Lecture 2

2.1 Administration

- Who is at WHOI?

- Collect problem set 1.

- Hand out problem set 2 (due next Tues., Sept. 16). Note: ex 3.1 <u>not</u> 3.2.

- Bill's new office hours: M and TR 1-3.

- **General Homework Comments:** The aim of this homework set is to get your brain working. It is not hard, though tedious (especially the KC01 problems).

- **Homework Hint:** For the last problem, use spherical coordinates, but do <u>not</u> use Kundu's spherical coordinates. Instead, use the geophysical way as is shown in figure **??**.
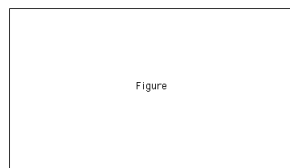

Figure 2.1: (`fig:GeophysicalSphericalCoordinates`) Diagram of geophysical spherical coordinates.

$$
\begin{aligned}
x &= r \cos \phi \cos \lambda & (2.1) \\
y &= r \cos \phi \sin \lambda & (2.2) \\
z &= r \sin \phi & (2.3)
\end{aligned}
$$

- Hand out tensor notes (I will stick with vector notation).

- Hand out KFF MS.

- Hand out dimensional analysis MS.

2.2 Math Review

Hand out tensor MS

2.2.1 Vector Stuff

Gradient vector: The gradient of a function f, often written ∇f ("del f"), in Cartesian coordinates is given by:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{2.4}$$

Note that the result is a vector.

Divergence of a vector field: The divergence of a vector field ($\mathbf{f}$) is a measure of the "flux", and in Cartesian coordinates is written:

$$\nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \tag{2.5}$$

$$\text{where} \qquad \nabla \;\; = \;\; \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{2.6}$$

$$\mathbf{f} \;\; = \;\; f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} \tag{2.7}$$

Here, f_x, f_y, and f_z denote the x, y, and z components of $\mathbf{f}$ respectively. The result of the divergence of a vector field is a scalar. Note: $\nabla \cdot \mathbf{f} = 0 \Rightarrow$ incompressible.

Curl of a vector field: The curl of a vector field is the vorticity per unit area of flux, and in Cartesian coordinates is written:

$$\nabla \times \mathbf{f} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\mathbf{k} \tag{2.8}$$

$$\begin{array}{ccccc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{j} & \mathbf{i} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ f_x & f_y & f_z & f_y & f_x \end{array} \tag{2.9}$$

Note that the result of the curl of a vector field is a vector. Note: $\nabla \times \mathbf{f} = 0 \Rightarrow$ irrotational.

2.2.2 Waves

Wave equations: Equations for waves in two dimensions can be written three different ways:

$$a \;\; = \;\; A\cos(kx + ly - \omega t + \phi) \tag{2.10}$$

or

$$a \;\; = \;\; A_1\cos(kx + ly - \omega t) + A_2\sin(kx + ly - \omega t) \tag{2.11}$$

$$A_1 = A\cos\phi$$

$$A_1 = -A\sin\phi$$

or

$$a \;\; = \;\; Re[A_c e^{i(kx+ly-\omega t)}] \tag{2.12}$$

$$A_c = A_1 - iA_2 = Ae^{i\phi} \qquad (e^{i\phi} = \cos\phi + i\sin\phi)$$

Where the variables are defined as follows:

$$
\begin{aligned}
A &\equiv \text{amplitude} \\
kx + ly - \omega t + \phi &\equiv \text{phase} \\
k, l &\equiv \text{wave numbers} \\
\omega &\equiv \text{frequency} \\
\phi &\equiv \text{reference phase} \\
t &\equiv \text{time}
\end{aligned}
$$

Wave numbers / wave length: With the wavelength defined as the crest to crest distance in a given direction, the wave numbers and wave lengths are related as follows:

$$k = \frac{2\pi}{\lambda_x}, \quad \lambda_x \equiv \text{wavelength in } x \tag{2.13}$$

$$l = \frac{2\pi}{\lambda_y}, \quad \lambda_y \equiv \text{wavelength in } y \tag{2.14}$$

$$K = \frac{2\pi}{\lambda}, \quad \lambda \equiv \text{shortest crest to crest distance} \tag{2.15}$$

$$K = \sqrt{k^2 + l^2} \tag{2.16}$$

Here, K is defined as the total wave number.

Frequency / period: The period, T, is the time over which the signal repeats and is related to the frequency by the following equation:

$$\omega = \frac{2\pi}{T} \tag{2.17}$$

Phase speed: The phase speed, c_p, is the speed of lines of constant phase, and given by the following equation:

$$c_p = \frac{\omega}{K} \tag{2.18}$$

In general, k, l, ω, and A are not independent. Thus,

$$c_p = c_p(k, l) = \frac{\omega(k, l)}{K} \tag{2.19}$$

Thus waves with different frequencies will travel at different speeds and tend to smear out coherent signals. This is called dispersion and is shown schematically in figure **??**.
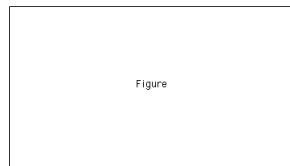

Figure 2.2: (`fig:SchematicDispersion`) Schematic picture of dispersion.

Group velocity: The group velocity is the speed at which amplitude modulation (or information) travels, and for the x direction is given by:

$$c_{g_x} = \frac{\partial \omega}{\partial k} \tag{2.20}$$

c_g gives the speed of a wave packet envelope and c_p gives the speed of the crests (see figure **??**). It is important to note that c_g need not equal c_p, and they can even have the opposite signs!
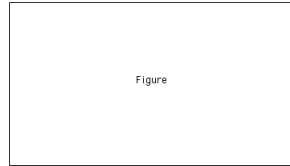

Figure 2.3: (`fig:CgVersusCp`) The group speed, C_g, is the speed of the amplitude modulation (dotted line) while the phase speed, C_g, is the speed of the crests (solid line).

2.3   Eigenvalues and eigenvectors

Eigenvalues and eigenvectors are interesting for at least two reasons:

1. The can be used to provide solutions to coupled differential equations.

2. They have be used to find the directions of maximum variance.

Generically, an eigenvector of a matrix $\mathbf{A}$ is only scaled, not rotated, when operated on by $\mathbf{A}$.

$$
\begin{aligned}
\mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \\
\mathbf{x} &\equiv \text{eigenvector} \\
\lambda &\equiv \text{eigenvalue}
\end{aligned}
\tag{2.21}
$$

Eigenvectors are the "preferred" or "natural" directions of the system described by $\mathbf{A}$. So what? Well, $\mathbf{A}$ might come from some coupled differential equations and the eigenvectors/values provide part of the solution to the equations. Consider the following ODE example:

$$
\begin{aligned}
\frac{dv}{dt} &= av - bw, & v(0) &= v_o \\
\frac{dw}{dt} &= cv - dw, & w(0) &= w_o
\end{aligned}
\tag{2.22}
$$

Rewriting this system using matrix notation yields:

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u} \quad \text{where} \ \ \mathbf{u}(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \ \ \text{and} \ \ \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.23}$$

Assuming a solution of the form $\mathbf{u}(t) = \mathbf{x}e^{\lambda t}$, and substituting into equation **??** yields:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad \text{or} \quad (\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = \mathbf{0} \tag{2.24}$$

Thus, to find the solution to the system (eigenvalues, λ, and their corresponding eigenvectors, $\mathbf{x}$), go through the usual drill:

- Find $\det(\mathbf{A} - \mathbf{I}\lambda)$.

- Roots of resulting polynomial are eigenvalues.

- Substitute eigenvalues back in $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = \mathbf{0}$ to solve for $\mathbf{x}$.

Another place one runs into eigenvectors and eigenvalues is in data analysis. Imagine a collection of points distributed in space as shown in figure **??**. While this picture has been drawn in two dimensions, any number of dimensions is realizable. A field in a 10^6 NWP forecast is nothing more than a single point in that space (figure **??**).
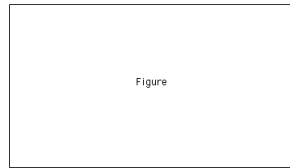
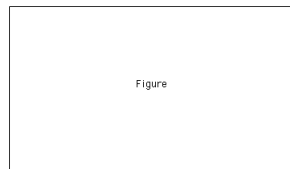Figure 2.4: (`fig:PointsInSpace`) Points in two dimensional space.

Figure 2.5: (`fig:NWPForecast`) Sample NWP forecast.

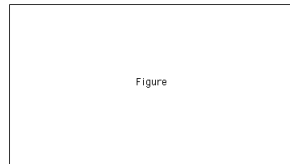A square matrix is found if the covariance of these points is computed.

Figure 2.6: (`fig:PointsInSpaceCovariance`) Covariance of points in two dimensional space (?).

The leading eigenvector (corresponding to the largest eigenvalue) of the covariance matrix is the direction in the cluster of points with the most variance. The eigenvalue is the variance in that direction. The second eigenvector (corresponding to the second largest eigenvalue) is the direction with the next most variance that is orthogonal to the first eigenvector. Again, the eigenvalue is the variance is the variance in that direction (figure **??**).

Why on Earth would one be interested in these directions? This approach is used all the time in the field to find so-called empirical orthogonal functions (EOFs) and so-called singular vectors. We might talk about them a little bit throughout the course, but the important point is that you realize that they are quite straight forward ideas.

2.4 Dimensional analysis and scale analysis

Dimensional analysis is a powerful tool, but in my opinion it requires a lot of experience to use is well. To show how useful it can be, I'll give a couple examples to try and access the importance of
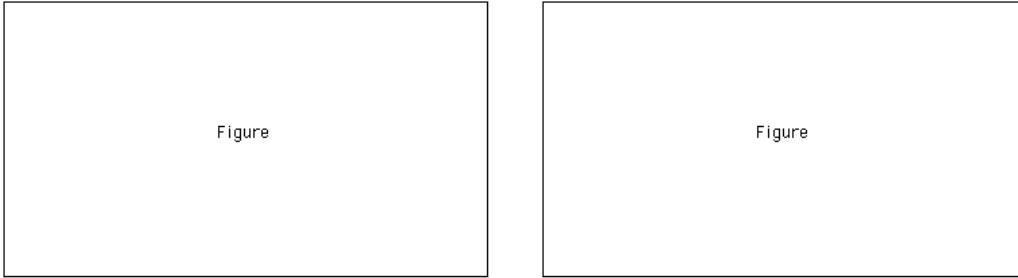
Figure 2.7: Eigenvector of the covariance matrix points in the direction of maximum variance (left panel), and the second eigenvector points in the direction of the next most variance that is orthogonal to the first eigenvector.

rotation and of stratification. The basic idea is as follows:

1. Take a <u>system of interest</u>.

2. Sit around scratching your head and figure out what <u>physical variables</u> are most relevant to the problem (gravity, density, velocities, etc.).

3. Express the physical variables in terms of their <u>fundamental quantities</u>: mass (M), length (L), time (T), temperature (θ), etc.

4. Determine the important <u>dimensionless groups</u> by grouping physical variables in such a way that the dimensions cancel. Buckingham-Pi tells you how many groups to expect.

 A classic example is the pendulum:

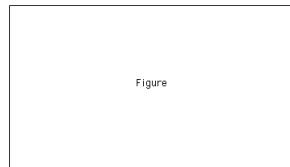1. System of interest: linear pendulum (see figure **??**).

Figure 2.8: (`fig:LinearPendulum`) Linear pendulum.

2. Fundamental quantities: Period (T), line length (l), gravity (g). Magic!

3. Physical variables in terms of their fundamental quantities: $T \equiv T$, $l \equiv L$, $g \equiv l/T^2$.

4. Group physical variables in such a way that the dimensions cancel:

$$T^a l^b g^c \equiv T^a L^b \left(\frac{L}{T^2}\right)^c$$
$$\Rightarrow a = 2, b = -1, c = 1$$
$$T^2 \frac{g}{l} = const$$
$$T\sqrt{\frac{g}{l}} = const \tag{2.25}$$

A famous example with a touch of legend is the dimensional analysis performed by G.I. Taylor on an atomic bomb blast in the late 40's. (Step 1) Legend had it Taylor saw some successive pictures of a test bomb blast (figure **??**).
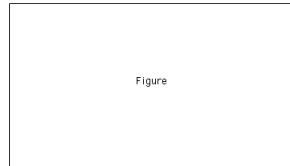

Figure 2.9: (`fig:BombBlast`) Nuclear bomb blast pictures used by G.I. Taylor.

(Step 2) Taylor scratched his head and decided that if he wanted to know the energy in the blast, the important physical variables are the radius of the blast as a function of time ($r(t)$), time (t), density of air (ρ), and of course energy (E). (Step 3) Expressed in fundamental quantities, $r \equiv L$, $t \equiv T$, $\rho \equiv M/L^3$, and $E \equiv ML^2/T^2$. (Step 4) This leads to the expression:

$$
\begin{aligned}
\frac{r^5 \rho}{Et^2} &= const \\
&\text{or} \\
r(t) &= const \left(\frac{E}{\rho}\right)^{1/5} t^{2/5}
\end{aligned}
\tag{2.26}
$$

Using the pictures in *Life* magazine, it is said, Taylor could measure radius as a function of time, and thereby get a very good estimate of the energy release by the bomb. Insert your own anecdote here about what the US government did when they found out that some Brit was running around telling everyone how powerful US bombs were... A top secret fact.

I'm not going to stand up here and try to get you to believe that thus dimensional analysis stuff is obvious or trivial. It is a powerful tool, but perhaps it is most useful in the hands of experts... Those who have enough experience to know what questions and what variables are important. That said, we can pretend that we know what variables are important in the GFD (geophysical fluid dynamics) case, and see what the resulting dimensionless groups tell us. We will look at rotation, stratification, and the combination of these effects.

2.4.1 Rotation

1. System of interest: rotation in fluids.

2. Important variables: rotation rate (Ω), length scale of motion (L), velocity scale of motion (U).

3. Physical variables in terms of their fundamental quantities: $\Omega \equiv 1/T$, $L \equiv L$, $U \equiv L/T$.

4. Group physical variables in such a way that the dimensions cancel: $U/\Omega L$. This does not look terribly useful at first. Rewrite as follows:

$$
\frac{U}{\Omega L} = \frac{1/\Omega}{L/U} \equiv \frac{\text{time scale of rotation}}{\text{time scale of motion}}
\tag{2.27}
$$

If the ratio is 1 or less, then rotation is important. This quantity ($U/\Omega L$) is important enough to warrant a name; it is called the Rossby number. It will appear numerous times in the course.

HW Note: (Can't read your writing)

2.4.2 Stratification

1. System of interest: stratification in fluids.

2. Important variables: scale height of fluid (H), density change over H ($\Delta\rho$), average density (ρ_o), fluid velocity (U), and gravity (g). Obvious, right?

3. Physical variables in terms of their fundamental quantities: ... Let's not bother.

4. Group physical variables in such a way that the dimensions cancel:

$$Fr = \frac{\rho_o U^2}{\Delta\rho H g} \tag{2.28}$$

This is the ratio of kinetic energy of the horizontal flow to the potential energy necessary to perturb a hunk of fluid over a height H. If the ratio is much less than 1, the stratification dominates. If it is of the order of 1, the stratification and the flow are of comparable importance. That is, the stratification can be modified by the flow, and the flow can be modified by the stratification. If the ratio is much greater than 1, the stratification has little effect.

This ratio is called the Froude number. Notice that $\Delta\rho g/\rho_o U$ has the units of frequency.

$$Fr = \frac{U}{NH} \quad \text{where} \quad N \equiv \text{Brunt} - \text{Vaisala frequency} \tag{2.29}$$

Notice U/NH looks a lot like $U/\Omega L$! There are many similarities between rotation and stratification.

2.4.3 Combined effects

What about the situation where both rotation and stratication matter?

$$\text{Rossby Number} = 1 \quad \Rightarrow \quad L \sim \frac{U}{\Omega}$$

$$\text{Froude Number} = 1 \quad \Rightarrow \quad U \sim \sqrt{\frac{\Delta\rho g H}{\rho}}$$

Eliminate U and you get the length scale:

$$L \sim \frac{1}{\Omega}\sqrt{\frac{\Delta\rho g H}{\rho}} \tag{2.30}$$

This is a variant of the Rossby radius of deformation and sets the length scale for motion when both rotation and stratification are important.

2.5 Reading for class 3

KC01: 3.1-3.5, 3.13
KFF: I, II, IV