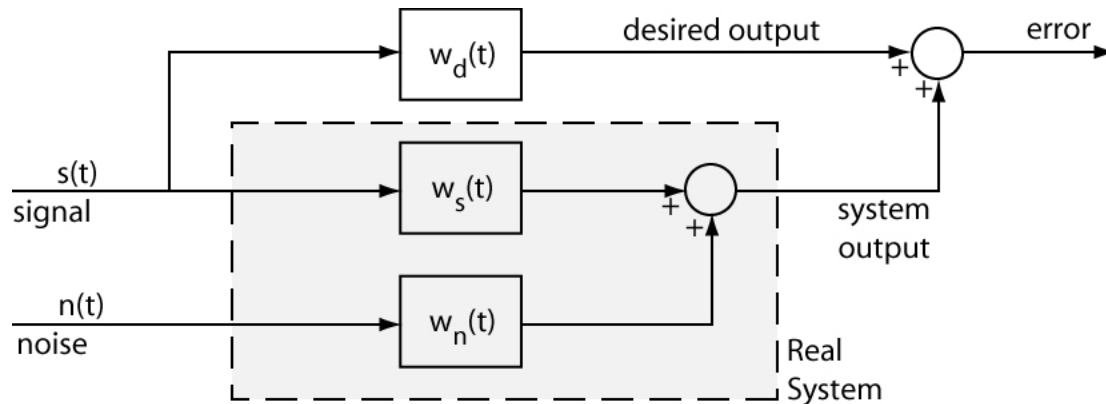# Lecture 17

Last time: Ground rules for filtering and control system design

General system

System parameters are contained in $w_s(t)$ and $w_n(t)$.

Desired output is generated by taking the signal through the desired operator. The difference between the actual output and the desired output is the error, whose mean squared value we want to minimize. We require a stable and realizable system. The error $e(t)$ is given by:

$$e(t) = \int_{-\infty}^{\infty} w_s(\tau_1)s(t-\tau_1)d\tau_1 - \int_{-\infty}^{\infty} w_d(\tau_1)s(t-\tau_1)d\tau_1 + \int_{-\infty}^{\infty} w_n(\tau_1)n(t-\tau_1)d\tau_1$$

$$= \int_{-\infty}^{\infty} w_e(\tau_1)s(t-\tau_1)d\tau_1 + \int_{-\infty}^{\infty} w_n(\tau_1)n(t-\tau_1)d\tau_1$$

where $w_e(\tau) = w_s(\tau) - w_d(\tau)$.

Using the replacement $w_e(\tau_1)$ reduces the number of terms in $R_{ee}(\tau)$ from nine to four.

$$R_{ee}(\tau) = \overline{e(t)e(t+\tau)}$$

$$= \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 w_e(\tau_1)w_e(\tau_2)R_{ss}(\tau+\tau_1-\tau_2)$$

$$+ \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 w_e(\tau_1)w_n(\tau_2)R_{sn}(\tau+\tau_1-\tau_2)$$

$$+ \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 w_n(\tau_1)w_e(\tau_2)R_{ns}(\tau+\tau_1-\tau_2)$$

$$+ \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 w_n(\tau_1)w_n(\tau_2)R_{nn}(\tau+\tau_1-\tau_2)$$

$$S_{ee}(s) = \int_{-\infty}^{\infty} R_{ee}(\tau)e^{-j\omega\tau}d\tau$$

$$= F_e(-s)F_e(s)S_{ss}(s)$$
$$+ F_e(-s)F_n(s)S_{sn}(s)$$
$$+ F_n(-s)F_e(s)S_{ns}(s)$$
$$+ F_n(-s)F_n(s)S_{nn}(s)$$

where $F_e(s) = F_s(s) - F_d(s)$.

We know the configuration so we can work out the form of each of the transforms F.

### Computation of the transfer functions $F_e(s)$ and $F_n(s)$

For many system configurations especially with minor feedback loops, manipulating the diagram into a standard input-output form is very time-conserving and subject to errors.

Usually just tracing signals around the loops is easiest.

Example: Find $F_n(s)$
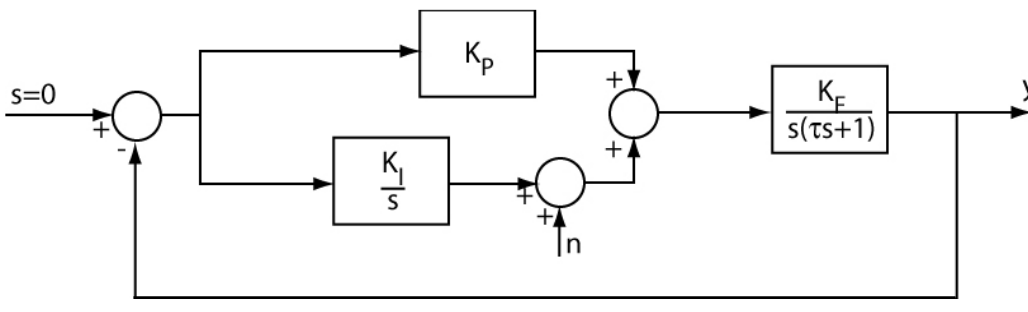
Don't try to manipulate this by block diagram algebra into the form:
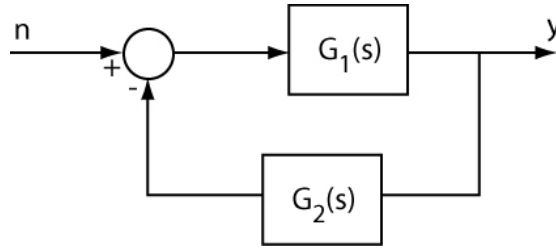
Rather:

$$y = \frac{K_F}{s(\tau s + 1)}\left[-K_P y + \left(-\frac{K_I}{s}y + n\right)\right]$$

Now the rest is just algebra,

$$\tau s^2 y + sy = K_F\left[-K_P y - \frac{K_I}{s}y + n\right]$$

Multiply by s and collect terms.

$$\left(\tau s^3 + s^2 + K_F K_P s + K_F K_I\right)y = K_F sn$$

$$\frac{y}{n} = \frac{K_F s}{\tau s^3 + s^2 + K_F K_P s + K_F K_I}$$

Now integrate to get $\overline{e^2}$, using $s = j\omega$:

$$\overline{e^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} S_{ee}(s)ds$$

Two methods:
Cauchy's Theorem:

$$\overline{e^2} = \frac{1}{2\pi j}2\pi j\sum\text{Res}\left(\text{poles of } S_{ee}(s) \text{ in LHP}\right)$$

$$= \sum\text{Res}\left(\text{poles of } S_{ee}(s) \text{ in LHP}\right)$$

Tabulated Integral (applicable only to rational functions):

$$S_{ee} = S_{ss} + (\ )S_{sn} + (\ )S_{ns} + (\ )S_{nn}$$

$$I_n = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{c(s)c(-s)}{d(s)d(-s)}ds$$

Just work out the coefficients $c(s)$ and $d(s)$ and use the table to find the integral I_n.

$(\ )S_{sn} + (\ )S_{ns}$ must be integrated together. For the rest of the semester, we will only deal with signals and noise which are uncorrelated, so these terms will be zero.

Having $S_{ee}(\omega)$ integrated to get an expression for $\overline{e^2}(p_1, p_2, ..., p_n)$, where p_i are the design parameters, we must determine the optimum set of values of the p_i.

Differentiate with respect to other variables than the parameters.
$$\frac{\partial \overline{e^2}}{\partial p_1} = 0$$
$$\frac{\partial \overline{e^2}}{\partial p_2} = 0$$
$$\vdots$$

If you have one parameter, just plot and find the minimum. For multiple parameters, the optimization becomes quite complex.

If the signal, noise and disturbance can be considered uncorrelated, then
$$\overline{e^2} = \overline{e_s^2} + \overline{e_n^2} + \overline{e_d^2}$$
and we can express these components of $\overline{e^2}$ in any direct or convenient way. ...

Example: A servo
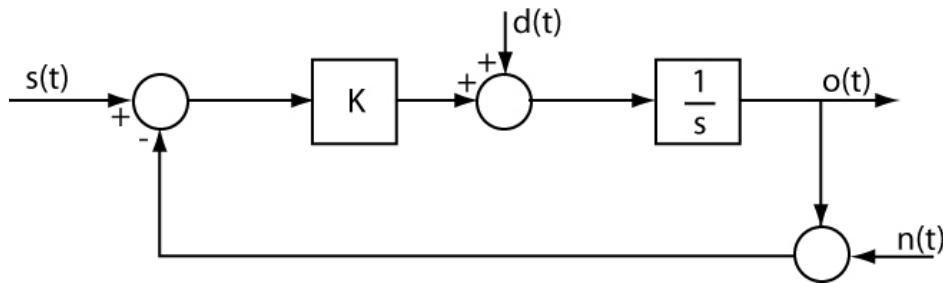
Signal $s(t)$ is a member of a stationary ensemble: $S_{ss}(\omega) = \dfrac{A}{\omega^2 + a^2}$

Disturbance $d(t)$ is a member of the ensemble of constant functions:
$$\overline{d(t)^2} = D$$
Noise $n(t)$ is a member of a stationary white noise ensemble: $S_{nn}(\omega) = N$

Desired output is the signal.

Find K which minimizes the steady-state mean squared error.

The inputs $s(t), d(t), n(t)$ are all <u>independent</u>. s and n have zero mean. Therefore, the three inputs are uncorrelated.

$$\overline{e^2} = \overline{e_s^2} + \overline{e_d^2} + \overline{e_n^2}$$

We'll need the error (signal minus desired) transfer function and the noise transfer function.
For the signal:

$$F_s(s) = \frac{\dfrac{K}{s}}{1 + \dfrac{K}{s}} = \frac{K}{s+K}$$

$$F_{desired}(s) = 1$$

$$F_e(s) = F_s(s) - F_{desired}(s) = \frac{K}{s+K} - 1 = \frac{-s}{s+K}$$

For the noise:

$$F_n(s) = \frac{-K}{s+K}$$

$$\overline{e_s^2} = \frac{1}{2\pi j} \int\limits_{-j\infty}^{j\infty} F_e(s)F_e(-s)S_{ss}(s)ds$$

$$S_{ss}(s) = \frac{A}{a^2 - s^2} = \frac{\sqrt{A}}{(a+s)}\frac{\sqrt{A}}{(a-s)}$$

The integrand:

$$F_e(s)F_e(-s)S_{ss}(s) = \frac{-s}{(K+s)}\frac{s}{(K-s)}\frac{\sqrt{A}}{(a+s)}\frac{\sqrt{A}}{(a-s)}$$

$$= \frac{\sqrt{A}s}{(K+s)(a+s)}\left[\frac{\sqrt{A}(-s)}{(K-s)(a-s)}\right]$$

$$= \frac{\sqrt{A}s}{s^2 + (a+K)s + aK}\left[\frac{\sqrt{A}(-s)}{s^2 + (a+K)(-s) + aK}\right]$$

$$= \frac{c(s)}{d(s)}\frac{c(-s)}{d(-s)}$$

$$\overline{e_s^2} = I_2 = \frac{c_1^2 d_0 + c_0^2 d_2}{2 d_0 d_1 d_2} \rightarrow \frac{c_1^2}{2 d_1 d_2}$$

$$= \frac{A}{2(a + K)}$$

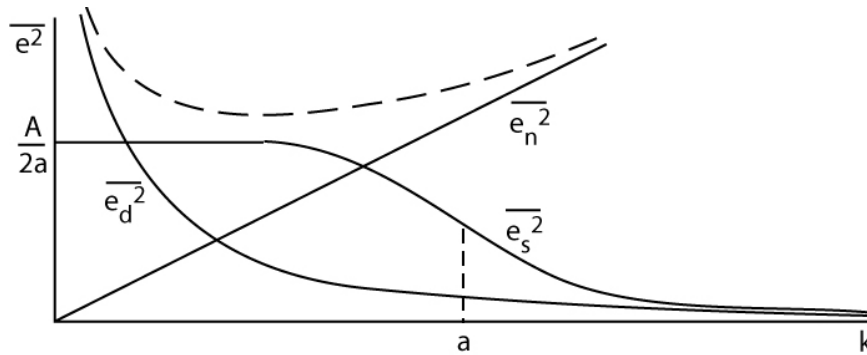For the disturbance: Steady state response O to constant input d

$$\frac{O}{d} = \frac{\dfrac{1}{s}}{1 + \dfrac{K}{s}} = \frac{1}{s + K}$$

$$O(s) = \frac{1}{s + K} \frac{d}{s}$$

$$O_{ss} = \lim_{s \to 0} \left(s O(s) \right)$$

$$= \lim_{s \to 0} \left(s \frac{1}{s + K} \frac{d}{s} \right) = \frac{d}{K}$$

$$\overline{O_{ss}^2} = \frac{\overline{d^2}}{K^2} = \frac{D}{K^2}$$

$$F_e(s) = F_s(s) - F_{desired}(s)$$

$$= \frac{K}{s + K} - 1 = \frac{-s}{s + K}$$

$$F_n(s) = \frac{-K}{s + K}$$

$$S_{ee}(s)_n = \frac{-K}{K + s} \frac{-K}{K - s} N$$

Using Cauchy's Theorem:

$$\overline{e_n^2} = \text{Res}\left(\text{pole at } s = -K \right)$$

$$= \frac{K^2 N}{(K + K)} = \frac{1}{2} KN$$

$$\overline{e^2} = \frac{\frac{1}{2}A}{K+a} + \frac{D}{K^2} + \frac{1}{2}NK$$

$$\frac{d\overline{e^2}}{dK} = \frac{-\frac{1}{2}A}{(K+a)^2} - \frac{2D}{K^3} + \frac{1}{2}N = 0$$

$$= \frac{1}{2}AK^3 - 2D(K+a)^2 + \frac{1}{2}NK^3(K+a)^2 = 0$$

$$= 5^{th} \text{ order polynomial in } K = 0$$

You should check the stability of your solution at your solution point. For
stability in this example, $K > 0$.

Semi-Free Configuration Design

(Free configuration design is a special case of this.)

Fixed transfer you have to deal with. You must drive the plant, $F(s)$. We will
design a compensator $C(s)$, which can be in a closed loop configuration, perhaps
with something in the feedback path, $B(s)$.