Lecture 9

9.1 Administration

- None.

9.2 Continuation of energy equation

From the last time, we ended with an expression for the energy equation

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) = \rho\mathbf{g}\cdot\mathbf{u} + \nabla\cdot(\tau\cdot\mathbf{u}) - \nabla\cdot\mathbf{q} \tag{9.1}$$

Where

$$
\begin{aligned}
\rho\mathbf{g}\cdot\mathbf{u} &\equiv \text{changes in potential energy} \\
-\nabla\cdot\mathbf{q} &\equiv \text{changes due to heat flux} \\
\nabla\cdot(\tau\cdot\mathbf{u}) &\equiv \text{a kinetic energy part and an internal heating part}
\end{aligned}
$$

Clearly, the stress term is the painful one:

$$\nabla\cdot(\tau\cdot\mathbf{u}) = \mathbf{u}\cdot(\nabla\cdot\tau) + \tau:\nabla\mathbf{u} \tag{9.2}$$

The first term, $\mathbf{u}\cdot(\nabla\cdot\tau)$, is the rate of work done by the surface forces. Imbalances in stress accelerate the fluid blob and change its kinetic energy. The second term, $\tau:\nabla\mathbf{u}$, is the work done by deformation. Not all stress moves the blob – some of it deforms and causes heat change. Note the double contraction, $\mathbf{A}:\mathbf{B}\Rightarrow$ scalar.

The $\mathbf{u}\cdot(\nabla\cdot\tau)$ terms is the easiest of the two since we came up with an expression for $\nabla\cdot\tau$ when we looked at momentum conservation.

$$
\begin{aligned}
\nabla\cdot\tau &= \nabla\cdot\left(2\mu\mathbf{e} - p - \frac{2}{3}\mu\nabla\cdot\mathbf{u}\right) \\
&= -\nabla p + \mu\nabla^2\mathbf{u} + \frac{\mu}{3}\nabla(\nabla\cdot\mathbf{u}) \\
\Rightarrow \quad \mathbf{u}\cdot(\nabla\cdot\tau) &= \mathbf{u}\cdot\left(-\nabla p + \mu\nabla^2\mathbf{u} + \frac{\mu}{3}\nabla(\nabla\cdot\mathbf{u})\right)
\end{aligned}
\tag{9.3}
$$

We're advecting τ through our fixed blob of fluid. It is bringing in kinetic energy associated with the stress on the blob. **(Jim – You have a note on the side here saying "think about this more"; I am not sure if this is directed at you or the students. In the least, "advecting τ" is a bit confusing.)**

Now the second term,

$$\tau:\nabla\mathbf{u} \qquad \text{double dot product}$$

Notice, there is no $\cdot$ between the ∇ and the $\mathbf{u}$. $\nabla\mathbf{u}$ is the gradient of the velocity... The velocity gradient tensor!

$$\nabla\mathbf{u} = \mathbf{G} \tag{9.4}$$

Thus, this term looks like

$$\tau : \mathbf{G} \tag{9.5}$$

But $\mathbf{G} = \mathbf{e} + \frac{1}{2}\mathbf{r}$,

$$\tau : \left(\mathbf{e} + \frac{1}{2}\mathbf{r}\right) \tag{9.6}$$

Recall that $\mathbf{r}$ is an antisymmetric tensor, and τ is a symmetric tensor. The double dot product of asymmetric and antisymmetric matrix is zero

$$\mathbf{A} : \mathbf{B} = \sum_i \sum_j A_{ij}B_{ij} \tag{9.7}$$

In 2D

$$
\begin{aligned}
\begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} : \begin{bmatrix} 0 & -\frac{1}{2}\omega \\ \frac{1}{2}\omega & 0 \end{bmatrix} &= \tau_{11}(0) + \tau_{12}\left(\frac{\omega}{2}\right) + \tau_{21}\left(-\frac{\omega}{2}\right) + \tau_{22}(0) \\
&= \tau_{12}\frac{\omega}{2} - \tau_{21}\frac{\omega}{2} \\
&= 0
\end{aligned} \tag{9.8}
$$

This last line is due to the fact that since τ is symmetric, $\tau_{12} = \tau_{21}$. Thus, you can get rid of the $\frac{1}{2}r$ term and end up with

$$\tau : \mathbf{e} \tag{9.9}$$

Substitute in our expression for τ,

$$
\begin{aligned}
\left(2\mu\mathbf{e} - \left[p + \frac{2}{3}\mu\nabla \cdot \mathbf{u}\right]\mathbf{I}\right) : \mathbf{e} \\
2\mu\mathbf{e} : \mathbf{e} - p\mathbf{I} : \mathbf{e} - \frac{2}{3}\mu\nabla \cdot \mathbf{u}\mathbf{I} : \mathbf{e}
\end{aligned} \tag{9.10}
$$

Lets start with an easy one

$$
\begin{aligned}
p\mathbf{I} : \mathbf{e} &= p\frac{\partial u}{\partial x} + p\frac{\partial v}{\partial y} + p\frac{\partial w}{\partial z} \\
&= p\nabla \cdot \mathbf{u}
\end{aligned} \tag{9.11}
$$

That is, change in energy due to compression or expansion. The last term is also pretty easy

$$
\begin{aligned}
\frac{2}{3}\mu\nabla \cdot \mathbf{u}\mathbf{I} : \mathbf{e} &= \frac{2}{3}\mu\nabla \cdot \mathbf{u}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\
&= \frac{2}{3}\mu\nabla \cdot \mathbf{u}(\nabla \cdot \mathbf{u}) \\
&= \frac{2}{3}\mu(\nabla \cdot \mathbf{u})^2
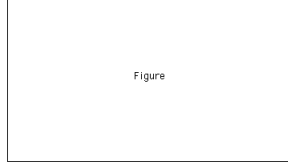\end{aligned} \tag{9.12}
$$

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Figure 9.1: (`fig:Lec9MuParameter`) μ parameterizes work associated in pulling apart molecules in a blob of fluid.

Finally, the first term $2\mu\mathbf{e} : \mathbf{e}$

$$
\begin{aligned}
\mathbf{e} : \mathbf{e} &= \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} : \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \\
&= \begin{aligned}[t] & e_{11}e_{11} + e_{12}e_{21} + e_{13}e_{31} + \\ & e_{21}e_{12} + e_{22}e_{22} + e_{23}e_{32} + \\ & e_{31}e_{13} + e_{32}e_{23} + e_{33}e_{33} \end{aligned}
\end{aligned} \tag{9.13}
$$

Note

$$
e_{11}e_{11} = \left(\frac{\partial u}{\partial x}\right)^2 \tag{9.14}
$$

$$
\begin{aligned}
e_{12}e_{21} &= \left[\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]^2 \\
&= \frac{1}{4}\left(\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2\right) \\
&= \frac{1}{4}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2
\end{aligned} \tag{9.15}
$$

etc.

Consider just the $e_{11}e_{11}$ term. Remember,

$$
2\mu\mathbf{e} : \mathbf{e} \quad \Rightarrow \quad 2\mu\left(\frac{\partial u}{\partial y}\right)^2 = 2\left(\mu\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial y} \tag{9.16}
$$

The $\mu\frac{\partial u}{\partial y}$ bit is associated with normal stress, and $\frac{\partial u}{\partial y}$ with the velocity gradient. Thus, this term has something to do with the work it takes to pull apart molecules in a blob of fluid (see figure 9.1). The other terms, such as $(\frac{\partial u}{\partial y})^2$ and $\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$, say something similar about shear and twisting. Clearly this is a bit of a mess. Thankfully, what is typically done is to sweep it all under the rug and define

$$
\begin{aligned}
\phi &= \left(2\mu\mathbf{e} : \mathbf{e} - \frac{2}{3}\mu(\nabla \cdot \mathbf{u})^2\right) \\
&\equiv \text{viscous component of the deformation work rate}
\end{aligned} \tag{9.17}
$$

This allows us to write

$$
\tau : \nabla\mathbf{u} = -p(\nabla \cdot \mathbf{u}) + \phi \tag{9.18}
$$

Tracking this all the way back to the energy equation gives:

$$
\rho\frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) = \rho\mathbf{g} \cdot \mathbf{u} + \mathbf{u} \cdot \left[-\nabla p + \mu\nabla^2\mathbf{u} + \frac{\mu}{3}\nabla(\nabla \cdot \mathbf{u})\right] - p(\nabla \cdot \mathbf{u}) + \phi - \nabla \cdot \mathbf{q} \tag{9.19}
$$

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9.3 Thermal energy (or heat) equation

What a mess. What games can we play to make this more comprehensible? Let's go back to a version of the momentum equation:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} + \mathbf{u} \cdot (\nabla \cdot \tau) \tag{9.20}$$

Multiply (dot) through by $\mathbf{u}$

$$\rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}^2\right) = \rho\mathbf{u} \cdot \mathbf{g} + \mathbf{u} \cdot (\nabla \cdot \tau) \tag{9.21}$$

This looks a little familiar. It is the mechanical energy equation. Subtract this from the total energy equation

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) = \rho\mathbf{g} \cdot \mathbf{u} + \mathbf{u} \cdot (\nabla \cdot \tau) - p(\nabla \cdot \mathbf{u}) + \phi - \nabla \cdot \mathbf{q}$$

$$- \qquad \rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}^2\right) = \rho\mathbf{u} \cdot \mathbf{g} + \mathbf{u} \cdot (\nabla \cdot \tau)$$

and we get the thermal energy (or heat) equation:

$$\Rightarrow \qquad \rho \frac{De}{Dt} = -p(\nabla \cdot \mathbf{u}) + \phi - \nabla \cdot \mathbf{q} \tag{9.22}$$

$$\text{OR} \quad \rho C_v \frac{DT}{Dt} = -p(\nabla \cdot \mathbf{u}) + \phi - \nabla \cdot \mathbf{q} \tag{9.23}$$

The second equation is obtained using $de = C_v dT$. Here

$$\begin{aligned} -p(\nabla \cdot \mathbf{u}) &\equiv \quad \text{change in thermal energy due to compression} \\ \phi &\equiv \quad \text{change in thermal energy due to viscosity} \\ -\nabla \cdot \mathbf{q} &\equiv \quad \text{change in thermal energy due to heat transfer} \end{aligned}$$

9.4 Approximations

Almost always it is safe to say that $\phi \ll 1$, and it is ignored (**Jim − Comparing ϕ to what other term that is O(1)?**).

$$\rho C_v \frac{DT}{Dt} + p(\nabla \cdot \mathbf{u}) = -\nabla \cdot \mathbf{q} \tag{9.24}$$

Here, $\rho C_v \frac{DT}{Dt}$ is temperature changes at constant volume, and $p(\nabla \cdot \mathbf{u})$ is temperature changes due to changing volume.

Remember the Boussinesq approximation? I stated it as

$$\rho = \rho_o + \rho'(x, y, z, t) \tag{9.25}$$

And it allowed us to simplify the continuity equation to

$$\nabla \cdot \mathbf{u} = 0 \tag{9.26}$$

It would be tempting to simply remove the $-p(\nabla \cdot \mathbf{u})$ term, leaving

$$\rho C_v \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} \tag{9.27}$$

But this isn't correct, because here $\nabla \cdot \mathbf{u}$ is similar in scale to $\rho C_v \frac{DT}{Dt}$, so we'll keep it. If we can't say $\nabla \cdot \mathbf{u} = 0$, let's substitute in the continuity equation instead

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{u} = -\frac{1}{\rho}\frac{D\rho}{Dt} \tag{9.28}$$

Substitution yields

$$\rho C_v \frac{DT}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} \;=\; -\nabla \cdot \mathbf{q} \tag{9.29}$$

Rewrite as

$$\rho C_v \frac{DT}{Dt} + \rho p \left(-\frac{1}{\rho^2}\frac{D\rho}{Dt} \right) \;=\; -\nabla \cdot \mathbf{q} \tag{9.30}$$

Using the fact that $-\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{D}{Dt}\frac{1}{\rho}$

$$\rho C_v \frac{DT}{Dt} + \rho p \frac{D\alpha}{Dt} \;=\; -\nabla \cdot \mathbf{q} \tag{9.31}$$

where $\alpha = \frac{1}{\rho} \equiv$ specific volume.

The equation of state for a perfect gas is

$$p = \rho R T \quad \text{or} \quad p\alpha = RT \tag{9.32}$$

Differentiate and rearrange:

$$p\frac{D\alpha}{Dt} + \alpha \frac{Dp}{Dt} \;=\; R\frac{DT}{Dt}$$

$$\Rightarrow \qquad p\frac{D\alpha}{Dt} \;=\; R\frac{DT}{Dt} - \alpha\frac{Dp}{Dt} \tag{9.33}$$

$$\tag{9.34}$$

Substitute:

$$\rho C_v \frac{DT}{Dt} + \rho R\frac{DT}{Dt} - \rho\alpha\frac{Dp}{Dt} \;=\; -\nabla \cdot \mathbf{q}$$

$$\rho(C_v + R)\frac{DT}{Dt} - \frac{Dp}{Dt} \;=\; -\nabla \cdot \mathbf{q}$$

$$\rho C_p \frac{DT}{Dt} - \frac{Dp}{Dt} \;=\; -\nabla \cdot \mathbf{q} \tag{9.35}$$

Where $C_v + R = C_p$. Here, $\rho C_p \frac{DT}{Dt}$ is temperature change from heating at constant pressure, and $\frac{Dp}{Dt}$ is the correction term for constant pressure. Thus the approximations made are ϕ small and an ideal gas (though structurally similar to case with any equation of state).

The Boussinesq approximation says that $\frac{Dp}{Dt}$ is small. Hence

$$\rho C_p \frac{DT}{Dt} \;=\; -\nabla \cdot \mathbf{q} \tag{9.36}$$

Further, using Fourier's Law of conduction

$$\mathbf{q} \;=\; -k\nabla T \tag{9.37}$$

which comes from measuring the heat flow through a block with different temperatures on each side, we get the following result

$$\rho C_p \frac{DT}{Dt} \;=\; \nabla \cdot (k\nabla T) \tag{9.38}$$

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Here $k \equiv$ thermal conductivity, and if constant we have

$$\rho C_p \frac{DT}{Dt} = k \nabla^2 T \tag{9.39}$$

(Jim – Here we have used an atmospheric approximation, ideal gas, and an oceanic one, Boussinesq; perhaps less than ideal).

This assumption that $\frac{Dp}{Dt}$ is small is okay for the ocean, but poor for the atmosphere. What can we do about that? Go back to

$$\rho C_p \frac{DT}{Dt} - \frac{Dp}{Dt} = -\nabla \cdot \mathbf{q} \tag{9.40}$$

Divide through by ρ

$$C_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = -\frac{\nabla \cdot \mathbf{q}}{\rho} \tag{9.41}$$

The source on the RHS has gone from a per volume to a per mass. Lets give it a new name, $\dot{Q}$.

$$C_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = \dot{Q} \tag{9.42}$$

Divide through by T

$$C_p \frac{1}{T}\frac{DT}{Dt} - \frac{1}{\rho}\frac{1}{T}\frac{Dp}{Dt} = \frac{\dot{Q}}{T} \tag{9.43}$$

Noting that $p = \rho R T \Rightarrow \frac{1}{pT} = \frac{R}{\rho}$

$$C_p \frac{1}{T}\frac{DT}{Dt} - R\frac{1}{p}\frac{Dp}{Dt} = \frac{\dot{Q}}{T} \tag{9.44}$$

Since $\frac{1}{x}\frac{dx}{dt} = \frac{d}{dt}\ln x$

$$C_p \frac{D}{Dt}\ln T - R\frac{D}{Dt}\ln p = \frac{\dot{Q}}{T} \equiv \frac{D\eta}{Dt} \tag{9.45}$$

Where $\eta \equiv$ entropy per mass, and $\delta\eta \equiv \frac{\delta Q}{T}$ where $Q \equiv$ heat per mass.

How on earth could that possibly be useful? Let me introduce a new variable

$$\theta = T\left(\frac{p_s}{p}\right)^{R/C_p} \equiv \text{potential temperature} \tag{9.46}$$

Here $p_s \equiv$ surface pressure. The potential temperature is the temperature a parcel would have if you moved it adiabatically (no heat enters or leaves the parcel) from whatever height it sits at, down to the surface. Take the ln

$$\begin{aligned} \ln\theta &= \ln T + \frac{R}{C_p}\ln\left(\frac{p_s}{p}\right) \\ C_p \ln\theta &= C_p \ln T - R\ln p + R\ln p_s \end{aligned} \tag{9.47}$$

Take the time derivative

$$C_p \frac{D}{Dt}\ln\theta = C_p \frac{D}{Dt}\ln T - R\frac{D}{Dt}\ln p \tag{9.48}$$

Compare with our expression for rate of change of entropy:

$$\frac{D\eta}{Dt} \quad = \quad C_p \frac{D}{Dt} \ln T - R \frac{D}{Dt} \ln p \tag{9.49}$$

So,

$$C_p \frac{D}{Dt} \ln \theta \quad = \quad \frac{D\eta}{Dt} \tag{9.50}$$

This tells us that a parcel that conserves entropy travels along a surface of constant θ. Why do we care? In a compressible atmosphere it isn't the density gradient with height that determines vertical stability, it is the entropy gradient that determine stability. The potential temperature analog in the ocean is potential density, defined as the density obtained if a parcel is taken to a reference pressure at constant salinity and adiabatically.

Back to the heat equation

$$C_p \frac{D}{Dt} \ln \theta = \frac{D\eta}{Dt} = \frac{\dot{Q}}{T} \quad = \quad \frac{-\nabla \cdot \mathbf{q}}{\rho T}$$

$$\frac{C_p}{\theta} \frac{D\theta}{Dt} \quad = \quad \frac{-\nabla \cdot \mathbf{q}}{\rho T}$$

$$C_p \frac{D\theta}{Dt} \quad = \quad -\frac{\theta}{\rho T} \nabla \cdot \mathbf{q} \tag{9.51}$$

No assumptions here except ideal gas and ϕ small. $\frac{Dp}{Dt}$ retained.

9.5   Reading for class 10

None assigned in class.