THE

SYSTEM OF THE WORLD,

BY

M. Le MARQUIS DE LAPLACE,

TRANSLATED FROM THE FRENCH,

AND

ELUCIDATED WITH EXPLANATORY NOTES.

BY THE

Rev. HENRY H. HARTE, F. T. C. D. M. R. I. A.

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THE

SYSTEM OF THE WORLD.

BOOK IV.

OF THE THEORY OF UNIVERSAL GRAVITATION.

Opinionum commenta delet dies, naturæ judicia confirmat.

CIC. DE NAT. DEOR.

HAVING, in the preceding Books, explained the laws of the celestial motions, and those of the action of forces producing motion, it remains to compare them together, to determine what forces animate the solar system, and to ascend without the assistance of any hypothesis, but by strict geometrical reasoning, to the principle of universal gravitation, from which they are derived. It is in the celestial regions, that the laws of mechanics are observed with the greatest precision; on the earth so many causes tend to complicate their results, that it is very difficult to unravel them, and still more difficult to submit them to calculation.

But the bodies of the solar system, separate by immense distances and subject to the action of a principal force, whose effect is easily calculated, are not disturbed in their respective motions, by forces sufficiently considerable, to prevent us from including under general formulæ, all the changes which a succession of ages has produced, or may hereafter produce in the system. There is no question here of vague causes, which cannot be submitted to analysis, and which the imagination modifies at pleasure, to accommodate them to the phenomena. The law of universal gravitation has this inestimable advantage, that it may be reduced to calculation, and by a comparison of its results with observation, it furnishes the most certain means of verifying its existence. We shall see that this great law of nature, represents all the celestial phenomena even in their minutest details, that there is not one single inequality of their motions, which is not derived from it, with the most admirable precision, and that it has frequently anticipated observations by revealing the cause of several singular motions, just perceived by astronomers, and which were either too complicated or too slow to be determined by observation alone, except after a lapse of ages. By means of it, empericism has been entirely banished from astronomy, which is now a great problem of mechanics, of which the elements of the motions of the stars, their figures, and masses are the arbitrary quantities, and these are the only indispensable data, which this science must derive from observation. The most profound geometry was required to establish these theories: I have collected them in *my Treatise of Celestial Mechanics.* I shall confine myself here to detail the principal results of this work, indicating the steps that lead to them, and explaining the reasons, as far as can be done, without the assistance of analysis.

CHAP. I.

Of the Principle of Universal Gravitation.

Of all the phenomena of the solar system, the elliptic motion of the planets and of the comets, seems the most proper to conduct us to the general law of the forces by which they are actuated. Observation has shewn that the areas described by the radii vectores of the planets and comets about the Sun, are proportional to the times. Now we have seen in the preceding Book, that for this to take place, the force which deflects the path of these bodies from a right line, must constantly be directed towards the origin of the radii vectores. The tendency of the planets and comets to the Sun, is therefore a necessary consequence of the proportionality of these areas to the times in which they are described.

To determine the law of this tendency, let us suppose that the planets move in circular orbits, which supposition does not greatly differ from the truth. The squares of their real velocities will then be proportional to the squares of the radii of these orbits, divided by the squares of the times of their revolutions. But by the laws of Kepler, the squares of these times are to each other as the cubes of the same radii. The squares of the velocities are therefore reciprocally as these radii. It has been already shewn that the (*a*) central forces of several bodies moving in circular orbits, are as the squares of the velocities, divided by the radii of the circumferences described; the tendencies therefore of the planets to the Sun are reciprocally, as the squares of the radii of their orbits supposed circular. This hypothesis, it is true, is not rigorously exact, but the constant relation of the squares of the times to the cubes of the greater axes of their orbits, being independent of their excentricities, it is natural to think it would subsist also in the case of the orbits being circular. Thus, the law of gravity towards the Sun, varying reciprocally as the square of the distance, is clearly indicated by this relation. Analogy leads us to suppose that this law, which extends from one planet to another, subsists equally for the same planet, at its different distances from the Sun, and its elliptic motion confirms this beyond a doubt. To comprehend this, let us follow this motion from the departure of the planet from its perihelion: its velocity is then at its maximum, (*b*) and its tendency to recede from the Sun, surpassing its gravity towards it, its radius vector augments and forms an obtuse angle with the direction of its motion. The force of gravity towards the Sun, decomposed according to this direction, continually diminishes the velocity, till it arrives at the aphelion; at this point, the radius vector becoming perpendicular to the curve, its velocity is a minimum, and its tendency to recede from the Sun, being less than its gravity towards it, the planet will approach it describing the second part of its ellipse. In this part, the gravity towards the Sun, increases its velocity in the same manner as it before diminished it, and the planet will arrive at its perihelion with its primitive velocity, and recommence a new revolution similar to the first. Now, the curvature of the ellipse at the aphelion and perihelion being the same, the radii of curvature are the same, and consequently the centrifugal forces of these two points are as the squares of the velocities. The sectors described in the same time being equal, the aphelion and perihelion velocities are reciprocally as the corresponding distances of the planet from the Sun; the squares of these velocities are therefore reciprocally as the squares of these same distances; but at the perihelion and aphelion the centrifugal (*c*) forces in the osculatory circumferences are evidently equal to the gravity of the planet towards the Sun, which is therefore in the inverse proportion of the squares of the distances from this star. Thus the theorems of Huygens on the centrifugal force, were sufficient to demonstrate the tendency of the planets towards the Sun: for it is highly probable that this law, which extends from one planet to another, and which is verified in the same planet, at its aphelion and perihelion, extends also to every part of the planetary orbit, and generally to all distances from the Sun. But to establish it in an incontestable manner, it was requisite to determine the general expression of the force which, directed towards the focus of an ellipse, makes a projectile to describe that curve. And it was Newton who demonstrated that this force was reciprocally as the square of the (*d*) radius vector. It was essential also to demonstrate rigorously that the force of gravity, towards the Sun, only varies from one planet to another, in consequence of their different distances from this star.

This great geometrician shewed, that this followed necessarily from the law of the squares of the periodic (*e*) times being reciprocally as the cubes of the greater axes of the orbits. Supposing, therefore, all the planets in repose at the same distance from the Sun, and abandoned to their gravity towards its centre, they would descend from the same height in equal times; this result should likewise be extended to the comets, notwithstanding the greater axes of their orbits are unknown, for we have seen in the second Book, chap. 6, that the magnitude of the areas described by their radii (*f*) vectores, supposes the law of the squares of their periodic times, proportional to the cubes of these axes.

An analysis, which in all its generalities, embraces every possible result from a given law, shews us that not only an ellipse, but any other conic section, may be described by virtue of the force, which retains the planets in their orbits; a comet may therefore move in an hyperbola, but then it would only be once visible, and would after its apparition recede from the limits of the solar system to approach other suns, which it would again abandon, thus visiting the different systems that are distributed through the immensity of the heavens. It is probable, considering the infinite variety of nature, that such bodies exist. Their apparition should be a very rare occurrence; the comets we usually observe, are these which, having reentrant orbits, return at the end of intervals more or less considerable, into the regions of space which are in the vicinity of the Sun. The satellites tend also, as well as the planets, perpetually to the Sun. If the Moon was not subject to its action, instead of describing an orbit almost circular round the earth, it would very soon abandon it; and if this satellite and those of Jupiter were not sollicited towards the Sun, according to the same law as the planets, sensible inequalities would result in their motions, which have not been recognized by observation. The planets, comets, and satellites are therefore subject to the same law of gravity towards the Sun. At the same time that the satellites move round their respective primary planets, the whole system of the planet and its satellites is carried by a common motion in space, and retained by the same force, round the Sun. Thus the relative motion of the planet and its satellites, is nearly the same as if the planet was at rest, and not acted on by any external force.

We are thus conducted without the aid of hypothesis, by a necessary consequence of the laws of the celestial motions, to regard the Sun as the centre of a force, which, extending indefinitely into space, diminishes as the square of the distance increases, and which attracts all bodies similarly. Every one of the laws of Kepler indicates a property of this attractive force. The law of the areas proportional to the times, shews us that it is constantly directed towards the centre of the Sun; the elliptic orbits of the planets shew that this force diminishes as the square of the distance increases; finally, the law of the squares of the periodic times proportional to the cubes of the distance, demonstrates that the gravity of all the planets towards the Sun is the same at equal distances; we shall call this gravity *the solar attraction*, for without knowing the cause, we may by one of those conceptions, common to geometricians, suppose an attractive power to exist in the Sun.

The errors to which observations are liable, and the small alterations in the elliptic motion of the planets, leave a little uncertainty in the results which we have just deduced from the laws of motion; and it may be doubted whether the solar gravity diminishes exactly in the inverse ratio of the square of the distance. But a very small variation in this law, would produce a very sensible difference in (*g*) the motions of the perihelia of the planetary orbits. The perihelion of the terrestrial orbit, would have an annual motion of 200″, if we only increased by one ten-thousandth part, the power of the distance to which the solar gravity is reciprocally proportional; this motion is only 36″4, according to observation, and of this we shall hereafter see the cause. The law of the gravity inversely as the square of the distance, is then at least, extremely near; and its extreme simplicity should induce us to adopt it, as long as observations do not compel us to abandon it. However we must not estimate the simplicity of the laws of nature, by our facility of conception; but when those which appear to us the most simple, accord perfectly with all the phenomena, we are justified in supposing them rigorously exact.

The gravity of the satellites towards the centre of their primary planet, is the necessary consequence of the proportionality of the areas described by their radii vectores to the times, and the law of the diminution of this force, according to the square of the distance, is indicated by the ellipticity of their orbits. But this can hardly be perceived in the orbits of the satellites of Jupiter, Saturn, and Uranus, which renders the law of the diminution of the force difficult to ascertain by the motion of any one single satellite; but the constant ratio of the squares of the times of their revolutions, to the cubes of their distances, indicates it beyond a doubt, by demonstrating, that from one satellite to another, the gravity towards the planet is reciprocally as the square of the distance from its centre.

This proof is wanting for the earth, which has but one satellite, but it may be supplied by the following considerations.

The force of gravity extends to the summits of the highest mountains, and the small diminution which it there experiences, does not permit us to doubt, but that at still greater altitudes it would also be sensible. Is it not natural to extend this to the Moon, and to suppose that this star is retained in its orbit by its gravity towards the earth, in the same manner as the solar gravity retains the planets in their orbits round the Sun? For in fact these two forces seem to be of the same nature: they both of them penetrate the most intimate parts of matter, animating them with the same velocities; for we have seen that the solar gravity sollicits equally all bodies placed at equal distances from the Sun, just as the terrestrial gravity causes all bodies to fall in a vacuo, through the same height in equal times.

A heavy body forcibly projected horizontally, from a great height, falls on the earth at a considerable distance, describing a curve which is sensibly parabolic, it will fall still farther if the force is greater; and if the velocity of projection was about seven thousand metres in a second, it would not fall to the Earth, but would setting aside the resistance of the air, circulate round it, like a satellite, its centrifugal force being then equal to its gravity. To form a moon of this projectile, it must be taken to the height (h) of that body, and there receive the same motion of projection.

But what completes the demonstration of the identity of the moon's tendency towards the earth with gravity, is that, to obtain this tendency, it is sufficient to diminish the terrestrial gravity according to the general law of the variation of the attractive force of the celestial bodies. Let us enter into the details suitable to the importance of this subject.

The force which at every instant deflects the Moon from the tangent of her orbit, causes it to move over, in one second, a space equal to the versed sine of the arc which it describes in that time; since this sine is the quantity by which the Moon, at the end of a second, deviates from the direction it had in the beginning. This quantity may be determined by the distance of the Earth, inferred from the lunar parallax, in parts of the terrestrial radius; but to obtain a result independent of the inequalities of the Moon, we must take (*i*) for the mean parallax, that part of it which is independent of these inequalites, and which corresponds to the semiaxis major of the lunar ellipse. Burgh determined, by a comparison of a great number of observations, the lunar parallax and it results that the part of which we have been speaking, is about 10541″, at the parallel of which the square of the sine of the latitude is equal to $\frac{1}{3}$. We select this parallel, because the attraction of the Earth, on the corresponding points of its surface is, as at the distance of the Moon, very nearly equal to the mass of the Earth, divided by the square of the distance from its centre of gravity. The radius drawn from a point of this parallel to the centre of gravity of the Earth is 6369809 metres, from whence it may be computed that the force which sollicits the Moon towards the Earth, causes it to fall $0^{me}.00101728$ in one second of time. It will be shewn hereafter, that the action of the Sun diminishes the lunar gravity by a $\frac{1}{358}$th part. The preceding height must therefore be augmented a $\frac{1}{358}$th part, to render it independent of the action of the Sun; it then becomes $0^{me}.00102012$. But in its relative motion round the Earth, the Moon is sollicited by a force equal to the sum of the masses of the Earth and Moon, divided by the square of their mutual distance; therefore to obtain the height which the Moon would fall through in one second by the action of the Earth alone, the preceding space must be diminished in the ratio of the mass of the Earth to the sum of the masses of the Earth and Moon. But a great number of phenomena depending on the action of the Moon, have given the mass of the Moon equal to $\frac{1}{75}$th of that of the earth, multiplying therefore this space by $\frac{75}{76}$, we have $0^{me}.0010067$ for the height which the Moon falls through in one second, by the action of the Earth.

Let us now compare this height, with that which results from observations made on the pendulum. At the parallel above mentioned, the length of the pendulum vibrating seconds is (by Chapter XIV, Book I.) equal to $3^{me}.65631$. But on this parallel, the attraction of the Earth is less than the force of gravity, by $\frac{2}{3}$ of the centrifugal force due (h) to the motion of rotation of the Earth at the equator; and this force is the $\frac{1}{288}$th part of that of gravity; the preceding space must therefore be augmented a $\frac{1}{432}$d part, to have the space due to the action of terrestrial gravity alone, which on this parallel is equal to the mass divided by the square of the terrestrial radius, we shall therefore have $3^{me}.66477$ for this space. At the distance of the Moon, it should be diminished in the ratio of the square of the radius of the terrestrial spheriod to the square of the distance of the Moon: for this it is sufficient to multiply it by the square of the sine of the lunar parallax, or by 10541″, this will give $0^{me}.00100464$ for the height which the Moon should fall through in one second by the attraction of the Earth. This quantity derived from experiments on the pendulum, differs very little from that which results from direct observation of the lunar parallax; to make them coincide, it is sufficient to diminish by about 2″ the preceding value. This small difference being within the limits of the errors of observation, and of the elements employed in the calculation, it is certain, that the principal force which retains the Moon in its orbit is the terrestrial gravity diminished in the ratio of the square of the distance. Thus the law of the diminution of gravity, which, in planets accompanied by several satellites, is proved by a comparison of their periodic times with their distances, is demonstrated for the Moon, by comparing its motion with that of projectiles at the surface of the Earth.

The observations of the pendulum made on the summits of mountains, had already indicated this diminution of the terrestrial gravity; but they were insufficient to discover the law, because of the small height of the most elevated mountains, compared with the radius of the Earth: it was requisite to find a body very remote from us, as the Moon, to render the law perceptible, and to convince us that the force of gravity on the Earth, is only a particular case of a force which pervades the whole universe.

Every successive phenomenon elucidates and confirms the laws of nature. It is thus that the comparison of experiments on gravity, with the lunar motion, shews us, that the origin of the distances of the Sun and of the planets in the calculation of their attractive forces, should be placed in their centres of gravity; for it is evident that this takes place for the Earth, whose attractive force is of the same nature as that of the Sun and planets.

The striking similarity between the Sun and the planets which are attended by satellites, and those which have none, should induce us to extend to them this attractive force. The spherical figure common to all these bodies, indicates that their particles are united round their centers of gravity, by a force which, at equal distances, equally sollicits them towards these points; this force is also indicated by the perturbations which planetary motions experience; but the following considerations leave no doubt on this subject.

We have seen that if the planets and the comets were placed at the same distance from the Sun, their gravity towards it would be in proportion to their masses: now it is a general law in nature, that action and reaction are equal and contrary, all these bodies therefore react on the Sun, and attract it in proportion to their masses; they are therefore endowed with an attractive force proportional to their masses, and inversely as the square of the distances. By the same principle, the satellites attract the planets and the Sun according to the same law. This attractive property then is common to all the celestial bodies: it does not disturb the elliptic motion round the Sun, when we consider only their mutual action; for the relative motion of the bodies of a system, are not changed by giving them a common velocity: by impressing therefore, in a contrary direction to (*l*) the Sun and to the planet, the motion of the first of these two bodies, and the action which it experiences on the part of the second, the Sun may be considered as immoveable; but the planet will be sollicited towards it, with a force reciprocally as the squares of the distance, and proportional to the sum of the masses: its motion round the Sun will therefore be elliptic. And we see by the same reasoning, that it would be so if the planet and Sun were carried through space, with a motion common to each of them. It is equally evident that the elliptic motion of a satellite is not disturbed by the motion of translation of its planet, nor would it be by the action of the Sun, if it was always exactly the same on the satellite and planet. Nevertheless, the action of a planet on the Sun influences the length of its revolution, which is diminished as the mass of the planet is more considerable, so that the relation of the square of its periodic time to the cube of the major axis of its orbit, is proportional to the sum of the masses of the Sun and planet. But since this relation is nearly the same for all the planets, their masses must evidently be very small compared with that of the Sun, which is equally true for the satellites with respect to their respective primary planets. This is what is confirmed by the volumes of these different bodies.

The attractive property of the heavenly bodies, does not only appertain to them in the aggregate, but likewise belongs to each of their particles. If the Sun only acted on the centre of the Earth, without attracting in particular every one of its particles, there would arise in the ocean oscillations incomparably more considerable, and very different from those which we observe. The gravity of the Earth therefore to the Sun is the result of the gravity of all its particles which consequently, attract the Sun in proportion to their respective masses; besides each body on the earth, tends towards its centre proportionally to its mass, it (m) reacts therefore on it, and attracts it in the same ratio. If that was not the case, and if any part of the Earth, however small, attracted another part without being attracted by it, the centre of gravity of the earth would move in space in virtue of the force of gravity, which is inadmissable.

The celestial phenomena compared with the laws of motion, conduct us, therefore, to this great principle of nature, namely, *that all the particles of matter mutually attract each other, in the ratio of their masses, divided by the squares of their distances.*

Already we may perceive in this universal gravitation, the cause of the perturbations to which the heavenly bodies are subject; for as the planets and comets are subject to the action of each other, they must deviate a little from the laws of elliptic motion, which they would otherwise exactly follow, if they only obeyed the action of the Sun. The satellites also, deranged in their motions round their planets, by their mutual action and that of the Sun, deviate a little from these laws.

We perceive, then, that the particles of the heavenly bodies, united by their attraction, should form a mass nearly spherical; and that the result of their action at the surface of the body, should produce all the phenomena of gravitation. We see, moreover, that the motion of rotation of the celestial bodies should slightly alter their spherical figure, and flatten them at the poles: and then the resulting force of all their mutual actions not passing through their centres of gravity, should produce in their axes of rotation motions similar to those discovered by observation. Finally, we may perceive why the particles of the ocean, unequally acted on by the Sun and Moon, should have oscillations similar to the ebbing and flowing of the tides. But these different effects of the principle of gravitation, must be particularly developed, to give it all the certainty of which physical truth is susceptible.

CHAP. II.

Of the Perturbations of the Elliptic Motion of the Planets.

If the planets only obeyed the action of the Sun, they would revolve round it in elliptic orbits, but they act mutually upon each other and upon the Sun, and from these various attractions, there result perturbations in their elliptic motions, which are to a certain degree perceived by observation, and which it is necessary to determine to have exact tables of the planetary motions. The rigorous solution of this problem, surpasses the actual powers of analysis, and we are obliged to have recourse to approximations. Fortunately, the smallness of the masses compared to that of the Sun, and the smallness of the excentricity and mutual inclination of their orbits, afford considerable facilities for this object. It is still, however, sufficiently complicated, (a) and the most delicate and intricate analysis is requisite to detect among the infinite number of inequalities to which the planets are subject, those which are sensible to observation, and to assign their values.

The perturbations of the elliptic motion of the planets may be divided into two distinct classes. Those of the first class affect the elements of the elliptic motion of the planets, they increase with extreme slowness, and are called *secular inequalities*. The other class depends on the configurations of the planets, both with respect to each other and to their nodes and perihelia, and being re-established every time these configurations become the same, they havebeen termed *periodical inequalities* to distinguish them from the secular inequalities, which are equally periodic, but whose periods are much longer, and independent of the mutual configurations of the planets.

The most simple manner of considering these various perturbations, consists in imagining a planet to move according to the laws of elliptic motion, upon an ellipse, whose elements vary by imperceptible gradations, and conceiving at the same time the true planet to oscillate round the imaginary planet in a (*b*) small orbit, the nature of which must depend on its periodic inequalities.

Let us first consider those secular inequalities which, by developing themselves in the course of ages, should change at length, both the form and position of the planetary orbits. The most important of these inequalities is that which may affect the mean motion of the planets. By comparing together, the observations which have been made since the restoration of astronomy, the motion of Jupiter appears to be quicker and that of Saturn slower, than by a comparison of the same observations, with those of the ancient astronomers: from which astronomers have concluded force due (h) to the motion of rotation of the Earth at the equator; and this force is the $\frac{1}{288}$th part of that of gravity; the preceding space must therefore be augmented a $\frac{1}{432}$d part, to have the space due to the action of terrestrial gravity alone, which on this parallel is equal to the mass divided by the square of the terrestrial radius, we shall therefore have $3^{me}.66477$ for this space. At the distance of the Moon, it should be diminished in the ratio of the square of the radius of the terrestrial spheriod to the square of the distance of the Moon: for this it is sufficient to multiply it by the square of the sine of the lunar parallax, or by 10541″, this will give $0^{me}.00100464$ for the height which the Moon should fall through in one second by the attraction of the Earth. This quantity derived from experiments on the pendulum, differs very little from that which results from direct observation of the lunar parallax; to make them coincide, it is sufficient to diminish by about 2″ the preceding value. This small difference being within the limits of the errors of observation, and of the elements employed in the calculation, it is certain, that the principal force which retains the Moon in its orbit is the terrestrial gravity diminished in the ratio of the square of the distance. Thus the law of the diminution of gravity, which, in planets accompanied by several satellites, is proved by a comparison of their periodic times with their distances, is demonstrated for the Moon, by comparing its motion with that of projectiles at the surface of the Earth.

them), all its terms would destroy each other. Calculation confirmed this supposition, and shewed me that, in general, the mean motions of the planets and their mean distances from the Sun are invariable; at least when we neglect (c) the fourth powers of the excentricities and of the inclinations of the orbits, and the squares of the perturbating masses, which is more than sufficient for the actual purposes of astronomy. Lagrange has since confirmed this result, and shewn, by a beautiful method, that it is even true, when the powers and products of any order whatever, of the excentricities and inclinations, are taken into the calculation. M. Poisson has shewn by an ingenious analysis, that the same result subsists even when the approximations are extended to the squares and products of the masses of the planets. Thus the variations of the mean motions of Jupiter and Saturn, do not depend on their secular inequalities.

The permanency of the mean motions of the planets and of the greater axes of their orbits, is one of the most remarkable phenomena in the system of the world. All the other elements of the planetary ellipses are variable, these ellipses approach to and depart insensibly from the circular form; their inclination to a fixed plane or to the ecliptic augments and diminishes, and their perihelia and nodes are continually changing their places. These variations, produced by the mutual actions of the planets on each other, are performed with such extreme slowness, that for a number of centuries they are nearly proportional to the times. They have already become apparent by observation; we have seen, in the first Book, that the perihelion of the Earth's orbit has a direct annual motion of 36″, and that its inclination to the equator diminishes every century 148″. It was Euler who first investigated the cause of this diminution, which all the planets contribute to produce, by the respective situation of the planes of their orbits. In consequence of these variations of the orbit of the earth, the perigee of the Sun coincided with the equinox of spring at an epoch to which we can ascend by analysis, which is anterior to our æra by about 4089 years. It is remarkable that this astronomical epoch is nearly that at which chronologists have fixed the creation of the world. The ancient observations are not exact enough, and the modern are too near each other to fix the exact quantity of these great changes of the planetary orbits, nevertheless they combine to prove their existence, and to shew that their progress is the same as would result from the law of gravitation. If we knew exactly the masses of the planets, future observations might be anticipated, and the true values assigned to the secular inequalities of the planets; and one of the surest means of determining them, will be the developement of these inequalities in the progress of time. We may then in imagination look back to the successive changes which the planetary system has undergone, and foretell those which future ages will offer to astronomers, and the geometrician will at once comprehend in his formulæ both the past and future states of the world.

Many interesting questions here present themselves to our notice. Have the planetary ellipses always been, and will they always be nearly circular. Among the number of the planets have any of them ever been comets whose orbits have gradually approached to the circular form, by the mutual attractions of the other planets? Will the obliquity of the ecliptic continually diminish till at length it coincides with the equator, and the days and nights become equal on the earth, throughout the year? Analysis answers these questions, in a most satisfactory manner. I have succeeded in demonstrating that whatever be the masses of the planets, in as much as they all move in the same direction, in orbits of small excentricity, and little inclined to each other; (*c*) their secular inequalities will be periodic, and contained within narrow limits, so that the planetary system will only oscillate about a mean state, from which it will deviate but by a very small quantity; the planetary ellipses therefore always have been, and always will be nearly circular, from whence it follows that no planet has ever been a comet, at least if we only take into account the mutual action of of the bodies of the planetary system. The ecliptic will never coincide with the equator, and the whole extent of its variations will not exceed three degrees.

The motions of the planetary orbits and of the stars will one day embarrass astronomers, when they attempt to compare precise observations separated by long intervals of time; already this difficulty begins to be apparent; it would be interesting therefore to find some plane that should remain invariable, that is, constantly parallel to itself. We have given at the end of the preceding book, a simple means of determining a similar plane, in the motion of a system of bodies which are only subject to their mutual action; this method when applied to the solar system, gives the following rule. If at any instant of time (h) whatever, and upon any plane passing through the centre of the Sun, we draw from this point straight lines to the ascending nodes of the planetary orbits referred to this plane, and if we take on these lines, reckoning from the centre of the Sun, lines equal to the tangents of the inclinations of these orbits to this plane, and if at the extremities of these lines, we suppose masses equal to the masses of the planets multiplied respectively into the square roots of the parameters of the orbits, and by the cosines of their inclinations; and lastly, if we determine the centre of gravity of this new system of masses, then the line drawn from the centre of the Sun to this point will be the tangent of the inclination of the invariable plane, to the assumed plane; and continuing this line to the heavens, it will there mark its ascending node.

Whatever changes the succession of ages may produce in the planetary orbits, and whatever be the plane to which they are referred, the plane determined by this rule, will always be parallel to itself. It is true, its position depends on the masses of the planets; but these will soon be sufficiently known to determine it with exactness. In adopting the values of these masses which will be given in the following chapter, we find that the longitude of the ascending node of the invariable plane was 114°,7008 at the commencement of the nineteenth century, and at the same epoch its inclination to the ecliptic was 1°,7565. In this computation we have neglected the comets, which nevertheless ought to enter into the determination of the invariable plane, since they constitute a part of the solar system. It would be easy to include them in the preceding rule, if their masses and the elements of their orbits were known. But in our present ignorance of the nature of these objects, we suppose their masses too small to influence the planetary system, and this is the more probable, since the theory of the mutual attraction of the planets, suffices to explain all the inequalities observed in their motions. But if the action of the comets should become sensible in the progress of time, it should principally affect the position of the plane, which we suppose invariable, and in this new point of view the consideration of this plane will still be useful, if the variations of this plane could be recognised, which would be attended with great difficulties.

The theory of the secular and periodic inequalities of the motions of the planets, founded on the law of universal gravitation, has been confirmed by its agreement with all observations ancient and modern. It is particularly in the motions of Jupiter and Saturn, that these inequalities are most sensible, but they present themselves under a form so complicated, and the length of their periods is so considerable, that it would have required several ages to have determined their law by observations alone, which has in this instance been anticipated by theory.

After having established the invariability of the mean motions of the planets, I suspected that the alterations observed in the mean motions of Jupiter and Saturn, proceeded from the action of comets. Lalande had remarked in the motion of Saturn, irregularities which did not appear to depend on the action of Jupiter: he found its returns to the vernal exquinox, more rapid than its returns to the autumnal equinox, although the positions of Jupiter and Saturn, both with respect to each other, and to their aphelia, were nearly the same. Lambert likewise observed that the mean motion of Saturn, which seemed to diminish from century to century by the comparison of ancient with modern observations, appeared on the contrary, to accelerate by the comparison of modern observations with each other, at the same time that Jupiter presented phenomena exactly contrary. All this seemed to indicate that causes independent of the action of Jupiter and Saturn on each other, had altered their motions. But on mature reflection, the order of the variations observed in the mean motions of these planets, appeared to me to agree so well with the theory of their mutual attraction, that I did not hesitate to reject the hypothesis of a foreign cause.

It is a remarkable result of the mutual action of the planets on each other, that if we only consider (*i*) the inequalities which have very long periods, the sum of the masses of every planet, divided respectively by the greater axes of their orbits considered as variable ellipses, is always pretty nearly constant. From this it follows, that the squares of the mean motions, being reciprocally as the cubes of these axes, if the motion of Saturn is retarded by the action of Jupiter, that of Jupiter should be accelerated by the action of Saturn, which is conformable to observation. I perceived, moreover, that the law of these variations was the same as corresponded to the preceding theory. In supposing with Halley the retardation of Saturn to be 256″94 for the first century, reckoned from 1700, the corresponding acceleration of Jupiter should be 109″80, and Halley found it to be 106″02 by observation. It was therefore very probable that the variations observed in the mean motions of Jupiter and Saturn, were the effects of their mutual action; and since it is certain that this action cannot produce any inequality either constantly increasing or periodic, but of a period independent of the configuration of these planets, and that it cannot effect in it any irregularities but what are relative to this configuration, it was natural to think that there existed in their theory a considerable inequality of this kind, of a very long period, and which was the cause of these variations.

The inequalities of this kind, although very small and almost insensible in differential equations, augment considerably in the integrations, and may acquire very great values in the expressions of the longitudes of the planets. (*k*) I easily recognized the existence of similar inequalities, in the differential equations of the motions of Jupiter and Saturn. These motions are very nearly commensurable; so that five times the mean motion of Saturn differs very little from twice that of Jupiter: from which I concluded that the terms which have for their argument five times the mean longitude of Saturn, minus twice that of Jupiter, might by integration become very sensible, although multiplied by the cubes and products of three dimensions of the excentricities and inclinations of the orbits. I considered therefore that these terms were the probable cause of the variations observed in the mean motions of these planets. The probability of this cause, and the importance of the object, determined me to undertake the laborious calculation, necessary to determine this question. The result of this calculation fully confirmed my conjecture; and it appeared, that in the first place there exists in the theory of Saturn a great inequality of 8895″7 at its maximum, of which the period is 929 years; and which ht to be applied to the mean motion of this ɜt; and secondly, that the motion of Jupiter is subject to a similar inequality, whose period and law are the same, but affected with a contrary sign, its amount is only 3662″41. The magnitude of the coefficients of these inequalities and the duration of their period are not always the same, they participate in the secular variations of the elements of the orbits on which they depend, I have determined with especial care, those coefficients and their secular diminution. It is to these two inequalities, formerly unknown, that that we must attribute the apparent retardation of Saturn, and the apparent acceleration of Jupiter. These phenomena attained their maximum about the year 1560; since this epoch, their mean apparent motions have approximated to their true mean motions, and they were equal in 1790. This explains the reason why Halley, in comparing the ancient with modern observations, found the mean motion of Saturn slower, and that of Jupiter more rapid than by the comparison of modern observations with each other, instead of which these last indicated to Lambert an acceleration in the motion of Saturn, and a retardation in that of Jupiter. And it is very remarkable that the quantities of these phenomena, deduced from observation alone by Halley and Lambert, are very nearly the same as result from the two great inequalities which I have just mentioned. If astronomy had been revived four centuries and a half later, observations would have presented the direct contrary phenomena. The mean motions which the astronomy of any people have assigned to Jupiter and Saturn, should afford us information concerning the time of its foundation. Thus it appears that the Indian astronomers determined the mean motions of these planets, in that part of the period of the preceding inequalities, when the motion of Saturn was the slowest, and that of Jupiter the most rapid. Two of their principal astronomical epochs, the one 3102 A. C. the other 1491 A. C. answer nearly to this condition. The nearly commensurable relation that exists in the motions of Jupiter and Saturn, occasions other very perceptible inequalities, the most considerable of which affects the motion of Saturn; it would be entirely confounded in the equation of the centre, if twice the mean motion of Jupiter was exactly equal to five times that of Saturn. The difference observed in the last century in the intervals of the returns of Saturn to the equinoxes of spring and autumn, arises principally from this cause.

In general, when I had recognised these various inequalities, and examined more carefully than had been done before, those which had been submitted to calculation, I found that all the observed phenomena of the motions of these two planets adapted themselves naturally to the theory; before they seemed to form an exception to the law of universal gravitation; they are now become one of the most striking examples of its truth. Such has been the fate of this brilliant discovery of Newton, that every difficulty which has arisen, has only furnished a new subject of triumph for it, which is the most indubitable characteristic of the true system of nature.

The formulæ which I have obtained for representing the motions of Jupiter and Saturn, satisfy with remarkable precision the last oppositions of these two planets, which have been observed by the most skilful astronomers with the best meridian telescopes and the greatest quadrants of circles, the error never amounted to 40″; and twenty years ago the errors of the best tables sometimes surpassed four thousand seconds. These formulæ also represent with the same accuracy as observations themselves, the observations of Flamstead, those of the Arabians, and the observations cited by Ptolomy. This great precision with which the two largest planets of our system, have obeyed from the most remote period, the laws of their mutual attraction, evinces the stability of this system, since Saturn, of which the attraction to the Sun is about an hundred times less than the attraction of the earth to the same star, has not since the æra of Hipparchus to the present day, experienced any sensible derangement from the action of extraneous causes.

I cannot in this place, refrain from making a comparison of the real effects of this relation between the mean motions of Jupiter and Saturn, with those which astrology had attributed to it. In consequence of this relation, the mutual conjunctions of these two planets are renewed after an interval of twenty years, but the point of the heavens to which they arrive, retrogrades by about a third of the zodiac, so that if the conjunction of the two planets arrives in the first point of Aries, it will in twenty years afterwards take place in Sagittarius, and in twenty years afterwards in Leo, to return then to the sign of the ram at ten degrees from its original position. It will continue to take place in these three signs, for nearly two hundred years. In the same manner, in the next two hundred years, it will go through the signs Taurus, Capricornus, and Virgo. In the next two hundred years, it will proceed through the signs Gemini, Aquarius, and Libra; and finally, in the last two hundred years, it will describe the remaining signs, Cancer, Pisces, and Scorpio; after which it will again begin with the sign Aries as before. From hence arises a great year, each season of which is equal to two centuries. They attributed different temperatures to the different seasons of this year, as likewise to the signs which belonged to them. The assemblage of these three signs was called a *trigon*. The first trigon was that of Fire, the second of Earth, the third of Air, and the fourth of Water.—We may easily imagine that astrology made great use of these trigons, which even Kepler himself describes with great exactness, in several of his works: but it is very remarkable that sound astronomy, while it dissipated the imaginary influence that was supposed to attend this relation in the motion of the two planets, should have recognised in this relation, the source of the greatest perturbations of the planetary system.

The planet Uranus, though lately discovered, offers already incontestable indications of the perturbations which it experiences from the action of Jupiter and Saturn. The laws of elliptic motion do not exactly satisfy its observed positions, and to represent them, its perturbations must be considered. Their theory, by a very remarkable coincidence, places it in the years 1769, 1756, and 1690, in the same points of the heavens, where Monnier, Mayer, and Flamstead, had determined the position of three stars, which cannot be found at present: this leaves no doubt of the identity of these stars with the new planet.

The small planets which have been discovered, are subject to very great inequalities, which will throw new light on the theory of the attractions of the heavenly bodies, and will enable us to render it perfect; but hitherto we have been unable to recognize these inequalities by means of observations. It is only three centuries since Copernicus first introduced into the astronomical tables the motion of the planets about the Sun: about a century after, Kepler took into account the laws of elliptic motion, which had been discovered by means of the observations of Tycho Brahe; this led Newton to the discovery of universal gravitation. Since these three epochs, which will be always memorable in the history of the sciences, the improvements in the infinitesimal calculus have enabled us to subject to computation the numerous inequalities of the planets which arise from their mutual attraction, and by this means the tables have acquired a degree of precision which could never have been anticipated; formerly their errors amounted to several minutes, they are now reduced to a small number of seconds, and very often, it is probable, that their apparent deviations arise from the inevitable errors of the observations.

CHAP. III.

Of the Masses of the Planets, and of the Gravity at their Surface.

The ratio of the mass of a planet to the mass of the Sun, being the principal element of the theory of the perturbations which it produces, the comparison of this theory with a great number of very precise observations, ought to give its value so much the more accurately, as the perturbations of which it is the cause are more considerable. It is in this manner (*a*) that the following values of the masses of Venus, of Mars, of Jupiter and of Saturn, have been determined. The masses of Jupiter, of Saturn, and of those planets which have satellites, may be determined in the following manner.

It follows from the theorems on centrifugal force, given in the preceding book, that the gravity of a satellite towards its primary is to the gravity of the Earth towards the Sun, as the mean radius of the orbit of the satellite divided by the square of the time of its sidereal revolution, is to the mean distance of the Earth (*b*) from the Sun, divided by the square of a sidereal year. To reduce these gravities, to the same distance from the bodies which produce them, they must be multiplied respectively by the squares of the radii of the orbits which they describe. And as at equal distances, the masses are proportional to their attractions, the mass of the planet is to that of the Sun, as the cube of the mean radius of the orbit of the satellite, divided by the square of the time of its sidereal revolution, is to the cube of the mean distance of the Earth from the Sun, divided by the square of the sidereal year. This result supposes that the mass of the satellite relatively to that of the planet has been neglected, and also the mass of the planet with respect to that of the Sun, which may be done without any sensible error, it will become more exact if we substitute in place of the mass of the planet, the sum of the masses of the planet and of its satellite, and instead of the mass of the Sun, the sum of the masses of the Sun and planet, since the force which retains a body in its relative orbit, about that which attracts it, depends on the sum of their masses. Let this result be applied to Jupiter; the mean radius of the orbit of the fourth satellite, such as it has been given in the second book, seen at the mean distance of the Earth from the Sun, would appear under an angle of 7964″75; the radius of the circle contains 636619″8: thus the mean radii of the orbits of the fourth satellite, and of the terrestrial orbit, are in the proportion of these two last numbers. The duration of the sidereal revolution of the fourth satellite (c) s $16^{d}\cdot6890$, and the sidereal year is $365^{d}\ 2564$. Setting out from these data, the mass of Jupiter is found to be $\frac{1}{1067{,}09}$, that of the Sun being represented by unity. To obtain greater exactness, it is necessary to diminish by unity the denominator of this fraction; the mass of this planet then is $\frac{1}{1066{,}09}$. I have determined by the same method, the masses of Saturn and of Uranus, equal respectively to $\frac{1}{3359{,}4}$, $\frac{1}{19504}$.

The perturbations which these three large planets experience from their reciprocal attractions, furnish an accurate method of obtaining the values of their masses. Mr. Bouvard, from a comparison of the formulæ which are given in the Celestial Mechanics, with a great number of observations carefully discussed, constructed new tables of Jupiter, of Saturn, and of Uranus. He has formed for this important object equations of condition, in which he left as indeterminate, the masses of these planets, and from a resolution of these equations, he obtained the following numerical values for these masses, $\frac{1}{1070{,}5}$, $\frac{1}{3512}$; $\frac{1}{17918}$. If we consider the great difficulty of measuring the elongations of the satellites of Saturn and Uranus, and our ignorance of the ellipticity of the orbits of these satellites; the little difference which exists between the values inferred from these elongations, and those which result from the perturbations, is really astonishing. These last values include for each planet, its mass and those of its satellites, to which it is necessary to add, in the case of Saturn (*d*), that of its ring. But every thing induces us to think that the mass of the planet is far superior to that of the bodies which surround it; at least this is certainly the case for the earth and Jupiter. But by applying the theory of probabilities to the equations of condition of M. Bouvard, it has been found, that it is a million to one, that the value given above for the mass, does not differ by a hundreth part from its true value. There is eleven thousand to one, that this is the case with respect to the mass of Saturn. Since the perturbations which Uranus produces in the motion of Saturn are inconsiderable, a great number of observations is required to obtain its mass with the same probability, but in the actual state of the case, it is 2500 to 1, that the preceding result does not differ from its true value by a fourth part. The perturbations which the earth experiences from the attractions of Venus and Mars, are sufficiently sensible to indicate the masses of these planets. M. Buckhardt, to whom we are indebted for excellent tables of the Sun, founded on four thousand observations, has concluded that the values of these masses are respectively $\frac{1}{405871}$ and $\frac{1}{2546320}$.

We may obtain in the following manner, the mass of the earth. If the mean distance of the Earth from the Sun be assumed equal to unity, the arc described by the Earth in a second of time, will be the proportion of the circumference to radius, divided by the number of seconds in the sidereal year, or by 36525636″1; dividing the square of this arc by the diameter, we shall get

$\frac{479565}{10^{20}}$ for its versed sine, it is the quantity which the Earth falls towards the Sun, during one second, in consequence of its relative motion round it. It has been seen, in the preceding chapter, that upon the terrestrial parallel, of which the square of the sine of the latitude is $\frac{1}{3}$, the attraction of the Earth causes bodies to fall through $3^{me.}$ 66477 in one second. To reduce this attraction to the mean distance of the Earth from the Sun, it must be multiplied by the square of the sine of the solar parallax, and then the product should be divided by the number of metres contained in this distance. Now the terrestrial radius, at the parallel we are considering, is 6369809 metres; dividing this number, therefore, by the sine of the solar parallax, or by 26″54, we shall get the mean radius of the terrestrial orbit, expressed in metres. It follows from hence, that the effect of the Earth's attraction, at the mean distance of this planet from the Sun, is equal to the product of the fraction $\frac{3,66477}{6369809}$ by the cube of the sine of 26″54, it is consequently equal to $\frac{4,16856}{10^{20}}$: taking this fraction from $\frac{1479565}{10^{20}}$ we shall have $\frac{1479560}{10^{20}}$ for the effect of the Sun's attraction at the same distance (*e*). The masses of the Sun and Earth are therefore in the proportion of the numbers 1479560,8 and 4,16856; from whence it follows that the mass of the Earth is $\frac{1}{354936}$. If the parallax of the Sun is a little different from what we have supposed, the value of the mass of the Earth should vary as the cube of this parallax compared to that of 26″54.

The mass of Mercury has been determined by its volume, supposing the densities of this planet and of the Earth, inversely as their mean distances from the Sun. An hypothesis indeed very precarious, but which corresponds with sufficient exactness to the respective densities of the Earth, Jupiter and Saturn. It will be necessary to rectify all these values, when in the progress of time the secular variations of the celestial motions shall be determined more correctly.

Masses of the Planets, that of the Sun being taken as unity.

Mercury	$\frac{1}{2025810}$
Venus	$\frac{1}{405871}$
The Earth	$\frac{1}{354936}$
Mars	$\frac{1}{2546320}$
Jupiter	$\frac{1}{1070,5}$
Saturn	$\frac{1}{3512}$
Uranus	$\frac{1}{17918}$.

The densities of bodies are proportional to their masses divided by their volumes, and when they are nearly spherical, their volumes are as the cubes of their radii. The densities therefore are as the masses divided by the cubes of the radii; but to obtain greater accuracy, that radius of a planet must be taken, which corresponds to the parallel, the square of the sine of whose latitude is $\frac{1}{3}$. It was stated in the first book, that the semidiameter of the Sun seen at its mean distance from the earth, subtends an angle of $2966''$, and at the same distance the, radius of the earth would appear under an angle of $26'',54$. It is easy to infer from this that the mean density of the solar globe being assumed equal to unity, (f) that of the earth is 3,9326. This value is independent of the parallax of the Sun; for the volume and the mass of the earth increase respectively, as the cube of this parallax. The semidiameter of the equator of Jupiter seen at its mean distance from the Sun, is according to the accurate measures of Arrago, equal to $56'',7''02$; the semiaxis passing through the poles is 53,497, therefore the radius of the spheroid of Jupiter, corresponding to the parallel of which the square of the sine of the latitude is $\frac{1}{3}$, will subtend at the same distance, an angle of $55,''967$, and seen at the mean distance of the Earth from the Sun, it will be $291'',185$. Hence it is easy to infer that the density of Jupiter is equal to 0,99239. The density of the other planets may be determined in the same manner, but the errors of which the measures of their apparent diameters, and the estimation of their masses are also susceptible, will cause considerable uncertainty in the results of the computation; if the apparent diameter of Saturn seen at its mean distance from the Sun be supposed equal to $50''$, its density will be equal to 0,55, that of the Sun being unity. A comparison of the respective densities of the earth, of Jupiter and of Saturn, indicates that they are smaller for the more distant planets; Kepler was led to the same result from his notions of suitableness and harmony, and he supposed the density of the planets to be reciprocally proportional to the square roots of their distances. But he concluded from the same considerations that the Sun was the densest of all the stars, which is not the case. The planet Uranus, of which the density appears to surpass that of Saturn, is an exception to the preceding rule. In consequence of the uncertainty which hangs over the measures of his apparent diameter, and the measures of his greatest elongations, we cannot pronounce with certainty on this subject.

To obtain the intensity of gravitation at the surface of the Sun and planets, it has been proved that if Jupiter and the Earth were exactly spherical, and deprived of their rotatory motion, gravity at their equators would be proportional to the masses of these bodies, divided by the squares of their diameters; now at the mean distance of the Sun from the Earth, Jupiter's apparent semidiameter is 291″,185, and that of the Earth's equator is 26″,541; representing then the weight of a body at the terrestrial equator by unity, the weight of this body transported to the equator of Jupiter would be 2,716, but this weight must be diminished by about a ninth part (*g*) from the effcts of the centrifugal force, due to the rotation of these planets. The same body would weigh 27,9 at the equator of the Sun, and falling bodies would describe one hundred and two metres in the first second of their descent. The immense interval which separates us from these great bodies, seemed for ever to debar us from obtaining a knowledge of the effects of gravity at their surface: but the connexion of truths leads to results which appear inaccessible, when the principle on which they depend is unknown. It is thus that the measure of the intensity of gravity at the surface of the Sun and of the planets, is rendered possible by the discovery of universal gravitation.

CHAP. IV.

Of the Perturbations of the Elliptic Motion of Comets.

The action of the planets produces in the motion of comets, inequalities which are principally sensible in the intervals of their returns to the perihelion. Halley having remarked that the elements of the orbits of the comets observed in 1531, 1607, and 1682, were nearly the same, concluded that they belonged to the same comet which in the space of 151 years, had made two revolutions. It is true, that the period of its revolution is thirteen months longer from 1537 to 1607, than from 1607 to 1682. But this great astronomer thought, and with reason, that the attraction of the planets, particularly of Jupiter and Saturn, might have occasioned this difference, and after a vague estimation of this action during the course of the following period, he judged that it should retard the return of the comet, and he fixed it for the end of 1758, or the commencement of 1759. This prediction was too important in itself, and too intimately connected with the theory of universal gravitation, not to excite the curiosity of all those who were interested in the progress of the sciences; and in particular of a theory which already accorded with a great number of phenomena. Astronomers, uncertain of the epoch at which it ought to return, sought it about the year 1757; and Clairaut, who had been one of the first to solve the problem of the three bodies, applied his solution to the determination of the inequalities which the comet had sustained by the action of Jupiter and Saturn. On the 14th November, 1758, he announced in the academy of sciences, (*a*) that the interval of the return of the comet to its perihelion, would be 618 days longer in the present period than in the former one, and consequently, the comet would pass its perihelion, about the middle of April 1759. He observed, at the same time, that the small quantities neglected in this approximate calculation, might advance or retard this term, a month. He remarked also that a body which passes into regions so remote, and which escapes our sight during such long intervals, may be subject to the action of forces entirely unknown, as the attraction of other comets, or even of some planet, whose distance is too great to be ever visible to us. This philosopher had the satisfaction of seeing his prediction accomplished; the comet passed its perihelion the 12th March 1759, within the limits of the errors of which he thought his results susceptible. After a new revision of his calculations, Clairaut fixed this passage at the 4th of April, and he would have brought it to the 24th

March, if he had employed the mass of Saturn, such as is given in chap. III.; that is, within thirteen days of the actual observation. This difference will appear very small, if we consider the great number of quantities neglected, and the influence which the planet Uranus might produce, whose existence was at that time unknown.

Let us remark, for the honour of the human understanding, that this comet, which in the last century only excited the most lively interest among astronomers and mathematicians, had been regarded in a very different manner, four revolutions before, when it appeared in 1456. Its long tail spread consternation over all Europe, already terrified by the rapid success of the Turkish arms, which had just destroyed the great empire. Pope Callixtus, on this occasion, ordered a prayer, in which both the comet and the Turks were denounced in the same anathema.

In those times of ignorance, mankind were far from thinking that nature obeyed immutable laws, and according as phenomena succeeded with regularity or without apparent order, they were supposed to depend either on final causes or on chance; so that whenever any thing happened which seemed out of the natural order, they were considered as so many signs of the anger of heaven.

To the terrors which the apparition of comets then inspired, succeeded the apprehension, that of the great number which traverse the planetary system in all directions, one of them might overturn the earth.

They pass so rapidly by us, that the effects of their attraction are not to be apprehended. It is only by striking the earth that they can produce any disastrous consequences. But this circumstance, though possible, is so little probable in the course of a century, and it would require such an extraordinary combination of circumstances for two bodies, so small in comparison with the immense space they move in, (*b*) to strike each other, that no reasonable apprehension can be entertained of such an event.

Nevertheless, the small probability of this circumstance may, by accumulating during a long succesion of ages, become very great. It is easy to represent the effect of such a shock upon the earth: the axis and motion of rotation would be changed, the waters abandoning their antient position, would precipitate themselves towards the new equator; the greater part of men and animals drowned in a universal deluge, or destroyed by the violence of the shock given to the terrestrial globe; whole species destroyed; all the monuments of human industry reversed: such are the disasters which the shock of a comet ought to produce, if its mass was comparable to the mass of the earth.

We see then why the ocean has covered the highest mountains, on which it has left incontestible marks of its former abode: we see why the animals and plants of the south may have existed in the climates of the north, where their relics and impressions are still to be (*c*) found: *lastly*, this explains the short period of the existence of the moral world, whose earliest monuments do not go much farther back than five thousand years. The human race reduced to a small number of individuals, in the most deplorable state, occupied only with the immediate care for their subsistence, must necessarily have lost the remembrance of all sciences and of every art; and when the progress of civilization had created new wants, every thing was to be invented again, as if mankind had been just placed upon the earth. But whatever may be the cause assigned by philosophers to these phenomena, I repeat it, we may be perfectly at ease with respect to such a catastrophe during the short period of human life, especially since it appears that the masses of the comets are extremely small, and therefore their shock ought only to produce local changes.

But man is so disposed to yield to the dictates of fear, that the greatest consternation was excited at Paris, and thence communicated to all France in 1773, by a memoir of Lalande, in which he determined, of those comets which had been observed, the orbits that most nearly approached the earth; so true it is, that error, superstition, vain terrors, and all the evils of ignorance, are ever ready to start up, when the light of science is unfortunately extinguished.

The observations of the comet which was first perceived in 1770, have conducted astronomers to a very remarkable result. After having in vain attempted to subject these observations to the laws of parabolic motion, which have hitherto represented the motions of the comets with sufficient accuracy, they at length recognized that it described during its appearance, an ellipse the duration of whose revolution did not surpass six years. Lexel, who first made this curious remark, satisfied on this hypothesis, a great number of observations of the comet. But so very short a duration could not be admitted, (*d*) except after incontrovertible proofs, founded on a new and profound discussion of the observations of the comet, and of the positions of the fixed stars to which it was compared. The Institute therefore proposed this discussion for the subject of a prize, which Buckhardt gained, and his investigations has conducted us to very nearly the same result, as Lexel, on which there ought not now to remain any doubt. A comet, of which the period was so short ought frequently to appear, notwithstanding which it was not observed previously to 1770, nor has it been seen again, since that period. To account for this twofold phenomenon, Lexel remarked that in 1767 and 1779, this comet was very near to Jupiter, of which the powerful attraction diminished in 1767, the perehelion distance of its orbit, so as to render this star visible in 1770, which was before invisible, and then in 1779 it increased this same distance, so as to render this comet perpetually invisible. But it was necessary to demonstrate the possibility of the two effects which have been ascribed to the attraction of Jupiter, by shewing that the elements of the ellipse described by the comet, ought to satisfy them. This I have accomplished by subjecting this question to analysis, and by this means the preceding explanation has been rendered very probable. Of all the comets, this approached the nearest to the earth, consequently it ought to experience a sensible action from it, if its mass was comparable to that of the earth.

These two masses being supposed to be equal, the sidereal year would have been increased 11612″, by the action of the comet. By a computation of a great number of observations which Delambre and Buckhardt made in order to construct the tables of the Sun, we may be assured that since 1770, the sidereal year has not increased 3″, consequently the mass of the comet is (*e*) not the $\frac{1}{5000}$ part of the mass of the earth; and if we consider that in 1767 and 1779 this star traversed the system of the satellites of Jupiter without producing the slightest derangement, it will be evident that it must be even less. The smallness of the masses of the comets is in general indicated by their insensible influence on the motions of the planetary system. These motions are represented by the sole action of the bodies of the system, with such remarkable precision, that the small abberrations of our best tables may be ascribed to the sole errors of approximations and of observations. But very exact observations continued for a great number of years, and compared with the theory, can alone throw light on this important point in the system of the world.

CHAP. V.

Of the Perturbations of the Motion of the Moon.

The Moon is attracted at the same time by the Sun and by the Earth; but its motion round the Earth is only disturbed by the difference of the actions of the Sun, upon these two bodies: if the Sun was at an infinite distance, it would act equally upon them, and in the direction of parallel lines; their relative motion, therefore, would not be affected by an action which was common to both; but its distance, though (*a*) very great compared with that of the Moon, cannot be considered as infinite: the Moon is alternately nearer and farther from the Sun than the Earth, and the straight line joining the centres of the Sun and Moon, forms angles more or less acute with the terrestrial radius vector. Thus the Sun acts *unequally* and in *different* directions on the Earth and Moon; and from this diversity of action, inequalities must necessarily arise in the lunar motion, depending on the respective positions of the Moon and Sun. This constitutes the famous problem of the three bodies, the exact solution of which surpasses the powers of analysis, (*b*)but from the proximity of the Moon, compared with its distance from the Sun, and from the comparative smallness of its mass, an approximation may be obtained extremely near the truth. Nevertheless, the most delicate analysis is necessary to extricate all the terms, whose influence becomes sensible.

Their discussion is the most important point of this analysis, when it is proposed to perfect the lunar theory, which indeed ought to be its principal object; there are various ways of reducing this problem of the three bodies to an equation; but its principal difficulty consists in discriminating in the differential equations, and determining exactly, the terms which, though extremely small in themselves, acquire by successive integrations a sensible value; this requires a judicious selection of coordinates, delicate considerations on the nature of the integrals, approximations accurately conducted, computations carefully made and frequently verified. I have endeavoured to fulfil these conditions in the theory of the Moon, which has been explained in the Celestial Mechanics, and I have the satisfaction of seeing my results coincide with those found by Mason and Burgh from a comparison of near five thousand observations of Bradley and Maskeline, and which have given to the lunar tables a precision which it will be difficult to surpass, and to which geography, and nautical astronomy are principally indebted for their progress. It is due to Mayer, one of the greatest astronomers that ever lived, to observe that he was the first who brought the tables to the degree of perfection which is necessary for this important object. Mason and Burg have adopted the form which he gave to them; they have corrected the coefficients of his inequalities, and have added to them some others indicated by his theory. Mayer moreover, by the invention of the repeating circle, which has been considerably improved by Borda, has brought observations made at sea, to the same accuracy, to which he has reduced the lunar tables. Finally, M. Burkhardt has rendered the lunar tables nearly perfect, by assigning to their arguments a simple and more commodious form, and by determining their coefficients from a great collection of modern observations. The object of my theory has been to shew, in the sole law of universal gravitation, the source of all the inequalities of the lunar motion, and to make use of this law, to perfect the lunar tables, and to infer from them, several important elements in the system of the world, such as the secular equations of the Moon, its parallax, that of the Sun and the compression of the earth. Fortunately, while I was occupied in these investigations, Burg on his part was endeavouring to perfect the lunar tables. My analysis indicated to him several new and extremely sensible equations; and from a comparison of them with a great number of observations, he has ascertained their existence, and thrown great light on the elements, of which I have been speaking. The motions of the nodes, and of the lunar perigee, are the principal effects of the perturbations which this satellite experiences. A first aproximation had only given to geometers, half of the second of these motions; from (c) which Clairaut concluded that the law of attraction was not quite so simple as had been imagined; and he supposed it to consist of two parts, of which one varying inversely as the square of the distances, is sensible only at the great distances of the planets from Sun, and that the other, increasing in a greater ratio as the distance diminished, became sensible at the distance of the Moon from the Earth. This conclusion was vehemently opposed by Buffon: he maintained that since the primordial laws of nature should be the most simple possible, they could only depend on one *modulus*, and their expression, therefore, must consist of one single term. This consideration should no doubt lead us not to complicate the law of attraction, except in case of extreme necessity; at the same time our ignorance respecting the nature of this force, does not permit us to pronounce with certainty as to the simplicity of its expression. However this may be, the metaphysician was in the right, this time, in his contest with the mathematician, who retracted his error on making this important discovery, that by carrying on the approximation farther than had been done at first, the law of attraction, reciprocally as the squares of the distances, gave the motion of the lunar perigee, exactly conformable to observation, which has since been confirmed by all those who have occupied themselves with this subject. The motion which I inferred from my theory, differs from the true motion by its four hundreth and fortieth part, and this difference is only the three hundredth and fiftieth part, with respect to the motion of the nodes. Although analysis may be indispensible to make known how all the inequalities of the Moon, result from the action of the Sun combined with that of the earth, it is possible, nevertheless, without analysis, to explain the cause of the annual equation of the Moon, and its secular equation. I shall the more willingly stop to explain the causes of these equations, because it will be seen that from them are derived the greatest inequalities of the Moon, which the course of ages may develope to observers, but which up to the present period have been almost insensible.

In its conjunctions with the Sun, the Moon is nearer to it than the Earth, and experiences from it a more considerable action: the difference of the attractions of the Sun upon these two bodies, tends to diminish the gravity of the Moon towards the Earth. In a similar manner, in the oppositions of the Moon to the Sun, this satellite being farther (*d*) from the Sun than the Earth, is more weakly attracted: thus the difference of the actions of the Sun, tends also in this case to diminish the gravity of the Moon to the Earth. In each case, the diminution is very nearly the same, and equal to twice the product of the mass of the Sun, by the quotient of the radius of the lunar orbit, divided by the cube of the distance of the Sun from the Earth. In the quadratures, the action of the

Sun upon the Moon, decomposed in the direction of the lunar orbit, tends to augment the gravity of the Moon to the Earth: but this augmentation is only half the value of the diminution which it experienced in the syzygies. Thus, from all the actions of the Sun upon the Moon in the course of a synodical revolution, there results a mean force in the direction of the lunar radius vector, which diminishes the gravity of this satellite; and it is equal to half of (*e*) the product of the mass of the Sun, by the quotient of the radius of the lunar orbit, divided by the cube of the distance of the Sun from the Earth.

To find the ratio which this product bears to the gravity of the Moon, we may observe, that this force which retains it in its orbit, is nearly equal to the sum of the masses of the Earth and Moon, divided by the square of their mutual distance; and the force which retains the Earth in its orbit, is very nearly equal to the mass of the Sun divided by the square of its distance from the Earth. According to the theory of central forces, explained in the third Book, these two forces are as the radii of the orbits of the Sun and of the Moon, divided respectively by the squares of the times of their revelutions. Hence it follows that the preceding product is to the gravity of the Moon, as the square of the time of the sidereal revolution of the Moon, is to the square of the time of the sidereal revolution of the Earth. This product therefore is very nearly the $\frac{1}{179}$th of the lunar gravity, which by the mean action of the Sun is thus diminished by its 358th part.

In consequence of this diminution, the Moon is sustained at a greater distance from the Earth, than if it was abandoned (*f*) entirely to the action of its own force of gravity. The sector described by its radius vector is not altered, since the force which produces it, is in the direction of this radius, but its real velocity and angular motion are diminished; and it is easy to see, that by placing the Moon at a greater distance, so that its centrifugal force might equal its gravity, diminished by the action of the Sun, and that its radius vector might describe the same sector, that it would have described without this action; this radius would be augmented by its 358th part, and its angular motion diminished by a 179th part.

These quantities vary reciprocally as the cubes of the distances of the Sun from the Earth. When the Sun is in perigee, its action being most powerful, dilates the lunar orbit, but this orbit contracts again, as the Sun approaches its apogee; thus the Moon describes in space, a series of epicycloids whose centres are on the terrestrial orbit, and which dilate and contract as the Earth approaches to or recedes from the Sun. From hence an inequality (*g*) arises in the angular motion, very similar to the equation of the centre of the Sun, with this difference, that it retards this motion when that of the Sun augments, and that it accelerates it, when the motion of the Sun diminishes. These two equations are therefore always affected with contrary signs. The angular motion of the Sun is, as we have shewn in the first Book, reciprocally as the square of its distance, at the perigee, this distance being $\frac{1}{60}$th less than the mean distance, its angular velocity is augmented $\frac{1}{30}$th; the diminution of $\frac{1}{179}$th produced by the action of the Sun in the lunar motion, is then greater by a twentieth; the increase of this diminution is therefore the 3580th part of this motion. Hence (*h*) it follows that the equation of the centre of the Sun, is to the annual equation of the Moon, as a thirtieth of the solar motion is to the 3580th of the lunar motion, which gives 2398″ for the annual equation. It is about an eighth part less according to observation; this difference depends on the quantities that have been neglected in this first calculation.

The secular equation of the Moon, is produced by a cause similar to that of the annual equation. Halley first remarked this equation, which Dunthorn and Mayer have confirmed by a profound discussion of the observations. These two learned astronomers have proved that the mean motion of the Moon cannot be reconciled with modern observations, and with the eclipses observed by the Chaldeans and Arabians. They have attempted to represent them, by adding to the mean longitudes of this satellite a quantity proportional to the square of the (*i*) number of centuries before or after the year 1700. According to Dunthorn, this quantity is 30″9, for the first century: Mayer made it 21″6, in his first tables, which he increased to 27″8, in his last. And since that time, Lalande, after a new investigation of the subject, was led nearly to the same result as Dunthorn. The Arabian observations which have been chiefly made use of, are two eclipses of the Sun and one of the Moon, observed by Ibn Junis, near Cairo, towards the end of the tenth century, and extracted some time ago, from a manuscript of this astronomer's existing in the library at Leyden. Doubts have arisen concerning the reality of these eclipses; but the translation which M. Caussin has lately made of the part of this valuable manuscript, which contains the observations, has dissipated these doubts; it has moreover made us acquainted with twenty-five other eclipses observed by the Arabians, and which confirm the acceleration of the mean motion of the Moon. Besides, our modern observations compared with those of the Grecians and of the Chaldeans, are sufficient to establish the existence of the secular equation of the Moon. In fact, Delambre and Bouvard and Burg, have determined, by means of a great number of observations of the two preceding centuries, the actual secular motion, with a precision that leaves a very slight uncertainty: they found it six or seven hundred seconds greater than what is given by a comparison of antient and modern observations. The lunar motion is therefore accelerated since the time of the Chaldeans, and the Arabian observations being made in the interval that separates them and confirming this supposition, it is impossible any longer to question the truth of it.

Now, what is the cause of this phenomenon? Does the theory of universal gravitation, which has so well explained the numerous inequalities of the Moon, account likewise for its secular variation? These questions are the more interesting to resolve, beeause if we succeed, we shall obtain the law of the secular variations of the motion of the Moon, for it is evident that the hypothesis of an acceleration proportional to the time, as admitted by astronomers, is only on approximation, and cannot extend to an indefinite period.

This object has greatly occupied the attention of geometricians, but their researches were for a long time fruitless, having discovered nothing either in the action of the Sun or planets on the Moon, nor in the figures not exactly spherical of of this satellite and the Earth, that could change the mean motion of the Moon, some rejected the secular equation altogether, others to explain it, had recourse to various hypotheses, such as the action of comets, the resistance of an ether, and the successive transmission of gravity. Yet the correspondence of the other celestial phenomena with the theory of gravitation is so perfect, that we could not observe without regret, that the secular variation of the Moon appeared to refuse to submit to it, and continued the only exception to a general and simple law, whose discovery, by the grandeur and variety of the objects which it embraces, does so much honour to the human understanding. This consideration having determined me to reconsider this phenomenon, after several attempts I was at last so fortunate as to discover its cause. *The secular equation of the Moon arises from the action of the Sun upon this satellite, combined with the variation of the excentricity of the terrestrial orbit.* To form a just idea of this cause, we must recollect that the elements of the orbit of the Earth, are subject to alterations from the action of the planets; its greater axis remains always the same, but its excentricity, its inclination to a fixed plane, and the position of its nodes and of its perihelion, are incessantly changing. It must also be considered, that the action of the Sun upon the Moon diminishes by $\frac{1}{179}$, its angular velocity, and that its numerical co-efficient varies reciprocally as the cube of the distance of the Earth from the Sun. Now in expanding the inverse third power of this distance, into a series arranged according to the sines and cosines of the mean motions of the Moon, (*k*) and of their multiples, taking for unity the semi-major axis of the terrestrial orbit; it is found that this series contains a term equal to three times the half of the square of the excentricity of this orbit; the diminution of the angular velocity of the Moon, contains therefore a term equal to the 179th part of this velocity, multiplied by this term. If the excentricity of the terrestrial orbit was constant, this term would be confounded with the mean angular velocity of the Moon; but its variation, though very small, has nevertheless in progress of time a sensible influence on the motion of the Moon. It is evident that this motion will be accelerated, when the excentricity diminishes, which has been (*l*) the case ever since the most ancient observations to the present time, this acceleration will be changed into a retardation, when the excentricity having arrived at its *minimum*, will cease to decrease, and begin to augment.

In the interval from 1750 to 1850, the square of the excentricity of the terrestrial orbit diminishes 0.00000140595, the corresponding increase in the angular velocity of the moon is therefore 0.0000000117821 of this velocity: this increase taking place successively and proportionally to the time, its effect on the Moon's motion is only half what it would be, if during the whole course of the century, it was the same as at the end. To determine therefore this effect, or the secular equation of the Moon at the end of a century, reckoning from 1700, we must multiply the secular motion of the Moon by the half of the very small increase in its angular velocity; but in a century, the motion of the Moon is 5347405406″, which gives 31″,5017 for its secular equation.

As long as the diminution of the square of the excentricity of the terrestrial orbit may be supposed proportional to the time, the secular equation of the Moon will increase sensibly as the square of the time; it would be sufficient therefore to multiply 31″,5017 by the square of the number of centuries contained between the time for which the calculation is made, and the commencement of the nineteenth century. But I have found that in going back to the observations of the Chaldeans, the term proportional to the cube of the times, in the expression in a series, of the secular equation of the Moon, becomes sensible, this term is equal to $0'',057214$ for the first century; it should be multiplied by the cube of the number of centuries reckoned from 1801, the product being taken as negative for the centuries anterior to this epoch.

The mean action of the Sun upon the Moon depends also on the inclination of the lunar orbit to the ecliptic, and we might suppose that the position of the ecliptic being variable, there should result inequalities in the motion of this satellite, similar to those produced by the diminution of the excentricity of the terrestrial orbit; but I have recognised by analysis, that the lunar orbit is constantly brought back by the action of the Sun, to the same inclination to that of the Earth, so that the greatest and least declinations of the Moon are, in consequence of the secular variations in the obliquity of the eliptic, subject to the same changes as the declinations of the Sun.

This constancy in the inclination of the lunar orbit, is confirmed by all observations both ancient and modern.

The excentricity of this orbit experiences in like manner only an insensible alteration, from the change of the excentricity of the terrestrial orbit.

It is not thus with the variations of the motion of the nodes and perigee, to which it is indepensably necessary to pay attention in investigations, the object of which is to perfect the lunar tables. In submitting these variations to analysis, I have found that the influence of the terms depending on the square of the perturbating force, and which, as we have seen, double the mean motion of the perigee, is yet greater on the variation of this motion. The result of this intricate analysis, has given me a secular equation, triple of the secular equation of the mean motion of the Moon, to be subtracted from the mean longitude of the perigee, so that the mean motion of the perigee is retarded, when that of the Moon is accelerated. I have found likewise in the motion of the nodes of the lunar orbit upon the true ecliptic, a secular equation to be added to their mean longitude, and equal to 735 thousandths of the secular equation of the mean motion. Thus the motion of the nodes is retarded, like that of the perigee, when that of the Moon augments, and the secular equations of these three motions, are constantly in the proportion of the numbers 0,735, 3, 1. It is easy to infer from this, that the three motions of the moon, with respect to the sun, to its perigee and its nodes, continually increase, and that their secular equations are as the three numbers 1, 4, 0,265.

Future ages will develope these great inequalities, which will produce one day variations at least equal to a fortieth of the circumference, in the secular motion of the Moon, and to a thirteenth of the circumference in that of its perigee.

These inequalities do not always continue increasing; they are periodical, like those of the excentricity of the terrestrial orbit on which they depend, and do not re-establish themselves till after millions of years.

They must at length, alter the periods which have been devised for the purpose of comprehending complete numbers of revolutions of the Moon, relatively to its nodes, to its perigee, and to the Sun, periods which differ sensibly in different parts of the immense period of the secular equation.

The luni-solar period (*m*) of six hundred years, has been rigorously exact at a certain epoch, which it would be easy to find by analysis, if the masses of the planets were accurately determined; but this determination, so desirable for the perfection of our astronomical theories, is yet wanting. Fortunately Jupiter, whose mass we know exactly, is the planet which has the greatest influence on the secular equation of the Moon, and the values of the other planetary masses, are sufficiently accurate, for us to be certain that there cannot exist a sensible error in the magnitude of this equation.

Already ancient observations, notwithstanding their imperfection, confirm these inequalities, and we may trace their progress, either in these ancient observations, or in the astronomical tables which have succeeded them to the present time. We have seen that the ancient eclipses, made known the acceleration of the Moon's motion, before the theory of gravity had developed the cause.

In comparing modern observations, and the eclipses observed by the Arabians, Greeks, and Chaldeans, with this theory, we find an agreement between them that appears surprising, when we consider the imperfection of ancient observations, the vague manner in which they have been transmitted to us, and the uncertainty which still exists concerning the variations of the excentricity of the earth's orbit, and from the obviously imperfect manner in which the masses of Venus and Mars have been determined. The developement of the secular equations of the Moon, will be one of the most proper data to determine these masses.

It was particularly interesting to verify the theory of gravity, relatively to the secular equation of the motions of the perigee of the lunar orbit or to that of the anomaly, four times greater than the secular equation of the mean motion. From its discovery I have inferred that the actual motion of the perigee, made use of by astronomers, and which they inferred from a comparison of ancient and modern observations, must be diminished by from fifteen to sixteen minutes. In fact, when they did not take into account its secular equation, they should find this motion too rapid in the same manner, as they assumed too small a mean motion to the Moon, when they did not take its secular equation into account; this is what Bouvard and Burgh have confirmed by determining the actual secular equation of the lunar perigee, by means of a great number of modern observations; moreover Bouvard has found the same motion, by means of the most ancient observations, and by those of the Arabians, if its secular equation be taken into account of which the existence is by this means incontestably established.

The mean motions and the epochs of the tables of the Almageste and of the Arabians, indicate evidently these three secular equations of the lunar motion. The tables of Ptolemy are the result of immense calculations made by this astronomer and by Hipparchus; the work of Hipparchus has not come down to us: we only know from the evidence of Ptolemy, that he had taken the greatest care to select eclipses the most advantageous for the determination of the elements of which he was in search. Ptolemy, after two centuries and a half of new observations, found very little to change in these elements; there is therefore reason to believe that those which he made use of in his tables, have been determined by a great number of eclipses, of which he only preserved those that appeared to him to coincide most with the mean results which had been obtained by Hipparchus and himself. Eclipses only make known correctly the mean synodical motion of the Moon, and its distances from its nodes and its perigee: we can only then depend upon these elements in the tables of the Almageste: now in going back to the first epoch of these tables, by means of motions determined only by modern observations, we do not find the mean distances of the Moon from its nodes, its perigee, and from the Sun, that are given in these tables at this epoch. The quantities which must be added to these distances, are very nearly those which result from the secular equations; therefore, the elements of these tables at the same time, confirm the existence of these equations, and the values which I have assigned to them.

The motions of the Moon relative to its nodes, to its perigee, and to the Sun, being slower in the tables of the Almageste, than in our days, indicate also in these motions an acceleration, equally indicated both by the corrections that Albatenius, eight centuries after Ptolemy, made to the elements of these tables, and also by the epochs of the tables which Ibn Junis constructed about the year one thousand, from the collection of the Chaldean, Greek and Arabian observations.

It is remarkable that the diminution of the excentricity of the terrestrial orbit should be much more sensible, in the lunar motion than in itself. This diminution which, since the most ancient eclipse we are acquainted with, has not altered the equation of the Sun's centre 15′, has produced a variation of two degrees in the Moon's longitude, and nearly a variation of eight degrees in its mean anomaly; we could hardly suspect it from the observations of Hipparchus and Ptolemy. Those of the Arabians indicated it with much probability; but the ancient eclipses, compared with the theory of gravitation, leave no doubt on this subject. This reflexion, if I may make use of the term, of the secular variations of the orbit of the Earth on the lunar motion, in consequence of the action of the Sun, has place also for the periodic inequalities. It is in this manner that the equation of the centre of the Earth's orbit, reappears in the lunar motion with a contrary sign, and reduced to about a tenth of its value; in like manner the inequality produced in the motion of the Earth by the action of the Moon, is reproduced in the motion of the Moon, but diminished in the ratio of five to nine. Finally, the action of the Sun in transmitting to the Moon, the inequalities which the planets produce in the motion of the Earth, renders this indirect action of the planets on the moon, more considerable than their direct action on this satellite.

Here we see an example of the manner in which phenomena as they are developed, lead us to the knowledge of their true causes. When only the acceleration of the mean motion of the Moon was known, it might be attributed to the resistance of ether, or to the successive transmission of gravity. But analysis proves that these two causes cannot produce any sensible alteration in the mean motion of the nodes and of the lunar perigee, and that alone would suffice to exclude them, even when the true cause of the variations observed in these motions was unknown.

The agreement of theory with observations, proves that if the mean motions of the Moon are altered by causes foreign to the principle of universal gravitation, their influence is very small, and hitherto insensible.

This agreement evinces in a decisive manner, the invariability of the duration of the day, which is an essential element in astronomical theories. If this duration was greater now by the hundredth part of a second, than in the time of Hipparchus, the duration of the present century would be greater than at that time, by 365″25 ; in this interval the Moon describes an arch of 534″,6 ; therefore the actual secular mean motion of the Moon would appear to be increased by this interval, which would increase its secular equation by 13″,51 for the first century, commencing from 1801, and which by what goes before is 31″,5017. Observations do not permit us to suppose so considerable an increase. We may therefore be assured that since the time of Hipparchus, the duration of the day has not varied by the hundredth part of a second.

One of the most important equations in the lunar theory, in as much as it depends on the compression of the earth, is relative to the motion of the Moon in latitude. This inequality is proportional to the (*n*) sine of the true longitude of this satellite. It arises from a nutation in the lunar orbit produced by the action of the terrestrial spheroid, and corresponding to that which the Moon produces in our equator, so that one of these nutations is the reaction of the other ; and if all the molecules of the Earth and Moon were connected firmly together by inflexible lines, void of mass, the entire system would be in equilibrio about the centre of gravity of the earth, in consequence of the forces which produce these two nutations ; the force which actuates the Moon, compensating its smallness, by the length of the lever to which it is attached. This inequality in latitude may be represented, by conceiving that the orbit of the Moon, instead of moving uniformly on the ecliptic with a constant inclination, moves according to the same conditions on a plane a little inclined to the ecliptic, and passing always through the equinoxes, between the ecliptic and equator; this phenomenon is produced in a more sensible manner in the motions of the satellites of Jupiter, in consequence of the very great compression of that planet. Thus, this inequality diminishes the inclination of the orbit of the Moon to the ecliptic, when its ascending node coincides with the equinox of spring, it increases it when the node coincides with the equinox of autumn, which being the case in 1755, renders the inclination too great, which Mason determined by the observations of Bradley, made between the intervals of 1750 and 1760. In fact Burg, who determined it by observations made during a much longer interval, and by taking into account the preceding inequality, found the inclination to be smaller by about $11'',\frac{1}{2}$; this astronomer undertook at my request to determine the coefficient of this inequality, and from a great number of observations he found it equal to—$24'',6914$. Burchardt, by employing for this purpose a still greater number of observations, arrived at the very same result, which gives the compression of the earth equal to $\frac{1}{304,6}$.

This compression may also be determined by means of an inequality in the Moon's motion in longitude, which depends on the longitude of the Moon's node. Observation indicated it to Mayer, and Mason fixed its quantity at 23,″765; but as it did not appear to result from the theory of gravity, the greater number of astronomers neglected it. This theory pointed out to me that its cause existed in the compression of the earth. Burgh and Burchardt from a great number of observations, fixed it at 20″,987, which answers to a compression of $\frac{1}{305,05}$, very nearly the same as is given by the preceding inequality of the motion in latitude. Thus the Moon by the observation of its motion, renders sensible to astronomy when brought to a state of perfection, the ellipticity of the earth, the round form of which it first made known to astronomers, by its eclipses.

The two preceding inequalities demand the greatest attention of observers. They have an advantage over geodesical observations, in as much as they give the compression of the earth, in a manner less dependant on the inequalities of the surface of the earth. If the earth was homogeneous, they would be much greater than what observation determines them to be, consequently the earth is not homogeneous. It follows also from this that the attraction of the Moon towards the Earth arises from the attractions of all the molecules of this planet; which is a new proof of the mutual gravitation of all the parts of matter.

Theory combined with the experiments of the pendulum and the measures of degrees on the earth, assigns to the parallax, as we have seen in the first chapter of this book, a quantity very nearly conformable (*o*) to observations, so that conversely, the magnitude of the earth might be inferred from these observations.

Finally, the parallax of the Sun might be inferred with accuracy from a lunar equation which depends on the simple angular distance of the Moon from the Sun. For this purpose, I have computed with great care, the coefficient of this equation, and by putting it equal to that, which Burgh and Burchardt concluded from a long series of observations, I concluded that the mean parallax of the Sun was 28″,56 the same which several astronomers deduced from the last transit of Venus.

It is worthy of remark that an astronomer without leaving his observatory, by merely comparing observations with analysis, can determine exactly the magnitude and compression of the earth, and its distance from the Sun and from the Moon, which elements have been determined by long and troublesome voyages in the two hemispheres. The agreement of the results obtained by these two different methods is one of the most striking proofs of the theory of universal gravitation.

The numerous comparisons which Bouvard and Burgh made of the lunar tables, with the observations of the end of the seventeenth century by La Hire and Flamstead; of the middle of the eighteenth century by Bradley, and with the uninterrupted series of observations of Maskeyline from Bradley to this day, furnish a result which we would be far from anticipating. The observations of La Hire and of Flamstead, compared with those of Bradley, indicate a secular sidereal motion of the Moon, greater at least by one hundred and thirty seconds, than what results from the observations of Bradley compared with the last of Maskeline; and the observations made during the last twenty years proves that the diminution of the secular motion of the Moon has been greater still during this interval; so that the existence of an anomaly in the mean motion of the Moon is at least very probable. Hence arises the necessity of perpetually retouching the epochs of the tables, until we can determine the cause or the law of this remarkable anomaly. It is evidently connected with one or more unknown inequalities, with long periods, of which theory alone can indicate the laws.

The best lunar tables are founded on theory and observation combined. They borrow from theory the arguments of the inequalities which it would have been difficult to know by means of observation alone. I have determined in my Treatise of Celestial Mechanics, the coefficients of these arguments in a very approximate manner, but in consequence of the slowness of the convergance of these approximations, combined with the difficulty of extricating from among the immense number of terms which the analysis developes, those which can acquire from integration a sensible value, this investigation is extremely troublesome. Nature itself furnishes in the collections of observations, the results of those integrations so difficult to obtain by analysis. Messrs. Burckhardt and Burgh have employed for their determination several thousand observations, and by this means have rendered their lunar tables extremely accurate. Being anxious to banish all empericism, and that other geometers should discuss several intricate points of the theory to which I first arrived, such as the secular equations of the motions of the Moon; I induced the Academy to propose for the subject of its mathematical prize for the year 1820, the formation by theory alone of lunar tables equally perfect with these which have been inferred from theory and observation combined. Two pieces were crowned by the Academy, the author of one of them, M. Damoiseau, accompanied it with tables which compared with observations, have represented them with the accuracy of our best tables. The authors of the two pieces agree on the periodical and secular inequalities of the motions of the Moon. They differ a little from my result on the secular equation of mean motion; but instead of the numbers 1; 4; 0,265 by which I represented the ratios of the secular inequalities of the motion of the Moon relatively to the Sun, to the perigee of the lunar orbit and to its nodes, they have found the numbers 1; 4,6776; 0,391. M. Domoiseau in his essay has made the second of these numbers, very nearly equal to 4; but after mature reflection on his analysis, he has arrived at the same result as

Messrs. Plana and Carlini, the authors of the other essay. As they extended their approximations a considerable way, their numbers appear preferable to those which I have determined. Finally, from those approximations the mean motions of the perigee and of the nodes of the orbit, have been inferred exactly conformable to observations.

It follows indubitably from what we have seen, that the law of universal gravitation is the sole cause of all the inequalities of the Moon, and if we consider the great number and the extent of these inequalities, and the proximity of this satellite to the earth, it will be agreed on, that of all the heavenly bodies, it is the best adapted to establish this great law of nature, and the power of analysis, of that wonderful instrument, without which it had been impossible for the human mind to penetrate into a theory so complicated, and which may be employed as a means of discovery, equally certain with observation itself.

Some partizans of final causes, have imagined that the Moon was given to the Earth, to afford it light during the night. But in this case, nature would not have attained the end proposed, since we are often deprived at the same time of the light of each of them. To have accomplished this end, it (*p*) would have been sufficient to have placed the Moon at first in opposition to the Sun and in the plane of the ecliptic, at a distance from the Earth equal to the one hundredth part of the distance of the Earth from the Sun, and to have impressed on the Earth and Moon, parallel velocities proportional to their distances from the Sun. In this case, the Moon being constantly in opposition to the Sun, would have described round it an ellipse similar to that of the Earth. These two stars would then constantly succeed each other, and as at this distance the Moon could not be eclipsed, its light would always replace that of the Earth. Other philosophers struck with the singular opinion of the Arcadians that they were older than the Moon, thought that this planet was originally a comet which, passing near to the Earth, was forced by its attraction to accompany it. But by reascending by means of analysis to the most remote periods, we shall find that the Moon always moved in an orbit nearly circular about the Earth, as the planets move about the Sun, therefore neither the Moon nor any satellite was originally a comet.

As the gravity at the surface of the Moon, is much less than at the surface of the Earth, and as this star has no atmosphere which can oppose a sensible resistance to the motion of projectiles, we may conceive that a body projected with a great force, by the explosion of a lunar volcanoe, may attain and pass the limit, where the attraction of the Earth commences to predominate over that of the Moon. For this purpose it is sufficient that its central velocity in the direction of the vertical may be 2500 metres in a second; then in place of falling back on the Moon, it becomes a satellite of the Earth, and describes about it an orbit more or less elongated.

The direction of its primitive impulsion may be such as to make it move directly towards the atmosphere of the earth ; or it may not attain it, till after several and even a great number of revolutions, for it is evident that the action of the Sun, which changes in a sensible manner the distances of the Moon from the Earth, ought to produce in the radius vector of a satellite which moves in a very excentrick orbit, much more considerable variations, and thus at length so diminish the perigean distance of the satellite, as to make it penetrate our atmosphere. This body traversing it with a very great velocity, and experiencing a very sensible resistance, might at length precipitate itself on the Earth : the friction of the air against its surface would be sufficient to enflame it, and make it detonate, provided that it contained ingredients proper to produce these effects, and then it would present to us all those phenomena which meteoric stones exhibit. If it was satisfactorily proved that they are not produced by volcanoes, or generated in our atmosphere, and that their cause must be sought beyond it, in the regions of the heavens, the preceding hypothesis, which likewise explains the identity of composition observed in meteoric stones, by an identity of origin, will not be devoid of probability.

CHAP. VI.

Of the perturbations of the satellites of Jupiter.

Of all the satellites, the most interesting, after that of the earth, are the satellites of Jupiter. The observations of these stars, the first which the telescope discovered in the heavens, are not older than two centuries, and it is only about a century and a half since their eclipses have been observed. But in this short interval they have presented, by the quickness of their revolutions, all the great changes which time would not develop except with great slowness in the planetary system, of which that of the satellites is only an epitome. The inequalities produced by their mutual attraction, do not differ materially from those of the planets and of the Moon: however the relations which exist between the mean motions of the three first satellites give rise to some inequalities of considerable magnitudes, which have a great influence on their theory. We have seen in the second book, that the differences between the mean motions of the first and second is very nearly twice the difference between the mean motions of the second and third, and that they are subject to very sensible inequalities, of which the periods though different one from the other, are in the eclipses transformed into one sole period of $437^d,659$.

The first inequalities which observation discovered in the motion of these bodies, are also the first which are derived from the theory of universal gravitation of the satellites. This theory not only determines these inequalities, but it shews us also, what observation seemed to indicate with great probability, namely, that the inequality of the second sateltite is the result of two inequalities, of which one being caused by the action of the first satellite, varies as the sine of the excess of the longitude of the first satellite (*a*) above that of the second; and of which the other, produced by the action of the third, varies as the sine of double the excess of the longitude of the second satellite above that of the third. Thus the second satellite experiences a perturbation from the action of the first, similar to that which itself causes in the third; and it experiences from the third a similar perturbation to that which itself causes in the first.

These two inequalities are combined into one in consequence of the relation which exists between (*b*) the mean motions and the mean longitudes of the three first satellites, according to which the mean motion of the first satellite *plus* twice that of the third, is equal to three times that of the second; and the mean longitude of the first satellite, *minus* three times that of the second, *plus* twice that of the third, is constantly equal to a semi-circumference: but will these relations always exist, or are they only approximate, and will the two inequalities of the second satellite which at present are combined, be separated in the course of time? It is to theory that we must apply for a solution to this question.

The approximate manner with which the tables furnished the preeeding relations, made me suppose that they were rigorously exact, and that the small quantities by which they still differed, depended on the errors to which they were liable; for it was against all probability that chance should have originally placed the three first satellites at the precise distances and positions suitable to the above relation: it was therefore extremely probable that it arose from some particular cause; I looked therefore for this cause in the mutual action of the satellites. A scrupulous investigation of this action, has shewn me that it has rendered these relations rigorously exact: from whence I concluded, that in determining again by the examination of a great many distant observations, the mean motions and the mean longitudes of the three first satellites, it would be found that they would approximate still more to these relations, to which the tables should be made exactly to agree. I had the satisfaction of seeing this consequence of the theory confirmed, with remarkable precision, by the researches which Delambre has made concerning the satellites of Jupiter. It is not necessary that these relations should have taken place exactly at their origin, it was enough that they did not greatly differ, then the mutual actions of the satellites upon each other were sufficient to subject them to this law, and to maintain it unaltered; but the little difference between this and the primitive relation, has given rise to a small inequality of an arbitrary extent, which is distributed among the three satellites, and which I have designated by the name of *libration*. The two constant arbitrary quantities of this inequality, replace whatever arbitrary quantity is made to disappear by the two preceding relations, in the mean motions and in the epochs of the mean longitudes of the three first satellites; for the number of arbitrary (c) quantities included in the theory of a system of bodies is necessarily sextuple the number of bodies: as observation does not indicate this inequality, it must evidently be very small, and even insensible.

The preceding relations would still subsist, even if the mean motions of the satellites were subject to secular variations analogous to that of the motion of the Moon. They would subsist also in the case of these motions being altered by the resistance of a medium, or by other causes, of which the effects would not be perceived until after a long time. In all these cases, the secular equations so arrange themselves by the reciprocal action of the satellites, that the secular equation of the first plus twice that of third is equal to three times that of the second: even their inequalities, which increase with extreme slowness, approach so much the more to coordinate themselves thus, as their periods are more considerable. This libration, in consequence of which the motions of the three first satellites are balanced in space according to the laws which we have just announced, extends also to their motions of rotation, if as all observations appear to indicate these motions are equal to those of revolution. The attraction of Jupiter must then maintain this inequality, by impressing on the motions of rotation, the same secular equations as affect the motions of revolution. Thus, the three first satellites of Jupiter consitute a system of bodies connected together by the preceding inequalities and relations, which their mutual action will maintain uninterruptedly, unless some extraneous cause should abruptly derange their respective motions and positions.

Such would be the effect of a comet, which traversing this system, as the first comet of 1770 appears to have done, would impinge on one of these bodies. It is probable that such rencounters have taken place, in the immensity of ages, which have lapsed, since the commencement of the planetary system. The shock of a comet, of which the mass was only the hundreth millioneth part of that of the earth, would be sufficient to render the libration of the satellites sensible. As this inequality has not been recognised, notwithstanding all the care of Delambre to detect it in his observations, we ought to conclude that the masses of any of the comets which might have (*d*) impinged on the three satellites of Jupiter must have been extremely small; which confirms what has been already observed on the smallness of the masses of the comets.

The orbits of the satellites experience changes analogous to the great variations which the planetary orbits undergo; their motions are in like manner subjected to secular equations similar to those of the Moon. The developement of all these inequalities in the progress of time, will furnish the most advantageous data for determining the masses of the satellites and the compression of Jupiter. The great influence which this last element has on the motions of the nodes, determines its value with more accuracy than direct measurement. By this means it is found that the ratio (*e*) of the lesser axis of Jupiter to the diameter of his equator, is equal to 0,9368, which differs very little from the ratio, sixteen to seventeen, which is given by a mean of the most accurate measures of the compression of this planet. This agreement is a new proof that the gravity of the satellites towards the primary planet, arises from the attractions of all its molecules.

One of the most remarkable consequences of the theory of the satellites of Jupiter is the knowledge of their masses, which would appear to be interdicted by their extreme smallness and by the impossibility of measuring their diameters. I have selected for this purpose, the data which in the actual state of astronomy have appeared to me the most advantageous, and I apprehend that the following values, which I have inferred from them, are very accurate.

Masses of the satellites of Jupiter, that of the planet being assumed equal to unity.

I Satellite	0,0000173281.
II Satellite	0,0000232355.
III Satellite	0,0000884972.
IV Satellite	0,0000426591.

These values should be corrected, when in the progress of time, we become better acquainted with the secular variations of the orbits.

Whatever be the perfection of the theory, an immense labour is reserved for the astronomer to convert the analytical formulæ into tables. These formulæ contain thirty-one constant arbitrary quantities, namely, the twenty-four arbitrary (f) quantities of the twelve differential equations of the motion of the satellites, the masses of these stars, the compression of Jupiter, the inclination of his equator, and the position of his nodes. In order to obtain the values of all these unknown quantities, we should discuss a very great number of eclipses of each satellite, and combine them in the manner best adapted to make each element arise. Delambre has performed this important work with the greatest success; and his tables, which represent observations with the accuracy of the observations themselves, afford to the navigator a sure and easy means of obtaining immediately by the eclipses of the satellites, especially by those of the first, the longitude of the places at which he can land. The following are the principal elements of the theory of each satellite, which result from a comparison which was made by Delambre of my formulæ, with observations.

The orbit of the first satellite moves uniformly with a constant inclination on a fixed plane, which passes constantly between the equator and the orbit of Jupiter, through the mutual intersection of these two last planes, of which the respective inclination is according to observations, equal to 3,°4352. The inclination of this fixed plane with the equator of Jupiter, is only 20″ by theory; it is consequently insensible. The inclination of the orbit of the satellite on this plane is in like manner insensible to observations; thus the first satellite may be supposed to move in the plane of the equator of Jupiter. An excentricity peculiar to this orbit has not been recognised, which only participates a little in the excentricities of the orbits of the third and fourth satellites, for in virtue of the mutual action of all these bodies, the excentricity proper to each orbit is diffused over the others, but more feebly as they are more distant. The sole inequality of this satellite which is sensible is that of which the argument, is double of the excess of the mean longitude of the first satellite above that of the second, and which produces in the recurrence of the eclipses, an inequality of $437^{d},659$; it is one of the data which I have made use of to obtain the masses of the satellites, and as it arises from the action of the second alone, it determines its mass with great accuracy.

The eclipses of the first satellite of Jupiter, gave rise to the (*g*) discovery of the successive transmission of light, which the penomenon of aberration has ascertained with still greater precision. It appeared to me that as the theory of the motion of this satellite is now better known, and as the observations of its eclipses are become more numerous, their discussion should give the quantity of aberration more exactly than direct observation. Delambre, who undertook this investigation at my request, found the entire quantity of aberration $62''5$, which is exactly that which Dr. Bradley derived from his observations. It is very curious to observe such a perfect agreement in results which have been obtained by such very different methods.

It follows from this agreement, that the velocity of light is uniform (*h*) through the whole space comprehended by the terrestrial orbit. In fact, the velocity of light given by the aberration is that which subsists at the circumference of the terrestrial orbit, and which, being combined with the motion of the Earth produces this phenomenon. The velocity of light, as given by the eclipses of the satellites of Jupiter, is determined by the time which light emknow the law, without the aid of analysis. It is curious to see these remarkable phenomena, which observation indicated, resulting from the analytical formulæ; but which arising from the combination of several simple inequalities are too complicated for Astronomers to discover their laws. The excentricity of the orbit of the fourth satellite is much greater than those of the other orbits, its perijove has a direct annual motion of 7959″; it is the fifth data which I employed in determining the masses. Each orbit participates a little in the motion of the others. The fixed planes to which we have referred them are not strictly speaking fixed; they move very slowly with the equator and orbit of Jupiter, always passing through the mutual intersection of those last planes, and preserving on the equator of Jupiter inclinations which, though variable, have to each other, and to the inclination of the orbit of the planet on its equator, a constant ratio.

Such are the principal results of the theory of the satellites of Jupiter compared with numerous observations of their eclipses. Observations of the ingress and egress of their shadows on the disk of Jupiter would throw considerable light on several elements of their theory. This kind of observations, hitherto too much neglected by Astronomers, ought, as it appears to me, to attract their attention, for it seems that the interior contact of the shadows would determine the time of conjunctiom more accurately than eclipses. The theory of the satellites is now so far advanced, that whatever tor and orbit of Jupiter through their mutual intersection, of which the inclination to this equator is 201″. The orbit of the satellite is inclined by 5152″ to its fixed plane, and its nodes have on this plane a retrogade tropical motion, of which the period is $29^{yrs},9142$: this period is one of the data which I have made use of in determining the masses of the satellites. Observation has not made known the excentricity peculiar to this orbit; but it participates a little in the excentricities of the orbits of the third and fourth satellite. The two principal inequalities of the second satellite depend on the actions of the first and of the third satellite. The ratio existing between the longitudes of the three first satellites always combines those inequalities (*l*) into one sole, of which the period in the recurrence of the eclipses is $437^{d},659$, and of which the value is the third quantity which I made use of in determining the masses. The orbit of the third satellite moves uniformly with a constant inclination, on a fixed plane, which passes constantly between the equator and the orbit of Jupiter, through their mutual intersection, and of which the inclination on this equator is 931″. The orbit of the satellite is inclined by 2284″ to its fixed plane, and its nodes have on this plane a retrograde tropical motion, of which the period is $141^{yrs},739$. Astronomers supposed the orbits of the three first satellites to move in the plane of the equator itself of Jupiter, but they deduced from the eclipses of the third satellite, a smaller inclination of this equator to the orbit of the planet, than what was collected from those of the two others. This difference, of which they did not know the cause, arose from this, that the orbits of the satellites do not move with a constant inclination to the equator, but on different planes, of which the inclination is greater for those satellites which are more distant. Our moon presents a similar result, as we have observed in the preceding chapter ; it is on this, that the lunar inequality in latitude depends, from which the compression of the earth has been inferred, perhaps with more accuracy than from the measures of the degrees of the meridian.

The excentricity of the orbit of the third satellites exhibits singular anomalies, of which theory has indicated the cause. They depend on two distinct equations of the centre. The one peculiar to this orbit respects a perijove, of which the annual sidereal motion is about 29010″. The other, which may be regarded as an emanation from the equation of the centre of the fourth satellite, respects the perijove of this last body. It is one of the data from which I have determined the masses. These two equations form by their combination a variable equation of the centre respecting a perijove, of which the motion is not uniform. They coincided and combined their effects in 1682 and their sum amounted to 2458 ;″ in 1777 the effect of one was taken from that of the other, and the difference amounted to 949″. Wargentin endeavoured to represent th   ›ns by means of two equations of the centre, but as he did not refer one of them to the perijove of the fourth satellite, he was obliged, by observations, to abandon his hypothesis, and he had recourse to that of one variable equation of the centre, of which he determined the changes by observations, which conducted him very nearly to what we have indicated. Finally, the orbit of the fourth satellite moves uniformly with a constant inclination on a fixed plane, inclined by 4457″ to the equator of Jupiter, and which passes through the line of the nodes of this equator, between this last plane and that of the orbit of the planet; the inclination of the orbit of the planet to its fixed plane is 2772″, and its nodes have on this plane a retrograde tropical motion, of which the period is 531 years. In consequence of this motion the inclination of the orbit of the fourth satellite on the orbit of Jupiter varies continually. Having attained its *minimum* towards the middle of the last century, it has been nearly stationary, and about 2°,7 from 1680 to 1760. In this interval its nodes have a direct motion in a year of 8′ very nearly. This circumstance, which observation indicated, was for a long time made use of by astronomers, who were employed in the tables of these satellites; it is a consequence of the theory which gives the inclination and the motion of the nodes very nearly the same, as Astronomers found them by a discussion of the eclipses. But in these last years the inclination of the orbit has undergone a considerable increase, of which it was difficult to know the law, without the aid of analysis. It is curious to see these remarkable phenomena, which observation indicated, resulting from the analytical formulæ; but which arising from the combination of several simple inequalities are too complicated for Astronomers to discover their laws. The excentricity of the orbit of the fourth satellite is much greater than those of the other orbits, its perijove has a direct annual motion of 7959″; it is the fifth data which I employed in determining the masses. Each orbit participates a little in the motion of the others. The fixed planes to which we have referred them are not strictly speaking fixed; they move very slowly with the equator and orbit of Jupiter, always passing through the mutual intersection of those last planes, and preserving on the equator of Jupiter inclinations which, though variable, have to each other, and to the inclination of the orbit of the planet on its equator, a constant ratio.

Such are the principal results of the theory of the satellites of Jupiter compared with numerous observations of their eclipses. Observations of the ingress and egress of their shadows on the disk of Jupiter would throw considerable light on several elements of their theory. This kind of observations, hitherto too much neglected by Astronomers, ought, as it appears to me, to attract their attention, for it seems that the interior contact of the shadows would determine the time of conjunctiom more accurately than eclipses. The theory of the satellites is now so far advanced, that whatever deficiency is required to complete this theory, can only be determined by the most exact observations; it is therefore necessary to try new modes of observations, or at least to be certain that those which we make use of, deserve the preference.

CHAP. VII.

Of the Satellites of Saturn and of Uranus.

THE extreme difficulty of observing the satellites of Saturn renders their theory so imperfect, that we hardly know with any precision their revolutions and their mean distances from the centre of this planet; it is therefore as yet unnecessary to consider their perturbations. But the position of their orbits presents a phenomenon worthy of the attention of Geometers and Astronomers. The orbits of the six first satellites appear to be in the plane of the ring, while the orbit of the seventh satellite deviates from it sensibly. It is natural to think that this depends on the action of Saturn, which, in consequence of his compression, retains the first six orbits and its rings in the plane of its equator. (*a*) The action of the sun tends to make them deviate from it, but this deviation increasing very rapidly and very nearly as the fifth power of the radius of the orbit, it only becomes sensible for the last satellite. The orbits of the satellites of Saturn, like those of Jupiter, move in planes, which pass constantly between the equator and orbit of the planet, through their mutual intersection, and which are always more inclined to this equator according as the satellites are farther from Saturn. This inclination is considerable relatively to the last satellite, and about 24°,0, if we refer to observations already made; the orbit of the satellite is inclined by 16°,96 to this plane, and the annual motion of its nodes on the same plane is 940′. But as these observations are cxtremely uncertain, the preceding results can only be considered as a very imperfect approximation.

We are even less informed with respect to the satellites of Uranus. It solely appears from the observations of Herschel, that they move in the same plane, almost perpendicular to that of the orbit of the planet; which evidently indicates a similar position in the plane of its equator. Analysis shews that the ellipticity of the planet, combined with the action of the satellites, can very nearly maintain their different orbits, in the same plane. This is all which can be affirmed of these stars, which in consequence of their distance and inconsiderable magnitude, will be for a long time inaccessible to the most extended researches.

CHAP. VIII.

Of the Figure of the Earth and Planets, and of the Law of Gravity at their Surface.

WE have detailed in the First Book, what has been indicated by observations on the figure of the Earth, and of the planets: let us compare these results with those of universal gravitation.

The force of gravity towards the planets, is composed of the (*a*) attractions of all their particles. If their masses were in a state of fluidity, and without motion, their figure and those of the different strata would be spherical, those nearer the centre being more dense. The force of gravity at their exterior surface, and at any distance whatever, without the sphere, would be exactly the same, as if the whole mass of the planet was conndensed into the centre of gravity. It is in consequence of this remarkable property, that the Sun, the planets, comets, and satellites, act upon each other, very nearly (*b*) as if they were so many material points. At very great distances the attraction of the particles of a body of any figure, which are the most remote, and those which are nearest the particle attracted, (*c*) compensate each other in such a manner, that their total attraction is very nearly the same as if they were united in the centre of gravity; and if the ratio of the dimensions of the body to its distance from the attracted point, be considered as a very small quantity of the first order, this result will be exact to quantities of the second order. But in a sphere, it is rigorously true, and in a spheroid differing but little from a sphere, the error is of the same order as the product of its excentricity, by the square of the ratio of its radius, to its distance from the point attracted. This property of the sphere, of attracting as if its mass was concentered in its centre, contributes greatly to the simplicity of the motions of the heavenly bodies. It does not belong exclusively to the law of nature, it equally appertains to the law of attraction varying proportionably to (*d*) the simple distance, and cannot belong to any other law but those formed by the addition of these two. And of all the laws which render the force of gravity nothing at an infinite distance, that of nature is the only one in which the sphere possesses this property.

According to this law, a body placed within a spherical stratum of uniform thickness, is equally attracted by all its parts, so as to remain at rest in the midst of the various attractions which act upon it. The same circumstance takes place in an elliptic stratum, when the exterior and interior surfaces are similar and similarly situated. Supposing therefore the planets to be homogeneous spheres, the force of gravity in their interior, must diminish as the distance from the centre; for the exterior part, relatively to the attracted particle, contributes nothing to its gravity, which is only produced by the attraction of the internal sphere, whose radius is equal to the distance of this point from the centre. But this attraction is equal to the mass of the sphere, divided by the square of the radius, and the mass, is as the cube of this same radius. The force of gravity on the attracted particle, is therefore proportional to the radius. But if, (as is probably the case) the strata are more dense as they are nearer to the centre, the force of gravity will diminish in a less ratio, than in the case of homogeneity. The rotatory motion of the (*e*) planets causes them to deviate a little from the spherical figure. The centrifugal force arising from this motion, causing the particles situated at the equator to recede from the centre, and thus to produce a flattening of the poles.

Let us consider first the effects of this compression in the simplest case, namely, that in which the Earth is considered as an homogeneous fluid, the gravity residing in its centre and varying reciprocally as the square of the distance from this point. It is then easy to prove that the terrestrial spheroid is an ellipsoid of revolution; for if we conceive two columns of fluids, communicating with each other at the centre, and terminating, the one at the pole, the other at any point on the surface, these two columns ought to be in equilibrio. The centrifugal force does not alter the weight of the column directed to the pole, but it diminishes the weight of the other column. This force is nothing at the centre of the Earth, and at the surface it is proportional to the radius of the terrestrial parallel, or very nearly, to the cosine of the latitude; but the whole of this force is not entirely employed in diminishing the force of gravity; for these two forces making an angle with each other, *(f)* equal to the latitude, the centrifugal force, decomposed according to the direction of gravity, is weakened in the ratio of the cosine of this angle to radius. Thus, at the surface of the Earth, the centrifugal force diminishes the force of gravity, by the product of the centrifugal force at the equator, by the square of the cosine of the latitude; therefore the mean value of this diminution in the length of a fluid column, is the half of this product, and since the centrifugal force is $\frac{1}{289}$ of the force of gravity at the equator, this value is the $\frac{1}{578}$th part of the force of gravity, multiplied by the square of the cosine of the latitude. And since it is necessary, for the maintenance of the equilibrium, that the column by its length should compensate the diminution of its weight, it ought to surpass the polar column by a $\frac{1}{578}$th of its length, multiplied by the square of the above cosine. Thus the augmentation of the radii, from the pole to the equator, is proportional to the squares of these cosines, from which it is easy to conclude, that the Earth is an ellipsoid of revolution, the equatorial and polar axis of which are in the proportion of 578 to 577.

It is evident that the equilibrium of the fluid mass would still subsist, even if a part should be supposed to consolidate itself in the interior, provided the force of gravity remains the same.

To determine the law of gravity at the surface of the Earth, we may observe that the force of gravity to any point on this surface, is less than that at the pole, from its being situated farther from the centre. This diminution is nearly equal to double the augmentation of the terrestrial radius; it is equal therefore to the product of the $\frac{1}{289}$th part of the force of gravity by the square of the cosine of the latitude. The centrifugal force diminishes likewise the force of gravity by the same quantity; thus by the union of these two causes, the diminution of gravity from the pole to the equator, is $= 0{,}00694$, multiplied by the square of the cosine of the latitude, the force of gravity at the equator being taken as unity.

It has been shewn in the First Book, that the measures of meridional degrees, assign to the Earth an ellipticity greater than $\frac{1}{578}$, and that the measures of the pendulum indicate a diminution in the force of gravity, from the poles to the equator, less than 0,00694, and equal to 0,00567. The measures of the degrees and of the pendulum concur, therefore, to prove that the force of gravity is not directed to a single point, which confirms *a posteriori* what has been antecedently demonstrated, namely, that the gravity is composed of the attractions of all the particles of the Earth.

This being the case, the law of gravity depends on the figure of the terrestrial spheroid, which depends itself on the law of gravity. It is this mutual dependance of the two unknown quantities on each other, that renders the investigation of the figure of the earth so extremely difficult. Fortunately, however, the elliptic figure, the most simple of all the re-entering figures next to the sphere, satisfies the condition of the equilibrium of a fluid mass, subject to a motion of rotation, and of which all the particles attract each other reciprocally, as the squares of the distances. Newton, upon this hypothesis, and supposing the earth a homogeneous fluid, found the ratio of the equatorial to the polar axis, to be 230 to 229.

It is easy to determine the law of the variation of the force of gravity on the earth upon this hypothesis. For this purpose let us consider two different points situated on the same radius, drawn from the centre to the surface of an homogeneous fluid, in equilibrio. All the similar elliptic strata, which cover any one amongst them, contribute nothing to its gravity. The resulting force of all the attractions which act on it, is derived entirely from the attraction of the interior spheroid, similar to the entire spheroid, and whose surface passes through the point in question. The similar and similarly situated particles of these two spheroids, attract the interior (g) point, and the corresponding point of the exterior surface, proportionally to their masses, divided by the squares of their distances. These masses are in the two spheriods, as the cubes of their similar dimensions, and the squares of their distances, are as the squares of these dimensions. The attractions on similar particles, are proportional therefore to these dimensions; from which it follows, that the entire attractions of the two spheroids, are in the same ratio, and their directions are parallel. The centrifugal forces of the two points, now under consideration, are likewise proportional to the same dimensions. Therefore the force of gravity in each of them, being the result of these two forces, will likewise be proportional to their distances from the centre of the fluid mass.

Now, if we conceive two fluid columns directed as before, to the centre of the spheroid, one from the pole, and the other from any point on the surface, it is evident, if the ellipticity of the spheroid is very small, that is, if it differs but little from a sphere, that the force of gravity, decomposed according to the directions of these columns, will be nearly the same as the total gravity. Dividing, therefore, the length of these columns into an equal number of parts, infinitely small and proportional to their lengths, the weights of the corresponding parts will be to each other as the products of the lengths of the columns, by the force of gravity at the points of the surface where they terminate. The whole weight of the columns will therefore be to each other in this ratio; and as these weights must be equal, to be in equilibrio, the force of gravity at their surface must consequently be reciprocally, as the length of these columns. Thus the length of the radius of the equator, surpassing the radius at the pole a 230th part, the force of gravity at the pole should likewise exceed that at the equator a 230th part.

This supposes the elliptic figure sufficient for the equilibrium of a homogeneous fluid mass. Maclaurin has demonstrated this in (*h*) a beautiful manner, from which it results, that the equilibrium is rigorously possible; and that, if the ellipsoid differs little from a sphere, the ellipticity will be equal to $\frac{5}{4}$ of the quantity, which expresses the proportion of the centrifugal force, to that of gravity under the equator.

Two different figures of equilibrium may correspond to the same motion of rotation. But the equilibrium cannot exist with every motion of rotation. The shortest period of rotation of an homogeneous fluid in equilibrio, of the same density as the earth, is 0.1009 of a day, and this limit varies reciprocally, as the (*i*) square root of the density. When the motion of rotation increases in rapidity, the fluid mass becoming more flattened at the poles, its period of rotation becomes less, and ultimately falls within the limits suitable to a state of equilibrium. After a great many oscillations, the fluid, in consequence of the friction and resistances which it experiences, fixes itself at last in that state which is *unique*, and determined by the primitive motion; and whatever may have been the primitive forces, the axis drawn through the centre of gravity of the fluid mass, and relative to which the moment of the forces was a maximum, at the origin, becomes the axis of rotation.

The preceding results furnish an easy method of verifying the hypothesis of the homogeneity of the earth. The irregularity of the measured degrees, may be supposed to leave too much uncertainty on the ellipticity of the earth to enable us to decide, if it is really such as the above hypothesis requires. But the regular increase of the force of gravity, from the equator to the pole, is sufficient to throw great light upon this subject.

By taking as unity the force of gravity at the equator, its increase at the pole, according to the hypothesis of homogeneity, should be equal to 0.00435. But by observations on the pendulum, this increase is 0.0054 : the earth therefore is not homogeneous. And indeed it is natural to suppose, that the density of the strata increases as they approach the centre. It is even necessary, for the stability of the equilibrium of the waters of the ocean, that their density should be less than the mean density of the earth ; otherwise, when agitated by the winds and other causes, they would overflow their limits, and inundate the adjoining continents.

The homogeneity of the earth being thus excluded by observation, we must, to determine its figure, suppose the sea covering a nucleus, composed of different strata, diminishing in density from the centre to the surface. (*k*) Clairaut has demonstrated, in his beautiful work, that the equilibrium is still possible, on the supposition that the surface, and strata of the interior nucleus, have an elliptic figure. In the most probable hypothesis, relative to the law of the densities and ellipticities of these strata, the ellipticity of the earth is less than in the case of homogeneity, and greater than if the force of gravity was directed to a single central point. The increase of gravity from the equator to the poles is greater than in the first case, and less than in the second. But there exists between the total increase of the force of gravity, taken as unity at the equator, and the ellipticity of the earth, this remarkable analogy, that in all the hypotheses relative to the constitution of the internal nucleus, which the sea incloses, the ellipticity of the earth is just so much less than that which would take place in the case of homogeneity, as the increase of the force of gravity exceeds that which should exist, according the same supposition, and reciprocally, so as that the fractions expressing the sum of the ellipticity and of the increment, make a constant quantity equal to five times the half of the ratio of the centrifugal force, to the force of gravity at the equator, which, for the earth is $\frac{1}{115 \cdot 2}$.

In attributing an elliptic figure to the strata of the terrestrial spheroid, the increase of its radii, and of the force of gravity, and the diminution of the degrees, from the pole to the equator, will vary as the squares of the cosines of the latitude, and these (*l*) are connected with the ellipticity of the earth, in such a manner, that the total increase of the radii is equal to the ellipticity. The total diminution of the degree, is equal to the ellipticity, multiplied by three times the degree at the equator; and the total increase of the force of gravity, is equal to the force of gravity at the equator, multiplied by the excess of $\frac{1}{115,2}$, above the ellipticity.

Thus the ellipticity of the Earth may be determined, either by direct measurement of degrees, or by observations on the length of the pendulum.

A consideration of a great number of observations of the pendulum give 0,00561, for the increase of the force of gravity, which taken from $\frac{1}{115,2}$ gives $\frac{1}{34,8}$, for the ellipticity of the Earth. If the hypothesis of the ellipse be conformable to nature, this ellipticity should agree with the measures of degrees; but it implies errors that are altogether improbable: and this circumstance, joined to the difficulty of reconciling all these measures to the same elliptic meridian, proves that the figure of the earth is much more complicated than had been supposed. This will not appear surprising, if we consider the different depths of the sea, the elevation of the continents, and islands above its level, the heights of mountains, and the unequal density of the water, and different substances which are at the surface of this planet.

To embrace, in the most general manner possible, the theory of the figure of the Earth and planets, it is necessary to determine the attraction of spheroids, differing little from spheres, and formed of strata, variable both in figure and density, according to any law whatever.

It will be also necessary to determine the figure which is suitable to the equilibrium of a fluid, expanded over its surface, for we must imagine the planets covered with a fluid in equilibrio similar to the case of the Earth, or their form would be entirely arbitrary. Dalembert has given, for this purpose, an ingenious method, which extends to a great number of cases, but which is deficient in that simplicity so desirable, in such complicated investigations, and which constitutes their principal merit.

A remarkable equation of partial differences relative to the (*l*) attraction of spheroids, led me, without the aid of integrations, and by differential methods only, to general expressions, for the radii of the spheroids; for the attractions upon any points whatever, either within the surfaces, or without them; for the condition of equilibrium of the fluids that surround them; for the law of gravity, and for the variation of the degrees at the surface.

All these quantities are connected with each other, by analogies extremely simple, from which results an easy method of verifying all the hypotheses that may be formed to represent either the variation of the force of gravity, or that of the values of different degrees of the meridian.

Thus Bouguer, with a view of reconciling the degrees measured at the equator, in France and in Lapland, supposed the Earth to be a spheroid of revolution, in which the increase of the degrees, from the equator to the pole, was proportional to the fourth power of the sine of the latitude. It is found that this hypothesis does not satisfy the increase of the force of gravity from the equator to Pello.—An increase, which according to observation, is equal to forty-five ten millionths of the whole gravity, and which would be only twenty-seven ten millionths on this hypothesis.

The above mentioned expressions give a direct and general solution of the problem, the object of which is to determine the figure of a fluid mass in equilibrio, supposing it subject to a motion of rotation, and composed of an infinity of fluids, of different densities, whose particles attract each other directly as their masses, and inversely as the squares of their distances.

Legendre had already solved this problem by a very ingenious analysis, which supposes the mass homogeneous. In the general case, the fluid necessarily takes the form of an ellipsoid of revolution, of which all the strata are elliptic, whose densities diminish at the same time that their ellipticities increase, from the centre to the surface.

The limits of compression of the whole ellipsoid, are $\frac{5}{4}$ and $\frac{1}{2}$ of the ratio of the centrifugal force, to the force of gravity at the equator. The first limit is relative to the hypothesis of homogeneity, and the second, to the supposition of the strata, indefinitely near to the centre, (m) being infinitely dense, and consequently the whole mass of the spheroid acting as if concentered in that point. In the latter case, the force of gravity being directed to a single point, and varying inversely as the square of the distance, the figure of the Earth would be such as has been above determined; but in the general hypothesis, the line which determines the direction of the force of gravity from the centre to the surface of the spheroid, is a curve, every element of which is perpendicular to the stratum through which it passes.

The analysis to which I have adverted, supposes that the terrestrial spheroid is entirely covered by the sea; but as this fluid leaves a considerable part of this spheroid uncovered; the analysis, notwithstanding its generality, does not represent nature exactly, and it is necessary to modify the results obtained on the hypothesis of a general inundation. Indeed the mathematical theory of the figure of the Earth presents on this supposition greater difficulties; but the progress of analysis particularly in this department, furnishes us with the means of surmounting them, and of considering the seas and continents such as they appear to observers. By thus adhering to nature, we get glimpses of several phenomena which natural history and geography present; which may thus diffuse great light on these two sciences, by connecting them with the theory of the system of the world. These are the principal results of my analypsis.

of the most interesting is the following theorem, which incontestably establishes the heterge-neity of the terrestrial strata.

"If to the length of a pendulum vibrating seconds at any point of the surface of the terrestrial spheroid, be added the product of this length into half the height of this point above the level of the sea, determined by observations made on the barometer, and divided by the semiaxis of the pole; the increase of this length thus corrected will be, from the equator to the poles on the hypothesis that the density of the earth to an inconsiderable depth is constant, the product of this length at the equator, into the square of the sine of the latitude, and by five fourths of the ratio of the centrifugal force to the gravity at (*n*) the equator, or by 43 ten thousaandths."

This theorem, to which I was conducted by a differential equation of the first order, which belongs to the surface of homogeneous spheroids, differing little from spheres, is generally true whatever may be the density of the sea and the manner in which it covers part of the earth. It is remarkable, in as much as it does not suppose a knowledge of the figure of the terrestrial spheroid, nor of that of the sea, figures which it would be impossible to obtain.

Experiments on the pendulum made in the two hemispheres, agree in giving to the square of the sine of latitude a coefficient greater than 43 ten thousandths, and very nearly equal to 54 ten thousands of the length of the pendulum at the equator. It is therefore satisfactorily proved by experiments, that the earth is not homogeneous in its interior. It appears moreover, by comparing them with analysis, that the densities of the terrestrial strata continually increase from the surface to the centre.

The regularity with which the observed variation of the lengths of the pendulum vibrating seconds, follows the law of the square of the sine of the latitude, proves that these strata are regularly arranged about the centre of gravity of the earth, and that their form is very nearly an ellipse of revolution.

The ellipticity of the terrestrial spheroid, may be determined by measures of degrees of the meridian. The different measures which have been made, compared two by two, give ellipticities which are sensibly different, so that the variation of degrees does not follow as exactly as gravity, the law of the square of the sine of latitude. This depends on the second differentials of the terrestrial radius, which the expressions of the degrees of the meridian and of the osculating circle contain, while the expression for the gravity contains only the first differentials of this radius, of which the small deviations from the elliptic radius, increase by successive differentiations. But if degrees at a considerable distance from each other be compared, such as those of France and the equator, their anomalies must be insensible on their difference; and it is found by this comparison that the ellipticity of the terrestrial spheroid is $\frac{1}{308}$.

But a more certain means of obtaining this ellipticity, consists, as has been already observed, in comparing with a great number of observations, the two lunar inequalities which are due to the compression of the earth, the one in longitude and the other in latitude. They agree in making the compression of the terrestrial spheroid very nearly equal to $\frac{1}{305}$, and what is very worthy of remark, each of the two inequalities leads to this result, which as we have seen differs very little from that furnished by a comparison of degrees in France and at the equator.

As the density of the sea is only the fifth part of the mean density of the earth; this fluid ought to have very little influence on the variations of degrees, and of gravity, and on the two inequalities of which we have spoken. Its influence is still more diminished by the smallness of its mean depth, which is thus proved. Conceive the terrestrial spheroid to be deprived of the ocean, and suppose that in this state the surface became fluid and was in equilibrio; we shall have its ellipticity by subtracting from five times the half of the ratio of the centrifugal force to the gravity at the equator, the coefficient assigned by experiments to the square of the sine of the latitude in the expression of the length of the pendulum which vibrates seconds; this length at the equator being assumed equal (n) to unity. By this means it is found that the compression of the terrestrial spheroid is $\frac{1}{304,8}$, the trifling influence which the action of the sea has on the variation of the gravity being neglected. The little difference which exists between this compression, and those furnished by the measures of terrestrial degrees and of the lunar inequalities, proves that the surface of this spheroid would be very nearly one of equilibrium, if it became fluid. On this account, and because the sea leaves vast continents uncovered, it is inferred that its depth is inconsiderable, and that its mean depth is of the same order as the mean height of continents and isles above the level of the sea, which height does not surpass a thousand metres. This depth is therefore a small fraction of the excess of radius of the equator above that of the pole, which excess does not surpass twenty thousand metres. But as high mountains are spread over some parts of the continents, so there may be great cavities in the bottom of the seas. However, it is natural to suppose that their depth is less than the elevation of high mountains: as the depositions of rivers and the remains of marine animals carried along by currents, must at length fill these cavities.

This is an important result for natural history and geology. There can be not the least doubt but that the sea covered a great part of our continents on which it has left incontestable proofs of its existence. The successive subsidence of isles, and of a part of the continents, followed by extended subsidences of the bason of the sea which have uncovered parts previously submerged, appear to be indicated by the different phenomena which the surface and strata of the existing continent present to us. In order to explain these subsidences, it is sufficient to assign more energy to causes, similar to those which have produced the subsidences of which history has preserved the record. The subsidence of one part of the bason of the sea, renders visible another part, so much the more extensive as the sea is less profound. Thus great continents might emerge from the ocean without producing great changes in the figure of the terrestrial spheroid. The property, which this figure possesses, of differing little from that, which its surface would assume if it became fluid, requires that the depression of the level of the sea, should be only a small fraction of the difference of the two axes of the pole and of the equator. Every hypothesis founded on a considerable displacement of the poles on the surface of the earth, must be rejected as incompatible with the property of which I have been speaking. Such a displacement has been suggested, in order to explain the existence of elephants, of which fossel remains are found in such great abundance in northern climates, where living elephants cannot exist. But an elephant, which is with great probability supposed to be cotemporaneous with the last flood, was found in a mass of ice well preserved with its skin, and as the hide was covered with a great quantity of hair, this species of elephant was guarantied by this means, from the cold of the northern climates, which it might inhabit and even select as a place of residence. The discovery of this animal has therefore confirmed what the mathematical theory of the earth had shewn us, namely, that in the revolutions which have changed the surface of the globe and destroyed several species of animals and vegetables, the figure of the terrestrial spheroid, and the position of its axis of rotation on its surface, have undergone only slight alterations.

Now what is the cause which has given to the strata of the earth forms very nearly elliptical, with densities increasing from the surface to the centre, which has arranged them regularly about their common centre of gravity, and which has rendered its surface very little different from what it would be, if it had been primitively in a fluid state? If the different substances which compose the earth had been primitively, by the effect of great heat in a fluid state, the most dense must have been carried towards the centre: all would have assumed elliptic forms, and the surface would have been in equilibrio. These strata in consolidating having changed their figure very little, the earth should at present exhibit the phenomena of which I have been speaking. This case has been amply discussed by geometers. But if the earth was homogeneous in the chymical sense, *i. e.* if it was composed of one sole substance in its interior, it might also exhibit these phenomena. In fact, we may conceive that the immense weight of the superior strata, should increase considerably the density of the inferior strata. Hitherto geometers have not taken into account in their investigations on the figure of the earth, the compression of the substances of which it was composed; although Daniel Bernoulli in his essay on the tides had already pointed out the cause of the increase of density of the strata of the terrestrial spheroid. From the analysis which I have applied in the eleventh book of the celestial mechanics, it appears that it is possible to satisfy all the observed phenomena, on the hypothesis of the earth being composed of one sole substance in its interior. The law of the densities which the compression of the earth assigns to the strata of this substance not being known, we can only make suppositions on this subject.

It is known (*o*) that the density of gases increases proportionally to their compression, when the temperature remains the same. But this law does not appear to agree to liquid and solid bodies; it is natural to think that these bodies resist the compression, so much the more as they are more compressed. This is in fact confirmed by experiment, so that the ratio of the differential of the pressure to the differential of the density, instead of being constant as in the case of gas, increases with the density. The simplest expression of this ratio, supposed variable, is the product of the density by a constant quantity. This is the law which I have adopted, since it combines to the advantage of representing in the simplest possible manner, what we know respecting the compression of bodies, that of adapting itself easily to the calculus in the investigation of the figure of the earth; my object in this investigation being only to shew that this manner of considering the interior constitution of the earth, may be reconciled with all the phenomena, which depend on this constitution, at least if the terrestrial spheroid had been primitively fluid. In the solid state, the adherence of the molecules, diminishes extremely their mutual compression, and it prevents the entire mass from assuming the regular figure which it would have in the fluid state, if it had primitively deviated from it.

Therefore in this very hypothesis on the constitution of the earth, as in all others, the primitive fluidity of the earth appears to me to be indicated by the regularity of gravity and by the figure at its surface.

All astronomers have assumed the invariability of the axis of rotation of the earth, and the uniformity of this rotation. The duration of a revolution of the earth about its axis is the standard of time; it is therefore of great importance to appreciate the influence of all the causes which may alter this element. The axis of the earth moves about the poles of the ecliptic, but since the epoch, at which the application of the telescope to philosophical instruments furnished the means of observing terrestrial latitudes with precision, no variation has been recognized in these latitudes, but what may have arisen from the errors of observation, which proves that since that epoch, the axis of rotation has existed very nearly on the same point of the terrestrial surface; it therefore appears that this axis is invariable. The existence (*p*) of a similar axis on solid bodies has been known for a long time. We know that each of these bodies has three principal rectangular axes, about which it may revolve uniformly; the axis of rotation remaining invariable. But does this remarkable property appertain to bodies, which, like the earth, are partly covered with a fluid? The condition of the equilibrium of the fluid must be then combined with the conditions of the principal axes: it changes the figure of the surface, when the axis of rotation is changed. It is therefore interesting to know whether among all the possible changes there is one, in which the axis of rotation and the equilibrium of the fluid remain invariable. Analysis proves that if we make to pass very near to that centre of gravity of the terrestrial spheroid a fixed axis about which it may revolve freely, the sea may always assume on the surface of the spheroid a constant state of equilibrium. I have given in the eleventh book already cited, in order to determine this state, a method of approximation arranged according to the powers of the ratio of the density of the sea to the mean density of the earth, and as this ratio is only $\frac{1}{5}$, the approximation is extremely converging. The irregularity of the depth of the sea, and of its contour, does not permit us to obtain this approximation. But it is sufficient to recognize the possibility of this circumstance, in order to be assured of the existence of a state of equilibrium of the sea. The position of the fixed axis of rotation being arbitrary, it is natural to think, that among all the positions which this axis may be made to undergo, there is one in which the axis passes through the common centre of gravity of the sea and of the spheroid which it covers, so that this fluid being in equilibrio, and congealed in that state, this axis should be a principal axis of rotation of the terrestrial spheroid and of the sea, considered as one body; it is evident that if its fluidity be restored to the congealed mass, the axis will be always an invariable axis for the entire earth; I have shewn that such an axis is always possible, and I have given the equations which determine its position. By applying these equations to the case in which the sea covers the entire spheroid, I have arrived at the following theorem.

" If the density of each stratum be supposed " to be diminished by the density of the sea; and " if through the centre of gravity of this imagi- " nary spheroid, we conceive a principal axis of " the spheroid to be drawn, the earth being " made to revolve about this axis, if the sea be " in equilibrio, this axis will be the principal axis " of the entire earth, of which the centre of " gravity will be that of the imaginary spheroid."

Thus the sea which partly covers the terrestrial spheriod not only does not render impossible the existence of a principal axis of rotation, but it even by its mobility, and by the resistances which its oscillations experience, would restore to the earth a permanent state of equilibrium, if any causes should derange it.

If the sea was sufficiently profound to cover the surface of the terrestrial spheroid, supposing it to turn successively round the three principal axes of the terrestrial spheroid, each of these axes would be a principal axis for the entire earth. But the stability of the axis of rotation has not place, as in the case of a solid body, but relatively to the two principal axes, for which the moment of inertia is a *maximum* or a *minimum*. However there is this difference between the earth and a solid body, that in the case of the solid body, if the axis of rotation be changed, the figure of the solid body will not be changed, whereas in the case of the earth the surface of the sea assumes another figure altogether. The three figures, which this surface assumes in revolving, successively with the same angular velocity of rotation, about each of the three axes of rotation of the imaginary spheroid, have very simple relations which, I have determined; and it follows from my analysis, that the mean radius between the radii of the three surfaces of the (q) sea, corresponding to the same point of the surface of the terrestrial spheroid, is equal to the radius of the surface of the sea in equilibrio on this spheroid, and deprived of its motion of rotation.

In the fifth book of the Celestial Mechanics, I have discussed the influence of interior causes, such as volcanoes, earthquakes, winds, currents of the sea, &c. on the duration of the rotation of the earth, and I have shewn by means of the principle of (r) areas, that this influence is insensible, and that in consequence of these causes, it is necessary in order that a sensible effect might be produced, that considerable masses should be transported considerable distances; which has not been the case since the periods of which history has preserved the records, but there exists an interior cause of alteration of the day, which has not been yet considered, and which, considering the importance of this element, deserves a particular discussion. This cause is the heat of the terrestrial spheroid. If, as every thing induces us to think, the earth had been primitively fluid, its dimensions have diminished successively with its temperature; its angular velocity of rotation has increased gradually, and it will continues to increase until the earth arrives at the constant state of the mean temperature of the space through which it moves. In order to form a just conception of this movement of angular velocity, suppose in a space of a given temperature, a globe (r) of homogeneous matter to revolve on its axis in a day. If this globe be transported into a space of which the temperature is less by the hundredth part of a degree, and if we suppose that the rotation is not altered, either by the resistance of the medium, or by friction; its dimensions will diminish with the diminution of temperature; and when at length it shall have assumed the temperature of the new space, its radius will be diminished by a quantity, which I shall suppose the hundred thousandth part, which is the case very nearly for a globe of glass, and which may be admitted for the earth. The weight of the heat is inappreciable to all experiments which have been made to (*s*) measure it; it appears therefore like to light to produce no sensible variation in the mass of bodies, consequently, in the new space two things may be supposed the same as in the first, namely, the mass of the globe and the sum of the areas described in a given time, by each of its molecules referred to the plane of its equator. The molecules approach to the centre of the globe by a hundreth thousand part of their distance from this point. The areas which they describe on the plane of the equator (*t*) being proportional to the square of this distance, will diminish therefore very nearly by a fifty thousandth part, if the angular velocity of rotation does not increase; hence it follows, that in order that the sum of the areas described in a given time may be constant, the increment of this velocity, and consequently the diminution of the duration of rotation, ought to be a fiftieth thousandth part; such is therefore the final diminution of this duration. But previous to its attaining this final state, the temperature of the globe continually diminishes, and more slowly at the centre than at the surface, so that from observation of this diminution, compared with the theory of heat, we can determine the epoch when the globe was transported into the new space. The earth appears to be in a similar state. This follows from thermometrical observations made in profound mines, and which indicate a very sensible increase of heat, according as we penetrate into the interior of the earth. The mean of the observed increments appears to be a centesimal degree for a depth of 32 metres, but a very great number of observations will make its value known very accurately, which cannot be the same for all climates. It was necessary, in order to obtain the increment of the earth's rotation, to know the law of the diminution of heat from the centre to the surface. This I have investigated in the eleventh book of the Celestial Mechanics, for a globe primitively warmed in any manner, and besides subjected to the heating action of an exterior cause. The law in question, which I published in 1819, in the Connaisance des temps, and which M. Poisson has since confirmed by a learned analysis, is represented by an infinite series of terms, which have for factors constant quantities, which are always less than unity, and of which the exponents increase proportionably to the time. The length of the time makes these terms to disappear the one after the other; so that before the establishment of the final temperature, only one of those terms which produces the increase of temperature in the interior of the globe, is sensible. I have supposed the earth to have attained this state, from which it is perhaps still far removed. But as I only wish to give here a general idea of the influence of the diminution of the interior heat on the duration of the day I have adopted this hypothesis, and I have inferred from it, the increase of the velocity of revolution.

It is necessary in order to reduce this increment to numbers, to determine numerically the two constant arbitrary quantities of which one depends on the conducting power of the earth with respect to heat, and the other on the elevation of temperature of its superficial stratum above the temperature of the ambient space. I have determined the first constant by means of the variations of the annual heat at different depths, and for this purpose I have made use of the experiments of M. Saussure, which this philosopher has cited in No. 1422 of his voyage to the Alps. In these experiments, the annual variation of the heat at the surface has been reduced to a twelfth part at the depth of $9^m,6$. I have afterwards supposed that in our mines, the increase of heat is a centesimal degree for a depth of 32 metres, and that the linear dilatation of the earth's strata is a hundred thousand part of each degree of temperature. I have found by means of these data, that the duration of the day has not increased by half a hundredth of a centesimal second for the last two thousand years, which is chiefly owing to the magnitude of the earth's radius. Indeed I have supposed that the earth is homogeneous, and it is certain that the densities of its strata increase from the surface to the centre. But it should be observed here that the quantity of heat and its internal motion would be the same in a heterogeneous substance, if in the corresponding parts of the two bodies, the heat and the property of conducting it were the same. The matter may here be considered as a vehicle of heat, which may be the same in substances of different densities. This is not the case for dynamical properties, which depend (u) on the mass of the molecules. Thus we can in this conception of the effects of terrestrial heat on the duration of the day extend to the earth, considered as heterogeneous, the data relative to heat considered as homogeneous. In this manner it is found, that the increment of the density of the strata of the terrestrial spheroid diminishes the effect of heat on the duration of the day, which effect since the time of Hipparchus has not increased this duration by $\frac{1''}{300}$.

The term on which the increment of the interior heat of the earth depends, does not now add the fifth of a degree to the mean temperature of its surface. Its annihilation, which a very long series of ages ought to produce, will not consequently cause any species of organized beings actually existing to disappear, at least as long as the proper heat of the sun, and its distance from the earth, do not experience any sensible alteration.

In fine, I am far from thinking that the preceding suppositions obtain in nature; besides, the observed values of the two constants of which I have spoken, depend on the nature of the soil which in different countries has not the same qualities with respect to heat. But the sketch which I have given, suffices to shew that the phenomena which have been observed on the heat of the earth, may be reconciled with the result which I have deduced from a comparison of the theory of the secular inequalities of the moon, and observations of ancient eclipses, namely, that since the time of Hipparchus, the duration of the day has not varied by the hundredth part of a second.

But what is the ratio of the mean density of the earth, to that of a known substance at its surface? The effect of the attractions of mountains, on the oscillations of the pendulum, and on the direction of the plumb line, ought to conduct us to the solution of this interesting problem.

It is true, that the highest mountains are always very small, in proportion to the Earth; but we may approach very near to the centre of their action, and this joined to the precision of modern observations, ought to render their effects perceptible.

The mountains of Peru, (*v*) the highest in the world, seemed the most proper for this object. Bouguer did not neglect so important an observation in the journey which he undertook, for the measure of the meridional degrees at the equator.

But these great bodies being volcanic and hollow in their interior, the effect of their attraction was found to be much less than might be expected from their size. However it was perceptible; the diminution of the force of gravity at the summit of Pichincha, would have been 0,00149, without the attraction of the mountain, and it was observed to be 0.00118. The effect of the deviation of the plumb-line, from the action of another mountain, surpassed 20″. Dr. Maskelyne has since measured, with great care, a similar effect produced by the action of a mountain in Scotland: the result was, that the mean density of the Earth, is double that of the mountain, and four or five times greater than that of common water. This curious observation deserves to be repeated several times on different mountains, whose interior constitution is well known. Cavendish determined this density by the attraction of two metallick globes of a great diameter, and he succeeded in rendering it sensible by a very ingenious process. It follows from these experiments that the mean density of the earth, is to that of water, very nearly in the ratio of eleven to two, which agrees with the preceding ratio as well as could be expected from such delicate observations and experiments.

I proceed here to present some considerations on the level of the sea, and on the reductions to this level. Conceive an extremely rare fluid of a uniform density throughout, and of an inconsiderable elevation, to surround the earth; let it, however, embrace the highest mountains; such would be very nearly our atmosphere if reduced to its mean density. Analysis shews that the corresponding points of the two surfaces, of the sea, and of this level, are separated by the same interval. If we conceive the surface of the sea to be prolonged below the continents and the surface of the fluid, so that the two surfaces may be always separated by this interval, this will be what is termed *the level of the sea*. It is the ellipticity of those two surfaces, that is determined by the measures of degrees; it is also the variation of gravity at the surface of the supposed fluid, which added to the ellipticity of this surface, gives a constant sum equal to $\frac{5}{2}$ of the ratio of the centrifugal force to the gravity at the equator. It is therefore to this surface or to the surface of the sea prolonged in the manner above specified, that it is necessary to refer the measures of degrees, and of the pendulum observed on the continents. But it is easily proved that the gravity does not vary from a point on the continent to the corresponding point of the surface of the fluid, but in consequence of the distance of those two points, when the slope to the sea is inconsiderable. Therefore in the reduction of the length of the pendulum to the level of the sea, we ought only to consider the height above this level such as we have defined it. In order to render this sensible by the results of the calculus in a case which I have subjected to analysis, conceive that the earth is an ellipsoid of revolution partly covered by the sea, of which we shall suppose the density to be very small relatively to the mean density of the earth. If the ellipticity of the terrestrial spheroid be less than that which corresponds to the equilibrium of the surface of the supposed fluid, the sea will cover the terrestrial equator to a certain latitude. The degrees measured on the continents, and increased in the ratio of their distance from the surface of the supposed fluid, (the radius of the earth being assumed equal to unity), will be those which are measured on this surface. The length of the pendulum which vibrates seconds diminished by twice this ratio, will be that which is observed on this surface; and the ellipticity determined by the measures of degrees, will be the same as would be obtained by subtracting from $\frac{5}{2}$ of the ratio of the centrifugal force to the gravity at the equator, the excess of the polar over the equatorial gravity being assumed equal to unity.

Let us apply the preceding theory to Jupiter.

The centrifugal force due to the motion of rotation of this planet, is nearly $\frac{1}{12}$ of the force of gravity at its equator; at least, if the distance of the fourth satellite from its centre, as given in the second Book, be adopted.

If Jupiter was homogeneous, (x) the diameter of its equator might be obtained, by adding five-fourths of the preceding fraction to its shorter axis taken as unity, these two axes would, therefore, be in the proportion of 10 to 9,06. According to observation, their proportion is that of 10 to 9,43. Jupiter, therefore, is not homogeneous. Supposing it to consist of strata, of which the densities diminish from the centre to the surface, its ellipticity should be included between $\frac{1}{24}$ and $\frac{5}{48}$, the observed ellipticity being within these limits, proves the heterogenity of its strata, and by analogy that of the strata of the terrestrial spheroid, already rendered very probable from the measures of the pendulum, and which have been confirmed by the inequalities of the Moon depending on the ellipticity of the Earth.

CHAP. IX.

On the Figure of the Ring of Saturn.

It was shewn in the first book, that the ring of Saturn consisted of two concentric rings of very small thickness. By what mechanism do these rings sustain themselves about the planet? It is not probable that this should take place from the simple adhesion of their particles. Since, were this the case, the parts nearest to Saturn, sollicited by the constantly renewed action of gravity, would be at length detached from the rings, which would, by an insensible diminution, finally disappear, like all those works of nature which have not had sufficient force to resist the action of external causes. These rings support themselves then without effort, and by the sole laws of equilibrium. But for this it is requisite to suppose them endowed with a rotary motion about an axis perpendicular to their plane, and passing through the centre of Saturn, so that their gravitation towards the planet, may be balanced by the centrifugal force due to this motion.

Let us imagine a homogeneous fluid spread about Saturn in the form of a ring, and let us see what ought to be its figure, for it to remain in equilibrio, in consequence of the mutual attraction of its particles, of their gravitation towards Saturn, and their centrifugal force. If, through the centre of the planet, a plane is imagined to pass, perpendicular to the surface of the ring, the section of the ring by this plane, is what I shall call the *generating curve*. Analysis proves that if the magnitude of the ring is small in (*a*) proportion to its distance from the centre of Saturn, the equilibrium of the fluid is possible, when the generating curve is an ellipse of which the greater axis is directed towards the centre of the planet. The duration of the rotation of the ring, is nearly the same as that of the revolution of a satellite, moved circularly at the distance of the centre of the generating ellipse, and this duration is about four hours and a third, for the interior ring. Herschel has confirmed by observation this result, to which I had been conducted by the theory of gravitation.

The equilibrium of the fluid would also exist, supposing the generating ellipse variable in size and position, within the extent of the circumference of the ring; provided that these variations are sensible only at a much greater distance, than the axis of the generating section. Thus, the ring may be supposed of an unequal breadth in its different parts, it may even be supposed of double curvature. These inequalities are indicated by the appearances and disappearances of Saturn's ring, in which the two arms of the ring have presented different phenomena. They are even necessary to maintain the ring in equilibrio about the planet, since if it was perfectly similar in all its parts, its equilibrium would be deranged by the slightest force, such as the attraction of a satellite, and the ring would finally precipitate itself upon the planet.

The rings by which Saturn is surrounded, are consequently irregular solids, of unequal breadth in the different points of their circumference, so that their centres of gravity do not coincide with their centres of figure. These centres of gravity may be considered as so many satellites, moving about the centre of Saturn, at distances dependant on the inequalities of the rings, and with angular velocities equal to the velocities of rotation of their respective rings.

We may conceive, that these rings, sollicited by their mutual action, by that of the Sun, and of the satellites of Saturn, ought to oscillate about the centre of this planet, and thus produce the phenomena of light, of which the period comprises several years. It might likewise be supposed, that sollicited by different forces, they should cease to exist in the same plane; but Saturn having a rapid rotatory motion, and the plane of its equator being the same with that of its ring, and of its six first satellites, its action retains the system of these different bodies in the same plane. The action of the Sun, and of the seventh satellite, only changes the position of the plane of Saturn's equator, which in this motion carries with it the ring, and the orbits of the six first satellites.

CHAP. X.

On the Atmosphere of the Celestial Bodies.

The thin, transparent, compressible, and elastic fluid which surrounds a body, and rests upon it, is called its *atmosphere*. We conceive, with great appearance of probability, that a similar atmosphere surrounds every celestial body; and the existence (*a*) of such a fluid, relatively to the Sun and Jupiter, is indicated by observations. In proportion as the atmospherical fluid is elevated above the surface of a body, it becomes thinner, in consequence of its elasticity, which dilates it so much the more, as it is less compressed. And if the particles of its exterior surface were (*b*) perfectly elastic, it would extend itself indefinitely, and would eventually dissipate itself in space.

It is then requisite that the elasticity of the atmospherical fluid should diminish in a greater proportion than the weight which compresses it; in order that there may exist a state of rarity, in which it may be without elasticity. It should be in this state at the surface of the atmosphere.

All the atmospheric strata should acquire, after a time, the rotatory motion, common to the body which they surround. For the friction of these strata against each other, and against the surface of the body, should accelerate the slowest motions, and retard the most rapid, till a perfect equality is established among them. In these changes, and generally in all those, which the atmosphere undergoes, (*c*) the sum of the products of the particles of the body, and of its atmosphere, multiplied respectively by the areas, which their radii vectores projected on the plane of the equator, describe round their common centre of gravity, are always equal in the same times.

Supposing then, that by any cause whatever, the atmosphere should contract itself, or that a part should condense itself on the surface of the body, the rotatory motion of the body, and of its atmosphere, would be accelerated, because the radii vectores of the areas, decribed by the particles of the primitive atmosphere becoming smaller, the sum of the products of all the particles, by the corresponding areas, could not remain the same, unless the velocity of rotation is increased.

At its surface the atmosphere is only retained by its weight, and the form of this surface is such, that the force which results from the centrifugal and attractive forces of the body (*d*), is perpendicular to it. The atmosphere is flattened towards the poles, and distended at its equator, but this ellipticity has limits, and in the case where it is the greatest, the proportion of the axis of the pole to that of the equator is as two to three.

The atmosphere can only extend itself at the equator, to that point where the centrifugal force exactly balances the force of gravity, for it is evident that beyond this limit, the fluid would dissipate itself. Relatively to the Sun, this point is distant from its centre by the length of the radius of the orbit of a planet, the period of whose revolution it equal to that of the Sun's rotation.

The Sun's atmosphere then does not extend so far as Mercury, and consequently does not produce the zodiacal light, which appears to extend beyond even the terrestrial orbit. Besides, this atmosphere, the axis of whose poles should be at least two-thirds of that of the equator, is very far from having the lenticular form which observation assigns to the zodiacal light.

The point where the centrifugal force balances gravity, is so much nearer to the body, in proportion as its rotatory motion is more rapid. Supposing that the atmosphere extends itself as far as this limit, and that afterwards it contracts and condenses itself from the effect of cold at the surface of the body, (*e*) the rotatory motion would become more and more rapid, and the farthest limit of the atmosphere would approach continually to its centre: it will then abandon successively in the plane of its equator, fluid zones, which will continue to circulate about the body, because their centrifugal force is equal to their gravity. But this equality, not existing relatively to those particles of the atmosphere, distant from the equator, they will continue to adhere to it. It is probable that the rings of Saturn are similar zones, abandoned by its atmosphere.

If other bodies circulate round that which has been considered, or if it circulates itself round another body, the limit of its atmosphere (*f*) is that point where its centrifugal force, *plus* the attraction of the extraneous bodies, balances exactly its gravity. Thus the limit of the Moon's atmosphere, is the point where the centrifugal force due to its rotatory motion, *plus* the attractive force of the Earth, is in equilibrio with the attraction of this satellite. The mass of the moon being $\frac{1}{75}$ of that of the earth, this point is therefore distant from the centre of the Moon, about the ninth part of the distance from the Moon to the Earth. If, at this distance, the primitive atmosphere of the Moon had not been deprived of its elasticity, it would have been carried towards the Earth which might have retained it. This is perhaps the cause why this atmosphere is so little perceptible.

CHAP. XI.

Of the Tides.

It was Newton, who first gave the true explanation of the tides, by shewing that they arose from the great principle of universal gravitation. Kepler had recognised the tendency of the waters of the sea towards the centres of the sun and moon; but being ignorant of the law of this tendency, and of the methods necessary to subject it to computation, he could only assign a very probable conjecture on this object. Galileo in his dialogues on the system of the world, expresses his astonishment and regret, that this conjecture, which appeared to bring back into natural philosophy the occult qualities of the ancients, had been suggested by such a man as Kepler. He explained the ebbing and flowing of the sea, by the diurnal changes which the rotation of the earth, combined with its revolution about the sun, ought to produce in the absolute motion of each molecule of the sea. This explanation appeared to him so incontestable, that he gave it, as one of the principal proofs of the Copernican system, for the defence of which he was afterwards so persecuted. Further discoveries have confirmed the conjecture of Kepler, and overturned the explication of Galileo, which is inconsistent with the laws of the equilibrium and motion of fluids.

The theory of Newton appeared in 1687, in his Treatise on the Mathematical Principles of Natural Philosophy. He there considered the sea as a fluid of the same density as the earth which it entirely covers, and he supposed that it assumed at each instant, the figure in which it would be in equilibrio under the action of the sun. If then this figure be supposed to be that of an ellipsoid (*a*) of revolution, of which the greater axis is directed towards the sun; he determined the ratio of the two axes, in the same way as he determined the ratio of the two axes of the earth, compressed by the centrifugal force of its motion of rotation. The greater axis of the aqueous ellipsoid being constantly directed towards the sun, the greatest height of the sea in each port, ought to happen when the sun is on the equator at midday and midnight, and the greatest depression ought to be at the rising and setting of this star.

Let us consider the manner in which the sun acts on the sea, when it deranges its equilibrium. It is evident, that if the Sun acted on the centre of gravity of the Earth, and of every particle of the ocean, by exerting equal and parallel forces, (b) the whole system of the terrestrial spheroid would obey these forces by a common motion, and the equilibrium of the waters would not be at all altered. This equilibrium then, is only deranged by the difference of these forces, and by the inequality of their directions. A particle of the ocean, placed directly under the Sun, is more attracted than the centre of the Earth. It tends therefore, to separate itself from it, but it is retained by its gravity, which this tendency diminishes. Half a day afterwards, this particle is opposite to the Sun, which attracts it less forcibly than it does the centre of the Earth; the surface of the terrestrial globe therefore tends to separate itself from it, but the gravity of the particles retains it. This force is therefore diminished also in this case by the solar attraction. But since the distance of the Sun is very great, compared with the radius of the Earth, it is easy to see that the diminution of gravity in each case is very nearly the same. A simple decomposition of the action of the Sun upon the particles of the ocean, is sufficient to shew, that in any position of this body, relatively to these particles, its action in disturbing their equilibrium, becomes the same after half a day.

The law according to which the water rises and falls, may be thus determined. Let us conceive a vertical circle, whose circumference represents half a day, and whose diameter is equal to the whole tide, or to the difference between the height of high and low water, and let the arcs of this circumference, (*c*) reckoning from the lowest point, express the time elapsed since low water, the versed sines of these arcs will express the heights of the water, corresponding to these times. Thus, the ocean in rising, covers in equal times, equal arcs of this circumference.

The greater the extent of the surface of the water, the more perceptible are the phenomena of the tides. In a fluid mass, the impressions which a fluid particle receives, are communicated to the whole. It is thus that the action of the Sun, which is insensible on an insulated particle, produces on the ocean such remarkable effects. Let us imagine, at the bottom of the sea, a curved canal, terminated at one of its extremities by a vertical tube, rising above the surface of the water, and which, if prolonged, would pass through the centre of the Sun.

The water will rise in (*d*) this tube by the direct action of the Sun, which diminishes the gravity of its particles, and particularly by the pressure of the particles enclosed in the canal, which all make an effort to unite themselves beneath the Sun. The elevation of the water in the tube, above the natural level of the sea, is the integral of all these infinitely small efforts. If the length of this canal is increased, this integral also becomes greater, because it extends over a larger space, and because there will be a greater difference in the quantity and direction of the forces, by which the extreme particles are sollicited.

By this example we see the influence which the extent of the sea has upon the phenomena of the tides, and the reason why they are insensible in the seas of inconsiderable extent, as the Euxine and the Caspian. The magnitude of the tides depends also much on local circumstances. The oscillations of the ocean, when confined in a narrow channel, may become extremely great, and these may be augmented by the reflection of the waters from the opposite shore. It is thus, that the tides, very small in the South Sea islands, are very considerable in our harbours.

If the ocean covered a spheroid of revolution, and experienced no resistance to its motion, the instant of high water would be that of the passage of the Sun over the superior or inferior meridian; but it is not thus in nature; local circumstances produce great variations in the times of high water, even in harbours that are very near each other. To have a just idea of these variations, we may suppose a large canal communicating with the sea, and extending into the land; it is evident that the undulations which take place at its entrance, will be propagated successively through its whole length, so that the figure of its surface will be formed by the undulations of large waves in motion, which will be incessantly renewed, and will describe their length in the interval of half a day. These waves will produce at every point of the canal, a flux and reflux, which will follow the preceding laws, but the hours of the flowing will be retarded, in proportion as the points are farther from the entrance of the canal. What we have here said of a canal, may be applied to rivers whose surfaces rise and fall by similar waves, notwithstanding the contrary motion of their streams. These waves are observed in all rivers near to their entrance; they extend to considerable distances in great rivers, and at the straights of Pauxis in the river of the Amazons, they are as sensible at the distance of eighty myriameters from the sea.

The action of the Moon on the sea produces an ellipsoid similar to that produced by the action of the Sun, but it is more elongated, because the lunar action is more powerful than that of the Sun. In consequence of the inconsiderable excentricity of these ellipsoids, we may conceive (*e*) them to be placed the one over the other, so that the radius of the surface of the sea is half the sum of the corresponding radii of their surfaces.

From hence arise the principal varieties of the tides. In the syzygies, the greater axes coincide, and the greatest elevation happens at the instant of mid-day and mid-night, and the greatest depression at the rising and setting of these stars. In the quadratures, the greater axis of the lunar ellipsoid and the lesser axis of the solar ellipsoid coincide; the full tide happens therefore at the rising and setting of these stars, and it is the least high water: the low water happens at the instants of mid-day and mid-night, and it is the greatest of low waters. If therefore the action of each star be expressed by the difference of the semiaxes of its ellipsoid, which is evidently proportional to it, when the place is situated at the equator, the excess of the greatest syzygial tide over the low water in syzygies will express the sum of the solar and lunar actions, and the excess of the least high water, which is in quadrature, over the greatest low water, which is likewise, (as we have seen in quadrature), will express the difference of these actions. If the harbour be not in the equator, this excess should be multiplied by the square of the cosine (f) of latitude. Therefore the ratio of the action of the Moon to that of the Sun may be determined by observing the heights of the tides in syzygies and in quadratures. Newton inferred from some observations made near Bristol, that this ratio is that of four and a half to unity. The distances of those stars from the centre of the earth influence all these effects; the action of each star being reciprocally as the cube of the distance.

As to the intervals between high water from one day to another, Newton observed that it is least in syzygies, and that it increases from syzygy to the following quadrature, that at the first octant it is equal to a lunar day, and that it attains its *maximum* at the quadrature; that it afterwards diminishes, becoming equal to a lunar day at the subsequent octant, and that it finally resumes its *minimum* at the syzygy. Its mean value being a lunar day, there are as many high waters as there are passages of the Moon over the superior or inferior meridian.

Such would be, according to the theory of Newton, the phenomena of the tides, if the sun and moon moved in the plane of the equator. But it appears from observation, that the highest tides do not arrive at the very moment of the syzygy, but a day and a half later. Newton ascribed this retardation to the oscillatory motion of the sea, which remains some time after the sun and moon cease to act. The exact theory of the undulations of the sea, produced by this action, shews that, without the accessory circumstances, the highest tides would coincide with the syzygies, and the lowest would coincide with the quadratures. Consequently their retardation at the moments of these phases cannot be attributed to the cause assigned by Newton, it therefore must depend, as also the hour of high water, in each harbour, on accessary circumstances. This example shews that we ought to distrust the most specious conjectures, when they are not confirmed by a rigorous analysis.

However the consideration of two ellipses, superimposed the one over the other, may also represent the tides, provided that the greater axis of this ellipsoid be conceived to be directed towards a fictitious sun, always equally elongated from the true sun. The axis of the lunar ellipsoid should be likewise always directed towards an imaginary moon equally elongated from the true.

but at such a distance that the conjunction of the two imaginary stars, does not arrive until a day and a half after the syzygy.

This consideration of the two ellipsoids, extended to the case, in which the stars move in orbits inclined to the equator, cannot be reconciled with observations. If the harbour be situated in the equator it gives near the *maximum* of the tides, the two high waters in the morning and in the evening, very nearly equal, whatever may be the declinations of these stars; only the action of each star is diminished in the ratio of the square of the cosine of its declination (g) to unity. But if the place is not on the equator, these two high waters may be extremely different, and when the declination of the stars is equal to the obliquity of the ecliptic, the evening tide at Brest should be eight times greater than that of the morning. However it appears from numerous observations made at this port, that these two tides are very nearly equal, and their greatest difference is not the thirtieth part of their sum. Newton ascribed the smallness of this difference to the same cause, by means of which, he explained the retardation of the high water beyond the moment of the syzygy, namely to a motion of oscillation in the sea, which, according to him, bringing back a great part of the evening tide on the subsequent morning tide, renders these tides very nearly equal. But the theory of the undulations of the sea shews that this explanation is not exact, and that without accessary circumstances the two consecutive tides would not be equal, unless the sea had every where the same depth.

In 1738, the Academy of Sciences proposed the cause of the ebbing and flowing of the sea, as the subject of the mathematical prize, which it decided in 1740. Four essays were crowned, the three first, founded on the principle of universal gravitation, were those of Daniel Bernouilli, of Euler, and Maclaurin. The Jesuit Cavalleri, the author of the fourth, adopted the system of vortices. This was the last honour paid to this system by the Academy, which was then composed of many geometers, whose successful labours contributed so powerfully to the advancement of the celestial mechanics.

The three essays which were founded on the law of universal gravitation, are developements of the theory of Newton. They depend not only on this law, but also on the hypothesis adopted by this great geometer, namely, that the sea assumes at each instant the figure in which it would be in equilibrio, under the star which attracts it.

The essay of Bernouilli contains the most extensive developements. He, like Newton, ascribed the retardation of the *maxima* and of the *minima* of the tides, after the instants of the occurrence of the syzygies and the quadratures, to the motion of the waters of the ocean ; and he adds, perhaps, a part of this retardation is owing to the time the action of the moon takes to arrive at the earth. But I have ascertained that, between the heavenly bodies all attractions are transmitted with a velocity, which, if it be not infinite, surpasses several thousand times the velocity of light; and we know that the light (*h*) of the moon reaches the earth in less than two seconds.

D'Alembert, in his treatise on the general course of the winds, which bore away the prize, proposed on the subject by the Academy of Sciences in Prussia, considered the oscillations of the atmosphere produced by the attractions of the sun and moon. And on the hypothesis that the earth is deprived of its motion of rotation, the consideration of which he judged to be totally useless in his investigations, and supposing the atmosphere every where equally dense, and acted on by a star at rest, he determined the oscillations of this fluid. But when he wished to consider the case of a star in motion, the difficulty of the problem obliged him to have recourse, in order to simplify his results, to a precarious hypothesis, and even with such restrictions the results cannot even be considered as approximations. His formulæ gave a constant wind blowing from east to west, of which the expression depends on the initial state of the atmosphere; now the quantities depending on this state ought long since to have disappeared, in consequence of all the causes which would reestablish the equilibrium of the atmosphere, if the action of the stars should cease; consequently we cannot thus explain the trade winds. The treatise of D'Alembert is particularly remarkable for the solutions of some problems on the integral calculus of partial differences, which solutions he successfully applied a year afterwards, to explain the motion of vibrating chords.

The motion of the fluids which cover the planets was a subject almost entirely new, when I undertook in 1772 to discuss it. Assisted by the discoveries made in the calculus of partial differences, and in the theory of the motion of fluids discovered in a great measure by D'Alembert, I published in the Memoirs of the Academy of Sciences for the year 1775, the differential equations of the motions of the fluids which being spread over the earth, are attracted by the Sun and Moon. I first applied these equations to the problem which D'Alembert in vain essayed to resolve, namely, that of the oscillations of a fluid spread over the entire earth, supposed spherical, and without rotation, the attracting star being supposed to be in motion about this planet. I gave the general solution of this problem, whatever might be the density of the fluid and its initial state, supposing that each fluid molecule experiences a resistance proportional to its velocity, which shews that the primitive conditions of motion are at length annihilated by the friction and the small viscosity of the fluid. But an inspection of the differential equations shewed me very soon, that I ought to take into account the rotatory motion of the earth. I therefore considered this motion, and I applied myself particularly to the determination of the oscillations of the fluid, which are independent of its initial state, and the only ones which are permanent. These oscillations are of three kinds. Those of the first kind are independent of the motion of rotation of the earth, and their determination presents few difficulties. The oscillations depending on the motion of rotation of the earth, and of which the period is about a day, constitute the second species; finally, the third species is composed of oscillations, of which the period is very nearly half a day. They surpass the others considerably in our harbours. I have accurately determined those different oscillations, in the case in which it can be determined, and by very convergent approximations, in the other cases. The excess of two consecutive high waters, one over the other in the solstices, depends on the oscillations of the second species. This excess, which is hardly sensible at Brest, ought, according to the theory of Newton, to be very considerable. This great geometer and his successors attributed, as I have already stated, this difference between the formulæ and observations, to the inertia of the waters of the ocean. But analysis shews that it depends on the law of the depth of the sea. I therefore investigated the law which would render this excess nothing, and I found that the depth of the sea ought to be every where constant. The figure of the earth being then supposed to be elliptical, which would render to the sea an elliptic figure of equilibrium, I have given the general expression of the inequalities of the second species: and I have deduced this remarkable proposition, namely, that the motions of the earth's axis are exactly the same as if the sea consituted a solid mass with the earth, which was contrary to the opinion of geometers, and particularly of D'Alembert, who in his (*i*) important Treatise on the precession of the Equinoxes, asserted that, in consequence of the fludity of the sea, it had no influence on this phenomenon. My analysis also indicated to me the general condition of the stability of the equilibrium of the sea. The geometers who considered the equilibrium of a fluid spread over an elliptic spheroid, remarked that if its figure be a little compressed, it does not tend to revert to its first state, except in the case in which the ratio of its density to that of the spheroid, was below $\frac{6}{3}$; and they have inferred from this condition, that of the stability of the equilibrium of the fluid. But in this investigation, it is not sufficient to consider a state of quiescence of the fluid, very near to the state of equilibrium, it is necessary to assign to this fluid some initial motion very small, and then to determine the condition necessary, in order that this motion may be always confined within very narrow limits. By considering the problem in this general point of view I have found, that if the mean density of the earth surpass that of the sea, this fluid, when deranged by any causes from its state of equilibrium, will never deviate from it, except by small quantities; but that the durations may be very considerable, if this condition be not satisfied. Finally, I have determined the oscillations of the atmosphere, on the ocean which it covers, and I have found that the attractions of the Sun and Moon cannot produce the constant motion from east to west, which is observed under the name of *trade winds*. The oscillations of the atmosphere produce (*k*) in the height of the barometer, small oscillations, of which the extent at the equator being only half a millimetre, demands the utmost attention of observers. The preceding observations, though extremely general, are still far from representing accurately the tides, which have been observed in our harbours. They suppose that the surface of the terrestrial spheroid is entirely covered by the sea; now it is evident that the great irregularities of its surface ought to modify considerably the motion of the waters, with which it is only partly covered. Experience shews in fact, that accessary circumstances produce considerable varieties in the heights, and in the hours of high water in the harbours, which are very near to each other. It is impossible to subject these varieties to the calculus, since the circumstances on which they depend are not known, and even if they were, we would not be able to solve the problem, in consequence of its extreme difficulty. However, in the midst of the numerous modifications of the motion of the sea, arising from these circumstances, this motion preserves, with the forces which produce it, relations which are proper to indicate the nature of those forces, and to verify the law of the attractions of the Sun and Moon on the sea. The investigations of these relations between causes and their effects, is not less useful in natural philosophy than the direct solution of problems, as well in verifying the existence of these causes, as also in determining the laws of their effects: we can frequently apply it, and it is like the calculus of probabilities a fortunate supplement to the ignorance and imperfection of the human mind.

In the present question, I make use of the following principle, which may be useful on various occasions. "The state of a system of bodies, in "which the primitive conditions of motion have "disappeared in consequence of the resistances "which this motion experiences, is periodic, like "the forces which actuate the system."

From this I have inferred, that if the sea is sollicited by a periodic force expressed by the cosine of an angle which increases proportionally to the time; there will result from it a partial tide expressed by the cosine of an angle increasing in the same manner, but of which the constant contained under the sign *Cosine*, and the coefficient of this cosine, may be in consequence of accessory circumstances, very different from the same constant quantities in the expression of the force, so that they can be determined by observation only. The expressions of the actions of the Sun and Moon on the sea may be developed into a convergent series of similar cosines. Hence arise so many partial tides, which in consequence of the coexistence of the small oscillations, combine together to form the total tide which is observed in any harbour. It is in this point of view that I have considered the tides in the fourth book of the Celestial Mechanics. In order to connect together the different constants of the partial tides, I have considered each tide as produced by the action of a star, which moves uniformly in the plane of the equator; the tides, of which the period is about half a day, arise from the action of stars, of which the proper motion is very slow, with respect to the rotatory motion of the earth; and as the angle of the cosine, which expresses the action of one of these stars, is a multiple of the rotation of the earth, plus or minus a multiple of the proper motion of the star, and since, besides the constants of the cosines, which express the tides of the two stars, would have the same ratio to the constants of the cosines which express their actions, if the proper motions were equal; I have assumed that the ratio varies from one star to another, proportionally to the difference of the proper motions. The error of this hypothesis, if there be any such, has no sensible influence on the principal results of my computations.

The greatest variations of the height of the tides in our harbours, arise from the action of the sun and moon, being supposed to move uniformly in the plane of the equator. But in order to have the law of these variations, it is necessary to combine the observations in such a manner, that all the other variations may disappear from their result. This is obtained by considering the height of high waters, above the neighbouring very low waters, in the syzygies, and the quadratures, assuming an equal number, near to each equinox and solstice. By this means the tides, independent of the rotation of the earth, and those of which the period is about a day, disappear, and likewise the tides produced by the variation of the distance of the sun from the earth. By considering three consecutive syzygies or solstices, and by doubling the intermediate, the tide produced by the variation of the distance of the moon from the earth is made to disappear; since if this star be in perigee at one of her phases, it is very nearly in apogee at the other eorresponding phase, and the compensation is the more exact, according as a greater number of observations is employed. By this process the influence of the winds on the result of observation becomes very nearly nothing, for if the wind raises the height of one high water, it elevates very nearly by the same quantity the neighbouring low water, and its effect disappears in the difference of those two heights. It is thus that by combining the observations in such a manner that their sum may present only one element, we are enabled to determine successively all the elements of the phenomena. The analysis of probabilities furnishes for the determination of these elements (*l*) a method still more certain, and which may be termed *the most advantageous method.* It consists in forming between these elements, as many equations of condition as there are observations. By the rules of this method, the number of these equations is reduced to that of the elements, which are determined by resolving the equations thus reduced. It is by this process that M. Bouvard has constructed his excellent tables of Jupiter, Saturn and Uranus. But observations relative to the tides are far from having the same accuracy as astronomical observations; the very great number of those which it is necessary to employ, in order, that the errors may compensate each other, does not permit us to apply to them *the most advantageous method.* At the suggestion of the Academy of Sciences, observations on the tides were made in the harbour of Brest, during the space of six consecutive years. It is to those observations published by Lalande, that I have compared my formulæ, in the book already cited. The situation of this harbour is very favourable to this kind of observation. It communicates with the sea by a vast canal, at the extremity of which this port has been constructed. Therefore the irregularities in the motion of the sea, when they arrive at this harbour, are very much diminished, just as the oscillations, which the the irregular motion of a ship produces in a barometer, are lessened by a contraction made in the tube of this instrument. Besides, the tides at Brest being considerable, the accidental variations are only an inconsiderable part of them. Thus, considering the fewness of the observations relative to the tides, a great regularity is observed, which are not affected by the little river which empties itself into the immense road of this harbour. Struck with this regularity, I suggested to government to order a new series of observations relative to the tides to, be made at Brest, which might be continued during the period of the motions of the nodes of the moon. This has been undertaken. These new observations commenced on the first of June, 1806, and they have been continued uninterruptedly since that period. We have examined those of 1807, and of the fifteen following years. The immense computations which the comparison of my analysis with observations required, are due to the indefatigable zeal of M. Bouvard, for every thing which concerns astronomy, near six thousand observations are employed, in order (*m*) to obtain the height of the high waters and their variation, which, near to the *maximum*, is proportional to the square of the time. I have considered near to each equinox and solstice, three consecutive syzygies, between which the equinox or the solstice were included; and the results of the intermediate syzygy, were doubled in order to destroy the effects of the lunar parallax, at the occurrence of each syzygy, the height of the evening high water above the low water of the morning, was taken, on the day which preceded the syzygy, on the very day of the syzygy, and on the four succeeding days; because the *maximum* of the tides occurs very nearly in the middle of this interval. The observations of these heights, made during the day, became more certain and exact. During each of the sixteen years, the sum of the heights of the corresponding days of the equinoctial syzygies has been taken, and a like sum relative to the solstitial syzygies, and from hence the *maxima* of the heights of the high waters, near to the equinoctial and solstitial syzygies has been inferred, and the variations of these heights near to their *maxima*. From an inspection of these heights, and of their variations, the regularity of this kind of observation on the harbour of Brest is immediately apparent.

In the quadratures a similar process has been pursued, with the sole difference that the excess of the height of the morning over the low water of the evening, has been taken on the day of the quadrature, and on the three succeeding days. The increment of the tides at the quadratures departing from their *minimum*, being much more rapid, than the diminution of the syzygial tides in departing from their *maximum;* the law of the variation proportional to the square of the time, ought to be restricted to a shorter interval.

All those heights evidently indicate the influence of the declinations of the Sun and Moon, not only on the absolute heights of the tides, but also on their variations. Several philosophers, and particularly La Lande, has questioned this influence, because instead of considering a great number of observations, they attended only to isolated observations in which the sea, by the effect of accidental causes, was elevated to a great height in the solstices. But the simplest application of the calculus of probabilities, to the results of Mr. Bouvard, is sufficient to shew that the probability of the influence of the declination of the stars is very great, and far superior to that of a great number of facts respecting which there does not exist any doubt.

From the variations of the high waters near to their maxima and minima, the interval at which these maxima and minima follow the syzygies and quadratures, has been inferred, and this interval has been found to be a day and a half very nearly, which perfectly accords with what I deduced from ancient observations in the fourth book of the Celestial Mechanics. The same agreement obtains relative to the magnitude of these *maxima* and *minima*, and with respect to the variations of the heights of the tides, in departing from these points, so that nature after the lapse of a century is found agreeing with herself. The interval to which I allude, depends on the constant quantities involved under the signs of *Cosines*, in the expressions of the two principal tides due to the actions of the Sun and Moon. The corresponding constant quantities of the expressions of the forces are differently modified by accessory circumstances; at the moment of the syzygy, the lunar tide precedes the solar tide, and it is not till a day and a half after that, (*m*) (the lunar tide retarding each day on the solar tide,) these two tidies coincide and thus produce the *maximum* high water. We shall have an adequate conception of the retardation of the highest tides at the instant of syzygy, if we conceive in the plane of a meridian, a canal at the mouth of which the highest tide arrives at the moment of the occurrence of the syzygy, and that it employs a day and a half to arrive at the port situated at the extremity of this canal. A similar modification obtains in the constant quantities, which multiply the cosines, and there results from it an increase in the action of the stars on the sea. I have given in the fourth book of the Celestial Mechanics the means of recognizing this increment, which by the ancient observations I have found to be a tenth part; but although the observations of the tides in the quadratures accord with the observations of the syzygial tides on this subject, I have stated that an element so delicate as this, requires a much greater number of observations. The computations of Mr. Bouvard have confirmed the existence of this increment, and made it very nearly equal to a fourth part, in the case of the Moon. The determination of this relation is necessary to enable us to infer from the observations of the tides, the true relations between the actions of the Sun and of the Moon, on which the phenomena of the precession of the equinoxes and of the nutation of the earth's axis depend. The actions of the stars on the sea being corrected by the increments due ($\varkappa$) to accessory circumstances, the nutation is found equal to 9″,4 in sexagesimal minutes; the lunar equation of the tables of the Sun is found equal to 6″8, and the mass of the Moon comes out to be a 75th of the mass of the earth.

These are very nearly the results furnished by astronomical observations. The agreement of values obtained from such different sources is extremely remarkable. It is from a comparison of my formulæ with the *maxima* and *minima* of the observed heights of the seas, that the actions of the Sun and Moon and their increments have been determined. The variations of the heights of the tides near to these points, is a necessary consequence of them; therefore by substituting the values of these actions in my formulæ, we ought to find very nearly the observed variations. This is in fact the case. This agreement is a striking confirmation of the law of universal gravitation. It receives an additional confirmation from observations of the syzygial tides near to the apogee and perigee of the Moon. In the work cited, I have only considered the difference of the heights of the tides in those two positions of the Moon. Here I have moreover considered the variations of these heights in departing from their *maxima*, and on these two points my formulæ coincide with the observations.

The times of high water, and the retardations of the tides from one day to another, present the same varieties as their heights. M. Bouvard constructed tables of them for the tides, which he employed in the determination of the heights. The influence of the declinations of the stars, and of the lunar parallax, are very evident in them. These observations, compared with my formulæ, exhibit the same agreement as the observations relative to the heights. The small anomalies which observations still present may be made to disappear, by a suitable determination of the constant quantities of each partial tide; the principle by which these various constant quantities have been connected together cannot be rigorously exact. Perhaps also, the quantities which have been neglected in adopting the principle of the coexistence of oscillations, become sensible in the great tides. I have barely adverted to those slight inaccuracies, in order to direct those who might wish to resume the computations when observations of the tides, which are making at Brest, and which are deposited at the Royal Observatory, will be sufficiently numerous to enable us to determine with certainty whether these anomalies arise from the errors of observations. But previously to making any modification in the principles which I have employed, it will be necessary to extend farther our analytical approximations. Finally, I have considered the tide, of which the period is about half a day. From a comparison of the differences of the two high and the two consecutive low waters, among a great number of syzigeal solstices, the magnitude of this tide and the hour of its *maximum*, in the harbour of Brest, have been determined. Although its magnitude is not the thirtieth part of the magnitude of the semidiurnal tide, still the forces which produce these two tides are very nearly equal, which shows how differently accessory circumstances affect the magnitude of the tides. We shall not be surprized at this, if we consider that even in the case in which the surface of the earth was regular, and entirely covered by the sea, the daily tide would disappear if the depth of the sea was constant.

The accessory circumstances may also cause the semidiurnal inequalities to disappear, and render the diurnal inequalities very sensible. Then we shall have on each day, but one tide, which disappears when the stars (*o*) are in the equator. This is what takes place at Batsham, a harbour in the kingdom of Tonquin, and in some islands of the south sea.

With respect to those circumstances, it may be observed that the one appertains to the entire sea, and refers to causes operating at a considerable distance from the harbour where they are observed, for instance, there can be no doubt but that the oscillations of the Atlantic ocean, and of the south sea, being reflected by the eastern side of America, which extends almost from one pole to another, has a considerable influence on the tides at the harbour of Brest. It is chiefly on these circumstances that the phenomena depend which are nearly the same in our harbours. Such appears to be the retardation of the highest tides at the moment of syzygy. Other circumstances more nearly connected with the ports, such as the shores or neighbouring straits, may produce the differences, which are observed between the heights and hour of port in harbours which are very near to each other. Hence it follows, that the partial tide has not with the latitude of the harbour, (p), the relation indicated by the force which produces it; since it depends on similar tides corresponding to very distant latitudes, and even to another hemisphere. Therefore the sign and magnitude of the tide can be determined by observation alone.

The phenomena of the tides which I have considered depend on terms arising from the expansion of the action of these stars, divided by the cubes of their distance, which are the only ones that have been hitherto considered. But the Moon is sufficiently near to the Earth to have the terms of the expression of its action divided by the fourth power of its distance, sensible in the results of a great number of observations; for we know from the theory of probabilities, that the number of observations compensates for their want of accuracy, and includes inequalities much less than the errors, of which each observation is susceptible. We can even by this theory, assign the number of observations necessary to acquire a great probability, when the error of the result which has been obtained, is contained within narrow limits. It therefore occurred to me that the influence of the terms of the Moon's action, divided by the fourth power of the distance, might be apparent in the collection of the numerous observations which have been discussed by M. Bouvard. The tides, which correspond to the terms divided by the cube of the distance, do not assign any difference between the high waters of full moon and those of new moon. But those of which the devisor is the fourth power, produce some difference between these tides. They produce a tide, of which the period is about the third part of a day, and observations discussed under this point of view, indicate with a great degree of probability the existence of this partial tide. They also unquestionably prove that the action of the Moon to raise the sea at Brest, is greater when its declination is southern, than when it is northern, which can only arise from the terms of the lunar action, divided by the fourth power of the distance.

It appears from the preceding *expose*, that the investigation of the general relations between the phenomena of the seas, and the actions of the Sun and Moon on the ocean, most fortunately supplies the impossibility of integrating the differential equations of this motion, and our ignorance of the data necessary to determine the arbitrary functions which occur in their integrals; it also follows, that these phenomena have one sole cause, namely, the attraction of these two stars conformably to the law of universal gravitation.

If the earth had no satellite, and if its orbit was circular and situated in the plane of the equator, we should only have in order to recognize the action of the Sun on the ocean, the hour of high water, (which would be always the same,) and the law according to which the sea rises. But the action of the Moon by combining with that of t'

Sun, produces in the tides, varieties relative to its phases, the agreement of which with observations, renders the theory of universal gravitation extremely probable. From all the inequalities in the motion, in the declination, and in the distance of these two stars, there arise the phenomena indicated by observation, which places this theory beyond all doubt ; it is thus that varieties in the actions of causes establish their existence.

The action of the Sun and Moon on the Earth, a necessary consequence of the universal attraction, demonstrated by all the celestial phenomena, being directly confirmed by the phenomena of the tides, ought to leave no uncertainty on the subject. It is indeed brought now to such a degree of perfection, that not the least difference of opinion exists upon the subject, among men sufficiently learned in the science of geometry and mechanics, to comprehend its relation with the law of universal gravitation.

A long series of observations, more precise than have hitherto been made, and continued during the period of the revolution of the nodes, will rectify the elements already known, fix the value of those which are uncertain ; and develope phenomena which before were obscured in the errors of observation. The tides are not less interesting to understand than the inequalities of the heavenly bodies, and equally merit the attention of observers. We have hitherto neglected to follow them with sufficient precision, because of the irregularities they present. But I can assert, after a careful investigation, that these irregularities disappear by multiplying the observations; nor is it necessary that their number should be extremely great, particularly at Brest, of which the situation is very favourable to this species of observation.

I have now only to speak of the method of determining the time of high water, on any day whatever. We should recollect, that each of our ports may be considered as the extremity of a canal, at whose *embouchure* the partial tides happen at the moment of the passage of the Sun and Moon over the meridian, and that they employ a day and a half to arrive at its extremity, supposed eastward of its embouchure, by a certain number of hours. This number is what I call the *fundamental hour* of the port. It may easily be computed from the hour of the establishment of the port, by consider- the former as the hour of the full tide, when it coincides with the syzygy. The retardation of the tides, from one day to another, being then 2705″, it will be 3951″ for one day and a half, which quantity is to be added to the hour of the establishment, to have the fundamental hour. Now, if we augment the hours of the tides at the *embouchure* by the fifteen hours, *plus* the fundamental hour, we shall have the hours of the corresponding tides in our ports. Thus, the problem consists in finding the hours of the tides in a place whose longitude is known, on the supposition that the partial tides happen at the instant of the ~~

sage of the Sun and Moon over the meridian. For this purpose analysis affords very simple formulæ, which are easily reduced to tables, and very useful to be inserted in the ephemerides that are destined for the use of navigators.

The great tides have frequently produced in harbours, and near shores, disastrous effects which might have been foreseen, if we were previously apprised of the height of these tides. The winds may have on this phenomenon an influence which it is impossible to anticipate. But we can predict with certainty the influence of the Sun and Moon, and this is sufficient most frequently to secure us from the accidents which high tides may occasion, when the direction and force of the wind is combined with the action of regular causes. In order that the maritime departments may participate in the advantage produced by the sciences, the Bureau of longitude publishes each year in its Ephemerides, the table of the syzygial tides, the mean height in the syzygies of the equinoxes being assumed equal to unity.

I have dwelt more particularly on the theory of the tides, because of all the effects of the attraction of the heavenly bodies, it is the most obvious, and most within our reach; besides it appeared of consequence to shew, how by means of a great number of observations, although inaccurate, we can recognize and determine the laws and the causes of the phenomena, the analytical expressions of which it is impossible to determine by the formation and integration of their differential equations. Such are the effects of the solar heat on the atmosphere, in the production of the trade winds and monsoons, and in the regular variations both annual and diurnal, of the barometer and thermometer.

CHAPTER XII.

Of the Oscillations of the Atmosphere.

THE action of the Sun and Moon on the ocean must previously traverse the atmosphere, which must necessarily be subject to their influence, and experience motions similar to those of the sea. Hence result periodic variations in the height of the barometer, and of the winds, the direction and intensity of which are also periodic. These winds are inconsiderable, and nearly insensible in an atmosphere already very much agitated from other causes: the extent of the oscillations of the barometer is not a millimeter at the equator itself, where it is greatest.

In the fourth book of the Celestial Mechanics I have given the theory of all these variations, and I have directed the attention of observers to this subject. It is at the equator that observations on the variations in the height of the barometer ought to be made, not only because they are greater than in any parallel, but also because the changes arising from irregular causes are smaller there. However, as local circumstances considerably increase the heights of the tides in our harbours, they may produce a similar effect in the oscillations of the atmosphere, and also in the corresponding variations of the barometer, it is therefore of importance to be assured of them by observations.

The atmospheric tide is produced by the three following causes; the first is the direct action of the sun and moon on the atmosphere; the second is the periodic elevation and depression of the ocean, which is the moveable base of the atmosphere; finally, the third is the attraction of this fluid by the sea, the figure of which varies periodically. These three causes arise from the same attractive forces of the sun and moon; they have, like their effects, the same periods as these forces, (*a*) conformably to the principle on which I have founded my theory of the tides. The atmospheric tide is therefore subject to the same laws as the tides of the ocean; it is, like to the latter, the combination of two partial tides produced, the one by the action of the Sun, the other by the action of the Moon. The period of the atmospheric solar tide is half of a solar day; and that of the lunar tide is half of a lunar day. The action of the Moon on the sea at Brest being triple (*b*) of the action of the sun, the atmospheric lunar tide is at least double of the solar tide. These observations should guide us in the selection of observations proper to determine such small quantities, and also in the modes of combining them together, so as to abstract as much as possible from the influence of causes which produce great variations in the height of the barometer. For several years the heights of the barometer and thermometer have been observed at nine o'clock A. M., at mid-day, at three o'clock P. M. and at nine o'clock P. M. These observations being made with the same instrument, and almost by the same observer, are from their precision, and their great number, very proper to indicate an atmospheric tide, if it be sensible. In the results of these observations, a diurnal variation of the barometer is indicated very plainly: one month only is sufficient to manifest it. The excess of the greatest observed height of the barometer, which occurs at nine o'clock, A. M., over the least height, which happens at three o'clock P. M., is at Paris eight-tenths of a millemetre, according to the mean result of observations made each successive day during six consecutive years. As the height of the barometer due to the solar tide, becomes the same at the same hour of each day; this tide is confounded with the diurnal variation, which it modifies, so that it cannot be distinguished by observations made at the Royal Observatory. This is not the case with respect to the barometric heights due to the lunar tide, and which regulating itself by lunar hours, does not become the same at the same solar hours, until after the lapse of half of a month. The observations of which I have spoken being compared from one half month to another, are arranged in the most advantageous manner for indicating the lunar tide. If, for example, the *maximum* of this tide occurs at one o'clock A. M. on the day of the syzygy, its *minimum* will happen towards three o'clock P. M. The contrary will be the case on the day of the quadrature. This tide will therefore increase the daily variation of the first of these days; it will diminish the daily variation of the second; and the difference of these variations (*e*) will be twice the height of the atmospheric lunar tide. But as the *maximum* of this tide does not take place at nine o'clock A. M. in the syzygy, it is necessary, in order to determine its magnitude, and the hour it happens, to employ barometrical observations made at nine o'clock A. M., at midday, and at three o'clock P. M. for each day, both of the syzygy and of the quadrature: We may likewise make use of observations made on the days which precede or which follow those phases by the same number of days, and make all the observations of the year concur in the determination of these delicate elements.

An important observation may be made here, without which it would have been impossible to recognise so inconsiderable a quantity as the lunar tide, in the midst of the great variations of the barometer. The more the observations approach to each other, the less sensible will b the effects of these variations; it is almost nothing on a result inferred (d) from observations made on the same day, and in the short interval of six hours. The barometer varies sufficiently slow, as not to derange in a sensible manner the effects of regular causes. This is the reason why the mean result of the daily variation of each respective year is always very nearly the same, although differences to the amount of several millemetres may exist in the mean absolute barometrical heights of different years: so that if the mean height of nine o'clock A. M. of one year, be compared with the mean height of three o'clock P. M. of another year; a diurnal variation will result frequently very erroneous, and even sometimes affected with a sign the contrary of the true sign. It is therefore of importance, in order to determine such very small quantities, to deduce them from observations made on the same day, and to take the mean between a great number of observations thus obtained. Consequently we cannot determine the lunar tide, except by a system of observations made on each day, at three different hours at least according to the method followed at the observatory.

M. Bouvard wished to insert in his registers, barometric observations made on the respective days of each quadrature and syzygy, and also on the day which precedes those phases, and on the first and second days which follow it. They embrace the eight years which have lapsed from the first of October, 1815, to the first of October,

1823. I have made use of the observations of nine o'clock in the morning, of midday, and of three o'clock P. M. However I did not take into account observations made at nine o'clock P. M., in order to diminish as much as possible, the interval at which observations are made. Besides those of the three first hours, which have been specified, were made more exactly at the time pointed out, than those made at nine o'clock P. M., and moreover the barometer being illuminated by the light of the day at the three first hours, the difference (e) which may arise from the different manner in which the instruments are illuminated, disappears. From a comparison of these numerous results (which embrace an interval of 1584 days,) with my observations, I have found that the magnitude of the lunar atmospheric tide is an eighteenth part of a millimetre, and the time of its maximum, on the evening of the day of the syzygy, is three hours and a quarter.

It is here particularly that the necessity is apparent of employing a great number of observations, of combining them in the most advantageous manner, and of having a method for determining the probability that the errors of the results, which (f) are obtained, are confined within narrow limits, without which we would be liable to present as laws of nature the effects of irregular causes, which is frequently the case in meteorology. I have given this method in my analytical theory of probabilities. And in the application of it to observations, I have determined the law of the anomalies of the diurnal variation of the barometer, and I have ascertained that we cannot without every appearance of improbability, attribute the preceding results to these anomalies solely: it is probable that the lunar atmospheric tide diminishes the diurnal variation in the syzygies, and that it increases it in the quadratures, but it is so inconsiderable that in the limits, this tide does not produce a variation in the height of a barometer of an eighteenth of a millimetre, more or less; which shews, that the action of the Moon on the atmosphere, is nearly insensible at Paris. Although these results have been obtained from 4752 observations, the method already adverted to, shews that in order to secure the requisite probability, and to obtain with sufficient accuracy such a small element as the lunar atmospheric tide, it is necessary to employ at least forty thousand observations. One of the principal advantages of this method is, that it indicates to what extent it is necessary to multiply observations, in order that no reasonable doubt may rest on their results.

It follows from the laws of the anomalies of the diurnal variation of the barometer, which I obtained, that there is a probability of $\frac{1}{2}$, or of one to one, that the daily variation from 9 o'clock A. M. to three o'clock P. M. will be constantly positive in its mean result for each month of 30 days, during 75 consecutive months. I have requested M. Bouvard to examine whether this is the case for each of the 72 months of the six years which have lapsed from the first of January 1817 to the first of January 1823, from which he inferred that the mean diurnal variation was equal to $0^{m}.,^{m}801$. A comparison of his observations has given the most probable result, namely, that the mean diurnal variation of each month has been always positive.

What is the respective influence on the lunar tide, of the three causes already cited of the atmospherical tide; it is difficult to give an answer to this question. However the little density of the sea comparatively to the mean density of the earth, does not permit us to ascribe a sensible effect to the periodic change of its figure. Without local circumstances, the direct effect of the action of the Moon would be insensible in our latitudes. These circumstances have indeed a great influence on the height of the tides in our harbours; but as the atmospheric fluid is diffused about the earth, much less irregularly than the sea, their influence on the atmospheric tide must be much less than on the tide of the ocean. From these considerations I am induced to (*f*) consider the periodic elevation or depression of the sea, as the principal cause of the lunar atmospheric tide in our climates. Barometric observations made every day in the harbours, where the sea ascends to a considerable height, would throw considerable light on this curious point of meteorolgy.

It may be remarked here, that the attraction of the Sun and Moon, does not produce either in the sea or in the atmosphere, any constant motion from east to west; that which is observed between the tropics, under the name of *trade* *winds*, must therefore arise from some other cause, the following appears to be the most probable.

The Sun, which for greater simplicity, we shall suppose in the plane (h) of the equator, rarifies by its heat the strata of the air, and makes them to ascend above their true level; they must therefore in consequence of their greater weight subside, and move towards the poles in the higher regions of the atmosphere; but at the same time a fresh current of air must arrive in the lower regions from the poles, in order to supply that which has been rarified at the equator. There is thus established two currents of air, blowing in opposite directions, the one in the inferior, and the other in the higher region of the atmosphere; but the actual velocity of the air, arising from the rotation of the earth, is always less according as it is nearer to the pole; it must therefore, as it approaches towards the equator, revolve slower than the corresponding parts of the earth, and bodies placed on the surface of the earth, must strike it with the excess of their velocity, and thus experience from its reaction, a resistance contrary to their motion of rotation. Therefore to an observer, who considers himself as immoveable, the air appears to blow in a direction opposite to that of the earth's rotation, *i. e.* from east to west; this is in fact the direction of the trade winds.

If we consider all the causes which derange the equilibrium of the atmosphere, its great mobility arising from its elasticity and mobility, the influence of heat and cold on its elasticity, the immense quantity of vapours with which it is alternately charged and unloaded, finally, the changes which the rotation of the earth produces in the relative velocity of its molecules, from this alone that they are displaced in the direction of the meridians, we will not be astonished at the variety of its motions, which it will be extremely difficult to subject to certain laws.

CHAP. XIII.

Of the Precession of the Equinoxes, and of the Nutation of the Axis of the Earth.

Every part of nature is linked together, and its general laws connect phenomena with each other, which appear to be altogether distinct. Thus, the rotation of the terrestrial spheroid compresses the poles, and this compression, combined with the action of the Sun and Moon, produces the precession of the equinoxes, which, before the discovery of universal gravitation, did not appear to have any connection with the diurnal motion of the Earth.

Let us suppose this planet to be an homogeneous speroid, protuberant at the equator, it may then be considered as composed of a sphere of a diameter equal to the axis of the poles, and of a meniscus surrounding the sphere, of which the greatest thickness corresponds with the equator of the spheroid. The particles of this meniscus may be considered as so many small moons adhering together, and making their revolutions in a period equal to the revolution of the Earth on its axis.

The nodes of all their orbits should therefore have a retrograde motion, arising from the action of the Sun, in the same manner as the nodes of the lunar orbit; and from the connection of these bodies together, there should arise a motion of the whole meniscus which would make its points of intersection with the ecliptic to retrograde, but this meniscus imparts to the sphere to which it is attached, its retrograde motion, which, for this reason, becomes slower; the intersection of the equator and the ecliptic, that is to say, the equinoctial points, should consequently have a retrograde motion. Let us endeavour to investigate both the law and the cause of this phenomenon.

And first let us consider the action of the Sun upon a ring, situated in the plane of the equator. If we conceive the mass of the Sun to be distributed uniformly over the circumference of its orbit, (supposed circular) it is evident that the action of this solid orbit will represent the mean action of the Sun. This action, on every one of the points of the ring above the ecliptic, being decomposed into two, one in the plane of the ring, and the other perpendicular to it, (*a*) it follows that the resulting force, arising from these last actions, on all the particles of the ring, is perpendicular to its plane, and situated on that diameter of the ring, which is perpendicular to the line of its nodes. The action of the solar orbit, on the part of the ring below the ecliptic, produces also a resulting force, perpendicular to the plane of the ring, and situated in the inferior part of the same diameter. These two resulting forces combine to draw the ring towards the ecliptic, by giving it a motion round the line of nodes; its inclination, therefore, to the ecliptic, would be diminished by the mean action of the Sun, the nodes all the time continuing stationary; and this would be the case but for the motion of the ring, which we now suppose to revolve in the same time as the Earth. In consequence of this motion, the ring is enabled to preserve a constant inclination to the ecliptic, and to change the effect of the action of the Sun, into a retrograde motion of the nodes. It gives to the nodes a variation, which otherwise would be in the inclination, and it gives to the (*b*) inclination a permanency, which otherwise would rest with the nodes. To conceive the reason of this singular effect, let us suppose the situation of the ring varied by an infinitely small quantity, in such a manner, that the planes of its two positions may intersect each other, in a line perpendicular to the line of the nodes.

At the end of any instant whatever, we may decompose the motion of each of its points into two, one of which should subsist alone in the following instant, the other being perpendicular to the plane of the ring, and which should therefore be destroyed. It is evident that the resulting force of these second motions relative to all the points of the upper part of the ring, will be perpendicular to its plane, and placed on the diameter which we just now considered, and this is equally true for the lower part of the ring. That this resulting force may be destroyed by the action of the solar orbit, and that the ring, by virtue of these forces, may remain in equilibrio on its centre, it is requisite that these forces should be contrary to each other, and their moments, relatively to this point, equal. The first of these conditions requires that the change of position, supposed to be given to the ring, be retrograde; the second condition determines the quantity of this change, and consequently the velocity of the retrograde (*c*) motion of the nodes. And it is easily demonstrated, that this velocity is proportional to the mass of the Sun, divided by thecube of its distance from the Earth, and multiplied by the cosine of the obliquity of the ecliptic.

Since the planes of the ring, in its two consecutive positions, intersect each other in a diameter perpendicular to the line of its nodes, it follows that the inclination of these two planes to the ecliptic is constant; therefore the inclination of the ring does not vary in consequence of the mean action of the Sun.

That which has been explained relatively to a ring, may be demonstrated by analysis, to hold true in the case of a spheroid, differing but little from a sphere. The mean action of the Sun produces in the equinoxes a motion proportional to its mass, divided by the cube of its distance, and multiplied by the cosine of the inclination to the ecliptic. This motion is retrograde when the spheroid is flattened at the poles; its velocity depends on the compression of the spheroid, but the inclination of the equator to the ecliptic always remains the same.

The action of the Moon produces likewise a similar retrogradation of the nodes of the terrestrial equator in the plane of its orbit; but the position of this plane and its inclination to the equator incessantly varying, by the action of the Sun, and as the retrograde motion of the nodes of the equator on the lunar orbit, produced by the action of the Moon, is proportional to the cosine (*d*) of this inclination, this motion is consequently variable.

Besides, even supposing it uniform, it would, according to the position of the lunar orbit, cause a variation both in the retrograde motion of the equinoxes, and in the inclination of the equator to the ecliptic. A calculation, by no means difficult, is sufficient to show, that the action of the Moon, combined with the motion of the plane of its orbit, produces. 1st A (*e*) mean motion in the equinoxes, equal to that which it would produce if it moved in the plane of the ecliptic. 2dly. An inequality *subtractive*, from this retrograde motion, and proportional to the sine of the longitude of the ascending node of the lunar orbit. 3dly. A diminution in the obliquity of the ecliptic, proportional to the cosine of this same angle. These two inequalities are represented at once by the motion of the extremity of the terrestrial axis (prolonged to the heavens) round a small elipse, conformably to the laws explained in Chap. XII. of Book I. The greater axis of this ellipse is to the lesser, as the cosine of the obliquity of the ecliptic is to the cosine of double this obliquity. We may comprehend from what has been said, the cause of the precession of the equinoxes, and of the nutation of the Earth's axis, but a rigorous calculation, and a comparison of its results with observation, is the best test of the truth of a theory. That of universal gravitation is indebted to d'Alembert, for the advantage of having been thus verified in the case of the two preceding phenomena. This great mathematician first determined, by a beautiful analysis, the motions of the axis of the Earth, on the supposition that the strata of the terrestrial spheroid were of any density or figure whatever, and he not only found his results exactly conformable to observation, but obtained an accurate determination of the dimensions of the small ellipse described by the pole of the Earth, with respect to which the observations of Bradley had left some little doubt.

The influence of a heavenly body, either upon the motion of the axis of the Earth, or upon the ocean, is always proportional to the mass of that body, divided by the cube of its distance from the Earth. The nutation of the Earth's axis being due to the action of the Moon alone, while the mean precession of the equinoxes arises from the combined actions of the Sun and Moon, it follows that the observed values of these two phenomena, should give the ratio of their respective actions (*f*). If we suppose, with Bradley, the annual precession of the equinoxes to be 154″4, and the entire extent of the nutation equal to 55″6, the action of the Moon would be found to be double that of the Sun. But a very small difference in the extent of the nutation, produces a very considerable one in the ratio of the actions of these two bodies. The most accurate observations give for this extent $58'',02$ hence it results that $\frac{1}{75}$ expresses the ratio of the mass of the Moon to that of the Earth.

The phenomena of the precession and of the nutation, throw a new light on the constitution of the terrestrial spheroid. They give a limit to the compression of the earth supposed elliptic, for it appears from them that this compression does not exceed $\frac{1}{247,7}$, which accords with the experiments that have been made on pendulums. We have seen in Chap. VII. that there exists in the expression of the radius vector of the terrestrial spheriod, terms, which, but little sensible in themselves, and on the length of the pendulum, cause the degrees of the meridian to deviate considerably from the elliptic figure. These terms disappear entirely in the values of the precession and nutation, and for this reason, these phenomena agree with the experiments on pendulums. The existence, of these terms, therefore reconciles the observations of the lunar parallax, those of the pendulums and degrees of the meridian, and the phenomena of precession and nutation.

Whatever figure and density we may suppose in the strata of the Earth, whether or not it be a solid of revolution, provided it differs little from a sphere, we can always assign an elliptic solid of revolution, with which the precession and nutation will always be the same. Thus in the hypothesis of Bouguer, of which we have spoken in Chap. VII, and according to which the increase of the degrees varies as the fourth power of the sine of the latitude, these phenomena are exactly the same as if the Earth was an ellipsoid, whose ellipticity was $\frac{1}{183}$, but we have seen that observations do not permit us to suppose a greater ellipticity than $\frac{1}{247,7}$, so that these observations, and the experiments on pendulums, combine to disprove the hypothesis of Bouguer.

We have hitherto supposed the Earth entirely solid, but this planet being covered in a great part by the waters of the ocean, ought not their action to change the phenomena of the precession and nutation? It is of importance to consider this question.

The ocean, in consequence of its fluidity, is obedient to the action of the Sun and of the Moon. It seems at first sight that their re-action should not affect the axis of the Earth. D'Alembert and every subsequent mathematician, who has investigated these motions, have entirely neglected it, they have even commenced from that point, to reconcile the observed quantity of the precession and nutation, with the measures of the terrestrial degrees. Nevertheless, a more profound examination of this question has shewn us, that the fluidity of the waters of the sea is not a sufficient reason why their effect in the precession of the equinoxes should be neglected; for if on one hand, they obey the action of the Sun and Moon, on the other, the force of gravity tends to bring them back without ceasing, to a state of equilibrium, and consequently permits them to make but small oscillations; it is therefore possible, that by their attraction and pression on the spheriod which they cover, they may communicate, at least in part, the same motion to the axis of the Earth, which they would, if they could possibly become solid. Besides, we may, by simple reasoning, be convinced that their action is of the same order as action of the Sun and Moon, on the solid part of the Earth.

Let us suppose this planet to be homogeneous and of the same density as the ocean, and moreover, that the waters assume at every instant the figure that is requisite for the equilibrium of the forces that animate them. If in these hypotheses the Earth should suddenly become entirely fluid, it would preserve the same figure, all its parts would remain in equilibrio, and the axis of the Earth would have no tendency to move; now it is evident that the same should be the case, if a part of this mass formed by becoming solid, the spheroid which the ocean covers. The preceding hypotheses serve as a foundation to the theories of Newton, relatively to the figure of the Earth (*g*) and of the tides.

It is remarkable, that among the infinite number of those which may be chosen on this subject, this great geometrician has selected two which neither give the precession nor the nutation; the re-action of the waters destroying the effect of the action of the Sun and Moon upon the terrestrial nucleus, whatever may be its figure. It is true that these two hypotheses, particularly the last, are not conformable to nature, but we may see, *à priori*, that the effect of the re-action of the waters, although different from that which takes place in the hypothesis of Newton, is nevertheless of the same order.

The investigations which I have made on the oscillations of the ocean, have enabled me to determine this effect of the re-action of the waters in the true hypotheses of nature, and have led to this remarkable theorem.

Whatever may be the law of the depth of the ocean, and whatever be the figure of the spheroid which it covers, the phenomena of the precession and nutation will be the same as if the ocean formed a solid mass with this spheroid.

If the Sun and Moon acted only on the Earth, the mean inclination of the equator to the ecliptic would be constant, but we have seen that the action of the planets continually changes the position of the terrestrial orbit, and produces a diminution of its obliquity to the equator, which is fully confirmed by observations ancient and modern, the same cause gives to the equinoxes, a direct annual motion of 0″9659 ; thus the annual precession produced by the action of the Sun and Moon, is diminished by this quantity in consequence of the action of the planets; without this action it would be 155′5927. These effects of the action of the planets are independent of the compression of the terrestrial spheroid, but the action of the Sun and Moon upon this spheroid, modifies these effects and changes their laws.

If we refer to a fixed plane, the position of the orbit of the Earth, and the motion of its axis of rotation, it will appear, that the action of the Sun in consequence of the variations of the ecliptic, will produce in this axis an oscillatory motion similar to the nutation, but with this difference, that the period of these variations being incomparably longer than that of the variations of the plane of the lunar orbit, the extent of the corresponding oscillation in the axis of the Earth, is much greater than in the nutation. The action of the Moon produces in this same axis a similar oscillation, because the mean inclination of its orbit to that of the Earth, is constant. The displacement of the ecliptic, by being combined with the action of the Sun and Moon upon the Earth, produces upon its obliquity to the equator, a very different variation from that which would arise from this change of position only: the entire extent of this variation would be, by this alteration of the ecliptic, about twelve degrees, however in consequence of the action of the Sun and Moon, it is reduced to about three degrees.

The variation in the motion of the equinoxes, produced by these same causes, changes the duration of the (*h*) tropical year in different cenutries, The duration diminishes as this motion augments, which is the case at present, so that the actual length of the year is now shorter by about 13′, than in the time of Hipparchus. But this variation in the length of the year has its limits, which are also restricted by the action of the Sun and Moon, upon the terrestrial spheroid. The extent of these limits which would be about 500″, in consequence of the alteration in the position of the ecliptic, is reduced to 120″ by this action.

Lastly, the day itself, such as we have defined it in the First Book, is subject by the displacement of the ecliptic, combined with the action of the Sun and Moon, to very small variations, which though indicated by the theory, are quite insensible to observation. According to this theory, the rotation of the Earth is uniform, and the mean length of the day may be supposed constant, an important result for astronomy, as it is the measure of time, and of the revolutions of the heavenly bodies. If it should undergo any change, it would be recognized by the durations of these revolutions, which would be proportionably increased or diminished, but the action of the heavenly bodies does not produce any sensible alteration.

Nevertheless, it might be imagined that the trade winds which blow constantly from east to west between the tropics, would diminish the velocity of the rotation of the Earth, by their action on the continents and mountains. It is impossible to submit this action to analysis; fort nately it may be demonstrated that this action on the rotation of the Earth is nothing, by means of the principle of the conservation of areas, which we have explained in the Third Book. According to this principle, (*i*) the sum of all the particles of the Earth, the ocean and the atmosphere, multiplied respectively by the areas which their radii vectores describe round the centre of gravity of the Earth, projected on the plane of the equator, is constant in a given time.

The heat of the Sun can produce no effect, because it dilates bodies equally in every direction ; now it is evident, that if the rotation of the Earth should diminish, this sum would be less. Therefore the trade winds, which are produced by the heat of the Sun, cannot alter the rotation of the Earth. The same reasoning shews us that the currents of the sea ought not to produce any sensible change. To produce any perceptible alteration in its period, some great change must take place in the parts of the terrestrial spheroid : thus a great mass taken from the poles to the equator, would make this rotation longer, it would become shorter if the denser materials were to (*k*) approach the centre or axis of the Earth ; but we see no cause that can displace such great masses to distances considerable enough to produce any variation in the length of the day, which may be regarded as one of the most constant elements in the system of the world. This is likewise the case with respect to the points where the axis of rotation meets the surface. If the Earth revolved successively about different diameters, making with each other considerable angles, the equator and the poles would change their positions on the Earth ; and the ocean, flowing continually towards the new equator, would alternately overwhelm and then abandon the highest mountains : but all the investigations which I have made upon this change of position in the poles, have convinced me that it is insensible.

CHAP. XIV.

On the Libration of the Moon.

We have now only to explain the cause of the libration of the Moon, and of the motion of the nodes of its equator.

The Moon, in virtue of its motion of rotation, is a little flattened at its poles; but the attraction of the Earth must have lengthened a little that axis which is turned towards it. If the Moon was homogeneous and fluid, it would (to be in equilibrio) assume the form of an ellipsoid, of which the lesser axis passed through the poles of rotation; (*a*) the greater axis would be directed to the Earth, and in the plane of the lunar equator, and the mean axis would be situated in the same plane, perpendicular to the other two. The excess of the greatest above the least axis would be quadruple the excess of the mean above the least, and nearly equal $\frac{1}{27640}$, the least axis being taken as unity.

We may easily conceive that if the greater axis of the Moon deviates a little from the direction of the radius vector, which joins its centre with that of the Earth, the terrestrial attraction will tend to bring it down to this radius, in the same manner as gravity brings a pendulum towards the vertical. If the primitive motion of rotation of this satellite had been sufficiently rapid to have overcome this tendency, the period of its rotation would not have been perfectly equal to that of its revolution, and the difference would have discovered to us (*b*) successively every point in its surface. But at their origin the angular motions of rotation and revolution having differed but little, the force by which the greater axis of the Moon tended to deviate from the radius vector, was not sufficient to overcome the tendency of this same axis towards the radius, due to the terrestrial gravity, which by this means has rendered their motions rigorously equal, and in the same manner as a pendulum, drawn aside from the vertical by a very small force, continually returns, making small vibrations on each side of it, so the greater axis of the lunar spheroid ought to oscillate on each side of the mean radius vector of its orbit. Hence would arise a motion of libration, of which the extent would depend on the primitive difference between the angular motions of rotation and revolution of the Moon. This difference must have been very small, since it has not been perceived by observation.

Hence we see that the theory of gravitation explains in a sufficiently satisfactory manner, the rigorous equality of these two mean motions of rotation and revolution of the Moon. It would be against all probability to suppose that these two motions had been at their origin perfectly equal, but for the explanation of this phenomenon, it is enough to assume that their primitive difference was but small, and then the attraction of the Earth would establish the equality which at present subsists.

The mean motion of the Moon being subject to great secular inequalities, which amount to several circumferences, it is evident that if its mean motion of rotation was perfectly uniform, this satellite would, by virtue of these inequalities, present successively to the Earth every point on its surface, and its apparent disk would change by imperceptible degrees, in proportion as these inequalities were developed; the same observers would see pretty nearly the same hemisphere, and there would be no considerable difference, except to observers separated by an interval of several ages. But the cause which has thus established an equality between the mean motions of revolution and rotation, must take away all hope from the inhabitants of the Earth, of seeing the opposite side of the lunar hemisphere. The terrestrial attraction, by continually drawing towards us the greater axis of the Moon, causes its motion of rotation to participate in the secular inequalities of its motion of revolution, and the same hemisphere to be constantly directed towards the Earth.

The same theory ought to be extended to all the satellites, in which an equality between their motions of rotation and of revolution round their primary, has been observed.

The singular phenomenon of the coincidence of the nodes of the equator of the Moon, with those of its orbit, is another consequence of the terrestrial attraction. This was first demonstrated by Lagrange, who by a beautiful analysis was conducted to a complete explanation (*c*) of all the observed phenomena of the lunar spheroid. The planes of the equator and of the orbit of the Moon, and the plane passing through its centre parallel to the ecliptic, have always very nearly the same intersection; the secular motions of the ecliptic neither alter the coincidence of the nodes of these three planes, nor their mean inclination, which the attraction of the Earth constantly maintains the same.

We may observe here, that the preceding phenomena cannot subsist with the hypothesis in which the Moon, originally fluid and formed of strata of different densities, should have taken the figure suited to their equilibrium. They indicate between the axes of the Moon, a greater inequality than would take place in this hypothesis. The high mountains which we observe at the surface of the Moon, have without doubt a sensible influence on these phenomena, and so much the greater as its ellipticity is very small, and its mass inconsiderable.

Whenever nature subjects the mean motions of the celestial bodies to determinate conditions, they are always accompanied by oscillations, whose extent is arbitrary. Thus the equality of the mean motions of revolution and rotation produces a real libration in this satellite. In like manner the coincidence of the mean nodes of the equator and lunar orbit, is accompanied by a libration of the nodes of this equator round those of the orbit, a libration so small as hitherto to have escaped observation. We have seen that the real libration of the greater lunar axis is insensible, and it has been observed, (Chap. VI.) that the libration of the three satellites of Jupiter is also insensible. It is remarkable, that these librations, whose extent is arbitrary, and which might have been considerable, should nevertheless be so very small; we must attribute this to the same causes which originally established the conditions on which they depend.

But relatively to the arbitrary quantities, which relate to the initial motion of the rotation of the celestial bodies, it is natural to think that without foreign attractions, all their parts, in consequence of the friction and resistance which is opposed to their reciprocal motion, would, in process of time, acquire a permanent state of equilibrium, which cannot exist but with an uniform motion of rotation round an invariable axis; so that observation should no longer indicate in this motion, any other inequalities than those derived from these attractions. The most exact observations show that this is the case with the Earth, the same result extends to the Moon, and probably to the other celestial bodies.

If the Moon had encountered a comet (which according to the theory of chances ought to happen in the immensity of time), their masses must have been very minute; for the impact of a comet, which would only be the hundredth millioneth part of that of the earth, would be sufficient to render the real libration of this satellite sensible, which however is not perceived by observations. This consideration, combined with those which we (*d*) have presented in the fourth chapter, ought to satisfy those astronomers who apprehend that the elements of their tables may be deranged by the action of these bodies.

The equality of the motions of rotation, and of revolution, furnishes the astronomer, who may wish to describe its surface, a universal meredian, (*e*) suggested by nature, and easy to be found at all times, an advantage which geography has not in the description of the earth. This meridian is that which passes through the Poles of the Moon, and through the extremity of its greater axis, always very nearly directed towards us. Although this extremity is not distinguished by any spot, still its position at each instant may be fixed, by considering that it coincides with the line of the mean nodes of the lunar orbit, when the line itself coincides with the mean place of the Moon. The situation of different spots of the Moon have been thus determined as exactly as that of many of the most remarkable places on the earth.

CHAP. XV.

Of the proper motions of the fixed stars.

After having considered the motions of the bodies composing the solar system, it remains to examine those of the stars, all of which ought in consequence of the universal gravitation of matter, to tend towards each other, and describe immense orbits. Already observations have indicated (*a*) these great motions, which probably in part arise from the motion of translation of the solar system, which motion, according to the laws of optics, is transferred in a contrary direction to the stars. When a great number of them are considered together, as their real motions have place in every direction, they ought to disappear in the expression of the motion of the Sun, which is inferred from a consideration of their proper observed motions taken collectively. By this means we have recognised that the system of the Sun, and of every thing which surrounds it, is carried towards the constellation Hercules, with a velocity at least equal to that (*b*) of the earth in its orbit. But very exact and multiplied observations, made for the interval of one or two centuries, will determine exactly this important and delicate point of the system of the world.

Besides these great motions of the Sun and of the Stars, we observe particular motions in several stars, which are called *double*. Thus two stars are termed, which being very near, appear to constitute but one, in telescopes whose magnifying power is inconsiderable. Their apparent proximity may arise from their being very nearly in the same visual ray. But a similar disposition is itself an index of their real proximity; and if moreover their proper motions are considerable, and differ little in right ascension and declination, it becomes extremely probable that they constitute a system of two bodies very near to each other, and that the small differences of their proper motions arise from a motion of revolution of each of them, about their common centre of gravity: without this, the simultaneous existence of these three circumstances, (*c*) namely the apparent proximity of these two stars, and their motions both in right ascension and declination being nearly equal, would be altogether improbable.

The 61^{me} of the swan and the star next to it, combine these three conditions in a remarkable manner: the interval which separates them is only 60″; their proper annual motions from the time of Bradley to the present day, have been 15″75, and 16″,03 in right ascension, and 10″,24 and 9″,56 in declination: it is therefore very probable that these two stars are very (*d*) near to each other, and that they revolve about their common centre of gravity in the period of several ages. The direction of their proper motions being almost contrary to that of the motion of the solar system, seems to indicate that they are at least in a great part an optical illusion due to this last motion; and as they are very considerable, the annual parallax of those two stars ought to be one of the greatest. If we could succeed in determining it, we would obtain by the time of their revolution, the one about the other, the sum of their (*e*) masses relatively to those of the Sun and of the Earth.

The contemplation of the heavens exhibits also several groups of brilliant stars comprised in a very small space; such is that of the Pleades. A like disposition indicates, with much probability, that the stars of each group are very near, relatively to the distance which separates them from the other stars, and that they have about their common centres of gravity, motions which the progress of time will make known.

BOOK THE FIFTH.

SUMMARY OF THE HISTORY OF ASTRONOMY.

Multi pertransibunt et augebitur scientia.

BACON.

THE principal phenomena of the system of the world have been detailed in the preceding books, according to the simplest and most direct analytical order. The appearances of the celestial motions were first considered, and then their mutual comparison conducted us to the discovery of the real motions which produced them. In order to arrive at the principal regulator of those motions, it was necessary to know the laws of the motion of matter, and accordingly, these have been developed in all their detail. By applying them to the bodies and motions observed in the solar system, it was ascertained that there exists not only between these bodies, but also between their smallest molecules, an attraction which varies as the respective masses divided (*a*) by the square of their mutual distance. Finally, proceeding in a reverse order, from this universal force to its effects, it was shewn that not only all the known phenomena, and also those *merely* perceived by astronomers, but likewise a great number of others *entirely* new, which subsequent observation has verified, arise from this source. This indeed is not the order according to which those results were discovered. The preceding method supposes that we have exhibited before our view the entire series of ancient and modern observations, and that in comparing them together, and in deducing from them, the laws of the heavenly motions, and the causes of their inequalities, we have employed all the resources which are now furnished by analysis and mechanics. But as our knowledge in these two departments of science has advanced concurrently with the improvements made in Astronomy, their condition at its various epochs, must necessarily have influenced our astronomical theories. Several hypotheses have been successively adopted, although directly contrary to the known laws of mechanics; but of many of those laws, even to this very day we are ignorant, so that it should not be a matter of surprize if, in consequence of this ignorance, difficulties have been raised against the true system of the world, interspersed as it is on all sides with such complicated phenomena. Henee the progress of our astronomical knowledge has been frequently embarrassed, and the evidence of our acquirement in this science has been rendered doubtful, from the truths with which it was enriched, being combined with errors, which nothing but time, observation, and the progress of the other sciences could separate from it. We proceed to give, in the following book, a summary of its history, and in this account we shall have occasion to observe how, after remaining for a long series of years, in its infancy, it sprung up and flourished in the Alexandrian school; that then it remained stationary, until the time of the Arabs, who improved and advanced it by their observations; and that, finally passing from Asia and Arabia, where it originated, it settled in Europe, where in less than three centuries it has obtained the eminence which it now holds among the sciences. This detail of the most sublime of the natural sciences will furnish the best excuse for the aberrations of the human mind in the invention of Astrology, which from the remotest antiquity has every where occupied the attention of ignorant and timid man, but which the improvements in this science have for ever dissipated.

CHAP. I.

Of the Astronomy of the Ancients, till the Foundation of the Alexandrian School.

THE view of the firmament must at all times have arrested the attention of mankind, and more particularly in those happy climates, where the serenity of the air invited them to observe the stars. Agriculture required, that the seasons should be distinguished and their returns known. It could not be long before it was discovered that the rising and setting of the principal stars, when they are immersed in the Sun's rays, or when they are again extricated from his light, might answer this purpose. Hence we find that among most nations, this species of observations may be traced back to such early times, that their origin is lost. But some rude remarks on the rising and setting of the stars, could not constitute a science. Astronomy did not commence till anterior observations being registered and compared, and the celestial motions examined with greater care, some attempt was made to explain their motions and their laws.

The motion of the Sun in an orbit inclined to the equator; the motion of the Moon, the cause of its phases and eclipses, the knowledge of the planets and their revolutions, and the sphericity of the Earth, were probably the objects of this ancient astronomy; but the few monuments, that remain of it, are insufficient to determine either its epoch or its extent. We can only judge of its great antiquity, by the astronomical periods which have come down to us, and which suppose a series of observations so much the longer, as they were more imperfect. Such has been the vicissitude of human affairs, that printing, the art, by which alone the events of past ages can be transmitted in a durable manner, being of modern invention, the remembrance of the first inventors in the arts and sciences has been entirely effaced. Great nations, whose names are hardly known in history, have disappeared, without leaving in their transit any traces of their existence.

The most celebrated cities of antiquity have perished with their annals, and the language itself which the inhabitants spoke; with difficulty can the scite of Babylon be recognised. Of so many monuments of the arts and of industry, which adorned their cities and passed for the wonder of the world, there only remains a confused tradition, and some scattered wrecks, of which the origin is for the most part uncertain, but of which notwithstanding the magnitude attests the power of the people who have elevated these monuments.

It appears that the practical astronomy of these early ages was confined to observations of the rising and setting of the principal stars, with their occultations by the Moon and planets, and of eclipses. The path of the Sun was followed, by means of the stars, the light of which was obscured by the twilights, and perhaps by the variations in the meridian shadow of the gnomon. The motion of the planets was determined by the stars which they came nearest to, in their course. To recognize all these stars and their various motions, the heaven was divided into constellations; and that celestial zone from which the Sun, Moon and planets were never seen to deviate, was called the Zodiac. It was divided into the twelve following constellations: Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricornus Aquarius and Pisces. These were called *signs*, because they served to distinguish the seasons. Thus the entrance of the Sun into Aries, in the time of Hipparchus, marked the commencement of the spring, after which it described the other signs, Taurus, Gemini, Cancer, &c. but the retrograde motion of the equinoxes changed, though slowly, the coincidence of the constellations with the seasons of the year, and at the æra of this great astronomer it was already very different from what it was at the origin of the zodiac; nevertheless since astronomy, according as it became more perfect, had need of signs to indicate the motion of the stars, they still continued as in the time of Hipparchus to denote the commencement of the spring by the entrance of the Sun into sign of Aries. Afterwards they distinguished the signs of the zodiac from the constellations, the first being ideal, and serving only to designate the course of the heavenly bodies. Now that we endeavour to refer our ideas to the most simple expressions, we no longer use the signs of the zodiac, but mark the positions of the heavenly bodies on the ecliptic, according to their distance from the equinoctial point.

The names given to the constellations of the zodiac were not assigned to them fortuitously; for they express relations which were the object of a great number of investigations and of systems. Some of these names appear to relate to the motion of the Sun. Cancer and Capricorn, for cxample, seem to indicate the retrogradation of this body from the solstices, and Libra denotes the equality of the day and night. The other names seem to refer to the climate and agriculture of those nations to whom the zodiac owes its origin. Capricorn, or the constellation of the goat, appears to be more properly placed at the highest than at the lowest point of the Sun's course. In this position, which goes backward fifteen *(b)* thousand years, the balance was at the equinox of spring; and the constellations of the zodiac had striking relations with the climate of Egypt and with its agriculture. All these relations would also subsist if the constellations of the zodiac, instead of being named from their rising with the Sun, or the commencement of the day, had been denominated from their setting, at the beginning of night; if, for example, the setting of libra had at this moment indicated the commencement of spring. The origin of the zodiac, which would not then go farther back than two thousand years before our æra, agrees much better than the preceding, with the little data which we possess of the antiquity of the sciences, and particularly of astronomy.

The Chinese are, of all people, those who furnish the most ancient astronomical observations. The first eclipses of which mention is made cannot be made use of in chronology, in consequence of the indeterminate manner in which they are detailed; notwithstanding this, these eclipses evince that when the Emperor Yao lived, which was more than two thousand years before our æra, astronomy was cultivated in China as the basis of their ceremonies. The calender, and the announcement of eclipses, were important objects for which a mathematical tribunal was created. At that period the length of the meridian shadows of the gnomon, at the time of the solstices, and the passage of the stars over the meridian, were measured; time was measured by means of clepsydræ, and the position of the Moon, with respect to the stars at the eclipses, was determined, which would give the sidereal positions of the Sun and of the solstices. They also constructed instruments for measuring the angular distances of the stars. From a combination of these means, the Chinese ascertained that the duration of the solar year exceeded, by a quarter of a day very nearly, three hundred and sixty-five days, and they fixed its commencement at the winter solstice. Their civil year was lunar, and in order to reduce it to the solar year, they made use of a period (*c*) of nineteen solar years, corresponding to two hundred, and thirty five lunations, which is exactly the same period as that which Calippus, sixteen centuries afterwards, introduced into the Grecian calendar. Their months consisted alternately of twenty-nine and thirty days, and their lunar year consisted of three hundred and fifty-four days; it was consequently shorter than their solar year by eleven days and a quarter; but in the year when the sum of these differences exceeded a lunation, they intercalated one month. They divided the equator into twelve immoveable signs, and into eighteen constellations, in which they carefully determined the position of the solstices. The Chinese, instead of a century, made use of a period of sixty years; and instead of a week, a period of sixty days; but this short cycle of seven days, which was in use throughout the entire east, was known to them from the most remote periods. The division of the circumference was always in China, subordinated to that of the length of the year, so that the Sun described exactly a degree every day; but the subdivisions of the degree, of the day, of weights, and of every kind of lunar measure, were decimal; and this precedent, furnished for upwards of four thousand years at least, by the most populous nation on the earth, evinces that these divisions, which besides offer so many advantages, may at length by use become extremely popular. The first observations which were useful to astronomy are those of Tcheou-Kong, whose memory is still held in the highest veneration in China, as one of the best princes who ever swayed the sceptre. Being brother of Ou Ouang, who founded the dynasty of Tcheou, he governed the empire after his death, during the minority of his nephew, from the year 1104 to the year 1098 before our æra. Confucius, addressed in the Chou-King, the book held in the highest veneration by the Chinese, through this great prince, to his pupil, the wisest maxims of government and morality. Tcheou-Kong himself, with his astronomers, made a great number of observations, three of which have fortunately come down to us, and they are of inestimable value, from their great antiquity. Two of them are about the meridian lengths of the gnomon, which were observed with the greatest care at the summer and winter solstice, in the town of Loyang; they assign an obliquity to the ecliptic, at this remote period, which perfectly corresponds to the the theory of universal gravitation. The other observation is relative to the position of the winter solstice in the heavens at the same epoch. It likewise answers to the theory, as far as can be expected from the means employed, to determine such a delicate element. This remarkable agreement does not permit us to doubt of the authenticity of these observations.

The burning of the Chinese books, commanded by the emperor Chi Hoanti, about the year 213, before our æra, destroyed all vestiges of the ancient methods of computing eclipses, and several interesting observations, so that in order to discover those which may be useful to the Atronomer, it is necessary to descend to four centuries after Tscheou-Kong, and to pass over to Chaldea. Ptolemy has transmitted several to us; the most ancient are three eclipses of the moon observed at Babylon in the years 719 and 720 before our æra, and which he made use of in determining the motions of the moon. Unquestionably, Hipparchus and he were not in possession of the most ancient, which were sufficiently accurate to be employed in these determinations, as their precision is always proportional to the interval which separates the extreme observations. This consideration should diminish our regrets on account of the loss of the Chaldean observations, which Aristotle, according to Porphyry, as cited by Simplicius, caused to be communicated by the interference of Callisthenes, and which went back to nineteen centuries before Alexander. But the Chaldeans could not discover, except after a long series of observations, the period of 6585 days and $\frac{1}{3}$, during which the moon makes 223 revolutions with respect to the sun, 239 anomalistic revolutions, and 241 revolutions with respect to its nodes. They added $\frac{4}{135}$ of the circumference, in order to obtain the sidereal revolution of the sun in this interval, which supposes that the length of the sidereal year is 365 days and $\frac{1}{4}$. Ptolemy, in recording this period, attributes it to the most ancient mathematicians; but the astronomer Geminus, who was cotemporary with Sylla, affirms that the Chaldeans discovered this period, and he explains the manner, in which they deduced from it, the diurnal motion of the moon, and the method by which they computed the lunar anomaly. His testimony should remove every doubt on the subject, if it be considered that the Chaldean *saros*, consisting of 223 lunar months, which brings back the moon to the same position with respect to the nodes, its perigee, and the sun, makes a part of the preceding period. Thus, the eclipses observed during one period, furnish a simple means of predicting those which ought to occur in subsequent periods. This period, and the ingenious manner in which they computed the principal lunar inequality, required a great number of observations, skilfully discussed; it is the most remarkable astronomical monument before the foundation of the Alexandrian school. The preceding is all we know with certainty respecting the Astronomy of a people whom all antiquity consider as the most advanced in the science of the heavens. The opinions of the Chaldeans respecting the system of the world have been various, as must necessarily be the case, concerning objects respecting which observation and theory had previously furnished so little information. However, some of their philosophers, more fortunate than others, or guided by juster views of the order and immensity of the universe, have thought that the comets were, like the planets, subject to motions regulated by immutable laws.

We have very little positive information respecting the Astronomy of the Egyptians. The exact direction of the faces (*d*) of their pyramids towards the four cardinal points, gives us a favourable notion of their mode of observing; but none of their observations have reached us. It is surprising that the astronomers of Alexandria were obliged to make use of the Chaldean observations, either because the record of the Egyptian observations had been lost, or that the Egyptians did not wish to communicate them, from a feeling of jealousy, which might excite the favour of the kings for the school which they had founded. Previously to this epoch the reputation of their priests had attracted to Egypt, the first philosophers of Greece. Thales, Pythagoras, Eudoxus and Plato, journeyed thither to acquire from them the knowledge with which they enriched their own country; and it is extremely probable that the school of Pythagoras is indebted to them for the sound notions which they taught respecting the constitution of the world. Macrobius expressly attributes to them the suggestion of the motions of Mercury and Venus about the sun. Their civil year consisted of three hundred and sixty five days, and they added at the end of each year five complimentary days called επαγομενα. But according to the ingenious remark of M. Fourrier, the observation of the heliacal rising of Sirius, the most brilliant of all the stars, might have taught them that the return of these risings would then be retarded each year by a fourth part of a day; and on this remark they founded (*e*) the Sothiac period of 1461 years, which would very nearly reduce their months and fetes to the same seasons. This period is renewed in the year 139 of our æra. If it had been preceded by a similar period, as every thing induces us to suppose, the origin of this anterior period would go back to an epoch when we may, with great probability, suppose that the Egyptians gave names to the signs of the zodiack, and when consequently their Astronomy was founded. They had observed, that in twenty-five of their years there were three hundred and nine returns of the moon to the sun, which assigns a very accurate value to the length of the month. Finally we may perceive, from what remains of their zodiacks, that they observed with great care the position of the solstices in the zodiacal constellations. According to Dion Cassius the week is due to the Egyptians. This period is founded on the most ancient system of Astronomy, which placed the Sun, the Moon, and the Planets in the following order of distances from the earth, commencing with the greatest; Saturn, Jupiter, Mars, the Sun, Venus, Mercury, the Moon: the successive parts of the series of days, divided respectively into twenty-four parts, were consecrated in the same order to these stars. Each day took the name of the star corresponding to its first part; the week is found in India among the Bramins with our denominations; and I am satisfied that the days denominated by them and by us in the same manner, correspond to the same physical instant. This period, which was made use of by the Arabians, by the Jews, the Assyrians, and throughout the entire East, is uninterruptedly renewed, and always the same, pervading all nations and changes of empires. It is impossible, among such a variety of nations, to ascertain which was its inventor; we can only affirm that it is the most ancient monument of astronomical knowledge. The civil year of the Egyptians consisted of 365 days; it is easy to perceive that if the name of its first day was assigned (*d*) to each year; the names of these years would be invariably those of the days of the week. It is thus that weeks of years might be formed, which was in use among the Hebrews, but which evidently belonged to a nation whose year was solar and consisting of 365 days.

The knowledge of astronomy appears to have constituted the basis of all the theogonies, the origin of which is thus explained in the simplest possible manner. In Chaldea and ancient Egypt, astronomy was only cultivated in their temples, and by priests, who made no other use of their knowledge than to consolidate the empire of superstition, of which they were the ministers. They carefully disguised it under emblems, which presented to credulous ignorance, heroes and gods, whose actions were only allegories of celestial phenomena, and of the operations of nature; allegories which the power of imitation, one of the chief springs of the moral world, has perpetuated to our own days, and mingled with our religious institutions. The better to enslave the people, they profited by their natural desire of penetrating into futurity, and invented astrology. Man being induced, by the illusions of his senses, to consider himself as ths centre of the universe, it was easy to persuade him, that the stars influenced the events of his life, and could prognosticate to him his future destiny. This error, dear to his self-love, and necessary to his restless curiosity, seems to have been co-eval with astronomy. It has maintained itself through a very long period, and it is only since the end of the last century, that our knowledge of our true relations with nature, has caused them to disappear.

In Persia and India, the commencement of astronomy is lost in the darkness which envelopes the origin of these people.

The Indian tables indicate a knowledge of astronomy cosiderably advanced, but every thing shews that it is not of an extremely remote antiquity. And here, with regret, I differ in opinion from a learned and illustrious astronomer, whose fate is a terrible proof of the inconstancy of popular favour, who, after having honoured his career by labours useful both to science and humanity, perished a victim to the most sanguinary tyranny, opposing the calmness and dignity of virtue, to the revilings of an infatuated people, of whom he had been once the idol.

The Indian tables have two principal epochs, which go back, one to the year 3102, the other to the year 1491 before our æra. These epochs are connected with the mean motions of the Sun, Moon, and planets, in such a manner, that setting out from the position which the Indian tables assign to all the stars at this second epoch, and reascending to the first by means of these tables, the general conjunction which they suppose at this primitive epoch, is found. Baillie, the celebrated astronomer, already alluded to, endeavours, in his Indian astronomy, to prove, that the first of these epochs is founded on observation. Notwithstanding all the arguments are brought forward, with that perspicuity he so well knew how to bestow on subjects the most absract, I am still of opinion, that this period was invented for the purpose of giving a common origin to all the motions of the heavenly bodies in the zodiac. Our last astronomical tables being rendered more perfect by the comparison of theory with a great number of observations, do not permit us to admit the conjunction supposed in the Indian tables; in this respect indeed they made much greater differences than the errors of which they are still susceptible, but it must be admitted that some elements in the Indian astronomy have not the magnitude which they assigned to them, until long before our æra; for example, it is necessary to ascend 6000 years back to find the equation of the centre of the Sun. But, independently of the errors to which the Indian observations are liable, it may be observed, that they only considered the inequalities of the Sun and Moon relative to eclipses, in which the annual equation of the Moon is added to the equation of the centre of the Sun, and augments it by a quantity which is very nearly the difference between its true value and that of the Indians. Many elements, such as the equations of the centre of Jupiter and Mars, are very different in the Indian tables from what they must have been at their first epoch.

A consideration of all these tables, and particularly the impossibility of the conjunction, at the epoch they suppose, prove, on the contrary, that they have been constructed, or at least rectified in modern times. This also may be inferred from the mean motions which they assign to the Moon, with respect to its perigee, its nodes, and the Sun, which being more rapid than according to Ptolemy indicate that they are posterior to this astronomer, for we know, by the theory of universal gravitation, that these three motions have accelerated for a great number of ages. Thus this result of a theory so important for lunar astronomy, throws great light on chronology. Nevertheless, the ancient reputation of the Indians does not permit us to doubt, but that they have always cultivated astronomy.

When the Greeks and Arabs began to devote themselves to sciences, they drew their first elements from India. It is there that the ingenious manner of expressing all numbers in ten characters originated, by assigning to them at once an absolute and a local value, a subtle and important conception, of which the simplicity is such that we can with difficulty, appreciate its merit. But this very simplicity and the great facility with which we are enabled to perform our arithmetical computations place it in the very first rank of useful inventions; the difficulty of inventing it will be better appreciated if we consider that it escaped the genius of Archimedes and Appollonius, two of the greatest men of antiquity.

The Greeks did not begin to cultivate astronomy till a long time after the Egyptians, of whom they were the disciples.

It is extremely difficult to ascertain the exact state of their astronomical knowledge, amidst the (*e*) variety of fable which fills the early part of their history. Their numberless schools for philosophy produced not one single observer, before the foundation of the Alexandrian school. They treated astronomy as a science purely speculative, often indulging in the most frivolous conjectures.

It is singular, that at the sight of so many contending systems, which taught nothing, the simple reflection, that the only method of comprehending nature is to interrogate her by experiment, never occurred to one of these philosophers, though so many were endowed with an admirable genius. But we must reflect, that as the first observations only presented insulated facts, little suited to attract the imagination, impatient to ascend to causes, they must have succeeded each other with extreme slowness. It required a long succession of ages to accumulate a sufficient number, to discover, among the various phenomena, such relations, which by extending themselves should unite with the interest of truth, that of such general speculations as the human understanding delights to indulge in.

Nevertheless, in the philosophic dreams of Greece, we trace some sound ideas, which their astronomers collected in their travels, and afterwards improved. Thales, born at Miletus, 640 years before our æra, went to Egypt for instruction: on his return to Greece he founded the Ionian school, and there taught the sphericity of the Earth, the obliquity of the ecliptic, and the true causes of the eclipses of the Sun and Moon; he even went so far as to predict them, employing no doubt the periods which had been communicated to him by the priests of Egypt.

Thales had for his successors—Anaximander, Anaximenes, and Anaxagoras; to the first is attributed the invention of the gnomon and geographical charts, which the Egyptians appear to have been already acquainted with.

Anaxagoras was persecuted by the Athenians for having taught these truths of the Ionian school. They reproached him with having destroyed the influence of the gods on nature, by endeavouring to reduce all phenomena to immutable laws. Proscribed with his children, he only owed his life to the protection of Pericles, his disciple and his friend, who succeeded in procuring a mitigation of his sentence, from death to banishment. Thus, truth, to establish itself on earth, has almost always had to combat established prejudices, and has more than once been fatal to those who have discovered it. From the Ionian school arose the chief of one more celebrated. Pythagoras, born at Samos, about 590 years before Christ, was at first the disciple of Thales. This philosopher advised him to travel into Egypt, where he consented to be initiated into the mysteries of the priests, that he might obtain a knowledge of all their doctrines. The Brachmans having then attracted his curiosity, he went to visit them, as far as the shores of the Ganges. On his return to his own country, the despotism under which it groaned, obliged him again to quit it, and he retired to Italy, where he founded his school. All the astronomical truths of the Ionian school, were taught on a more extended scale in that of Pythagoras; but what principally distinguished it, was the knowledge of the two motions of the earth, on its axis, and about the Sun. Pythagoras carefully concealed this from the vulgar, in imitation of the Egyptian priests, from whom, most probably, he derived his knowledge; but his system was more fully explained, and more openly avowed by his disciple Philolaus.

According to the Pythagoricians, not only the planets, but the comets themselves, are in motion round the Sun. These are not fleeting meteors formed in the atmosph[illegible] the eternal works of nature. Th[illegible] perfectly correct, on the system of the universe, have been admitted and inculcated by Seneca, with the enthusiasm which a great idea, on a subject the most vast of human contemplation, ought naturally to excite in the soul of a philosopher.

" Let us not wonder," says he " that we are " still ignorant of the law of the motion of comets, " whose appearance is so rare, that we can nei- " ther tell the beginning nor the end of the revo- " lution of these bodies, which descend to us from " an immense distance. It is not fifteen hundred " years since the stars have been numbered in " Greece, and names given to the constellations. " The day will come, when, by the continued " study of succesive ages, things which are now " hid, will appear with certainty, and posterity " will wonder that they have escaped our notice."

In the same school, they taught that the planets were inhabited, and that the stars were suns, distributed in space, being themselves centres of planetary systems. These philosophic views ought from their grandeur and justness, to have obtained the suffrages of antiquity ; but having been taught combined with systematic opinions, such as the harmony of the heavenly spheres, and wanting, moreover, that proof which has since been obtained, by the agreement with observations, it is not surprising that their truth, when opposed to the illusions ofthe senses, should not have been admitted.

The only observation which the history of Grecian Astronomy furnishes us with, previously to the foundation of the school of Alexandria, is that of the solstice of the summer of the year 432, before our æra, by Meton and Euclemon. The former of these Astronomers is celebrated for the cycle of nineteen years, which he introduced into the calendar, corresponding to the two hundred and thirty-five lunations already mentioned. The simplest method of measuring time, is that which makes use of solar revolutions, but in the infancy of society, the phases of the moon presented to their ignorance so natural a division of time, that it was universally adopted. They regulated their fetes and games by the return of those phases, and when the necessities of agriculture compelled them to have recourse to the sun, in order to distinguish the seasons, they did not give up the old custom of measuring time by the revolutions of the moon, the age of which may be thus determined by the days of the month. They endeavoured to establish between the revolutions of this star and those of the sun, an agreement depending on the number of periods, which contain entire numbers of these revolutions. The simplest is that of nineteen years. Meton therefore established this cycle of nineteen years, of which twelve were common, or consisting of twelve months, the seven remaining consisted of thirteen. These months were unequal, and so constituted, that in two hundred and thirty-five months of this cycle, one hundred and ten contained twenty-nine days, and one hundred and twenty-five thirty days. This arrangement was proposed by Meton to the

Greeks assembled to celebrate the Olympic games, and was unanimously adopted. But it was not difficult to perceive that at the end of each period, the new calendar retarded about the fourth part of the day on the new moon. Calippus proposed to quadruple the cycle of nineteen years, and to form a period consisting of seventy-six years, at the termination of which one day was to be subtracted. This period was denominated the Calippean, from the name of its inventor; and although not so ancient as the *Saros* of the Chaldeans, it is inferior to it in accuracy. About the time of Alexander, Pythias rendered Marseilles, his country, celebrated by his works as an Astronomical Geographer. We are indebted to him for an observation on the meridian length of the gnomon in this town, at the summer solstice; it is the most ancient observation of this kind after that of Tscheou-Kong. And it is extremely important, in as much as it confirms the continued diminution of the obliquity of the ecliptic. It is to be regretted that the ancient Astronomers did not make a greater use of the gnomon, which produces much more accuracy than their armillæ. By taking some easy precautions to level the surface on which the shade is projected, they might have left us observations on the declinations of the sun and moon, which would be at this day extremely useful.

CHAP. II.

Of Astronomy, from the Foundation of the Alexandrian School to the Time of the Arabs.

Hitherto the practical astronomy of different people has only offered us some rude observations relative to the seasons and eclipses; objects of their necessities or their terrors. Their theoretical astronomy consisted in the knowledge of some periods, founded on very long intervals of time, and of some fortunate conjectures, relative to the constitution of the universe, but mixed with considerable error. We see, for the first time, in the school of Alexandria, a connected series of observations; angular distances were made with instruments suitable to the purpose, and these were calculated by trigonometrical methods. Astronomy then assumed a new form, which the following ages have adopted and brought to perfection. The positions of the fixed stars were determined with more accuracy than before, the paths of the planets were carefully traced, the inequalities of the Sun and Moon were better known, and, finally, it was the school of Alexandria that gave birth to the first system of astronomy that ever comprehended an entire series of celestial pheuomena. This system was, it must be allowed, very inferior to that of the school of Pythagoras, but being founded on a comparison of observations, it afforded, by this very comparison, the means of rectifying itself, and of ascending to the true system of nature, of which it was an imperfect sketch.

After the death of Alexander, his principal generals having divided his empire among themselves, Ptolemy Soter received Egypt for his share. His munificence, and love of the sciences, attracted to Alexandria, the capital of his kingdom, a great number of the most learned men of Greece. Ptolemy Philadelphus, who inherited, with the kingdom, his father's love of the sciences, established them there under his own particular protection. A vast edifice, in which they were lodged, contained both an observatory and that magnificent library, which Demetrius Phalereus had collected with such trouble and expence. Being supplied with whatever books and instruments were necessary to their pursuits, they devoted themselves without distraction to their studies; and their emulation was excited by the presence of a prince, who often came amongst them to participate in their conversation and their labours. The impulse given to the sciences by this school, and the great men which it produced, or which were cotemporary with them, constitutes the epoch of the Ptolemies one of the most memorable in the history of the human mind.

Arystillus and Thimocares were the first observers of the Alexandrian school; they flourished about the year 300 before the Christian æra. Their observations of the principal stars of the zodiac enabled Hipparchus to discover the precession of the equinoxes, and served as the basis of a theory which Ptolemy gave of this phenomenon.

The next astronomer which the school of Alexandria produced, was Aristarchus of Samos. The most delicate elements of astronomy were the subjects of his investigation, unhappily they have not come down to us. The only one of his works which remains is his Treatise *on the magnitudes and distances of the Sun and of the Moon*, where he gives an account of the ingenious manner in which he endeavoured to determine the ratio of these distances. Aristarchus measured the angle contained between the Sun and the Moon, at the moment he judged half of the lunar disk to be illuminated by the Sun, at this instant the visual ray drawn from the eye of the observer to the centre of the Moon is perpendicular to the line which joins the centre of the Moon and Sun, and having found the angle of the observer smaller than a right angle by about the thirtieth part of this angle, he concluded that the Sun was nineteen times farther from us than the Moon. Notwithstanding the inaccuracy of this result, it extended the boundaries of the universe much farther than had been done before. In this treatise Aristarchus supposes the apparent diameters of the Sun and Moon, equal to each other, and to the 180th part of the circumference, which value is much too great; but he afterwards corrected this error, as we learn from Archimede that he made the diameter of the Sun equal to about the 720th part of the zodiac, which is a mean between the limits which Archimede himself, a few years afterwards, assigned by a very ingenious process to this diameter. This correction was unknown to Pappus, a celebrated geometer of Alexandria, who lived about the fourth century, and commented on the treatise of Aristarchus. This induces us to apprehend that the burning of a considerable part of the library of Alexandria during the siege which Cesar sustained in this city, had already distroyed the greater part of the writings of Aristarchus, and also a number of other works equally precious. Aristarchus revived the opinion of the Pythagoricians, relative to the motion of the Earth. But as his writings have not been transmitted to us, we are ignorant to what extent he carried this theory in his explanation of the celestial phenomena. We only know, that this judicious astronomer, from the consideration that the motion of the Earth produced no change in the apparent position of the stars, placed them at a distance incomparably greater than the Sun. Thus it appears, that of all the ancient astronomers, Aristarchus had formed the most just notions of the magnitude of the universe. They have been transmitted to us by Archimede in his Treatise on the *Arenarea*. This great geometer had discovered the means of expressing all numbers, by conceiving them formed of successive periods of myriads of myriads, the units of the first being simple units; those of the second being myriads of myriads, and so on. He denoted the parts of each period by the same characters as the Greeks employed, as far as an hundred millions. In order to evince the advantage of this method, Archimede proposed to express the number of grains of sand which the celestial sphere could contain, a problem of which he increased the difficulty by selecting the hypothesis which assigns to this sphere the greatest extent: it is with this view, that he adduces the opinion of Aristarchus.

The celebrity of his successor, Eratosthenes, is principally due to his measure of the Earth, and of the obliquity of the ecliptic. It is probable that the measurement of the earth was undertaken a long time before, but there only remained of these observations some evaluations of the terrestrial circumference, which it was sought by some approximations, more ingenious than certain, to reduce to the same value, very nearly agreeing with the result of modern observations. Having, at the summer solstice, remarked a deep well, whose whole depth, was illuminated by the Sun, at Syene, in Upper Egypt, he compared this with the altitude of the Sun, observed at the same solstice at Alexandria. He found the celestial arc, contained between the zeniths of these two places, equal to the fiftieth part of the whole circumference; and as their distance was estimated at five hundred stadia, he fixed at two hundred and fifty thousand stadia, the length of the whole terrestrial circumference. It is not at all probable that for such an important result, this astronomer would be content with the rough observation of a well illuminated by the Sun. This consideration, and the account given by Cleomedes, authorises us to suppose that he made use of observations of the meridian lengths of the gnomons at the summer and winter solstices at Syene and Alexandria. This is the reason why the celestial arc between these two places, as determined by him, differs little from the results of modern observations. Eratosthenes erred in supposing that Syene and Alexandria existed under the same meridian; he also erroneously supposed that the distance between these two cities was only five thousand stadia, if the stadium which he most probably employed contained three hundred cubits of the nilometer of Elephantinus. Then the two errors of Eratosthenes would be very nearly compensated, which would lead us to conclude that this astronomer only employed a measure of the earth, formerly executed with great care, the origin of which was lost.

The observation of Erastothenes on the obliquity of the ecliptic, is very valuable, inasmuch as it confirms the diminution of it, determined *à priori*, by the theory of gravitation. He found the distance between the tropics equal to eleven parts of the circumference, divided into eighty-three parts. Hipparchus and Ptolemy found no reason to alter this result by new observations. It is remarkable, that if we suppose, with the Alexandrian astronomers, the latitude of this city equal to thirty-one sexagesimal degrees; this measure of the obliquity places Syene exactly under the tropic, agreeably to the opinion of antiquity.

But of all the astronomers of aniquity, the science is most indebted to Hipparchus of Nice, in Bithynia, for the great number and extent of his observations, and by the important results he ohtained, from a comparison of them with those that had been formerly made by others; and for the excellent method which he pursued in his researches. He flourished at Alexandria in the second century before our æra. Ptolemy, to whom we are principally indebted for a knowledge of his work, and who recurs always to his observations and his theorems, pronounces him, with justice, an astronomer of great skill, of rare sagacity, and a sincere friend of truth. Not content with what had already been done, Hipparchus determined to recommence every thing, and not to admit any results but those founded on a new examination of former observations, or on new observations, more exact than those of his predecessors.

Nothing affords a stronger proof of the uncertainty of the Egyptian and Chaldean observations on the Sun and stars, than the circumstance of his being compelled to recur to the observations of the Alexandrian school, to establish his theories of the Sun, and of the precession of the equinoxes. He determined the length of the tropical year, by comparing one of his observations of the summer solstice with one made by Aristarchus of Samos, 381 years before our æra. This duration appeared to him less than the year of 365¼ days, which had been hitherto adopted, and he found that at the end of three centuries we should subtract one day. But he remarks himself on the little reliance that can be placed on a determination from solstitial observations, and on the advantage of employing observations of the equinoxes. Those which he made in an interval of nearly thirty-three years led him to the same result very nearly. Hipparchus recognized also that the two intervals from one equator to another, were unequally divided by the solstices, so that 94 days and a half elapse from the vernal equinox to the summer solstice, and 92 days and a half from this solstice to the autumnal equinox.

To explain these differences, Hipparchus supposed the Sun to move uniformly in a circular orbit; but, instead of placing the Earth in the centre he supposed it removed to the twenty-fourth part of the radius from the centre, and fixed the apogee at the sixth degree of Gemini. From these data he formed the first solar tables to be found in the History of Astronomy. The equation of the centre, which they suppose, was too great; and it is very probable, that a comparison of the eclipses, in which this equation is augmented by the annual equation of the Moon, confirmed Hipparchus in his error, or perhaps even led him into it. For this error, which surpasses a sixth of the entire value of the equation, is reduced to one sixteenth of this value in the computation of these phenomena. He was mistaken also in supposing the orbit of the Sun, which is really elliptical, to be circular, and that the real velocity of this body was constantly uniform. The contrary is now demonstrated by direct measures of the Sun's apparent diameter; but such observations were impossible at the time of Hipparchus, whose solar tables, with all their imperfections, are a lasting monument of his genius and which Ptolemy so respected, that he subjected own observations to them.

This great Astronomer next considered the motions of the moon. He determined, by a comparison of a great number of eclipses, selected in the most favourable circumstances, the durations of their revolutions relatively to the stars, to the sun, to its nodes, and to its apogee. He found that an interval of $126007^{d}\frac{1}{24}$ contained 4267 months, 4573 returns of the anomaly, 4612 sidereal revolutions of the moon minus $\frac{15}{720}$ of the circumference. He found moreover, that in 5458 months, the moon returns 5923 times to the same node of its orbit. These results are perhaps the most precious of ancient astronomy from their accuracy, and becaus they represent at this epoch the perpetually variable durations of its revolutions (Note IV). Hipparchus determined also the excentricity of the lunar orbit and its inclination to the ecliptic, and he found them very nearly the same as those which have now place in eclipses, in which we know that the one and the other of these elements are diminished by the evection, and the great inequalities of the motion of the moon in latitude. This constancy of the inclination of the lunar orbit to the plane of the ecliptic, notwithstanding the variations which this plane experiences relatively to the stars, and which by the ancient observations are sensible on its obliquity to the equator, is, as we have seen in the fourth book, a result of universal gravitation which the observations of Hipparchus confirm. Finally, from the determination of the parallax of the moon, he endeavoured to conclude that of the Sun, by the breadth of the cone of the terrestrial shadow, (*a*) in an eclipse at the moment it was traversed by the Moon, which led him nearly to the same result as had been obtained by Aristarchus. He made a great number of observations on the planets; but too much the friend of truth to explain their motions by uncertain theories, he left the task of this investigation to his successors. A new star which appeared in his time induced him to undertake a catalogue of the fixed stars, to enable posterity to recognize any changes that might take place in the appearances of the heavens. He was sensible also of the importance of such a catalogue for the observations of the Moon and the planets. The method he employed was that of Arystillus and Timochares, which we have already explained in the third chapter of the First Book. The reward of this long and laborious task, was the important discovery of the precession of the equinoxes; in comparing his observations with those astronomers, he discovered that the stars had changed their situation with respect to the equator; but had preserved the same latitude with respect to the ecliptic; he at first supposed that this was only true for the stars situated in the zodiack, but having observed that they all preserve the same relative position, he concluded that this phenomenon was general. To explain these different changes, he assigned a direct motion to the celestial sphere round the poles of the ecliptic, which produces a retrograde motion in longitude of the equinoxes with respect to the stars, which appeared to him to be for each century the three hundred and sixtieth part of the zodiack. But he announced his discovery with some reserve, being doubtful of the accuracy of the observations of Arystillus and Timochares. Geography is indebted to Hipparchus for the method of determining places on the Earth, by their latitude and longitude, for which he first employed the eclipses of the Moon. Spherical trigonometry, also, owes its origin to Hipparchus, who applied it to the numberless calculations which these investigations required. His principal works have not been transmitted to us, and we are only acquainted with them through the Almagest of Ptolemy, who has transmitted to us the principal elements of the theories of this great Astronomer, and some of his observations. Their comparison with modern observations having shewn their accuracy, and their use even to astronomers at the present day, makes us regret others, and particularly those which he made on the planets, of which there remains very few ancient observations. The only work of Hipparchus which has come down to us is a critical commentary on the sphere of Eudoxus, described in a poem of Aratus; it is anterior to the discovery of the precession of the equinoxes. The positions assigned to the stars on this sphere are so erroneous, and thay gave for the epoch of its origin such different results, that it is astonishing to see Newton establish on these imperfect positions a system of chronology, which besides deviates considerably from dates assigned with much probability to several ancient events. The interval of near three centuries, which separated these two astronomers, presents to us Geminus and Cleomedes, whose works have come down to us; and some observers, as Agrippa, Menelaus and Theon of Smyrna. We may also notice in this interval the reformation of the Roman calendar by Julius Cæsar, for which purpose he made Sosthenes come to Alexandria, and the precise knowledge of the ebbing and flowing of the sea. Possidonius observed the law of this phenomenon, which appertains to astronomy by its evident relation to the motion of the Sun and Moon, and of which Pliny the naturalist has given a description remarkable for its exactness.

Ptolemy, born at Ptolemais in Egypt, flourished at Alexandria about the year 130 of our æra. Hipparchus had given, by his numerous works, a new face to Astronomy, but he left to his successors the care of rectifying his theorems by new observations, and of establishing those which were deficient. Ptolemy continued this labour, and has given a treatise on this science in his great work entitled the Almagest.

His most important discovery is that of the evection of the Moon. Astronomers previously had only considered the motion of this body relatively to eclipses; in which it was solely sufficient to have regard to the equation of the centre, especially if we suppose with this astronomer that the equation of the centre of the Sun is greater than its true value, which in part replaces the annual equation of the Moon. It appears that Hipparchus had recognized that this did not represent the motion of the moon in its quadratures, and that observations presented great anomalies in this respect. Ptolemy carefully followed these anomalies, determined its law and fixed its value with great accuracy. In order to represent it, he supposed the moon to move on an epicycle carried by a moveable excentrick, of which the centre revolved about the earth in a contrary direction to the motion of the epicycle.

It was a general opinion of the ancients, that the uniform circular motion being the most simple and natural, was necessarily that of the heavenly bodies. This error maintained its ground till the time of Kepler, and for a long time impeded him in his researches. Ptolemy adopted it, and, placing the Earth in the centre of the celestial motions, he endeavoured to represent their inequalities in this false hypothesis. Conceive to move on a circumference, of which the Earth occupies the centre, that of another circumference, on which moves that of a third, and so on, up to the last circumference, on which the body is supposed to move uniformly. If the radius of one of these circles surpasses the sum of the others, the apparent motion of the body round the Earth, will be composed of a mean uniform motion, and of several inequalities depending on the proportions these several radii, the motions of their centres, and of the Star, have to each other. By increasing their number, and giving them suitable dimensions, we may represent the inequalities of this apparent motion. Such is the most general manner of considering the hypothesis of epicycles and excentrics. For an excentric may be considered as a circle of which the centre moves about the earth with a greater or less velocity, and which vanishes if it is immoveable. The Geometers who preceded Ptolemy were occupied with the appearances of the motions of the planets on this hypothesis, and it appears in the Almagest that the great geometer Appollonius had already resolved the problem of their stations and retrogradations. Ptolemy supposed the Sun, Moon, and planets in motion round the Earth in this order of distances—the Moon, Mercury, Venus, the Sun, Mars, Jupiter, Saturn; each of the planets superior to the Sun, was moved on an epicycle, of which the centre described an excentrick about the earth, in a time equal to that of the revolution of a planet. The period of the motion of the star on the epicycle was that of the solar revolution; and it was always found in opposition to the Sun, when it attained the point of the epicycle which was nearest to the earth. Nothing in this system determines the actual magnitude of the circles and of the epicyles. Ptolemy had only occasion to know the ratio which the radius of each epicycle had to that of the circle described by its centre. In like manner he made each inferior planet, to move on an epicycle of which the centre described an excentrick about the earth; but the motion of this point was equal to the solar motion, and the planet described its epicycle in a time which in modern astronomy is that of the revolution of the Sun: the planet was always in conjunction with it when it arrived at the lowest point of its epicycle. Here also nothing determines the absolute magnitude of the circles and of the epicycles. Astronomers anterior to Ptolemy were divided in their opinions as to the position of Mercury and Venus; Ptolemy followed the most ancient opinion, and placed them below the Sun; others placed them above, and finally, the Egyptians made them move round it. It is singular, that Ptolemy does not mention this hypothesis, which is equivalent to making the excentrics of those two planets equal to the solar orbit. If moreover he supposed the epicycles of the superior planets equal and parallel to this orbit, his system would make all the planets to move about the Sun, while this star revolves about the earth, and then there was but one step to make, in order to arrive at the true system of the world. This manner of determining the arbitrary quantities in Ptolemy's system, by supposing the circles and epicycles described in an annual motion equal to the solar orbit, renders the agreement of this motion with that of the Sun evident. By thus modifying this system, he can exhibit the mean distances of the planets from this star, in parts of its distance from the earth; for those distances are the ratios of the radii of the excentricks to those of the epicycles for the superior planets, and of the radii of the epicycles, to the radii of the excentricks for the two inferior. Such a simple and natural modification of the system of Ptolemy escaped all astronomers till the time of Copernicus. None of them appeared to have sufficiently considered the relations which subsist between the geocentrick motion of the planets and that of the Sun, to have investigated its cause; none of them were curious to know their respective distances from the Sun and Earth, they were content with connecting by new observations the elements determined by Ptolemy, without making any change in his hypothesis. But even, if by means of epicycles we could represent the inequalities of the motions of the heavenly bodies, still it would be impossible to represent the variations in their distances. In the time of Ptolemy, these variations were almost insensible in the planets, whose apparent diameters could not then be measured. But his observations on the Moon should have taught him that his hypothesis was erroneous, according to which the perigean diameter of the Moon in the quadratures, should be double of the apogean diameter in the sysigies. Besides every new inequality which the improvements in the art of observing discovered, incumbered this system with an additional epicycle, which, instead of being confirmed by the progress of the science, has only grown more and more complicated; and this should convince us, that it is not that of nature. But in considering it as a method of adapting the celestial motions to calculation, this first attempt of the human understanding towards an object so very complicated, does great honour to the sagacity of its author. Such is the weakness of the human understanding, that it frequently requires the aid of hypotheses to connect phenomena together, and to determine their laws; and if hypotheses are restricted to this use, by avoiding to ascribe any reality to them, and by restifying them perpetually by new observations, we arrive finally at the true causes, or at least we can supply them, and conclude from the observed phenomena those which given circumstances ought to develope. The history of philosophy furnishes us with several examples which hypotheses may procure in this point of view, and of the errors to which it is exposed when they are realized.

Ptolemy confirmed the motion of the equinoxes discovered by Hipparchus, by comparing his observations with those of this great astronomer. He established the respective immobility of the Stars, their invariable latitude to the ecliptic, and their motion in longitude, which he found conformable to what Hipparchus had suspected.

We now know that this motion is much more considerable; which circumstance, considering the interval between the observations of Ptolemy and Hipparchus, implies an error of more than one degree in their observations. Notwithstanding the difficulty which attended the determination of the longitude of the Stars, when observers had no exact measure of time, we are surprised that so great an error should have been committed, particularly when we observe the agreement of the observations with each other, which Ptolemy cites as a proof of the accuracy of his result. He has been reproached with having altered them, but this reproach is not well founded; his error, in the determination of the motion of the equinoxes, seems to have been derived from too great confidence in the result of Hipparchus, relative to the length of the tropical year. In fact, Ptolemy determined the longitudes of the stars, by comparing them either with the Sun, or with the Moon, which was equivalent to a comparison with the Sun, since the synodical revolution of the Moon was well known by the means of eclipses. Now, Hipparchus having supposed the year too long, and consequently the motion of the Sun, with respect to the equinoxes, too slow, it is clear that this error diminished the longitudes of the Sun employed by Ptolemy. The motion in longitude, which he attributed to the Stars, ought to be increased by the arc described by the Sun in the time, equal to the error of Hipparchus in the length of the year, and then it comes out very nearly what it ought to be. The sidereal year being the tropical year increased by the time necessary for the Sun to describe an arc equal to the annual motion of the equinoxes, it is evident that the sidereal year of Hipparchus and of Ptolemy ought to differ from the true year; in fact the difference is only the tenth part of that which exists between their tropical year and ours.

This remark has led to the examination of another question. It has been generally believed, that the catalogue of Ptolemy, was that of Hipparchus, reduced to his time by means of a precession of one day in ninety years. This opinion is founded on the circumstance, that the constant error in longitude of his Stars, disappears when reduced to the time of Hipparchus. But the explanation which we have given of the cause of this error, justifies Ptolemy from the reproach which has been imputed to him, of having taken the merit of Hipparchus to himself; and it seems fair to believe him, when he asserts that he has observed all the Stars of his own catalogue, even to the stars of the sixth magnitude. He adds, at the same time, that he found very nearly the same position of the Stars, relatively to the ecliptic, as Hipparchus, and we are always more induced to think so, as Ptolemy continually endeavours to make his results approximate to those of this great astronomer, who was in fact a much more accurate observer.

Ptolemy inscribed on the temple of Serapis at Canoeum the principal elements of his system; this astronomical edifice subsisted near fourteen centuries, and now that it is entirely destroyed, his Almagest considered as a depositary of ancient observations, is one of the most precious monuments of antiquity. Unfortunately it contains but a small number of the observations anterior to his æra. The author only related those which were necessary to explain his theory. The astronomical tables being once formed, he judged it useless to transmit with them to posterity the observations which Hipparchus and he employed for this purpose, and his example has been followed by the Arabs and the Persians. The great collections of precious observations collected solely for themselves, and without any application to theories, belong to modern astronomy, and is one of the fittest means of rendering it perfect. Ptolemy has not rendered less service to geography, in collecting all the known longitudes and latitudes of different places, and laying the foundation of the method of projections, for the construction of geographical charts. He composed a great treatise on optics, which has not been preserved, in which he explained the astronomical refractions: he likewise wrote treatises on the several sciences of chronology, music, gnomonics, and mechanics. So many labours, and on such a variety of subjects, manifest a very superior genius, and will ever obtain him a distinguished rank in the history of science. On the revival of astronomy, when his system gave way to that of nature, mankind avenged themselves on him for the despotism he had so long maintained; and they accused Ptolemy of having appropriated to himself the discoveries of his predecessors. But the honourable mention which he makes of Hipparchus, whom he frequently cites to support his theories, fully justifies him from this charge. At the revival of letters among the Arabs, and in Europe, his hypotheses combining the attraction of novelty with the authority of antiquity, were generally adopted by minds desirous of knowledge, and who were anxious at once to obtain possession of that which antiquity had acquired after long labour. Their gratitude elevated Ptolemy too high, whom they afterwards too much depressed. The fame of Ptolemy has met with the same fate as that of Aristotle and Descartes. Their errors were no sooner reeognized, than a blind admiration gave way to an unjust contempt, for even in science itself, the most useful revolutions are not always exempt from passion and prejudice.

CHAP. III.

Of Astronomy from the time of Ptolemy to the period of its restoration in Europe.

The progress of astronomy in the school of Alexandria terminated with the labours of Ptolemy. This school continued to exist for five centuries, but the successors of Ptolemy and Hipparchus contented themselves with commenting on their works without adding to their discoveries. The phenomena of the heavens continued unobserved during a period of more than six hundred years. Rome, for a long time, the seat of valour, glory, and learning, did nothing useful to science. The consideration that was always attached by the republic to eloquence and military talents, attracted all talents to those pursuits: and science, offering no advantage, was necessarily neglected in the midst of conquests undertaken by ambition, and of internal commotions, in which liberty expired, and gave way to the despotism of the emperors. The division of the empire, the necessary consequence of its vast extent, brought on its fall, and the light of science, extinguished by the barbarians, was only again revived among the Arabians.

This people, actuated by a wild spirit of fanaticism, after having extended their religion and arms over a great part of the Earth, had no sooner reposed in peace, than they devoted themselves with ardour to letters and science.

It, however, was but a short time before that they destroyed their most beautiful ornament, by burning the famous library of Alexandria.

In vain the philosopher Philoponus exerted himself for its preservation. If these books, replied Omar, are conformable to the alcoran, they are useless; if they are contrary to it, they are detestable. Thus (*a*) perished this immense treasure of erudition and genius. Repentance and regret soon followed this barbarous execution, for the Arabians were not long before they perceived their irreperable loss, and that they had deprived themselves of the most precious fruits of their conquests.

About the middle of the eighth century, the caliph Almansor gave great encouragement to astronomy; but among the Arabian princes who distinguished themselves for their love of the sciences, the most celebrated in history was Almamoun, of the family of the Abassides and son of the famous Aaron-al-Rashid, so celebrated throughout Asia. Almamoun reigned in Bagdat in 814; having conquered the Greek emperor

Michael III., he imposed on him, as one condition of peace, that he should have delivered to him the best books of Greece ;—the Almagest was among the number ; he caused it to be translated into the Arabian language, and thus diffused the astronomical knowledge which had formerly acquired so much celebrity for the Alexandrian school. Not content with encouraging learned men by his liberality, he was himself an observer, and determined the obliquity of the ecliptic; he likewise caused a degree of the meridian to be measured on the vast plain of Mesopotamia. He did more still, he wished to render the science more perfect, and for this purpose he collected together several distinguished astronomers, who after making a great number of observations, published new tables of the Sun and Moon, more accurate than those of Ptolemy, and for a long time celebrated in the East, under the name of the *verified tables*. In this table, the solar perigee has the position which it ought to have, the equation of the centre of the Sun, which, according to Hipparchus, is considerably greater, is reduced to its true value; but this precision became then a source of error in the computation of the eclipses, in which the annual equation of the Moon, partly corrected the inaccuracy in the equation of the centre of the Sun, which was adopted by this astronomer. The duration of the tropical year is much more exact than that of Hipparchus, it is however too short by almost two minutes, but this error arises from this; that the authors of the verified table compared their observations with those of Ptolemy; it would have been nearly nothing if they had employed the observations of Hipparchus. This is also the reason why they supposed the precession of the equinoxes too great.

Almamon caused also to be measured, with great care, in the extensive plane of Mesopotamia, a terrestrial degree which he found equal to two hundred thousand five hundred cubits. This measurement exhibits the same uncertainty as that of Eratosthenes, relatively to the length of the modulus made use of. These measures cannot now interest us, unless their modulus is made known; but the errors to which these observations were then liable do not permit us to draw from thence the advantage, which can only result from the accuracy of modern operations, by means of which we can always find our measures if in the course of time their standards should alter.

The encouragement given to astronomy by this prince and his successors, produced a great number of astronomers, among whom Albategnius deserves to be placed the first. His Treatise on *The Science of the Stars* contains several interesting observations, and the principal elements of the theory of the Sun and Moon; they differ little from those of the astronomers of Almamon. His work being for a long time the only known treatise of Arabian astronomy, the advantageous changes which were made in the tables of Ptolemy have been attributed to him. But a precious fragment, extracted from the astronomy of Ebn. Junis, and translated by Caussin, evinces that these changes are due to the authors of the verified tables. Besides it has furnished us with precise and very accurate notions of the Arabian astronomy. Ebn. Junis, astronomer of Hakenn, caliph of Egypt, observed at Cairo about the year one thousand. He arranged a great treatise of astronomy, and constructed tables of the celestial motions, which were celebrated through the East for their accuracy, and which appear to have served as the foundation of tables formed afterwards by the Arabians and the Persians. We perceive in this fragment, from the age of Almanon to the time of Ebn. Junis, a long series of observations of eclipses, of equinoxes, of solstices, of conjunctions of planets, and of occultations of stars; observations important for the perfection of astronomical theories, inasmuch as they have enabled us to recognise the secular equation of the Moon, and have thrown considerable light on the great variations of the system of the world. (Note 5). These observations are still only a small part of those of the Arabian astronomers, of which the number has been prodigious. They perceived the inaccuracy of the observations of Ptolemy on the equinoxes, and by comparing their observations either together, or with those of Hipparchus, they determined very exactly the true length of the year; that of Ebn. Junis only exceeds ours by thirteen seconds, and it ought to exceed it by five seconds.

It appears by this work, and by the tables of several manuscripts existing in our libraries, that the Arabians were particularly occupied with the perfection of astronomical instruments, the treatises which they left on this subject shew the importance which they attached to it, which importance is confirmed by the accuracy of their observations. They also paid particular attention to the measure of time by clepsydræ, by immense solar dials, and also by the vibrations of the pendulum. Notwithstanding this, their observations of the eclipses exhibit the same uncertainty as those of the Greeks and of the Chaldeans; and their observations on the Sun and Moon are far from having over those of Hipparchus that superiority, which can compensate the advantage of the distance which separates us from this great astronomer. The activity of the Arabian astronomers is confined to observations: it is not extended to the investigation of new inequalities, and in this point of view they have added nothing to the hypotheses of Ptolemy. That lively curiosity which attaches us to phenomena till their laws and cause are perfectly known, is what characterises the learned of modern Europe. (Note 5.)

The Persians, after having for a long time submitted to the same sovereigns as the Arabians, and professing the same religion, about the middle of the eleventh century shook off the yoke of the Caliphs. About this time their calendar received a new form, by the care of the astronomer Omar Cheyam; it was founded on an ingenious intercalation, which consists in making in every thirty-three years, eight of them bissextile ; Dominick Cassini, at the end of the seventeenth century, suggested the adoption of this intercalation as more exact and simple than the Gregorean: not knowing that the Persians had for a long time employed it. In the thirteenth century Holagu Ilecoukan, one of their last sovereigns, assembled the most learned astronomers at Maragha, where he constructed a magnificent observatory, the direction of which he entrusted to Nassireddin. But no prince of this nation distinguished himself more for his zeal for Astronomy than Ulugh-Beigh, whom we ought to place in the first rank of great observers. He himself formed at Samarcand, the capital of his states, a new catalogue of the stars, and of the best astronomical tables which we had before Tycho Brahe. He measured in 1437, with a great instrument, the obliquity of the ecliptic, and his results, when corrected by refraction and the erroneous parallax which he employed, gives this obliquity greater by seven minutes than at the commencment of this century, which confirms its successive diminution.

The annals of China furnish us with the most ancient astronomical observations. They present to us also twenty-four centuries after, the most accurate observations which have been made previously to the restoration of Astronomy, and even before the application of the telescope to the quadrant of the circle. We have seen that the astronomical year of the Chinese commenced about the winter solstice, and that to fix its origin, they repeatedly observed the meridian shades of the gnomon near the solstices. Gaubil, one of the most learned and judicious Jesuit missionaries sent to this empire, has made us acquainted with a series of observations of this kind, which extend from the year 1100 before our æra, to 1280 years after. These indicate with great clearness the diminution of the obliquity of the ecliptic, which in this long interval has been the thousandth part of the circumference. Tsou-tchong, one the most skilful astronomers of China, by a comparison of the observations made at Nankin in 461, with those which were made at Loyang in the year 173, determined the magnitude of the tropical year more exactly than the Greeks had done, or even the astronomers of Almamon. He found it $365^{\text{days}}{\cdot}24282$, the same very nearly with that of Copernicus. While Holagu Ilecoukan made astronomy to flourish in Persia, his brother Cobelai, who in 1271 founded the dynasty of Yuun, granted the same protection to it in China: he named Cocheou King, the first of the Chinese Astronomers, chief of the tribunal of mathematicians. This great observer constructed instruments much more exact than those hitherto made use of; the most valuable of all being a gnomon of forty Chinese feet, terminated by a plate of brass which was vertical, and pierced by a hole of the diameter of a needle. It is from the centre of this opening that Cocheou King reckoned the height of the gnomon; he measured the shade to the centre of the image of the Sun. "Hitherto," says he, "the higher limb of the Sun has been observed, and the extremity of the shade can with difficulty be distinguished; besides the gnomon of eight feet, which has been constantly made use of, is too short. These reasons have induced me to use the gnomon of forty feet, and to take the centre of the image." Gaubel, from whom we have these details, has recommended to us several observations made from 1277 to 1286, and they are precious for their accuracy, and prove unquestionably the diminutions of the obliquity of the ecliptic, and of the excentricity of the earth's orbit, from that epoch to our days. Cocheou King determined with remarkable precision the position of the lunar solstice with respect to the Stars in 1280, he made it to coincide with the apogee of the Sun, which took place thirty years before. The length which he assigned to the year is exactly that of our Gregorian year.

The Chinese methods for the computations of eclipses are inferior to those of the Arabians and of the Persians; the Chinese have not profited by the knowledge acquired by these people, notwithstanding their frequent communications with them; they have extended to Astronomy itself their constant attachment for their ancient customs.

The history of America, before its conquest by the Spaniards, exhibits some traces of astronomy; for the most elementary notions of this science have been amongst all nations the first fruits of their civilization. The Mexicans had, instead of the week, a short period of five days. Their months were each twenty days, and eighteen of these months constituted a year, which commenced at the winter solstice, and to which they added five complementary days. There is reason to suppose they composed from the combination of one hundred and four years, a great cycle, in which they intercalated twenty-five days. This supposes a duration of the tropical year more exact than that of Hipparchus; and what is very remarkable, it is nearly the same as that of the astronomers of Almamon. The Peruvians and the Mexicans carefully observed the shades of the gnomon at the solstices and at the equinoxes; they had even elevated for this purpose columns and pyramids. However, when we consider the difficulty of obtaining such an exact determination of the length of the year, we are induced to think that it was not accomplished by them, and that they obtained it from the ancient continent. But from what people or by what means did they receive it? Wherefore, if they received it from the north of Asia, have they a division of time so different from those in use in this part of the world? These are questions which it appears impossible to determine. There exist in the numerous manuscripts which our libraries contain, several ancient observations yet unknown, which would throw great light on astronomy, and especially on the secular inequalities of the heavenly motions. To their discussion, the attention of the learned skilled in the eastern languages ought to be directed; for the great variations which have taken place in the system of the world are not less interesting to man than the revolutions of empires. Posterity, which can compare a long series of very exact observations with the theory of universal gravitation, will much more enjoy the agreeement of these results than we, to whom antiquity has left observations for the most part very uncertain. But those observations, subjected to a sound discussion, can at least, in part, compensate for the errors to which they are liable, and supply the place of exact observations; as in geography, in order to fix the position of places, we compensate for the want of astronomical observations, by comparing together the different relations of travellers. Thus, though the account which the series of observations from the ancient times to the present day presents to us, be very imperfect; still we may perceive, in a sensible manner, the variations of the excentricity of the orbit of the Earth, and of the position of its perigee; those of the secular motions of the Moon, with respect to its nodes, to its perigee, and to the Sun; finally, the variations of the elements of the orbits of the planets. The successive diminution of the obliquity of the ecliptic during a period of nearly three thousand years, is particularly remarkable in the comparison of the observations of Tcheou Kong, of Pytheas, of Ebn Junis, of Cocheou King, of Uleugh-Beigh, and of the moderns.

CHAP. IV.

Of Astronomy in modern Europe.

It is to the Arabians that modern Europe is indebted for the first rays of light that dissipated the darkness in which it was enveloped during twelve centuries. They have transmitted to us the treasure of knowledge which they received from the Greeks, who were themselves disciples of the Egyptians; but by a deplorable fatality the arts and sciences have disappeared among all these nations, almost as soon as they had communicated them.

Despotism has for a long period extended its barbarism over those beautiful countries where science first had its origin, so that those names which formerly rendered them so celebrated, are now utterly unknown to them.

Alphonso, king of Castille, was one of the first sovereigns who encouraged the revival of astronomy in Europe, but he was ill seconded by the astronomers, whom he had assembled at a considerable expense, and the tables which they published did not answer to the great cost they had occasioned.

Endowed with a correct judgment, Alphonso was shocked at the confusion of the circles and epicycles, in which the celestial bodies were supposed to move; he felt that the expedients employed by nature ought to be more simple. "*If the Deity,*" said he, "*had asked my advice, these things would have been better arranged.*" By these words, in which he was charged with impiety, he meant to express that mankind were still far from knowing the true mechanism of the universe.

In the time of Alphonso, Europe was indebted to the encouragement of Frederic II. Emperor of Germany, for the first Latin translation of the Almagest of Ptolemy, which was made from the Arabic version.

We are now arrived at that celebrated epoch when astronomy, emancipating itself from the narrow sphere in which it was hitherto confined, advanced by a rapid and continued progress to its present exalted eminence. Purbach, Regiomontanus, and Walterus, prepared the way to these prosperous days of the science, and Copernicus gave them birth by the fortunate explanation of the celestial phenomena, by means of the motion of the Earth on its axis, and round the Sun.

Shocked, like Alphonso, at the extreme complication of the system of Ptolemy, he tried to find among the ancient philosophers a more simple arrangement of the universe. He found that many of them had supposed Venus and Mercury to move round the Sun: that Nicetas, according to Cicero, made the Earth revolve on its axis, and by this means freed the celestial sphere from that inconceivable velocity which must be attributed to it to accomplish its diurnal revolution. He learnt from Aristotle and Plutarch that the Pythagoreans had made the Earth and planets move round the Sun, which they placed in the centre of the universe. These luminous ideas struck him; he applied them to the astronomical observations which time had multiplied, and had the satisfaction to see them yield, without difficulty, to the theory of the motion of the Earth. The diurnal revolution of the heavens was only an illusion due to the rotation of the Earth, and the precession of the equinoxes is reduced to a slight motion of the terrestrial axis. The circles, imagined by Ptolemy, to explain the alternate direct, and retrograde motions of the planets, disappeared. Copernicus only saw in these singular phenomena, the appearances produced by the combination of the motion of the Earth round the Sun, with that of the planets; and he concluded, from hence, the respective dimensions of their orbits, which, till then, were unknown. Finally, every thing in this system announced that beautiful simplicity in the expedients of nature, which delights so much when we are fortunate enough to discover them. Copernicus published it in his work, *On the Celestial Revolutions*; not to shock received prejudices, he presented it under the form of an hypothesis. "Astronomers," said he, in his dedication to Paul III., "being permitted to "imagine circles, to explain the motions of the "stars, I thought myself equally entitled to ex- "amine if the supposition of the motion of the "Earth would render the theory of these appear- "ances more exact and simple."

This great man did not witness the success of his work. He died suddenly, by the rupture of a blood vessel, at the age of seventy-one years, a few days after receiving the first proof. He was born at Thorn, in Polish Prussia, the 19th of February, 1473. After learning the Greek and Latin languages, he went to continue his studies at Cracovia. Afterwards, induced by his taste for astronomy, and by the reputation which Regiomontanus had acquired, he undertook a journey to Italy, where this science was taught with success: being greatly desirous to render himself illustrious in the same career, he attended the lectures of Dominique Maria, at Bologna. When arrived at Rome, his talents obtained him the place of professor, where he made several observations: he afterwards quitted this city, to establish himself at Fravenberg, where his uncle, then Bishop of Warmia, made him a canon. It was in this tranquil abode that, by thirty-six years of observation and meditation, he established his theory of the motion of the Earth. At his death he was buried in the cathedral of Fravenberg, without any pomp or epitaph; but his memory will exist as long as the great truths which he taught with a clearness that eventually dissipated the illusions of the senses, and surmounted the difficulties which ignorance of the laws of mechanics had opposed to them.

These truths had yet to vanquish obstacles of another kind, and which, arising from a respected source, would have extinguished them altogether, if the rapid progress of all the mathematical sciences had not concurred to support them.

Religion was invoked to destroy an astronomical system, and one of its defenders, whose discoveries did honor to Italy, was harassed by repeated prosecutions. Rethicus, the disciple of Copernicus, was the first who adopted his ideas; but they were not in great estimation till towards the beginning of the seventeenth century, and then they owed it principally to the labours and misfortunes of Galileo.

A fortunate accident had made known the most wonderful instrument ever discovered by human ingenuity, and which, by giving to astronomical observations a precision and extent hitherto unhoped for, displayed in the heavens new inequalities, and new worlds. Galileo hardly knew of the first trials of the telescope, before he bent his mind to bring it to perfection. Directing it towards the stars, he discovered the four satellites of Jupiter, which shewed a new analogy between the Earth and planets; he afterwards observed the phases of Venus, and from that moment he no longer doubted of its motion round the Sun. The milky way displayed to him an infinite number of small stars, which the irradiation blends to the naked eye, into a white and continued light; the luminous points which he perceived beyond the line which separated the light part of the Moon from the dark, made him acquainted with the existence and height of its mountains. Finally he observed the singular appearances occasioned by Saturn's ring, and by the spots and rotation of the Sun. In publishing these discoveries, he showed that they proved incontestibly the motion of the Earth; but the idea of this motion was declared heretical by a congregation of cardinals; and Galileo, its most celebrated defender, was cited to the tribunal of the inquisition, and compelled to retract this theory, to escape a rigorous prison.

One of the strongest passions in a man of genius, is the love of truth. Full of the enthusiasm which a great discovery inspires, he burns with ardour to disseminate it, and the obstacles which ignorance and superstition, armed with power, oppose to it, only stimulate and increase his energy; besides, the subject is of the highest importance to us, from the rank which it assigns to the globe which we inhabit. Galileo, more and more convinced by his own observations of the motion of the Earth, had long meditated a new work, in which he proposed to develope the proofs of it. But to shelter himself from the persecution of which he had escaped being the victim, he proposed to present them under the form of dialogues between three interlocutors, one of whom defended the system of Copernicus, combated by a Peripatetician. It is obvious that the advantage would rest with the defender of this system;

but, as Galileo did not decide between them, and as he gave as much weight as possible to the objections of the partisans of Ptolemy, he had a right to expect that tranquillity which his age and labours merited.

The success of these dialogues, and the triumphant manner with which all the difficulties against the motion of the Earth were resolved, roused the inquisition. Galileo, at the age of seventy, was again cited before this tribunal. The protection of the Grand Duke of Tuscany could not prevent his appearance. He was confined in a prison, where they required of him a second disavowal of his sentiments, threatening him with the punishment incurred by contumacy, if he continued to teach the system of Copernicus.

He was compelled to sign this formula of abjuration:

"*I Galileo, in the seventieth year of my age,* "*brought personally to justice, being on my knees,* "*and having before my eyes the holy evangelists,* "*which I touch with my own hands; with a sin-* "*cere heart and faith, I abjure, curse, and detest,* "*the error, and heresy, of the motion of the Earth,*" "&c.

What a spectacle! A venerable old man, rendered illustrious by a long life, consecrated to the study of nature, abjuring on his knees, against the testimony of his own conscience, the truth which he had so evidently proved. A decree of the inquisition condemned him to a perpetual prison. He was released after a year, at the solicitations of the grand duke; but, to prevent his withdrawing himself from the power of the inquisition, he was forbidden to leave the territory of Florence.

Born at Pisa, in 1564, he gave early indications of those talents which were afterwards developed. Mechanics owe to him many discoveries, of which the most important is the theory of falling bodies, the most splendid discovery of his genius.

Galileo was occupied with the libration of the Moon, when he lost his sight; he died three years afterwards, at Arcetre, in 1642, regretted by all Europe, which he left enlightened by his labours, and indignant at the judgment passed against so great a man, by an odious tribunal.

While this passed in Italy, Kepler, in Germany, developed the laws of the planetary motions. But, previous to the account of his discoveries, it is necessary to look back and to describe the progress of astronomy in the north of Europe, after the death of Copernicus.

The history of this science presents at this epoch a great number of excellent observers. One of the most illustrious was William IV., Landgrave of Hesse-Cassel. He had an observatory built at Cassel, which he furnished with instruments, constructed with care, and with which he observed a long time. He procured two celebrated astronomers, Rothman and Juste Byrge; and Tycho was indebted to his pressing solicitations for the favours which were conferred on him by Frederic King of Denmark.

Tycho Brahe, who was one of the greatest observers that ever existed, was born at Knucksturp, in Norway. His taste for astronomy was manifested at the age of fourteen years, on the occasion of an eclipse of the Sun, which happened in 1560. At this age, when reflection is so rare, the justice of the calculation which announced this phenomenon, inspired him with an anxious desire to know its principles; and this desire was still further increased by the opposition of his preceptor and family. He travelled to Germany, where he formed connexions of correspondence and friendship with the most distinguished persons, who pursued astronomy either as a profession, or amusement, and particularly with the Landgrave of Hesse-Cassel, who received him in the most flattering manner.

On his return to his own country, he was fixed there by his sovereign, Frederic, who gave him the little island of Huene, at the entrance of the Baltic. Tycho built a celebrated observatory there, which was called Uranibourg. There, during an abode of twenty-one years, he made a prodigious number of observations, and many important discoveries. At the death of Frederic, envy, then unrestrained, compelled Tycho to leave his retreat. His return to Copenhagen did not appease the rage of his persecutors; the Minister, Walchendorp, (whose name, like that of all men who have abused the power intrusted to them, ought to be handed down to the execration of posterity,) forbad him to continue his observations. Fortunately, Tycho found a powerful protector in the of the grand duke; but, to prevent his withdrawing himself from the power of the inquisition, he was forbidden to leave the territory of Florence.

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Emperor Rodolph II. who settled on him a considerable pension, and lodged him commodiously at Prague. He died suddenly at this city, on the 24th of October, 1601, in the midst of his labours, and at an age when astronomy might have expected great services from him.

The invention of new instruments, and great improvements made in the old ones, a much greater precision in observations; a catalogue of stars much more accurate than those of Hipparchus, and Ulugh Beigh; the discovery of that inequality of the Moon, which is called variation; that of the inequalities of the motion of the nodes, and of the inclination of the lunar orbit; the interesting remark, that the comets are beyond this orbit; a more perfect knowledge of astronomical refractions; finally, very numerous observations of the planets, which have served as the basis of the discoveries of Kepler, are among the principal services which Tycho Brahe has rendered astronomy. The accuracy of his observations, to which he was indebted for his discoveries on the lunar motion, shewed him also, that the equation of time with respect to the Sun and the planets is not applicable to the moon, and that the part depending on the anomaly of the Sun, and even a quantity much greater must be deducted from it. Kepler, carried away by his imagination to investigate the relations and the cause of those phenomena, thought that the moving virtue of the Sun caused the Earth to move more rapidly on itself in its perihelion than in its appelion. The effect of this variation of diurnal motion could not be recognized by the observations of Tycho, except in the motions of the Moon, where it is thirteen times more considerable than in that of the Sun. But clocks brought to perfection by the application of the pendulum, shew that this effect vanishes in this last motion, and that the rotation of the earth is uniform. Flamstead transferred to the Moon itself the inequality depending on the anomaly of the Sun, which he had regarded as only apparent. This inequality, which Tycho first perceived, is that which is termed the *annual equation*. By this example we may perceive how observations, in becoming more perfect, discover to us inequalities till then enveloped in errors. The researches of Kepler furnishes us also with a still more remarkable example. Having shewn in his commentary on Mars, that the hypothesis of Ptolemy necessarily differs from the observations of Tycho by eight sexagesimal minutes, he adds; "This difference is less than the uncertainty of the "observations of Ptolemy, which uncertainty, by "the confession of this Astronomer, is less than "ten minutes, but the divine goodness giving "us Tycho Brahe, a very exact observer, it is "meet to be grateful for the kindness of the divi- "nity, and to render him thanks for it. Being "now convinced of the error of the hypothesis "which we made use of, we ought to direct all "our efforts to the true laws of the heavenly mo- "tions. These eight minutes which I am no longer "permitted to neglect have enabl[illegible] form

" all Astronomy, and indeed constitute the mate-" rials of the greatest part of this work." Struck with the objections which the adversaries of Copernicus made to the motion of the Earth, and perhaps influenced by the vanity of wishing to give his name to an astronomical system, he mistook that of nature. According to him the Earth is immoveable in the centre of the universe; all the Stars move every day round the axis of the world; and the Sun, in its annual revolution, carries with it the planets. In this system, which ought, in the natural order of things, to precede that of Copernicus, the appearances are the same as in the theory of the motion of the Earth. For we may, in general, consider any point we choose, for example, the centre of the Moon as immoveable, provided that we assign the motion with which it is animated, to all the stars in a contrary direction.

But, is it not physically absurd to suppose the Earth immoveable in space, while the Sun carries along the planets in the midst of which the Earth is included? How could the distance from the Earth to the Sun, which agrees so well with the duration of its revolution in the hypothesis of the motion of the Earth, leave any doubt of the truth of this hypothesis in a mind constituted to feel the force of analogy? Ought we not to confess with Kepler, that nature proclaims with a loud voice the truth of this hypothesis. Indeed it must be admitted, that Tycho, though a great observer, was not fortunate in his research after causes; his unphilosophical mind had even imbibed the prejudices of astrology, which he tried to defend.

It would be, however, unjust to judge him with the same rigor, as one who should refuse at present to believe the motion of the Earth, confirmed by the numerous discoveries made in astronomy since that period.

The difficulties which the illusions of the senses opposed to this theory, were not then completely removed. The apparent diameter of the fixed stars, greater than their annual parallax, gave to these stars in this theory, a real diameter, greater than that of the terrestrial orbit. The telescope, by reducing them to luminous points, made this improbable magnitude disappear. It could not be conceived how these bodies, detached from the Earth, could follow its motion. The laws of mechanics have explained these appearances; they have proved, what Tycho, deceived by an erroneous experiment, had refused to admit, that a body, falling from a considerable height, and abandoned to the action of gravity alone, ought to fall very nearly in a vertical line, only deviating towards the east by a quantity difficult to estimate accurately by observation, from its minuteness, so that at present there is as much difficulty in proving the motion of the Earth by the direct experiment of a falling body, as formerly existed to prove that it should be insensible.

The reformation of the Julian calendar is also to be traced up to the æra of Tycho Brahe. It is convenient that the months and festivals should be attached to the same seasons, to make them to be remarkable epochs for agricul' to secure this inestimable advantage to the inhabitants of a country, it was necessary by the regular intercalation of a day, to compensate the excess of the solar year above the common year of three hundred and sixty-five days. The simplest mode of intercalation was that employed by Julius Cæsar; it consisted in making a bissextile to succeed three common years. But as the length of the year, which this intercalation supposes, was too great, the vernal equinox always preceded it, so that after the interval of fifteen centuries, it had approached to the commencement of the year by above eleven days and a half. In order to remedy this inconvenience, Pope Gregory decreed in 1582, that the month of October of that year should only consist of twenty-one days; that the year 1600 should be bissextile; and that henceforth for years which terminate each century, only three of them should be bissextile in four centuries. Even this intercalation assigns too great a length to the year, so that the equinox would anticipate it by about a day in four thousand years; but if the bissextile which terminates this interval is considered as a common year, the Gregorian intercalation would be very nearly correct. In other respects the Julian calendar has not been altered. It would then have been easy to fix the origin of the year, at the winter solstice, and to render the length of the months more uniform, by assigning thirty-one days to the first, and twenty-nine days to the second month in common years, and thirty days in bissextile years. and by making the other months alternately thirty and thirty-one days; it would have been convenient also to denote them by their numerical order, which would have done away with the improper denominations of the four last months of the year. If then this intercalation adopted by Gregory was corrected in the manner specified above, the Gregorian calendar would be as perfect as could be desired. But is it requisite to give it all this perfection? When it is considered that this calendar has been now adopted by almost every nation in Europe and America, and that it required two centuries, and all the influence of religion, to secure to it this advantage, it will be immediately apparent that it ought to be retained, notwithstanding some imperfections which attach to it, and which, it may be observed besides, are comparatively of trifling importance. For the principal object of a calendar is, to connect by a simple mode of intercalation events to the series of days, and to make the seasons for a great number of years to correspond to the same months of the year, which conditions are sufficiently well secured in the Gregorian calendar. As the part of this calendar which refers to the fixing of Easter is foreigu to the science of Astronomy, I have not adverted to it here.

Towards the close of his life, Tycho Brahe had Kepler for a disciple and assistant. He was born in 1571, at Viel, in the duchy of Wirtemberg, and was one of these extraordinary men whom nature grants now and then to the sciences, to bring to light those grand theories which have been prepared by the labour of many centuries.

The career of the sciences did not appear to him adequate to satisfy the ambition he felt of rendering himself illustrious; but the ascendancy of his genius, and the exhortations of Mæstlin, led him to astronomy; and he entered into the pursuit with all the avidity of a mind passionately desirous of glory.

The philosopher, endowed with a lively imagination, and impatient to know the causes of the phenomena which he sees, often obtains a glimpse of them, before observation can conduct him to them. Doubtless he might, with greater certainty, ascertain the cause from the phenomena; but the history of science proves to us, that this slow progress has not always been that of inventors.

What rocks has he to fear, who takes his imagination for his guide!

Prepossessed with the cause which it presents to him, instead of rejecting it when contradicted by facts, he alters them to make them agree with his hypotheses; he mutilates, if I may be allowed the expression, the work of nature, to make it resemble his imagination, without reflecting that time destroys with one hand these vain phantoms, and with the other confirms the results of calculation and experience.

The philosopher who is really useful to the cause of science, is he, who, uniting to a fertile imagination, a rigid severity in investigation and observation, is at once tormented by the desire of ascertaining the cause of the phenomena, and by the fear of deceiving himself in that which he assigns.

Kepler owed the first of these advantages to nature, and the second to Tycho Brahe, who gave him useful advice, from which he too frequently deviated, but which he followed in all cases where he could compare his hypotheses with observations, which, by the method of exclusion, conducted him from hypothesis to hypothesis to the laws of the planetary motions. This great observer, whom he went to see at Prague, and who had discovered the genius of Kepler in his earliest works, notwithstanding the mysterious analogies of numbers and figures with which they were filled, exhorted him to devote his time to observation, and procured him the title of imperial mathematician.

The death of Tycho, which happened a few years afterwards, put Kepler in possession of his valuable collection of observations, of which he made a most noble use, founding on them three of the most important discoveries that have been made in natural philosophy.

It was an opposition of Mars which determined Kepler to employ himself to examine, in preference, the motions of this planet. His choice was fortunate in this circumstance, that the orbit of Mars, being one of the most eccentric of the planetary system, the inequalities of his motion were more perceptible, and therefore led to the discovery of their laws with greater facility and precision. Though the theory of the motion of the Earth had made the greater part of those circles with which Ptolemy had embarrassed astronomy disappear, yet Copernicus left several to remain, in order to explain the real inequalities of the celestial bodies.

Kepler, deceived like him, by the opinion that their motions ought to be circular and uniform, tried a long time to represent those of Mars, in this hypothesis. Finally, after a great number of trials, which he has related in detail in his famous work *de Stella Martis*, he surmounted the obstacle, which an error, supported by the suffrage of every period, had opposed to him; he discovered that the orbit of Mars is an ellipse, of which the Sun occupies one of the foci, and that the motion of the planet is such, that the radins vector, drawn from its centre to that of the Sun, describes equal areas in equal times. Kepler extended these results to all the planets, and published from this theory, in 1626, the Rudolphine tables, for ever memorable in astronomy, as being the first founded on the true laws of the planetary motions, and freed from all the circles with which anterior tables were encumbered.

If we separate the astronomical investigations of Kepler from the chimerical ideas with which they were frequently accompanied, we will perceive that he arrived at those laws in the following manner: He first ascertained that the equality of the angular motion of Mars had sensibly place about a point situated beyond the centre of his orbit, with respect to the Sun. He recognised the same thing for the Earth, by comparing together select observations of Mars, of which the orbit, by the magnitude of its annual parallax, is proper to make known the respective dimensions of the orbit of the Earth. Kepler, from these results, concluded that the real motion of the planets were variable, and that at the points of the greatest and least velocities, the areas described in a day, by the radius vector of a planet, are the same. He extended this equality of areas to all the points of the orbit, which gives him the law of the areas, proportional to the times. Afterwards, the observations of Mars, near his quadratures, showed him that the orbit of this planet is an oval, elongated in the direction of the diameter which joins the points of the extreme velocities. Finally, he concluded from this the elliptic motion.

Without the speculations of the Greeks, on the curves formed from the section of a cone by a plane, these beautiful laws might have been still unknown. The ellipse being one of these curves, its oblong figure gave rise, in the mind of Kepler, to the idea of supposing the planet Mars, whose orbit he had discovered to be oval, to move on it, and soon, by means of the numerous properties which the ancient geometricians had found in the conic sections, he became convinced of the truth of this hypothesis. The history of the sciences offers us many examples of these applications of of pure geometry, and of its advantages; for every thing is connected in the immense chain of truths, and often a single observation has been sufficient to show the connection between a proposition apparently the most sterile, and the phenomena of nature, which are only mathematical results of a small number of immutable laws.

The perception of this truth probably gave birth to the mysterious analogies of the Pythagoricians: they had seduced Kepler, and he owed to them one of his most beautiful discoveries. Persuaded that the mean distances of the planets from the Sun, ought to be regulated conformably to these analogies, he compared them a long time, both with the regular geometrical solids, and with the intervals of tones. At length, after seventeen years of meditations and calculation, conceiving the idea of comparing the powers of the numbers which expressed them, he found that the squares of the times of the planetary revolutions, are to each other as the cubes of the major axes of their orbits; a most important law, which he had the advantage of observing in the system of satellites of Jupiter, and which extends to all the systems of satellites.

After having determined the curves which the planets describe about the Sun, and discovered the laws of their motions, Kepler was too near the principle whence those laws are derived, not to anticipate it. The investigation of this principle frequently exercised his active imagination; but the moment of making this last step was not yet arrived, which supposes the invention of dynamics, and of the infinitesmal analysis. Far from approaching this end, Kepler deviated from it by vain speculations on the moving cause of the planets. He supposed in the Sun a motion of rotation on an axis perpendicular to the ecliptic; immaterial species emanating from this star, in the plane of its equator, and endowed with an activity decreasing in the ratio of the distances, and preserving their primitive motion of rotation, cause each planet to participate in this circular motion. At the same time the planet, by a sort of instinct or magnetism, approaches and recedes alternately from the Sun, elevates itself above the solar equator, or is depressed below it, so as to describe an ellipse always situated in the same plane, passing through the centre of the Sun. In the midst of those numerous errors, Kepler was nevertheless led to sound views on universal gravitation in the introduction of the work *de Stella Martis*, in which he presents his principal discoveries.

"Gravity," says he, "is only a mutual corpo-
"real affection between bodies, by which they tend
"to unite. The gravity of bodies is not directed
"towards the centre of the world, but towards that
"of the bodies of which they make a part; and if
"the Earth was not spherical, heavy bodies placed
"on different parts of its surface would not fall
"towards the same centre. Two isolated bodies
"are carried towards each other, as two magnets,
"in running to join, describe spaces inversely as
"their masses. If the Earth and Moon were not

" retained at the distance which separates them " by an animal force, or by some other equi- " valent force, they would fall towards each " other; the Moon would fall $\frac{53}{54}$ of the way, and " the Earth would describe the rest, supposing " them to be equally dense. If the Earth ceased " to attract the waters of the ocean, they would " flow towards the Moon, in virtue of the attrac- " tive force of this star. This force, which ex- " tends to the Earth, produces there the pheno- " mena of the tides." Thus the important work which we have cited contains the first germs of the celestial mechanics which Newton and his successors have so happily developed.

We may be astonished that Kepler should not have applied the general laws of elliptic motion to comets. But, misled by an ardent imagination, he lost the clue of the analogy, which should have conducted him to this great discovery. The comets, according to him, being only meteors, engendered in ether, he neglected to study their motions, and thus stopped in the middle of the career which was open to him, abandoning to his successors a part of the glory which he might yet have acquired. In his time, the world had just begun to get a glimpse of the proper method of proceeding in the search of truth, at which genius only arrived by instinct, frequently connecting errors with its discoveries. Instead of passing slowly by a succession of inductions, from insulated phenomena, to others more extended, and from these to the general law of nature, it was more easy and more agreeable, to subject all the phenomena to the relations of suitableness and harmony, which the imagination could create and modify at pleasure.

Thus, Kepler explained the disposition of the solar system by the laws of musical harmony. It is a humiliating sight for the human mind to behold this great man, even in his latest works, amusing himself with these chimerical speculations, even so far as to regard them as the "*life and soul*" of astronomy. These being blended with his true discoveries was unquestionably the cause why the astronomers of his age, Des Cartes himself and Galileo, who might have drawn the most advantageous consequences from his laws, do not appear to have perceived their importance. They were not generally admitted till after that Newton made them the base of his theory of the system of the world.

Astronomy likewise owes to Kepler many useful works. His treaties on optics are full of new and interesting matter; he brought the telescopic theory to perfection; he there explains the mechanism of vision, which was unknown before him. He assgned the true cause of the *lumiere cendrée* of the Moon; but he gave the honour of this discovery to his master, Maestlin, entitled to notice from this discovery, and from having recalled Kepler to astronomy, and converted Galileo to the system of Copernicus.

Finally, Kepler, in his work entitled *Stereometria Doliorum,* has presented some conceptions on infinity, which have influenced the revolution experienced by geometry towards the end of the last century; and Fermat, whom we ought to regard as the true inventor of the differential calculus, has founded on them his beautiful method *de maximis et de minimis.*

With so many claims to admiration this great man lived in misery, while judicial astrology, every where honoured, was magnificently recompensed.

Fortunately the enjoyment which a man of genius receives from the truths which he discovers, and the prospect of a just and grateful posterity, console him for the ingratitude of his contemporaries.

Kepler had obtained pensions which were always ill paid: going to the diet of Ratisbon to solicit his arrears, he died in that city the 15th of November 1630. He had in his latter years the advantage of seeing the discovery of logarithms, and making use of them. This was due to Nepier, a Scottish baron; it is an admirable contrivance, an improvement on the ingenious algorithm of the Indians, and which, by reducing to a few days the labour of many months, we may almost say doubles the life of astronomers, and spares them the errors and disgusts inseparable from long calculations;—an invention so much the more gratifying to the human mind, as it is entirely due to its own powers: in the arts, man makes use of the materials and forces of nature to increase his powers, but here all is his own work.

The labours of Huygens followed soon after those of Kepler and Galileo. Very few men have deserved so well of the sciences, by the importance and sublimity of their researches. The application of the pendulum to clocks is one of the most beautiful acquisitions which astronomy and geography have made, and to which fortunate invention, and to that of the telescope, the theory and practice of which Huygens considerably improved, they owe their rapid progress.

He discovered, by means of excellent object-glasses which he succeeded in constructing, that the singular appearances of Saturn were produced by a very thin ring with which this planet is surrounded: his assiduity in observing enabled him also to discover one of the satellites of Saturn. He published these two discoveries in his *Systema Saturneum*, a work which contains some traces of the Pythoragean notions which Kepler had so much abused, but which the genuine spirit of science, which in this celebrated age has made so much progress, has for ever effaced. This satellite of Saturn rendered the number of satellites equal to that of the planets then known. Huygens judging this equality necessary for the harmony of the system of the world, dared to affirm, that there were no more satellites to discover; and Cassini, a few years afterwards discovered four new satellites belonging to the same satellite.

He made numerous discoveries in geometry, mechanics and optics ; and if this extraordinary genius had conceived the idea of combining his theorems on centrifugal forces with his beautiful investigation on involutes, and with the laws of Kepler, he would have preceded Newton in his theory of curvilinear motion, and in that of universal gravitation. But it is not such approximations that constitute invention.

Towards the same time, Hevelius rendered himself useful to astronomy by his immense labours. Few such indefatigable observers have existed ; it is to be regretted that he would not adopt the application of telescopes to quadrants, an invention for which we are indebted to Picard, which gives to observations a precision previously unknown to astronomy, and has rendered the greater number of those of Hevelius useless.

At this epoch astronomy received a new impulse from the establishment of learned societies.

Nature is so various in her productions and phenomena, that it is extremely difficult to ascertain their causes, hence it is requisite for a great number of men to unite their intellect and exertions in order to comprehend and develope her laws. This union is particularly requisite when the progress of the sciences multiplying their points of contact, and not permitting one individual to penetrate them all; they can only receive from several learned men the mutual support which they require.

It is then that the natural philosopher has recourse to geometry, to arrive at the general causes of the phenomena which he observes, and the geometrician in his turn interrogates the philosopher, in order to render his own investigation useful, by applying them to experience, and to open in these applications a new road in the analysis. But the principal advantage of learned societies is the philosophical feeling on every subject which is introduced into them, and from thence diffuses itself over the whole nation. The insulated philosopher may resign himself without fear to the spirit of system; he only hears contradiction at a distance; but in a learned society the shock of systematic opinions at length destroys them, and the desire of mutually convincing each other, establishes between the members an agreement only to admit the results of observation and calculation. Hence experience has proved that since the origin of these establishments, true philosophy has been generally extended.

By setting the example of submitting every opinion to the test of severe scrutiny, they have detroyed prejudices which had so long reigned among the sciences, and in which the highest intellects of the preceding age had participated. Their useful influence on opinion has dissipated the errors accumulated in our own time, with an enthusiasm which at other periods would have perpetuated them. Equally removed from the credulity which admits every thing, and the prejudices which would induce us to reject every thing which was at variance with preconceived notions, they have always in difficult questions and in extraordinary phenomena, sagely waited for the answers of observation and of experience, exciting them by prizes, and by their proper works, regulating their estimation as much by the greatness and difficulty of a discovery, as by its immediate utility; and convinced, by several instances, that the most barren in appearance may have one day the most important consequences. They have encouraged the investigation of truth on all subjects, only excluding those which, by the limits of the extent of human understanding, will be for ever inaccessible. Finally, it is among them that those grand theories have been formed which are placed above the reach of the vulgar by their comprehensiveness; and which, extending themselves by the numerous occasions in which they are applicable, to nature and to the arts, are inexhaustible sources of delight and intelligence. Wise governments, convinced of the utility of learned societies, and viewing them as one of the principal foundations of the glory and of the prosperity of empires, have instituted and placed them near themselves, to illuminate by their information, from which they have frequently derived great advantages.

Of all the learned societies, the two most celebrated for the number and importance of their discoveries in astronomy, are the Academy of Sciences at Paris, and the Royal Society in London.

The first was founded in 1666, by Louis XIV. who foresaw the lustre which the arts and sciences were to diffuse over his reign. This monarch, worthily seconded by Colbert, invited many learned strangers to fix themselves in his capital. Huygens availed himself of this flattering invitation; he published his admirable work, *De horologio oscillatorio*, in the midst of the academy, of which he was one of the first members. He would have finished his days in this country, had it not been for the disastrous edict which, towards the end of the last century, widowed France of so many valuable citizens. Huygens, departing from a country in which the religion of his ancestors was proscribed, retired to the Hague, where he was born the 14th of April, 1629, and died there the 15th of June, 1695.

Dominic Cassini was likewise induced to go to Paris, by the advantages held out by Louis XIV. During forty years of useful labours, he enriched astronomy with a crowd of discoveries: such are the theory of the satellites of Jupiter, the motions of which he determined from observations of their eclipses; the discovery of the four satellites of

Saturn, that of the rotation of Jupiter, of the belts parallel to his equator, of the rotation of Mars, of the zodiacal light, a very approximate knowledge of the Sun's parallax, a very exact table of refractions, and, above all, a complete theory of the libration of the Moon.

Galileo had only considered the libration in latitude. Hevelius explained the libration in longitude, by supposing that the Moon always presents the same face to the centre of the lunar orbit, of which the Earth occupies one of the foci. Newton, in a letter addressed to Mercator in 1675, rendered the explanation of Hevelius perfect, by reducing it to the simple conception of a uniform rotation of the Moon on itself, while it moves unequally about the Earth. But he supposed with Hevelius the axis of rotation always perpendicular to the plane of the ecliptic. Cassini, by his own observations, recognized that it was inclined to the plane of the ecliptic at a small angle of an invariable magnitude; and to satisfy the condition already observed by Hevelius, according to which all the inequalities of libration are re-established after each revolution of the nodes of the lunar orbit, he made the nodes of the lunar equator to coincide constantly with them. Such has been the progress of opinions on one of the most curious points of the system of the world.

The great number of astronomical academicians of extraordinary merit, and the limits of this historical abridgment, do not permit me to give an account of their labours; I shall content myself with observing, that the application of the telescope to the quadrant, the invention of the micrometer and heliometer, the successive propagation of light, the magnitude of the Earth, its ellipticity, and the diminution of gravity at the equator, are all discoveries due to the Academy of Sciences.

Astronomy does not owe less to the Royal Society of London, the origin of which is a few years anterior to that of the Academy of Sciences. Among the astronomers which it has produced, I shall cite Flamstead, one of the greatest observers that has ever appeared; Halley, rendered illustrious by his travels, undertaken for the advantage of science, by his beautiful investigations concerning comets, which enabled him to discover the return of the comet in 1759; and by the ingenious idea of employing the transit of Venus over the Sun, in order to determine its parallax: I shall mention, lastly, Bradley, the model for observers, and who will be for ever celebrated for two of the most beautiful discoveries that have been made in astronomy, namely, the aberration of the fixed stars, and the nutation of the axis of the Earth.

When the application of the pendulum to clocks, and of telescopes to quadrants, had rendered the slightest changes in the position of the celestial bodies perceptible to observers, they endeavoured to determine the annual parallax of the fixed stars; for it was natural to suppose, that so great an extent as the diameter of the terrestrial orbit, would be sensible even at the distance of these stars. Observing them carefully, at every season of the year, there appeared slight variations; sometimes favourable, but more frequently contrary to the effects of parallax. To determine the law of these variations, an instrument of great radius, and divided with extreme precision, was requisite. The artist who executed it, deserves to partake of the glory of the astronomer who owed his discovery to him. Graham, a famous English watch-maker, constructed a great sector, with which Bradley discovered the aberration of the fixed stars, in the year 1727. To explain it, this great astronomer conceived the fortunate idea of combining the motion of the earth with that of light, which Roemer had discovered at the end of the last century, by means of the eclipses of Jupiter's satellites. We should be surprised that in the interval of half a century, which intervened between this discovery and that of Bradley, none of the distinguished philosophers who then existed, and who knew the motion of light, should have paid any attention to the very simple effects which result from it, in the apparent position of the fixed stars. But, the human mind, so active in the formation of systems, has almost always waited till observation and experience have acquainted it with important truths, which its powers of reasoning alone might have discovered.

It is thus that the invention of telescopes has followed by more than three centuries that of lenses, and even then it was solely due to accident.

In 1745, Bradley discovered by observation, the nutation of the terrestrial axis and its laws. In all the apparent variations of the fixed stars, observed with extraordinary care, he perceived nothing which indicated a perceptible parallax. We are also indebted to this great man for the first sketch of the principal inequalities of the satellites of Jupiter, which was soon afterwards extended. Finally, he left an immense number of observations of all the phenomena which the heavens presented towards the middle of the last century, for more than ten consecutive years. The great number of these observations and the accuracy which distinguishes them, form by their collection one of the principal foundations of modern Astronomy, and the epoch whence we ought to set out in all the delicate investigations of this science. He has given us a model for several similar collections, which being rendered successively more perfect by the progress of the arts, are so many signs placed in the path of the heavenly bodies to denote their periodic and secular changes.

At the same epoch Lacalle flourished in France and Tobias Mayer in Germany, both of them indefatigable observers and laborious compilers; they have rendered the tables and astronomical theories perfect, and from their own observations have formed catalogues of the stars, which, compared wiih those of Bradley determine with great precision the state of the heavens in the middle of the last century.

The measures of the degrees of the terrestrial meridian, and of the pendulum, (repeated in different parts of the globe, of which France gave the example, by measuring the whole arc of the meridian, which crosses it, and by sending academicians to the north and to the equator, to observe the magnitude of these degrees, and the intensity of the force of gravity ;) the arc of the meridian, comprised between Dunkirk and Barcelona, determined by very precise observations, and forming the base of the most natural and simple system of measures; the voyages undertaken to observe the two transits of Venus over the Sun's disk, in 1761 and 1769, and the exact knowledge of the dimensions of the solar system, which has been derived from these voyages; the discovery of achromatic telescopes, of chronometers, of the sextant, and repeating circle, invented by Mayer, and brought to perfection by Borda; the formation by Mayer of lunar tables sufficiently exact for the determination of the longitude at sea; the discovery of the planet Uranus, by Herschel, in 1781; that of its satellites, and of the two new satellites of Saturn, due to the same observer; these, with the discoveries of Bradley, are the principal obligations which astronomy owes to our century, which, with the preceding, will always be considered as the most glorious epoch of the science.

The present age has commenced under the most favourable auspices for astronomy: its first day is remarkable for the discovery of the planet Ceres, made by Piazzi, at Palermo; and this discovery was soon followed by those of the two planets, Pallas and Vesta, by Olbers, and of the planet Juno, by Harding.

CHAP. V.

Of the Discovery of universal Gravitation.

After having shewn by what successive efforts the human mind has attained the knowledge of the laws of the celestial motions, it only remains to consider the means by which it has arrived at the general principle, on which these laws depend. Descartes was the first who endeavoured to reduce the motions of the heavenly bodies to some mechanical principle. He imagined vortices of subtle matter, in the centre of which he placed these bodies. The vortex of the Sun forced the planet, its satellites, and their vortices, into motion; that of the planet, in the same manner, forced its satellite to revolve round it. The motion of comets traversing the heavens in all directions, destroyed these vortices, as they had before destroyed the solid heavens, and the whole apparatus of circles imagined by the ancient astronomers. Thus, Descartes was no happier in his mechanical, than Ptolemy in his astronomical theory. But their labours have not been useless to science. Ptolemy has transmitted to us, through fourteen centuries of ignorance, the few astronomical truths which the ancients had discovered, and which he had also increased. When Descartes appeared on the stage, the impulse given by the invention of printing, by the discovery of the New World, by the Reformation, and by the Copernican system, rendered people eager for new discoveries. But this philosopher, by substituting in the place of ancient errors, others most seducing, and resting on the authority of his geometrical discoveries, was enabled to destroy the empire of Aristotle, which might have stood the attack of a more careful philosopher; and his system of vortices, at first received with enthusiasm, being founded on the motion of the planets and Earth about the Sun, contributed to make these notions be generally adopted; but by establishing as a principle, that we should begin by doubting of every thing, he himself warned us to adopt his own system with great caution, and his astronomical system was very soon overturned by later discoveries, in which his own, combined with those of Kepler and Galileo; and also, with the just philosophical notions then entertained on all subjects, has rendered his age, already illustrious for so many chefs d'œuvre in literature and the fine arts, likewise the most remarkable epoch in the history of the human mind. It was reserved for Newton to make known the general principles of the heavenly motions. Nature not only endowed him with a profound genius, but placed his existence in a most fortunate period. Descartes had changed the face of the mathematical sciences, by the application of algebra to the theory of curves and variable functions. Fermat had laid the foundation of the geometry of infinites, by his beautiful method *de maximis* and *de minimis*, and of tangents. Wallis, Wren, and Huygens, had discovered the laws of motion; the discoveries of Galileo, on falling bodies, and of Huygens on evolutes and on the centrifugal force, led to the theory of motion in curves; Kepler had determined those described by the planets, and had formed a remote conception of universal gravitation; and finally, Hook had distinctly perceived that their motion was the result of a projectile force, combined with the attractive force of the Sun. The science of celestial mechanics wanted nothing more to bring it to light, but the genius of a man, who, by generalizing these discoveries, should be capable of deducing from them the law of gravitation: it is this which Newton accomplished in his immortal work on the mathematical principles of natural philosophy. This philosopher, so deservedly celebrated, was born at Woolstrop, in England, in the latter end of the year 1642, the year in which Galileo died. His first success in mathematics announced his future reputation; a cursory perusal of elementary books, was sufficient for him to comprehend them; he next read the geometry of Descartes, the optics of Kepler, and the arithmetic of infinites, by Wallis; but soon aspiring to new inventions, he was, before the age of twenty-seven, in possession of his method of fluxions, and of his theory of light. Anxious for repose, and averse to literary controversy, which he had better avoided by sooner making known his discoveries, he delayed publishing his works. His friend and preceptor, Dr. Barrow, exerted himself in his favour, and obtained for him the situation of professor of mathematics in the university of Cambridge; it was during this period, that, yielding to the request of Halley, and the solicitations of the Royal Society, he published his *Principia.* The university of Cambridge, whose privileges he strenuously defended when attacked by James II., chose him for their representative, in the conventional parliament of 1688, and for that which was convened in 1701. He was knighted and appointed director of the mint, by Queen Anne: he was elected president of the Royal Society in 1703, which dignity he enjoyed till his death, in 1727. During the whole of his life he obtained the most distinguished consideration, and the nation to whose glory he had so much contributed, decreed him at his death, public funeral honours.

In 1666, Newton, retired into the country, for the first time, directed his thoughts to the system of the world. The descent of heavy bodies, which appears nearly the same at the summit of the highest mountains as at the surface of the Earth, suggested to him the idea, that gravity might extend to the Moon, and that being combined with some motion of projection, it might cause it to describe its elliptic orbit round the

Earth. To verify this conjecture, it was necessary to know the law of the diminution of gravity. Newton considered, that if the Moon was retained in its orbit by the gravity of the Earth, the planets should also be retained in their orbits by their gravity towards the Sun, and demonstrated this by the law of the areas being proportional to the times. Now it results from the relation of the squares of the times to the cubes of the greater axes of their orbits, discovered by Kepler, that their centrifugal force, and consequently their tendency to the Sun, diminishes inversely as the squares of the distances from this body. Newton, therefore, transferred to the Earth this law of the diminution of the force of gravity, and reasoning from the experiments of falling bodies, he determined the height which the Moon, abandoned to itself, would fall in a short interval of time. This height is the versed sine of the arc which it describes in the same interval; and this quantity the lunar parallax gives in parts of the radius of the Earth, so that, to compare the law of gravitation with observation, it was necessary to know the magnitude of this radius; but Newton having, at that time, an erroneous estimate of the terrestrial meridian, obtained a different result from what he expected, and suspecting that some unknown forces operated concurrently with the gravity of the Moon, he abandoned his original idea. Some years afterwards, a letter from Dr. Hook induced him to investigate the nature of the curve described by projectiles round the centre of the Earth. Picard had lately finished the measure of a degree in France, and Newton found, by this measure, that the Moon was retained in its orbit by the force of gravity alone, supposed to vary inversely as the square of the distance. From the action of this law he found that bodies in their fall, describe ellipses, of which the centre of the Earth occupies one of the foci, and then, considering that the planetary orbits are likewise ellipses, having the Sun in one of their foci, he had the satisfaction to see, that the solution which he had undertaken from curiosity, could be applied to the greatest objects in nature. He arranged the several propositions relative to the elliptic motions of the planets, and Dr. Halley having induced him to publish them, he composed his grand work, the *Principia*, which appeared in 1687. These details, which have been transmitted to us by his friend and cotemporary, Dr. Pemberton, prove that this great philosopher had, so early as 1666, discovered the principal theorems on centrifugal force, which Huygens published sixteen years afterwards, at the end of his work De Horologio Oscillatorio; indeed it is extremely probable that the author of the method of fluxions, who seems then to have been at that time in possession of it, should easily have discovered these theorems. Newton arrived at the law of the diminution of gravity, by the relation which subsists between the squares of the periodic times of the planets, and the cubes of the greater axes of their orbits, supposed circular. He demonstrated that this relation exists in elliptic orbits generally, and that it indicates an equal gravity of the planets towards the Sun, supposing them at an equal distance from its centre. The same equality of gravity towards the principal planet, exists likewise in all the systems of satellites, and Newton verified it on terrestrial bodies, by very accurate experiments. Whence it results, that the development of gases, of electricity, of heat, and of affinities, in the mixture of several substances contained in a closed vessel, do not alter the weight of the system, neither during, nor after the mixture.

This great geometrician, by considering the question generally, demonstrated that a projectile can move in any conic-section whatever, in consequence of a force directed towards its centre, and varying reciprocally as the square of the distances. He investigated the different properties of motion in this species of curves; he determined the conditions requisite for the section to be a circle, an ellipse, a parabola, or an hyperbola, which conditions depend entirely on the velocity and primitive position of the body.

Any velocity, position, and initial direction of a body being given, Newton assigned the conic section which the body should describe, and in which it ought consequently to move, which refutes the objectiou advanced by John Bernouilli against him of not having demonstrated, that the conic sections are the *only* curves which a body, solicited by a force varying reciprocally as the squares of the distance, can describe. These investigations, applied to the motion of comets, informed him that these bodies move round the Sun, according to the same laws as the planets, with the difference only of their ellipses being very excentric ; and he indicated the means of determining by observations the elements of these ellipses.

He learned from the comparison of the distances and durations of the revolutions of the satellites, with those of the planets, the respective densities and masses of the Sun, and of planets accompanied by satellites, and the intensity of the force of gravity at their surface.

By considering that the satellites move round their planets very nearly, as if the planets were immoveable, he discovered that all these bodies obey the same force of gravity towards the Sun.

The equality of action and reaction, did not permit him to doubt, but that the Sun gravitated towards the planets, and these towards their satellites ; and even that the Earth is attracted by all the bodies that gravitate on it. He extended this proposition afterwards to all the celestial bodies, and established as a principle, *that each particle of matter attracts all others directly as its mass, and inversely as the square of its distance from the attracted particle.*

Arrived at this principle, Newton saw that the great phenomena of the system of the world might be deduced from it. By considering the gravity at the surfaces of the celestial bodies, as the result of the attractions of all their particles, he ascertained this remarkable and characteristic property of the law of attraction varying inversely as the square of the distance, namely that two spheres composed of concentrical strata and of densities varying according to any given law, attract each other mutually as if their masses were united in their centres; hence the bodies of the solar system act on each other, and likewise on the bodies placed at their surfaces, as so many attracting points, which result contributes to the regularity of their motions, and enabled this great geometer to recognise the terrestrial gravity in the force which retains the moon in its orbit.

He proved that the motion of rotation of the Earth ought to have flattened it in the direction of the poles, and he determined the law of the variation of the degrees and of gravity at its surface.

He demonstrated that the action of the Sun and Moon on the terrestial spheroid combined with its rotatory motion, ought to produce the retrograde motion of the equinoxes, to elevate the waters of the ocean, and to produce in this great fluid mass the oscillations which are observed under the name of tides.

Lastly, he was convinced that the lunar inequalities and the retrograde motion of the nodes were produced by the combined action of the Sun and Earth on this satellite. This principle is not simply a hypothesis that satisfies phenomena, which may be otherwise explained, as the equations of an indeterminate problem may be satisfied in different ways. Here the problem is determined by laws observed in the celestial motions, of which this principle is a necessary result. The gravity of the planets towards the Sun is demonstrated by the proportionality of the areas to the times; the diminution in the inverse ratio of the squares of the distances is proved by the ellipticity of the planetary orbits; and the law of the squares of the times of the revolutions, proportional to the cubes of the greater axes, demonstrates that the solar gravity would act equally on all bodies supposed at the same distance from the Sun, of which the weight would therefore be proportional to the masses. The equality of action to reaction shews that the Sun gravitates in its turn towards the planets, proportionably to their masses divided by the squares of their distances from this star; the motions of the satellites prove that they gravitate at the same time to the Sun and to the planets, which reciprocally gravitates towards them, so that there exists between all the bodies of the solar system a mutual attraction directly proportional to the masses, and inversely as the squares of the distances. Finally their spherical figure, and the phenomena of gravity at the surface of the earth, do not permit us to suppose that this attraction appertains to the bodies considered in mass, but belongs to each of their particles.

Considering then the elevation of the earth at the equator as a system of satellites adhering to its surface, he found that the combined actions of the sun and moon tended to make the nodes of the circles which they describe about the axis of the earth, to retrograde, and that all these tendencies being communicated to the entire mass of the planet ought to produce, in the intersection of the ecliptic and equator, that slow retrogradation which is termed *the precession of the equinoxes.*

Thus the cause of this phenomenon depends on the compression of the earth, and on the retrograde motion which the action of the Sun communicates to the nodes, two facts which Newton first announced, and which could not before his time be suspected. Kepler himself, carried along by an ardent imagination to explain every thing by hypotheses, was obliged to admit the inutility of his efforts on this subject. But, with the exception of what concerns the elliptic motion of the planets and comets, the attraction of spherical bodies, and the intensity of gravity at the surface of the Sun, and of those planets that are accompanied by satellites, all these discoveries were only sketched by Newton. His theory of the figure of the planets is limited by the supposition of their homogeneity: his solution of the problem of the precession of the equinoxes, though very ingenious, is, notwithstanding the apparent agreement of his result with observation, in many respects defective; among the great number of the perturbations of the celestial motions, he has only considered those of the lunar motion, of which the most considerable, the evection, escaped his investigation. He perfectly established the existence of the principle which he discovered, but the developement of its consequences and its advantages, has been the work of the successors of this great geometrician. The state of imperfection in which the infinitesimal calculus must have been in the hands of its inventor, did not permit him to resolve completely the difficult problems which the theory of the system of the world presents; and he has been often obliged to give conjectures, always uncertain, till they have been verified by a rigorous calculation. Notwithstanding these inevitable defects, the importance and extent of his discoveries, the great number of original and profound conceptions, which have been the germ of the most brilliant theories of the geometricians of this century, and arranged with much elegance, insures to his *Principia* a pre-eminence over all other productions of the human intellect.

The case is not the same with the sciences as with general literature: this has limits which a man of genius may reach, when he employs a language brought to perfection; he is read with the same interest in all ages; and time only adds to his reputation by the vain efforts of those who try to equal him.

The sciences, on the contrary, like nature herself, without bounds, indefinitely increase by the labours of successive generations, the most perfect work; and thus by raising them to a height from which they can never again descend, gives birth to new discoveries, which produce in their turn new works, which efface those from which they originated. Others will present in a point of view more general and more simple, the theories detailed in the *Principia*, and all the truths which it has brought to light; but it will ever remain as an eternal monument of the profundity of that genius which has revealed to us the greatest law of the universe.

This work, and the equally original treatise by the same author on optics, have still the merit of being the best models which can be proposed in the sciences, and in the delicate art of making experiments and submitting them to calculation. We there see the most beautiful applications of the method, which consists in tracing the principal phenomena to their causes, by a succession of inductions, and afterwards in redescending from these causes, to all the details of the phenomena.

General laws are impressed in all individual cases, but they are complicated with so many extraneous circumstances, that the greatest address is often necessary to develope them. The phenomena most proper for this object must be chosen, and these must be multiplied by varying the attendant circumstances, so that whatever they have in common may be observed.

We thus ascend successively to relations more and more extended, until we arrive at length at general laws, which are verified either by proofs or by direct experiment, if that is possible, or by examining if they satisfy all the known phenomena.

This is the most certain method by which we can be guided in the search of truth. No philosopher has adhered more closely to this method than Newton; none ever possessed, in a higher degree, that felicitous tact of discerning in objects the general principles involved in them, and which enabled him to recognise in the fall of bodies, the principle of universal gravitation. Other philosophers in England, cotemporaries of Newton, adopted the method of inductions by his example, which thus became the basis of a great number of excellent works in physics and analysis.

The philosophers of antiquity following a contrary path, and considering themselves as the source of every thing, imagined general causes to explain them.

Their method, which was only productive of vain systems, had not greater success in the hands of Descartes. In the time of Newton, Leibnitz, Malebranche and other philosophers employed it with as little advantage.

At length the inutility of the hypotheses to which it led its followers, and the progress for which the sciences are indebted to the method of inductions, have recalled all philosophers to this last method, which was explained by Chancellor Bacon, with the whole force of reason and eloquence, and which Newt[illegible] more strongly recommended by h[illegible]

At the period of their [illegible] scartes had substituted for the o[illegible] ne Peripatetics, the intelligi[illegible], of impulse, and centrifuga[illegible] ious system of vortices, founded on these ideas, were received with avidity by the learned, who rejected the obscure and trifling doctrines of the school; and they thought that they perceived to arise in the doctrine of universal gravitation, those occult qualities which the French school had so justly proscribed. It was not till after the vagueness of Descartes' explanation was recognised, that attraction was considered as it ought to be, *i. e.* as a general fact, to which Newton was led by a series of inductions, and from which he descended again to explain the heavenly motions. This great man would justly have merited the reproach of re-establishing the occult qualities, if he was content to ascribe to universal attraction, the elliptic motion of the planets and of the comets, the inequalities of the motion of the Moon, those of terrestrial degrees and of gravity, the precession of the equinoxes and the ebbing and flowing of the sea, without demonstrating the connection of his principle with the phenomena. But as the Geometers who corrected and generalized these his demonstrations, and compared all the observations to the same principle, found the most perfect agreement between them and the results of analysis; they therefore have unanimously adopted his theory of the system of the world, which has thus become, by their researches, the basis of all Astronomy.

This analytical connection of particular with general facts, is what constitutes a theory. It is thus that having deduced, by a rigorous calculus, all the effects of capillary action, from the sole principle of a mutual attraction between the particles of matter, which is only sensible at imperceptible distances, we may presume that we have found out the true theory of these phenomena. Some philosophers, struck with the advantages which the admission of unknown causes have produced in several branches of the natural sciences, have brought back the occult qualities of the ancients, and their trifling explanations. Viewing the Newtonian philosophy under the same point of view which made it reject the Cartesians, they have assimilated their doctrines to it; which, however, have nothing common in the most essential circumstance, namely, the rigorous agreement of the results with the phenomena.

It is by means of synthesis that this great geometrician has explained his theory of the system of the world. It appears, however, that he discovered the greater part of his theorems by analysis, the limits of which he has considerably extended, and to which he allows himself to have owed his general results on the quadratures of curves.

But his great predilection for synthesis, and his esteem for the geometry of the ancients, has induced him to represent his theorems, and even his method of fluxions, under a synthetic form. And it is evident, by the rules and examples which he has given of these transformations in many works, how much importance he attached to it. We may regret with the geometricians of his time, that he has not followed in the exposition of his discoveries, the path by which he arrived at them; and that he has suppressed the demonstration of many results, such as the equation of the solid of least resistance, preferring the pleasure of leaving it to be divined, to that of enlightening his readers.

The knowledge of the method which has guided a man of genius is not less serviceable to the progress of the sciences, and even to his own glory, than his discoveries. This method is frequently the most interesting part; and if Newton, instead of merely announcing the differential equation of the solid of least resistance, had, at the same time, furnished the analysis of it, he would have the honour of giving the first essay on the method of variations, one of the most fruitful branches of modern analysis; and his example has perhaps prevented his countrymen, from contributing as much as they might to the advancement, which astronomy has made, from the application of analysis to the principle of universal gravitation.

The preference of Newton for the synthetical method, may be explained by the elegance with which he connected his theory of curvilinear motion with the investigations of the ancients on the conic sections, and the beautiful discoveries which Huygens had published according to this method. Geometrical synthesis has besides the property of never losing sight of its object, and of enlightening the whole path which leads from the first axioms to their last consequences, while algebraic analysis soon makes us forget the principal object, to occupy ourselves with abstract combinations, and it is only at the end that it brings us back to it. But though it thus separates itself from the object of investigation, after having assumed what is indispensably necessary to arrive at the required result; still by directing our attention to the operations of analysis, and reserving all our forces to overcome the difficulties which present themselves, we are conducted by the universality of this method, and by the inestimable advantage of thus transferring the train of reasoning into mechanical processes, to results often inaccessible to synthesis. Such is the fecundity of analysis, that if we translate particular truths into this universal language, we shall find a number of new and unexpected truths arise merely from the form of expression. No language is so susceptible of the elegance which arises from the developement of a long train of expressions connected with each other, and all flowing from the same fundamental idea. Analysis unites to all these advantages, that of always being able to conduct us to the most simple methods. Nothing more is requisite than to apply it in a convenient manner by a judicious selection of unknown quantities, and by giving to the results the form most easily reducible to geometrical construction, or to numerical calculation. Newton himself furnishes many examples in his Universal Arithmetic. The geometricians of this century, convinced of its superiority, have principally applied themselves to extend its domain, and enlarge its boundaries.

However, geometrical considerations ought not to be abandoned; they (*a*) are of the greatest utility in the arts. Besides, it is curious to show how the different results of analysis may be represented in space; and reciprocally, to read all the affections of lines and surfaces, and all the variations in the motions of bodies, in the equations which express them. This connection of geometry and analysis, diffuses a new light over the sciences; the intellectual operations of the latter, rendered perceptible by the images of the former, are more easy to comprehend, and more interesting to pursue; and when observations realizes, and transforms these geometrical results into laws of nature, and when these, embracing the whole universe, display to our view its present and future state, the view of this sublime spectacle presents to us one of the most noble pleasures reserved for mankind.

About fifty years passed after the discovery of the theory of gravitation, without any remarkable addition to it. All this time had been requisite for this great truth to be generally understood, and to surmount the obstacles opposed to it by the system of vortices, and the authority of geometricians contemporary with Newton, who combated it perhaps from vanity, but who have nevertheless accelerated its progress by their labours on the infinitesmal analysis.

Among the contemporaries of Newton, Huygens, who appears more than any other to have appreciated the value of this discovery, admits the gravitation of the heavenly bodies towards each other in the inverse ratio of the squares of the distances, and all the results which Newton deduced relative to the elliptic motion of the planets, of the satellites and comets, and relative to the gravity at the surfaces of planets, which are accompanied by satellites. On these points he rendered to Newton all the justice to which he was entitled. But his erroneous notions respecting the cause of gravity, made him to reject the mutual attraction of molecules, and the theories of the figure of the planets and of the variation of gravity at their surface, which depends on it. It must however be observed, that the law of universal gravitation had not, for Newton himself and his cotemporaries, all the certainty which the subsequent progress of observations and of mathematical sciences has secured to it. Euler and Clairault, who first with D'Alembert applied analysis to the perturbations of the celestial motions, did not deem it sufficiently established, to attribute to the inaccuracies of approximations and computations, the differences which were found to exist between observation and their results, on the motions of Saturn and the Lunar Perigee. But these three great Geometers having rectified these results, perfected the methods, and carried the approximation as far as is necessary, succeeded in explaining by the sole law of universal gravitation all the phenomena of the system of the world, and have thus assigned to the tables and astronomical theories a precision which could not be anticipated.

It is about three centuries since Copernicus first introduced into his tables the motions of the Earth, and of the planets round the Sun; about a century after Kepler introduced the laws of elliptic motion, which depend on the solar attraction alone; now they contain the numerous inequalities, which arise from the mutual attraction of all the bodies of the solar system, so that all empiricism is banished, and they only borrow from observations indispensable data.

It is principally in the application of analysis that the power of this wonderful instrument is evinced, without which it would be impossible to penetrate a mechanism so complicated in its effects, at the same time that it is so simple in its cause. The Geometer now embraces in his formulæ the entire of the solar system and its successive variations; he can ascend in imagination to the various changes which this system has undergone at the remotest periods, and he can redescend to all those which time will reveal to observers. He perceives those great changes, of which the entire developement requires millions of years, to be repeated in a few centuries, in the system of the satellites of Jupiter, by the quickness of their revolutions, and thus to produce those remarkable phenomena, just conjectured by Astronomers, but which were too complicated or too slow to enable them to determine the laws. The theory of gravity becomes, by so many applications, a means of discovering, as certain as observation itself; it has made known a great number of new inequalities, of which the most remarkable are the inequalities of Jupiter and Saturn, and the secular inequalities of the Moon with respect to its nodes, to its perigee and the Sun. By this means the Geometer has known to derive from his observations, as from a fruitful source, the most important elements of the system of the world, which would remain for ever concealed, without the aid of analysis. He has determined the respective values of the masses of the Sun, of the planets, and of the satellites, by the revolutions of these different bodies, and by the developement of their periodic and secular inequalities: the velocity of light, and the ellipticity of Jupiter have been made known to him by the eclipses of the satellites with more precision than by direct observation: he has inferred the rotation of Uranus, of Saturn, and of its ring, and the ellipticity of these two planets, from the respective position of the orbits of their satellites; the parallaxes of the Sun and of the Moon, and the ellipticity itself of the earth, are indicated in the lunar inequalities; as we have already seen that the moon by its motion reveals to astronomy, when brought to perfection, the compression of the earth, of which it made known the round form to the first observers, by its eclipses. Finally, by a fortunate combination of analysis with observations of the Moon, which seems to have been given to the Earth to illuminate it in the night, it has become the surest guide to the mariner, which guards him from the

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Finally, by a fortunate combination of analysis with observations of the Moon, which seems to have been given to the Earth to illuminate it in the night, it has become the surest guide to the mariner, which guards him from the should attract as it is attracted, and that consequently the gravity is the resultant of the attractions of all the molecules of the attracting body. The principle of action being equal to reaction, is embarrassing, when the mode of action of the forces is unknown. Thus Huygens, who had founded on this principle his investigations on the collisions of elastic bodies, did not find it sufficient to establish the mutual attraction of each molecule. It was therefore necessary to confirm this attraction by observation, in order to remove every doubt on this important point of the Newtonian theory. The celestial phenomena may be divided into three classes. The first comprehends all those which depend on the mutual tendency of the heavenly bodies towards each other; such are the elliptic motions of the planets and satellites, and their reciprocal perturbations, which are independent of their figures. Under the second class are contained those phenomena which are produced by the tendency of the molecules of the attracted body towards the centres of the attracting bodies; such are the ebbing and flowing of the tide, the precession of the equinoxes, and the libration of the Moon. Finally, I have arranged under the third class, the phenomena which depend on the action of the molecules of the attracting bodies, on the centres of those which are attracted, and on their own molecules. The two lunar inequalities which arise from the compression of the Earth, the motion of the orbits of the satellites of Jupiter and Saturn, the figure of the Earth and the variation of gravity at its surface, are phenomena of this kind. The Geometers who, in order to explain the cause of gravity, surround each of the heavenly bodies with a vortex, ought to admit the Newtonian theories relative to the phenomena of the two first classes; but they ought to reject, as Huygens did, the theories of the phenomena of the third class, founded on the action of the molecules of the attracting bodies. The perfect agreement of these theories with all observations, ought now to remove every doubt of the mutual attraction of the molecules. The law of attraction, inversely as the square of the distance, is that of emanations which proceed from a centre. It appears to be the law of all forces, of which the action is sensible at a distance, as has been recognised in electrical and magnetic forces. Hence, as this law corresponds exactly to all the phenomena, it should be regarded from its simplicity and generality, as rigorously true. One of its remarkable properties is, that if the dimensions of all the bodies in the universe, their mutual distances and velocities, increase or diminish proportionably, they would describe curves entirely similar to those which they at present describe; so that if the universe be successively reduced to the smallest imaginable space, it would always present the same appearances to observers. These appearances are consequently independent of the dimensions of the universe, as they are also, in consequence of the law of the proportionality of the force to the velocity, independent of the absolute motion which it may have in space. The simplicity of the laws of nature therefore only permits us to observe the relative dimensions of the universe. (*b*)

In the law of attraction, the heavenly bodies attract each other very nearly as if their masses were united in their centres of gravity; their surfaces and orbits also assume in this law the elliptical form, which is the simplest after the spherical and circular, which last the ancients deemed to be essential to the stars and their motions.

Is the attraction communicated instantaneously from one body to another? The time of its transmission, if it was sensible to us, would be particularly evinced in a secular acceleration of the Moon's motion. I suggested this as a means of explaining the acceleration which is observed in this motion; and I have found, that in order to satisfy observations, we must ascribe to the force of gravity, a velocity seven million of times greater than that of a ray of light. As the cause of the secular equation of the Moon (*c*) is now well ascertained, we may affirm that the attraction is transmitted fifty millions of times more rapidly than light. We can therefore assume, without any apprehension of error, that its transmission is instantaneous.

The attraction may also produce, and continually maintain the motion in a system of bodies which were primitively in repose; for it is not true, as some philosophers have asserted, that it must at length reunite them all about their common centre of gravity. The only elements which must always remain equal to nothing, are the motion of this centre, and the sum of the areas described about it, in a given time, by all the molecules of the system projected on any plane whatever.

CHAP. VI.

Considerations on the system of the World, and future progress of Astronomy.

The preceding summary of the history of Astronomy presents three distinct periods, which referring to the phenomena, to the laws which govern them, and to the forces on which these laws depend, point out the career of this science during its progress, and which consequently ought to be pursued in the cultivation of other csiences. The first period embraces the observations made by Astronomers antecedently to Copernicus, on the appearances of the celestial motions, and the hypotheses which were devised to explain those appearances, aud to subject them to computation. In the second period, Copernicus deduced from these appearances, the motions of the Earth on its axis and about the Sun, and Kepler discovered the laws of the planetary motions. Finally in the third period, Newton, assuming the existence of these laws, established the principle of universal gravitation; and subsequent

Geometers, by applying analysis to this principle, have derived from it all the observed phenomena, and the various inequalities in the motion of the planets, the satellites, and the comets. Astronomy thus becomes the solution of a great problem of mechanics, the constant arbitraries of which are the elements of the heavenly motions. It has all the certainty which can result from the immense number and variety of phenomena, which it rigorously explains, and from the simplicity of the principle which serves to explain them. Far from being apprehensive that the discovery of a new star will falsify this principle, we may be antecedently certain that its motion will be conformable to it; indeed this is what we ourselves have experienced with respect to Uranus and the four telescopic stars recently discovered, and every new comet which appears, furnishes us with an additional proof.

Such is unquestionably the constitution of the solar system. The immense globe of the Sun, the focus of these motions, revolves upon its axis in twenty-five days and a half. Its surface is covered with an ocean of luminous matter. Beyond it the planets, with their satellites, move, in orbits nearly circular, and in planes little inclined to the ecliptic. Innumerable comets, after having approached the Sun, recede to distances, which evince that his empire extends beyond the known limits of the planetary system. This luminary not only acts by its attraction upon all these globes, and compels them to move around him, but imparts to them both light and heat; his benign influence gives birth to the animals and plants which cover the surface of the Earth, and analogy induces us to believe, that he produces similar effects on the planets; for, it is not natural to suppose that matter, of which we see the fecundity develope itself in such various ways, should be sterile upon a planet so large as Jupiter, which, like the Earth, has its days, its nights, and its years, and on which observation discovers changes that indicate very active forces. Man, formed for the temperature which he enjoys upon the Earth, could not, according to all appearance, live upon the other planets; but ought there not to be a diversity of organization suited to the various temperatures of the globes of this universe? If the difference of elements and climates alone causes such variety in the productions of the Earth, how infinitely diversified must be the productions of the planets and their satellites? The most active imagination cannot form any just idea of them, but still their existence is, at least, extremely probable.

However arbitrary the elements of the system of the planets may be, there exists between them some very remarkable relations, which may throw light on their origin. Considering it with attention, we are astonished to see all the planets move round the Sun from west to east, and nearly in the same plane, all the satellites moving round their respective planets in the same direction, and nearly in the same plane with the planets. Lastly, the Sun, the planets, and those satellites in which a motion of rotation have been observed, turn on their own axes, in the same direction, and nearly in the same plane as their motion of projection.

The satellites exhibit in this respect a remarkable peculiarity. Their motion of rotation is exactly equal to their motion of revolution; so that they always present the same hemisphere to their primary. At least, this has been observed for the Moon, for the four satellites of Jupiter, and for the last satellite of Saturn, the only satellites whose rotation has been hitherto recognized.

Phenomena so extraordinary, are not the effect of irregular causes. By subjecting their probability to computation, it is found that (*a*) there is more than two thousand to one against the hypothesis that they are the effect of chance, which is a probability much greater than that on which most of the events of history, respecting which there does not exist a doubt, depends. We ought therefore to be assured with the same confidence, that a primitive cause has directed the planetary motions.

Another phenomenon of the solar system equally remarkable, is the small excentricity of the orbits of the planets and their satellites, while those of comets are very much extended. The orbits of this system present no intermediate shades between a great and small excentricity. We are here again compelled to acknowledge the effect of a regular cause; chance a[illegible] have given a form nearly circul[illegible] of all foci of the principal forces which actuate them are constant; all the secular inequalities are periodic.

The most considerable are those which affect the motions of the Moon, with respect to its perigee, to its nodes and the Sun; they amount to several circumferences, but after a great number of centuries they are reestablished. In this long interval all the parts of the lunar surface would be successively presented to the earth, if the attraction of the terrestrial spheroid, which causes the rotation of the Moon to participate in these great inequalities, did not continually bring back the same hemisphere of this satellite to us, and thus render the other hemisphere (*d*) for ever invisible. It is thus that the primitive attraction of the three first satellites of Jupiter originally established, and maintains the relation which is observed between their mean motions, and which consists in this, that the mean longitude of the first satellite minus three times that of the second, plus twice that of the third is equal to two right angles. In consequence of the celestial attractions the duration of the revolution of each planet is always very nearly the same. The change of inclination of its orbit to that of its equator being confined within narrow limits, only produces slight changes in the seasons. It seems that nature has arranged every thing in the heavens, to secure the continuation of the planetary system, by views similar to those which she appears to follow so admirably on the earth,

the motion of the solar system is extremely complicated. The Moon describes an orbit nearly circular about the earth, but seen from the Sun, it appears to describe a series of epicycles, of which the centres exist on the terrestrial orbit. In like manner, the earth describes a series of epicycles, of which the centres lie on the curve, which the Sun describes about the common centre of gravity of the group of stars, of which it makes a part. Finally, the Sun himself describes a series of epicycles, of which the centres lie on the curve described by the centre of gravity of this group, about that of the universe. Astronomy has already made an important step, in making us acquainted with the motion of the earth, and the epicycles which the Moon and the satellites describe on the orbits of their respective primary planets. But if ages were necessary in order to know the motions of the planetary system, what a great length of time must be required for the determination of the motions of the Sun and the stars; notwithstanding this, such motions appear to be already indicated by observations. From all of them considered together, it has been inferred, that the bodies of the solar system are in motion towards the constellation Hercules; but however they at the same time seem to prove that the apparent motions of the stars result from a combination of their proper motions with that of the Sun. (*f*)

There are also remarked some singular motions in the *double* stars: we have denominated such, those stars which, when seen through a telescope,

appear to be composed of two neighbouring stars. These two stars revolve about one another in a manner sufficiently sensible to enable us to determine for some of them, by means of a few years observation, the duration of their revolutions.

All these motions of the stars, their parallaxes, the periodic variations of the light of the changeable stars, and the durations of their motions of rotation; a catalogue of those stars which just appear and then disappear, and their position at the instant of their transient passage; finally, the successive changes in the figure of those nebulæ which are already sensible in some of them, and particularly in the beautiful nebula of Orion, will be, relatively to the stars, the principal objects of Astronomy in subsequent ages. Its progress depends on these three things: the measure of time, that of angles, and the perfection of optical instruments. The two first are nearly as perfect as we could wish; it is therefore to the improvement of the latter that our attention should be directed, for there can be no doubt but that if we succeeded in enlarging the apertures of our achromatic telescopes, they would enable us to discover in the heavens, phenomena which have been hitherto invisible, especially if we were able to remove them to the pure and rare atmosphere of the high mountains of the equator.

There are also numerous discoveries to be made in our own system. The planet Uranus and its satellites, which have been lately disco-

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appear to be composed of two neighbouring stars. These two stars revolve about one another in a manner sufficiently sensible to enable us to determine for some of them, by means of a few years observation, the duration of their revolutions.

All these motions of the stars, their parallaxes, the periodic variations of the light of the changeable stars, and the durations of their motions of rotation; a catalogue of those stars which just appear and then disappear, and their position at the instant of their transient passage; finally, the successive changes in the figure of those nebulæ which are already sensible in some of them, and particularly in the beautiful nebula of Orion, will be, relatively to the stars, the principal objects of Astronomy in subsequent ages. Its progress depends on these three things: the measure of time, that of angles, and the perfection of optical instruments. The two first are nearly as perfect as we could wish; it is therefore to the improvement of the latter that our attention should be directed, for there can be no doubt but that if we succeeded in enlarging the apertures of our achromatic telescopes, they would enable us to discover in the heavens, phenomena which have been hitherto invisible, especially if we were able to remove them to the pure and rare atmosphere of the high mountains of the equator.

There are also numerous discoveries to be made in our own system. The planet Uranus and its satellites, which have been lately disco-

the planets. It is therefore necessary that the cause which determined the motions of these bodies, rendered them also nearly circular. This cause then must also have influenced the great excentricity of the orbits of comets, and their motion in every direction; for, considering the orbits of retrograde comets, as being inclined more than one hundred degrees to the ecliptic, we find that the mean inclination of the orbits of all the observed comets, approaches near to one hundred degrees, which would be the case if the bodies had been projected at random. (*b*)

What is this primitive cause ? In the concluding note of this work I will suggest an hypothesis which appears to me to result with a great degree of probability, from the preceding phenomena, which however I present with that diffidence, which ought always to attach to whatever is not the result of observation and computation.

Whatever be the true cause, it is certain that the elements of the planetary system are so arranged as to enjoy the greatest possible stability, unless it is deranged by the intervention of foreign causes. From the sole circumstance that the motions of the planets and satellites are performed in orbits nearly circular, in the same direction, and in planes which are inconsiderably inclined to each other, the system will always oscillate about a mean state, from which (*c*) it will deviate but by very small quantities. The mean motions of rotation and of revolution of these different bodies are uniform, and their mean distances from the foci of the principal forces which actuate them are constant; all the secular inequalities are periodic.

The most considerable are those which affect the motions of the Moon, with respect to its perigee, to its nodes and the Sun; they amount to several circumferences, but after a great number of centuries they are reestablished. In this long interval all the parts of the lunar surface would be successively presented to the earth, if the attraction of the terrestrial spheroid, which causes the rotation of the Moon to participate in these great inequalities, did not continually bring back the same hemisphere of this satellite to us, and thus render the other hemisphere (*d*) for ever invisible. It is thus that the primitive attraction of the three first satellites of Jupiter originally established, and maintains the relation which is observed between their mean motions, and which consists in this, that the mean longitude of the first satellite minus three times that of the second, plus twice that of the third is equal to two right angles. In consequence of the celestial attractions the duration of the revolution of each planet is always very nearly the same. The change of inclination of its orbit to that of its equator being confined within narrow limits, only produces slight changes in the seasons. It seems that nature has arranged every thing in the heavens, to secure the continuation of the planetary system, by views similar to those which she appears to follow so admirably on the earth, for the preservation of individuals and the perpetuity of the species. It is principally to the attraction of the great bodies which are placed in the centre of the system of the planets, and the system of the satellites, that the stability of these systems is due, which the mutual action of all the bodies of the system, and extraneous attractions tend to derange. If the action of Jupiter ceased; his satellites, which now appear to move with such admirable regularity, would be immediately disturbed, and each would describe about the Sun a very excentric ellipse; (*e*) others would recede indefinitely in hyperbolic orbits. Thus an attentive inspection of the solar system evinces the necessity of some paramount central force, in order to maintain the entire system together, and secure the regularity of its motions.

These considerations of themselves will be sufficient to explain the disposition of this system, unless the Geometer extends his view farther, and seeks, in the primordial laws of nature, the cause of the most remarkable phenomena of the universe. Some have been already reduced to these laws. Thus the stability of the poles of the Earth, and that of the equilibrium of the seas, which are both necessary for the preservation of organised beings, are simple consequences of the rotation of the earth and of universal gravitation. By its rotation the earth has been compressed, and its axis of revolution is become one of the principal axes about which the motion of rotation is invariable. In consequence of this gravity the denser strata are nearer to the centre of the earth, of which the mean density thus surpasses that of the waters which surround it, which is sufficient to secure the stability of the equilibrium of the seas, and to put a check to the fury of the waves; in fine, if the conjectures which I have proposed on the origin of the planetary system have any foundation, the stability of this system is also a consequence of the laws of motion. (*f*) These phenomena, and some others which are explained in a similar manner, induce us to think that every thing depends on these laws by relations more or less concealed; but of which it is wiser to avow our ignorance than to substitute imaginary causes, for the sole purpose of dissipating our anxiety. I must here remark how Newton has erred on this point, from the method which he has otherwise so happily applied. Subsequently to the publication of his discoveries on the system of the world and on light, this great philosopher abandoned himself to speculations of another kind, and inquired what motives induced the author of nature to give to the solar system its present observed constitution. After detailing in the scholium which terminates the principles of natural philosophy, the remarkable phenomenon of the motions of the planets and of the satellites in the same direction, very nearly in the same plane, and in orbits Q. P. circular, he adds, all these motions, so very regular, do not arise from mechanical causes, because the comets move in all regions of the heavens, and in orbits very excentric. (*g*) "This admirable arrangement of the Sun, of the planets, "and of the comets, can only be the work of an "intelligent aud most powerful being." At the end of his optics he suggests the same thought, in which he would be still more confirmed, if he had known that all the conditions of the arrangement of the planets and of the satellites are precisely those which secure their stability. "A blind "fate," says he, "could never make all the pla- "nets to move thus, with some irregularities "hardly perceivable, which may arise from the "mutual action of the planets and of the comets, "and which, probably, in the course of time will "become greater, till in fine the system may re- "quire to be restored by its author." But could not this arrangement of the planets be itself an effect of the laws of motion; and could not the supreme intelligence which Newton makes to interfere, make it to depend on a more general phenomenon? such as, according to us, a nebulous matter distributed in various masses throughout the immensity of the heavens. Can one even affirm that the preservation of the planetary system entered into the views of the Author of Nature? The mutual attraction of the bodies of this system cannot alter its stability, as Newton supposes; but may there not be in the heavenly regions another fluid besides light? Its resistance, and the diminution which its emission produces in the mass of the Sun, ought at length to destroy the arrangement of the planets, so that to maintain this, a renovation would become evidently necessary. And do not all those species of animals which are extinct, but whose existence Cuvier has ascertained with such singular sagacity, and also the organization in the numerous fossil bones which he has described, indicate a tendency to change in things, which are apparently the most permanent in their nature? The magnitude and importance of the solar system ought not to except it from this general law; for they are relative to our smallness, and this system, extensive as it appears to be, is but an insensible point in the universe. If we trace the history of the progress of the human mind, and of its errors, we shall observe final causes perpetually receding, according as the boundaries of our knowledge are extended. These causes, which Newton transported to the limits of the solar system, were, in his time, placed in the atmosphere in order to explain the cause of meteors: in the view of the philosopher, they are therefore only an expression of our ignorance of the true causes.

Leibnitz, in his controversy with Newton, relative to the invention of the infinitesmal calculus, attacks him with great force on account of his introducing the divinity to restore order into the solar system. "It is," says he, "to have too "confined notions of the wisdom and power of "the Deity." Newton rejoined by an equally severe critique on the preestablished harmony of Leibnitz, which he denominated a continual miracle. Subsequent ages have not admitted these vain hypotheses; they have, however, rendered the most ample justice to the mathematical labours of these two great men; the discovery of

the motion of the solar system is extremely complicated. The Moon describes an orbit nearly circular about the earth, but seen from the Sun, it appears to describe a series of epicycles, of which the centres exist on the terrestrial orbit. In like manner, the earth describes a series of epicycles, of which the centres lie on the curve, which the Sun describes about the common centre of gravity of the group of stars, of which it makes a part. Finally, the Sun himself describes a series of epicycles, of which the centres lie on the curve described by the centre of gravity of this group, about that of the universe. Astronomy has already made an important step, in making us acquainted with the motion of the earth, and the epicycles which the Moon and the satellites describe on the orbits of their respective primary planets. But if ages were necessary in order to know the motions of the planetary system, what a great length of time must be required for the determination of the motions of the Sun and the stars; notwithstanding this, such motions appear to be already indicated by observations. From all of them considered together, it has been inferred, that the bodies of the solar system are in motion towards the constellation Hercules; but however they at the same time seem to prove that the apparent motions of the stars result from a combination of their proper motions with that of the Sun. (*f*)

There are also remarked some singular motions in the *double* stars: we have denominated such, those stars which, when seen through a telescope,

ways every where the same. All these stars, after they become invisible, do not change their place during their appearance. Therefore there exists, in the immensity of space, opaque bodies as considerable in magnitude, and perhaps equally numerous as the stars.

It appears that far from being distributed at distances which are nearly equal, the stars are disposed in groups, some of which contain thousands of these objects. Our Sun, and the most brilliant stars, probably constitute part of one of those groups, which, seen from the earth, appear to surround the earth, and form the milky way. The great number of stars, which are seen in the field of a powerful telescope, directed towards this way, evinces its immense distance, which is a thousand times greater than the distance of Syrius from the earth, so that it is probable that rays emanating from these stars have employed several centuries to reach the earth. To a spectator at an immense distance from the milky way, it would present the appearance of an uninterrupted band of white light, having a very inconsiderable diameter, for the irradiation which subsists even in our best constructed telescopes, would not cover the interval between the stars. It is therefore probable, that amongst the nebulæ several consist of groups of a great number of stars, which, viewed from their interior, appear similar to the milky way. If now we reflect on the profusion of stars and nebulæ distributed through the heavenly regions, and on the immense intervals between them, the imagination, struck with astonishment at the magnitude of the universe, will find it difficult to assign any limits to it.

Herschel, while observing the nebulæ by means of his powerful telescopes, traced the progress of their condensation, not on one only, as their progress does not become sensible until after the lapse of ages, but on the whole of them, as in a vast forest we trace the growth of trees, in the individuals of different ages which it contains. He first observed the nebulous matter diffused in several masses, through various parts of the heavens, of which it occupied a great extent. In some of these masses he observed that this matter was fully condensed about one or more nuclei, a little more brilliant. In other nebulæ, these nuclei shine brighter, relatively to the nebulosity which environs them. As the atmosphere of each nucleus separates itself by an ulterior condensation, there result several nebulæ constituted of brilliant nulcei very near to each other, and each surrounded by its respective atmosphere; sometimes the nebulous matter being condensed in a uniform manner, produces the nebulæ which are termed *planetary*. Finally, a greater degree of condensation transforms all these nebulæ into stars. The nebulæ, classed in a philosophic manner, indicate, with a great degree of probability, their future tranformation into stars, and the anterior state of the nebulosity of existing stars. Thus, by tracing the progress of condensation of the nebulous matter, we descend to the consideration of the Sun, formerly surrounded by an immense atmosphere, to which consideration we can also arrive, from an examination of the phenomena of the solar system, as we shall see in our last note. Such a marked coincidence, arrived at by such different means, renders the existence of this anterior state of the Sun extremely probable.

Connecting the formation of comets with that of nebulæ, they may be considered as small nebulæ, wandering from one solar system to another, and formed by the condensation of the nebulous matter which is so profusely distributed throughout the universe. The comets will be thus, relatively to our system, what the meteoric stones appear to be relatively to the earth, to which they do not appear to have originally belonged. (*e*) When these stars first become visible, they present an appearance perfectly similar to the nebulæ; so much so, that they are frequently mistaken for them, and it is only by their motion, or by our knowing all the nebulæ contained in our part of the heavens, that we are able to distinguish one from the other. This hypothesis explains, in a satisfactory manner, the increase of the heads and tails of the comets, according as they approach the sun, and the extreme rarity of their tails, (which, notwithstanding their great depth, do not sensibly diminish the brilliancy of the stars seen through them;) the motions of the comets, which are performed in every direction, and the great excentricity of their orbits.

From the preceding considerations, which are founded on telescopic observations, it follows, that

the motion of the solar system is extremely complicated. The Moon describes an orbit nearly circular about the earth, but seen from the Sun, it appears to describe a series of epicycles, of which the centres exist on the terrestrial orbit. In like manner, the earth describes a series of epicycles, of which the centres lie on the curve, which the Sun describes about the common centre of gravity of the group of stars, of which it makes a part. Finally, the Sun himself describes a series of epicycles, of which the centres lie on the curve described by the centre of gravity of this group, about that of the universe. Astronomy has already made an important step, in making us acquainted with the motion of the earth, and the epicycles which the Moon and the satellites describe on the orbits of their respective primary planets. But if ages were necessary in order to know the motions of the planetary system, what a great length of time must be required for the determination of the motions of the Sun and the stars; notwithstanding this, such motions appear to be already indicated by observations. From all of them considered together, it has been inferred, that the bodies of the solar system are in motion towards the constellation Hercules; but however they at the same time seem to prove that the apparent motions of the stars result from a combination of their proper motions with that of the Sun. (*f*)

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All these motions of the stars, their parallaxes, the periodic variations of the light of the changeable stars, and the durations of their motions of rotation; a catalogue of those stars which just appear and then disappear, and their position at the instant of their transient passage; finally, the successive changes in the figure of those nebulæ which are already sensible in some of them, and particularly in the beautiful nebula of Orion, will be, relatively to the stars, the principal objects of Astronomy in subsequent ages. Its progress depends on these three things: the measure of time, that of angles, and the perfection of optical instruments. The two first are nearly as perfect as we could wish; it is therefore to the improvement of the latter that our attention should be directed, for there can be no doubt but that if we succeeded in enlarging the apertures of our achromatic telescopes, they would enable us to discover in the heavens, phenomena which have been hitherto invisible, especially if we were able to remove them to the pure and rare atmosphere of the high mountains of the equator.

There are also numerous discoveries to be made in our own system. The planet Uranus and its satellites, which have been lately disco-

vered, give grounds to suppose that other planets, as yet not observed, exist. It has been even conjectured that one must exist between Jupiter and Mars, in order to satisfy the double progression, which obtains (*g*) very nearly, between the intervals of the planetary orbits, to that of Mercury. This conjecture has been confirmed by the discovery of four small planets, whose distances from the Sun differ little from that, which this double progression assigns to a planet intermediate between Jupiter and Mars. The action of Jupiter on these planets increased by the magnitude of the excentricities and of the inclinations of the intersecting orbits, produces considerable inequalities in their motions, which throw new light on the theory of the celestial attractions, and will enable us to render them more perfect. The arbitrary elements of this theory, and the convergence of its approximations, depend on the precision of observations and on the progress of analysis, and this should thereby acquire every day more and more accuracy. The great secular inequalities of the heavenly bodies, which is a consequence of their mutual attractions, and which has been already indicated by observation, will be developed in the course of ages. By means of observations on the satellites, made with powerful telescopes, we shall be able to render their theory more perfect, and perhaps to discover new satellites. By accurate and repeated measures of the Earth, all the inequalities of the figure of the Earth, of gravity at its surface, will be determined, and in a short time all Europe will be covered with a chain of triangles, which will accurately determine the position, the curvature, and the magnitude of all its parts. The phenomena of the tides, and their remarkable varieties in the two hemispheres, will be determined by a long series of observations, compared with the theory of gravity. We will ascertain whether the motions of rotation and revolution of the Earth are sensibly changed by the changes which it experiences at its surface, and by the impact of meteoric stones, which, according to all probability, come from the depths of the heavenly regions. The new comets which will appear; those which, moving in hyperbolic orbits, wander from one system to another; the returns of those which move in elliptic orbits, and the changes in the form and intensity of light, which they undergo at each appearance, will be observed; and also the peturbations which all those stars produce in the planetary motions, those which they experience themselves, and which, at their approach to a large planet, may entirely derange their motions; finally, the changes which the motions and the orbits of the planets and satellites experience from the action of the stars, and perhaps likewise from the resistance of ethereal media; such are the principal objects which the solar system offers to the investigations of future Astronomers and Mathematicians.

Astronomy, from the dignity of the subject, and the perfection of its theo-- the most beautiful monument of the human mind—the noblest record of its intelligence. Seduced by the illusion of the senses, and of self-love, man considered himself, for a long time, as the centre about which the celestial bodies revolved, and his pride was justly punished by the vain terrors they inspired. The labour of many ages has at length withdrawn the veil which covered the system. And man now appears, upon a small planet, almost imperceptible in the vast extent of the solar system, itself only an insensible point in the immensity of space. The sublime results to which this discovery has led, may console him for the limited place assigned to the Earth, by showing him his proper magnitude, in the extreme smallness of the base which he made use of to measure the heavens. Let us carefully preserve, and even augment the number of these sublime discoveries, which constitute the delight of thinking beings.

These indeed have rendered important services to navigation and astronomy; but their great benefit consists in their having dissipated the alarms occasioned by extraordinary celestial phenomena, and thus exterminating the errors arising from the ignorance of our true relation with nature; errors and apprehensions which would speedily spring up again, if the light of the sciences was extinguished.

NOTES.

NOTE I.

A SEPARATE history of the Chinese Astronomy was published by the Jesuit Gaubel, who appears to have been particularly well acquainted with the subject. He discussed again in much detail, in the 26th letter of the Instructive Letters, the ancient part of this history. I published in the Connaisance des Tems for the year 1809, an invaluable manuscript of the same Jesuit, on the solstices and meridian shadows of the gnomon observed at China. From these treatises it appears that Tcheou Kong observed the meridian shadows of a gnomon eight Chinese feet long, at the solstices in the city of Loyang, now called Honan Fou, in the province of Honan. He carefully traced the meridian, and levelled the earth on which the shadow was projected. He found the length of the meridian shadow to be one foot and a half at the summer solstice, and thirteen feet at the winter solstice. In order to infer from these observations the obliquity of the ecliptic, he applied several corrections to them; the most considerable is that of the Sun's semi-diameter, for it is evident that the extremity of the shadow of the gnomon indicates the height of the upper limb of this star; it is therefore necessary to subtract from this height the apparent semidiameter of the Sun, in order to obtain the height of its centre. It is strange, that so simple and essential a correction should have escaped the observation of all the old Astronomers of the Alexandrian school; it must have caused their geographical latitudes to have erred by a quantity very nearly equal to this semidiameter. A second correction respects the astronomical refraction, which not being observed, may without sensible error be supposed such as would correspond to a temperature of ten degrees, and to a height of the barometer equal to 0,76. Finally, a third correction respects the parallax of the Sun, and reduces these corrections to the centre of the Earth. By applying these three corrections, to the preceding observations, the height of the centre of the Sun, referred to the centre of the Earth, is found to be equal to 87°,9049 at the summer solstice, and to 34°,7924 at the winter solstice. These heights assign 38°,6513 for the height of the pole at Layang, which result differs very little from the mean between all the observations of the Jesuit missionary on the latitude of this city: they make the obliquity of the ecliptic at the epoch of Tcheou-Kong to be about 26,6563. This epoch may without sensible error be fixed at the year 1100 before our æra. If by means of the formula given in the sixth book of the treatise on Celestial Mechanics, we go back to this epoch, we shall find that the obliquity ought then to be equal to 26,5161. The difference 402″ will appear very inconsiderable, if we consider the uncertainty which exists relative to the masses of the planets, and that which the observations on the gnomon present, especially on account of the penumbra, which renders the umbra itself very indistinctly terminated.

Tcheou-Kong also observed the position of the winter solstice, with respect to the stars, and he fixed it at two Chinese degrees of *Nu*, a Chinese constellation, which commences with ϵ of Aquarius. In China, the division of the circumference was always regulated by the length of the year, so that the Sun described a degree every day; and the year at the epoch of Tcheau-Kong being supposed equal to $365^{d}\frac{1}{4}$; two degrees correspond to $2^{\circ},1905$ of the decimal division of the quadrant of the circle. The stars having been at the same epoch referred to the equator, the right ascension of the star was, according to this observation, about $297^{\circ},8096$. By the formulæ of the celestial mechanics it ought to be $298^{\circ},7265$, in the year 1100 before our æra. In order to get rid of the difference 9169″, it is sufficient to go back fifty-four years beyond this, which is inconsiderable if we consider the uncertainty of the precise epoch at which this great prince made his observations, and particularly that of the observations themselves. There also exists an observation on the instant of the solstice, but the greatest error to apprehend is in the manner of referring the solstice to the star ι of Aquarius; whether Tcheou-Kong made use of the difference in time, between the passages of the star and Sun over the meridian, or whether he measured the distance of the Moon from this star, at the moment of the occurrence of a lunar eclipse, two means employed by the Chinese Astronomers.

NOTE II.

By means of a long series of observations, the Chaldeans recognised that in 19756 days, the Moon made 669 revolutions with respect to the Sun, 717 anomalistic revolutions, *i. e.* with respect to the points of its greatest velocity, and 726 revolutions with respect to its nodes. They added $\frac{4}{45}$ of a revolution to the position of those two stars, in order to obtain in this interval 723 sidereal revolutions of the Moon, and 54 of the Sun. Ptolemy, in explaining this period, attributes it to the ancient Astronomers, without specifying the Chaldeans; but Geminus, a cotemporary of Sylla's, whose treatise on Astronomy has come down to us, removes all doubt on this head, for he not only attributes this period to the Chaldeans, but he even gives their method for computing the anomaly of the Moon. They supposed that from the least to the greatest velocity of the Moon, its angular motion accelerated by a third of a degree every day, during one half of the anomalistic revolution, and that it retarded by the same quantity during the other half. He is mistaken in supposing that the increments, which are proportional to the cosines of the distance of the Moon from its perigee, are constant. Notwithstanding this error, the preceding method is creditable to the sagacity of the Chaldean Astronomers; it is the only monument of this kind which remains previously to the foundation of the Alexandrian school. The period of which I have spoken supposes that the sidereal year is very nearly equal to $365\frac{1}{4}$; that of 365,2576, which Albaterius ascribed to the Chaldeans, cannot only belong to times posterior to Hipparcus.

NOTE III.

In the second book of his Geography, chap. iv., Strato states, that according to Hipparcus, the proportion of the shadow at Byzantium to the gnomon, is the same which Pythias asserts that he observed it to be at Marseilles; and in the 5th chapter of the same book he quotes from Hipparchus, that at Byzantium at the summer solstice, the proportion of the shadow to the gnomon is that of 42 minus $\frac{1}{5}$ to 120. It is unquestionable from this observation, that Ptolemy, in the 6th chapter of the second book of the Almagest, makes the parallel on which the duration of the longest day of the year is, five-eighths of the astronomical day, to pass through Marseilles; which supposes that the proportion of the meridian shadow to the gnomon at the summer solstice, is that of 42 minus $\frac{1}{5}$ to 120. Pytheas was at the latest, a contemporary of Aristotle; therefore we may without sensible error refer his observation to the year 350 before our æra. By correcting it for the refraction, the parallax of the Sun, and its semidiameter, it makes the zenith solstitial distance of the centre of the Sun from the zenith of Marseilles, equal to 21,6386. The latitude of the Observatory of this city is 48,1077. If the preceding distance be subtracted from it, the obliquity of the ecliptic at the time of Pytheas comes out equal to 26,4691. This obliquity, when compared with that given in the time of Tcheou-Kong, indicates already a diminution in this element. From the formula given in the Celestial Mechanics, the obliquity of the ecliptic 350 years before our æra comes out equal to 26,4095; the difference 596″ between this result and that of the observation of Pytheas, is within the limits of the errors of this kind of observation.

NOTE IV.

Hipparcus found, from comparing together a great number of eclipses of the Moon, 1st, that in the interval of 126007^d plus $\frac{1}{24}$ of a day, the Moon performs 4267 revolutions with respect to the Sun, 4573 revolutions relatively to its perigee, and 4612 revolutions relatively to the fixed stars, minus eight degrees and one-third; 2dly, that during 5458 synodic months it performs 5923 revolutions relatively to its nodes. According therefore to this result, the motions of the Moon in the interval of $126007^d\frac{1}{24}$ are

with respect to the Sun 1706800°
with respect to the perigee 1829200°
with respect to the node 1852212°, 89368.

A comparison of these motions with those which have been determined by combining together all the modern observations, should render their acceleration, which is indicated by the theory of universal gravitation, very sensible. In fact, those who have thus determined it for the commencement of this century, assign, for the same interval of time, the preceding quantities increased respectively by + 2657″,0; + 10981″,9; + 432″,8. The acceleration of these three motions from the time of Hipparchus to the present, is evident: we see, moreover, that the acceleration of the motion of the Moon with respect to the Sun, is about four times less than that of its motion with respect to the perigee, whilst it surpassses considerably the acceleration of its motion with respect to the node. This is very nearly conformable to the theory of gravity, according to which, these accelerations are Q. P. in the ratio of the numbers 1; 4,70197; 0,38795. Hipparchus supposed that Babylon was more eastward than Alexandria, by 3472″ of time. According to the observations of Beauchamp it was still more eastward by *557″*. This ought to increase a little the mean lunar motions, which Hipparchus inferred from a comparison of his observations, with those of the Chaldeans.

Ptolemy has not transmitted to us the epochs of the lunar motions of Hipparchus; but from the slight changes which he made in these motions, and from his always endeavouring to make his results approximate to those of this great Astronomer, we are justified in supposing that the epochs of Hipparchus differ very little from those of the tables of Ptolomy, which assign at the epoch of Nobonassar, *i. e.* the 26th of February of the year 746 before our æra, at mid-day, mean time of Alexandria,

distances from the Moon	to the Sun	78°, 4630
	to the perigee	98 , 6852
	to the ascending node	93 , 6111

If we go back to this epoch, by means of the mean motions determined for the commencement of this century, from the comparison of modern observations solely; if, moreover, we suppose, agreeably to the latest observations, that Alexandria is more eastward than Paris by 7731″,48 of time, we shall find the distances less than the preceding by the respective quantities — 1°,6316; — 7°,6569; — 0°,8205. These differences, which are much too great to be ascribed to the errors of either ancient or modern determinations, evince incontrovertibly the acceleration of the lunar motions, and the necessity of admitting the secular equations. The secular equation of the distance of the Sun from the Moon, which equation is the same as that of the mean motion of the Moon, since that of the Sun is uniform, becomes at the epoch of Nabonassar, 2°, 0480. In order to obtain those of the distance of the Moon from its perigee and its ascending node at the same epoch, it is necessary to multiply the preceding, by the numbers 4,70197 and 0,38795 respectively. Therefore the three secular equations will be 2,°480; 9°,6299; 0°,7949. By adding them to the three preceding differences, they are reduced to the three following +4164″; +19730″; —260″. These differences thus reduced may depend on the errors of ancient and modern observations, for the secular mean motion of the node being determined, for example, by a comparison of the observations of Bradley with those made since his time, *i. e.* by the observations made in the last half century; there may exist in its value, an uncertainty of half a minute at least.

NOTE V.

The Astronomers of Almamon found, by their observations, the greatest equation of the centre of the Sun equal to 2°,2037, greater than ours by 635″. Albatenius, Ebn Junis and a great number of other Arabian astronomers, make very slight changes in this result, which evinces incontrovertably, the diminution of the excentricity of the terrestrial orbit from their time to the present. The same astronomers found the longitude of the apogee of the Sun to be 830, equal to 91,°8333; which corresponds very nearly with the theory of gravity, according to which the longitude at the same epoch ought to be 92°,047. This theory assigns 36,″44 for the annual motion of this apogee with respect to the fixed stars; and the preceding observation gives the same motion to within a few seconds. Finally, from a comparison of their observations of the equinoxes with those of Ptolemy, they found the duration of the period of the tropical year to be $365^d,240706$. About the year 803, which is more than twenty-five years before the formation of the verified table, the Arabian astronomer found by comparing his observations with those of Hipparchus, a much more exact duration of the year; he determined it to be 365,242181. Almost all the Arabian astronomers supposed that the obliquity of the ecliptic was about 26,2037; but it seems that this result is influeneed by the erroneous parallax which they assigned to the Sun; at least it is certainly the case with respect to the observations of Ebn Junis, which when corrected for this erroneous parallax, and for the refraction, make this obliquity 26,1932 for the year 1000. Theory makes it at this epoch, $26,^\circ 2009$, the difference $-77''$ is within the limits of the errors of the Arabian observations. The epochs of the astronomical tables of Ebn Junis, confirm the secular equations of the motions of the Moon; the great inequalities of of Jupiter and Saturn are likewise confirmed by these epochs, and by the conjunction of these two planets, observed at Cairo by this astronomer. This observation, one of the most important in Arabian astronomy, was made on the 31st of October, 1007, at $0^d,16$ of mean time at Paris. Ebn Junis found the excess of the geocentric longitude of Saturn above that of Jupiter, equal to $4444''$. The tables constructed by M. Bouvard, according to my theory, and from a comparison of all the observations made by Bradley, Maskeline, and at the Royal Observatory, make this excess 5191″; the difference 747″ is less than the error of which this observation is susceptible.

NOTE VI.

The observations of the meridian shadows of the gnomon, made by Cocheou-King, and inserted in the Connaissance des Tems of the year 1809, assign 2°,1759 as the greatest equation of the Sun for the year 1280, which exceeds its actual value by about 377″. They likewise make the obliquity of the ecliptic at the same epoch, about 26°,1489, which is greater by 757″, than the actual obliquity. Hence it appears that the diminution of these two elements is demonstrated by these observations.

An observation of the obliquity of the ecliptic by Ulug-Beigh, when corrected for refraction and parallax, makes the obliquity in 1437 equal to 26°,1444; it is smaller than the preceding, as it ought to be, on account of the interval of 157 years, which separates the corresponding epochs. The following table clearly points out the successive diminution of this element in an interval of 2900 years.

Tcheou King, 1100 years before our æra - -	26°,5563	402″
Pythias, 350 years before our æra - - -	26°,4691	596″
Ebn Junis, the year one thousand - - -	26°,1932	—77
Cocheou-King, 1280 -	26°,1489	—62
Ulug-Beigh, 1437 -	26°,1444	130
In 1801 - - -	26°,0732.	

The second row of numbers indicates the excess of this obliquity over the results of the formulæ given in the Celestial Mechanics.

NOTE VII. AND LAST.

From the preceding chapter it appears, that we have the five following phenomena to assist us in investigating the cause of the primitive motions of the planetary system. The motions of the planets in the same direction, and very nearly in the same plane; the motions of the satellites in the same direction as those of the planets; the motions of rotation of these different bodies and also of the Sun, in the same direction as their motions of projection, and in planes very little inclined to each other; the small eccentricity of the orbits of the planets and satellites; finally, the great eccentricity of the orbits of the comets, their inclinations being at the same time entirely indeterminate.

Buffon is the only individual that I know of, who, since the discovery of the true system of the world, endeavoured to investigate the origin of the planets and satellites. He supposed that a comet, by impinging on the Sun, carried away a torrent of matter, which was reunited far off, into globes of different magnitudes and at different distances from this star. These globes, when they cool and become hardened, are the planets and their satellites. This hypothesis satisfies the first of the five preceding phenomena; for it is evident that all bodies thus formed should move very nearly in the plane which passes through the centre of the Sun, and through the direction of the torrent of matter which has produced them: but the four remaining phenomena appear to me inexplicable on this supposition. Indeed the absolute motion of the molecules of a planet ought to be in the same direction as the motion of its centre of gravity; but it by no means follows from this, that the motion of rotation of a planet should be also in the same direction. Thus the Earth may revolve from east to west, and yet the absolute motion of each of its molecules may be directed from west to east. This observation applies also to the revolution of the satellites, of which the direction in the same hypothesis, is not necessarily the same as that of the motion of projection of the planets.

The small eccentricity of the planetary orbits is a phenomenon, not only difficult to explain on this hypothesis, but altogether inconsistent with it. We know from the theory of central forces, that if a body which moves in a re-entrant orbit about the Sun, passes very near the body of the Sun, it will return constantly to it, at the end of each revolution. Hence it follows that if the planets were originally detached from the Sun, they would touch it, at each return to this star; and their orbits, instead of being nearly circular, would be very eccentric. Indeed it must be admitted that a torrent of matter detached from the Sun, cannot be compared to a globe which just skims by its surface: from the impulsions which the parts of this torrent receive from each other, combined with their mutual attraction, they may, by changing the direction of their motions, increase the distances of their perihelions from the Sun. But their orbits should be extremely eccentric, or at least all the orbits would not be *q. p.* circular, except by the most extraordinary chance. Finally, no reason can be assigned on the hypothesis of Buffon, why the orbits of more than one hundred comets, which have been already observed, should be all very eccentric. This hypothesis, therefore, is far from satisfying the preceding phenomena. Let us consider whether we can assign the true cause.

Whatever may be its nature, since it has produced or influenced the direction of the planetary motions, it must have embraced them all within the sphere of its action; and considering the immense distance which intervenes between them, nothing could have effected this but a fluid of almost indefinite extent. In order to have impressed on them all a motion *q. p.* circular and in the same direction about the Sun, this fluid must environ this star, like an atmosphere. From a consideration of the planetary motions, we are therefore brought to the conclusion, that i- consequence of an excessive heat, the sola- mosphere originally extended beyond the of all the planets, and that it has successively contracted itself within its present limits.

In the primitive state in which we have supposed the Sun to be, it resembles those substances which are termed nebulæ, which, when seen through telescopes, appear to be composed of a nucleus, more or less brilliant, surrounded by a nebulosity, which, by condensing on its surface, transforms it into a star. If all the stars are conceived to be similarly formed, we can suppose their anterior state of nebulosity to be preceded by other states, in which the nebulous matter was more or less diffuse, the nucleus being at the same time more or less brilliant. By going back in this manner, we shall arrive at a state of nebulosity so diffuse, that its existence can with difficulty be conceived.

For a considerable time back, the particular arrangement of some stars visible to the naked eye, has engaged the attention of philosophers. Mitchel remarked long since how extremely improbable it was that the stars composing the constellation called the Pleiades, for example, should be confined within the narrow space which contains them, by the sole chance of hazard; from which he inferred that this group of stars, and the similar groups which the heavens present to us, are the effects of a primitive cause, or of a primitive law of nature. These groups are a general result of the condensation of nebulæ [illegible] nuclei; for it is evident that the ne[illegible] being perpetually attracted by [illegible] nuclei, ought at length to form a [illegible]ce to that of the Pleiades. The condensation of nebulæ consisting of two nuclei, will in like manner form stars very near to each other, revolving the one about the other like to the double stars, whose respective motions have been already recognized.

But in what manner has the solar atmosphere determined the motions of rotation and revolution of the planets and satellites? If these bodies had penetrated deeply into this atmosphere, its resistance would cause them to fall on the Sun. We may therefore suppose that the planets were formed at its successive limits, by the condensation of zones of vapours, which it must, while it was cooling, have abandoned in the plane of its equator.

Let us resume the results which we have given in the tenth chapter of the preceding book. The Sun's atmosphere cannot extend indefinitely; its limit is the point where the centrifugal force arising from the motion of rotation balances the gravity; but according as the cooling contracts the atmosphere, and condenses the molecules which are near to it, on the surface of the star, the motion of rotation increases; for in virtue of the principle of areas, the sum of the areas described by the radius vector of each particle of the Sun and of its atmosphere, and projected on the plane of its equator, is always the same. Consequently the rotation ought to be quicker, when these particles approach to the centre of the Sun. The centrifugal force arising from this motion becoming thus greater; the point where the gravity is equal to it, is nearer to the centre of the Sun. Supposing therefore, what is natural to admit, that the atmosphere extended at any epoch as far as this limit, it ought, according as it cooled, to abandon the molecules, which are situated at this limit, and at the successive limits produced by the increased rotation of the Sun. These particles, after being abandoned, have continued to circulate about this star, because their centrifugal force was balanced by their gravity. But as this equality does not obtain for those molecules of the atmosphere which are situated on the parallels to the Sun's equator, these have come nearer by their gravity to the atmosphere according as it condensed, and they have not ceased to belong to it, inasmuch as by this motion, they have approached to the plane of this equator.

Let us now consider the zones of vapours, which have been successively abandoned. These zones ought, according to all probability, to form by their condensation, and by the mutual attraction of their particles, several concentrical rings of vapours circulating about the Sun. The mutual friction of the molecules of each ring ought to accelerate some and retard others, until they all had acquired the same angular motion. Consequently the real velocities of the molecules which are farther from the Sun, ought to be greatest. The following cause ought likewise to contribute to this difference of velocities: The most distant particles of the Sun, and which, by the effects of cooling and of condensation, have collected so as to constitute the superior part of the ring, have always described areas proportional to the times, because the central force by which they are actuated has been constantly directed to this star; but this constancy of areas requires an increase of velocity, according as they approach more to each other. It appears that the same cause ought to diminish the velocity of the particles, which, situated near the ring, constitute its inferior part.

If all the particles of a ring of vapours continued to condense without separating, they would at length constitute a solid or a liquid ring. But the regularity which this formation requires in all the parts of the ring, and in their cooling, ought to make this phenomenon very rare. Thus the solar system presents but one example of it; that of the rings of Saturn. Almost always each ring of vapours ought to be divided into several masses, which, being moved with velocities which differ little from each other, should continue to revolve at the same distance about the Sun. These masses should assume a spheroidical form, with a rotatory motion in the direction of that of their revolution, because their inferior particles have a less real velocity than the superior; they have therefore constituted so many planets in a state of vapour. But if one of them was sufficiently powerful, to unite successively by its attraction, all the others about its centre, the ring of vapours would be changed into one sole spheroidical mass, circulating about the Sun, with a motion of rotation in the same direction with that of revolution. This last case has been the most common;

however, the solar system presents to us the first case, in the four small planets which revolve between Mars and Jupiter, at least unless we suppose with Olbers, that they originally formed one planet only, which was divided by an explosion into several parts, and actuated by different velocities. Now if we trace the changes which a farther cooling ought to produce in the planets formed of vapours, and of which we have suggested the formation, we shall see to arise in the centre of each of them, a nucleus increasing continually, by the condensation of the atmosphere which environs it. In this state, the planet resembles the Sun in the nebulous state, in which we have first supposed it to be; the cooling should therefore produce at the different limits of its atmosphere, phenomena similar to those which have been described, namely, rings and satellites circulating about its centre in the direction of its motion of rotation, and revolving in the same direction on their axes. The regular distribution of the mass of rings of Saturn about its centre and in the plane of its equator, results naturally from this hypothesis, and, without it, is inexplicable. Those rings appear to me to be existing proofs of the primitive extension of the atmosphere of Saturn, and of its successive condensations. Thus the singular phenomena of the small eccentricities of the orbits of the planets and satellites, of the small inclination of these orbits to the solar equator, and of the identity in the direction of the motions of rotation and revolution of all those bodies with that of the rotation of the

Sun, follow from the hypothesis which has been suggested, and render it extremely probable. If the solar system was formed with perfect regularity, the orbits of the bodies which compose it would be circles, of which the planes, as well as those of the various equators and rings, would coincide with the plane of the solar equator. But we may suppose that the innumerable varieties which must necessarily exist in the temperature and density of different parts of these great masses, ought to produce the eccentricities of their orbits, and the deviations of their motions, from the plane of this equator.

In the preceding hypothesis, the comets do not belong to the solar system. If they be considered, as we have done, as small nubulæ, wandering from one solar system to another, and formed by the condensation of the nebulous matter, which is diffused so profusely throughout the universe, we may conceive that when they arrive in that part of space where the attraction of the Sun predominates, it should force them to describe elliptic or hyperbolic orbits. But as their velocities are equally possible in every direction, they must move indifferently in all directions, and at every possible inclination to the ecliptic; which is conformable to observation. Thus the condensation of the nebulous matter, which explains the motions of rotation and revolution of the planets and satellites in the same direction, and in orbits very little inclined to each other, likewise explains why the motions of comets deviate from this general law.

The great eccentricity of the orbits of the comets, is also a result of our hypothesis. If those orbits are elliptic, they are very elongated, since their greater axes are at least equal to the radius of the sphere of activity of the Sun. But these orbits may be hyperbolic; and if the axes of these hyperbolæ are not very great with respect to the mean distance of the Sun from the Earth, the motion of the comets which describe them will appear to be sensibly hyperbolic. However, with respect to the hundred comets, of which the elements are known; not one appears to move in a hyperbola; hence the chances which assign a sensible hyperbola, are extremely rare relatively to the contrary chances. The comets are so small, that they only become sensible when their perihelion distance is inconsiderable. Hitherto this distance has not surpassed twice the diameter of the Earth's orbit, and most frequently, it has been less than the radius of this orbit. We may conceive, that in order to approach so near to the Sun, their velocity at the moment of their ingress within its sphere of activity, must have an intensity and direction confined within very narrow limits. If we determine by the analysis of probabilities, the ratio of the chances which in these limits, assign a sensible hyperbola to the chances which assign an orbit, which may without sensible error be confounded with a parabola, it will be found that there is at least six thousand to unity that a nebula which penetrates within the sphere of the Sun's activity so as to be observed, will either describe a very elongated ellipse, or an hyperbola, which, in consequence of the magnitude of its axis will be as to sense confounded with a parabola in the part of its orbit which is observed. It is not therefore surprising that hitherto no hyperbolic motions have been recognised.

The attraction of the planets, and perhaps also the resistance of the ethereal media, ought to change several cometary orbits into ellipses, of which the greater axes are much less than the radius of the sphere of the solar activity. It is probable that such a change was produced in the orbit of the comet of 1759, the greater axis of which was not more than thirty-five times the distance of the Sun from the Earth. A still greater change was produced in the orbits of the comets of 1770 and of 1805.

If any comets have penetrated the atmospheres of the Sun and planets at the moment of their formation, they must have described spirals, and consequently fallen on these bodies, and in consequence of their fall, caused the planes of the orbits and of the equators of the planets to deviate from the plane of the solar equator.

If in the zones abandoned by the atmosphere of the Sun, there are any molecules too volatile to be united to each other, or to the planets, they ought in their circulation about this star to exhibit all the appearances of the zodiacal light, without opposing any sensible resistance to the different bodies of the planetary system, both on account of their great rarity, and also because their motion is very nearly the same as that of the planets which they meet.

An attentive examination of all the circumstances of this system renders our hypothesis still more probable. The primitive fluidity of the planets is clearly indicated by the compression of their figure, conformably to the laws of the mutual attraction of their molecules; it is moreover demonstrated by the regular diminution of gravity, as we proceed from the equator to the poles. This state of primitive fluidity to which we are conducted by astronomical phenomena, is also apparent from those which natural history points out. But in order fully to estimate them, we should take into account the immense variety of combinations formed by all the terrestrial substances which were mixed together in a state of vapour, when the depression of their temperature enabled their elements to unite; it is necessary likewise to consider the wonderful changes which this depression ought to cause in the interior and at the surface of the earth, in all its productions, in the constitution and pressure of the atmosphere, in the ocean, and in all substances which it held in a state of solution. Finally, we should take into account the sudden changes, such as great volcanic eruptions, which must at different epochs have deranged the regularity of these changes. Geology, thus studied under the point of view which connects it with astronomy, may, with respect to several objects, acquire both precision and certainty.

One of the most remarkable phenomena of the solar system is the rigorous equality which is obsered to subsist between the angular motions of rotation and revolution of each satellite. It is infinity to unity that this is not the effect of hazard. The theory of universal gravitation makes infinity to disappear from this improbability, by shewing that it is sufficient for the existence of this phenomenon, that at the commencement these motions did not differ much. Then, the attraction of the planet would establish between them a perfect equality; but at the same time it has given rise to a periodic oscillation in the axis of the satellite directed to the planet, of which oscillation the extent depends on the primitive difference between these motions. As the observations of Mayer on the libration of the Moon, and those which Bouvard and Nicollet made for the same purpose, at my request, did not enable us to recognize this oscillation; the difference on which it depends must be extremely small, which indicates with every appearance of probability the existence of a particular cause, which has confined this difference within very narrow limits, in which the attraction of the planet might establish between the mean motions of rotation and revolution a rigid equality, which at length terminated by annihilating the oscillation which arose from this equality. Both these effects result from our hypothesis; for we may conceive that the Moon, in a state of vapour, assumed in consequence of the powerful attraction of the earth the form of an elongated spheroid, of which the greater axis would be constantly directed towards this planet, from the facility with which the vapours yield to the slightest force impressed upon them. The terrestrial attraction continuing to act in the same manner, while the Moon is in a state of fluidity, ought at length, by making the two motions of this satellite to approach each other, to cause their difference to fall within the limits, at which their rigorous equality commences to establish itself. Then this attraction should annihilate, by little and little, the oscillation which this equality produced on the greater axis of the spheroid directed towards the earth. It is in this manner that the fluids which cover this planet, have destroyed by their friction and resistance the primitive oscillations of its axis of rotation, which is only now subject to the nutation resulting from the actions of the Sun and Moon. It is easy to be assured that the equality of the motions of rotation and revolution of the satellites ought to oppose the formation of rings and secondary satellites, by the atmospheres of of these bodies. Consequently observation has not hitherto indicated the existence of any such. The motions of the three first satellites of Jupiter present a phenomenon still more extraordinary than the preceding; which consists in this, that the mean longitude of the first, minus three times that of the second, plus twice that of the third, is constantly equal to two right angles. There is the ratio of infinity to one, that this equality is not the effect of chance. But we have seen, that in order to produce it, it is sufficient, if at the commencement, the mean motions of these three bodies approached very near to the relation which renders the mean motion of the first minus three times that of the second, plus twice that of the third, equal to nothing. Then their mutual attraction rendered this ratio rigorously exact, and it has moreover made the mean longitude of the first minus three times that of the second, plus twice that of the third, equal to a semicircumference. At the same time, it gave rise to a periodic inequality, which depends on the small quantity, by which the mean motions originally deviated from the relation which we have just announced. Notwithstanding all the care Delambre took in his observations, he could not recognise this inequality, which, while it evinces its extreme smallness, also indicates, with a high degree of probability, the existence of a cause which makes it to disappear. In our hypothesis, the satellites of Jupiter, immediately after their formation, did not move in a perfect vacuo; the less condensible molecules of the primitive atmospheres of the Sun and planet would then constitute a rare medium, the resistance of which being different for each of the stars, might make the mean motions to approach by degrees to the ratio in question; and when these movements had thus attained the conditions requisite, in order that the mutual attraction of the three satellites might render this relation accurately true, it perpetually diminished the inequality which this relation originated, and eventually rendered it insensible. We cannot better illustrate these effects than by comparing them to the motion of a pendulum, which, actuated by a great velocity, moves in a medium, the resistance of which is inconsiderable. It will first describe a great number of circumferences; but at length its motion of circulation perpetually decreasing, it will be converted into an oscillatory motion, which itself diminishing more and more, by the resistance of the medium, will eventually be totally destroyed, and then the pendulum, having attained a state of repose, will remain at rest for ever.

NOTES TO CHAPTER I.

(*a*) THE uniform velocities are proportional to the circumferences of the circles described, divided by the periodic times, or times of their description, *i. e.* $v = \frac{2r\pi}{P}$; and as by hypothesis $P^2 \backsim r^3$ $\therefore$ $v^2 \backsim \frac{r^2}{r^3} \backsim \frac{1}{r}$, hence F which is $::$ l to $\frac{v^2}{r}$ varies as $\frac{1}{r^2}$; this, however, only proves, that if the orbits of the planets were circular, the forces by which they are retained in their respective circumferences vary inversely as the squares of their distances from the sun.

(*b*) The areas being proportional to the times, the bases described in the interval *dt* are inversely as the altitudes or perpendiculars let fall from the centre of force on the tangents to the curve described; but as *dt* is assumed indefinitely small, the velocities with which the bases are described may be considered as uniform, and therefore proportional to the bases, consequently they are reciprocally, as the perpendiculars from the centre of forces; hence, as at the perihelion, the distance is least, the velocity at this point must be a maximum. As the body is supposed to describe an ellipse, its tendency to recede from the sun at the perihelion must be greater than its gravity towards it, for, otherwise, if they were equal, the body would describe a circle about the sun; and if this tendency was less at the perihelion than the gravity towards it, the body would fall within this circle, which is contrary to the hypothesis; this is also evident from the ratio which the centripetal bears to the centrifugal force at the same distance; it is likewise apparent from the circumstance of the velocity in the ellipse decreasing in a greater ratio than the inverse subduplicate ratio of the distance, and therefore in a greater ratio than the velocities of bodies moving in circles at the same distance (*see* note (*d*) of this chapter;) therefore, as the velocities decrease in a greater ratio than the velocities of bodies moving in circles at the same distance, the velocity of the body moving in the ellipse, continually approaches to the velocity of a body moving in a circle at the same distance; there is a certain point in the curve where the velocity becomes equal to the velocity in a circle at the same distance; this is at the mean distance of the body from focus; but though the velocities are in this case the same, the curves described will not coincide, for as the angle of projection in the ellipse is obtuse, the body will continue to recede from the centre, until it arrives at the point where the direction of its motion is at right angles to the radius vector, and as at this point the velocity is less than in a circle at the same distance, the path described by the body will fall within the circle, and the body will return to the perihelion, tracing a curve precisely equal and similar to that by which it arrived at aphelion.

(*c*). *See* note (*b*) page 256 of first volume. Let, as in notes page 248 of first volume, x and y represent the rectangular coordinates of the planets, the origin being in the sun to which the force soliciting the bodies is directed, (it is not necessary to introduce a third coordinate, because the areas being proportional to the times, the curve described is of single curvature,) by what is stated in the notes already adverted to, $o=\frac{d^2x}{dt^2}+\mathrm{P}$; $o=\frac{d^2y}{dt^2}+\mathrm{Q}$, multiplying the first equation by $-y$, and the second by x, and then adding them together, we have $d\left(\frac{xdy-ydx}{dt}\right)+x\mathrm{Q}-y\mathrm{P}=o$; but the first member being the differential of $\frac{xdy-ydx}{dt}$, which is constant and = to cdt, $x\mathrm{Q}-y\mathrm{P}=0$; $\therefore x:y::\mathrm{P}:\mathrm{Q}$, and the force is directed to the origin of the coordinates; if the first of the preceding equations be multiplied by dx, and the second by dy, and then added together, we obtain $\frac{dx.d^2x+dy\,d^2y}{dt^2}+\mathrm{P}dx+\mathrm{Q}dy=0$, and therefore (by including the constant arbitrary under the sign $\int$) $\frac{dx^2+dy^2}{dt^2}+2\int(\mathrm{P}dx+\mathrm{Q}dy)=0$, substituting for dt its value $\frac{xdy-ydx}{c}$, this equation becomes

$$\frac{c^2(dx^2+dy^2)}{(xdy-ydx)^2}+2\int(\mathrm{P}dx+\mathrm{Q}dy),$$

but if r be the radius vector, and v the angle which r makes with the axis of x, we have $x=r.\cos.v$, $y=r.\sin.v$, and therefore $dx^2+dy^2=r^2.dv^2+dr^2$, $x.dy-y.dx=r^2dv$, $\mathrm{P}=\phi\cos.v$, $\mathrm{Q}=\phi\sin.v$, ϕ being $=\sqrt{\mathrm{P}^2+\mathrm{Q}^2}$, $\therefore$ by substituting we obtain

$$c^2.\frac{(r^2.dv^2+dr^2)}{r^4.dv^2}+2\int\phi\,dr, \text{ and therefore}$$

$dv=\frac{c\,dr}{r\sqrt{-c^2-2r^2.\int\phi\,dr}}$, when ϕ is given in terms of r, we can obtain v in a function of r, by the method of quadratures; but if ϕ be unknown, and the nature of the curve described be given, we obtain (by differentiating the preceding expression)

$\phi=\frac{c^2}{r^3}-\frac{c^2}{2}.d.\frac{\frac{dr^2}{r^4.dv^2}}{dr}$, now, as the planetary ellipses, $\frac{1}{r}=\frac{1+e.\cos.(v-\varpi)}{a(1-e^2)}$, $\therefore$

$\frac{dr^2}{r^4.dv^2}=\frac{2}{ar(1-e^2)}-\frac{1}{r^2}-\frac{1}{a^2(1-e^2)}$, consequently

$\phi=\frac{c^2}{a(1-e^2)}\cdot\frac{1}{r^2}$; and conversely, if ϕ varies as $\frac{h}{r^2}$, the preceding equation will satisfy the differential equation, which expresses the value of ϕ, for then $h=\frac{c^2}{a(1-e^2)}$, is an equation of condition between a and e, and therefore the three quantities a e ϖ are reduced to two distinct quantities, which is enough, as the differential equation between r and v is only of the second order. The coefficient $\frac{c^2}{a(1-e^2)}$ determines the *intensity* of the force ϕ for *each* planet and comet; but it is easy to show that this is the same in passing from one planet to another, for from the proportionality of the areas to the times of their description, we have $\frac{cdt}{2}:\pi a^2.\sqrt{1-e^2}$ (which expresses the area of the ellipse) as dt : P the periodic time,

$\therefore c=\frac{2\pi a^2.\sqrt{1-e^2}}{P}$; but since the squares of the periodic times are as the cubes of the greater axes of the ellipses, we have $P^2=k^2.a^3$, k being the same for all the planets, and therefore we have by substituting,

$c=\frac{2\pi.\sqrt{a(1-e^2)}}{k}$; $\therefore$ as $2a(1-e^2)$ expresses the principal parameter of the orbits traced by the planets, c which is $\div 1$ to the areas traced in equal times, varies as the square root of the parameters. In the case of the comets, as their orbits are parabolic, the preceding value of c becomes $\frac{2\pi\sqrt{2D}}{k}$, D being the perihelion distance; in this case the preceding proportion becomes

$\frac{2\pi.\sqrt{2D}}{2k}.dt : \pi a^{\frac{3}{2}}\sqrt{2D} :: dt : P$, and $\therefore P^2 = k^2 a^3$, therefore when P is known we can determine a. The value of c gives

$\frac{c^2}{a(1-e^2)} = \frac{4\pi^2}{k^2}$ and $\therefore \phi = \frac{4\pi^2}{k^2}.\frac{1}{r^2}$; which shows that ϕ varies from one planet to another, only in consequence of the change of distance, and therefore at equal distances from the sun the accelerating force of all the planets is the same, and the moving force varies as their masses; therefore, if all the planets fell at the same instant from different points of the same spheric surface towards the sun, they would reach it in the same time, just as all bodies near to the surface of our earth are equally accelerated by the force of terrestrial gravity, and the weight of the planets to the sun is proportional to their masses divided by the squares of their distance from the sun. The greatest and least values of r in the ellipse, correspond to $r-\bar{\omega}=\pi$, $r-\bar{\omega}=0$, therefore they are respectively $a(1+e)$, $a(1-e)$, consequently they lie in directum, hence it follows that when ϕ varies as $\frac{1}{r^2}$, the apsides are 180° distant, and *vice versa* if the apsides are 180 distant, the force varies as $\frac{1}{r^2}$. *See* Principia Math. book 1, prop. 45; and note (i) chapter 5. From the equation $c^2 = h.a(1-e^2)$, it follows that the synchronous areas vary generally as the square root of the absolute forces into the square roots of the parameters of the orbits described, and therefore if the absolute forces be different, we have

$\sqrt{h.a(1-e^2)}.\, dt : \pi a^2.\sqrt{1-e^2} :: dt : P$, and $\therefore P^2 = \frac{\pi.a^3}{h}$

i.e. the square of the periodic time varies as the cube of the distance divided by the absolute force. Generally speaking, the quantity c, which results from the integration of the equation $d.\left(\frac{x\,dy - y\,dx}{dt}\right) = o$, is common to all laws of central forces, therefore it does not depend on the law of the attractive force, but on its absolute quantity, and it will serve to determine the ratio of the central force of the sun, to every other; v the velocity

$$= \frac{\sqrt{dr^2 + r^2 \cdot dv^2}}{dt} = \frac{\sqrt{dr^2}}{dt} + \frac{c^2}{r}$$

$= (\text{as } d.(r^2.dv) = 0,)\ 2\mu.\left(\frac{2}{r} - \frac{1}{a}\right)$, (Celestial Mechanics, Nos. 18, 26.), therefore v is a maximum when r is a minimum, and *vice versa*; if U denotes the velocity which the body would have if it described a circle about the sun at the unit of distance, then

$$r = a = 1, \text{ and } \therefore \mathrm{U}^2 = \mu, \therefore \mathrm{v}^2 = \mathrm{U}^2\left(\frac{2}{r} - \frac{1}{a}\right),$$

hence given the velocity of projection and distance, we can determine the axis major a, as a is positive in the ellipse, infinite in the parabola, and negative in the hyperbola, the section described will be an ellipse, a parabola, or hyperbola, according as

$$\mathrm{v} \text{ is } \angle = \text{or} > \text{than } \mathrm{U}.\sqrt{\frac{2}{r}},$$

it is remarkable that the direction of projection does not influence the *species* of conic section, for $\frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mathrm{U}^2}$, therefore, when r and v are given, a and therefore P remain the same; as $\mathrm{U}\sqrt{\frac{1}{r}}$ = the velocity in a circle at the distance of r from the sun, in the ellipse the velocity at any point is to that in a circle at the same distance, in a less ratio than that of $\sqrt{2} : 1$; in the parabola this ratio is that of $\sqrt{2} : 1$; in the hyperbola the ratio is greater than that of $\sqrt{2} : 1$; in the ellipse, when v diminishes r increases, and when $\mathrm{v} = o$, $r = 2a$, in which case $e = 1$; in the hyperbola, when r is infinite, the limit of the velocity is $U^2 . \frac{1}{a}$ = the square of the velocity in a circle at the distance of a from the focus, when $r=a$, $v=U\sqrt{\frac{1}{r}}$ = the velocity in a circle at the same distance, and in general

$$v : U.\sqrt{\frac{1}{r}} :: \sqrt{2a-r} : \sqrt{a};$$

hence we see the truth of what is stated in notes page 372. As $\frac{dr}{dt}$ expresses the velocity resolved in the direction of the radius, it is = to v. cos. ε, ε. being the angle which the radius vector makes with the tangent, therefore

$$\frac{dr^2}{dt^2} = \mu . \left(\frac{2}{r} - \frac{1}{a}\right) . \cos.^2\varepsilon, \text{ but as } \frac{\mu}{a} - \frac{2\mu}{r}$$

$$+ \frac{dx^2+dy^2}{dt^2} = 0,$$

$\mu . a(1-e^2) = 2\mu r - \frac{\mu r^2}{a} - \frac{r^2\, dr^2}{dt^2}$, and substituting for $\frac{dr^2}{dt^2}$, its value, we obtain

$a(1-e^2) = r^2 \sin.^2\varepsilon \left(\frac{2}{r} - \frac{1}{a}\right)$, $a(1-e^2)$ expresses the parameter, which when r and a are given, varies as the square of the sine of projection; ∴ the parameter, when every thing else remains the same, depends on that part of the velocity which acts perpendicularly to the radius vector, it is termed the paracentric velocity, and is evidently a maximum at the extremity of the focal ordinate.

From the expression $a(1-e^2)=r^2 . \sin.^2\varepsilon . \left(\frac{2}{r} - \frac{1}{a}\right)$, it follows that sin. $^2\varepsilon$ varies inve[illegible] $\frac{r}{-}\Big)$, but as

$r + 2a - r$ is given, their product is a maximum, and $\therefore$ the sine of projection the least possible when $r = a$, *i. e.* at mean distance.

(*f*) It appears from what has been established with respect to the relation which exists between the velocity in a circle and the velocity in a conic section at the same distance, that the hyperbola and ellipse are equally possible, *with* this sole difference, that the hyperbola supposes a greater velocity than the ellipse; the parabola is infinitely less probable than the two other conic sections, since it supposes an *unique* case, the circle likewise requires a *perfect equality*. The parabolas may be considered as the asymptotes to which very excentric ellipses perpetually approach in the perihelion. It is on this supposition that the investigation of the cometary motions is founded. See Vol. I., page 394.

As $r^2 dv$ expresses the elementary area, it follows that $dv = \frac{cdt}{r^2}$, *i. e.* the angular velocity, varies as the square root of the parameter or of the sychronous areas divided by the square of the distance, therefore the angular velocity in a conic section is to that in a circle at the same distance r, as $c : \sqrt{r}$, and they are equal at the extremity of the focal ordinate, as

$$c = \frac{2\pi a^2 . \sqrt{1-e^2}}{P}, \quad \frac{dv}{dt} = \frac{2\pi a^2 \sqrt{1-e^2}}{P . r^2};$$

if a circle is described at the unity of distance in a time equal to P, we have $\frac{2\pi}{P}$ = the mean angular velocity in the ellipse, therefore when the angular velocity in the ellipse is equal to the mean angular velocity, we have

$\frac{2\pi}{P} = \frac{2\pi a^2 \sqrt{1-e^2}}{P r^2}$, and $\therefore r = a(1-e^2)^{\frac{1}{4}}$ = a mean proportional between the semiaxes; in this position the equation of the centre is a *maximum*.

(g) See Celestial Mechanics, No. 58, and also Princip. Math. Section 9. If X represents the quantity by which the force deviates from the inverse ratio of the square of the distance, then the distance between the apsides

$$= 180. \frac{\sqrt{1+X}}{\sqrt{1+3X}} = 180(1-X),$$

the square of X being neglected.

If a represent the mean distance of the satellite from the centre of Jupiter, P′ its period, expressed in seconds, $\frac{2a\pi}{P'}$ will represent the arc described in a second; and $\frac{2a\pi^2}{P'^2}$ which is equal to the versed sine of the arc described, is the space through which the attractive force of the planet causes the body to descend in a second; and if a' P″, &c. represent the same quantities for another satellite, the ratio of $\frac{2a\pi^2}{P'^2}$ to $\frac{2a'\pi^2}{P''^2}$, expresses the ratio of ϕ to ϕ', the attractive forces of Jupiter at the distances a, a'; but as by observation

$$P'^2 : P''^2 :: a^3 : a'^3, \text{ we have } \phi : \phi' :: \frac{1}{a^2} : \frac{1}{a'^2}.$$ See page 10 of the text.

In note (u), page 356 of the first volume, we showed how the number of oscillations performed by a pendulum in a given time indicated the diminution of gravity.

(h) In note (t), page 426, we gave the method of determining the velocity which should be impressed on a projectile, in order that, setting aside the resistance of the air, it might perpetually revolve about the earth.

It may be shown, by a comparison of the apparent angular motion of the moon with her apparent diameter, that she describes equal areas in equal times about the earth; and therefore that the force by which she is retained in her orbit is directed to the earth. See notes page 315, Vol. I. Indeed, as will be shown in the fifth chapter, if great accuracy be required, the observations ought to be made in the syzygies and quadratures; for in the other points of the orbit the disturbing action of the sun is not directed to the centre of the earth. Newton shows, from the small quantity by which the apsides are observed to prograde, that the force must be nearly inversely as the square of the distance; for if the orbit was elliptical, the earth being in one of the foci, the distance between the apsides would be 180°, and the force by which the moon would be retained in her orbit would vary as $\frac{1}{d^2}$. See notes, page 375. Now the apsides are observed to prograde 3°, 3′ every month, and the law of the force which would produce such a progression must vary inversely as some power of the distance, intermediate between the square and the cube, but which is nearly sixty times nearer to the square; consequently, on the hypothesis that the progression is produced by a deviation from the law of elliptical motion, it must be nearly in the inverse ratio of the squares of the distance; but as Newton proves this motion of the apsides to arise from the disturbing action of the sun, it follows that the force varies accurately as $\frac{1}{d^2}$. See Luby's Physical Astronomy, page 197.

(*i*) He computes the space through which the moon would fall in a second, in consequence of the action of the force by which she is retained in her orbit; which force, in consequence of the proportionality of the areas to the times, is directed towards the centre of gravity of the earth; and assuming that the force decreases in the inverse ratio of the square of the distance, he determines, from knowing the space described by a body falling near the earth's surface in a second, the space through which, in consequence of the action of the same force diminished in the inverse ratio of the square of the distance of the moon from the earth, a body would fall at this distance; and as this space comes out equal to that by which the moon is deflected from the tangent to the orbit, he justly concludes, that this force is the terrestrial gravity diminished in the ratio of the square of the distance.

In consequence of the disturbing action of the sun, the moon's distance and motion are subject to several inequalities, which are detailed in Chapter IV. of the First Volume. The particular explanation of the most remarkable of them will be given in notes to Chapter V. of this Volume.

Knowing the parallax and radius of the earth, it is easy to obtain the distance.

In determining the space through which the moon falls in a second, in consequence of the force which sollicits it, there are two corrections applied; one arising from the disturbing action of the sun, which, taking into account the entire orbit, diminishes the lunar gravity $\frac{1}{358}$th part, see note (f) Chapter V.; and in consequence of this the result obtained should be increased a $\frac{1}{358}$th part. The other correction arises from this, that in the relative motion of the moon about the earth, the point about which it really revolves is the common centre of gravity of the earth and moon, and the central force which should exist in the centre of the earth, which would cause the moon to revolve about this centre in the same time in which she actually revolves about the common centre of gravity of the earth and moon, should be equal to $m+m'$, the sum of the masses of the earth and moon; for if a be the distance of the earth from the moon, y the distance at wh the moon would revolve about the earth by itself, considered as quiescent,

$$=\frac{a.m^{\frac{1}{3}}}{(m+m')^{\frac{1}{3}}},$$ (Princip. Math. prop. 59, book 1.), $\therefore$

$P^2 = \frac{y^3}{m^3} = \frac{a^3}{m+m'}$, *i. e.* if a be the distance, the central force should be $m+m'$, $\therefore$ as the versed sine of the arc described in a second is the space through which the moon descends in consequence of the combined actions of the earth and moon, it must be multiplied by

$\frac{m}{m+m'}$, *i. e.* $\frac{75}{76}$, to obtain the space described by the sole action of m.

(k) Let a b represent the major and minor semiaxes of the terrestrial spheroid, its solid content $= \frac{4\pi\, a^2 b}{3}$, and if r be the radius of the equicapacious sphere, its content

$= \frac{4\pi r^3}{3}$, $\therefore a^2 b = r^3$, $\therefore$ as $r = \frac{b}{\overline{1-e^2.\cos.^2\lambda)}}^{\frac{1}{2}}$ we have

$a^{\frac{4}{3}}.b^{\frac{2}{3}} = \frac{b^2}{1-e^2.\cos.^2\lambda}$, *i. e.* $a^{\frac{4}{3}} = b^{\frac{4}{3}}.(1+e^2.\cos.^2\lambda)$ very nearly; and if $a = b(1+\epsilon)$, ϵ being a very small quantity of which the square may be neglected, then

$$1 + \frac{4}{3}.\epsilon = 1+2\epsilon.\cos.^2\lambda, \text{ and } \therefore \cos.^2\lambda = \frac{2}{3};$$

now, as the efficient part of the centrifugal force at any parallel of latitude λ diminishes as $\cos.^2\lambda$, and as the centrifugal force at the equator is the $\frac{1}{288}$th part of gravity, the centrifugal force at the parallel in question

$= \frac{2}{3}.\frac{1}{288} = \frac{1}{432}$th part of gravity, $\therefore$ if p represent the lunar parallax, v the versed sine of the arc described by the moon, and s the space fallen through near the surface of the earth in the same time,

$v = \frac{2a\pi^2}{p^2}$, $a = \frac{6369809}{\text{sin.}\, p}$, ∴ applying the corrections specified above,

$v = \frac{2.63.69809^m}{\text{sin.}\, p.\, P^2} \cdot \frac{369}{368} \cdot \frac{75}{76}$, and s when diminished in the ratio of the square of the distance

$= 3^m.65631 \left(1 + \frac{1}{432}.\right) . \text{sin.}\, ^2p$; now these two expressions come out $q.p$ equal, ∴ it follows that the force varies as $\frac{1}{d^2}$. In the Celestial Mechanics, the identity of terrestrial gravity with the force deflecting the moon, is proved from the equality of p, the lunar parallax, as determined by observation, and from the preceding equation.

With respect to the diminution of the force of gravity on the summits of the highest mountains, *see* note (v) Chapter VIII.

(l) In consequence of the equality of action and reaction, whatever motive force is produced in the planet by the action of the sun, an equal and contrary force is produced in the sun by the planet's reaction; now, if M m represent the respective masses of the sun and planet, and d their mutual distance, the motive force of any planet is ∻l to $\frac{M.m}{d^2}$; ∴ at equal distances from the sun, the motive force towards that body is proportional to the masses of the planets; and, therefore, as the accelerating force of the planets is the same at equal distances from the sun, it follows that the moving force is ∻l to the ma‐‐ same is true for bodies near the earth's surfdent from experiments made with pendulums. The accelerating force of the planet is expressed by

$\frac{M}{d^2}$, and the accelerating force of the sun $= \frac{m}{d^2}$; now, if we impress on the sun and planet in a contrary direction to the motion of the sun, an accelerating force equal to $\frac{m}{d^2}$, the sun will be at rest, their relative motion is evidently not affected, and the planet will be actuated by the accelerating force

$\frac{M}{d^2}$, and also by $\frac{m}{d^2}$, *i. e.* by $\frac{M+m}{d^2}$, and as P the periodic time $= \frac{d^{\frac{3}{2}}}{\sqrt{M+m}}$, it will be less than if the sun was immoveable in the ratio of $\sqrt{M+m}$, to $\sqrt{M}$; $\therefore$ the ratio of P^2 to d^3, is, strictly speaking, different for each planet; however, as in point of fact, this ratio is nearly the same for all the planets, it follows that the masses of the planets must be very small compared with that of the sun. This comparative smallness is also evinced by the circumstance of Kepler being able to announce his laws, for from the universality of gravitation each body is attracted by every other body, therefore those laws do not accurately obtain; still their effect must be *q.p* small, as the elliptic orbit satisfies the observations.

In addition to what is stated in page 15, it is to be remarked, that every computation founded on this hypothesis, if it satisfies all the observed phenomena, furnishes an additional proof of the truth of the theory of universal gravitation; and in this way, all physical astronomy, and in particular the theory of perturbations, by means of which the modern tables accord so perfectly with observation, is one of its most satisfactory confirmations.

We shall, in the subsequent Chapter, find this fact of the universality of attraction, confirmed by numerous astronomical observations, the precession of the equinoxes, the nutation of the earth's axis, and the compression of the planets, have been computed on this hypothesis, and its truth is evinced by the accurate agreement of the results of the computation with actual observation. *See* Chapters V. and VI. of this Volume, pages 48 and 55.

(*m*) For suppose the attracted body to approach towards the earth until it came in contact with it, if the reaction was not exactly equal to action, the two bodies would move with a common velocity in the direction of the pressure which predominates, therefore the centre of gravity of the two bodies would have a rectilinear motion in some direction, in consequence of the force of gravity, which is contrary to what is established in notes page 440, Vol. I.

This principle of reaction is of the greatest consequence in physics. Let, as before, M m represent the masses of two attracting bodies, V v the velocities which they communicate to each other, A B the intensities of their forces, d their mutual distance, we shall have, in consequence of reaction,

$$\mathrm{MV} = m\mathrm{v} \therefore \mathrm{V} : \mathrm{v} :: m : \mathrm{M}, \ i.e. \text{ as}$$

$$\mathrm{V} = \frac{\mathrm{B}}{d^2}, \mathrm{v} = \frac{\mathrm{A}}{d^2}; \ \mathrm{B}. : \mathrm{A} :: \mathrm{V}. : \mathrm{v} : \mathrm{B} : \mathrm{A} :: m : \mathrm{M}; \ i.e.$$

the intensity of the forces is $\div\!\!\div$l to the masses, and the velocities communicated are inversely as the masses. It also appears that if the bodies do not receive an initial impulse, they will approach each other in a right line, and meet in their common centre of gravity.

NOTES TO CHAPTER II.

(*a*) The formulæ which analysis has furnished for the determination of the perturbations, are composed of two different descriptions of terms. The first are proportional to the sines or cosines of certain angles; the second are proportional to the angles themselves. The first description of terms have periods, at the term of which they attain their greatest or least values, without ever passing these limits, so that they can never accumulate, and thus, at the end of millions of years they will not be more considerable than they are at present. As very accurate observations are required to detect them, it is only recently that they have been observed, with the exception of the inequalities of the Moon, and one relating to the motion of Jupiter and Saturn, which have been known a considerable time. The other terms, which are not proportional to the sines, but to the arches, or to the time in which these arches are described, have no period, and, therefore, continually accummulate. When the interval between the time of making observations is not considerable, these progressive perturbations are confounded with those which are periodical; but they at length become more and more detached from them, so that eventually they are so different, that it is impossible to mistake the one for the other. Like the mean motions of the planets, they may be determined by means of observations separated by considerable intervals from each other, though not with the same accuracy. Such, in particular, is the manner in which the motion of the nodes and of the apsides has been determined, which, in one hundred years, amounts to more than one degree. The observations of several ages would, in this way, indicate whether the change of the longitudes of the aphelia and of the nodes corresponds to the retrogradation of the equinoctial points, or if the apsides have a proper motion of their own; by means of such observations, we might determine with great accuracy the *progressive* perturbations, and then theory has only to account for them; they are useful in this respect, in the investigation of the *periodical* perturbations, inasmuch, as being easily observed, they enable us to determine the constant coefficients, masses, &c. But although the *coefficients* are furnished in this way by observation, the *form* of the equations which indicate the periodic pertubations can only be determined by theory; and this form, in the case in which the coefficients are unknown, is of the greatest importance in the *empirical* determination of the equations; for, by pointing out the arguments on which they depend, they suggest a mode according to which the observations should be made. They also indicate the period of each inequality, and consequently the epoch and situation in which it is at its maximum, and therefore easiest to be observed, and thus furnish a means of separating one perturbation from the other, and of determining them separately by observation. As the progressive perturbations, or those which are proportional to the times or the arcs, are commonly so small, that they do not become apparent until after the lapse of one or of several ages; their value is generally indicated for every hundred years; hence it is that they have be-denominated *secular* inequalities. The following is a brief outline of the method by which the different perturbations of the planetary system are investigated. The differential equations of the second order, which are given in page 420, Volume I., furnish at once both the elliptic motion and the perturbations: the first integration of these equations, which is not difficult, consists of differential equations of the first order, and determines the *variations* of all the elements. The second integration, if it is expressed by polar coordinates, would determine the longitude and latitude, and the other elements; but it has hitherto baffled the skill of mathematicians, who have had recourse to various artifices to effect it. They have only ascertained, that if even the exact integral could be obtained, it would be so complicated, that in order to render the result applicable, it should be developed into a series. Thus, the only practicable method is to develope the result in an *approximative* manner, in which the quantities, which are extremely small with respect to others, are neglected; and the integral should be also exhibited in a series, the form of which is indicated by that of the differential equation, and its coefficients are determined by comparing the differential of the supposed integral with the given differentials. Therefore, the solution of the problem consists in expressing the integral by a convergent series, which necessarily supposes that the mass or distance, or, in short, that the attraction of one of the bodies is inconsiderable relatively to that of the other, the last body being termed the *central*, and the first the *disturbing* body. In such a case the disturbed orbit will deviate very little from the laws of Kepler. It can be considered as a variable ellipse, subject to these laws, as Lagrange has proved. Indeed, if the disturbed orbit deviated considerably from an ellipse, which, for instance, would be the case with the moon if she was four times farther from the earth than she is, (in which case the sun might be regarded as the central body equally as the earth,) the computation of its orbit would surpass the powers of our analysis. Fortunately the solar system is so arranged that we may assume the elliptic motion as the basis of each disturbed orbit.

The constant arbitrary quantities which are introduced at each integration, are the elements of the planetary ellipses. They are data which cannot be determined by theory, but solely from observation. They do not affect the differential equations, or general laws of motion, but solely the arbitrary modifications of the elliptic orbits, which, for each planet that moves according to the laws of Kepler, may be indefinitely varied. There are, in general, six constant arbitrary quantities independent of each other. See Book 3, Vol. I., page 185. For, as is mentioned in page 428, each body being referred to three rectangular coordinates, and then putting the second differentials of the coordinates, divided by the square of the element of the time, equal to the attractions which the body experiences from the other bodies, we shall have the three differential equations of the second order which determine the motion of the body. As each body of the system furnishes three similar equations, the entire number of these equations is triple that of the bodies; therefore their *complete* integrals contain six times as many arbitrary quantities as there are bodies. These constant quantities are determined by the initial coordinates of each body, and by the initial velocities resolved in the direction of the coordinates. The bodies of the system are almost always referred to one principal one; and this is done by subtracting the differential equations of its motion resolved in the direction of each coordinate from the corresponding differential equations of the motions of the other bodies; by this means the differential equations relative to their motions about the principal body will be obtained. See Celestial Mechanics, Vol. II. page 259. By means of these differential equations, there have been obtained seven integrals relatively to the motion of a system; three of these refer to the motion of the centre of gravity of the bodies of a system which are not acted on by any extraneous forces. The four remaining integrals, which are furnished by the principles of the conservation of areas and of living forces, are differentials of the *first order*. They are, as has been noticed in the Fourth Book, the generalization of the law of the areas, proportional to the times, and of the expression for the square of the velocity, which Newton announced in the motion of the system of two bodies. The determination of the motion in the case of two bodies, is reduced to the integration of differential equations of the first order, which is easily effected; but when there is a greater number of bodies the problem becomes extremely complicated, and we are obliged to have recourse to approximations.

(*b*) If, as is stated in the text, a body A be supposed to describe about the sun an ellipse, the elements of which vary by insensible gradations, and if the planet B be supposed to describe an epicyle about it, as the satellites do about their respective primaries, the motion of A would represent the primitive orbit changed imperceptibly by the secular inequalities, while the motion of B in the epicyle would represent the periodic inequalities.

(*c*) Calling a the mean distance, nt $n't$ the mean motions, the value of $\frac{1}{a} = \frac{2im'nK}{\mu(i'n'-in)} . \cos. (in't-int+A)$, and ζ the mean motion

$= \frac{3im'an'K}{\mu(i'n'-in)^2} . \sin. (i'n't-int)$. In this case the disturbing action of the planet m' on m is solely considered; and as it appears that $i'n'-in$ does not vanish, the quantities a and ζ only contain periodical inequalities, the approximation being continued as far as the first power of the disturbing force; and as i i' are integral numbers (See No. 54, Celestial Mechanics,) the equation $i'n'-in=0$, cannot have place when the mean motions of m and m' are incommensurate, which is the case of the planets. Here the disturbing action of m' on m was only considered; but if the disturbing actions of all the planets m' m'' m''', &c. were taken into account, we should have, instead of $i'n'-in=0$, $in+in'$ $i'n''$ &c. $=0$, which is still more improbable than the equation $i'n'-in=0$. In the Supplement to the third volume, Laplace extended this conclusion in the manner mentioned in the text; however, what is stated here may suffice to point out the principle of the method. It follows, consequently, from this, that the greater axes of the orbits of the planets and their mean motions are only subject to periodical inequalities, depending on their mutual configuration, and therefore if these are neglected, the greater axes are invariable, and the mean motions are uniform. It is not, perhaps, generally known, that Mr. Simpson made this observation respecting the inequalities of the planets. See Miscellaneous Tracts, 179.

(c) Calling $e\,e'\,e''$, &c. the eccentricities of the orbits of $m\,m'\,m''$, &c., it is proved, in No. 57, Book 2, of the Celestial Mechanics, that

$$0 = ede.m\sqrt{a} + e'de'.m'\sqrt{a'} + e''de''.m''\sqrt{a''} + \&c.;$$

now, as $a\,a'\,a''$, &c. have been shown to be constant, if we integrate this expression, we shall have

$$e^2m\sqrt{a} + e'^2m'\sqrt{a'} + e''^2.m''\sqrt{a''} + \&c. = C \text{ a constant}$$

quantity, and as the planets all revolve in the same direction, the signs of $\sqrt{a}$, $\sqrt{a'}$, $\sqrt{a''}$, &c. must be the same; therefore each of the terms of the first member of the preceding equation is positive, and consequently less than C; hence, if at any given epoch the eccentricities $e\,e'\,e''$, &c. are very small, the constant quantity C will be very small; therefore each of the terms of the equations will *always* remain very small; consequently the orbits will always remain $q.p$: circular. See Celestial Mechanics, p.

332. The eccentricities of the planetary orbits are therefore subject to this condition, scilicet, that the sum of their squares, multiplied respectively by their masses, and by the square roots of their greater axes, is constantly the same. By a similar analysis, if ϕ, ϕ', ϕ'', &c. represent the inclinations of the orbits of m, m', m'', &c. to a fixed plane, we can obtain the equation

$C' = \tan.^2\phi.m\sqrt{a} + \tan.^2\phi'.m'\sqrt{a'} + \tan.^2\phi''m''\sqrt{a''} +$ &c.; and as, by hypothesis, the orbit is inclined at a very small angle to the fixed plane, it may be shown that its inclination to this plane will be always inconsiderable, and consequently the system is always stable, for the inclinations, as well as for the eccentricities.

(h) It is evident, from what has been stated in page 501, Vol. I., that if γ be the inclination of the invariable plane to the plane xy and $\bar{\omega}$, the longitude of its ascending node $\tan.\gamma \sin.\bar{\omega}$,

$= \dfrac{c''}{c'}$, but we have $\dfrac{x\,dy-y\,dx}{dt} = \sqrt{a(1-e^2)}$, $=$ the

square root of the parameter. See page 374. But if the area be referred to a fixed plane, it should be multiplied by the cosine of its inclination to this plane,

$$\therefore \frac{x\,dy-y\,dx}{dt} = \cos.\phi.\sqrt{a(1-e^2)} = \frac{\sqrt{a(1-e^2)}}{\sqrt{1+\tan.^2\phi}}$$

in like manner,

$$\frac{x'dy'-y'dx'}{dt} = \frac{\sqrt{a'.(1-e'^2)}}{\sqrt{1+\tan.^2\phi'}}, + \text{\&c.}$$

$\therefore$ neglecting quantities of the order $m m'$, &c.

$$c = m.\sqrt{\frac{a(1-e^2)}{(1+\tan.^2\phi)}} + m'.\sqrt{\frac{a'(1-e'^2)}{1+\tan.^2\phi'}} + \text{\&c.};$$

hence, if $p = \tan.\phi.\sin.\theta$, $q = \tan.\phi.\cos.\theta$, we have

$$c' = mq.\sqrt{\frac{a(1-e^2)}{1+\tan.^2\phi}} + m'q'\sqrt{\frac{a(1-e'^2)}{1+\tan.^2\phi'}} + \text{\&c.};$$

$$c'' = mp\sqrt{\frac{a(1-e^2)}{1+\tan.^2\phi}} + m'p'\sqrt{\frac{a'(1-e'^2)}{1+\tan.^2\phi'}} + \&c.,$$

$\therefore$ substituting for $\frac{c''}{c'}$, these values, and concinnating, we may arrive at the expression given in the text for the tangent of $\bar{\omega}$.

(*i*) It is shown in No. 9 of the Celestial Mechanics, if the central body of the system, which in this case is the sun, be considered as unity, that h a constant quantity

$$= \Sigma m.\frac{dx^2+dy^2 dz^2}{dt^2} - \frac{2.\Sigma m}{r},$$

we have also from No. 18,

$\Sigma\frac{m}{a} = \Sigma\frac{2m}{r} - \Sigma m\frac{dx^2+dy^2+dz^2}{dt^2}$, $\therefore$ multiplying by Σm, and neglecting quantities of the order m^2, &c. we obtain

$$h = \Sigma\frac{m}{a},\ i.\ e. = \frac{m}{a} + \frac{m'}{a'} + \frac{m''}{a''} + \&c.$$

(*k*) These inequalities which have very long periods, are expressed in this form

$$\zeta = \frac{3\,im'}{\mu}.\int\int a\,kn^2\,dt^2.\ \sin.\,(i'n't - int + \text{A}),$$

which double integration gives a term

$= -\frac{3\,im'.\,an^2k}{\mu\,(i'n'-in)^2}.\ \sin.\,(i'n't - int + \text{A})$, in which the denominator is very small, when $nt : n't$ very nearly in the ratio of $i' : i$; from which there result inequalities in ζ which increase very slowly, and which, on that account, might induce us to to suppose that the mean motions of m and m' were not uniform; n n' the mean motions of Jupiter and Saturn, are such that $5n' =$ very nearly $2n$, $\therefore$ the term of the expression for ζ, which depends on $5n't-$

$2nt$, though of the third order, becomes extremely sensible.

Besides the stability which is secured to our system by the law of gravity varying as $\frac{1}{d^2}$, it is likewise a peculiarity of that law, that the orbits of the heavenly bodies, their distances, &c. are independent of their *dimensions* and *absolute motion* in space; for if M represent the mass of a central body, d its distance, if the dimensions are changed in any ratio of 1 to $\frac{1}{n}$, every line such as d becomes $\frac{d}{n}$, and every mass m becomes $\frac{m}{n^3}$, if $\phi\ (d)$ be the function of the distance, which determines the law of attraction, so that $\frac{m}{\phi\,(d)}$ determines the action of m on a body revolving about m, the new force at the distance $\frac{d}{n}$ will be $\frac{m}{n^3\,\phi.\left(\frac{d}{n}\right)}$, but the orbits being supposed to be similar, the force must change in the ratio of 1 to $\frac{1}{n}$, because the lines which the forces cause the bodies to describe are ∺ to the forces, $\therefore \frac{m}{n^3.\,\phi.\left(\frac{d}{n}\right)} = \frac{1}{n}.\ \frac{m}{\phi\,(d)}$ or $n^2.\,\phi.\ \left(\frac{d}{n}\right) = \phi\ (d)$, *i. e.* $\phi\ (d)$ must be such a function of d, that if $\frac{d}{n}$ be substituted in place of d, the function after being multiplied by n^2 has the same value as before, *i. e.* $= \phi\ (d)$, $\therefore$ all the terms in which n occurs must respectively vanish, this only obtains when $\phi\,(d) = Cd^2$, for if $\phi\ (d) = Ad^m$, $\phi\ \left(\frac{d}{n}\right) = A.\ \frac{d^m}{n^m}$ and

$n^2 . \phi\left(\frac{d}{n}\right) = Ad^m . n^{2-m}$, this must be $= \phi\left(d\right) = Ad^m$, $\therefore$ $2 = m$, $\therefore$ the force with which m acts on a body at a distance $= d$

$= \frac{m}{\phi(d)} = \frac{m}{Cd^2}$. This law also gives the *simplest* possible expression; for spherical bodies made up of particles attracting according to this law, attract each other according to the same, which would not be the case if the attraction *decreased* according to any other law; likewise $\frac{m}{d^2}$, the expression for this force, is of one dimension, which should be the expression of a force reduced to its utmost simplicity; and the lines described by two bodies acting according to this law are always of the second order; therefore, as no other law could secure the same stability, neither could any other give the same simplicity.

NOTES TO CHAPTER III.

(*a*) In fact there are, as appears from the details of this Chapter, two ways of determining the ratio of the mass of a planet to that of the sun; either from a consideration of the inequalities and perturbations which the actions of these bodies produce, or from knowing accurately the periods and distances of the primary planets, and of the satellites which accompany them. In the former case, we may make use of either the periodic or secular inequalities; the latter, if accurately determined, would obviously give the most accurate results; but as these are as yet not sufficiently well determined, we are obliged to make use of those periodic inequalities, which are determined by a great number of exact observations.

(*b*) If $\mu\ \mu'$ represent the sum of the masses of the sun and earth, and planet and satellite, A a the respective distances of the earth and satellite from the sun and primary, and P p their respective periods, we have, by what is stated in page 384,

$$P^2 \backsim \frac{A^3}{\mu}, \text{ and } p^2 \backsim \text{as } \frac{a^3}{\mu'} \therefore \frac{\mu'}{\mu}, \ i.e. \ \frac{m+m'}{M+m} = \frac{a^3}{A^3} \cdot \frac{P^2}{p^2};$$

if m' be very small relatively to m, and m very small with respect to M, we have $\frac{m}{M} = \frac{a^3}{A^3} \cdot \frac{P^2}{p^2}$; this would be accurately true, if we had $M : m :: m : m'$, rejecting m' and retaining m, we have $\frac{m}{M+m} = \frac{a^3}{A^3} \cdot \frac{P^2}{p^2}$, *i.e.* $\frac{m}{M} - \frac{m^2}{M^2}$ $q.p = \frac{a^3}{A^3} \cdot \frac{P^2}{p^2}$, but $m = M \frac{a^3}{A^3} \cdot \frac{P^2}{p^2}$ very nearly, $\therefore \frac{m}{M}$ $= \frac{a^3}{A^3} \cdot \frac{P^2}{p^2} \cdot \left(1 + \frac{a^3}{A^3} \cdot \frac{P^2}{p^2}\right)$, now if δ be the sine of the angle under which m appears at the mean distance A from the sun, $\frac{m}{M} = \delta^3 \frac{P^2}{p^2} \left(1 + \frac{a^3}{A^3} \frac{P^2}{p^2}\right)$, relatively to the sun and earth, if π Π denote the horizontal parallaxes of the sun and moon, $\frac{a^3}{A^3} = \frac{\pi^3}{\Pi^3} \therefore \frac{m}{M} = \frac{P^2}{p^2} \cdot \frac{\pi^3}{\Pi^3} \therefore \frac{m}{M} \propto$ as $\frac{\pi^3}{\Pi^3}$, hence any error in the parallax produces an error three times as great in $\frac{m}{M}$.

(*c*) The duration of the several revolutions has been inferred by means of the formula $\frac{PT}{P+T}$, given in page 328 of the First Volume, corrected for the inequalities of light and motion of the apsides, &c.

(*d*) The action of the satellites, and also of the ring of Saturn, contribute to induce perturbations in the system, :lly to the quantity of matter which they contain. See page 17.

(*e*) Let $s =$ the arc described in $1''$, and x its versed sine, the mean distance of the sun from earth being assumed, $= 1$, $x = \frac{s^2}{2}$, but $s = \frac{2 \times 3.14159}{36225636'',1} \therefore \frac{s^2}{2} = \frac{1479565}{10^{20}}$, the space described at the latitude, the square of the sine of which $=\frac{1}{3}$, relatively to r the radius of this parallel, is $\frac{3.^{m}66477}{6369809}$, and relatively to R the radius of the earth's orbit it is $=\frac{3.^{m}66477}{6369809}\cdot\frac{r}{R}=\frac{3.^{m}66477}{6369809}\cdot\sin.\ \pi$,

$$\therefore \frac{3.^{m}66477}{6369809}\cdot\sin.\ \pi\,\frac{r^2}{R^2}\cdot=3.^{m}66477.\ \frac{\sin.\ \pi.\ \sin.\ ^2\pi}{6369809}=g,$$

is the earth's attraction reduced to the mean distance of the sun from the earth, for the number of metres in this distance : 6369809 :: 1 : sin. π;

but $\frac{M}{m}=\frac{s-g}{g}$, (see page 17.) and $\frac{m}{M}=g.\ \frac{1}{1-\frac{g}{s}}$.

As the cube of the parallax is involved in this expression of the ratio of the masses, it follows that it is of the greatest consequence to obtain this quantity, or, in fact, the dimensions of the solar system, as accurately as possible; we shall see, in the sequel, that the perturbations of the moon furnish perhaps the most accurate means of obtaining it. See note (o) in Chapter V.

(f) Let $\bar{\omega}=2966''$ the apparent diameter of the sun as seen from the earth, Δ $\vartriangle$ the respective densities of the sun and earth, V v their respective volumes, we have

$$\frac{\Delta}{\vartriangle}=\frac{M}{V}\cdot\frac{v}{m}\cdot=\frac{A^3}{a^3}\cdot\frac{p^2}{P^2}\cdot\frac{\pi^3}{\bar{\omega}^3}=\frac{P^2}{p^2}\cdot\frac{1}{\bar{\omega}^3},$$

hence it appears that the density is independent of the parallax of the sun, or of the magnitude of the solar system. This would not be the case if the law of attraction was different from that of nature. See last note of preceding Chapter.

(g) If g' represent the space described by a body in a second, at the latitude on the earth of which the square of the sine is $\frac{1}{3}$, the acceleration h, or the space which a body would describe at the distance r from the centre of m another planet, equal to the terrestrial radius is equal to $m.g'$, $\therefore$ G the space described at a distance, $= k$ the radius of the planet at the latitude of which the square of the sine is $\frac{1}{3} = \frac{hr^2}{k^2}$, but $k = \frac{\bar{\omega}.r}{\pi}$, $\therefore \mathrm{G} = \frac{hr^2}{k^2} = \frac{\mathrm{M}g'r^2}{k^2} =$

$\mathrm{M}.g'.\frac{\pi^2}{\bar{\omega}^2}$, $\bar{\omega}$ is the apparent diameter of m, seen from the earth.

In the numerical expressions for the masses, volumes, and densities of the planets, the only *absolute* quantity is the fall of heavy bodies at the surface of the planets. For, in the expressions for the masses and densities, their ratios to the mass and density of the earth is all that is given. In the Sixth Chapter there are several methods given of determining the mean density of the earth, which can be had relatively to that of water; but as the absolute density of water, or of any substance, is not given, this itself is only a ratio. See Book 5, p. 259, Vol. I. It is worth remarking, that the density of the sun, according to the preceding computation, does not differ much from that of water, and is considerably less than the mean density of the earth. The methods given in the text are not applicable to the moon. For obtaining the density and quantity of matter of this body, see Chapter VIII., note (g).

NOTES TO CHAPTER IV.

(*a*) WHAT constitutes the difficulty in investigating the motions of the comets, arises from the eccentricities of their orbits, and the inclinations to the ecliptic being so considerable, that the formulæ which furnished expressions for the disturbed orbit of the planet, are not at all applicable in this case; therefore, in the present state of our analysis, we cannot express these perturbations by analytical formulæ, which embrace, as in the case of the planets, an *indefinite* number of revolutions.

The following is the method which Laplace made use of to compute the perturbations which the comet of 1759 experienced in its successive revolutions, which he determined so exactly, that he was enabled to predict its next return to the perihelion, to within thirteen days of its actual appearance. From a careful discussion of the observations of this comet in 1682 and 1759, the elements of the orbit at these two epochs, on the hypothesis that it is an ellipse, of which the greater axis answers to the duration of a revolution from 1682 to 1759, were computed. Then, assuming the elements of 1682 as strictly accurate, he determines, by what is established in the Ninth Book, the changes which have taken taken place in the elements, and in the mean anomaly, in the three first quadrants of the excentric anomaly, *i.e.* from $u = 0$, to $u = 270°$. See Volume I., page 395. In order to determine the changes in the last 90°, it is better to go back from 1759 to the extremity of this quadrant, which is the same thing as if we fixed the origin of this angle at the perihelion of 1759, and then went back to 1682, making u negative, and commencing with the elements and epoch observed in 1759. As the comet is nearer to the disturbing planets, particularly to Jupiter, in the first and last quarters of the ellipse, than in the second and fourth, it is necessary to have as exactly as possible its position and distance from these planets, the attractions of which may make a change of a considerable number of degrees in the elongation of the comet. To obtain still greater accuracy, the changes in the elements and mean anomaly from 1682 should be computed, making use of the greater axis corresponding to this epoch, which is given by the preceding approximation; then, as far as 25° of excentric anomaly, we can employ the elements of the new ellipse, which answer to this anomaly; and afterwards, in this ellipse, thus rectified, compute the perturbations from 22½ to 45. The fundamental ellipse is rectified in the same manner, from 90 to 180; and the perturbations up to 270 of eccentric anomaly are then determined. The alterations in the last quadrant of the eccentric anomaly are then obtained by rectifying the ellipse to −22½—45 and −90; by this means the perturbations of the comet from 1682 to 1759 will be given much more accurately by a second approximation. Similar operations may be performed from 1759 to the next perihelion; but as the moment of the passage through this last point is unknown, when we arrive at 270, the ellipse is rectified for every 22½ up to 360. These computations, when carefully performed, ought to give, within a very few days, the instant of the passage of the comet through the next perihelion; the only uncertainty is with respect to the mass of Uranus, the determination of which is perhaps best determined by observing this passage.

(*b*) Lalande computed whether, among the sixty comets whose orbits and returns had been observed and discussed when he wrote, any of them had their nodes near to the circumference of the earth's orbit; and he found, that there were only eight of the sixty whose distance from the sun when at their node, did not differ much from that of the earth from the sun. In fact, the question comes to this, to determine whether, among the sixty known comets, it ever happens, that at the time their distance from the sun is equal to the distance of the earth from the sun, they are also in their node, and consequently in the plane of the ecliptic? for in that case it might happen, among an infinite number of revolutions, that the earth might be at that very part of its orbit at the moment of the comet's passing through the node; in which case there would be necessarily a collision between the two bodies. If even the distances were not precisely the same, still if the difference was not very great, the mutual attractions of the earth and comet, and also the actions of the other planets, might cause them to be exactly equal; and consequently produce an impact between them; and we know, from experience, that very considerable changes are frequently produced in the cometary orbits. If a comet, equal to the earth, approached three times nearer to it than the moon, the effects which it would produce in elevating the waters of the ocean would be such as entirely to inundate the earth; but then, when the rapid motion is taken into account, and also the inertia of the waters of the sea, it will be apparent that they would soon be beyond the effect of the earth's attraction. From all these circumstances taken into account, it appears there are the following conditions to be satisfied, 1st, That the exact coincidence of the node with the orbit of the earth, which is itself instantaneous, should occur at the very time that the comet passes through it: 2dly, Granting this coincidence, it is necessary that these two bodies of which the orbits accurately intersect, should meet at the same time in the very point of their intersection. The probability of this last might be estimated in the following manner, as the diameter of the earth seen from the sun is only 17″, it does not occupy more than the 76th thousand part of the circumference of its orbit; therefore, on the hypothesis that the comet traverses accurately the orbit of the earth, there is, at the instant it is in the node, 76 thousand to one that the earth is not in that point of its orbit where it can be struck. Besides, the passages through the nodes are of rare occurrence, since each revolution requires a considerable time, and thus thousands of revolutions may be performed without the nodes being accurately on the circumference of the earth's orbit.

With respect to the effect of the attraction of comets, which, though they do not actually impinge, approach to the earth so as to effect an elevation of its waters, Sejour shows, that in consequence of the inertia of the waters, if even the sea was diffused over the whole earth, it would take $10^h\ 52'$ to produce its entire effect; but then the true circumstances of the problem are not so favourable to those great perturbations; for, 1st, The comet is not always perpendicular to the same point of the earth, in consequence not only of the rotatory motion of the earth, but also on account of the very rapid motion of the comet itself; besides, the waters are not diffused over the entire earth, which necessarily diminishes the effect of the earth's attraction; and, 3dly, there is only a very short time (less considerably than $10^h\ 52'$) during which the comet is at the distance at which its effect might raise the waters of the sea. The velocity with which the comet moves at the distance of the earth from the sun is easily obtained, on the hypothesis that the orbit is parabolic, for this velocity is to that of the earth as $\sqrt{2}$ to 1.

(*c*) See note (*n*) to Chapter VIII. of this Volume. According to Cuvier, the appearances exhibited by various strata on the lowest parts of the earth, and also on the tops of mountains, where shells and various marine productions have been dug up, may be adduced as decisive proofs of a number of revolutions having taken place on the surface of our globe. That these revolutions have been very sudden, he infers, from the circumstance of the carcasses of some large quadrupeds having been arrested and preserved entire, with their skin, hair, and flesh, which could not be the case unless they were frozen almost as soon as they were killed. See note (*n*) Chapter VIII., where the real cause of these productions being found in these regions is assigned.

(*d*) From knowing the place of the ascending node, the inclination of the orbit of 1770, &c., Laplace determines the epoch at which the comet passes out from the sphere of Jupiter's attraction; and then, by means of these data, he computes the elements of the relative orbit of the comet about the sun, from which the elements of the ellipse at its entrance into the sphere of Jupiter's attraction are computed; and the value of the axis major $=13,293$, and of the perihelion distance $=5,0826$, shows that the comet is perpetually invisible. Buckardt, in like manner, determined the effect of the action of Jupiter on the comet, which appeared in 1779, at the moment it entered within the sphere of its attraction; and an investigation of its elements at that instant showed that they differed very little from the preceding; and then, by computing the effect of Jupiter's action, he found that the axis major became 6,388, and the perihelion distance 3,3346, at which distance the comet also remains invisible. Hence we see how the action of Jupiter may have rendered (Mechanique Celeste, Tom. II., p. 226.) this star visible in 1770, which was previously invisible, and render it then invisible from the year 1779. The changes produced in the orbit of this comet are the *greatest* instance of perturbation observed among the bodies of our system.

(*e*) n n' being the mean motions of the earth and comet, a' a the greater axes of their orbits, by assuming a' the radius of the earth's orbit = to unity, we have

$$\frac{\delta n}{n} = 104.791.m'.a, \ i.e. \text{ as } \frac{n^2}{n'^2} = \frac{a'^3}{a^3}, \ \frac{\delta n}{n} = 104.791.m'. \frac{n'^{\frac{2}{3}}}{n^{\frac{2}{3}}},$$

if T represents the duration of the comet's revolution, T′ that of the earth, and δT a variation corresponding to δn, we shall have $n\,T = 2\pi$

$$= (n + \delta n).(T + \delta T), \ \therefore \frac{\delta n}{n} = -\frac{\delta T}{T}, \text{ but } \frac{n}{n'} = \frac{T'}{T},$$

$$\therefore \delta T = -104{,}791.m' \left(\frac{T}{T'}\right)^{\frac{2}{3}}.T;$$ hence substituting for T′ and m', their numerical values, we find $\delta T = -2^d{,}046$; this is the quantity by which the action of the earth diminishes the period of the comet; and as, by No. 65 of the First Book of Celestial Mechanics,

$$\delta n' = -\frac{m\sqrt{a}}{m'\sqrt{a'}}.\delta n, \text{ and } \therefore \frac{\delta n'}{n'} = -\frac{m\sqrt{a}}{m'\sqrt{a'}}.\frac{n}{n'}.\frac{\delta n}{n}, \ i.e. \text{ as } \frac{n'^{\frac{1}{3}}}{n^{\frac{1}{3}}} = \sqrt{\frac{a}{a'}}, \text{ and as } \frac{\delta n}{n} = 104{,}791.\frac{m'.n'^{\frac{2}{3}}}{n^{\frac{2}{3}}},$$

$$\frac{\delta n'}{n'} = -104{,}791.m, \text{ and } \therefore \delta T' = 104{,}791\, m T',$$ hence if $m'=m$, we shall have $\delta T' = 0{,}11612$; and as accurate observations prove that $\delta T'$ does not exceed $2'',5$, it follows that m is not the $\frac{1}{5000}$th part of m.

NOTES TO CHAPTER V.

(a) Let the distance of the sun from the earth (which for the present we shall suppose to be constant) $= a$, and let y, z be the distances of the moon from the earth and sun, ϕ the elongation of the moon from from the sun, $\frac{\mu}{a^2} = \mathrm{F} =$ the force with which the earth is solicited towards the sun, $\frac{1}{y^2}$ $(= \mathrm{P},)$ $\frac{\mu}{z^2}$ are the forces with which the moon is sollicited to the earth and sun, (μ being the mass of the sun relatively to the earth represented by unity,) and it is evident, that if the sun was infinitely distant, $\frac{\mu}{a^2}$ and $\frac{\mu}{z^2}$ might be considered as $=$, and acting in parallel directions. If $\frac{\mu}{z^2}$ be resolved into two forces, one in the direction of y, and the other in the direction of a, they will be respectively $\frac{\mu y}{z^3} = \mathrm{Q}$, $\frac{\mu a}{z^3} = \mathrm{R}$, if $\mathrm{R} = \mathrm{F}$, the relative motion of the moon about the earth would not be changed, the only effect would be to make the sun approach the common centre of gravity of these two bodies. Hence it follows, that the moon is drawn from the earth parallel to a, in virtue of the force $\mu\, a. \left(\frac{1}{z^3} - \frac{1}{a^3}\right) = S$; now as the greatest value of $z = a + y$, and its least value $= a - y$, and consequently the mean value $= a$, it might seem that in general this force cannot alter the mean distance of the moon from the earth; but this is not the case. See note (g) of this Chapter. From what precedes, it appears, therefore, that the forces which affect the relative motion of the moon about the earth, are $P + Q$ in the direction of y the radius of its orbit, S in a direction parallel to a, the line connecting the earth and sun; therefore, if in a line drawn from the place of the moon in its orbit, parallel to a, a portion be assumed $= S$, and if this force be resolved into two, one in the direction of y, and the other in the direction of a tangent to the moon's orbit, we have N the whole central force of the moon

$$= \frac{1}{y^2} + \frac{\mu y}{z^3} - \mu . \frac{(a^3 - z^3)}{a^2 z^3} . \cos . \phi,$$ and M the force in the direction of the tangent $= \mu \frac{(a^3 - z^3)}{a^2 z^3} . \sin . \phi$.

(b) The manner in which the perturbations of the moon are investigated, is not essentially different from the manner employed to determine those of the planets; however, they are not exactly the same, for though the fundamental equations have the same form, still we cannot assume, as in the case of the planets, that the mass of the disturbing body is inconsiderable relatively to that of the central, for the sun is in this case the disturbing body, whose mass is 354790 times that of the earth; this quantity is indeed multiplied by the ratio of the radius of the lunar to that of the terrestrial orbit, *i. e.* by $\frac{1}{64000000}$, still, however, the product is too considerable to permit us to neglect its square, which we can do for the most part in case of the planets. Notwithstanding the facilities mentioned in the text, the theory of the moon is embarrassed with peculiar difficulty, owing in a great measure to the magnitude of its numerous inequalities, and also very much to the little convergence of the series which furnish them; hence results the necessity of a judicious selection of ordinates, &c., insisted on in the text. The solution of the problem of the three bodies which is given by Laplace, is called the direct method, in contradistinction to the more simple and indirect method which Newton followed, which is by no means so accurate as the first, in which the terms of the series by which the motions of the moon are expressed may be computed to an indefinite extent, or at least until the quantities omited are too small to affect observations, whereas, in the method of Newton, we cannot go farther than the first, or at most a *few of the leading* terms of each series.

(*c*) Though it is difficult to determine the precise quantity by which the apsides advance, it is easy to show that, in consequence of the disturbing force of the sun, they *must progràde;* for, as is evident from the nature of an elliptic orbit, if a body leaves an apsis, it will arrive at the other apsis after describing 180°; (see notes, page 375.;) but as the mean disturbing force in the direction of the radius vector tends, *on the whole*, to diminish the gravitation of the moon to the earth, (see note (*f*),) the portion of her path described in any instant will be less deflected from the tangent than if this disturbing force did not exist; therefore the actual path of the moon will be less incurvated than the elliptic orbit, which would be described if the moon was influenced solely by the force of gravity, and consequently it will not be brought to intersect the radius vector at right angles, until it has moved over a *greater arc than* 180°; therefore, in consequence of the action of the solar force, the apsides advance. The term which Clairault proposed to add to $\frac{1}{r^2}$, was one of which the effect could be only sensible in the motion of the apogee, and not on the other lunar inequalities. However, when he proceeded to fix the form of the quantity which would make this new supposition represent the motion of the apsides, he found it necessary to perform the computations more rigorously, and to extend his approximation farther than before, so as to include terms which he had previously neglected; and when these terms were taken into account, he found that it was not necessary to make any correction to the Newtonian theory; consequently it followed that no such term as $\frac{B}{r^m}$ was required, or, in other words, that a force varying inversely as the square of the distance was sufficient to explain the motion of the apsides. For an able and satisfactory analytical statement of the process by which Clairault arrived at the true expression for the motion of the lunar apsides, see Woodhouse's Astronomy, Vol. II., Chap. 13.

According to Buffon, it is more philosophical to suppose that every primordial law of nature depends on one sole function of the distance; for if two powers of the distance were introduced, the function expressing the law of attraction would necessarily contain one constant arbitrary quantity at least, so that the ratio of the attractive forces at two different distances of the attracting body would not depend solely on two different distances, but would also involve some parameter, which would modify and complicate this ratio; thus the attraction would depend not only *on the distance*, but also on this parameter, for the introduction of which there does not appear to be a sufficient reason.

As the method by which Clairault determined the precise motion of the lunar apsis was analytical, it was supposed by several mathematicians that the direct synthetic method of Newton was inadequate to the determination of its exact quantity, and it is thought that Newton himself was conscious of the insufficiency of his method, from the circumstance of his omitting, in the later editions of the Principia, any mention of the motion of the apogee. However, though Newton certainly did not mention the quantity of this progress, still there can be no doubt but that his method is fully equal to determining its exact quantity. See Stewart's Mathematical and Philosophical Tracts.

If ☾ ψ represent the relative motions of the moon and apsis, then it is easy to show that $d\psi$, the differential of the motion of the apsis $= \frac{3\mu y^3}{4a^3}$ (d ☾ $- 3nd\phi.\cos.2\phi$), when d ☾ is the periodic, and $d\phi$ the synodic motion of the moon, and $n:1$ the ratio of $d\psi$ to $d\phi$,

$\therefore \psi = \frac{3\mu y^3}{4a^3}.$ (☾ $-\frac{3}{2}n.\sin.2\phi$); $\frac{\mu y^3}{a^3}$ 270 is the mean motion of the apsides in a *periodic* month. The second term $-\frac{9\mu y^3}{8a^3}.n.\sin.2\phi$, gives the libration of the apsides; it evidently vanishes in syzygies and quadratures, is a positive *maximum* in the octants which precede the quadratures; and a negative maximum in the octants which precede the syzygies. This equation of the apsides produces a correction to be applied to the equation of the centre; the argument for which depends on the distance of the sun from the moon's apogee.

(d) In the conjunctions N, given in note (a), becomes, (as $\phi = 0$, and

$$x = a - y) = \frac{1}{y^2} + \frac{\mu.(y-a)}{(a-y)^3} + \frac{\mu}{a^2} = \frac{1}{y^2} - \frac{\mu(2ay-y^2)}{a^2(a-y)^2},$$

i. e. as y is very small relatively to $a = \frac{1}{y^2} - \frac{2\mu y}{a^3}$, in opposition $N = \frac{1}{y}$ $\quad \frac{\mu}{a^2} = \frac{1}{y^2} - \frac{2\mu y}{a^3}$, in the quadratures, cos. $\phi = 0$, and $z = a$, $\therefore$ N $= \frac{1}{y^2} + \frac{\mu y}{a^3}$,

hence in the syzygies the central force $\frac{1}{y^2}$, experiences the greatest diminution, and in the quadratures the greatest increase; and it is evident that the diminution in syzygies is twice the increase in quadratures. Conceive a perpendicular from the place of the moon on a, the part of a intercepted between the perpendicular and the centre of the sun is nearly

$= $ to z, $\therefore z = a - y.$ cos. ϕ, and $z^3 = a^3 - 3a^2 y.$ cos. ϕ,

hence, by substituting this value of z^3 in the expression for M, given in note (a), page 324, we obtain

$$\text{M} = \frac{3\mu y. \sin. \phi. \cos. \phi}{z^3} = \frac{3\mu y. \sin. 2\,\phi}{2\,a^3},$$

$$\text{N} = \frac{1}{y^2} + \mu y\,\frac{(1 - 3. \cos.^2\phi)}{z^3} = \frac{1}{y^2} - \frac{\mu y(1 + 3. \cos. 2\phi)}{2a^3},$$

hence it appears that M attains its greatest *positive* value in the octants which precede the syzygies, and its greatest negative value at the octants which precede the quadrature; and that the central force varies as $\frac{1}{y^2}$, when $1 + 3 .\cos. 2\,\phi = 0$, *i. e.* when $\cos. 2\phi = -\frac{1}{3}$, and $\therefore \phi = 54°, 44'$; $\phi = 125°, 16'$; $\phi = 135°, 16'$; $\phi = 234°, 44'$; $\therefore$ in these four points, two of which are nearly at 10° before the octants which precede the syzygies, and two at 10° after the octants which precedes the quadratures, the central force varies inversely at the square of the distance.

It is easy to show, that in the quadratures the force varies in a ratio less than that of the inverse square of the distance, for in that case N $= \frac{a^3 + \mu y^3}{a^3 y^2}$, and if y' differs very little from y, N′ being the central force at the distance y', then

$$N : N' :: \frac{1}{y^2}\left(1 + \frac{\mu y^3}{a^3}\right) : \frac{1}{y'^2}\left(1 + \frac{\mu y'^3}{a^3}\right), \textit{ i. e.} \text{ if}$$

$$\alpha\ \alpha' = \frac{\mu y^3}{a^3}, \frac{\mu y'^3}{a^3} \text{ respectively, } N : N' :: \frac{1+\alpha}{y^2} : \frac{1+\alpha'}{y'^2},$$

now, as $y = 60$ radii of the earth, $q.p$ all powers of y, whether integral or fractional, whose index is positive, are necessarily greater than unity; therefore we may assume $1 + \alpha = y^\gamma$ $1 + \alpha' = y'^\gamma$, γ 1 being a small positive fraction, and consequently

$$N : N' :: \frac{y^\gamma}{y^2} : \frac{y'^\gamma}{y'^2} :: \frac{1}{y^{2-\gamma}} : \frac{1}{y'^{2-\gamma}} :: \frac{1}{y^n} : \frac{1}{y'^n}, \therefore \text{ as } \gamma \text{ is}$$

a positive fraction near to the quadratures, the force varies in a less ratio than the square of the distance; in like manner, at the syzygies

$$N : N' :: \frac{a^3 - 2\mu y^3}{y^2} : \frac{a^3 - 2\mu y'^3}{y'^2} :: \frac{1-\beta}{y^2} : \frac{1-\beta'}{y'^2}$$

$$:: \frac{y^{-\lambda}}{y^2} : \frac{y'^{-\lambda}}{y'^2} :: \frac{1}{y^{2+\lambda}} : \frac{1}{y'^{2+\lambda}}, \text{ when } \beta = \text{a very small}$$

fraction, $\therefore$ as $1+\alpha = y^\gamma$, and $1-\beta = y^{-\lambda}$, we have $q.p$ $\beta = 2\alpha \therefore \lambda = 2\gamma$, $\therefore$ if $N = \frac{1}{y^m}$, m is $>$ than 2, and $m-2=2(2-n)$, therefore in the syzygies the central force varies in a greater ratio than the inverse square of the distance; $\frac{1}{y^2} - N$ is a maximum, and $= \frac{2\mu y}{a^3}$ in the syzygies; but its actual value depends on the situation of the apsides; as N : N′ near to the syzygies are as $\frac{a^3 - 2\mu y^3}{y^2} : \frac{a^3 - 2\mu y'^3}{y'^2}$, which would be the inverse :: of the squares of the distances, if the numerators were $=$, $\therefore$

it will be the more deranged, according as y y' differ more from each other; but the difference is evidently a maximum, when the line of apsides coincides with that of the syzygies; therefore the ellipse suffers the greatest derangement in this case; from the first and last quadratures to the syzygies, M increases the velocity of the moon, and from the syzygies to quadrature, M retards by the same quantity the velocity of the moon; hence the velocity is a maximum in the syzygies, and a minimum in the quadratures, and at its mean value in the octants, since the velocity is a maximum, and the central force a minimum in the syzygies, the moon in these points deviates less from the tangent; therefore the moon approaches the earth least in syzygies, and most in quadratures; and as the orbit is a curve returning into itself, it follows that, as the moon commences to recede from the earth at the syzygies, and to approach the earth in the quadratures, that the distance of the moon from the earth in an orbit which, without the disturbing force action of the sun, would be circular, is greatest in quadratures and least in syzygies. If the orbit be an ellipse, *i. e.* if the force varied as $\frac{1}{y^2}$, then if, in going from apogee to perigee, the force increases in a greater ratio (see page 341), the *true* orbit will fall within the ellipse, and the perigean distance will be less than for the ellipse, consequently the eccentricity will increase so much the more as the axis major diminishes; for a like reason, if the moon departs from the perigee, and the force decreases in a greater $\div$ than the inverse square of the distance, the moon, when in the apogee, will have receded farther from the earth than if the orbit described was an ellipse; therefore, in the other half of the orbit the eccentricity will be also increased, and the contrary to this will obtain if the force varies in a less ratio than the inverse square of the distance. Now, as the force varies in a greater or less ratio than $\frac{1}{y^2}$, according as the apsides coincide with the syzygies or the quadrature, it follows that the eccentricity is a maximum in the former, and a minimum in the latter case; *i.e.* when the greatest equation of the centre coincides with the quadratures the eccentricity is a maximum, when this equation occurs near to the syzygies this eccentricity is a minimum, and generally in the progress of the apsides from the syzygies to quadratures the eccentricity diminishes, and from quadratures to syzygies the eccentricity increases. This is the explanation of the phenomenon known by the name of evection. See note (*t*) Chapter IV., Vol. I.; and Princip. Math. Prop. 66., Cor. 9.

(*e*) In order to compute the *mean* quantity of the force $\frac{\mu y}{a^3} . (1-3 \cos.^2\phi)$, which is continually directed to or from the centre of the earth, if we multiply it by $d\,\phi$, the differential of the arc of elongation, we have $\frac{\mu y}{a^3} . (d\,\phi - 3\,d\,\phi \cos.^2\phi)$, the integral of which
$= \frac{m y}{a^3} . \left(- \phi + \frac{3}{2} . \sin. \phi . \cos. \phi \right)$, which, when extended to the whole orbit, *i. e.* when ϕ is four right angles, becomes $\frac{\mu y}{a^3} \times - \frac{\pi}{2}$, which therefore expresses the sum of the forces for an entire revolution; and ∴ when divided by π gives the mean force $\frac{-\mu y}{2\,a^3}$, which being negative, shows that the mean effect of the solar force is to diminish the gravitation of the moon to the earth.

Calling $\tau\,\tau'$ the periods of the earth and moon, F the gravity of the moon, we have

$\frac{\mu}{a^2} : \mathrm{F} :: \frac{a}{r^2} : \frac{y}{r'^2}$, $\therefore \frac{\mu y}{a^3} = \frac{\mathrm{F}.r'^2}{r^2} = \frac{\mathrm{F}}{179}$, for $\frac{r'^2}{r^2} = \frac{1}{179}$,

and $\therefore \frac{-\mu y}{2a^3} = \frac{-\mathrm{F}}{358}$.

(*f*) Though the mean area is therefore not altered, since the radius vector is increased $\frac{1}{358}$th part, inasmuch as the angular velocity in *this case* is inversely as the square of the distance, its diminution will be *q.p* half the increase of the radius vector, and $\therefore \frac{1}{179}$th part.

(*g*) It is evident, from what precedes, that every thing else being the same, the numerical coefficient $\frac{1}{179}$ varies inversely as the cube of the distance; therefore, making $a = 1 + \varepsilon$, we shall have the mean disturbing force in the direction of $y = -\frac{\mathrm{F}}{358} \cdot (1+\varepsilon)^{-3}$; now, as the magnitude of the area is the same, we have

$$\delta.(y.^2\delta\phi) = 0, \text{ } i.e. \text{ } 2\delta y\delta\phi + y\delta^2\phi = 0; \therefore$$

$\frac{\delta^2\phi}{\delta\phi} = -\frac{2\delta y}{y} = (1+\varepsilon).^{-3} \frac{2}{358} = \frac{(1+\varepsilon)^{-3}}{179}$, $\therefore$ on the hypothesis that the moon's orbit is circular, and

$$\therefore \delta\phi = n\delta t, \text{ we have } \delta^2\phi = -\frac{n\,dt}{179}(1+\varepsilon)^{-3},$$

hence, if e be the eccentricity of the lunar orbit, and l the mean anomaly, we have

$$a = 1+\varepsilon = 1 + \frac{e^2}{2} - e.\ \cos.\ l - \frac{e^2}{2}.\cos.\ 2l + \text{, \&c.}$$

See Celestial Mechanics, page 146.

$\therefore (1+\varepsilon)^{-3} = 1 - 3\varepsilon + 6\varepsilon^2$, &c. $= 1 + 3e(\cos.\ l - \frac{e}{2} + \frac{e}{2}.$

$\cos.\ 2l) + 6e^2 \cos.\ ^2l$, &c. $= 1 + \frac{3}{2}e^2 + 3e.\ \cos.\ l + \frac{9}{2}e^2 \cos.$

$2l$; if m represent the mean anomalistic motion of the sun, so that $\delta a = m\delta t$, we shall have $\delta^2\phi$

$$= \frac{-n\delta t}{179}\cdot\left(1+\frac{3}{2}e^2\right) - \frac{3en\delta a}{179m}\cdot\left(\text{cos. } l + \frac{3}{2}e \text{ cos. } 2l\right),$$

$$\therefore \int\delta^2\phi = \frac{-nt}{179} - \frac{3n}{358}\int e^2\delta t - \frac{3en}{179m}\left(\text{sin. } l + \frac{3}{4}e.\ \text{sin. } 2l\right).$$

The first term of this expression is included under the mean motion of the moon, namely nt; in fact, it is the mean diminution of the lunar motion, constituting, as appears, the 179th part of the primitive motion or $\frac{-nt}{179}$; the second term would also belong to the mean motion, if e was constant; but as the eccentricity of the earth is changed in consequence of the action of the planets, there results from it a *secular equation* of the mean motion of the moon $= -\frac{3n}{358}\int e^2\delta t$; the third term gives the *annual* equation of the moon; and if their values 13,37, and $\frac{1}{60}$ be substituted for $\frac{n}{m}$ and e, we shall obtain the value of the greatest equation given in Chapter IV., Vol. I., note (u); and it is evident, that is, if the same form, only affected with an opposite sign, as the equation of the centre of the sun. See note (u), Chapter IV., Vol. I.

(h) If the mean distance $=1$, e equal $\frac{1}{60}$, and A, A$+a$ are the respective angular velocities, we have
$A : (A+a) : 1 : \left(1+\frac{1}{60}\right)^2 :: 1 : 1 + \frac{1}{30}$, $\left(\frac{1}{60}\right)^2$ being neglected, $\therefore a = \frac{A}{30}$; in like manner the mean value of the diminution of the mean motion $= \frac{-nt}{179}$, and as this

varies inversely as the cube of the distance, at perigee it is $= \frac{-nt}{179} \cdot (1+e)^{-3}$, $\therefore$ neglecting e^2 and e^3, and substituting its numerical value for e, the increase of the diminution $= \frac{nt}{20.179} = \frac{nt}{3580}$, $\therefore$ as the arguments of the equation of the centre and of the annual equation are the same, they are as their coefficients, *i. e.* as $\frac{mt}{30}$ to $\frac{nt}{3580}$.

(i) If the mean motion of the moon, as determined by observations made at considerable intervals from each other, varies, its correction a, as it depends on the time, must be given by a series of the form $at+bt^2+ct^3+$, &c.; but all terms of the form at, being already included in the mean motion, we shall have $a=bt^2+ct^3+$, &c.; therefore the secular equation, or rather the most considerable of its terms is proportional to the square of the time, which is indeed otherwise evident from the following consideration, the force, whatever it be, which accelerates the motion of the moon, must be considered as constant, otherwise it could not produce a *real secular* equation; therefore δv, the increment of velocity communicated at *each instant*, may be considered as constant; hence $\delta v = \gamma \delta t$, and $v = c + \gamma t$, $\therefore$ if s be the mean motion in the time t, we shall have $\delta s = v\delta t = c\delta t + \gamma t \delta t$, and $s = ct + \frac{\gamma t^2}{2}$, cv being the mean motion, therefore the acceleration or the secular equation is proportional to the square of the time, hence appears the reason of what is stated in the text, page 65, that as the increase takes place successively, and proportionally to the time, the effect on the moon's motion is half what it would be, if, during the entire century it was the same as at the end, and also the reason why the secular equation may be considered as increasing proportionally to the square of the time, as long as the diminution of the square of the eccentricity of the earth's orbit, is supposed to be proportional to the time.

If X be the mean motion of the moon between two observations, at the interval of n years, then $\frac{X}{n}$ and $\frac{100X}{n} = x$ will be the annual and secular motions. If y be the number of seconds by which the mean motion of the moon is greater than x in the following century, and less than x in the preceding, the correct mean motion in the following century will $x+y$, and in m centuries $= mx+m^2y$; hence, if C be the mean longitude at the commencement of the epoch, the mean longitude m centuries *after* the epoch $= C+mx+m^2y$, and for m centuries *before* the epoch the mean longitude $= C-mx+m^2y$, for in the last case m must be taken negatively, and $(-m)^2=m^2$; hence in *both* cases the *secular* equation is *additive*.

(k) The second term of the value of $\int \delta^2\phi = -\frac{3n}{358}. \int e^2\delta t = -\frac{3n}{179}\int\frac{e^2}{2}\delta t$, hence, as has been remarked already in page 416, since e is variable, this expression will not remain always the same. If ε be value of e at the commencement of the present century, then the value of e at any subsequent time $t = \varepsilon + t.\frac{\delta\varepsilon}{\delta t} + \frac{t^2}{1.2}.\frac{\delta^2\varepsilon}{\delta t^2}$ and $e^2 = \varepsilon^2 + 2t\,\varepsilon.\frac{\delta\varepsilon}{\delta t} + t^2\left(\frac{\delta e^2}{\delta t^2} + \varepsilon.\frac{\delta^2 e}{\delta t^2}\right)$, by means of this equation we can deduce the value of $\frac{\delta^2 e}{\delta t^2}$ in terms of ε and $\frac{\delta e}{\delta t}$, which are respectively given, $\frac{\delta e}{\delta t}$ being the variation of the eccentricity, and ∴ known; consequently, by substituting their numerical values, we obtain

$$e^2=\varepsilon^2-2t.\,0.''14671-t_{\prime}.^2 0.''0006027, \text{ and } \therefore \frac{3n}{358}.\int e^2\delta t =$$

$$\frac{-3\varepsilon^2}{358}.\,nt+\frac{3n}{358}.\,t_{\prime}^2 0.''14671+\frac{n}{358}t'^3.\,0.''006027,$$

i. e. substituting for n its value, and neglecting the first term of this expression, which is included under the mean motion, we have the part of the mean motion which depends on the variation of the eccentricity

$$=t.^2 10,''18+t.^3 0,''018538,$$

hence appears the reason why, as stated in page 66, he adds a term proportional to the cube of the time.

(*l*) As $\frac{3n}{358}\int e^2\delta t$ is affected with a negative sign, it is evident that the motion will be accelerated when e^2, or the eccentricity of the earth's orbit is diminished, and that it will be retarded when the quantity to be subtracted increases with the increase of e^2, or the eccentricity of the earth's orbit.

(*m*) In reference to what is stated in page 67, it may be observed, that after a most complicated investigation, and by substituting their numerical values, the secular equation of the perigee comes out = to $3{,}03.\ \frac{3}{2}m^2\int e.^2 n\delta t$, and has a contrary sign to the secular equation of the longitude; the secular equation of the motion of the nodes comes out = to $0{,}735452.\ \frac{3}{2}m^2\int e^2\ n.\delta t$, and as in the same circumstances, the secular equation of the mean motion $=-\frac{3}{2}m^2,\int e^2 n\delta t$; it follows that when the two first are accelerated, the last is retarded, and their ratio in numbers is that given in the text.

(*n*) In consequence of the attraction of the terrestrial spheroid, a nutation arises in the orbit of the moon, corresponding to that which the attraction of the moon produces in our equator, so that one of these nutations may be shown to be the reaction of the other; (see Chapter XIII. notes;) as the extent of this oscillation or nutation depends on the *compression* of the earth, it can throw considerable light on this important element. The existence of this inequality in the latitude of the moon, was indicated by observations long before the law which it observed was discovered. It can be perfectly represented by the expression 8″ sin. L, if the compression of the earth was assumed $= \frac{1}{334}$, whereas, if the compression was that which resulted from the hypothesis of homogeneity, namely, $\frac{1}{230}$, (see Chapter VIII., page 108,) the expression of this inequality would be —13.5. sin. L. The manner in which the quantity of the compression is determined is as follows: the *theoretic* expression for this inequality, which involves the compression of the earth, is compared with the value furnished from a careful discussion of a number of observations, and then substituting numerical values for sin. L, the value of the compression is thence deduced. In like manner, the inequality of the motion of the moon in longitude, which depends on the compression of the earth, is = 6″,8 cos. L (L expressing, as before, the longitude of the ascending node) on the hypothesis of a compression $= \frac{1}{334}$, which is exactly conformable to observation; whereas, on the hypothesis of homogeneity, this inequality would be = 11.″5. sin. L, contrary to observation. There is, according to theory, a certain given relation between the coefficient of the lunar inequality in longitude and latitude, from which the value of the coefficient of the inequality in longitude may be determined; and on the supposition that the coefficient of that in latitude is 8″, namely, that which results from the hypothesis of a compression of the earth $= \frac{1}{334}$, the coefficient of the inequality in longitude comes out 6,846, very nearly that which is given by observation. A phenomenon analogous to the preceding, and arising from the same cause, is produced in the orbit of Jupiter's satellites. See Chapter VI. Besides these inequalities depending on the compression of the earth, Laplace also investigated whether the difference which is known to exist in the quantity of land distributed over the northern and southern hemispheres, had any sensible influence; but a careful discussion showed that this effect was altogether inappreciable.

(*o*) In an inequality depending on the true distance of the moon from the sun, which Laplace terms the parallactic inequality, (see Mechanique Celest. Tom. 7., page 281,) the argument is $v-mv$; it depends on the ratio of the moon's distance from the earth to the sun's distance from the same, *i. e.* on the ratio of their horizontal parallaxes, which, as that of the moon is determined in the Second Chapter, it is easy to find; the result ought to be considered as extremely accurate, inasmuch as the approximation is extended to quantities of the fifth order inclusively. The only point in the theory of the moon's motion which remains to be cleared up after the delicate investigations of Messrs. Plana, &c. and Damoiseau, is a small change which astronomers have thought they discovered in the mean motion of the moon. However, as the existence of this change, though extremely probable, is not incontrovertably established, a greater number of accurate observations, made in the most favourable circumstances, is required before there will be occasion to ascertain its cause. The imperfect manner in which accurate observations have been made and transmitted to us, may also explain why we have not been hitherto able to appreciate the changes produced in the motions of the planets and satellites, in consequence of the attraction of comets, and also of the impact of meteoric stones, which are observed sometimes to impinge on our earth, and which appear to come from the depths of celestial space. The only thing which can throw light on this subject is a series of accurate observations.

(*p*) Laplace, in the Sixth Chapter of the Second Part of his Tenth Book, investigates in what cases we can rigorously obtain the motion of a *system* of bodies, which mutually attract each; and as, in order that this may be secured, it is necessary that the resultants of the forces by which each of the bodies of the system is actuated should pass through their common centre of gravity, and be proportional to the respective distances of the bodies from that point, he shows, that if the position of the bodies of the system be such, that the lines connecting them constitutes a polygon, existing in the same plane, which remains always similar to that formed by joining the bodies at the commencement of their motion, then (the law of attraction being proportional to any power of the distance between the bodies) the resultants of the forces by which the bodies are actuated must pass always through the common centre of gravity; but it is evident that these resultants, at the commencement, being supposed to pass through the centre of gravity, and to be proportional to the distances from that centre, they will always remain so, if, on the several bodies of the system, velocities be impressed in directions equally inclined to those distances, and respectively proportional to them; then the polygons formed at each instant by lines connecting the bodies will be *always* in the same plane, and similar, and the curves which the bodies will describe about their common centre, and about each other, will be similar to each other, and they will be of the same species with that which a body, actuated by the same law of force, would describe about a fixed point. See Princip. Math., Sect. 11, Prop. 58. Let the preceding conclusions be applied to the case of three bodies, $m\,m'\,m''$, acting on each other. If s be the distance between m and m', s' the distance of m from m'', and s'' the distance between m' and m'', it is easy to perceive that the force by which m is actuated parallel to the axis of x, is

$$m'\frac{\phi(s)}{s}(x-x')+m''\frac{\phi(s')}{s'}.(x-x''),$$

(the attractive force being proportional to $\phi(s)$.) and parallel to the axis of y will be

$$m'\frac{\phi(s)}{s}.(y-y')+m''.\frac{\phi(s')}{s'}.(y-y'');$$

similar expressions may be obtained for the forces parallel to these axis, acting on m' and m''; now, as by hypothesis, the resultant of the two forces parallel to the axes of x and y, which act on m, passes through the centre of gravity, we have

$$\frac{m.\phi(s)}{s}.(x-x')+m''\frac{\phi(s')}{s'}.(x-x'')$$

$$=\mathrm{K}x,\ \frac{m\phi(s).}{s}(y-y')+m''\frac{\phi(s').}{s'}(y-y'')=\mathrm{K}y;$$

therefore the force which solicits m towards the centre of gravity is $\mathrm{K}.\sqrt{x^2+y^2}$; in like manner, it might be shown, that the force soliciting m' towards this point is $\mathrm{K}'\sqrt{x'^2+y'^2}$, and as, by hypothesis, the forces are as the distances, $\mathrm{K}=\mathrm{K}'$; therefore, for the forces acting on $m\,m'\,m''$ parallel to the axis of x, we have the three following equations:

$$m'.\frac{\phi(s)}{s}.(x-x')+m''\frac{\phi(s')}{s'}.(x-x'')=\mathrm{K}x$$

$$m.\frac{\phi(s)}{s}.(x'-x)+m''.\frac{\phi(s'')}{s''}.(x'-x'')=\mathrm{K}x' \qquad (a)$$

$$m.\frac{\phi(s')}{s'}.(x''-x)+m'.\frac{\phi(s'')}{s''}.(x''-x')=\text{K}x''.$$

Similar equations may be obtained for the forces parallel to y, changing $x\ x'\ x''$ into y, y', y'', &c.

Multiplying the preceding equations by $m\ m'\ m''$ respectively, and then adding them together, we obtain

$$0=mx+m'x'+m''x'';$$

which shows that the point to which the forces are directed must be the centre of gravity; by combining this equation with the first of the equations (a), we obtain

$$x\left(m.'\frac{\phi(s)}{s}+(m+m'')\frac{\phi(s')}{s'}\right)+m''x'\left(\frac{\phi(s')}{s'}-\frac{\phi(s)}{s}\right)=$$

Kx, which, if $s=s'$, gives $\text{K}=(m+m'+m'').\frac{\phi(s)}{s}$, the same value of K will be obtained, if we suppose $s=s''$, $\therefore$ if $s=s'=s''$, this expression satisfies the equations (a), and the corresponding ones in $y y' y''$, $\therefore$ if on this supposition $r\ r'\ r''$ represent the respective distances of m, m', m'', from the centre of gravity of the system, the forces which solicit m, m', m'', towards this point are Kr, Kr', Kr''; and if, on these bodies, velocities be impressed proportional to r, r', r'', respectively, and in directions equally inclined to $r\ r'\ r''$, we will have, during the motion $s=s'=s''$, *i. e.* the three bodies will always exist in the vertices of an equilateral triangle formed by connecting them, and they will describe similar curves about each other, and about their common centre of gravity; Princip. Math. Prop. 52, Sect. 11. He next proceeds to determine the expression for K in a function of r, which will evidently determine the law of force Kr in a function of r. For this purpose, let the origin of the coordinates be any point different from X, Y, the centre of gravity; as, for instance, the centre of m, then $x\ y$ are = to cypher, $\text{X}^2+\text{Y}^2=r^2=$ generally

$$\Sigma m.\frac{(x^2+y^2)}{\Sigma m}-\Sigma m m'\frac{(x-x')^2+(y'-y)^2}{(\Sigma m)^2}, \textit{ i. e. as}$$

$xy=0$; and $s^2=s'^2$, *i. e.* $(x'-x)^2+(y'-y)^2=$

$$(x''-x)^2+(y''-y)^2,\ r^2=\frac{(m'+m'').s^2}{m+m'+m''}$$

$$-\frac{(mm'+mm''+m'm'')\,s^2}{(m+m'+m'')^2},\ \therefore s=\frac{(m+m'+m'').r}{\sqrt{m'^2+m'm''+m''^2}},$$

$\therefore$ as $K=(m+m'+m'').\dfrac{\phi(s)}{s}$, we have

$$Kr=\sqrt{m'^2+m'm''+m''^2}.\phi\left(\frac{m+m'+m''.r}{\sqrt{m'^2+m'm''+m''^2}}\right),$$

hence, as we have the expression for the law of force, we can, by what is stated in notes page 373, determine the nature of the curve described, when the form of ϕ is given; if $\phi(s)=\dfrac{1}{s^2}$, then the force which solicits the body m towards the centre of gravity

$=\dfrac{\overline{m'^2+m'm''+m''^2}|^{\frac{3}{2}}}{(m+m'+m'')^2.r^2}$, $\therefore$ the three bodies will describe similar conic sections about the centre of gravity of the system, the lines connecting them constituting always an equilateral triangle, the sides of which continually vary, and become infinite, if the section be a parabola or hyperbola, which circumstance depends on the initial velocity. But if $s\ s'\ s''$ are not $=$, then we have

$$x.\left(m'.\frac{\phi(s)}{s}+(m+m'').\frac{\phi(s')}{s'}\right)+m''x'\left(\frac{\phi(s')}{s'}-\frac{\phi(s)}{s}\right)=$$

Kx. As a similar equation obtains between y and y', we have $x : x' :: y : y'$, $\therefore m\ m'$ exist in the same right line with the centre of gravity; therefore $m\ m'\ m''$ are in the same right line. Suppose that this right line is the axis of the abscissæ, and that the bodies are ranged in the order $m\, m'\, m''$, their common centre being between m and m', let $x' = -\mu x$, $x'' = -Vx$, then if $\phi(s) = s^n$, as $s = (1+\mu).x$, $s' = (1+V)x$, from this and the equations (a) we obtain $K = x^{n-1}.\{m'(1+\mu)^n + m''(1+V)^n\}$, $\mu\{m'(1+\mu)^n + m''(1+V)^n\} = m(1+\mu)^n - m^n.\ (V-\mu)^n$; let $V-\mu = (1+\mu).z$, then $1+V = (1+\mu).(1+z)$; consequently, $\mu\{(m'+m'').(1+z)^n\} = m - m''z^n$, and as the equation $mx + m'x' + m''x'' = 0$, gives $m - m'\mu - m''V = 0$;

$\therefore \mu = \dfrac{m - m''z}{m' + m''(1+z)}$; therefore we have

$(m - m''z).\{m' + m''(1+z)^n = (m'+m'').(1+z)\}.(m' - m''z^n)$; when $n = -2$, this equation becomes

$$mz^2.\{(1+z)^3 - 1\} - m'\{(1+z)^2(1-z^3)\} - m''\{(1+z)^3 - z^3\} = 0,$$

which is of the fifth degree, therefore it has one real root, which is necessarily positive, for when $z=0$, the first member of this equation is negative, and when $z = \infty$, this first member is positive. If m be the sun, m' the earth, and m'' the moon, then, as z and m', m'' are very small quantities relatively to m, we have $3m\, z^3 = m' + m''$ very nearly, and $z = \sqrt[3]{\dfrac{m'+m''}{3m}}$, which, by substituting their values, gives $z = \frac{1}{100}\ q.p$; $\therefore$ if, as is stated in the text, the earth and moon were placed in the same right line, at distances from the sun proportional to 1 and $1 + \frac{1}{100}$, and if velocities were impressed on these bodies in parallel directions, and proportional to their distances from the sun, the moon would be always in opposition, and these two luminaries would succeed each other alternately; and as the extent of the earth's shadow ranges from 213 to 220 semidiameters of the earth, and therefore is much less than the $\frac{1}{100}$th part of the earth's distance from the sun, the moon would be never eclipsed, consequently, during the night, its light would succeed that of the sun; it is assumed here that the sole use of the moon was to afford light in the absence of the sun, but though this may be one use, there are others equally important, such as to elevate the waters of the ocean and air, and thus produce a continual circulation of the sea and of the atmosphere, &c.

This opinion of the Arcadians, mentioned in page 80, that their ancestors inhabited the earth before the moon was a satellite, has been transmitted to us by Ovid and other authors. In his Fasti, speaking of Arcadia, Ovid has these remarkable words:

> " Orta prior Luna, de se si creditur ipsi
> A magno Tellus Arcade nomen habet.
> Ante jovem genitum terras habuisse feruntur
> Arcades et Luna gens prior illa fuit."

And in Apollonius Rhodius we have

> αρκάδἑς οἱ και πρόσθε σεληναιης υδεονται.

It was in consequence of these authorities, combined with the appearance which the moon presents through a telescope, and its almost total absence of atmosphere, that some philosophers fancied they perceived on the surface of the moon, vestiges of a body burned up by the sun, and this led them to think that the moon might one time or other have been a comet, which passing very near the earth after the perihelion, was forced by the attraction of the earth to become its satellite. However, it is easy to show, by means of what has been established in notes to Chapter I. of this Volume, that no comet moving in a parabolic or in a hyperbolic orbit can become a satellite of the earth. If a comet, moving in an elliptic orbit becomes a satellite, it must, at the moment it enters within the sphere of the earth's attraction, be at right angles to the extremity of the upper apsis of the ellipse which the comet describes about the sun; for if, instead of being perpendicular, it made an acute angle with it, it is evident, that however small the velocity of the comet at this point, the comet cannot remain always within the sphere of the earth's attraction; for when, by the nature of conic sections, the comet arrives at a distance from the earth equal to that which it had when it commenced to be subject to the action of the earth, the radius vector of the comet is situated relatively to the axis of the conic section, in a manner similar to the radius vector by which the comet entered within the sphere of attraction, and the tangential velocities will be equal; but in the first case the direction of the motion makes an acute, and in the second case an obtuse angle with the radius vector, and therefore will cause the comet to move out of the sphere of the earth's attraction; consequently the comet must, at the time of its entrance within the sphere of the earth's attraction, be at its highest apsid; hence it appears how extremely improbable it is, that a comet, moving in an elliptic orbit, can ever become a satellite of the earth; but with respect to our moon the improbability is still greater, as it is considerably within the sphere of the earth's attraction; besides, it appears to be firmly connected with the earth: its motions, rigorously computed and estimated, by going back to the remotest periods, do not present any circumstances from which we can infer that it could be in a condition to cease to revolve about the earth.

Knowing the quantity of matter and magnitude of the earth and moon, it is easy to estimate the point of equal attraction. If these two bodies were at rest, a body projected from the surface of the moon, with the velocity of 12,000 feet in a second, would be carried beyond the point of equal attraction, if the moon's mass was $\frac{1}{39.78}$, which was Newton's estimation; but this estimation is now admitted to be too great: see notes to Chapter X., where its true value $\frac{1}{75}$, is assigned, therefore a force a little more than half of the above power would be sufficient to produce that effect, *i. e.* a force capable of projecting a body with a velocity a little more than a mile and a half in a second ; but cannon balls have been propelled with a velocity of 2500 per second, which is upwards of half a mile; and in the experiments of Perkins, the balls were driven by the force of steam with a still greater velocity; therefore a projectile force, causing a velocity three times greater than that with which a cannon ball is projected, would move a body beyond the point of equal attraction, and cause it to reach the earth; and there can be no doubt but a force equal to that is exerted by volcanoes on the earth, and also by the steam produced by subterraneous heat, for huge masses of rock, many times larger than cannon balls, are thrown much higher; and a like cause of motion exists in all probability in the moon, as well as in the earth, and that it is even in a superior degree, is probable from the circumstance that there is no sensible atmosphere to resist or retard the motion of bodies, as at the surface of the earth; and besides, the appearances observed in the moon indicate traces of more powerful and extensive volcanoes than on the surface of the earth. After the body passes the point of equal attraction, the *path* which it describes in approaching the earth must in a great measure depend, as is stated in the text, on the direction of the primitive impulsion; for as, besides this impulsion, it also participates in the absolute motion of the moon, it must, when it reaches the point of equal attraction, be actuated by the tangential velocity of the moon, combined with the force drawing it to the centre of the earth ; these two would cause it to describe an ellipse about the earth, and the sun's action would disturb its motion in the same manner as the moon's motion is disturbed; when the body reaches our atmosphere it has not lost much of its heat, inasmuch as the space which it traversed b paratively a vacuum, it enters the upper regi mosphere with little diminution of its original temperature, from which circumstance, combined with its very great velocity, which is *then* more than ten times greater than that of a cannon ball, and passing through a part of the atmosphere consisting chiefly of inflammable gas, (see notes, page 373, Volume I.,) it is easy to conceive how the body will be suddenly ignited.

These stones consist always of the same ingredients, namely, silex, magnesia, sulphur, iron in a *metallic* state, nickel, and a small quantity of chromium. As these are invariably the constituents of these stones, it has been justly concluded that they have a *common origin*, besides, iron is never found in a metallic state in terrestrial bodies; even what is found in volcanic eruptions is always oxidized; nickel is likewise very seldom met with, and never on the surface of the earth; and chromium is rarer still. At the period when they burst forth, they are a considerable height about the earth's surface, as appears from estimating their parallaxes, by means of simultaneous observations, made at the instant of their explosion. Beside the threefold opinion of their origin, given in the text, namely, a lunar, a volcanic, and atmospheric, some philosophers have supposed that they were small planets, or fragments of planets, like those lately discoved, revolving in space, and which, meeting with the earth's atmosphere, are ignited by the friction which they experience in the earth's atmosphere.

NOTES TO CHAPTER VI.

As Jupiter and his satellites are considerably more distant than the earth from the sun, and as the mass of Jupiter is much greater than that of the earth, the part of the sun's force which disturbs the motions of the satellites is much less than the corresponding part of the sun's force in case of the earth; therefore the principal cause of the inequalities in the motions of the satellites arises from their mutual attraction.

(*a*) It is impossible to give a perfect explanation of the different inequalities of these satellites without discussing the theory of these bodies more in detail than the limits which these notes admit of. The reader will find them satisfactorily accounted for in the Sixth Chapter of the Second Book, and in the Eighth Book of the Celestial Mechanics.

$m\, m'\, m''\, m'''$ being the masses of the satellites, in the order of their distances from the sun, $n\, n'\, n''\, n'''$, &c. their respective mean motions, $r\, r'\, r''$, &c. their radii vectores, and $v\, v'\, v''$ their longitudes, then, as has been observed in notes, page 405 of the First Volume, since the mean motions of the three first satellites constitute very nearly a duplicate progression, $n-2n'$, $n'-2''n$ must be very small fractions of n, and their difference $n-2n'-(n'-2n'')$, or $n-3n'+2n''$, is incomparably less than either of them. It is easy to prove that the action of m' on m produces in the radius vector r, and the longitude v, a very sensible inequality, depending on the argument $2(n't-nt+\epsilon'-\epsilon)$; the terms relative to this inequality have for a divisor

$$4(n'-n)^2-n^2=(n-2n')\,.\,(3n-2n'),$$

which, in consequence of the smallness of the factor $n-2n'$, is very sensible. It appears also, from a consideration of the same expressions, that the action of m on m' produces in r' and v' an inequality depending on the argument

$$n't-nt+\epsilon'-\epsilon,$$

which, as its divisor is

$$(n'-n)^2-n^2, \text{ or } n(n-2n'),$$

is also extremely sensible. In like manner, the action of m'' on m', and *vice versa*, of m' on m'', produces respectively, in their respective longitudes and radii vectores, inequalities depending on the arguments $2(n''t-n't+\epsilon''-\epsilon')$, and $n''t-n't+\epsilon''-\epsilon'$; (see as above;) therefore the value of $\delta v'=m''F''.\sin.2(n''t-n't+\epsilon''-\epsilon')+mH.\sin.(n't-nt+\epsilon'-\epsilon)$.

(*b*) By hypothesis we have $2n''t+2\epsilon''-2n't-2\epsilon'=\pi+n't-nt+\epsilon'-\epsilon$, $\therefore\ m''F'.\sin.\ 2(n''t-n't+\epsilon''-\epsilon')=-m''F.''\sin.(n't-nt+\epsilon'-\epsilon)$, consequently the value of $\delta v'$ becomes $(mH-m''F'')\sin.(n't-nt+\epsilon'-\epsilon)$, in which we see how the two inequalities are made to coalesce into one.

The manner in which it may be shown that the mutual action of the satellites rendered this proportion, which was originally only approximate, accurately true, is as follows: assuming $V=nt-3n't+2n''t+\epsilon-3\epsilon'+2\epsilon''$, it is easy to prove that $\frac{d^2V}{dt^2}=Cn.^2\sin.V$; C being a constant coefficient depending on the masses of $m\,m'\,m''$, n being likewise supposed constant, and by integrating we have

$$dt = \pm \frac{dV}{\sqrt{c-2Cn^2 \cos. V}}; \text{ now, as } \left(\frac{dV}{dt}\right)^2 =$$

$$(n-3n'+2n'')^2, \; c-2\,Cn^2 \cos. V = \left(\frac{dV}{dt}\right)^2 =$$

$(n-3n'+2n'')^2$; and if this last quantity be greater than $\mp 2Cn^2.(1-\cos. V)$, c must be positive and $>$ than $2Cn^2$, in which case, as the radical can never vanish, V, or its equivalent, should increase continually, and become $=$ 2π, 4π, 6π, &c.; but this is not the case, for let $\bar{\omega}=\pi-V$, and we have

$$dt = \frac{d\bar{\omega}}{\sqrt{c+2Cn^2. \cos. \bar{\omega}}}; \text{ and when } c \text{ is not less than } 2Cn^2,$$

$\sqrt{c+2Cn^2 \cos. \bar{\omega}}$ is $>$ than $\sqrt{2Cn^2}$ from $\bar{\omega}=0$ to $\bar{\omega} = \frac{\pi}{2}$;

therefore t the time in which the angle $\bar{\omega}$ passes from 0 to a right angle, is $\angle$ than $\frac{\pi}{2n\sqrt{2C}}$; and this last angle, by substituting for C and n comes out $\angle$ than two years; but as $\bar{\omega}$ has always remained insensible, this last case, namely that of c, not less than $2Cn^2$, is not the case of Jupiter and his satellites. If c is $\angle$ than $\mp 2Cn^2$, V will oscillate about a mean state either of two right angles, if C be positive, in which case $\sqrt{c-2Cn^2 \cos. V}$ becomes imaginary, when $V=0$, or 2π, 4π, &c., $\therefore$ $\bar{\omega}$ can never become equal to cypher, its value is therefore periodic, oscillating about a mean state $=\pi$. If, in the same circumstances, C be negative, then the radical is imaginary, when $V=\pi$, 3π, 5π, therefore $\bar{\omega}$ can never reach π, and its mean value is cypher. Now all observations of Jupiter and its satellites give a positive value to C; therefore in the case of Jupiter, V oscillates about a mean state $= \pi$. From the equation

$$n't-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''=V=\pi-\bar{\omega},$$

we deduce, by putting the parts which are not periodic separately = to cypher, $nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''=\pi$, $\therefore n-3n'+2n''=0$; therefore the mean motion of the first satellite + twice that of the third, minus three times that of the third = 0, and $\varepsilon-3\varepsilon'+2\varepsilon''=\pi$, *i.e.* the mean longitude of the first + twice that of the third, minus three times that of the second = π; and since according to observation the angle $\bar{\omega}$ in the equation

$dt=\dfrac{d\bar{\omega}}{\sqrt{c+2Cn^2\cos.\bar{\omega}}}$ must be always very small, we can assume $\cos.\bar{\omega}=1-\dfrac{\bar{\omega}^2}{2}$, and the preceding equation will give by integrating $\bar{\omega}=\lambda.\sin.(nt\sqrt{C}+\gamma)$, where λ and γ are two constant arbitrary quantities.

(*c*) As three differential equations of the second order are necessary to determine the motion of each body of the system, and as the integration of each equation involves two constant arbitrary quantities, there are in the determination of the motion of each body six constant arbitraries; therefore, in general, as is stated in the text, the number of arbitrary quantities is sextuple of the number of bodies; consequently, in the case of the four satellites of Jupiter, there are twenty-four arbitrary quantities, which are reduced to twenty-two, in consequence of the two relations between the epochs of mean longitudes and also the mean motions of the three first satellites, which is established by the two preceding theorems; but these two are supplied by the new arbitraries, which the expression of $\bar{\omega}$ contains.

If the satellites were affected either by a secular inequality analogous to that of the moon, as stated in the text, or by one arising from the resistance of a medium, it would be necessary to add to $\dfrac{d^2V}{dt^2}$ a quantity of the form $\dfrac{d^2\psi}{dt^2}$, which can only become sensible by integrations; therefore if $V=\pi-\bar{\omega}$, when $\bar{\omega}$ is very small, the differential equation in V will become of the form

$$0=\frac{d^2\bar{\omega}}{dt^2}+Cn^2\bar{\omega}+\frac{d^2\psi}{dt^2},$$

as the period of the angle $nt\sqrt{C}$ embraces but five years, while the quantities contained in $\frac{d^2\psi}{dt^2}$ are either constant, or extend to several centuries, we shall obtain, very nearly, by integrating,

$$\bar{\omega}=\lambda.\ \sin.(nt\sqrt{C}+\gamma)-\frac{d^2\psi}{Cn^2.dt^2},$$

therefore the value of $\bar{\omega}$ will be always very small, and the secular equations of the mean motions of the three first satellites will be so coordinated by their mutual action, that the secular equation of the first plus twice that of the third, is equal to three times that of the second.

(d) It has been already stated, in notes, page 405, that from the circumstance of the length of the year not having been altered 2″,8 by the action of the comet of 1770, its mass is not the $\frac{1}{5000}$th part of that of the earth; and if, as is stated in the text, in the lapse of ages these bodies have more than once impinged on the satellites, the effect would be particularly perceptible in a real libration of these satellites, and also of the moon; for, as will be stated hereafter, in notes to Chapter XIV., it is by no means probable that the equality which obtains between the motion of rotation and revolution subsisted at the very origin of the planetary system.

(e) The compression of Jupiter has also a considerable influence on the motion of the apsides of the satellites, as well as on the motion of the nodes; and it is from the accuracy with which these quantities have been determined, that we are able to deduce such an exact expression for the compression of Jupiter.

The masses of the satellites, and the value of the compression, are determined relatively to the mass and equatorial diameter of Jupiter. In order to determine these five unknown quantities, it is necessary to have five data furnished by observation. Those are selected in which the quantity required to be known has most influence. As the ratio of Jupiter's mass to the earth's is given in page 42, we can have the mass of the satellites relatively to that of the earth, and by substituting their numerical values, it is found that the mass of the third satellite, which is nearly double of the fourth, is also double of the moon's mass.

(*f*) Three differential equations of the second order are necessary to determine the circumstances of the motions of each satellite; and for the integrations of each of them, two constant arbitrary quantities are introduced; this gives twenty-four constant arbitrary quantities, in addition to which, the masses of the four satellites, the compression of Jupiter, the inclination of his equator, and the position of his nodes, furnish seven more, which make thirty-one in all. See note (*c*.)

(*g*) See notes to Chapter II., Book 2., Volume I.

(*h*) Assuming that the velocity of the light which comes from the stars was such as is given from a comparison of the eclipses of Jupiter's satellites, the quantity of the aberration comes out exactly equal to what is deduced from actual observation, which shows that the velocity of the light which comes from the stars is equal to that which is *reflected* from Jupiter's satellites. The uniformity of the velocity through the diameter of the earth's orbit might be evinced by taking into account the effect of the ellipticity of the orbit of the earth. It also follows, that the velocity of a ray of light which emanates from stars of *different* magnitudes, and at *different* distances, is uniform and the same through the diameter of the earth's orbit.

(*i*) It is easy to show that the velocities of pulses which are propagated in any elastic medium, are in the direct subduplicate ratio of F the elasticity, and the inverse subduplicate ratio of *d* the density of the medium; consequently when F varies as *d*, the velocity must be uniform. (See Princip. Math. Prop. 48., Book 2.) It is to be observed here, that on the hypothesis of the materiality of light, it is supposed that light is emitted from stars of different magnitudes with the same velocity.

(*l*) See note (*b*) of this Chapter.

NOTES TO CHAPTER VII.

(*a*) THE expression for the part of Saturn's compression which disturbs the satellite at the distance a from Saturn,

$= \frac{\rho - \frac{\phi}{2}}{a^3}$, where ϕ is the centrifugal force at the equator of Saturn, and $\therefore = \frac{T^2}{t^2 a^2}$, t being the time of Saturn's rotation, and ρ the compression; and as the compression of the earth is to $\frac{1}{289} :: \rho : \phi$, we have

$\rho - \frac{\phi}{2} = \frac{m.T^2}{t^2.a^3}$, $\therefore \frac{\rho - \frac{\phi}{2}}{a^2} = \frac{m.T^2}{t^2} \cdot \frac{1}{a^5}$. as is stated in the text; therefore, the more distant the satellite from Saturn the less will be this quantity; and for the last satellite, it is so small, that the disturbing action of the sun predominates over it, causing the satellite to deviate from the plane of Saturn's equator.

Laplace also infers from this deviation of the last satellite, that its mass must be inconsiderable, for otherwise its action on the last but one would cause it to deviate from the plane of Saturn's equator, in which, however, it accurately moves.

The preceding formula is not applicable to the case of Uranus's satellites; for we do not know the precise amount of t, the rotation of the satellite. If the orbits are perpendicular to the plane of Uranus's orbit, the theoretic discussion of the disturbing action of the sun on these satellites would require formulæ different from those used to investigate and express the disturbances produced in the planets, moon, and other bodies of our system.

NOTES TO CHAPTER VIII.

(*a*) SEE note (*d*) page 453 of Volume I., where it is proved, that the equation of the figure which the fluid affects in the hypothesis of the text, is that of a sphere; and as the density must be a function of the pressure, the fluid will be arranged in strata of equal pressure, and the same density, having the strata which are nearer to the centre denser than those which are more remote.

If the earth was fluid, and at rest, it would necessarily assume a spherical shape; for the mutual attraction of the particles would so collect them together, that if any particles were more protuberant than the others, the direction of gravity would not be perpendicular to its surface, and as it would not remain in such a form, the projecting parts would flow down; in consequence of the centrifugal force induced by the motion of rotation, all the particles have a tendency to recede from the axis of rotation, which, as it is greater near the equator, will enlarge the earth more than near the poles, and thus make the earth assume a spheroidical shape.

(*b*) A particle placed without a sphere of which each of the particles attracts in the inverse ratio of the square of the distance, is urged to the *centre* of the sphere with a force which varies in the inverse ratio of the square of the distance from the centre; (see Princip. Math. Book I., Section 12., Prop. 74.;) therefore, if f denotes the distance of the particle from the centre of the sphere, the attraction of the sphere upon the particle will be expressed by $\frac{A}{f^2}$; (A being a constant quantity, which can be determined by the actual attractive force, at any determinate distance from the centre;) if r denote the radius of the sphere, and M its mass, since no part of the matter of the sphere is nearer the attracted particle than $f-r$, and none more remote than $f+r$, the attraction of the sphere on the particle will be $>$ than

$\frac{M}{(f+r)^2}$, and $<$ than $\frac{M}{(f-r)^2}$, $\therefore \frac{A}{f^2}$ being always contained between those limits, A must be $=$ to M, for if $A > M$, such values of f could be found as would make $\frac{A}{f^2} =$ or $>$ than $\frac{M}{(f-r)^2}$, and if A is $<$ than M, such values of f could be found as would render $\frac{A}{f^2} =$ or $<$ than $\frac{M}{(f+r)^2}$, which can never be the case; $\therefore$ $A = M$, and the attraction of the sphere $= \frac{M}{f^2}$, or the same as if all the matter were collected in the centre. Now, if ρ denotes the density of the matter contained in the sphere, we have $M = \frac{4\pi r^3 \rho}{3}$, $\therefore$ the attraction at the distance $f = \frac{4\pi r^3 \rho}{3.f^2}$, and at the surface, where $r = f$ the attraction $= \frac{4\pi r.\rho}{3}$.

(*c*) Suppose a system of bodies $m\ m'\ m''$, &c., whose mutual distances from each other are inconsiderable relatively to their respective distances from the attracted point, let the origin of the coordinates be in the attracted point, $x\,y\,z$, $x'\,y'\,z'$, x'', &c. being the coordinates of the $m\,m'\,m''$, &c. X Y Z, those of the centre of gravity, we have

$$x=X+x_{,}\,,\; y=Y+y_{,}\,,\; z=Z+z_{,}\,,\; x'=X+x'_{,}\,,\ \&c.$$

$x_{,}\,y_{,}\,z_{,}$ being the coordinates of m with respect to the centre of gravity; if $r\,r'$, &c. denote the distances of the attracted point from $m\,m'$, &c. and R the distance of the centre of gravity from the same, the action of m on the attracted point resolved parallel to $x=\frac{m\,x}{r^3}$, $\therefore$ the sum of the attractions of $m\,m'\,m''$, &c. resolved parallel to $x=\Sigma.\frac{m\,x}{r^3}$,

but $\frac{x}{r^3}=\frac{X+x_{,}}{\left((X+x_{,})^2+(Y+y_{,})^2+(Z+z_{,})^2\right)^{\frac{3}{2}}}=$

neglecting very small quantities of the second order, as is stated in the text, namely, the products and squares of $x_{,}\,y_{,}\,z_{,}\,x'_{,}$ &c.

$$X.(X^2+2Xx_{,}+Y^2+2Yy_{,}+Z^2+2Zz_{,})^{-\frac{3}{2}}$$

$$+\,x_{,}.\,(X^2+Y^2+Z^2)^{-\frac{3}{2}}=X.(X^2+Y^2+Z^2)^{-\frac{3}{2}}-\frac{3}{2}.$$

$$X(2Xx_{,}+2Yy_{,}+2Zz_{,})R^{-5}+x_{,}.\,(X^2+Y^2+Z^2)^{-\frac{3}{2}}=$$

$\frac{X+x_{,}}{R^3}-3.X\frac{(Xx_{,}+Yy_{,}+Zz_{,})}{R^5}$, and as $\Sigma mx_{,}=0$, $\Sigma my_{,}=0$; $\Sigma m\,z_{,}=0$; $\Sigma\frac{m\,x}{r^3}=\Big(\frac{X.\Sigma m}{R^3}+\frac{\Sigma mx_{,}}{R^3}-$

$3\,X.\frac{(X.\Sigma mx_{,}+Y.\Sigma my_{,}+Z.\Sigma mz_{,})}{R^5}\Big),=\frac{X.\,\Sigma m}{R^3}$, which

is the same as if the bodies were united in the centre of gravity. Now, if m, m', m'', &c., are so near as to be parts of the same attracting body, this will be more accurately true, as in the case of spheroids differing little from a sphere; then, as in a sphere, an exterior point is attracted, as if the whole mass was united in the centre; in a spheroid differing little from a sphere, the error or the difference from what would be the case if the body was a sphere, is of the same order as the eccentricity for all points contiguous to the surface; for if a sphere be supposed to be described concentrical with the given spheroid whose radius is equal to the distance of any assumed point on its surface from centre of spheroid, then this sphere attracts as if the entire matter was collected in the centre; therefore the difference between its attractions and that of the spheroid must be of the same order as the eccentricity; and as for very distant particles, in estimating the effect of the attraction of a body of any figure whatever on them, in showing that its action is nearly the same as if the entire mass was collected in the centre of gravity; by what has been just established, the ratio of the quantities which are neglected to those which are retained is that of the square of the radius to the square of the distance of the point attracted; the error for a spheroid in the case of very distant points, must be the product of the excentricity into the square of this ratio. It would not be difficult to show, that if the force varied directly as the distance, then a point outside a body of any figure *whatever*, is attracted as if the entire mass was condensed into the centre of gravity.

(*d*) In order to demonstrate this property, it may be observed, that if a homogeneous sphere attracts a point placed without it, as if all the matter was united in its centre, the same result will have place for a spherical stratum of a uniform thickness; for if we take away from the sphere a spherical stratum of a uniform thickness, we shall obtain a new sphere of a smaller radius, which will possess the property equally with the first sphere of attracting, as if the entire mass was united in its centre; but it is evident, that if this property belongs to the two spheres, it belongs also to their difference; therefore the problem reduces itself to determine the laws of attraction, according to which a spherical stratum of a uniform and indefinitely small thickness attracts an exterior point, as if the entire mass was collected in its centre. Let r represent the distance of the attracted point from the centre of the stratum, u the radius of the stratum, θ the angle contained between r and u, $\bar{\omega}$ the angle which the plane passing through r and u makes with a fixed plane, then it is easy to prove that $u^2 du.d\bar{\omega}.d\theta.\ \sin.\ \theta$ is $=$ to the element of the spherical stratum; and if f be the distance of this element from the attracted point, $f^2 = r^2 - 2ru.\ \cos.\ \theta + u^2$,

$\therefore \frac{df}{dr} = \frac{r - u.\ \cos.\ \theta}{f}$; $d\theta.\ \sin.\ \theta = \frac{fdf}{r.u}$; if $\phi\,(f)$ expresses the law of attraction, then the action of the element resolved parallel to r, and directed towards the centre $=$

$u^2 du.d\bar{\omega}.d\theta.\ \sin.\ \theta.\ \frac{(r - u.\ \cos.\ \theta)}{f}\ .\ \phi\,(f)$, which $\Big($since $\frac{r - u.\ \cos.\ \theta}{f} = \left(\frac{df}{dr}\right)\Big)$ assumes the form $u^2.du.d\bar{\omega}.d\theta.\ \sin.$ $\theta.\left(\frac{df}{dr}\right).\ \phi\,(f)$, *i. e.* if we denote $\int d\,(f).\ \phi.\ (f)$ by $\phi_{,}(f)$,

we obtain the entire action of the spherical stratum, by means of the integral $u^2 du.\int d\bar{\omega}.d\theta.\ \sin.\ \theta.\ \phi_{,}(f)$, differenced with respect to r, and divided by $dr.$; relatively to $\bar{\omega}$, the preceding integral, should be taken from $\bar{\omega} = 0$, to $\bar{\omega} = 2\pi$, the circumference, *i. e.* the preceding integral, becomes $2\pi.u^2 du\int d\theta.\ \sin.\ \theta.\ \phi_{,}f =$ (by substituting for

$d\theta.\sin.\ \theta$,) $2\pi.\ \frac{udu}{r}\ .\int fdf.\ \phi_{,}(f)$. Now, as relatively to θ, the integral should be taken from $\theta = 0$ to $\theta = \pi$, which corresponds to $f = r - u$, $f = r + u$, when $\frac{fdf}{du}$ is substituted for $d\theta.\ \sin.\ \theta$, the integral relatively to f must be taken from $f=r-u$ to $f=r+u$; $\therefore$ if we make $\int f df. \phi_{r}(f)=\psi(f)$, we shall have

$$\frac{2\pi udu}{r}\int f df.\phi_{,}(f)=\frac{2\pi udu}{r}(\psi(r+u)-\psi(r-u)); \text{ now}$$

if $\phi(f)=\frac{1}{f^2}, \int df.\phi(f)=\phi_{,}(f)=-\frac{1}{f}$, and $\int df.f.\phi_{,}(f)$ $=\psi(f)=-f=$ at the limits $-r-u$, $r-u$, $\therefore \psi(r+u)-\psi(r-u)=-2u$, $\therefore$ the differential coefficient of the second member of this equation with respect to r, (which, as has been observed, gives the attraction of the spherical stratum,) $=\frac{-2\pi.udu}{r^2}.(-2u)=\frac{4\pi u^2du}{r^2}$, *i. e.* as $\pi u^2=$ the area of a circle whose radius is u, $4\pi u^2=$ the *surface* of the spherical stratum, and $4\pi u^2du=$ the mass of the stratum whose thickness $=du$; $\therefore$ when $\phi(f)=\frac{1}{f^2}$, an exterior point is attracted, as if the whole mass was united in its centre; but to determine $\phi(f)$ *generally*, when the attraction of the stratum is the same as if the mass was collected in its centre; in that case the attraction would be $=4\pi.u^2du.\phi(r)$ which, by hypothesis,

$$=2\pi udu.\left(\frac{d.\left(\frac{1}{r}[\psi(r+u)-\psi(r-u)]\right)}{dr}\right), \text{ integra-}$$

ting with respect to r, and dividing by $2\pi udu$, we shall have

$$\psi(r+u)-\psi(r-u)=2ru.\int dr.\phi(r)+rU,$$

U being a function of u, and of constant quantities, let $\psi(r+u)-\psi(r-u)=R$, and then differentiating twice with respect to r, we obtain

$$\frac{d^2R}{dr^2}=4u.\phi(r)+2ru.\frac{d.\phi(r)}{dr},$$

and differentiating twice with respect to u, we obtain

$\frac{d^2R}{du^2} = r. \frac{d^2U}{du^2}$, but from the nature of the function R we have

$$\frac{d^2R}{dr^2} = \frac{d^2R}{du^2}; \therefore 2u. \left(2\phi\, r + \frac{r.d.\phi(r)}{dr}\right) = r. \left(\frac{d^2U}{du^2}\right),$$

$$i.e. \frac{2\phi(r)}{r} + \frac{d\phi(r)}{dr} = \frac{1}{2u} \cdot \frac{d^2U}{du^2},$$

(see Celestial Mechanics, Book 2, page 68,) therefore as the first member is independent of u, and the second member is independent of r, they must each be equal to a constant quantity, which denoting by 3A, we have

$\frac{2\phi(r)}{r} + \frac{d\phi(r)}{dr} = 3A$, which, by multiplying both sides by r^2dr, gives

$$2r.dr.\phi(r) + r^2.d\phi(r) = 3Ar.^2dr, \therefore r^2\phi(r) = Ar^3 + B, \therefore$$

$\phi(r) = Ar + \frac{B}{r^2}$; $\therefore$ as is stated in the text, all the laws in which the sphere acts on an exterior point, as if the whole mass was condensed in the centre, are comprised in the general formula $Ar + \frac{B}{r^2}$; in fact, this value of $\phi(r)$ will satisfy the equation given in the preceding page. If the point be situated within a spherical stratum of uniform thickness, then since u is $> r$, the expression for the attraction of the stratum whose thickness $= du$ is

$2\pi udu. \left(\frac{d. \frac{1}{r}\left(\psi(u+r) - \psi(u-r)\right)}{dr}\right)$. In order that this function should vanish, we should have $\psi(u+r) - \psi(u-r) = rU$, and it is evidently the case when $\phi(f) = \frac{B}{f^2}$, but to show that this vanishes *only* when $\phi(f) = \frac{1}{f^2}$, if

$$\psi'(f) = \frac{d.\psi(f)}{df}, \psi''(f) = \frac{d.\psi'(f)}{df}, \text{\&c. \&c.}$$

$$\frac{d\psi(u+r)-d\psi(u-r)}{dr} = \psi'(u+r)-\psi'(u-r) = \text{U},$$

$$\frac{d^2\psi(u+r)-d^2\psi(u-r)}{dr^2} = \psi''(u+r)-\psi''(u-r) = \frac{d\text{U}}{dr} = 0,$$

$\therefore$ as $\psi''(u+r)$ is always $= \psi''(u-r)$, each of them must be equal to a constant quantity, $\therefore \psi''(f) =$ a constant quantity, and $\psi'''(f)=0$; but as $\psi'(f) = f\phi_{,}(f)$, $\psi'''(f) = 2\phi(f)+f.\phi'(f)$, *i. e.* $0=2\phi(f)+f.\phi'(f)$, or $\psi(f)=\int f.df\phi_{,}(f)$, $\psi'(f)=f.\phi_{,}(f)$, and $\psi''(f)=\phi_{,}(f)+f.\phi(f)$, and $\psi'''(f)= \phi(f)+\phi(f)+f\phi'(f)=0$; multiplying by fdf we obtain

$$2f\phi(f).df+f^2\phi'f.df=0, \therefore f^2\phi(f)=\text{B}, \text{ and } \phi(f)=\frac{\text{B}}{f^2}; \therefore$$

a point situated within the interior of a spherical stratum is equally attracted in every direction when the force of gravity is inversely as the square of the distance; the same is true, for a spheroid in the circumstances specified in the text, for any common chord to the two spheroids drawn through the interior point, has the portions of it which are intercepted between the two spheroidical surfaces equal; therefore, if the point in the interior be conceived to be the vertix of two *similar* pyramids, whose common axis is the chord, its gravitation to the pyramid whose axis is the distance of the point from the exterior surface, is equal and *opposite* to the gravitation to the matter contained in the *frustrum*, of the similar cone, whose axis being the part of the chord intercepted between the two spheroids at the other side, is equal to the axis of the first cone, and as this is true *whatever be* the direction of the chord drawn through the given point, this point is in equilibrio in every direction. From the expression $\frac{4\pi r\rho}{3}$ given in page 441, it follows that when the density ρ is given, the force of gravity is proportional to r; but if the strata nearer the centre are denser, then this force varies evidently in a less ratio.

(*e*) As the centrifugal force at the equator is greatest, the weight of a column of water extending from the surface to the centre, must be less than the weight of an equal column at the poles, reaching from surface to centre; therefore to compensate for the loss of weight produced by the centrifugal force, the equatorial columns should be longer than the polar.

(*f*) What is here stated may be analytically expressed in the following manner: calling m the equatorial, and n the polar diameter, r the radius belonging to any parallel λ, then if A and F denote the gravity and centrifugal force at the equator, the gravity at $\lambda = \frac{A.m^2}{r^2}$, and the efficient part of the centrifugal force $= \frac{Fr. \cos.^2\lambda}{m}$, (see notes Volume I., page 427.,) therefore the gravity diminished by the centrifugal force $= \frac{Am^2}{r^2} - \frac{F.r.\cos.^2\lambda}{m}$, and conceiving a canal, of an indefinitely small thickness, to extend from the surface at λ to the centre, the weight of an element at the surface $= Am^2.\frac{dr}{r^2} - \frac{Frdr.\cos.^2\lambda}{m}$, therefore, by integrating, the weight of the entire canal $= -\frac{Am^2.}{r} - \frac{Fr^2.\cos.^2\lambda}{2m}$, at the equator and at the poles these quantities become respectively $-Am - \frac{Fm}{2}, -\frac{A.m^2}{n}$, and as in the case of equilibrium, these quantities must be equal, we have $\frac{n}{m} = \frac{2A}{2A+F} = \left(\text{as } \frac{F}{A} = \frac{1}{289.} \text{ (see notes, Vol. I., p. 427.)}\right) \frac{578}{579}$; and in general as $\frac{Am^2}{n} = (\text{if } m=n+e)\ An + A2e$

and as $\left(A + \frac{F}{2}\right) . m = \left(A + \frac{F}{2}\right)(n + e)$, we have $\Big($since $\left(A + \frac{F}{2}\right) . m = A . (n+2e),\Big)$ $A : \frac{F}{2} :: m : e$; now if $n + dn = r$, G the gravity at the pole is to G′ the gravity at any parallel $:: (n+dn)^2 : n^2$, *i. e.* $n+2dn : n$, hence it appears that the diminution of gravity is nearly twice the increase of the terrestrial radius; and as the centrifugal force *at the surface* is equal to the same quantity, we can obtain *the whole* diminution which arises from the two causes of decrease.

(*g*) As the spheroids are by hypothesis similar, if they be supposed to be divided into an indefinite number of similar and similarly situated particles, the attractions are as the quantities of matter, and inversely as the squares of the distances, *i. e.* directly as their similar dimensions, which vary by hypothesis as the distances from the centre. In like manner, the centrifugal forces are as the radii of the respective circles described by the two particles, which are by hypothesis as their distances from the centre. Hence as the two forces which affect the particle, namely, the centrifugal and the attractive forces are respectively in the same ratio, the directions in which the two particles will have a tendency to move will be parallel.

(*h*) It is to be remarked here, that Newton did not prove that if the earth revolved on an axis, it would necessarily have the figure of an ellipsoid of revolution. He assumed it was an elliptic spheroid, differing little from a sphere; and he then estimated, in a manner similar to that indicated in the text, the weight of a column extending from the pole to the centre, and the weight of another column extending from the equator to the centre, and as the equatorial column must compensate for its loss of weight, by its greater length, by making the difference of the weights of the two columns equal to the sum of the centrifugal forces of the parts of the equatorial column, he has at once the ratio of the axes. (As for every particle in a column which reaches from the surface at the equator to the centre, the diminution is proportional to the distance from centre; the whole diminution must be the same as if each particle of the column lost half as much as the outermost particle loses.)

Since the weights of the corresponding parts are to each other as the magnitudes of the parts into the force of gravity at the points where they terminate, they are (as the parts are proportional to the whole lengths) as the whole lengths into the force of gravity.

M'Laurin's proof consists in the demonstration of the three following particulars: *1st*, That the direction of gravity affected by the centrifugal force of rotation is every where normal to the surface of the spheroid, for otherwise the fluid would flow off towards that quarter to which the gravity inclines; *2dly*, That all canals from the centre to the surface, must balance at the centre, otherwise the preponderating column would subside, and pressing up the other, would produce a change in the surface; *3dly*, Any particle of the whole mass must be in equilibrio, being equally pressed in every direction; and he shows that these conditions will be *secured in an homogeneous spheroid revolving on its axis, if the gravity at the pole be to the equatorial gravity diminished by the centrifugal force arising from the rotation, as the radius of the equator to the semiaxis*; which was the conclusion Newton arrived at. It would be impossible, in these notes, to demonstrate these points in all their details; we shall only advert to some inferences which M'Laurin makes from establishing the first condition; namely, that the *sensible* gravity of any particle at the surface, is to the polar gravity, as the part of the normal *terminated by the axis* to the radius of meridional curvature at pole; and it is to the equatorial gravity as part of the normal, *terminating in the equator* to the radius of meridional curvature at the equator; from which it follows, that the sensible gravity is every where inversely as perpendicular from centre on tangent, and the gravity estimated in the direction of a radius to the centre is inversely as the distance from the centre. Hence it follows, from what has been established in Volume I., page 348, with respect to the decrements of the radii vectores, that the sensible increment of gravity, which, as we have seen, varies as the decrement of distance, is proportional to the square of the sine of the latitude. It is to be observed here, that M'Laurin, no more than Newton, does not prove that a fluid sphere, revolving on an axis, *must* assume the form of an elliptical spheroid, but only that it is a possible form. In fact, all that M'Laurin demonstrates is, that whatever be the proportion between the axes of an oblate spheroid, there is a certain velocity of rotation which will induce such a relation between the diminished equatorial and polar gravity, as is required in order to satisfy the three conditions of equilibrium above adverted to. Clairault, indeed, requires other conditions to be satisfied, such as: 1*st*, That a canal of any *form whatever*, must be every where in equilibrio; 2*ndly*, That such a canal, reaching from any one part to the other, shall exert no force at its extremities; 3*dly*, That a canal of any form, returning into itself, shall be in equilibrio through its whole extent. But it is not difficult to show, as Professor Robinson remarks, that these conditions are contained in the three previous ones.

What has been just established, only proves, as has been remarked, the competency of an elliptical spheroid for the rotation of the earth; another point remains to be determined, namely, given the velocity of rotation to determine the corresponding proportion which exists between the diameters. In order to determine this, M'Laurin investigated the gravitation of a particle at the pole of a spheroid, to the matter which is redundant over the inscribed sphere, and of a point in the equator to the excess of matter by which the circumscribed sphere exceeds the spheroid. The former comes out ∷1 to $\frac{8}{15}.e$, e being $=$ the difference between the equatorial and polar diameters, and the latter is half this quantity, and very nearly $= \frac{4}{15}.e$; now, the gravity at the pole to the inscribed sphere is ∷1 to $\frac{2}{3}\pi.(r-x)$, add to this $\frac{8}{15}.\pi e$, the gravitation to the redundant matter, and the sum is $=$ $\frac{2}{3}\pi r - \frac{2}{15}\pi.e$, and the gravitation of a particle at the equator to a sphere, whose radius $= r = \frac{2}{3}\pi r$, from this substract $\frac{4}{15}.\pi e$, the deficiency of gravitation, and the *undiminished* equatorial gravity comes out $= \frac{2}{3}\pi r - \frac{4}{15}.\pi e$, therefore, dividing by $\frac{2}{3}\pi r$, the ratio of E, the equatorial to P, the polar gravity comes out that of $r - \frac{1}{5}.e$ to $r - \frac{2e}{5}$, *i. e.* as e is very small relatively to r, as $r : r - \frac{e}{5}$, or *q.p*, as $r + \frac{e}{5} : r \therefore P = E + \frac{Ee}{5r}, \therefore E - c : E + \frac{Ee}{5r} :: r : r + e$, *i. e.* $E : E + \frac{Ee}{5r} + c :: r : r + e, \therefore E : \frac{Ee}{5r} + c : r : e, \therefore Ee = \frac{Ee}{5} + rc$, and $\frac{4Ee}{5} = rc; \therefore e = \frac{5rc}{4E}$ and $\frac{e}{r}$, or the ellipticity $= \frac{5c}{4E}$, as is stated in the text; hence, when c and E are given, we can determine $\frac{e}{r}$; $\therefore$ as $c = \frac{r}{T^2}$ and $E = r.\rho$, $\frac{e}{r} = \frac{5}{4}.\frac{1}{T.^2\rho}$. (See Princip. Math., Book 3, Prop. 19.)

(i) In No. 18, Book 3, Laplace investigates the figure which satisfies the equilibrium of an homogeneous fluid mass endowed with a motion of rotation; and assuming that the figure is an ellipsoid of revolution, if the forces which result from this hypothesis, when substituted in the equation $0 = \mathrm{P}da+\mathrm{Q}db+\mathrm{R}dc$, which is the equation of equilibrium at the free surface of a fluid, give the differential equation of the surface of an ellipsoid, the elliptic figure satisfies the equilibrium of a fluid mass endowed with a rotatory motion, then by substituting for P Q R, their values in the case of an ellipsoid of revolution affected with a rotatory motion, he obtains the following equation, $\frac{(9+2q\lambda^2).\lambda}{3(3+\lambda^2)} - \text{arc tan. } \lambda = 0$, where $q = \frac{g}{\frac{4}{3}\pi\rho}$, and $\lambda^2 = \frac{1-m}{m}$, m being the coefficient of y in the equation of an ellipsoid $x^2+my^2+nz^2=k^2$.

If this equation $\frac{9\lambda+2\lambda^3}{9+3\lambda^2} - \text{arc tan. } \lambda = 0, = \phi$, is susceptible of several real roots, then several figures of equilibrium may correspond to the same motion of rotation, it is evident that if $\lambda=0$ it is satisfied, but this is not the case of nature; if λ is very small, then arc tan. $\lambda=a$, becomes $=\lambda$; therefore in this case, ϕ is positive; if a is $=\frac{\pi}{4}$, $\lambda=1$, which renders ϕ negative; now suppose a curve of which λ is the abscissa, and ϕ the ordinate, this curve intersects the axis when $\lambda=0$, the ordinates will then be positive, and increasing until they attain their maximum, after which they will diminish; and when the abscissa has that value of λ which renders $\phi=0$, *i. e.* which corresponds to the state of equilibrium of the fluid, the curve will cut the axis a second time; the ordinates will afterwards become negative; and since they are positive when $\lambda=\infty$, it is necessary that the curve should meet the axis a third time, which intersection determines a second value of λ, which renders $\phi=0$, or satisfies the condition of equilibrium, $\therefore$ for the same value of q, or for a given rotatory motion, there are several figures with which the equilibrium is possible; in order to determine the number of figures, we should determine how many maxima exist between the roots, and by taking the derivitive function of ϕ, we find only two maximum ordinates on the positive side, and the same number on the side of the negative abscissæ; therefore, on this side, the curve intersects its axis in three points only, of which the origin is one, consequently there are only *two* figures which satisfy the equilibrium; for as even powers of λ only occur in the determination of these figures, those furnished by the positive and negative abscissæ are identically the same. If q is very small, as in the case of the earth, the equation $\phi=0$, may be satisfied either by making λ^2 very small, or very great; in the first case, as has been already remarked, we have $\lambda^2 = q(2{,}5 + 5{,}35'7.q+23q^2 + 123{,}q^3 +$, &c.) which, by substituting the value of q, gives $1 + \lambda^2 = 1{,}008746123$; therefore we can obtain λ, and consequently the ratio of the axes which comes out $\frac{230}{231}\cdot q.p$; in the second case, arc tan. λ is nearly $=\frac{\pi}{2}$; $\therefore \lambda = \frac{\pi}{2} - a$, where a is very small, so that its tangent is $= q.p\frac{1}{\lambda}$, $\therefore a = \frac{1}{\lambda} = \frac{1}{3\lambda^3} + \frac{1}{5\lambda^5} -$ &c.; $\therefore$ arc tan. $\lambda = \frac{\pi}{2} - \frac{1}{\lambda} + \frac{1}{3\lambda^3} -$ &c. $= \frac{9\lambda+2q\lambda^3}{9+3\lambda^2}$, and by reversion of series we can obtain λ, which, by substituting for q its value, give the ratio of the axes $=680$.

If two of the roots of $\phi=0$ were equal, then we would have evidently $\phi=0$; $d\phi=0$; in which case the curve will touch the axis at the origin; therefore the value of ϕ can never become negative at the side of the *positive* abscissæ; consequently the value of q, determined by these two equations $\phi=0$, $d\phi=0$, will be the limit of those with which the equilibrium can subsist; and if q has a greater value, the equilibrium would be impossible; for in that case the curve would not meet the line of the abscissæ. It follows, from what precedes, that there is only one value of q which satisfies the equations $\phi=0$, $d\phi=0$, these equations give the following values,

$$q=\frac{6\lambda^2}{(1+\lambda^2)(9+\lambda^2)};\ 0=\frac{7\lambda^5+3\lambda^3+27\lambda}{(1+\lambda^2).(3+\lambda^2).(9+\lambda^2)}-$$

arc tan. λ; the value of λ which satisfies the last equation, is $\lambda=2,5292$, $\therefore q=0,337007$, and $\sqrt{1+\lambda^2}=2,7197$, as in case of the earth $q=,00344957$, this value corresponds to a rotation $=0,^d99727$, but $q \backsim \frac{1}{T^2.\rho}$, note ($h$) page 453, $\therefore$ for a mass of the same density as the earth, T the time of rotation, which corresponds to $0,337007=0,10090$, which is the limit; for if the time of rotation be less than this, the equilibrium is impossible; if it be greater, there are two figures which satisfy the equilibrium. It appears also, as is stated in page 106, that the time of rotation varies generally inversely as the square root of the density.

If the rotatory motion should increase so as to be greater than that which answers to the limit of q, it does not necessarily follow that the fluid cannot be in equilibrio with an elliptic figure; for we may suppose, that according as the compression increases, the motion of rotation will become less rapid; therefore, if there exists between the molecules of the fluid mass a force of tenacity, this mass, after a great number of oscillations, may at length arrive at a motion of rotation, comprised within the limits of equilibrium, and fix itself in that state. In fact, when the rotation is increased, the spheroid becomes more oblate, and the fluids having less velocity of rotation than the equator, accumulate about that circle, and retard the motion; this goes on for some time, until the true shape is overpassed, and then the accumulation relaxes. Now, the motion is too slow for the accumulation, and the waters flow back towards the poles; in this way an oscillation is produced, which, however, in consequence of the mutual tenacity of the particles of the fluid, gradually subsides, and the appropriate form is eventually assumed.

In order to determine the ellipticity of Jupiter, resuming the equation $0 = \frac{9\lambda + 2\,q\lambda^3}{9+3\lambda^2} = \text{arc tan. } \lambda,\ \sqrt{1+\lambda^2}$ is the ratio of the equatorial to the polar diameter, and k the axis; $\therefore$ to determine λ we must have q; now if D be the distance, and P the periodic time of the fourth satellite, t the time of Jupiter's rotation, M the mass of Jupiter, and F the centrifugal force, we have F to the force retaining the satellite in its orbit, *i. e.* $\frac{M}{D^2}$ as $\frac{1}{t^2} : \frac{D}{P^2}$, $\therefore F = \frac{MP^2}{t^2.D^3}$, $M = \frac{4}{3}\pi k^3.(1+\lambda^2)$, $\therefore$ as $D = 26.63$ of the radius of Jupiter's equator, $\therefore \frac{k.\sqrt{(1+\lambda^2)}}{D} = \frac{1}{26,63}$; and as $t = 0,41377$, $P = 16,.^d 68902$, $\frac{F}{\frac{4}{3}\pi} = q = \frac{k^3.(1+\lambda^2).P^2}{t^2.D^3} = 0,0861450.(1+\lambda^2)^{-\frac{1}{2}}$, hence the preceding equation in λ, becomes

$$0 = 9\,\lambda. + \frac{0,172290.\lambda^3}{\sqrt{1+\lambda^2}} - (9+3\lambda^2).\text{arc tan. }\lambda. \text{ and } \therefore \lambda =$$

0,481, and if the polar diameter be unity, axis of equator $= 1,10967$. This is the ratio of the axes on the hypothesis of homogeneity; but the observed proportion being 1,0769, as deduced from actual observation, and also from the motion of the nodes of the satellites, (see page 452,) it follows that Jupiter is not homogeneous, but his density increases as we approach towards the centre. The limits of the ellipticity of the planets, if they were primitively fluid, are $\frac{5c}{4.g}$, $\frac{c}{2g}$; (see pages 107 and 459;) and as the observed ellipticity is within these limits, it follows that the density increases towards the centre.

From the expressions given in page 454, we can compare the ellipticities of Jupiter and the earth on the hypothesis of homogeneity; or even when the densities at distances proportional to their diameters are in a given ratio. In the first case, $\frac{e}{r} = \frac{5c}{4g} = \frac{5}{4} \cdot \frac{1}{\rho.t^2}$, for g is proportional to ρr, and $c = \frac{r}{t^2}$; in the other case, $\frac{e}{r} = \frac{5c}{2g} \cdot \frac{n}{5n.-3.f}$. See note ($x$) of this Chapter.

And from what has been established in the notes, page 493., Volume I., it appears, that whatever be the manner in which the fluid particles act on each other, whether by their tenacity, their mutual attraction, or even by impinging on one another, in which case they experience finite changes of motion; if through the centre of gravity of this fluid, supposed immoveable, we conceive a plane to pass, with respect to which the sum of the areas described on this plane by each molecule, and multiplied respectively by their corresponding molecules, is at the origin a maximum, this plane will always possess the property; therefore, when, after a great number of oscillations, the fluid mass assumes a uniform motion of rotation about a fixed axis, this axis will be perpendicular to the preceding plane, which will be, from what is stated in notes page 512., Volume I., the plane of the equator, and the motion of rotation will be such, that the sum of the areas described in the instant dt by the molecules projected on this plane, will be the same as at the commencement of the motion; and the axis in question is that with respect to which the *sum of the moments of the primitive forces of the system is a maximum*; it evidently preserves this property during the motion of the system, and finally becomes the axis of rotation. The actual velocity of rotation, as well as the axis of the ellipsoid of revolution which the fluid assumes, are determined by this *maximum;* and from what has been established in page 454, there is evidently only one possible figure of equilibrium.

(k) As it would be impossible, in the limits of these notes, to give the complete investigation of the figure of the earth, when the density increases towards the centre, we shall confine ourselves to pointing out some remarkable consequences which follow from the results, as given by Clairault and others. If n denotes the density of the nucleus, and f that of the rarer fluid which is spread over it, the value of $\frac{e}{r}$, is $\frac{5c}{2g} \cdot \frac{n}{5n-3f}$; if the density of the interior part be infinitely greater than that of the ambient fluid, (which is the hypothesis of page 101.,) $\frac{e}{r} = \frac{c}{2g}$; if $n = f$ then $\frac{e}{r} = \frac{5c}{4g}$, as we have before deduced; therefore the ratio of the ellipticity when the spheroid is homogeneous, to the ellipticity when the nucleus is infinitely denser than the fluid, is that of 5 : 2. These, as we shall see presently, are the extreme cases; likewise it appears, from the above expression, that according as f becomes less with respect to n, the ellipticity diminishes, and conversely. In the preceding hypothesis, it is easy to show that the expression of the increase of the force of gravity from the equator to the poles is expressed by the fraction $\frac{5c}{2g} \cdot \frac{4n-3f}{5n-3f}$; the sum of this and of $\frac{5c}{2g} \cdot \frac{n}{5n-3f} = \frac{5c}{2g}$, which is double the ratio of the centrifugal force at the equator to the force of gravity at the equator; or of the ellipticity of a homogeneous spheroid; for, in the case of homogeneity, the ellipticity and increase of the force of gravity are both expressed by the same fraction.

(l) Hence, if we can find the ratio of the equatorial and polar gravities, the ellipticity will be had by substracting the fraction expressing this ratio from twice the ellipticity of a homogeneous spheroid; and as a mean of a great number of observations made with the pendulum, gives 0,00561 for the increase of the force of gravity, $\frac{1}{115{,}2}$ — ,00561 $= \frac{1}{34{,}8}$, which shows that the *mean density* of the interior parts of the earth is > than the exterior; therefore, if l be the length a pendulum vibrating seconds at the equator, and $l+d$ the length of an isochronous pendulum at the pole, which is easily determined from the length of an isochronous pendulum at any latitude λ, and from knowing that the increments of the lengths are as sin. ${}^2\lambda$, then $\frac{d}{l} = \frac{5c}{2g} \cdot \frac{4n-3f}{5n-3f}$, $\therefore \frac{4n-3f}{5n-3f} = \frac{2gd}{5cl}$, and $\frac{n}{f} = \frac{15cl-6gd}{20cl-10gd}$, &c. See notes, page 347, Volume I.

The inequalities observed in the measurement of contiguous arcs of the meridian, which, according to Laplace, are to be attributed to the earth's not being spheroidical, arise in some measure also as well from the unequal distribution of the rocks which compose it, as from inequalities in the surface of the earth; which, according to Playfair, account for the discrepancy observed between the preceding ellipticity and that deduced from the measurement of degrees of the meridian. As a remarkable instance of this discrepancy, the spheroid which best agrees with the degrees measured in France, is one of which the ellipticity $= \frac{1}{152}$, which is very nearly double of what may be reckoned the mean ellipticity.

The equation here adverted to, in (l) page 110, is that which is given in No. 11 of the Second Book, and is detailed at greater length in the Second Chapter of the Third Book of the Celestial Mechanics. It is of the following form:

$$\left(\frac{d^2V}{dx^2}\right) + \left(\frac{d^2V}{dy^2}\right) + \left(\frac{d^2V}{dz^2}\right) = 0;$$

and may be generally announced in the following manner; that the sum of the three partial differences of the second order of the function V, which expresses the sum of the attracting molecules of a spheroid, divided respectively by their distances from the attracted point, (of which function the partial differences with respect to any line, is the resultant of its attractions decomposed according to this line,) is constantly equal to cypher. By combining this fundamental equation with a differential equation of the first order, which the preceding function must satisfy when the attracted point is at the surface of a homogeneous spheroid, which differs little from a sphere, Laplace obtained by developing, the attraction of a spheroid composed of fluid or solid strata of any density whatever, and endowed with a motion of rotation; the molecules being supposed to attract each other inversely as the square of the distance. The general and simple relations between the attractions and the figure of the spheroids, which are furnished by this expression, enabled Laplace *directly* to determine the figure of the fluid strata in the case of equilibrium, and the law of gravity at their surface. From the fecundity of the fundamental equation, which is the basis of his analysis, and is reproduced in the theory of the fluids, and in that of heat, Laplace was induced to think that the formulæ which he obtained were the simplest and most general which could be obtained.

(m) See note (k) of this Chapter.

In the general hypothesis, the strata increase in density and diminish in ellipticity from the surface; therefore, if a line be conceived to be drawn from the surface to the centre, the tangents drawn to the strata at the intersection of this line with them, will not be parallel, and consequently the perpendicular to these tangents which indicate the direction of gravity will not be parallel; now, if the number of these strata be increased indefinitely, these perpendiculars will form a curve. In fact, the direction of gravity being a curve line, all those elements are perpendicular to the strata of level which it traverses; this curve is the trajectory which intersects at right angles all ellipses which, by their revolution, form these strata.

The following is the analytical expression of what is stated in (*n*) p. 117, $p'' = \mathrm{P}.\left(1 - \frac{1}{2}\,a.(l - y'') + \frac{5}{4}\,a\,\phi.\mu^2\right)$, when p'' is the force of gravity at the surface of the spheroid, P the force of gravity at the surface of the sea and at the equator, a a constant coefficient, so small that its square and higher powers are neglected, and $a(l-y'') =$ the depth of the sea, and $a\phi =$ the ratio of the centrifugal force to the force of gravity at the equator; p'', P are determined by means of isochronous pendulums, and ay' is the elevation of points of the surface of the spheroid above the surface of the sea, $a\,y''$ is the elevation of corresponding points of the atmosphere, $a\,l = a\,y' - a\,y''$, and $a\,l$ and $a\,y''$ can be always determined by means of barometrical measurement, μ^2 is the square of the sine of the latitude, and $\therefore\ \frac{5}{4}\,a\phi\mu^2$ is the increase of gravity at any latitude μ; now, from what has been established in page 107., it appears, that $\frac{5}{4}\phi = 0{,}004325$, $\therefore$ the increase is $0{,}004325.\mathrm{P}\mu^2$, which being less than the observed increment, it follows that the earth is not homogeneous.

In the Eleventh Book, besides the causes mentioned in the text, and which are detailed at length in the Third Book of the Celestial Mechanics, another source of deviation from the law of the square of the sine of the latitude, arises from the errors to which the observations of the amplitudes of the measured arcs are liable, which are, relatively to the measured arc much more considerable than the errors of the pendulum; the reason of which appears to be, that the intensity of gravity is much less affected by local variations than its direction; for the inequalities on the earth's surface, and unequal distribution of the rocks which compose it, must produce great local irregularities in the direction of the plumb line, which, in all probability, are the causes of the inequalities observed in the measurement of contiguous arches of the meridian, reduced to the level of the sea.

It may not be unnecessary to mention, that in general there are three modes of determining the ellipticity of the earth given in the text, either by observing the lengths of isochronous pendulums, or by measuring the arcs of degrees; or, thirdly, by means of some lunar inequalities. By means of the observed quantity of the precession of the equinoxes, Laplace shows, in the Eleventh Book, that D, the mean density of the earth $= 1,587(\rho)$ where (ρ) denotes the density at the surface; and as this density is, by the experiments of Maskeylyne and Cavendish, which will be detailed in note (r) of this Chapter, three times that of water, we have $D = 4,761$; that of water being unity, which agrees very well with the conclusions of Maskeylene.

(n) In the Eleventh Book the author shows, that the radius of the terrestrial spheroid $= 1 + a\bar{h}\left(\mu^2 - \frac{1}{3}\right) + ax$, where ax is a very small quantity with respect to $a\bar{h}$, and of the same order as the mean elevation of the continents; in like manner, the expression for the radius of the surface of the sea is $al - a(h'+h).\left(\mu^2 - \frac{1}{3}\right) + ax'$, where al is a constant quantity and ax' of the same order as ax.

The depth of the sea is very nearly = the difference of these radii, and $\therefore = al - ah'\left(\mu^2 - \frac{1}{3}\right) + ax' - ax$; at the equator the continents occupy a great extent, for which this expression becomes negative; but the sea occupies a still greater extent, for which this expression is positive. In the first case, $al + \frac{1}{3}ah'$ is $<$; and in the second case it is $>$ than $ax - ax'$; consequently $al + \frac{1}{3}ah'$ is of the same order as ax; very near to the north pole, where $\mu = 1$, the sea covers part of the terrestrial spheroid, and leaves another part uncovered; in the first case, $al - \frac{2ah'}{3}$ is $>$, and in the second case it $<$ than the value of $ax - ax'$ corresponding to $\mu = 1$. $\therefore$ as $al + \frac{ah'}{3}$, $al - \frac{2ah'}{3}$ are respectively of the order ax, their difference ah' and also the constant quantity al are of the same order; consequently the depth of the sea must be inconsiderable, and of the same order as the elevations of continents above the level of the sea; but as there are mountains which rise very high above the level of the adjacent continents, so there may be some parts of the sea of very considerable depths. Hence it follows, that the surface of the terrestrial spheroid is *q.p* elliptic, for, by what precedes, the equation of the equilibrium of the surface of the sea, would become that of the equilibrium of the surface of the terrestrial spheroid supposed fluid, if the sea was to disappear. It is generally admitted, that at least two-thirds of the surface of the earth is at the present time fluid, and from this circumstance, combined with what is stated in the text, it would seem to follow that the earth was primitively *fluid*.

(o) If Π represents the pressure, and ρ the density, the equation adverted to in the text may be expressed as follows, $\frac{d\Pi}{d\rho} = 2k\rho$, $2k$ being constant, $\therefore \Pi = k.(\rho^2 - (\rho^2))$ (ρ) being the density at the surface, where $\Pi = 0$; and as it is proved in No. 30, of the Third Book of the Celestial Mechanics, that $\frac{d\Pi}{\rho} = -4\pi . \frac{da}{a^2} . \int \rho a^2 da$, where a denotes the radius of the stratum of which the pressure $= \Pi$, we have $\frac{d\rho}{da} = -\frac{n^2}{a^2} . \int \rho a^2 da$, where $n^2 = \frac{2\pi}{k}$, if $\rho' = ar$, then $a^2 d\rho = ad\rho' - \rho.'da$, $\therefore \frac{ad\rho'}{da} - \rho' = -n^2 . \int \rho' ada$, $\therefore$ by differentiating $\frac{d^2\rho'}{da^2} + n^2\rho' = 0$; and the integral of this equation is $\rho' = \text{A} \sin. an + \text{B}. \cos. an$, A and B being constant arbitrary quantities, therefore $\rho = \frac{\text{A}}{a} \sin. an + \frac{\text{B}}{a} . \cos. an$; as ρ is not = to infinity at the centre where a vanishes, B must be $= 0$, and $\therefore \rho = \frac{\text{A}}{a} . \sin. an$. This is the law of the density of the strata of the terrestrial spheroid, on the hypothesis that $\frac{d\Pi}{d\rho} = -2k\rho$, at the surface $a = 1$ and $\rho = (\rho)$, $\therefore (\rho) = \text{A}. \sin. n$; $\frac{-\left(\frac{d\rho}{da}\right)}{(\rho)} = 1 - \frac{n}{\tan. n.}$, and as D is the mean density of the earth, $\int \rho a.^2 da = \text{D}. \int a^2 da = \frac{\text{D}}{3}$; but at the surface the equation $\frac{d\rho}{da} = \frac{-n^2}{a^2} . \int \rho a^2 da$, becomes

$$(\rho) . \left(1 - \frac{n}{\tan. n.}\right) = n^2 . \int \rho a^2 da; \therefore \frac{\text{D}}{(\rho)} =$$

$\frac{3}{n^2} \cdot \left(1 - \frac{n}{\tan. n}\right) = \frac{3q}{n^2}$, q being $= 1 - \frac{n}{\tan. n}$, $\frac{D}{(\rho)}$ the ratio of the mean density of the earth to the density of its surface, and $n^2 = \frac{2\pi}{k}$, hence we can determine k and $\therefore n^2$, when D is known, and *vice versa.* From these results, Laplace obtains expressions for the gravity, ellipticity, &c. which accord sufficiently well with observation; from whence he infers, that it is extremely probable, the internal constitution of the earth is conformable to the preceding hypothesis. It is worthy of remark here, though Laplace infers, in page 120, that the primitive fluidity of the earth is clearly indicated by the regularity of gravity, and by the figure at its surface, Playfair, in his Outlines, asserts the express contrary: he states, that the approximation, which, notwithstanding the irregularities in the measured degrees, the figure of the earth has made to the spheroid of equilibrium, cannot, in consistency with other appearances, be ascribed to its having been once in a fluid state, for though the action of water may be evidently traced in the formation of those stratified rocks which constitute a large proportion of the earth's surface, it is of water depositing the *detritus* of solid bodies: with respect to those rocks which contain no such detritus, but have the character of crystallization in a greater or less degree, it is not evident that they are of aqueous formation. Indeed, the only action of water of which we have any distinct evidence in the natural history of the globe, is partial and local, and therefore insufficient to account for the spheroidical figure of the earth.

(p) See note (c) page 465, Volume I.; and No. 27 of the First Book of the Celestial Mechanics.

(q) It follows from the theorem announced in the text, that the three principal axes of rotation of the imaginary spheroid, are the principal axes of the earth.

The expressions for these radii, according as the earth revolves about the first, second, or third principal axes,

$$\text{are, } 1+al+au-\frac{5}{2}\cdot\frac{a\phi\cdot\left(\mu^2-\frac{1}{3}\right)\cdot\int\rho\, d.a^3}{5.\int\rho d.a^3-3},$$

$$1+al+au+\frac{5}{4}\cdot\frac{a\phi\cdot\left(\left(\mu^2-\frac{1}{3}\right)-(1-\mu^2).\cos.\,(2\bar{\omega}'-2\Pi)\right)}{5.\int\rho.d.a^3-3.},$$

$$1+a.l+a.u+\frac{5}{4}\cdot\frac{a\phi\left(\left(\mu^2-\frac{1}{3}\right)+(1-\mu^2)\cdot\cos.\,(2\bar{\omega}'-2\Pi)\right)}{5.\int\rho.d.a^3-3},$$

therefore, if these be added together, their mean value is $1+a.l+au$, so that it is independent of the centrifugal force $a.\phi$, as is stated in the text.

(r) The principal of areas in reference to the present subject, may be announced in the following manner: if we project, on a fixed plane, each molecule of a system of bodies which react on each other, and if, moreover, we draw from these projections to a fixed point assumed on this plane, lines which we shall term radii vectores, the sum of the products of each molecule by the area which its radius vector describes in a given time, is proportional to the time; so that if A denotes this sum, and t the time, we shall have $A=ht$, h being constant. Now, in the case of earthquakes, volcanoes, &c. it is easy to show, that while these phenomena diminish the motion of the earth in one way, there exist simultaneous causes which produce the contrary effect, so that the value of A remains the same: but if, as is stated in the text, considerable masses are brought from the poles to the equator, the radii vectores *increase;* therefore, in order that the value of A may remain unvaried, the other factor must diminish, consequently the rotatory motion of the earth must diminish.

(s) If we counterpoise a quantity of ice in a delicate balance, and then leave it to melt, the equilibrium will not be in the slightest degree disturbed; or if we substitute for the ice, boiling-water or red-hot iron, and leave them to cool, the result will be precisely the same; and if a pound of mercury be placed in one scale, and a pound of water in the other, and if they then be heated or cooled through the same number of degrees, although thirty times more heat either enters or leaves the water than the mercury, in consequence of its different capacity for heat, they will still balance each other; likewise if a beam of solar light, be condensed by means of a burning glass, and then made to fall upon the scale of a delicate balance, it will not depress the scale, as would be the case if the beam of light had the least inertia or weight.

(*t*) If a be the arc described in a given time, r the radius, and V the angle, we have $V = \frac{a}{r}$, but the area $= a.r = V.r^2$, therefore if the angular velocity of rotation does not increase, the areas described in the plane of the equator are proportional to r^2; and as the decrement of r is the 100,000th part, the decrement of r^2 will be very nearly double of this, or the 50,000th part.

If this diminution of r arose from a decrease of temperature equal to one degree; and if the duration of the earth's rotation be 100,000 *decimal* seconds, the duration of rotation will be diminished $2''$ in this hypothesis; now, as it appears from the comparison of observations with the theory of the secular equation of the moon, that the duration of rotation since the time of Hipparchus has not varied $\frac{1}{100}$th of a second, the variation of the internal heat of the earth since that time is insensible. Indeed, the dilatation, specific heat, and greater or less permeability to heat, which are all unknown, may not be the same in the earth and the glass globe, in which a diminution of $\frac{1}{100}''$ in a day corresponds to a diminution $\frac{1}{200}$ of a degree in temperature. But still this difference can never increase from $\frac{1}{100}$ of a degree to $\frac{1}{10}$ of a degree, the loss of terrestrial heat corresponding to a diminution of $\frac{1}{100}$ of a second in the duration of a day; but a diminution of $\frac{1}{100}$ of a degree near the surface, supposes a much greater diminution in the temperature of the inferior strata; for, eventually, the temperature of all the strata diminishes in a geometric progression, so that the diminution of a degree near the surface, implies a much greater diminution in the strata which are nearer to the centre; therefore the dimensions and moment of inertia of the earth diminish more than in the case of the sphere of glass. From what precedes, it follows, that if, in the progress of time, any change is observed in the mean height of a thermometer placed at the bottom of a deep cavern, it must be ascribed to a change in the climate of the place, and not to a variation in the mean temperature of the earth. It is worthy of remark, that the discovery of the true cause of the secular equation of the moon, makes known at the same time the invariability of the duration of the day, and of the mean temperature of the earth. Connaissance des Temps, 1813, compared with text.

According to M. Fourier, who has discussed the subject of the interior temperature of the globe, the heat distributed within the earth is susceptible of three distinct modifications, arising, 1*st*, from the rays of the sun, which penetrating the globe, cause diurnal and annual variations in its temperature. These periodical variations cease to be perceptible at a certain distance beneath the surface. Beyond that depth, and even to the greatest accessible excavations, the temperature due to the sun has long since become fixed and stationary; the whole quantity of solar heat which regulates the periodical variations, oscillates in the exterior shell of the earth, descending further within the surface during one portion of the year, and rising up to be dissipated into space during the opposite or the winter season. Secondly, the temperature for deep excavations, which, though constant for any one place, varies for localities more or less distant from the equator; so that the *solar heat* penetrates farther at the equinoctial zones, to reascend and be dissipated at the polar regions. But besides the external focus of heat, there is also to be considered the proper or intrinsic heat of the earth; and if, as the experiments mentioned in p. 471 seem to prove, the temperature of the deep recesses of the earth becomes perceptibly greater according as we penetrate farther into the interior, it is impossible to ascribe this increase to the heat of the sun; it can only arise from a primitive heat, with which the earth was endowed at its origin, and which may diminish with greater or less celerity, by diffusion from its surface; it is evident that the increase will not be always the same in amount as at present, it will diminish progressively; but a number of ages must elapse before it is reduced to half of its present value; in general the extent of this diffusion will be proportional to its primitive intensity, and to the conducting quality of the surrounding materials.

If V be the heat of a molecule at the surface, it is proved by analysis, that the increment of heat at the depth z' relatively to r the radius of the earth, is equal to the product of this depth by the elevation of the temperature of the surface of the earth above the mean state of temperature, *i. e.* $= z'\left(\frac{dV}{dr}\right)$ which becomes $= f z' V$, when we only consider in V the part of the heat, which is independent of the action of the heating causes at the exterior.

It is to be remarked here, that at all distances to which we can penetrate, the temperature of the *sea decreases*, and at the equator at a depth of 600 metres, the temperature of the water was 7°,5 of the centrigrade thermometer, while that at the surface was 30°; but this is not inconsistent with what is stated in the text, as this decrease of temperature is owing to currents of water coming from the poles to the equator.

It is the opinion of geologists, that originally there existed in the interior of the crust of the earth, a great magazine of fire, which, according to them, was the cause of the deluge, and the numerous catastrophies to which an accurate examination of the various appearances of the internal constitution of the earth proves that our globe has experienced, previous to the deluge, particularly the alternation of marine and fresh-water products. According to them, this heat was much more intense formerly than at present; and as in consequence of the fluidity of the earth in its primeval state, very little heat was lost in its transmission from the interior to the surface, any warmth imparted to the bottom of the ocean would be transmitted without sensible loss to the surface. In this order of things, a genial climate would exist over the whole surface of the earth, from one pole to another; and, in like manner, this intrinsic source of heat would, when its diffusive energy was thus slightly obstructed, predominate over the solar, so that the position of the sun with respect to the equator would act a comparatively subordinate part in modifying climate; therefore, as in this case, the difference of the temperature at the pole and equator would be comparatively small, a considerable uniformity of temperature would thus obtain over the whole earth; and this may explain why animals and plants which are now peculiar to the tropical regions, might have formerly existed as far north as the arctic and antartic circles; (see page 88.)

According as the deposits after *each successive catastrophe to which the earth was subjected*, thickened, there was a progressive interception from the ocean of the subjacent heat; but besides the thickenings of the deposits of the ocean, a great mechanical change took place on the terraqueous constitution, the influence of which, in refrigerating climates, is considerable; for, at every catastrophe, the area of the land in proportion to the sea would be diminished, and that of the sea increased, with a proportionate diminution of depth, *i. e.* the cooling surface would be increased, and the ocean would rest on a cooler bed, because it is more distant from the central heat of the earth; and besides these two, there is a third cause of the decrease of heat, namely, that which arises from its diffusion into the ambient space.

The elephant mentioned in p. 117, whatever its hide may have been, required necessarily for its subsistance an enormous supply of vegetable food, which necessarily implied a luxurious herbage in the northern regions; and the freshness of his carcass proves that the animal perished at once, with its kindred, in a sudden revolution, accompanied by a *sudden* change of climate, which prevented the decomposition of *its* flesh, and of the bones of its kindred, which are found in great abundance on the banks of the Tanais.

Suppose that three thousand metres beneath an extensive plane, there existed a vast reservoir of water, produced by rain water; at this depth it would acquire, from the heat of the earth, a temperature very nearly equal to that of boiling-water; and if now, in consequence of the pressure of the adjacent columns of water, or from the action of vapours, which ascend in the reservoir, these waters ascend to the height of the inferior part of the surface from which they had flowed down, they will constitute a source of warm water, impregnated with such substances of the strata through which it flowed, as were soluble by it. This furnishes an extremely probable explanation of the natural tepid waters which are found in different parts of the earth.

(*v*) It is easy to estimate what would be the effect of the attraction of a *spherical* and *homogeneous* mass near to which a plumb-line is suspended, for if g denote the force of gravity, and x the angle which the direction of the plumb-line makes with the vertical, g sin. x expresses the force of gravity resolved perpendicular to the direction of the plumb-line; and if y denote the distance of the centre of gravity of the attracted body from the centre of the homogeneous sphere, μ the mass of the attracting body, and f the intensity of the attractive force at the unity of distance, and for the unity of mass; $\frac{\mu f}{y^2}=$ the attraction of the spherical mass on the suspended body; now, if a denote the distance of the point of suspension from the centre of the sphere, and α the angle which this line makes with the verticle, $\alpha-x$ is the angle, which a makes with the plumb-line; the cosine of the angle which a perpendicular to the direction of the plumb-line makes with y, $=$ sine of angle which y makes with plumb-line $=$ $\frac{a.\sin.(\alpha-x)}{y}$, $\therefore \frac{\mu f}{y^2}$ resolved in the direction of this perpendicular $=\frac{\mu f\, a.\sin.(\alpha-x)}{y^3}$, which, when the plumb-line is at rest $=g.\sin.x$; if y be supposed $=a$, we have $\frac{\mu f}{ga^2}=\frac{\sin.x}{\sin.(\alpha-x)}$, by means of this equation, when μ, f and a are given, we can determine the deviation x. Now, if m denote the mass of the earth, ρ its mean density, and r its radius, and ρ', r' the density and radius of the attracting body, $\mu : m :: r'^3\rho' : r^3.\rho$, and $\frac{\mu}{m}=\frac{\rho' r'^3}{\rho.r^3}$, and $mf=g.r^2$, $\therefore \frac{\mu f}{g}=\frac{\rho' r'^3}{\rho.r}$, $\therefore$ from what precedes we have $\frac{\rho' r'^3}{\rho r a^2}=\frac{\sin.x}{\sin.(\alpha-x)}$. The value of x will be so much the greater as a diminishes, and as α approaches to a right angle; and as the least value of a is r', it follows that the deviation from the vertical will be a *maximum* when $a=r'$ and $\alpha=90°$, in which case we have $\tan.x=\frac{\rho' r'}{\rho.r}$, $\therefore$ if ρ', ρ r', r be given, we can determine the deviation and conversely, it is easy to show that if $\rho'=\rho$ the radius of the sphere, which would cause a deviation $= 1°$, should be $= 30^m,866$. But in nature it is not easy to determine the ratio of ρ to ρ', &c. The manner in which Maskelyne determined the value of x is as follows, by observation of the zenith distances of the stars on the north and south sides of Schehallien, he determined the difference of the latitudes of two stations; from a trigonometrical survey of the mountain, the distance between the same two points was ascertained; and thence, from the known length of a degree of the meridian at that parallel, the difference of the latitudes of the two stations was again inferred, and it was found less by $11''6$, than by astronomical observations. This could only arise from the zeniths of the two places being separated from each other by the attraction of the mountain on the plummets. From the quantity of this change of direction, the ratio of the attraction of the mountain to that of the earth was concluded to be that of 1 to 17804; and from the magnitude and figure of the mountain, which was given by the survey, it was inferred that ρ' was to $\rho :: 5 : 9$, $\therefore \rho$ is nearly double of ρ' the density of the rocks which compose the mountain; indeed these last appear to be considerably more dense than the mean of those which compose the exterior crust of the earth, and at least two or three times more dense than water, $\therefore \rho$ is four or five times more dense than water; hence, as the earth is four times denser than the sun, it follows that the density of the sun is nearly = that of water. See notes, page 399.

However, as was already remarked in page 399, it is to be observed here, that the density obtained is only relative, as we do not know the absolute density of water; and indeed, we are so far from knowing the actual mean density of the earth, that there are considerable discrepancies in the results which determine the ratio of its mean density to that of water.

Though the Cordillieries exhibit evident traces of their being volcanic, and therefore hollow in their interior, there is no reason to suppose that Schehallien is of that nature; on the contrary, it is very probable that it is an extremely dense mountain. The universality of the attraction of every particle of matter is clearly established by this deflection; also it follows, that the force of gravity varies inversely as the square of the distance, for if the attraction of the hill was to that of the earth only as their respective masses, the effect of its attraction would be altogether insensible, in consequence of the comparative smallness of its mass.

We might, by means of the oscillations of the pendulum, determine the length of the pendulum, which vibrates seconds at the level of the sea, for if l be the length of a pendulum vibrating seconds at the height h of the Cordellieries, the length of an isochronous pendulum at the level of the sea $= l.\frac{(r+h)^2}{r^2} = l + \frac{2hl}{r}$, omitting $\frac{lh^2}{r^2}$ as a very small fraction; now, it is observed, that the value of the correction of the length of the pendulum, determined by observation, is less than what theory assigns to it from a diminution of distance, which can only arise from the action of the mountain itself making the difference less than it ought to be, from its increased distance from the centre of the earth. The ratio deduced from the effect of the mountain in deflecting the plummet, is considerably less than the estimation by means of the attraction of two leaden balls, the effect of which Cavendish rendered sensible by the balance of torsion, an instrument by means of which we can determine very small, and apparently inappreciable forces; it consists of a very delicate metallic thread, attached to a fixed point, at the extremity of which is suspended an horizontal lever; while the thread is not twisted, the lever quiesces in a certain position, termed the line of repose; according as it deviates from this position the thread becomes twisted, and this torsion tends to cause the lever to revert to the line of repose; therefore, in order to retain it in this position, it is necessary to apply to its extremities equal and contrary forces, existing in the horizontal plane, and acting perpendicular to its length; the common value of these forces will be the measure of the force of torsion, which, when the thread remains the same, is proportional to the angle through which the lever is deflected. Now, if two leaden balls be brought near to the opposite extremities of this line, their attraction will cause the lever to deviate from the line of repose, and according as the deviation increases, the force of torsion increases, and there exists a position in which this force constitutes an equilibrium with the attraction of the two spheres; but as the lever attains this position with an accelerated velocity, it will pass beyond it, and will perform oscillations on each side, like a *horizontal* pendulum; from observing the relation between the length of this pendulum and that of an ordinary isochronous pendulum, we can infer the ratio of the attraction of each sphere to that gravity, and consequently the proportion of the mass of this sphere to that of the earth. As it would be impossible here to enter into all the details of this experiment, we shall give the resulting equation, *i. e.* $\frac{m}{\mu} = \frac{l'.r^2.a.\sin.\alpha}{l c^3 b}$. Where m, μ denote the masses of the earth and sphere, $l\,l'$ the lengths of the lever and isochronous pendulum, a the distance of the centre of the attracting body from the point of bisection of the lever, α the angle which c the line from the centre of attracting body to quiescent extremity of the lever subtends at its point of bisection, and b a constant arbitrary quantity; as all the terms of the second member are given, we can determine the ratio of m to μ, and as we know the magnitudes, we have the ratio of the densities.

(x) If $t\,t'$ denote the times of the earth's and Jupiter's rotation, $c\,c'$ the respective centrifugal forces at their equators, of which the radii are $r\,r'$, and $g\,g'$ their gravities, we have, on the hypothesis of homogeneity,

$e : e' :: \frac{c}{g} :: \frac{c'}{g'} :: \frac{r}{g.t^2} : \frac{r'}{g'\,t'^2}$, (see page 452,) *i. e.* because g varies as $\rho.r$, $e : e' :: \frac{1}{g.\rho} : \frac{1}{g'.\rho'}$; if C′ be the centrifugal force of the fourth satellite, whose distance from the centre of Jupiter is given in terms of r', $c' : C' :: \frac{1}{t^2} : \frac{D}{T^2}$ (T being the period of the satellite), but $C' = \frac{M}{D^2}$, M being the mass of Jupiter, $\therefore c' = \frac{MT^2}{t^2.D^3}$; therefore knowing the ratio of the centrifugal force to that of gravity, we can obtain the proportion of the axes, on the hypothesis of homogeneity, in the manner indicated in the text; which proportion not agreeing with that furnished by accurate observations, it follows that Jupiter is not homogeneous; this also follows from the proportion $e : e' :: \frac{1}{g\rho} : \frac{1}{g'.\rho'}$ for the ellipticity deduced from the preceding proportion, by substituting for e, g, g', ρ, ρ', does not agree with observation.

NOTES TO CHAPTER IX.

(*a*) SUPPOSE a to be the distance of the centre of this ellipse from that of Saturn, which by hypothesis is very great relatively to the dimensions of the ellipse, and let c represent the centrifugal force due to the motion of rotation at the distance of unity from the axis of rotation, then if the coordinates of a molecule of the ring referred to its centre as origin be u, z, the centrifugal force of this molecule, multiplied by the element of its direction, will be equal to $(a+u).cdu$, and the attraction of Saturn on the same molecule is $\frac{S}{(a+u)^2+z^2}$, (S being the mass of Saturn, supposed to be condensed into its centre,) and as the element of its direction is $-d.\sqrt{(a+u)^2+z^2}, = \frac{-(du.a+du.u+dzz)}{\sqrt{(a+u)^2+z^2}}$ if the squares of z and u be neglected, by multiplying by $\frac{S}{a^2+2au}$, we obtain $-\frac{S\,du}{a^2}+\frac{2Sudu}{a^3}-\frac{S.zdz}{a^3}$, the attractions which the same molecule experiences from the ring itself, when multiplied by the element, $-du$, $-dz$ are given by the expressions $-\frac{4\pi udu}{\lambda+1}$, $-\frac{4\pi\lambda zdz}{\lambda+1}$, the equation of the generating ellipse being $u^2+\lambda^2z^2=k^2$; and therefore its differential equation is $0=udu+\lambda^2zdz$, which being compared with the preceding, gives the two following,

$c = \frac{S}{a^3}$; $\frac{4\pi\lambda}{\lambda+1} + \frac{S}{a^3}$ divided by $\frac{4\pi}{\lambda+1} - \frac{3S}{a^3} = \lambda^2$; the first equation determines the rotatory motion of the ring; the second determines the ellipticity of its generating figure, making $e = \frac{S}{4\pi . a^3}$, we obtain, by means of the second equation, $e = \frac{\lambda(\lambda-1)}{(\lambda+1).(3\lambda^2+1)}$, and since e is positive, λ must be greater than unity: as the axis of the ellipse directed towards Saturn, which measures the breadth of the ring, is $= 2k$, the axis which is perpendicular to it $= \frac{2k}{\lambda}$, and as it measures the thickness of the ring, it must be less than its breadth; as $e=0$, when $\lambda=0$, and also when $\lambda=\infty$, it follows that for the same value of e, there are two different values of λ; but we should select the greatest, which gives the most compressed form to the ring; when, therefore, e is a maximum, $\lambda=2{,}594$, in which case $e=0{,}0543026.$, and as $S = \frac{4}{3}\pi . \rho . R^3$, ρ being the density, and R the radius of Saturn, $e = \frac{\rho R^3}{3a^3}$, $\therefore$ the greatest value of which ρ is susceptible is $0{,}1629078. \frac{a^3}{R^3}$; but this limit is not well defined, in consequence of the difficulty of obtaining the exact ratio of a to R, owing to the effects of irradiation, and the smallness of the apparent magnitudes; if $\frac{a}{R} = 2$ for the innermost ring, this limit = very nearly $\frac{13}{10}$. It is probable that the irradiation increases considerably the apparent magnitude of the ring, and it is likely that, in consequence of it, several rings are blended into one. As the centrifugal force c, arising from the motion of rotation $= \frac{S}{a^3}$, the motion of rotation is evidently equal to that of a satellite whose distance from the centre of Saturn is a. Robison shows, from a consideration of the distance and period of the second satellite, that the period of the ring is not the same as that of a satellite at the same distance. See Mechanical Philosophy, page 514.

Relatively to what is stated in page 135, it is shown in page 165 of the Third Book of the Celestial Mechanics, that if the ring was circular, the attraction of Saturn on an element of the ring is always negative, whatever may be the distance of the centre of Saturn from that of the ring; hence then it follows, that the centre of Saturn always repels that of the ring, consequently the curve which the second centre describes about the first is always convex towards Saturn, therefore eventually the second centre is elongated more and more from that of the planet, until its circumference touches the surface; and as a ring perfectly symmetrical in all its parts would be composed of an infinity of circumferences similar to that which we have just considered, its centre would be repelled by that of Saturn, provided that these two centres ceased to coincide, and then the ring would eventually be attached to the surface of Saturn.

Laplace's theory of the ring has been severely criticised by Professor Robison, who is so far from admitting Laplace's conclusions, that he thinks the inequalities in the form of the ring are incompatible with the equilibrium of forces among incoherent bodies, such as, according to our author, the parts composing the ring are: besides, as by supposition, there is no cohesion in it, any inequalities in the constitution of its different parts cannot influence the general motion of the whole in the manner he assumes, but merely by an inequality of gravitation, the effect of which would be to destroy the permanency of its construction, without securing, as Laplace imagines, the steadiness of its position; likewise, as he thinks, that the equilibrium of the fluid ring is one of instability, any, the slightest disturbance, would derange it. Robison supposes that the ring consists of coherent matter, the cohesive force being considerable, in order to counteract the centrifugal force, which is greater than the weight; its substance, according to him, is viscid, like to melted glass, and if the ring is not uniform, which is indicated from a consideration of its spots, but more massive on one side of the centre than the other, then the planet and the ring may revolve about a common centre, very nearly, but not accurately coinciding with the centre of the ring.

NOTES TO CHAPTER X.

(a) As the density ρ of the atmosphere is a function of the pressure p, according as we ascend in the atmosphere p and therefore ρ diminishes. See note (k) page 365, Volume I.

(b) It should follow from this, that at the surface of the atmosphere, the force of gravity would be equal to the centrifugal force arising from the motion of rotation. See as above.

(c) See notes page 492, Volume I. As r^2dv expresses the elementary area described by a molecule projected on the plane of the equator; (see page 467;) if r diminishes, dv and therefore the angular velocity of rotation must increase.

(d) See notes page 454 of Volume I. The mutual attraction of the molecules of the atmosphere is not taken into account here; however, it is easy to perceive, that in consequence of the rarity of the atmosphere, this attraction is inconsiderable. If r be the distance of a molecule dM of the atmosphere from the centre of gravity of the earth, (which we shall suppose spherical,) θ the angle which r makes with the axis of rotation, n the angular velocity of rotation, the centrifugal force of $d\text{M}=n^2r.\sin.\theta$, and the element of its direction $= d.(r.\sin.\theta)$, therefore the integral of this force into the element of its direction, is $\frac{1}{2}n^2r^2.\sin.^2\theta$, $\therefore$ as $\frac{dp}{\rho} = \text{P}\delta x+\text{Q}\delta y+\text{R}\delta z$; page 454, Volume I., we have $\int \frac{dp}{\rho} = C + V + \frac{1}{2} n^2 r^2 . \sin.^2\theta$, V representing the sum of the molecules of the earth divided by their respective distances, *i. e.* as the earth is supposed to be spherical, $V = \frac{m}{r}$, being the integral of

$-\int \left(\frac{dV}{dr}\right) . dr$, now at the exterior surface $p = 0$, $\therefore$ we shall have $C = \frac{m}{r} + \frac{n^2}{2} . r^2 \sin.^2\theta$, $= \therefore \frac{2C}{m} = \frac{2}{r} + \frac{n^2}{m} . r^2$ $\sin.^2\theta$, or $c = \frac{2C}{m} = \frac{2}{r} + a r^2 \sin.^2\theta$, a denoting $\frac{n^2}{m}$, *i. e.* the ratio of the centrifugal force at the earth's equator to the force of gravity, the radius of the equator being supposed $= 1$, if R denotes the radius of the pole of the atmosphere, we have $c = \frac{2}{R}$, for θ then vanishes, $\therefore \frac{2}{R} = \frac{2}{r} + ar.^2 \sin.^2\theta$, and if R′ denote the radius of the equator of the earth's atmosphere, we have, as $\theta = 90$, $\frac{2}{R} = \frac{2}{R'} + aR'^2$, $\therefore aR'^3 = \frac{2.(R'-R)}{R}$; the greatest value of which R′ is susceptible is evidently that which belongs to the point in which the centrifugal force is equal to gravity, in which case we have

$$\frac{m}{R'^2} = n^2.R', \ i. e. \ \frac{1}{R'^2} = aR',$$

and therefore $1 = aR'^3$, consequently $\frac{R'}{R} = \frac{2}{3}$; this ratio of R′ to R is the greatest possible, for supposing $aR'^3 = 1 - z$, z being necessarily positive or cypher, we shall have $\frac{R'}{R} = \frac{3-z}{2}$; it is evident that r increases with θ, and is a maximum at the equator, for differentiating

$c = \frac{2}{r} + ar^2 \sin.^2\theta$ the equation of the surface, we obtain

$dr = \frac{ar.^3\, d\theta.\, \sin.\, \theta.\, \cos.\, \theta}{1 - ar^3.\, \sin.\, ^2\theta}$; now the denominator of this fraction is always positive, for as $am.r.\, \sin.\, \theta$ is the centrifugal force of a molecule whose radius $= r$, (am being$=n^2$), this force resolved in the direction of r, $= amr.\, \sin.\, ^2\theta$, and as it must be less than the gravity $\frac{m}{r^2}$, we have $ar^3.\, \sin.\, ^2\theta < 1$, therefore r increases with θ, and consequently is a maximum at the equator.

The atmosphere has only one possible figure of equilibrium, for making the equation of the surface of the atmosphere to assume the form

$$r^3 - \frac{2r}{a\mathrm{R}.\, \sin.\, ^2\theta} + \frac{2}{a\, \sin.\, ^2\theta} = 0,$$

the values of r, from what precedes, which satisfy the problem, must be positive, and such that $1 - ar^3.\, \sin.\, ^2\theta$, is greater than cypher, but there is but one root of this kind, for if $r'\, r''\, r'''$ be the three values of r given by the preceding equation, and if two of them are positive, which is the greatest number that can be so, inasmuch as the absolute quantity in the preceding equation is positive, then as $1 - ar^3\, \sin.\, ^2\theta$ is >0, both r' and r'' must be positive and $<$ than $\frac{1}{\sqrt[3]{a.\, \sin.\, ^2\theta}}$; and as the second term is wanting in the given equation $r''' = -r' - r''$, $\therefore r'''$ is negative, and as it is $= -\, r' - r''$ it must be less than $\frac{-2}{\sqrt[3]{a\, \sin.\, ^2\theta}}$, $\therefore -r.'r.''r''$ must be less than $\frac{2}{a\, \sin.\, ^2\theta}$; but as the absolute quantity is always equal to the product of the roots with the sign changed, $-r.'r.''r'''$ should be $= \frac{2}{a.\, \sin.\, ^2\theta}$, hence it appears that the supposition of their being two positive values of r is impossible, and therefore there is only one possible figure of equilibrium.

(*e*) The solar atmosphere only extends to the orbit of a planet which would circulate about the sun in a time equal to that of the rotation of this star, namely, in twenty-five days and one half; and according as the rotatory motion increases, the limit of atmosphere must be continually contracted. See note (VI.) of this Volume.

(*f*) Knowing the mass of the moon, and also the time in which it revolves on its axis, and likewise the mass of the earth, it is easy to obtain this distance; for if a be the distance of the earth from the moon, x the required distance, m the mass of the earth, and n the angular velocity of the moon, we have $\frac{m}{75.x^2} = \frac{m}{(a-x)^2} + n^2x.$

NOTES TO CHAPTER XI.

(*a*) In notes, page 414, the ratio of the disturbing force of the sun to the force of gravity was determined. Let f, f' represent the disturbing force of the sun on the moon, and on a particle of the terrestrial spheroid, of which the radius is r', r being the distance of the moon from the earth, c being the force which retains the moon in her orbit, and p, P the periods of the earth and moon, we have

$$f : c :: p^2 : \mathrm{P}^2, \text{ and } c : f' :: \frac{r}{p^2} : \frac{r'}{\mathrm{P}^2}, \therefore f : f' :: r : r';$$

and as we have the ratio of f to g, the force of gravity, (see notes page 448,) we can obtain the ratio of f' to g. According to Newton's estimation, this ratio expressed in numbers is that of 1 to 386046000, this gives the value of the additious force in places 90° distant from the sun, the ablatitious force in places to which the sun is vertical, and in their antipodes is twice greater; therefore the sum of the forces is to the force of gravity as 1 : 1286200; and this sum is the whole force which the sun exerts to raise the waters of the sea; for the effect is precisely the same whether the additious force depresses the water at places 90° from the sun, or elevates the water in the places beneath the sun, and in their antipodes. Now, as we have the ratio of c' the centrifugal force to g the force of gravity, and as we have the ratio of $g : 3f'$, we have the ratio of c' to $3f$, namely, that of 1 to 44527; hence, as according to Newton's theory, the centrifugal force makes the height of the water at the equator exceed the height at the pole by 85472 feet, by a proportion we find the height of the water under the sun, and in the opposite regions, 1 foot 11 $\frac{1}{30}$th, of an inch. He determines this elevation somewhat differently in the Systema Mundi, and makes the height = 9.2 inches.

Newton deduces the force of the moon to move the sea from its proportion to that of the sun, which proportion he infers from the proportion of the motions of the sea, which arise from these forces. If L represent the force of the moon in the equator, and at its mean distance from the earth, S that of the sun in the same circumstances; as at the conjunction and opposition, the height of the tide is the sum of S and L, and in the quadratures it is produced by their difference, we have $L+S : L-S :: 45 : 25$, these numbers expressing the mean of the observed heights in syzygies and quadratures. If the sea covered the entire earth, supposed spherical, the figure which it would assume in consequence of the action of each luminary separately, would be that of an *oblong* spheroid, in which the elevation above the equicapacious sphere is double of the depression below this sphere; for in this case the capacity of the spheroid $\frac{4}{3}\pi.ab^2=\frac{4}{3}\pi.r^3$, r being the radius of the equicapacious sphere, then $a-s=r=b+\delta$, we have $\frac{4}{3}\pi.(r+s)(r-\delta)^2=\frac{4}{3}\pi.r^3$, $\therefore$ neglecting the squares and higher powers of δ and s, we have $r^3=r^3+r^2s-2r^2\delta$, $\therefore s=2\delta$, if the spheroid was oblate we would have $\delta=2s$, $\therefore$ in the first case, $r=\dfrac{a+2b}{3}$, in the other, $r=\dfrac{2a+b}{3}$.

And it is evident, from the $\quad\dfrac{b^2}{1-e.^2\cos.^2\lambda}=$

$q.p.\ b.^2(1+e.^2 \cos.\ ^2\lambda)$, that the difference between the elevation at any point λ and the greatest elevation, varies as $\sin.\ ^2\lambda$; and it is easy to show, that the elevation of any point about the greatest depression, varies as $\cos.\ ^2\lambda$, therefore it is easy to show, that the elevation of any point above the equicapacious sphere $= \delta. \left(\cos.\ ^2\lambda - \frac{1}{3}\right)$, and the depression of any point beneath this sphere $= \delta. \left(\sin.\ ^2\lambda - \frac{2}{3}\right)$. See page 382.

(*b*) Let S represent, as before, the mass of the sun, a its distance from the earth, r the radius of the terrestrial spheroid, and ϕ the distance of any place from the point where the sun is vertical, it is evident, from notes, page 410, that a particle of matter at that place is drawn towards the moon by a force $= \frac{3\,Sr}{a^3}.\cos.\ \phi$, and besides, its gravity towards the earth is increased by another force $= \frac{Sr}{a^3}$; and since, in the hemisphere opposite to the sun, ϕ is $> \frac{\pi}{2}$, π being the semicircumference, and therefore $\cos.\ \phi$ is negative; the effect of the force is to draw the particle from the earth in a direction opposite to that which it has in the other hemisphere. As $\frac{Sr}{a^3}$, is nearly always the same, it does not disturb the equilibrium of the waters. In order to obtain the whole force by which the action of the sun diminishes the gravity of a molecule, we should resolve $\frac{3\,Sr.\ \cos.\ \phi}{a^3}$ into a force in the direction of the radius vector, and another at right angles to the radius, then the whole force $=$

$\frac{3\,Sr.\ \cos.\ ^2\phi - Sr}{a^3}$, and the other force draws the particles tangentially or horizontally, and it is $= \frac{3Sr}{a^3} \cdot \sin.\phi.\cos.\phi$; if a' ϕ', represent corresponding quantities for the moon, its force in the direction of the radius $= \frac{3Lr}{a'^3} \cdot (\cos.^2\phi' - Lr)$. Newton does not take into account the effect of $\frac{3}{2}Lr^2.\sin.^2\phi$; its effect is to increase the previous quantities.

(*c*) It follows from this construction, that near to high and low water the difference of the depths from those of high and low water, are as the squares of the times since high or low water.

(*d*) Besides, the value of $\cos.^2\phi$ relatively to different parts of the same sea, must be considerably different, in order that an oscillation may be produced; for the disturbance of the equilibrium of the waters of the sea is only produced by the inequality of action on different parts of the mass of waters; and this, combined with what is stated in page 143, explains why the tides of the Caspian and other inland seas are so inconsiderable.

(*e*) If L, S represent the actions of the moon and sun, or the difference between the respective semiaxes of the ellipsoids mentioned in the text, it is evident that in syzygies the total rise of the water arises from L+S, and in the quadratures this height is produced by L—S, for the height is regulated by the situation of the moon. From this it follows, that L is more than twice as great as S; indeed, it was observed by Newton, as stated in page 486, that L : S :: 7 : 2; or more accurate observations since his time, make

$$L+S : L-S :: 2 : 1, \text{ and } \therefore L : S :: 3 : 1.$$

Hence we can obtain the mass of the moon, for $\frac{S}{a^3}$ $\frac{L}{a'^3}$, are the forces of the sun and moon to move the waters;

therefore $\frac{S}{a^3} : \frac{L}{a'^3} :: 1 : 3$, and consequently $L = \frac{3Sa'^3}{a^3}$,

L in this way is found to be $\frac{1}{75.5}$ of the mass of the earth. In a given distance A of the sun from the moon, it is easy to determine the point where the elevation produced by the combined action of these luminaries is a maximum; for in that case we have

$$\delta.\left(\cos.^2\lambda - \frac{1}{3}\right) + \delta'.\left(\cos.^2\lambda' - \frac{1}{3}\right),$$

a maximum, (see note page 486;) and consequently $\delta.\, d\lambda.\, \sin.\, 2\lambda + \delta'.d\lambda'.\, \sin.\, 2\lambda' = 0$, but as $\lambda + \lambda' = A$, $d\lambda = -d\lambda'$, and $\therefore \delta.\, \sin.\, 2\lambda = \delta' \sin.\, 2\lambda'$; hence, if twice A be divided into two parts, such that the ratio of the sines may be given, half of each part will give the distances of S and L from the high water.

(*f*) If the harbour be not in the equator, it follows, from the expression for the elevation above the equicapacious sphere in page 487, that the difference of the semiaxes must be multiplied by $\cos.^2\lambda$.

It is evident, that as the place of high water coincides with the moon in the syzygies, and in the following quadrature, and is always between her place and that of the sun, that it must for some time be gradually left behind, and afterwards overtake the moon. To determine when the separation of the moon from the place of high water is a maximum, call x the distance between L and S, and y, the distance of the moon from the place of high water, then $\sin.\, 2y : \sin.\, (2x - 2y) :: \delta' : \delta$, therefore we have $\tan.\, 2y = \frac{\delta' (\sin.\, 2x)}{\delta + \delta' \cos.\, 2x}$, and $\therefore$ is a maximum when $2x = 90$, *i. e.* when $x = 45$, therefore in this case $2y$ is a maximum, and $dy = 0$, *i. e.* the motion of high water, or its separation from the sun to the eastward, is equal to the moon's easterly motion, *i. e.* in the octants the tide day is equal to a lunar day; and as the height of the lunar tide is proportional to $Sr. \cos.^2 \phi$, its momentary diminution is proportional to $\sin. \phi \cos. \phi$, or to $\sin. 2\phi$.

(*g*) According to Newton's theory, which we are at present assuming to be correct, the water at every instant assumes the figure of an oblong spheroid, of which the greater axis is directed to the luminary, when a luminary has north declination, the duration and magnitude of the superior tide will be greater than the duration and magnitude of the inferior tide; if the declination of the luminary was equal to the colatitude of the place, there would be only one tide in the day. For places in the equator, whatever be the place of the luminaries, the superior and inferior tides of the same day are the same, though from one day to another they differ, their value is $L. \cos.^2 d$, d being the declination, at the pole there is no *daily* tide, but there is a gradual subsidence and rising by the moon's declining from the equator.

(*h*) The manner in which Laplace estimates the velocity of the propagation of gravity, is as follows: he supposes a force which, like light, though acting in a contrary direction, rushes towards the sun with an immense rapidity; the resistance which the planet experiences from this current in the direction of the tangent, he conceived to produce a perturbation in the elliptic motion, like to the aberration of light. Calling v the velocity of the gravific fluid which acts in the direction of r a radius vector drawn towards the sun, if δs represent the arc described by a planet in an inconceivably short interval of time, then the planet will be actuated by two forces in the direction of r and δs, which are respectively as v and $\frac{\delta s}{\delta t}$, the force in the direction of r being $\frac{M}{r^2}$, the resistance in the direction of the tangent $= \frac{M \delta s}{r.^2 v. \delta t}$, and if this force be resolved into two, one in the direction of r, and the other perpendicular to r, they will be respectively $\frac{M\,\delta r}{r^2 v.\delta t}$ and $\frac{\delta\phi}{r.v.\delta t}$, for the arc perpendicular to $r = r\delta\phi$, ϕ being the angle which r makes with the axis of x; therefore the entire force directed towards the centre $= \frac{M}{r^2}.\left(1+\frac{\delta r}{v.\delta t}\right)$; if this force be resolved into two, parallel to x and y, they will be $= \frac{M}{r^2}.\sin.\phi\left(1+\frac{\delta r}{v.\delta t}\right)$; $\frac{M}{r^2}.\cos.\phi\left(1+\frac{\delta r}{v.\delta t}\right)$, and the force $\frac{M\delta\phi}{r.v.\delta t}$, resolved parallel to x and y, gives

$\frac{M\delta\phi.\cos.\phi}{rv.\delta t}$; $\frac{M\delta\phi.\sin.\phi}{r.v.\delta t}$; therefore the entire force in the directions of x and y, are

$$\frac{M}{r^2}\left(\cos.\phi+\frac{\delta r.\cos.\phi - r\delta\phi.\sin.\phi}{v\delta t}\right)$$

$$\frac{M}{r^2}\left(\sin.\phi+\frac{\delta r.\sin.\phi + r\delta\phi.\cos.\phi}{v\delta t}\right);$$

therefore, by means of the equations in 273, we obtain

$$2\delta r.\delta\phi + r\delta^2\phi = -\frac{2gM}{rv}\delta t\delta\phi;\ \delta r^2 - r\delta\phi^2 =$$

$-\frac{2gM}{r^2}\delta t^2\left(1+\frac{\delta r}{v\delta t}\right)$; if v be supposed to be infinite, these equations would give those of elliptic motion; multiplying by r and integrating, we obtain $r^2\delta\phi = A\delta t - 2gM\delta t\int\frac{\delta\phi}{r}$, if v is not infinite it is probably a function of r, however, relatively to the small variations of distance for each planet, we may assume it as constant, particularly as the integral $\int\frac{\delta\phi}{r}$ is extremely small, $\therefore \int\frac{\delta\phi}{v} = \frac{\phi}{v}$, and consequently $\delta\phi = \frac{A\delta t}{r^2} - \frac{2gM\phi\delta t}{r^2 v}$, $\therefore$ squaring and neglecting $\frac{1}{v^2}$, we obtain $\frac{r\delta\phi^2}{\delta t^2} = \frac{A^2}{r^3} - \frac{4gAM\phi}{r^3 v}$, which being substituted in the second of the foregoing equations, will give $0 = \frac{\delta^2 r}{\delta t^2} - \frac{A^2}{r^3} + \frac{4gAM\phi}{r^3 v} + \frac{2gM}{r^2}$, $\frac{\delta r}{v\delta t}$ being neglected as inconsiderable; making $\phi = nt + a\zeta, r = a(1+au)$, nt will be the mean motion, a the mean distance, and $a\zeta$, au very small numbers, $nT=2\pi$, T being the time of a revolution, and as $\pi = \frac{AT}{2a^2}$, $n = \frac{A}{a^2}$; thus the preceding equations will become $q.p$

$$0 = n + \frac{a\delta\zeta}{\delta t} - \frac{A}{a^2}(1-2au) + \frac{2gMnt}{a^2 v},$$

$$0 = \frac{\delta^2 u}{\delta t^2} - \frac{A^2}{aa^4}(1-3au) + \frac{4g.A.Mnt}{a.a^4 v} + \frac{2gM}{a.a^3}(1-2au), \text{ or}$$

by substituting $A^2 = 2gMa$,

$$0 = \frac{\delta\zeta}{\delta t} + 2nu + \frac{n^3 at}{av}; \quad 0 = \frac{\delta^2 u}{\delta t^2} + n^2 u + \frac{2n^4 at}{av};$$

the form of the integral of the second of these equations is $u = D. \cos. (\beta t + b) + Et$, $\therefore$ as $r = a\ (1+au)$, aD is the eccentricity, and $\beta t + b$ is the anomaly of the ellipse; if the epoch from which we reckon is the time of passing through perihelion, b must be $= 0$; hence if γ' denote the eccentricity, $u = \frac{\gamma'}{a} \cos. \beta t + Et$, and $\therefore \frac{\delta^2 u}{\delta t^2} = -\frac{\gamma.}{a}$ $\beta^2. \cos. \beta t$, and $0 = \frac{n^2 - \beta^2}{a} \gamma \cos. \beta t + n^2 t \left(E + \frac{2n^2 a}{av}\right)$, $\therefore$ $\beta = n$; $E = -\frac{2n^2 a}{av}$, and $u = \frac{\gamma}{a}. \cos. nt - \frac{2n^2}{v}. t$, and by substituting we obtain

$$\delta\zeta = \frac{3n^2}{v} t\delta t - \frac{2n}{a}\gamma\, \delta t. \cos. nt, \text{ and } \zeta = \frac{3n^3}{2v}. t^2 - \frac{2\gamma}{a}. \sin. nt,$$

and consequently $r = a\left(1 + \gamma.\cos.\beta t - \frac{2n^2a}{v}t\right)$, and $\phi =$ $nt - 2\gamma\sin.nt + \frac{3n^2a}{2v}t^2$; nt being the mean motion, $2\gamma.$ $\sin.nt$ the first term of the equation of the centre, and $\frac{3n^2a}{2v}t^2$ the secular equation proportional to the square of the time, which would appear to explain the secular equation of the moon. See page 63. If the secular equation of the moon was known, on the hypothesis *that it arose from this cause*, we could determine v, for if i be the number of months in the time t, then $nt = 2\pi i$, and the secular equation $= \frac{6na\pi^2 i^2}{v}$; in 2000 years $i = 2000.\frac{525969}{39343}$, $\therefore \zeta$ the secular equation for 2000 years $=$

$$\frac{6na\pi^2(2000).^2(525969)^2}{(39343)^2 v};$$ but if this ζ is $= 1°$, then $v =$

$$\frac{6n}{1°}.\pi^2.\ 4000000.\left(\frac{525969}{39343}\right)^2,$$ now $n = 32'', 94$ if the time is expressed in seconds, and $a = 0{,}0025138\,a$, a being the distance of the sun from the earth; therefore

$$\frac{6n}{1°} = \frac{32{,}94}{600} = 0{,}0549,$$ and for one minute,

$$v = \pi^2 549.1{,}00552.\left(\frac{525969}{39343}\right)^2 a = 973753\,a,$$ *i.e.* in a minute of time the gravific fluid passes over one million of the earth's semidiameters; and its velocity is 8000000 greater than that of light, as the secular equations of the different planets, $\zeta = \frac{6na\pi^2 i^2}{v}$ vary as nai^2 or as n^3a, we can find the secular equation for all planets, knowing that of any one.

(i) From the characteristic property of fluids, namely, the perfect mobility of its particles, it follows, that when a fluid mass is in equilibrio, each of its particles must likewise be in equilibrio, in consequence of the forces which solicit it. This is the general principle which Laplace applies to determine the relation which must exist between the forces which solicit the system when this condition is satisfied; and in determining the figure of the earth, he applies it to determine the equilibrium of a homogeneous fluid mass spread over a solid nucleus of any figure whatever. In the theory of the tides he introduces into the differential equations of the motion of fluids, the forces which disturb the equilibrium, namely, the attraction of the sun and moon; and secondly the attraction of the aqueous stratum, of which the interior radius is that of the spheroid of equilibrium, and the exterior that of the disturbed spheroid. The integrations of these differential equations present almost insuperable difficulties, even in the case in which the depth of the sea is assumed to be a function of the latitude; for even then the determination of the radius of the troubled spheroid would lead to a linear differential equation which cannot be integrated; however, the integration of this equation is not necessary, it is sufficient if we are able to satisfy it, for that part of the oscillations which depend on the primitive state of the sea must disappear very soon, from the action of exterior obstacles, so that, as without the action of the sun and moon, the sea would long since have attained a permanent state of equilibrium, it is only the action of these two stars which causes them to deviate from this state, and therefore it is solely necessary to consider the oscillations which depend on this action, now if the terms which produce these be developed, the part of the action of the star which disturbs the fluid molecule, is (neglecting the fourth powers of $\frac{1}{r}$, See page 166.)

$$\frac{3L}{2r^3}\left((\cos.\theta.\sin.v+\sin.\theta.\cos.v.)\cos.(nt+\varpi-\psi)^2-\frac{1}{3}\right),$$

where r is the distance of the attracting body from the centre of the earth, v its declination, ψ it right ascension, nt the rotatory motion of the earth, and ϖ the angle which a plane passing through x and r makes with the plane x, y, the preceding expression is equivalent to the three following:

$$\frac{L}{4r^3}\left(\sin.^2 v - \frac{1}{2}\cos.^2 v\right).(1+3\cos. 2\,\theta)$$

$$+\frac{3L}{r^3}.\sin.\theta.\cos.\theta.\sin. v.\cos. v.\cos.(nt+\varpi-\psi)$$

$$+\frac{3}{4}.\frac{L}{r^3}\sin.^2\theta.\cos.^2 v.\cos. 2.(nt+\varpi-\psi).$$

Now, (as has been remarked,) the only oscillations which it is necessary to consider, are those which depend on the action of the sun and moon, for those which depend on the primitive state of the sea, must long since have disappeared, from the resistance which the waters of the sea have experienced in their motion. As r v and ψ vary with extreme slowness relatively to nt, the three preceding terms give rise to three different species of oscillations. The periods of the oscillation of the first species are very long; they are independent of the rotatory motion of the earth, and depend solely on the motion of L in its orbit. The periods of the oscillation of the second species depend principally on nt, the motion of rotation of the earth, their duration is very nearly a day. Finally, the periods of the oscillations of the third species, depend principally on $2nt$, their duration is about half a day. As the resulting equation which determines the oscillation of the sea is linear, it follows, from what is stated in page 286, Volume I., that these oscillations mix, without interfering with each other, therefore we may consider each separately.

With respect to the oscillations of the first species, they can be obtained in an approximate manner, if the spheroid covered by the sea is an ellipsoid of revolution, in which case the depth of the sea must be a function of the latitude. The part of these oscillations which depends on the motion of the nodes of the lunar orbit, may be very considerable; however, in consequence of the resistance which the waters of the ocean experience, the oscillations of this species are very much diminished, and their extent becomes very inconsiderable, so that, in virtue of these resistances, the oscillations are very nearly the same as if the sea should be in equilibrio under the attracting star.

With respect to the oscillations of the second species, they can be determined when the depth of the sea is supposed to be very nearly constant. The difference of the two tides of the same day depends on these oscillations; now it appears from observations that this difference is very small, and as it would seem to follow from the expression for the difference, that the height of the superior tides is >r than that of the inferior, the depth of the sea is greater near to the poles than at the equator; but this depends on an hypothesis which we know not to be true, namely, that the sea is spread over the entire earth.

With respect to the oscillations of the third species, these also are easily determined, if the depth of the sea be supposed every where the same; according as the depth is increased, these oscillations approach to what they would be if the sea was in equilibrio under the attracting body.

In reference to what is stated in page 154, as to the effect of local circumstances, he shows, in the Thirteenth Book, that in consequence of the rotation of the earth, and of these local circumstances, the daily tide is reduced very nearly to a third, while the semidiurnal tide becomes at least sixteen times greater; however, when it is considered that the rotation of the earth destroys, in a sea of a *uniform* depth, the daily tide altogether, and likewise that if the depth of the sea was $\frac{1}{720}$th of the earth's radius, the height of the semidiurnal sea in the syzygies would be 11 metres, we should not be surprised at these results. See note (x) and Chapter XII.

It is easy to show, that in the hypothesis of a great depth, the two tides of the same day would be very different at Brest, if the declinations of the sun and moon were considerable; in fact, one tide would be eight times greater than the other; but according to observation they are very nearly equal, therefore the hypothesis of a great depth of sea is inadmissible.

Laplace proves, that the value of the quantity by which the sea is elevated, in consequence of the action of extraneous attracting bodies, ceases to be periodic, (which is a condition necessary in order to insure an equilibrium,) when the density of the sea surpasses that of the nucleus over which it is spread; when the contrary is the case, the equilibrium is stable, whatever may be the original agitation; but if otherwise, the stability of the equilibrium depends on the original disturbance.

He likewise shows, from the relations which exist between the depth of the sea and the oscillations of the second species, that these oscillations must disappear for the entire earth when the depth of the sea is constant; but no admissible law of the depth of the sea can render the oscillations of the third species equal to nothing for the whole earth.

In some harbours the oscillations of the second species may be insensible, while in others the oscillations of the third species can hardly be recognized.

The reason why the principle stated in page 155 is applicable to the tides, is, that the forces become the same after the interval of half a day.

This principle being combined with that of the coexistence of very small oscillations already adverted to, enables us to obtain an expression for the height of the tides, of which the arbitrary quantities comprise the effect of the local circumstances of the port and each harbour; for this purpose Laplace reduced into a series of sines and cosines of angles, increasing proportionally to the time, the expression of the solar and lunar forces. He considers each term of the series as representing the action of a particular star, which moves uniformly at a constant distance in the plane of the equator; hence arise several species of partial tides, of which the periods are nearly half a day, an entire day, half a year, an entire year, eighteen years and a half.

When the sun and moon do not move in the plane of the equator, then the effect produced may be conceived to be made up of the action of *several* stars respectively moving in the plane of the equator at different distances and at different periods; and the total tide due to the action of the sun is the combination of the partial tides due to the action of each of those stars.

(*l*) Each observation has for its analytical expression a function of the elements which we want to determine, and if these elements are very nearly known, this function becomes a linear function of their corrections. By putting it equal to an observation, we form what is called an equation of condition; and if there be a considerable number of like observations, they are combined so as to form as many final equations as there are elements; and then, by resolving these equations, we determine the corrections of the elements. The artifice consists in combining the equations of condition in the most advantageous manner; for this purpose, it is to be observed, that the formation of a final equation by means of equations of condition, is effected by multiplying each of them by an indeterminate factor, and then combining these products; but it is necessary to select the system of factors which gives the smallest error; now it is evident, that if we multiply each error of which an element determined by a system is still susceptible by the probability of this error, *the most advantageous system* is that in which the sum of these products, taken positively, is a *minimum;* for a positive or negative error may be considered as a loss. Therefore, by forming this sum of products, the condition of the *minimum* will determine the most advantageous system of factors, and the *minimum* of error to be apprehended on each element. In the analytic theory of probabilities, Laplace shows that this system is that of the coefficients of the elements in each equation of condition, so that a first final equation is formed by multiplying respectively each equation of condition by the coefficient of its first element, and then combining all these equations thus multiplied. A second final equation is formed by employing the coefficients of the second element, and so on. In the same work he gives the expression of the minimum of error, whatever may be the *number* of the elements. This *minimum* gives the probability of the errors of which the corrections of these elements are still susceptible, and which is proportional to the number of which the hyperbolic logarithm is unity, raised to a power of which the exponant is the square of the errors taken negatively, and divided by the square of the minimum of the error, multiplied by 2π. The coefficient of the negative square of the error may therefore be considered as the modulus of the probability of errors, since the error remaining the same, the probability decreases with rapidity, when it increases, so that the result obtained *inclines* towards truth so much the more as the modulus is greater. Laplace, for this reason, terms this modulus *the weight of the result;* and by a remarkable analogy of those weights with those of bodies, referred to their common centre of gravity, it happens that if the *same* element is furnished by different compound systems, each consisting of a great number of observations, the most advantageous mean result of them all taken together is the sum of the products of each partial result by its weight, this sum being divided by the sum of all the weights; moreover, the total weight of different systems is the sum of their partial weights, so that the probability of the errors of the mean result of their aggregate sum is proportional to the number of which the hyperbolic logarithm is unity, raised to a power of which the exponent is the square of the error taken negatively, and multiplied by the sum of the weights. Indeed, each weight depends on the law of probability of the errors in each system, and almost always this law is unknown; but Laplace fortunately succeeded in eliminating the factor which contains it, by means of the sum of the squares of the deviations of the observations of the system from their mean result. It were therefore desirable, in order to perfect our information on the results obtained from a collection of a great number of observations, that at the side of each result the weight which corresponds to it should be written. In order to facilitate the computation, Laplace developed the analytical expression when there were only four elements to determine. But as the number of elements increases, this expression becomes more and more complicated. He gives a very simple means of determining the weight of a result, whatever be the number of elements, and then a regular process of arriving at our object is preferable to the employment of analytical formula. When by this means the exponential which represents the law of the probability of the errors of the result is obtained, the integral of the product of this exponential by the differential of the error, being taken within definite limits, will give the probability that the error of the result is comprised within those limits, by multiplying it by the square root of the weight of the result divided by 2π. See Celestial Mechanics, page 82, and Volume I., page 473.

(m) Observation agrees with theory in making the diminution of the total tide, reckoning from the maximum, to be proportional to the square of the times. Likewise the solstitial tides are less than those in equinoxes in the proportion of the square of the cosine of declination to radius, which is exactly the proportion between them which can be inferred from theory. In like manner, agreeable to the formula in page 494, the variations of distance must have some influence on the height and retardation of the tides, in which there is also a perfect conformability between theory and observation.

The ratio of the action of the moon to that of the sun can be determined either from the syzygial heights compared with the heights in quadratures, or from the variation of retardation in syzygies and quadrature, or from the actual diminution of the heights in these positions of the sun and moon.

If e be the proportion of the mass of the moon, divided by the cube of its mean distance from the earth to the mass of the sun, divided by the cube of its mean distance from the earth, it is $q.p=3$.

The proportion of the solar to the lunar action is ⅓d in the harbour of Brest; but it would be nothing at a harbour constructed at the extremity of two canals, whose embouchures being near each other, are so situated that the solar tide employs a day and a half to arrive by one canal at the harbour, and only a quarter of a day to arrive by the other. The low water of the second corresponds to the high water of the first canal; therefore, if, at the common termination of the two canals, the tides are of equal height, the sea, as far as the action of the sun is concerned, will then be stationary; but as the lunar day surpasses the solar, the low lunar tide of one canal does not correspond with the high lunar tide of the other, so that at their common extremity they will not destroy each other.

The number of observations from which Laplace deduced the ratio of the heights in solstitial syzygies to those in the syzygies of quadratures, in the Fourth Book of the Celestial Mechanics, was twenty-four, made respectively in the quadratures and syzygies of these luminaries, whereas the number from which he deduced the corresponding proportions in 1820, were 128 in each; therefore a greater degree of accuracy was to be expected from the last; however, an inspection of the results from ancient and modern observation, shows that there is a perfect conformity between them.

(*o*) Suppose a canal communicating by means of its two extremities with the ocean, the tide in any harbour situated on the banks of this canal will be the result of undulations transmitted by its two *embouchures*, but its situation may be such, and the undulations of the tides may arrive at it at such different times, that the *maximum* of the one may coincide with the minimum of the other; and if they are equal, it is evident that, in consequence of these undulations, there is no tide in this harbour, but there will be a tide produced by the oscillations of second species, of which, as the period is twice as long, will not so correspond that the maximum of those which arrive by one embouchure may correspond with the minimum of those which come by the other. In this case there will be no tide on the day when the sun and moon are in the plane of the equator, but when the moon has declination, there will be only one tide in the lunar day, so that, if the high water is at the rising, the low water will happen at the setting of the sun, and *vice versa*. See Princip. Math., Vol. III. Prop. 24.

(*p*) See notes, page 489.

If, as is stated in page 166, there is any tide depending on the fourth power of the distance of the moon from the earth, it would be evinced in the difference between the action of the moon, in the new compared with its action in full moon, and between its action in the northern and southern quadratures; and it is certain, from the theory of probabilities, that the increased number of observations can supply their want of accuracy, so that, by means of them, we can appreciate inequalities much less than the errors of which they are susceptible. The differences above-mentioned ought to be sensibly indicated in the numerous observations of the height of the tides discussed by Bouvard. The terms divided by the cube of the distance, which are the only ones hitherto considered, do not indicate any difference between the lunar tides of full and new moon: but a comparison of a great number of observations proves, that the terms divided by the fourth power of the distance indicate an excess of the full moon tides over those of the new moon, both in the equinoxes and also in the solstices; and, conformably to theory, the excess is greater in the equinox than in the solstices.

Bouvard having separated, in the computation of the solstitial syzygies, the tides in which the declination of the moon was southern, from the tides in which the declination of the moon was northern, found, from taking the sum of a great number of each, that the action of the southern moon on the sea exceeded the action of the northern moon.

Newton thus accounts for this phenomenon: there are two inlets to this port; and if, through one of those inlets, a tide arrives at Batsha at the third hour after the moon passes the meridian, and through the other, six hours after, if these tides are equal, as one is flowing while the other is ebbing, the water must stagnate, this is the case when the moon is on the equator, but when the moon declines to the north of the equator; the morning tide exceeds the evening, as appears by what is already stated in notes, page 490, so that two greater and two lesser tides arrive at Batsha by turns. The difference of these will produce an ebbing and flowing, which will attain its maximum at the middle, between the two greatest tides, and be lowest at the middle, between the two lowest tides; therefore, at the setting of the moon it is high, and at the rising it is low water; when the moon is at the other side of the water, or the evening exceeds the morning tide, the case is reversed, and it is high water as the moon rises, and low water when she sets.

NOTES TO CHAPTER XII.

(*a*) Besides the oscillations of the atmosphere due to the attractions of the sun and moon, there are also movements excited in it by the variations of the solar heat; but it is impossible to subject these last to analysis. The first mentioned oscillations are given by an analysis similar to that which determines the oscillations of the sea when the depth is uniform.

The oscillations in the atmosphere ought to produce corresponding oscillations in the heights of the barometer; and indeed it is only by means of the variations of the barometer that the existence of the very inconsiderable wind, which is produced by the action of the sun and moon in an atmosphere already considerably agitated by other causes, can be indicated. These barometric observations ought to be made within the tropics, where, as is stated in page 173, the changes arising from irregular causes are fewer; indeed, the gravity of the mercury in the barometer must be affected, however, not so much as the more distant air.

The principle referred to here, is that stated in page 155.

(*b*) See notes to the preceding Chapter, page 501.

(*e*) Since, on the day of the syzygy, the lunar action combines with the greatest diurnal variation, and on the day of quadrature, it is greatest when the diurnal variation is least, the difference of these heights must be evidently equal to twice the lunar action, and therefore equal twice the height of the atmospheric lunar tide.

The diurnal variation which has been observed being regulated by the *solar* day, indicates evidently that this variation is due to the action of the sun; however, when we consider the smallness of the effects due to the *combined* attractions of the sun and moon, the attractive force of the sun *alone* must be considered as almost insensible, therefore it must be by the *action of heat*, that the sun produces the daily variation of the barometer. It is, however, as has been already remarked, impossible to submit to analysis the effects of this action on the height of the barometer; it is principally apparent at the equator; however, notwithstanding the inconstancy of our climates, it is also indicated, though less sensibly, to observations without the tropics; besides the maximum and minimum mentioned in the text, there is a second maximum at eleven o'clock, P. M., and a second minimum at four o'clock, A. M. See *Essai Philosophique sur les Probabilities*, page 123, *5me* edition.

(*d*) By comparing the heights at nine A. M., with those of the *same days*, at three P. M., he found that its mean value for each month remained constantly positive for each of seventy-two months, reckoning from the 1st of January 1817 to the 1st of January 1823, its mean value in these seventy-two months is very nearly $\frac{8}{10}$ of a millimeter, which is much less than at the equator; it is remarkable that the mean result of the diurnal variations of the barometer from nine A. M., to three P. M., is only 0,5428 for the three months of November, December, and January, and that it increases to 1^{m},0563 for the three following months; nothing similar to this occurs in the following six months.

(*e*) As there is a *calorific* quality accompanying the *colorific* action of light in the spectrum, so in every modification of the rays of light a calorific quality is a concomitant. Its existence is clearly established by means of the photometer, an instrument which is contrived to point out the power of illumination by the slight elevation of temperature which it occasions. It consists of a differential thermometer, having one of its balls *diaphonous*, and the other blown of a deep black enamel, and when the light incident on the two balls is of the same intensity, the temperature of the black ball will rise more than that of the other, owing to its absorbing a greater number of calorific rays; and *vice versa*, if the two balls were precisely the same, it is evident that the one which was most *illuminated* would be that whose temperature would be most *increased*.

(*f*) Before he applied the calculus of probabilities to this phenomenon, he determined the law of the probability of the anomalies of the diurnal variation, which may arise from chance, and then, by applying it to the observations of this phenomenon, he found that there was more than 300,000 to 1 that it was produced by a regular cause. The following is the outline of the method for determining the probability of the mean error of a great number of values of the diurnal variation: let n denote a great number of values of the diurnal variation of the barometer, the sum of them all divided by n gives the mean value; if e denotes the sum of the squares of the differences of this mean value from each of their values, and u the mean error of a great number s of values of the diurnal variation, the probability of u will be proportional to $c^{-s.\frac{n}{2e}.u^2}$, as in this case, $n=1584$, and $\therefore e = 5473{,}98$, and $\therefore \frac{n}{2e} =$ 0,144685, and if s expresses the number of diurnal variations near the syzygies, we have $s=792$, and the probability of the mean error u will be proportional to $e^{-114{,}59u^2}$, and the probability of a mean error u' near the quadratures, is proportional to $e^{-114{,}59u'^2}$; $\therefore$ if z denotes the excess of u' over u, by the method of the work already cited, the probability of z will be proportional to $-114{,}59.\frac{z^2}{2}$.

It is to be observed here, that the observations employed by Laplace, are taken without any reference to the time of year, therefore the partial lunar tides which would depend on the declinations of the moon and on its parallax, disappear in the collection of these observations. The analytic expression for the lunar tide, like to that for the sea, is expressed by the formula

$$R. \cos. \{2nt+2\bar{\omega}-2mt-2(m't-mt)-2\lambda'\}$$

R depends on the action of the moon on the atmosphere, whether direct or transmitted by the sea, mt $m't$ represent the mean motions of the sun and moon, nt the rotation of the earth, $\bar{\omega}$ the longitude of the place, $nt+\bar{\omega}-mt$ is the horary angle of the sun, and λ' is an indeterminate constant quantity.

The combined action of the sun and moon must cause a tendency in the air as well as the ocean to move westward; however, as the rate is, according to Laplace, only *four miles* during each revolution of the earth on its axis, it is evidently too small to be subjected to observation.

(*h*) At the parallel of 25°, the mean temperature is 4° of the centigrade thermometer lower than at the equator. This difference of heat may be supposed to graduate through the atmosphere to the height of 10,000 feet; therefore the expansion of air at the equator, which draws to it a meridional wind, will amount to a column of 100 feet. The velocity of the current thence produced, must be $8.\sqrt{100}$, or 80 feet in a second, *i. e.* 54 miles in an hour; but as the velocity of a point in a parallel of 24° is seven miles an hour faster than on the parallel of 25°, when the wind arrives at the parallel of 24, it will seem to a spectator to have acquired a tendency of seven miles an hour to the west; at its arrival at the parallels of 23°, 22°, 21°, &c. it will gain continual though decreasing additions to its apparently westerly course, which, at the equator, will be increased to 104 miles in an hour. The same obtains also for the southern hemisphere; however, as the mean temperature for a given latitude is greater on the northern than on the southern side of the equator, inasmuch as a larger land surface is presented to the action of the solar rays in the northern hemisphere, the mean path of the easterly current of the air is 3° to the north of the equator. It is also to be observed, that the sun is not always vertical to the same place; therefore, though the hottest region for the entire year is 3° north of the equator, still its position must in some measure be dependent on the seasons. In the summer months it shifts towards the tropic of Cancer; during winter the hottest parallel passes to the other side of the equator; hence, in the progress of summer the trade wind oscillates about a point towards the north, and it declines towards the south with the advance of winter. But the trade winds experience a much more considerable modification, arising from the circumstance of the sun acting more powerfully upon the land within the torrid zones than upon the water; hence, when he moves towards the northern hemisphere great heat is communicated to the desarts of Africa, the consequence of this greater heat acquired in the sands of these desarts than in the seas which lie to the east and north east of them, is a rarefaction in the columns of air incumbent on them, and therefore a tendency in the adjacent columns which are more moderately heated to flow in and displace the heated air; this changes the direction of the wind. These periodical winds are called monsoons; and on the north side of the equator, in the Arabian and Indian seas, it is north west during the summer months, from April to October, and in the opposite direction, or south east, during the winter months; on the south side of the equator it is the direct contrary, being north east in summer, and south west in winter. In order that the equilibrium between the parts of the atmosphere may be preserved, it is necessary that in the upper regions of the atmosphere there should be a perpetual current towards the poles. As these streams, after they pass the tropics, descend towards the surface, with the celerity due to the equatorial regions, they will appear to blow to the west with the excess of their previous velocity over that of the parallel which they reach. This is the reason why, in places above the latitude of 30° the prevailing wind is westerly, and this is also the reason why westerly winds are generally warm, as coming from a warmer region; and on the same principle the north and east winds are cold, as they originate in regions nearer to the arctic circle.

Jupiter's atmosphere must be much more agitated than ours is by the moon, from the joint attractions of the *four* satellites; however, the effect of the sun's action cannot be so considerable, in consequence of its much greater distance.

NOTES TO CHAPTER XIII.

(*a*) THERE are two cases in which there would be no precession of the equinoxes, namely, first if the earth was a perfect sphere; in which case the solar force, on any particle in the hemisphere turned towards the sun, which is proportional to the distance of the particle from the plane of the circle of light and darkness, is equal and contrary to the force by which similarly situated particles in the opposite hemisphere are drawn; therefore the solar forces in the opposite hemispheres balance each other; see page 407, from which it is evident, that the mean quantity of the solar force is $\frac{3S}{a^3}.r.\cos.\delta$, where S denotes the mass of the sun, a the mean distance of the sun from the earth, r the radius of the equator, and δ the declination of the sun. The part of this force, which is perpendicular to the plane of the ring $= \frac{3S}{a^3}.r.\sin.\delta.\cos.\delta.$; or secondly, if the axis of the earth was always perpendicular to the ecliptic, in which case the action of an external body would be absolutely equal on the two parts of the spheroid above and below the ecliptic, therefore it would not produce any alteration in any of its motions. Since to each of the moons mentioned in the text we can apply what has been stated respecting the lunar orbit, which, in consequence of the solar action, intersects at each of its revolutions the plane of the ecliptic in a point anterior to that in which it met the ecliptic at a previous revolution, which causes the lunar nodes to retrograde; in the very same manner, each point of the ring intersects the ecliptic in a point anterior to that at which it had intersected it twenty-four hours before; and from the action of all these moons on the globe of the earth, there will result every day a small retrogradation or angular motion of the intersection of the equator and ecliptic, which, on account of the rapidity of the earth's revolution, and the greatness of its mass relatively to that of the ring, must be very small; however, as this retrogradation is repeated 365 times in the course of the year, there results at the end of the year a retrograde motion of several seconds, produced by the sole action of the sun. The part of the solar action which is perpendicular to the ring $= \frac{3\,S}{a^3} . r.$ sin. δ. cos. δ. Hence, if $F = \frac{S}{a^2}$ the force with which the sun acts on a particle at the centre of the earth, and if t T represent the times of the diurnal and annual revolutions of the earth, and e the centrifugal force we have

$$F : e :: \frac{a}{T^2} : \frac{r}{t^2} \text{ and } \therefore \; F = \frac{e.t.^2 a}{T.^2 r}$$

and the part of the solar action perpendicular to the plane of the ring

$$= \frac{3t^2}{T^2} \cdot \frac{ea}{r} \cdot \text{sin. } \delta. \text{ cos. } \delta.$$

(*b*) Besides the motion round the line of the nodes which the force $\frac{S}{3\,a^3} . r$ sin. δ. cos. δ has a tendency to produce, the ring in twenty-four hours revolves on an axis perpendicular to its plane; therefore, since these two forces act on it simultaneously, the consequence will be, that the ring will neither revolve on this axis, nor on the line of the nodes, but on an axis which lies in the same plane with each, dividing the angular distance between them in such a manner that the sine of the angular distance between them is inversely as the angular velocity about that axis; for the composition of angular motions follows the general law of the composition of forces; but as the axes are perpendicular to each other, the sine of the angular distance of the new axis from the line of the nodes is equal to the cosine of the angular distance of the first axis of rotation from the second. Hence, if π $\bar{\omega}'$ represent the angles which, in consequence of the earth's rotation and of the solar force, the equator and axis of the earth describe in an indefinitely small portion of time, the axis of the earth will be changed by the simultaneous action of the two forces, by an angle of which the tangent $= \frac{\bar{\omega}}{\pi}$; but these forces are not of the same kind, for that which produces $\bar{\omega}'$ acts incessantly, while the other acts only once; hence it follows, that as the quantity $\bar{\omega}$ is continually renewed, the position of the earth's axis is continually changing. However, though this axis is continually shifting its position, neither the angular velocity of the axis or its inclination would undergo any change, if $\frac{3S}{a^3} \cdot r. \cos. \delta.$ was constant. For if $\bar{\omega}$ as before, represents the angular velocity of a body, and if $\frac{3S}{a^3} r. \cos. \delta$ would generate in $1''$ an angular velocity $= a$, then if $1''$ be divided into n parts, the velocity produced in each of these parts $= \frac{a}{n}$, hence, from what has been just stated, by compounding the angular velocities $\bar{\omega}$ and $\frac{a}{n}$, of which the axes are at *right* angles to each other, the resulting angular velocity $= \sqrt{\bar{\omega}^2 + \frac{a^2}{n^2}}$, compounding this with the angular velocity generated in the second, third, fourth, &c. intervals, the compound angular velocity becomes

$$\sqrt{\bar{\omega}^2+\frac{2a^2}{n^2}}\ \sqrt{\bar{\omega}^2+\frac{3a^2}{n^2}},\ \&c.\ \sqrt{\bar{\omega}_2+\frac{na^2}{n^2}}=$$

$\sqrt{\bar{\omega}^2+\frac{a^2}{n}}=$ (when n is indefinitely increased) $\bar{\omega}$, hence it follows, that when the axis of rotation is *at right angles* to the axis about which $\frac{3S}{a^3}r.$ sin. δ cos. δ has a tendency to produce a motion of rotation, the angular velocity is uniform, when $\frac{3Sr \cos. \delta}{a^3}$ is *a uniform force.* Neither is the inclination to the line of the nodes altered. For suppose this force to generate an angular velocity a in $1''$, if this time be divided into n parts, then $\frac{a}{n}$ will be the velocity generated in each of them; consequently, from what has been just established, it follows, that the tangent of the angle contained between two successive positions of the axes of rotation $=\frac{a}{n\bar{\omega}}$, which, when n is increased indefinitely, is the expression for the arc between them, and since, by what preceds, $\bar{\omega}$ remains constant, at every successive interval, angles $=\frac{a}{n\bar{\omega}}$ will be added to this angle; therefore, at the end of $1''$ the two axes will be inclined at an angle $=\frac{na}{n\bar{\omega}}=\frac{a}{\bar{\omega}}$, and as this obtains for each successive interval $=1''$, the axis of rotation will shift its position with an angular velocity $=\frac{a}{\bar{\omega}}$, and as the angle $\frac{a}{\bar{\omega}}$ is very small, the axis of rotation at the end of $1''$ will deviate from the solstitial colure by an angle which is indefinitely small with respect to $\frac{a}{\bar{\omega}}$, therefore this axis will describe a circle round the pole of the ecliptic, moving *in antecedentia* with an angular velocity equal to that of the line of the equinoxes. This would be the case if $\frac{3S}{a^3}$ r. sin. δ cos. δ was constant, which, however, is not the case, for it is 0 at the equinoxes, besides, as the arc described by the pole is $\perp$ to the plane passing through the sun and the earth's axis, it is not always in the direction of a tangent to the circle whose centre exists in a perpendicular to the plane of the ecliptic; hence, strictly speaking, neither the angular motion of the pole of the equator, nor its inclination to the ecliptic, is invariable, however, the changes are confined within very narrow limits. This is the cause of the solar inequality, of precession, &c.

The decomposition of motion adverted to in page 184, is, in fact, an application of the principle of D'Alembert, explained in page 287, Vol. I.

(*c*) Differentiating the expression $\frac{3t^2}{T^2} \cdot \frac{ca}{r}$. sin. δ cos δ. with respect to t and δ, and then integrating, the precession for the entire year, comes out $= 360. \frac{3t}{2T} \cdot \frac{ca}{r}$. cos. of obliquity, it appears from this expression, that in order to obtain the exact quantity of the precession, we should know the compression of the earth.

(*d*) It appears, from what has been already stated, that (every thing else being the same) the retrogradation is proportional to the cosine of the inclination of the plane of the ring to that in which the external body moves; and as, in the case of the moon, this inclination is continually varying, the precession and inclination of the axis is subject to continual change from the lunar action. It also follows, from this, that the greatest inclination of the ecliptic to the equator is in the new moon of spring, and the full moon of autumn, the moon being at the same time

$$\sqrt{\bar{\omega}^2 + \frac{2a^2}{n^2}}\ \sqrt{\bar{\omega}^2 + \frac{3a^2}{n^2}}, \text{ \&c. } \sqrt{\bar{\omega}^2 + \frac{na^2}{n^2}} =$$

$\sqrt{\bar{\omega}^2 + \frac{a^2}{n}} =$ (when n is indefinitely increased) $\bar{\omega}$, hence it follows, that when the axis of rotation is *at right angles* to the axis about which $\frac{3S}{a^3} r.$ sin. δ cos. δ has a tendency to produce a motion of rotation, the angular velocity is uniform, when $\frac{3Sr \cos. \delta}{a^3}$ is *a uniform force*. Neither is the inclination to the line of the nodes altered. For suppose this force to generate an angular velocity a in $1''$, if this time be divided into n parts, then $\frac{a}{n}$ will be the velocity generated in each of them; consequently, from what has been just established, it follows, that the tangent of the angle contained between two successive positions of the axes of rotation $= \frac{a}{n\bar{\omega}}$, which, when n is increased indefinitely, is the expression for the arc between them, and since, by what preceds, $\bar{\omega}$ remains constant, at every successive interval, angles $= \frac{a}{n\bar{\omega}}$ will be added to this angle; therefore, at the end of $1''$ the two axes will be inclined at an angle $= \frac{na}{n\bar{\omega}} = \frac{a}{\bar{\omega}}$, and as this obtains for each successive interval $= 1''$, the axis of rotation will shift its position with an angular velocity $= \frac{a}{\bar{\omega}}$, and as the angle $\frac{a}{\bar{\omega}}$ is very small, the axis of rotation at the end of $1''$ will deviate from the solstitial colure by an angle which is indefinitely small with respect to $\frac{a}{\bar{\omega}}$, therefore this axis will describe a circle round the pole of the ecliptic, moving *in antecedentia* with an angular velocity equal to that of the line of the equinoxes. This would be the case if $\frac{3S}{a^3} r.$ sin. δ cos. δ was constant, which, however, is not the case, for it is 0 at the equinoxes, besides, as the arc described by the pole is $\perp$ to the plane passing through the sun and the earth's axis, it is not always in the direction of a tangent to the circle whose centre exists in a perpendicular to the plane of the ecliptic; hence, strictly speaking, neither the angular motion of the pole of the equator, nor its inclination to the ecliptic, is invariable, however, the changes are confined within very narrow limits. This is the cause of the solar inequality, of precession, &c.

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It follows, ∴ from the rotation being given by one sole term, compared with what is stated in Vol. I., page 465, that if the earth revolves on a principal axis the rotation is perfectly uniform. Even if the axis of rotation was not a principal one, still the actions of sun and moon would not affect its motion, as appears from what is just stated; but in this case, in consequence of the centrifugal forces, the rotation cannot be uniform. However, as from the observation of a long series of years, no irregularity has been discovered in the rotation of the earth, we must conclude that it revolves about a principal axis, which is confirmed from a consideration of the variations of the obliquity and of the precession which result from it; for if the axis of rotation deviated $1''$ from the principal axis, the obliquity and precession, or what is the same thing, the latitudes and longitudes of the stars would experience, in the course of six months, variations of $2''$ and $5''$, which would be indicated by observations. The uniformity of rotation is likewise proved from the following consideration, namely, that if it was deranged by the actions of the sun and moon, the centrifugal force which depends on the rotation, would experience, in the course of a month and year, variations depending on the different positions of the sun and moon; therefore the gravity which is diminished by the centrifugal force, and consequently the length of a pendulum which vibrates seconds, would be liable to analogous variations; but no change has been observed in the length of a pendulum vibrating seconds under a given latitude.

(*k*) When a body descends from a considerable height, or moves from the equator towards the poles, it brings into its new situation more velocity than it can retain, consequently it must impart some of it to the general mass of the earth; the contrary obtains when a body recedes from the axis of the earth. In general, the momentum of rotathis axis will describe a circle round the pole of the ecliptic, moving *in antecedentia* with an angular velocity equal to that of the line of the equinoxes. This would be the case if $\frac{3S}{a^3} r. \sin. \delta \cos. \delta$ was constant, which, however, is not the case, for it is 0 at the equinoxes, besides, as the arc described by the pole is $\perp$ to the plane passing through the sun and the earth's axis, it is not always in the direction of a tangent to the circle whose centre exists in a perpendicular to the plane of the ecliptic; hence, strictly speaking, neither the angular motion of the pole of the equator, nor its inclination to the ecliptic, is invariable, however, the changes are confined within very narrow limits. This is the cause of the solar inequality, of precession, &c.

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(*d*) It appears, from what has been already stated, that (every thing else being the same) the retrogradation is proportional to the cosine of the inclination of the plane of the ring to that in which the external body moves; and as, in the case of the moon, this inclination is continually varying, the precession and inclination of the axis is subject to continual change from the lunar action. It also follows, from this, that the greatest inclination of the ecliptic to the equator is in the new moon of spring, and the full moon of autumn, the moon being at the same time

in its ascending node; or in the full moon of spring, and new moon of autumn, the moon being then in the descending node. The least obliquity has place in the first and last quarter, at the beginning of summer or winter, the moon being at this time 90° from her node.

(*e*) Strictly speaking, the inequalities produced by the action of the moon are of two kinds, the period of the first being equal to that of the moon in her orbit, and that of the second equal to the time of a revolution of the moon's nodes. Hence it follows, that there are limits within which the variations of the precessional motion and obliquity of the ecliptic are contained; the inclination to the ecliptic returning to its former value in the time of a revolution of the moon's nodes.

(*f*) Subtracting the expression for the lunar precession from the entire annual precession produced by the combined action of the sun and moon, we obtain the ratio of the solar annual precession to that of the lunar; which, as it involves the ratio of the sun's to the moon's mass, enables us to determine the relative proportions of these qualities.

In reference to what is stated in page 189, it is to be remarked, that the cause of D'Alembert's error arose from his supposing that as the molecules of the sea, with which the earth is in a great measure covered, yield to the action of the stars, they could not contribute to the motions of the earth's axis, so that, in computing those motions, he employed the ellipticity of the spheroid, which was covered by the ocean, which ellipticity he supposed to be less than that of the surface of the sea. But Laplace, by subjecting to analysis the oscillations of the fluid spread over the terrestrial spheroid, and also the pressure which it exerts on the surface of the spheroid, proved that this fluid transmits to the terrestrial axis the same motions as if it constituted a solid mass with the earth. He also, by means of the principle of the conservation of areas, showed that the action of the stars on the sea, in whatever manner it was spread over the spheroid, produced on the nutation and precession the same effects as if the sea consolidated itself about the spheroid.

(*g*) The theories of Newton relatively to the figure of the earth and the seas, are those which suppose the earth homogeneous, the sea having the same density as the earth which it covers, and that the waters of the ocean assume every moment the figure in which they would be in equilibrio under the action of the sun.

(*h*) The effect of the action of each of the planets is to induce a motion of the common section of the planes of the two orbits of the earth and planet, while their mutual inclination is not altered; see page 25, Vol. II. In the case of precession and nutation, the variation is in the equator and earth's axis; but in this case the variation is in the ecliptic, to which the axis is referred. Laplace proved, by a careful analysis, that if the earth was perfectly spherical, the variation of the obliquity of the true ecliptic to the equator, which is caused by the attractions of the planets, would be much more considerable than they are, and from the same cause the variation of the length of the tropical year, which would be caused by the sole motion of the ecliptic, is reduced to a fourth of what it would be if the earth was a sphere. However, the *sidereal* year remains invariable.

(*i*) If the right ascension of a star reckoned from the true equinox be converted into time, it will be expressed by two terms, one of which gives the mean rotation, and the other is the correction, which is variable; this would seem to imply that the rotation of the earth was variable. However, as has been remarked, this is only an illusion, for the term which is added to the mean rotation, is independent altogether of the motion of the earth on its axis.

NOTES TO CHAPTER XIV.

(*a*) What is termed the mean axis in the text, is the second principal axis of rotation; and it is shown, in the Celestial Mechanics, Vol. II., page 370, that if the moon was homogeneous, the excess of the first above the second is to the excess of the second above the third, as 40 : 10, *i.e.* 4 : 1. But as the ratio of these axes deduced from substituting numerical values for the terms of the proportion, and of the principal moments referred to them, does not agree with observation, it follows that the moon is not homogeneous: see note (*e*).

(*b*) Laplace obtained the difference between the motion of rotation and revolution of the moon, by the integration of a differential equation of the second order; this quantity is composed entirely of *periodic* terms, and contains two constant arbitrary quantities; therefore, there results from this a libration, of which the extent is also arbitrary. Hence it follows, that the *mean* motion of rotation of the moon is equal to her mean motion of revolution. The difference between the motions of rotation and revolution should be comprised between the greatest and least of the values of which the periodic quantity was susceptible. It is necessary, to secure the stability of equilibrium, that the periodic terms which multiply the time should be real; for if they were imaginary, the arguments which depend on them would be changed into exponentials and arcs of circles susceptible of indefinite increase, or at least the tion of the entire mass of the earth is to the change of the momentum of rotation of the displaced body, as the velocity of diurnal rotation to the variation in that velocity, arising from the motion of the body. In this way, the continual degradation of mountains and alluvial deposits produced by rain, &c. which is incessantly going on, should cause an increase in the length of the day.

In addition to what is stated in page 194, it may be remarked, that there is a general compensation of the effects produced by the current of air from the poles to the equator, (which is the cause of the trade winds,) which tends to diminish the motion of the earth, by a contrary current in the upper regions of the air, which sets in from the equator to the poles.

Some geologists maintain, that the level of the sea was once 15000 feet higher than at present, from which it follows, that a mass equal to the 440th of the whole earth must have been degraded from being above the level of the present sea, to being underneath it; and if the density of water was equal to the mean density of the earth, it would be easy to show that, in consequence of this degradation, the duration of a revolution on the earth's axis must have been deminished by 5′,682; see preceding page. As, however, the mean density is to that of water as 4.71 : 1, this acceleration is reduced to 1.′12″ This change on the surface, or even in the interior of the earth, would also produce great changes in the position of the axis of rotation; it may, if an explosive force existed in the interior of the earth, as was suggested in notes, page 470, have changed by the action of such a force continually its position, and with it that of the earth's equator; and that such a force was formerly in very active operation, appears to be indicated by many facts in the natural history of the earth, and of the mineral kingdom.

given relative to the moments of inertia of the lunar spheroid; and by a comparison of them with those *furnished* by the theory of the figure of this spheroid, it appears that these conditions cannot be satisfied by supposing the moon homogeneous and fluid, nor on the hypothesis that it is originally fluid, and of a variable density; hence it follows that the moon has not the figure which it would have, if it was primitively fluid, consequently it must have been at its origin a hard body of irregular figure, which is confirmed by a consideration of its spots. Newton determines the ratio of the greater to the lesser lunar axis, (on the supposition that the moon is fluid,) from knowing the height to which the sea is elevated by the lunar action; for the force of the earth to raise the lunar fluid is to the corresponding force of the moon to raise the waters of our ocean, in a ratio compounded the accelerating gravity of the moon to the earth to the accelerating gravity of the earth to the moon; and of the diameter of the moon to that of the earth, which by substituting numerical values become the ratio of 1081 to 100; and as the tide by the lunar action alone is raised $8\frac{5}{8}$ feet, the lunar fluid ought to be raised 93 feet, ∴ the major axis should exceed the minor by 186 feet, and as the lunar equator is inclined at a very inconsiderable angle to the plane of its orbit, the effect of the rotation on its axis ought to increase this excess.

(*d*) See notes page 435.

(*e*) The position of this meridian being determined, we are enabled to establish every circumstance connected with the moon's rotation, from a computation of a great number of longitudes and latitudes as seen from the centre of the moon, by means of observations of a spot made at different epochs, it is found that these longitudes and latitudes differ from each other, and vary with the time; hence it follows that the moon revolves on an axis inclined to the ecliptic. As a comparison of the latitudes indicates but inconsiderable changes, the axis of rotation does not differ much from that of the ecliptic, i. e. the lunar equator is inclined at an inconsiderable angle to the ecliptic.

NOTES TO CHAPTER XV.

(*a*) According to what is established in page 290, vol. X, it follows that the common centre of gravity is either in perfect repose, or has a uniform rectilinear motion in space. But there is *only one case* in which the centre would remain in perfect repose, while there is an infinite number in favour of a motion in some one direction or other with some determinate velocity; it is ∴ much more probable that our sun and the fixed stars, which are bodies of the same nature, have a proper motion in space, than that they are absolutely at rest. With respect to the sun, its motion of translation may be, with great probability, inferred from its rotatory motion; it is likewise probable for the stars, as will appear from the following note.

(*b*) Herchell found, that if we suppose the sun to be in motion towards that region of the heavens in which the constellation Hercules is situated, there should arise a separation between several stars situated on that side, while, on the other hand, there would arise a contraction between those which are situated on the opposite side; and he found, that out of forty-two stars which appear to have experienced particular motions, there were upwards of thirty, *part* of whose motions corresponds to what should result from the motion of our sun towards the one, and from the other. He specified that only *part* of their motions arose from this, for as the stars have proper motions of their own in different directions, it is evident that their apparent motion results from their true motion combined with that of the sun. It is evident from the principle of universal gravitation, adverted to in the text, that the stars, which we may consider as the centres of so many different systems, must revolve about some common centre, for otherwise, as they exert attractive forces on each other, they must tend to approach towards each other; and though in consequence of their immense distance, this tendency may be extremely feeble, still as it would be caused by a motion continually accelerated, after a great lapse of time, they would all meet *in the common centre* of gravity. See page 446, vol. 1.

When the proper motion of the star is in an opposite direction to that of the sun, it is in the most favourable circumstances to be be observed, for in that case the apparent motion is = to the sum of these two motions. Suppose, then, that a star = to our sun moved with an = and contrary motion, they will be at the same distance from the centre of our system, and the apparent motion from the sun, considered as immoveable, will be double of the true motion, hence A the arc described in any time = half f, the distance of the star from the sun multiplied into ϕ, the apparent motion of the star in that time, i. e. $A = \frac{f \cdot \phi}{2}$, in 50 years ϕ is observed to be $= 45''$, and $\therefore$ in one year $\phi = 0'',9$ and $\frac{\phi}{2} = 0'',45. = 0,00000218166$, which multiplied by f (= 300000 semidiameters of the earth's orbit) gives A = 0,654, in one year, but an arc of the earth's orbit = 0,654 subtends an angle at the sun = 37°, 30′, which is described by the earth in 38 days $\therefore$ the velocity of the star will be to that of the earth inversely as the times *i. e.* :: 38 : 365 $\therefore$ 1 : 9; now it appears from what is stated in Notes, page 371, that the velocities of

NOTES TO CHAPTER I. BOOK V.

(*b*) In fact, on the supposition that the Zodiac originated in Egypt, and that it was first invented in order to serve as a sort of kalendar to point out the different circumstances of the rural year, then there are two ways of reconciling what is indicated by the signs of the Zodiac with the climate of Egypt and its agriculture, either by making the Zodiac to have originated at a period long anterior to that at which it is at present supposed to commence; or by supposing that the constellations of the Zodiac are named, not from their rising with the sun, or the commencement of the day, but from their setting, or the beginning of night. The former hypothesis would make the world to be created at a time long anterior to that which we know from all history both sacred and profane, and also from contemporary records, it actually was; besides it would assign to the human race a duration longer than what Laplace himself admits it had, see page 50. Likewise, it may be remarked, that in these rude times, when the observations of the stars were made by the naked eye, it is much more likely that the stars were observed at night, when they are easily seen, and not in the day-time, when they are with difficulty discerned; the latter then is the true mode of reconciling the names of these signs with the different circumstances of the year: indeed on the first hypothesis if we consider the

they appear to have described a considerable part of their orbit round the common centre of gravity.

(*e*) From a consideration of the observations of previous astronomers he inferred the position for the year 1800, also the annual motion, the time of revolution, which he thought to be $= 400$, the semiaxis major which he assumed $= 25''$, and the annual parallax $= 0'',46$: it is evident from the formulæ previously established in page 374, that if the axis major and period are known we can obtain the ratio of the sum of their masses to that of the earth.

Note to page 205.

(*a*) It thus appears that the laws of motion and general properties of matter are the same in every part of the universe, and that all are explained by the *one* principle of the mutual gravitation of bodies; it is likewise evident that the existence of this force was not hypothetically assumed, but was deduced as a necessary consequence of the laws of Kepler, combined with the laws of motion.

NOTES TO CHAPTER I. BOOK V.

(*b*) In fact, on the supposition that the Zodiac originated in Egypt, and that it was first invented in order to serve as a sort of kalendar to point out the different circumstances of the rural year, then there are two ways of reconciling what is indicated by the signs of the Zodiac with the climate of Egypt and its agriculture, either by making the Zodiac to have originated at a period long anterior to that at which it is at present supposed to commence; or by supposing that the constellations of the Zodiac are named, not from their rising with the sun, or the commencement of the day, but from their setting, or the beginning of night. The former hypothesis would make the world to be created at a time long anterior to that which we know from all history both sacred and profane, and also from contemporary records, it actually was; besides it would assign to the human race a duration longer than what Laplace himself admits it had, see page 50. Likewise, it may be remarked, that in these rude times, when the observations of the stars were made by the naked eye, it is much more likely that the stars were observed at night, when they are easily seen, and not in the day-time, when they are with difficulty discerned; the latter then is the true mode of reconciling the names of these signs with the different circumstances of the year: indeed on the first hypothesis if we consider the

positions which the signs are observed to have in the Zodiac, their names do not indicate any thing connected with the climate of Egypt; for if Capricornus was originally at the lowest point, then the sign Virgo, representing a gleaner, could not indicate the harvest; for three thousand years ago, the principal star of this sign rose for Memphis, 45 days after the summer solstice, and set about 15 days before the autumnal equinox, during which time Egypt was inundated by the Nile. Besides, Capricornus is represented half goat and half fish, and Aquarius, in the most ancient Zodiac, by a simple urn. The sign Pisces, from their very denomination, can only designate the rainy season and an abundance of waters, and notwithstanding all this, the principal stars belonging to these constellations, rise and set heliacally at the very time Egypt is most dry. But according to the second of the preceding suppositions, there is a striking correspondence between these signs and the different circumstances of the Egyptian year, for then Aries is placed at the autumnal equinox, and Libra at the vernal; Capricornus at the summer solstitial point, then Aquarius, and after them Pisces; the Nile begins to rise in June, or a little before it; now this phenomenon, combined with the motion of the sun, through the highest point of his course, could not be better indicated than by an animal half a fish and half a quadruped, remarkable for seeking always the highest points of the mountains. The months of August and September, during which Egypt is overflowed, could not be better designated than by the Urn and Pisces. To these signs succeeds Aries, symbol of the reviving of nature, which excites animals to reproduction: the Bull, emblematic of labour, which in Egypt begins in November; after this sign comes that of Twins, which is a symbol of the regeneration of all natural productions. The return of the sun or its retrogradation, was thus represented by the Cancer, which is vulgarly supposed to march backwards. The Lion, which in its ascending node; or in the full moon of spring, and new moon of autumn, the moon being then in the descending node. The least obliquity has place in the first and last quarter, at the beginning of summer or winter, the moon being at this time 90° from her node.

(*e*) Strictly speaking, the inequalities produced by the action of the moon are of two kinds, the period of the first being equal to that of the moon in her orbit, and that of the second equal to the time of a revolution of the moon's nodes. Hence it follows, that there are limits within which the variations of the precessional motion and obliquity of the ecliptic are contained; the inclination to the ecliptic returning to its former value in the time of a revolution of the moon's nodes.

(*f*) Subtracting the expression for the lunar precession from the entire annual precession produced by the combined action of the sun and moon, we obtain the ratio of the solar annual precession to that of the lunar; which, as it involves the ratio of the sun's to the moon's mass, enables us to determine the relative proportions of these qualities.

In reference to what is stated in page 189, it is to be remarked, that the cause of D'Alembert's error arose from his supposing that as the molecules of the sea, with which the earth is in a great measure covered, yield to the action of the stars, they could not contribute to the motions of the earth's axis, so that, in computing those motions, he employed the ellipticity of the spheroid, which was covered by the ocean, which ellipticity he supposed to be less than that of the surface of the sea. But Laplace, by subjecting to analysis the oscillations of the fluid spread over the terrestrial spheroid, and also the pressure which it exerts on the surface of the spheroid, proved that this fluid transmits to the terrestrial axis the same motions as if it constituted a solid mass with the earth. He also, by means of the principle of the conservation of areas, series of terms, one consisting of ten and the other of twelve terms; the first of the one are combined with the first of the other, so that as one series has ten terms, and the other twelve, after the first series is exhausted, its first term is combined with the eleventh term of the second series, and the second term of the first series with the twelfth term of the second series, and this goes on until the first term of the first series concurs with the first term of the second series; but this, as is evident from the theory of combinations, does not take place until after sixty different combinations with respect to the days; the first day of each year bears the name of the year, after which we reckon them by the names composed of the sexagenary period which is recommenced whenever it is necessary.

The Luni Solar period of 600 years, to which we adverted in page 68, was invented by the Chaldean astronomers. This supposes a tolerably accurate knowledge of the solar year, and also of a lunation; for in 600 years, each consisting of $365^d\ 5^h\ 51',\ 36''$ there are exactly 7421 lunations, each of which consists of $29^d\ 12^h\ 44'\ 3''$, but if the motions of the sun and moon were the same then as at the present day, at the end of this period there would be a considerable aberration.

(*d*) Such an exact situation of the pyramids could not be the effect of chance; we infer from it that they had accurate means of finding the meridian line, which is extremely difficult to trace accurately, as is evident from the error which Tycho Brache committed, in tracing the meridian line at the observatory of Uraniburgh. According to some historians, the Pyramids were observatories from which the Egyptian priests surveyed the heavens.

(*e*) The rising and inundation of the Nile, an event which excited the attention of all Egypt, was at the commencement of this empire announced by the Heliacal rising of Sirius. It is probable that it was on this acc that they made their years to commence then, whic cording to their estimation of the length of the year, made its commencement continually to retrograde, so that if it commenced for any one year at the summer solstice, four years after it would commence a day sooner, on the hypothesis that the true length of the year exceeded 365^d by the fourth part of a day; in this way the commencement of the year would retrograde continually, and in 1461 years take place at every season of the year, at the end of which time it would recommence at the summer salstice, for $\frac{1461}{4} = 365 + \frac{1}{4}$.

(*d*) *to page* 219.] For $\frac{365}{7} = 52 + \frac{1}{7}$, hence it appears that the last day of the year is of the same denomination as the first, and ∴ if the first day of the week denotes the first year, the second day of the week will represent the second year, and so on.

(*e*) In fact, previously to the time of Thales, who derived all his information on these subjects from the Egyptian priests, their astronomy consisted only in having given denominations to some constellations, and in having noted the heliacal rising and setting of certain stars. This is all which is furnished by Hesiod and Homer, their most ancient writers. Thales, at his return from Egypt, made them acquainted with some of the important astronomical truths known to the ancients.

NOTES TO CHAPTER II.

This constancy of the inclination of the lunar orbit to the plane of the ecliptic, adverted to in page 238, was remarked by Kepler at the conclusion of his Epitome of the Copernican Astronomy; but the reason which he assigned for it was very remarkable. "It is agreed," says he, "that the moon, a secondary planet and satellite of the earth, is inclined at an invariable angle to the plane of the earth's orbit, whatever be the variations which this plane experiences in its position with respect to the fixed stars; and if ancient observations on the greatest latitudes of the moon, and on the obliquity of the ecliptic are irreconcileable with this hypothesis, it should be rejected sooner than call them in question." Here the reasons of suitableness and harmony have conducted Kepler to a just result; but how often have they bewildered him; when we give ourselves up to imagination and conjecture, it is only by a lucky chance that we can light on truth; but the almost total impossibility of arriving at it in the midst of the errors with which it is almost always encumbered, ought to induce us to ascribe all the merit of its discovery to him who establishes it solidly by observation and computation, the sole bases of human knowledge.

(*a*) Knowing the duration of a total and central eclipse of the moon, and also the periodic time of the moon, the angle which the semisection of the shadow subtends at the earth is known; hence, as the apparent diameter of the sun, and also the horizontal parallax of the moon are known, we obtain an expression for the horizontal parallax of the sun.—See Brinkley's Astronomy, page 255.

NOTE TO CHAPTER III.

(*a*) A celebrated peripatetic philosopher, John, surnamed the Grammarian, who was in high favour with the Saracen general who took the city, requested as a present the royal library. The general replied, that it was not in his power to grant such a request without the knowledge and consent of the Caliph; he accordingly wrote to Omar, who was then Caliph, and the answer has been given in the text. This account is, however, now doubted.

NOTES TO CHAPTER V.

(*a*) The very first applications of analysis to the motions of the moon furnish an example of this superiority. For they give with the greatest ease not only the inequality of the variation, which is obtained with the greatest difficulty by the synthetic method, but likewise the evection which Newton did not even suppose was caused at all by the law of gravity. It would certainly be impossible to obtain by means of synthesis, the numerous lunar inequalities, the values of which, determined by analysis, represent observations as exactly as our very best tables, which are formed by combining an immense number of observations with theory.

(*b*) The endeavours of geometers to demonstrate Euclid's twentieth axiom about parallel lines, have been hitherto unsuccessful. However no person questions the truth of this axiom, or of the theorems which Euclid has deduced from it. The perception of extension contains ∴ a peculiar property, which is self-evident, without which we could not rigorously establish the doctrine of parallels. The motion of a limited extension, for example of a circle, does not involve any thing which depends on its absolute magnitude; but if we conceive its radius to be diminished, we are forced to diminish also in the same proportion its circumference, and the sides of all the inscribed figures.

This proportionality was, according to Laplace, an axiom much more obvious than that of Euclid. It is curious to observe, that agreeably to what is stated in page 322, this axiom is pointed out in the results of universal gravitation.

ERRATA.

Page 37	line 14	*after the words* may be *read* also.
—— 86	——	5 *from bottom, for* millioneth *read* thousandth.
—— 157	——	9 *for* solstices *read* quadratures.
—— 175	——	6 *for* one *read* nine.
—— 201	——	5 *for* millioneth *read* thousanth.
—— 236	——	17 *for* their *read* its.
—— 300	——	18 *after* fall *read* through.
—— 330	——	11 *for* each *read* some.
—— 347	——	11 *for* Albatenus *read* Albatenius.
—— ib.	——	16 *for* Strato *read* Strabo.
—— 466	——	9 *for* principal *read* principle.

NOTES TO CHAPTER I. BOOK V.

(*b*) In fact, on the supposition that the Zodiac originated in Egypt, and that it was first invented in order to serve as a sort of kalendar to point out the different circumstances of the rural year, then there are two ways of reconciling what is indicated by the signs of the Zodiac with the climate of Egypt and its agriculture, either by making the Zodiac to have originated at a period long anterior to that at which it is at present supposed to commence; or by supposing that the constellations of the Zodiac are named, not from their rising with the sun, or the commencement of the day, but from their setting, or the beginning of night. The former hypothesis would make the world to be created at a time long anterior to that which we know from all history both sacred and profane, and also from contemporary records, it actually was; besides it would assign to the human race a duration longer than what Laplace himself admits it had, see page 50. Likewise, it may be remarked, that in these rude times, when the observations of the stars were made by the naked eye, it is much more likely that the stars were observed at night, when they are easily seen, and not in the day-time, when they are with difficulty discerned; the latter then is the true mode of reconciling the names of these signs with the different circumstances of the year: indeed on the first hypothesis if we consider the