails to excite.

Example 12.3 The magnetisation curve for a separately-excited generator when driven at constant speed is given by the following readings (table 12.1):

table 12.1

I_f (amperes)	0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6
E (volts)	8	40	80	117	153	181	202	221	233

Determine the voltage to which the machine will excite if the generator is shunt connected and the total resistance of the field circuit is 160 Ω. Estimate also the critical resistance of the field circuit above which the generator will fail to excite.

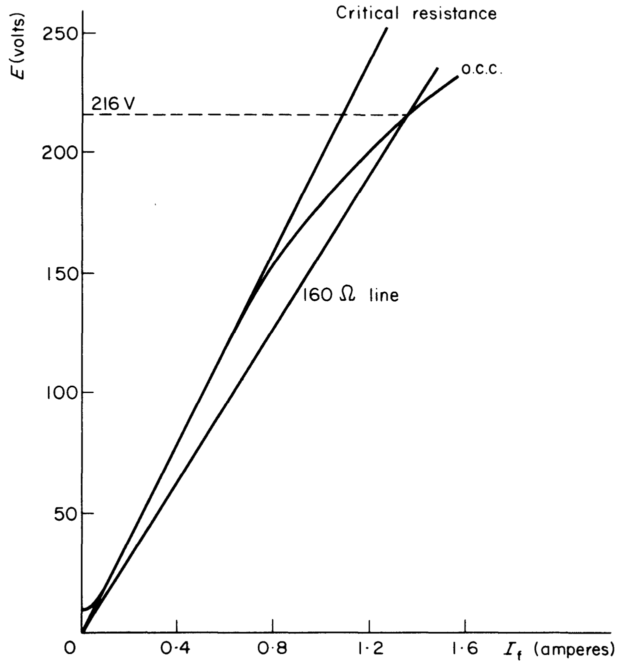


figure 12.14

The magnetisation curve is drawn in figure 12.14, and by selecting a point with co-ordinates 160 V, 1·0 A, the 160 Ω resistance line is drawn. This intersects the o.c.c. at a voltage of 216 V; hence

Voltage to which generator excites
= 216 V

A tangent drawn to the curve passes through the co-ordinates 200 V, 1·0 A, representing a resistance line for 200 Ω, hence

Critical field circuit resistance
= 200 Ω

The shunt generator on load

An increase in load current causes an increase in armature current which, as in the separately-excited generator, leads to a decrease in terminal voltage because of *IR* drop in the armature. But the fall in terminal voltage now leads to a reduction in field current, thus reducing the generated EMF and leading to a further fall in terminal voltage. The fall in terminal voltage is thus more pronounced than in a separately-excited generator. An increase in load current is produced by a decrease in load resistance, and if the load resistance is continuously reduced, the increased effects of the armature *IR* drop and reduction in flux eventually causes the generated EMF and terminal voltage to fall more rapidly than the load resistance, with the result that the load current actually decreases, and finally the generator can be short-circuited. When short-circuited, some current flows in the external circuit because an EMF is generated as a result of the residual flux. The shape of the load characteristic is shown in figure 12.15. Normal full load current is much less than the maximum possible current.

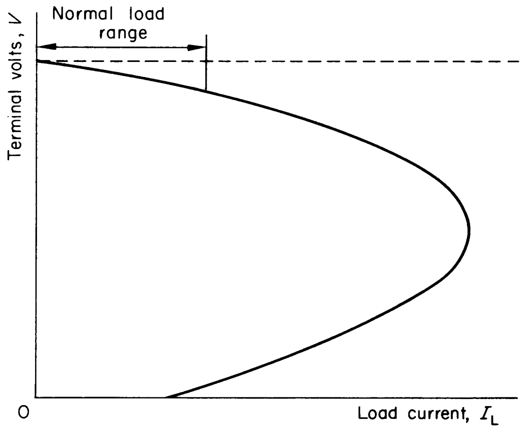


figure 12.15

The series-excited generator

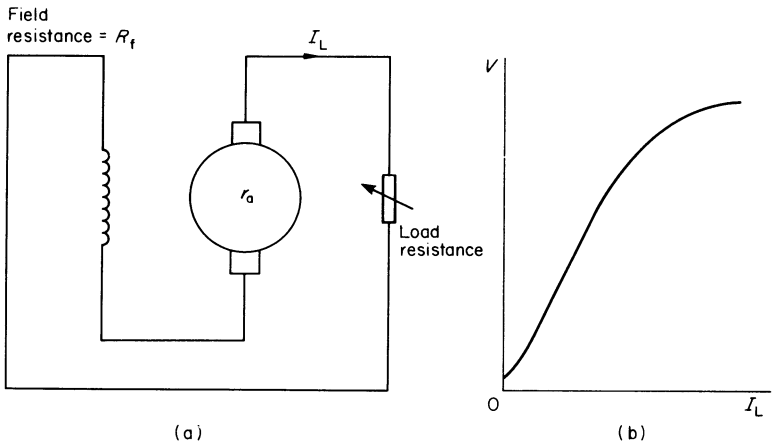


figure 12.16

Figure 12.16a shows the connections for the series-excited generator, where it is seen that the field circuit is connected in series with the external load circuit. As the field coils carry the load current, they consist of a few turns of heavy gauge conductor. The generated EMF is dependent upon the load current, and until saturation is approached, is almost proportional to it. At any given load current, the terminal voltage is given by

$$V = E - I_L(r_a + R_f) \tag{12.6}$$

The general shape of the load characteristic shown in figure 12.16b makes the series-excited generator unsuitable for normal service.

The compound-excited generator

The fall in terminal voltage of the shunt generator can be compensated by a series winding, arranged to assist the shunt winding when the machine is supplying a load. The generator is then compound-excited, and arranged in this way, is described as *cumulatively-compounded.* Various degrees of compounding are possible, depending upon the number of series turns used: if *over-compounded*, increasing load gives a rising terminal voltage characteristic that is useful when, for example, it is required to maintain a constant voltage at the end of a long feeder; if *level-compounded,* the terminal voltage on full-load is equal to the no-load terminal voltage; and if *under-compounded,* the result is a falling terminal voltage characteristic.

The series field may also be connected so that its ampere-turns oppose those of the shunt field, and the generator is then described as *differentially-compounded.* The terminal voltage then falls even more rapidly than for a shunt-excited machine.

The above characteristics are shown in figure 12.17, which also shows the shunt generator characteristic for comparison.

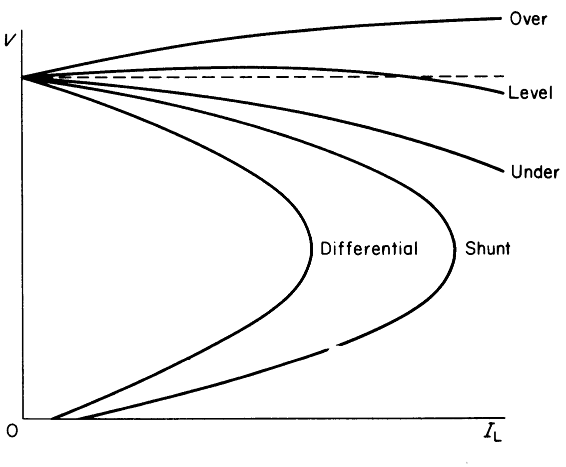


figure 12.17

THE D.C. MOTOR

When a separately-excited d.c. machine is driven mechanically by a prime-mover, the generated EMF causes an armature current I_a to flow, so that electrical power is supplied to a load. The terminal voltage V has been seen to be less than the generated EMF E by the volts drop $I_a r_a$ in the armature, so that

$$V = E - I_a r_a$$

which is the equation relating to the machine acting as a *generator*. Rearranging the equation, the armature current delivered by the generator is given by

$$I_a = \frac{E - V}{r_a}$$

The machine may be connected to an electrical supply system and thus be working in parallel with a number of other generators. The current delivered to the supply system is seen to be dependent upon the difference between the generated EMF and the terminal voltage, the terminal voltage being constant. To increase the current and power supplied, the power output of the prime mover is increased. This tends to increase the speed of the generator, resulting in an increase in the generated EMF, and the machine converts the mechanical energy supplied by the prime mover into electrical energy. If, on the other hand, the generated EMF is reduced to a value less than that of the constant terminal voltage, a reversal in the direction of current (and thus of energy) flow takes place, and the machine converts electrical energy into mechanical energy. The armature current is now given by

$$I_a = \frac{V - E}{r_a}$$

from which

$$V = E + I_a r_a \tag{12.7}$$

which is the equation relating to the machine acting as a *motor*. The current flowing in the armature conductors now produces an armature flux that interacts with the main flux to produce a torque on the armature. Only a proportion of the armature conductors (namely, those under the poles at any instant) contribute to this torque, as shown in figure 12.18.

The torque produced is proportional to the armature current and to the main flux, that is

$$T \propto \Phi I_a \qquad \text{or} \qquad T = k\Phi I_a \tag{12.8}$$

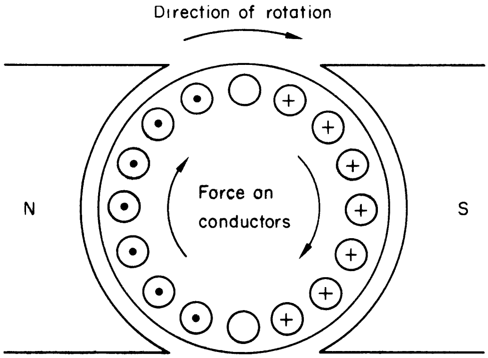


figure 12.18

The armature thus rotates in the direction shown, and electrical energy is now supplied to it. No prime mover is required. Instead, a mechanical load may be applied to the armature. An increase in mechanical load necessitates an increase in the current supplied to the armature. The armature speed thus decreases, slightly reducing the generated EMF. As $I_a = (V - E)/r_a$, the armature current I_a increases, until the electrical power supplied to the armature is sufficient to provide for the mechanical power required by the load. The EMF induced in the armature conductors when they cut the main flux Φ is in the opposite direction to the current flow and to the applied voltage, and is thus often referred to as a 'back EMF'. The value of this EMF is given by equation 12.4, that is

$$E = \Phi Z n \frac{p}{a} \quad \text{volts}$$

from which

$$E = \mathrm{k}\Phi n \tag{12.9}$$

since Z, p and a are constants.

Power output of the armature

Equation 12.7 relating to the machine acting as a motor gave

$$V = E + I_a r_a$$

Multiplying through by I_a gives

$$VI_a = EI_a + I_a^2 r_a$$

VI_a is the product of the supply voltage and armature current and is thus the electrical power to the armature. $I_a^2 r_a$ is the ohmic power loss in the armature winding. Hence, the total power output of the armature is given by the generated (or back) EMF times the armature current.

Example 12.4 The armature of a 4-pole motor is 30 cm in diameter and has a wave winding with 378 conductors, each of effective length 20 cm. The armature rotates at a speed of 750 rev/min, and at any time 70 per cent of the conductors lie under the poles, in a field of density 0·6 T. Calculate the torque developed and the power output of the armature, when the armature current is 80 A.

Armature is wave-wound with two paths in parallel

∴ Current per conductor

$$= \frac{80}{2} = 40\ \text{A}$$

Force on each conductor when in the field $= BIl$

$$= 0{\cdot}6 \times 40 \times 0{\cdot}2 = 4{\cdot}8\ \text{N}$$

$$\text{Total torque } T = \text{force} \times \text{radius} \times \text{no. of effective conductors}$$

$$= 4{\cdot}8 \times \frac{0{\cdot}3}{2} \times 0{\cdot}7 \times 378$$

$$= 190{\cdot}5\ \text{Nm}$$

$$\text{Power output} = 2\pi nT$$

$$= 2\pi \times \frac{750}{60} \times 190{\cdot}5 \quad \text{watts}$$

$$\simeq 15\ \text{kW}$$

Example 12.5 A d.c. motor has its field and armature circuits connected in parallel to a 240 V d.c. supply. The armature resistance is 0·6 Ω, and the load is such that the armature current is 20 A. Calculate the value of the induced EMF, and the total power output of the armature.

$$E = V - I_a r_a$$

$$= 240 - (20 \times 0{\cdot}6)$$

$$= 228\ \text{V}$$

$$\text{Power output} = EI_a$$

$$= 228 \times 20 \quad \text{watts}$$

$$= 4{\cdot}56\ \text{kW}$$

D.C. motor characteristics

The chief characteristics are those that connect speed with armature current and those that connect torque with armature current. The characteristics depend upon the method of excitation used, and these are the same as for generators, except that there is no need to distinguish between separate-excitation and shunt-excitation.

Shunt-excitation

Figure 12.19 shows the circuit for the shunt-excited motor, in which the field circuit is connected in parallel with the armature to the constant voltage supply.

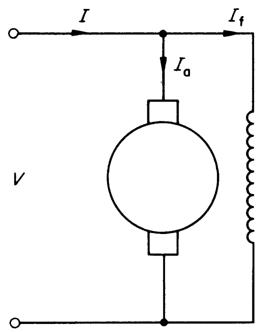


figure 12.19

The total current taken from the supply is the sum of the field and armature currents; and because the field current is constant the flux is also constant. An increase in load torque requires an increase in armature current, which, in accordance with $I_a = (V - E)/r_a$, requires a decrease in generated EMF. This decrease in EMF is not large (because the armature resistance is always low) and is brought about by the small decrease in speed which naturally occurs when a mechanical load is applied. The small decrease in EMF permits a large increase in armature current, and the speed of the motor automatically adjusts itself to the load so that the torque produced by the armature is equal to the load torque (including friction and windage). As the flux is constant, the torque produced by the armature is proportional to the armature current. In figure 12.20, the speed/armature-current characteristic is shown at (a) and the torque/armature-current characteristic at (b). By plotting corresponding values of torque and speed for various values of armature current, the speed/torque characteristic shown at (c) is obtained. As the torque is proportional to the armature current this characteristic is similar to that of the speed/armature-current.

If the flux produced by the field system is reduced, by reducing the field current by means of a series resistance, the generated EMF momentarily decreases. This allows a large increase in armature current, which increases the torque produced to a value greater than that required by the load. The

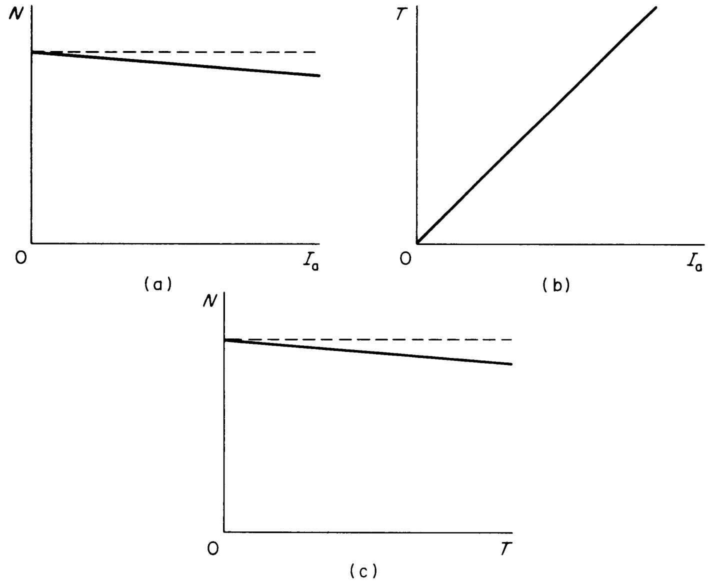


figure 12.20

surplus torque accelerates the armature and load until the generated EMF rises and thereby reduces the armature current to a steady value. The motor is now running at a higher speed in order to generate the same EMF with the reduced flux. Hence, weakening the field of a shunt motor increases the speed, and the relationship between flux and speed is given approximately by

$$N \propto \frac{1}{\Phi} \tag{12.10}$$

Example 12.6 A 240 V d.c. shunt motor has field and armature resistances of 120 Ω and 0·2 Ω respectively. When running unloaded at 500 rev/min, the total current taken from the supply is 7 A. Calculate the speed if the motor is loaded and taking a total current of 52 A.

$$I_f = \frac{240}{120} = 2\text{ A}$$

$$I_a \text{ on no-load} = 7 - 2 = 5\text{ A}$$

$$\begin{aligned} E \text{ on no-load} &= V - I_a r_a \\ &= 240 - (5 \times 0{\cdot}2) \\ &= 239\text{ V} \end{aligned}$$

$$I_a \text{ on full load} = 52 - 2 = 50\text{ A}$$

$$\begin{aligned} E \text{ on full load} &= 240 - (50 \times 0{\cdot}2) \\ &= 230\text{ V} \end{aligned}$$

As flux is constant,

$$N \propto E$$

$$\therefore \text{ Speed at full load} = \frac{230}{239} \times 500 = 481 \text{ rev/min}$$

***Example** 12.7* A 200 V shunt motor has an armature resistance of 0·2 Ω and takes an armature current of 60 A when running at 600 rev/min, the flux per pole being 0·4 mWb. If the resistance of the field circuit is increased so that the flux per pole decreases instantaneously to 0·36 mWb, calculate the new speed, assuming the load torque remains constant.

$$E_1 = 200 - (60 \times 0{\cdot}2) = 188 \text{ V}$$

When the flux is suddenly reduced, the generated EMF falls to

$$E = 188 \times \frac{0{\cdot}36}{0{\cdot}4} = 169{\cdot}2 \text{ V}$$

Resultant armature current

$$I_a = \frac{200 - 169{\cdot}2}{0{\cdot}2} = 154 \text{ A}$$

Since $T \propto \Phi I_a$, the torque increases momentarily to 0·36/0·4 × 154/60 = 2·31 times its original value. Hence the motor accelerates until the armature current is again just sufficient to provide for the load. Since the load torque is constant, $\Phi_2 I_{a_2} = \Phi_1 I_{a_1}$

$$\therefore \qquad I_{a_2} = 60 \times \frac{0{\cdot}4}{0{\cdot}36}$$

$$= 66{\cdot}7 \text{ A}$$

$$E_2 = 200 - (66{\cdot}7 \times 0{\cdot}2) = 186{\cdot}7 \text{ V}$$

As the speed is proportional to E and inversely proportional to Φ,

$$\text{New speed} = 600 \times \frac{0{\cdot}4}{0{\cdot}36} \times \frac{186{\cdot}7}{188} = 662 \text{ rev/min.}$$

The change in generated EMF is seen to be small and the speed is thus inversely proportional to the flux.

Series excitation

Because the field circuit is connected in series with the armature to the supply, the armature current is also the field current, and the value of the

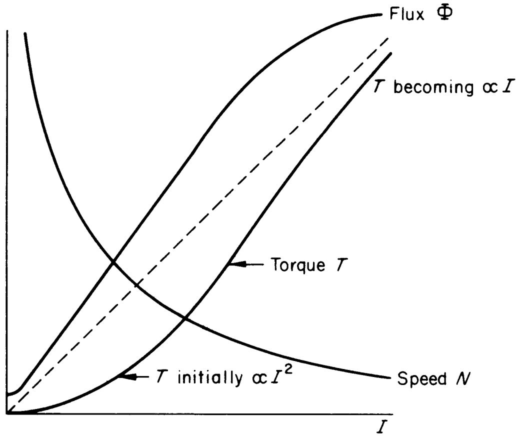


figure 12.21

flux varies with this current in accordance with the magnetisation curve shown in figure 12.21.

As $N \propto 1/\Phi$ (neglecting change in generated EMF caused by the motor resistance), the shape of the speed/current curve may be obtained by plotting reciprocals of flux against current. On light load, the current is small and the field is weak, and there is a possibility of dangerously high speeds. Series motors must therefore be coupled to a mechanical load either directly or through gearing but never by belt.

Now

$$T \propto \Phi I$$

but initially

$$\Phi \propto I$$

$\therefore$

$$T \propto I^2$$

When the iron becomes magnetically saturated, Φ becomes almost constant, so that $T \propto I$. Before the iron is saturated, however, the motor exerts a torque that is proportional to the square of the current. The starting torque is therefore high, which makes the motor especially suitable for driving cranes and trains, where large masses have to be quickly accelerated, and where light running is impossible.

Example 12.8 A 460 V series motor runs at 500 rev/min, taking a current of 40 A, and developing a torque of 100 Nm. Calculate the speed and the torque if the load is reduced so that the motor is taking 30 A. Total resistance of armature and field circuit is 0·8 Ω. Assume the flux to be proportional to the field current.

$N \propto (E/\Phi)$ and since $\Phi \propto I$, then $N \propto (E/I)$

$$E \text{ when } I \text{ is } 40\text{ A} = 460 - (40 \times 0{\cdot}8) = 428\text{ V}$$

$$E \text{ when } I \text{ is } 30\text{ A} = 460 - (30 \times 0{\cdot}8) = 436\text{ V}$$

$\therefore$ Speed when current is 30 A

$$= 500 \times \frac{436}{428} \times \frac{40}{30}$$

$$= 679 \text{ rev/min}$$

$T \propto \Phi I$ and since $\Phi \propto I$ then $T \propto I^2$

$\therefore$ Torque when current is 30 A

$$= 100 \times \left(\frac{30}{40}\right)^2$$

$$= 56{\cdot}25 \text{ N m}$$

Compound excitation

As with the generator, the series field may be connected either to oppose or to assist the shunt field. With the first method of connection the motor is described as *differentially compounded*; with the second method of connection it is described as *cumulatively compounded.*

When differentially compounded, an increase in load causes a decrease in flux and hence an increase in speed. The number of series turns may be so adjusted that speed is approximately constant over the normal load range. However, differential compound motors are rarely used, because the speed characteristics of the shunt motor satisfy most normal load requirements.

With the cumulative compound motor, increasing load causes an increase in flux and therefore a decrease in speed, though this is not so marked as in the plain series motor. This type of motor is used for driving continuously running rolling mills, when it is also coupled to a fly wheel. When a billet is inserted between the rollers, the increased load causes the motor to slow down, and the flywheel then gives up some of its kinetic energy to help drive the billet through the rollers. When the billet has passed through the rollers, the resultant decrease in load causes both the motor and flywheel to speed up, so that energy continues to be drawn from the supply and to be stored as kinetic energy in the flywheel. The tendency is thus towards an even flow of energy taken from the supply.

The speed/armature-current characteristics for the various types of d.c. motors are shown together in figure 12.22.

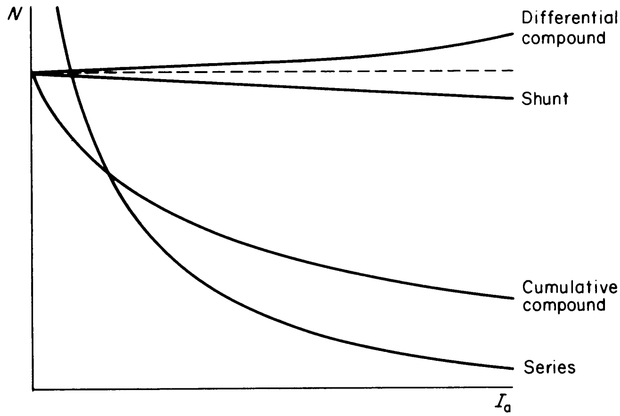


figure 12.22

MISCELLANEOUS EXAMPLES

Example 12.9 State Faraday's law of electromagnetic induction and use it to explain the electrical effects that occur in a conductor that is cutting a magnetic field.

A rectangular coil of length 40 cm and breadth 20 cm and containing 1000 turns is able to rotate in a uniform magnetic field about an axis in the plane of the coil and joining the middle points of the two short sides. The field of 0·015 T is perpendicular to the axis. The coil rotates at a uniform speed of 10 rev/s.

Show that as the open-circuited coil rotates a sinusoidal EMF is generated in it, and find the trigonometrical expression representing this EMF.

[$e = 24\pi \sin 20\pi t$ volts]

(CGLI Principles B, 1964)

Example 12.10 Describe the principle of a simple generator and show how a commutator rotating with its shaft provides a d.c. output. What factors in the design of a d.c. generator determine the output voltage at its full load? Explain why, when the field winding is shunt connected, the output voltage varies as the load changes, although the speed of the armature is constant.

(CGLI Principles B, 1964)

Example 12.11 Describe with the aid of sketches the principle of the simple d.c. generator. Explain

(a) the function of the commutator
(b) the meaning of the term *shunt-connected* field coils.

A 2-pole d.c. generator has a flux per pole in the air gap of 5 mWb. The rotating armature coil has 240 turns (that is, 480 conductors) connected in series. Calculate the average EMF at the brushes, when the machine is running at 1000 rev/min with negligible current load.

[40 V]

(CGLI Principles B, 1967]

Example 12.12 The curve of induced EMF for a separately-excited d.c. generator when driven at 975 rev/min on open-circuit is given by table 12.2.

table 12.2

EMF (volts)	10	44	73	98	113	122	127
Field current (amps)	0	0·2	0·4	0·6	0·8	1·0	1·2

Plot the curve of induced EMF against field current for the generator when driven at 750 rev/min. To what voltage will the generator excite when shunt-connected and driven at 750 rev/min if the resistance of the field circuit is 100 Ω?

[91 V]

Example 12.13 A shunt generator delivers 50 kW at 250 V and 400 rev/min. The armature and field resistances are 0·02 Ω and 50 Ω respectively. Calculate the speed of the machine running as a shunt motor and taking 50 kW at 250 V.

[387 rev/min]

Example 12.14 What factors determine the starting torque of a d.c. motor? Explain why a d.c. series connected motor has a high starting torque.

The armature of a d.c. series motor has 400 uniformly spaced conductors (that is, 200 turns). The diameter of the armature is 25 cm and the conductors are each 25 cm long. If the starting current taken by the motor is 10 A, giving a magnetic field in the air gap of density 0·5 T, calculate the starting torque given by the motor. Assume that the pole faces of the field coils embrace the whole surface of the armature.

[62·5 N m]

(CGLI Principles B, 1967)

Example 12.15 A 250 V d.c. series motor has the magnetisation characteristic defined by table 12.3 when driven as a separately-excited generator at 750 rev/min.

table 12.3

I_f (amperes)	10	30	50	70
E (volts)	91	210	252	264

Calculate and plot to a base of input current, the speed and torque characteristics of the machine when running as a motor from a 250 V supply. Armature and field winding resistances total 0·4 Ω.

[Speed (rev/min)	2030	850	685	630]
[Torque (N m)	11·6	80·2	161	236]

Example 12.16 A 250 V d.c. shunt motor has an armature resistance of 0·5 Ω and a field circuit resistance of 125 Ω. It takes a current of 30 A when driving a load at 1000 rev/min, the load torque being constant. If the flux and field current can be assumed to be proportional, calculate the final speed and armature current if the field circuit resistance is slowly increased to 150 Ω. Why is a sudden change of field circuit resistance undesirable? (B)

[1186 rev/min; 36 A]

13 Transmission lines

(B) BASIC CONCEPTS AND EXPRESSIONS

Electric signals are conveyed over a distance from one point to another by means of transmission lines. In their simplest form these consist of either two parallel conductors or two concentric conductors, uniformly spaced throughout the length of the line. If a source is connected to one end (termed the *sending end* of such a line) a voltage does not suddenly appear at the other (that is, at the *receiving end* of the line). Let a number of voltmeters, considered ideal in the sense that they measure instantly a potential difference without in any way affecting it, be imagined connected at progressive intervals along the line. At time $t = 0$, let the signal source apply a steady potential difference V volts to the sending end, and at the same time imagine an observer moving along the line at a velocity approaching 3×10^8 m/s, to record the voltmeter readings. Such an observer would find that the voltage is established along the line progressively and uniformly, though successive voltmeters may record an ever-decreasing voltage. The finite value of the velocity with which the voltage wave is propagated down the line occurs because the line conductors have series inductance and shunt capacitance. The reduction, or attenuation, of the voltage occurs because the line conductors have series resistance, and the dielectric between the conductors has leakage conductance. These properties of a line are uniformly distributed along its length. In order to determine the velocity of wave propagation, it is thus necessary to obtain expressions for the inductance and capacitance of the line.

Self-inductance of two parallel conductors

Let two long parallel cylindrical conductors A and B each of radius r metres be spaced at a distance S metres between conductor centres as shown in figure 13.1. Let S be large compared with r, and let the conductors carry current I amperes in opposite directions as shown.

Because of the way in which current is flowing, a magnetic field exists between the conductors, of the form already shown in figure 2.3c. The current in each conductor, however, produces its own magnetic field with circular lines of force centred upon itself, as in figure 2.3a. The total flux

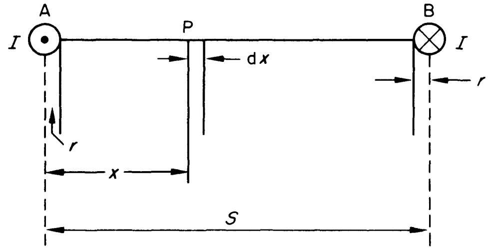


figure 13.1

linking with the circuit formed by the two conductors is thus given by the sum of the fluxes produced by each current (considered to be acting independently) in the space between the conductors, magnetic flux within the conductors being ignored. It is convenient to determine the inductance per unit length; that is, per metre, of the line. Consider a point P on the line joining the centres of the conductors, at distance x from the centre of conductor A. Then

Magnetising force H at P due to current in A

$$= \frac{I}{2\pi x} \quad \text{ampere-turns/metre.}$$

Magnetising force H at P due to current in B

$$= \frac{I}{2\pi(S-x)} \quad \text{ampere-turns/metre.}$$

Total magnetising force at P

$$= \frac{I}{2\pi}\left(\frac{1}{x} + \frac{1}{S-x}\right) \quad \text{ampere-turns/metre.}$$

Flux density at P $= \mu H$

$$= \frac{\mu I}{2\pi}\left(\frac{1}{x} + \frac{1}{S-x}\right) \quad \text{where } \mu = \mu_r\mu_0.$$

∴ Flux in an elemental area of width dx metre and length 1 metre

$$= \frac{\mu I}{2\pi}\left(\frac{1}{x} + \frac{1}{S-x}\right)\mathrm{d}x \quad \text{tesla}$$

∴ Total flux between conductors per metre length

$$= \frac{\mu I}{2\pi}\int_r^{S-r}\left(\frac{1}{x} + \frac{1}{S-x}\right)\mathrm{d}x$$

$$= \frac{\mu I}{2\pi}\Big[\ln x - \ln(S-x)\Big]_r^{S-r}$$

$$= \frac{\mu I}{2\pi}[\ln(S-r) - \ln r] - [\ln r - \ln(S-r)]$$

$$= \frac{\mu I}{\pi} \ln \frac{S-r}{r} \quad \text{weber}$$

Inductance = Flux linkages/ampere

$$\therefore \qquad L = \frac{\mu}{\pi} \ln \frac{S-r}{r} \quad \text{henry/metre} \tag{13.1}$$

Self-capacitance of two parallel conductors

Referring again to figure 13.1, let conductors A and B carry charges of $+Q$ and $-Q$ coulombs per metre length respectively, the p.d. between the conductors being V volts.

Flux density D at P due to A alone

$$= \frac{Q}{2\pi x} \quad \text{coulomb/metre}^2$$

Electric force E at P due to A alone

$$= \frac{Q}{2\pi x \epsilon} \quad \text{volts/metre} \qquad \text{where } \epsilon = \epsilon_r \epsilon_0$$

Electric force E at P due to B alone

$$= \frac{Q}{2\pi(S-x)\,\epsilon} \quad \text{volts/metre}$$

Total electric force E at P

$$= \frac{Q}{2\pi\epsilon}\left(\frac{1}{x} + \frac{1}{S-x}\right) \quad \text{volts/metre}$$

Total p.d. between conductors

$$= \frac{Q}{2\pi\epsilon}\int_r^{S-r}\left(\frac{1}{x} + \frac{1}{S-x}\right)\mathrm{d}x$$

$$= \frac{Q}{\pi\epsilon} \ln \frac{S-r}{r} \quad \text{volts}$$

$$\text{Capacitance} = \frac{Q}{V}$$

$$\therefore \qquad C = \frac{\pi\epsilon}{\ln \dfrac{S-r}{r}} \quad \text{farad/metre} \tag{13.2}$$

Self-inductance of coaxial cylinders

Let r metre be the radius of a long cylindrical conductor A, and let R be the inner radius of another cylindrical conductor B which is coaxial with the first one, as shown in figure 13.2.

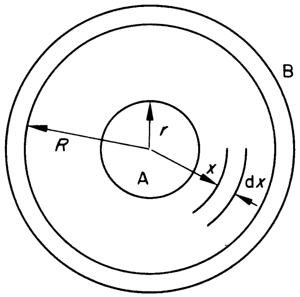


figure 13.2

Let the dielectric between the cylinders have absolute permeability $\mu_r\mu_0$ and let current I flow in opposite directions in the two conductors. Ignoring again any flux within the conductors, the self-inductance is due to the flux between the conductors, this flux linking with the current in A only and therefore being produced by this current alone. Consider a point P at radius x, the magnetising force at this point being given by

$$H = \frac{I}{2\pi x} \quad \text{ampere-turns/metre}$$

Then, flux density at P

$$= \frac{\mu I}{2\pi x}$$

Flux in an elemental area of width dx metre and length 1 metre

$$= \frac{\mu I}{2\pi x}\,\mathrm{d}x$$

Total flux between cylinders

$$= \frac{\mu I}{2\pi} \int_r^R \frac{1}{x}\,\mathrm{d}x$$

$$= \frac{\mu I}{2\pi} [\ln x]_r^R$$

$$= \frac{\mu I}{2\pi} \ln \frac{R}{r} \quad \text{weber}$$

Inductance = Flux linkages/ampere

$$\therefore \qquad L = \frac{\mu}{2\pi} \ln \frac{R}{r} \quad \text{henry/metre} \tag{13.3}$$

Self-capacitance of coaxial cylinders

Let cylinders A and B of figure 13.2 carry charges of $+Q$ and $-Q$ coulombs-per-metre-length respectively, the p.d. between the cylinders being V volts. Let the dielectric have absolute permittivity, $\epsilon = \epsilon_r \epsilon_0$.

Flux density D at radius x

$$= \frac{Q}{2\pi x} \quad \text{coulomb/metre}^2$$

Electric force E at radius x

$$= \frac{Q}{2\pi x \epsilon} \quad \text{volts/metre}$$

Total p.d. between conductors

$$= \frac{Q}{2\pi\epsilon} \int_r^R \frac{1}{x} \, dx$$

$$= \frac{Q}{2\pi\epsilon} \ln \frac{R}{r} \quad \text{volts}$$

$$\text{Capacitance} = \frac{Q}{V}$$

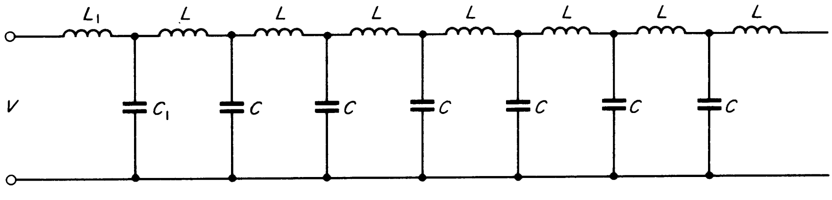
$$\therefore \qquad C = \frac{2\pi\epsilon}{\ln (R/r)} \quad \text{farad/metre} \tag{13.4}$$

THE LOSS-FREE LINE

Let a direct voltage V be applied to the sending end of a line that possesses series inductance and shunt capacitance only, so that no attenuation of the signal occurs. Although in such a line, the inductance and capacitance are uniformly distributed along its length, they may be represented by a large number of discrete components as shown in figure 13.3a.

Voltage and current

Upon the application of the voltage, a current I begins to flow through L_1 into C_1. Momentarily, the current I cannot flow beyond C_1 because it is charging C_1 and thus establishing an electric flux at that point. At the same time, the voltage V is balanced by the EMF of self-inductance due to the



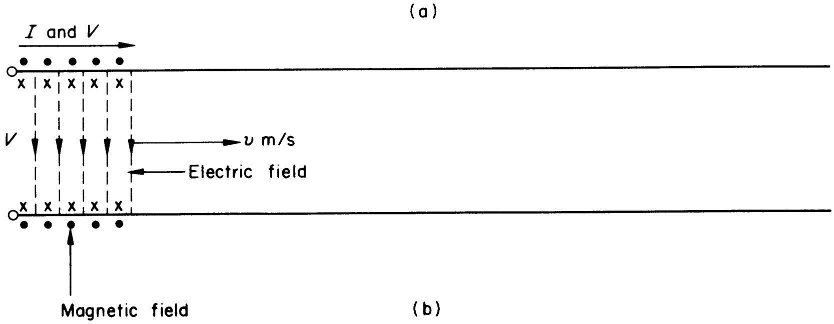


figure 13.3

increase of magnetic flux in L_1. As soon as these small increments of electric and magnetic flux have been produced, the voltage V is available to send I further along the line. The voltage V and current I thus progress at a uniform velocity v, establishing the electric and magnetic fields instantaneously and progressively along the length of the line as shown in figure 13.3b. In a short time dt, the voltage wave travels a short distance $dl = v\,dt$, establishing a magnetic flux in this section. The current in this section increases from 0 to I in time dt, the rate of increase being I/dt. The inductance of the section is $L\,dl = Lv\,dt$, where L is the inductance per unit length of line. Hence the EMF induced in this section (given by the inductance multiplied by the rate of change of current) is $e = -Lv\,dt(I/dt) = -LvI$. This EMF is balanced by the voltage between the conductors, hence

$$V = LvI \tag{13.5}$$

At the same time, the voltage wave establishes an electric flux, the voltage across the section increasing from 0 to V in time dt, with a rate of increase, V/dt. The capacitance of the section is $C\,dl = Cv\,dt$ where C is the capacitance per unit length of line. The current flowing in the section (given by the capacitance multiplied by the rate of change of voltage) is $I = Cv\,dt(V/dt)$.

$$\therefore \quad I = CvV \tag{13.6}$$

Velocity of propagation

Multiplying equation 13.5 by equation 13.6

$$V \times I = LvI \times CvV$$

$$v^2 = \frac{1}{LC}$$

that is, velocity of propagation, v

$$= \frac{1}{\sqrt{(LC)}} \text{ metres/second} \qquad (13.7)$$

For two parallel conductors in air, where $\mu_r = 1$ and $\epsilon_r = 1$

$$L = \frac{\mu_0}{\pi} \ln \frac{S-r}{r} \quad \text{henry/metre}$$

and

$$C = \frac{\pi\epsilon_0}{\ln \frac{S-r}{r}} \quad \text{farad/metre}$$

$\therefore$

$$LC = \mu_0\epsilon_0$$

$$= 4\pi \times 10^{-7} \times \frac{1}{4\pi \times c^2 \times 10^{-7}}$$

where c is the velocity of electromagnetic radiation in free space in metres per second.

$$= \frac{1}{c^2}$$

$\therefore$

$$v = \frac{1}{\sqrt{(1/c^2)}} = c \triangleq 3 \times 10^8 \text{m/s} \qquad (13.8)$$

The velocity of propagation of the voltage wave along a parallel conductor transmission line in air, having no resistance and no leakage conductance, is thus equal to the velocity of electromagnetic radiation in free space.

For a concentric cable having a solid dielectric, $\mu_r = 1$ but ϵ_r is greater than unity, depending upon the material of the dielectric, so that

$$L = \frac{\mu_0}{2\pi} \ln \frac{R}{r} \quad \text{henry/metre}$$

and

$$C = \frac{2\pi\epsilon_r\epsilon_0}{\ln \frac{R}{r}} \quad \text{farad/metre}$$

$$\therefore \quad LC = \mu_0\epsilon_0\epsilon_r = 4\pi \times 10^{-7} \times \frac{1}{4\pi \times c^2 \times 10^{-7}} \times \epsilon_r$$

$$= \frac{\epsilon_r}{c^2}$$

and

$$v = \frac{c}{\sqrt{(\epsilon_r)}} \simeq \frac{3 \times 10^8}{\sqrt{(\epsilon_r)}} \quad \text{metres/second} \qquad (13.9)$$

Characteristic impedance

Consider an infinitely long loss-free line with a signal source applying a p.d. of V volts to the sending end, so that a voltage wave V moves along the line and a finite current I flows into the line. Evidently the line does not behave to the source as an open circuit, but presents an impedance $Z_0 = V/I$ to the generator, where Z_0 is termed the *characteristic impedance* of the line. Dividing equation 13.5 by equation 13.6

$$\frac{V}{I} = \frac{LvI}{CvV} \qquad \therefore \frac{V^2}{I^2} = \frac{L}{C}$$

$$\therefore \quad \frac{V}{I} = Z_0 = \sqrt{\left(\frac{L}{C}\right)} \quad \text{ohms} \qquad (13.10)$$

Open-wire line

For an open-wire line

$$L = \frac{\mu}{\pi} \ln\left(\frac{S-r}{r}\right) \quad \text{henry/metre} \qquad \text{(as in equation 13.1)}$$

Substituting $\mu = \mu_0\mu_r = 4\pi \times 10^{-7} \times 1$ for a non-magnetic medium such as air, and converting the logs to base 10

$$L = \frac{4\pi}{10^7} \times \frac{2{\cdot}303}{\pi} \log_{10}\left(\frac{S-r}{r}\right)$$

$$= 0{\cdot}92 \log_{10}\left(\frac{S-r}{r}\right) \quad \text{microhenry/metre} \qquad (13.11)$$

Also

$$C = \frac{\pi\epsilon}{\ln\dfrac{S-r}{r}} \quad \text{farad/metre} \qquad \text{(as in equation 13.2)}$$

Substituting

$$\epsilon = \epsilon_0\epsilon_r = \frac{1}{4\pi \times c^2 \times 10^{-7}} \times 1$$

$$= \frac{1}{36\pi \times 10^9} \quad \text{for air dielectric}$$

Converting the logs to base 10 as before,

$$C = \frac{\pi}{36\pi \times 10^9 \times 2{\cdot}303 \log_{10}\left(\frac{S-r}{r}\right)}$$

$$= \frac{12{\cdot}1}{\log_{10}\left(\frac{S-r}{r}\right)} \quad \text{picofarad/metre} \tag{13.12}$$

Therefore, as

$$Z_0 = \sqrt{\left(\frac{L}{C}\right)}$$

$$Z_0 = \log_{10}\left(\frac{S-r}{r}\right)\sqrt{\left(\frac{0{\cdot}92}{10^6} \times \frac{10^{12}}{12{\cdot}1}\right)}$$

that is, characteristic impedance of an open-wire line is

$$Z_0 = 276 \log_{10}\left(\frac{S-r}{r}\right) \quad \text{ohms} \tag{13.13}$$

Example 13.1 An overhead line 48 km in length is to be constructed of two parallel wires which are 2 mm in diameter. The total inductance of the line must not exceed 0·1H. Estimate the maximum permissible spacing of the conductors.

Since

$$L = \frac{0{\cdot}1 \times 10^6}{48 \times 10^3} \quad \mu\text{H/m}$$

$$= 2{\cdot}0833 \ \mu\text{H/m}$$

$$= 0{\cdot}92 \log_{10}\left(\frac{S-r}{r}\right)$$

$$\log_{10}\left(\frac{S-r}{r}\right) = \frac{2{\cdot}0833}{0{\cdot}92} = 2{\cdot}2645$$

$$\therefore \qquad \frac{S-r}{r} = 183{\cdot}9$$

from which it will be found that $S = 0{\cdot}1849$ m; that is, the wires must not be separated by more than 0·185 metre.

Coaxial line

By a similar method to that used in producing equation 13.13, it will be found that the characteristic impedance of a coaxial line is given by

$$Z_0 = \frac{138}{\sqrt{(\epsilon_r)}} \log_{10}\frac{R}{r} \quad \text{ohms} \tag{13.14}$$

where ϵ_r = relative permittivity of the solid dielectric used to enclose the centre core.

Example 13.2 A sample of VHF coaxial feeder is required to have a characteristic impedance of 75 Ω. If the inner conductor is 1 mm in diameter, calculate the diameter of the outer conductor given that the dielectric has a relative permittivity of 2·56.

From equation 13.14

$$Z_0 = \frac{138}{\sqrt{(2{\cdot}56)}} \log_{10} \frac{R}{0{\cdot}5} = 75\ \Omega$$

where R is the radius of the outer conductor in mm.

$$\therefore \qquad \log_{10} \frac{R}{0{\cdot}5} = \frac{75\sqrt{(2{\cdot}56)}}{138} = 0{\cdot}87$$

$$\therefore \qquad R = 0{\cdot}5 \times 7{\cdot}4\ \text{mm}$$

that is, the outer conductor must have a diameter of 7·4 mm.

Travelling waves

Consider now a sinusoidal voltage as shown in figure 13.4a to be applied to the sending end of an infinitely long, loss-free transmission line, and assume the voltage to be applied at the instant $t = 0$ when the voltage is at a positive peak. This peak voltage travels progressively along the line as a maximum instantaneous voltage on each capacitor element, accompanied by a flow of current which establishes the magnetic field and charges the line capacitances. At $t = 0$, the voltage exists only at the sending end as shown in figure 13.4b. Consider next an instant $t = t_1$ in the input voltage cycle, when the voltage has fallen from V_M to some value V_1. The peak voltage V_M has at that instant moved down the line a distance $d_1 = vt_1$ where $v = 1/\sqrt{(LC)}$ metres/second. The reduced voltage V_1 is accompanied by a decrease in electric flux, and the reduced current by a decrease in magnetic flux. The decrease in magnetic flux induces an EMF in an elemental length dl of line, of inductance $L\,dl$, which balances the voltage difference across that section due to the falling voltage. At the same time, the decrease in electric flux leads to a reduction in charge on the capacitance $C\,dl$ of the elemental section; this reduction in charge being accompanied by a discharge current which, together with the reduced current flowing through the elemental length dl, supplies the necessary charging current for the succeeding elemental capacitance. Figure 13.4c shows the voltage, and current, distribution along the line at this instant $t = t_1$.

At time $t_2 = T/4$ where T is the periodic time of the input voltage, the peak voltage V_M has moved down the line a distance $d_2 = vt_2 = v(T/4)$ metres,

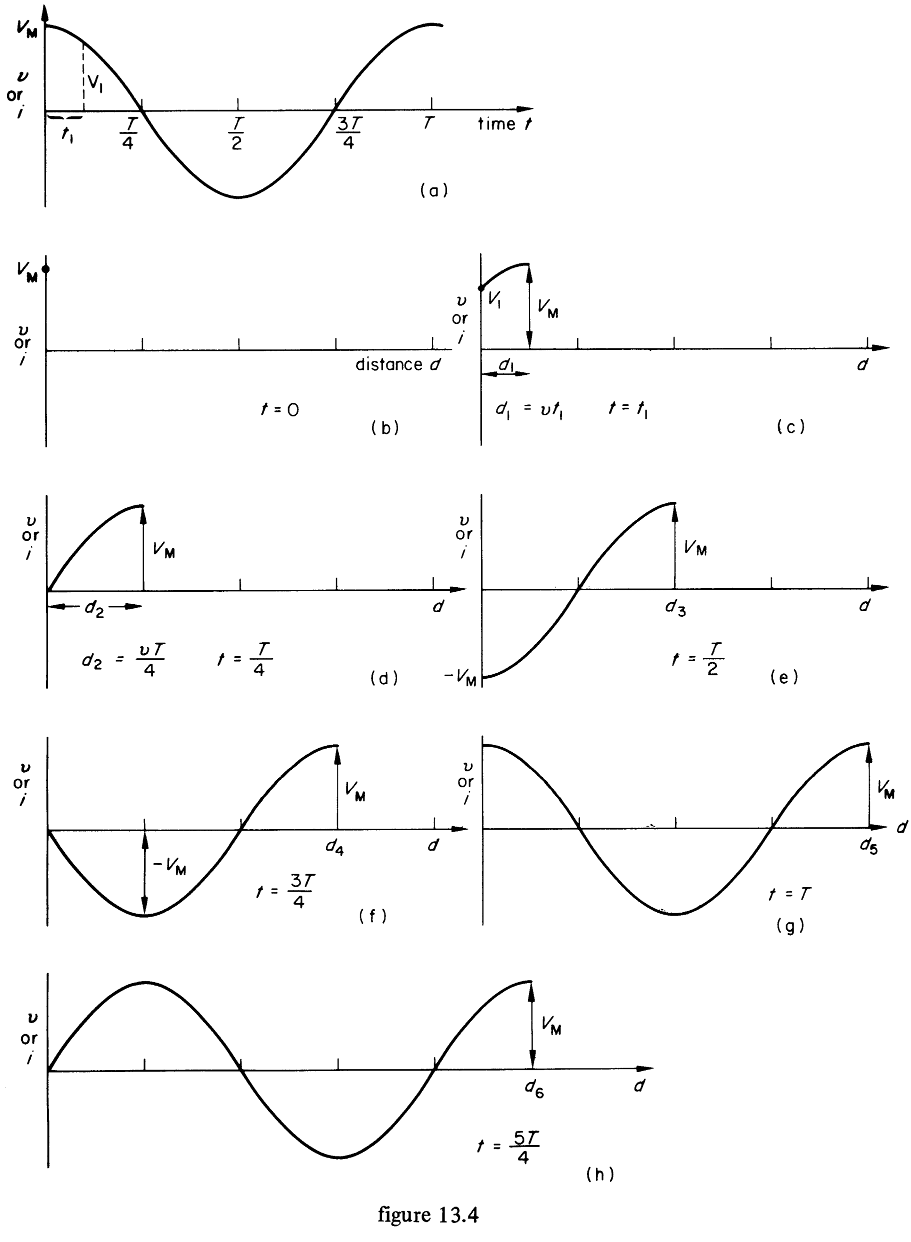


figure 13.4

the voltage at the sending end at this instant now being zero. The voltage, and current, distribution along the line is now as shown in figure 13.4d.

The voltage at the sending end now reverses, accompanied by a reversal in the flow of current, the elemental capacitances of the line thus progressively charging up in the negative direction. Sections (e), (f), (g), and (h) of figure 13.4 show the voltage, and current, distributions along the line at times $t = T/2$, $t = 3T/4$, $t = T$ and $t = 5T/4$ respectively. A travelling wave of voltage, and current, is seen to move down the line. The distance along the line occupied by one complete wave is termed the *wavelength* and is represented by the symbol λ. There is a continuous recession in phase (or phase change) at any point along the line relative to the voltage, or current, at the sending end. In one wavelength there is a lag of 2π radians; in two wavelengths the lag is 4π radians, and so on. The phase displacement per unit distance is termed the *phase constant* and is represented by the symbol β. Over a distance l, the phase change is thus βl. By definition, as the phase-change is 2π in a distance equal to the wavelength of the line

$$\frac{1}{\lambda} = \frac{\beta l}{2\pi}$$

Hence

$$\lambda = \frac{2\pi}{\beta} \qquad (13.15)$$

A positive peak of voltage is seen to travel a distance λ in the time taken for the applied voltage to complete one cycle. The velocity of propagation of the wave is thus given by

$$v = \frac{\lambda}{T} = f\lambda$$

Hence, from equation 13.15

$$v = \frac{2\pi f}{\beta} \quad \text{metres/second} \qquad (13.16)$$

Also, from equation 13.7

$$v = \frac{1}{\sqrt{(LC)}}$$

$$\therefore \quad \frac{2\pi f}{\beta} = \frac{1}{\sqrt{(LC)}}$$

$$\therefore \quad \beta = 2\pi f\sqrt{(LC)} \quad \text{rad/metre} \qquad (13.17)$$

Example 13.3 A transmission line of negligible loss has a series inductance of 1·2 mH/km and a shunt capacitance of 0·075 μF/km. The input voltage is at a frequency of 796 Hz. Calculate the propagation velocity, the wavelength and the characteristic impedance.

$$L = 1{\cdot}2\ \text{mH/km} = 1{\cdot}2 \times 10^{-6}\ \text{H/m}$$

$$C = 0{\cdot}075\ \mu\text{F/km} = 0{\cdot}075 \times 10^{-9}\ \text{F/m}$$

Propagation velocity v

$$= \frac{1}{\sqrt{(LC)}} = \frac{1}{(1{\cdot}2 \times 10^{-6} \times 0{\cdot}075 \times 10^{-9})^{1/2}}$$

$$= 1{\cdot}054 \times 10^{8}\,\mathrm{m/s}$$

Wavelength λ

$$= \frac{v}{f} = \frac{1{\cdot}054 \times 10^{8}}{796} \text{ metres} \quad = 132{\cdot}5\,\mathrm{km}$$

Characteristic impedance Z_0

$$= \left(\frac{L}{C}\right)^{1/2} = \left(\frac{1{\cdot}2 \times 10^{-6}}{0{\cdot}075 \times 10^{-9}}\right)^{1/2} = 127\,\Omega$$

MISCELLANEOUS EXAMPLES

Example 13.4 Calculate the velocity of wave propagation in the following transmission lines:

(a) A parallel open-wire line in which the inductance and capacitance per loop kilometre are 0·975 mH and 0·0114 μF respectively.
(b) A coaxial line having $L = 0{\cdot}75\,\mu\mathrm{H/m}$ and $C = 80\,\mathrm{pF/m}$.

Find also the relative permittivity of the insulation used in the manufacture of line (b).

[3×10^{5} km/s; $1{\cdot}29 \times 10^{8}$ m/s; 5·4]

Example 13.5 A certain grade of coaxial feeder has inner and outer diameters of 0·9 mm and 5·1 mm. Estimate the relative permittivity of the polythene dielectric if the characteristic impedance is 52 Ω.

[4]

Example 13.6 A transmission line has the following constants per loop kilometre when measured at a frequency of $200/\pi$ Hz:

Inductive reactance = 1 Ω, capacitive reactance = 125 kΩ. Calculate Z_0 and λ of this line at a frequency of 1 kHz.

[353 Ω; $100\sqrt{2}$ km]

Example 13.7 An open-wire line is terminated with its own characteristic impedance of $710\underline{/-14^\circ 15'}\,\Omega$. A signal generator of 2 V EMF and $600\underline{/0^\circ}\,\Omega$ internal impedance is connected to the sending end. Calculate the power injected into the line.

[1·63 mW] (B)

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