Christoph J. Scriba
Peter Schreiber

# 5000 Years of Geometry

## Mathematics in History and Culture

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Christoph J. Scriba
University Hamburg
Hamburg, Germany

Peter Schreiber
University Greifswald
Greifswald, Germany

Translated by Jana Schreiber

Editor: Project Group "History of Mathematics" of Hildesheim University
H.W. Alten, K.-H. Schlote, H. Wesemüller-Kock

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## Preface of the editor of the German edition

Geometry (from the Greek word for 'measuring the Earth', the modern scientific discipline of which is now called geodesy), branch of science which deals with regular patterns, shapes and solids, was one of the first human attempts, after counting, to concern themselves with the emerging science mathematics. This is evident from the spirals on megalithic graves, incisions in stone and patterns on clay fragments.

In this book, you will learn how geometry has developed over the millennia from these earliest origins in distant times and much more. Geometry is an indispensible aid for building and surveying, and became an axiomatic science of plane and spatial shapes in Ancient Greece. It served as a basis for astronomical observations and calculations, for Islamic decorative art, and the building of medieval Christian cathedrals. Furthermore we will look at the discovery of perspective and its application in Renaissance art, at the disputes regarding the Euclidean parallel postulate, the discovery of non-Euclidean geometries in the $19^{th}$ century, and, finally, the theory of infinite-dimensional spaces and contemporary computer graphics.

This book is edited by the project group "History of Mathematics" at the University of Hildesheim as part of the series *Vom Zählstein zum Computer* (From Pebbles to Computers). Other titles in this series published by Springer Publishing Heidelberg are: *4000 Jahre Algebra* (4000 Years of Algebra) [Alten et al. 2003], and *6000 Jahre Mathematik* (6000 Years of Mathematics) [Wußing, in two volumes 2008/09]. To the series 'From Pebbles to Computers' two video films have been produced (University of Hildesheim): '*Mathematik in der Geschichte – Altertum*' (Mathematics in History – Antiquity) [Wesemüller-Kock/Gottwald 1998] and '*Mathematik in der Geschichte – Mittelalter*' (Mathematics in History – Middle Ages) [Wesemüller-Kock/Gottwald 2004]. Following multiple reprints and the second edition in 2004 we now present the third edition of *5000 Jahre Geometrie* including new research results on circular ditches in the Stone Age and the Nebra Sky Disk, as well as many illustrations in colour.

In this book, we will reflect on the development of geometry as part of our cultural history over the course of five millennia. Both authors have succeeded in portraying the origins and growth of this branch of mathematics, which is often thought of as dry and jejune, in a tremendously lively manner. They uncover the origins and impulses for the development of geometric notions and methods, and present how they are related to historical events and personal fates. Moreover, they describe the applications of geometrical knowledge and methods in other areas and the interdependencies that resulted from them. Finally, they emphasize their importance for other disciplines.

At the heart of this book series is portraying the history of mathematics as an integral part of the history of mankind, particularly as a fundamental part of our cultural heritage. Both authors have done justice to this task in an impeccable manner. They have depicted the genesis of geometry and its interlacing with cultural developments in other areas, such as literature, music, architecture, visual arts and religion, by a standard far higher than usual in mathematical-historical presentations. They also describe the implications of geometrical findings and methods for other areas. As such, the authors also deal far more extensively than usual with the development of geometry in other cultures, mainly in the ancient oriental cultures, in Islamic countries, as well as in India, China, Japan and the old American cultures. Tables at the beginning of each chapter give an overview of important political and cultural events of each cultural area and era dealt with. Tables at the end summarise the main geometrical contents of each chapter in note form.

Moreover, the authors compare views of ancient and medieval mathematicians with modern mathematical findings and link those to contemporary mathematics and related sciences, for example, references to computer sciences regarding the description of Euclid's "algorithmic accomplishment". Furthermore, they highlight the specifications of geometrical examinations of different eras and cultural areas and the changes in content, methods and approaches geometry has faced as a proto-physics within three-dimensional or even infinite-dimensional spaces. They discuss the relationship of geometry with other branches of mathematics, for instance with algebra, analysis, and stochastics. Refreshing asides with biographical highlights and references to unexpected relations, as well as text excerpts in the appendix, bring this book to life.

Chapters 1 through 4, with the exception of sub-chapter 2.3 (Euclid), were written by Dr. Christoph J. Scriba, professor emeritus for the history of the natural sciences in the former Institute for History of Natural Sciences, Mathematics and Engineering at the University of Hamburg. Euclid's accomplishments and the development of geometry in modern times from Chapters 5 through 8 were described by Dr. Peter Schreiber, professor for geometry and the foundations of mathematics at the University of Greifswald.

We are also grateful to the authors for numerous illustrations and the texts for the appendixes. The figures that have been added to support geometrical theorems that are not referenced were drawn by the authors themselves. They also thought of the summarising problems for every sub-chapter at the end of each chapter (cf. Introduction). They often differ from ordinary tasks in regard to type and size and also vary in level of difficulty. Thus, solving them requires very different background knowledge, as well as the use of secondary literature at times. Hence, to solve some of the problems of Chapters 1 through 4, you will mainly need knowledge gained in junior high school, while other problems will require highschool knowledge, whereas some problems to Chapters 5 through 8 demand insight into notions and methods taught at university. This is due to the nature of the subject, since mathematics has grown more and more complex and difficult over the course of the centuries and understanding modern mathematics usually assumes knowledge of the mathematics of past eras. Therefore, you will occasionally find hints to solutions within the text and also the literature. However, the solutions themselves have not been included in the appendix to avoid the following: first, we do not want you to look up the solutions too quickly; second, the solutions most often are not the result of calculations, but require the description of approaches for solving the problem at hand or retracing more or less extensive considerations.

All this has been done intentionally in order to attract as large a readership as possible. Cursory readers or those that are in a hurry should not simply skip the problems, since they include many interesting historical remarks and additions to the text, which is why reading the problems carefully will benefit everyone. The extensive bibliography and index of names invite the reader to study further.

I thank both authors sincerely for the multifaceted and intensive work in particular their dedication to setting new accents with this book integrating geometry in cultural history and composing many interesting problems.

I further express my gratitude to my colleagues Dauben, Flachsmeyer, Folkerts, Grattan-Guinness, Kahle, Lüneburg, Nádeník und Wußing for their scholarly advice and critical reviewing and thank H. Mainzer for advice on historical details and Lars-Detlef Hedde (University of Greifswald), Thomas Speck and Sylvia Voß (University of Hildesheim) for converting the manuscripts, illustrations and figures into printable electronic formats.

Moreover, I wish to thank media educator Anne Gottwald, who helped us clear the licensing for printing the illustrations, and each publisher for authorising the printing rights.

I also remain grateful to the director of the Centre for Distance Learning and Extension Studies (ZFW), Prof. Dr. Erwin Wagner, the present and former directors of the Institute for Mathematics and Applied Computer Science, Prof. Dr. Förster and Prof. Dr. Kreutzkamp, the deans Prof. Dr. Schwarzer and Prof. Dr. Ambrosi and the administration of the University of Hildesheim.

Last but not least, I wish to thank the members of the project group "History of Mathematics" of ZFW: the historian of mathematics Dr. Alireza Djafari Naini and the media expert and sociologist Heiko Wesemüller-Kock, for the great and intensive teamwork while planning and preparing this book. I express my gratitude to Springer Publishing Heidelberg for taking my requests into account and the excellent design of this book.

I hope that this volume will inspire many readers to study the history of mathematics more intensively, and to learn about the background of the origins and incredibly exciting development of geometrical notions and methods. Hopefully, this will result in the reader viewing geometry not just as a mathematical discipline or as an indispensible aid for architects, robot engineers and scientists, but also as a valuable part of our culture that we encounter everywhere and that makes the world in which we live so much richer.

On behalf of the project group

Hildesheim, August 2009 Heinz-Wilhelm Alten.

## Preface of the editor of the English edition

This is the first volume of our series ‘From Pebbles to Computers’ that appears in English. It is a translation of the $3^{rd}$ edition (2010) of *5000 Jahre Geometrie*, again updated, supplemented and enriched with many illustrations by the author P. Schreiber and the editors H.-W. Alten, K.-H. Schlote, and H. Wesemüller-Kock.

Meanwhile the book *4000 Jahre Algebra* appeared 2014 in its $2^{nd}$ edition, in 2011 *3000 Jahre Analysis* was published by Springer Berlin Heidelberg, and we are now preparing its translation *3000 Years of Analysis* to be published by Springer Basel. Some other volumes of this series will also be published in English. Besides the film ‘Mathematik in der Geschichte – Mittelalter’ has now been produced in English as *History of Mathematics – Middle Ages*.

All of us have been affected by the death of our author Prof. Dr. C. J. Scriba in 2013. We are grateful for his support over many years and glad to be able to present Chapters 1 through 4 of this book with the supplements he wrote before his death as part of his scientific legacy. We shall miss his advice in future.

The translation of this book was done by Jana Schreiber, the daughter of the author P. Schreiber. We are very grateful to her because she has done this with great efficiency and commitment in a short time.

After the corrections and supplements of the editor, language copy editing by the publisher, and proof reading by the author and editor we now present this volume.

I thank the members of the project group for their intensive teamwork: A. K. Gottwald for clearing the licenses (now world-wide) for printing the illustrations, H. Wesemüller-Kock for his involvement inserting new illustrations with his comments and the index of illustrations, proof reading and preparing the graphic design and layout for the whole book, the historian of mathematics Dr. K.-H. Schlote for many comments and for transferring the index of names and the subject index, Prof. Dr. K.-J. Förster and Prof. Dr. E. Wagner for providing financial support.

We are grateful for the help of our secretaries B. David and R. Falso, the students J. Schönborn and N. Westphal for preparing the text, illustrations and indexes ready for printing.

Last, but not least we thank Springer Publishing Basel AG and its editor Dr. A. Mätzener for her kindly support and the excellent layouting of this book.

I hope that this book will please, inspire and benefit many readers all over the world.

On behalf of the project group

Hildesheim, August 2014 Heinz-Wilhelm Alten.

## Table of Contents

# Introduction

It is certainly not easy to define the content and nature of mathematics briefly. Formal explanations, which are possible nowadays due to the general notion of structure and other logical notions, neglect not only the historical development, but also the instinct and experience of a mathematician, who knows what is “substantial” and “interesting” and what is not. However, within the given understanding of mathematics, it is even more complicated to explain what geometry is and what, in turn, is part of its history. The dominant views on the subject of geometry, as well as its position and meaning within mathematics, have not only changed repeatedly over the course of time. As mathematics became increasingly sophisticated, mathematicians also took opposing positions while trying to find answers to these questions. We will look at all these aspects in this book.

Even though geometry was mainly considered as one application of a primarily arithmeticaly oriented mathematics amongst many others in the earliest cultures (such as in ancient Egypt, Mesopotamia, India, China, etc.), it became the core and main interest of mathematics in Ancient Greece. It was there and then that vague notions and procedures justified only by trial and error were transformed into a theory with definitions, axioms, theorems and proofs. The heritage stemming from this period was so powerful for over two thousand years that mathematicians were usually called geometricians. Furthermore, the axiomatic-deductive method of cognition assurance, which was based on the Greeks’ methods of dealing with geometric matters, was referred to as “mos geometricus”, and the implementation in other sciences, including other realms of mathematics, “more geometrico”, in other words ‘in geometrical fashion’, became a rarely achieved scientific and theoretical program. This agenda influenced, for example, Newton in the $17^{th}$ century, as he re-founded mechanics, Galois at the beginning of the $19^{th}$ century, when criticising the contemporary situation of algebra, and Hilbert, while encouraging the scientific community to axiomatise further branches of physics in his famous speech in 1900.

As a result of the European Renaissance, geometry was flooded by an extraordinary wave of inspiration and applications in the fields of astronomy, geodesy, cartography, mechanics, optics, architecture, visual arts and, hence, leading to a wealth of new challenges. The efforts made to solve these new challenges essentially led to the development of the four pillars of the “modern” mathematics in the $17^{th}$ century. These pillars are: the concept of function, coordinate-systems, differential calculus and integral calculus. Geometry gave birth to these pillars, and then was superseded and lost its leading position to them in a very subtle manner. Formulae and calculus took over increasingly in the $18^{th}$ century and pushed visualisation and logical argumentation aside.

The $19^{th}$ century led to an enormous growth in the size and meaning of geometry. Projective, descriptive and $n$-dimensional geometry, vector calculus, non-Euclidean geometry, intrinsic differential geometry, topology, and also numerous "buds" in other areas that would only come to blossom in the $20^{th}$ century, such as geometrical probability and measure theory, graph theory and general polyhedral theory, began developing at first without any recognisable relationship to one another. This "explosion" of geometrical disciplines, which led to the century being named the "geometrical century" according to mathematicians, was accompanied by the disintegration of the then dominant understanding of geometry as a science of "true physical space". We will look at how the different approaches for dealing intellectually with the new situation in geometry crucially coined the whole view of mathematics that was dominant until the invention of the computer and its rising popularity. However, we must also examine how geometry lost its central position within mathematics over the course of the first half of the $20^{th}$ century. This has been a development that still negatively influences the organisation of mathematics in secondary and further education nowadays, despite the fact that geometry has achieved a higher than ever level in regards to its theoretical width and depth as well as its practical significance.

At the end of the $20^{th}$ century, geometry was, on the one hand, a huge pool of facts on the "ordinary two and three dimensional Euclidean space" and an even bigger pool of unanswered questions on those. On the other hand, geometry was not really thought of as being part of mathematics in the ordinary sense nowadays, but rather considered a way of thinking, which is more or less useful and necessarily found in almost every realm of mathematics, depending on the scientist's personal approach. Thus, there is a geometrical theory of numbers, a geometrical theory of functions, algebraic geometry and geometrical stochastics. There are geometrical methods within variational calculus, discrete and combinatorial geometry, as well as computer geometry. The latter is not to be confused with computational geometry, which basically refers to a "theory of complexity of geometrical algorithms".

The dichotomy of geometry suggested here has established itself very well in the meantime. The three dimensional Euclidean space remains the appropriate model for all "ordinary" problems, even though it is only a very rough approximation of reality according to the findings of physics. Within the Euclidean plane we create "pictures" of everything we want to "look at" and understand. Their meanings are associated with the dominance of seeing amongst the human senses. Inside the $n$-dimensional Euclidean space, mathematics embeds functions, relations and, almost all other examined objects by using coordinates, for example. Furthermore, geometry predominates in all those areas where a number of possibly very abstract objects are viewed as a "space" by using in broader sense terms taken from geometry, such as topology, metrics, dimension and linearity, with the intention of inspiring our imagination and to use analogies. To what extent one may want to practise this is – as already pointed out – a matter of style. It is an intellectual technique, without which modern mathematics in the form described here could not have developed.

To what degree the latter can really be considered geometry and to what extent the applied branches of geometry belong to mathematics or are already part of engineering is debatable. In the following, we will also defend the concept that there is an "unconscious" unprofessional mathematics that coexists with professional, deductive mathematics. The former manifests itself in the intuitive use of notions, shapes, methods, knowledge and know-how, which is difficult to put into words, but exists as a material product of engineering, handcrafts and the arts. Hence, this book will also serve as a reflection on the historical development of geometry, which will include many, often unusual aspects. We intend to contribute to the clarification of the position and meaning of geometry within mathematics and to raise interest in it.

The critical reader, that we would like to have, may pose the question how a history of geometry fits in a series called '*From Pebbles to Computers*'. What computers have to do with geometry is investigated in detail in Chapter 8.5. With regards to 'pebbles' (accounting tokens) we refer to the Pythagoreans, who got some simple pre-numbertheoretical results from patterns of geometrically ordered stones. Thus they could realize why $ab$ is forever equal to $ba$ and why the distance between two square numbers $n^2$ and $(n+1)^2$ is always $2n+1$.

This book features problems added chapter by chapter, most of which are not historical problems strictly speaking, but problems that result from the history presented here. For instance, questions without answers when they first occurred; questions that just simply did not come to mind, but were possible; old problems that nowadays are much easier to solve given modern methods; and suggestions that result from old problems. Most of the problems are reduced to special cases, contain hints or are asked in a manner that will require only a highschool or slightly more advanced mathematical background to be solved. However, a few questions are more difficult and "open-ended". Here, the reader is invited to probe and explore.

We have avoided the use of first names and the inclusion of the dates births and deaths within the main text apart from a few, well-reasoned exceptions. As far as we could determine those data, they are available in the index of names at the end of the book.

The pictures of the people at the beginning of each chapter are of different styles. We cannot rely on authentic portraits from antiquity or the non-European Middle Ages. (One reason being that people in Islamic countries were often not portrayed due to religious reasons.) However, we must acknowledge that later eras felt the necessity to make pictures of their most important personalities. In this book, a "picture" can be an imagined portrait or a symbolic graphic representation. In this respect, stamps can also serve as a cultural document of the history of science. Multiple books have been devoted to this exact subject [Gjone 1996, Schaaf 1978, Schreiber, P. 1987, Wußing/Remane 1989]. For example, a picture of Euclid (not shown here) was taken from a manuscript of Roman field surveyors (agrimensores). Here, two things are striking. First, these agrimensores thought of Euclid, the master of the logical-axiomatic approach, as their forefather, and, second, the picture has an almost oriental ambience. Considering the mix of peoples and cultures in Alexandria at 300 BC, this may appear more realistic than some neo-classically influenced pseudo-antique art.

From the European Middle Ages onwards, portraits began to appear intentionally more similar to the individual persons, as artists started relying on themselves as models. For example, the portrait of Piero della Francesca is an alleged self-portrait. It comes from his Fresco “Resurrection” (around 1465) located in his hometown of Borgo Sansepolcro.

The picture of René Descartes presented here was painted by Frans Hals shortly before the philosopher departed for Sweden. It is not only one of the very few cases in which a genuinely famous painter portrayed a genuinely famous mathematician (a second example is the portrait of Felix Klein painted by Max Liebermann), but multiple copies of this painting were subsequently made in the $17^{th}$ century reflecting varying facial expressions, which since then partially even flipped horizontal have haunted encyclopaedias and the science-historical literature as images of Descartes.

Peter Schreiber

## Advice for the reader

Round brackets (...) contain additional insertions, translation of original titles or information on illustrations or problems.

Square brackets [...] contain information on literature within the text, explanations or references below illustrations.

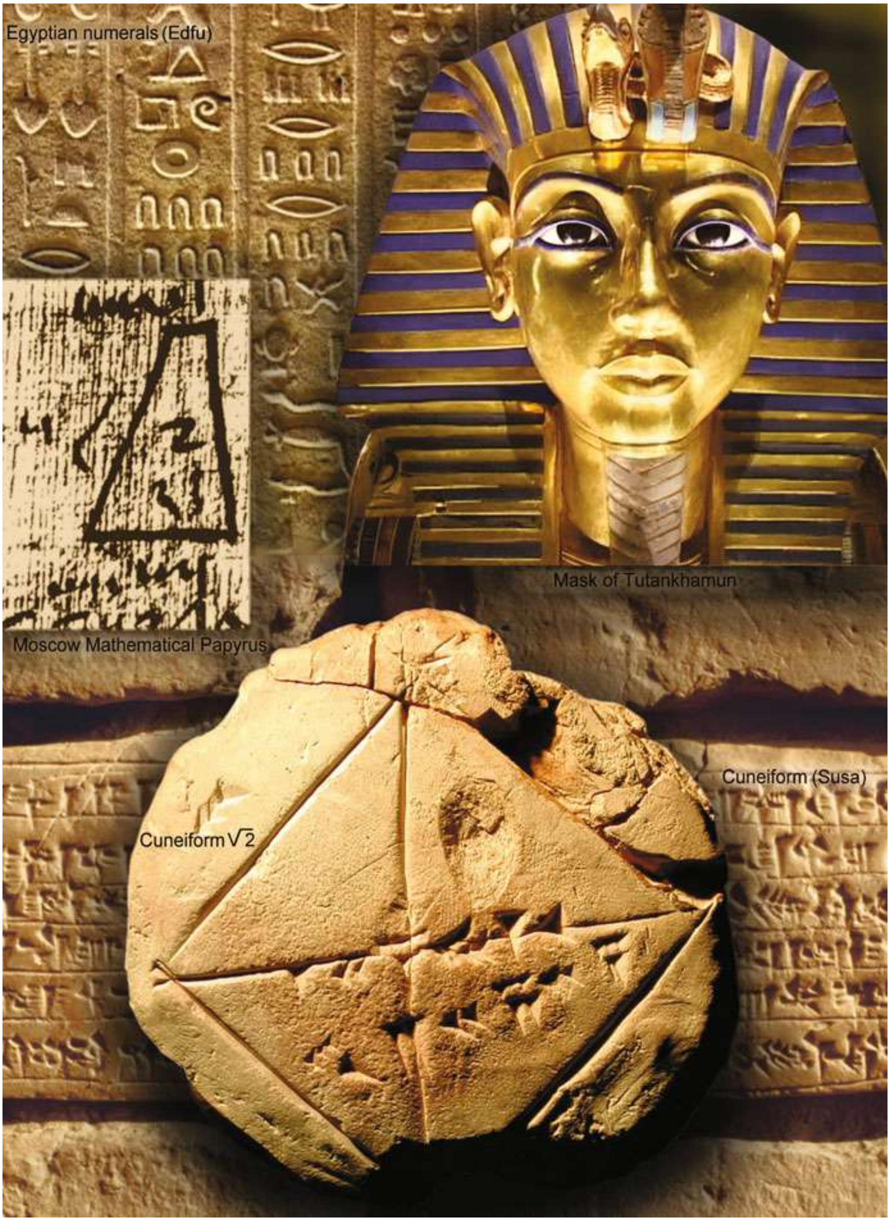
Illustrations have been numbered according to sub-chapters, e.g. illustration 7.4.3 is the third illustration of Part 4 of Chapter 7.

In order for the reader to find related texts more easily, problems have been summarised at the end of each chapter and been numbered according to sub-chapters, e.g. problem 7.3.6 is the sixth problem of Part 3 of Chapter 7.

The problems are of different sizes and vary in level of difficulty. Problems or partial problems, which the publisher believes to be especially challenging, have been marked with an*. However, we would like to point out that such a judgement is clearly subjective and depends on the reader's individual knowledge and skills.

The kind of quotations and the references follow the style of the author P. Schreiber.

# 1 The beginnings of geometrical representations and calculations

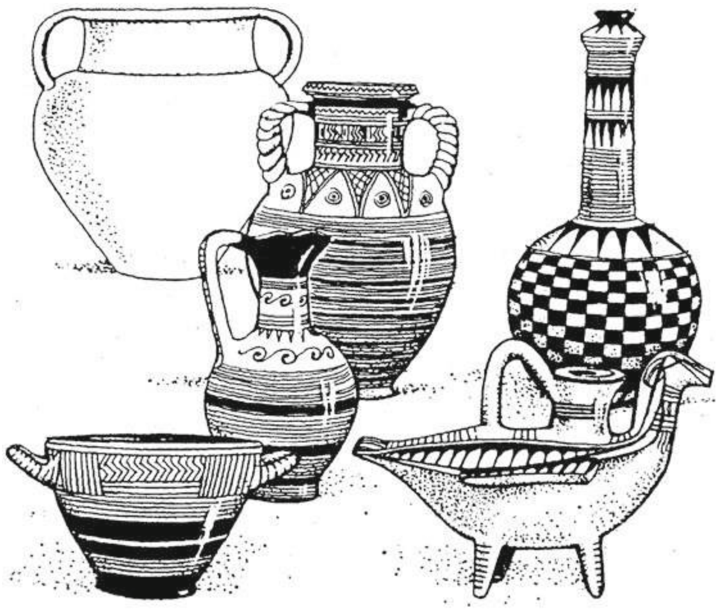





C.J. Scriba, P. Schreiber, *5000 Years of Geometry*,
DOI 10.1007/978-3-0348-0898-9_1

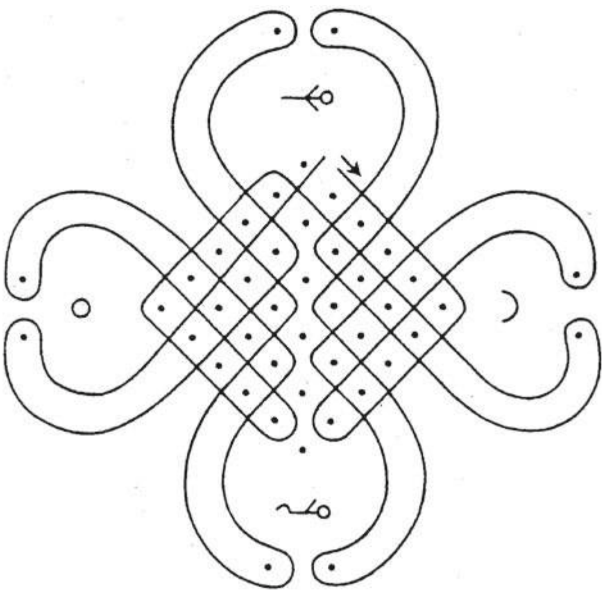
## 1.1 Primal Society

Long before writing was developed, mankind may have realised and systematically used geometrical structures. Nature offers the eye multiple curved lines, and a blade of grass or a tree trunk can symbolise the thought of a straight line as well as the idea of a circle (as a cross-section). When weaving or braiding we generate simple two-dimensional patterns, which then are purposely modified or also replicated as decoration on clay pots. There is evidence that such purposely geometrically shaped ornaments existed already in 40 000 BC. They can be so characteristic for such cultural societies that prehistorians can reconstruct their migrations by digging up and analyzing clay fragments. For instance, we can find in the Cretan culture patterns of folded strips on Neolithic clay pots, or six congruent circles, aligned around a central circle of the same size and touching each on two neighbouring circles. The equilateral triangle, the square (with four right-angled corners) or also the regular hexagon must have been noticed very early as special cases of plane shapes, awakening playful interest as well as first theoretical considerations (cf. e.g. [Kadeřávek 1992]).



**Illus. 1.1.1** Geometrical ornaments on prehistoric ceramic
[Drawing by Hubert J. Pepper from "The Dawn of Civilization" edited by Stuart Piggott, Thames and Hudson Ltd., London]

Needs and activities of everyday life provided further inspiration: when constructing ditches, dams or houses, and land surveyingelementary geometrical ratios were required. Men probably did not realise this at first until their first logical considerations set in. Without three-dimensional solids (cuboids, cubes, pyramids, columns) building was impossible. Observing the course of the stars suggested a transition from the plane triangle to the spherical triangle. It seemed to be obvious that the diagonal bisects the square or rectangle as does the diameter the circle. All pre-Greek cultures have been aware of such immediately insightful relations and applied them in practise. Only the Greeks started probing and asking for reasons. They finally arrived at an axiomatic construction of a geometric theory that has been passed down to us by Euclid's 'Elements'. If we want to focus primarily on Egyptian and Babylonian geometry in the following, we must emphasize that there is no culture that does not reflect the versatile use of geometrical elements. Designing jewellery is often heavily influenced by religious ideas: Pots devoted to the gods would feature more abundant decorations, the altars would feature special shapes and rituals (including dances) which would be conducted in a special manner. We also must not neglect play as a source of engaging with geometrical properties. This goes beyond just board games, which are almost always sources for symmetrical patterns.



**Illus. 1.1.2** Single course line concerning the cosmogonic myth of Jokwe in Angola: The course of the sun (left), moon (right) and man (below) to god (above) [Africa counts: Number and Pattern in African Culture, ©1973 by Claudia Zaslavsky. Publ. by Lawrence Hill Books, an imprint of Chicago Review Press Inc.]

**Illus. 1.1.3** Goseck circle (near Halle, Germany), Nebra sky disk (State Museum of Prehistory, Halle/Saale) [Photo: H. Wesemüller-Kock]

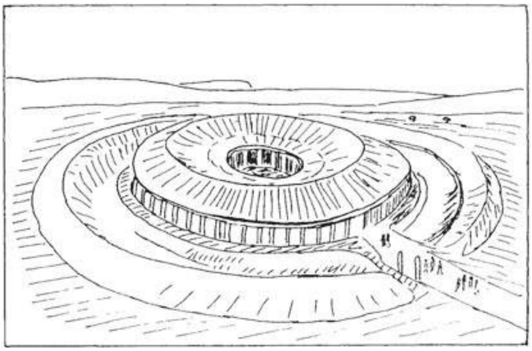
Ethnomathematics, which recently has turned towards the implicit mathematical ideas of primitive people, yields some astonishing research results. For example, there is an African tribe in Angola, whose people draw a shape freehandedly from a single curve, which interlaces elaborately, when telling their cosmogonic myth. This indicates thorough geometrical considerations, if the desired outcome with its symmetrical properties is to be achieved (Illus. 1.1.2). Since we have been aware of its changes, the starry sky has provided men with further inspiration to make basic geometrical observations. The movements of the shadow of a tree trunk or towering stone, taken over the course of a day or a year, form the basis for a simple sundial. Drawing the course of the shadow lace systematically on the ground, the result is a projection of the course of the sun on the sky in plane curves, which encourage us to think about it. In the 1990s, in Goseck (near Halle, Germany) a set of concentric circular ditches, dating back to approx. 4800 BC, was discovered, archaeologically researched and reconstructed. It is the earliest sun observatory currently known worldwide (Illus. 1.1.3). Circular ditches were constructed in Central Europe close to settlements around 4800 to 4500 BC. Goseck's circle features a dual ring of palisades with three gates, one each facing north, southeast (sunrise on 21 December) and southwest (sunset on 21 December). The distance between the palisades grows wider around 21 June. This configuration allowed farmers of 7000 years ago to determine, by means of position of the sun, the most propitious times to sow and harvest over the course of the year. However, as findings indicate, circular ditches were also used for cultural purposes. Only about 2000 years later the most famous construction of the megalithic culture ($3^{rd}$ and $2^{nd}$ millennium BC), Stonehenge near Salisbury in the south of England, was erected, which has been interpreted as a sun observatory and a cult site [Gericke 1984], (Illus. 1.1.4).

Research of the last decades has shown that Stonehenge not only reflects use of astronomical knowledge but also basic geometrical ratios, e.g. Pythagoras's theorem. However, we can only assume that the Pythagorean triangle with side lengths 3, 4, 5 (for instance, it is possible to mark them with knots on a rope of length 12) was used that early to generate right angles. Researchers

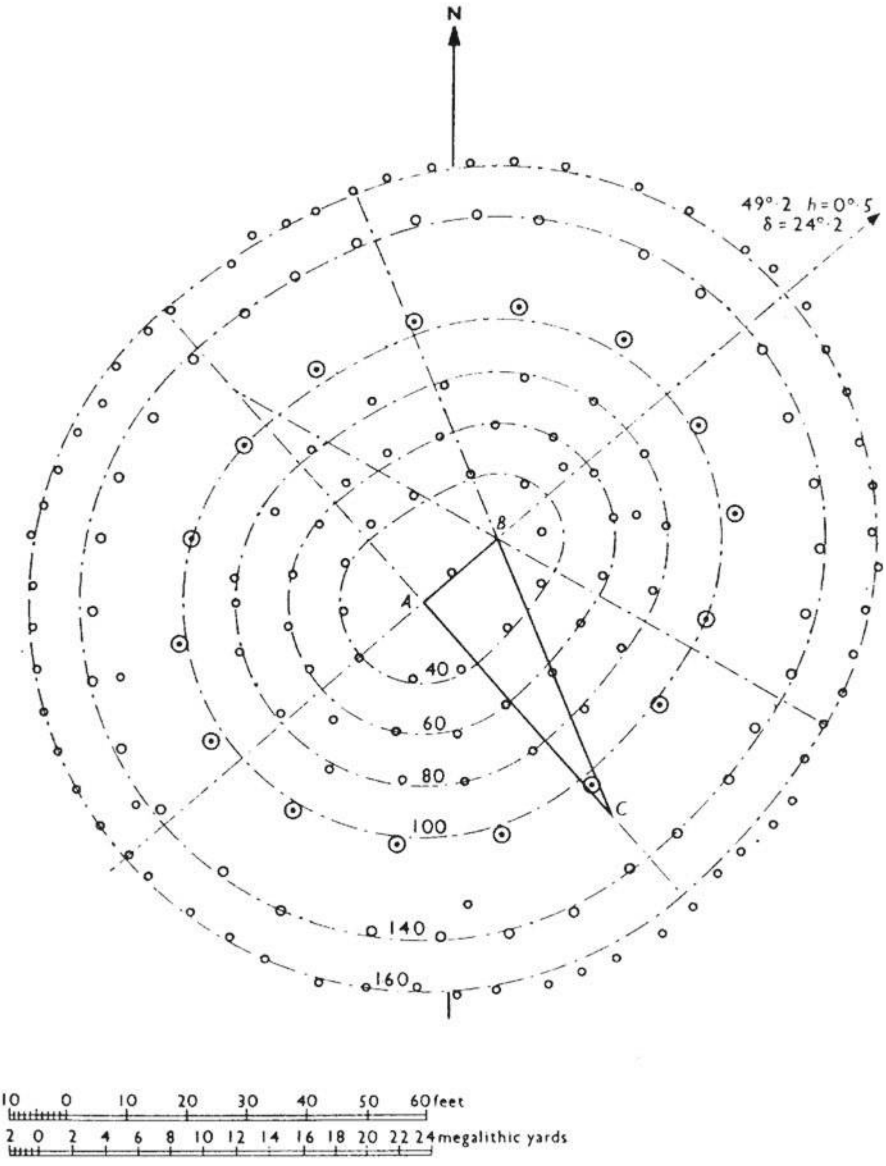
**Illus. 1.1.4** Stonehenge (South England): The biggest preserved stone monument in Europe from the 3rd/2nd millennium (diameter of outer ring approx. 100 m) [Photo: H.-W. Alten]

argue that they can prove that the wood construction of Woodhenge (approx. 1800 BC) was built by applying the Pythagorean triangle 12, 35, 37 (Illus. 1.1.5, 1.1.6).

For Stonehenge see [North 1996]; for a critique on the hypothesis of the right angle view [Knorr 1985]. The bronze Nebra sky disk, found just recently near Halle (Germany), comes from approx. the same time as Woodhenge. Its constellation of the stars with the Pleiades is taken to be the first sky representation [Schlosser 2004]. This disk has been the source of lively debates in regards to theories of interpretation and meaning, whose final outcomes are expected in the near future.



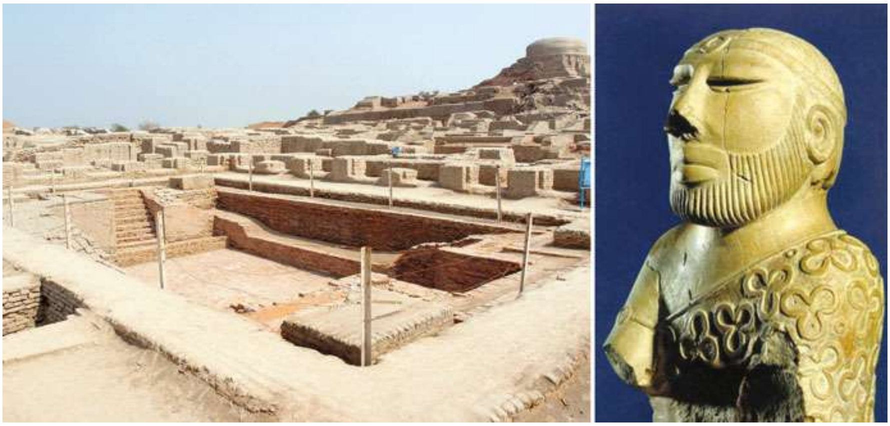
**Illus. 1.1.5** Reconstruction of Woodhenge
[Ashbee, P.: The Bronze Age Round Barrow in Britain, Phoenix House Ltd, London 1960]




**Illus. 1.1.6** Ground plan of Woodhenge
[Thom, A.: Megalithic Sites in Britain, Oxford, Clarendon Press 1967, Fig. 6.16 p. 74, by permission of Oxford University Press]

## 1.2 Old river valley civilisations

| | | |
|---|---|---|
| 3000-2000 | Town civilisations at the Indus valley: Harappa and Mohenjo Daro | Script not yet deciphered |
| 3000-2700 | Union of kingdoms at Nile | Hieroglyphs invented |
| 3000-2700 | Sumerian city states | Cuneiform on clay tablets developed |
| 2700-2170 | Old kingdom in Egypt | Pyramids built |
| 2700-2100 | Akkadian invasion and reign | Nomographs |
| 2170-2040 | First intermediate period of Egypt | |
| 2040-1794 | Middle kingdom in Egypt | Mathematical papyri |
| 2100-1900 | Several kingdoms in Mesopotamia | |
| 1900-1600 | Old Babylonian kingdom | |
| 1728-1668 | King Hammurabi in Babylon | Tablets of law |
| 1794-1550 | Second intermediate period of Egypt | |
| 1550-1070 | New kingdom in Egypt | Temple of Hatschepsut |
| 1290-1224 | Pharaoh Ramses II | Amun temple in Karnak |
| 1285 | Battle of Kadesh | Graves in Valley of the Kings |
| 1600-625 | Hittites, Kassites, Assyrians rule in Mesopotamia | Mathematical scripts in cuneiform |
| 1070-525 | Late period in Egypt: Libyans, Ethiopians, Assyrians rule at Nile | |
| 625-539 | New Babylonian kingdom | Astrology and astronomy prosper |
| 539 | Cyrus the Great conquers Babylon | |
| 525 | Persians conquer Egypt | |
| 332 | Alexander the Great conquers Egypt | |
| 323-30 | Egypt reigned by Ptolemy Dynasty | Egypt trade and cultural centre of the world |
| | | Eratosthenes of Cyrene director of Library, Euclid and Apollonius in |
| 47 BC | Library of Alexandria on fire | Alexandria |
| 30 BC | Egypt becomes Roman province | Hero of Alexandria |
| 391 AD | Library of Alexandria is destroyed | Pappus and Proclus work in Alexandria |
| | | Mathematician Hypatia murdered by pagan persecution |
| 395 | Egypt becomes part of the Eastern Roman Empire (Byzantium) when the Roman Empire is divided | |



**Illus. 1.2.1** Mohenjo-Daro. Exvacated ruins of one of the largest Settlements of the ancient Indus Valley Civilisations [Photo: Saqib Qayyum, 2014]; stone statue of a 'Priest-King', found in 1927 AD in Mohenjo-Daro (National Museum, Karachi, Pakistan) [Photo: Mamoon Mangal]

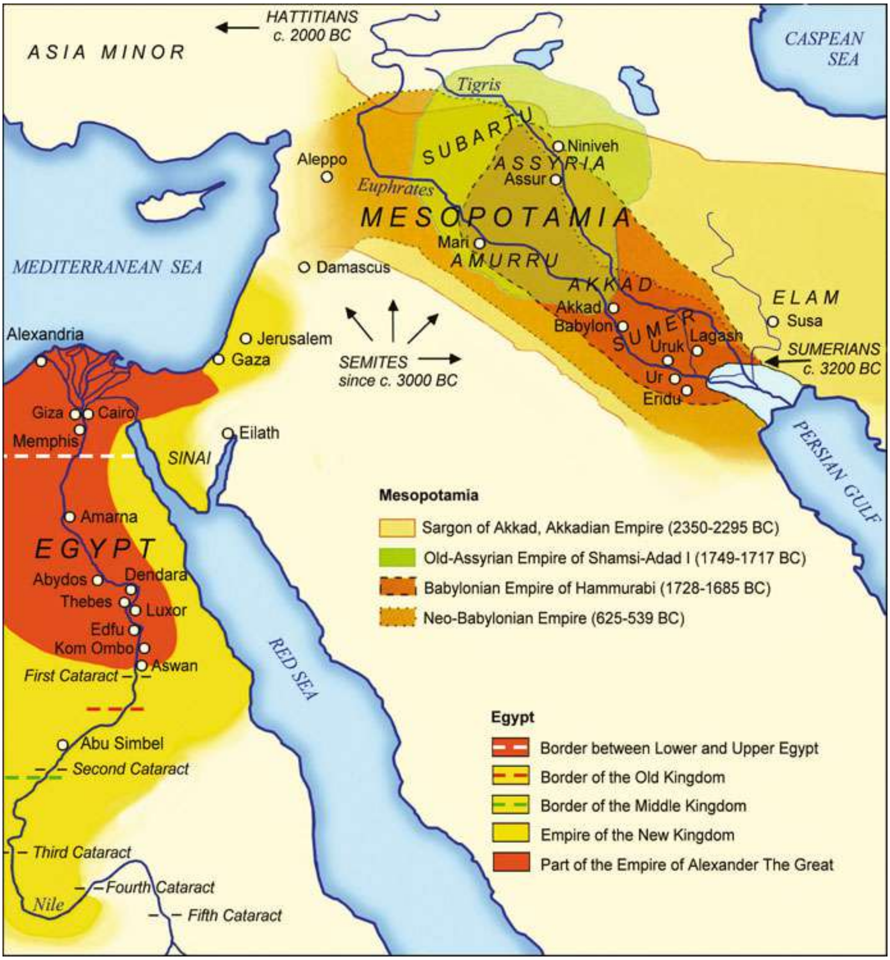
### 1.2.1 Indus civilisations

One of the oldest advanced civilisations of mankind is the settlement Mohenjo-Daro at the Indus. The town belonging to the Harappa culture with approx. 40 000 inhabitants experienced its heyday around 2500 BC. It was almost as old as the Egyptian kingdom located along the Nile and Mesopotamia situated between the river valleys of Euphrates and Tigris. In all archaeological sites of this culture, bricks feature the same side lengths with a ratio of 1:2:4, streets follow the outline of a chessboard and weights were standardised. Since excavations and interpretations of the findings of Mohenjo-Daro (located in todays Pakistan) are still continuing, we are not able to reach final conclusions on the role of geometry in this cultural area.

### 1.2.2 Egyptian mathematics

We have gained better insights into the geometrical knowledge of old Egypt and Mesopotamia (also called Babylonia), since both civilisations have their origins in the Neolithic age, and have left written sources behind, which have been studied in great depth since the middle of the $19^{th}$ century.

Hieroglyphs had been developed since approx. 2900 BC in the strictly organized and centrally administrated Egypt. Next to the impressive constructions of the pyramids, two mathematical papyri from the time of the middle kingdom ($11^{th}$ to $13^{th}$ dynasty) have served particularly well as sources for our knowledge of Egyptian geometry. Their content reflects the level of knowl-




**Illus. 1.2.2** Egypt and Mesopotamia in ancient times
[Map: H. Wesemüller-Kock]

edge for approx. or shortly after 2000 BC. The two most important ones are the Rhind Mathematical Papyrus and the Moscow Mathematical Papyrus. They constitute collections of problems with relevant approaches to solving them. They seem to be texts, which have been written by teachers (writers) at schools for officials to serve as teaching handbooks. The Rhind Mathematical Papyrus was originally 5.34 m long, but only 33 cm wide. The Moscow Mathematical Papyrus was 5.44 m long, but only 8 cm wide. The latter contains 25, the former 84 problems ordered according to factual aspects, which sometimes feature visualising drawings. Thereby, geometrical solids are represented by their top or side views, since perspective drawing was unheard-of in Egypt of that time. Sometimes the same drawing even demonstrates the most important aspect in a top view and individual parts in front views, e.g. the representation of a rectangular pond with trees on the edge, the trees are folded over to the left side (Illus. 1.2.3).

Relief designs and other wall pictures provide evidence that surveying the ground of a temple was a holy act accompanied by many ceremonies, which only the pharaoh or the highest priests were allowed to carry out. The holy and mysterious aspects of the art of surveying and constructing were reflected by conserved amulets, which have the shape of simple geometric instruments. However, it does not seem likely that they drew the construction and transferred them to the building true to scale. Top and front views of columns and ledges in original size have been found on suitable plane surfaces of stone. Realisations of these can be found in surrounding buildings [Kadeřávek 1992].

**Illus. 1.2.3** "Pond in a Garden" Change of perspective in the same picture, fresco from the Tomb of Nebamun, Thebes, c. 1400 BC [British Museum London, MDID Collection]

One of the simplest geometric problems is the calculation of area $A$ of rectangles, trapeziums and triangles. The approximation formula for any quadrilateral with sides $a, b, c, d$ is

$$A = \frac{(a+c)}{2} \cdot \frac{(b+d)}{2}. \tag{1.2.1}$$

Hence, it entails dual averaging of the opposite sides. Interestingly, this rule has also been applied to a triangle by zeroising the fourth side (better: omitted because not existing, since the Egyptians did not know the concept of the number zero). A peculiar instruction is applied when calculating area A of a circle by means of a given diameter $d$: deduct 1/9 of its length and multiply the result with itself and the outcome is

$$A = (\frac{8}{9}d)^2. \tag{1.2.2}$$

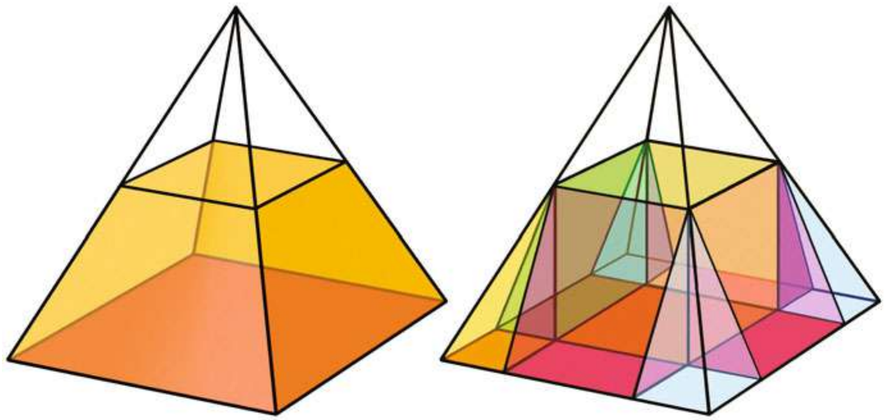
As usual, there is no reason for the astonishingly accurate method. However, problem 48 of the Rhind Mathematical Papyrus contains a drawing showing a square of side length 9, which is turned into an octagon by cutting off the edges. This can be interpreted as a circle approximation. This shape inspired Kurt Vogel in 1928 to interpret the Egyptian instruction (see Problem 1.2.1).

Apart from plane shapes, in Egyptian texts also volumes are calculated, when structurally engineered problems or calculation of the holding capacity of pots and basins are concerned. Hereby, the mention of a layer measure for volumes is remarkable. Similarly, there is a stripe measure for calculating areas. It suggests more of a calculation of the volume of a brick by multiply inserting a layer, which equals its base and whose height constitutes the unit measure, on top of one another (like when making plywood boards), rather than calculating the volume of a brick by means of filling it with unit cubes (since we use the latter method nowadays to multiply length, width and height). All problems are calculated like recipes and only with concrete numerical values. In these early times, men had neither a method to express formulae nor abstract quantities.

When calculating volumes, they mainly dealt with cuboid-shaped or cylindrical containers, whereby the mentioned formula for circular areas was used. The great pyramids suggest that the old Egyptians must have also known the capacity formula for pyramids. However, there is no proof of this. (As proven by Max Dehn in 1900, a strict derivation of this formula for any pyramid is impossible without a limit process. See also Problem 1.2.2 for special cases.).

In contrast, Problem 14 of the Moscow Mathematical Papyrus contains the correct instructions to calculate the volume of a square truncated pyramid according to the correct formula

$$V = \frac{h}{3} \cdot (a^2 + ab + b^2) \tag{1.2.3}$$



**Illus. 1.2.4** Regarding the calculation of the volume of a truncated square pyramid [Design: H. Wesemüller-Kock]

($V$ = Volume, $a$ = length of basis edge, $b$ = length of top edge, $h$ = height). You can arrive at this formula, if the one for the volume of a pyramid is known (see Problem 1.2.3). As pointed out, there is no evidence prominent in the sparsely preserved Egyptian texts that this formula was used.

Sometimes the Egyptians approximated the square truncated pyramid by calculating an average, i.e. they treated it like a cuboid, whose basis $B$ was chosen to be the arithmetic means of the basis area and top surface area:

$$B = \frac{1}{2}(a^2 + b^2). \tag{1.2.4}$$

leads to

$$V = \frac{h}{2}(a^2 + b^2). \tag{1.2.5}$$

Historian of mathematics Kurt Vogel pointed out that the Egyptians may have realised their mistake and, as a result, have inserted a median area unit $a \cdot b$:

$$B = \frac{(a^2 + ab + b^2)}{3}. \tag{1.2.6}$$

This way, they discovered the correct calculation instruction from an incorrect formula by means of unproven generalisation. (Beyond: If we view a pyramid as a truncated pyramid with the top surface area $b^2 = 0$, the formula for the capacity of the truncated pyramid delivers the correct formula for the volume of the pyramid.)

**Illus. 1.2.5** Cheops Pyramid of Giza, the tallest of all pyramids [Photo: H.-W. Alten]

**Illus. 1.2.6** Cheops Pyramid and Sphinx 1858 AD. The Sphinx deeply covered by sand [Photo: Francis Frith 1858]

### 1.2.3 Babylonian mathematics

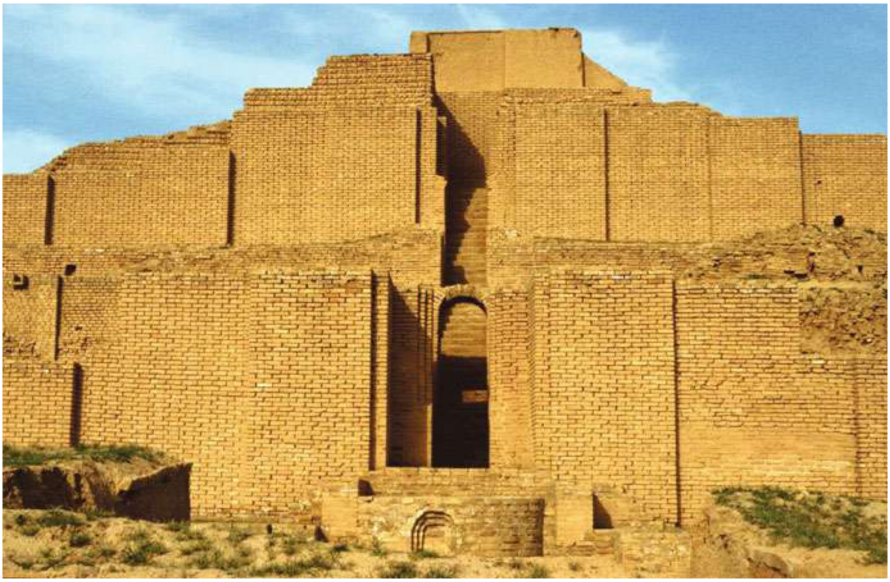
The sources of Babylonian mathematics are much richer than those of the Egyptians, since Mesopotamian clay tablets were used to write on. They have survived time much better than the perishable papyrus. Numerous texts date back to the time of the old Babylonian kingdom (ca. 1900 – 1600 BC), which followed the time of the Sumerian city states (approx 3000 – 2700 BC) and the Akkadian reign (approx. 2700 – 2100 BC). However, findings of the following centuries, in which Mesopotamia experienced a lot of political turmoil (Assyrian and Chaldean reign, rule of the Hittites), show that after the initial development of mathematics hardly anything changed for a long time. The next advancements occurred in the period of the Seleucids (the last pre-Christian centuries), especially in astronomy, since just as in Egypt, Mesopotamian mathematics was applied practically and developed within this context: economy, trade, building industry and sky observations led to mathematical considerations, which reached a higher peak than in Egypt. Researchers were especially amazed when they found the Pythagorean Theorem and a method for calculating square roots in texts in 1916 AD.

Field plans, ground views of houses or technical constructions of dams or channels are often attached to the appropriate calculation instructions and enable us to gain initial insights into the practical nature of the problems. A special terminology is partially missing. Instead they have borrowed words from everyday language to refer to, for instance, a wall, a dam, a ditch etc. Of course, when they wanted to calculate the area of regular polygons and, consequently, scratched matching geometrical drawings into tablets, there seems to be early theoretical interest, which exceeded immediate everyday needs, just as in the so-called Babylonian algebra (see Illus. 1.2.9).

The frequent calculation of the diagonals of rectangles by means of the Pythagorean Theorem is particularly striking, given that it was centuries before Pythagoras was even born. Thereby, Babylonian mathematicians chose numeric values in a manner that would guarantee rational sides. However, they were also able to calculate square roots by means of approximation either through iteration or applying Hero's formula,

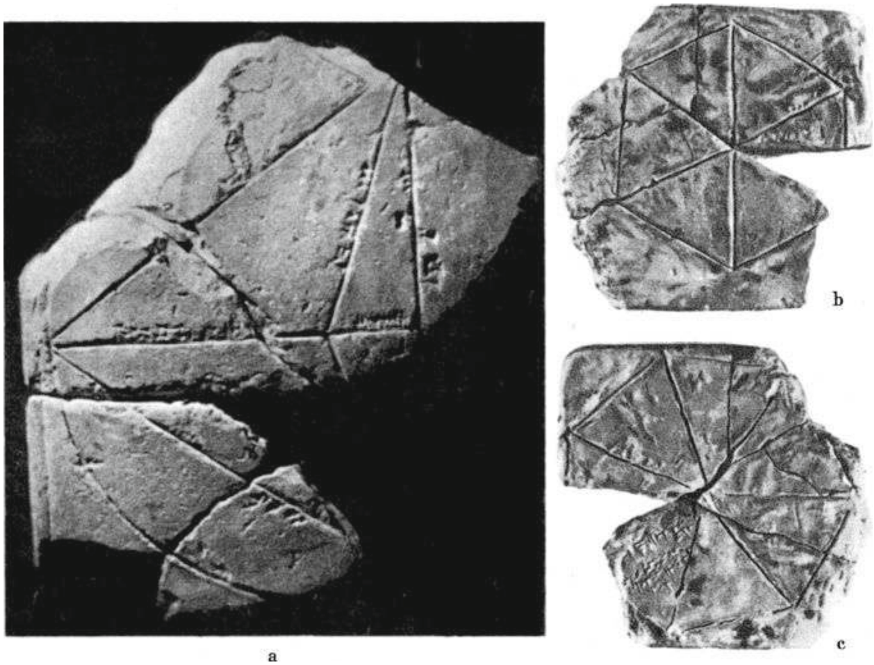
$$\sqrt{n} = \sqrt{a^2 \pm r} \approx a \pm \frac{r}{2a}, \tag{1.2.7}$$

whereby $n$ is broken down into the nearest square $a^2$, either increased or reduced by rest $r$. One problem of this type, not just known from contemporary school teaching, but also occurring in Chinese and Indian mathematics, as well as the European Middle Ages, is manifested in a problem the Seleucids wrote down about a bar leaning against a wall (BM 34568, British Museum London). When first leaned against the wall perpendicularly it reaches an unknown height. Then the foot of the bar is moved nine cubits from the wall, which brings the top of the bar three cubits lower. The aim is to calculate bar



**Illus. 1.2.7** Ziggurat of Chogha Zambil. The gradual tower made of clay bricks has the typical shape of a temple erected by the Sumerians, Babylonians, Assyrians and Elamites. The five-stage Ziggurat of Chogha Zambil built around 1250 BC is the best preserved building of its kind
[Photo: H.-W. Alten]

**Illus. 1.2.8** Elamite cuneiform text at the Ziggurat of Chogha Zambil
[Photo: H.-W. Alten]




**Illus. 1.2.9** Babylonian polygons. [Kurt Vogel: Vorgriechische Mathematik (Pre-Greek Mathematics), part II. Illus. 22a-c, p.69: Mémoires de la Mission Archéologique française en Iran, Tome XXXIV p. 12]
[Bruins and Rutten, published by Paul Geuthner, Paris 1961]

length ($x$). Hence, we must calculate quantity $x$ by means of the following formula for a Pythagorean triangle: $x^2 = (x-3)^2 + 9^2$ (see Problem 1.2.4).

The widespread division problems also form a part of geometry. For instance, if we want to divide any four-sided field with the sides $a, b, c, d$ into two parts of equal area by means of a transversal $x$, which runs from $b$ to $d$, we follow the instruction below:

$$x = \sqrt{\frac{(a^2 + c^2)}{2}}. \tag{1.2.8}$$

You could interpret this approximation as forming a median square of both squares above sides $a$ and $c$, whose side in this case is taken to be the quantity of the transversal. However, as you can see, this method neglects the lengths of sides $b$ and $d$. Thus, this instruction can only be used to deliver a proximately correct value for certain field forms (see Problem 1.2.5).

In order to calculate circles, Babylonians used a novel method, which was completely different from those applied by the Egyptians. Area $A$ of a circle was calculated by taking a detour and using its circumference c. Thus, we were supposed to take a twelfth of the square of the circumference,

$$A = \frac{c^2}{12}. \tag{1.2.9}$$

Thereby, they accepted the triple diameter $d$ as the circumference. If we apply this, the result is $A = \frac{9d^2}{12} = 3r^2$. The question arises as to why the Babylonians calculated the circular area in this peculiar manner, even though it suggests using either the diameter or the radius. First, we need to understand that two factors of proportionality occur when studying the circle. On one hand, there is a fixed ratio between the two lengths of diameter and circumference. On the other hand, there is a fixed ratio between the two areas of diameter or radius square and circular area. Only Archimedes showed that both factors are identical by stretching out the periphery of the circle or, in other words, by strictly proving the ratio (see section 2.4.2)

$$A = \frac{1}{2} \cdot c \cdot r. \tag{1.2.10}$$

Since $c = 3d = 6r$ means that you can approximate the circumference by means of the periphery of the inscribed hexagon, you might think that you could calculate the circular area by viewing the circle sector as a triangle of baseline c/6 and height r approximated from the internal angle by 60°. As a result, $A = 6 \cdot (\frac{1}{2} \cdot \frac{c}{6} \cdot r) = 6 \cdot (\frac{1}{2} \cdot \frac{c}{6} \cdot \frac{c}{6}) = \frac{c^2}{12}$. (Another explanation would be that they calculated the arithmetic average between the circumscribed square $d^2$ and the inscribed square $\frac{d^2}{2}$, hence $\frac{3d^2}{4} = 3r^2$, which then afterwards was calculated back from the radius to the circumference.)

The Babylonians also dealt with circle segments cut off from the circle through $a$ chord $c$ (cf. Illus. 1.2.5), whose height (the line segment, which stands perpendicularly on the middle of the chord between chord and circumference), also called sagitta $s$, was calculated with diameter $d$ and the chord $c$ according to the formula

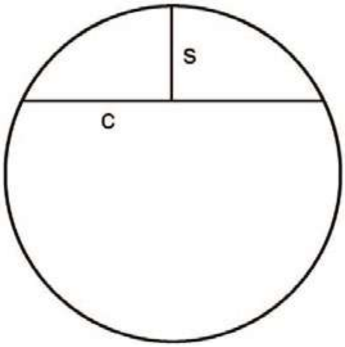
$$s = \frac{1}{2}(d - \sqrt{d^2 - s^2}). \tag{1.2.11}$$

The chord or segment base c was calculated according to

$$c = \sqrt{d^2 - (d - 2s)^2}. \tag{1.2.12}$$

(see Problem 1.2.6).

We are dealing here with the first steps of chord geometry, which was further developed by Hipparchus later on. Ptolemy placed this subject at the beginning of his great astronomic textbook (called "Almagest" by the Arabs; cf. section 2.5.4). However, a notion of angle in general had not yet been developed in Babylonian mathematics; only the right angle occurs implicitly




**Illus. 1.2.10** Circle segment with chord c and sagitta s

in regards to rectangles and Pythagorean triangles. It was possible to measure the inclination of oblique planes, by which e.g. dams were bordered, by a so-called rebound by one cubit. They stated on which height section the rebound by one cubit occurred.

Within the realm of problems of spatial geometry occurring everyday the calculation of cuboids, perpendicular prisms and cylinders did not constitute any problems (if necessary, the base was calculated by means of the already discussed approximation formula before multiplying it by height). Neither in Egyptian sources nor in Mesopotamian texts the formula for the volume of a pyramid has been found until now. The Babylonians used the same approximation formula for the truncated pyramid, which we have already encountered in Egypt (1.2.5). This certainly confirms that calculating the average is one of the earliest and most widespread mathematical considerations. Text BM 85194,28 could contain the same formula as we know it from Greek sources, if factor $(a-b)/3$ was omitted, which in this case equals 1 due to the special numeric values of $a$ and $b$.

$$V = [(\frac{a+b}{2})^2 + \frac{1}{3} \cdot (\frac{a-b^2}{2})] \cdot h = \frac{a^2+ab+b^2}{3} \cdot h. \qquad (1.2.13)$$

In one case the frustum of a cone is also calculated by forming the average according to the formula

$$V = \frac{1}{2}(F_1 + F_2) \cdot h. \qquad (1.2.14)$$

Repeatedly had been pointed out that Babylonian mathematicians used the Pythagorean Theorem - or perhaps better said: the validity of $a^2 + b^2 = c^2$ in rectangular triangles. An old Babylonian text published by Otto Neugebauer and Abraham Sachs in 1945 was long taken to be particularly remarkable. This text can be found in the Plimpton Collection of Columbia University in New York and quickly became known as “Plimpton 322”

**Illus. 1.2.11** Plimpton 322; Old Mesopotamian text in cuneiform (Plimpton Library, Columbia University, New York). The text contains a list of right-angled triangles with integer sides h, w and d. A few columns on the left have broken off. The second and third column feature width w and the diagonal (hypotenuse) d in integers. The last column specifies the ongoing row numbering. In rows 11 and 15 values w and d have a common factor; in all other instances they are coprime

[Neugebauer/Sachs 1945]. It seemed that this text contained a type of trigonometric table, fitting in the great number of tables which we have from Mesopotamia.

Apart from the title, the text of the table (there may be a piece missing on the left side) consists of five columns and fifteen rows, which indicate only numbers (see illus. 1.2.7). The column on the right edge shows an ongoing row numbering from 1 (indicated by a perpendicular wedge) to 15 (next to the wedge for one, which can be repeated up to nine times, we also find the 'Winkelhaken' ('hook'), the Babylonian symbol for ten, from row 10 onwards.) The preceding column always reflects the same term, i.e. it indicates the following row numbering. The analysis of the mostly multi-digit sexagesimal numbers in the first three columns showed that the first column indicates the ratios, which are ordered according to decreasing values:

$$\frac{d^2}{h^2} = \frac{(h^2 + w^2)}{h^2} = 1 + \frac{w^2}{h^2}. \tag{1.2.15}$$

This led Neugebauer and Sachs to interpret the table as a systematically arranged sequence of 15 right-angled triangles. The one in the first row would be almost isosceles; the last one would have the approximate angles of 30° and 60°. This interpretation led to the sensational conclusion that Plimpton 322 was indeed a trigonometric table from the Babylonian era. Other researchers attempted a number-theoretic interpretation based on reciprocal number pairs. In both cases, we would be dealing with a highly abstract text, which until then had not been demonstrable in Mesopotamian mathematics. Besides, our knowledge of the Babylonian angle concept contradicts a trigonometric interpretation. Since the view had been accepted during the last decades that such texts should not be analysed in isolation but under consideration of the entire cultural environment, both interpretations were doubted, especially the one by Eleanor Robson [Robson 2001]. The tablet written in the middle of the $18^{th}$ pre-Christian century comes from the town of Larsa in present Iran and belongs to a collection of texts, which were categorized as accounting and bureaucracy. The header of the tablet and the composition of the table make this evident. They remind us of other clay tablets, on which writers composed exercises and collections of examples for teaching purposes. In this case, we would be dealing with a collection of problems on Pythagorean triples, for which in the second column preferably simple numeric values were chosen, whose geometrical relation is yet unclear. This interpretation eliminates the special position, which tablet Plimpton 322 had been assigned by the previous description of Mesopotamian mathematics. The belief that trigonometry already existed approx. 4000 years ago has been reduced to a mere myth. However, this tablet remains a unique compilation of Pythagorean triples to date, which are connected to each other by a quadratic relation.

**Illus. 1.2.12** Cuneiform text with quantities on a claybull with seal ($18^{th}$ century BC), Old Babylonian Period(l.); Cylinder with tablet of quantities according to capacity, Old Babylonian Period (r.) (Vorderasiatisches Museum Berlin) [Photo: H. Wesemüller-Kock]

## 1.3 Problems to 1

**Problem 1.2.1:** Reconstruction of the Egyptian approximation formula for a circular area

a) Reconstruct Kurt Vogel's interpretation of the Egyptian approximation formula for the circular area (1.2.2): A circle with a diameter of $d = 9$ has a square circumscribed, which is divided into nine squares with a side length of 3. Each of the four corner squares has the outer half (along a diagonal) cut off, which gives you an octagon approximated to the circle. Calculate its area and approximate it to the most proximate square root, from which you extract the root! Accordingly, this root is the side length of a square, which together with the circle of diameter $d = 9$ are almost of equal area. To what extent does the Egyptian formula result from that?

b) In order to judge the quality of this approximation formula, calculate the resulting value for $n$ with two digits after the comma!

**Problem 1.2.2:** Volume of a pyramid

Prove the formula for calculating the volume of a pyramid $V = \frac{1}{3} \cdot B \cdot h$ ($B$ = Base, $h$ = height) for two special cases:

a) If we draw the four spatial diagonals in a cube with side lengths $a$, the solid is divided into six congruent perpendicular pyramids with a square base, whose apex is located in the centre of the cube.

b) Consider that this cube can also be divided into three congruent oblique pyramids of height a (only use one spatial diagonal)!

c) Is it possible to apply this train of thought to a cuboid with sides $a, b, c$?

**Problem 1.2.3:** Volume of a pyramid frustum

Derive the formula for calculating the volume of a pyramid frustum by using the formula for calculating volume $V$ of a pyramid, $V = \frac{1}{3} \cdot B \cdot h$ ($B$ = base, $h$ = height), and by dividing the frustum (base = $a^2$, top surface area = $b^2$, height = $h$) into a cuboid, four lateral rest prisms and four corner pyramids! (see Illus. 1.2.4)

**Problem 1.2.4:** Applying the binomial formula

In the text from the Seleucids' era (BM 34568), obviously composed for teaching purposes, occurs a rectangle with the sides $a = 4$, $b = 3$, for whose diagonals $d$ we are given the terms $d = a/2 + b$ and $d = b/3 + a$. Then we are supposed to calculate the three pieces $a, b$ and $d$ by means of the quantities $a + d$ and $b$ or $b + d$ and $a$. Other than that, we can use $a - b = 1$ and $A = a \cdot b = 12$, or $a + b = 7$ and $A = a \cdot b = 12$.

a) Solve these problems according to the modern method!

b) Retrace the Babylonian steps of calculation by using the term $(a+b)^2 - 4A = (a-b)^2$ passed down to us on clay tablets, in order to, first of all, calculate the difference $a-b$ from the sum $a+b$ and thereafter the sum and difference of quantities $a$ and $b$ themselves!

**Problem 1.2.5:** The transversal formula

Derive the transversal formula (1.2.8) $x = \sqrt{(\frac{a^2+c^2}{2})}$ by using the area formula (1.2.1) for any quadrilateral (which also occurred in Egypt):

$$A = \tfrac{a+c}{2} \cdot \tfrac{b+d}{2}.$$

a) Sketch the following: in the original field divide transversal $x$, sides $b$ and $d$ in $b_1$ and $b_2$ or $d_1$ and $d_2$, whereby the new field $A_1$ is embedded by line segments $b_1, a, d_1, x$ and the new field $A_2$ by line segments $b_2, c, d_2, x$. Consequently, $4A_1 = 4A_2$ and $4A_1 + 4A_2 = 4A$ must be valid for the quadruple areas, from which it is possible to derive the transversal formula.

b) Show that this formula does not apply only to rectangles (given that the transversal is parallel to $a$ and $c$), but also to trapeziums, if $x$ is also parallel to $a$ and $c$.

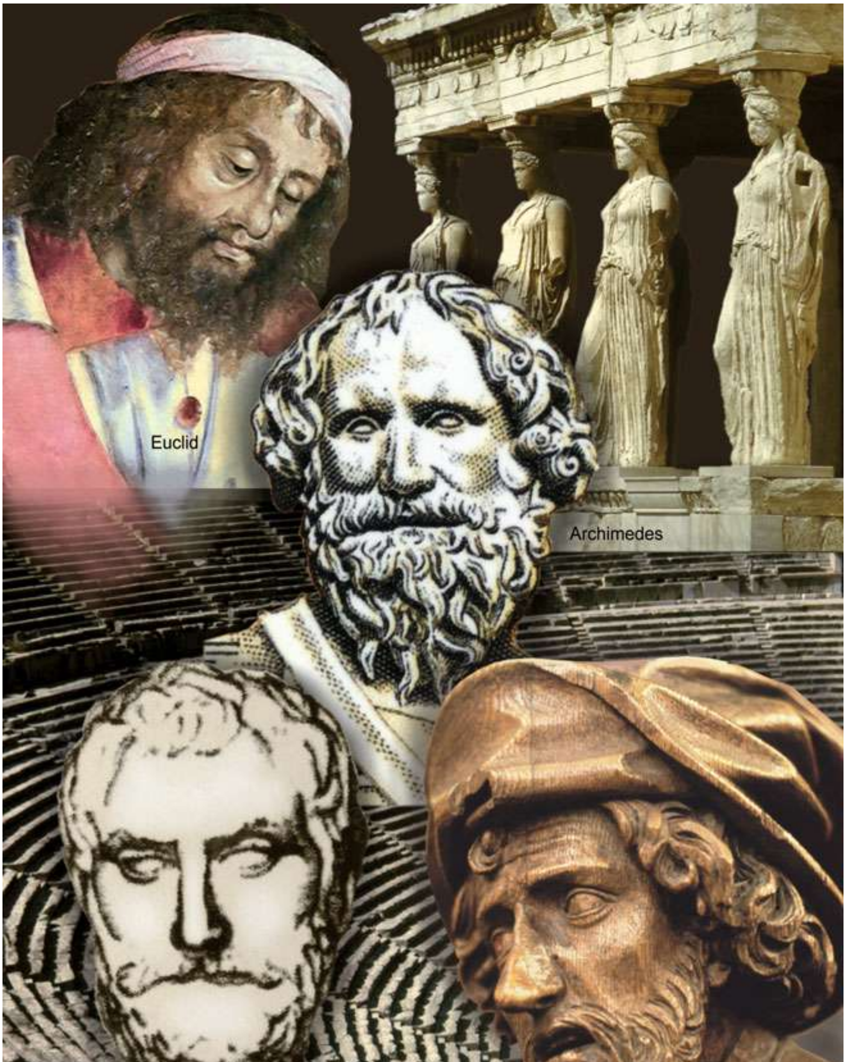
**Problem 1.2.6:** Segment of a circle

Show that it is possible to derive both formulae (1.2.11) and (1.2.12) for sagitta $s$ and chord $c$ of a circle segment (see Illus. 1.2.10) by means of the Pythagorean theorem.

**Problem 1.2.7:** A Pythagorean triangle in the sexagesimal system

Show that the famous Pythagorean triangle (3, 4, 5) in row 11 of the Plimpton tablet (the row with the smallest number in Illus. 1.2.11) is presented multiplied by one factor. The numbers stated there are: $\frac{w^2}{d^2} = 33, 45, b = 45, d = 1, 15$. Consider that the Babylonian manner of writing numbers was not familiar with decimal points – better: sexagesimal points. Thus, value 1 can also refer to 60.

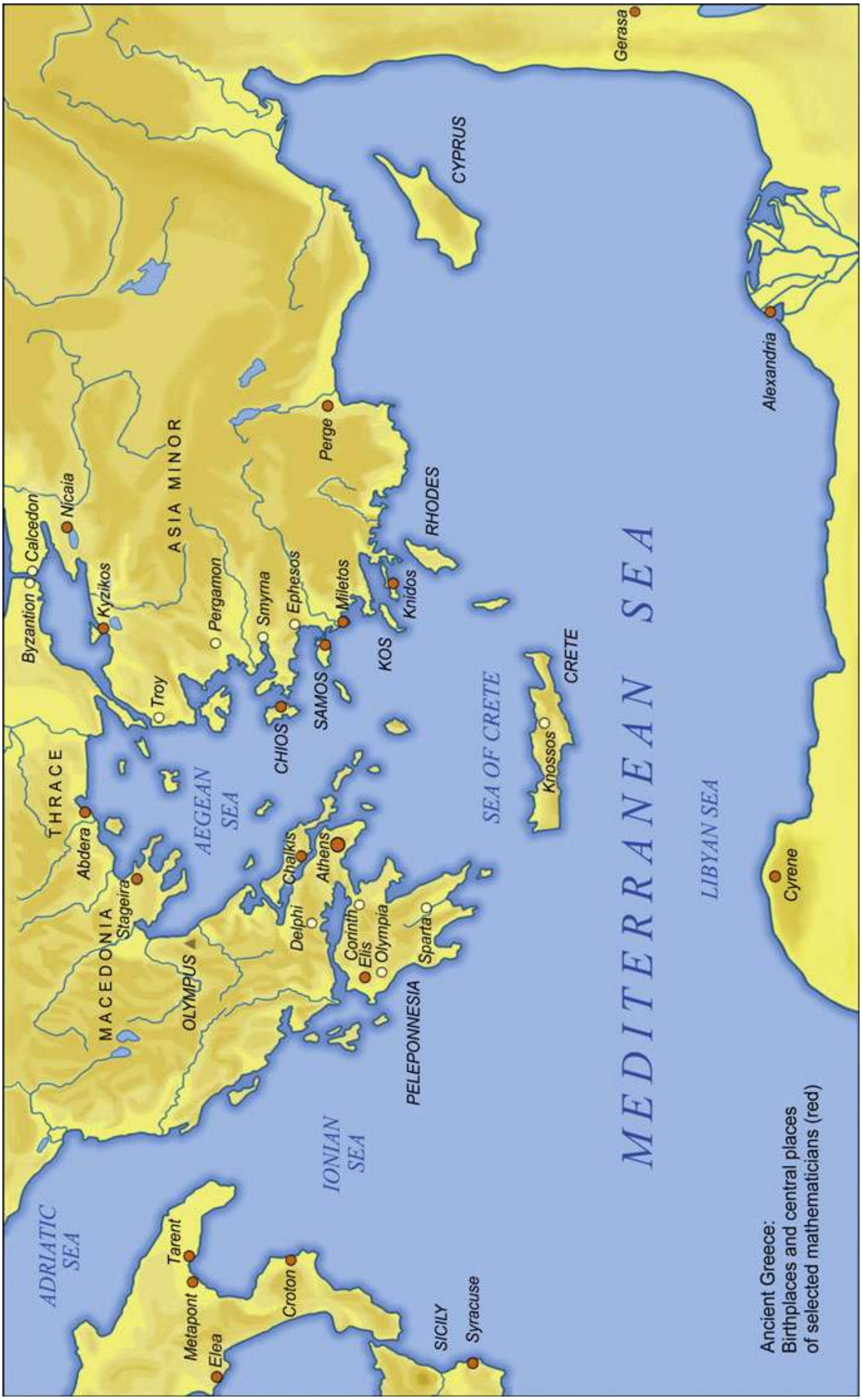
# 2 Geometry in the Greek-Hellenistic era and late Antiquity






C.J. Scriba, P. Schreiber, *5000 Years of Geometry*,
DOI 10.1007/978-3-0348-0898-9_2

| | | |
|---|---|---|
| $3^{rd}/2^{nd}$ milennium | Mycenae reign over Peloponnese and Crete | Minoan and Mycenaean culture |
| Approx. 1000 BC | Doric migration | Geometric arts |
| ~900 BC | Greeks settle down in Aegean islands and west coast of Asia Minor | Greeks adopt Phoenician alphabet |
| $8^{th}$–$6^{th}$ century BC | Founding of Greek colonies in Sicily, lower Italy, Libya and at the Black Sea | Archaic arts, monumental plastics and architecture, vase painting, Homer's epics, Hesiod's teaching poems |
| Approx. 600 – approx 450 | Ionian era | Natural philosophy: Thales, Anaximander, Hecataeus, Anaximenes |
| 490 | Battle at Marathon | Pythagoras, Heraclitus |
| 490–448 | Persian wars | Parmenides, Empedocles, Anaxagoras |
| Approx. 450–300 | Classical Greece (Athenian era) | Sophists |
| 462–429 | Golden age of Athens under Pericles | Socrates, Plato, tragedies of Aeschylus, Sophocles, Euripides |
| 431–404 | Peloponnesian war between Athens and Sparta | Origins of Dorian, Phrygian, Lydian music keys |
| 387 | Founding of Platon's Academy | Acropolis of Athens is built, classical sculptures of Phidias and Polycleitus |
| From 338 | Greece under Macedonian rule | |
| 335 | Aristotle founds Lyceum | |
| 334–323 | Campaigns of Alexander the Great to conquer Persia, Egypt and India | Hippocrates of Cos founder scientific medicine |
| 311 | Division of the Alexandrian Empire | Hellenistic culture develops due to mix of Greek and Oriental cultures |
| Approx. 300–150 | Hellenistic (Alexandrian) era | Stoicism, Epicureanism, Scepticism; teaching poems and epigrams as poetic categories; late Baroque period of Greek art; mathematical and natural scientific findings by Euclid, Aristarchus, Archimedes, Apollonius of Perga |


Ancient Greece:
Birthplaces and central places
of selected mathematicians (red)
ADRIATIC SEA
IONIAN SEA
AEGEAN SEA
SEA OF CRETE
MEDITERRANEAN SEA
LIBYAN SEA
THRACE
MACEDONIA
ASIA MINOR
OLYMPUS
PELEPONNESIA
SICILY
CHIOS
SAMOS
KOS
RHODES
CRETE
CYPRUS
Elea
Metapont
Tarent
Croton
Syracuse
Abdera
Stageira
Delphi
Chalkis
Athens
Corinth
Elis
Olympia
Sparta
Troy
Byzantion
Calcedon
Nicaia
Kyzikos
Pergamon
Smyrna
Ephesos
Miletos
Knidos
Perge
Knossos
Cyrene
Alexandria
Gerasa

## 2.0 Introduction

Generally speaking, the Greeks are accepted to be the founders of the natural sciences, in other words, of rational explanations of natural phenomena based on principles and systems. At the same time, it was they who systemised and accounted for rules and instructions passed on (partially by the Oriental cultures) for counting, measuring and solving equations by means of a self-developed logic. These were summarised into a system of theories, which made them the founders of scientific mathematics.

Sometime near the end of the $2^{nd}$ or beginning of the $1^{st}$ millennium BC, as part of their migration, the Dorians invaded Greek areas (especially within Peloponnese), apparently as a consequence of the downfall of the Mycenaean state, which had been highly developed culturally and strictly administrated. The Dorians advanced further and further from the Northwest from the Albanian-Dalmatian coastal area, where they had settled down originally, and colonised the Greek home country, which had been populated by the Achaeans for a millennium. The natives were conquered or drew back to the islands and the west coast of Asia Minor as part of the so-called Ionian migration. As a result, within a very geologically and geographically confined space divided into small and very small areas, there was a plenitude of different tribes and peoples, each of which took its own course and development. The structure of the city states (polities) became determinant politically as well as culturally. Colonists settled down as farmers and tradesmen, especially in the colonies of Miletus in Asia Minor, at the south coast of the Black Sea, and at the Nile Delta, all of which were home to centrally organised empires. As a result, they were influenced intellectually and culturally by various Oriental aspects. They became familiar with collections of observations and rules of conduct, which provided them with material for the gradual organisation of scientific thinking.

This period in Greek mathematics is called here and treated as Ionian era (approx. 600–450 BC) and followed by Classical Greece (Athenian era, approx. 450–300 BC), whereas these eras both are usually subsumed under Classical Greece (about 600–300 BC).

## 2.1 Ionian era

### 2.1.1 The early natural philosophers

The Ionian era (approx. 600 – approx. 450 BC), which is followed by Classical Greece in the middle of the $5^{th}$ century, is usually cited as the beginning of the "discovery of the mind" (book title by philologist Bruno Snell [Snell 1946]). The feudalism of the aristocracy was replaced by the polity/town structure during this era. Next to the Oriental empires with centralist ruling, independent Ionian trading towns blossomed. Due to the merchants' practical thinking and the small confined areas of political structure and administration, citizens took part in public life to a greater extent. These towns became the centres of classic Greek culture and science. The peripheral areas of the Mediterranean Sea and the Black Sea were also Hellenised due to the founding of colonies. The Ionian era is particularly known for the first great natural philosophers of all time: Thales, Anaximander and Anaximenes. It is here that we find the origins of European thinking and, also, of the deductive method in mathematics, which developed in this era in close correlation with logic.

It is not possible to reconstruct in detail this unique process due to the almost complete absence of immediate sources, especially since the reports passed down to us later on are often written from a certain perspective and, therefore, tend to be subjective. Proclus ($5^{th}$ century AD) mentioned Thales of Miletus (ca. 600 BC) a few times in his commentary on Euclid, relying on historical communications written by Aristotle's student Eudemus (ca. 320 BC). Apparently, Thales was not only the first Greek philosopher, but also the first mathematician, bringing mathematics from Egypt to Greece and making a lot of discoveries himself. According to Herodotus, who lived in a temporally proximate time, Thales had Phoenician roots; approx. 300 years before his time, the Greeks had adopted the Phoenician alphabet. Babylonian astronomical knowledge may have permitted Thales to forecast a solar eclipse that took place in 585 during the battle between the Lydians and the Persians at Halys River and caused the fight to be abandoned.

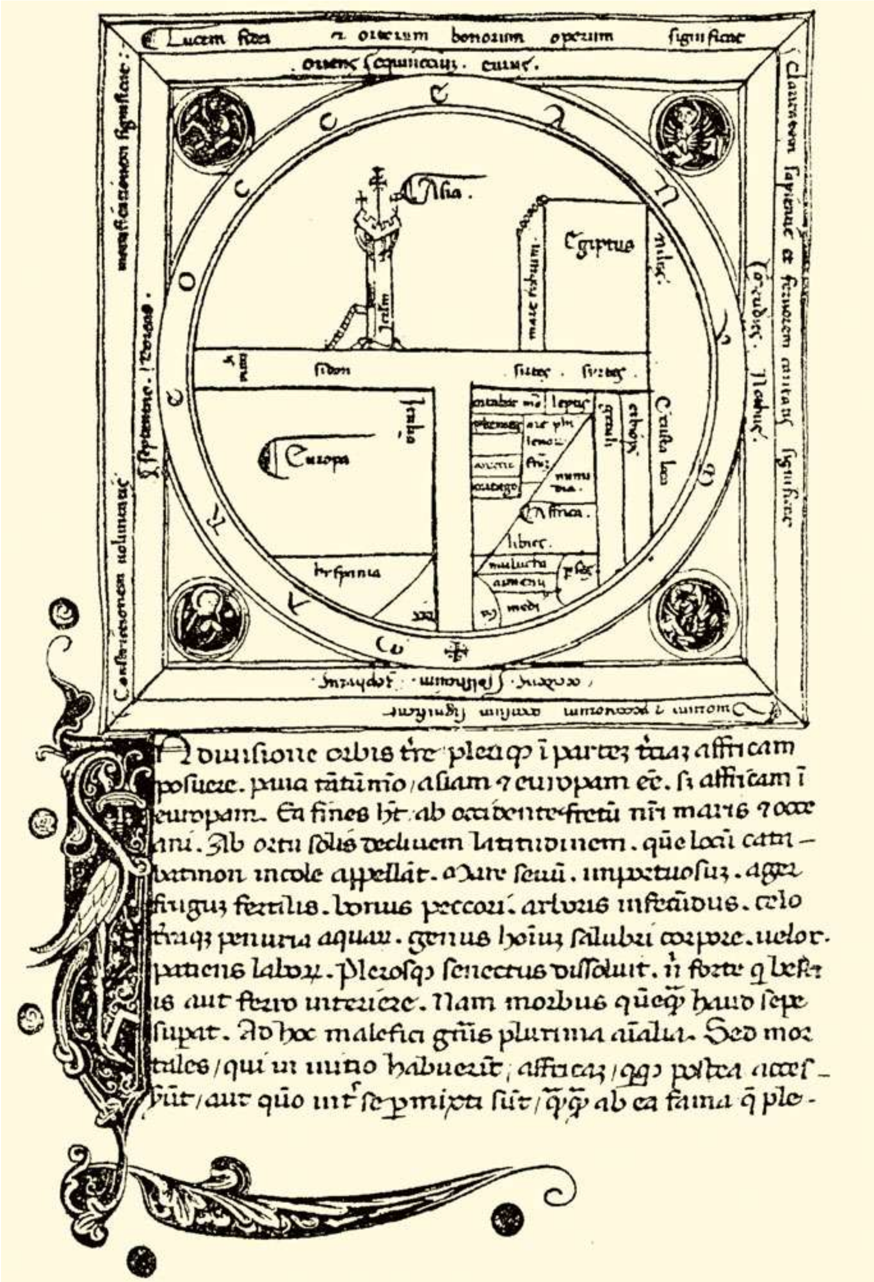
The Ionian natural philosopher Anaximander of Miletus, who worked until the middle of the $6^{th}$ pre-Christian century, was a little younger than Thales. He is said to have brought the gnome (an instrument used to measure sun shadow and made of a perpendicular bar fixed onto a horizontal board) from Babylon to Greece. Furthermore, Anaximander developed a mathematically organised world system: the Earth located at the worldly centre is said to have the shape of a column drum, the height of which was a third of its diameter; fixed-star sky, moon and sun orbit the Earth like turning wheels with distances of 1-3-3, 2-3-3, 3-3-3 Earth diameters (i.e., moon and sun outside of fixed-star sphere!). We recognise how such system concepts geometrically determined by elementary number ratios coined this early view of the world. Geometry and symmetry are means of visualisation of the natural regularities and reoccurring laws of nature.

Here, at the beginning of Greek natural science, the early philosophers could not make use of an accomplished mathematical theory. Yet, they accessed its so to speak first building blocks, when attempting to clarify their view of the world by means of simple numeric rules and elementary geometric shapes. In his representation of ancient science, which is worth reading [Krafft 1971, p. 200], Fritz Krafft argues that we "cannot refer to mathematics as an aid

**Illus. 2.1.1** Theatre of Miletus
[Photo: H.-W. Alten]

to recognition of nature and its forms (...), since its mathematical field as a science itself only has developed and existed in the human mind since the middle of the $5^{th}$ century." Thereby, he implies that "mathematics" is a theory that has been well developed by the human mind and can be used as an "aid". The development of this "own realm of mathematics" took place in close correlation to external reality, as we can see many times during the course of the history of geometry. Anaximander's model allows us to take a look at the early phase of this development. The analysis of structures found in nature led repeatedly to new mathematical theories, which were then used to explain or visualise further natural (later also economic, social and other) phenomena. The mutual interlocking of empirical observation and scientific theoretical formation is not just characteristic for natural sciences, but also for the historical development of mathematical thinking. Thereby, we do not want to deny that there were repeated phases, in which inspiration for developing mathematical theories originated from mathematics itself. Furthermore, mathematical impulses from within were very determining, whereas in other cases external challenges provided stimuli for new developments.

But now let us go back to the early natural philosophers, who are also often referred to as Pre-Socratic philosophers. The generation after Anaximander was dominated by Hecataeus of Miletus (born in approx. 560). One long journey led him to Egypt, others to the Persian Empire and Scythia at the north coast of the Black Sea. Hecataeus (following Anaximander, who is accepted to have designed the first picture of Earth) drew a map of the Ecumene,




**Illus. 2.1.2** World map with OT representation. Tip: North is on the left. The large surrounding circle represents the circularly drawn ocean; the horizontal crossbar of the T reflects the Nile on the right; the Bosporus and the access to the Black Sea on the left; the trunk of the T indicates the Mediterranean Sea. This OT form determined the design of maps of Earth within the European area until the Middle Ages (Concerning the tradition of Ptolemaic maps cf. paragraph 5.2.)

[Sallust, manuscript from $14^{th}$ century] the contemporary term for the known world, by means of written credentials supplemented by personal experience. Since Hecataeus only had scant information on many regions, he was forced to schematise. Thereby, his use of geometrical concepts is characteristic. He divided Earth, which was imagined to be a circular disc, with a diameter into two semi-circles.

“I have to laugh, when I see some people drawing maps of the world, which yet do not really know how to explain the shape of Earth. They draw Oceanus running around Earth and as regularly as a circle” [Krafft 1971, p. 175].

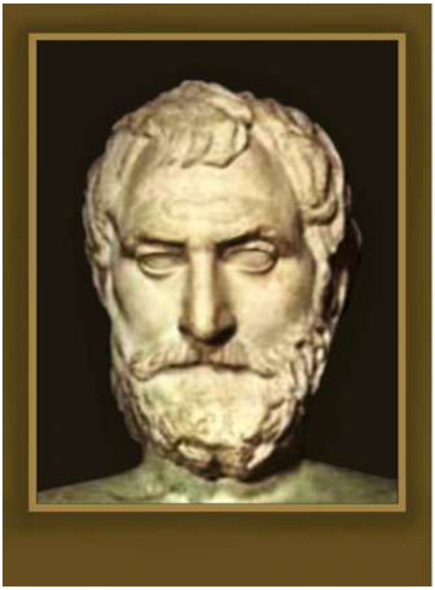
It was not the highest aim of this type of representation to remain truthful to details, as we expect from modern geographical maps, but rather to visualise principles and generalities in a memorable manner. Hecataeus described Scythia and the north of Libya in a similar geometrical manner by means of rectangles which were segmented by parallel stripes and which were home to different tribes.

### 2.1.2 Thales

Thales has been referred to as one of the Seven Sages of Greece since the $4^{th}$ century BC. The late ancient Neo-Platonist Proclus, who, as already pinpointed, could indirectly utilise the lost mathematical history of Eudemus of Rhodes, passed down the first four theorems, which have been accredited to Thales as a mathematician. These theorems are often represented in short as follows:

1. The base angles in an isosceles triangle are equal (Euclid, ‘Elements’ I, 5).
2. The vertical angles between two intersecting line segments are equal (Euclid, ‘Elements’ I, 15).
3. A triangle is determined by one side and both adjacent angles or, in other words, two triangles, which agree in one side and the adjacent angles, thereby agreeing in all units (Euclid I, 26).
4. The diameter halves the circle.

To compare, let us look at how, for instance, the first and third theorem are worded by Proclus:
“May the old Thales be blessed, the discoverer of many other and especially these theorems! It is said that he was the first to recognise and say aloud that the base angles in every isosceles triangle are equal, but that he used ‘similar’ instead of ‘equal’ in an old-fashioned manner.” [Proclus/Morrow 1992, p. 341f.].




**Illus. 2.1.3** Thales of Miletus (l.); detail from the Market Gate of Miletus (Vorderasiatisches Museum Berlin SMBI) [Photo: H.-W. Alten]

"But Eudemus states in his 'History of geometry' that this theorem is to be accredited to Thales. Given the manner, with which Thales is said to have determined the distance of ships at high sea [from the shore?], Eudemus concluded that there was no other way than to use this theorem." [Proclus/Morrow 1992, p. 409].

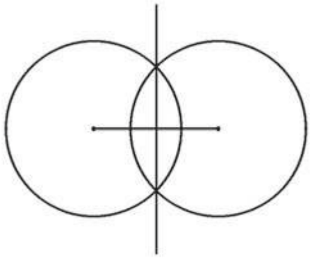
We notice instantly that the theorems mostly refer to elementary symmetric ratios. The fact that Thales is said to have been the first to 'find' and say these theorems aloud can mean at most that he was the first Greek to have phrased these theorems explicitly. It was also considered to mean that he may have presented us with the first reasons to prove their correctness, i.e., considerations of how to prove these theorems. The second of the Proclus quotes shows that such assumptions stand on rather wobbly ground. Therein, Thales is accredited with knowledge of a method to determine the distance of ships at high sea from the shore (cf. Problem 2.1.1). This must have had a geometrical background, from which Eudemus concluded that the third theorem is also to be accredited to Thales. However, he could have brought this method with him from Egypt.

Moreover, there are two further theorems, which are related to each other. The second one especially is often linked to Thales:

5. The diagonals of a rectangle are equal and halve each other.
6. The peripheral angle in a semi-circle is a right one.

The theorem stated last, known as Thales' theorem, has been passed on by the female historian Pamphile ($1^{st}$ century AD), as reported by Diogenes Laertius ($3^{rd}$ century). In order to express his gratitude for recognising this fundamental fact, Thales is said to have sacrificed an ox to honour the gods!

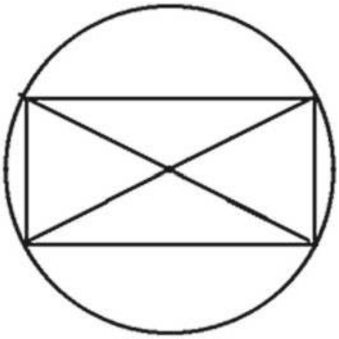
As we already know from the paragraphs on pre-Greek mathematics, the line segment and/or the straight line and the circle (next to the point) belong to the oldest geometrical elements. This is supplemented by Thales' mentioning of the notion of the angle, which is created by intersecting two straight lines. They also must have noticed early that the right angle ($R = 90°$) amongst all others has a special position. It occurs when two line segments so intersect each other that each one is an axis of symmetry of the other one. The relevant construction is stated in Euclid's 'Elements' I, 9–11. Principally, we can construct this playfully by taking a given line segment and drawing two circles with a compass from the extremities using the same opening. (The connecting line of both circle intersections is placed perpendicularly onto the initial line segment "for symmetrical reasons".):



**Illus. 2.1.4** Construction of a right angle

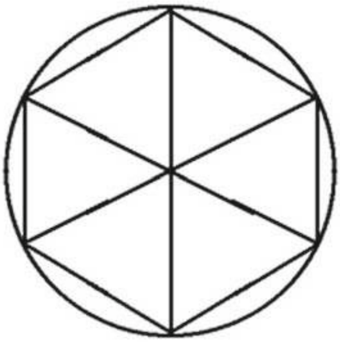
From this, it is easy to conclude the construction of a rectangle with four right angles and the fact that each of its opposing sides is of equal length. If we add both diagonals, they are also of equal length and halve each other. Thus, a circle drawn around the intersection of the diagonals touching one corner of the rectangle will also touch the remaining three corners. (Illus. 2.1.5)

Using this shape we can demonstrate almost all theorems that are connected to Thales. The equality of the vertical angles according to Theorem 2 would result from the relevant arc of the circle, which again is determined by the equality of the opposing sides of a rectangle and leads to the equality of the diagonals (Theorem 5). Theorem 1, concerning base angles within an isosceles triangle, would again be the result of the symmetry of a rectangle, as well as Theorem 3 (at least, concerning the two special cases of the isosceles and the right-angled triangle). Theorem 6 is further verified by the fact that the same shape, half a rectangle in each, is found in the semi-circles divided by a diagonal. Hereby, the construction begun with any rectangle and the resulting circumstances are equal for all of them.



**Illus. 2.1.5** Rectangle with circumcircle

Thus, we can derive the theorem immediately by looking at the shape: "The sum of angles in a triangle equals two right angles" (first, for all right-angled triangles, then also for any triangle in which we still need to draw a height $h$). However, this consequence has not been accredited to Thales in the ancient tradition, but to 'the elder', i.e., probably the subsequent generation. Besides, the theorem was proved independently for both the equilateral and the isosceles triangle. Concerning the latter, the rectangular shape in the circle could have served again as the basis of evidence and the isosceles triangle itself suggests consideration of the hexagon inscribed in the circle, which can be viewed as one of the original geometrical shapes:



**Illus. 2.1.6** Regular hexagon inscribed in the circle

Every angle at the centre here equals $60°$ or a sixth of four right angles. Since each angle at the centre lies opposite to a triangle side of length $r$ ($r$ = radius), the other two angles also equal $60°$, thus, $3 \times 60° = 180° = 2R$. The symmetric ratios form the basis for the logical conclusion in this case, too: since the three sides are of equal length, there is no reason for the three angles not to be of equal size as well. Such simple symmetric ratios form the starting point of every geometrical theory.

Finally, the ancient sources report that Thales determined the height of tall buildings by means of their shadow. To do so, he is said to have waited for the time of day at which his own shadow was as long as he himself. This implies that he knew how to deal with similarity contemplations (see Problem 2.1.2).

### 2.1.3 Pythagoras and the Pythagoreans

Unfortunately, reliable texts are scarce concerning the early development of mathematics in ancient Greece and subsequent eras, if we ignore the fact that Pythagoras of Samos chronologically follows Thales. Pythagoras is said to have acquired his wealth of knowledge on his great journeys, during which it is believed that he gained his mathematical expertise from Babylonian sources. Soon after the middle of the $6^{th}$ century, he immigrated to Croton in lower Italy, where he founded a religious/philosophical community and soon turned into a mythical figure. It is unclear nowadays which geometrical contributions were made by Pythagoras himself and which by his followers (he died in approx. 500 BC). Aristotle was already referring to them as "Pythagoreans"; this school of thought died out during the course of the $4^{th}$ century. (We must not confuse this movement with the so-called Neo-Pythagoreans, who followed up on the old school of thought from 100 AD onwards. The mathematicians Porphyry and Iamblichus formed part of this later movement.) Eudemus's historical report, mentioned above, which unfortunately has been passed on only indirectly, states that Pythagoras is said to have turned geometrical knowledge into 'free teaching'. This supposedly means that geometry was studied for its own sake, in contrast to acting according to rules given by necessities, practical life or sacred purposes. Hence, here lies the beginning of the development of pure mathematics.

The number was at the core of Pythagoras's religious-philosophical teaching. Based on the unit (monas) and the natural numbers 2, 3, 4, ..., the Pythagoreans developed the first foundations of number theory as well as a musical theory based on simple numeric ratios.

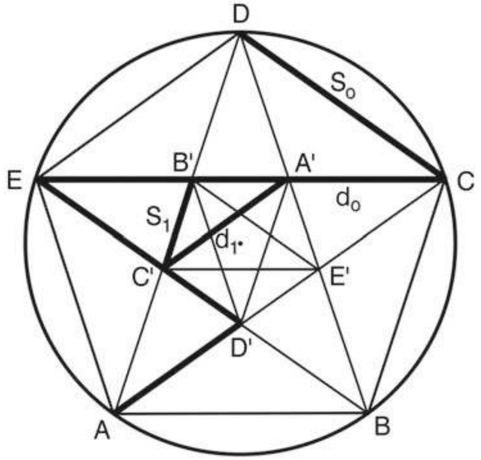
When dealing with mathematical questions, they had the devastating cognition that irrational ratios exist. According to many researchers, this resulted in the first crisis in regards to the foundation of mathematics. Hippasus of Metapontum is said to have made this tragic discovery in approx. 450 BC. According to the legend, as a result of revealing the secret, he was banned from the Pythagorean community and perished at sea.

A possible trigger could have been studying the ratio of side $s$ and diagonal $d$ of a square: the Pythagorean Theorem immediately results in $d^2 = 2 \cdot s^2$. Hereby, we can quickly derive a contraction, if we assume $s$ and $d$ as numbers and use the natural number properties 'even' and 'uneven'. (In Euclid's manuscript, 'Elements', this proof is located at the end of Book X; Aristotle referred to this proof of contradiction repeatedly throughout his texts.)


**Illus. 2.1.7** Memorial of Pythagoras on the isle of Samos
[Photo: R. Tobies]

**Illus. 2.1.8** Pythagoras of Samos (medieval wooden sculpture in the choir stalls of the Minster of Ulm) [Photo: H. Wesemüller-Kock]

This contradiction can also be proved purely geometrically, and not only with the square, the simplest shape. Above all, the regular pentagon offers itself as first evidence of the existence of non-rational ratios, since the regular pentagram (a five-pointed star) used to be the symbol of the Pythagorean order. The pentagram is drawn by marking the five diagonals inside a pentagon (Illus. 2.1.9). As with the square, we cannot rationally relate the length of a diagonal to the length of the side of a pentagon, since this method, derived from measuring the anthyphairesis (see below), leads in both cases to a never-ending process: the exciting discovery of the existence of irrational ratios. Since we can assume that the Pythagoreans had intensively researched the properties of the symbol of their order, it is not farfetched to suppose that their belief that everything could be expressed in integers was only destroyed when studying pentagrams or regular pentagons. The method of anthyphairesis (which corresponds to the arithmetical Euclidean algorithm) is a further development of the normal measuring procedure: we deduct the shorter of the two line segments as often as possible from the longer one. If this is possible without remainder, then we have an integer ratio (a factor). If there is a remainder, which then must be shorter than the shorter line segment, we deduct this rest as often as possible from the shorter line segment. If this works without an additional remainder, then this remainder is a common measure for both line segments and both stand in a rational ratio to each other. Otherwise, the procedure, if necessarily multiplied, is repeated with the previously obtained, smaller remainder. If at some point there is no




**Illus. 2.1.9** Regular pentagon inscribed in a circle

remainder left, the previous remainder is a common measure for all line segments generated via this method, including both initial ones. These are called commensurable and they can be measured exactly with a common measure (in other words, they stand in a rational ratio to each other). However, if the anthyphairesis does not come to an end, the two initial line segments are incommensurable to each other (they do not relate to each other rationally).

Hippasus could have proven the irrationality of diagonal and side by using the regular pentagon in the following manner (the terms refer to Illus. 2.1.9): Each of the five diagonals (of length $d_0$) of the initial pentagon run parallel to one side ($s_0$), and internally they form a new regular pentagon, the diagonals of which (of length $d_1$) also take the same five directions. Consequently, $CDED'$ and $AC'\ A'D'$ are two rhombi (varying in size), each with four equal sides. Hence, the greater section of a diagonal, e.g., $ED'$, has the length of a side $s_0$ of the initial pentagon, whereas the smaller section has the same length as a diagonal $d_1$ of the inner pentagon. Since the smaller section of $d_0$ is identical to side $s_1$ of the inner pentagon, anthyphairesis takes its course as

$$\begin{gathered} d_0 - s_0 = d_1 < s_0, \quad s_0 - d_1 = s_1 < d_1, \\ d_1 - s_1 = d_2 < s_1, \quad s_1 - d_2 = s_2 < d_2, \\ d_2 - s_2 = d_3 < s_2, \ldots. \end{gathered}$$

As shown, it does not end and, therefore, does not deliver a common measure for side and diagonal (see Problem 2.1.3).

Since all core Pythagorean discoveries and theories within the realm of geometry are part of Euclid's 'Elements', which will be looked at in detail in paragraph 2.3, we will now turn to the so-called 'Golden Age' of Greek natural sciences and mathematics.

## 2.2 Classical Greece (Athenian era)

As already implied by the name, the heart of mathematical research lies within this era in Athens, lasting for approx. 150 years (approx. 450 until approx. 300BC). In the philosophical school of thought of Plato's (429–348) academy, mathematics was shaped as ideal case of purely deductive science, which has influenced the development of this science enormously up to the present day. Plato argued that mathematics had an intermediate position between the realm of mere ideas and the world of empirical objects. Within the surroundings of the academy, many theories originated that have been passed down to us in a systematised form in Euclid's 'Elements'.

In *Politeia* (Republic), designed as a dialogue between Socrates, Glaucon and other dialogue partners, Plato describes the high requirements that a wise statesman has to fulfill. On top of everything, there is philosophy, the true science that leads to true epistemological cognition (last but not least, of the good and the beautiful). Nonetheless, it is clear for Plato that a statesman must also be skilled in the art of war: "And our guardian is both warrior and philosopher? – Certainly." Socrates enquires about those matters of education that lead to cognition of reason and, thus, open the path to the world of essential being (known as 'ousia').

**Illus. 2.2.1** Famous witness of classical Greek architecture on the Acropolis of Athens: Propylaei (437-432 BC)
[Photo: H.-W. Alten]

**Illus. 2.2.2** The Parthenon (447-438 BC), built with pentelian marble, looked upon as perfect masterpiece of Greek architecture
[Photo: H.-W. Alten]

First, he mentions the art of counting and calculating; we will skip its justification here. Then, the dialogue partners agree on geometry as the second science. Again, its use for commanders of war is cited first: "Clearly, he said, we are concerned with that part of geometry which relates to war; for in pitching a camp, or taking up a position, or closing or extending the lines of an army, or any other military manoeuvre, whether in actual battle or on a march, it will make all the difference whether a general is or is not a geometrician. (...), but for that purpose a very little of either geometry or calculation will be enough." That is why Socrates stresses that we must check whether studying the subject of geometry more extensively and thoroughly is beneficial "to make more easy the vision of the idea of good". This is affirmed, since it is epistemological cognition of the never changing being. Hence, the citizens, consequently the coordinators of the ideal state, "should by all means learn geometry". Moreover, there are extra benefits to geometrical knowledge: first, the practical gain for war; second "and in all departments of knowledge, as experience proves, any one who has studied geometry is infinitely quicker of apprehension than one who has not."

The third indispensible science is astronomy, according to Plato and Socrates, and the fourth one the study of harmonies. The former opens the eyes to laws and principles of celestial movements, the latter, the ears for the movements which show themselves in harmonic tones. Hence, these sciences "are sister sciences – as the Pythagoreans say". (An extract of the *Politeia* can be found in Appendix A. 1 p. 565)

### 2.2.1 Eudoxus

The most genius mathematician amongst Plato's contemporaries was Eudoxus of Cnidus (408?–355?). He developed his theory of irrationality based on an ambition to extend the notion of ratio to include irrational ones. His second great accomplishment was developing the method of exhaustion (Euclid, 'Elements', Book XII), the basis for determining the capacity of curved areas and volumes, which was expertly applied by Archimedes one century after Euclid. The Pythagoreans' basic assumption that all numbers are composed of one unit that, as such, is indivisible, enabled them to develop an elementary theory of proportions. Accordingly, the ratio of two given numbers $a$, $b$ was rational, if there was a common measure $k$, of which both are an integer factor: $a = n \cdot k,\ b = m \cdot k;\ n, m \in \mathbb{N}$. If in according manner for a number pair $c, d$ it is true that $c = n \cdot k',\ d = m \cdot k'$, then $a : b = c : d$. In this case. $ma = nb$ and $mc = nd$. But if $m'a < n'b$ for any two integers $m', n'$, then also $m'c < n'd$ and likewise for $>$. This is worded in Euclid's 'Elements' in definition 20 of Book VII as follows:

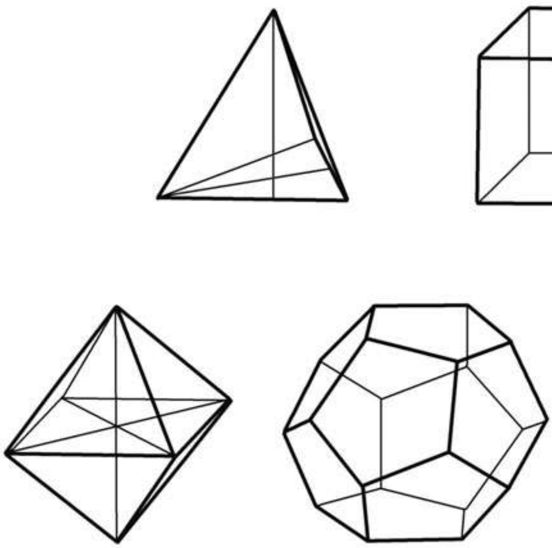
> "Numbers are proportional if the first is the same multiple, or the same part or the same set of parts of the second that the third is of the fourth."

In contrast to (natural) numbers, which consist of units, line segments are infinitely divisible; this is why we cannot necessarily assume a common measure, which consequently makes this approach to proportionality fail. Eudoxus solved this dilemma by means of constructing a new approach, which applied to rational as well as irrational quantities. The considerations adumbrated above may have shown him the way, since it is unsure whether a rational ratio is given, the definition of equality of ratio of two pairs of magnitudes is based on these ratios. This has been described in definition 5 and 6 at the beginning of Book V of 'Elements' as follows:

> "Quantities are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order."

> "Let quantities which have the same ratio be called proportional."

This way, Eudoxus freed geometry of the Pythagorean shackles of restriction to rational numbers. The so-justified proportionalism is systematically established in Book V of 'Elements'.



**Illus. 2.2.3** The five Platonic solids: tretrahedron, hexahedron (cube), octahedron, dodecahedron, icosahedron

The reader may have noticed that the simpler definition (for proportions made of natural numbers) is offered in Book VII of 'Elements', whereas the considerably more sophisticated definition already appears in Book V. This is one of the examples which demonstrates Euclid's approach to historians: his work unites different models and, apparently, uses an older definition in Book VII, which was actually outdated by the one introduced in Book V proposed by Eudoxus. Another explanation which offers itself in such cases lies within the assumption that the parts might have been confused in the long process of tradition. Such questions do not only arise concerning Euclid: the texts passed on are duplications of duplications, adaptations, translations or all of the above, never the self-written manuscripts of the authors themselves. The text passages, which afterwards were taken to be 'definite versions', are the result of careful comparison and critical pondering by the editors, who had to take into account the updated results of contemporary research.

Apart from Eudoxus, we must, above all, mention the extremely talented Theaetetus (415?–369?), to whom Plato dedicated a particular dialogue. He accomplished a systematic construction of quadratic irrationalities ('Elements', Book X) and, based on this, the existential proof of the five regular (or Platonic) solids ('Elements', Book XIII, wherein it is also proven that there cannot be more than five of these solids).

Aristotle (384–322) assigned mathematics a central role in science too, even if his opinion on the nature of mathematics differs greatly from the Platonic. He defended the view that mathematical objects are obtained by abstraction from perception. He was especially interested in mathematics in relation to his elaboration of syllogistics. In his book ‘Physics’, he began examining the infinite (which, according to him, could only exist potentially, but not actually) and the continuum (like space and time), which, according to him, cannot consist of points. Classical Greece ended with the great mathematical systematist Euclid, even if he did most of his work at the Musaeum in Alexandria. Due to his importance to geometry, we have dedicated an own section to him (2.3).

However, since Euclid’s ‘Elements’ only deals with compass and straightedge constructions, we must emphasize the discovery of conic sections and the development of the relevant theory here. Nowadays we usually describe curves by means of their algebraic or transcendent equations (as taught by Descartes during the $17^{th}$ century; cf. Chapter 6), which means that the circle is a type of conic section. However, the Greeks believed that the means of construction were a crucial criterion when systematizing geometry for the first time. As a result, compass and straightedge constructions have determined the field of elementary geometry up to the present day.

### 2.2.2 The so-called classical problems of mathematics

In accordance with nature, the Greeks had occasionally to encounter problems, which, despite all efforts, could not be exactly solved by these two (theoretically imagined) instruments. Three of these problems have been given a special meaning in the history of geometry and are often summarised as ‘the three classical problems of geometry’. These are the Delian problem or doubling the cube, angle trisection and squaring the circle. However, since the decision on the real ‘nature’ of these problems (possible only in the $19^{th}$ century) was made with the help of algebraic means, it is customary nowadays to deal with all three of them within the realm of algebra. The reader is most likely to know the outcome of this analysis: doubling the cube and angle trisection lead to cubical equations; in contrast, the problem of squaring the circle is transcendental; in other words, it cannot be grasped by an algebraic equation, regardless of how high the order. From today’s view, it means that the Greeks, in order to solve these problems, had either to introduce further curves (exceeding straight line and circle, in part influenced by designing more complicated instruments for drawing) or see themselves forced to use the straightedge in a manner otherwise not permitted in geometry. The following presents an overview of their diverse efforts. Some examples concerning the different possibilities shall be introduced here  partly in anticipation of later paragraphs presented chronologically in order to preserve coherence. A detailed account can be found, for example, in [Heath 1921, vol. 1, Chap. VII].

### Doubling the cube

The problem of doubling the cube has a legendary origin, which varies from version to version. According to Theon of Smyrna, the story is as follows: troubled by plague, the Delians consulted the Oracle of Apollo, who told them that the god demanded they build an altar double the size of the one already built to him (cf. paragraph 3.3.1). This is said to have caused great awkwardness amongst the architects, since they did not know how to double a solid. According to another tradition, people were trying to adhere to the demand to double Glaucus's cubical tomb by means of doubling the length of the sides. Hereby, they stumbled upon their mistake and began searching for the right solution (cf. account of the texts in [Waerden 1962]). Speaking purely geometrically, we can deal with this issue easily when generalising the problem of doubling the square (cf. paragraph 3.3.1). Speaking algebraically, we are dealing with the extension of extracting the square root to extracting the cubic root, which is an issue that had already been addressed by the Babylonians.

Hippocrates of Chios traced the problem back to determining two median proportionals $x, y$ between the cube side $a$ and their double:

$$a : x = x : y = y : 2a \tag{2.2.1}$$

This formation of two geometrical means $x, y$ between two given quantities (here: $a, 2a$) corresponds to a pure cubic equation, as springs to mind easily. Basically, all subsequent attempts to find a solution originated from this version of the problem.

The relation for doubling the cube (2.2.1) found by Hippocrates inspired Eudoxus's student Menaechmus to determine these two proportionals by means of a parabola and an equilateral hyperbola and to solve the Delian problem constructively in this manner (see Problem 2.2.1).

It is not certain if Menaechmus had already realised that parabolae and hyperbolae were sections of the perpendicular circular cone. He could have simply (stated in a modern fashion) investigated how the transformation of all rectangles, the one side length of which equals $2a$, into squares of equal area of side length $y$ follows from the functional relation, or how $xy = 2a^2$ results in the transformation of a rectangle $2a \cdot a$ into all other possible ones, which are of equal area, respectively. In order to do so, he could have used the method of constructing areas described in Book II of Euclid's 'Elements'.

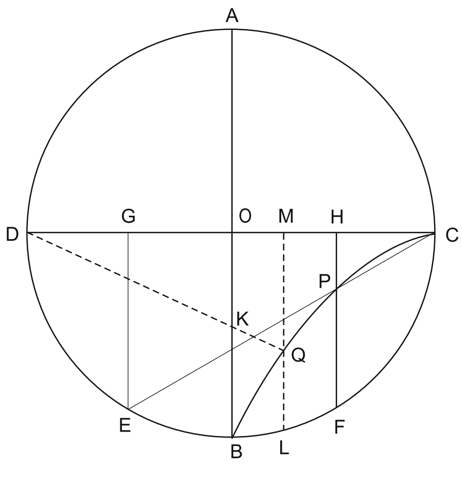
Aristaeus knew (approx. 330) the correlation of these curves with the circular cone. He constructed the hyperbola (to be exact: a branch thereof), the parabola and also the ellipse as sections perpendicular to a generating line of an obtuse, a right-angled and an acute (perpendicular) circular cone. He dealt with these curves in a (lost) treatise on such "bodily" loci thus called because they were derived from the solid of a cone. All other curves of higher order, including the transcendent ones, were referred to as "linear loci" (not to be confused with the contemporary term "linear").

Also a treatise by Euclid on conic sections has been lost. The reason for losing both should be the same: they were superseded by an approach to conic sections taught by Apollonius of Perga in a series of eight books. Apollonius, however, already belongs to the Alexandrian Era together with Archimedes, who was even more outstanding.

Another approach to solving this problem based on determining two median proportionals is reflected by the Cissoid of Diocles, discovered in approx. 180 BC, and of which Proclus reports in his commentary on Euclid more than 500 years later. We draw the perpendicular diameter $AB$ and the horizontal diameter $CD$ in a circle. Then we mark the points $E$ and $F$ on the arcs $\widehat{BD}$ and $\widehat{BC}$, whereby the arcs $\widehat{BE}$ and $\widehat{BF}$ are of equal length. Then we draw $EG$ and $FH$ perpendicularly onto $DC$. The connecting straight line $EC$ intersects $FH$ in $P$, a point on the cissoid. We obtain the other points of the cissoid by varying the position of $E$ and $F$ (Illus. 2.2.4).

Now it has to be proven that point P on the constructed cissoid by means of line segments $FH$ and $HC$ delivers two mean proportionals between $DH$ and $HP$, which means that it is true that

$$DH : HF = HF : HC = HC : HP \tag{2.2.2}$$




**Illus. 2.2.4** The Cissoid of Diocles

This can be easily shown, if we consider the equality of both triangles $DHF$ and $CGE$, as well as the fact that $HF$ is the geometrical mean of $DH$ and $HC$ (altitude theorem in the right-angled triangle $DCF$). If we introduce a coordinate system with the axes $OC, OB$ and origin $O$ equate $OH = x,\ HP = y$ and circle radius $OB = OC = OD = r$, we arrive at the Cartesian equation of the cissoid as

$$y^2 = \frac{(r-x)^3}{r+x}. \tag{2.2.3}$$

It has an apex at $C$, and the tangent line to the circle at $D$ is its asymptote simultaneously.

If we take the cissoid to be given and want to find both mean proportionals for two given line segments $a, b$, we have to choose point $K$ on $OB$ so that $DO : OK = a : b$. (Notice that only the ratio of the two given line segments matters at this point, since radius $r$ of the circle on which the cissoid is based is already given!) Then we connect $DK$ and extend it until intersection $Q$ with the cissoid. We draw the ordinate $LM$ through $Q$ perpendicularly to $DC$. Then $LM$ and $MC$ are the two mean proportionals for $DO$ and $OK$:

$$DM : LM = LM : MC = MC : MQ$$

In addition,

$$DM : MQ = DO : OK = a : b$$

If we want to obtain the mean proportionals belonging to both given line segments $a, b$, we have to extend (or shorten) similarly the line segments $DM, LM, MCandMQ$ according to level of measurement $DM : a$.

### Angle trisection

The classical problem of dividing **any** angle into three equal parts kept many Greek mathematicians busy. Since they were not able to solve this problem by means of compass and straightedge alone, they devised sophisticated methods with other resources.

Only the methods of modern algebra proved that trisecting any angle by means of compass and straightedge alone is an impossible task. We can easily see that constructing $\alpha$ from $3\alpha$ is equivalent to constructing $cos\alpha$ from $cos3\alpha$. Trigonometry then yields

$$cos3\alpha = 4cos^3\ \alpha - 3cos\ \alpha \tag{2.2.4}$$

or with $cos\ \alpha = x$ and $cos\ 3\alpha = a$

$$4x^3 - 3x - a = 0 \tag{2.2.5}$$

(cf. Problem 2.2.5). It is likely that this problem occurred amongst mathematicians when attempting to construct a table of chords for astronomical purposes. As further details on Ptolemy's table of chords show in paragraph 2.5.5, we encounter the problem there, which cannot be dealt with by elementary geometry, to obtain a chord for $1°$ from the chord for $3°$. The systematic construction of regular polygons beyond the pentagon and hexagon also requires non-trivial divisions of angles: $360° : 7$ for heptagons, $120° : 3$ *or* $60° : 3$ for nonagons.

In order to divide an angle into $n$ parts, Hippias of Elis thought of a curve, which was later named ‘quadratrix’, since it solved the problem of squaring the circle, too. Hence, it is clear that we must be dealing with a transcendental curve! Yet, we can easily describe it, since it is constructed by two simple movements. Imagine a square, the upper side of which moves parallel to the starting position with constant speed to the lower side. At the same time, the left side of the square turns clockwise around the lower corner point with constant angle speed in a manner such that both movements start and end simultaneously. In this case, the upper end of the turning line segment describes a quadrant within the square. Accordingly, in the end position both turning sides collapse with the lower square side.

It is advisable for us to introduce a right-angled coordinate system, the $x$-axis of which is the lower side and the $y$-axis the left side of the square (side length is $a$). The intersection $P(x, y) = P(\varrho, \Phi)$ of both moved line segments describes the quadratrix ($\varrho =$ distance to origin, $\Phi =$ angle between $x$-axis and revolving side, decreasing from $\frac{\pi}{2}$ until $0°$). Hence, $P$ moves from the upper left corner to the right and down. The consequence of this movement instruction is, for instance, that the side which moves down from the top has already moved a third of its way (parallel to itself) after a third of the time, while simultaneously the left side turns a third of $90°(30°)$. If we imagine the quadratrix now, the left side of the square parts into $n$ equal parts and draws a parallel through the top point of division to the upper side. As a result, we obtain a point of intersection on the curve, the link of which to the origin delivers an axis of length $p$, which has been cut off the fraction $\frac{1}{n}$ of angle $\frac{\pi}{2}$. From the proportionality of the movements, we get

$$y : a = \Phi : \frac{\pi}{2} = \varrho \; sin\Phi : a. \tag{2.2.6}$$

The polar equation, which can be immediately derived from this, is transcendental. As far as the sources tell us, this transcendental curve, the oldest of its kind, was discovered earlier than conic sections!

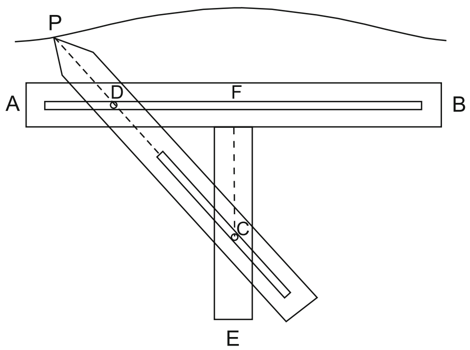
Approximately two generations after its inventor Hippias, Dinostratus recognised that it was also possible to use the quadratrix in order to square the circle (cf. Problem 2.2.2).

A method used more frequently by the Greeks is an insertion, the so-called neusis construction from the Greek. Hereby, the basic principle consists of fitting a given line segment in between given curves in a certain manner. Hereby, we can mark the extremities of this line segment on a straightedge, which then can be moved until both points have reached the desired position. Since marking a straightedge is not permitted in elementary geometry, this construction is also referred to as “paper strip construction”.

Pappus described such a construction from classical times concerning angle trisection. We embed angle $\alpha = \measuredangle AOB < \frac{\pi}{2}$, which is to be trisected, in a rectangle in such a manner that the vertex $O$ in the lower left corner comes to lie, one arm collapses with the lower side and the other forms diagonal $OB$ of the rectangle (such rectangles can always be found). The corners are to be named anti-clockwise $O, A, B, C$. We now mark both line segments $DF = FE = OB$ on a paper strip, place $E$ on straight line $CB$, which goes beyond $B$, in such a manner that $D$ comes to lie on side $AB$ and the paper strip simultaneously runs through $O$. Once this position has been found experimentally, the new artificial line $ODFE$ parts the given angle in the ratio $2 : 1$; in other words, it is trisected (cf. Problem 2.2.3).

For another trisection of angles by neusis, ascribed to Archimedes, see section 2.4.2 and Problem 2.2.4. It offers a simple and generalisable possibility to derive the trisection equation (2.2.5) (or the $n$-section equation) for the given angle. The reader will find advice for this in Problem 2.2.5).

Nicomedes conceived of an instrument that accomplished this condition mechanically (cf. [Scriba 1992]). Due to their shape, the curves constructed with this instrument were given the name 'conchoid' or 'cochleoid', which literally translates to 'shell curve'. The instrument consists of two straightedges that have the shape of a 'T' and are firmly attached to each other. A third straightedge can move on those in a certain manner (Illus. 2.2.5).



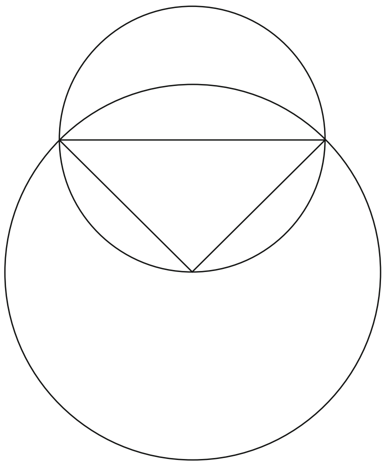

**Illus. 2.2.5** Nicomedes' conchoid and the instrument required for its mechanical construction

The first straightedge $AFB$ has a slot longitudinally. We perpendicularly attach the second straightedge $FE$ at its middle $F$, in which we fit a fixed pen $C$ in distance $b$ of $F$. The third straightedge ends in apex $P$ and carries a pen $D$ in fixed distance $a$ of $P$ (i.e. $PD = a$), which can move within the slot of the first straightedge. Another slot starts perpendicularly on this straightedge longitudinally beyond $D$, in which pen $C$ hitches. When moving the third straightedge, extremity $P$ describes a conchoidal line. Obviously, the crossbar of the $T$-shaped device (more precisely: the straight lines running through $A, F$ and $B$) is an asymptote.

In order to compare Nicomedes' instrument with Pappus's neusis construction, we must identify pen $C$ with corner $O$ and slot $AFB$ of the first straightedge with rectangle side $BA$, on which the fixed line segment $a = DE$ is supposed to start. If we now draw the conchoid with this device, the extended upper rectangle side $CB$ will intersect the wanted point, which will lead us to the solution (cf. Problem 2.2.6).

### Squaring the circle

The problem in constructing a square of exactly equal area to a given circle is a matter of integration according to present understanding. Thus, this issue is addressed by mathematical analysis. The Greeks predominantly saw it as a geometrical construction problem. We will list a few examples.
Already, Hippocrates' attempts in the $5^{th}$ century BC to square circular moons may have been triggered by the problem of turning the area of a circle into a square. He managed to do so in three cases, the easiest being the one whereby the moon is bordered by a semi-circle and a quadrant (see Illus. 2.2.6).



**Illus. 2.2.6** Hippocrates' squaring a moonlet

However, this result did not contribute to squaring the full circle. The indirect question arising here is if there are further squared moonlets beyond those of Hippocrates that can be squared by means of elementary geometry. However, this question was only answered fully in 1947 (with algebraic methods): there are exactly five such circular moons [Scriba 1988]. Approx. in 430 BC, Antiphon attempted to exhaust the circle area by means of inscribed regular $3 \cdot 2^n - gons$ or $4 \cdot 2^n - gons$. A little later, Bryson utilised simultaneously inscribed and circumscribed regular polygons, whereby he, however, used a simple intermediate value theorem, which led to criticism. Approximately in the middle of the $4^{th}$ century, the brothers Dinostratus and Menaechmus realised how they could utilise Hippias's curve to square the circle (cf. Problem 2.2.2). Archimedes provided the crucial proof that the constant $\pi$ determines the ratio of circumference to diameter as well as the ratio of circle area to square. He also delivered both bounds, which are often used as approximate values:

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}$$

## 2.3 Euclid

### 2.3.1 Euclid's Elements

Euclid's era was between Classical Greece (Athenian era) and the Hellenistic era of Greek mathematics, in regards to content and time. We hardly know anything about him as a person, but his works were so influential for the whole of mathematics, especially for geometry, that his name was often synonymously used for both and is still constantly used in correlation to Euclidean space, Euclidean geometry, Euclidean metric and Euclidean ring.

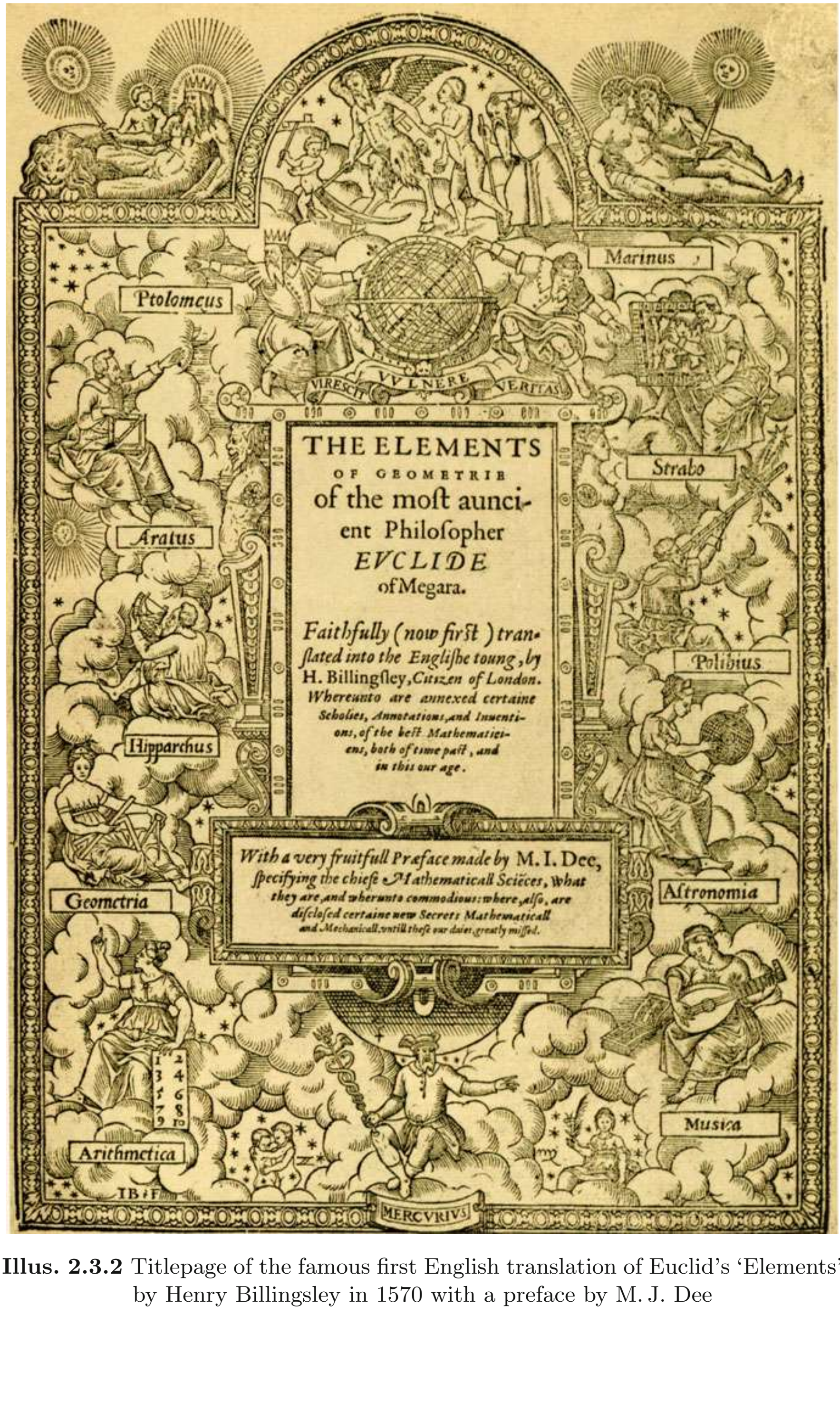
Euclid's main treatise 'Elements' is the oldest of the larger mathematical texts passed down from ancient Greece. Our modest knowledge of the developments preceding 'Elements' comes from different sources: fragmented, unreliable and subjective reports of scholars of the late Antiquity and the Islamic Middle Ages; from what we can extract from the text of 'Elements' by means of content and critical language analysis, or what we believe to be able to do, the smallest part comes from pre-Euclidean text fragments. A wide

**Illus. 2.3.1** Fragment of Euclid's 'Elements' (Papyrus from 75-125 AD, one of the oldest diagrams from Euclid's Elements of Geometry. The diagram accompanies Proposition 5 of Book II of 'Elements')

field of speculations, hypotheses and debatable questions opens up whenever a meagre source and a certain world-view of philosophical pertinence coincide, as was the case with the early history of occidental mathematics. Thus, secondary literature on Euclid and his 'Elements' has become almost unmanageable by now and continues to grow constantly. To present all of this even briefly would fill a book on its own. ([Schreiber 1987a], [Artmann 1999], [Schönbeck 2003], also see the article on Euclid by [Wußing/Arnold 1989].) We have to limit this chapter to a summarising overview on the content and (obviously subjectively chosen) aspects, as well as some details on the history of tradition. Further information on content and history of tradition will be inevitably looked at in the following chapters (see, e.g., 5.1, 6.1.1 and 7.5).

'Elements' is composed of 13 'Books', traditionally numbered in Roman letters, whereby one 'Book' corresponds to one papyrus roll, i.e., one chapter of a modern book. The following list gives a first overview on content and supposed origin:

(I) Beginning of an axiomatic construction of plane geometry until the Pythagorean group of theorems, and far reaching knowledge of the Ionian era, especially of the Pythagoreans
(II) Foundations of algebraic operations with geometrical quantities (line segments, areas, volumes), as well as Pythagorean ideas, traditionally interpreted as a reaction of the discovery of incommensurable line segment pairs. As a result, mathematicians turned towards a notion of geometrical quantity. The term 'geometric algebra', suggested by the Danish mathematics historian H. G. Zeuthen around the end of the $19^{th}$ century, has become widely accepted.
(III) Circle theory, supposedly Pythagorean
(IV) Construction of inscribed and circumscribed regular polygons
(V) Eudoxian theory of proportionality in its general form, i. e., not limited to geometrical quantities
(VI) Application of the theory of proportionality to plane geometry
(VII) Theorems on natural numbers, not further considered here
(VIII) Theorems on natural numbers, not further considered here
(IX) Theorems on natural numbers, not further considered here
(X) A very challenging algebraic theory of compass and straightedge constructible quantities, especially classification according to the number of (stated in a modern fashion) nested square roots, which are necessary to generate them. Apparently, going back to Theaetetus.
(XI) Basics on spatial geometry
(XII) Theorems on volumes, partially by Eudoxus (cf. Problem 2.3.1)
(XIII) Construction of the five regular polyhedra based on the radius of the circumsphere (according to Theaetetus) by consulting the results of Book $X$ on characterising the relevant edge length. Book $XIII$ concludes with the (relatively trivial) proof that apart from the ones addressed here there are no further regular polyhedra (cf. Problem 2.3.2).




**Illus. 2.3.2** Titlepage of the famous first English translation of Euclid's 'Elements' by Henry Billingsley in 1570 with a preface by M. J. Dee

Apart from the preliminary works of Thales, Oenopides, Hippocrates, Leon, the Pythagoreans, Eudoxus, Theaetetus, etc., Euclid's 'Elements' in the given form are hardly imaginable without Plato's philosophy and Aristotle's methodology. The former arises from the kind of demonstration that explicitly relates to neither the material world nor any other applications; possibly also from the fact that construction means were limited to compass and straightedge. (Of course, Euclid never talks of the instruments themselves anywhere, only of the objects line segment and circle, which are obtained by their means.) The latter concerns not just deductive composition as a whole but also partition of conditions into axioms (these are general basic principles, whose truth is undisputable) and postulates (these are theory-specific basic principles, whose permission is debatable. In Euclid's work, they essentially concern supposed feasibility of certain constructive basic operations, of which all solutions of construction problems can be combined.) Most books start with definitions, particularly Book I. Some of those definitions reduce derived notions to basic notions in a manner also customary in modern logic, e.g., Def. I, 17: A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle; Def. I, 23: Parallel straight lines are straight lines which, being in the same plane and being generated indefinitely in both directions, do not meet one another in either direction.

Others attempt to describe a basic notion, e.g., Def. I, 1: A point is that which has no part; Def. I, 2: A line is breadthless length. It is clear that it is impossible to prove a mathematical theorem concerning the defined notions with the 'definitions' of this second type. Furthermore, they are unnecessary from a modern point of view and are not used by Euclid anywhere else. This criticism, however, presupposes that Euclid really intended something like a formal axiomatic composition of geometry (like in the sense of Hilbert's famous book *Grundlagen der Geometrie* (Foundations of Geometry from 1899), of which we are by no means certain. Other mathematicians, philosophers and translators have debated and attempted to improve these definitions for centuries, whereby the difference between definitions of type I, 23 and such of type I, 1 went mostly unnoticed.

In light of so many roots and sources of 'Elements', Euclid was multiply denied his own scientific accomplishments and described as merely a 'textbook writer' and 'didact', if yet a very successful and clever one. Both have to be questioned: there are justified doubts that Euclid did not write the extant text himself, that his teaching was not oral discourse, following the traditions of the time, and that the text was developed based on his students' notes. In this case, Euclid could have been a good didact, from whose teaching only dry facts were written down (as is still the practice of many modern students). Moreover, his debatable definitions may have just been badly written notes of an extensive propaedeutic explanation. If, however, he wrote 'Elements' himself, we must clearly criticise his dry style, the stolid listing of definition, theorem and proof, as well as the lack of motivation and examples. Quite a few didacts of modern times and authors of most diverse eras have argued that mathematics as a subject at school was constantly unpopular due to Euclid's dominance within its teaching. (Also see, e.g., [Fladt 1927], extract at appendix, or the polemic book title *Los von Euklid* (Free of Euclid) [Kusserow 1928].)

Concerning Euclid's own scientific accomplishments, we must consult the so-called compendium of smaller works, which are hardly known nowadays, while also keeping in mind how clear it is that the constructive aspect of 'Elements', detailed below, forms a red line throughout his work, which is otherwise so heterogeneous in subject and content. It is not exactly easy to ascribe this aspect to the different sources. Here the original scientific style is evident. Especially from the view of the end of the $20^{th}$ century, Euclid comes across as an excellent pioneer of an algorithmic culture of mathematics that had been lost for a long time. Hence, nobody had looked for it in Euclid's work and, consequently, not found it [1].

The building blocks of 'Elements', called propositions, are divided into theorems followed by proofs and problems with the relevant solutions. Every task deals with given objects (points, line segments, circles), for which certain conditions are made, and wanted objects, which are supposed to stand in certain relation to the given objects, e.g., Book I, prop. 2 (translation here and in the following according to Joyce):

> To place a straight line [line segment] equal to a given straight line [line segment] with one end at a given point (i.e., to construct a point D for given points $A, B, C (B$ unequal $C)$, so that $AD$ congruent to $BC$).

The solution to such a problem always consists of describing a method (algorithm), the input of which are the given objects and the output the wanted objects, and the proof that this algorithm delivers the objects with the wanted properties by means of the given conditions constituted by the objects. This method was not just overall paradigmatic for all areas of mathematics of all subsequent eras, but also details many remarkable future-indicating trends, only some of which can be discussed here.

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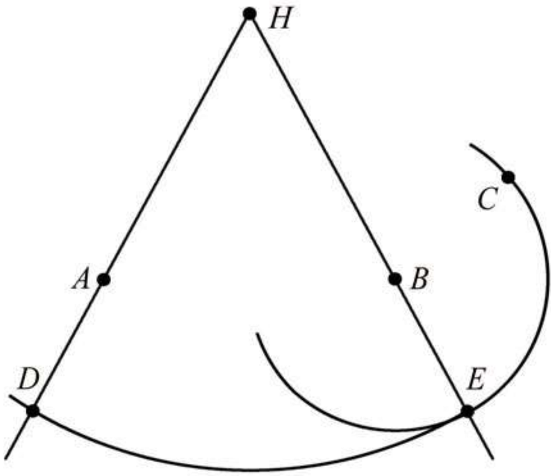
[1] In order to prove that this view of Euclid is shared nowadays by renowned mathematicians, we will quote from the preface of the book [Preparata/Shamos 1985, Computational Geometry, 1985, p. 1]: "It is popularly held that Euclid's chief contribution to geometry is his exposition of the axiomatic method of proof, a notion that we will not dispute. More relevant to this discussion, however, is the invention of Euclidean construction, a schema which consists of an algorithm and its proof, intertwined in a highly stylized format. The Euclidean construction satisfies all of the requirements of an algorithm: it is unambiguous, correct, and terminating. After Euclid, unfortunately, geometry continued to flourish, while analysis of algorithms faced 2000 years of decline. This can be explained in part by the success of reduction ad absurdum, a technique that made it easier for mathematicians to prove the existence of an object by contradiction, rather than by giving an explicit construction for it (an algorithm)."

The steps permitted to formulate construction algorithms can be found in the postulates (after definitions and axioms) at the beginning of Book I:

Required is

1. To draw a straight line [line segment] from any point to any point.
2. To generate a finite straight line continuously in a straight line.
3. To describe a circle with any centre and radius.
   [Euclid a]
   ...



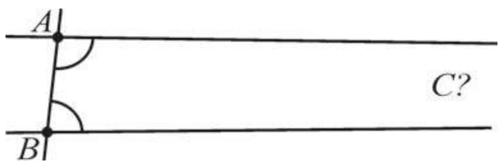

**Illus. 2.3.3** Regarding proposition I, 2 of Euclid's 'Elements'

In the first two problems it is shown (without actually explaining) that we can reduce the operation to construct a circle of radius $BC$ around $A$ with the given points $A, B, C (B$ unequal $C)$, to a more specific operation by means of a 'subprogram' (stated in a modern fashion) to construct the circle around $A$ through $B$, whereby we clarify the otherwise existing ambiguity of the third postulate. To do so, we must first construct a point $H$ by means of the auxiliary problem (subprogram!) solved in I, 1 in a manner, which allows $A, B$ and $H$ to form an isosceles triangle. We extend $HA$ and $HB$ via $A$ resp. $B$ until we have found the following intersections: intersection $E$ of $HB$ with the circle around $B$ of radius $BC$, which is located on the side of $B$, which is turned away from $H$; intersection $D$ of $HA$ with the circle around $H$ of radius $HE$, which is located on the side of $A$, which is turned away from $H$ (Illus. 2.3.3). Now we come to understand proposition I, 2 as quoted above: Since $AD$ is congruent to $BC$, the circle around $A$ through $D$ corresponds to the circle around $A$ with radius $BC$.

Algorithms were also taught as part of old Egyptian and old Mesopotamian mathematics, but only by means of a series of numeric examples. Euclid's works are the first to refer to algorithms in general. For this we need (stated in a modern fashion) addresses for the input objects, intermediate results and output objects. Euclid used (Greek) capital letters for points, referred to straight lines by means of two points located on the straight line and to circles by means of three points located on the circle. This is a unique, pioneering achievement, but partially due to factual circumstances: geometrical objects are, in contrast to numbers, bound to a fixed location. If we name a concrete geometrical object, then this name does not identify the object on its own without a corresponding mapping, in contrast to number-naming systems. As a result, the names of geometrical objects, which are unavoidable in the description of an algorithm, basically assume the characters of variables or (stated in a modern fashion) addresses. However, we shall remark peripherally that Euclid also used his 'programming technique', which he developed successfully for his geometrical algorithms, in the number-theoretic books of 'Elements' here also, for the first time, by using variables in order to formulate algorithms to solve number-theoretic problems, e.g., the famous Euclidean algorithm for determining the greatest common divisor.

The notion of an infinite straight line is strange to Euclid. There are only straight line segments, which, according to postulate 1, can be constructed as a connection of their extremities. Their gradual extension is permitted as an elementary operation by the second postulate. Whereas intersections of circles can be determined without theoretical (axiomatic) justification everywhere, where their existence is graphically evident, intersections of two straight lines prove to be a challenge under these conditions, since we are only given constructed line sections, which we have to extend unpredictably, often depending on their position to each other in order to finally obtain an intersection. Euclid – analogous to the other basic operations – is just consistent when formulating a condition, under which this procedure is assumed to be feasible:

> $5^{th}$ postulate: That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if generated indefinitely, meet on that side having the angles less than the two right angles. (Illus. 2.3.4)

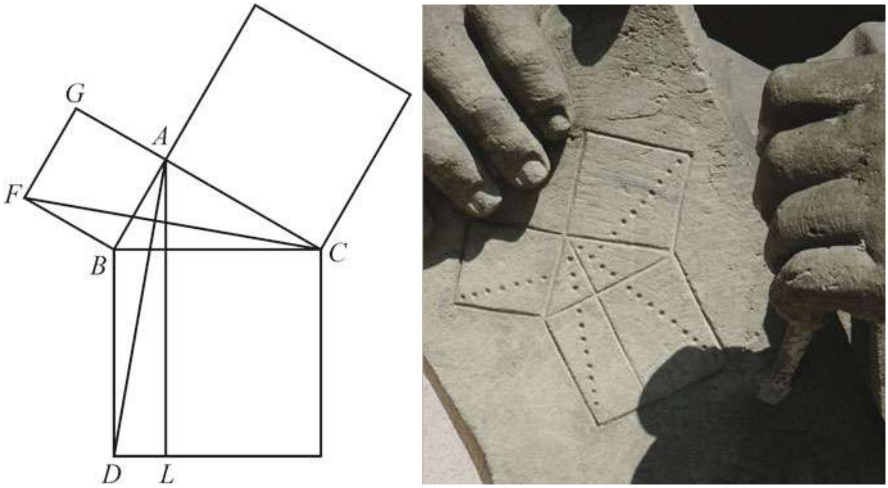



**Illus. 2.3.4** Regarding the $5^{th}$ postulate of Euclid's Elements

Proposition I, 17 says “in any triangle the sum of any two angles is less than two right angles”. Accordingly, the condition of the existence of an intersection (which is the third corner of a triangle ABC) listed in the $5^{th}$ postulate is necessary. Thus, the $5^{th}$ postulate eliminates the potential difficulty by regulating that this necessary condition shall also be sufficient. Unfortunately, we will never be able to clarify if Euclid and maybe even his predecessors really recognised a problem here and contemplated it thoroughly or if the $5^{th}$ postulate was introduced to the world as a, so to speak, improvisation. Since this formulation is noticeably complex compared to other postulates, criticism arose already during Antiquity and attempts were made to prove it as a theorem.

As a result, non-Euclidean geometry developed via numerous intermediate stages during the $19^{th}$ century. That its consistency was proven around 1870 clarifies that the $5^{th}$ postulate is neither mandatory nor capable of proof by means of the other axioms and postulates of Euclidean geometry. We must also remark that – with increasing distance to Aristotle’s methodology – the difference between axioms and postulates started to be blurry. Postulates are, after all, explicitly debatable, not mandatory basic propositions; we can either accept or reject them. Only once the difference between axioms and postulates (the European Middle Ages occasionally renumbered the $5^{th}$ postulate into the $11^{th}$ axiom) had been eliminated did equalling the postulates with the current customary formulation (coming from Ptolemy, but popularised by John Playfair in 1796) of the uniqueness of parallels to a given straight line through a given point become acceptable. Both expressions are indeed equal as axioms, in other words, each is capable of proof by means of the other and the remaining axioms. However, Euclid stated more with his $5^{th}$ postulate; namely, he assumed that the relevant operations are feasible, i.e., the cyclic process of repeating the extension of both line segments always delivers an intersection under the given conditions (to clarify: given these weak conditions, we would have good reason to reject the feasibility of the straightedge operation, postulate 1, without having to question the validity of the theorem of the unique existence of the connecting straight lines, e.g., because the points are too close to or too far from each other).

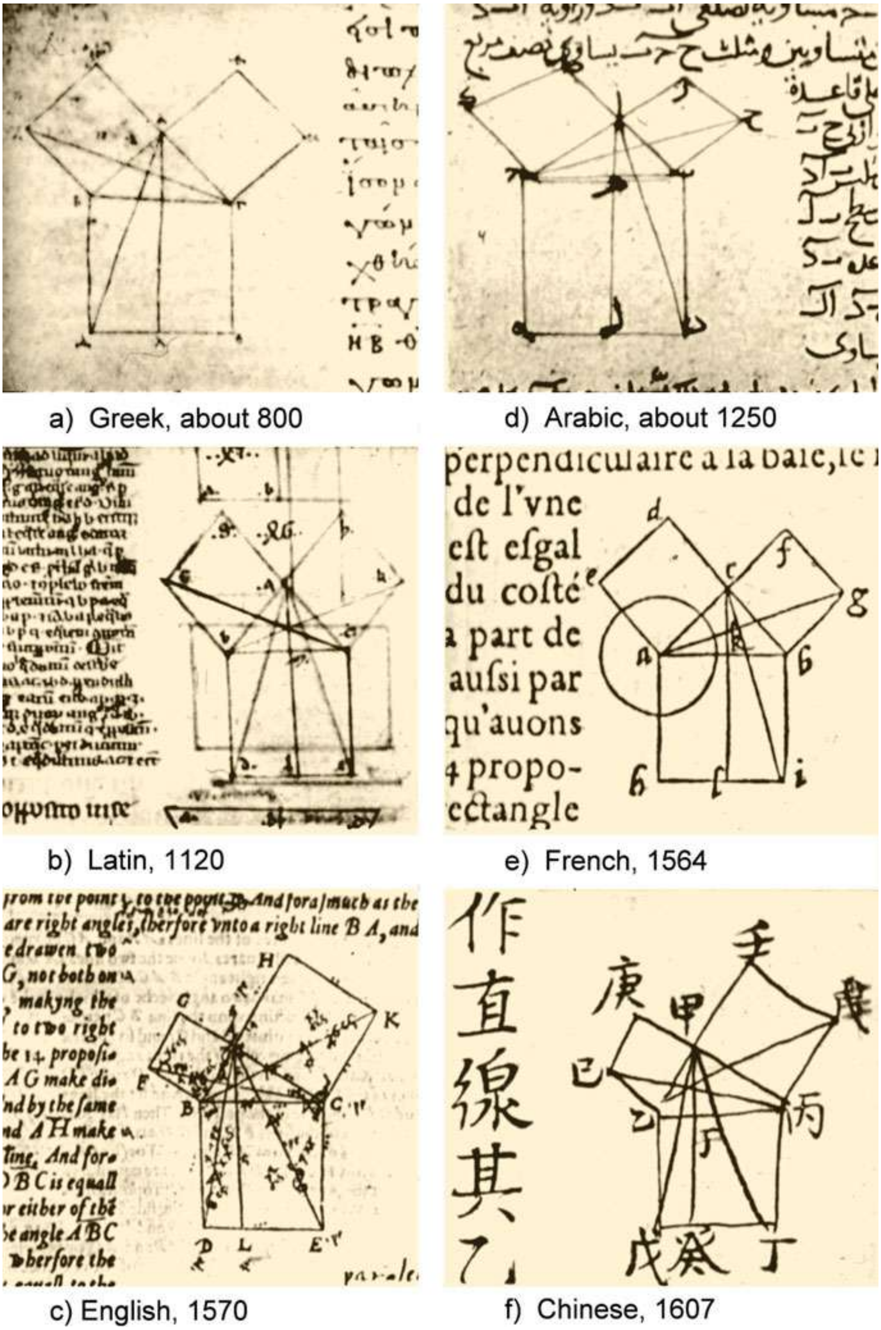
Concerning the further geometrical content of ‘Elements’, many theorems are proven in the same manner and order as is customary nowadays. The gaps remaining in Euclid’s proofs almost always refer to questions of order and can, in most cases, be easily eliminated. His ideas of proof are universally feasible and often tricky, which suggests a long maturing process of the subject matter. We sometimes miss elementary subject matters, which we would intuitively expect in ‘Elements’ (e.g. the fourth theorem of congruence or the construction of the tangent common to two circles), since it has been added by later generations. Propositions often prove to be theorems regarding construction problems after they and the location in which they occur have been inspected more closely, e.g., by verifying conditions as necessary or solutions as uniquely determined. Having studied this intensively, we may




**Illus. 2.3.5** Figure for the Euclidean proof of the theorem of the catheti (I, 47); tablet with this figure in the hands of personified geometry in front of 'Zeughaus' in Berlin [Photo: Hollewood Media OHG]

come to believe (although this is subjective) that Euclid did not intend to compose an encyclopaedia of everything known back then, but planned exemplarily to demonstrate a (highly constructivist) scientific program, which, especially within the realm of geometry, exhausted the compass and straightedge "program package". The overwhelming impact that Euclid's 'Elements' had on subsequent eras has all in all somewhat deformed the picture of ancient geometry. Compass and straightedge were not as dominant, as suggested by 'Elements', which does not mention conic sections, other special curves of different types, approximate methods, determining of capacity and other important aspects of ancient geometry.

Euclid's proof of the *theorem of the catheti* for right-angled triangles is one of the much-admired didactic pieces of art, from which we can easily derive Pythagoras's theorem. The difficulty was constituted by the fact that the customary proof of the similarity of the triangular parts generated by the height on the hypotenuse to the whole triangle, which was probably already known by the Pythagoreans, required theorems of similarities of triangles and, thus, knowledge of proportionality. The Eudoxian approach to proportionality forms, however, part of the most challenging chapter of 'Elements' and is only addressed in Book V. In contrast, Pythagoras's theorem was meant to be accessible as early as possible. Illus. 2.3.5 (also see 2.3.6) belongs to those Euclidean proofs that seemingly do not require proportionality. The triangles $ADB$ and $FCB$ are congruent, i.e., of equal area, due to the congruence of SAS (Side-Angle-Side). Since, as proven before, every triangle is of equal area to every rectangle made of one triangle side and half of the belonging height,




**Illus. 2.3.6** The same figure in different versions of 'Elements'
[a) Ms. Vat. Grec. 204; b) Adelard of Bath; c) Billingsley, London 1570; d) so called pseudo-Tusi; e) Forcadel, Paris 1564; f) Ricci 1607]

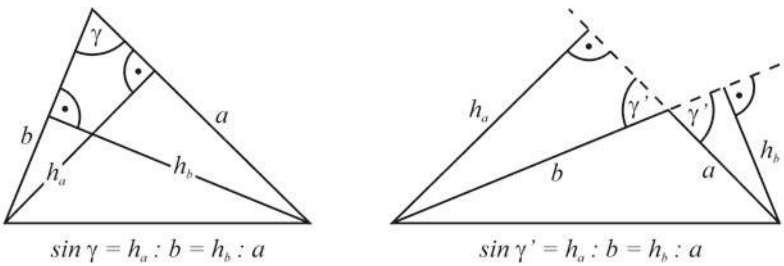
it follows that triangle $ADB$ is of equal area to half the rectangle made of base $DB$ and height $DL$, and triangle $FCB$ is of equal area to half the rectangle made of base $FB$ and height $FG$. (Speaking from experience, pupils tend to struggle with viewing triangle $FCB$ from this unusual perspective, i.e., locating the height belonging to base $FB$!) All in all, employing customary marking, we obtain: $b = AB, c = BC, p = DL : b^2 = pc$. We invite the reader to round down this fine train of thought where appropriate whilst referring to each axiom used.

1. Things that are equal to the same thing are also equal to one another.
2. Things that are doubles of each other are equal to each other.
3. Things that coincide with one another equal one another.

This way, we indeed obtain a method to prove Pythagoras's theorem without proportions. Nonetheless, we must inspect sharply what we have, in fact, just proven:

The square above the hypotenuse is equivalent by decomposition to the union of both squares above the catheti.

We must not consider area, since otherwise proportions enter again through the backdoor, as, in order to define the notion of area, we must prove that the product of base and height of a triangle is a quantity independent of what base is selected, i.e., (Illus. 2.3.7) $a \cdot h_a = b \cdot h_b$ or $a : h_b = b : h_a$, which can only be proven via the similarities of the participating triangles, i.e., by proportionality.




**Illus. 2.3.7** Regarding the independence of the product side times height in a triangle

The oldest known parchment manuscript of 'Elements' (D'Orville 301, named after an early owner and kept at the Bodleian Library in Oxford) was written in Byzantium in 888, approx. 1200 years after the assumed date of origin of 'Elements'. As known today, the parchment represents a version of the text that was written down by the Alexandrian mathematician Theon in approx. 370. Only in 1808 another version was found at the Vatican Library (Ms. 'P' after its discoverer, Peyrard), and even though it was written down later (from the $10^{th}$ century), this version resembles an older text and always serves as

**Illus. 2.3.8** A page from Manuscript P from Vatican Library [Codex Vat. Grec. 190 © Biblioteca Apostolica Vaticana]

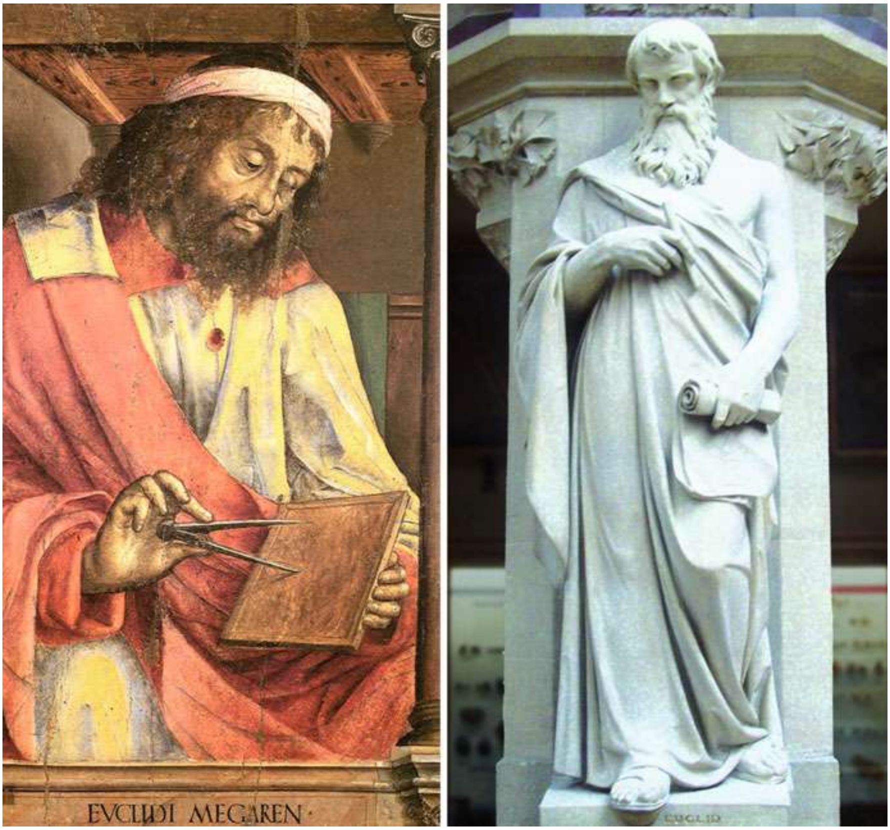
the basis for the modern versions of ‘Elements’ (Illus. 2.3.8). However, this version is probably also not the original, if there ever has been one. All in all, 120 lines of texts on shards and papyrus fragments have been found up to now, all much older than the named manuscripts. In regards to content, they belong to ‘Elements’, but their wording constantly deviates from the text accepted nowadays as ‘canonical’.

‘Elements’ resembles a cultural legacy of the first order, which has received due attention at all times. However, two dividing approaches were chosen to deal with this text. One group (it would be too simple to call them "the mathematicians") addressed the content without much respect for the wording, but were inspired and developed it further. Every era found new interesting aspects of 'Elements'; a very modern view was shown above. The other group thought of 'Elements' as an educational asset, material for language analysis, texts for reading and as historical documentation, and they pursued the phantom of the authentic wording. This proved to be difficult due to the pronounced ambiguity of the ancient Greek language. This group, who could not truly grasp the meaning of what was intended to be said (let alone prove that they had succeeded in doing so), stood little chance of accomplishing a well done translation or even an interpretation. But, to be absolutely clear, the approaches of both groups were legitimate. Mathematics, as well as historiography has a lot for which to thank them.

### 2.3.2 Euclid's further geometrical works

Apart from 'Elements', Euclid has been accredited some smaller works, which have been passed on partially in Greek or Arabic; some have become lost or been labelled as 'fake' by modern research, i.e., not composed by Euclid. Some of these writings have been studied, translated and printed for centuries together with 'Elements' and have belonged to the general mathematical education legacy as much as 'Elements' itself. Nowadays, these texts are mainly known by specialists, and are difficult to access at times, since they are available in the Greek-Latin standard edition of the works of Euclid by Heiberg/Menge, but have rarely been translated into modern languages or reprinted. However, they are crucial if we want to paint a well-rounded picture of Euclid's personality and accomplishments. Just as with 'Elements', they hold some inspiring insights for modern mathematics. A complete overview can be found in [Schreiber 1987a] and the article on Euclid in [Wußing/Arnold 1989]. Below, we will present only those stages that belong to geometry in the broader sense. These are:

- A lost theory of conic sections, the content of which approximately corresponds to the first four books of Apollonius of Perga's theory of conic sections, which we will look at later. (It is likely that the latter extended and pushed aside Euclid's 'Conics' in a manner similar to the way 'Elements' superseded some similarly titled works by earlier writers.)
- A lost script on geometrical loci in space. (It could have dealt with areas, such as planes, spheres and spheroids in space and also loci, i.e., curves in such areas.)
- A text on catoptrics, a theory of mirrors and reflections, of which we know today that the included text was not written by Euclid, although he is said to be the author of 'Catoptrics' according to reports from late Antiquity.




**Illus. 2.3.9** Euclid l.: Tableau of Justus of Ghent; r.: Statue, Oxford University Museum of Natural History [Photo: Th. Sonar]

- A treatise called ‘Phaenomena’ about the beginnings of spherical geometry made to fit the needs of astronomy
- One text called ‘Data’, one ‘Optics’, one ‘On divisions of figures’ and a lost writing called ‘Porisms’.

We will report in a little more detail on the four last mentioned texts. However, the title overview already shows that Euclid was very different from the person portrayed in the mathematical and historical literature. His writings basically cover all areas of applied geometry relevant back then, apart from building trade and geography.

‘Data’ (from the Greek ‘dedomena’ meaning ‘given’) stands in closest relation to ‘Elements’. In fact, they had been printed together until the beginning of modern history, of which there is a modern English translation [McDowell/Sokolik 1993]. At first glance, it offers little new compared to ‘Elements’. It seems to be a kind of compendium and was also mostly interpreted as such. However, it seems that Euclid attempted here to deal with construction problems, the given objects of which are not empirically perceivable objects like in 'Elements', i.e., points, straight lines and circles according to Greek opinion, but equivalence classes of such objects in regards to different equivalence relations. Namely, he started by defining when a geometrical object is given according to size, shape or position. Thus, the propositions are of the following nature, e.g. prop. 39:

> "If all sides of a triangle are given according to size, then this triangle is given according to form."

This can be interpreted in the following manner:

> Given are three 'lengths'; each one, of course, by means of a constructed line segment, i.e., a representative of the relevant equivalence class of line segments of the same length. The object to be constructed is an equivalence class too, namely of congruent triangles. As in 'Elements', construct a concrete triangle as a representative of the wanted class. This problem and its solution are, as said in modern mathematics, independent of the representatives, i.e., if we change the representatives of the given classes, we would influence the concrete result, but not the equivalence class in which it is located.

Analogously, 'Data' deals with problems of ratios, which – stated in a modern fashion – are equivalence classes of fractions. For instance, the result of the problem of finding $d$ for quantities $a, b, c$ given $a : b = c : d$, does not depend on the concrete representative $(a, b)$, but only on the class of all pairs $(a', b')$ equivalent to $(a, b)$, i.e., the ratio. The climax of the theoretical script, although rather trivial in regards to practical execution, is reflected by grasping the notion of power of a point regarding a circle as an abstraction. Although this abstraction is already given by the product of the chord sections of any circle chord running through the point, it does not depend on the arbitrary selection of this chord.

Euclid's 'Optics' is the oldest remaining text concerning this topic. In ancient times, optics was the "geometry of seeing" and was based on experiencing straight dispersion of "visual rays" via the air between object and eye. The fact that there already were two opposing theories on the direction of these visual rays (Euclid assumes the direction from the eye to the object) did not concern the addressed geometrical issues. The notion and content of optics gradually changed in a process over many centuries. Whereas 'dioptrics' was still concentrating on studying gauges via "dioptre" in the $1^{st}$ century $AD$, it changed its focus to the visual rays passing through the boundary layers between different media. During the Islamic and European Middle Ages, questions regarding the anatomy and physiology of the eye arose, including psychological questions of vision, e.g., an extended classification of optical illusions. Optics became a branch of physics and only focussed specifically on studying the nature of light and its dispersion very late, such as in Kepler's work.

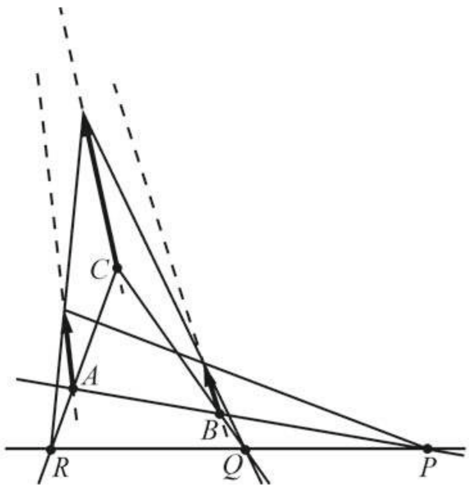
On the one hand, Euclid's 'Optics' is extremely primitive in comparison to nowadays' view; on the other hand, it contains a great scientific concept, which again – as almost always is the case in Euclid's work – is only demonstrated by example. Clearly, those people who he could not personally introduce to his intellectual world did not grasp his idea. By means of many geometrical, mostly very elementary examples, Euclid showed what conclusions we can and cannot draw from the immediately observable (in this case, the angles of vision) of the observed by mathematical means (Problem 2.3.5). Hereby, he leaves no doubt that he first devised this idea as a geometrical auxiliary science for astronomy. Aristarchus of Samos's clever conclusions (cf. 2.4.1) agree here, and the biographical data of both open the possibility that Aristarchus may have been Euclid's direct student. (More details on Euclid's 'Optics' are in [Schreiber 1995].)

The text 'On divisions on figures' has only been passed on in an Arabic version, which features all 36 problems, but only four solutions to the problems of actual interest (also see Problem 2.3.3). They address the issue of "cutting" certain elementary geometric figures in given area ratios by means of straight incisions, which satisfy certain side conditions (e.g., run through a given point). Some of these problems are highly suitable for understanding the systematically taught geometrical algebra, as described in Book II of 'Elements', and can be used to find a solution with compass and straightedge by means of analysis and synthesis in a manner analogous to the later Cartesian coordinate method. Other problems require a clever trick and, to a degree, are only presented to demonstrate this trick (Problem 2.3.4). All these together do not really carry any practical meaning, which is why it is even more astonishing that they belonged to the standard content of texts on practical geometry (art of land surveying and similar uses) over the centuries. (More details are in [Schreiber 1987a], [Schreiber 1994].)

The script 'Porisms' belongs to those that were lost. We know of its existence and some of its content due to reports written by Pappus and Proclus. However, its actual importance as well as the meaning of the word 'porism' is largely unclear despite many attempts to reconstruct it by renowned mathematicians (including Fermat and Chasles). Pappus's description in Book VII of his 'Synagogue' or 'Collection' suggests that 'Porisms', amongst others, basically contained Desargues' theorem, though in a peculiar dynamic wording:

> Given six points $A, B, C, P, Q, R$ of which $P, Q, R; A, B, P$ as well as $B, C, Q$ and $A, C, R$ each are located on a mutual straight line, we fixate $P, Q, R$ and move $A$ and $B$ along two straight lines, then $C$ moves on a straight line (illus. 2.3.10).

We could conclude from this that Euclid's 'Porisms' intended to grasp something like functional relations by means of geometrical figures by fixating some parts of the figure (we would call them parameters) and subjecting other parts (we would call them independent variables) to arbitrary change.




**Illus. 2.3.10** Desargues' theorem in Euclid's and Pappus's version

The manner of the reaction of the remaining parts of the figure (dependent variables) is expressed as a type of proposition. However, we know far too little about the original content to construct this hypothesis wholeheartedly. Pappus had already joined different Euclidean theorems together to form one and then generalised them, which is why it is hard to distinguish between his additions and Euclid's intention.

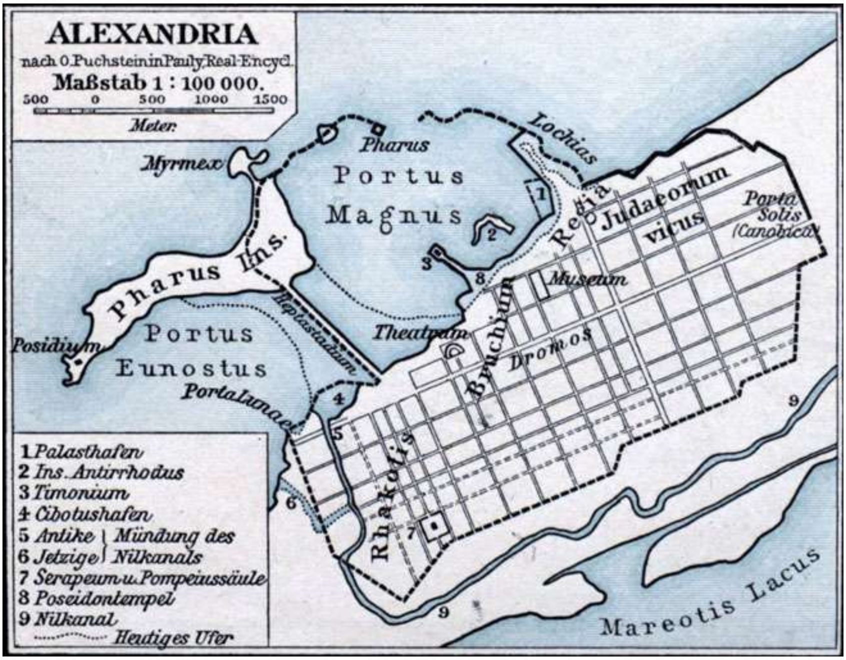
## 2.4 Era of Alexandria (Hellenistic era)

Alexander the Great (356–323) was tutored by Aristotle, and as general of the League of Corinth, took on the Persian wars, occupied Asia Minor, subdued Syria, Egypt and Palestine, conquered Babylon and Persia and advanced as far as India. As a result, countless cities were founded. The foundation of Alexandria in 332/331 at the west border of the Nile delta was of crucial importance for the development of science and mathematics. From 304 until 30 BC, the government of the Ptolemaic dynasty of Macedonian origin was located in this city. The famous Library of Alexandria, the most influential of ancient times, was attached to the Musaeum there, in which artists and scholars lived and worked together. There, scientists had access to and based their work on hundreds of thousands of scrolls, which covered almost the entirety of literature written until then. Thus, it is not surprising that, within the history of science, we also speak of the era of Alexandria in addition to using "Hellenistic era" (derived from the process of Hellenisation, the development of a unified culture with Greek as their world language). Its golden era lasted approximately until the middle of the $2^{nd}$ century BC;

the following decline of science at the end of Antiquity was enlightened only occasionally by remarkable individual accomplishments. Nonetheless, Alexandria remained attractive to students for a long time. Euclid is likely to have worked at the Library of Alexandria, the director of which, amongst others, was Eratosthenes of Cyrene (276?–194?), who accomplished an astonishingly accurate determination of the circumference of the Earth based on an arc measurement between Alexandria and Syene (Aswan) and was Archimedes' contemporary, living approximately one generation after Apollonius. Euclid, Archimedes and Apollonius form an outstanding triad of the golden era of Greek mathematics. The work of Aristarchus falls in the time between Euclid and Archimedes. Aristarchus is known as the first astronomer, having founded the heliocentric model. In the $16^{th}$ century, Copernicus referred to this ancient predecessor, who also deserves to be mentioned in the history of geometry.

### 2.4.1 Aristarchus

Aristarchus's treatise 'On the sizes and distances of the sun and the moon' starts when exactly 50% of the part of the moon's surface that is turned towards Earth is illuminated by the sun. Hereby, the midpoints of the three celestial bodies – sun, moon, and Earth – form a triangle with a right angle




**Illus. 2.4.1** Old Alexandria in ancient times (c. 30 BC until first century AD)

**Illus. 2.4.2** Sphinx and column of Pompeius in Alexandria, city with the major bibliotheca of Antiquity and many scholars. The column is called after Pompeius because his grave was supposed to be here. After an inscript at the socle it was probably erected by the Roman Emperor Diocletian
[Photo: H.-W. Alten]

at the centre of the moon. Aristarchus assumed that the acute angle of this triangle, located at the Earth (in this time very hard to measure exactly), is $87°$ (in reality $89°50'$). During a lunar eclipse, he observed that the diameter of the moon was half of the shadow of the Earth at this distance. (The vanishing of the moon in the shadow of the Earth took the same time as its total invisibility.) Aristarchus's further considerations follow the model of Euclid's 'Optics' in regards to composition and precision. Beyond these aspects the text evokes further interest since it basically addressed trigonometric ratios. He systematically estimated these ratios, as did Archimedes in a similar manner later when measuring the circle. Aristarchus used two theorems:

1. If $\alpha$ is the arc measure for an angle of $\alpha < \frac{\pi}{2}$, then the ration $\frac{sin\ \alpha}{\alpha}$ decreases and ratio $\frac{tan\ \alpha}{\alpha}$ increases, while $\alpha$ rises from 0 until $\frac{\pi}{2}$.
2. If $\beta$ is the arc measure for a second angle of $\beta < \frac{\pi}{2}$ and $\alpha > \beta$, then:
$$\frac{sin\ \alpha}{sin\ \beta} < \frac{\alpha}{\beta} < \frac{tan\ \alpha}{tan\ \beta}. \tag{2.4.1}$$


**Illus. 2.4.3** Aristarchus of Samos a Greek atronomer and mathematician (Statue Aristoteles-University, Thessaloniki) [Photo: Dr. Manuel, 2005]; Eratosthenes of Cyrene, mathematician geograph, astronomer, poet, director of the famous Library of Alexandria

Based on this, he proved:

1. The distance between Earth and sun is greater than 18 times, but smaller than 20 times, of the distance from Earth to moon.
2. The diameter of the sun relates to the diameter of the moon in the same manner.
3. The ratio of the diameter of the sun to the diameter of the Earth is greater than 19 : 3, but smaller than 43 : 6.

Even though the used values of the observations do not reflect the real ratios, as we know today, we must admire the ingenuity with which Aristarchus developed and executed his method (more details in Problem 2.4.1).

### 2.4.2 Archimedes

Archimedes (287?–212), who was of noble descent, grew up in the seaport town of Syracuse, which was founded by the Corinthians at the eastern shore of Sicily about 733 BC. The Syracusans had to defend themselves multiple times against Carthaginian attacks. Archimedes was the son of an astronomer and is said to have studied in Alexandria before returning to his hometown. Apart from mathematics, he focused on mechanics, optics, hydrostatics and engineering (Illus. 2.4.4).

**Illus. 2.4.4** Archimedes sets fire with parabolic mirror on Roman ships. (Copperplate print at the title page of the Latin edition of *Thesaurus opticus* by Alhacen)

He is said to have crucially contributed to the two-year-long defence of his town against the Roman occupation during the second Punic war by means of constructing especially effective weapons. Contrary to General Marcellus's order, Archimedes was slain as an old man by a Roman soldier when Syracuse was conquered in 212 BC. According to the legendary tradition, this happened when he was busy drawing a geometrical figure and shouted to the approaching Roman soldier: "Do not disturb my circles!" (Illus. 2.4.5)

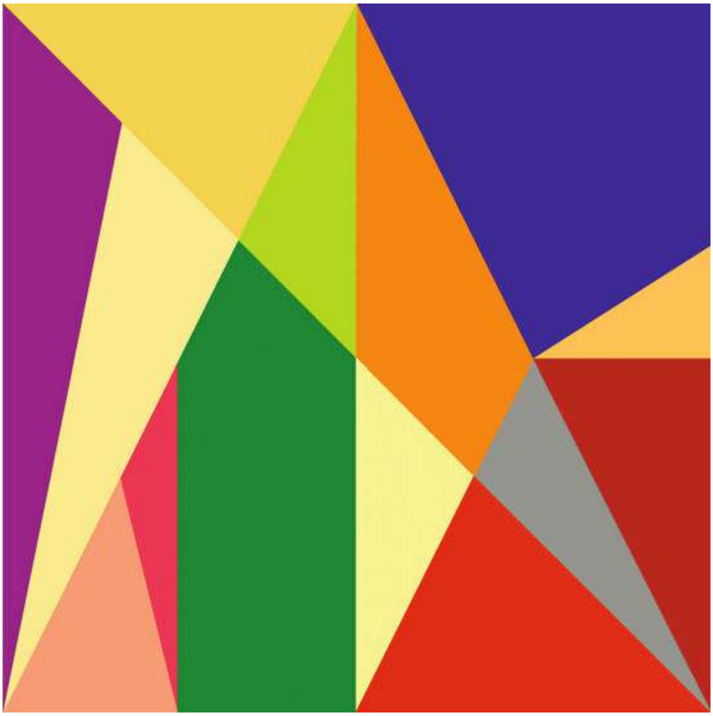
Perhaps Archimedes' most genius accomplishment lies within shaping the prehistory of infinitesimal mathematics. In a text devoted to Eratosthenes on the 'method', which was only rediscovered in 1906 in a duplicate from the $10^{th}$ century, he explains heuristic methods to determine areas and volumes based on mechanic considerations. (The duplicate is a so-called palimpsest, i.e., a manuscript parchment, the text of which was washed off and newly written over in the $13^{th}$ century. Fortunately, it was still possible to identify Archimedes' original text. This unique manuscript was purchased at a spectacular auction on 29 October 1998 for an astonishing two million US Dollars! [Archimedes d]). It was examined again with modern methods at Walters Art Museum in Baltimore and shown there in an exhibition. Parts of the palimpsest were presented in an exhibition called "The Archimedes code" in the Roemer- and Pelizaeus-Museum at Hildesheim, Germany 2012 (see [Netz/Noel 2007] and Illus. 2.4.6).

**Illus. 2.4.5** Death of Archimedes (Mosaic, Municipal Gallery Francfort)

**Illus. 2.4.6** One page of the Archimedes Palimpsest after preparation and examination at Walters Art Museum, Baltimore
[Archimedes exhibition in Roemer- und Pelizaeus- Museum, Hildesheim 2012, Walters Art Museum Baltimore, Photo: Wesemüller-Kock]

On the last folio of the palimpsest Archimedes started his treatise called 'stomachion'. It contains the introduction to a combinatorial problem: What is the number of different possibilities to arrange 14 pieces of geometric shape within the canonical "stomachion puzzle" (see Illus. 2.4.7) to a square?
In this only preserved folio of the 'stomachion' Archimedes supposed the sought number of possibilities may be very great. He started with a simple proposition and the beginning of a second as preliminary practice. Unfortunately nothing is delivered about real mathematical calculation on the problem because the following folios were lost.

With the modern trick of "pseudo-colour-pictures" the damaged and mouldy text of the preserved folio could be read and the problem was solved afterwards with two methods: with a special computer program by Bill Cutler from Illinois and by mathematicians with 'pencil and paper' by rotating or substituting some of the 14 pieces or combinations of them within the square. The exact number of possibilities is 17 152! Thus the 'stomachion' can be looked upon as the first example of combinatorics or as a very difficult problem of geometry.
The name 'stomachion' was probably given to the problem because studying it caused stomachache (see [Netz/Noel 2007]).



**Illus. 2.4.7** The so-called Stomachion of Archimedes in its canonical form [Roemer-Pelizaeus Museum, Hildesheim]

Geometry as such was not Archimedes' focus, but took on an indispensible auxiliary function when investigating infinitesimal problems (e.g., squaring the parabola). The only text of his that was entirely dedicated to geometry dealing with semi-regular polyhedra – was lost. His treatise 'On spirals' [Archimedes c] serves as an example of how Archimedes used geometrical considerations in his research. Archimedes dedicated this particular work to the mathematician Dositheus. Hereby, his introduction was written as a letter to Dositheus and referred to other works and proofs that had been communicated previously (cf. extract in Appendix A. 2, p. 566).

Then Archimedes continued as follows:

"*After these came the following propositions about the spiral, which are as it were another sort of problem having nothing in common with the foregoing; and I have written out the proofs of them for you in this book. They are as follows. If a straight line [ray] of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line revolves, a point move at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane. I say then that the area* *bounded by the spiral and the straight line which had returned to the position from which it started is a third part of the circle described with the fixed point as centre and with radius the length traversed by the point along the straight line during the one revolution... .*" [Archimedes a, p. 153f]

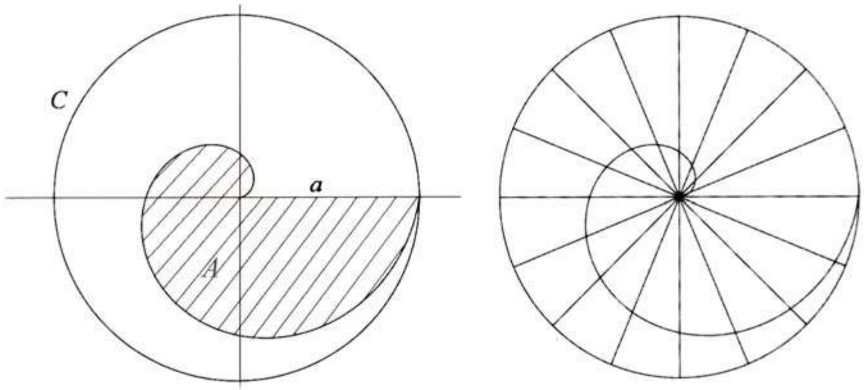
We first observe that Archimedes defined the spiral (nowadays named after him) based on geometrical movements as a combination of a revolving and a linear movement; it is impossible to construct this "mechanically" generated movement with compass and straightedge. Then we notice that the quoted sentence by him should be ascribed to integral calculus. So if we translate the definition into the equation

$$r = a\Phi, \tag{2.4.2}$$

he seems to claim the following (since an infinitesimal part of the area in polar coordinates has the quantity of $\frac{1}{2}r \cdot rd\Phi$ as a triangle):

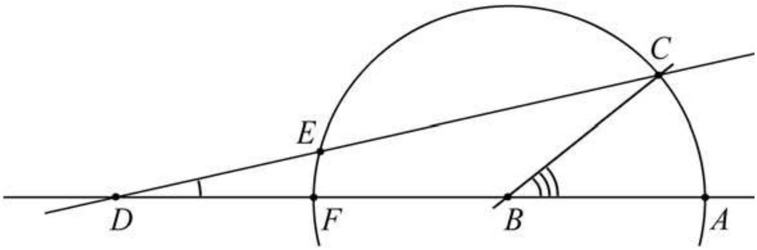
$$\int_0^{2\pi} \frac{1}{2}a^2\Phi^2 d\Phi = \frac{4}{3}\pi^3 a^2. \tag{2.4.3}$$

After one rotation, the fixed extremity of the ray has the distance $2a\pi$ from the centre. Hence, the whole circle has an area of $4\pi^3 a^2$, which proves Archimedes' claim.




**Illus. 2.4.8** Regarding the calculation of the Area $A$ under the spiral; l.: Area A, r.: division of circle and spiral in $n$ parts

Archimedes also studied the tangents of the spiral (cf. Problem 2.4.2). If we draw a straight line perpendicularly to a position vector $r = OP$ through centre $O$ and imagine drawing tangent $t = PT$ in curve point $P$, whereby $T$ shall be its intersection with this straight line, then line segment $PT$ is called polar subtangent $s_t$. Archimedes proved that $s_t = \frac{r^2}{a}$.




**Illus. 2.4.9** Regarding the trisection of an angle by Archimedes

Arabic tradition also ascribes two further accomplishments to Archimedes, which cast an interesting light on his mechanically-oriented mathematical approach to thinking. In both cases we are dealing with the applied version of an intuitive intermediate value principle for continuous insertions, referred to as neusis by the Greeks. When trisecting an acute angle $ABC$ (Illus. 2.4.9), we need to position a straightedge, on which a line segment of length $AB$ is marked, in a manner so that it passes through $C$ and, thereby, cuts out line segment $DE$ of marked length $AB$ between the straight line $AB$ extended in direction of $BF$ and circle $ACF$. It is easy to see that angle FDE forms the third part of angle $ABC$. The argument, which is not explicitly stated in the Arabic text, for the possibility of such a position of the straightedge is based on the fact that the applicable section between line $AB$ and circle $ACF$ increases monotonously and continuously from zero to infinity when gradually turning up the straightedge around point $C$ from the initial position $CF$ until it has reached the position parallel to $AB$. Consequently, there must be a position (which is easy to find by drawing) in which it has the exact length $BC = AB$. In contrast, Archimedes' solution to construct a regular heptagon is also based on an argument of continuity, but cannot really practically be executed by means of drawing, since it concerns locating a position in which both triangles have the same area by turning a straight line around a point, whereby a certain triangle decreases and a certain other one increases simultaneously [Scriba 1992] [Archimedes/Schreiber 2009]. Both problems are, as known since the $19^{th}$ century, not resolvable by means of compass and straightedge.

### 2.4.3 Apollonius

Apollonius of Perga (260?–190?) is almost an entire generation younger. He also studied in Alexandria, taught by Euclid's successors, and worked there for a long time. As mentioned before, Apollonius's theory on conic sections pushed older texts aside. Only the first four books (chapters) have been passed on in Greek, three further ones in Arabic translation, while the last one has not been preserved. In contrast to Euclid, Apollonius followed Archimedes by laying out his intentions in prefaces in each of the individual books. The text starts with a letter addressed to Eudemus:

**Illus. 2.4.10** Cicero discovers the tomb of Archimedes
[Oil painting, Benjamin West 1797, Yale University Art Gallery]

"If you are in good health and circumstances are in other respects as you wish, it is well; I too am tolerably well. When I was with you in Pergamum, I observed that you were eager to become acquainted with my work in conics; therefore I send you the first book which I have corrected, and the remaining books I will forward when I have finished them to my satisfaction. I daresay you have not forgotten my telling you that I undertook the investigation of this subject at the request of Naucrates, the geometer at the time when he came to Alexandria and stayed with me, and that, after working it out in eight books, I communicated them to him at once, somewhat too hurriedly, without a thorough revision (as he was on the point of sailing), but putting down all that occurred to me, with the intention of returning to them later. Wherefore I now take the opportunity of publishing each portion from time to time, as it is gradually corrected...

Now of the eight books the first four form an elementary introduction... (An overview of each book is given.) The rest [of the books] are more by way of surplusage... . When all the books are published it will of course be open to those who read them to judge as they individually please. Farewell." [Apollonius/Heath 1896, 1961]

In letters accompanying later books, Apollonius partially provides more details on the prehistory of the addressed topics and notes which parts reflect results of his own research. Apollonius's work differs greatly compared to his predecessors in regards to the generality with which he phrases and proves theorems in 'Conics'. He purposely follows Euclid by systematically establishing the basics of the entire theory of conic sections in the first four books before turning towards more specialised problems in the second part.

Apollonius generated all the different conic sections by means of a single oblique circular cone (in case of a hyperbola, this cone was extended beyond the apex to generate a double cone), by making the even cuts under varying angles. As seen, Aristaeus had obtained the curves by means of perpendicular cuts applied to three different circular cones. Thereby, Apollonius discussed the geometrical relations, which we nowadays would extract from the vertex equation of the conic sections (constant $r =$ latus rectum, $t =$ latus transversum):

$$y^2 = rx(1 \pm \frac{x}{t}). \tag{2.4.4}$$

From this we can derive the names of the conic section, which are still customary nowadays: if the second member is missing on the right side, we would be dealing with a parabola (from the Greek meaning ‘equality’); if this member turns out to be positive, we would be dealing with a hyperbola (from the Greek meaning ‘surplus’); if it is negative, we obtain an ellipse (from the Greek “elleipsis” meaning ‘deficit’ or ‘fault’).

**Illus. 2.4.11** Perga, metropolis of Pamphylia in Antiquity. Row of columns round the ‘agora’ (market place). View through the southern gate of the city-wall at the two round towers of the ‘Hellenistic Gate’. It was here that Apollonius lived and worked [Photo: H.-W. Alten]

Worded in this manner, we can afterwards interpret these three relations as coordinates of the three (not degenerate) conic sections in an oblique coordinate system, as introduced by Descartes in the $17^{th}$ century. The axes of this coordinate system are formed by one of the diameters and the tangent in the intersection with the conic section. Whereas Archimedes had defined conic section equations as ratio equations, Apollonius – as well as Menaechmus – used the tradition of establishment of area as described in Euclid's second book. Having obtained a characteristic relation (symptom) for each conic section curve, Apollonius did not need the stereometric origin anymore. Properties of tangents and asymptotes, the harmonic properties of pole and polar and relations of foci of conic sections became the subject of further investigations. Since Apollonius distinguished between the hyperbolae made of a branch and two hyperbolae "belonging together", and did not include the circle within his definition of ellipse, he, for instance, phrased a proposition that comes close to that of pole and polar, as follows:

> If we draw a straight line through the intersection of two tangents of a conic section, a circle or two together belonging hyperbolae, which intersect the curve in two points, the line segment, which lies on that curve between these two points, is divided by the connecting line of the osculation points of the tangent and the tangent intersection in an equal ratio.[2]

In contrast to Euclid's 'Elements', which also formed the basis of geometrical teaching during the Middle Ages until well into the modern era (of course, often just in extracts), Apollonius's 'Conics' was only taught in the Islamic area. It first gained more significance in Western Europe in the $17^{th}$ century after Kepler had recognised the ellipse as the true orbit of planets around the sun and Newton had deducted this orbit shape from the general law of gravity.

Apollonius is author of a series of further texts, of which, however, only "Cutting of a ratio" has been preserved in an Arabic translation. Different treatises on sections deal with properties of projective point ranges, others with neusis constructions. He also investigated the following problem, which has been named after him: given three circles in a plane or their degeneracy to points or straight lines (seen as circles with infinite radius), construct those circles with compass and straightedge, which touch all the given ones (cf. Problem 2.4.3).

In some of his contributions, Apollonius followed up on Archimedes' works, e.g., when investigating the concave mirror (geometrical optics) and regu-

[2] Stated in a modern fashion, this means: If $P$ and $Q$ are osculation points of two tangents on a conic section, $T$ is its intersection, $R'$ is nearest to $T, R$ is the other intersection of a straight line through $T$, which intersects the conic section twice, and $S$ is the intersection of the straight line through $T$ with the straight lines $PQ$, then the line segment $R'R$ through $S$ and $T$ is divided internally and externally in an equal ratio, i.e. divided harmonically: $RS : SR' = RT : TR'$.

lar solids (as we have seen, Archimedes had dealt with semi-regular ones). Here, he succeeded in delivering the proof that the surface areas of regular dodecahedra and icosahedra, which are inscribed into the same sphere, have the same ratio as their volumes. Hypsicles of Alexandria (approx. 180 BC) included this result in his work on regular polyhedra which was attached to Euclid's 'Elements' as Book XIV.

## 2.5 Late Antiquity, Rome and Byzantium

After these three great theorists, Euclid, Archimedes and Apollonius, no more mathematicians of comparable significance followed during the Hellenistic era. Besides, neither Rome in the west nor Byzantium in the east, which superseded this era, is known for any comparably famous mathematician. Therefore, it seems to be justified to summarise the time from the middle of the $2^{nd}$ century BC until the ruin of the Roman Empire during the migration period ($3^{rd}$ to $6^{th}$ century AD). Finally, we will look at the late prosperity of mathematics in Byzantium, which lasted from approx. the $10^{th}$ until the $14^{th}$ century, since in regards to content, it developed in close relation to Greek mathematics, which is also due to linguistic reasons.

**Illus. 2.5.1** Statue of Ptolemy II. in front of the new library of Alexandria. The old famous library was founded by Ptolemy I. in the $3^{rd}$ century BC. Nothing remained of the ancient building after being damaged or destroyed by several fires [Photo: K.A. Gottwald]

| | | |
|---|---|---|
| 146 BC | Romans destroy Carthage and subdue Greece | |
| 47 BC | | Library of Alexandria burns down, Alexandria remains centre of Greek-Hellenistic culture |
| 44 BC | Caesar is murdered | |
| 27 BC–14 AD | Octavius as Augustus first Roman emperor | Roman law and culture define life of the people of the Roman Empire |
| 27 BC–395 AD | Roman Empire | |
| 7 BC–30 AD | Jesus of Nazareth | Art of surveying and cartography are developed |
| 66–70 AD | First Jewish-Roman war, Jerusalem destroyed under Titus | Monumental architecture (Colosseum, Pantheon, Castel Sant'Angelo, Palatine Hill, Imperial Fora) |
| 176 AD | Christianity forbidden in the Roman Empire | Wall paintings, mosaic art (Herculaneum, Pompeii), classical poetry (Horace, Virgil, Ovid) |
| 290 | Climax of Persecution of Christians | Historiography (Livy, Tacitus, Pliny the Elder) |
| 391 | Christianity becomes state religion in the Roman Empire | Library of Alexandria on fire again |
| 395 | Division in Western and Eastern Roman Empire | |
| 476 | Ruin of Western Roman Empire | |
| 529 | Academy in Athens closed by Justinian | Mathematicians collect and comment on earlier outcomes in Constantinople (Byzantium) |
| 1453 | Constantinople conquered by the Turks, end of Eastern Roman Empire | |

### 2.5.1 Hero

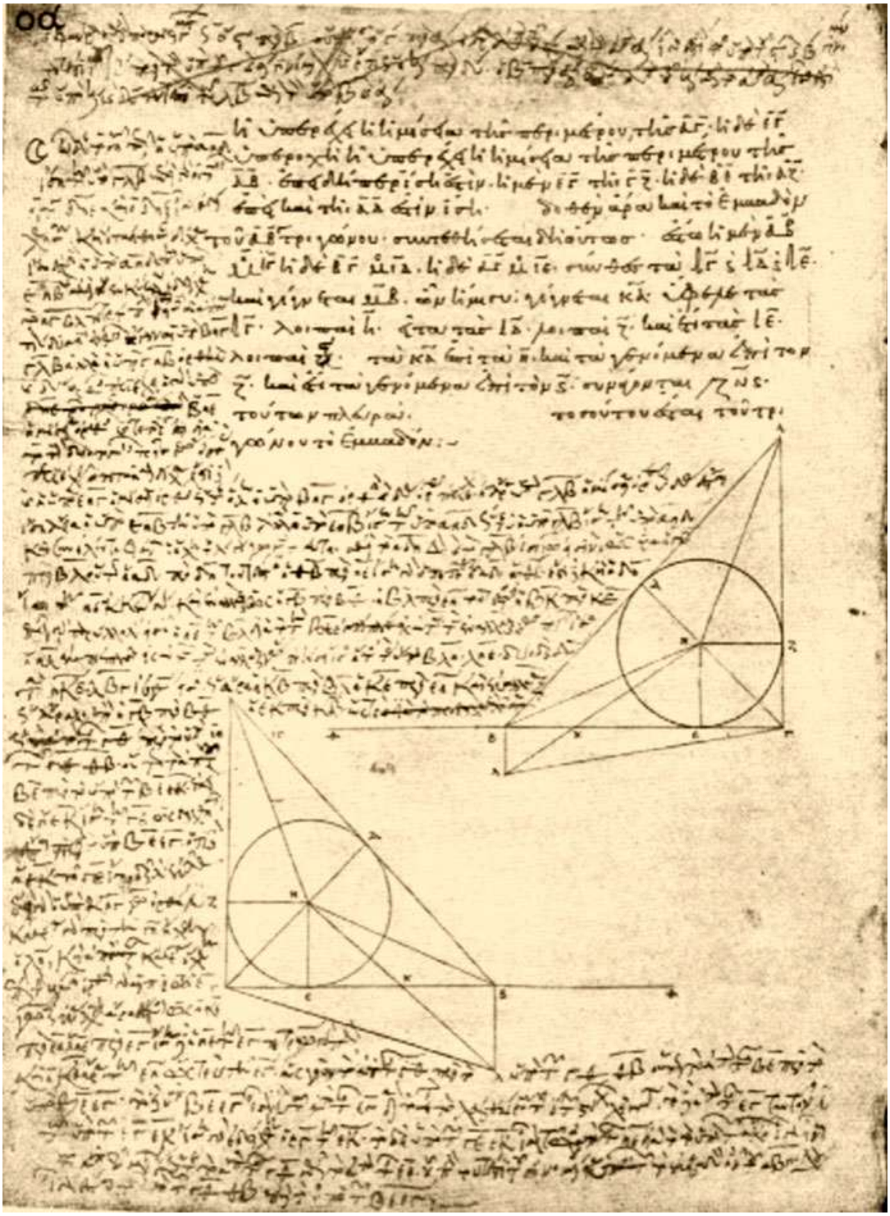
Hero was a mathematician and technician of whom we know very little. He is said to have lived anytime between 150 BC and 200 AD (or even a little later). Nowadays, he is assumed to have worked in Alexandria at the end of the $1^{st}$ century AD. As with Archimedes, he was also active within the area of mechanics and mathematics. However, his interest in the latter was preferably due to practical aspects. Hence, any comparison to the brilliant theorist Archimedes would be absolutely inappropriate. In fact, he focussed on encyclopaedic work, thereby following the ambition of his era to collect and preserve the classical legacy. Many of his texts and lectures were widely spread and often heavily revised. This was not just due to the fact that his works centred on applications common in his era.

Hero did not just follow up on the classical tradition (e.g., Euclid's 'Elements' and multiple texts of Archimedes), but also on the Babylonian–Egyptian tradition, whereby he adopted or further developed some only approximately valid methods. He replaced many indirect Euclidean proofs with direct ones in a commentary on Euclid (partially known from references in Arabic texts). His work 'Metrica' (the Greek text was rediscovered at the end of the $19^{th}$ century) was primarily addressed to applied scientists and deals with approximation formulae apart from many constructing and measuring instructions of plane and spatial figures, e.g., to calculate square and cubic roots. The third book (chapter) focuses on dividing areas and solids according to prescribed ratios including Archimedes' approach to dividing a sphere.

Nowadays, we still associate Hero's formula of calculating the area of a triangle with Hero's name. If three sides $a, b, c$ of any triangle are given, it is possible to calculate the area $A$ of the triangle by means of the auxiliary variable $s = \frac{1}{2}(a + b + c)$ without having to calculate one of its heights first:

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{2.5.1}$$

Although named after Hero, Archimedes already knew this formula according to Hero's proof (cf. Problem 2.5.1). As shown by Illus. 2.5.2, Hero made use of the incircle of the triangle (radius $r$): the three line segments drawn from the corners to the centre divide the triangle into three smaller triangles of height $r$. Inserting two right-angled triangles makes calculating $r$ possible. Apart from the two figures, only the text block about Hero's formula forms part of the displayed page; the remainder are marginalia of a reader, who also writes in Greek. They hint at how much the reader struggled when trying to comprehend the proof. It was unusual for that time that Hero worked with the product of four lengths (to be more precise: the product of two areas) when proving his formula. This enabled him to first calculate the square of the wanted triangle area without interference from square roots calculation. Hero missed citing all required Euclidean lemmas in this case. So, for instance, the writer of the manuscript (Illus. 2.5.2) did not immediately realise that the



**Illus. 2.5.2** Manuscript page of Hero's 'Metrica' containing the proof of Hero's formula: Codex Constantinopolitanus Palatii Veteris No. 1 ($11^{th}/12^{th}$ century), folio $71^v$.

connecting lines of the corners and the centre of the inscribed circle equal the bisecting lines of the angles! He was further fooled by the thought that both figures seemed to refer to a special case: a right-angled and isosceles triangle instead of the general case.

### 2.5.2 Pappus

Pappus, who was significant for the tradition of history, worked in Alexandria around 320, during late Antiquity next to other mathematicians, who dealt more with other branches of mathematics (e.g., the number theorist Diophantus, who was probably active in Alexandria around 250) and the founders of trigonometry, which we will look at further on. He belonged to the circle of the neo-Platonists. Pappus's 'Collections' is a collective work in eight books; all of them have been preserved apart from the first and the appendix of the second. In the early modern ages, European mathematicians found a lot of inspiration in 'Collections', as it contains many extracts from (partially conserved, partially lost) texts by Euclid, Apollonius, Archimedes and other mathematicians. Pappus expanded these extracts, critically commented on them, and made his own remarkable additions.

He added theorems on projective figures, remarks on extreme value problems, and the so-called Guldinus theorem, or Pappus's centroid theorem for solids of revolution (published by Paul Guldin in 1641!). Pappus phrased this theorem with a generality that was very rare during Antiquity and he himself remarked: "These propositions, which are practically a single one, contain many theorems of all kinds for curves and surfaces and solids, all at once and by one proof (...)." [Pappus/Jones 1986]. He also summarised and generalised a lost work by Apollonius on the division of two, three or four straight lines into given ratios by another straight line through a given point. Whereas Apollonius discussed the problem for three or four straight lines, Pappus extended the investigation to any $n$ straight lines, a problem that challenged several mathematicians to study it in the $17^{th}$ century.

### 2.5.3 Proclus

Among all the numerous collective and commenting literature composed during late Antiquity, we will also mention the detailed commentary of Proclus Diadochus (410–485) on Book I of Euclid's 'Elements'. Having studied in Alexandria, Proclus became head of the academy in Athens and wrote commentaries on a series of Plato's dialogues. He opened his commentary on Euclid with an explanation of the neo-Platonic philosophy of mathematics. Yet, amongst all other things, his numerous valuable historical remarks, which he included in his commentary on Euclid, became most important for posterity. (They mainly refer to the lost works by Eudemus and Geminus.) Proclus engaged thoroughly with the principles of geometry, discussed the relation between postulates, axioms and hypotheses, explained the difference between theorems and problems, and debated the structures of the proofs and terminology. In the one instance in which he attempted to go beyond Euclid, he intended to prove the parallel postulate after he had rejected an attempt of proof by Ptolemy. It is unclear if Proclus also composed commentaries on the other books of 'Elements' as he had intended.

We will discuss three more topics in this section: the origins of trigonometry, the texts of the Roman surveyors, and the Byzantine accomplishments after the turn of the millennium.

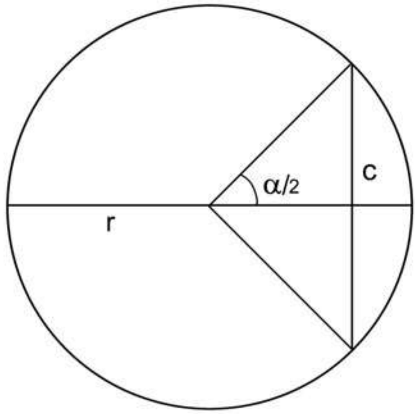
### 2.5.4 Chord geometry

Trigonometry was thought of as belonging to astronomy until Copernicus (1473–1543). Accordingly, it is addressed in astronomical works. The first independent textbook on trigonometry was written by Regiomontanus around 1464. However, it was only published posthumously almost 70 years later (1533). We have already been introduced to the first elements in pre-Greek mathematics. Stating the rebound in order to mark the inclination of a plane surface (e.g., of a pyramid) is core to a tangent function. Hereby, we must, of course, consider that the Babylonians expressed both necessary indicated dimensions (magnitude of rebound at given height difference) in different units and that they did not combine them to a common superordinate concept as reflected by the tangent function. Their ambition to grasp a circle segment by chord and sagitta can also be ascribed to a pre-stage of trigonometry, since plane trigonometry was based on the circle and the study of the relations of the circle chords before the sine and tangent function were introduced. (For this reason, it is also better to speak of chord geometry and to reserve the notion 'trigonometry' for the branch of geometry that applies to right-angled triangles, including their side ratios, i.e., the trigonometric functions.)

Calculating with semi-chords, which commonly known angle functions are based on, is more convenient than using chords. We will look at these in section 3.4.4 when addressing Indian geometry, since they were first introduced in India. The traditional Greek literature does not mention them. After the sine function had been well spread across and beyond India, those rather incomprehensible chord calculations disappeared from mathematics.

However, in order to understand chord geometry a little better, it is useful first to clarify the relation between chord and sine (Illus. 2.5.3). If we draw chord $c$ to the angle $\alpha$ at centre in a circle of radius $r$, half the chord $\frac{c}{2}$ divided by $r$ is the sine of $\frac{\alpha}{2}$. If we employ the commonly used abbreviation 'crd' for chord (Latin: chorda, corda), then: $sin\frac{\alpha}{2} = \frac{s}{2} : r = \frac{s}{2r}$. As a result:

$$c = crd\alpha = 2r \cdot sin\frac{\alpha}{2}. \tag{2.5.2}$$




**Illus. 2.5.3** The relation between chord and sine geometry

### 2.5.5 Ptolemy

We know that the astronomer Hipparchus calculated a table of chords in the $2^{nd}$ century BC, which was passed on as Ptolemy's table of chords. It forms part of his astronomical main work "Mathēmatikē Syntaxis", also called "Megalē Syntaxis" (the mathematical/great collection), which was the leading astronomical textbook until Copernicus. It is better known under the Arabic title "Almagest".

When publishing this table and its calculations, Claudius Ptolemy (ca. 100–ca. 160), who also worked in Alexandria, made a note that he would only contribute the smallest possible amount of theorems required, all of which were previously known. Based on the calculation of the side lengths of the first regular polygons in a circle, he aimed to obtain the chord values with a distance of each half a degree to the next. Thereby, he assumed the length of the diameter to be 120 parts (*partes* = $p$) in order to avoid too small sexagesimal fractions, since Ptolemy also used the sexagesimal system, as was customary within astronomy since the Babylonians. This method delivered the chords for the following regular polygons with $n$ edges by means of elementary geometrical considerations:

| $n$ | 3 | 4 | 5 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $\Phi$ | 120° | 90° | 72° | 60° | 45° | 36° | 30° |

By means of his chord theorem (an equivalent for the subtraction formula of the sine function: cf. Problem 2.5.2) he could calculate the chord of 12°, from which he obtained the chord of $1\frac{1}{2}^0$ by means of repeated halving. After remarking, that an exact construction of the angle $1^0$ is impossible, Ptolemy derives an approximate value for the chord of $1^0$ from the chords of $1\frac{1}{2}^0$ and $\frac{3}{4}^0$ which is sufficient for any practical use.


**Illus. 2.5.4** Ptolemy observing the constellation of stars (from Gregor Reisch: Margarita Philosophica ca. 1503, edition Straßburg 1504). It was normal to show Ptolemy with a crown until the Renaissance because it was assumed that he was a member of the ruling family of Ptolemean Egypt. The personified astronomy goes back to the Egyptian tradition of wall painting whereby the priest was always assisted by a goddess behind him (invisible for the people) when constructing the foundation of a temple or executing other holy acts

Accordingly, Ptolemy cites the relevant theorem as follows:

> If we draw two unequal chords in a circle, then the ratio of the greater chord to the smaller chord is smaller than the ratio of the arc on the greater chord to the arc on the smaller chord.

In other words, since it is not possible to construct the angle trisection elementarily, Ptolemy used here the linear interpolation in order to finally obtain the chord of 1°. He found: $crd.1^\circ = 1^p 2' 12''$ (written in sexagesimal). In his table, he stated the lengths of the chords for angles between $\frac{1}{2}^0$ and 180° in half a degree intervals, which corresponds to a sine table with an angle difference of 15' (see Problem 2.5.2).

As the reader will have noticed, Ptolemy had obtained the chord of 12° from the chords of 72° and 60°, i.e., from the sides of a regular pentagon and hexagon. His construction of the side of the regular pentagon is of interest in this respect (see Problem 2.5.3), since it differed from Euclid's.

Around 370, Theon of Alexandria composed a commentary on the first two books of 'Almagest'. Thereby, he was supported by his educated daughter Hypatia, as he himself mentioned. Hypatia, who also lectured at the Musaeum of Alexandria, was famous for her education and eloquence. She is said also to have written a commentary on Apollonius's conic sections (as well as on Diophantus's *Arithmetica*). Unfortunately, these works have not been passed on and Hypatia was murdered by a Christian mob in 415 because she was a follower of neo-Platonism. Her fate has inspired some novels, tales, films and paintings.

### 2.5.6 Menelaus

Next to plane trigonometry, astronomers also needed the spherical version, since the starry sky presents itself to the observer in the shape of a sphere, so that the simplest figure is the spherical triangle made of great circular arcs. The first collection of theorems from this area was composed by Theodosius of Bithynia around 100 BC. The basics for *spherical* trigonometry were written down then by the astronomer Menelaus, who worked in Alexandria, in his *Sphaerica* around 100 AD. There we find the notion and a definition of a spherical triangle for the first time. Menelaus presented relevant theorems in analogy to those which Euclid had phrased concerning plane triangles. Thereby, his rule of six great circular arcs was central. However, this does not occur in Menelaus's work in the form of a statement about the spherical triangle, which is intersected by a transversal (neither in Ptolemy's work, who also proved the proposition), but he refers to two great circular arcs $ADB$ and $AEC$ (both smaller than a semi-circle), which are intersected by two other great circular arcs $DFC$ and $BFE$ (also smaller than a semi-circle) in $D, B$ or $C, E$. The result is stated in a term equal to the following formula:

$$\frac{sin\ CE}{sin\ EA} = \frac{sin\ CF}{sin\ FD} \cdot \frac{sin\ DB}{sin\ BA}. \tag{2.5.3}$$

Nowadays, we would rather relate this to a spherical triangle $ADC$ (with an extended side $ADB$), which is intersected by the great circle $BEF$ in $B, E, F$ and phrase a proposition, in which the product of the three sines is equal to the product of the three other sines (apart from the algebraic sign):

$$sin\ AE \cdot sin\ CD \cdot sin\ DB = sin\ EC \cdot sin\ FD \cdot sin\ BA \tag{2.5.4}$$

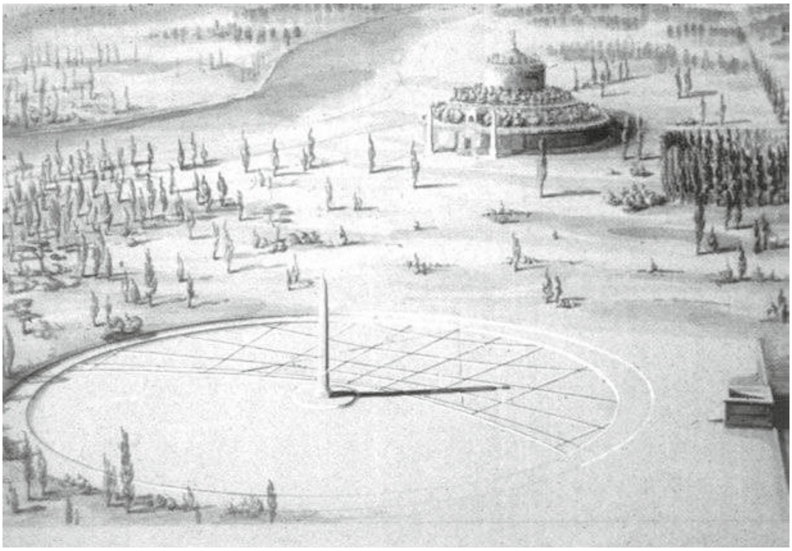
Similar to Ptolemy's chord theorem, simpler rules follow from the rule of six quantities (whose application presumes the knowledge of five quantities, if we want to calculate the $6^{th}$), if one or even more great circular arcs are greater than 90°. However, we leave it up to the reader to look at such special cases.

### 2.5.7 Sundial, analemma

The idea to link movements along spherical orbits and plane curves must have arisen quite early, i.e., as soon as one started to observe the simplest type of sundial, the gnomon, more closely. The Egyptians used it in the form of an obelisk. As already mentioned, Anaximander is said to have introduced this instrument, which he supposedly got to know in Babylon, in Greece in the first half of the $6^{th}$ century BC. If we mark the amount the shadow of the peak of the perpendicular bar moves around during an entire day on a (plane) base area, we obtain the picture of the daily circular arc of the course of the sun. If we do so in regular weekly or monthly intervals, these lines also reflect the changes of the annual course. Thus, we can also determine the time of the solstices, the incline of the ecliptic and the geographical latitude of the observation point. Thus, the gnomon became an indispensible instrument for astronomers and geographers, and engaging with the relevant theory, the origin of the theory of geometrical representations, was mandatory.

In his *Naturalia Historica* (Natural History), Pliny describes in detail the greatest ancient construction, the Solarium Augusti at Campus Martius in Rome, with its 30 m high obelisk. The analemma, the plane dial face, covered a field of approx. 65 m by 175 m. As partial excavations from some years ago show, it contained hour lines, meridians (noon lines with day marks) and month lines, as well as engraved zodiac names and details on the season ([Buchner 1982] cf. Illus. 2.5.5).

The geometrical-mechanical method to generate the line net on differently inclined receiving areas of sundials under consideration of the geographical latitude was probably developed by Eudoxus of Cnidus. Ptolemy dedicated this problem its own text, 'Analemma' (only preserved in a Latin translation). His treatise 'Planispherium' has only been passed on in Arabic. Having



**Illus. 2.5.5** Solarium and Mausoleum of Augustus at Campus Martius in Rome (Reconstruction) [E. Buchner, in: Archälogische Entdeckungen. Die Forschungen des Deutschen Archälogischen Instituts im 20. Jh., Vol. II, p. 180; Illus. 202 Mainz: Philipp von Zabern 1999]

been revised by al-Majrītī in 1143, it was translated into Latin by Hermann von Kärnten. It describes the representations of celestial circles in a plane. Ptolemy selected the south pole of the celestial sphere to be the projection centre and laid the plane through the equator. Regarding this representation, the stereographical projection has introduced the most popular astronomical instrument of the Middle Ages, the (plane) astrolabe, as well as the later displays of astronomical clocks. Since it represents circles by circles (or meridians by straight lines), it is especially suitable for astronomical purposes.

### 2.5.8 Cartography

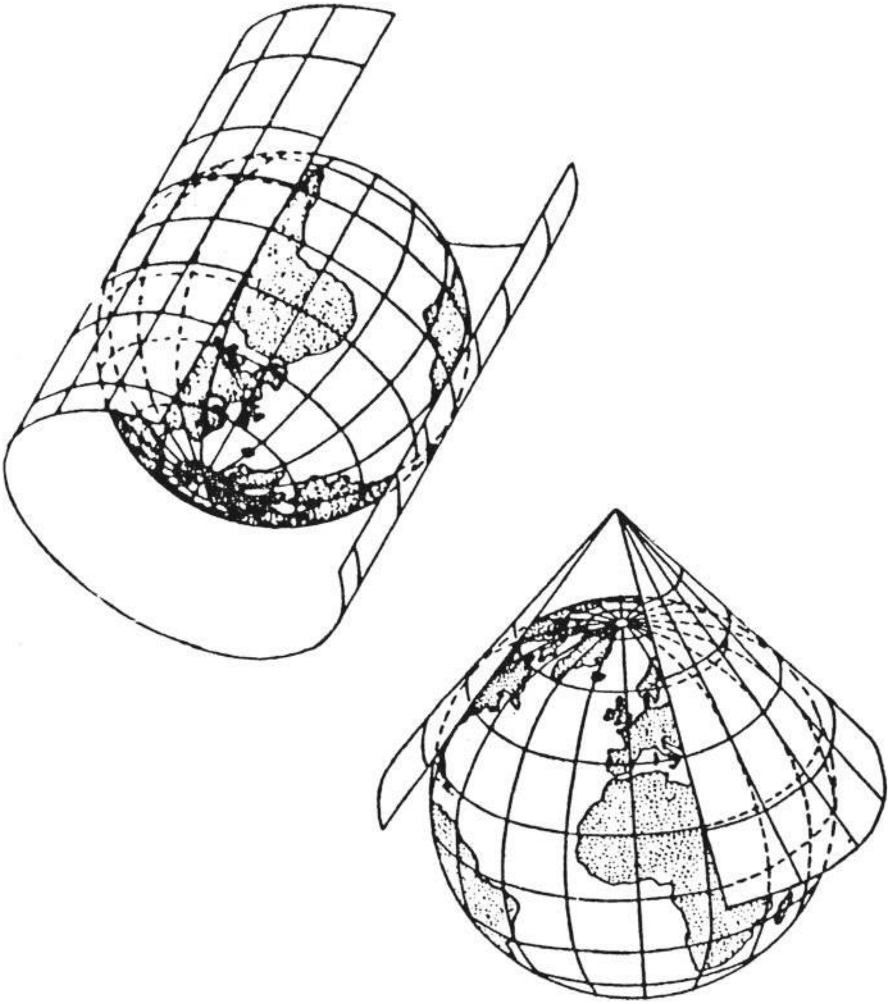
The geographers also encountered representational issues as soon as they attempted to represent greater areas or even the whole Ecumene (again, their term for the section of the inhabited world known to them). Erastothenes is to thank for introducing the right-angled coordinate system made of parallel circles and meridians, which pass through fixed points (towns, whose coordinates had been determined). He selected a circle of latitude, which runs through the Pillars of Hercules (Gibraltar), as the centre line. Although some aspects of this draft were criticised by astronomer Hipparchus (who suggested observing lunar eclipses for the difficult determination of longitude), the method applied by Erastothenes to allocate points on the sphere to their picture point in the plane has been clearly determined from a geometrical perspective. The geographer Strabo (64/63 BC–20 AD) used this as the foundation of a draft of a world map in a right-angled coordinate system in 'Geographica', his historically-oriented encyclopaedia of mostly regional geography.

Claudius Ptolemy's 'Geography' had a similarly significant role for geographical science as his 'Almagest' had for astronomy, being a summary and tradition of the geographical knowledge of his era. He dealt thoroughly with the theory of projection and criticised the use of the cylindrical projection used by his predecessor, Marinus of Tyre (beginning in the $2^{nd}$ century AD), following Eratosthenes due to the strong distortions near the poles. Instead, he propagated two versions of the conic projection, which are much more suitable in this respect. His great catalogue is ordered according to continents and countries and contains the longitudinal and latitudinal coordinates for 8100 locations. Hence, it provides the possibility of marking the respective locations in any constructed coordinate system. It is not known if he himself constructed a map according to these specifications.

Marinus had attempted to represent the Ecumene on a plane map by projecting Earth's surface on a cylinder, which touches the globe at the equator, whereby it generates a right-angled coordinate system. Ptolemy's idea was basically to equate Earth to a conical cap, which touches the globe in a circle of latitude, lying in the approx. middle of the known inhabited zone. (Ptolemy selected the circle of 54° that runs through Rhodes.) As a result, the strip of the Earth's surface containing this circle is hardly distorted in this representation. The meridians remain straight lines in this projection and the circles of latitude are represented as arcs of a circle. Only the northern and southern border zones are noticeably stretched in a north-south direction (Illus. 2.5.6). In order to eliminate this deficit as well, Ptolemy added a second cone from the south opposing the cone put on top of the northern hemisphere. This leads to a bend-over at the equator in the map representation (cf. Illus. 5.2.4).

Nonetheless, Ptolemy was not satisfied and developed a modified, mathematically more challenging conic projection. The circles of latitude are again represented as circles of an arc. This time, both circles of latitude are meant to pass through Thule (27°) and Aswan ($66°17'$). There is also a third, which runs through a location situated in the same distance as Meroë as a reflection south of the equator with $106°42'$ distance to the pole. All three locations are mapped true to scale. Thus, there are three points given for each circle of longitude, which are connected with each other via the determined arc of a circle. If we did not just want to map three but indeed all circles of latitude true to scale, the circles of longitude would be transformed into transcendent curves.

Hence, Ptolemy chose an approximation, which can be constructed with compass and straightedge (cf. Illus. 2.5.6; the stereographical projection was not considered by Ptolemy to generate maps of Earth.)



**Illus. 2.5.6** Principle of Marinus' cylindrical projection and Ptolemy's conic projection

Historical maps from the Renaissance, which were used for Ptolemy's conic projection, are displayed in section 5.2. (From a geometrical point of view, the development of cartography since the Renaissance has been determined by the fact that the cartographer aims to fulfil three requirements: truth in scale, angle and area. Their incompatibility can be demonstrated mathematically by means of differential geometry as developed in the $18^{th}$ century. Thus, the wealth of the projections used by now for producing maps is a consequence of deciding which of these properties can be neglected in favour of another one. Also cf. section 5.2)

### 2.5.9 Agrimensores

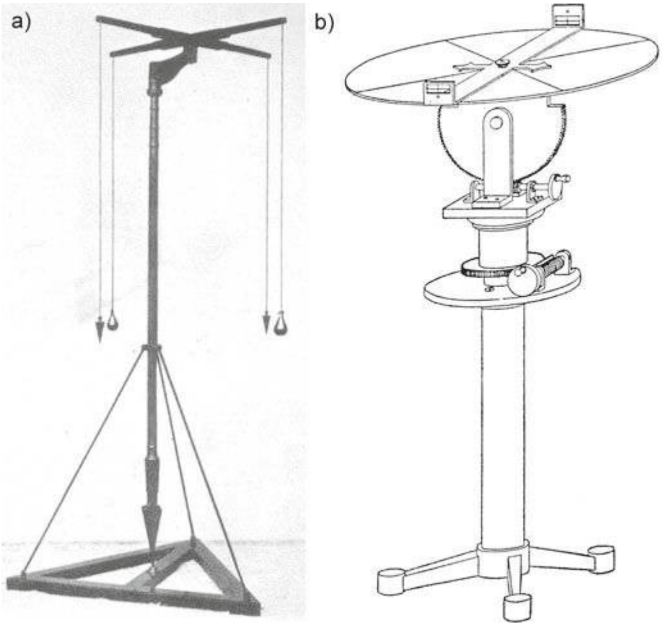
It seems to be a widespread opinion that mathematical theory developed from practical need. History until now has repeatedly indicated that the Greeks adapted a lot of knowledge and practical rules from many pre-Greek cultures, systematised all this, accounted for and joined to form a network of theories. However, we often miss the fact that there is not just one path from practice to theory, from prescriptions and rules to scientific textbooks. The influence is effective in the other direction as well. However, mathematical research has often failed to realise this. Fragments of mathematical theories can find their way back to practice. Depending on who got hold of them first, the user was aware of the inner correlation and the scope of validity. Otherwise, the knowledge of the structure and the logical composition that distinguished the theory got lost again. It is also possible that the adopted elements were only understood partially and applied in a way that did not necessarily do justice to the practical needs in an optimal manner.

The Roman surveyors, or agrimensores, as they were called, constitute an example of such a transfer. (Another one is indicated by the medieval masons' lodges and their tradition, which lasted until Albrecht Dürer; cf. Chap. 4). Cicero, reporting in *De Divinatione* I, 17 that Romulus had founded and built Rome according to the rules of the art of surveying (which, as known, is said to have taken place in 753 BC), rather indicates that this profession was highly regarded during the era of Cicero (106-43), a somewhat dubious assertion, since, initially, different 'professions' were responsible for surveying in the Roman Empire. Priests were in charge of surveying temple areas, the military took care of surveying the troop camps, and architects surveyed the construction of aqueducts. Apparently, Greek experts were called to aid in the first empire survey ordered by Caesar (conducted 37-20 BC). Because land had to be allocated after successful Roman conquests an agrarian legislation was necessary. As known, Rome has received great credit concerning the extension of the justice system. As a result, "guardians of justice" were required who were responsible for transfers true to scale. Such professionals were also needed when founding new towns within the colonies.

The rank of agrimensor had only fully emerged by the end of the $1^{st}$ century AD. Their own literature also developed during this time, the gromatic texts, named after a simple measuring tool, the groma or gruma (a surveyor's pole), which was further developed by Hero into an instrument of precision, the dioptra (see Illus. 2.5.7). The specialist literature was combined as *Corpus Gromaticorum* in the $3^{rd}$ century.

Cassiodorus (480?–575?) colourfully described how the agrimensores pursued their profession after he noticed that the theoretical nature of other sciences resulted in those professors having smaller numbers of students:

"But the agrimensor is entrusted with the adjudication of a boundary dispute that has arisen, so that there may be an end to wanton quarrelsomeness. He is a judge, at any rate of his own art; his law-court is deserted fields; you might think him crazy, seeing him walk along tortuous paths. If he is looking for ev-




**Illus. 2.5.7** Reconstruction a) of the Groma to determine right angles and b) Hero's Dioptra [O. A. W. Dilke: The Roman land surveyors. Newton Abbot, David & Charles 1971; p. 50, 75]

idence among rough woodland and thickets, he doesn't walk like you and me, he chooses his own way. He explains his statements, puts his learning to the proof, decides disputes by his own footsteps, and like a gigantic river takes areas of countryside from some and gives them to others." [Dilke 1971, p. 45-46]

Our modern knowledge of the Roman surveyors' geometrical skills mainly comes from collective manuscripts, which now belong to the Herzog August Library in Wolfenbüttel (Germany). It concerns both *Codices Arcerianus* A and B (named after Johannes Arcerius, who owned them from 1566 until 1604) and the *Codex Gudianus* (after the manuscript collector Marquard Gude, 1635–1689). The 'Arcerianus' was written in the $5^{th}$–$6^{th}$ century and is, thus, the oldest completely preserved ancient mathematical manuscript. The 'Gudianus' comes from the $9^{th}$ century. We are dealing with duplicates of texts in both cases, which refer to the surveyors' or agrimensores' work duties.

Noteworthy authors who informed us of the details of Roman surveying were Vitruvius (writing around Christ's birth), the former officer Columella, who composed twelve books on country life in approx. 65 AD, Frontinus, the architect and supervisor of water engineering and management in Rome (died approx. 100 AD), and the surveyors Hyginus and Balbus. Apart from land surveying, the agrimensores were also in charge of jurisdiction as far as it concerned real estate disputes (an example can be found in Appendix A. 3, p. 568). Consequently, the surveyors' texts are not just simple handbooks of applied geometry, but represent legal guidelines at the same time. For instance, they contain a distinguished specialised terminology to refer to different types of fields and their borders, which goes far beyond the terms customary in geometry to refer to plane shapes. Whereas Euclid ('Elements', Book I, def. 13) defines: "A boundary is that which is an extremity of anything", the surveyor Balbus writes: "Boundary is that, until which freehold is effective." However, the basic elements of geometry were often introduced based on Euclid's or Hero's definitions.

Here are some examples from the Wolfenbüttel Manuscript Cod. Guelf. 105 Gud. Lat. (cf. Illus. 2.5.8):

*Rectum est cuius longitudinem sine latitudine metimur.*
(A straight line is that which is measured as length without width.)

However, following Euclid's first definition,

*Punctum est, cuius part nulla est.*
(A point is that which has no part.)

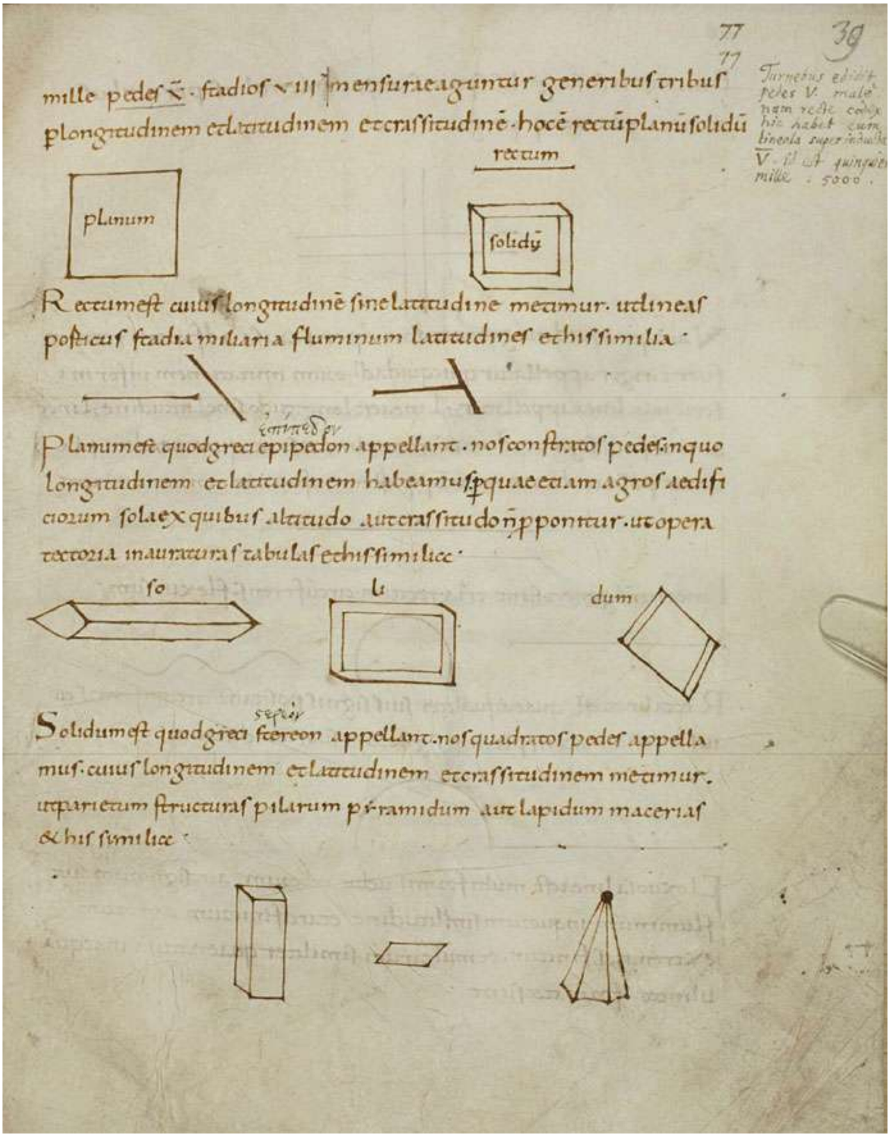
he wrote the second:

*Linea autem sine latitudine longitudo.*
(A line [is] breadthless length.)

Hence, a surveyor explicitly includes the practical appliance of surveying in the definition, which would only be an interference for a theorist like Euclid. Defining area (*planum*) and solid (*solidum*) – pay attention to the drawing – was done analogously. Three types of lines are introduced on the following page: straight, circular and flexible, i.e., bendable in any direction. With their help, the most diverse of figures can be named and later calculated. The approximation formula used by the agrimensores can in part be traced back to the pre-Greek cultures. This applies not only to measuring methods but also to calculation approaches. Sometimes we can also observe improvements. For instance, when the area of a circle segment is calculated by means of chord $c$ and sagitta $s$ (cf. Formula 1.2.11) according to the rule:

$$A = \frac{c+s}{2} \cdot s + (\frac{c}{2})^2/14 \tag{2.5.5}$$

The first part, which indicates the substitute of a segment by a trapezoid, can also be found in an Egyptian papyrus ($3^{rd}$ century BC or earlier). The correcting term is part of an extensive work on agriculture (64 AD) by Columella, which is meant to teach agriculturists basic knowledge of surveying. Hero discussed this correction in great detail. The denominator 14 indicates


77 39

mille pedes V̄ · stadios VIII mensurae aguntur generibus tribus
p longitudinem et latitudinem et crassitudinē · hoc ē rectū planū solidū

Turnebus edidit pedes V. male nam recte codex hic habet cum lineola super inducta V̄. id est quinque mille : 5000.

rectum

planum

solidū

Rectum est cuius longitudinē sine latitudine metimur · ut lineas posticus stadia miliaria fluminum latitudines et his similia ·

Planum est quod greci epipedon appellant · nos constratos pedes in quo longitudinem et latitudinem habeamus per quae etiam agros aedificiorum sola ex quibus altitudo aut crassitudo non ponitur · ut opera tectoria in auraturas tabulas et his similiter ·

so li dum

Solidum est quod greci stereon appellant · nos quadratos pedes appellamus · cuius longitudinem et latitudinem et crassitudinem metimur · ut parietum structuras pilarum pyramidum aut lapidum macerias et his similiter ·

**Illus. 2.5.8** Manuscript by agrimensores
[Cod. Guelf. 105 Gud. Lat., fol. 77r: (geometrical definitions), Herzog August Library Wolfenbüttel]

**Illus. 2.5.9** Agrimensor manuscript
[Cod. Guelf. 105 Gud. Lat., fol. 99r: Layout of quadrilateral fields, Herzog August Library Wolfenbüttel]

the use of Archimedes' value 22/7 for $\pi$ and, thus, shows how a value obtained by theoretical consideration advances into practice, even though the applied scientist is usually not able to recognise the reasoning of the rule (cf. Problem 2.5.4).

To conclude the section on agrimensores, we present another page from the Gudianus (Illus. 2.5.9). It shows how rectangular fields were meant to be constructed in front of a town wall in a plane located between sea and mountains. We also want to mention some scholars from the realm of Roman culture from the $5^{th}$ to $6^{th}$ century, Martianus Capella, Boethius and Cassiodorus. The mathematical parts of their texts, together with elementary sections of Euclid's 'Elements', constituted an essential foundation for the medieval teaching of geometrical knowledge within the scope of the quadrivium (the four mathematical sciences: arithmetics, geometry, astronomy, and harmony or music). These will be introduced in Chap. 4.

### 2.5.10 Byzantium

The Alexandrian school of mathematics finally fell victim to the persecution by the Christians, whereby – as mentioned – Hypatia lost her life in 415. The academy of Athens was closed by Emperor Justinian I in 529 (he ruled from 527 until 565). In Constantinople (independent of Rome since the Emperor Theodosius's death in 395), scholars were able to work a little longer and to continue their collecting and commenting work. The Hagia Sophia was erected there by Anthemius of Tralles (died 534) and Isidore of Miletus, who, after Anthemius's death, took charge of the construction. The former wrote a treatise on foci, in which he proved that parallel incident rays are collected by a parabolic reflector at the focus. Thereby, he also looked for a configuration that reflects a beam of light, which comes from the sun and enters a dimmed room through a small hole in a manner that makes it pass through a fixed point independent of time of day or year. Anthemius succeeded in creating a point-by-point construction for the shape of the reflecting surface, known as an ellipsoid. He concluded the possibility of a string construction of the ellipse based on the properties of the ellipsoid concerning a fixed plane (constant sum of distance of two fixed points) (see Problem 2.5.5).

The short text on platonic solids, later referred to as Book XV of 'Elements', probably comes from the Byzantine mathematics of the $6^{th}$ or $7^{th}$ century. Its author remains a mystery. The text consists of three parts. The first part contains methods of how to use platonic solids to generate another platonic solid by means of inscribing, linking edge medians and likewise. The second part describes ratios between the numbers of edges, edges originating from a corner, the number of edges of each face and likewise. The third part deals with determining the angles between adjacent faces of solids.

The topics are not addressed systematically, but perfunctorily. Despite the triviality of the content compared to the works of Euclid, Archimedes and


**Illus. 2.5.10** Emperor Justinian (Mosaic in San Vítale, Ravenna) [Photo: H.-W. Alten]

Apollonius, especially the first two parts, offer approaches to combinatory thinking, completely unheard-of in Greek mathematics until then.

Eutocius of Ascalon (born around 480), who was friends with Anthemius, is to thank for a number of valuable commentaries on some texts from Archimedes and Apollonius, wherein he, amongst other things, reports of solutions for the problems of doubling the cube, which have not been passed on in other forms. The first four books of Apollonius's conic sections were only preserved in their original language as a result of Eutocius's commentary. However, this last centre, which tried to preserve Greek mathematics, experienced a major decline in the era of iconoclasm (726–843).

In 863, Leo the Mathematician and Philosopher became vice chancellor of the University of Constantinople, newly founded by Bardas, after he had rejected an offer to work at the caliph's court in Bagdad. He made his students duplicate classic philosophical and mathematical texts. A Euclidean manuscript, which was created back then, is kept in Oxford nowadays. An Archimedes duplicate belonging to the popes had crossed the court of the lower Italian Normans and was studied intensively in the $16^{th}$ century before being lost. The most beautiful of all the preserved manuscripts by Ptolemy was also duplicated in Byzantium at the time.

From the $10^{th}$ until the $14^{th}$ century, a thin upper class re-emerged gradually in Constantinople who were interested in philosophical and mathematical studies and consciously used the Greek language. The scientists were again

**Illus. 2.5.11** The Abbasid caliph al-Ma'mun sends an envoy to the Byzantine Emperor Theophilos (lat. Theophilus). During the Middle Ages has been an exchange between Byzantine and Islamic science.
(unknown, $13^{th}$ century, History of John Skylitzes, Biblioteca Nacional de España)

focussed on collecting, preserving and continuing the Greek legacy in this phase of Byzantine mathematics. However, new influences were also noticeable in this era. These were from the Latin Middle Ages, from Arabic culture and science, and some touched on Indian calculating with digits and other Indian methods.

Around 1050, Michael Psellos became the leading professor of the renewed university, at which Plato's and Aristotle's philosophy and science were discussed and critically commented upon – last but not least by means of the logical texts of Aristotle. Psellos joined Plato's and Proclus's view that mathematics is the link between the world of ideas and bodily objects, and simultaneously a means to teach students abstract thinking. The church was finally divided (in 1054) into western and eastern Roman during his lifetime. One and a half centuries later during the fourth crusade (1202–1204), Byzantium was conquered and plundered by the crusaders and was ruled by the Latin Empire until 1261.

In the $12^{th}$ century, the Byzantines made first contact with Indian numbers, as marginalia in a Euclidean manuscript show. Indian calculation is first explained in an anonymous text from 1252; a later account was composed by Maximus Planudes (1255?–1310). He was already living under the reign of the Palaiologan dynasty, which began in 1261 and lasted for approx. 200 years, constituting the actual humanistic golden age of Byzantine mathematics. Nicolaus Rhabdas (died 1350) edited the work of Maximus Planudes in 1340 and composed a description of finger calculation in the meantime.

Theodore Metochites (1260?–1332) worked within the area of geometry. Amongst other things, he focussed on Euclid's Elements, and the writing of Apollonius and Ptolemy.

**Illus. 2.5.12** Mausoleum of Galla Placidia in Ravenna. Byzantine mosaics with stars and geometrical ornaments decorate the ceiling of the tomb built in the $5^{th}$ century for the daughter of Theodosius I (the Great), who was the last Emperor to rule the whole Roman Empire.
[Photo: H.-W. Alten]

At the same time, Johannes Pediasimus proposed an approach to applied geometry in Hero of Alexandria's style. The bilingual Basilian monk Bernhard Barlaam (1290?–1348?) from Calabria, who, as a Byzantine messenger, negotiated the reunion of the western and eastern Roman churchs with pope Benedict XII in 1339, interpreted Book II of Euclid's 'Elements' in an arithmetic-algebraic manner. Isaak Argyros (1310?–1371) wrote scholia on Book I to VI of 'Elements'. Being faced with downfall after negotiations between the popes for the reunion of the church had broken apart, and with the Turkish threat a growing menace, many scholars immigrated to Italy and passed on their knowledge to the prospering science of the Renaissance (cf. Chap. 5).

To finish, we must emphasize that the research on Byzantine mathematics is by no means concluded, as many manuscripts have only recently begun to be looked at and evaluated. Thereby, the collaboration of mathematical historians and Byzantinists, as demonstrated excellently by Kurt Vogel and Herbert Hunger around 1960, is crucial. An intensification of this research would be highly desirable based on the position of Byzantine mathematics at the intersection of the mutual influence of east and west.

**The classical problems of Greek mathematics**

| | | |
|---|---|---|
| ≈ 585 BC | Thales | Diameter halves circle, right-angled triangle in semi-circle |
| ≈ 550 | Pythagoras?, Pythagoreans | Regular pentagon in circle |
| ≈ 450 | Hippasus | Dodecahedra; discovery of irrational numbers |
| ≈ 440 | Hippocrates | Squaring of circular moonlets; insertion of two intermediate proportionals $x, y$ to solve the Delian problem |
| ≈ 434 | Anaxagoras | Attempt of squaring the circle (how?) |
| ≈ 430 | Antiphon | Squaring the circle by means of inscribed polygons of $3 \cdot 2^n$ or $4 \cdot 2^n$ edges |
| ≈ 420 | Hippias | Angle section by means of $\frac{y}{\alpha} = \frac{\Phi}{\pi/2}$ |
| ≈ 410 | Bryson | Squaring the circle by means of inscribed and circumscribed polygons; intermediate value theorem |
| ≈ 390 | Archytas | Insertion of two geometrical means, solved stereometrically |
| ≈ 380 | Theaetetus | Five platonic solids with circumscribed spheres |
| ≈ 370 | Eudoxus | Theory of irrationalities; 'platonic' movement-geometrical insertion of two geometrical means? |
| ≈ 350 | Brothers Deimonstratus and Menaechmus | Squaring the circle by means of Hippias' curve ('Quadratrix'); construction of parabola and hyperbola by two geometric means |
| ≈ 330 | Euclid | 'Elements'; compass and straightedge constructions |
| 287–212 | Archimedes | Squaring the circle, the parabola; sphere volume; heptagon construction (only preserved in Arabic); angle trisection (paper strip construction) |
| ≈ 240 | Eratosthenes | Mechanical solution for the insertion of two median proportionals |
| ≈ 210 | Apollonius | Theory of conic sections; two geometric means; 'platonic' solids |
| ≈ 180 | Diocles | Cissoid |
| ≈ 180 | Nicomedes | Conchoid (shell line) |
| ≈ 75 AD | Hero | Neusis construction for two geometric means |
| 85?–165? | Ptolemy | Table of chords (angle trisection) |
| ≈ 320 | Pappus | 'Collections' with historical reports |
| ≈ 370 | Theon | Euclid adaptation |
| ≈ 460 | Proclus | Commentary on book I of Euclid's 'Elements' |
| ≈ 520 | Simplicius | Extracts from Eudemus (around 320BC): report on Hippocrates' squaring the moonlets |
| ≈ 520 | Eutocius | Commentary on Archimedes 'On the sphere and the cylinder': Report of the history of the Delian problem |

## 2.6 Problems to 2

**Problem 2.1.1:** Measuring distance according to Thales

Since we cannot measure the distance of a ship at sea directly, Thales is said to have located it once from the peak of a tower and once from the foot of the tower. The tower represented a side of a triangle, the two sight lines the other two sides.

a) To what extent is the geometric theorem cited above required to justify this method?
b) Which formula for calculating the distance of a ship follows from this?

**Problem 2.1.2:** Versions of measuring distance according to Thales

How could Thales have determined the distance of a ship at sea, if there was no tower high enough on the shore? (Under the condition that he knew all theorems accredited to him, two different methods are possible.)

**Problem 2.1.3:** The golden ratio applied to the pentagon

As is well known, the diagonal and side of a regular pentagon stand to each other in the golden ratio (cf. Illus. 2.1.9). Derive the square equation, which exists between $d_0$ and $s_0$ and the ratio as the solution. How is its irrationality expressed?

**Problem 2.2.1:** Solving the Delian problem by means of conic sections

a) Show why Hippocrates' approach leads to the solution of doubling the cube.
b) Sketch the three conic sections contained therein in a $(x, y)$ – coordinate system and determine quantities $x$ and $y$.

**Problem 2.2.2:** Hippias' quadratrix

a) Draw the quadratrix by constructing some points according to the rule of motion geometry (e.g., by using a protractor or by continued halving of line segment and angle).
b) Where exactly are the limitations of elementary geometrical construction exceeded when using the quadratrix for division of angles?
c) Write down the equation of the quadratrix in the polar form of $\rho = f(\Phi)$.
d) The extremity of the quadratrix on the $x$-axis is indeterminate, since the side that is moving downwards and the revolving side collapse in this position. Determine the limit point of the quadratrix on the $x$-axis, i.e., it's distance $\rho$ from the origin (the $y$-coordinate equals 0 here) and verify that it is $\frac{2}{\pi}$. (Hint: Use the rule of de l'Hospital!)
e) How can you use this result for squaring a circle?

**Problem 2.2.3:** Paper strip construction for angle trisection according to Pappus

a) Draw the figure the paper strip construction is based on, as described in the text.
b) In order to prove that this is correct, add the auxiliary line $FB$ and drop the perpendicular from $F$ to $BE$, whose foot shall be named $G$. Why does $BF$ now also equal $OB = DF = FE$? If we refer to the smaller of both separated angles $\measuredangle AOD$ as $\beta$, then $\measuredangle OFB = 2\beta$. Why? Also why does $\measuredangle BOE = 2\beta$? Consequently, $\alpha = 3\beta$, as was meant to be proven.

**Problem 2.2.4:** Angle trisection by means of neusis construction according to Archimedes

a) Add in Illus. 2.4.9 the auxiliary line $BE$ and prove that angle $FDE$ is the third part of angle $ABC$.
b) How can we extend the basic idea of this construction to find the fifth, seventh, ... part of a given angle by means of a suitable aid?

**Problem 2.2.5:** Nicomedes' conchoid

a) Assure yourself that the polar equation $\rho = f(\varphi)$ of the conchoid has the following form ($\varphi = \measuredangle DCF$):

$$\rho = a + \frac{b}{cos\ \varphi} = a + b\ sec\ \varphi \tag{2.6.1}$$

b) What is the distance between any point $P$ of the conchoid and the straight line $AFB$ (Illus. 2.2.5)?
c) Prove that this straight line is an asymptote.

**Problem 2.3.1:** Volume of a pyramid according to Euclid

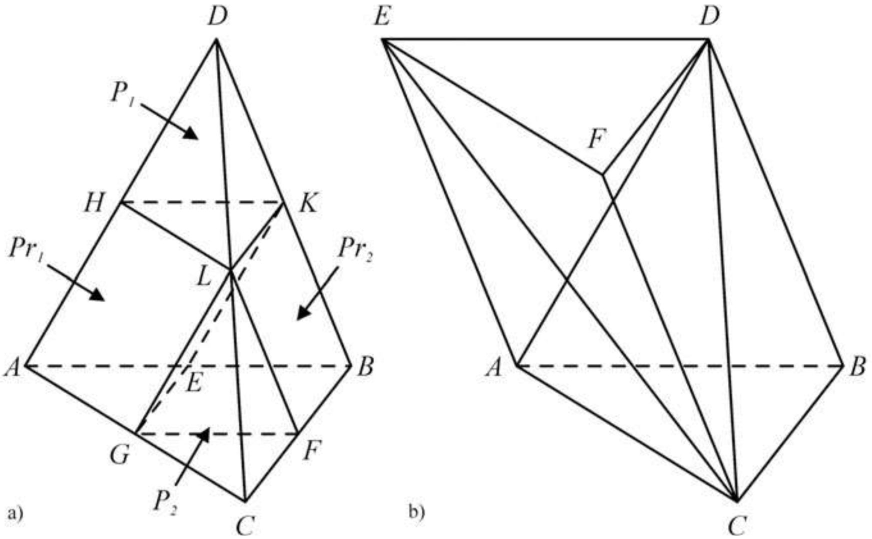
a) Book XII, 3 of 'Elements' describes how any three-sided pyramid $ABCD$ with base $\mathcal{G}$ and height $h$ is decomposed in the manner shown in Illus. 2.6.1a) by cuts through the centres $E, F, G, H, L, K$ of the edges in two pyramids $P_1 = HKLD$, $P_2 = GFCL$, which are similar to the initial pyramid, and two triangular prisms $Pr_1 = AEGHKL$ and $Pr_2 = EBKGFL$, of which we already know at this point that their volume is the product of base and height, so together they have the volume of $\frac{1}{4}\ \mathcal{G}h$.
(Concerning $Pr_1$, $\frac{1}{4}\ \mathcal{G}$ is the base and $\frac{1}{2}\ h$ is the height. Regarding $Pr_2$, first imagine an affinely distorted cuboid with base $\frac{1}{2}\ \mathcal{G}$ and height $\frac{1}{2}\ h$ by means of a congruent exemplar, which is added upside down, and then take half of this product!) Since you can proceed analogously with the

union of both remaining pyramids, whereby we must put the factor $\frac{1}{4}$ in front now (as a product of 2, the quartering of the base and the halving of the height), the modern mathematicians would conclude without a doubt that the pyramid volume is given by an infinite series

$$V = \mathcal{G} \cdot h \cdot (\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n + \cdots) \qquad (2.6.2)$$

Account for the formula (2.6.2) and the value $\frac{1}{3}\ \mathcal{G}h$ of the sum of the series.

b) Euclid's approach to the volume formula is much more circumstantial. By showing that the prismatic parts split off the pyramid constitute more than half of the overall pyramid volume (Which follows from?), he first of all indicates in Book XII, 4-5 that the volumes of three-sided pyramids with equal heights act like their bases. He does so by means of an indirect proof (similar to the approach Archimedes chose when deriving volume formulae). Finally, Book XII, 7 demonstrates that each three-sided pyramid $ABCD$ adds up to an oblique triangular prism (Illus. 2.6.1b) by means of two other three-sided pyramids $DEFC$ and $ACED$, whereby each two of these have a base and the respective height in common.




**Illus. 2.6.1** Figure to problem 2.3.1

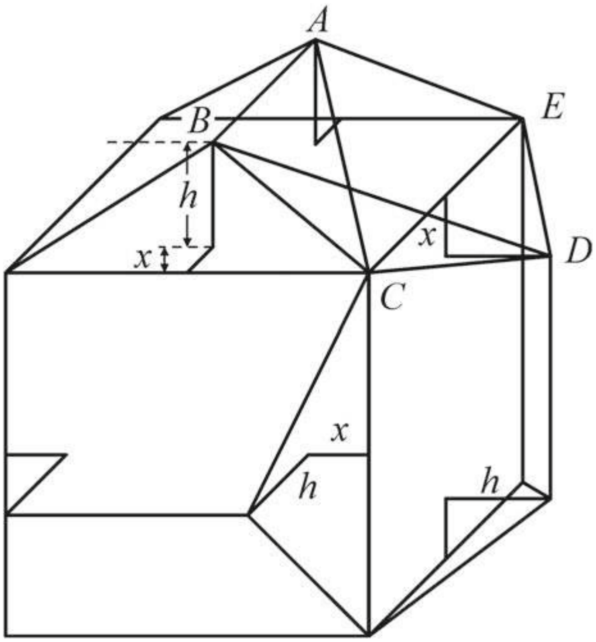
As a result, we obtain the formula for volume. The text passage in 'Elements' in which this conclusion is explicitly stated may, however, be a later addition. Given the basic idea of XII, 7, which is not obvious at all, the formula is, in fact, almost too trivial to be mentioned.

c) However, there is another strange method for obtaining the volume formula directly by means of decomposing the pyramid as in XII, 3. Namely, if we presuppose that the pyramid volume is proportional to both base and height, then we only have to determine the constant $c$ in the formula $V = c\,\mathcal{G}\,h$. Then, we obtain a linear equation for the constant $c$ from the decomposition stated in XII, 3. The solution would be $c = \frac{1}{3}$ [Schreiber 1994]. Establish the equation!

**Problem 2.3.2:** Construction of the regular dodecahedra from a cube

a) Book XIII of 'Elements' shows that we can obtain a regular dodecahedron from a cube by attaching hipped roofs to the sides of the cube as shown in Illus. 2.6.2. (This is to date the most elegant method for proving the existence of the regular dodecahedron). Take into account that this proof is necessary and that the existence of the regular dodecahedron is not as self-evident as that of the regular tetrahedron and the cube.
b) If we take the edge lengths of the given cube to be a unit, any hipped roof is determined by the two parameters $x(0 < x < \frac{1}{2})$ and $h(> 0)$. However, these two parameters must meet the following four conditions:
   1) The trapezoid of a roof must lie in a plane with the adjacent triangle of the adjacent roof. (Transfer this into an equation for $x$ and $h$!)
   2) All edges of the created pentagons must be of equal length, i.e., the roof ridge $AB$ must be as long as edge $AE$ from the ridge to the corner of the cube. (Transfer this also into an equation for $x$ and $h$!)
   3) The pentagon angles must be equal. (If 1) and 2) are already met, it suffices that each diagonal of type $BD$ and each diagonal of type $AC$ are equal to diagonal $CE$, i.e., equal 1.)

   Verify that the solution $x, h$, which has already been definitely determined by conditions 1) and 2), also meets both conditions of 3).
c) From the described construction of the regular dodecahedra, it follows conversely that each dodecahedron has five inscribed cubes. If we view the regular pentagon $ABCDE$ from Illus. 2.6.2 as the side of a regular dodecahedron, then each of the five diagonals could take on the role of the cube edge $CE$. A congruent mapping of the regular dodecahedron onto itself is obtained i) by determining a map corner $f(A)$ for any arbitrarily selected corner A, ii) by choosing one of the three edges originating from $f(A)$ as a map edge for one of the three edges $k$ originating from $A$ and iii) by choosing one of the two possible map faces for one of the faces, which borders on the chosen edge $k$. Accordingly, there are 20 (number of corners of the dodecahedron) times 3 times 2 equals 120 mappings of the regular dodecahedron onto itself. Half of them (60) preserve the orientation of the regular dodecahedron (These are revolutions around its centre). This group is isomorphic to group $A_5$ of the even permutations of the five inscribed cubes. State a one-to-one mapping between the group of revolutions of the dodecahedron and the group of permutations of the




**Illus. 2.6.2** Figure to problem 2.3.2

cubes, and prove that it is an isomorphism of both groups. This correlation proved to be of great significance for the irresolvability of group $S_5$ and, consequently, for the Abel-Ruffini theorem regarding the insolvability of the general equation of degree five in radicals. This shows us again how theories and theorems of modern mathematics are often closely connected with concepts sometimes even dating back to Euclid.

**Problem 2.3.3:** Division of triangles in a given area ratio

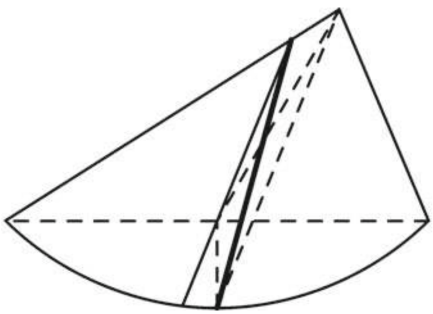
Problem 19 of Euclid's treatise 'On divisions of figures' requires us to divide a given triangle $ABC$ in two segments of the same area by means of a straight line, which runs through a given point $D$.

a) Reduce this problem down to determine a line segment, the knowledge of which thereof enables the construction of the wanted straight line, by means of the sides $a, b, c$ of the given triangle and two further line segments $d, e$, which characterise the position of $D$ regarding the triangle.
b) Solve this reduced problem in an algebraic manner and conclude the resolvability of the problem by means of compass and straightedge.
c) Contemplate how the solution found in b) can be accomplished with compass and straightedge based on Book II of 'Elements'.

d) Investigate (beyond Euclid), how the resolvability and the number of solutions depends on the position of point $D$ regarding the triangle (e.g., if $D$ is the centre of gravity of the triangle, then there are three straight lines of the wanted type).
e) How does everything change if, instead of halving the area, division is required based on a different ratio? (Euclid deals with the division given any rational ratio according to the example 1 : 3.) Does this ratio have to suffice under any conditions (e.g., due to an arbitrarily divided line segment) in order for the problem to remain solvable with compass and straightedge? Advice: the historical solution to problems a) and b) can be found in [Schreiber 1987a], the solution to d) in [Schreiber 1994].

**Problem 2.3.4:** Halving a triangle with an attached circle segment

Another problem (no. 28) from 'On divisions of figures' requires us to divide a figure that consists of a triangle and a circle segment, which has been outwardly attached to one of the triangle sides, into two halves of equal area by means of a straight line, which runs through the midpoint of the arc of the circle. Illus. 2.6.3 shows the historical solution to this problem without a rationale. We refer those who are not able to reconstruct the rule of construction and/or their justification to [Schreiber 1987a].



**Illus. 2.6.3** Figure to problem 2.3.4

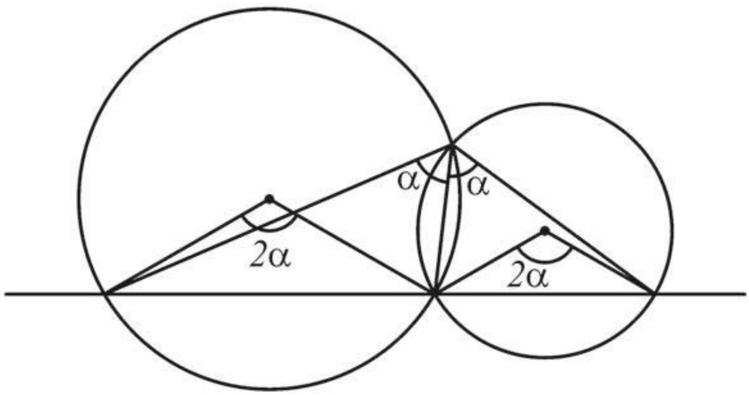
**Problem 2.3.5:** Apparent centre of a line segment depending on the view of the observer

In 'Optics', Euclid raised the following question: from which points can you view a line segment $AC$, which has been divided unequally by point $B$, in a manner that makes it appear halved? The fact that there are an infinite number of such points follows immediately from the inscribed angle theorem. It is even possible to indicate the common visual angle $\alpha$, which is meant to appear under the section of the line segment, beforehand (Illus. 2.6.4). This problem may be the origin of Apollonius's theorem, since he characterised circles as geometrical loci of such points in a plane [Schreiber 1994]. If $A, B, C$ are given in this order on a straight line and we have a visual angle $\alpha$ under which both line segments are meant to appear, then each of the line segments $AB$ and $BC$ together with angle $2\alpha$ determine a circle $k_1$ and $k_2$ respectively, the peripheral angle of which concerning each chord $AB$ and $BC$ equals $\alpha$ (Illus. 2.6.4). The wanted point $P$ is located in the intersection of both circles differing from $B$. If $M$ refers to the homothetic centre of both circles, then$MA : MB = MB : MC$, thus $MB^2 = MA \cdot MC$. Consequently, $M$ is solely determined by $A, B, C$ and independent of $\alpha$ and $P$, respectively. Since we can also obtain $P$ by reflection of $B$ on straight line $g$ through the centres of both circles, $PM = BM$, i.e., the variable point $P$ is always located on the circle with centre $M$ and radius $MB$ (Illus. 2.6.5). In order to arrive at this characterisation of this circle by means of $A, B, C$, which is expressed by Apollonius's theorem, we refer to $AB$ as $a$, $BC$ as $b$, $PA$ as $c$ and $PC$ as $e$ (Illus. 2.6.5). According to the law of sine:

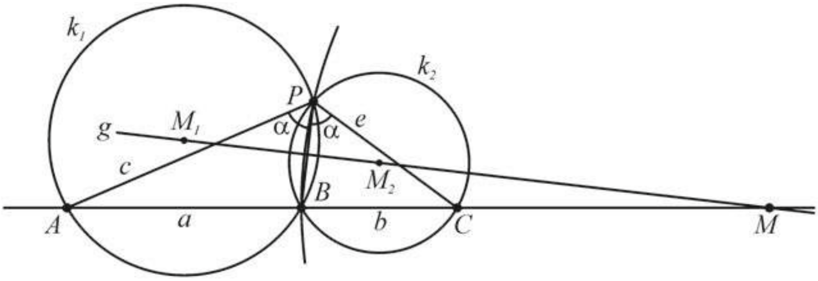
In triangle $ABP$  $a : c = sin\ \measuredangle APB : sin\measuredangle ABP$,
In triangle $BPC$  $b : e = sin\ \measuredangle BPC : sin\measuredangle CBP$.

The latter is equal to the sine of the adjacent angle of the triangle $\measuredangle ABP$. As a result, $\measuredangle APB = \measuredangle BPC$ (equal in meaning to $sin\measuredangle APB = sin\measuredangle BPC$) exactly when $PA : PC = c : e = a : b$. Consequently, the circle, which is characterised as a locus of all points $P$ with $\measuredangle APB = \measuredangle CBP$, can also be described as the set of all points $P$ with $PA : PC = a : b$ (Apollonius). Retrace these considerations!

Two further "freshly" composed problems regarding Euclid's 'Optics' can be found in [Schreiber 1995].




**Illus. 2.6.4** Figure to problem 2.3.5




**Illus. 2.6.5** Figure to problem 2.3.5

**Problem 2.4.1:** Aristarchus of Samos (approx. 310 – approx. 230 BC) realised a part of the program adumbrated in Euclid's 'Optics'.

He calculated the real sizes and distances of objects far away by taking a geometrical approach to analysing observed facts. The reader is asked to prepare unifilar drawings in respect to the following considerations and to retrace Aristarchus's conclusions.

a) If exactly half the moon appears to be illuminated seen from Earth ("half-moon"), moon, sun and Earth (more precisely: the observer's exact location on Earth) form a right-angled triangle at the moon, whose shape or side ratios are fully determined by the visual angle, which is to be measured, between moon and sun. The practical difficulties are due to the following:
   1) It is not easy to determine the exact time of half-moon.
   2) It is difficult to recognise the moon, if the sun shines simultaneously.
   3) The visual angle is close to 90° and, consequently, small errors in measurement can greatly impact the side ratios.

   Aristarchus found the following for the distance ratio: Earth-sun: Earth-moon has the lower bound 18 : 1 and the upper bound 20 : 1, thus, a relative value much too small for the distance of the sun, although he tried to correct this estimate by means of a mathematical trick from Euclid's 'Optics'.
b) Given the assumption to be observed that the moon almost completely covers the sun during a total solar eclipse, we also obtain a value of approx. 19 : 1 for a ratio of the diameter of both celestial bodies.
c) We observe that the moon takes approx. the same time to enter the umbra of Earth as it takes to reappear immediately afterwards on the other side of the shadow. Accordingly, the umbra of Earth is approx. double as wide as the diameter of the moon, where it is crossed by the moon. (Account for this and compare it to the phenomenon of a train, which takes exactly the same time to completely disappear in a tunnel as to reappear on the other side!) What can we conclude here regarding the diameter of Earth and moon under consideration of the results of a) and b)?

d) Since the common value of the visual angle under which the sun and moon appear is approx. 0.5 degrees (Aristarchus used a greater estimate), the circumference of both circuits corresponds to approx. 720 diameters of each respective celestial body. If we know (according to Erastosthenes) an approximate value for the diameter of Earth, we also have an absolute approximate value for the diameters and the distances of sun and moon. Although Aristarchus obtained a value far too small for the diameter and distance of the sun based on the angle measured in a), he arrived at the qualitatively correct result that the sun is considerably larger than Earth. Aristarchus concludes from this that it is highly unlikely that the huge sun revolves around the small Earth and assumed the contrary. During the Antiquity, this view stood no chance against the highly superior Aristotelian physics. However, Copernicus knew of his predecessor and mentioned him explicitly.

**Problem 2.4.2:** The Archimedean spiral

a) State the subset of those points on the Archimedean spiral that can be constructed with compass and straightedge.
b) To what extent can the Archimedean spiral be used as an aid to rectification (extension) of the circumference and, hereby, indirectly to construct $\pi$?
c) The trisection of any angle can also be accomplished, if we take an Archimedean spiral as given. How do we proceed?
d) Confirm Archimedes' result for the subtangent by means of differential calculus. As a result, how can we obtain a rule of construction for the tangent at the Archimedean spiral?
e) If we draw the normal $n$ in a curve point $P$ and bring it to the intersection (intersection shall be $N$) with the subtangent extended beyond $O$, then line segment $ON = s_n$ is the polar subnormal. Prove that $s_n = a$. (Archimedes does not state this reference!) Which tangent construction do we obtain?

**Problem 2.4.3:** Apollonian problem

a) In the special case, regarding Apollonius's problem, at which the three given circles have degenerated to points, there is exactly one circle touching the three given points. How do we locate its centre?
b) Determine the touching circles for the special case, at which two points and a circle are given, the centre of which has the same distance from the given points.
c) How do we have to proceed if three congruent circles are given that do not intersect each other, the centre of which is formed by the corners of an isosceles triangle? How many touching circles are there in this case?

d) How do we have to proceed in the degenerated case that three straight lines are given?

A complete elementary solution for all cases can be found in H. Begander's article *Das apollonische Berührungsproblem* (The Apollonian problem of touching); Mathematical Student Journal *alpha* 1980, Issue 5, published by Volk und Wissen.

**Problem 2.5.1:** Hero's Formula

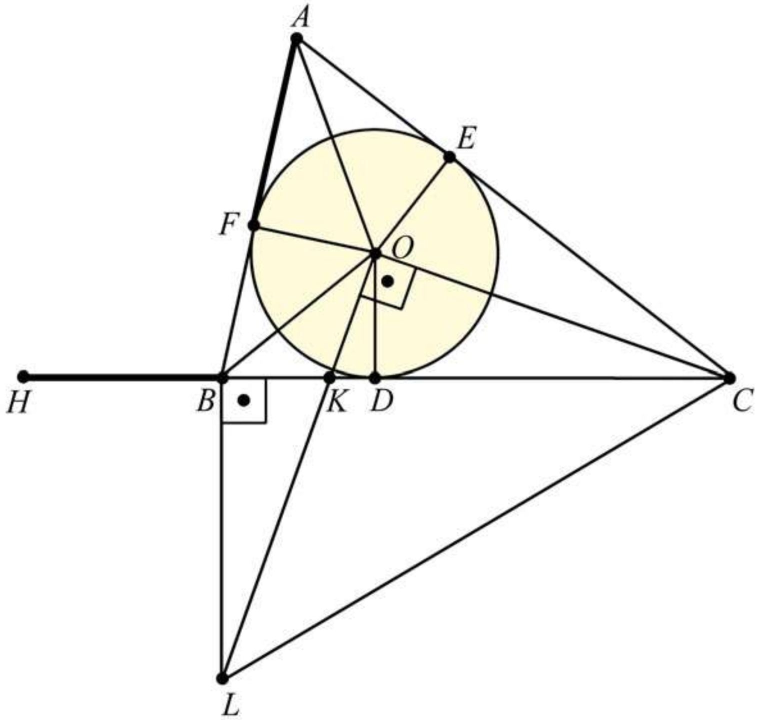
Prove Hero's formula for area $\Phi$ of any triangle
$\Phi = \sqrt{s(s-a)(s-b)(s-c)}$ according to Hero's model (see Illus. 2.6.6).
Draw the inscribed circle of triangle $ABC$ (in order to maintain the agreement of the figures in the manuscript, name the corners with $A, B, C$ starting at the top anticlockwise). Then connect its centre $O$ with the three corners $A, B, C$ and drop the perpendicular from $O$ to the three sides $a, b, c$, the feet of which shall be $D, E, F$. As a result, the perimeter of the triangle $2s = a+b+c$ is divided into 6 pairwise even parts, the lengths of which are presented by line segments $s-a, s-b, s-c$, those quantities that occur next to $s$ in the formula that is to be proven. The following method of proof is determined by the fact that the four quantities are represented geometrically and expressed appropriately.

Extend the triangle side $CDB$ beyond $B$ by line segment $AF$ until $H$. As a result (why?): $CH \cdot OD = \Phi$ and, thus, $\Phi^2 = CH^2 \cdot OD^2$. Now construct the perpendicular $OL$ on $OC$ (which intersects $BC$ in $K$), the perpendicular $BL$ on $BC$ and finally link $C$ with $L$. Why do points $C, O, B, L$ lie on one circle now and, consequently, $\measuredangle COB + \measuredangle CLB = 2R$? Since $\measuredangle COB + \measuredangle AOF = 2R$ (why?), it follows that $\measuredangle AOF = \measuredangle CLB$. Hence, $\triangle AOF \sim \triangle CLB$. This leads to $CH : HB = BD : DK$. Now look at $CH^2 : (CH \cdot HB) = (BD \cdot DC) : OD^2$ (Why does that apply?). Now insert this in the relation $\Phi^2 = CH^2 \cdot OD^2$ derived above:

$$\Rightarrow CH^2 \cdot OD^2 = CH \cdot HB \cdot BD \cdot DC = s(s-a)(s-b)(s-c).$$

**Problem 2.5.2:** Ptolemy's chord theorem

a) Prove Ptolemy's chord theorem! In a closed inscribed quadrilateral of a circle with sides $a, b, c, d$ and diagonals $e$ and $f$, we have $ef = ac + bd$. (Draw the respective quadrilateral $ABCD$ with the diagonals $e = AC$ and $f = BD$. Why are the angles $\measuredangle BDC$ and $\measuredangle BAC$ equal to each other? Link $B$ with a point $E$ that is located on $AC$, which is chosen in a manner that leads to $\triangle ABE \sim \triangle DBC$. However, $\triangle ABD \sim \triangle EBC$. Confirm that $AB \cdot CD = AE \cdot BD$ or respectively $AD \cdot BC = CE \cdot BD$, from which we can conclude the chord theorem by means of addition.)




**Illus. 2.6.6** Figure to problem 2.5.1

b) Consider the special case in which both diagonals are simultaneously (different) diameter of the circle. Which proposition follows?
c) Side $d$ shall be the diameter of a circle. Refer to the arc above $f$ as $\varphi$ and to the one above $a$ as $\psi$. If we introduce abbreviation $s$ for the chord for the sake of shortening it, so that the respective chords become $s(\varphi)$ and $s(\psi)$, and if we, according to Ptolemy, set the diameter at 120p, then $s(180° - \psi) \cdot s(\varphi) = s(\psi) \cdot s(180° - \varphi) + s(\varphi - \psi)120^p$. Having introduced the further abbreviation $s(180° - \varphi) = c(\varphi)$, how does the result differ from the difference formula of the sine function?

**Problem 2.5.3:** Constructing a regular pentagon according to Ptolemy

Construct the side of a regular pentagon according to Ptolemy using the unit circle and confirm algebraically that the length is the quantity

$$s_5 = \tfrac{1}{2}\sqrt{10 - 2\sqrt{5}}.$$

Draw a circle with centre $O$ and a (horizontal) diameter $AOB$, construct the perpendicular upwards from $O$, which intersects the circle in $C$. Then halve radius $OB$ in $D$. Position the compass at $D$ and transfer line segment $DC$

in line segment $DE$ on diameter $AB$. As a result, $E$ is positioned between $O$ and $A$. Hence, $EB$ is divided by $O$ in the ratio of golden section (Proof?). Claimed is $s_5 = CE$. Consequently, we just need to transfer this line segment to the circle by means of a compass positioned in $C$. This way, we construct the first side of the pentagon in the desired position.

**Problem 2.5.4:** Hero's Formula for a segment of a circle

Contemplate how Hero (or whoever was the original inventor) may have found the formula (2.5.5) for the area of the circle segment!

**Problem 2.5.5:** Derivation of the ellipse formula from the string construction

Derive the canonical equation for the ellipse based on the underlying relation of the string construction stating that the sum of the connecting line segments of a curve point and two fixed points (foci) is constant. To do so, select the two foci $F_1$ and $F_2$ on the $x$-axis with the coordinates $-e, +e$. Refer to the constant sum of the distances of the curve points $P = (x, y)$ from them as $2a$. Also consider that the relation $e^2 + b^2 = a^2$ applies to the ellipse, whereby $a$ is the great semi-axis, $b$ the smaller one and $e$ refers to linear eccentricity.

# 3 Oriental and old American geometry




C.J. Scriba, P. Schreiber, *5000 Years of Geometry*,
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## 3.0 Introduction

Before we deal with the European Middle Ages in the following chapter, we will present an overview in this chapter of the development of geometry in the oriental countries up to the $15^{th}$ century, in Japan until the end of its seclusion in 1868, and in the old American cultures. We will show that, despite all the outstanding problems with research, it is possible to define certain characteristics for mathematics in China, Japan, India and the Islamic countries. Nonetheless, due to the specific focus of this book – geometry – we can only sketch general statements rather than document those characteristics by means of extensive examples from all branches of the mathematical sciences. Whereas Chinese and Indian mathematics developed in a long-lasting process of domestic maturation and Japanese Wasan-mathematics experienced a short-lived, artificial golden age almost as if under a bell jar, mathematics in the Islamic countries was not built upon a unified, independent development from ancient times onwards. It depended on the adaptation and further development of calculating and measuring methods from neighbouring cultures from the $8^{th}$ century onwards when it began growing. They may have been particularly open to studying Greek theoretical mathematics. As a result of diverse influences, 'Arabic mathematics' emerged. For instance, they adopted the deductive method for proofs, which is characteristic and distinguishes them from the remaining oriental mathematical cultures.

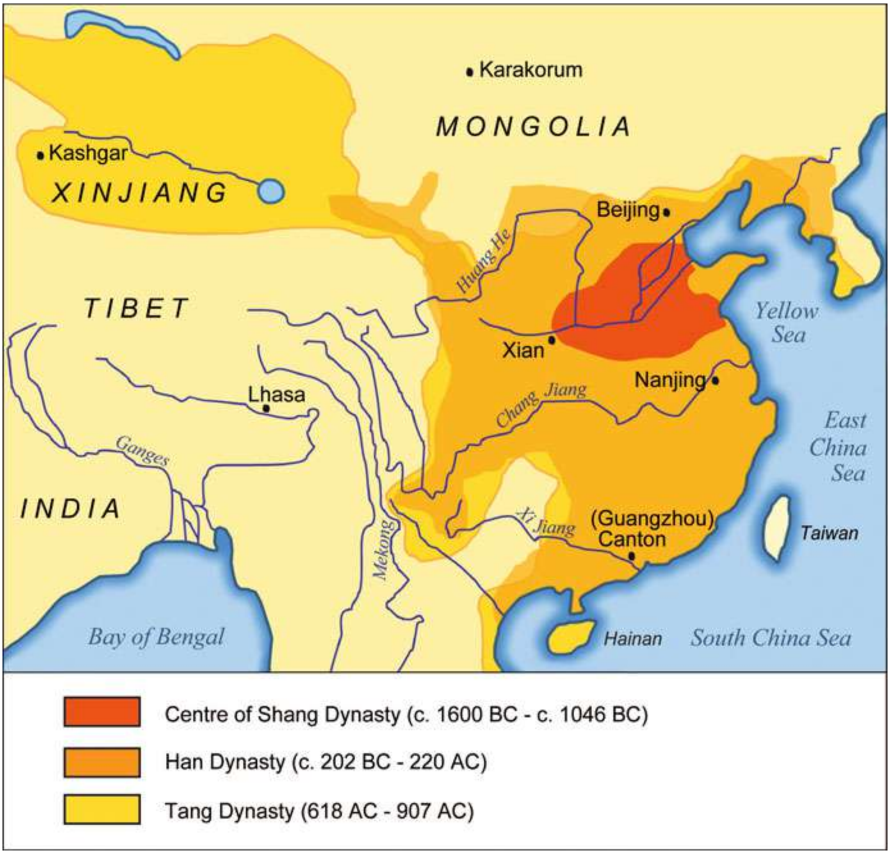
The name 'Arabic mathematics' is just as unsuitable as the customary term 'Muslim mathematics', since not all mathematicians working under Arabic sovereignty were necessarily Arabs or Muslims. Even if we use the adjective 'Arabic' to refer only to the language, we must not forget that scientific texts were also written in the Persian language from the $10^{th}$ century onwards.

The names and book titles of oriental scholars and their works are represented differently in the literature. Hence, this chapter occasionally contains spellings that differ from each other or details from different versions.

Due to the fact that the written documents of the old American cultures are only sparsely preserved, we will look at the development of geometrical elements and constructions in pre-Columbian America by examining and illustrating remarkable examples of the cultures of the Inuit and Ojibwa hunting civilisations in North America and the advanced Aztec and Mayan cultures in Middle America, as well as the Incas in South America.

## 3.1 China

| | | |
|---|---|---|
| 3000–1500 | | Early river valley civilisations at Huang He and Yangtze River |
| Approx. 2000–1500 | Legendary Xia Dynasty founded by mythical Emperor Yu | Bronze engravings in pre-classical language |
| 1500–1030 | Shang Dynasty | First songs, sacral bronze vessels |
| 1030–481 | Zhou Dynasty | Confucius (551–479) proclaims his Ethics, Laozi (around 300) founds Taoism |
| 221–207 | Qin Shi Huang unites empires and becomes first emperor | Great Wall of China (2540 km) is built |
| 206 BC – 220 AD | Han Dynasties | Golden age of mathematics, astronomy, philosophy and rhyming prose |
| 221–280 | Three Kingdoms period | The Sea Island Mathematical Manual |
| 280–618 | Further decline, different dynasties | Advances of Buddhism, temples and convents with Buddha figures are built |
| 618–906 | Tang Dynasty, Climax of Chinese power | Golden age of economy and culture (painting, verses by Li Bai, Du Fu and Bai Juyi) |
| 907–960 | Five dynasties period | |
| 960–1278 | Song Dynasties | Paintings of animals and plants |
| 1278–1368 | Yuan Dynasty (Mongolian Rule) | $2^{nd}$ golden age of mathematics; blue and white procelain, lacquer works, paintings; Marco Polo's travels (1271–1291) |
| 1368–1644 | Ming Dynasty | Monumental buildings; Ming Dynasty Tombs |
| 1644–1911 | Qing Dynasty | French and British influences and conflicts |
| 1911 | Proclamation of the Republic | |
| 1949 | Mao Zedong proclaims People's Republic of China | |
| 1965–1969 | Culture Revolution | |
| 1976 | Death of Mao Zedong | |
| 1977 | Deng Xiaoping establishes China's advance to economic power | |
| 1997 | British Crown colony Hongkong falls back to China | |
| 1999 | Portugese colony Macáo falls back to China | |
| 2011 | China rises to worldwide second economic power | |




**Illus. 3.1.1** China in Antiquity and Middle Ages
[Map: H. Wesemüller-Kock]

### 3.1.0 Historical introduction

Within the scope of a brief presentation on the contributions of Chinese mathematics to the development of geometry, it is neither possible nor wise to look in great detail at the history (according to tradition, 4000 years long) of the huge area China covers. It is customary to arrange it according to dynasties (cf. table above). The early ages (from approx. 2000 BC) are rather legendary. It is said that from -600 until -300, early mathematical-astronomical texts were composed in China, none of which, however, have been passed on. Those old Chinese scripts, which are still known nowadays, were written within a period of approx. 500 years, the beginning of our Christian calculation of time in Europe falls rather accurately in its middle. Hence, their age

**Illus. 3.1.2** The Chinese Wall
[Photo: H.-W. Alten]

corresponds roughly to the age of the Alexandrian Hellenistic texts from the European cultural area. Chinese mathematics experienced a second golden era during the $13^{th}$ century, even though late works were still created in the $16^{th}$ century. With the arrival of the Jesuit astronomer and mathematician Matteo Ricci in Peking in 1601, European science started slowly to penetrate China. As a result, the era of uninfluenced development of Chinese mathematics ended at that time. Nonetheless, there was much resistance to this during the $17^{th}$ century. Only with the beginning of the second half of the $19^{th}$ century, when Chinese scholars again came into intensive contact with western science, did they begin to be open-minded toward the often strange ways of thinking.

There is a special feature that is characteristic for both the ancient and the medieval era of mathematics in China, namely that a certain number of mathematical works was ennobled to canons to which later mathematicians were required to adhere. This was possible because mathematics was mainly taught within the scope of educating officials – China kept a strictly organised system for officials in force. Next to the officials' customary administrative duties, which would be considered standard and required elementary mathematical skills, there were also more mathematically-challenging tasks that were addressed by trained specialists, e.g., celestial observations and calendar calculations by astronomers, land surveys by geodesists. As a result, in 656, officials finally compiled a collection of works of varying levels of difficulty under the title 'The Ten Computational Canons' (or handbooks, Suanjing Shi Shu). From then onwards, this compilation determined education and, hence, also mathematical accomplishments in China. 'The Ten Computational Canons' (or parts thereof) were commented on again and again and, thereby, supplemented and extended. Simultaneously, people were scared to leave tradition behind with these completely newly designed books. Nevertheless, there was a wealth of Chinese mathematical literature. A catalogue of such works created in 1936 in Peking's libraries listed more than one thousand titles. And we know only the names of many further works from the $13^{th}$ century. In the $15^{th}$ century, the ten classics became part of an encyclopaedia. However, after that it became more and more difficult to get hold of a complete text. 'The Ten Computational Canons' were reprinted at the end of the $18^{th}$ century (the first known reprint circulated in 1084!), after the original texts of the ten books had been successfully rediscovered.

We will present here the development of Chinese geometry within the scope of its mathematical and cultural surroundings in chronological order:

a) From the beginning until the Sanguozhi era (San Kuo; the division of China into three kingdoms between 221 and 280)
b) From the division until the beginning of the Song Dynasty (960)
c) In the midst of the Song Dynasty (960–1278), Yuan Dynasty (Mongolian rule, 1278–1368) and Ming Dynasty (1368–1644)

### 3.1.1 From the beginning until the division of China into three kingdoms between 220 and 280

Archaeological excavation showed objects of everyday use from the $13^{th}$ and $12^{th}$ centuries BC, such as ornaments decorated with penta-, hepta-, octa- or nonagons. However, the regularity of these shapes does not seem to have inspired Chinese mathematicians to further research, in contrast with Greek and Arabic mathematicians. The case of the relations between the sides of a right-angled triangle was different, as they apparently sparked the interest of Chinese mathematicians just a few centuries later. However, very little is known of these early times since the tradition of old Chinese science was brutally interrupted by an order of the despotic 'Yellow Emperor' Huangdi in 213 BC. He ordered that all books were to be burned and had many scholars executed. Following generations struggled greatly to re-obtain some of the old texts.

**Zhou Bi Suan Jing (Chou Pei Suan Ching)**

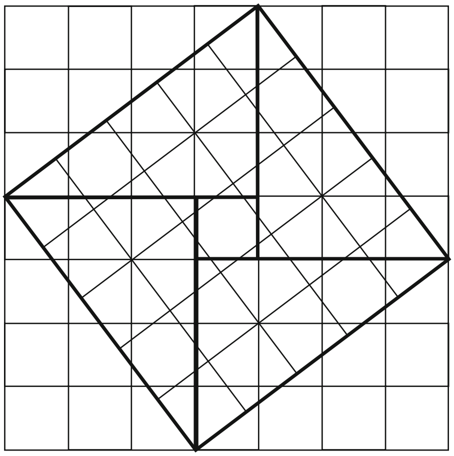
We intend to introduce here three especially important works of this era. The oldest one is called *Zhou Bi Suan Jing* (Arithmetical classic of the Chou Gnomon – Needham [Needham 1959] literally 'The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven'). In the past, this work has been back dated until approx. -1100. However, today we assume that it was composed no earlier than the $4^{th}$ century BC and may contain older parts from the late $6^{th}$ and $5^{th}$ centuries BC [Needham 1959, p. 19, 210, 257], [Martzloff 1997, p. 124]. Essentially, it is a book on astronomy and calendar calculation, and the featured mathematics is only taught as a means of assisting these endeavours. It deals with the dialogue form properties of the right-angled triangle, Pythagoras's theorem, square and circle, measuring heights and distances, and applying the gnomon. The book presumes mathematical skills in fractions and extracting square roots (which are supposed to be acquired before engaging with geometry). (Initially, the Chinese used counting rods when counting and calculating. These rods represented either ones, tens, or hundreds in the decimal system. Around the middle of the $14^{th}$ century, the abacus made its entrance – a ball calculator, called, in Chinese, 'suanpan' – and spread quickly across the whole country and beyond to Japan and Korea. Due to its similarity to the Roman abacus, there are speculations that the suanpan could have come to China from Europe; how is unclear.)

Deductive proofs in the Euclidean sense are missing from the 'Arithmetical classic of the Chou Gnomon'. Pythagoras's theorem is demonstrated by a method referred to as 'the putting together of rectangles' by using a concrete example as means of illustration. However, this is meant to apply generally. Summarised, the description says: Draw the right-angled triangle with the sides 3, 4, 5, the square above the hypotenuse and add congruent triangles (3, 4, 5) on top of the three sides, so that the hypotenuse square is fitted at a right angle in a square of side length 7. In the enclosed figure (Illus. 3.1.3), both squares are divided into $5 \cdot 5$ respectively $7 \cdot 7$ unit squares, by two nets crossing each other. The inner square equals 25 unit squares; the outer one is split into four rectangles of $3 \cdot 4$ unit squares and the central unit square (see Illus. 3.1.3). Generally expressed this delivers the formula

$$c^2 = 4 \cdot \frac{ab}{2} + (a-b)^2 = a^2 + b^2. \tag{3.1.1}$$

Thus, this oldest Chinese "proof" combines a visual construction and considerations that carry out implicit algebraic transformations. This also pinpointed the direction that geometrical development took in China: geared to applications in practice, geometry continued to develop within an algebraic scope. Attempts were made to express geometrical questions as equations in order to apply known methods of equation solving to these problems [Needham 1959, p. 22-23], [Libbrecht 1973, p. 96ff.].

The astronomical parts, which, of course, form a crucial component of this work and are indispensible for an overall evaluation, will not be discussed here. Nonetheless, we must not conclude from this that the ancient authors had already split the two disciplines of geometry and astronomy in such a manner.



**Illus. 3.1.3** Chinese figure for Pythagoras's theorem

However, it is striking that the Zhou Bi Suan Jing contains nothing on astrology and fortune-telling, which was blossoming in China at that time. Celestial and earthly occurrences were addressed highly factually without the slightest reference to superstition (cf. quote in appendix A. 4, p. 568).

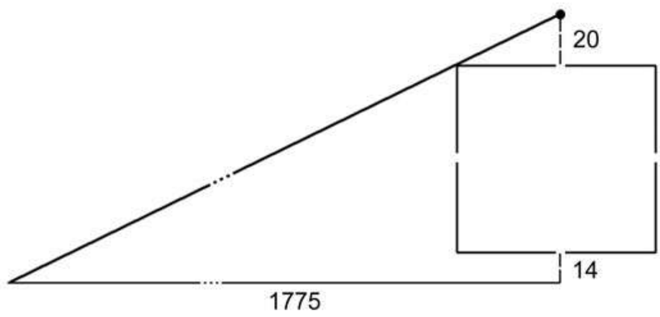
**Jiu Zhang Suanshu (Chiu Chang Suan Shu)**

The second oldest work that we will address here is part of the most well-known Chinese mathematical books of all times: the *Jiu Zhang Suanshu* (Chiu Chang Suan Shu, The Nine Chapters on the Mathematical Art) [Vogel K. 1968]. The origins of this work have been verified as having been found in the early Han era (-202 until +9) and older drafts were used to compose this work. In contrast to the book mentioned before, this is a completely mathematical text: a collection of 240 problems with solution instructions. Since it was declared an official textbook in 656, meaning it had to be studied by all civil servants and engineers, it became the most influential of all Chinese mathematical books. Due to its laconic shortness, which was thought of as difficult, it represented one of the advanced works among 'The Ten Computational Canons', of which it became a part at the same time. Furthermore, some parts went far beyond the needs of lower officials and merchants. More recent investigations provided evidence that 'The Nine Chapters on the Mathematical Art' had been influenced by Babylonia. This is not surprising, since the Chinese were already entertaining a legation in Babylon around 200 BC.

The first, fourth, fifth and ninth chapters especially address geometrical problems. The other chapters contain problems from the areas of applied calculation and diverse applications, which are important for officials (grain mixtures, allocation problems, taxes and charges, and also questions about fractions and radicals, motion problems, Euclidean algorithm, linear equation systems, and indeterminate equations). As expected, the first chapter offers calculations of simple plane figures: rectangle, triangle, trapezoid, as well as circle and circle segment. The circular area is calculated according to the Babylonian rule (cf. p. 21 (1.2.9)) or by means of $\frac{3d^2}{4}$. Concerning the circle segment, we find a formula, which has been discussed by Hero, but rejected afterwards due to being too imprecise ($A$ = area, $c$ = chord, $s$ = sagitta; cf. Illus. 1.2.10):

$$A = \frac{(c+s)s}{2} \tag{3.1.2}$$

Chapter 4, titled 'The lesser breadth', deals with area transformations, whereby the breadth of a given area is meant to be decreased and the length to be increased whilst maintaining the area. Chapter 5, titled 'Consultations of works', contains problems on calculations of volumes of walls, dams, channels, etc.; in other words, stereometrical problems in wrapped form. In regards to solids, the chapter addresses prisms, pyramids, tetrahedra, wedges, cylinders, circular cones, and conic frusta. Given that the problems of right-angled triangles are located in Chapter 9, this late classification suggests that the individual parts are of different origin. Concerning this passage, we can infer a close connection to the book *Zhou Bi Suan Jing* described beforehand. Some of the problems, e.g., the bent bamboo pole, may have reached India later perhaps communicated by Buddhist monks before finally arriving in Western Europe. Let us look at some examples of this last book [Vogel K. 1968, p. 90ff.]. Problem 4 requires us to saw a rectangular bar of given strength out of a circular tree trunk. The solution implicitly presumes that the angle at circumference above the diameter is a right one. The following problem requires to calculate the overall length of a helix, which is introduced to us as a twining plant revolving around a tree. The rule for solving this problem




**Illus. 3.1.4** Length of the side of a square town wall in 'The Nine Chapters on the Mathematical Art'

is based on the unwinding in a plane. The problem of a bar leaning obliquely against a wall, the foot of which is moved away by a certain line segment, whereby the peak sinks by a certain amount, can also be found in other cultures, as well as the problem of the bent bamboo pole.

Problem 20 addresses a town with a square top view that needs a gate on each side in the middle of the wall (cf. Illus. 3.1.4). There is a tree at a distance of 20 *bu* (Chinese measure of length) to the north from the northern gate. If we walk 14 *bu* from the southern gate to the South and then 1775 *bu* to the West, the tree becomes just visible behind the North-West corner of the town wall. We have to calculate the length of the side of the town wall (the side of the square). We obtain the square equation $x^2 + px = q, p, q > 0$. A derivation and solution approach is not given, only a rule, which utilizes the algorithm to extract a square root onto this case (cf. Problem 3.1.1).

The chapter contains several versions of this town problem, which were picked up again by later authors (e.g. Qin Jiushao in the $13^{th}$ century; [Juschkewitsch 1964, p. 50]).

**Haidao Suanjing (Hai Tao Suan Ching)**

This last chapter is followed by a small, but important book, *Haidao Suanjing* by Liu Hui [Swets 1992]. It was published in 263. Its title means 'The Sea Island Mathematical Manual'. Supplementing his commentary on the 'Nine Chapters', Liu Hui described measuring heights and distances. To accomplish this, he used a level pole, which, if necessary, featured a perpendicular cross-bar at the top end. These methods are most often based on the observation of similar right-angled triangles. (Thus, it is not justified to speak of applied trigonometry, as Mikami did in his representation of Chinese and Japanese mathematics from 1913 [Mikami 1913], since neither angle properties nor angle functions are featured. Moreover, there are no algebraic generalisations, just concretely stated problems.)

The last chapter of *Jin Zhang Suanshu* (nine chapters) concludes with some surveying problems after the described town problem. Liu Hui believed that the method described there was insufficient in the case of inaccessible objects. As a result, he also explained the 'Chong Cha method' of doubled measuring, which was already widely spread in his time (Illus. 3.1.5).

The woodcut clearly illustrates the situation of the island that gave his work not just its name, but also the addressed set of problems. A towering, inaccessible mountain peak on an island is measured by two observation points via the end of a pole. We are asked to calculate the height of the mountain and its distance to the front pole. We know the distance between the two measuring poles. Thus, this method basically amounts to constructing a triangle given a line segment and both adjacent angles and to calculate its height afterwards. The approach chosen by Liu Hui is explained in Problem 3.1.2.

Since Chinese astronomers had early on been interested in the distance of the sun from Earth, this approach could have been developed within this scope and then been transferred to earthly circumstances. This method marks the climax of early Chinese surveying theory; it was explained in many later works and often illustrated through Liu Hui's phrasing of the problem (cf. [Swets 1992]).

### Volume calculations

Within the scope of his commentary on the 'Nine Chapters', Liu Hui's calculation of the constant $\pi$ deserves a special mention. The astronomer and philosopher Chang Héng had claimed approx. 150 years before him that the ratio of the square of the circumference [i.e., $(2\pi r)^2$] to the square of the circumference of the circumscribed quadrangle of the circle [i.e., $(8r)^2$] is $5 : 8$. As a result, $\pi \approx \sqrt{10} = 3.162...$ . (This approximation was also known by Brahmagupta in the $7^{th}$ century and al-Khwārizmī in the $9^{th}$ century.)

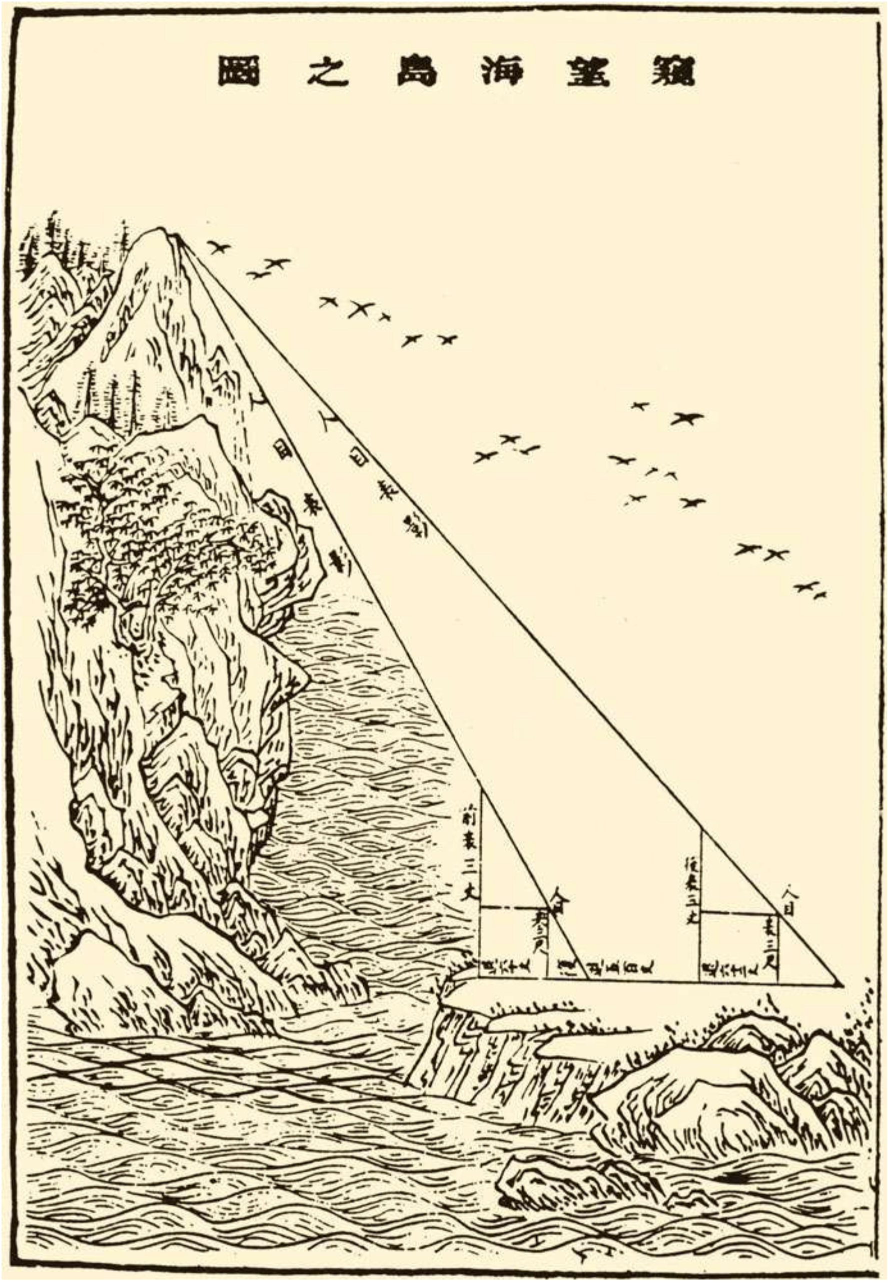
In order to calculate the perimeter, Liu Hui started off with an inscribed regular hexagon $f_6$, the side number of which is increased to 192 by repeated doubling, whereby he determined the approximation $\pi \approx \frac{157}{50}$. He derived better approximations by means of exhaustion of the circular area (cf. Illus. 3.1.6) by approximating its quantity by means of

$$f_{n+1} < f < f_n + 2 \cdot (f_{n+1} - f_n). \tag{3.1.3}$$

He obtained a fraction from the $6 \cdot 2^9 - -$ polygon, which corresponds to $\pi \approx 3.14159$ and, therefore, exceeds Ptolemy's accuracy of 3.14166 from around 150 AD (see Problem 3.1.3). Two centuries later, Zu Chongzhi (Tsu Ch'ung-Chih) even obtained the approximate fraction $\frac{355}{113}$ and the bounds $3.1415926 < \pi < 3.1415927$.

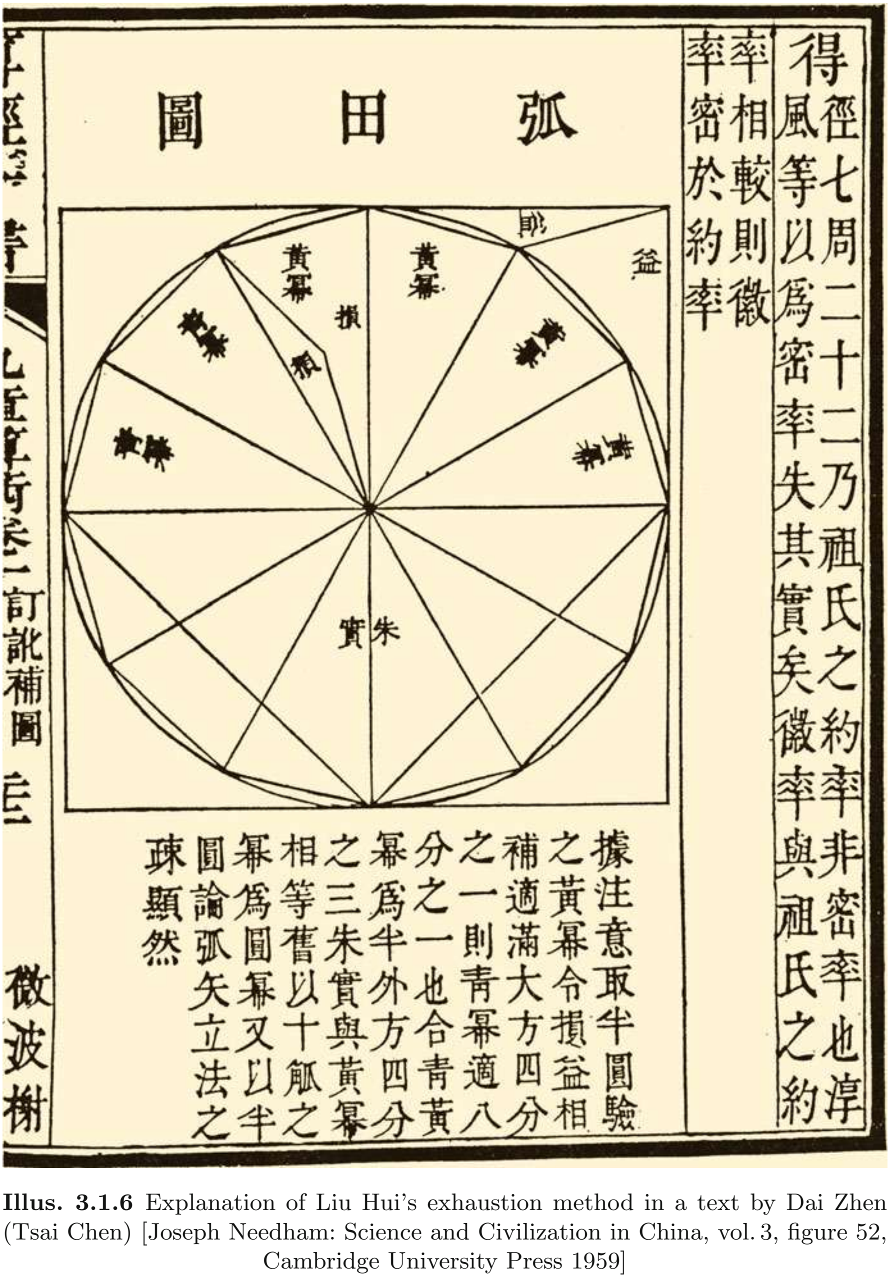
As part of the 'Nine Chapters', the $5^{th}$ book addresses calculations of cuboids and straight prisms, but also several complicated solids, which are confined by plane areas, e.g., rising dams with inclined sides and solids, similar to pyramid frusta. Some remind us of corresponding rules found in Babylonian mathematics; deconstructing such solids into solid parts and, if necessary, also using calculation of averages. In the difficult case of the pyramid, which cannot be dealt with by elementary means, Liu Hui did not succeed by means of deconstruction alone: he also had to deconstruct rest pyramids again and to apply this method repeatedly, and so basically carry out a limit process (cf. [Wagner 1979]).

Most interesting are the solids with curved surfaces: cylinder, cone and conic frustum. Whereas we are used to calculating the volume of a cylinder by means of base and height, the Chinese rule dictates that we square the circumference of the base circle, then take a twelfth thereof and multiply the result by the height of the cylinder. Thus, it corresponds entirely to the Babylonian method for calculating the circular area ($A = \frac{C^2}{12}$ by means of value 3 for $\pi$; cf. 1.2.9). Accordingly, the volume of the cone is determined (multiplication by $\frac{h}{3}$) and of the conic frustum is determined to be




**Illus. 3.1.5** Illustration of the method of double measurements (Woodblock printing from the encyclopaedia Gu jin tu shu ji cheng (1726))

[Frank G. Swets: The Sea Island Mathematical Manual: Surveyings and Mathematics in ancient China, p. 10, Fig. 3, 1992, University Park, PA: The Pennsylvania State University Press]




**Illus. 3.1.6** Explanation of Liu Hui's exhaustion method in a text by Dai Zhen (Tsai Chen) [Joseph Needham: Science and Civilization in China, vol. 3, figure 52, Cambridge University Press 1959]

$$V = \frac{C_1C_2 + {C_1}^2 + {C_2}^2}{12} \cdot \frac{h}{3}. \tag{3.1.4}$$

($C_1, C_2$ circumference of lower rsp. upper area of the conic frustrum)
No rule was given for calculating the volume of a sphere (the reasons are discussed in [Fu 1991]). Nevertheless, the book states how to extract radicals of the sphere diameter $d$ by means of sphere volume $V$ according to $d = \sqrt[3]{\frac{16}{9}V}$, which is equivalent to $V = \frac{9}{16}d^3$. Thus, we obtain $\pi \approx \frac{27}{8}$. As always, no reason is given. However, we could imagine that we have estimated the sphere volume to three quarters of the volume of the circumscribed cylinder. Hereby, Liu Hui's commentary states the estimate $\frac{8}{16}d^3 < \pi < \frac{9}{16}d^3$, which corresponds to $3 < \pi < 3\frac{1}{8}$. Of course, it is left unanswered as to whether there was a mathematician back then who recognised the relation between the constant $\pi$ and the constant featured in this sphere calculation.

### 3.1.2 From the division in three Kingdoms to the beginning of the Song Dynasty (960)

Development progressed relatively slowly in the following centuries. Presumably the *Sun zi suan jing* (The Mathematical Classic of Master Sun) was composed in the $4^{th}$ or $5^{th}$ century. This manual has been significant due to its description of calculation with bamboo digits. Apart from explaining calculation (including a demonstration of how to extract cube roots), it also contains a compilation of customary measures and weights. (An English translation can be found in [Lam/Ang 1992, p. 149-182].)
The *Zhui shu* (Method of interpolation) by Zu Chongzhi from the $5^{th}$ century has been accepted as the most important book of this era. Unfortunately, however, it has been lost. It was said to be one of the era's most difficult books – probably because $\pi$ was calculated to seven decimals and the difference method (interpolation) was described for astronomical calculations.

#### Contacts with India

By the end of the $5^{th}$ century, first contact had been established with Indian scholars. This is indicated by a series of titles of lost works, which feature the addition 'Brahmin'. Indian Buddhists also influenced Chinese thinking within mathematics, for example, regarding the reproduction of large numbers. During the $7^{th}$ and $8^{th}$ centuries, Indian scholars worked at the astronomical office of the Chinese capital and improved the calendar. One of them, known for a significant astronomical work, was even promoted to director. Those Indian scholars probably even brought an early form of trigonometry with them from their home country. However, it would be inaccurate to conclude from this that there was a continuous, intensive scientific communication between both countries. Rather, new developments were limited to the discoverer and his circle of students. A report from 855 describes the efforts of the far-reaching travels to which talented young men had to subject themselves if they wanted to receive further training and education by one of the rare great masters [Needham 1959, p. 202-206]. This, as well as the inner turmoil during the $7^{th}$ and $8^{th}$ centuries that led ultimately to the division of the huge empire into five dynasties, may have been amongst the causes that resulted in the stagnation of the subsequent era, since the court was not just a political, but also a cultural and scientific centre of highest order.

### 3.1.3 The dynasties Song (960-1278), Yuan (Mongolian reign, 1278-1368) and Ming (until 1644)

It wasn't until the $13^{th}$ and $14^{th}$ centuries that Chinese mathematics experienced a strong boom again. The second half of the $13^{th}$ century is especially said to have been the actual golden era of Chinese mathematics, as several excellent mathematicians composed important works.

**Qin Jiushao (Ch'in Chiu-Shao)**

Due to the narrow scope of this book, we have selected just a few noteworthy mathematicians, of whom we will first mention Qin Jiushao. He completed his work *Shushu jiuzhang* (Shu-shu chiu-chang, Mathematical Treatise in Nine Sections) in 1247, ten years after the Mongols had invaded his home country. The turmoil had long prevented him from engaging with mathematical problems [Libbrecht 1973]. The often discussed 'Nine Sections' (Shushu jiuzhang) mentioned above was his main source. Qin Jiushao also chose a structure in nine sections. However, he structured the subject matter differently and often created difficult problems as a model – nine in each section (= chapter) for a total of eighty-one. The questions, which are often asked in an applied manner, are followed by answers, then general explanation, and finally numeric solutions.

One stands out amongst the rules for calculating elementary plane figures: a calculation that corresponds to Hero's formula for calculating the area of a triangle by means of its three sides (see Problem 3.1.4). This formula also occurs in Brahmagupta's work (around 625) and al-Karajī's (around 1015). Qin Jiushao used the formula in the following form:

$$A = \sqrt{\frac{1}{4}[c^2a^2 - (\frac{c^2+a^2+b^2}{2})^2]} \tag{3.1.5}$$

The stated approaches to the solution are not always the easiest ones, a situation that reoccurs. We cannot exclude the possibility that tradition was subject to some misunderstanding at times or that an author just took a problem from a different source, which he himself then did not interpret correctly. Below is an example of an 'applied' problem [Libbrecht 1973, p. 107]:

"Find the side of a square camp in which 99 companies are encamped. Each company has at its disposal a square area with a side of 90 feet; between any two companies there must be a distance equal to the side of the same square. We take the area for each company as being 4 times the occupied area; we put the rectangle $BGFE$ below the rectangle $DHJI$. The width of the rectangle $ABKE$' is $x$ and the length $x+2$, giving an area $x \cdot (x+2)$; the area is also equal to $4 \cdot 99 + 3$. This gives the equation $x(x+2) = 4 \cdot 99 + 3$ or $x^2 + 2x - 399 = 0$ with the positive solution $x = 19$."

Also see Problem 3.1.5!

The volumes of pyramid and cone frusta are calculated by means of the rule that we already know from the Moscow Mathematical Papyrus:

$$V = (a^2 + ab + b^2) \cdot \frac{h}{3} \tag{3.1.6}$$

($V$ = volume, $a$ = length of base edge, $b$ = length of upper edge, $h$ = height). We also find observations on compound figures.

The height of a distant mountain is measured by locating the peak from two points on a plane. In order to calculate this, we need the distance of both observation points, the gradient of both visual rays, and theorems on similar triangles. Evidence suggests that this known method was found in China first in 263 (see Illus. 3.1.4). Qin Jiushao applied this method differently depending on the nature of the applied problem. Here is a final text passage for such an 'applied' problem: 'Observing the Distance of the Enemy' [Libbrecht 1973, p. 147-149]:

"The enemy pitches a circular camp on a sandy plain north of a river. We do not know the number of men. Spies report that the space occupied by each soldier in this camp is a square with a side of 8 feet. Our army is south of the river at the foot of a hill. Below the hill we set up a gnomon 80 feet high so that its top is at the height of a plateau at the edge of the mountain. We stretch a cord from the top of the gnomon to this plateau. This cord horizontal to the observer's viewpoint shall be 30 paces long. At this point, the northern border of the camp appears to the observer to be in one line with the top of the gnomon. Afterwards, the observer focuses on the southern border of the camp in a view line, which touches on the gnomon eight foot underneath its top. The eye of the observer is 4.8 feet above the ground. We make use of the precise value of $\pi (= \frac{22}{7})$ and apply the ch'ung-sh'a method. Find the numerical strength of the enemy."

U. Libbrecht, who analysed the text in 1973, laconically remarked in regards to the solution approach: "This is entirely incorrect" [Libbrecht 1973, p. 147-149]. Perhaps the spies would have been better off counting the soldiers than measuring the space they occupied!

**Li Ye (Li Zhi)**

In 1248, Li Ye (Li Yeh, originally Li Zhi), who held highly rated administrative offices only temporarily, published his book *Ceyuan Heijing* (Tshe Yuan Hai Ching, Sea mirror of circle measurements). It does not address the calculation of $\pi$, but deals partially with properties of a circle, which is inscribed in a triangle. However, above all the author engaged with the solving of equations by means of algebra, whereby he marked negative coefficients with a crossbar on top. It seems that Li Ye did not know Qin Jiushao's work, which was finalised one year before, but printed much later. This is not surprising given that Li Ye lived in the North and Qin Jiushao in the South of the huge country of China.

**Yang Hui**

Thirteen years later, Yang Hui's commentary on 'The Nine Chapters of the Mathematical Art' was published. Yang Hui purposely avoided common fractions by expressing them in decimal fractions. This way, he could represent them on a calculating frame analogously to whole numbers and calculate with them just as well. (In Europe, calculating with decimal fractions was only introduced at the end of the $16^{th}$ century by Stevin.) He also found an approach that is similar to a theoretical, proving geometry. He used the figure on which Problem 3.1.1 is based to prove that both partial rhomboids, which are determined by the diagonal and located in a rhomboid, are of equal area.

Zhu Shijie (Chu Shih-Chieh) earned his living as a travelling scholar. He published an introduction to mathematical studies in 1299. Four years later, his famous work *Siyuan yujian* (Ssu Yuan Yü Chien, 'Jade Mirror of the Four Unknowns') followed.

**Guo Shojing (Kuo Shou-Shing)**

Under the Yuan Dynasty, the Mongolian sovereignty, Guo Shojing (Kuo Shou-Shing) emerged as a mathematician, astronomer and engineer. He developed a theory on spherical triangles. However, it is unsure if he used angle functions. Since all his texts have been lost, we only know of their content indirectly. Thus, it may not be justified to speak of 'trigonometry' here. It seems he was particularly interested in problems concerning the movements of celestial bodies. He thought of a method for determining the length of a circular arc, which amounts to a complicated square equation between diameter $d$, sagitta $s$ and arc $b$:

$$d^2 \cdot (\frac{b}{2})^2 - d^2 \cdot s - (d^2 - bd) \cdot s^2 + s^4 = 0. \tag{3.1.7}$$

His (lost) method was referred to as 'the study of right-angled triangles, chords and sagittae, squares and rectangles, which are all contained in circles, obliquely or perpendicularly'. He used differences of second order to provide exact descriptions of the changing angular velocity of the sun.


**Illus. 3.1.7** Guo Shojing (Kuo Shou-Shing) [Photo: Shizhao 2006], Stamp of Guo Shojing (China 1962)

Since Guo worked at the court of the Mongolian ruler Kublai Khan, who also employed Muslim specialists, this suggests that he could have come across their mathematical knowledge. Persians, Syrians and scholars from other countries also worked at the Mongolian court, which, as a result, offered plenty of opportunity for scientific exchange. In 1368, during the transition from the Yuan to the Ming Dynasty, an independent Muslim astronomy office was opened on top of the already existing domestic observatory. When the Jesuits came to Peking at the end of the $16^{th}$century, the followers of these 'Arabic' astronomers were still working at the observatory. Hence, we cannot exclude the possibility that Chinese astronomy and mathematics were influenced by the famous observatories in Samarkand and Maragha. Some even believe that Arabic translations of Euclid's 'Elements' and Ptolemy's 'Almagest' arrived in China earlier. However, historical research has still not found clear evidence of this.

Between 1400 and 1500, Chinese mathematics lost its high position, but recovered in the $16^{th}$ century. For instance, in 1552 Gu Yingxian (Ku Ying-Hsiang), governor of Yunnan, published a book, in which he systematically compiled the rules found until then to calculate circular arcs and segments ([Needham 1959, p. 51]). It contains, e. g., the instructions corresponding to the following formulae ($d$ = diameter, $c$ = chord, $s$ = sagitta, $b$ = arc, $A$ = area of a circle segment):

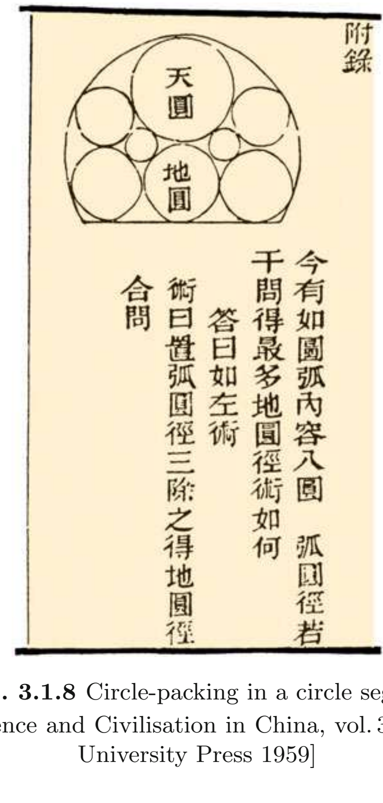
$$s = \frac{d}{2}\sqrt{(\frac{d}{2})^2 - (\frac{c}{2})^2}, \quad b = \frac{2s^2}{d} + c, \tag{3.1.8}$$

$$d = s + \frac{(\frac{c}{2})^2}{s}, \quad A = \frac{1}{2}(s + c)s. \tag{3.1.9}$$

Even though the Chinese had a passion for solving equations and equation systems (by means of Pascal's triangle and a method similar to the Horner scheme), some of their previously acquired knowledge in this area was forgotten. Only long after the Jesuits had introduced European algebra did medieval Chinese algebra spring back into mathematical minds and lead to the utilisation of former customary domestic methods in the $18^{th}$ century.

**Circle- and sphere-packings**

In contrast to the Greeks, Chinese mathematicians did not show any further interest in conic sections. To the contrary, they pursued a problem that had hardly been addressed in Europe: neatly fitting circles touching each other into given figures, such as crescents, ellipses and other shapes. This process is called circle-packing (Illus. 3.1.8). (This problem also interested Japanese mathematicians. Their perspective will be discussed in the section on Japanese mathematics.) Furthermore, the Chinese were neither interested in the Greeks pursuit of understanding regular and semi-regular solids, nor in the so-called classical problems. The only exception was squaring the circle.




**Illus. 3.1.8** Circle-packing in a circle segment
[Joseph Needham, Science and Civilisation in China, vol. 3, Figure 73, Cambridge University Press 1959]

In order to round out the picture of Chinese mathematics illustrated herein, we want to emphasise that the Chinese preference of calculation algorithms was an important feature of their mathematics. The individual steps were executed with bamboo sticks on a calculation frame. The algorithms were highly developed and suggest that negative numbers were already used very early, although they were not justified by theoretical considerations (in the texts), but introduced by means of examples.

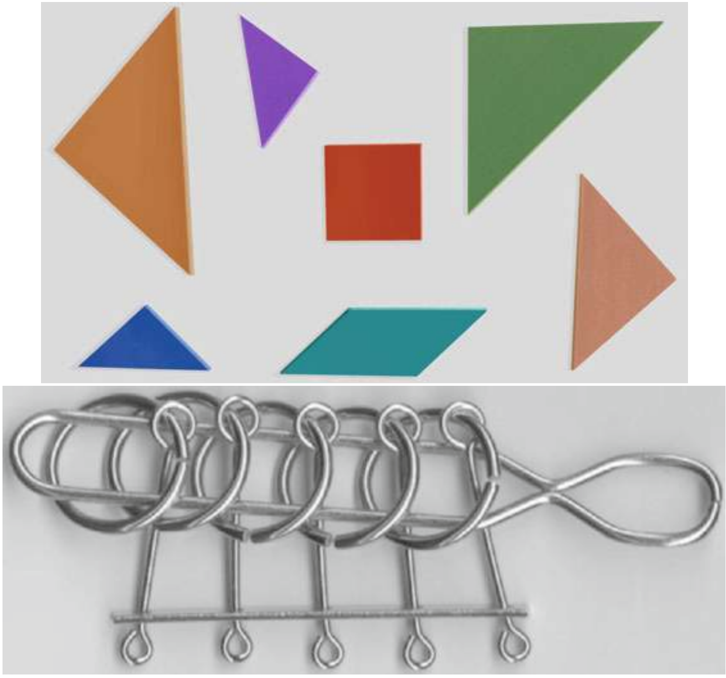
The same applies to the geometrical theorems mostly directed at applications. Heuristic methods were introduced by means of concrete examples. By applying them inductively, Chinese mathematicians obtained a wealth of further results. In contrast, the Greek approach of viewing geometry as a deductive system and configuring it accordingly only became known in China when the Jesuits arrived. The initial translation of the first six books of Euclid's 'Elements', initiated by Matteo Ricci and finalised in 1607 in collaboration with Xu Guangqi (Hsu Kuang-Chhi) (taken from the Latin version of Euclid (1574, among other dates) by the Jesuit Christoph Clavius [Knobloch 1990a]), had a highly missionary character due to its religious overtones and, as a result, was not fully embraced. (The complete Chinese translation of all books of 'Elements' was only introduced in 1857!) Nonetheless, the deductive approach was finally accepted. The Chinese ordered the examination of the prospective officials in traditional mathematics in favour of European methods [Martzloff 1997, p. 273ff.], [Li/Dú 1987, p. 190ff.].

The Jesuits also introduced trigonometry to China. In collaboration with Xu Guangqi, Matteo Ricci published the first modern trigonometry book in the Chinese language in 1607. Whereas geometry was needed particularly for astronomy and calendar calculation, trigonometry assisted practical land surveyors. The fifteen surveying problems addressed in this book are narrowed down to relatively simple problems and could not compete with the ingenuity of Liu Hui's 'Sea Island Mathematical Manual' discussed above. To add to such statements on the size of mathematical knowledge or the geometrical content of such books, we must stress again that geometry, as was the case with arithmetics, was not thought of as a separate discipline back then, but as an integral component of a more extensive science, as the quote in A.4 suggests.

**Games**

To conclude this chapter, we will comment on mathematical games, which are referred to as 'Chinese'.

The most well-known game in Europe went by the name 'Tangram': a square wooden board is split into a (smaller) square, an oblique quadrilateral and five triangles of different size. Using these parts, we are meant to construct a series of different shapes. Nothing is known for sure about the origins of this game. The oldest known printed comment on this game comes from the beginning of the $19^{th}$ century. However, the two Chinese historians of mathematics, Liu Dun and Guo Zhengyi, found a source from 1617 that gives a short description of a similar game of splitting figures. One version, called 'Yizhitu' (literally: the forms of growing wisdom), consists of a square split into 15 segments – amongst them two semi-circles [Martzloff 1997, p. 367-369].



**Illus. 3.1.9** Tangram and Chinese rings

The art of origami (which, amongst other things, also allows us to illustrate Pythagoras's theorem), is mentioned in a well-known poem by Du Fu (Tu Fu) [Needham 1959, p. 112]. The origin of the Chinese rings (also known under a series of other names: Baguenaudier, Cardan's Suspension, Cardano's Rings, Devil's needle or five pillars puzzle) has not yet been clarified. In Europe, G. Cardano mentioned this puzzle, which is based on topological linking, in his 1550 work *De subtilitate libri XXI.* In 1693, J. Wallis published a mathematical analysis as part of his work *Opera* (vol. 2, p. 472). The novel *Hongloumeng* (Dream of the red chamber), which is widespread in China and was published in 120 chapters in 1791, peripherally mentions this game in Chapter 7. Nonetheless, this does not exclude the possibility of a European origin [Martzloff 1997, p. 366-370]. The reference to 'Chinese' does not tell us anything about the origin, since it was not uncustomary in Europe to raise interest in such puzzles by giving them exotic-sounding names.

**Essential contents of Chinese geometry**

| | |
|---|---|
| $6^{th}$ – $4^{th}$ century | *Zhou Bi Suan Jing*<br>'The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven': Theories on triangles and circles, Pythagoras's theorem, height and distance measurements, simple astronomical applications; no proofs |
| Early Han era 202 BC – 9 AD | *Jiu Zhang Suanshu*<br>'The Nine Chapters on the Mathematical Art':<br>240 problems; calculation of simple plane figures, approximations for circular areas and circle segments, volume calculations including pyramid, circular cone and cone frustum (without proof), applied problems (including applications of Pythagoras's theorem) |
| 263 | Liu Hui: *Haidao Suanjing*<br>'The Sea Island Mathematical Manual':<br>Climax of early Chinese theory on surveying; calculation of circumference based on the regular polygon of 92 corners |
| End of $5^{th}$ century | First contact with Indian scholars: Early form of trigonometry, calendar calculations |
| 656 | *Suanjing shi shu*<br>'The Ten Computational Canons':<br>Standardisation of mathematical problems |
| 1261 | Yang Hui:<br>Commentary on the 'Nine Chapters'; first attempts of proofing mathematics |
| $13^{th}$ century | New golden age of Chinese mathematics:<br>Further development of problems from the 'Nine Chapters' (in particular Qin Jiushao: *Shushu jiuzhang*): Hero's formula for calculating the area of a triangle, teaching of spherical triangle; interest in circle packing |
| 1601 | Jesuit Matteo Ricci takes western astronomy and mathematics to China |

## 3.2 Japan

| | |
|---|---|
| 2/11/660 BC | Mikado (Emperor) Jimmu Tenno founds the empire |
| 440 AD | Japan adopts Chinese script |
| From $7^{th}$ century | Rise of royal court, absolute meritocracy |
| From $13^{th}$ century | Rise of the shoguns (military dictators), who took over power, military nobleness, feudality (Samurai) |
| 1637 | Seclusion from external world, golden age of arts and science, Japanese mathematics |
| 1867 | Absolute monarchy taken over from Shogun rule |
| 1868 | Japan opens for trade with other states |
| 1894 | Sino-Japanese War |
| 1904 | Russo-Japanese War |
| 1910 | Korea conquers Japan |
| 9/27/1940 | Japan signs Tripartite Pact |
| 12/7/1941 | Japan attacks USA at Pearl Harbour |
| 1945 | Nuclear bombs dropped on Hiroshima and Nagasaki |
| 1950-1999 | Rise to greatest world power of Asia |
| From 1990 | Economic crisis, loss of economical leadership in Asia |

- *Japanese Art and Culture*: Coined by Buddhist and Chinese influences, temple constructions according to Chinese model, silk and brush paintings, calligraphy; Buddha and god sculptures according to Chinese model; lacquerware, pottery (for tea), porcelain
- *Literature:*

| | |
|---|---|
| 750–800 | *Man'yōshū*, ("Collection of Ten Thousand Leaves"), 4500 poems |
| Around 1010 | *Monogatari* (legends, fairy tales, stories), *Nikki Bungaku* Diary literature form |
| Around 1000 | *Zuihitsu* (essays), *Noh* (class. drama with music and dance), *Kabuiki* (folk theatre and puppet shows) |
| Since 1868 | Influences of European literature |

- *Philosophy:* Confucian ethics in 3 waves as base of feudalism; Buddhist philosphy practised by upper priesthood (Zen cult); occidental philosophy first conveyed by Dutchmen

### 3.2.0 Historical introduction

Since a detailed history of Japanese mathematics in a modern western language does not exist, we need basically to focus on Mikami's account from 1913 and the one by Smith and Mikami from the following year. The latter describes the historical development chronologically in 14 chapters, with the authors dividing the totality of Japanese history into six eras:

| | |
|---|---|
| Until 552 | A time only minutely and indirectly influence by Chinese mathematics |
| 552 until 1603 | Penetration of Chinese science, first via Korea, then directly: the Japanese Middle Ages |
| 1603 until 1675 | Renaissance of Japanese mathematics: Renewed penetration of Chinese science, first encounters with Europe science |
| 1675 until 1775 | The teaching of Seki Kōwa and his student Takabe lays the groundwork for domestic Japanese mathematics |
| 1675 until 1868 | Climax of independent Japanese mathematics, which is already influenced by Europe to a small degree |
| Since 1868 | End of Japan's seclusion, connection to western mathematics. |

The more extensive book *Science and Culture in Traditional Japan* (1978) by M. Sugimoto and D. L. Swain characterises the historical development from 552 onwards by means of describing cultural-scientific 'waves', which are embedded into the times of seclusion or periods of a (relatively) open-minded attitude. Besides, it also contains an insightful description of the cultural and social background of Japanese wasan mathematics. Their book presents us with the following picture:

| | |
|---|---|
| Approx. 600 until 894 | First Chinese wave |
| 894 until 1401 | Semi-secluded era |
| 1401 until 1854 | Second Chinese wave, which is overlapped |
| from 1639 until 1854 | by a time of isolation and superimposed |
| from 1543 until 1639 | by a first western wave and |
| from 1729 until 1854 | a second western wave |

### 3.2.1 Dawn and Middle Ages

We know nothing for sure about the ancient days of mathematical development in Japan. It is only certain that the decimal system was known in China as well as in Japan. Hence, we will look directly at the 'Japanese middle ages'. This era began with the introduction of Buddhism, which arrived in Japan via China and Korea and was absorbed there by means of Chinese mathematical books, the Chinese calendar and the Chinese measuring system. There is evidence that suggests that bamboo sticks had been used for calculation since 600 and mobile balls on the abacus (known in Japanese as a 'soroban') since the $7^{th}$ century.

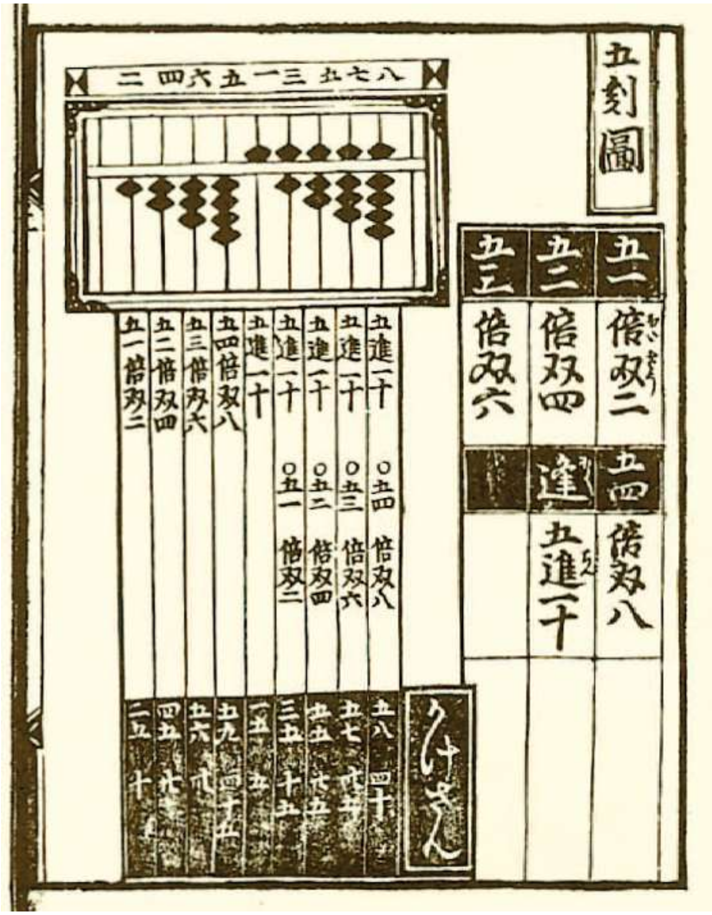
Japanese cultural heritage already encompassed a wealth of painting, literature, music, architecture and garden design in the Early Middle Ages, but medicine, natural sciences and mathematics were neglected. At the end of the $7^{th}$ century, a university was founded in Kyoto, an institute for fortune-telling and another for medicine. The university's function was the training of civil servants. At the beginning, only Confucian philosophy and mathematics within the scope dictated by the administrative requirements were taught. The instruction at the fortune-telling institute included astronomy, astrology and calendar composition. We only know of a small number of mathematicians from the $12^{th}$ and $13^{th}$ century onwards, although their texts were not preserved.

### 3.2.2 Renaissance of Japanese mathematics

More detailed facts on the history of Japanese mathematics are known beginning from the $17^{th}$ century onwards. In 1603, feudalism came to an end and was superseded by the centrally administrated state under Tokugawa. Trade relations with the Portuguese already existed from 1543 onwards, and in 1609 Japan also began trading with the Netherlands. The Jesuits started to proselytise in Japan in 1549. We can see that the phenomenon of the 'Renaissance of Japanese mathematics', as referred to above, took place when Japan had closer connections to Europe. Hence, science developed and conducted in this time was influenced by Europe.

However, countermovement soon occurred: the Jesuits had already been banned from Japan in 1587, and by the first third of the $17^{th}$ century, a massive suppression of Christians had begun. Japan secluded itself from all external influences (and only opened up again in 1868!). This seclusion, benefitted by the fact that Japan is not on the mainland, resulted in unique development of all aspects of life, including mathematics: an individual mathematics culture was formed called wasan, which means 'Japanese mathematics'. This form of study was of interest mainly to intellectual samurai, who engaged with the subject for pastime and recreational reasons; its connection to real life applications was too limited. Wasan mathematics was neither based on axiomatic-deductive theory, as we saw in Greece, nor connected to natural scientific concept formation and philosophical reflections, as had been customary in Europe since the $17^{th}$ century. On the contrary, it was founded upon the early encounters with the West and partially influenced by European findings (including those of the early modern era). Its interest in Chinese mathematics, its original source, started to decrease soon, and only maintained a sporadic level. Thus, trapped like in a greenhouse, wasan mathematics took its own course and developed an individual character. We want to devote our special attention to this period. The beginning of the Renaissance of Japanese mathematics during the $17^{th}$ century was influenced by the Chinese.

Elementary calculation on calculation frames was at the centre of Japanese publications, the numbers of which increased noticeably and which were only circulated as printed versions (of course, often in small number). Nonetheless, more difficult operations, such as extracting square or cube roots by means of calculation sticks, were also taught. The interest in surveying increased, along with calculations of areas and volumes, as explained in the section on Chinese geometry. Thus, we also find values 3.16 or $\sqrt{10}$ for $\pi$ here. These applied mathematics encompassed all the skills that craftsmen, master-builders, merchants and administrative officials needed.



**Illus. 3.2.1** Calculation with the Soroban in *Jinko-ki* (1641) of Yoshida Kōyū

**Geometry as part of wasan mathematics**

In contrast to the applied scientists, Yoshida Shichibei Kōyō, also known as Mitsuyoshi, came from a respected and very wealthy family. He travelled to China himself and studied the Chinese language thoroughly in order to engage with the more challenging Chinese mathematical texts before composing his own works. He wrote his first text in 1627 under the title 'Jinko-ki', which has been accepted as one of the most significant textbooks from the period of the Ming Dynasty. The title was also adopted by other authors after Yoshida's death (it translates as 'Treatise on numbers from the largest to the smallest'). Whereas Yoshida took on many applied problems in the first edition, he addressed numerous problems from the realm of recreational mathematics in the abundantly illustrated five-volume second edition. Later authors liked to use Yoshidas collection of problems as their role model, which is why this book became so significant as the foundation of developing wasan mathematics. Thereby, the custom emerged of adopting unsolved problems as challenges in the authors' own works. Their followers solved them according to their knowledge and simultaneously presented new or adapted problems of greater difficulty to their colleagues.

A work by Imamura Chishō published in 1639 gained its author many followers, as it addresses the calculation of regular polygons (from the triangle to the decagon)  a topic that became quite popular. A problem from a collection by Isomura Kittoku, published in five books in 1660, shows the finesse with which these problems were designed. It followed Yoshidas collection and was re-published with added notes by the author in 1684. The problem at hand requires us to place nine smaller circles in a large circle with a diameter of 3 feet whilst ensuring that each of the smaller circles is exactly 0.2 feet away from the central, adjacent and the given outer circle. Our goal is to determine the diameter of the inner circle placed in the centre and the diameter of the remaining eight circles placed around the central circle in a ring shape (the centres of which then form a regular octagon; cf. Illus. 3.2.2).

This concept is related to another problem that requires us to fit nineteen smaller circles into a given circle in a ring shape whilst ensuring that each of these circles touches its two neighbours and the given larger circle. On top of that, we must add nineteen circles tangentially around the outside of the larger circle under the same conditions. We are asked to calculate the radii of both kinds of circles. (Thus, we are implicitly dealing with a regular nineteenagon.) In contrast, other mathematicians were satisfied with calculating the side lengths of a great number of regular polygons, which are inscribed in a circle, as accurately as possible [Smith/Mikami 1914, p. 77/78].

It was common to wrap problems so that they would appear to have practical relevance. Yoshida's followers, for example, engaged with the following problems [Smith/Mikami 1914, p. 66/67]:


**Illus. 3.2.2** A version of the nine 'floating circles' in Seki: the circles arranged in a ring shape touch each other, but neither the inner nor the outer circle.
[Takakazu Seki: Collected Works, Ed. with Explanations, Osaka Kyoiku Tosho 1974]

(a) A mound of Earth has the shape of a circular conic frustum. The circumference at the top is 40 units, the one at the bottom 120 units. The mound is 6 units high. How high is the mound, if we reduce the amount of earth symmetrically from the top by 1200 units?
(b) A circular area of land with a diameter of 100 units is meant to be divided by two parallel chords and to be allocated to three people, who should receive, respectively, 2900, 2500 and 2500 units. What are the lengths of the chords and the height of the segments (see Problem 3.2.1)?

One of the most famous Japanese mathematicians is Seki Kowa, who was probably born in the same year as Newton. He anticipated the development of determinant calculation to some extent (almost simultaneously to Leibniz's approaches) and improved the method of dealing with algebraic equations and systems of equations. He derived the latter by means of similar geometrical questions, like the afore-mentioned circle arrangements, the conditions of which he analysed step by step. His most significant student, Takebe Hikojirō Kenkō, particularly emphasised this systematic approach connected to a special notation in a description of his teacher.

### The circle principle

The yenri (also enri) or circle principle of Japanese mathematics deserves a special mention. This principle concerns a peculiar method of determining the length of a circle arc and the circumference of a circle, respectively, in order to calculate constant $\pi$ at high accuracy. Who exactly thought of this concept first is historically controversial: older assignations to Seki Kowa cannot be verified. It is probable that the composition of the method, at least, is to be accredited to his student Takebe, who devoted himself to solving the problem


**Illus. 3.2.3** Seki Kowa Takakazu (taken from Masuhito Fujiwara) and stamp (Japan 1992)

of squaring the circle with fanatical persistence. Later Japanese mathematicians greatly praised Takebe, saying that he was one man in a thousand years and the light of the Land of the Rising Sun. He also seems to have been exceptional in another respect: he considered mathematical objects philosophically. Furthermore, he distinguished between analytical mathematicians (in which group he counted himself) and those that choose an intuitive approach. He argued that some problems could be better solved by means of the first group and others by means of the second.

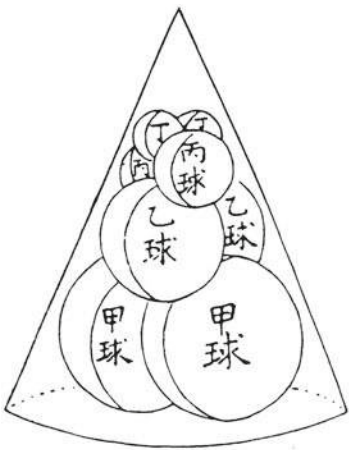
According to modern classification, the yenri principle almost belongs to infinitesimal mathematics. However, the interesting geometrical approach justifies discussing Takebe's peculiar method developed around 1720. The aim of this method is to calculate accurately the length of an arc $b$ in a circle of a given diameter $d$. The belonging chord is $c$, and the sagitta (distance between arc middle and chord middle) is $s$. Takebe chose $d = 10, s = 0.000001$, which allowed him to assume that the tiny arc is straight. Then he calculated its square by means of Pythagoras's theorem to 53 decimals, exactly to 0.00000 00000 33333 35111 11225 ... . If we calculate the square of the semi-arcs in the same manner for the sagittae $s = 1, s = 0.1$ and $s = 0.00001$, then we obtain 10, 1 and 0.0001 or, expressed differently, the product of the diameter with the belonging sagitta: $d \cdot s$. Hence, Takebe chose $d \cdot s = 0.00001$ as his first approximation for the square of the very accurately calculated arc and as first difference $d_1 = \frac{1}{3}s^2$. Repeating this procedure lead to the second difference $d_2 = \frac{8}{15}{d_1}^2$. Continuing and inserting each preceding difference into the one that follows ultimately delivers the infinite series

$$\frac{1}{4} \cdot b^2 = d \cdot s[1 + \sum_1^\infty \frac{2^{2n+1}(n!)^2}{(2n+2)!} \cdot (\frac{s}{d})^n]. \tag{3.2.1}$$

Since the sagitta does not behave differently from the versine (= 1 − cos), the square of the inversed sine is expressed here by the versine. Takebe derived a number of further series by means of similar considerations tailor-made for each case. Thus, the yenri or circle principle is not a general method, as represented by, for example, the Taylor series, but a special chain of reason that applies to the circle, the outcome of which agrees with infinite series, which were known in Europe at the end of the $17^{th}$ century.

A later historiographically interesting problem is associated with the yenri principle. The Jesuit Pierre Jartoux arrived in Peking in 1700 and lived there until he died in 1720. Whilst working as an astronomer, he corresponded with Leibniz and was familiar with the differential calculus developed by him. He is said to have derived some series expansions, of which three arrived in Japan. Since the explanations passed on by Takebe are not completely comprehensible, Smith and Mikami (1914) did not think it was impossible that he encountered one or more series imported from the West and then aimed to explain those as best he could. He encouraged other Japanese mathematicians to turn towards research in this field. They derived a whole number of further series expansions by means of clever considerations in the $18^{th}$ century.

The applied method by Kittoku and Takebe for determining the surface of a sphere is also based on an idea that bears resemblance to infinitesimal mathematics. Traditionally, the Japanese used as its measurement one quarter of the square of the circumference: $S = \frac{1}{4}(2\pi r)^2 = (\pi r)^2$. Isomura recognised the inadequacy of the rule and thought of calculating the surface by means of the difference of the volumes of two concentric spheres, which only differ

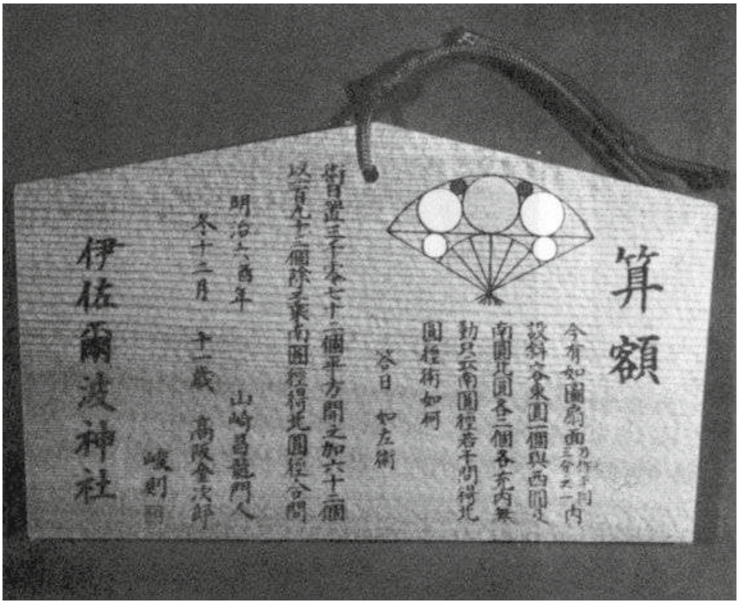



**Illus. 3.2.4** Sphere-packing in a cone from Fujita Sadasuke: Seijo Sampo (1779) [Smith/Mikami: *A History of Japanese Mathematics.* Chicago Open Court Publ. Co. 1914, p. 184]

slightly. He took a sphere with a diameter of 10 and a second one with a diameter of 10.0002, calculated the volume of the spherical shell bound by both spheres to be 0.03141 66283 24118 8 and divided it by its thickness 0.0001. Then, he repeated the calculation for two spheres with a diameter of 10 and 9.9998 and formed the average from both results: 314.16000 00418 88. (Due to $r = 5$, it is a hundredfold of $\pi$ since the Japanese back then did not know the surface formula $S = 4\pi r^2$!) Takebe refined the method in 1722 and, thereby, realised that he had obtained the one hundredfold of $\pi$. As a result, he concluded from his number ratio that the relation between $d$ and $S$ must be $S = d^2\pi$ (see Problem 3.2.2). Whereas Isomura and Takebe were restricting their considerations to a concrete numeric case, it was generalised using the example of the surface of an ellipsoid.

During the 18$^{th}$ century, the problem of circle-packing in the plane was also generalised to sphere-packing: given certain side conditions, the balls had to be packed in a given sphere (or cone) whilst making sure that they touched the wrapping solid and their neighbours (Illus. 3.2.4).

In the last third of the 17$^{th}$ century, the custom had already emerged of hanging mathematical problems with their solutions, without indicating the approach, on votive tablets in front of temples (Illus. 3.2.5). This tradition was then practised for more than two centuries. Originally intended to thank a Shinto deity or Buddha, the tablets also represented challenges of the col-

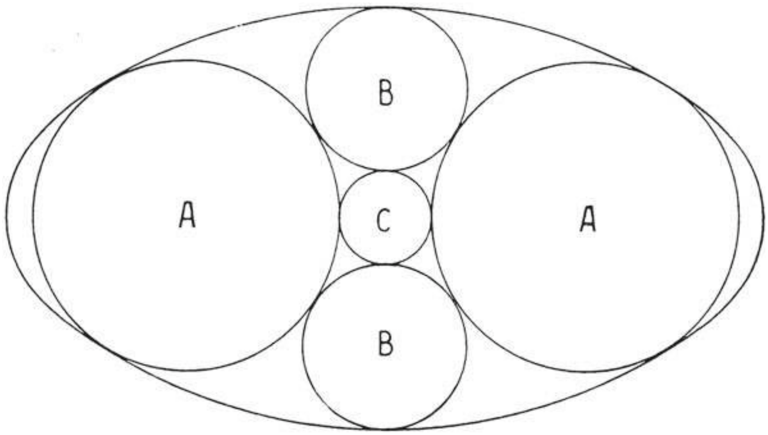



**Illus. 3.2.5** Example of a temple problem and how it was dealt with in the magazine "SUT Bulletin" 1987, No. 5 [Science University of Tokyo, SUT Bulletin 1987, No. 5, p. 11]

leagues. This custom reminds us of the habit of some European mathematicians of that time of competing with other mathematicians by means of problems spread by flyers.

Some examples of such problems with a geometrical background are below:

(a) We have two circles, one inscribed in a quadrilateral and one circumscribed. We know the diameter of the circumscribed circle and the product of the diagonals of the quadrilateral. We have to find the diameter $i$ of the inscribed circle. (In 1795, there was a mathematician who cited the formula $i\sqrt{(u+e)} = e$ to solve this, whereby $e$ is the product of the diagonals and $u$ the diameter of the circumscribed circle).
(b) Five circles are symmetrically inscribed in an ellipse with a great axis $a$ and a small axis $b$ (cf. Illus. 3.2.6). Calculate the diameter of circle $A$.
(c) Two spheres of size $A$, two of size $B$ and two of size $C$ are fitted into a sphere whilst touching each other. Given are the diameters of $A$ and $C$. Calculate the diameter of $B$.
(d) A circle segment is halved by its sagitta. A square as large as possible is inscribed in the left half (one side lies on the chord, the following one on the sagitta). A circle as large as possible is fitted into the right half. Given are the sum of chord, sagitta and diameter of the circle, the side of the square and the sum of the three quotients sagitta to chord, circle diameter to sagitta and side of the square to circle diameter. Find the individual magnitudes. (Displayed at Gion Temple in Kyoto, it became known as the Gion temple problem due to its high level of difficulty. It leads to an equation for the chord of degree 1024. Simplifications resulted in an equation of degree 46 and finally to one of degree 10.)




**Illus. 3.2.6** A Japanese temple problem (around 1800): Five circles inscribed in an ellipse [Smith/Mikami: *A History of Japanese Mathematics*. Chicago Open Court Publ. Co. 1914, p. 186]

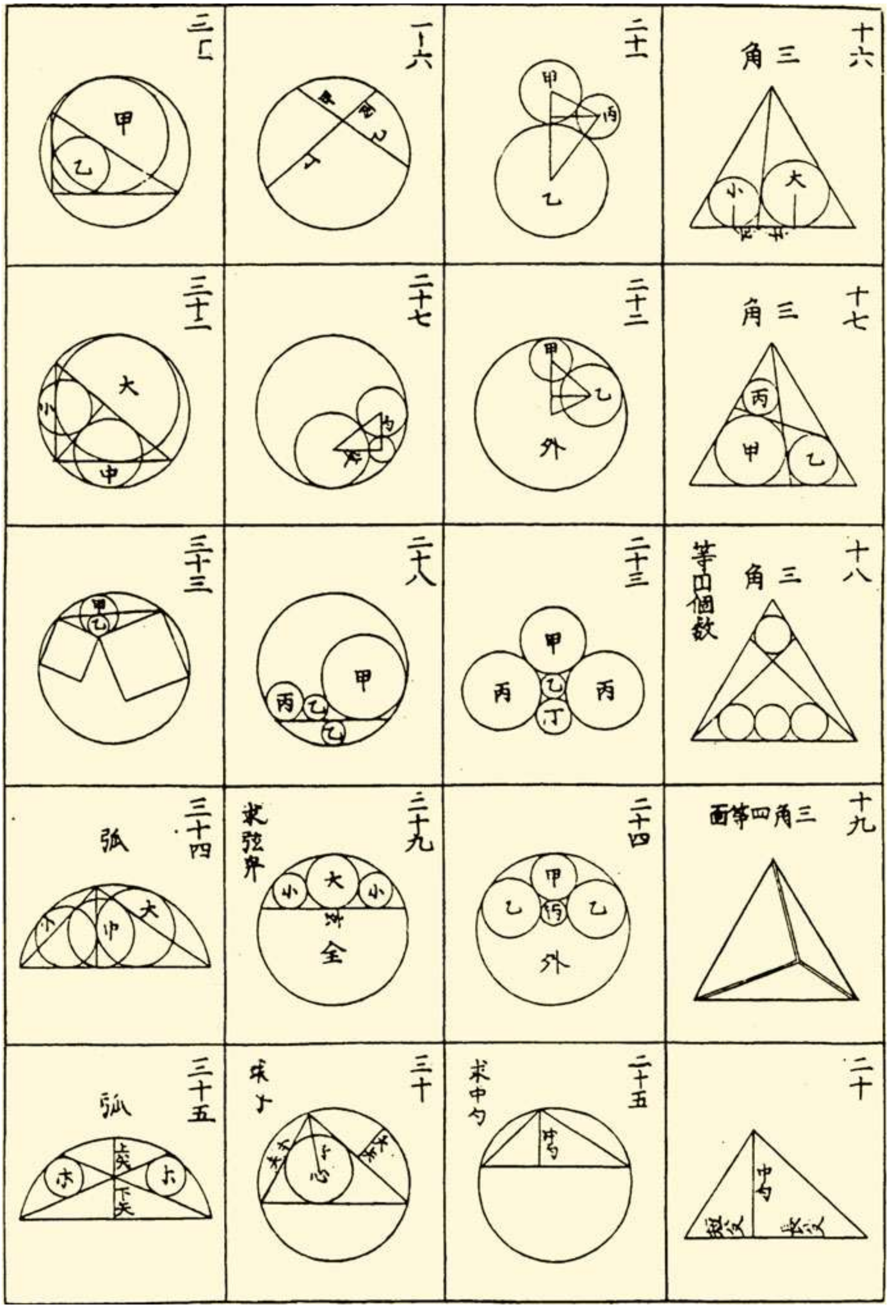
At the beginning of the $19^{th}$ century, Aida Ammei devoted an extensive work to the ellipse and its associated problems. As a consequence, other mathematicians were also inspired to engage with the ellipse and, thus, studying ellipses within a wider scope (also cf. problems mentioned above) became a popular subject. Aida described the string construction, as well as an instrument with changeable distance of the foci to draw all kinds of ellipses. Since it is said that an instrument for drawing ellipses arrived in Japan from abroad around this time, it is possible that Aida had heard of it or maybe even seen it. He derived the ellipse equation as an affine map of the circle, calculated the area of an ellipse segment and also the length of an elliptical arc.

To determine the latter, he suggested a peculiar approach. Possibly inspired by the method described above to calculate the circumference, he started drawing a series of chords in a quarter of an ellipse that approached it more and more. The first one resembled the connection between the extremities (the two points at which the great and the small semi-axis meet the elliptical arc). Then, he constructed the ordinate at the centre of the great semi-axis and drew the two chords from its intersection with the elliptical arcs to the extremities. Afterwards, he quartered the great semi-axis and obtained four chords in the same manner, which approached the arc more closely, etc. (The method fully corresponds to Archimedes' approach when geometrically squaring the parabola.) $n$ narrow ellipses segments lie between the chords of the $n^{th}$ repetition and the elliptical arc. If we continue this construction far enough, the area of the segments practically disappears and their sum strives against the length of the quarter elliptical arc. When calculating the ellipses segments, Aida substituted the small elliptical arcs by cleverly chosen circular arcs. Nonetheless, he obtained terms so complicated that he had to leave the summation to his successors.

Other Japanese mathematicians attempted to calculate the surface of an ellipsoid. They decomposed those either into rings, which they treated like conic frusta by means of parallel sections, or into sectors (shaped like orange peel) by means of central, plane sections, which have a common axis and are twisted against each other in small angles. These sectors also had to be approximated in a suitable manner. This led to long and complicated series expansions, which they addressed very aptly by also using tables to represent coefficients. The result is said to have been first published in print in 1844.

The authors' infatuation with mathematical artistry, distant from applications, self-sufficient, and untouched by the ambition of developing unified theories, was obvious. It seems that Japan's long-lasting seclusion crucially contributed to the fact that mathematics was treated like a game of glass beads. All the attention was directed at a few types of problems that mathematicians attempted to advance to perfection.

It would be inapt to use our notion of science here. Wasan served a recreational purpose, an art comparable to the Japanese tea ceremony or floristry. The impulse to perceive and investigate different problems brought about by external influences is missing in wasan mathematics, the followers of which




**Illus. 3.2.7** Figures for 20 problems from Yamamoto Kazen: Sampo Yoyutsu (1841) [Smith/Mikami, *A History of Japanese Mathematics*. Chicago Open Court Publ. Co. 1914, p. 246, Fig. 57]

mainly came from the upper classes. Even in the first two thirds of the $19^{th}$ century, when western knowledge slowly penetrated the country due to imported books, the Japanese stuck to their path and continued to fine-tune their traditionally developed equipment (see Illus. 3.2.7).

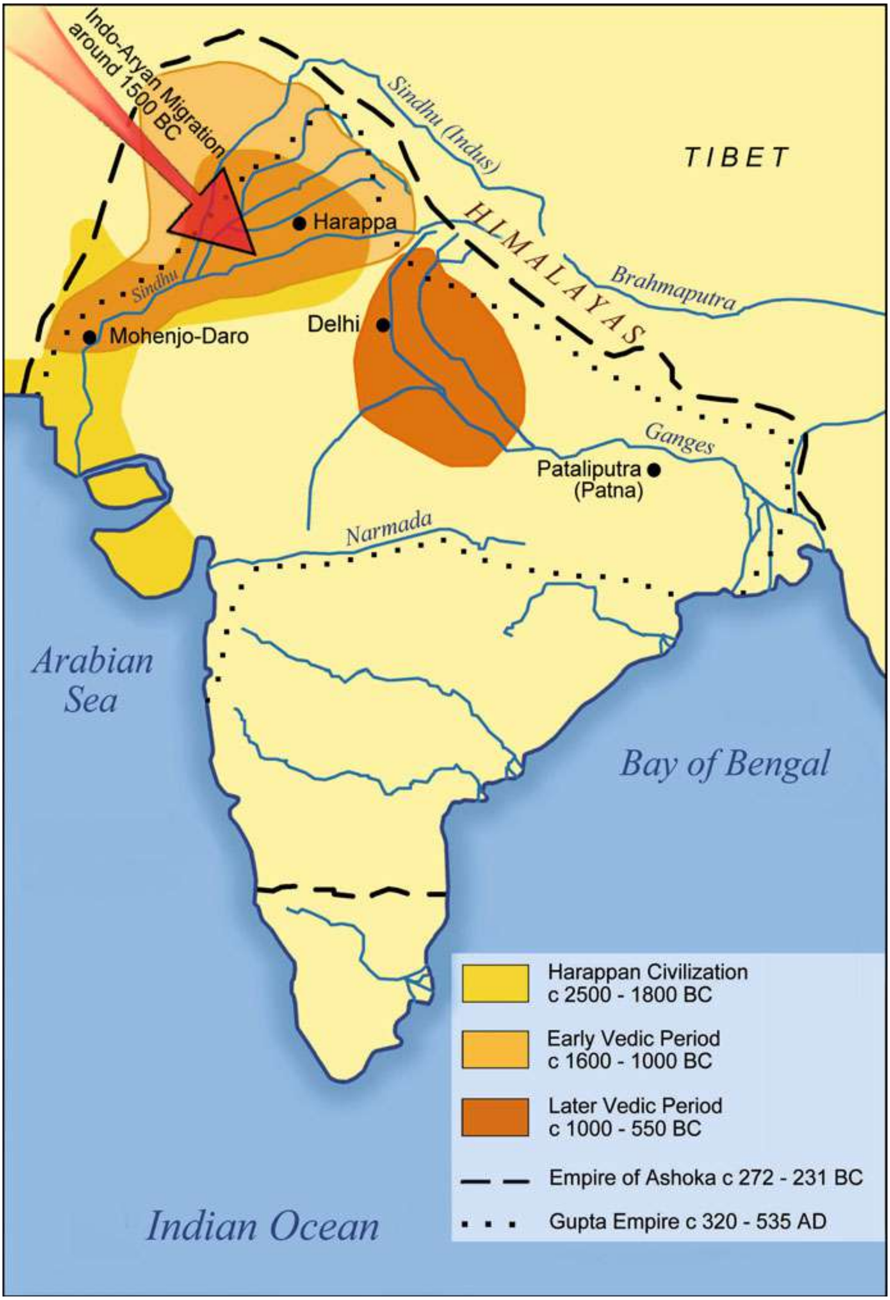
Things only started to change after Japan had ended its seclusion in 1868. A new generation of mathematicians managed to follow up on international mathematics, although some representatives of the traditional methods refused to relinquish their belief in the superiority of the domestic approaches. Nevertheless, wasan mathematics died and domestic traditions were forgotten. It has only been in the past few decades that mathematical historians have become increasingly interested in Japan's peculiar development of mathematics.

**Essential contents of Japanese geometry**

| | |
|---|---|
| Dawn and Middle Ages | Influenced by China (e.g. decimal system), rich cultural life, but little interest in natural sciences and mathematics |
| $17^{th}$ century | After contact with Portugal and the Netherlands in 1603, feudalism is superseded by a centrally administrated state; beginning of seclusion; origins of own mathematical culture "wasan" under samurais; little reference to applied science, intellectual pastime |
| 1627 | Yoshida Shichibei Koyu, also called Mitsuyoshi: *Jinkoki*: Collection of problems, taken from applied and recreational mathematics; role model for many works of the same title |
| 1639 | Imamura Chishō studied regular polygons (triangle until decagon); theory of circle-packing |
| 1640/2 – 1708 | Seki Kowa Takakazu and his student |
| 1664 – 1739 | Takebe Hikojirō Kenkō: Yenri principle for calculating a circular arc: series expansions; calculating surface of a sphere by means of volume differences of two concentric spheres |
| From approx. 1660 | Temple problems |
| $18^{th}$ century | Generalisation of circle-packing to sphere-packing; mathematical artistry, no ambition for unified theories, lack of external impulses |

## 3.3 India

| | | |
|---|---|---|
| $3^{rd}$ millennium BC | Towns established in Indus valley: Mohenjo-Daro, Harappa | Streets at right angle with canalisation |
| Since 2000 BC | Indo-Arians invade | Language: Sanskrit |
| 1500-200 | Vedic era | Indo-Arian-Brahman culture and religion, Rigveda, heroic epic Mahabharata, Kharosthi script and number symbols |
| Approx. 560-480 | Buddha founds new religion | Buddhism in Northeast India |
| Approx. 545-470 | Mahāvīra founds new religion | Jainism spreads across Northwest India |
| 327-325 | Alexander the Great in India | Hellenistic influence |
| 322-184 | Empire of Mauryan Dynasty | Temple and stone sculptures (the Great Stupa at Sanchi) |
| 272-231 | Emperor Ashoka: greatest Expansion of Mauryan Empire | Buddhism becomes state religion |
| 184-320 AD | Different dynasties rule microstates | Hellenistically-influenced Gandhara culture |
| 320-535 | Empire of Gupta Dynasty | 'Golden age' of science and art |
| 480-1525 | Many dynasties in Indian constituent states | Revival and strengthening of Brahman Hinduism, temples with sculptures and geometrical decor, golden era of Indian mathematics |
| Approx. 505-587 | Varāhamihira | Astronomical observations and calculations |
| 712 | Islam begins gradual conquest of the subcontinent | Penetration of Arabic and Persian aspects of culture |
| 1498 | Vasco da Gama arrives in Calicut | Encounter with Christian religion and occidental culture |
| 1525-1754 | Mughal emperors | 'Golden age' of so-called Mughal architecture and painting with geometrical ornaments |
| 1600 | Dutch East India Company founded | Increasingly influenced by west-European culture |
| From 1757 | England terminates reign of Hindu Princes | |
| 1877 | Queen Victoria becomes Empress of India | |
| 1920 | Ghandi invokes non-violent resistance | |
| 1947 | Independence and partition in India and Pakistan | |
| Since 1957 | Conflict with Pakistan over Kashmir | |
| $2^{nd}$ half of $20^{th}$ century | Rise to nuclear and economic power | Developments to become industrialised state |




**Illus. 3.3.0** Cultures and states in India in Antiquity and Middle Ages
[Map: H. Wesemüller-Kock]

### 3.3.0 Historical introduction

Just as in China and Japan, the origins of mathematics in India are almost a complete mystery. (It may be possible that the excavations in Mohenjo-Daro and Harappa in the Indus valley, which hide ruins of a town dating back to around 3000 BC, will reveal some insights, since the pre-Indian script found there has not yet been decrypted.) Around 2000 BC, Indo-Germanic tribes whose language was Sanskrit began to advance. The Vedic era (polytheistic worshipping of natural deities noted down in the Vedas), which began around 1500 BC, is mathematically characterised by the so-called Śulbasūtras (string rules). They were composed around 800 BC, yet were only passed on in very late, annotated editions around 300 AD. They contain instructions for the construction of sacrificial altars; these are to be treated here as first geometrical objects.

Buddhism and Jainism spread around 500 BC. Both were opposed to Vedic sacrificial rites for several centuries, which is why the string rules gradually lost their use. There is evidence that the Karosthi script and numbers had existed since around 400 BC.

Very few texts have been preserved from the post-Vedic era, which ended around 400 AD. Hinduism had been pushing Buddhism aside since around 700. With the Arabic invasion in 712, India started being Islamised, although Islam did not succeed in blocking Hinduism.

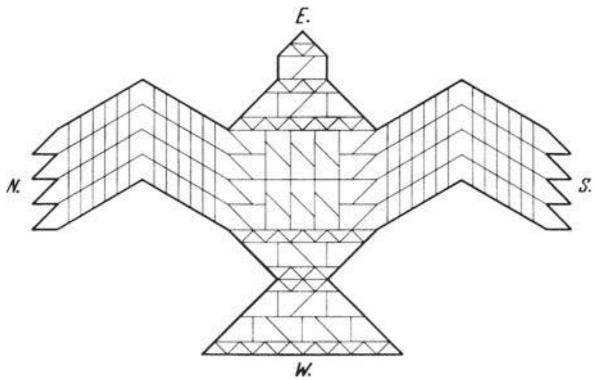
During the Indian Early Middle Ages, the time between 400 and 1200, mathematics became independent in India. The first so-called Siddhāntas, astronomical-mathematical Hindu-texts, seem to come from the $5^{th}$ century AD. The 'golden age' lasted from the $6^{th}$ until the $12^{th}$ century. Here, many Indian mathematicians showed how creative they were. The following 400 years, which are also referred to as the Late Middle Ages of Indian mathematics, were mainly characterised by commentaries on preceding mathematicians.

In accordance with the subject of this book, we will, all in all, focus mainly on the geometrical aspect of mathematical development in India. Further remarks are kept to a minimum. Although geometry only took on a minor role compared to the Indians' accomplishments in the areas of algebra and number theory, it still offers sufficient subject matter and is worth an independent observation. Even though India never developed a theoretical geometry similar to the scope of Euclid's work, their colourful intuitive treatment is a tempting topic and took several unexpected paths at times, giving Indian geometry its own special character.

### 3.3.1 Antiquity

#### Śulbasūtras

The ancient string rules or Śulbasūtras fall under the classification of sacral geometry. They concern instructions (passed on in several versions) for Vedic priests as to how to arrange sacrificial altars pointing in an East-West direction and how to construct them by using specially shaped bricks. Depending on the ritual to be executed, there were altars in the shape of squares, rectangles, trapezoids, circles and semi-circles; there were even altars for special occasions in the shape of a hawk, constructed by means of these shapes, and even wheel-shaped altars (see Illus. 3.3.1). There were supposed to be three simple fire altars in every household (square, circular or semi-circular), which were made of bricks of given shapes. The amount of brick rows [Srinivasiengar 1988, p. 6] also took on a mystical meaning. One version of the Śulbasūtras even emphasises: "We will describe the rules for marking the ground, on which the altar is to be built." Hence, teaching mathematics is not the primary aim, but fulfils an auxiliary purpose. The text describes the geometrical requirements needed to follow the religious instructions: area transfers, similarity relationships, theorem for the supplementary parallelogram, and Pythagoras's theorem. The problems resemble some analogies to Babylonian problems. However, their execution methods differed. We can only notice some similarities to Chinese mathematics concerning the fact that Indians as well as Chinese were fond of mathematical algorithms. These texts, however, do not show how the methods are derived, but simply state rules of how to proceed. The thesis that geometry developed out of ritual developments rather than practical needs was founded on the seemingly mythical-religious background visualised by these early Indian texts [Seidenberg 1962].




**Illus. 3.3.1** An altar in the shape of a hawk. The shapes of the bricks to be used are drawn in.

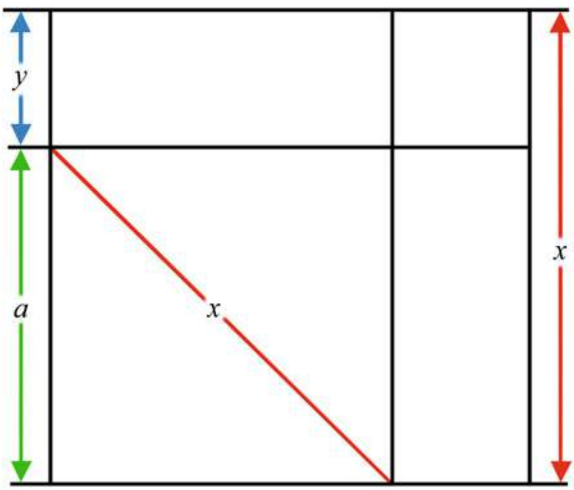
It is only partially possible to reconstruct the methods by means of the given outcomes. To state a numerical example, imagine we have to calculate the value for $\sqrt{2}$ by following the rule $1+\frac{1}{3}+\frac{1}{3\cdot4}+\frac{1}{3\cdot4\cdot34}$; in Babylon this would have been done by repeated application of Equation (1.2.7): $\sqrt{a^2+r}\approx a+\frac{r}{2a}$ (see Problem 3.3.1).

A group of propositions deals with the increase in size of a square altar area: it is meant to be doubled, tripled, ..., sextupled. To do so, the Śulbasūtras state the following fundamental geometrical proposition: "The string positioned across the square yields a (ground) area double the size." This proposition corresponds in its contents to Pythagoras's theorem for right-angled isosceles triangles. Comparing the given square $a^2$ with the new one $x^2$ leads to the gnomon figure. If we insert the quantity $y = x - a$ instead of the difference of diagonal and side, we have to attach the two rectangles of the quantity $a \cdot y$ and the smaller square $y^2$ at the corner. This is phrased in the Śulbasūtras as follows (cf. Illus. 3.3.2):

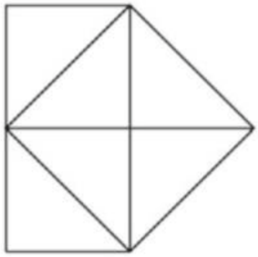
> Henceforth the general instruction: The (rectangle), which is drawn twice with the extension (of a side of the square) and the side itself, is attached on two sides (of the square) and the square, which is produced of it (i.e. the extension) is added at a corner.

Thus, we can geometrically construct the formula

$$a^2 + (2ay + y^2) = (a + y)^2 \tag{3.3.1}$$




**Illus. 3.3.2** Gnomon figure regarding doubling the square



**Illus. 3.3.3** Pythagoras's theorem for right-angled isosceles triangles

In other words, the gnomon is represented by the area $2ay + y^2$. By means of this gnomon figure we can also explain the method for determining $\sqrt{2}$ mentioned above. (There is no hint whatsoever that India discovered the irrationality of $\sqrt{2}$ in this context.)

Doubling the area of the square altar reminds us not just of the classical Greek problem of doubling the cube, but could also have triggered the discovery of Pythagoras's theorem concerning right-angled isosceles triangles. We only have to put two examples of the initial square together to form a rectangle, then halve them by two diagonals with a common extremity and reposition the two outer triangles to generate a new square (see Illus. 3.3.3).

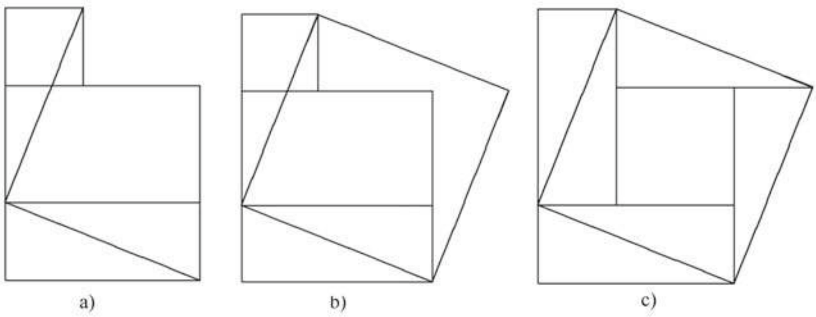
All in all, the Śulbasūtras feature six concrete Pythagorean triangles, namely those with sides (3, 4, 5), (12, 5, 13), (15, 8, 17), (7, 24, 25), (12, 35, 37) and (15, 36, 39). However, the theorem regarding the sum of the square of the catheti is not stated in a general manner. Nonetheless, the concrete triangles are cited at least once to prove that the theorem concerning the diagonal of the rectangle is correct. The Indian mathematicians' habit of basing their constructions on concrete measures from the very beginning may have inspired them to study numerical relations when contemplating general geometrical relations.

The Śulbasūtras also state how to generalise the concept of a square to apply to any rectangle: "The string (laid) across the rectangle yields both (areas together), which the long side and the broad side yield for each on their own."

If we are required to triple the area of the square altar, we can add the simple and the doubled square together, according to the following rule:

> If we want to unite two different large squares, we must elevate a (parallel) strip to the larger square with the side of the smaller square. The string (laid) across this strip is the side of both united squares.

Illus. 3.3.4 a) illustrates what is meant, and the following one (b) shows the completed new square. We obtain the known figure c), which illustrates Pythagoras's theorem in the form of $c^2 = 2ab + (b-a)^2 [= a^2 + b^2]$, by altering the auxiliary lines.



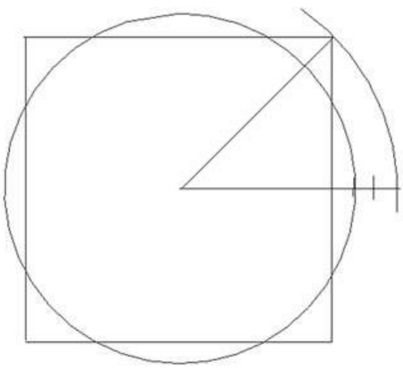

**Illus. 3.3.4** Addition of two unequal squares to form a third square

The religious instructions also demanded the construction of altars for special occasions, whereby equality of area took on a different form. As a result, the problem of circling the square arose (configuring the radius of a circle that is of equal area to a given square):

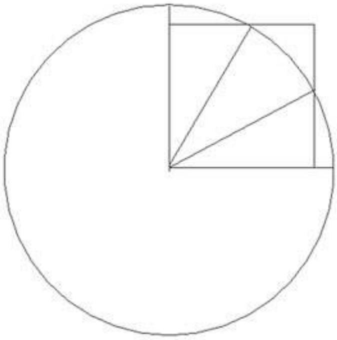
> If we want to make a circle out of a square, we must position (a string) at the middle of the square and link it to a corner. Then pull it around in direction to a side (of the square). Together with a third, what is left-over of it (what lies outside the square), draw the circle (cf. Illus. 3.3.5).

In another version the following remark is added: "Add as much as you take." (cf. Problem 3.3.2, [Jaggi 1986, p. 182])

The Śulbasūtras state two rules for executing the better known reverse problem of squaring the circle:



**Illus. 3.3.5** Circling the square according to the Śulbasūtras



**Illus. 3.3.6** Squaring the circle according to the Śulbasūtras

1. If we want to square a circle, divide the diameter in 8 parts, divide *one* of these parts in 29 parts, deduct 28 parts (of these 29 parts) and, furthermore, deduct the $6^{th}$ part reduced by its $8^{th}$ part from the (remaining *one*) part.
2. Or, divide (the diameter) into 15 parts and deduct 2 (parts). The latter (string) is usually used as the side of the square [Müller 1930, p. 179-190]. (cf. Problem 3.3.3)

Of course, the question arises as to how the composers of the Śulbasūtras thought of the stated fractions. Conrad Müller could derive the second value of $\frac{13}{15}$ under the assumption that the quarter circle arc was divided into three, and a vertical or horizontal line, respectively, was drawn through both points of division. Together with the bound radii of the quarter circle, these lines then formed a square, the area of which seemed roughly to correspond to the quarter circle (Illus. 3.3.6).

The square side can be calculated by means of Pythagoras's theorem as $s = \frac{\sqrt{3}}{4} \cdot d$ ($d$ = circle diameter). Although, in contrast to $\sqrt{2}$, the conserved Śulbasūtras do not state a fraction for $\sqrt{3}$, we can easily obtain the approximation $\sqrt{3} \approx \frac{5}{3} + \frac{1}{15} = \frac{26}{15}$ in two steps by means of the Indian method, which answers the question.

We can also provide a reconstruction for the more complicated first instruction, given we consult the fraction representation for $\sqrt{2}$, as cited above. Hereby, Müller stresses that the obvious question of accuracy, or which of the two methods yields the most precise value, did not arise back then. The Indians neither had an exact construction with which to compare it, nor were they aware of the relations between circumference and area of a circle at this time. (The Śulbasūtras state once that circumference is $3d$!)

**Jaina geometry**

A so-called Jaina geometry had begun to develop around 400 BC. This geometry seemed to be mainly interested in the trapezoid as a representation of the universe and the continents, and the circle as a reflection of Earth and as the orbits of celestial bodies. The so-called Ganita texts describe Earth and celestial orbits. Thereby, they state that the ratio of circumference to its diameter is $\sqrt{10}$, next to $\pi = 3$ (without reason). This can also be found in Brahmagupta's work. Besides, Jaina geometry talks of circle segments. However, not one mathematical or geometrical Jaina text has been preserved. It is said that they also contained details on everyday practice.

### 3.3.2 Middle Ages

The revival of mathematical studies in India in the Middle Ages, between approx. 400 and 1200, coincides with the so-called classical Indian era or the Renaissance of Brahmanism, which led to a golden age of literature and philosophy. The penetration of astronomical knowledge from Alexandria may have helped to awaken the interest in mathematical problems. It is striking, hereby, that the mathematicians of that time did not refer back to the old Śulbasūtras – in contrast to China, which, as we saw, highly appreciated the old works, which were passed on, and commented on over and over again. Creation of the decimal position system is, without a doubt, the most successful aspect of the global development of mathematics. The earliest evidence of its existence comes from the $7^{th}$ century. It is certain that great assistance was given by the calculation frame, which was used all over the oriental world, as its division in columns anticipates the meaning of the place value within the numerical script. The Arabs got to know this digit system via an Indian astronomical work, which finally spread across the whole world.

#### The Bakhshāli Manuscript

The Bakhshāli Manuscript, possibly from the $6^{th}$ century, is one of the oldest preserved collections of problems and economical texts with calculations. It was written on birch bark and was only excavated in Northwest India in 1881. It contains around 70 pages and seems to be a commentary on an older lost work. Explanations of elementary calculation operations, including fractions and extracting square roots, form the main part of this work. On top of that, the text addresses series, calculating profit and loss, interest and the rule of three. The numerical examples and their solutions are represented in the decimal system. Just as in all later mathematical texts of India, the interest in algebraic topics prevails in the Bakhshāli Manuscript: it looks at linear equations and systems of equations, quadratic equations (negative quantities are permitted, too) and the wrong approach. Contrary to the norm, a topic that otherwise always reoccurs is missing: a discussion of the gnomon shadow, which was important for astronomy and calculation of time. Some authors believe that this proves that this text was composed very early [Jaggi 1986, p. 130-131].

### The Sūrya Siddhāntas

The Sūrya Siddhāntas, a collection of astronomical texts, are at least as old as the Bakhshāli Manuscript. The older texts work with chord geometry, which was conceived by Hipparchus and Ptolemy and passed on to India by Alexandrian sources. It was up to the Indians to overcome the inconveniences of chord calculation – no Hellenistic astronomer had attempted to improve this means in any way. The sine, the essential function for modern trigonometry, first occurred in an Indian handbook from the $4^{th}$ or $5^{th}$ century [Berggren 1986, p. 132]. The Indian scholars improved and updated the construction of observatories. In order to obtain angle and shadow lengths as accurately as possible, they erected huge 'instruments'. Some were refurbished or reconstructed. Illus. 3.3.8 shows the observatory; Illus. 3.3.9 represents an instrument of the observatory in Jaipur.

The Sūrya Siddhāntas state in Sanskrit verses the values of the sine function up to 90° in intervals of $3\frac{3^\circ}{4} (= 60^\circ : 2^4)$. At the beginning of the $6^{th}$ century, Varāhamihira used trigonometric formulae in his astronomical work "Pancha Siddhāntika". Therein, he presented an annotated overview of all five Siddhāntas. We will reproduce here one of the simpler instructions for determining geographical latitude (Illus. 3.3.7):

विषुवद्दिनसममध्यच्छायावर्गात् स'वेदकृतरूपात्' ।
मूलेन शतं विंशं विषुवच्छायाहतं छिन्द्यात् ।। २० ।।
लब्धं विषुवज्जीवा चापमतोऽक्षोऽथवैवमिष्टदिने ।
मेषाद्यपक्रमयुतस्तुलादिषु विवर्जितः स्वाक्षः ।। २१ ।।

'Measure the midday shadow on the day when the Sun is at the equinoxes (the equinoctial shadow). Square it, add 144, and find the square root. By this, divide the product of the shadow multiplied by 120. The result is the sine of the latitude of the place, called *viṣuvajjyā*.'

**Illus. 3.3.7** Varāhamihira's instruction to determine the geographical latitude by means of the height of the sun at noon

[Subbarayappa/Sarma: Indian Astronomy, p. 184, No. 15.5.2]

If $s$ is the length of the shadow, $g = 12$ the height of the gnomon and $l$ the distance from its image point (the end of the shadow) to the top of the gnomon, then divide $120s$ by $l$. (The factor 120 is the result of the assumption that we imagine the right-angled triangle inscribed into a circle with a diameter of 120). Hence, the quotient to be formed is the sine of the angle at the top of the gnomon. At the time of equinox, when the sun is exactly above the equator, the quotient (or the belonging arc, respectively) equals the geographical latitude. The second part of the instruction refers to the case that this observation is made on a different date than the equinox.

The Indians did not just replace the chord by the sine, they also introduced cosine and versine ($sin\ vers\ \alpha = 1 - cos\ \alpha$). The latter occurs in formulae such as

$$sin\ \frac{\alpha}{2} = \sqrt{\frac{1 - cos\ \alpha}{2}}. \tag{3.3.2}$$

Manjula studied the functions sine, cosine and versine in all four quadrants around 930. We can even find early hints of the laws of sines and cosines in spherical trigonometry.

**Āryabhaṭa I**

The frequent linking of astronomy and mathematics in India can already be found in the work Āryabhaṭa by Āryabhaṭa I, written in very condensed verses in 499. The author, who came from the very South of India, composed this work at the age of 23. The mnemonics can often only be understood if one is already familiar with the subject matter at hand (astronomical or mathematical propositions). They were meant to serve as memory hooks for oral teaching. Similarly to the Śulbasūtras, supplements are required when translating. These usually come in brackets. Āryabhaṭa I deals with the following aspects in four parts:

- His own alphabetical representation of numbers (nothing to do with the position system) and next to astronomical details, the sine function including a sine table with 24 values (i.e., in intervals of $3\frac{3^\circ}{4}$, in the form of first differences)
- Art of calculating (including some geometry) in 33 verses
- Calculation of time
- Spherics.

Along with ancient Indian influences, we can also prove Greek ones, especially in the astronomical section (deferent and epicycle). Hereby, the Āryabhatīya was already translated into Arabic around 800 AD.

The geometrical problems contain formulae concerning triangular area, pyramid volume, circular area ($F = r \cdot \frac{C}{2}$) and spherical surface. It was believed until recently that Āryabhaṭa stated incorrect pyramid and sphere volumes. K. Elfering provided an interpretation in his German translation from 1975, which proves the accuracy of the instruction for the calculation of pyramid volume and argued that the verse concerning the sphere, in fact, relates to the surface [Elfering 1975]. Accordingly, we would translate it as follows:

> Half the circumference multiplied by half the diameter is the circular area. This (i.e., the circumference) multiplied by his determining base (r) is the surface of the semi-sphere, and, in fact, exactly.

This interpretation is further supported by a late Indian commentary. Unfortunately, the brief mnemonics contain no hints how Āryabhaṭa I (or his sources) arrived at this conclusion.

**Illus. 3.3.8** The reconstructed observatory in Jaipur
[Photo: H.-W. Alten]

In another verse Āryabhaṭa I states a value for $\pi$:

> One hundred and four times eight, add sixty two thousand; this is approximately the circumference for the diameter of a pair of ten thousands.

This means: $\pi \approx \frac{62832}{20000}$. This value can be reconstructed by taking the regular hexagon and repeatedly halving the chords until we obtain a polygon with 384 edges [Gericke 1984, p. 185], [Elfering 1975, p. 87]. The verse concerning Pythagoras's theorem is very insightful for the theoretical environment, in which Indians gained and also originally phrased mathematical understandings:

> Having added the square of the measure of a gnomon to the square of its shadow, its square root is the radius of the 'celestial circle'.

Two further verses summarise elementary similarity relations and the altitude theorem in the right-angled triangle, and another one deals with the common chord of two intersecting circles. What follows concerns the summation of series, solving determinate and indeterminate equations and other topics from the area of algebra and number theory.

**Illus. 3.3.9** One of the reconstructed instruments of the observatory Jaipur [Photo: H.-W. Alten]

### Brahmagupta

Apart from the usual instructions for calculating straight bounded, plane areas and elementary solids (it would be redundant to repeat those here), we can also find some remarkable particularities in the Indian texts. For instance, Brahmagupta (598 – after 665) states a formula (rule) for calculating the area of a general quadrilateral by means of its sides $a, b, c, d$ and half the perimeter $p$, which reminds us of Hero's (better: the Archimedean) formula (2.5.1) concerning the triangular area:

$$A = \sqrt{((p-a)(p-b)(p-c)(p-c))} \tag{3.3.3}$$

However, he does not state that we are dealing with an approximation formula, which only applies exactly to the special case of an inscribed quadrilateral in a circle. (The correct formula would be:

$$A = \sqrt{((p-a)(p-b)(p-c)(p-d) - abcd \cdot cos^2\ \alpha)}, \tag{3.3.4}$$

whereby, $\alpha$ is half the sum of two opposite angles.) Later mathematicians adopted this formula as well as the special case for $d = 0$, as done by Brahmagupta. Only Āryabhaṭa II highlighted around 950 that the formula (3.3.3) does not yield the correct value for every quadrilateral.

We also find Ptolemy's theorem of chords for the cyclic quadrilateral $a, b, c, d$ with the two diagonals $e, f$ in the circle in Brahmagupta's work:

$$e \cdot f = a \cdot c + b \cdot d \tag{3.3.5}$$

Additionally, it is the first time that we find the following proportion in his work:

$$\frac{e}{f} = \frac{ad + bc}{ab + cd} \tag{3.3.6}$$

Compare this with Regiomontanus's efforts in the $15^{th}$ century to generate a formula concerning the area of an inscribed quadrilateral or with the fact that W. Snellius encountered Brahmagupta's formula in Europe only in 1615, for which Philip Naudé the Younger could finally provide a proof in 1727(!).

Around 850, the Indian Jaina scholar Mahāvīra also stated the formula 3.1.2, obtained from the old Chinese texts, concerning the area of a circle segment: $A = \frac{(c+s)\cdot s}{2}$ ($A$ = area, $c$ = chord, $s$ = sagitta). In his work, he followed up on almost all the problems that had kept his predecessors busy. However, as with most Indian mathematicians, concerns of numerical questions, solutions to equations and infinite series dominated [Juschkewitsch 1964, p. 86].

### Three-dimensional coordinate geometry

We find starting points of a three-dimensional coordinate geometry in Vācaspati's work from the $9^{th}$ century. He developed this idea in a philosophical commentary when examining the position of an atom in space. To do so, he thought of three axes, one in an East-West direction between the point of sunrise and sunset at the horizon, one North-South perpendicular to the first one, and a third leading from the crossing of both points to the respective position of the sun at noon (normally, it does not stand perpendicularly to the plane of the two other axes). He then described the distance of two atoms by means of the three directions [Bag 1979, p. 169-170].

### Euclid's influence

Determining to what degree Euclid's 'Elements' influenced Indian mathematics is a difficult historical undertaking. It seems that single definitions or propositions became known from the $6^{th}$ century onwards. It is certain that 'Elements' was accessible in India beginning in the $14^{th}$ century. However, as already mentioned, the encounters with Euclid (in Arabic or Persian translation) did not result in the adoption of proof by means of deduction.

### Bhāskara II

Bhāskara II, to whom we owe several ingenious mathematical contributions, lived in the $12^{th}$ century. Above all, he is known for inventing the 'chakravala' method, a cyclic algorithm for solving indeterminate quadratic equations in two unknowns. Hermann Hankel referred to this as the finest accomplishment of number theory before Lagrange. We will now sketch his calculation of the spherical surface, which introduces us again to a new method and shows the various approaches taken during the course of history to attack the same problem. Bhāskara II imagined a sphere with a circumference of 96 units placed at the equator. This sphere was subdivided into 96 equal parts. He drew 48 meridians through these points of division, each one of which was also subdivided into 96 equal arcs. He laid the circles of latitude through the points of division of equal altitude. This resulted in small fields (spherical trapezoids and spherical triangles at the poles, respectively), which he held to be plane. Their size decreases from the equator to the poles and proves to be proportional to the sine of the angle distance of the pole, if we view the fields within a strip (shaped like orange peel) from the pole to the equator. This way, Bhāskara II was able to yield a good approximate value for the spherical surface [Bag 1979, p. 296]. The chosen interval size required him to yield sine values in an interval of $\alpha = \frac{2\pi}{90} = 3\frac{3}{4}^\circ$ (cf. Problem 3.3.4).

To conclude, we will look at another interesting version for determining the spherical surface, which had earlier been wrongly ascribed to Āryabhaṭa I. It is based on the following consideration, which perhaps was initiated by his school of thought. (It has been passed on in the work *Yuktibhāṣā* by an unknown author from the $16^{th}$ century.) Imagine the surface of a sphere divided into circles by parallel (latitudinal) circles. Then, approximate the area of one such ring by means of the lateral surface of the conic frustum, which is determined by the lower and upper parallel circle. Finally, add all these strips together by imagining unwinding the conic frusta in the plane and viewing their surfaces as trapezoids. This idea also leads to the exact formula $S = 4\pi r^2$, if we, according to the modern view, proceed to the boundary [Bag 1979, p. 296], [Sarasvati Amma 1979, p. 213-215]; (cf. Problem 3.3.5).

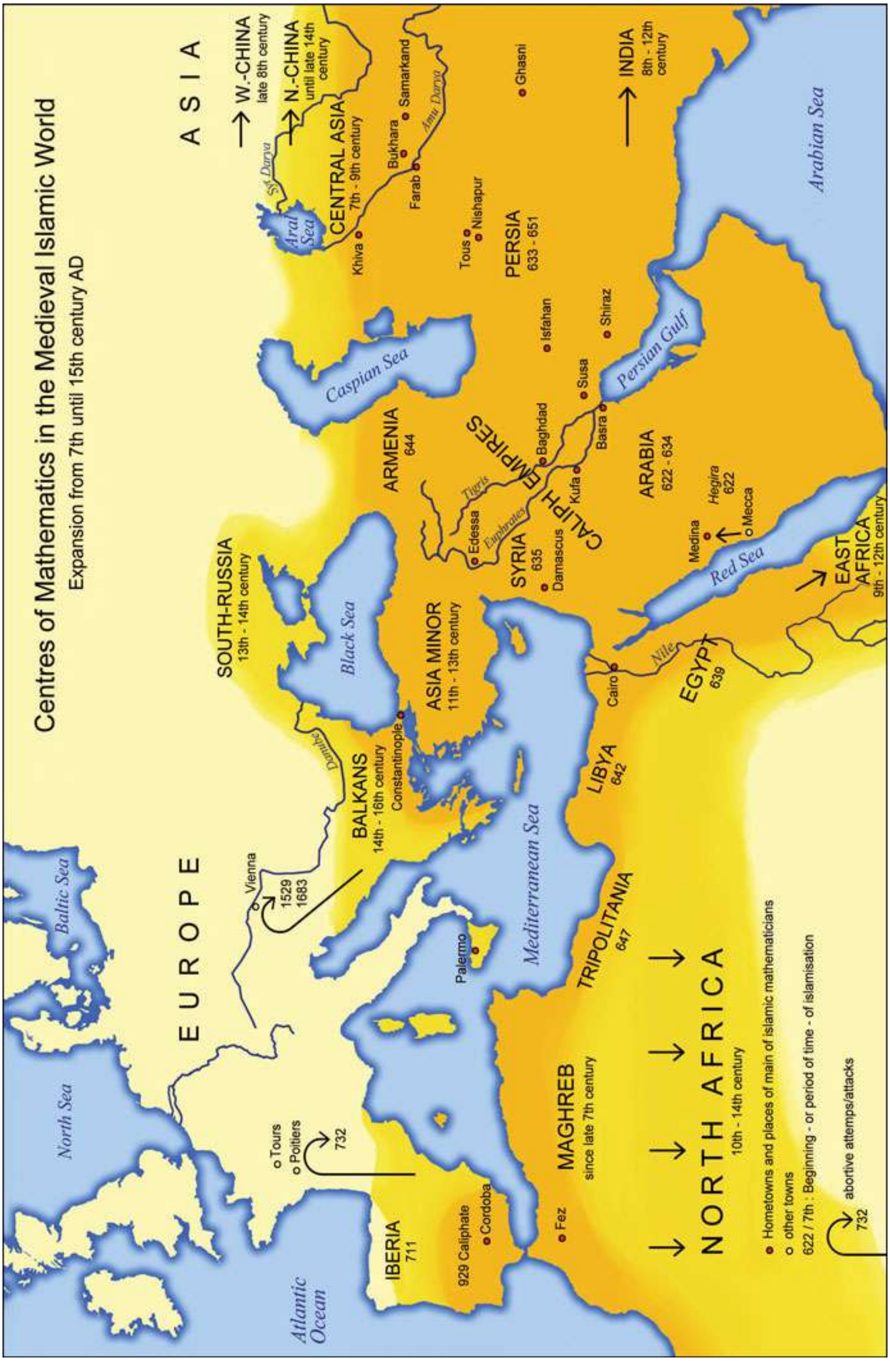
We have repeatedly pointed at the Indian mathematicians' preference for treating geometrical problems based on algebraic methods. Their main accomplishments were not of geometrical, but of algebraic nature – number theory (divisor problem, continued fraction algorithm, Diophantine equations) –as well as the development of numerous infinite series and considerations, which point in the direction of infinitesimal methods. Nevertheless, they have also shown a remarkable wealth of ideas in the realm of geometry in ancient and medieval times, as the examples demonstrate.

**Essential elements of Indian geometry**

| | |
|---|---|
| 700–500 BC | *Śulbasūtras* (String rules) |
| (Vedic era) | Altar constructions by means of triangles and rectangles; Pythagoras's theorem; square additions; squaring the circle and circling the square |
| approx. $5^{th}$ century | *Siddhāntas* (astronomical, mathematical texts) Introduction of trigonometric functions: first sine and cosine (relation to gnomon); around 930 in all four quadrants |
| 400–1200 (Hindu Middle Ages) | Āryabhaṭa I (born 476):<br>*Āryabhaṭīya* (mathematical mnemonics) Elementary geometry, spherical geometry, deferent and epicycle (Greek influence)<br>Brahmagupta (born 598):<br>Generalisation of Hero's triangle formula to include inscribed quadrilaterals; Ptolemy's theorem for inscribed quadrilaterals<br>Bhāskara II (1114–1185?):<br>*Siddhānta-siromani* (Wreath of sciences):<br>Climax of Indian mathematics (chakravala method (cyclic algorithm) in number theory); calculation of spherical surface by decomposition |
| 1200–1600 (Late Middle Ages) | Numerous commentators on the works of Bhāskara II; general characteristics: calculating geometry, touching on Islamic mathematics |

## 3.4 Islamic Countries

| | | | |
|---|---|---|---|
| 622 | Mohammed emigrates from Mecca to Medina (begin of Islamic calendar) | | |
| 632 | Mohammed dies | | |
| 634–644 | Caliph Umar (book burning) | | |
| 635 | Damascus and Mesopotamia are conquered | | |
| 635–651 | Sassanid Empire (Persia since 226) smashed | | |
| 642 | Egypt is conquered (Alexandria!) | | |
| 712 | Chorasmia (between Caspian Sea and Aral Sea) conquered, advances until Indus | | |
| | **West** | | **East** |
| | | 661–750 | Umayyad Caliphate, based in Damascus |
| 711 | Crossing at Gibraltar, Visigothic Kingdom in Spain destroyed | 717–720 | Caliph Umar II: 718 scholars relocated from Musaeum in Alexandria to Antiochia |
| 732 | Karl Martell beats Arabs at Tours and Poitiers | 750–1517 | Abbasid Caliphate, based in Bagdad from 763–1258, after Mongol attack 1261–1517 in Cairo |
| 756–1031 | Umayyad Emirate, based in Córdoba, Caliphate since 929 | $9^{th}$ century | Arabs at Crete and Sicily |
| $8^{th}-10^{th}$ century | Translations into Arabic | 754–775 | Caliph al-Manṣūr, founder of Bagdad (763) |
| | | 768–809 | Caliph Harūn ar-Rashīd (1001 Nights) |
| $11^{th}-13^{th}$ | | 813–833 | Caliph al-Ma'mūn |
| century | Translations into Latin | 833–843 | Caliph al-Mutazīm |
| | | 847–861 | Caliph al-Mutawakkil |
| | | 892–903 | Caliph al Mutadīd |
| 912–961 | Emir, from 929 Caliph Abd ar-Raḥmān | | |
| 961–976 | Caliph al-Hakam II, Library at Córdoba | 969–1171 | Fatimids in Egypt, Capital Cairo (founded in 969) |
| 1031 | Caliphate broken down into principalities | 1206–1227 | Dschingis Khan, Mongol attack |
| | | 1258 | Hūlāgū Khan conquers Bagdad, gains title of Ilkan |
| | | 1409–1449 | Ulugh Beg in Samarkand (Uzbekistan) |
| | | 1453 | Osmans conquer Constantinople |
| 1492 | Downfall of Granada, the last Moorish kingdom | 1517 | Turks conquer Cairo |


Centres of Mathematics in the Medieval Islamic World
Expansion from 7th until 15th century AD
ASIA
W.-CHINA
late 8th century
N.-CHINA
until late 14th century
CENTRAL ASIA
7th - 9th century
Aral Sea
Syr Darya
Amu Darya
Khiva
Bukhara
Samarkand
Farab
Ghasni
INDIA
8th - 12th century
Tous
Nishapur
PERSIA
633 - 651
Isfahan
Shiraz
Caspian Sea
Persian Gulf
Susa
Arabian Sea
ARMENIA
644
CALIPH EMPIRES
Baghdad
Kufa
Basra
Tigris
Euphrates
ARABIA
622 - 634
Hegira
622
Medina
Mecca
Red Sea
EAST AFRICA
9th - 12th century
SYRIA
635
Edessa
Damascus
SOUTH-RUSSIA
13th - 14th century
Black Sea
ASIA MINOR
11th - 13th century
Constantinople
BALKANS
14th - 16th century
Danube
Cairo
Nile
EGYPT
639
LIBYA
642
Mediterranean Sea
EUROPE
Vienna
1529
1683
Baltic Sea
Palermo
TRIPOLITANIA
647
NORTH AFRICA
10th - 14th century
MAGHREB
since late 7th century
Fez
North Sea
Tours
Poitiers
732
IBERIA
711
929 Caliphate
Cordoba
Atlantic Ocean
Hometowns and places of main of islamic mathematicians
other towns
622 / 7th : Beginning - or period of time - of islamisation
732 abortive attemps/attacks

### 3.4.0 Historical information

Mathematics in the Islamic countries, henceforth mostly referred to as Arabic mathematics, is based on three pillars: mathematics of old Mesopotamia, contemporary Indian mathematics, and, in particular, Greek mathematics.

It is not necessarily wrong to stress that, above all, Arabic mathematics brought the decimal position system and Greek mathematics (working with proofs) to the occidental world and, therefore, the entire present civilisation. However, this is a very narrow way of looking at the matter, since it relies completely on the contemporary state of development as its benchmark. This should not be the guideline for a historical representation, the duty of which is primarily to draw as objective a picture as possible regarding development of contemporary civilisation and culture (in our case, of mathematics and, particularly, geometry). Thereby, it should attempt to carve out the mutual influences and characterise the role that these activities played in that development. Looking at the entire picture, the relations to religion fulfilled a relatively subordinate function. The majority of investigations belonged to the field of the so-called pure mathematics.

The prophet Mohammed from Mecca had his first visions around 610. His epiphanies are noted in the Koran. He resettled in Medina with his followers in 622, where more and more Bedouin tribes joined him. Soon, the religion he was preaching had found a new home across the entire Arabic peninsula. Three years after Mohammed died, the Arabs conquered Damascus and proclaimed it home of the Caliph. Two years later, they subdued Persia, reached Kabul in 664, and Bukhara and Samarkand in 674. Bagdad was re-founded as the base of the Caliph of the Abbasids in 762/63 (140 years after Mohammed had fled from Mecca to Medina, also known as the "hijra", which was the beginning of Islamic calculation of time). Bagdad fast grew into an influential cultural centre. Particularly, the Caliphs al-Manṣūr (ruler from 754 until 775), Harūn ar-Rashīd (786–809) and al-Ma'mūn (813–833) took a great interest in science. The latter caliph founded the House of Wisdom in Bagdad, which was equipped with an elaborate library and an observatory.

As part of the expansion to the West, the Arabic army had already conquered the greatest part of the Iberian Peninsula before Bagdad had been re-founded. Having subdued North Africa at Gibraltar (Jabal Tariq, "mountain of Tariq") in 711, commander Tariq stepped on European ground for the first time. The emirate of Cordóba was founded here in 756. Islamic culture blossomed here, especially in the $10^{th}$ century after the city had grown to be the second largest in Europe (after Byzantium), with paved streets that were illuminated at night.

Yet, the difference between the cultures of the eastern and western parts was extensive. The influences of the three named cultural circles crossed in Bagdad. Even some degree of domestic Syrian and Persian scholarship could be continued. In contrast, culture, arts and science from the eastern parts of the country only gradually penetrated the West Arabic provinces in

North Africa and Spain. Hereby, they were united not just by religion, but also by the Arabic language. The Koran was only allowed to be read and cited in Arabic, which also immediately became the language spoken at the chancellery and administration, as well as the language of science.

The most important sources, according to our knowledge, are Arabic texts, which were copied between the $10^{th}$ and the $19^{th}$ century, often by writers who did not know much of mathematics. Non-mathematical texts (astronomy, optics, law, linguistics, and many other subjects) can also contain excerpts that belong to the realm of mathematics nowadays. Even though exploring Arabic mathematics started with the beginning of the $19^{th}$ century, we have by no means inspected all manuscripts and examined their meaning. The most significant collections of manuscripts are to be found at libraries in the Middle East, Europe, India and North Africa [Berggren 1986], [Rashed 1996].

### 3.4.1 Translation work

First of all, avid translation work into Arabic started in the East. Apart from some Indian works, all Greek classics have been accessible in good Arabic versions since 900. It was inevitable that the Islamic mathematicians would deal with this legacy in countless commentaries. In a second phase, which overlapped the first translation period, critical points and open questions triggered their own continuing investigations.

However, from the $11^{th}$ century onwards, Spain became the best place, apart from Sicily, where the small number of Western European scholars who were so interested could encounter Islamic culture and science. As a result, Spain became the important bridge that linked Greek and Oriental knowledge with Western Europe. In Toledo, and to a lesser degree in other cities of the Iberian Peninsula, significant schools of translators developed. Scientific works available in Arabic were mainly translated here into Latin, the language of scholars in the Middle Ages. Afterwards, they were copied and circulated in Europe. (However, we must not overlook the cultural differences between East and West. Many significant works of mathematicians working in Persia, Egypt and other countries of the Middle East were not known in the West back then, and, consequently, had no effect here. We owe our knowledge of those texts to modern historical research.)

This development went hand in hand with a second strand of communication, which took its course from Byzantium via Italy. We will examine this aspect in the subsequent chapter on the Middle Ages.

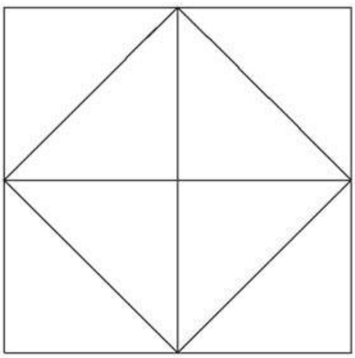
In the following passages, we will introduce key aspects and examples of independent investigations by Islamic mathematicians ordered according to the branches of theoretical geometry, applied geometry, and trigonometry. In this method, it is not always possible to avoid boundary crossings.

### 3.4.2 Theoretical geometry

Within the field of theoretical geometry, Arabic mathematicians had already started focussing on Euclid's 'Elements' around 820. Euclid's masterpiece was translated into Arabic multiple times, was repeatedly commented on, and represented the main foundation for further research of Arabic mathematicians. It seems that they began to show interest in mathematical questions for the sake of the subject itself very early, since applied science took on a rather subordinate role. As key aspects of theoretical geometry, which could be extended in any direction, we will introduce exemplarily the construction of regular polygons, circle calculations (as topics within the realm of the so-called three classical problems, whereby conic sections play a role, too) and their theory of parallels.

Although not in an independent work of its own, al-Khwārizmī, who worked at Caliph al-Ma'mūn's court in Bagdad, also dealt with geometrical problems in a section of his groundbreaking work in algebra at the beginning of Arabic mathematics. Therein, he partially followed up on Hero (including some of the numerical values, which occur in some problems). His selection of subject matter all in all corresponds to the content of the first two books of Euclid's 'Elements'. An illustrative proof (possibly inspired by an Indian source; see Illus. 3.4.1 and Problem 3.4.1) of Pythagoras's theorem concerning the case of isosceles, right-angled triangles is worth mentioning.

In the middle of the $9^{th}$ century, the Banu (brothers) Musa wrote their own work on geometry. These three brothers developed a vivid science of mechanics, astronomy, mathematics and the building of musical instruments in Bagdad. Their work, preserved in Latin translation by Gerard of Cremona, is known as "*Liber trium fratrum de geometria*". The authors had named it more distinctively 'Book on the Measurement of Plane and Spherical Figures'. (In the form of an edition by at-Tusi, the content has also been passed on in Arabic.) The three brothers added proofs to all propositions. Circle,



**Illus. 3.4.1** al-Khwārizmī's illustrative proof of Pythagoras's theorem for the isosceles, right-angled triangle

**Illus. 3.4.2** Brick ornaments at the Iwan of the Friday Mosque of Nain, Iran. An Iwan (also Eiwan or Liwan) is the overarching porch of a prayer room, which is opened towards a yard with its narrow side
[Photo: H.-W. Alten]

sphere and cone formed the most important objects of their investigation. Hence, they did not bow to the restrictions marked by Euclid's 'Elements'. They were the first within Arabic literature to describe the ancient Greek method of exhaustion. Their work was studied in the oriental world for centuries and also impacted European mathematicians in its Latin version.

### Construction of regular polygons

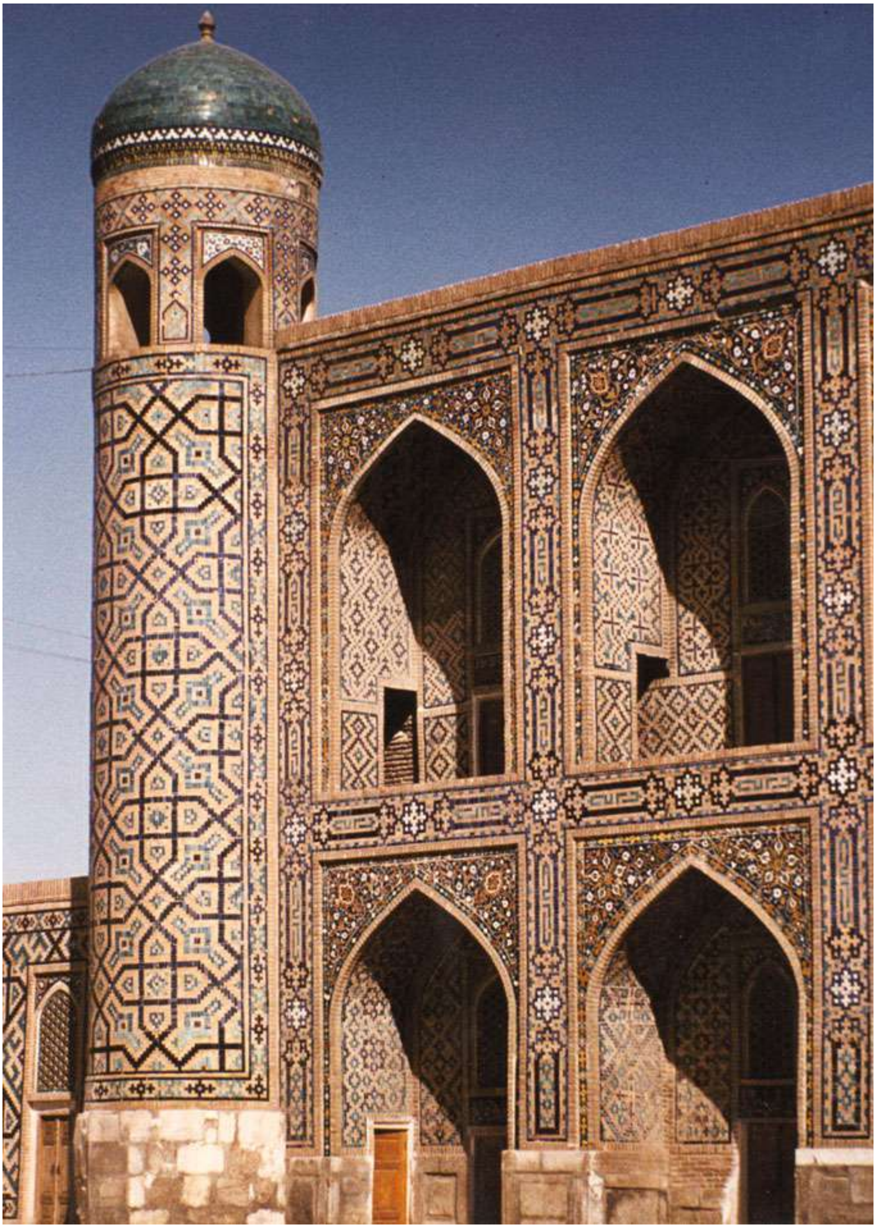
Beyond the known elementarily constructed shapes (triangle, square, pentagon, polygon with 15 corners and the respective polygons as the result of repeatedly doubling the sides), the question of constructing regular polygons was popular amongst geometers. The regular heptagon and the regular nonagon must have been the first ones to awaken interest. The construction of the latter is connected to the problem of angle trisection; the construction of the first leads to a cubic formula. Thus, both are on the same level of difficulty algebraically speaking, and suggest the same choice of path that the Greek mathematicians had already taken, namely to find solutions based on conic sections. Nonetheless, the Islamic mathematicians also conceived of brilliant neusis constructions [Hogendijk 1984], [Scriba 1985a].

Abū'l-Wafā was one of the scholars who took particular pleasure in addressing regular polygons. His versatile work was done in Bagdad in the $10^{th}$ century and he is author of a treatise called 'Book on What Is Necessary from Geometric Constructions for the Artisan'. Therein, he described many wonderful constructions of regular polygons up until the decagon. There were some constructions amongst those (except for the heptagon and the nonagon) that could be constructed by compass and straight edge. The proposal for the square is demonstrated in Problem 3.4.2.

He suggested a very simple approximation construction for the heptagon: take half of a side of an isosceles triangle inscribed in a circle to be the side (see Problem 3.4.3). Hero had already taught the same approach, whereas the neusis construction (though exact, not constructible by means of elementary geometry!) stated by Abū'l-Wafā for the regular heptagon came from Archimedes [Berggren 1986], [Gericke 1984], [Juschkewitsch 1964].

Another heptagon construction based on the construction of two conic section curves was developed by al-Kūhī at the end of the $10^{th}$ century. Imagine a heptagon inscribed in a circle. Name one corner $A$, the corner after the next one $B$ and the following one $C$ so that a triangle $ABC$ can be formed by means of a short chord ($AB$), a side ($BC$) and a long chord ($CA$) of the heptagon. Now imagine the triangle (heptagon) side $BC$ extended to the left by the short diagonal $AB$ and extended to the right by the long diagonal $CA$ and the extremities $E$ (left) and $D$ (right), determined this way, connected with $A$. Then we can show that the following applies to the line segment $EBCD$:

$$EB^2 = BC \cdot BD \quad and \quad CD^2 = EB \cdot EC \tag{3.4.1}$$



**Illus. 3.4.3** Madrasah (religious school) Tillya-Kari at Registan in Samarkand, Uzbekistan
[Photo: H.-W. Alten]

Thus, it is up to us to divide a given line segment $ED$ by two points $B$ and $C$, whilst fulfilling both equations (which were already known by Archimedes). By basing them on the construction of a parabola and a hyperbola and determining their intersections, al-Kūhī could construct the triangle $ABC$ by means of the required line segment division, construct a circle around this and draw further heptagon sides therein (see Problem 3.3.4).

The wealth of Arabic ornaments is especially well known. Whereas some indicate a high level of symmetry as a whole, others often demonstrate the artistic linking of regular polygons of different types. The delivered Islamic scholarly discussions concerning this topic have not yet been investigated extensively. Apart from Abū'l Wafā's often described text, we only know of a study written by the mathematician al-Kāshī around 1415, which addresses three-dimensional patterns shaped like honeycombs, called muqarnas.

Since we otherwise do not have any sources pointing at the deliberate application of geometrical knowledge concerning architecture or mosaic constructions (even though we can prove today that the latter reflected its applications in the diversity of the plane symmetric groups discovered in the $19^{th}$ century), it is justified to assume that we are dealing with empirically acquired insights, which were passed on and fine-tuned from generation to generation within the scope of artisan tradition (cf. Illus. 3.4.2 – 3.4.6).

### Circle calculation

As with all cultures we have looked at so far, Islamic countries could not resist dealing with the shape of the circle. The constructions found in Euclid's 'Elements' and relating to the circle were studied and attempts were made to develop further or apply them to additional questions. Thereby, the Islamic mathematicians tried to determine constant $\pi$ more precisely, showing very clearly that they followed up on both Greek and Indian mathematics. Concerning circle calculation, al-Khwārizmī already stated: $C = d \cdot (3 + \frac{1}{7})$ and $C = d \cdot \sqrt{10}$ or $C = d \cdot \frac{62832}{20000}$. The first value is Archimedean, the third one is said to have been used by astronomers. al-Khwārizmī could have taken the third, just as he did second, from Indian texts [Gericke 1984], [Juschkewitsch 1964].

To determine the area $A$ of a circle by means of diameter $d$, he stated that

$$A = d^2 - \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{7} d^2 \tag{3.4.2}$$

This formula comes very close to one stated by Hero (cf. Problem 3.4.5). Al-Khwārizmī's rule for calculating area $S$ of a circle segment by means of arc $b$, chord $c$ and the height of the segment (sagitta) $s$ is also peculiar. First, he determined the diameter $d$ as $d = \frac{c^2}{4s} + s$, then he applied a double rule:

For a segment that is smaller than a semi-circle, calculate according to the formula

**Illus. 3.4.4** Floral and geometrical Ornament in a tile of the Friday-Mosque of Yazd, Iran
[Photo: H.-W. Alten]

$$S = \frac{d}{2} \cdot \frac{b}{2} - (\frac{d}{2} - s)\frac{c}{2}. \tag{3.4.3}$$

For a segment that is greater than a semi-circle, calculate according to the formula

$$S = \frac{d}{2} \cdot \frac{b}{2} + (s - \frac{d}{2})\frac{c}{2}. \tag{3.4.4}$$

Since the terms he used are of Indian origin, al-Khwārizmī probably learned of these rules from an Indian text [Juschkewitsch 1964] (see Problem 3.4.6).

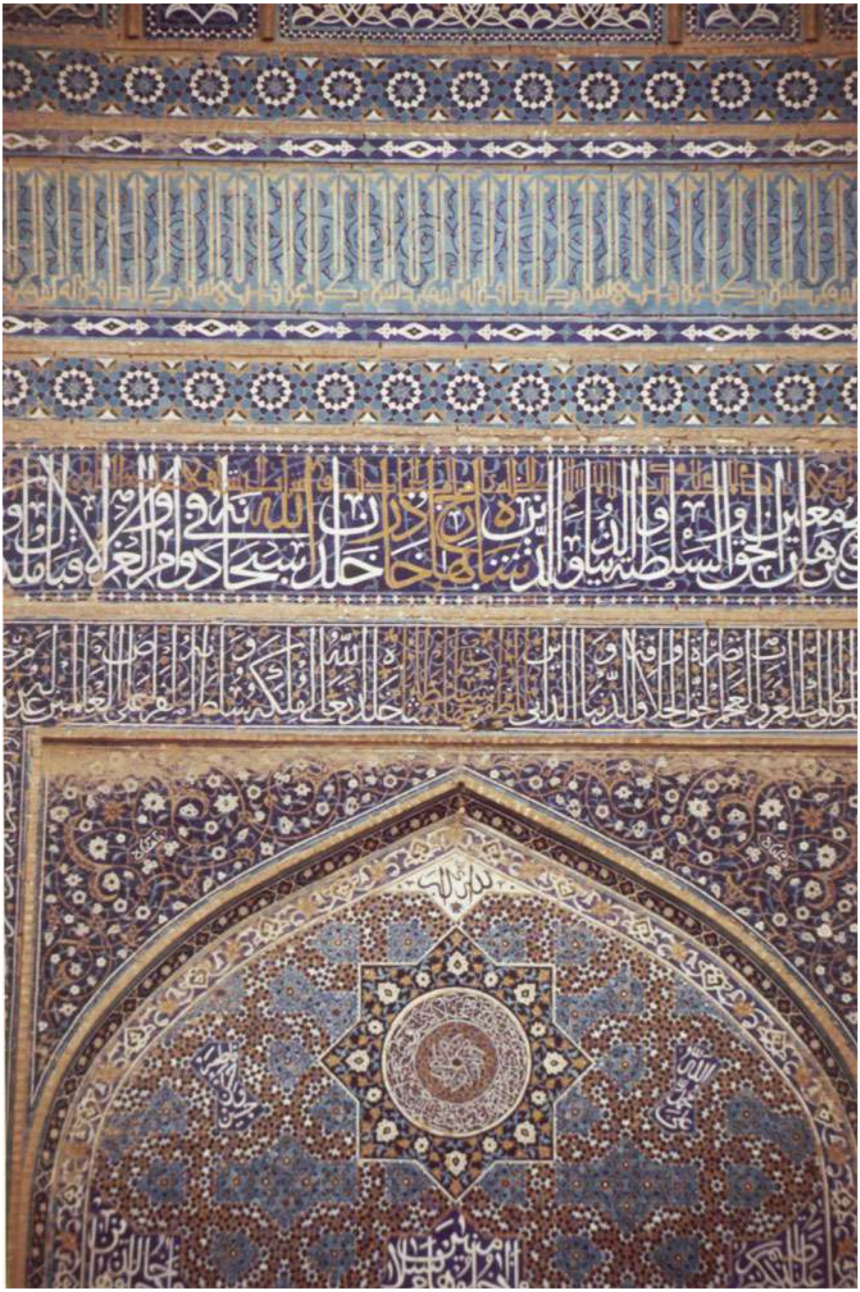
A special accomplishment of Arabic mathematics is the calculation of constant $\pi$ in al-Kāshī's 'Treatise on the circumference', which he concluded in 1424. Al-Kāshī had set himself the ambitious aim to calculate $\pi$ so exactly that the error concerning a circumference, the diameter of which is 600 000 Earth diameters, would not exceed a hair's breadth. He contemplated using a regular quadrilateral, the side of which fulfils the inequation $a < \frac{8}{60^4}$ for a circle with a radius of 60. His conclusion was to use an inscribed polygon of $3 \cdot 2^{28} = 805\,306\,368$ sides, or, respectively, a $1, 2, 8, 16, 12, 48$-gon in the sexagesimal system. Al-Kāshī conducted his calculation so cleverly that his result was exact to ten sexagesimals or 17 decimals. Not only did he state his result in both systems, it also was the first time decimal fractions were used in Islamic mathematics. In decimals, he found

$$2\pi = 6.283\,185\,307\,179\,586\,5.$$

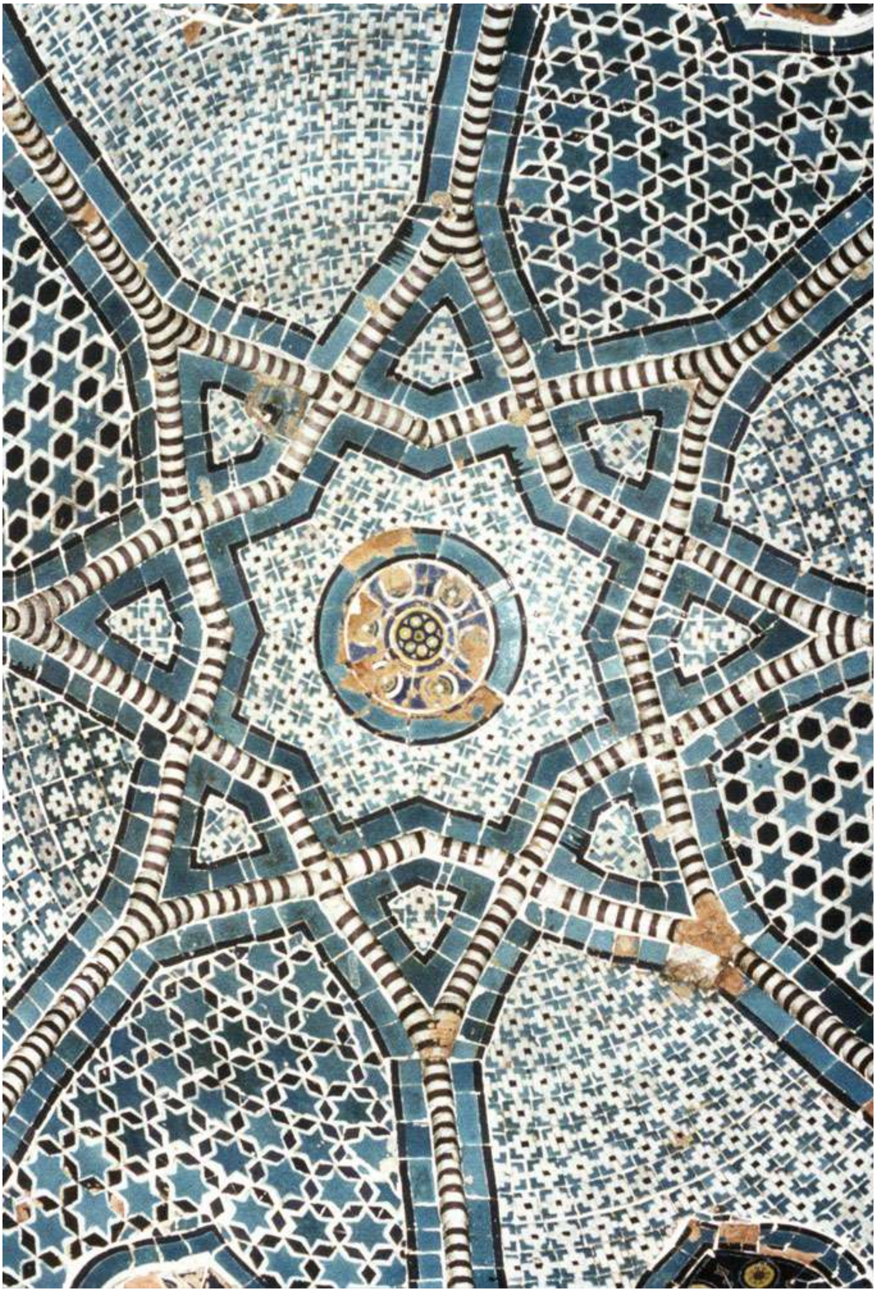
This calculation was only reproduced around 1600 by Adriaan van Roomen by means of the 230-gon. Just a little later, he was superseded by Ludolph van Ceulen, who first calculated 20 and then 32 decimals by means of a $60 \cdot 2^{29}$-gon. This is why $\pi$ is also sometimes called the 'Ludolphian number'.

### The parallel postulate

A topic particularly discussed by the Muslims was Euclid's parallel postulate, which had already given rise to debates back in ancient times. (Around 520, Simplicius had written about a supposed proof of a certain 'Aganiz' (Geminos?) in his commentary on Euclid, which has been lost. The foundation is said to have been the definition that "parallel straight lines are those of fixed distance".) Research regarding the parallel problem may be the example that points most to the future within the scope of pure geometrical studies in Islamic mathematics. Around 830, al-Jawhari followed up on the mentioned definition. One generation later, Ṯābit ibn Qurra referred to this before he consulted the notion of motion, which Euclid had avoided, as part of a second investigation. Around 900, al-Nayrīzī spread knowledge of the parallel postulate in an extensive commentary on the first ten books of Euclid's 'Elements'. He proved the parallel axiom based on the definition that parallel straight lines are lines which remain equidistant no matter how far they are extended. The versatile Arabic mathematician, astronomer, physicist and physician ibn al-Haytham, who was familiar with his predecessors' works, also worked with



**Illus. 3.4.5** Geometrical ornaments and friezes in cufic script at the Iwan of the Friday Mosque in Yazd, Iran
[Photo: H.-W. Alten]



**Illus. 3.4.6** Dome decoration of a Mausoleum in Shah-i Zinda, Uzbekistan [Photo: H.-W. Alten]

**Illus. 3.4.7** Sextant in the mural quadrant of the Ulugh Beg Observatory, Samarkand, Uzbekistan
[Photo: H.-W. Alten]

**Illus. 3.4.8** Ulugh Beg Observatory, Samarkand, Uzbekistan
[Photo: H.-W. Alten]

motions. He used a quadrilateral with three right angles (a Lambert quadrilateral; cf. Illus. 6.4.2) and proved that the fourth angle could neither be acute nor obtuse. (Thereby, he silently used the Archimedean axiom and Pasch's axiom.) Around 100 years later, the mathematician and poet Umar (or Omar) al-Khayyām worked with the Saccheri quadrilateral, which features two right and two equal angles (cf. Illus. 6.4.1), although he criticised the introduction of motions in geometrical proofs. In the $13^{th}$ century, Naṣīr ad-Dīn aṭ-Tūsī gave his opinion on the parallel problem by means of two Euclid editions initiated by him. Besides, he adopted large excerpts from his predecessors' works when writing his own text on this topic.

The investigations initiated by Islamic mathematicians between the $9^{th}$ and $15^{th}$ centuries all had something in common, namely that they excluded the non-Euclidean possibilities by means of circular reasoning and illusive proofs. Thus, they all reached the conclusion that the parallel postulate could be proven. As mentioned, they thereby encountered (partially implicit without the researchers being aware of it) very important axioms, e.g., the one by Archimedes/Eudoxus and the Pasch configuration [Gericke 1984, p. 204-214].

European mathematicians did not find out about all Arabic texts in time. However, it is certain that al-Nayrīzī's achievements concerning the parallel problem were studied by the influential Jesuit mathematician Christoph Clavius around 1600. Naṣīr al-Dīn aṭ-Tūsī's accomplishments were studied and further developed by John Wallis around 1650.

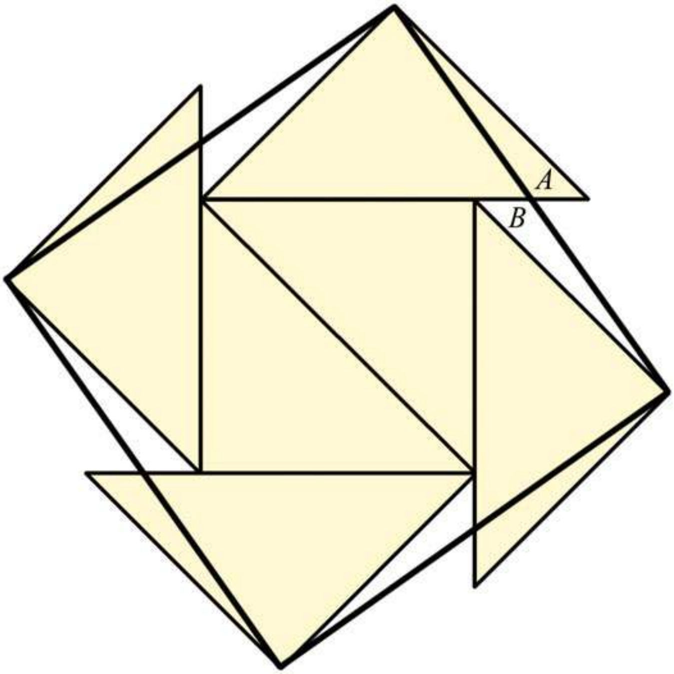
In relation to basic matters, geometry was also consulted in Arabia to account for algebraic operations. For instance, Ṯhābit ibn Qurra, based on Book II of Euclid's 'Elements', stated geometrical proofs for al-Khwārizmī's rules to solve quadratic equations. The efforts to solve cubic equations by means of conic sections starting in the $10^{th}$ century also belong within this context. An investigation by al-Khayyām gained much attention, but dealt insufficiently with the necessary and sufficient conditions of the existence of solutions. As the history of mathematics showed a quarter of a century ago, this was only accomplished in the $12^{th}$ century by Sharaf ad-Dīn aṭ-Tūsī (not to be confused with Naṣīr ad-Dīn aṭ-Tūsī) [Hogendijk 1989]. Other mathematicians made an effort to state a geometrical justification of the proportionality proposed by Euclid in his fifth book.

### 3.4.3 Practical geometry

Following the investigations within the realm of theoretical geometry, Islamic mathematicians also engaged with multiple questions from the area of applied geometry. Of course, geometry continued to form the basis for surveying. The geometrical chapter of al-Khwārizmī's algebra, mentioned above, bears some resemblance to the old Hebrew treatise *Mishnat ha-Middot* (Treatise on Measures). Its origins lie between the middle of the $2^{nd}$ century and the $9^{th}$ century. The unknown author had based his work on Hero and may have been familiar with Euclid's 'Elements'. It is possible that al-Khwārizmī had taken his problems from this Hebrew work. Another possibility is that both authors had access to the same sources, which have not been preserved.

The already cited work 'Book on What Is Necessary from Geometric Constructions for the Artisan' by Abū'l-Wafā from the $10^{th}$ century systematically describes constructions with a fixed span of the compass. Accordingly, they were already popular in Arabic crafts; we also find them in the European "Bauhütten" of the Middle Ages. Their practical advantage lies within the fact that they avoid errors, which are inevitable when using compasses instead of exact circle models and particularly with respected alteration in the span of the compass. (This may have also been a motive for geometers like Leonardo da Vinci, Tartaglia, Cardano, and later for Mascheroni and Steiner, who addressed such constructions from a theoretical point of view; more details in section 7.3.)

Having dealt with regular polygons constructible by means of elementary geometry in the second and third chapter, Abū'l-Wafā turned towards the problem of deconstructing a spherical surface into regular spherical polygons in the last chapter of his book. This is synonymous with the construction of regular polyhedra, which the author, however, did not mention. Therein, not only did he determine the five platonic solids, but also two of the 13 semi-regular solids found by Archimedes. Given the Arabs' fondness for geometrical ornaments (due to the religious prohibition of image representations in the mosques), we could assume that his intention was to provide craftsmen with instructions to design surfaces of curvature. At the beginning of




**Illus. 3.4.9** Abū'l-Wafā's highly illustrative solution to the problem of constructing a square of an area three times as big as the area of a given square. We only have to prove that area $A$ is congruent and, hence, of equal area to area $B$.

the $15^{th}$ century, in the chapter on measures in his book titled 'The Key to Arithmetic', al-Kāshī also included complicated calculations and constructions for pointed arches, vaults, domes and, lastly, the so-called stalactites, which were very characteristic of Arabic architecture.

The successful continuation of the methods adopted from the Greeks and, to a lesser degree, the Indians characterised the applied geometry of Islamic mathematics. Furthermore, it distinguished itself by extending Archimedes' approaches to systematically studying areas and solids of curvature, which led to infinitesimal observations. It translates geometrical considerations into the language of algebra and conducts calculations (including irrational quantities) by means of well-considered methods with high accuracy. However, the methods based on similarity observations with triangles and common during the Antiquity (and the European Middle Ages) were pushed into the background in favour of trigonometric methods.

### 3.4.4 Trigonometry

As known, astronomy, geography and geodesy had to fulfil important religious tasks in the Islamic world: the calendar, depending on the course of the moon, had to be forecasted by the astronomers, for which they had to know the date of the first visibility of the crescent after new moon. The time of the five daily prayers depended on the position of the sun and, thus, also depended on the respective geographical coordinates. Moreover, it was necessary to determine exactly the direction of the prayer to Mecca, known as the kibla or qibla, for each inhabited place. The direction was further indicated by the sundials and in every mosque.

By 800, in Bagdad already Alexandrian and Indian works on astronomy were known, which addressed trigonometric methods [Sesiano 1993]. Hence, there was an opportunity to compare the appropriate works by Hipparchus, Ptolemy and Menelaus with the semi-chord trigonometry developed in India [since the $6^{th}$ century (?)]. At the beginning of Arabic trigonometry, we find al-Khwārizmī again, who composed a sine table including explanations. The Greek chord trigonometry was increasingly pushed aside by sine trigonometry. The Muslims extended the two trigonometric basic functions put forward by the Indians, sine and cosine, to six such functions. Tangent and cotangent were developed first when studying the shadow cast by sundials: tangent as a relation of the shadow length of a standardised pole mounted horizontally to a wall, cotangent as the shadow of a gnomon (a vertical pole on horizontal ground). Since the late $10^{th}$ century, the possibility was known for using the circle radius as a unit length so that all functions can be viewed as ratios of line segments. These were soon also used for other problems. In addition, there were secant and cosecant (the ratios of hypotenuse to the adjacent or opposite side in the right-angled triangle). Tables had to be calculated for all these functions and the relations between them had to be investigated. For instance, al-Habashis said to have configured tables for some of the new basic functions in the $9^{th}$ century. Almost all astronomers in the Islamic area composed astronomical, trigonometric handbooks, known as 'ziges'. Mathematicians and astronomers spent a lot of time over the centuries working towards such calculations, including the improvement of the required methods (next to interpolation of first degree, there are also examples of second degree). Thereby, they maintained – just as Ptolemy had done – the sexagesimal system, which goes back to the Babylonians and the division of the circle into 360°. In Abū'l-Wafā's work 'zig almagisti' from the $10^{th}$ century, we can, for example, find the following wording for the addition and subtraction theorems regarding the sine function [Berggren 1986, p. 136]:

> "Calculation of the sine of the sum of two arcs and the sine of their difference when each of them is known. Multiply the sine of each of them by the cosine of the other, expressed in sixtieths, and we add the two products if we want the sine of the sum of the two arcs, but take the difference if we want the sine of their difference."

**Illus. 3.4.10** Armillar-Sphere, displayed in the first printed edition by Hajji Kalifahs Jihan Numa (mirror of the world)
[ARAMCO World, vol. 43, no. 3]

When proving this theorem, Abū'l-Wafā consulted a proposition of Book III of Euclid's 'Elements'.

**Illus. 3.4.11** Muslim astronomers with instruments, $16^{th}$ century [ARAMCO World, vol. 43, no. 3]

The Islamic mathematicians also introduced the trigonometric functions in spherics. For instance, the law of sines for spherical triangles was known in the $10^{th}$ century: the ratio of the sines of two sides equals the ratio of the sines of their opposite angles in a spherical triangle.

Naṣīr al-Dīn aṭ-Ṭūsī, whom A. P. Juschkewitsch [Juschkewitsch 1964, p. 304] called the most significant oriental scholar within the realm of trigonometry and for whom Mongol ruler Hūlāgū Khan had built an observatory in Maragha in Persia, systematically examined the application of the law of sines for all possible cases of plane triangles:

$$\frac{c}{sin\ \gamma} = \frac{b}{sin\ \beta} = \frac{a}{sin\ \alpha}. \tag{3.4.5}$$

He composed the first independent treatise on trigonometry: 'Book on the complete quadrilateral'. Menelaus's theorem, which refers to this figure, had already been consulted for triangle calculations by Islamic astronomers very early on. Aṭ-Ṭūsī dealt with spherical triangles with and without this theorem, whereas the law of sines was already known by his predecessors. Muslim astronomy and trigonometry reached their climax in the $15^{th}$ century at the famous, excellently equipped Ulugh Beg Observatory in Samarkand (see Illus. 3.4.7, 3.4.8). The ingenious al-Kāshī worked there, utilising a clever iteration method in order to calculate the sine of 1° with great accuracy by means of the equation for angle trisection. Basically, he proceeded in the following manner:

Since we can determine sin 3° as exactly as we wish (we can construct it by means of compass and straightedge using the difference of 36° at a pentagon and 30° at a hexagon), he applied the equation for angle trisection:

$$sin\ 3\alpha = 3sin\ \alpha - 4sin^3\ \alpha. \tag{3.4.6}$$

(We encounter this formula for the first time in this exact wording in Vieta's work at the end of the $16^{th}$ century.) It is of type $x^3 + q = px$. (The former classification presumed coefficients to be positive – here: $p = \frac{3}{4}, q = \frac{1}{4} sin\ 3°$.) Al-Kāshī calculated the first approximation to be $x_1 = \frac{q}{p}$ by means of $x = \frac{q+x^3}{p} \approx \frac{q}{p}$. This leads to the second approximation $x_2 = \frac{q+{x_1}^3}{p}$, etc. This, in turn, indicates the special feature of the ability to obtain a further exact sexagesimal with every step (cf. Problem 3.4.7). Converted into the decimal system, al-Kāshī's result delivers 18 decimals:

$$sin 1° = 0.017\,452\,406\,437\,283\,571. \tag{3.4.7}$$

This selection of geometry of Islamic mathematics is not narrowed down due only to the limited space of this book. A great number of unread Arabic manuscripts are located at oriental libraries, which is why researchers have not yet been able to gain a more complete picture of the development and accomplished knowledge. The future, here, may well hold some surprises.

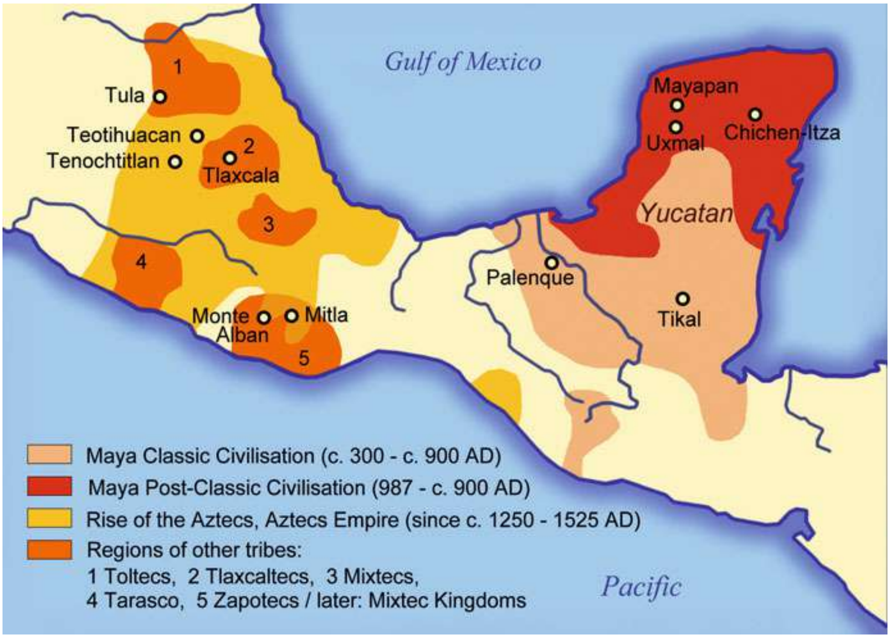
**Illus. 3.4.12** This ornament (Iran) is of special interest, because the principle of generating starpolygons by turning a crossbar may be generalized
[Photo: P. Schreiber]

**Essential contents of Islamic geometry**

| | |
|---|---|
| $8^{th}$–$10^{th}$ century | Translations of Greek, Persian and Indian works) especially in Bagdad) – before, no independent mathematical culture |
| Middle of $9^{th}$ – middle of $11^{th}$ | Independent further development of mathematics following Euclid and Hero |
| From $9^{th}$ century | Geometrical solutions of cubic equations (by means of conic sections and other) |
| $2^{nd}$ half of $9^{th}$ century | Banu Musa (three brothers): book on geometry: circle calculations, surfaces and volumes of solids, angle trisection with neusis; string construction of ellipse; book on conic sections |
| From $9^{th}$ century | Following up on Indians and Greeks, further development of trigonometry more strongly towards applied methods |
| Before 983 | al-Kūhī: construction of regular heptagon |
| $13^{th}$ century | Aṭ-Tūsī: first independent work on plane and spherical trigonometry |
| 1424 | Al-Kāshī: calculation of $\pi$ to 10 sexagesimals and 18 decimals by means of a regular polygon of $3 \cdot 2^{28} = 805\,306\,368$ sides |

## 3.5 Old American cultures

| | | |
|---|---|---|
| From | **America** | |
| 40000 | Populated by Asia via Bering Strait in multiple batches | |
| From | **North America** | *Area* |
| 9000 | Hunters of field prey and small animals | "Great Basin" |
| 8000 | Big game hunters | East and planes |
| | Gatherers of wild fruits and plants | West coast |
| 5000 | First settlements | East and planes |
| 3000 | First plant cultivations | Southwest |
| 2500 | First ceramics and farming | East and planes |
| 1500 | Permanent settlements | East |
| 600 | Barrows | East |
| 100 | First villages, ceramics | Pueblo area of southwest North America (Mogollon culture) |
| 0 | Basket makers | Southwest |
| 500 | Towns in area of pueblo | Southwest |
| 1000 | Temple and hill towns | Middle Mississippi and Southeast |
| From | **Middle America** | *Area* |
| 9000 | Mammoth hunters | Central Mexico |
| 3000 | First plantations | Northeast Mexico |
| 2500 | Plantations with settlements, ceramics | Central Mexico (Tehuacan sequence) |
| 1500 | First temple hills, Olmecs | Central Mexico (La Venta culture, 500–100) |
| 200 | First towns | Teotihuacán, Monte Albán |
| 0 | Teotihuacán-Toltecs: 'classical time' | Central Mexico |
| 0 | 'Classical' Maya | Guatemala, Southeast Mexico |
| 1000 | Aztecs and other tribes | Central Mexico |
| 1200 | Toltec Maya | Yucatán |
| 1525 | End of Aztec Empire | Central Mexico |
| 1546 | End of Maya Empire | Yucatán |
| From | **South America** | *Area* |
| 7000 | Hunters | Patagonia |
| 4000 | Hunters | Peru, Venezuela, Argentina |
| 3500 | First plant cultivations | Central Andes |
| 3200 | Oldest ceramics findings | Ecuador, Columbia |
| 2500 | Village settlements | Central Andes (Chillón sequence) |
| 2000 | Early fishers/planters | |
| 1000 | Begin of ceramics; Chavin culture | North coast and South coast |
| 0 | 'Classical time' | |
| 200 | First towns | Southern highlands |
| 300 | Moche civilisation | North coast |
| 400 | Nazca | South coast |
| 450 | Tiwanaku | South Andes |
| 1000 | Tiwanaku expansion | Southern highlands |
| 1200 | Inca Empire | Andes (Columbia until Chile) |
| 1500 | Chimú | Coast states |
| 1532 | End of Inca Empire | Andes |




**Illus. 3.5.0** Native Indian advanced cultures in Middle America
[Map: H. Wesemüller-Kock]

### 3.5.0 Historical introduction

In contrast to the preceding sections of this chapter, when looking at the ancient American (Indian) cultures, we are dealing with a great number of different developments. The strictly organised Mesoamerican cultures of the Nahuan and Mayan people are best known. The Aztecs and Mayans lived in central and southern Mexico and the adjacent parts of Guatemala, Belize, El Salvador and Honduras. At the time of its greatest expansion around 1500, the Incan Empire covered the area from the northern border of Ecuador until Rio Maule in Middle Chile. The ruins of towns and other relics of these large-scale cultures, which were characterised by intensive building activity, have served as rich sources for archaeological research for a long time. Spain came into contact with these advanced ancient American civilisations when discovering the 'New World' in the $16^{th}$ century. Our knowledge of the less settled Native Indians and Eskimos (Inuit) on the North American continent and the inhabitants of the tropical lowland in South America is much more incomplete. Whereas the matter of mathematical ideas and concepts of the early American inhabitants was just one amongst many for ethnological research, the so-called ethnomathematical research that grew strongly over the last decades has delivered extensive results, which are incomplete, but allow for an indicative description. Again, we can only illustrate a few examples here, which belong to the field of geometry.

Ethnomathematics is dedicated to the examination of opinions and concepts of the cultures of the native peoples, which are associated with numbers, logic and spatial perception, as well as the organisation of such into systems and structures. Thereby, we are by no means dealing with notions that can be identified as explicitly mathematical, but with cultural expressions connected to subject matter, which manifests itself as implicitly mathematical. They can also occur in a context that can well differ from the traditional western understanding. Special linguistic properties can also be consulted as sources for ethnomathematical research as well as artefacts or behaviours (exhibited in, for example, play, dance, religious rituals or craftsmanship). All in all, we extract from these cultures what we believe to be mathematical – or, in the context of this book, geometrical – with the eye of a European. Ethnomathematics teaches us to view mathematics as an abstract entity, the cultural background of which can be incredibly versatile.

The cultural developmental state of the ancient American inhabitants, which we will look at in this section, varies greatly. It is a more complicated undertaking, since early conquerors, tradesmen and missionaries, although often noting down the numerical systems that they saw used by the natives (an everyday life necessity for the conquerors as much as the natives), reported very little on geometrically interesting details. This, for instance, is reflected by the miscellany [Closs 1986] of thirteen articles in Native American Mathematics, ten more or less deal exclusively with numerical terms and symbols, and their composition and application. Another article already pinpoints the problem at hand with its title, *In Search of Mesoamerican Geometry.*

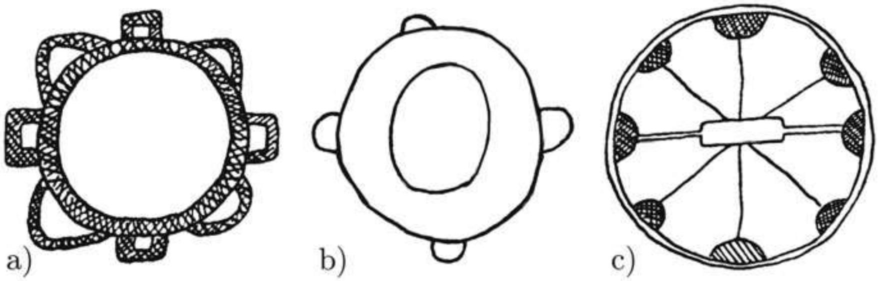
We must also consider a further difference: Settled people are generally able to erect great buildings and to decorate items of household and culture with geometrical patterns. Itinerant hunting and gathering people have few possessions and naturally rarely leave permanent traces behind. So what do we know nowadays of geometrically inspired games or dances or other temporary cultural expressions of the early inhabitants of the American double continent? We can only present here a small collection of examples of early American cultures, which remind us of geometrical thinking and acting in its broadest sense.

### 3.5.1 Hunting civilizations Inuit (Eskimo) and Ojibwa

The Ojibwa belong to the group of Asian immigrants who arrived in the area approx. 40 000 years ago and are related to many other Algonquin tribes. They hunted in the forests of North Ontario, around Lake Huron and Lake Superior. In contrast, the Inuit arrived from Asia only approx. 6000 years ago. They hunted in the open tundra, in the Arctic Polar Regions and the adjacent islands. Despite their differences, their mathematical concepts are very similar to each other and, thus, are seen as representative of hunting and gathering tribes, whose members itinerated and fed on wild plants and animals.

In contrast, settled people adapt their environment by means of agriculture, growing new types of plants and domesticating animals, which were not just eaten, but also used for labour. This leads to an additional specialisation that hunting tribes had no use for, which is the need for mathematics in its broadest sense: to conceive of numerical systems. Furthermore, ethnomathematicians have also shown that hunters required much less time to ensure their survival than farmers. As a result, they had more time for and were more willing to engage with recreational and leisure activities, e.g., spinning an extensive web of myths.

When discussing the Ojibwa and Inuit we will focus on examples of linguistic instances, since we can read off the existence of concepts thereof that we would accept as belonging to the area of geometry. It seems that the Ojibwa had categories that were fixed by prototypical elements, the boundaries of which were blurred. For example, the circle is a central or prototypical element of a category that different objects fall into: more or less round and two or three dimensional, such as the cross section of a tree, an egg or a potato (Illus. 3.5.1). Another category was 'longish'. Moreover, objects can also belong to two categories, e.g., a longish potato, which is also seen as round. Compared to pure geometrical concepts, the perception of dimension is just as blurry. A stick, an animal skin and an apple represent objects that, respectively, cover one, two and three dimensions. In contrast, all three items are three dimensional according to our concept of geometry.




**Illus. 3.5.1** Ojibwa representation of the worldly orb. Pictures of Earth with 'four tails' can be found in many Native Indian tribes of North America and reached their climax with the Ojibwa of Lake Superior. They were carved into the inner skin of birch bark, scratched into slate and wood or painted onto leather scrolls. The pictures were meant to preserve oral tradition and also represented symbols of the 'Great Spirit'. Earth by Ojibwa on scrolls: a) Red Lake, b) White Earth (Minnesota) [According to Hoffmann: The Midewin, Tables III A and IV.], c) Ojibwa drawing Aki of Earth [According to W. Jones: Ojibwa Texts II, p. 322 Table V, Brill, Leyden 1917]

That having been said, a stick can have a round or round-longish cross section. Additionally, it can have a long shape; it can be straight or bent. The Ojibwa language has individual terms for all these special properties.

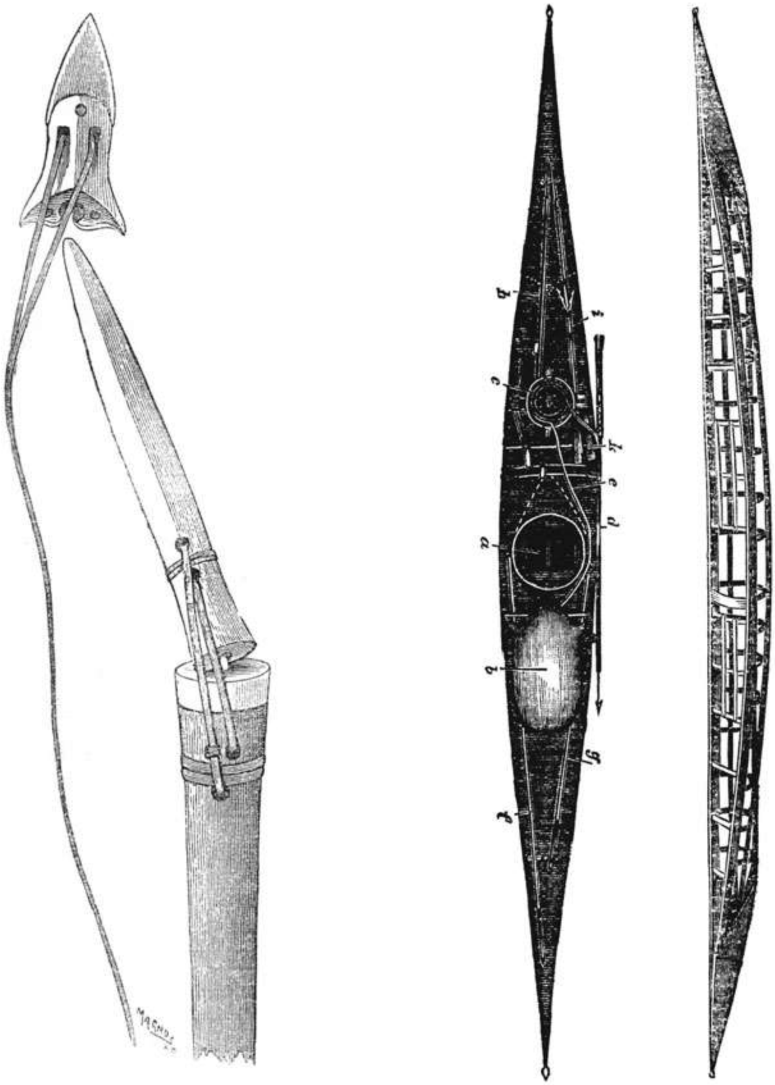
There is a linguistic advancement that is worth mentioning. It was discovered when the Ojibwa made first contact with the objects introduced by the Europeans. They only knew the round shape of the wigwam and were then made familiar with right-angled cabins. On top, the logs were pre-cut into squares by the settlers before assembling the walls. In order to grasp the concept of this new shape, the Ojibwa developed two new categories for angles, which occurred repeatedly. One concerned repeating right angles (to describe a square or rectangle, which feature four right angles), the other one stood for repeating acute or obtuse angles (strangely, they did not distinguish between those!). By means of the latter category they could, for example, describe a triangle, which is formed by the obtuse angle at the ridge of the roof and the two acute angles at the base of both roof sides.

The Inuit have properties for categories comparable to the Ojibwa shape categories (Illus. 3.5.2). The category 'round' can be modified more or less by means of an infix: *angmaluqtuq* means round in general, *angmalu-riq-tuq* completely round, and *angmalur-lak-tuq* round to some degree.

There is also a wealth of terms to describe the location of objects within the speakers surrounding. In English, we basically use the words here and there for this. However, there has been little research thus far into these categories and their applications by the Inuit.

The considerable span of the variety of categories in both languages corresponds to the typical forms and shapes that occur in nature. Nonetheless, it is also suitable to describe human products of those societies, such as arcs, knives, sleighs or moccasins. These linguistic features teach us that we must not limit our research of early geometrical notions to abstract objects like straight lines or circles. (J. P. Denny in chap. 6 *Cultural Ecology of Mathematics* by [Closs 1986]).

Hereby, there is an instantaneous correlation to geometrical skills, which travellers noticed when visiting the Arctic at the end of the $19^{th}$ century, namely the Inuits' skill for drawing highly detailed maps of great areas covering several hundreds of kilometres. These were drawn either in the snow or with pencil on paper. The relative position of individual objects and directions was reproduced with astonishing accuracy. The absolute distances were not right. (To supplement this, the Inuit added detailed descriptions of distinctive places, which were indispensible for orientation on their long itineraries.) It seems that their geometrical imagination enabled them to think topologically rather than metrically. It remains to be researched if the distances between the marked locations were determined by the amount of time it took to cross through them, rather than representing length as actual distance [Ascher 1991].



**Illus. 3.5.2** Inuit harpoon and kayak. The Inuit harpoon consists of a lance linked via a strap joint to a bone cone made of walrus tooth and the harpoon top made of walrus or narwhale tooth, and a lead to catch prey made of walrus or seal skin. The wooden skeleton of the kayak was originally made of light driftwood; the outside was covered with leather made of common or hooded seal fur. The slopes at the bow and the stern form an obtuse angle of approx. 140° together with the base of the kayak [Fritjof Nansen: *Eskimoleben* (Eskimo life), publishing house Georg Heinrich Meyer, Leipzig and Berlin 1903]

### 3.5.2 Aztec, Mayan and Incan advanced cultures

The fact that they structured their societies in classes is common to all those civilisations: a large population of farmers is ruled by a small number of aristocrats and priests. Apart from that, there were soldiers, who came from high-ranking families, and a small middle class of craftsmen and merchants. Above all, this social structure is reflected by the configuration of the towns and design of representative monumental buildings.

**Aztecs**

The Aztecs form the largest tribe of the Nahuatl language group in Mexico. After migrating for years, they founded their capital Tenochtitlán in 1325 (according to their own sources). It was located on an island in the area of today's megalopolis Mexico City and became the centre of Aztec expansion in the $15^{th}$ century. When Cortés arrived there in 1519, Tenochtitlán ruled all other towns. Power was divided between the king as head of the army, and the pontifex, a close relative, who functioned as a type of Prince of Peace. The economy was mainly focussed on farming corn, which is why each farmstead featured a big corn barn.

The Aztecs had a rebus-like writing, which also preserved historical events and tribute lists. Concerning arts, they adopted many aspects of preceding cultures: pyramid and temple architectures, playgrounds for ball games, lapidary and gold work. The temple of Malinalco in the highland of Mexico, which was carved out of rock between 1476 and 1520, is a special Aztec accomplishment. The entrance in the shape of a dragons mouth leads to the circular interior (diameter: 5.20m), which features a surrounding bench and a temple floor with animal illustrations, hinting at the Aztec warfare. Relics of the old Aztec culture have been preserved in remote valleys up to the present day. Artefacts indicate that talented Aztec artists planned and sketched objects in their spatial surrounding before they were painted and sculptured, probably by less gifted craftsmen. Thereby, they helped themselves with simple geometrical aids. Research has shown traces of using a simple compass for objects. A preserved list of tools of Aztec master builders cites, apart from the compass, the square (a right angle?), the plumb line, a straightedge, a brick trowel and a wedge. Long before the Aztecs settled down in Mexico, one of the most significant towns of Mesoamerica in that neighbourhood, in ruins nowadays, experienced its golden age: Teotihuacán, located 40 km to the North of Mexico City. This town was once capital of a great empire, which had its climax around 600; between 650 and 750 the town, which was arranged like a chess board, was abandoned. We will briefly sketch some of its characteristics before introducing the Aztec capital Tenochtitlán. The ceremonial area located at the centre of Teotihuacán is dominated by the Pyramid of the Sun and the Pyramid of the Moon. It is arranged around a central procession street – the Avenue of the Dead, 44 m wide and 5 km long – starting at the Pyramid of the Moon. This town, which expands across 22 square kilometres and grew considerably during the previous pre-Christian centuries, experienced its golden age from the first century to the middle of the seventh. It also influenced other temple structures in Mesoamerica built in the early classical era (approx. 300-550). The name Teotihuacán means ‘birthplace of the gods’. The pyramid of the Sun is approx. 61 m high, has a basal side length of 213 m and a volume of around one million cubic metres (Illus. 3.5.3).

The pyramid was erected upon a basalt cave accessible via a long tunnel, which was believed to be the entrance to the underworld. There are myths about the origins of sun and moon and the cycles of time surrounding this pyramid. Furthermore, it is said that water and rain ghosts lived there.

There has been much speculation concerning the motivations for the planning and execution of this enormous urban structure. Some suspect that we are not dealing with a ‘rational’ town planning here, but the belated effort to connect the most important ceremonial buildings in a sensible manner, whereby astronomically relevant events are supposed to have served as a basic indication. Others argue that this construction reflects the centralised, hierarchically-ordered rule, which had to integrate and hold together more than 100 000 inhabitants of different social classes. Linking religious and civil buildings by a unified construction would have primarily served this aim. Certainly, the art and architecture of Teotihuacán in conjunction with the practised cultural rites would have strengthened the peoples feeling of togetherness.

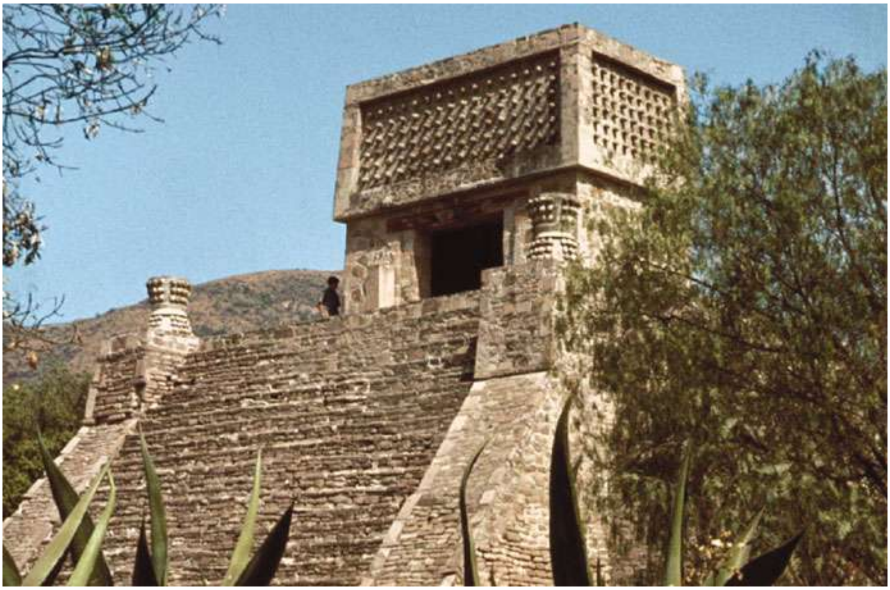
**Illus. 3.5.3** Pyramid of the Sun in Teotihuacán (Mexico). The step pyramid with the monumental staircase comes from the classical era of Teotihuacán culture (100–600 AD) before the rule of the Aztecs. [Photo: H.-W. Alten]

The geometrically arranged ‘Talud-Tablero’ profile, which features in the architecture of the walls of the building, is also quite striking with its change of inclined areas and vertical panels. This profile is found at other constructions as well. This profile existed evidently in pre-Christian times already and was repeated in pyramid buildings with upward reduced scale. The panels provided room for a wealth of decorative (partially leaping) sculptures. (J. K. Kowalski in chap. 4 *Natural Order, Social Order, Political Legitimacy, and the Sacred City* in [Kowalski 1999]).

Now let us turn towards the capital of the Aztec Empire, Tenochtitlán. It features a large temple area in the shape of a rectangle of 350 x 300 m. Apart from the main temple (Templo Major, Pyramid of the Sun), this area was the base for several smaller temples and further sacred buildings. The great pyramid-shaped temple of Tenochtitlán was accepted as the cosmic centre of the Aztec universe (E. M. Moctezuma in chap. 9 *The Templo Mayor of Tenochtitlán* in [Kowalski 1999]), since its components (e.g., the sacrificial altar) could be associated with the mythical Aztec gods. The upper sacred area was based upon an expanded platform representing the earthly area. An underground source was discovered below the pyramid, which was expanded several times. The part of the huge building located there was dedicated to the underworld, the world of the dead. Thus, the three worlds created by the gods – heaven, earth and the underworld – were symbolised by the stepped construction of the Pyramid of the Sun (cf. Illus. 3.5.3). The actual temple area was separated from the profane area by a “serpent wall” – a great wall made of heavy, worked cuboids with sculptured serpent heads. Relics of such walls were also found at other locations. Simply comparing these two great temple constructions illustrates the commonalities, which we also encounter at other sacred Aztec sites and are part of the cultures that preceded the Aztecs or developed parallel to them.

For instance, the core of the pyramid of Tenayuca (Illus. 3.5.4) on the outskirts of Mexico City ascribed to the Chichimeca comes from the $13^{th}$ century. In accordance with the custom of ancient Mexican peoples, it was encased four times in intervals of 52 years – a ‘Mexican century’, last under Aztec rule in 1507. Just like the Mayans, the Aztecs calculated with the vigesimal system (base 20) and, thus, divided the solar year into 18 months of 20 days each and 5 ‘empty’ days. This was important for sowing and harvest, as well as for calendar calculations. The days of a month were given names, the twenty glyphs of which are engraved in the famous Aztec Calendar Stone (Illus. 3.5.5) in a circle surrounding the symbols of the four preceding world ages encompassed by the sun.

The days of the solar year were marked by their names and the number of the month (or a symbol for the five ‘empty’ days, respectively). Apart from that, the Aztecs calculated with 260 days in the ‘ritual’ year, divided into 13 periods of 20 days each and also marked by day names and the number of the period. Due to L.C.M. $(260\,365) = 18\,980 = 52 \times 365$, the same constellation of both markings of one day only repeats itself after 52 solar years – an explanation for the “Mexican century” and the encasing of pyramids. The



**Illus. 3.5.4** Temple pyramid of Tenayuca
[Photo: H.-W. Alten]

**Illus. 3.5.5** Aztec calendar stone
[Photo: H.-W. Alten]

**Illus. 3.5.6** Serpent frieze at Coatepantli in Tollán (Tula, Mexico)
[Photo: H.-W. Alten]

**Illus. 3.5.7** "Stone mosaic" in the palace of the Great Seer in Mitla (Mexico)
[Photo: H.-W. Alten]

feathered serpent that served as a symbol of the god Quetzalcoatl can also be found in many Mesoamerican cultures. Myths tell us that the Great King Quetzalcoatl brought culture to the Toltecs in the $10^{th}$ century. Their temple town Tollán (today Tula) is encompassed by a wall, the middle ornament frieze of which displays moving serpents with skulls in their wide-open mouths (Illus. 3.5.6). Such "serpent walls" also decorated the Toltec Teotihuacán, the temple of the feathered serpent in Xochicalco, as well as the temple on the Pyramid of Kukulcan and the Temple of Warriors in Chichen Itzá at Yucatán, designed by the Toltecs and Mayans.

Purely geometrical ornaments cover the walls in the palace of the Great Seer in the temple town of Mitla (Illus. 3.5.7). The mosaic-like ribbons with step meanders are not made of mosaic stones, but were carved out of carefully assembled flagstone. Their patterns show clear relation to the textile weaving. They came from the Mixtecs, who were pushed aside by the Zapotecs in the high valley of Oaxaca between 1000 and 1400, who, in turn, superseded the advanced civilisation that had developed at Monte Albán since the $5^{th}$ century BC.

### Maya

The pre-European culture of the Mayans had developed in the lowland of North Guatemala, the Yucatán peninsula, and the adjacent areas since 1000 BC. Mayan culture has been accepted as the artistic and scientific climax of Native Indian culture. The stone temples and palaces, often built upon high step pyramids, formed the centre of the spaciously constructed towns. It is not certain yet to what extent those sites were constructed according to astronomical aspects, although we do know that the Mayan calendar was based on astronomical observations: one solar year was divided into 18 months of 20 days each and five intercalary days. Their writing, a mix of syllables, words and images, has been preserved in stone reliefs in many glyphs of the classical era (300–950). Three codices (manuscripts) of post-classical time (1000–1500) have been preserved, including numbers and calendar details. However, the apex of Mayan culture had already passed when Spain encountered them.

According to the multiply reproduced Dresden Code (the most extensively preserved Mayan text), the $13^{th}$ Mayan 400 year cycle since the creation of Earth ends on $21^{st}$ December 2012. However, a consequent apocalypse is not mentioned. Rather, the $14^{th}$ 400 year cycle started the following day.

The preference of symmetry in Mayan sculptures and buildings, which are openly constructed around a central axis, is striking. It is assumed that when composing the stelae at Tikal, Mayans worked with the overlapping of isosceles and Pythagorean triangles. However, there has been no definite evidence yet. Since the monuments were often seriously affected by weather conditions, it is, for instance, not possible in many cases to prove the use of exact angles (as they must occur, if determined by small Pythagorean numbers). Methodologically speaking, the question arises as to whether recognisable geometrical structures in artefacts are just the interpretation of modern researchers or actually intended by the maker.

This also applies to planning monumental buildings, temples and administrative buildings, as well as their relative position to one another. The sites of Tikal (Illus. 3.5.8, 3.5.9) and Copán were examined in regard to this aspect. For instance, it was shown that Tikal features an exact East-West line from the gateway of temple I (built around 700) to the gateway of temple III (built around 810). This line must have been defined by means of astronomical observations before erecting temple III. However, the orientation of the façades deviates by either 9° or 18°. We would have to find another explanation (e.g., astronomical) for this. Right-angled and isosceles triangles, as well as parallel lines, can also be found in the design of Tikal. Prominent locations, such as altars, stelae or platforms (e.g., for ball games), or rock reliefs, could have served as points of orientation when devising Copán.

Public buildings and sites could have provided strong indications for astronomically-determined orientations. There, not only was the course of the sun crucial, but also the exact boundary position of where Venus rises. This planet, so close to the sun, took on a special role in Mayan cosmology. Such astronomical aspects could explain why pure, right-angled sites, as we encounter in Teotihuacán, are a rather rare feature and often interrupted by seemingly irregular positioning.

The town of ruins Uxmal also offers impressive examples of Mayan architecture. It is located on the Yucatán peninsula in Mexico, was inhabited between the $7^{th}$ and $11^{th}$ centuries, and is accepted as one of the most important Mayan centres. The so-called House of the Nuns was constructed around a large, almost square place and seems to symbolise cosmological ideas (Illus. 3.5.10).

The Pyramid of the Magician, erected in multiple building phases, features an almost elliptical ground view and carries several temples (Illus. 3.5.11).

However, the (partially reconstructed) Governor's Palace is accepted to be the masterpiece of Mayan architecture. It was built in the late $9^{th}$ and early $10^{th}$ century during an economic golden age and served as both an elite residence and the administrative centre. It is the most outstanding among all impressive buildings in this spacious area (Illus. 3.5.12). The palace itself stands on a four-tier platform so as to remain visible at a distance and to reach a height comparable to those of some pyramid temples within the area. The most striking feature of the almost 100m long, right-angled construction is the huge amount of large, immaculately worked stone blocks in a unified structure stretched out along the front side. It is trisected by means of two high entrances, which end in acute triangles. Each part features smaller, rectangular accesses, which result in the front side being structured symmetrically. The proportional sequence 2-7-2 is immediately eye-catching. On top of the smooth cuboid wall, there is a wider and higher façade wall, which is decorated by wonderful examples of Mayan sculpture art. The design is strongly symmetrical. It features squares and both large and small rectangles, shifted against each other whilst emphasising the diagonal. Nonetheless, the complex underlines the monumentality of the overall view by means of

**Illus. 3.5.8** The northern Acropolis of Tikal (Guatemala)
[Photo: H.-W. Alten]

**Illus. 3.5.9** Pyramid II of Tikal (Guatemala)
[Photo: H.-W. Alten]

**Illus. 3.5.10** House of the Nuns in Uxmal (Yucatán, Mexico)
[Photo: H.-W. Alten]

**Illus. 3.5.11** Pyramid of the Magician in Uxmal (Yucatán, Mexico)
[Photo: H.-W. Alten]

**Illus. 3.5.12** Governor's Palace in Uxmal (Yucatán, Mexico)
[Photo: H.-W. Alten]

**Illus. 3.5.13** Ornaments and a round sculpture at House of the Nuns in Uxmal
[Photo: H.-W. Alten]

**Illus. 3.5.14** Kukulcan Pyramid and the Temple of Warriors in Chichén Itzá (Yucatán, Mexico) [Photo: H.-W. Alten]

a clear, net-like structure. The spirals in the rectangles or squares, which come forward three-dimensionally and are assembled by horizontal or vertical parts (they do not feature any curvatures) are peculiar. When looking at the neighbouring Pyramid of the Magician, we are attracted by a lattice work made of diagonally running rhombi. This lattice – just as with the spirals – is fit in between the characteristic Mayan sculptures. Other buildings also feature similar net-like structures and alternate between fields with geometrical patterns and those with sculptures (Illus. 3.5.13).

We can find especially impressive examples of Mayan culture in Chichén Itzá in the North of the Yucatán peninsula. They have their origins in the time when Chichén, already founded in 432 BC, was re-founded in the $10^{th}$ century AD by the Mayan tribe of Itzá, together with the Toltecs, who had been banished from Tollán. The great temple pyramid of Kukulcan (Illus. 3.5.14; kuk = quetzal, ul = feather, an = serpent) and the Temple of Warriors confirm their relation to the Toltec culture in the highland by means of their geometrical shape and the decoration dominated by serpent heads.

In contrast, a round building used as an observatory was named "caracol" (snail) by the Spanish due to the spiral staircase inside. These are witnesses set in stone for the Mayans' highly advanced astronomical observations and calculations.

The architects of these great buildings must have had a distinct feeling for geometrical shapes, the effect of symmetry as elements of building design, and their decorative use, as well as a gift for impressive architectural composition [Kowalski 1987].

**Inca**

The Incan Empire, a Native Indian State in the West of South America, existed from around 1200 until 1532, when the Inca Atahualpa was captured by Francisco Pizarro in Cajamarca. The destruction of the Incan culture started with the conquest of the Incas by the Spanish. They had covered the area from the Pacific coast to the Andes mountains and in a North-South direction from Ecuador to Middle Chile. Originally very different peoples lived there. However, the Incas started to establish themselves in 1400 and built a firmly organised state. The capital was Cuzco, at the eastern border of the Andes not far from the equator (today: Peru). Beneath the Inca in charge, there were four vice kings, who managed the four quarters of the state. Even though they had a common language, they did not use writing, in contrast to the Mayans. Hence, artefacts are the only genuine source of studying their culture. Additionally, there are the reports of the Spanish conquerors. The development of a unique, highly complex system of coloured knotted strings ("quipus", sometimes called 'talking knots') to record statistical information is remarkable. The mathematical elements of this symbolic system consist of numbers, spatial configurations and logic.

Cuzco was re-structured in the $15^{th}$ century, whereby the ideal plan of spatial relations had to be adapted to the natural surroundings. The town was arranged in four parts. Three extended families, who had a mutual ancestor (an earlier regent), lived in each part. Each quarter was in charge of settlements of further extended families in the surrounding areas. Thereby, ideal lines, radially originated from the capital, separated the areas of responsibility. Here, as elsewhere, the geometrical structure symbolised the power relationships. The Incas devised storage facilities, groups of silos located at big crossroads, which were supplemented by administrative and work buildings.

This took place in a very methodological manner by repeating the same basic pattern over and over and over again. The scheme for residential buildings, which were, e.g., erected around a square, was repeated likewise unmodified from unit to unit. The expansion of towns also took place according to the pattern. Thus, a newly designed area was basically a copy of an already existing one.

Public buildings and temples were erected by means of large, exactly worked polygonal blocks without mortar (Illus. 3.5.15). Although the basic pattern was that of a square or rectangle, the blocks varied in size and shape and, thus, provided the wall with extra stability. Sun-dried lay bricks were used for simple buildings.

**Illus. 3.5.15** Cyclopean wall in Cuzco (Perú). The Incan master builders designed true marvels of precision with these walls made of gigantic, polygonal stone blocks in the capital of the Incan Empire.
[Photo: H.-W. Alten]

**Illus. 3.5.16** Terraces at Pisac (Perú)
[Photo: H.-W. Alten]

**Illus. 3.5.17** Incan stronghold (l.) and temple gate (r.) of Machu Picchu (Peru)
[Photo: H.-W. Alten]

The tendency to repeat existing patterns as often as necessary is also reflected by Incan ceramics. The decorations of clay pots repeat the elements mostly made of small-scale geometrical patterns, until a strip or an area is completely covered. These elements could be squares, triangles, rectangles, trapezoids, parallel or diagonally running lines or circles, and also nested. Reflections, double reflections in vertical or horizontal direction, and rotational symmetry were used here, as well as in the design of buildings.

For instance, the trapezoid-shaped gate of the temple in Machu Picchu (Illus. 3.5.17, right) is a hallmark of Incan architecture and can be found in varying sizes at doors, windows and niches. The Incan stronghold Machu Picchu (Illus. 3.5.17, left) is thought to be the most spectacular construction by Incan master builders. It was only discovered in 1911 by Hiram Bingham, between the steep rock walls of the Urubamba Valley. It served as protection for the capital Cuzco and is towered over by the steep rugged rock cone of Huayna Picchu. At its peak, buildings offered last shelter, and terraces facilitated the cultivation of vegetables to feed to the refugees. The terraces were based on geometrical considerations and built with enormous effort through highly advanced techniques. Hence, they rise to vertiginous heights even at the steepest sides of the wild mountain world (Illus. 3.5.16). They helped to ensure food for the growing population.

Clothes made of wool or cotton played an important role in Incan culture. Changing one's clothes indicated important stages in one's life and was associated with ceremonial acts. The ponchos mainly featured decorations of small rectangles with geometrically arranged patterns [Ascher 1991].

The striped ornaments on clay pots found in graves constitute an interesting example of geometric decoration. The patterns, mostly made of simple elements bound by straight lines, repeat themselves in the stripes and can be dyed in one or two variable colours. Most times, these colours are firmly connected with the basic pattern, which is why dying does not result in new symmetric groups. Both translational and rotational symmetry are used frequently. In contrast, horizontal or glide reflection occur rarely. Next to simple stripes, there are also double striped patterns, e.g., made of trapezoid-shaped basic elements, which are reflected or contrasted by reversing colours.

Despite this wealth, all the striped patterns examined so far do not indicate all possible symmetric groups. The seven basic symmetric groups lead to 17 further ones when using two colours. However, these have only been partially realised in the Incan striped ornaments. Of course, the Incan artists' and craftsmen's interest in such decorations, which did not stem from a practical necessity, but arose from playful enjoyment of the design and creation, is more significant than this mathematical imperfection [Ascher 1991].

All in all, results of the research up to the present day justify the assumption that geometry was thought of as an integral part of the mix of religion, techniques and science in the Central American civilisations (F. Vinette in chap. 13 *In Search of Mesoamerican Geometry* by [Closs 1986]).

**Illus. 3.5.18** Clay vase with ornamental ribbons from the Nazca culture [Auction catalogue Richter & Kafitz, 2006, No. ET 041]

## 3.6 Problems to 3

**Problem 3.1.1:** Length of a Chinese town wall

Problem 20 of the $9^{th}$ chapter of '9 Chapters' [Kangshen Shen et al.1999](cf. Illus. 3.1.4) (and the description of the problem on page 125):

a) Sketch the situation of the square town with the tree and the observer's path. Choose quantity $a$ for the path North, $b$ for the path South and $c$ for the path West.

b) What is the best approach for obtaining the equation stated in the text, whereby $x$ determines the length (in paces) of a side of the town wall?

c) How do $p, q$ depend on $a, b, c$?

d) Is it also possible to interpret the quadratic equation as a relation between areas?

e) Solve the quadratic equation and then try to comprehend the rule stated in the text: (The Chinese mathematicians represented the coefficients of an equation by means of calculation sticks, which were used to carry out the necessary operations.) "The rule states: Multiply the number of paces walked towards the West with the number of paces stepped away from the North gate. Double this to obtain the dividend. Add the amounts of paces together, which were stepped from the South and North gates to obtain the amended divisor. Extract the square root from this to obtain the side of the square town." In the resulting equation $x^2+px = q$ (here $p = a+b,\ q = ac$) the Chinese call $q$ the 'dividend' and $p$ the 'amended divisor'. 'Extract the square root from this' means solve this quadratic equation. The Chinese did this with the 'Horner-scheme'. Because $p = 34,\ q = 71000$ they tried with $x_0 = 1$ and $x_0 = 250$ :

| | | | | |
|---|---|---|---|---|
| | 1 | 34 | -71 000 | |
| $x_0 = 1$ | 1 | 35 | -70 965 | |
| | | | 285 | remainder |

| | | | | |
|---|---|---|---|---|
| | 1 | 34 | -71 000 | |
| $x_0 = 250$ | 1 | 284 | 71 000 | |
| | | | 0 | remainder |

Therefore the solution is $x = 250$.

f) It is obvious that we are not dealing with an applied problem here. Rather, it is an exercise for establishing and solving quadratic equations (which we can also find in Euclid's work), which has been 'wrapped'.

**Problem 3.1.2:** The method of doubled measure according to 'The Sea Island Mathematical Manual'

The method represented by the woodcut in Illus. 3.1.5 is described in the first problem of 'The Sea Island Mathematical Manual' as follows: ([Li/Dú 1987, p. 76–78], [Swets 1992, p. 19–20, 42–43]):

"Observe an island, whose height and distance are unknown. Erect two measuring poles of height $h = 3$ *zhang*; the distance between both is $d = 1000bu$. The two poles and the (peak of) the island shall be one line (vertical plane). Go back $a_1 = 123bu$ from the first pole and observe the top of this pole and the peak of the mountain in one line from the ground. We see that both collapse into one another. Then go back $a_2 = 127bu$ from the second measuring pole and observe the peak of the mountain from the ground. The top of the measuring pole and the peak of the mountain collapse into one another. Determine the height of the island and its distance from the first pole." (The reader will have noticed that the woodcut does not exactly correspond to the description of 'The Sea Island Mathematical Manual', but an improved version thereof. The observer's eye is no longer located at ground level, but at the upper right corner of the added rectangles. Thus, the observer need not lie on the ground, but can observe whilst standing up. Of course, now we also must consider his eye level when calculating. The further description in the text concerns the more primitive case.)

(Answer:) "The height of the island is 4 li and 55 bu, its distance is 102 li and 150 bu."

(Method:) "Multiply distance $d$ with height $h$ of the poles. This will yield the numerator. Take the difference of the distances of the observing points $a_2 - a_1$ as the denominator, by which we have to divide the numerator. Add the height of the pole to the result. The result is the height of the island. – In order to obtain the distance of the island from the front pole, multiply line segment $a_1$, which was covered when going back from the first pole, with distance $d$ of both poles from each other. This will yield the numerator. Take the difference of the distances of the observing points $a_2 - a_1$ as the denominator, by which we have to divide the numerator. The result is the distance of the island from the front pole $y$."

(The letter labelling is, of course, absent from the description.) The following instructions result if expressed in formulae:

$$x = \tfrac{d \cdot h}{a_2 - a_1} + h, \ y = \tfrac{d \cdot a_1}{a_2 - a_1}.$$

a) Derive both formulae.
b) Generalise them to apply to the case shown in the illustration, which considers the observer's eye level.

**Problem 3.1.3:** Estimating the circle area according to Liu Hui

a) Make a drawing to make sure you understand what Liu Hui meant geometrically with the upper bound in the doubled estimation.
b) Which limits do we obtain for $\pi$, if we use $f_n + 1$ for the dodecagon and $f_n$ for the hexagon?
c) Which of the necessary steps for calculation cannot be executed exactly, i.e., require a rounding? Do we need to round up or down if we want to

make sure that in the case of a polygon with a high number of sides we do not exceed $F$ with the lower bound and, respectively, not undercut this value with the upper bound?

**Problem 3.1.4:** Calculating the triangular area according to Qin Jiushao

a) Verify that the formula (3.1.5) corresponds to the one named after Hero.
b) Attempt a geometrical interpretation of this formula.

**Problem 3.1.5:** Determining a square military camp

a) Make a sketch of the camp described on p. 131/132.
b) $BGFE$ shall be the last right column, which is made of blank squares and which is repositioned to be under the last lower row $DHIJ$ (which is also made of blank squares). The rectangle constructed this way is $ABKE'$.
c) How are the occupied fields arranged, if we assume that the upper left corner of the camp is occupied by a company?

**Problem 3.2.1:** Allocation of a circular land lot

Look at problem b) on p. 144 [Smith/Mikami 1914, 66]:

a) Which value for $\pi$ was implicitly used here?
b) Which segment division is advisable in this case?
c) Is it possible to solve this problem by means of a formula discussed by Hero concerning the area of a circle segment (cf. (2.5.5) and (3.1.2)?
d) What is the outcome if we use integral calculus?

**Problem 3.2.2:** Determining spherical surface according to Kittoku and Takebe

Execute the Japanese method of determining spherical surface by means of the difference of the volumes of two spheres imagined in a modern manner as a limit process. [Mikami 1913, p. 206/207]

**Problem 3.3.1:** Indian fraction representation for $\sqrt{2}$

a) Derive the stated Indian fraction representation for $\sqrt{2}$ by means of the Babylonian method.
b) How exact is the value (how many decimals)?

**Problem 3.3.2:** Circling the square

Execute the construction according to the instructions stated in the Śulbasūtras (string rules).

a) What is meant is that we must add $\frac{1}{3}$ of the difference from half the diagonal $d$ and half the side $s$ to half the side and use this to draw a circle around the centre of the square.
b) Calculate the area of this constructed circle and check by how much this circle deviates from the given square area.

**Problem 3.3.3:** Squaring the circle according to the Śulbasūtras

The first rule clarifies how difficult it often is to understand the very brief wording of the verses. In this case, they mean that the square side $a$ is obtained by means of the circle diameter $d$ according to the following calculation: $a = [\frac{7}{8} + \frac{1}{8 \cdot 29} - \frac{1}{8 \cdot 29}(\frac{1}{6} - \frac{1}{6 \cdot 8})] \cdot d$. Calculate the approximate values of $\pi$ by means of both instructions from p. 158.

**Problem 3.3.4:** Calculating spherical surface according to Bhaskara II

Look at Bhaskaras approach in general:

a) Divide the circumference $2\pi r$ into $n$ parts ($a = \frac{2\pi}{n}$). For the area of one of the small trapezoids, use the arc length along the meridian as height and the average of the arcs of the upper and lower circle of latitude as breadth. Then, add together the triangular areas at the pole and the trapezoid areas within half a strip from the pole until the equator. Double the outcome and then multiply by $n$.
b) Finally, stated in a modern fashion, calculate the limit for $n \to \infty$, i.e., $\alpha \to 0$. Confirm the following result:

$$S = 4\pi r^2 (\int_0^{\frac{\pi}{2}} sin\ \alpha d\alpha - \lim_{\alpha \to 0} \frac{a}{2}) = 4\pi r^2.$$

**Problem 3.3.5:** Spherical surface according to the Yuktibhāṣā Test

Calculate the spherical surface according to the Yuktibhāṣā description:

a) $r$ shall be the radius, $r_1 < r_2 < r$ the radii of both parallel circles of length $C_1 = 2\pi r_1$ or respectively $C_2 = 2\pi r_2$, which bound the strip. Imagine the strips as the external area of a disc cut out of a cone; it is approximated as a conic frustum. Its height shall be $h$; the distance measured along a circle of longitude between both bounding circles shall be $s (> h)$.
b) Consider that the untwined strip can be viewed as a trapezoid of height $s$ and the average length $\frac{C_1 + C_2}{2}$ and calculate its area in dependence of $r$ and $h$. (Contemplate two similar triangles, given that we draw a radius $r$ from the centre to a point at the middle of the strip.)
c) Add together the lateral areas of all discs (conic frustum).

**Problem 3.4.1:** Pythagoras's theorem according to al-Khwārizmī

Contemplate the idea that al-Khwārizmī's figure for proving Pythagoras's theorem can be interpreted in two ways (Illus. 3.4.1).

**Problem 3.4.2:** Abū'l-Wafā's square construction with a fixed span of the circle

Execute Abū'l-Wafā's construction of a square inscribed in a circle (also called "rusty compass construction").

a) A circle with a centre $S$ (drawn with a rusty compass) is given. Draw a diameter $ASG$ and mark the arcs $\widehat{AZ}, \widehat{AE}, \widehat{GT}, \widehat{GH}$ ($Z$ and $T$ at the same side of ASG) with the compass and draw the lines $ZE$ and $TH$, which intersect the diameter $ASG$ in $I$ and $K$. Connect $Z$ with $K$ and $I$ with $T$; the intersection of the connecting lines shall be $M$. Now draw the diameter intersecting the circle in $D$ and $B$ passing though $M$ and $S$. Then $ADGB$ is a square.

b) Consider the idea that the same construction for the same circle can also be done with another fixed span of the compass.

**Problem 3.4.3:** Heptagon construction according to Abū'l-Wafā

a) Construct a regular heptagon according to the rules by Abū'l-Wafā. (see p. 174). Can you see how it deviates from an exact heptagon with the naked eye?

b) By what percentage does the side length of the heptagon obtained this way deviate from the real one?

**Problem 3.4.4:** Heptagon construction according to al-Kūhī

Fully execute al-Kūhī's heptagon construction.

a) Why does the triangle $ABC$ described in the text (p. 174) feature angles $a$, $4a$ and $2a$?

b) How great are the angles in the three part triangles of the extended triangle $AED$?

c) Derive the two relations key for the construction:
$EB^2 = BC \cdot BD$ and $CD^2 = EB + EC$.

d) To construct a parabola and a hyperbola whilst keeping it simple, start with the line segment $ED$, which is already divided by $B$ and $C$ according to the rule (draw it horizontally in the middle of a sheet.). Draw a perpendicular upwards from $B : BZ = CD$, and one downwards from $B : BF = BC$. Add the rectangle $EBZT$. Now:

$$FZ \cdot BC = (BC + ET) \cdot BC = TZ^2$$
$$EC \cdot EB = (EB + BC) \cdot EB = (TZ + BC) \cdot TZ = ET^2.$$

e) If we confirm $BC = m$ as a fixed line segment and $TZ = y$, $ET = x$, then the first equation represents a parabola and the second one a hyperbola. Turn both into normal form and draw them into the figure – we only need the left branch of the hyperbola. The result is $EB = TZ = y_1$ and $CD = ET = x_1$ as coordinates of the intersections of both curves. Hence, we constructed the length of two other line segments, which together form a triangle similar to the triangle $BCA$, to a given line segment $m$. Consequently, we must, as mentioned by al-Kūhī, conduct a direct homothetic transformation in order to inscribe a regular heptagon into a given circle.

**Problem 3.4.5:** Calculating the circular area according to Hero and al-Khwārizmī

Hero had stated the following concerning the calculation of the circular area by means of the diameter: $A = d^2 - \frac{1}{7}d^2 - \frac{1}{14}d^2$.

a) How does this formula relate to Archimedes' value of $\pi$?
b) Theoretically speaking, what is behind al-Khwārizmī's rule? Does his approach constitute an improvement over Hero's formula?

**Problem 3.4.6:** al-Khwārizmī's formulae concerning the circle segment

a) Is the calculation for obtaining the circle diameter an approximation formula?
b) For the first case (segment smaller than a semi-circle), try to state a heuristic justification of the formula.
c) What does the formula yield in both extreme cases, in which $p = 0$ or respectively, $p = r$?

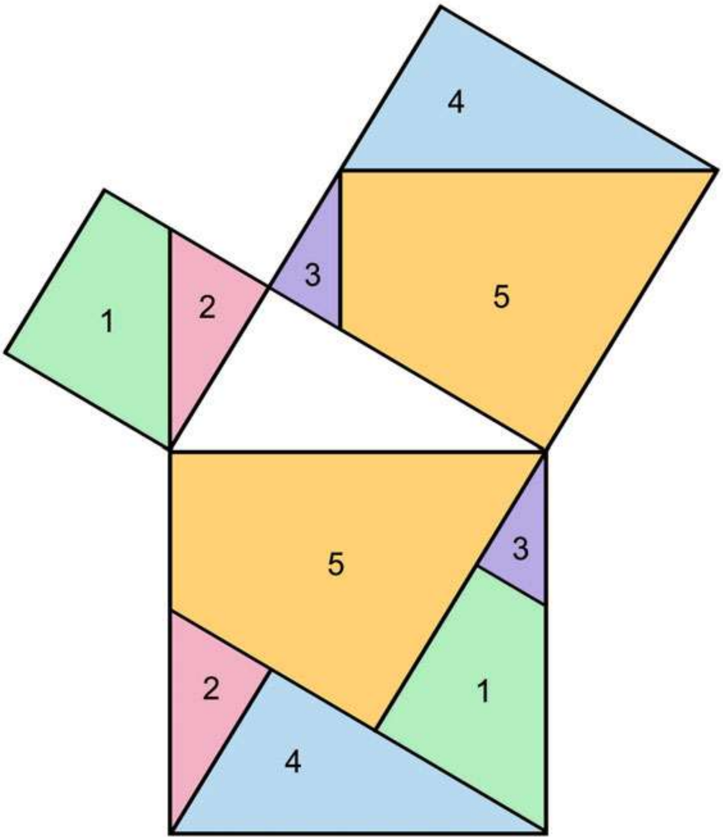
**Problem 3.4.7:** al-Kāshī's iteration method to calculate $sin$ 1°.

Examine the beginning of al-Kāshī's calculation method to determine $sin$ 1°. within the sexagesimal system. Consider hereby that he based his definition of the sine function on a circle radius of 60 and understood the sine as a line segment (semi-chord). Hence, we can expect the value to be close to 1 (the belonging circumference is $2\pi \cdot 60$). Besides, the trisection equation adopts the form of $Sin\ 3a = 3Sin\ a - 0;0.4 \cdot Sin^3 a(0;0,4 = \frac{0}{60} + \frac{4}{3600})$ if we use the Sine definition above and distinguish it from the ordinary one by Sin $a$. If we now say that $x = Sin\ a$ and with al-Kāshī
$Sin 3° = 3;8,24,33,59,34,28,16$, we obtain $3x = 3;8,24,33,59,34,28,15 + 0;0.4x^3$.

a) Assure yourself (whilst restricting yourself to the first positions) that this equation can be rewritten as:
$$x = \frac{47,6;8,29,53,37,3,45+x^3}{45,0}.$$
b) Since a value close to 1 is expected, al-Kāshī uses $x = 1;\ a, b, c...$ in the equation, whereby $a, b, c,$ ...represent the following sexagesimals, and deducts 1:
$0; a, b, c, ... = \frac{47,6;8,29,53,37,3,45+(1;a,b,c...)^3}{45,0} - 1 = \frac{2,6;8,29,53,37,3,45+(1;a,b,c...)^3}{45,0}.$
The first sexagesimal on the left side, $a$, must be equal to the first sexagesimal on the right. Due to the great denominator, the latter does not depend on $a$; it suffices to calculate $\frac{2,6;8,29...+1^3}{45,0} = 0;2,[49 \text{ or } 50], i.e., a = 2$.
c) In the second step, position $b$ is calculated accordingly by means of the approach $1;2,b,c,... = ....$ Confirm the result $b = 49$. Al-Kāshī repeated this elaborate iteration method until the $9^{th}$ sexagesimal!
d) Given $f(x) = \frac{47,6;8,29,53,37,3,45+x^3}{45,0}$, we recognise the thought behind this method. The function $f(x)$ increases so slowly close to 1 that the $n^{th}$ position of $f(x)$ does not depend on the $n^{th}$ position of $x$, but only on the first $n-1$ positions. However, al-Kāshī does not address the question as to whether this is generally justifiable.

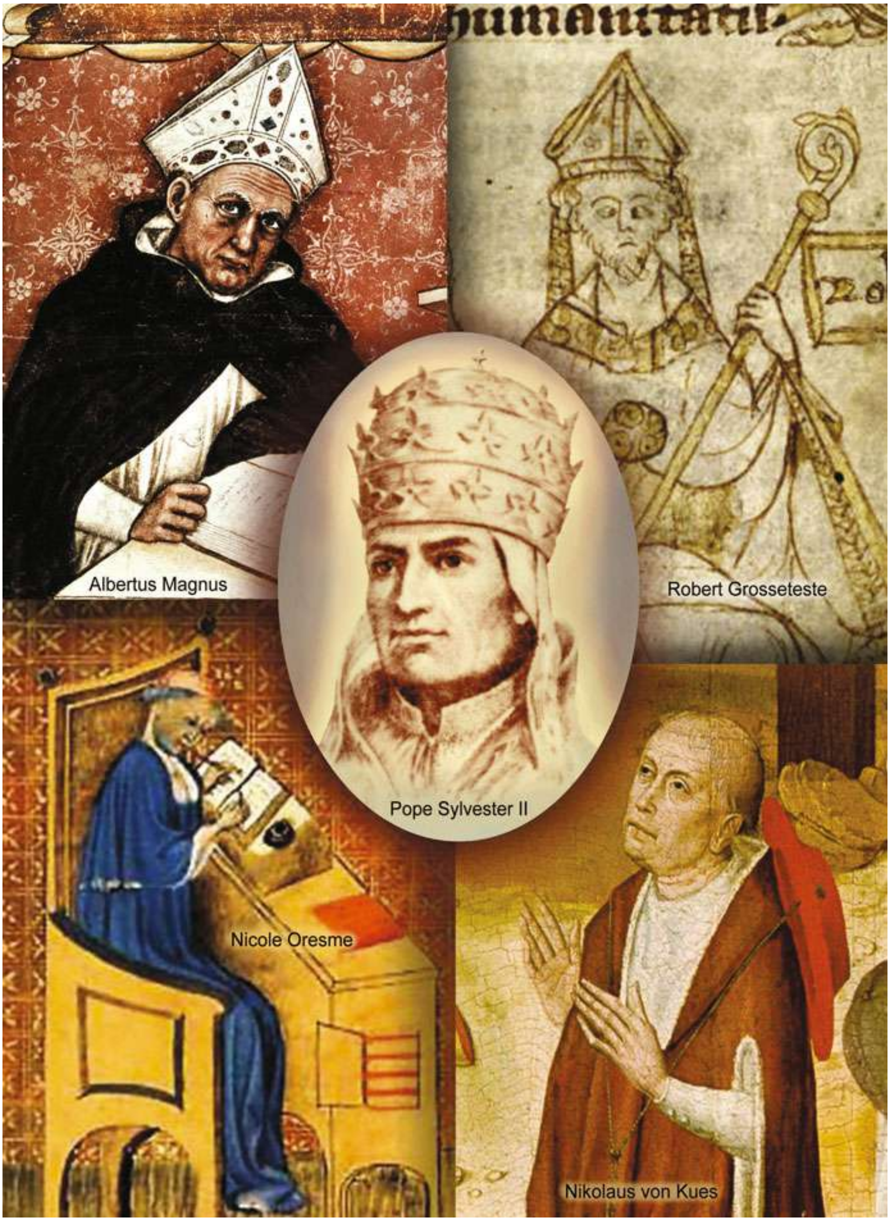
**Problem 3.4.8:** Proof of the Pythagorean theorem

Describe the construction of the decomposition (see Illus. 3.6.1) and justify that the tiles with equal numbers are congruent




**Illus. 3.6.1** Proof of the Pythagorean theorem according to an-Nayrīzī's commentary ($10^{th}$ century) on 'Elements'

# 4 Geometry in the European Middle Ages






C.J. Scriba, P. Schreiber, *5000 Years of Geometry*,
DOI 10.1007/978-3-0348-0898-9_4

| | |
|---|---|
| 375–568 | Period of migration of nations |
| 466 | Collapse of the Roman reign in Gaul |
| 486–751 | Merovingian Kingdom of the Franks |
| Around 500 | Angles, Saxons and Jutes immigrate to England |
| 711 | Arabs cross Strait of Gibraltar, conquer Iberian Peninsula apart from Asturias |
| 718–1492 | Reconquista (re-capture) of the Iberian Peninsula |
| 732 | Karl Martell beats the Arabs at Tours and Poitiers |
| 756 | Umayyads found Emirate (caliphate since 929) of Córdoba |
| 756 | Papal State is constituted |
| 800 | Charlemagne is crowned emperor in Rome |
| 843 | Treaty of Verdun: Carolingian Empire is divided into three parts |
| 870 | Treaty of Meersen: Lothar's (middle) kingdom is divided |
| From 862 | Normans (Varangians) rule Nowgorod |
| 911 | Duchy Normandy is formed at Seine mouth |
| 955 | Battle of Lechfeld (Hungarians defeated) |
| 962 | Otto I (the Great) is crowned Emperor of the Holy Roman Empire by the pope |
| 10$^{th}$century | Early Roman (Ottonian) architecture |
| 978–1328 | Capetians rule France |
| Around 1000 | Leif Eriksson discovers North America (Vinland) |
| 1024–1137 | Frankish-Salian emperors |
| 1066 | Battle of Hastings, Normans conquer England |
| 1077 | Henry IV's walk to Canossa |
| 1096–1270 | Crusades |
| 11$^{th}$/12$^{th}$century | Roman cathedrals, monasteries and sculptures in West Europe |
| 1130–1260 | Normans and House of Hohenstaufen rule in Sicily and Lower Italy |
| 1̃150–1535 | Hanseatic League expands and rules Baltic Sea |
| 12$^{th}$/13$^{th}$century | Upper Italian towns (Pisa, Venice, Genoa, Milan) flourish, early Gothic in Ile de France |
| 1138–1250 | House of Hohenstaufen rules in Germany |
| 13$^{th}$/14$^{th}$century | High Gothic cathedrals in West Europe |
| 1337–1453 | Hundred Years' War between England and France |
| 1453 | Ruin of the Eastern Roman Empire |
| 15$^{th}$century | Late Gothic in Germany, England and France |
| 1469 | Union of Castilia and Aragon by marriage of the "Catholic Kings" Isabella and Ferdinand |
| 1470 | Moscow becomes "Third Rome" |
| 1480 | Grand Prince Ivan III founds tsardom in Moscow |
| 1492 | Granada conquered, end of Reconquista, Columbus (re-)discovers America |
| Science and arts in the Middle Ages | Carolingian book paintings, gospel books and wall paintings in Ottonian era, glass paintings in Salian era and frescos in era of House of Hohenstaufen; golden age of glass paintings in Gothic time; scholastic philosophy (Albertus Magnus), arts and science practised in monasteries and cathedral schools(septem artes liberales), first universities founded |

## 4.0 Introduction

The time from the collapse of the Roman Empire as the result of the migration of nations ("Völkerwanderung") up until the Renaissance shall be summarised here as the era of the European Middle Ages. From the aspect of development within the mathematical, natural-scientific realm we end here with the beginning of the $15^{th}$ century and not only with the discovery of America. As a result, the two "newcomers" of mathematics, Nikolaus von Kues (Nicholas of Cusa) and Regiomontanus will open the next period.

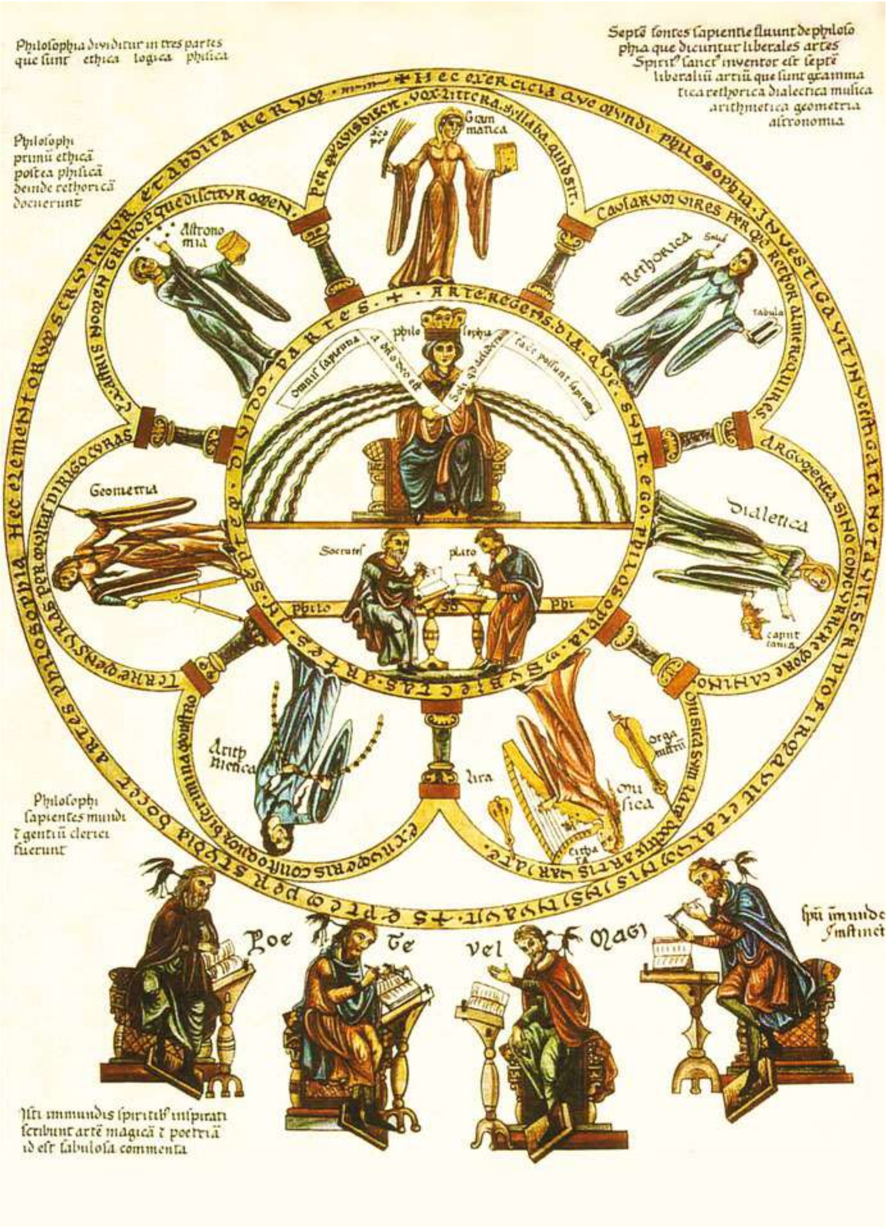
It is more difficult to determine precisely the beginning of the Middle Ages in regards to mathematical development. The Greek-Hellenistic era strongly affected the Islamic countries (which also passed on Indian knowledge); their influence on Europe in the Middle Ages (predominantly effective in Sicily and Spain) was supplemented by ancient mathematical knowledge communicated via the Eastern Roman Empire (Byzantium). Nonetheless, the collapsing Western Roman Empire also passed relics of ancient education and science directly to the Middle Ages. Hence, it is difficult to determine an exact date of the beginning of this "middle" era. Concerning the history of mathematics, it may be sensible to start this chapter with the $5^{th}$ century.

## 4.1 Geometry in the Early Middle Ages

### 4.1.1 The seven liberal arts

Some late Roman works became fundamental for teaching mathematics, which spread rather slowly. These works contained the so-called "artes liberales" or "Seven Liberal Arts" (or at least parts thereof). These seven sciences consisted of two parts: the "trivium", formed by grammar, rhetoric and dialectics (also called logics), and the "quadrivium", composed of arithmetics, geometry, astronomy and music. The trivium (the threefold path of language-orientated subjects), which taught the necessary knowledge of Latin, was the basis of all teaching at church schools and universities. The quadrivium was responsible for introducing the variety of mathematical sciences, which were already accepted by Plato as an educational foundation following Pythagorean scientific teaching.

The oldest preserved abstract of the liberal arts, valued so highly and connected to a family of sciences in Plato's 'Republic', was written by Martianus Capella after 450: 'On the Marriage of Philology and Mercury'. This neo-Platonic philosophical allegory consists of nine books and was partially written in prose and partially in verse. The author describes the wedding of the god of merchants, craftsmen and philology (in other words, the sciences, apart from philosophy, personified). The guests are the gods and allegoric




**Illus. 4.1.1** The Seven Liberal Arts were separated in two groups: trivium (grammar, rethoric, dialectic) and quadrivium (arithmetic, geometry, astronomy, music) [Herrad von Landsberg, Hortus Deliciarum, around 1180]

figures, such as the four cardinal virtues, united since Plato, prudence, temperance (restraint), fortitude (courage) and justice. Books 3 to 9 show, in order, the seven arts as bridesmaids, who give their scholarliness as presents. This gave Martianus the opportunity to represent the subject matter of these seven disciplines rather dryly in textbook style. Since this work was passed on in full, it became one of the most important sources for medieval school teaching. Needless to say, the part titled 'de geometria' is mainly reserved for geographical knowledge. It was only after a warning from the gods that Martianus attached a brief description of Euclidean geometry. In contrast, the book on astronomy is accepted as the best treatise in Latin of the Early Middle Ages.

The portraits of the mathematical disciplines by Boethius, the chancellor of Ostrogothic king Theoderich, are not completely preserved, but are qualitatively superior. He coined the term "quadrivium" for the four mathematical sciences (see below for more details on Boethius' geometry). We also owe him the Latin translations of some Aristotelian texts, which feature examples and comparisons of mathematical nature.


**Illus. 4.1.2** Cassiodorus [Gesta Theodorici; Flavius Magnus Aurelius Cassiodorus; Varie; and other texts. Latin, Manuscript on vellum. 186ff., Fulda, before 1176]

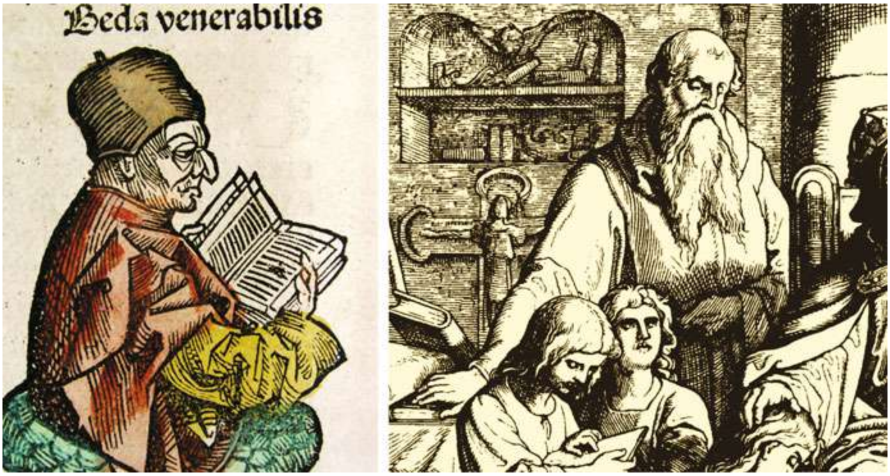
Cassiodor composed a very brief description of the artes liberales that was used fondly for centuries. He wrote his work in the middle of the 6th century after retreating from politics in 537 – following the recapture of Italy by Justinian – and moving into the monastery he founded at Vivarium. The monks were supposed to dedicate their time there to studying the holy texts and fostering the sciences. He outlined the subject matter of the arts in a textbook for the members of the order, titled *Institutiones divinarum et humanarum litterarum*. Geometry, of course, is summarised in approx. two pages (!). (His justification for also studying the worldly sciences can be found in the text passage in the Appendix: A. 5, p. 570)

Cassiodor explains the name 'geometry' as "terrae dimensio", earth expansion or surveying. He mentions the history of its origins in Egypt, the division of the year into months, determining the distance between Earth and moon/sun, as well as the size of Earth. Actual geometry is said to be the science of immobile quantities (in contrast to astronomy, which deals with movements). It is said to deal with quantities in the plane (countable, rational and irrational) and solid figures. The plane figures feature length and breadth, the solid ones, on top of those, height. Geometry examines the variety of these figures on Earth and in the sky. In Greece, this was done by Euclid, Apollonius, Archimedes and other authors; in Rome, Boethius published Euclid's work in Latin. These four short passages end with the hint that engaging with astronomy turns the soul towards the sky and, thus, towards the shape of the sphere or circle. They neither look concretely at just one figure, nor do they cite a single axiom or theorem from Euclid's 'Elements' or draw a figure. A few short geometrical additions are contained in around 20 manuscripts, which probably were taken from Boethius.

### 4.1.2 Venerable Bede and Alcuin

The monasteries were the first locations at which science was slowly revived. Ireland had already been introduced to Christianity in the $5^{th}$ century. First aspects of independently acquired knowledge were found in the works of the Irish monk Venerable Bede in Newcastle around 700. We do not just owe him the oldest preserved representation of the finger calculating method; he also addressed astronomical problems, such as moon phases, and described the relation between those and the tides at sea.

It was significant for the further distribution of education and science that the Frankish king Charlemagne summoned the educated monk Alcuin from York in 781, so he could take over the palace school at his court and supervise the development of a school system in Francia. The *Propositiones ad acuendos iuvenes*, 56 problems for sharpening the young, have become quite well-known amongst the texts that are attributed to him. The 13 preserved manuscripts were written between the end of the $9^{th}$ and the $15^{th}$ century. We cannot be absolutely certain that Alcuin is the author, since the text does not refer to a




**Illus. 4.1.3** Venerable Bede [Schedel, Nuremburg Chronicle, 1493]; Alcuin of York in the palatial school of Charles the Great, woodcut [Deutsche Geschichte, 1862]

writer. However, there are good indications that speak for him. M. Folkerts, who examined the manuscript and, together with H. Gericke, published a German translation of the problems [Folkerts/Gericke 1993], distinguished (next to some problems not easily classified, such as the well-known one of a wolf, a goat and a cabbage that have to be transported across the river) between three arithmetic-algebraic groups: equations and series, problems of composition, and problems of calculating geometry.

Whereas most problems belong to recreational mathematics, the geometrical problems follow the tradition of the Roman land surveyors and can be found in the same or a similar manner in the so-called *Geometria incerti auctoris*. We re-encounter the very old instruction for calculating a quadrilateral area, i.e., multiplying the averages of the opposite sides with each other. Below, we will demonstrate the example of Problem 25 concerning a round field, including both stated solutions [Folkerts/Gericke 1993], partially also [Hadley/Singmaster 1992]:

*Propositio de campo rotundo*

"There is a round field which contains 400 perticae in its circle. Tell me, how many aripenni ought it to hold?" (pertica = rod, aripennus = 12 rods times 12 rods)

Solution 1:
"A quarter of this field, which contains 400 perticae, is 100. If you multiply [100] by 100, you get 10 000, which you must divide into 12 parts. For indeed, a twelfth of 10 000 is 833, which when again partitioned into twelfths gives 69. This many aripenni are included in the field."

Solution 2:
"Take the fourth part of 400, which is 100. Further, the third part of 400, which is 133. Take the half of 100, which is 50. Take the half of 133, which is 66. Multiply 50 by 66, which is 3151 (possibly a typo? Correct would be 3300; It seems this value is used to continue the calculation.) Divide this into the twelfth part, which is 280 (3300:12 = 275). Again divide 280 into the twelfth part, which is 24. Multiply 24 times 4, which is 96. In total there are 96 aripenni." [Folkerts/Gericke 1993, p. 325/36]

Also see Problem 4.1.1.

Problem 29 is a continuation or further development of Problem 25, reflecting more of a joke problem than an issue of applied geometry: "Proposition concerning a round city":

"There is a round city which is 8000 feet in circumference. Let him say, he who is able, how many houses should the city contain, such that each [house] is 30 feet long, and 20 feet wide?"

Solution 1:
"This city measures 8000 feet around, which is divided into proportions of one-and-a-half to one, i.e. 4800 and 3200. The length and width of the houses are to be of these [dimensions]. Thus, take half of each of the above [measurements], and from the larger number there shall remain 2400, while from the smaller, 1600. Then, divide 1600 into twenty [parts] and you will obtain 80 times 20. In a similar fashion, [divide] the larger number, i.e. 2400, into 30 pieces, deriving 80 times 30. Take 80 times 80, making 6400. This many houses can be built in the city, following the above-written proposal." [Hadley/Singmaster 1992]

Solution 2:
"The circumference of this city is 8000 feet. Take the fourth part of 8000, which is 2000. Further, take the third part of 8000, which is 2666. Take the half of 2000, which is 1000, and the half of 2666, which is 1333. Take the 30th part of 1333 (which is 44, further the 20th part of 1000, which is 50. Multiply 50 by 44), which is 2200. Then multiply 2200 times 4, which is 8800. This is the number of houses." [Folkerts/Gericke 1993, p.330/32]

Also see Problem 4.1.2.

It seems clear that we are dealing here with problems for sharpening the mind rather than anything to be applied directly. Nonetheless, these calculation instructions clarify the level on which Carolingian applied geometry existed.

### 4.1.3 Gerbert d' Aurillac

Only poor fragments, which had been preserved until the end of the $1^{st}$ millennium and were mainly passed on by educated monks in monasteries, constituted the knowledge of scientific mathematics of the classical Antiquity. The first monumental encounter with Islamic science took place shortly before the turn of the millennium, when the French monk Gerbert came to Catalonia from 967 until 970. Since he was elected pope in 999 (ruling as Silvester II until 1003), his mathematical works were very influential, being the first known description of calculating by means of the calculation frame (abacus) in the occidental world. He introduced new numbered calculation stones, calculi, but did not use the customary Roman numbers. Instead, he used the West Arabic or Gobar digits. Although initially not very skilled in mathematics, he found parts of the Boethius Euclid translation as abbot of Bobbio in the monastery library and wrote a book on geometry himself, the original version of which, however, has been lost.

Editions from the $12^{th}$ century clarify just how little was known back then. The book only contains the simplest theorems on geometry and land surveying. Euclid's theory of parallels is not mentioned and the sum of angles in a triangle is obtained by experimentation.

**Illus. 4.1.4** Gerbert d'Aurillac l: as pope Silvester II. (detail from French stamp), r.: monument in Aurillac
[Photo: H.-W. Alten]

### 4.1.4 Boethius and pseudo-Boethius

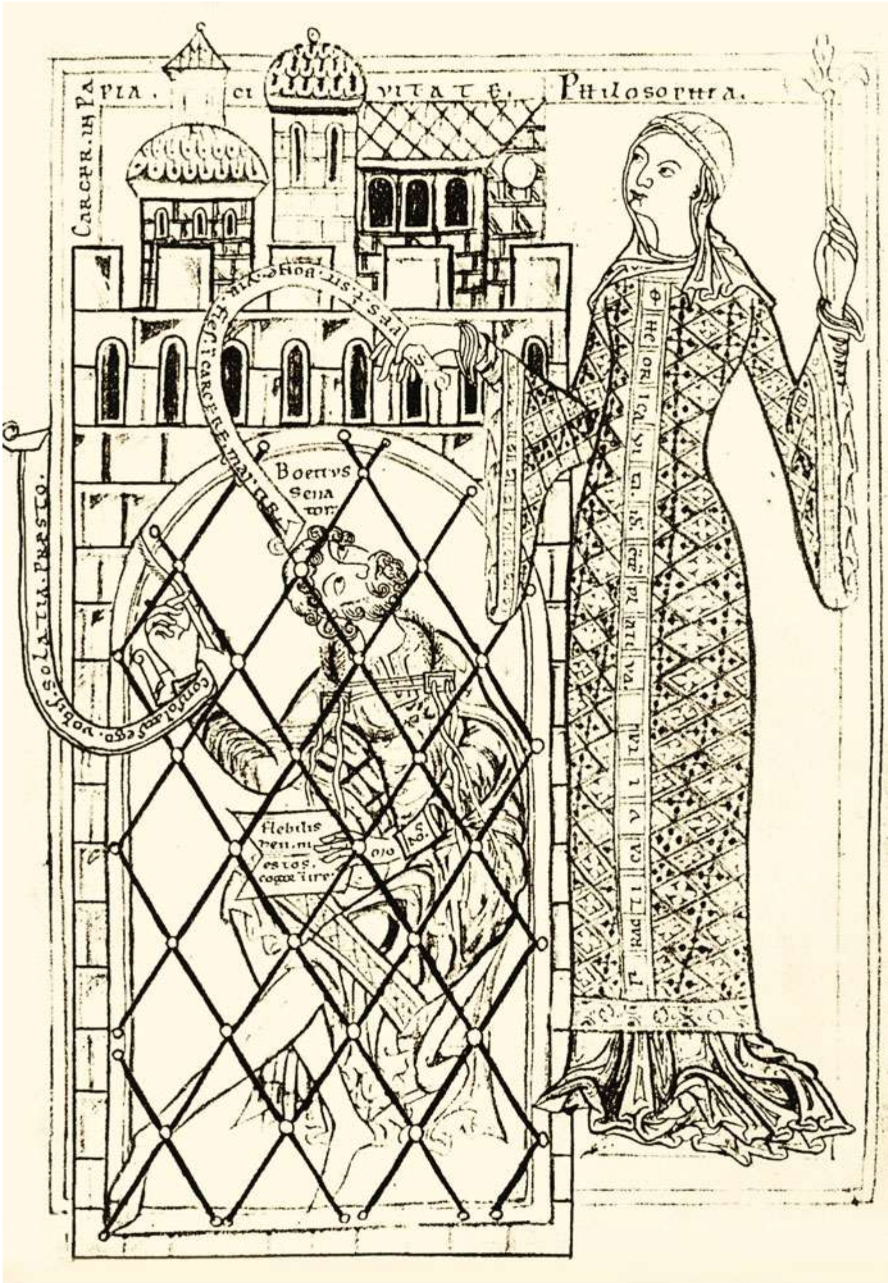
In the first half of the $11^{th}$ century, a compilation, which used to be ascribed to Boethius, was composed by two or three sources. Nowadays, it is referred to as Boethius' 'Geometry II', since the author is unknown. In contrast, 'Geometry I' does not just contain Euclidean excerpts, but also extensive parts of Boethius's arithmetics, agrimensor texts and other excerpts. It is probable that this work was composed in Corbie in the $8^{th}$ century. 'Geometry II' consists of two books. The first one partially covers the Euclidean excerpts from 'Geometry I' extended by a series of proofs and a section on calculating with the abacus. The second book mainly consists of texts of surveying literature, as described above, supplemented by a section on fractions. Although this geometry shows, on one hand, how low the level of geometrical knowledge that arrived in Europe in the $11^{th}$ century was (as was mathematical knowledge in general), it has maintained parts of the Euclid translation from Greek into Latin, which can probably be traced back to Boethius. Moreover, it belongs to the earliest works that show an abacus tablet and the Arabic digits. A selection of drawings (which precede those taken from the agrimensor texts) demonstrates which geometrical problems the anonymous author addressed in his work (Illus. 4.1.5).

### 4.1.5 Scholasticism

Before the occidental world became acquainted with the classical ancient texts on a large scale by means of translations from Arabic, the so-called scholastic method in philosophy and theology was formed under the influence of Anselm of Canterbury and Peter Abaelard around 1100.

Based on the intentions of the church fathers and other authorities, the opposing views were settled in discussions, and decisions were made by means of syllogisms. From around 1175 onwards, this method was linked to Aristotelian logic and, thus, formed a mental training thought of as a means of preparing for mathematical thinking. In the $13^{th}$ century, this scholastic mentality reached its climax under Thomas Aquinus with the successful melding of Christian teaching and Aristotelian philosophy.

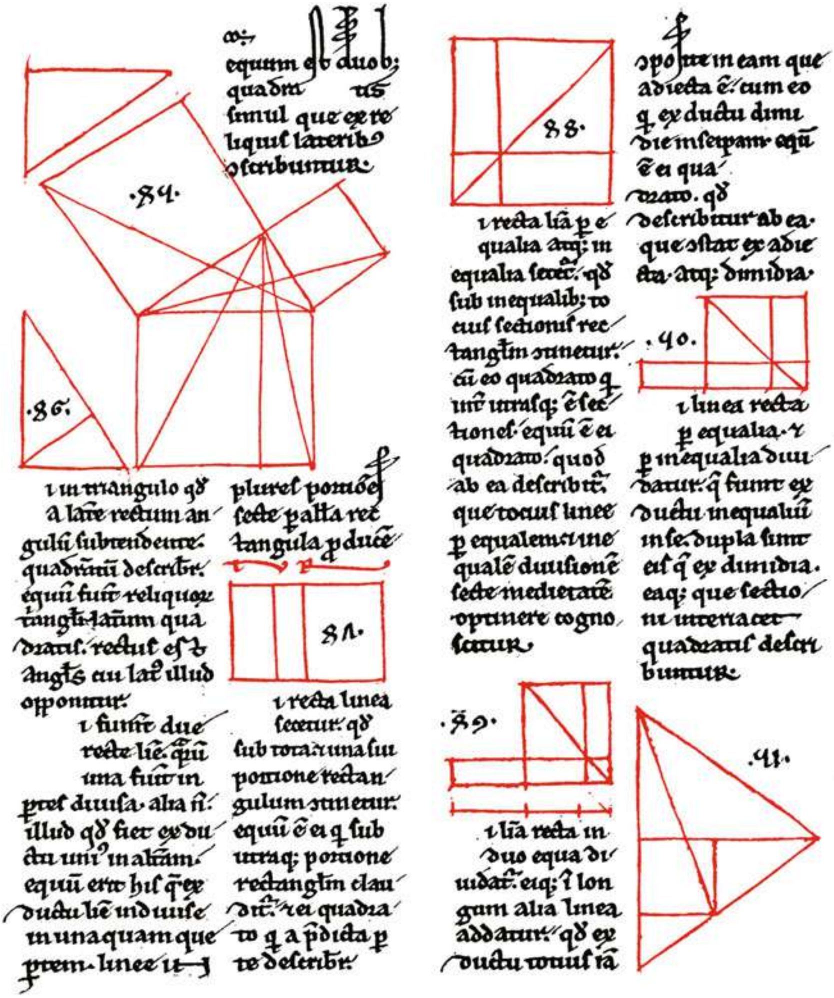
Monasteries and later the great churches had built their own schools to look after their theological offspring. This resulted in European universities being centres of study and education and the carriers of scholastic thinking from 1200 onwards.




**Illus. 4.1.5** Boethius in prison and the consolation of Philosophy (circa 524)
[Bayerische Staatsbibliothek München, Clm 2599, fol. 106v]

### 4.1.6 Translations from Arabic

Countless works of translation started in Spain soon after Toledo had been taken from the Moors as part of the reconquista in 1075. Translations of Arabic scientific and mathematical texts were also done in Southern France and Sicily, the latter of which was conquered by the Normans in 1091 after Arabic rule of 200 years. Translators also began to engage with these texts in a compiling and commentative manner.



**Illus. 4.1.6** Euclid manuscript from Lüneburg [M. Folkerts: *Ein neuer Text des Euclides Latinus*. (A new text of the Euclid Latinus) Facsimilie print of the manuscript from Lüneburg D4°48, $f \cdot 3^r - 17^\nu$. Gerstenberg, Hildesheim 1970]

The educated Adelard of Bath, who was proficient in both Arabic and Greek, translated, amongst other texts, Euclid's 'Elements' from Arabic into Latin. This Euclid translation, supposedly from around 1120 (usually referred to as Adelard I), is very likely to have been the first to feature all 15 books (including the two fake books, XIV and XV) in Latin. Before that, the Middle Ages only knew 'Elements' in the form of excerpts of Boethius's Euclid translation from around 500, whereas the Arabs had been engaging with the complete work since the $8^{th}$ century. Apparently independent of Adelard, Herman of Carinthia and Gerard of Cremona also translated 'Elements' in the $12^{th}$ century. The latter also translated the valuable commentary on Euclid by al-Nayrīzī (Anaritius), the Banu Musa's geometry, and other mathematical and astronomical works.

Until 1500, it was not the pure Euclid translations but edited versions that were most effective. One of them is usually called Adelard II, as it is based on Adelard I. However, it also makes remarkable references to the Boethius tradition. Nowadays, Robert of Chester (as the name indicates, also an English scholar) is accepted as the author. Close to the end of the $12^{th}$ century, efforts were already being made to combine Robert of Chester's compilation with Boethius's tradition. One example of this is a text that only comprises five pages and is only preserved in one manuscript. This text was written in Northern Germany around 1200 and has been owned by the council's library in Lüneburg (Germany) since at least the end of the $18^{th}$ century (see Illus. 4.1.6). Originally transferred to the Michaelis Monastery in Hildesheim by an otherwise unknown priest, Helmold of Bosau, this manuscript contains most definitions, postulates, axioms and theorems of the first four books of 'Elements' (without proofs). The definitions and postulates bear a general similarity to the so-called 'Geometry II' by "Boethius", with axioms and propositions mostly following Adelard II.

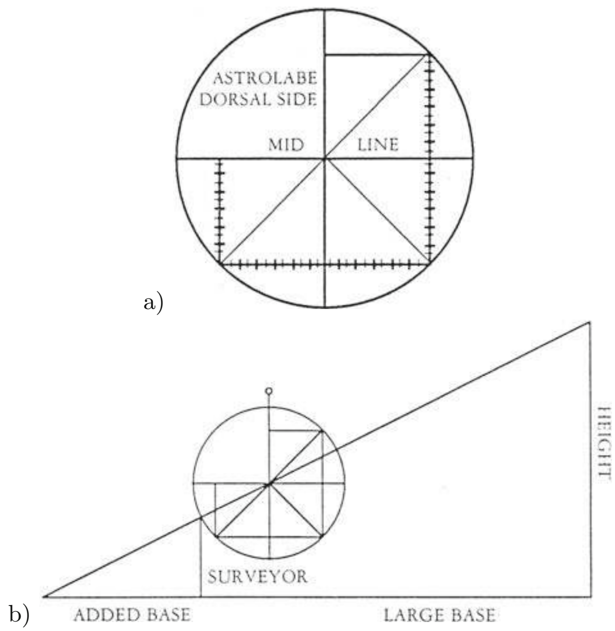
Campanus edited another version of Euclid shortly before 1260. The fact that there are still around 130 manuscripts of this edition preserved (compared to only 23 of Boethius's 'Geometry II') reflects how influential this version must have been. Campanus generally extended the proofs to a considerable degree and partially rewrote them into commentaries to attempt a didactically arranged representation (by means of using remarks or whole sentences by Jordanus Nemorarius).

## 4.2 Practical geometry

### 4.2.1 Hugo of Saint Victor

Next to the scientifically orientated strand of the Euclidean tradition, a series of texts on practical geometry were composed during the Middle Ages. Whereas the former was taught at universities, the latter followed from the agrimensor tradition. Hugo of Saint Victor's *Practica geometriae*, written in the first half of the $12^{th}$ century, became the model for the subsequent era. Hugo structured his work in three sections: altimetry (*altimetria*), calculation of area (*planimetria*), and *Cosmimetria* (calculation of the Earth's circumference, geographical latitude, size of the sun's diameter, solar orbit, ecliptic and some optical problems, such as determining the horizon). Therein, Hugo of Saint Victor, who was a scholar, not a craftsman, also named ancient forerunners. The prologue already shows how different Hugo's style is compared to Euclid's dry texts and his commentators [Homann 1991]:

"My goal is to teach practical geometry to our students, not as something new, but rather as a collection of older, scattered material. Say what you will, I think our predecessors worked miracles. They had immense energy, and tried to get at the truth. Hard work could not dampen their ardour, nor any obstacle deter their efforts. They had deep insights into marvellous and almost incredible matters, and even in lesser ones they provided many examples of wisdom. To equal them may not be possible; not to try would be a disgrace. But enough exhortation; let us address our task." The line is defined as an expansion from one point to another one in any direction, forwards, backwards, to the right or left, up or down; as long as there is expansion, nothing else would be required to suffice the nature and definition of the line. The point is capable of initiating a line in any direction or incorporating one.




**Illus. 4.2.1** Astrolabe according to Hugo of Saint Victor: a) construction, b) application[by permission: Practical Geometry, translated by F.A. Homann, Marquette University Press, Milwaukee, Wisconsin 1991. a) fig 11, p. 42; b) fig 14, p. 45]

The sources consulted by Hugo of Saint Victor do not describe the astrolabes (an instrument adopted from the Arabs) to measure angles and altitudes. A rotatable pole, the alidade (line of sight), is mounted onto a disc, which is to be held perpendicularly. This disc features engraved lines and partial lines as shown in Illus. 4.2.1.a). The alidade reaches the edge of the disc on both sides and features a reading device. Illus. 4.2.1.b) shows how it is applied practically: The geometer (or astronomer) holds the astrolabe in front of his eye along the alidade, which is positioned in a suitable manner, to allow him to focus on his target, e.g., the peak of a tower. As is clear, we must then consider the eye level. Hugo mentioned that the surveyors often carried a stick of this length for this reason (see Problem 4.2.1 and compare with Problem 3.1.2 and Illus. 3.1.5).

A Hebrew book on practical geometry by the Jewish mathematician Abraham bar Hiyya (Savasorda) called 'Treatise on Geometry' was translated into Latin by Plato of Tivoli in 1145 under the title *Liber embadorum*. In the last quarter of the $13^{th}$ century, Robertus Anglicus from Montpellier described the application of another Arabic measuring device, the quadrant (quarter circle) to measure angles.

### 4.2.2 Leonardo of Pisa

At the beginning of the $13^{th}$ century, Leonardo of Pisa ("Fibonacci"), who was primarily known for his extensive arithmetical, algebraic book *Liber abbaci*, referred back to the *Liber embadorum* by Abraham bar Hiyya when composing his *Practica geometriae*. However, in contrast to its title, this *Practica geometriae* is no special work on applied geodesic geometry, but features different theorems with proofs, which refer to measuring, planimetry and stereometry (different from Hugo of Saint Victor's work). It also contains problems on decomposing figures. Leonardo used relevant Greek and Arabic literature, but also added his own theorems or proofs.

### 4.2.3 Johannes de Sacrobosco (John of Holywood)

Neither mathematics nor astronomy at that time was capable of comprehending the difficult argumentation of the classical works of the Greek scholars, much less of advancing them. The elementary level of astronomy also at university vividly documents the popular small book, later reprinted many times, by Johannes de Sacrobosco. However, whereas Leonardo of Pisa was also influenced by oriental sources, Sacrobosco (originally John of Holywood) joined the Anglo-Saxon tradition. Three illustrations may communicate an insight into the quality of this elementary introduction to astronomy. In Illus. 4.2.2a), the division of the world is introduced. Within the fixed starry sky, the spheres of the five planets and those of the sun and moon are arranged around Earth in concentric circles. On the outside (represented as a double ring decorated with the signs of the zodiac), the ninth sphere, the Primum Mobile, surrounds all others and informs them about its motion. Illus. 4.2.2b) shows a lunar eclipse, and Illus. 4.2.2c) illustrates the development of a solar eclipse; Earth is at the centre; the moon, illuminated by the beaming sun, is shown in ten different phases on an eccentric orbit.

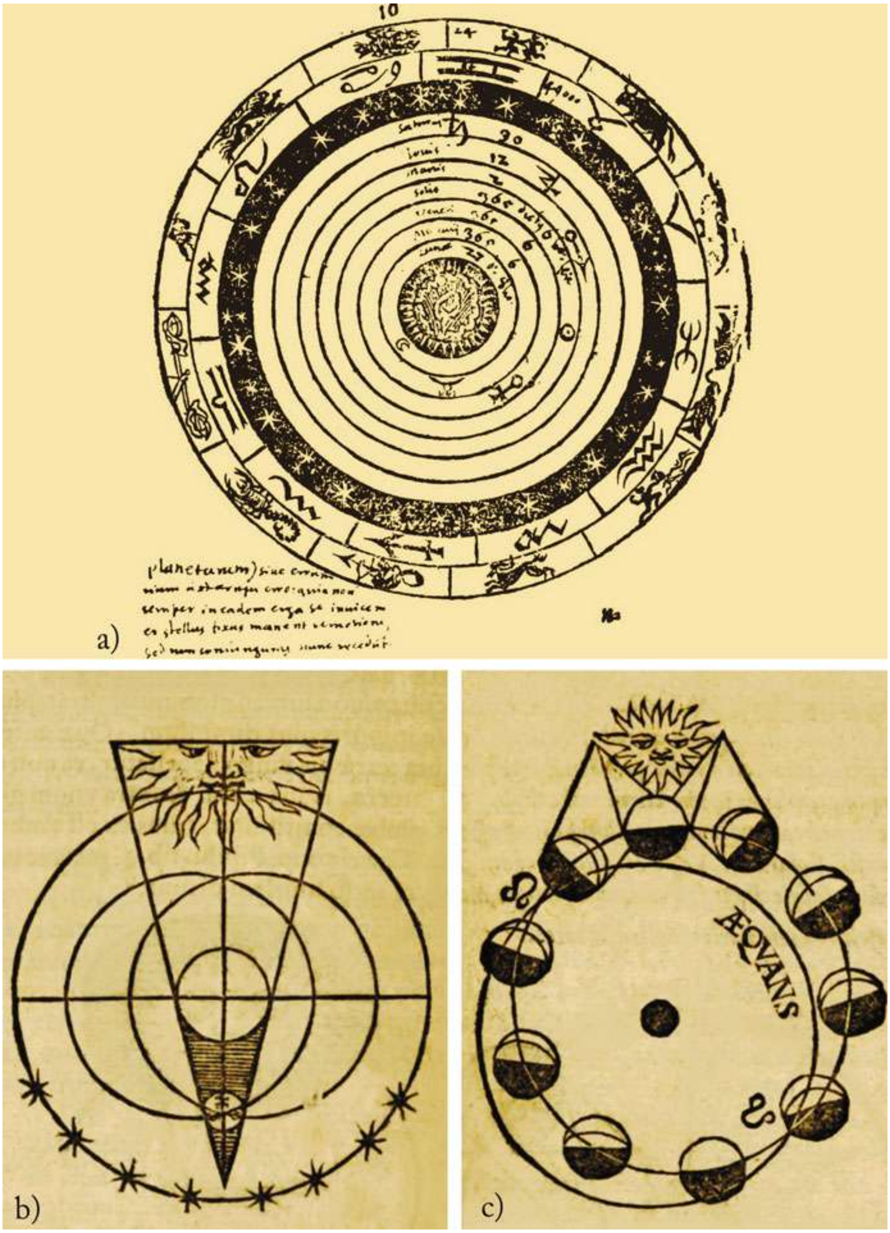
### 4.2.4 Trigonometry

Engaging with astronomy more thoroughly towards the end of the Middle Ages required sufficient familiarity with plane and spherical trigonometry. (Since Greek chord calculation lacks the idea of using right-angled triangles, it would actually be better to speak of triangular theory rather than trigonometry.) The occidental world learned the appropriate propositions from Greek and Arabic sources, whereby the latter had also adopted and advanced methods originally developed in India (cf. Chap. 3).

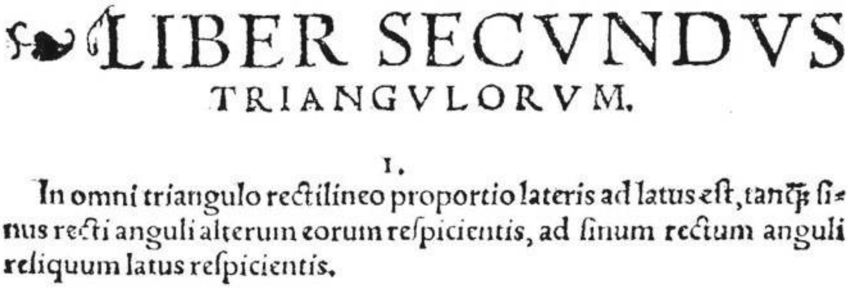
Tables, either chords tables following Ptolemy, later sine tables, or tables concerning other angle functions, were an important component of most trigonometric works. It would go too far to name all these individually and to describe their advancements (especially in regards to accuracy, but also methods of calculation); an approximate overview ought to suffice here.

The works on spherics by Theodosius of Bithynia and Menelaus, which feature the fundamental theorem on transversals for planes and spheres, were well known, after Gerard of Cremona translated them into Latin (cf. section 2.5.6). Ptolemy's *Almagest* was also translated into Latin twice in the $12^{th}$ century. As a result, the astronomers encountered Ptolemy's chord trigonometry. The functions sine, cosine, and versine, introduced by the Indians, feature in one of al-Khwārizmī's works, which had been edited by al-Majrīṭī in the $10^{th}$ century and was translated by Adelard of Bath in the $12^{th}$ century. However, he was not familiar with the Latin word "sinus" (sine), in contrast to Robert of Chester. Albategnius (al-Battānī) perfected the *Almagest*. He systematically used trigonometric lines instead of chords. This perfected version was then translated by Plato of Tivoli in the $12^{th}$ century. Both Richard of Wallingford in the $14^{th}$ century and Regiomontanus in the $15^{th}$ century were familiar with this work. Of course the different tables translated from Arabic were effective and influential, e.g., to Johannes de Lineriis in the $14^{th}$ and Johannes von Gmunden in the $15^{th}$ century. In 1534, Peter Apianus printed the work "Improvements" of the *Almagest* by Jabir ibn Aflah (Geber), translated by Gerard of Cremona. This work had already been used before by, for instance, Richard of Wallingford.

The reformation of astronomy brought about by the School of Vienna in the $15^{th}$ century (Johannes von Gmunden, Georg von Peurbach, Johannes Regiomontanus) was accompanied by a renewal and refining of trigonometry




**Illus. 4.2.2** Astronomical schemata by Johannes de Sacrobosco: a) Structure of the universe (Notice the marginalia of an early reader); b) Schema to explain a lunar eclipse; c) moon phases and solar eclipse [Joh. de Sacrobosco: Sphaera. Venice 1574, p.79, p.12]



LIBER SECVNDVS
TRIANGVLORVM.

I.

In omni triangulo rectilineo proportio lateris ad latus est, tanq̃ sinus recti anguli alterum eorum respicientis, ad sinum rectum anguli reliquum latus respicientis.

**Illus. 4.2.3** Johannes Regiomontanus: Wording of the law of sines [Joh. Regiomontanus: *De triangulis omnimodis.* Nuremburg 1533, p.46]

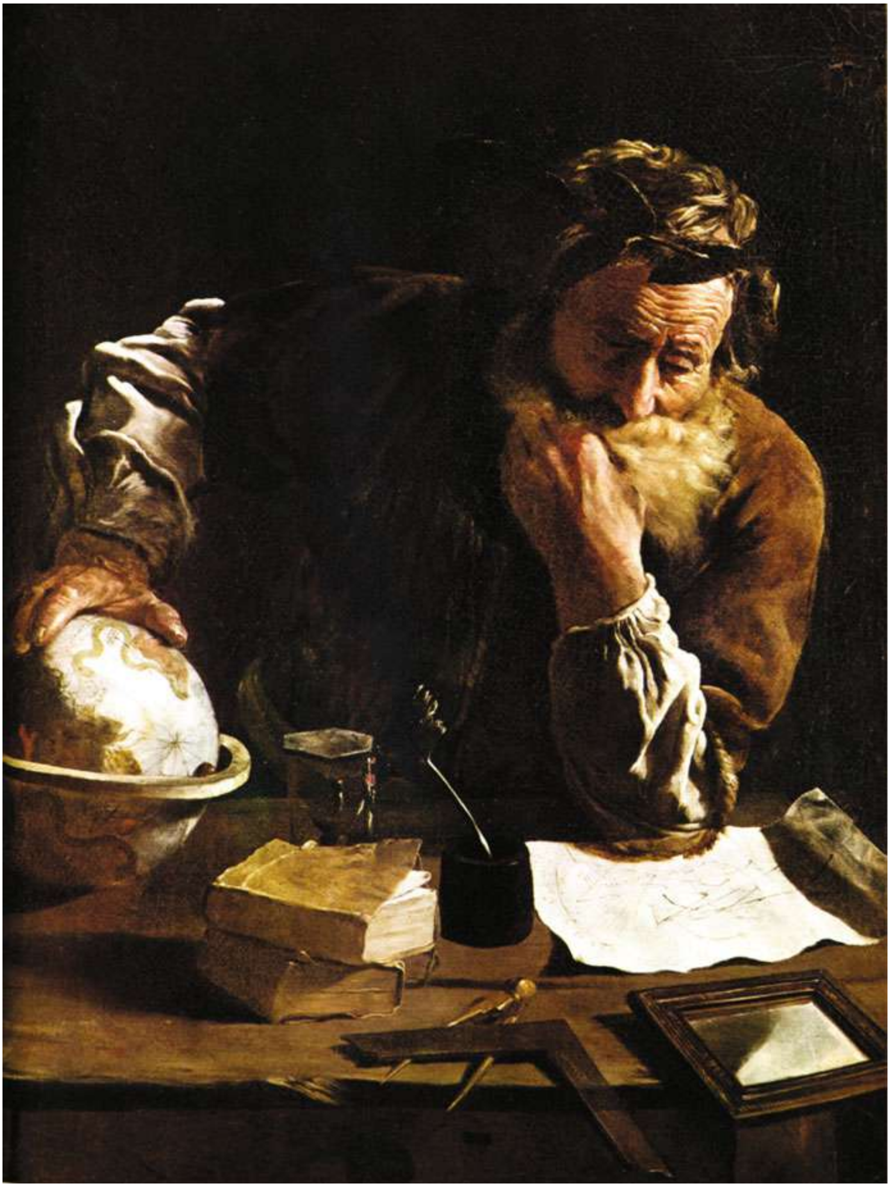
and table calculation. In 1437, Johannes von Gmunden described Ptolemy's and al-Zarqālī's calculation methods in *Tractatus de sinibus, chordis et arcubus* (Treatise on sines, chords and arcs). Georg von Peurbach continued this work. His student Regiomontanus even calculated three sine tables with increasing accuracy. He succeeded in doing so by increasing the circle radius (to avoid decimal or sexagesimal fractions), as was customary in his time: At first, 60 000, second time, 6 000 000, and the third time, $10^7$. Above all, however, Regiomontanus composed the first independent textbook on plane and spherical trigonometry in Western Europe. It was titled: *De triangulis omnimodis libri quinque* (Five books on triangles of all kinds, printed in Nuremburg in 1533). For instance, the book explicitly states the law of sines for triangles (see Illus. 4.2.3 and section 5.2).

## 4.3 Science on the move

### 4.3.1 Translation from Greek

Only after many works had been translated from Arabic did texts begin to be translated on a larger scale directly from Greek into Latin, as far as manuscripts were available. Not too long before, people had still believed that the Euclid translation by Zamberti at the end of the $15^{th}$ century had been the first to be translated directly from Greek. However, we know nowadays that 'Elements' (under Theon's compilation) had already been directly translated from Greek into Latin in Sicily in the $12^{th}$ century.

The most important translator to translate many Greek works into Latin was Wilhelm von Moerbeke. He was active in the $13^{th}$ century. He worked for his friend, the influential theologian and versatile scholar Thomas of Aquinus. Wilhelm travelled to Greece himself twice to look for Greek manuscripts. He translated numerous works at the pontifical court of Viterbo, amongst them mathematical works by Archimedes, Hero, Proclus and Ptolemy.



**Illus. 4.3.1** Archimedes (painting from Domenico Fetti, 1620; Gemäldegalerie Alter Meister, Staatliche Kunstsammlung Dresden)

### 4.3.2 Archimedes in the Middle Ages

The great influence Archimedes bore on the development of mathematics in the early modern ages has long been known. This is demonstrated by, for instance, the very fact that Galileo mentioned him in his works more than one hundred times and praised him highly. The role his works played in the Middle Ages had been less clear until the monumental collected edition *Archimedes in the Middle Ages* by Marshall Clagett since 1964 [Clagett 1964], was published, because, in contrast to Euclid, the number of preserved manuscripts featuring Archimedean works is small. Only in the $12^{th}$ century were Latin translations made, for example, a translation of 'On the Measurements of a Circle' by Gerard of Cremona before 1187. The overriding translation, which comprised the majority of Archimedes' works, comes from Wilhelm von Moerbeke (1269). Apart from one manuscript, he also had access to another, now lost text from Byzantium, which featured most of Archimedes' works in Greek. (Since there were only a few manuscripts by Archimedes in Byzantium, the Arabs also struggled to familiarise themselves with the complete mathematical works by Archimedes. However, they got hold of the crucial methods and knew much better than those in the European Middle Ages how to advance them.)

In 'On the Measurements of a Circle', Archimedes had already proved in Theorem 1 that the area $A$ of a circle equals the area of a right-angled triangle, the sides of which embedding the right angle have the length of the circle radius $r$ or, respectively, the length of circumference $c$. A number of medieval commentators thought it was necessary to carefully prove implied Archimedean assumptions by means of Euclidean theorems. For instance, they proved that more than half of the unexhausted area can be covered by doubling the number of sides of a regular polygon inscribed in a circle stated in a modern fashion, that the exhaustion method does indeed really converge.

Another commentator realised the difficulty lying within the assumption that we could equal the length of a side to the length of the circumference in the mentioned triangle. Thus, he consulted the Archimedean treatise 'On spirals', the first 18 theorems of which address this issue. As a result, he compiled a hybrid edition of both texts. Several manuscripts of this kind from the $13^{th}$ and $14^{th}$ centuries clarify how scholars of that time strove towards understanding and, where thought necessary, also improved Archimedes' works. This proves that there were already noteworthy attempts to engage with Archimedes' studies before the Renaissance (when the Greek text became available). A characteristic example of this is Johannes de Muris's attempt in the first half of the $14^{th}$ century to introduce the problem of measuring the circle in the $6^{th}$ chapter of his treatise *De arte mensurandi* (On the art of measuring):

"To measure the area of a given circle, following a previous estimation of the ratio of the diameter of a circle to its circumference. This ratio perhaps no one yet has truly reduced to number, although Archimedes, the most fervent searcher among the geometers, thought he had demonstrated the ratio of a straight line to a curve by means of spirals. I propose to explain his intention in this regard and concerning the quadrature of the circle in the eighth chapter of this work. Be content [for now] with the [approximate] concord between the prior [terms], which is that three times the diameter of the circle with the addition of a seventh part of it is equal to the circumference [i.e. $d \cdot 3\frac{1}{7} = c$], and that, if from the circumference $\frac{1}{22}$ part is taken, one-third of its remainder is equal to the diameter (i.e., $[c - (\frac{c}{22})] \cdot \frac{1}{3} = d$). I shall speak on the method of [determining] this concord in the following (i.e., in the beginning of the eighth chapter), it having been supposed that perhaps one ought to be satisfied that, if a circle were continuously and regularly moved on a plane until it completes a revolution, the line described in the plane is equal to a circular line. Then with these things assumed (until we demonstrate more fully the equality of a curve to a straight line), it is easy to measure the circle. The common method is this: Multiply (1) the radius by the semicircumference, or (2) the diameter by the semicircumference, keeping half of the product, or (3) either the diameter by the semiperiphery or the circumference by the radius with half the product [retained], or (4) the whole diameter by the whole circumference, with one-fourth of the product taken; or (5) take 11 times the square of the diameter and then assume 1/14 of the product, or (6) subtract one-seventh of the square of the diameter from the square of the diameter and [then subtract] one-half of the one-seventh [i.e., $d^2 - (\frac{d^2}{7}) - (\frac{d^2}{14}) = A$]. The result of produced by all of these methods will demonstrate the area of the circle. These methods, with which our ancestors have until now been satisfied, differ only in the dominion of numbers. And these deductions assume that from the product of a radius and a semicircumference a rectangle is generated which is equal to the circle. This will be evident in what follows."[Clagett 1964, vol. 3, p. 31-32]

As we can tell from the cited text, Johannes de Muris was also hesitant to compare the curvilinear circumference with a straight line segment. For sure, the Aristotelian dictum not to compare curves and straight lines with each other had an after-effect here. Thus, Johannes first tried to describe the circle untwining in the plane and then took the straight line produced during a revolution as length. The reference to the Archimedean spiral, defined as $r = \alpha\,\phi$ applies to its property of featuring exactly one radius vector of length $r = 2\alpha\pi$ after one revolution of the angle; hence, its first intersection with the positive x-axis has the same distance from the origin, which has a circle of radius $a$ as circumferential length.

Explanations as such show the effort by mathematically interested medieval scholars to understand other ancient authors beyond Euclid and to think the featuring problems through independently. Even though some aspects might come across as clumsy or, as here, list trivial alternatives for calculations, we should not ignore the striving towards theoretical clarification in a time that is often incorrectly called the dark Middle Ages.

### 4.3.3 The $14^{th}$ century

The universities of Paris and Oxford became the centres of mathematics and physics in the $14^{th}$ century. Robert Grosseteste, the first chancellor of the University of Oxford, and his student Roger Bacon devised a new ideal of science based on the natural scientific texts by Aristotle and Islamic authors: one's own judgement based on observation and experience and, if necessary, verified by experiments was to decide the truth of concepts of nature, not just the authority of renowned authors. These were the origins of a school interested in studying natural laws and principles of changes. This school was named after Merton College of the University of Oxford and was advanced in the respect that the school attempted to join physical and mathematical considerations just as the school of Paris did. However, their mathematics remained embedded in the general philosophical/theological concepts of late scholasticism.

### 4.3.4 Thomas Bradwardine

The most important representative of this school was Thomas Bradwardine, who later was elected Archbishop of Canterbury. Whilst teaching at Merton College, he composed several mathematical texts, portions of which were studied intensively far beyond England. A treatise on the speed of motion is dedicated to kinematics. The *Geometria speculativa* (theoretical geometry) followed up on the version of Euclid's 'Elements' by Campanus. It often refers to Boethius, but also contains the author's own contributions; for instance, the construction of star-shaped polygons by extending the sides of regular $n-gons$ $(n \geq 5)$ and a discussion of angles of contingence, i.e., angles between circle and tangent. Bradwardine, as well as Campanus before him, came to the conclusion that such angles stand in a certain irrational ratio to linearly bordered angles, however different from the irrationality that exists between the side and the diagonal of a square.


**Illus. 4.3.2** Inscription above an entrance of Oxford University [Photo: K.A. Gottwald]

**Illus. 4.3.3** Merton College of the University of Oxford, founded 1264
[Photo: K.A. Gottwald]

Bradwardine also studied the question of filling space without gaps by means of regular solids. His geometry was printed around 1500, published in several editions and highly appreciated by the subsequent generations of mathematicians. For what it is worth, we just want to mention here that he also established his own law regarding the speed of a motion in dependence on force and resistance, which contradicted the Aristotelian theory.

In the treatise *De continuo*, which touches on philosophical as well as physical and mathematical matters, he justifies his opinion that the continuum could not consist of indivisible atoms. If a line segment $d$ is only made of a finite number of points, which cannot be divided further, he went on to say that, for example, we could draw a semi-circle above this line segment as its diameter $d = 2r$ and erect a perpendicular in each of these points. Then, the semi-circular arc would be intersected in just as many points as the diameter; hence, the full circular arc $C$ in double as many points. Thus, we would have to conclude: $C = 2d$, which contradicts all our experience.

Nicole Oresme in Paris picked up and continued the investigations conducted in Oxford, whereby he also helped himself to geometrical interpretations of different motion types or quality changes. Hence, we speak of the 'formlatitudes': the change in latitude of a form illustrates the increase or decrease of a quality, e.g., of speed. The investigations of mathematicians in the $14^{th}$ century were often inspired by engaging with Euclid's work and represent the climaxes of mathematical work in the European Middle Ages. At the beginning of the $17^{th}$ century, these considerations, mainly from Oxford and Paris, found new interest: it is not by coincidence that the approaches of Galileo's motion theory and Cavalieri's principle, also called the method of indivisibles, bear similarities to the late scholastic schools.

**Illus. 4.3.4** Robert Grosseteste, Chancellor of Oxford University and Bishop of Lincoln ($1^{st}$ quarter pf the $14^{th}$ century [British Library Harley MS 3860, f.48])

## 4.4 Applied geometry in the High and Late Middle Ages

### 4.4.1 Villard d' Honnecourt

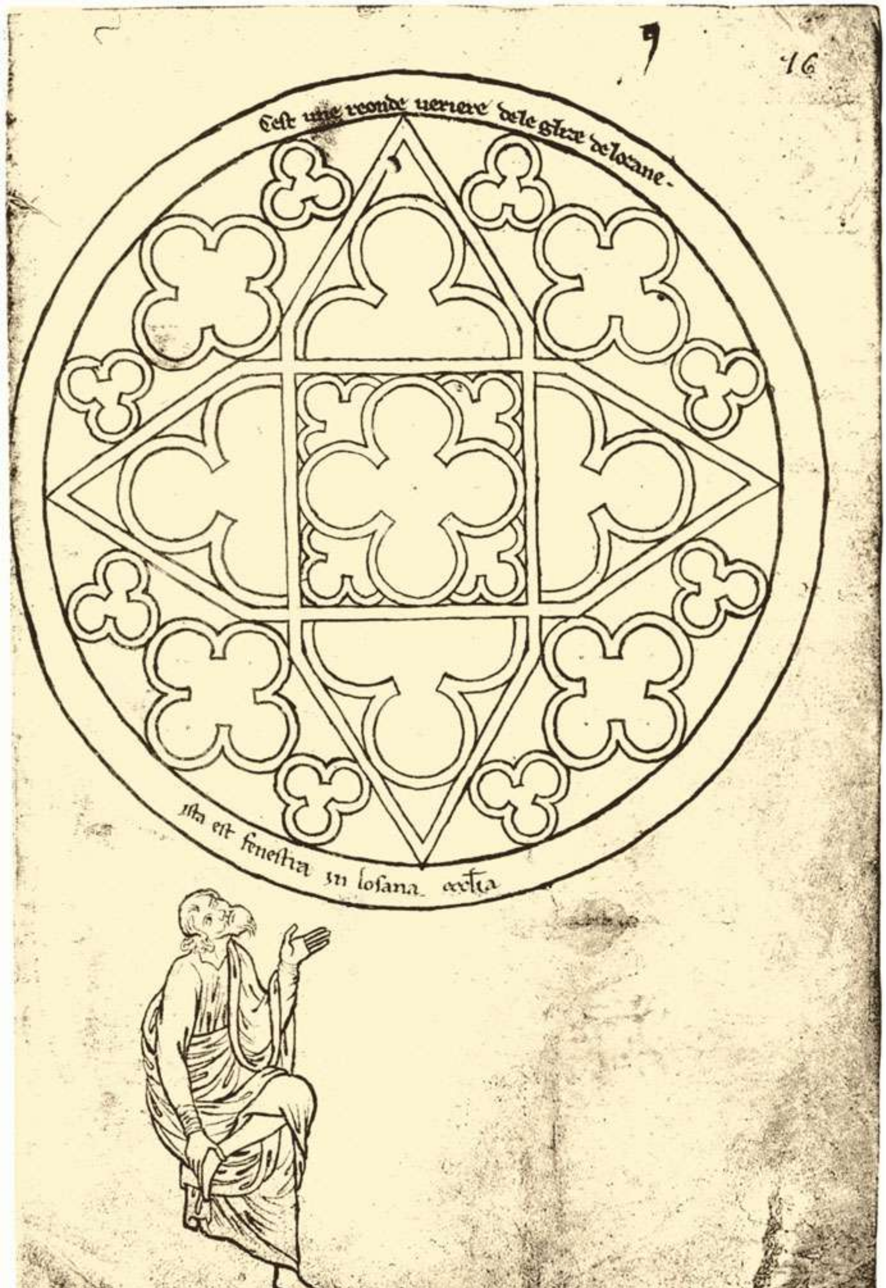
To conclude this chapter, we will again turn towards applied geometry; to be precise, to that aspect of geometry that formed the "tools" for the medieval "Bauhütten" (cathedral builders' guilds). However, the master builders' specialised knowledge was only passed on orally according to the rules of the guilds. Hence, apart from one exception for the Early and High Middle Ages, the sketchbook of Villard d' Honnecourt, there are no noteworthy sources regarding the geometrical knowledge of the master builders and masons. Thus, we still stand in front of the great medieval church and monastery buildings in amazement and admiration. These constructions silently certify their creators' skills, but also offer room for speculation due to the absence of written sources. We only know for sure that medieval architecture was strongly dominated by symbolic and mythical ideas.

The master builder Villard d' Honnecourt came from the province of Picardy (close to Cambrai) and travelled through France, Switzerland and Hungary numerous times in the $13^{th}$ century. He noted down numerous details of significant buildings in his sketchbook, which originally consisted of 63 parchments: building drawings, stone mason work, representations of technical aids, but also scattered bodily compositions (of humans and animals). Unfortunately, the sketchbook hardly contains any text next to the 325 preserved pen drawings. Illus. 4.4.1 and 4.4.2 will give the reader an impression of the quality of the drawings. The first picture shows a rose window of the Cathedral of Lausanne. (The upper French inscription says: "C'est une reonde veriere de leglize de Lozane", the lower Latin one states: "Ista est fenestra in Losana ecclesia" (in English: This is a [round glass] window of the Cathedral of Lausanne.)

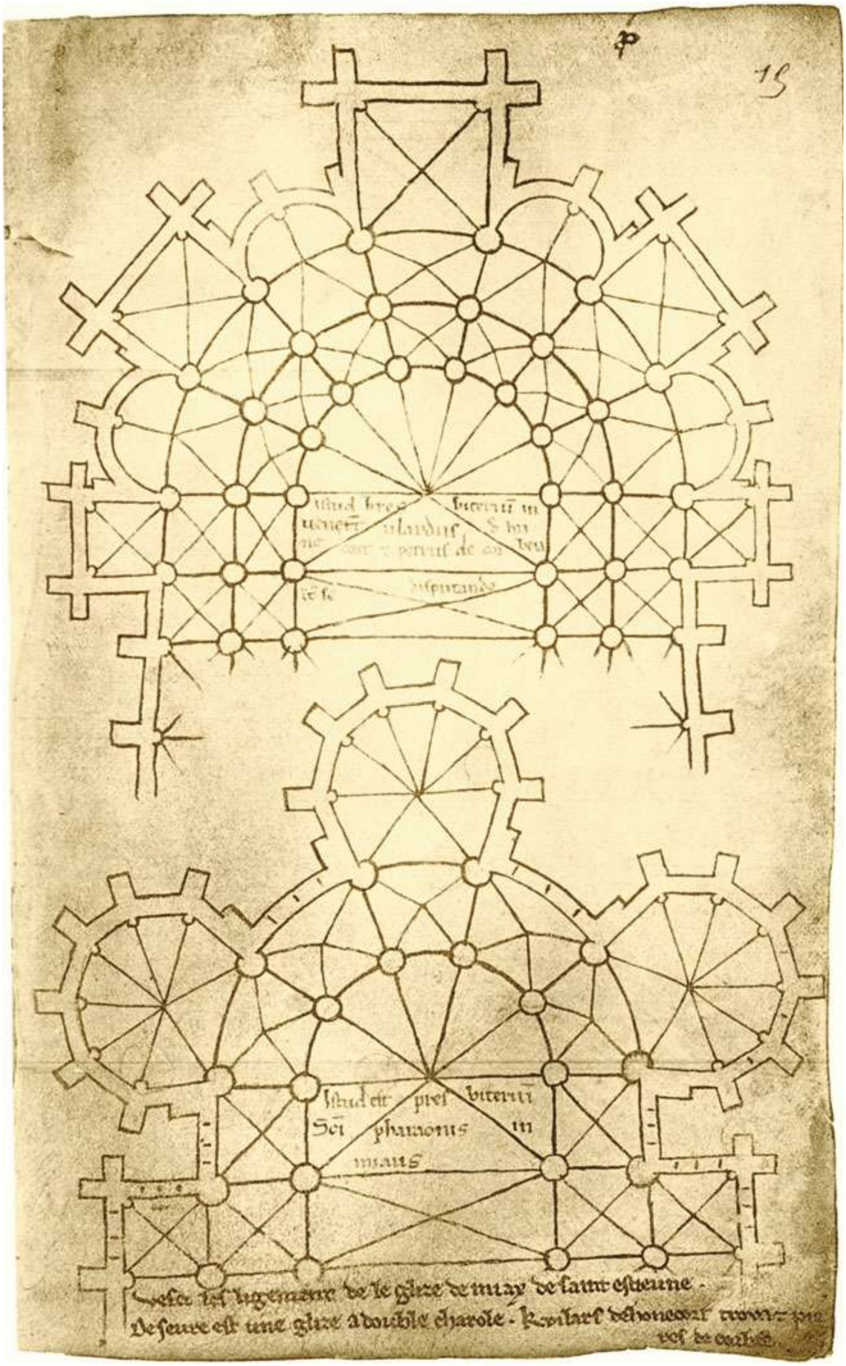
The second illustration (4.4.2) shows views of the chancel vault of two cathedrals. The inscription of the upper drawing says: "Istud bresbiterium invenerunt Ulardus de Hunecort et Petrus de Corbeia, inter se disputando" (This presbytery was designed by Villard d' Honnecourt and Petrus de Corbie when arguing with each other.)

### 4.4.2 The "Bauhütten" booklets

The four so-called "Bauhütten" booklets are structured differently. They were written in Southern Germany around 250 years later in the German language. Similar texts from the time before 1486 have not been preserved or, perhaps, just simply were not written, since the specialised knowledge of master builders, masons and carpenters was passed on orally from generation to generation. These four early printed works concern three small texts by Mathes



**Illus. 4.4.1** From Villard d' Honnecourt's sketchbook (1230-35): Church window of the Early Gothic Cathedral of Lausanne (consecrated in 1275) [Villard d' Honnecourt Album. Paris, nd]



**Illus. 4.4.2** From Villard d' Honnecourt's sketchbook (1230-35): Views for two church chancel rooms: Illustration of a rip vault
[Villard d' Honnecourt Album. Paris, nd]

Roriczer – *Büchlein von der Fialen Gerechtigkeit* (Booklet Concerning Pinnacle Correctitude), *Wimpergbüchlein* (Booklet on Gablets) and *Geometria Deutsch* (Geometry [in] German) and *Fialenbüchlein* (Booklet on Pinnacles) by Hans Schmuttermayer.

Pinnacles are small decorative turrets that crown buttresses and cap wimpergs, (Gothic ornamental gables with tracery over windows or portals, which were often accompanied by pinnacles; originally "wimperge" = protecting from the wind: protective gable). The pinnacle consists of a shaft or body and a spire, whereby the shaft can also serve as a tabernacle a hollow space to accommodate figures.

In this context, correctitude refers to the accurate draft of the drawings or views to which the masons had to adhere. The first "textbooks" for this decorative design at Gothic cathedrals are also significant from the perspective of the history of printing: they are rare incunables from the first decades of printing. They concern small texts: the first 'Booklet concerning pinnacle correctitude' comprises 16 pages (only four copies have been preserved, partially incomplete) and 'Geometry [in] German' is six pages long, with the last three pages featuring instructions as to how to construct wimpergs.

The family Roriczer fathered four master builders of cathedrals in three generations in Regensburg. The first, Wenzel, probably came from Bohemia and was trained by the famous master builder family of Parler in Prague. His work as a master builder at the Cathedral of Regensburg covered the time from 1411 until his death in 1419. His son, Lorenz, worked in Eichstätt and at the chancel construction of St. Lorenz in Nuremburg before he was put in charge of continuing the construction of the Cathedral of Regensburg in 1456. During his twenty years of work, he concluded the main portal, created a magnificent draft with one tower (not executed) and was consulted as an expert regarding the building of St. Stephen's Cathedral in Vienna as well as Frauenkirche (full name: 'Cathedral of Our Dear Lady') in Munich. His older son was Mathes Roriczer, the author of the three mentioned texts. He published them in his own printing factory the first domestic one in Regensburg in 1486/88. In the meantime, he had succeeded his father and become master builder of the Cathedral of Regensburg.

Shortly after Roriczer's booklet on pinnacles, Hans Schmuttermayer's 'Booklet on Pinnacles" was released in Nuremburg, also dealing with the construction of wimpergs. The author was a goldsmith in Nuremburg. He had to mount similar motives onto shrines and monstrances as the masons onto the cathedrals. Perhaps this provoked his interest in these geometrical constructions. As Roriczer did, Schmuttermayer referred to the master builder family Parler. Hence, both stand in the tradition of Gothic architecture. Only one copy of Schmuttermayer's text exists nowadays.

Mathes Roriczer started his booklet of pinnacles with a dedication to the Bishop of Eichstätt, who had encouraged him to publish. This dedication speech presents his intention in old-fashioned German:

**Illus. 4.4.3** Tracery and figures of the late Gothic period at the Cathedral of Rouen, France [Photo: H.W. Alten]

**Illus. 4.4.4** Apse of the Cathedral Notre Dame of Reims, France. In his sketchbook Villard d' Honnecourt has adhered the design of buttresses and their pinnacles [Photo: H.W. Alten]

"...since every art is materials, form and measure, I have to explain some of the touched art of geometry with God's help and first intend to explain the beginning of drawn stonework, how and by what means this is meant to basically originate from geometry and by means of using the compass and to be adjusted to a suitable extent. I also have drawn up a little depiction of the following forms. This is my intention, which is only meant to benefit the greater good." [Roriczer/Shelby]

Both authors agree that the pinnacle construction starts with the so-called "crossing over location". A square is put in a second square used as a floor plan by linking the centres of the sides. This process is repeated multiple times. By turning each second square around the centre by 45°, we obtain a series of nested squares with parallel sides (Schmuttermayer displays eight in his 'booklet on pinnacles'.) Two consecutive squares have a side ratio of $1 : \frac{\sqrt{2}}{2}$ ; they form the higher positioned cross sections of the pointy pinnacle turrets, which are to be transferred in order from the foot in exactly determined intervals. Having designed the body of the pinnacle, Roriczer describes the construction of the spires and finally the draft of the decorations (flowers and leaves), (cf. Illus. 4.4.5). The mason received his instruction in a recipe-like manner: mark certain line segments with the compass and transfer them onto the work piece.

Apart from the "crossing over location", a construction principle we already find in Plato's, Vitruvius's and Villard's works, and the halving of the line segments, Roriczer also sometimes speaks of trisection. First of all, he names all marked points with letters. Those points and all auxiliary lines are only eliminated at the end. Roriczer concluded:

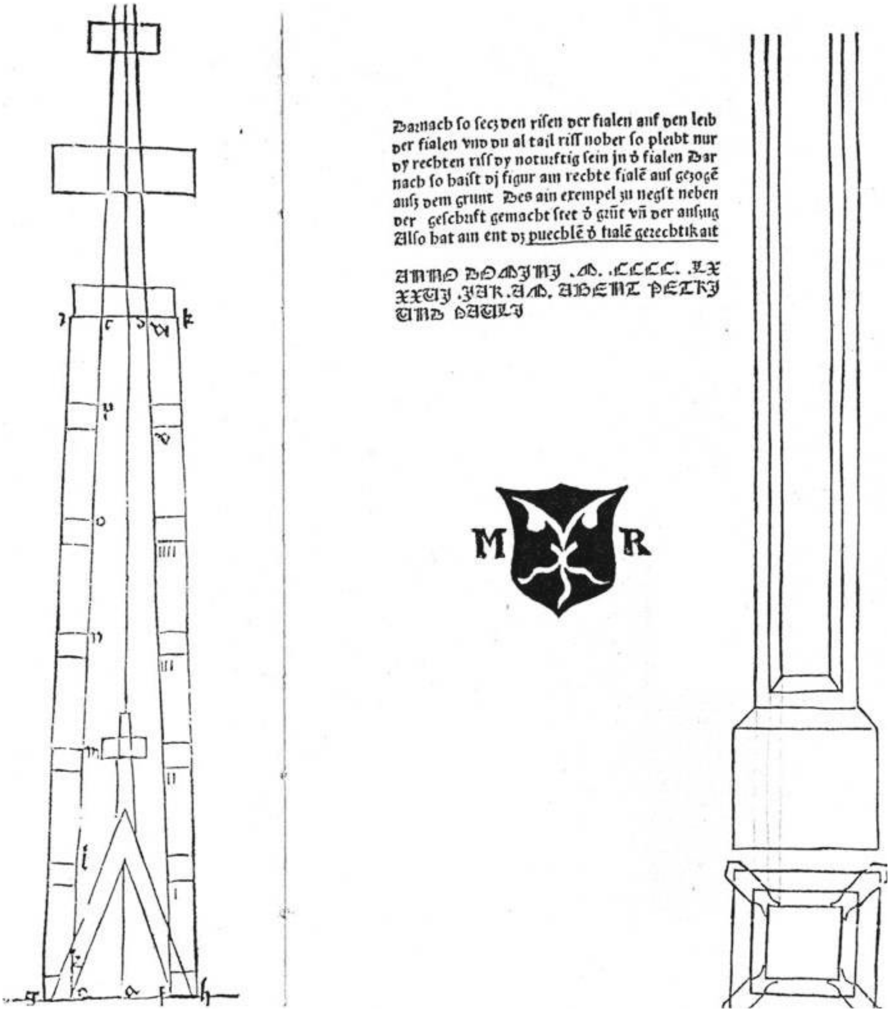
"Afterwards put the spire of the pinnacle on the body of the pinnacle and remove all part lines [auxiliary lines]. Thus, only the right lines remain, which are necessary for the pinnacle. Now the figure is called a right pinnacle [i.e. a right, appropriately constructed], drawn based on the top [view]. This [is] an example next to the text, i.e. the top and front view. Thus, the booklet of pinnacle correctitude comes to an end. Anno Domini M. CCCC. LXXXVJ. Jar. In the evening Petrj and Paulj." [Roriczer/Shelby]

Next to systematically using letters to refer to points and occasionally drawings too, it is remarkable that an entire construction can be executed without any calculations and, furthermore, that all other measures can be constructively derived from the initial square. If we consider how long it took to build cathedrals, the advantage of such a method becomes evident: they helped later generations of stone masons execute their predecessors' intentions and conclude unfinished constructions.

Simple versions of pinnacle decorations did not constitute a problem, since just the dimensional ratios had to be altered a little. However, the magnificent overall impression of the "Gothic style" remained.

Roriczer's 'Geometry [in] German', the first printed book on geometry in the German language, features "a number of chapters [meaning: single constructions] from geometry", e.g., the construction of a right angle, of a regular penta-, hepta- and octagon, and how to locate the "lost" centre of a circle (all in all, seven constructions). The pentagon construction is special, since it is done with a fixed span of the compass (see Problem 5.3.2). Albrecht Dürer also described this construction, which is useful for practitioners.

The heptagon construction is very simple: A chord of same length is put into a circle of radius r (i.e., die side of a regular hexagon). Then, we draw a radius, which meets this chord at its centre. The part between circle centre and chord centre is then transferred onto the circumference seven times, i.e., it will represent the side of the regular heptagon (see Problem 4.4.1). Consider that, in contrast to the pentagon, there is no heptagon construction that can be carried out exactly by means of compass and straightedge (cf. 2.4.2 and 3.4.2). Some of these constructions have a long history. The one with the right angle goes back to Proclus; the one with the octagon was stated by Hero.

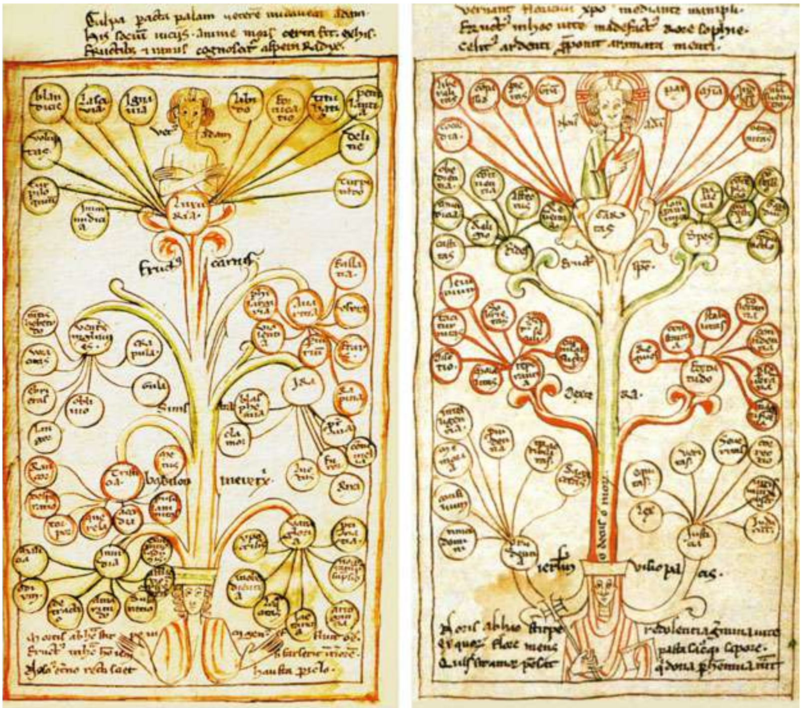



**Illus. 4.4.5** End of the 'Booklet Concerning Pinnacle Correctitude' by Mathes Roriczer: Two front views, ground view, end of text and printer's mark (in red). [Roriczer 1486]

Nonetheless, this small booklet on geometry, just as the booklets on pinnacles, stand in strong contrast to the ancient proving geometry. Therein lies a significant difference compared to the works of the Renaissance, such as Dürer's 'Four Books on Measurement'. These do not just contain recipe-like instructions, but also attempt to convey the mathematical justifications, at least rudimentarily. In other words, the leading "heads" are concerned with linking science and praxis and, thus, to provide praxis (as Leonardo da Vinci did with the arts) with a scientific foundation.

### 4.4.3 Visualisation

The speculative, philosophical approach to thinking in the Middle Ages benefited the early development of graphic representations of abstract entities, which cannot be seen themselves, and their mutual relations [Schreiber 2003]. This is a branch of applied geometry, which was abruptly interrupted by the fact that the Renaissance turned towards the worldly aspects of life and empirical perception. It was only revived in the $19^{th}$ century, but is nowadays indispensible in all branches of science and techniques. We especially find circular diagrams as well as directed and undirected graphs with vertices and edges labelling, for instance, the tree of sciences, the tree of virtues, and the tree of vices (see Illus. 4.4.6). The last mentioned example is particularly interesting, since both trees are identical from a graph-theoretic viewpoint (apart from the number of the top end knot) and, hence, both graphs yield a mapping between each virtue, or, respectively, sub-virtue, and the corresponding vice. We also want to refer the reader to the invention of the European notation of music as a predecessor of the plane coordinate principle (cf. remark on p. 326).



**Illus. 4.4.6** Tree of vices and tree of virtues (from a collective edition of the $14^{th}$ century, Hessian county and university library, Darmstadt, l. Hs 815, fol. 33v, r. Hs 815, fol. 34r)

**Essential contents of geometry in the European Middle Ages**

| | |
|---|---|
| Late Roman ($6^{th}$century) | Boethius,Cassiodor: Some elementary Euclidean geometry |
| $11^{th}$century | Boethius' 'Geometry II': A mathematical compendium with elementary Euclidean excerpts based on Boethius and excerpts from agrimensor manuscripts |
| Around 1120 | Adelard of Bath: Euclid translation from Arabic into Latin |
| $12^{th}/13^{th}$ | Hugo of Saint Victor, Leonardo of Pisa: Practical geometry, use of simple instruments (elementary geometry influenced by agrimensor praxis) |
| $13^{th}$century | Johannes de Sacrobosco: Very elementary spherical geometry as basis of explaining sky movements |
| Around 1235 | Villard de Honnecourt: Architectural sketchbook |
| $14^{th}$century | Thomas Bradwardine: "speculative" (=theoretical) geometry<br>Nicole Oresme: Theory of formlatitudes (geometrical representation of variable qualities) |
| $15^{th}$century | Johannes von Gmunden: Treatise on sine function, chord and arc |
| Around 1500 | Mathes Roriczer, Hans Schmuttermayer: "Bauhütten" booklets |

## 4.5 Problems to 4

**Problem 4.1.1:** Circular area based on circumference
Consider that the circumference is expanded to a square in the first solution, whereby the second solution reveals the Babylonian formula $A = \frac{c^2}{12}$ (1.2.9).

**Problem 4.1.2:** Alcuinus: rectangular houses in a circular town

a) Which geometrical concepts stand behind the first solution? To what extent is the shape of the house included in the course of the solution?
b) Of what nature is the connection of the second solution with the second solution of the preceding problem?
c) The respective problem in *Geometria incerti auctoris* requires us to deduct the $22^{nd}$ part of the circumference and to divide the rest by 3. This is then the diameter, the half of which is multiplied by half the circumference. The

result is divided by 600. Which considerations stand behind this instruction? [Folkerts/Gericke 1993, p. 332]

d) How would we solve this problem nowadays? Which of the three methods comes closest to the real solution?

**Problem 4.2.1:** Applying the astrolabe according to Hugo of Saint Victor

Hugo explained that four congruent triangles occur in Illus 4.2.1b). The ratio of both catheti (nowadays, we would say 'tangent') can be read off the scale.

a) Which measurements and which calculation steps are necessary to determine the height of the objects focussed on?

b) If it is not possible to measure the distance between the land surveyor and the object directly, Hugo recommends focusing on the target from two different standpoints (this method was also known by the agrimensores). Follow his instructions:

   "To measure the height of an object in front of you without moving from your place, raise the astrolabe to the object. Adjust the medicline until you can see the top of the object through both apertures. Then compare the medicline degree reading with the whole side of the square. The ratio of the medicline reading to the whole side (i.e., to twelve) is the ratio of height to intervening space with surveyor's height added either proportionally or exactly. If the intervening space is impassable because of an obstacle such as a river or a gorge, you can still get your result. Use the astrolabe where you are. Adjust the medicline to the top of the object until you can see it through both apertures. After this, note how many degrees of the side of the square below appear above the medicline. Compare them to the 12 degrees of the whole side. By rule, this is the ratio of height to intervening distance plus surveyor's height. [Hugo had explained before that "increased by the observer's height" is only taken to be literal in one measure in diagonal direction. Otherwise, this quantity is to be converted proportionally.]

   Next, move back some distance to a second position. Take the astrolabe, and sight the top along the medicline. Record the medicline degrees on the square side and compare them to the whole side. The ratio is now that between height and intervening space plus surveyor's height. Then compare the first and the second base, to determine how much the second exceeds the first. Now compute the length of the first base by means of the difference between first and second, i.e., the distance between your first and second positions.

   For example, suppose the medicline marker reading at the first station is four. Because twelve is the triple of four, the intervening space plus surveyor's height will be triple the object's height. Suppose the medicline marker reading at the second station is three. Because twelve is the quadruple of three, the space plus surveyor's height will be quadruple the object's height. So suppose the first station separation plus surveyor's height (the first base) is triple the object height (the perpendicular), and the second station separation plus surveyor's height (the second base) quadruple the object height. Clearly, the second base is one and a third times the first. A third part of the first base will be the excess of the second over the first. Evaluate this distance, and take it as just one third of the first base.

A warning. This distance is not always that from first to second station. The surveyor's height adjustment is not the same at both. Rather the distance is measured from the end point of the first addition (where the first base ends) to the end of the second addition (where the other ends). This gives the true difference between bases." [Homann 1991, p. 46-47].

c) This is how the instruction ends. How long is height $H$ of the object focussed on?

Tip: Name the distances between both observation points and the object $d_1$ and $d_2$, the difference $d = d_1 - d_2$, the surveyor's eye level $h$ and show that we obtain $H = 2d + h$ for the stated measured values.

**Problem 4.4.1:** Heptagon construction according to Roriczer

- Calculate the angle at centre, which belongs to one side of the heptagon as constructed by Roriczer, and compare it with the angle at centre $\frac{2\pi}{7}$, which occurs in the exact, regular heptagon.
- Compare this to the heptagon construction by Hero and Abū'l-Wafā addressed in Problem 3.4.3 on p. 216.

# 5 New impulses for geometry during the Renaissance




C.J. Scriba, P. Schreiber, *5000 Years of Geometry*,
DOI 10.1007/978-3-0348-0898-9_5

| | |
|---|---|
| $14^{th}$ century | Painter Giotto di Bondone (1266–1337) and poets Petrarca (1304–1374) and Boccaccio (1313-1375) mark the beginning of the Renaissance |
| 1370 | Hanseatic league first gains dominance over the Baltic Sea Area |
| 1397–1524 | Kalmar Union of Scandinavian countries under Danish leadership |
| 1419 | First Northern European university is founded in Rostock (Germany) |
| 1434–1498 | Florence under rule of House of Medici |
| 1436 | First treatise on perspective (Piero della Francesca) |
| Around 1445 | Beginning of printing with movable letters (Gutenberg) |
| 1452–1519 | Leonardo da Vinci |
| 1453 | Turks conquer Constantinople, end of Byzantine Empire |
| 1453 | End of Hundred Years' War between France and England |
| 1471–1528 | Albrecht Dürer |
| 1475–1564 | Michelangelo Buonarrotti |
| 1492 | First globe of Earth in Europe (Martin Behaim, Nuremburg) |
| 1492 | Columbus rediscovers America |
| 1494 | Spain and Portugal share New World (Treaty of Tordesillas) |
| Around 1510 | Peter Henlein makes first pocket watches in Nuremburg |
| 1517 | Luther's Ninety-Five Theses, beginning of reformation in Germany |
| 1518–1550 | Adam Ries' books on calculation |
| 1519–1522 | First sail around the world |
| 1543 | *De revolutionibus orbium coelestium* by Copernicus is printed |
| 1543 | Paracelcus founds modern medicine |
| 1547 | Ivan IV (the Terrible) becomes tsar |
| 1548–1603 | Elisabeth I rules England |
| 1560 | First European academy is founded in Naples |
| 1564–1616 | William Shakespeare |
| 1569 | Mercator's projection of the world (angle-preserving) |
| 1582 | Gregorian calendar takes over from Julian calendar (first in Catholic countries) |
| 1587 | First attempt at forming colonies in America (Virginia) |
| 1588 | Decline of Spanish Armada |
| 1609 | Kepler publishes the first two laws on planet movements |
| 1610 | Galileo publishes sensational astronomical discoveries with the telescope |
| 1614 | First logarithm tables (John Napier) |
| 1618 | Beginning of the Thirty Years' War |

## 5.0 Preliminary remarks

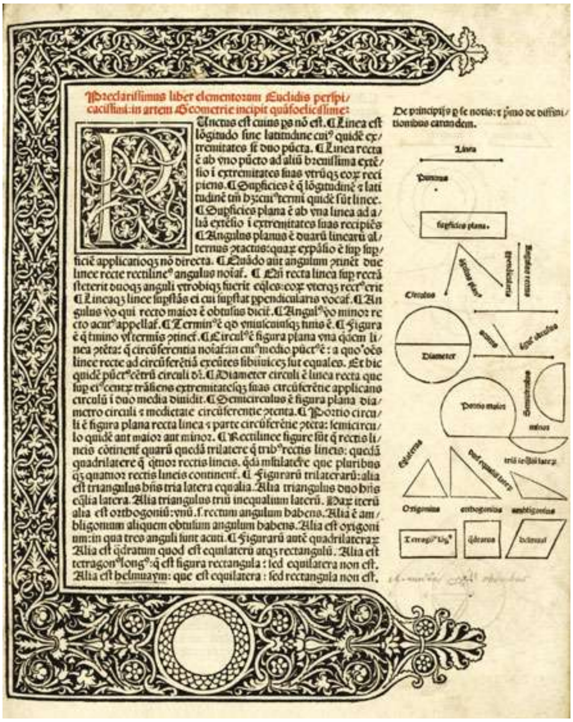
The following time period of around 230 years between approx. 1400 and approx. 1630 is usually referred to as the Renaissance (i.e., rebirth, namely of Antiquity) in the history of science, although this is not entirely accurate and cannot be reconciled with the art-historical periods. The time frame we have agreed on here differs from all other periods of the history of mathematics by combining two features:

- In contrast to the preceding time, a clear turn towards praxis and developing numerous new areas of applications and applications themselves. (The second half of the $20^{th}$ century has this in common with the Renaissance.)
- The essential advances often do not come from the educated scholars themselves, but partially from practitioners of all kinds: masters of calculation (Adam Ries), engineers (Simon Stevin), artists (P. della Francesca, Dürer), craftsmen (Bürgi), marine engineers, merchants (Thomas Gresham),..., and also partially from educated amateurs[4] enthusiastic about mathematics: physicians (Gemma Frisius, Robert Recorde, Cardano), jurists (Vieta), noble estate owners (John Napier), courtiers (G. B. Benedetti), ... This sort of accumulation is unique in the history of mathematics.

Mainly, the above-named practitioners and amateurs interested in the practical aspects of science started organising academies in the middle of the $16^{th}$ century to serve as a platform for the exchange of scientific ideas and concepts. These academies, which were only private at the beginning, were then transformed into sovereign institutions from the middle of the $17^{th}$ century or were re-founded by the sovereigns according to the model of those that already existed. Of course, parallel to this turn towards praxis, as emphasised above, secondary and higher education grew steadily quantitatively and qualitatively in this era and far beyond the medieval standard. “Professional” mathematicians, meaning those teaching at schools and universities, also wrote about practical problems of geometry on occasion. However, the Renaissance primarily fostered the scholar’s interest in the almost forgotten old Greek language and efforts to gain access to the original texts or to restore texts of ancient authors, which, until then, were only known through retranslations from Arabic, or from references of their existence in different sources. This was benefited by the fact that after the final ruin of the Byzantine Empire (1453, Turks conquer Constantinople), Greek scholars sought refuge in Italy and brought with them the Greek language and a series of manuscripts that were still unknown in the rest of Europe. Having said that, the invention of printing around 1445 made it possible to circulate scientific texts in larger quantities.

[4] The word ‘amateur’ can have a negative connotation nowadays. However, originally it comes from ‘amare’ (Lat. to love) and refers to a person who engages with something not professionally but - stated in a modern fashion - as a hobby, whereby material interest played a subordinate role at most.

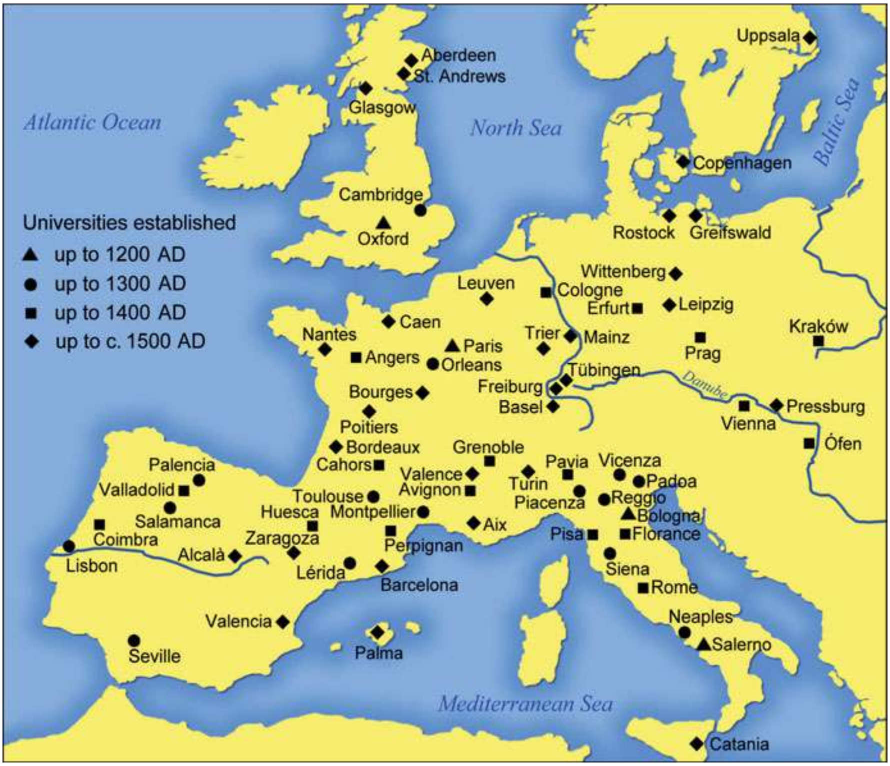
Indeed, 'Elements' was one of the first printed books (1482 in Venice by printer Erhard Ratdolt from Augsburg, Illus. 5.0.2). It immediately became a "bestseller", which, over the course of the next 500 years, lived through countless translations, versions and editions, of which we will report in due course. If we take into account that ancient mathematics was basically geometry, it was, of course, much easier in the Renaissance to advance in arithmetics, algebra and numerical mathematics beyond the ancient level than it was in geometry. If we measure the advancements only by considering the amount and difficulty of new geometrical theorems and solutions of constructive problems, we will soon see that the mathematicians of the Renaissance put a lot of effort into reacquiring the ancient knowledge and rising to that level. However, if we look at the variety of new practical challenges geometry had to face, the wealth of means conceived to solve such problems, and the contribution of geometry to the social effectiveness and acknowledgement of mathematics, the Renaissance appears as one of the most fruitful eras concerning the historical development of geometry.




**Illus. 5.0.2** Title page of the first printed version of 'Elements', Venice, 1482. It differs from all later versions in the aspect that it does not name a publisher, translator or printer and starts with the actual text, i.e., with the definitions of Book I, straight away after just one introductory sentence

## 5.1 Geometry at schools and universities, Euclid during the Renaissance

Following the formation of the first universities, the dates of which are often not precise, an era of regularity started in around 1360: sovereigns (including ruling bishops), as well as rich trading towns, demanded a state university. A pontifical bull permitted such formations. Thus, the following universities were founded (as well as many others): Prague (1348) Krakow (1364), Vienna (1365), Heidelberg (1386), Cologne (1388), Erfurt (1389), Leipzig (1409), Rostock (1419), Löwen (1425), Greifswald (1456), Ingolstadt (1472), Uppsala and Tübingen (1477) and Copenhagen (1479). The arising Protestantism (1517, Luther's 95 Theses) split the European education system deeply and sustainably. In reaction to the higher Catholic schools, which mainly originated from monastery or cathedral schools, the first Protestant and humanistic grammar schools were established. The Society of Jesus was founded in 1534 with the aim of re-catholicisation. Part of their strategy was to establish their own educational system with a high level. Jesuit lecture series were created




**Illus. 5.1.1** Foundation of European Universities in the Middle Ages [Map: H. Wesemüller-Kock]

depending on the situation of each town either within the already existing schools and universities or as competition. Hereby, Euclid took on an especially prominent role. A series of the most important printed Euclid editions of this time were published by the Jesuits and adopted to form part of the Jesuit educational institutions. Among them was the most extensive edition by Christopher Clavius, which, since its first publication in Rome in 1574, had gone through over twenty editions up to 1738. Clavius added, apart from the late ancient Books XIV and XV, a sixteenth book to the actual Euclid text and further extensive supplements, which covered almost all historical and mathematical issues that had surrounded this text for almost 1900 years. Many aspects we suspect to be part of 'Elements', but look for in vain, such as the fourth congruence theorem for triangles or the construction of the common tangent of two circles, are to be found in Clavius's version. Other Jesuit Renaissance Euclid editions come from Stephanus Gracilis (1557), Johann (Johannes) Lanz (1617) and Carolus Malapertius (1620), amongst others. Matteo Ricci, one of Clavius's students and one of the first missionaries in China, translated parts of Clavius's work into Chinese from 1603 to 1607. He received help therein from domestic literature experts. As a result, he was able to strengthen crucially the Jesuits' position at the Chinese Emperor's court. In 1594, the Jesuits even had an Arabic Euclid version printed in Rome (according to the so-called Pseudo-Tusi). This version was intended to serve as a tutorial for the Jesuits sent to Islamic countries.

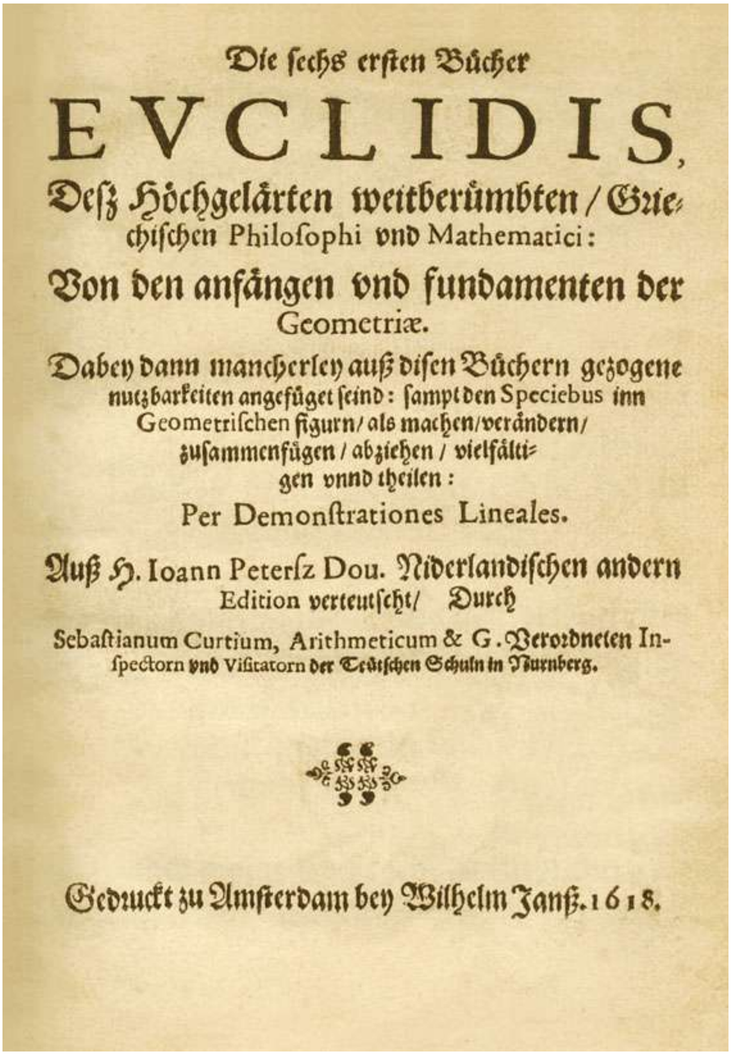
Whereas the first printed edition of 'Elements', already mentioned in 5.0, was based on the Latin text by Campanus of Novara from around 1260, a new Latin translation of 'Elements' as well as 'Data', 'Optics', 'Catoptrics' and 'Phenomena' by Bartholomeo Zamberti appeared in 1505, based on a Greek manuscript. This initiated a dispute about the nature and meaning of studying Euclid that continued throughout the centuries under different aspects. Campanus, just as his Arabic sources had done, strove for a text as mathematically meaningful as possible. Zamberti, who was more a philologist than a mathematician and also more aware of his original sources, insulted Campanus by calling him "the most barbarian of all translators": However, he failed just as often to convey the content accurately. Old Greek is an extremely equivocal language. Hence, it has been difficult up to the present day to grasp the meaning of ancient mathematical texts from a sole philological perspective. Nevertheless, Zamberti, who had been the first to make an "original" Greek Euclidean text generally available in print, represented the beginning of a new debate over Euclid's legacy: his works were more understood as cultural heritage, a classical tutorial and reading material, rather than a mathematical textbook. This, unfortunately, set the image of Euclidean geometry for many generations of pupils and students, often decreasing their interest in and understanding of mathematics.

As already stated, the revival of Old Greek as the language of science was part of the humanistic program and was especially practised at the mostly Protestant grammar schools. The first Greek edition of 'Elements' in print was published together with Proclus's commentary in Basel in 1533. It was published by the Hellenist Simon Grynaeus, who belonged to the circle of Erasmus of Rotterdam. Copies of this edition, as well as the oldest Latin printed versions, can still be found at libraries of very old universities or grammar schools nowadays. The mathematics professor J. A. Segner from Halle (Germany) was still defending the following opinion in a foreword of a new Latin school edition of 'Elements' from 1773: the withdrawal of Greek in favour of Latin school editions, enforced by the increasingly insufficient knowledge of old Greek, would lead to a sad loss of the students' oral and writing skills, as well as their ability to express themselves and to think logically (quote in [Schreiber 1987a, 124]).

To adumbrate the surrounding environment of the Euclidean tradition during the Renaissance, we must stress that 'Elements', in full and in parts, had been published in numerous popular, folk-like printed editions in modern languages of that time since the middle of the $16^{th}$ century: Italian in 1543, English in 1551 and 1570, German in 1555 and 1562, French in 1564 and 1616, Spanish in 1576 and Dutch in 1606. This continued until the $18^{th}$ century and ought to be distinguished from contemporary Euclid editions in modern languages, which usually cater to strict historical and philological demands and target a completely different readership. The folk-like literature on Euclid from the $16^{th}$ until the $18^{th}$ century, written by masters of calculation and educated amateurs, but also by university professors, omitted proofs in favour of detailed examples and applications, reduced Euclid's purely theoretical theory of proportions to numeric calculations, and displayed the authors' own scientific specialities regarding surveying, using proportional dividers, or even recreational mathematics. The practical use expected from buying these books and their simple comprehensibility were promoted tabloid-style in words and pictures on the title pages (Illus. 5.1.2).

Geometry was still part of the seven liberal arts in the Renaissance and, thus, belonged to the curriculum at the Faculties of Arts, the "seeds" of the later philosophical faculties. Graduating successfully was a requirement for studying theology, medicine or law. The students were quite young, compared to nowadays, when studying the "studium generale". There was no real difference between the level of mathematics at university and at other educational institutions (grammar schools, colleges, schools for squires, academies for knights). However, there were some universities that took an increased interest in mathematical sciences, such as Vienna or Krakow.

Geometry was divided into "geometria speculativa" (theoretically orientated) and "geometria practica" (practically orientated). The theoretically orientated part of Renaissance geometry was arranged around some preferred topics, such as squaring the circle or the problem of the "horn-shaped" angles: the starting point is an unclear addition to Proposition III.16 of 'Elements', which claims that the angle (of so-called contingency) between a circle and its tangent is smaller than any acute angle with straight arms. This text passage was sufficient for Clavius and others to speculate in page after page about



Die sechs ersten Bücher
EVCLIDIS,
Deß Höchgelärten weitberümbten / Grie-
chischen Philosophi vnd Mathematici:
Von den anfängen vnd fundamenten der
Geometriæ.
Dabey dann mancherley auß disen Büchern gezogene
nutzbarkeiten angefüget seind: sampt den Speciebus inn
Geometrischen figurn / als machen / verändern /
zusammenfügen / abziehen / vielfälti-
gen vnnd theilen:
Per Demonstrationes Lineales.
Auß H. Ioann Petersz Dou. Niderlandischen andern
Edition verteutscht / Durch
Sebastianum Curtium, Arithmeticum & G. Verordneten In-
spectorn vnd Visitatorn der Teutschen Schuln in Nürnberg.

Gedruckt zu Amsterdam bey Wilhelm Janß. 1618.

**Illus. 5.1.2** Title page of a German edition of books I – VI of 'Elements' [Staats- und Universitätsbibliothek Dresden, SLUB, Lit. Graec. B. 1569]

such quantities, which are "infinitely small, yet different to zero". It is by all means possible that these texts, still known by all mathematicians in the $17^{th}$ century, inspired the development of infinitesimal mathematics. Other mathematicians of the Renaissance, such as J. Peletier from France, vehemently rejected the idea that the angle of contingency is an infinitely small positive quantity. We can find first approaches to a theory on a structured character of an axiomatically arranged geometry embedded in speculations, which are difficult to comprehend nowadays, in the work of the Parisian scholar P. de la Ramée, who was predominantly involved with philosophy. He said it was not the duty of geometry to clarify the (physical) nature of objects, such as the point and the straight line. Postulates are not logically mandatory for geometrical considerations. However, assuming them should be motivated philosophically and/or didactically. Henry Savile, professor at the University of Oxford, who himself held lectures on 'Elements', founded a chair for geometry under the condition that each owner would have to engage with the "naevi" (birth defects) of 'Elements', namely the parallel problem and the notion of ratio concerning proportionality, which was used on an insufficient axiomatic basis. The passion for systematising and classifying brought about by scholasticism also rubbed off on geometry. For instance, when listing all possibilities of how to construct a triangle by means of given sides and angles, Clavius encountered the fourth theorem of congruence, which is not mentioned by Euclid. Other authors investigated how many different types of quadrilateral, penta-, hexa-, ...gons there are.

Although special chairs for mathematics were established at several universities, for which several mathematicians worked, not forgotten until today, it was quite common to have had mathematics taught along the way by professors of other specialities. Kepler's academic teacher in Tübingen, M. Maestlin, was also professor of Hebrew. J. J. Scaliger, who worked for a mathematical chair in Leiden at the end of the $16^{th}$ century, was highly distinguished within the area of chronology as part of the historical sciences. However, a solution to the problem of squaring the circle, which he had thought of as correct and also published himself, was strongly attacked by Vieta, Snellius, Ludolph van Ceulen and other contemporaries due to the inherent mistakes of his proposal. Scaliger rejected this criticism, arrogantly arguing that a renowned scholar like him could not be expected to deal with mathematics in the style of a fencing master (van Ceulen) or surveyor (Snellius) (cf. contribution from Vermij in [Hantsche 1996]).

Nonetheless, university professors also composed texts on practical aspects of geometry, such as geodesy, cartography, optics and perspective, astronomy and astrology, ballistics, architecture, and the construction of fortresses. Hereby, only questions with apparent practical relation, such as "On division of figures" according to Euclid, were addressed. Such works partially served as a simple means of topping off the mostly poor pay of university professors; for instance, the contributions of W. Snellius in Leiden and P. Apianus in Ingolstadt on geodesy, cartography and navigation, as well as many practical works by W. Schickard in Tübingen [Hantsche 1996]. Moreover, there is a *Geometria practica* in eight books by Clavius (1604), which is the first to describe, amongst other things, the nonius (a forerunner of the vernier scale) as a means of improving the accuracy of readings for linear measurements. (P. Nunes, Lat. Nonius, after whom the device was named, had proposed a similar device for angle measurements in 1542.) Clavius transformed the basic problems of spherical geometry into problems of plane geometry by means of stereographic projection in order to solve them. Clavius recalculated the


**Illus. 5.1.3** Christoph Clavius (Engraving after a painting by [Francesco Villamena 1606]) and Piero della Francesca (presumably a selfportrait; detail of fresco, Museo civico di Sansepolcro)

error concerning the approximate construction of the regular pentagon with a fixed span of the compass, as known from the “Bauhütten” praxis mentioned in the previous chapter and taught in *Geometria deutsch* (Geometry [in] German) by M. Roriczer and Dürer (1525) (cf. Problem 5.3.2). This is interesting since it proves that scholars like Clavius also acknowledged, at least partially, the literature written by artists and craftsmen.

Joachim Jungius chose his own path with his *Geometria empirica* from 1627 [Jungius 1627], a textbook with multiple editions for academic grammar schools. Therein, he tried to include the pupils’ empirical experience in the introduction to geometry by also permitting them to experiment with geometrical devices, such as compass and straightedge. His aim, hereby, was to foster the pupils’ insights into geometrical relations. Jungius’s German translation of his book, which, unfortunately, is only known incompletely, stressed this aim in its title, ‘The art of ripping or experiential figure introduction’ (original: *Reiß-Kunst* oder *Erfahrmessiger Figuhrkündigung*).

## 5.2 Geometry in astronomy, geodesy and cartography

Interest in astronomy had been one of the strongest incitements for engaging with mathematics since the oldest civilisations up to the beginning of the $19^{th}$ century. Although it was just pure curiosity at the beginning, next to cultic, religious motives and the needs of calendar calculation, astrology had increasingly been the main motive for advancing astronomy since the Late Antiquity. Many significant patrons of astronomically-orientated mathematics in the Middle Ages and Renaissance, from Alfonso X of Castile and Friedrich II of Hohenstaufen to the German emperors of the end of the $16^{th}$ century, were motivated to support mathematicians and later also to print mathematical works, mainly due to their belief in astrology. However, we also need to include the role of astronomy since the $15^{th}$ century as an auxiliary science for developing nautical science and geodesy, which, over the course of approx. three hundred years, will almost completely replace astrology as the motive for astronomy.

Astronomy, from a mathematical perspective, is first of all geometry of the celestial movements projected onto an imagined sphere. Hence, it is not surprising that spherical trigonometry developed coequally to plane trigonometry for a long time (although it does not form part of the contemporary mathematical school and general education). Trigonometry was not an independent branch of mathematics until the $15^{th}$ century, but inherently connected to its main astronomical application. However, astronomy was, of course, accepted as a mathematical science as part of the quadrivium. This did not just influence terminology considerably, e.g., umbra recta = shadow of the vertical gnomon on a horizontal plane for cosine, umbra versa = shadow of the horizontal pole on a vertical plane for sine (Illus. 5.2.2), azimuth and rectascensio to denote special angles, but also led to a limitation of the discussion of notions and problems over a long period of time to only include those that actually occur in astronomy. For instance, the law of cosines was phrased and proven as early as 1593 by Vieta, in clearer explanation by Bartholomaeus Pitiscus 1600, not because finding the respective polar principles is relatively difficult, but rather because determining a spherical triangle by means of its three angles is nowhere to be found in positional astronomy. For the same reason, the tangent function was introduced just as late.

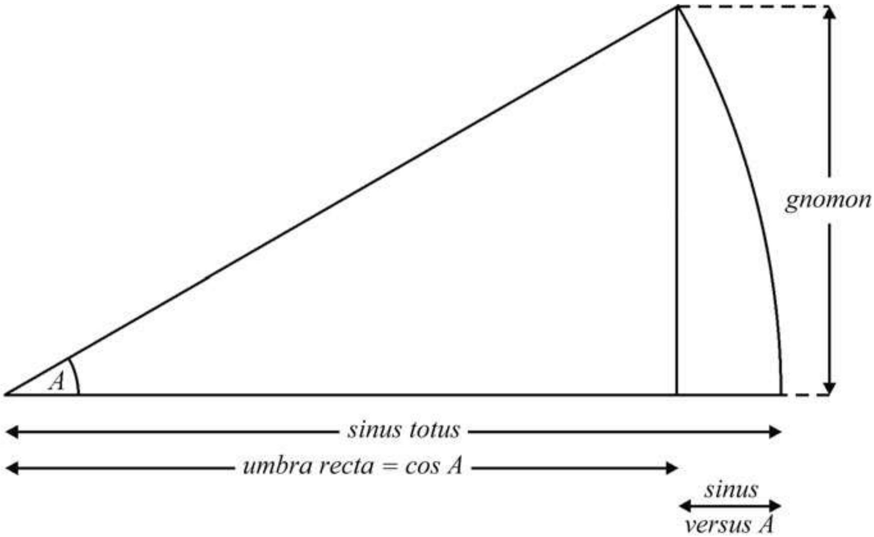
Apart from the difficulties already mentioned (lack of systematics, terminology nowadays being uncommon and clumsy, barely formalised notation), further obstacles hindered those that engaged with modern trigonometry when attempting to understand the trigonometry of the Renaissance. For example, the values of the numerous tables calculated back then (one line segment length in dependence of an angle measured by one degree, minute and second) did not relate to the unit circle, but to an assumed radius, which differs from case to case (and is then called sinus totus when the sine value of 90 degrees occurs). In order preferably to avoid fractions, this sinus totus was ascribed a very high measure, e.g., 60 000 (by Regiomontanus), 100 000 (by


**Illus. 5.2.1** Regiomontanus (Johannes Müller) and Vieta (François Viète)

Regiomontanus, Rheticus, Maurolico), 600 000 (by Johannes von Gmunden, Georg von Peurbach), 6 000 000 (by Regiomontanus), $10^7$ (by Napier) and $10^{15}$ (by Rheticus).

These general remarks will now be followed by a brief chronological overview. [Braunmühl 1900] offers a portrait that is very rich in details, but requires correction or supplements in many aspects. [Hamann 1980] and the updated English translation by [Zinner 1990] of the classic biography of Regiomontanus from 1938 represent more extensive and more recent literature on this subject.

A school of astronomy and trigonometry was established by Johannes von Gmunden at the University of Vienna soon after 1400. His student and successor Georg von Peurbach was mainly known as the last to advocate the old Ptolemaic astronomy before it was superseded by Copernicus' and Kepler's world view. His student and friend Regiomontanus (actually Johannes Müller) continued his works, which were unfinished due to his early death. Regiomontanus, who, after a long trip to Italy and a temporary stint working for the Hungarian king Matthias Corvinus, settled down in Nuremburg and pursued three tasks there: first, he ran a factory for producing instruments (partially invented or improved by himself); second, he engaged in the translation and printing of classic mathematical and astronomical texts; third, along with other works, he wrote the five books *'De triangulis omnimodis'* (On triangles of every kind), with which he founded trigonometry in Europe as a systematic discipline independent of astronomy (see Illus. 4.2.3). Regiomontanus has often been celebrated as the most significant German mathematician of the $15^{th}$ century. Accordingly, the literature on him




**Illus. 5.2.2** Geometrical representation for umbra recta, sinus totus and sinus versus

is extensive. However, it is clear that on top of his own contribution, he owed much of his acquired knowledge to ancient Islamic and also medieval European literature (here particularly by Levi ben Gerson).

Whereas the first four books (chapters) are based on his influential work on the law of sines of plane and spherical geometry and all triangle problems are basically solved by means of decomposing them into right-angled triangles, the fifth book phrases and proves the law of cosines of spherical geometry as a theorem that applies to any spherical triangle for the first time, although the wording is difficult for a modern audience to read, namely:

$$sinversA : (sinvers\ a - sinvers(b - c)) = sintotus^2 : sinb \cdot sinc. \tag{5.2.1}$$

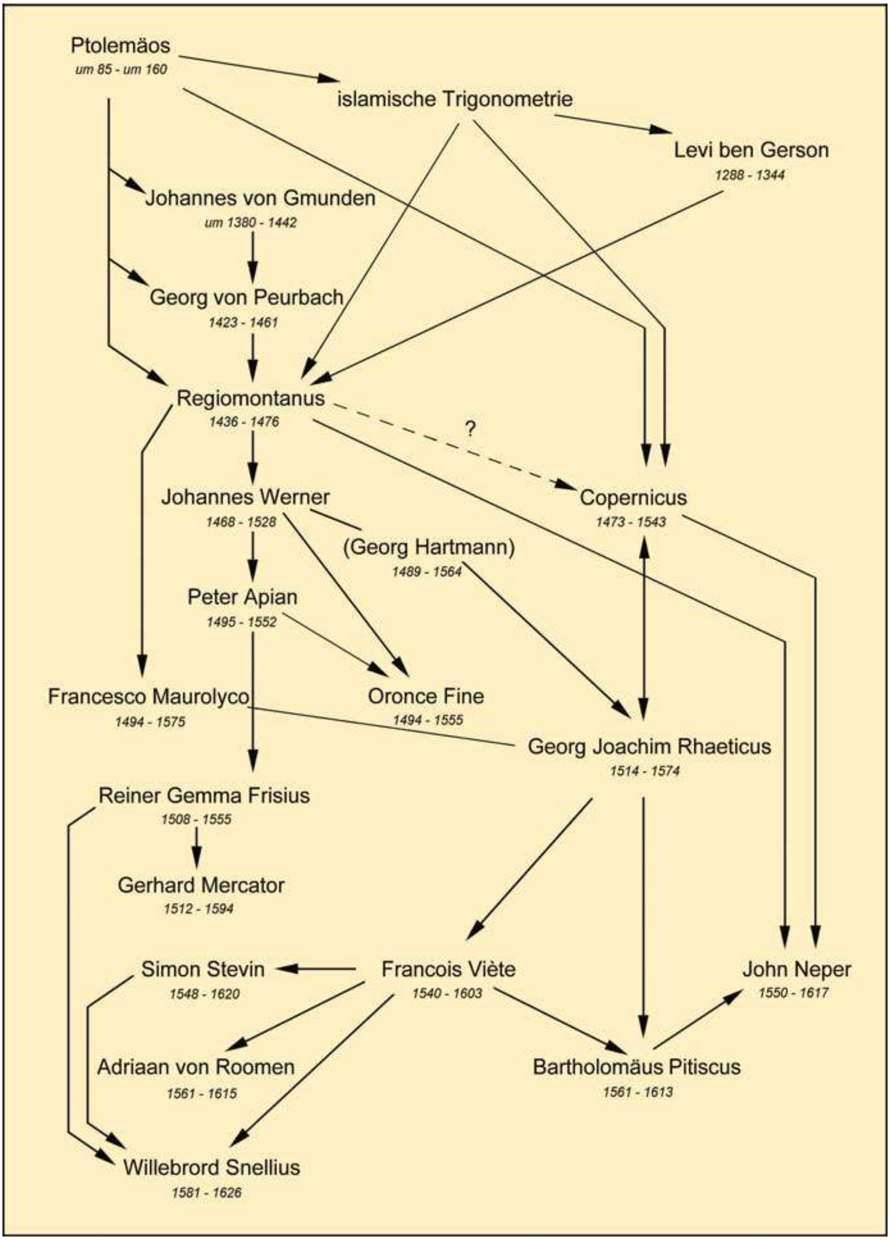
Thereby, $A$ is the angle nowadays referred to as $\alpha$ and $sinvers = versine = sinustotus - cosine$ is a rather sensible quantity geometrically speaking, the "height of the sagitta" of the arc belonging to the double angle (Illus. 5.2.2). Equation (5.2.1) can be transferred to the form customary nowadays by means of a simple calculation:

$$cosa = cosb \cdot cosc + sinb \cdot sinc \cdot cos\alpha. \tag{5.2.2}$$

However, we must keep in mind that equation 5.2.2 would be completely pointless in the type of mathematics influenced by Greek tradition, since it violates the principle of homogeneity if we grasp the participating quantities as line segments. In contrast, (5.2.1) adheres to the permitted form of a proportion: line segment: line segment = area: area. The role of sinus totus also becomes clear due to this consideration. (5.2.1), as well as (5.2.2), represents a relation between the three sides of a spherical triangle and any of its angles. However, Regiomontanus was still so under the spell of astronomy that he only used the theorem to calculate the third side with two given sides and one given angle, but not to obtain the angles by means of three given sides.

Whereas Regiomontanus's main work, which was only printed in 1533 (57 years after his death), significantly influenced Johannes Werner, Georg Joachim Rheticus, Peter Apianus, Francesco Maurolico and other specialists of trigonometry, it seems that Copernicus conceived of his trigonometry relatively independent of his direct predecessors. As a result, Copernicus's concept differs in some distinctive features, such as the introduction of the secant function (as the reversed cosine) or the use of the tangent function. Particularly, Rheticus and Bartholomaeus Pitiscus contributed greatly to the systematically and didactically skilled preparation of the gathered knowledge. Rheticus was the first to define all six trigonometric functions – sin, cos, tan, cot, sec, cosec – as side ratios in a right-angled triangle and tabled them in a manner in which each angle only varies between 0 and 45 degrees and there is one table each to read off a pair sin-cos, tan-cot, sec-cosec. Pitiscus introduced the word 'trigonometry' with the title *Trigonometriae sive dimensionae* of his book published in 1595.

The advances in trigonometry first achieved in Germany soon spread across Italy (Maurolico), France (Fine, Vieta), Great Britain (Napier) and the Netherlands (Gemma Frisius, Stevin, Snellius). Vieta was the first to succeed in deriving the law of cosines, whereby he, of course, only substitutes one corner of the triangle with the appropriate polar corner, instead of the presently customary polar triangle (introduced by Snellius, see Problem 5.2.1). Geodesic and other technical applications of trigonometry also gradually penetrated the textbooks: a Venetian manuscript from the middle of the $15^{th}$ century determines the shortest distance between the beginning and the end of a trip by means of the linear parts of the route and the changes of direction in the inflexion points. It seems that a text published by Johannes Werner in 1514 is the first in Europe to display the problem well-known from Islamic trigonometry for determining the shortest distance between two points on the terrestrial surface by means of geographical coordinates. Regiomontanus had similar problems. Pitiscus was the first to use an appendix featuring application of the building trade in the third edition (1612) of his book on trigonometry. This edition was translated into English in 1614. Rheticus's 'Chorography' could have made a significant contribution to the development of geodesy, but was not printed. Hence, the greatest impulse for geodesy came from the Dutch.




**Illus. 5.2.3** "Family tree" of European trigonometry
[P. Schreiber in German edition]

Gemma Frisius (actually Jemma Reinerszoon), a physician from the Frisian North of the Netherlands, was the first to describe the method of triangulation concerning purely terrestrial distance measuring in 1533: starting with a very exactly measured basic line segment, only angles are measured; thence the sides of joining triangles are calculated successively trigonometrically. In 1547, he was again the first to propose measuring geographical longitude by means of precise watches he took with him. The simplest principle is as follows: the difference of longitude of two locations is proportional to the difference of their local times. Hence, we only need to compare the astronomically determined local times at each location with the local time of the standardised time we have carried with us. Of course, to be able to realise this technically, we need watches that are not just accurate over long periods of time, but also unsusceptible to turbulence occurring whilst travelling at land or sea for longer periods. This was only put into practice by the British John Harrison around 1736 [Howse 1980, Sobel 1995].

Snellius, son and successor of a mathematics professor at Leiden, conducted the first arc measurement by means of triangulation[5] . Thereby, he determined the distance of around 130 km between the locations of Alkmaar and Bergen op Zoom in the Netherlands, both of which are located approximately on the same meridian, with remarkable precision. Apart from this survey, his main work *Eratosthenes Batavus* (the Dutch Eratosthenes) from 1617 was also the first to describe the so-called "recession"[6] , which was later named after Laurent Pothenot (1692) who worked as a professor in Paris around 1700. Thereby, the angles between each two of three points $A, B, C$ with known locations are measured from a standpoint $S$, which we need to determine. The circles through $S, A, B$ and $S, B, C$ are obtained from their peripheral angles in $S$, standpoint $S$ as that intersection of these circles different from $B$. As a result, this procedure fails if $S$ lies on the circle through $A, B, C$ and it is arbitrarily imprecise if $S$ approaches this "dangerous circle". Just such considerations are the messengers of a new type of geometrical thinking despite the theoretical triviality of the subject matter.

The $17^{th}$ century then became the century of the first large-scale land surveys, in which mathematicians like J. Kepler, W. Schickard, G. D. Cassini and M. Ricci participated. It seems that the mathematics professor W. Schickard from Tübingen engaged practically with triangulation and recession simultaneous to and independent of Snellius. We shall not go into detail here

[5] The first European arc measurement had already been conducted by Jean Fernel, personal physician to the French king Henry II, between Paris and Amiens in 1525, by passing the distance in a carriage whilst counting the amount of times the wheels revolved. Although his method had its mistakes, he still managed to state a relatively precise value. As a result, Fernel was accused of manipulating the result (also by Snellius) [Bialas 1982].

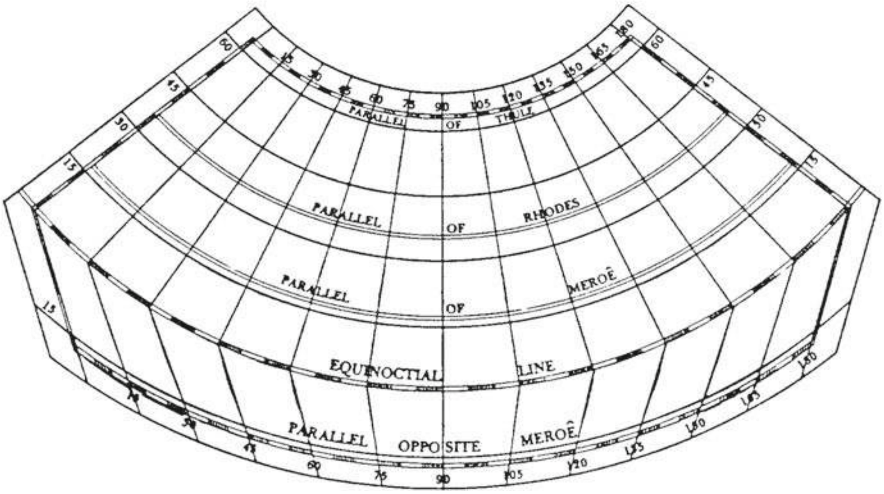
[6] When intersecting, we determine the location of this new point by means of focussing on the "new point" of two known locations. Such a procedure had already been ascribed to Thales.

concerning the degree to which the (re-) discovery of America and the other great expeditions of the $15^{th}$ and $16^{th}$ centuries revolutionised the medieval world view, or how the new view of the world also intensified the issue of the position of Earth within the cosmos and the question of its structure. In close correlation with the boom of astronomy, geography and geodesy, a further new field for application of geometry was developed, which, according to its nature, was most closely related to "actual geometry", that being cartography. Basically, Antiquity had only developed one mathematically interesting and exactly defined representation of the spherical surface in the plane; namely stereographic projection, the circle-preserving quality of which is the base of the functioning of astrolabes or, respectively, the mechanically-powered faces of astronomical clocks. Nobody before Johannes Werner seems to have conceived of also using this representation for geographical maps. Even Ptolemy's geography only addresses the representation of the known part of the world, for which Ptolemy developed the much simpler but also more illustrative maps that we nowadays identify as parts of conic projections (Illus. 2.5.6 and 5.2.4, 5.2.5).

During the European Middle Ages, the world maps – as with most other illustrative representations – had to fulfil a predominantly symbolic requirement. There were the so-called T and O maps with Jerusalem as the centre of the circular world disc, which sometimes were precisely quartered by waters, such as the Mediterranean Sea and the Red Sea, whereby one quarter represented Europe, a second one Africa and the remainder Asia (Illus. 2.1.2). Later maps featured many, mostly fantastic details. Sea maps mainly served as symbolised representations, indicating the compass courses or wind directions to reach a certain destination [Köberer 1982]. A geometrically-founded cartography was only established around 1500. However, even elementary basic notions, such as being true to area or angle, are not explicitly mentioned there. Relations, such as orthographic and stereographic projection, were only introduced in the influential 'Optics' by the Belgian Jesuit François d' Aguilon around 1600. The imperial court astronomer Johann Stab (Lat. Stabius) developed the image of the spherical surface true to area, which was multiply reproduced due to its distinctive heart-shaped form (Illus. 5.2.6):

Based on the straight picture NS true to the length of a "Prime Meridian", the circles of latitude are transferred to both sides true to length onto concentric circular arcs around the image N of the North Pole. As a result, we obtain the distinctive heart shape as the location of the extremity of these circular arcs and simultaneously as the double picture of the 180 degree meridian. We cannot know at this point if Stab conceived of this heuristic consideration himself for obtaining a picture true to area of the spherical surface by means of truth to length at the Prime Meridian and the circles of latitude transferred from there. The means to confirm such a hypothesis exactly, such as truth to area or angle, were only introduced in the $19^{th}$ century. Stab's idea was executed by the already-mentioned J. Werner in Nuremberg in 1514, but only published by Peter Apianus in 1530.

a)

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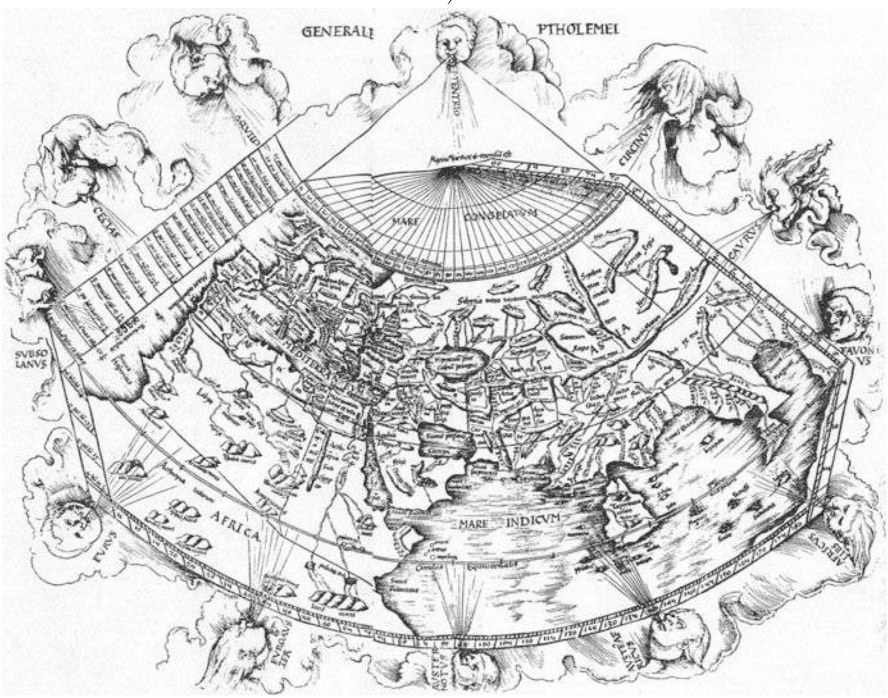
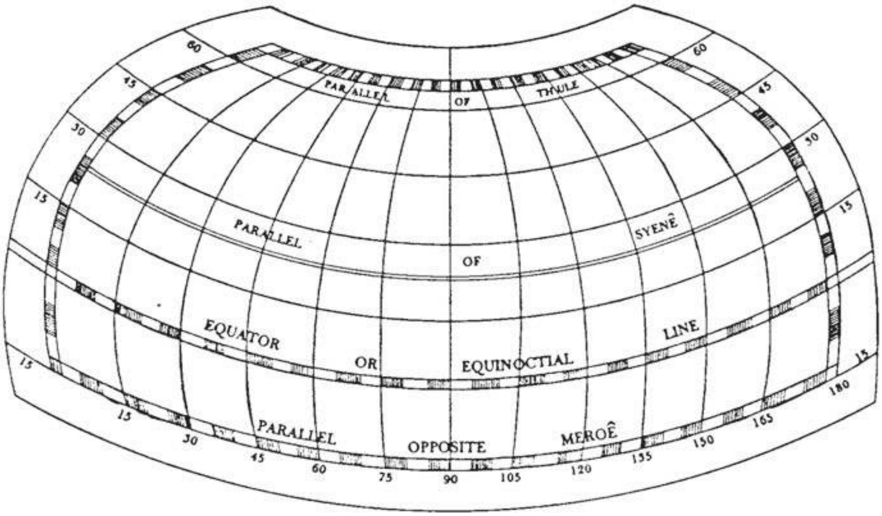


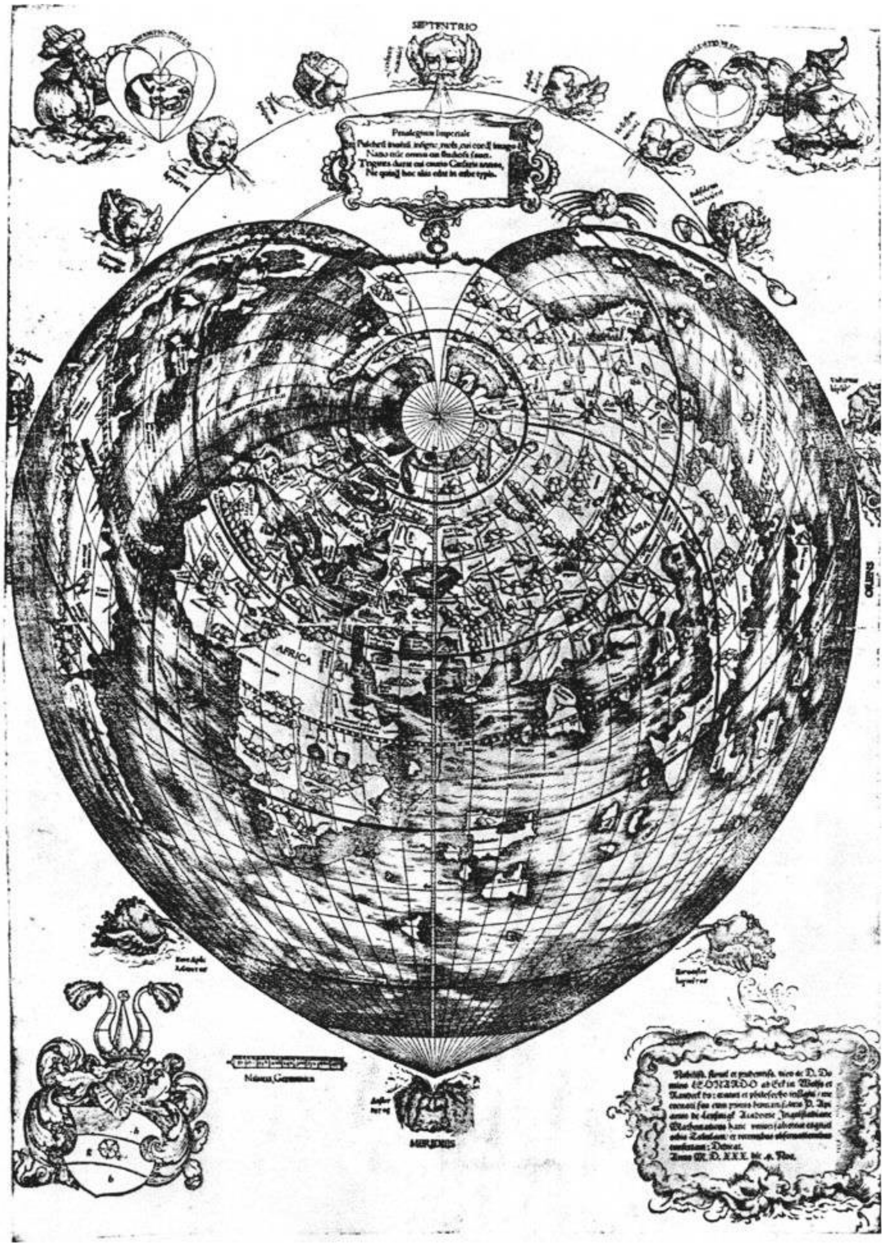

**Illus. 5.2.4** Ptolemy's world map a) Reconstruction of Ptolemy's conic projection, b) Map of the inhabited world in conic projection (Strasbourg 1513), reconstructed by means of the longitudes and latitudes from Ptolemy's handbook on geography [a) from Lloyd A. Brown: The Story of Maps, Bonanza Books, New York 1949 new ed. Dover Publ. Inc., Mineola 1990; b) Herzog August Library, Wolfenbüttel 1.2. 4.1 Geogr. 2°, Karte]

a)




b)

**Illus. 5.2.5** Ptolemy's modified spherical projection a) Reconstruction of Ptolemy's modified spherical projection, b) Nicolaus Germanus's world map from the first Ptolemy edition printed in Germany (Ulm 1482) [a) from Lloyd A. Brown: The Story of Maps, Bonanza Books, New York 1949 new ed. Dover Publ. Inc., Mineola 1990; b) Herzog August Library, Wolfenbüttel 2.2 Geogr. 2°, Karte]




**Illus. 5.2.6** Heart-shaped world map according to the principle of Stab(ius) and Joh. Werner [Published by Peter Apianus: Tabula orbis cogniti universalior, Ingolstadt, 1530]

Stab also designed a map of the terrestrial hemisphere by means of vertical parallel projection onto a tangential plane. Besides, Albrecht Dürer helped with the design of this map printed in 1515 by order of Emperor Maximilian (Illus. 5.2.7). Further cartographic illustration first proposed or used in this time are, amongst others:

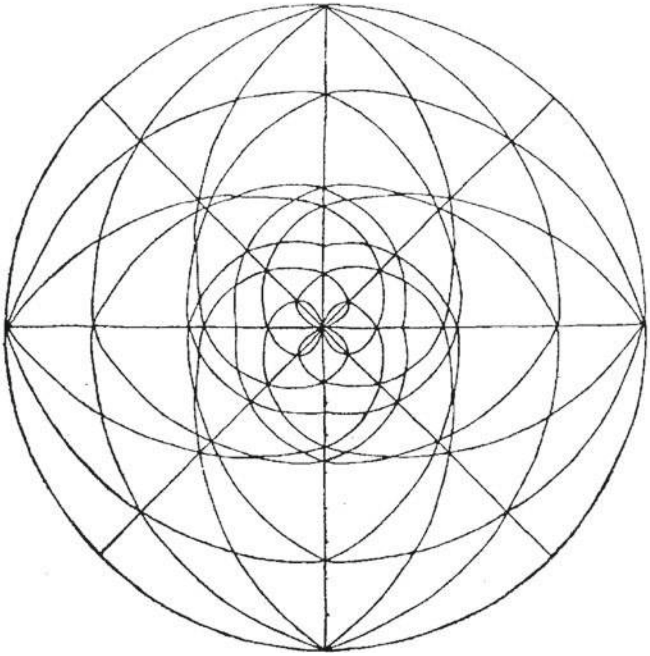
- Azimuthal equidistant projection (Cusanus, Snellius), in which the meridians are represented by rays originating in N and, the circles of latitude by concentric circles in the map, whilst making sure that its radius equals the pole distance measured in arc measure (the name of the method means that all points are represented in their true distance from the North Pole = centre of map).
- The draft first used by Gerard Mercator and later named after Sanson and Flamsteed, in which the circles of latitude are represented by parallel line segments true to distance and meridians stay true to area.

The climax of Renaissance cartography is, without a doubt, symbolised by the work of the Flemish cartographer Gerard Mercator, who later worked in Duisburg. The excellent mathematician Pedro Nunes (Nonius), who lived in Portugal, first addressed the curves of constant course (later called rhumb lines or loxodromes) on the globe, which are so important for seafaring; in other words, curves that are defined by the fact that they intersect all meridians in a constant angle. (In 1624, Snellius introduced the still customary name 'loxodrome' for these curves in his theory on navigation *Tiphys Batavus* and likewise the name 'orthodrome' for great circle arcs, i.e., shortest curve on the globe.) Nunes showed by means of approximate construction (only using eight meridians and approximation until the next meridian by means of circular arcs, Illus. 5.2.8) that these loxodromes, called "curvas dos rombos" (rhumb curves) by him, approximate both poles in a spiral manner without ever being able to reach them.

Some of these curves are displayed on a globe made by Mercator in 1541. In 1568, he finalised the first world map, which shows these curves as straight lines – and the Mercator projection was born. However, this projection was only circulated in 1595 in the printed world atlas, after Mercator's death. Ever since then, the literature has speculated as to how he could have achieved his map [Köberer 1982]. The reasons for this speculation are twofold. On one hand, the exact law, which says that the intervals of the images of the circles of latitude grow into the infinite with increasing latitude from the image of the equator, can only be found by solving an infinitesimal equation. On the other hand, the Mercator projection, which features the spherical surface (apart from the two poles) represented on a cylinder of infinite height, tangentially located at the equator and subsequently unwound, can by no means be explained as a 'projection' in an elementary geometrical manner. (In this respect, the picture on p. 84 in [Mainzer 1980] is also misleading.)

However, since the loxodromes – constructed pointwise – occurred on Mercator's globe first, it is most probable that he transferred them from there

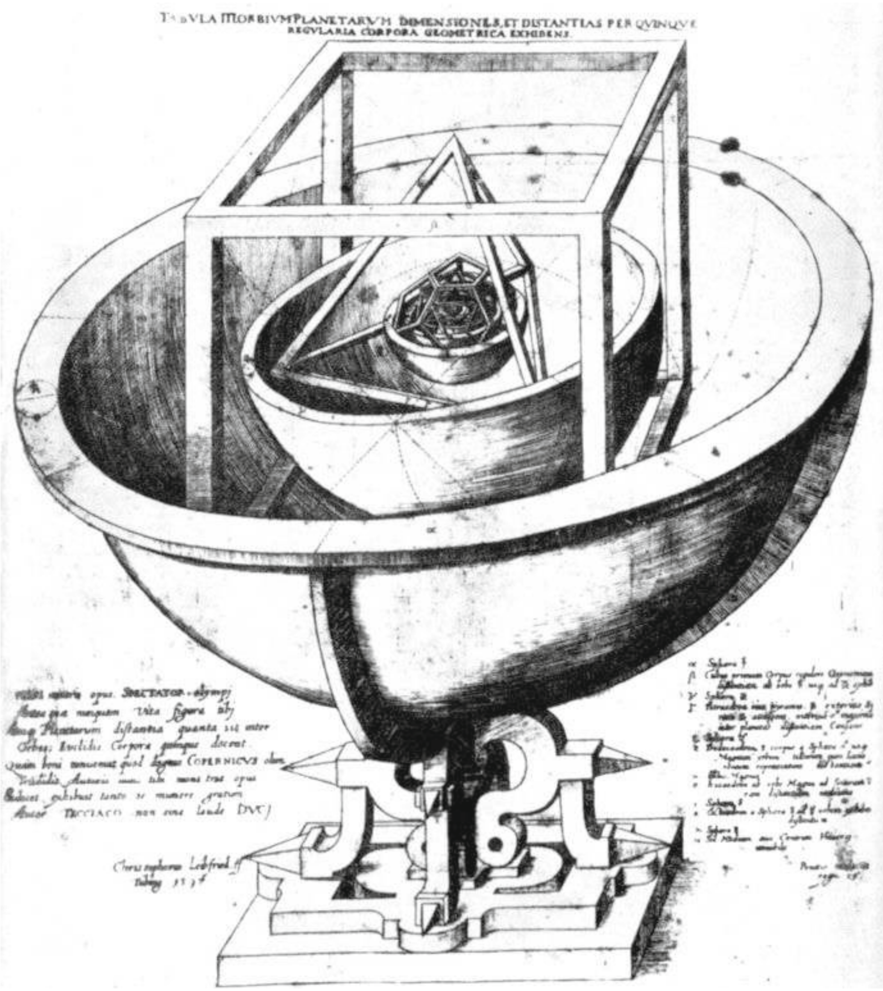
**Illus. 5.2.7** Stab(ius)-Dürer-map, 1515



**Illus. 5.2.8** Loxodrome diagram by P. Nunes from 1537
[Nunes: *Tratado em defensam da carta marear*. Lisbon 1537]

onto the approximately constructed map. In any case, the importance of this new invention for high-sea navigation resulted in many mathematicians engaging with the problem of exactly defining and calculating Mercator's map. Mercator, consequently, contributed a small stepping stone for the development of infinitesimal mathematics. (To make matters more confusing, one of these mathematicians, Nikolaus Kauffmann, also called himself Mercator.) An almost forgotten English mathematician, Thomas Harriot, whose remarkable and versatile works remained unpublished during his lifetime, seems to have been the first to prove the truth of angle of the stereographic projection around 1600 and, consequently, derived that the loxodromes intersect the mappings of all circles of latitude in a constant angle during a stereographic projection onto the equator plane. Thus, they are represented as logarithmic spirals around the pole. He also realised that the area of a spherical triangle is proportional to the spherical excess, i.e., the surplus of the sum of the angle over 180 degrees. In anticipation, we will mention here that it is natural to desire a convenient route between two locations on Earth far distant from each other and to navigate approximately orthodromic globally, but loxodromic locally. This leads to the very difficult problem, only solved at the end of the $19^{th}$ century, of determining the image of a great circle arc between two given points in a Mercator projection, which can then be approximated by means of a broken line corresponding to the local loxodromic line segments [Schreiber, O. 1908].

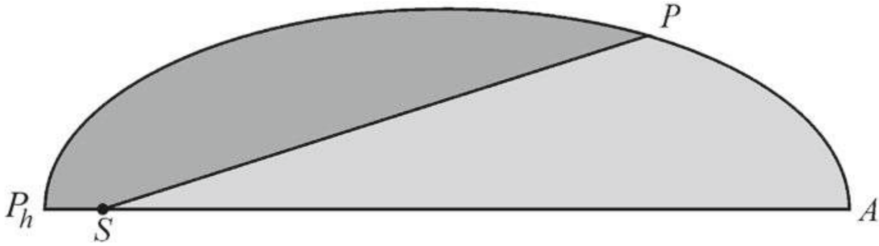
Trigonometry led us from astronomy to geodesy and cartography. However, we must now return to astronomy once more: At the end of the era introduced here is a mastermind, an intellectual giant, who stands with one leg in the Renaissance and the other one in the following era, which was characterized by the invention of the coordinate method and infinitesimal mathematics: Johannes Kepler. In this section, we will look, for now, at his astronomically motivated contributions to geometry. With his first work *Mysterium cosmographicum* (The cosmographic mystery, 1596), he put himself forward to the most significant astronomer of the preceding generation, Tycho de Brahe, as the sought-after genius who could make the observation data agree with a bold theory. Kepler had tried to account for the distances between the planets known at the time and the sun by means of interlacing the five regular solids so that the circumscribed sphere of a solid determines the orbit of a planet and simultaneously is the inscribed sphere of the following solid (Illus. 5.2.9). Hence, according to Plato's theory on atoms and in great proximity to it on an intellectual level, this would be the first time that the ensemble of all five platonic solids would have been ascribed a natural scientific meaning. We need not dwell upon about how pointless this speculation is from today's perspective. However, only a few know that Kepler held onto this basic idea his entire life, which is why we must stress this fact. Having crucially improved the Copernican model of circular orbits around the sun with his three laws, he published a second version of *Mysterium cosmographicum* in 1621, in which the concentric spheres were substituted for spherical shells of finite




**Illus. 5.2.9** Kepler's model of the world from *Mysterium cosmographicum* [M. Caspar (Transl. and Ed.): Das Weltgeheimnis (The cosmographic mystery), Augsburg 1923]

thickness between each two consecutive platonic solids in such a manner that the interspace between the circumscribed sphere of the inner solid and the inscribed sphere of the outer solid left just enough space for the elliptical orbit of each planet (see Problem 5.5).

We also will mention how Kepler's laws of planetary motion stimulated the purely geometrical theory of conic sections. Out of the wealth of problems, we will pick out the following one originating from Kepler's second law: planet $P$ is supposed to move on an elliptical orbit around the sun, which stands

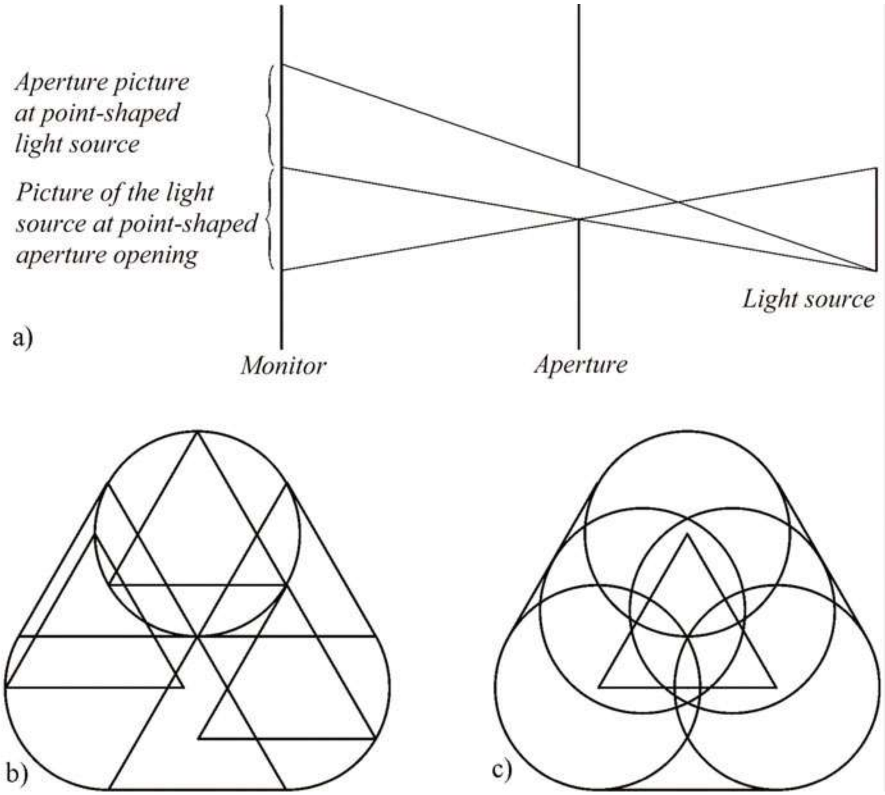



**Illus. 5.2.10** Kepler's problem: Based on Kepler's second law, the position of planet $P$ for a given ratio of the two shaded areas is to be determined

in a focus $S$. Since the radius vector $PS$ covers equal areas in equal times, we have to split the semi-ellipse $A$ (aphelion = farthest point from sun) $P$ $P_h$ (perihelion = closest point to sun) into two partial areas $ASP$ and $SP_hP$, which stand to each other in a ratio given by the time ratio, in order to determine the location of $P$ at a certain point of time $t$ (Illus. 5.2.10). Thereby, for example, a quarter year after aphelion the areas of $ASP$ and $SP_aP$ must be equal and one day after aphelion the area of $ASP$ must be around $\frac{1}{183}$ of the area of $SP_aP$, see Problem 5.2.6.

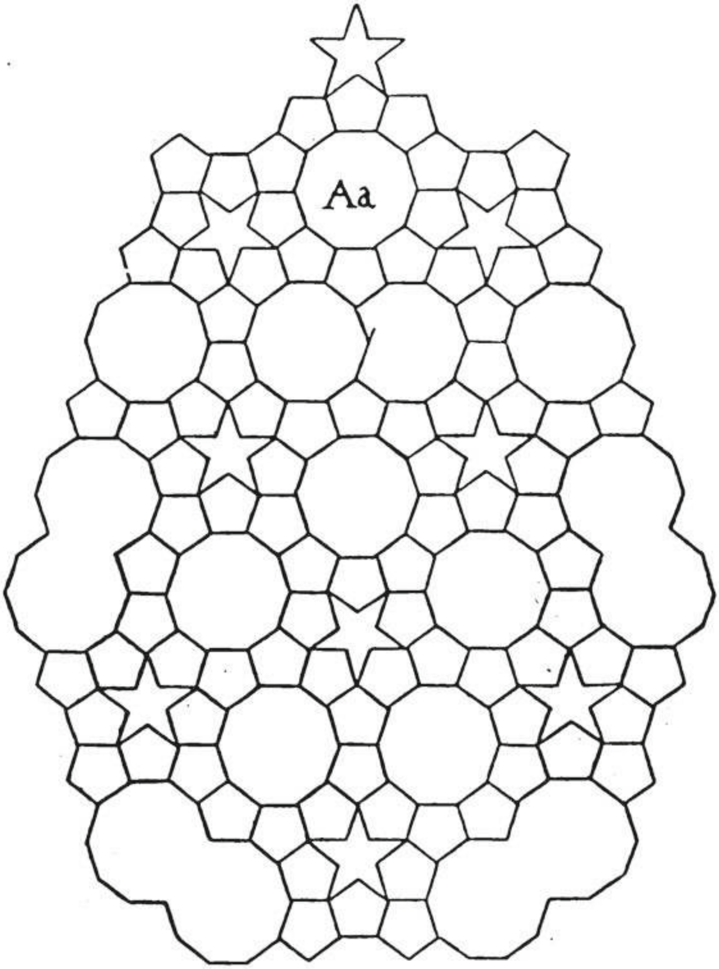
Newton's theory of gravity will clarify the physical reason for conic section orbits and, thus, the phenomenon of parabola and hyperbola-shaped orbits of celestial bodies. However, it was Kepler – as strange as that may be – who, in a completely different context – namely in his first text on optics (*Ad Vitellionem paralipomena quibus astronomiae pars optica, 1694*) - put into words for the first time the idea that the three naively different shapes of non-degenerated conic sections always merge as the second focus drifts into the infinite and re-approaches from the other side in a fixed focus. Kepler was motivated to take on this view due to the definition of a concave mirror as a reflector, in which the beams originating from one point run together again in a second point. As the title of the mentioned text states, Kepler viewed optics above all as an auxiliary science of astronomy (as did Euclid!). The trigger for Kepler's first text on optics was the question left behind by Tycho Brahe as to why the sun's diameter appears too large and the moon's diameter too small on the monitor of a camera obscura when observing a solar eclipse. Kepler generally clarified that, given a light source and a pinhole both finite in size, the illuminated picture is the union of all pictures of the light source with varying aperture point on one hand, and on the other hand, the union of all pictures of the pinhole with varying illumination point (Illus. 5.2.11, also cf. [Schreiber 1997]). Thus, by means of knowing the aperture diameter and the distance between monitor and aperture, we can eliminate the impact of the finite aperture diameter. From the view of elementary geometry, this is trivial and far below the level of ancient geometry. The liberal application of known geometry to manifold practical problems is new.

In a further text (Dioptrice, 1611), Kepler brought much to optics of the notion and content of its modern equivalent by addressing the optical path in the human eye and in different, systematically examined lens systems. However,



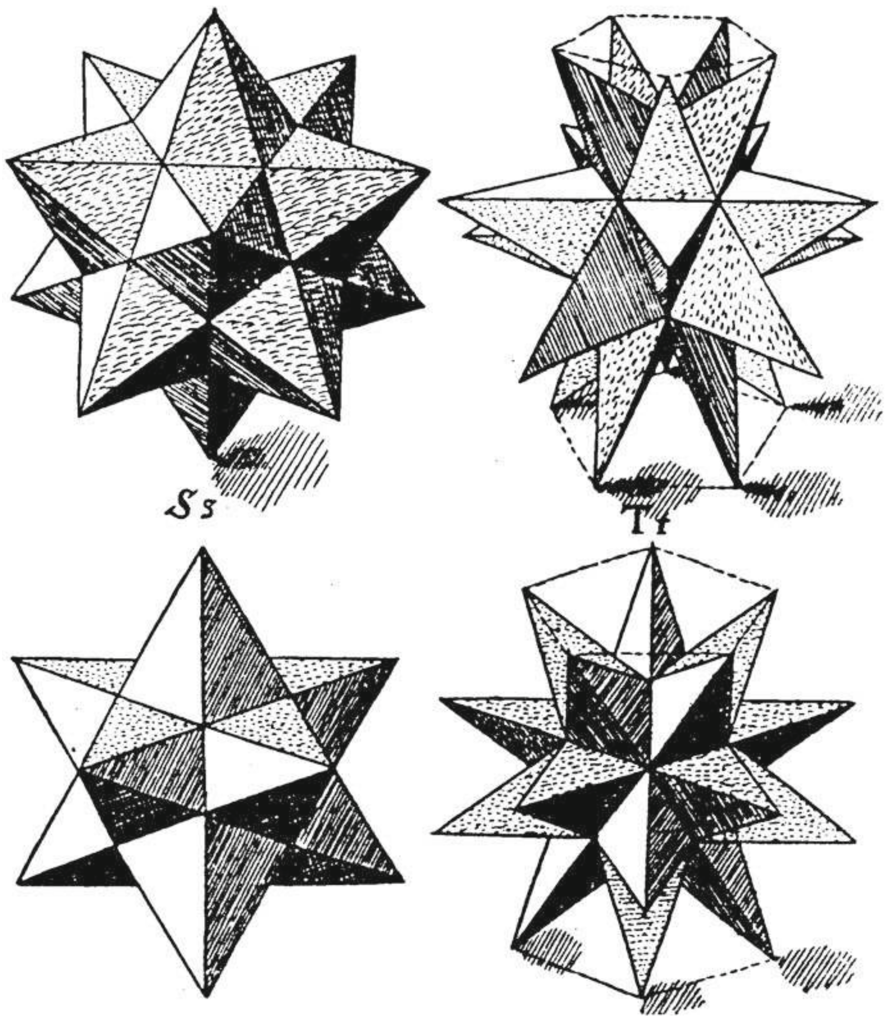

**Illus. 5.2.11** The Brahe problem: a) Composition of the light spot on the diffusing screen by means of the form of the light source and the form of the pinhole. b) If the light source is an isosceles triangle and the aperture a circle, then the picture is formed as the union of all triangular pictures, when a fixed point (for example, the upper peak of the triangle) runs through all points of the circular aperture or respectively, c) as the union of all circular aperture pictures, when the illuminated point runs through the triangular light source. We see here that a triangular aperture together with a circular light source would yield the same picture.

he just missed the exact wording of Snell's law of refraction despite intensive efforts. The approximation he had found for not too large apex angles, basically based on an approximation of the sine function, was, of course, sufficient for practical needs of that time. Simpson's rule (also called *Keplersche Fassregel*), which we will look at in due course, and his studies of tessellation and polyhedra contained in *Harmonice mundi* and the small, so-to-speak, popular scientific text 'On the Six-Cornered Snowflake' (1611) shows to what degree Kepler was a creative geometer beyond his astronomically-motivated works. Kepler's motive was his absolute conviction that the same geometrical harmony rules the world by both large and small measures. Hence, he attempted to contrast speculative cosmology of his "mysterium" to a matching "microcosm". Thereby, his systematic proceeding led him to find tessellations with star-shaped tiles and "monsters" (Illus. 5.2.12), star polyhedra (Illus. 5.2.13), of which it is often (wrongly) said that Kepler was the first to have known of them (cf. Chap. 5.3) and the first Catalan solids (Archimedean duals) (see Problem 5.2.4). Furthermore, he included the infinite families of prisms and anti-prisms in the discussion of Archimedean semi-regular polyhedra.




**Illus. 5.2.12** Kepler's tessellation from *Harmonice mundi*. The attempt to pave the ground with regular pentagons leads to a tessellation, which also features regular decagons, pentagrams and non-convex polygons with sixteen edges, which Kepler referred to as "monster" [From M. Caspar (Transl. and Publ.): *Weltharmonik* (Harmony of the world). Munich – Vienna 1939]

However, he dismissed them again since, according to his opinion, they turn "disc-shaped" and are no longer "sphere-shaped" given a sufficiently high number of corners (The Harmony of the World, 1619, Book II, def. XIII.) In 'On the Six-Cornered Snowflake', he discusses the question, amongst others, as to what extent the highly symmetrical shapes realised in nature are determined by laws of nature and/or maximal practicality.




**Illus. 5.2.13** The two star polyhedra displayed in Kepler's *Harmonice mundi*. The small star dodecahedron $S_S$ already occurs in around 1425 (Illus. 5.3.22). $T_t$ is nowadays referred to as a great star dodecahedron [From M. Caspar (Transl. and Publ.): *Weltharmonik* (Harmony of the world). Munich – Vienna 1939]

## 5.3 Geometry in Renaissance art

There is a highly extensive specialised literature on this topic at the border between history of science, culture and art often richly and attractively illustrated and reaching far into the $19^{th}$ century. However, we can only present a rough overview here (we recommend [Kemp 1990] and [Field 1997] as the latest extensive representations; on mathematical perspective, also see the works by K. Andersen listed in the bibliography and further works cited therein). Nonetheless, some of the following aspects have not been pointed out anywhere before. The art historians are often lacking knowledge of and interest in mathematically relevant details. As the true children of their time, the earlier generations of historians of mathematics little appreciated anything that did not fit into the picture of mathematics directed at proving theorems or explicitly wording algorithms. Besides, the modern mathematician coined by computer science lacked the time and opportunity for extensive historical studies in this field. However, the type of mathematics addressed here is "unconscious mathematics" (see introduction concerning this notion) to a special extent, meaning we must at least consult their artistic works coequally when judging the Renaissance artists' geometrical knowledge and skills from our present perspective next to what they may have written.

**Illus. 5.3.1** The School of Athens (fresco from Raffael 1510/11 in the 'Stanza della Segnatura' of Vatican). Many scholars of Greek Antiquity are to be found in this fresco. Plato and Aristotle are in the middle of the background, Aristotle presents his 'ethics'. Below left Pythargoras reads in a book. With Averroes also persons of the Renaissance are appreciated for the transfer of knowledge from antiquity

(This applies particularly to, for example, Paolo Uccello, Leonardo da Vinci and Wenzel Jamnitzer.) The intuitive, unverbalised knowledge of shapes, subject matter and algorithms expressed in material objects is displayed nowhere so clearly as in the arts. Finally, we must remark in advance that the frequent mentioning of Albrecht Dürer in due course is founded objectively. He is by far the "mathematical mastermind" amongst the artists of his time [Dürer c], [Schreiber 1999-2008].

With the beginning of the Renaissance, artists began turning away from the heavily symbolised representation of the Middle Ages and strove towards realism. Hence, the re-awakening[8] and development of central perspective was inevitable. However, it would be completely wrong to limit the relations between the era's geometry and art to this. Artists thought of themselves as craftsmen back then and not just in the best sense of the word; in other words, they placed practical know-how above programmatic demands, but also were incredibly versatile, being, respectively, engineers, architects, mechanics and natural scientists. Furthermore, within the scope of their possibilities, they were interested in re-acquiring ancient knowledge. However, in the foreground stood the hope of practical use, the will to apply and the necessary redesign and advancing. These desires were even stronger than those of the university scholars. Apart from perspective, the following aspects also played an important role:

- Here and there further examples of applying the method of top and front view to solve spatial, constructive problems, which was only conceived generally much later
- Geometrical constructions, whereby practical approximation methods often replace exact solutions, also in such cases in which an exact solution with compass and straightedge is possible; invention of mechanisms to solve geometrical problems
- Discovery of new geometrical shapes (curves, areas, solids)
- Approaches to study plane tessellation and ornaments
- Aesthetic fascination originating from regular and semi-regular polyhedra
- Attempts to grasp harmony and beauty by means of number ratios or other mathematically expressible laws
- Last but not least, first origins of a non-Latin geometrical terminology, since the artists mainly relied on their relevant national language if they were composing texts targeted at professional peers.

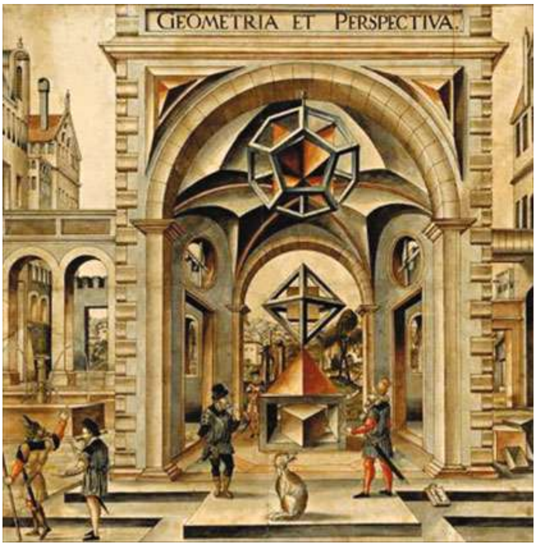
We can find all of this united primarily in Albrecht Dürer's and also in Leonardo da Vinci's work. Nonetheless, there are great differences: Dürer

[8] The pioneers of perspective in the Renaissance knew nothing about the very modest input of Antiquity. Even Vitruv's *De architectura*, which only mentions a few things on perspective, was first published in print in Italy in 1521 and in German only in 1548, cf. [Vitruv].

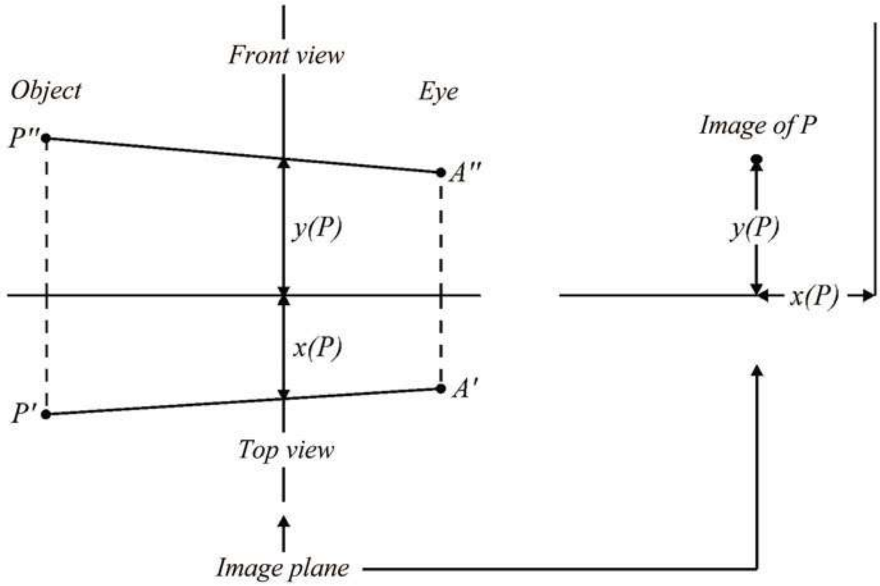
finalised three theoretical texts, which – printed multiply and in several editions up until today – are easily available. Leonardo did not conclude anything. His written legacy was scattered to the four winds after his death and only gradually put back together again in the $20^{th}$ century. As a result, imitations and errors appeared apart from the surprising, new discoveries of some of Leonardo's writings. ([Marinoni/Zammatio/Brizio 1980] does not just offer a good history of these manuscripts, but also a great introduction to their character.) Dürer tended to proceed theoretically, deductively and systematically. Leonardo "jumped" back and forth, rarely separated mathematics from nature and deduction from induction, and is often mistaken. His main achievements lie within anticipating technical principles and inventions, and in the subtle observation of nature.

### 5.3.1 Perspective

According to Giorgio Vasari's credential, Filippo Brunelleschi, known as the architect of the cathedral dome in Florence, is said to have invented a method for the pointwise construction of a view of correct perspective by means of the top and front view of a building ensemble. This method was later referred to as "method of intersection", since it is based on supplementing the top and front view by the assumed visual point and the assumed image plane, and then determining the intersection of the "viewing rays" by means of the image plane (Illus. 5.3.3). It is, after the astronomers' analemma method, the second



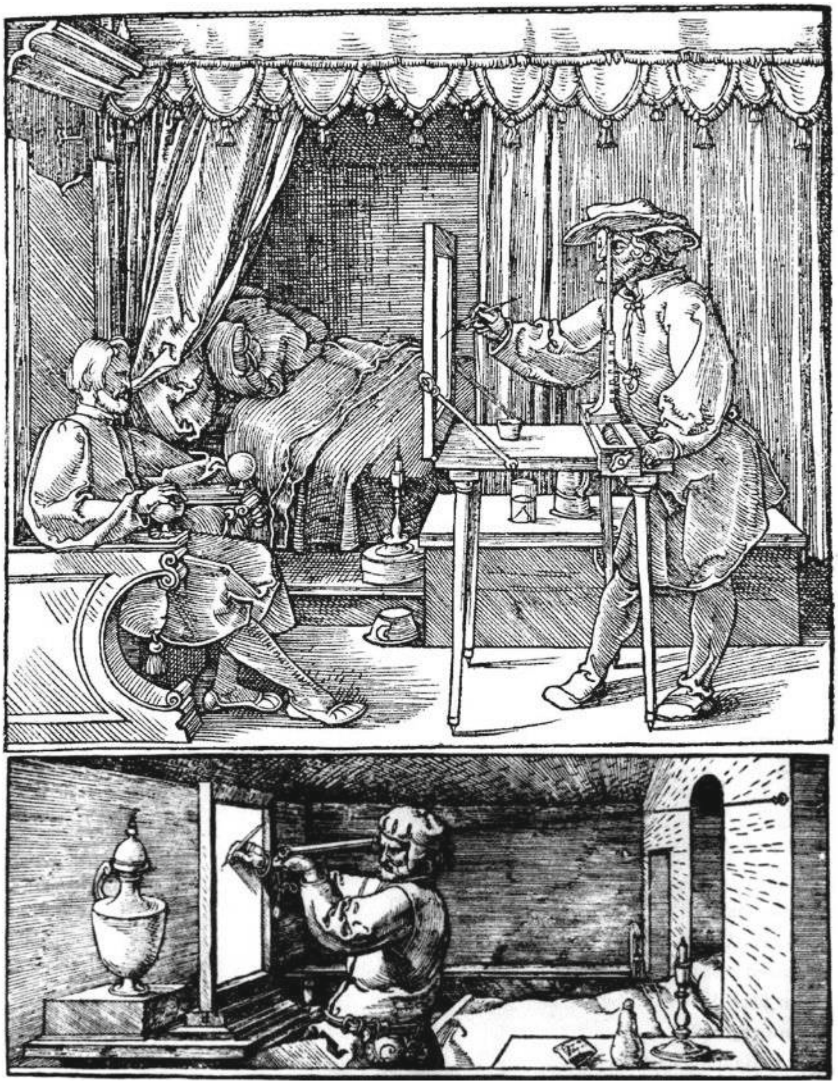

**Illus. 5.3.2** Layout of an unpublished title page of *Geometrica et Perspectiva*, [Lorenz Stoer, 1566, Inventar No. 21268Z, Staatliche Graphische Sammlung München]



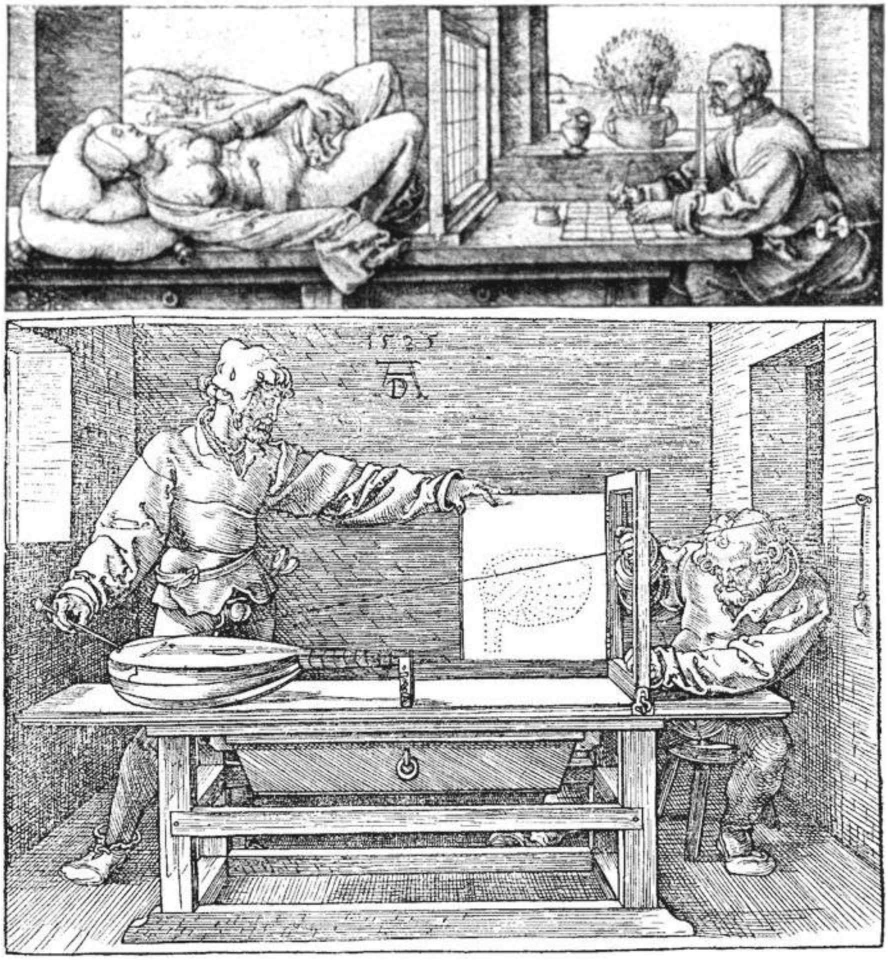

**Illus. 5.3.3** Schematic representation of the intersection method

special case of the solution of a constructive spatial problem by means of the multiplane method. The method is still suitable for deriving basic laws of perspective in an illustratively and mathematically correct fashion. However, it was also extremely tedious in its practical "pointwise" execution before the computer was used to effect it. Brunelleschi himself did not leave anything behind about his invention in writing and his priority is debated nowadays as a result.

On one hand, all further efforts were directed at mechanising the laborious construction process (Illus. 5.3.4 and 5.3.5). As a result, photography was developed via many intermediate steps. On the other hand, attempts were made to derive laws and principles by means of the intersection method, which would make the construction of pictures of correct perspective easier. This trend climaxed in the solution of the questions as to which parameters of a picture can be chosen freely under which conditions and how we can reconstruct the observer's location from a correct picture. The latter was discussed rudimentarily by Simon Stevin in his textbook on perspective *Van de deursichtighe* from 1605 and concluded by J. H. Lambert in the $18^{th}$ century. However, while Italian artists in favour of scholarliness, like Leone Batista Alberti (*Della pittura libri tre* 1436), Antonio Averlino Filarete (*Trattato della architectura* around 1460) and Piero della Francesca (*De prospectiva pingendi* around 1475), composed the first texts on perspective, the tricks known amongst painters were first treated as a professional secret. Several German and Dutch artists travelled to Italy in the $15^{th}$ and $16^{th}$ centuries in order to learn a little more about the "new perspective art" at the Italian masters' painting workshops.



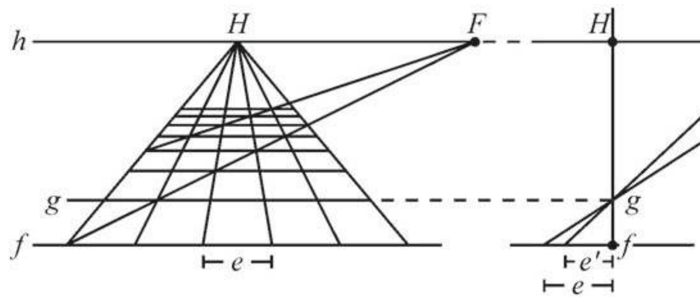
**Illus. 5.3.4** Dürer's proposals to make producing perspective pictures mechanically easier: Above: The essential element here is the adjustable pole to fix the visual point [Underweysung 1525]. Below: By connecting the pen with a nail hit into the rearward wall via a tight thread, the visual point can be located at a distance from the image plane that is greater than the drawer can reach [Underweysung 1538, $2^{nd}$ edition]



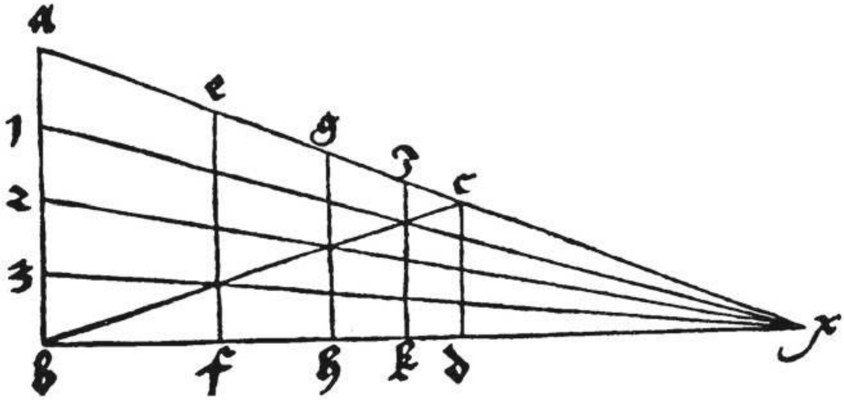
**Illus. 5.3.5** Further suggestions for producing perspective pictures: Above: By replacing the diffusing screen by a square grid, which is transferred onto the drawing surface, we can replace the uncomfortable drawing on the vertical image plane by working at the drawing table [Underweysung 1538]. Below: In contrast, the edition from 1525 shows a still very copious and tedious method for determining image points by means of their "coordinates", i.e., by two shiftable threads in the picture frame, which are perpendicular to each other. Of course, this picture has the didactic advantage for beginners of perspective that the visual line between object and picture can be illustrated by means of material realisation [Underweysung 1525]

The simplest aid, easily grasped by everyone, was the 'pavimento' at first, meaning a floor patterned like a chess board with an edge parallel to the lower front picture edge; by means of its correct construction, one could approximately correct design the minimisation against the picture background for the entire picture (Illus. 5.3.6). The construction of a pavimento that was not parallel to the front of the picture was also learned gradually, followed by how to scale any inclined straight line projectively (Illus. 5.3.7). In this context, notions, such as horizon, vanishing point, principal point, etc., were established. However, they were tainted with many obscurities, which are difficult to comprehend nowadays. For instance, the visual point (equals location of the observer's eye) was identified with the principal point (equals foot of dropped perpendicular from the visual point onto the image plane). It was characteristic that Leonardo juxtaposed three types of perspective coequally, since he thought as an observer of nature, not as a mathematician: the scaling down of distant objects, the increasing blurriness of contours and the increasing cloudiness of colours [Leonardo 1952, p. 767].

While Guidobaldo del Monte, Commandino's student, was writing the first notionally reasonably clear (but very copious and diffuse) textbook on perspective with strict proofs in Italy around 1600, the teaching of perspective spread to France (Jean Pélerin, called Viator, 1505, Jean Cousin, 1560, Jacques Perret, 1601), Germany (Dürer, 1525, 1538, Hieronymus Rodler, 1531, and others), the Netherlands (Simon Stevin, 1605) and finally, England (a translation from French, 1710, then Brook Taylor, 1715). Above all, the Jesuits laid hands on perspective towards the end of the $16^{th}$ century, perfecting it, but using it as an instrument of their strategy to re-catholicise people by means of overwhelming sensations.

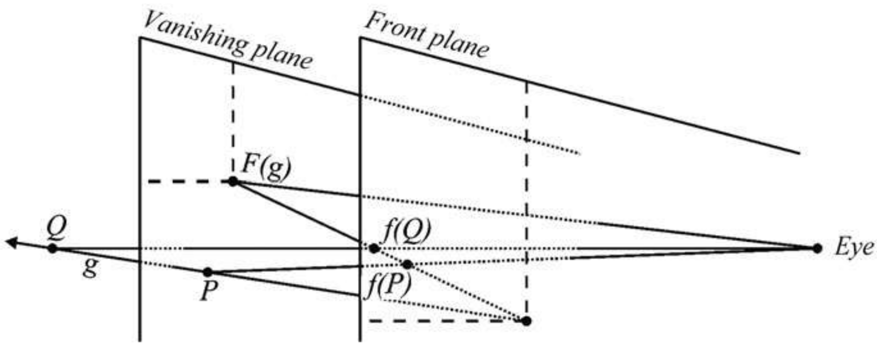



**Illus. 5.3.6** Pavimento method. Choosing horizon $h$ parallel to the lower picture edge and principal point $H$ on h fixes the perpendicular, on which eye $A$ must be located. Choosing the distance between $f$ and the first seam row g controls the eye distance $d = HA$, as the profile view shows. At the same time, it fixes vanishing point $F$ of the diagonal direction and, thus, allows the construction of all further seam rows



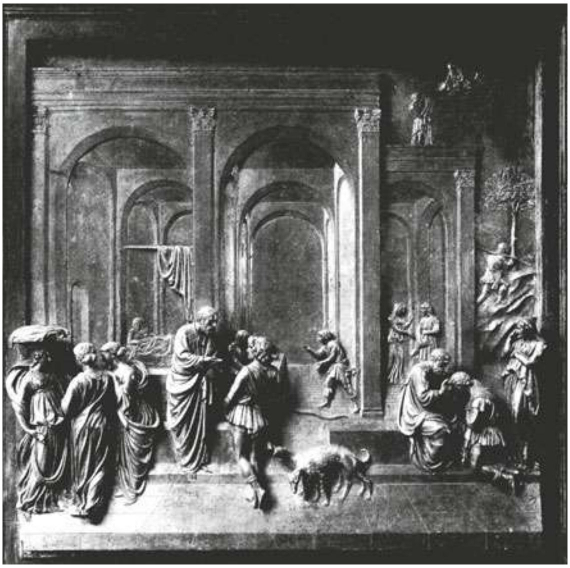

**Illus. 5.3.7** Construction of a projective scale according to Dürer [Underweysung 1538, $2^{nd}$ edition]

In addition, the relief and the anamorphosis, or "curious" perspective, were developed. Concerning the first, the half-space behind a "front plane" is mapped injectively onto the layer between this plane and a parallel vanishing plane (Illus. 5.3.8). It is said that this method was first used by Lorenzo Ghiberti around 1420 when sculpting the door of the baptistery of Florence Cathedral (Illus. 5.3.10). However, this is hard to believe given the early date. Later, it was mainly used in theatre perspective and the so-called coro finto (feigned choir), in other words, if the construction site of a church was not big enough for a large apse, it was simulated by a relief (also cf. Illus. 6.3.6). Anamorphosis concerns an ordinary picture of correct perspective that is constructed for an extreme observer's viewpoint. Only if we find this point can we fully grasp the content. Otherwise the picture has a chaotic effect; at best, it comes across as marbled. A special textbook on this curious perspective from the French Franciscan J. F. Niceron was published in 1638.




**Illus. 5.3.8** Principle of the relief perspective

**Illus. 5.3.9** Portrait of Luca Pacioli and his student Guidobaldo da Montefeltro (Duke of Urbino; Guido Ubaldo I reigned Urbino from 1482 – 1508 and was a great fan and supporter of art and science) by Jacopo de' Barbari, 1495 [Museo Nazionale di Capodimonte, Napoli]



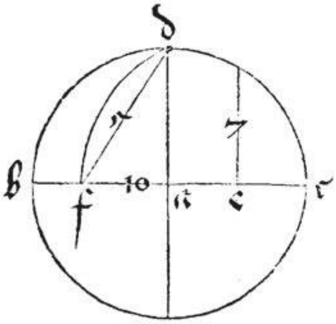
**Illus. 5.3.10** One of the ten fields of the Gates of Paradise by Lorenzo Ghiberti at Florence Baptistery at Florence Cathedral. Given the early development (around 1420), a correct, relief-perspective construction is highly unlikely. However, the work documents an early interest in naturalist or illusionist reliefs and, hence, the motive to solve the problem mathematically [Photo: A. Schreiber]

### 5.3.2 Constructions

The oldest German textbooks on constructive geometry still followed the tradition of the "Bauhütten". These are 'Booklet concerning Pinnacle Correctitude' (1486, reprinted in 1923, 1965) and 'Geometry [in] German' (thought of as anonymous for a long time, see [Steck 1948]) by Mathes Roriczer, 'Book on Pinnacles' (around 1490, reprinted in 1881) by Hans Schmuttermayer, and 'Instruction' by Lorenz Lechler (1516).

Dürer's first and most famous geometrical work is *Underweysung* (1525, a second posthumous edition, extended with many details, from 1538; in the following section, it will be referenced as 'Instruction', which is the corresponding meaning of *Underweysung*). The full title in English reads *A Manual of Measurement of Lines, Areas, and Solids by means of Compass and Ruler* [Strauss 1977]. The work is based on the tradition of the afore-mentioned predecessors. The second main work *Four Books on Human Proportions* (1528) and the so-called *Dresden Sketchbook* [Strauss 1972] are also important. However, Dürer differed from his predecessors, because he had at least partial direct knowledge of ancient sources (mostly by benefit of his educated friend W. Pirckheimer) and had an overall incomparably higher mathematical level. A copy of the Latin printed edition of 'Elements' by Zamberti (1505), which, according to Dürer's own record, he had bought for a ducat in Venice in 1507 and which featured his own marginal notes, is now located in the Herzog August Library in Wolfenbüttel. In 1523, he acquired a second copy of 'Elements' from Regiomontanus's inheritance. Dürer started his 'Instruction' with (seemingly) abundant modesty as follows: "The most sagacious of men, Euclid, has assembled the foundation of geometry. Those, who understand him well, can dispense with what follows here, because it is written for the young and for those who lack a devoted instructor..." [Strauss 1977]. However, the truth is that Dürer did not just introduce many details, sought in vain in Euclid's work (amongst them, ancient material from various sources, as well as his own contributions), but also pinpointed new directions and problem clusters of geometry.

Regarding the title of 'Instruction', we further want to remark that Euclid has been constantly linked to the "art of measuring" since the beginning of geometrical literature in German, in an all-too literal translation of the word geometry. As a result, measuring meant something like constructing at that time and is very different from the present meaning of the word. (Concerning the content of 'Instruction', see the following overview in a modern language). Of course, Dürer knew and also taught how to construct exactly a regular pentagon with compass and straightedge. However, he additionally recommended an approximation construction to his artistic peers already described in 'Geometry [in] German', which is easily done with a fixed span of the compass and a relatively low number of steps, and yet, the angles do not deviate more than 1.2 degrees from the exact value. The second book of 'Instruction' also mentions good approximate constructions for the regular 7-, 9-, 11- and 13-gon.




**Illus. 5.3.11** Dürer's original drawing for constructing a 5, 10 (exact) and 7-gon (approximate) [Underweysung 1525]

Up until then, the 9-gon construction seemed only to have been passed on orally in accordance with the craftsmen's tradition [Steck 1948, p. 49]. (All these constructions will be further looked at in the problem section.) Dürer's approximate angle trisection was especially appreciated by later mathematicians. His approach was compared with several other approximation solutions for the same problem in [Vogel, F. 1931]. It was shown that it never deviates from the exact value by more than 20 arc seconds and, hence, beats all other later solution suggestions. We must add that Dürer's construction idea is easy to iterate. Hence, it can be even more precise, although this can neither be practically realised nor is it necessary. In everything Dürer does, he is always aware of the fundamental difference between exact (he calls them "demonstrative") and approximate (he calls them "mechanice") solutions, and, thereby, distinguishes himself from most other professional mathematicians of his time.

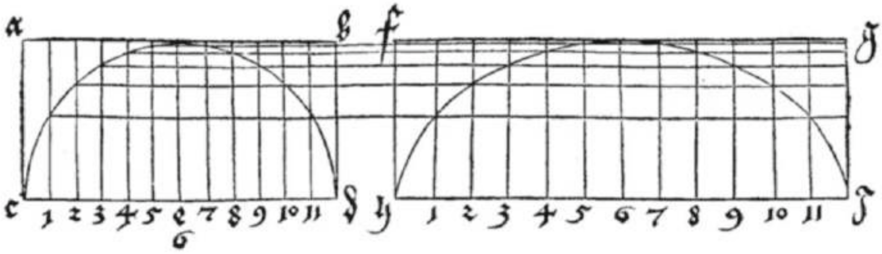
**Illus. 5.3.12** Albrecht Dürer's self-portrait from 1500 [Old Pinakothek Munich]

‘Instruction’ – Overview of Content

| | |
|---|---|
| Book 1 | Definitions of basic notions, special curves, special different spirals and (spatial) screws; Dürer’s conchoid (“shell curve”) and related curves; epicycloids; ellipses as affine transformation of the circle; ellipse, parabola, hyperbola as conic sections (by using the top and front view method); approach to functional thinking using the example of the circle (reconstruction of ordinate by means of given abscissa and vice versa); proportional line segment division; reconstruction of the centre of a given circle and other basic constructions. |
| Book 2 | Classification of curves, angles and areas: special different forms of circular arc triangles and quadrilaterals, exact (as possible) and approximate construction of regular $n$-gons for (in this order!) $n = 6, 3, 7, 14, 4, 8, 16, 5, 10, 7$ again, approx. $5, 15, 9, 11$ and 13. Very precise approximate circular arc (or angle) trisections, plane patterns made of circles; tessellations with isosceles triangles, squares, rhombi and regular pentagons, regular hexa- and heptagons, squares and octagons, ...transformations equal to area, especially approximately squaring the circle, Pythagoras’s theorem. |
| Book 3 | Columns and pyramids; graphic representation of “tangent function” (How is font size to be graded in dependence on the height of its fitting, if we want all lines to appear equally high?); construction of sundials; construction of different alphabets according to unified geometrical rules. |
| Book 4 | The five regular polyhedra in top/front view and unfolding into a net; polyhedral sphere (with “meridians” and “circles of latitude”); unfolding into a net; 7 Archimedean polyhedra as well as two further polyhedra, which are not semi-regular in the modern sense; problem of the cube multiplication of $n$ times next to the ancient tool to construct the two mean proportionals to two given line segments; brief demonstrations of central perspective, especially the construction of a view of a cube with shadow at light incidence; suggestions for mechanical aids (cf. Illus. 5.3.4 and 5.3.5). |

### 5.3.3 New forms

In the first book of ‘Instruction’, Dürer looks at a great range of partially known and partially novel construction principles for plane and spatial curves. Apart from different spirals and screw lines, there are also new ones like the conchoid he called “shell curve” (Problem 6.1.1), or the epicycloid he called “spider curve”. Thereby, the relation between pointwise construction



**Illus. 5.3.13** Dürer's ellipse construction based on the circle [Underweysung 1525]

of a curve and mechanical production by means of mechanisms is exemplarily demonstrated. We can also find Dürer's much discussed "intelligent mistake". He was aware of ellipses as affine transformation of circles and knew how to construct them (Illus. 5.3.13). However, he constructed the plane intersection of a straight circular cone pointwise as an "egg curve" by means of the completely regular two-plane method, despite not knowing anything about the identity of these two types of curves, which is not exactly self-evident. (After all, they do not occur in 'Elements'.[9]) When constructing the "egg curve" pointwise, he made an error in reasoning: Since the diameter of a cone is smaller at the top than at the bottom, the "egg curve" can only have one line of symmetry and must really look like an egg (Illus. 5.3.14). We could say here that Dürer was behind ancient mathematicians. However, he was, in fact, ahead of them, since his approach in constructing the "egg curve" can be generalised. He constructed the parabola and hyperbola in the same manner. He demonstrated a principle method for constructively obtaining curves as plane sections of solids by using the cone coincidentally, so to speak. As a result, he became Monge's remarkable predecessor. Apart from the Archimedean spiral, he also constructed an "Ionic snail" by arranging increasingly large circular arcs together whilst avoiding any breaks.

Analogously, he constructed a real egg-shaped closed curve by means of six circular arcs, which connect to one another "almost" without breaks[10] (Problem 5.3.7). After the coordinate method had been established, such curves arranged piece by piece were banned from mathematics for a long time. Nowadays we would tend to call Dürer the father of "splining".

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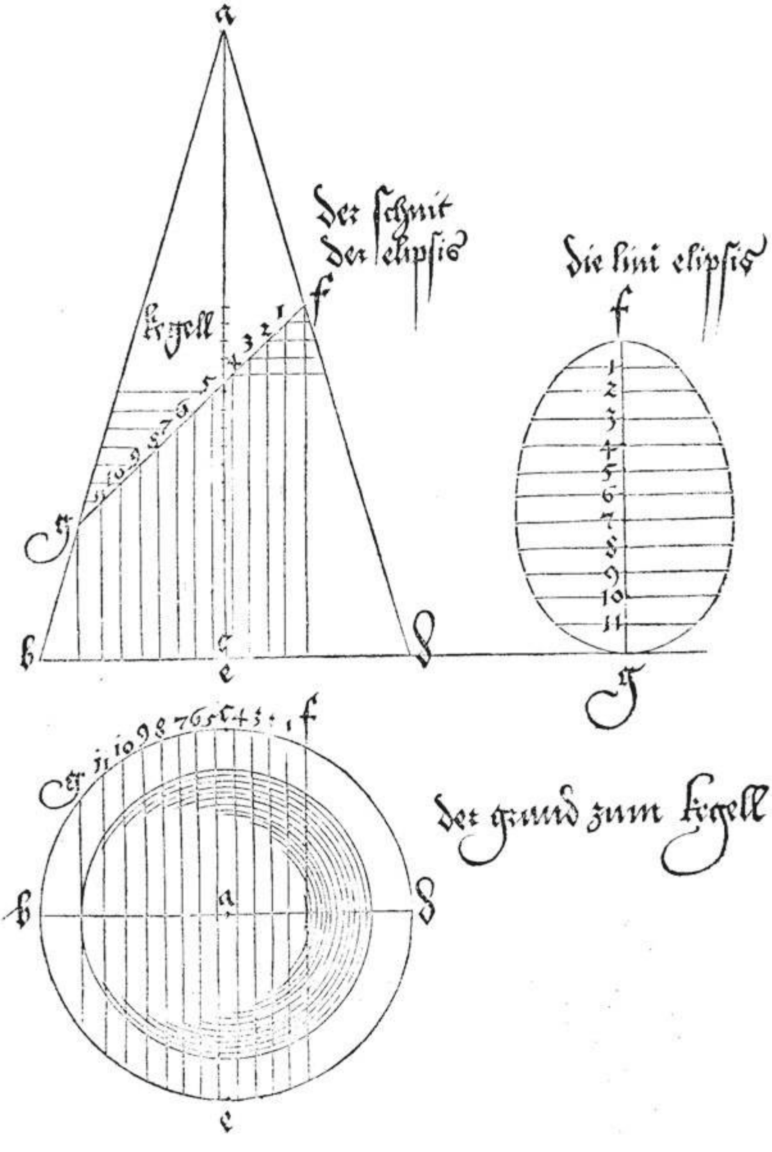
[9] J. Werner, who was close to Dürer at that time and surely owed him a lot of knowledge, had composed a text on conic sections, which – however strange that may be – only addresses parabolae and hyperbolae.

[10] This small break reveals the deficit of Renaissance mathematicians passed on by Antiquity and concerning the general tangent notion. It was mainly caused by the fact that the notion of tangent regarding the conic sections dominating in Antiquity, but only these, can only be grasped due to the existence of exactly one common point of curve and tangent without limit process.



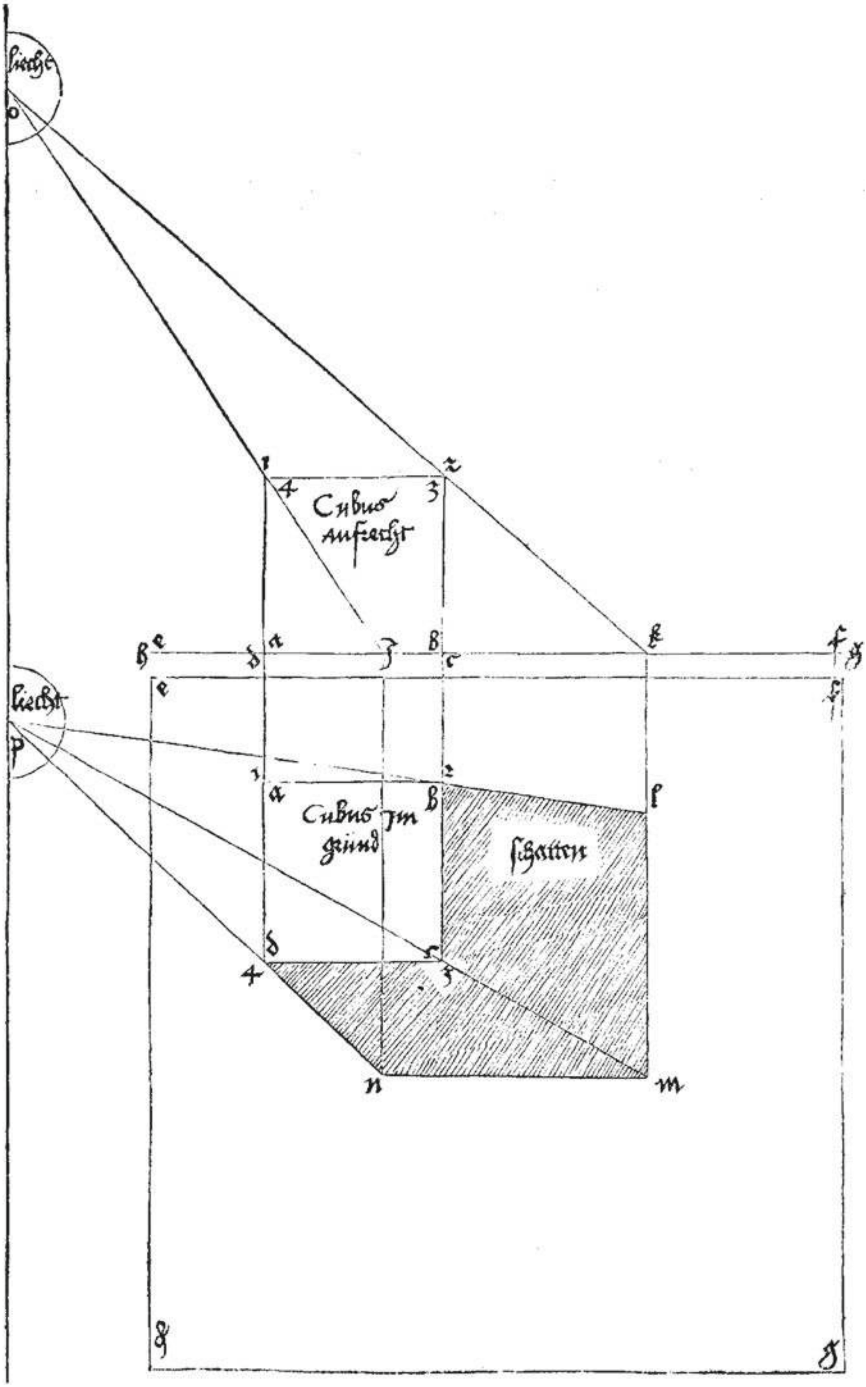

**Illus. 5.3.14** Dürer's ellipse construction based on the cone [Underweysung 1525] [Underweysung 1525, p. 34]

### 5.3.4 Top and front view method

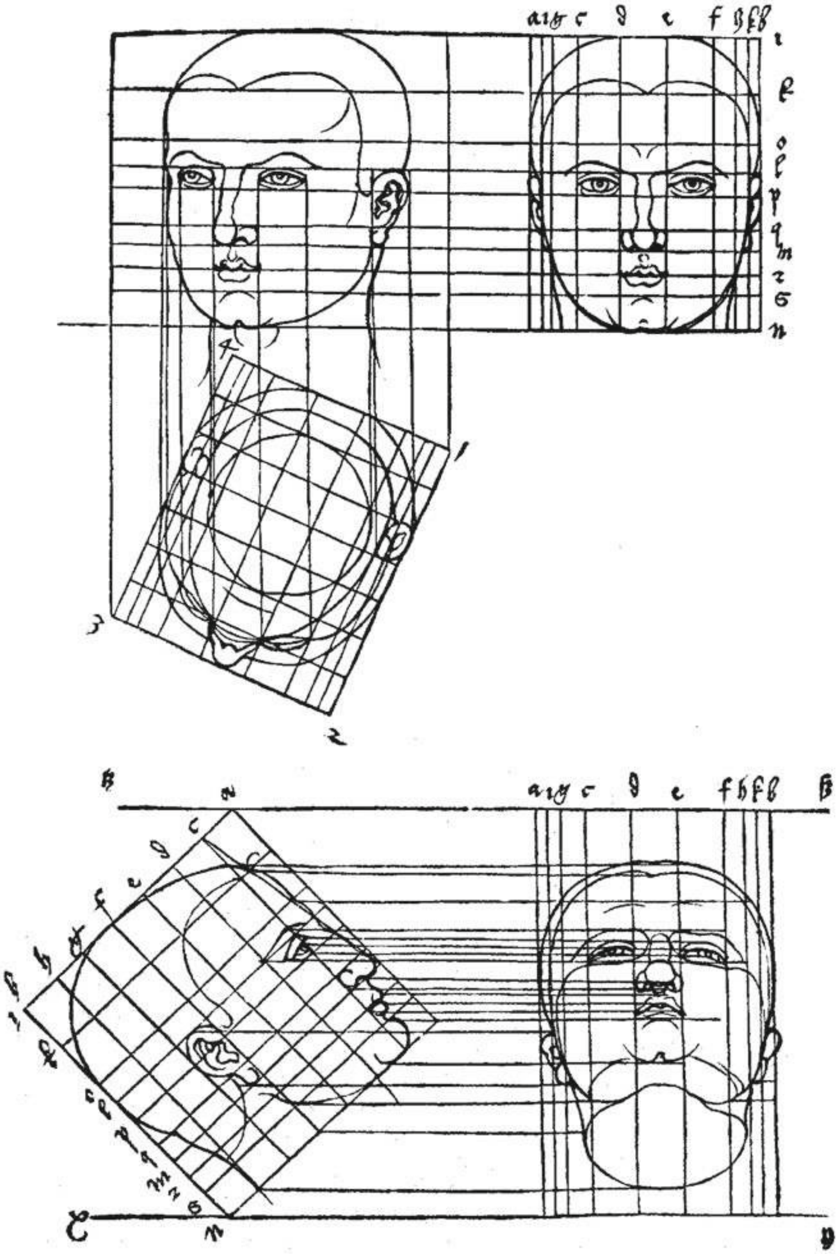
Piero della Francesca addresses the top and front view method with astonishing virtuosity in his *Prospettiva pingendi* (around 1475), which remained unpublished back then. (Some examples of this can be found in [Field 1997].) Dürer made some "technical suggestions" regarding central perspective in 1525 (see Illus. 5.3.4 and 5.3.5) that, due to their frequent reproduction in more recent books, contributed considerably to colouring general opinions of Dürer's mathematics and the contents of his 'Instruction'. His posthumous edition from 1538 mainly features a lot of supplements in this respect, but basically none of his own contributions. In 1525, he was mainly focused on the construction of perspective pictures by means of combining the method of intersection and the top and front view method. Here, Dürer already proved himself a master of the two-plane method in a manner that brings him quite close to the level of Monge's intellect, although this has been little recognised and appreciated until now. This becomes particularly clear when looking at the simple problem of constructing the shadow of a cube given a cube in top and front view and a point-shaped light source (Illus. 5.3.15). In his *Four Books on Human Proportion* from 1528, he applies the two and three-plane method in a versatile manner, for instance, in order to construct an inclined face by means of the front of a head and the turned top or profile view (Illus. 5.3.16). It also comes as a surprise to find the later standardised method of representing a body first in a very simple position in the top and front view and then to turn it gradually in increasingly general positions applied here to the cube and several other simple polyhedra as well (Illus. 5.3.17): Dürer intended to teach proportional representation of a human body in different positions and, thus, approximated the body in a polyhedral manner for this purpose. This seems to have affected the subsequent generations of artists stylistically [Schreiber 1999, 2005a, 2005b, 2007].

### 5.3.5 Ornaments and tessellations

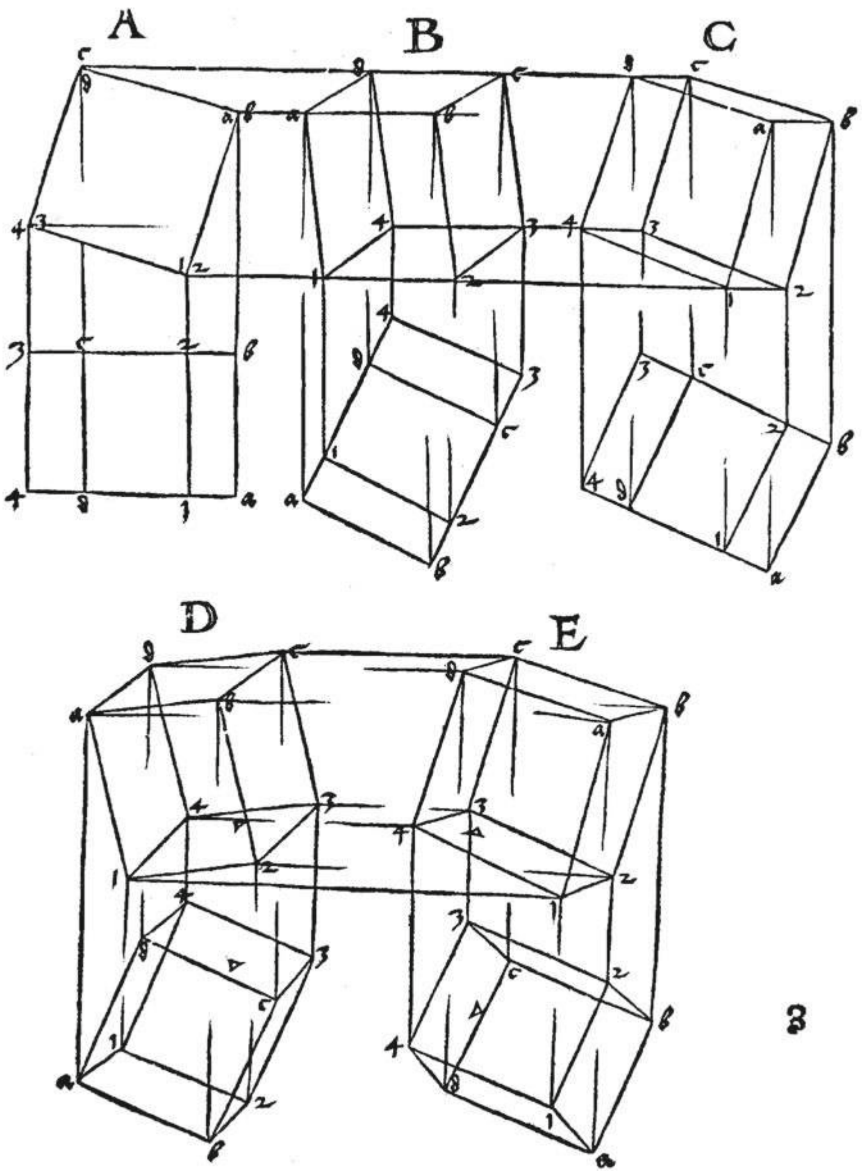
Can the Renaissance add something decorative regular ornaments to the overwhelmingly Gothic tracery? Leonardo, Dürer and others discovered the completely non-Gothic circular arc patterns (Illus. 5.3.18) and the multiply-interlaced knots, which seemingly were adopted from Islamic art (Illus. 5.3.19). Both experimented with the possibilities in assembling regular tessel-



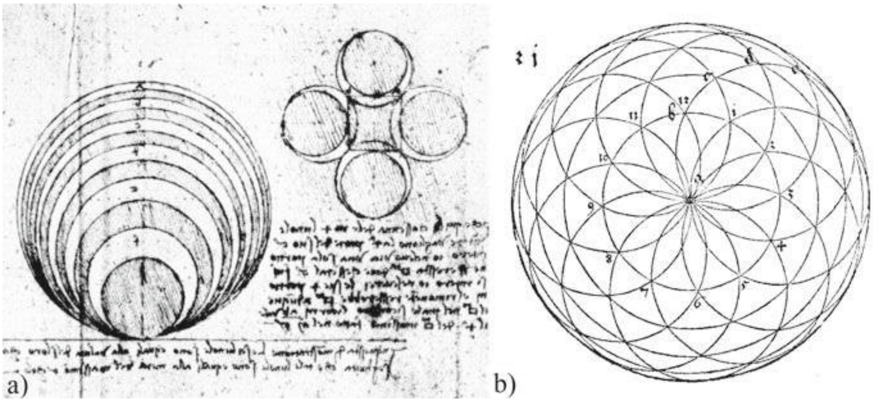

**Illus. 5.3.15** Shadow of the cube according to Dürer [Underweysung 1525]



**Illus. 5.3.16** Application of assigned orthogonal projections for constructing different views of the same human head [Dürer 1528, 'Four Books on Human Proportion']

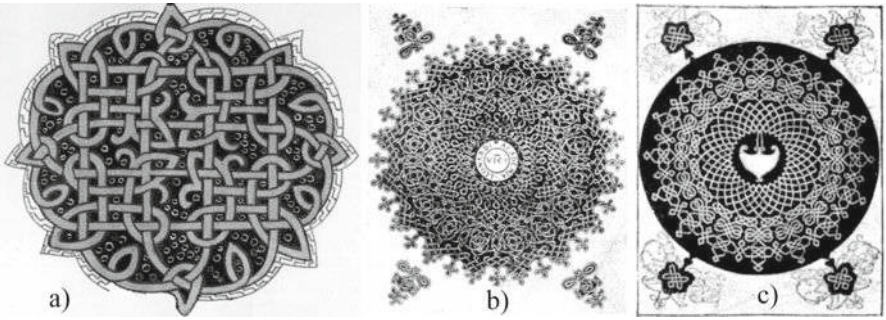



**Illus. 5.3.17** Gradual construction of top and front view of a cube in general position [Dürer 1528]




**Illus. 5.3.18** Circular ornaments a) Leonardo da Vinci b) Dürer

lations based on a few types of pairwise congruent "tiles"[11] (in broader sense). Dürer put forth some possibilities that were addressed neither before nor after him[12]. Kepler will have advanced this kind of geometry to a level that only the end of the $20^{th}$ century will be able to continue [Grünbaum/Shephard 1987]. This is an area in which many ideas had to be gathered before it could be systematised and theories could be formed. Nonetheless, the Middle Ages and the Renaissance were already in love with "floor geometry", partially only painted, partially realised in buildings (Illus. 5.3.20, Problem 5.3.8).




**Illus. 5.3.19** Knots a) in Islamic art b) Leonardo da Vinci c) Dürer [a) from A. Speltz: *Das farbige Ornament aller historischen Stile* (The coloured ornament of all historic styles, Leipzig: A. Schumanns Publishing 1915; b) and c) from Steck 1948]

[11] It is interesting that the modern English terms "tile" and "tiling" are hinted at by Dürer in this context: he uses the word "tillen" [Dürer/Strauss a, p. 170, line 4].

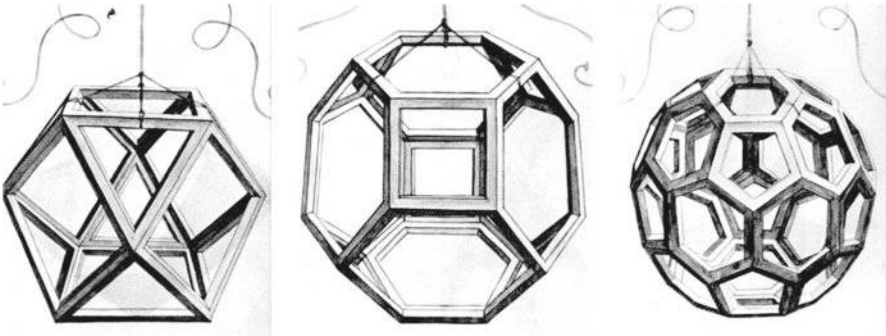
[12] Dürer did not notice that two of the tessellations he found made of squares and octagons merge if turned by 45 degrees.

**Illus. 5.3.20** Floor at Baptistery of Pisa Cathedral, $12^{th}$ century [A. Speltz 1915 l.c.]

### 5.3.6 Polyhedra

Apart from the common knowledge of 'Elements', Pacioli's *Divina proportione* (1509) created a general awareness of the regular and semi-regular polyhedra[13]. Leonardo illustrated this book in a manner that shows that he knew much more about mathematical perspective than we can assume based on his scattered written comments (Illus. 5.3.21). This book also indicates that the notion of semi-regularity was different from our modern meaning. It rather seems that the existence of a circumscribed sphere and a good approximation of this sphere were a crucial criterion for accepting something as semi-regular. Thus, apart from the five regular solids, Dürer also addresses only seven of the Archimedean solids, and in the same context some solids with circumscribed sphere , which are not semi-regular from today's modern perspective (Problems 5.3.10-12). The work *Perspectiva corporum regular-*

[13] Thereby, Pacioli followed up on a text on regular polyhedra by Piero della Francesca, whose student he seems to have been, at least for a while.



**Illus. 5.3.21** Leonardo's illustrations regarding *Divina Proportione* by L. Pacioli [Pacioli: *De Divina Proportione*. German by C. Winterberg. Vienna 1889]

*ium* (1668) by goldsmith Wenzel Jamnitzer, apparently meant to serve as a sample catalogue, offers an impressive amount of intuitively regular solids (Illus. 5.3.24). The style era of Mannerism following the Renaissance tended to choose complicated geometrical shapes for their own sake as the subject of their attention [Eimer 1956]. These were the origins of proper geometrical still lifes (Illus. 5.3.23). Paolo Uccello was a surprising early forerunner of this style (around 1397–1475). According to Vasari, he was heavily criticised by his painting peers. Uccello produced one of the great science-historical miracles by displaying the regular star solid, nowadays knows as "small star dodecahedron", in the marble floor of San Marco in Venice in 1426 (Illus. 5.3.22). Polyhedral net unfolding was first displayed in Pacioli's *Divina proportione* and in more detail in Dürer's 'Instruction'.

In 2006, whilst creating a new digital catalogue of the treasures of the depot of the famous Albertina in Vienna, a set of 40 wooden printing blocks was found representing the nets of all regular and semi-regular polyhedra (also including examples of the prisms and anti-prisms, but no other sphere-like solids!). This set also clarifies wordlessly how to obtain the semi-regular polyhedra in different manners by cutting off vertices and/or edges of regular polyhedra. Since some of the blocks exhibit the signing of Hieronymus Andreae (Dürer's collaborator and engraver, who died in 1556), there is only a small interval of time between 1538 (when Andreae edited and printed the second expanded posthumous edition of the 'Instruction') and 1556 in which these woodcuts could have been made. Unfortunately, there has been no serious hint or idea until now about the creator of this wordless theory of the full set of semi-regular polyhedra, which is even more modern than Kepler's, since it also accepts prisms and anti-prisms. We can only be reasonably certain that it was never printed [Schreiber 2008]. The next instance in which Archimedean polyhedra are dealt with from a remarkable mathematical perspective is not to be found until Simon Stevin's *Problemata geometrica* (1583).

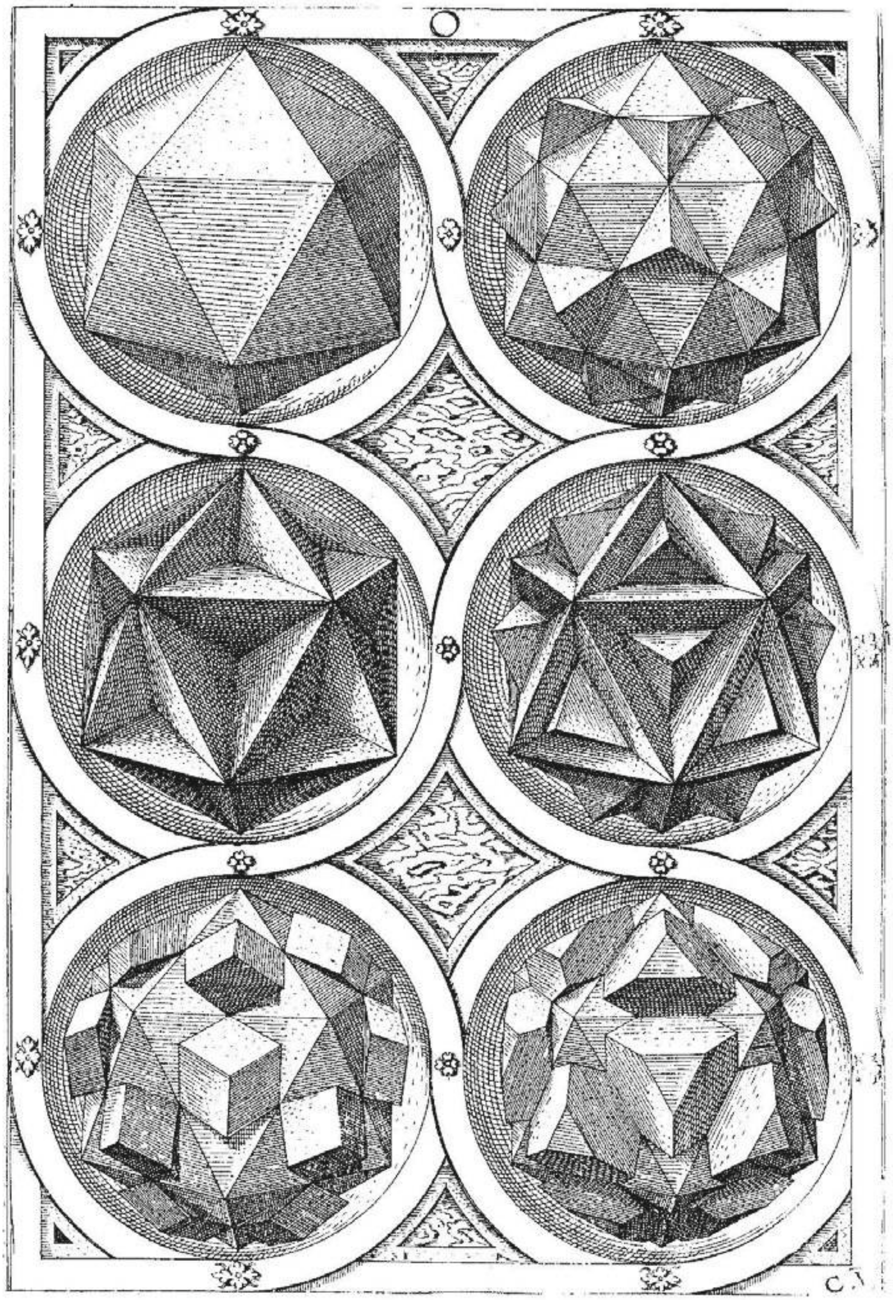
The art-historical literature of the $19^{th}$ and $20^{th}$ centuries is full of attempts to reconstruct mathematical design principles of the Renaissance artworks

**Illus. 5.3.22** Floor mosaic at the entrance area of San Marco, Venice, based on a draft by Paolo Uccello, around 1425. It clearly shows the star solid, nowadays called "small star dodecahedron" [Photo: A. Schreiber]

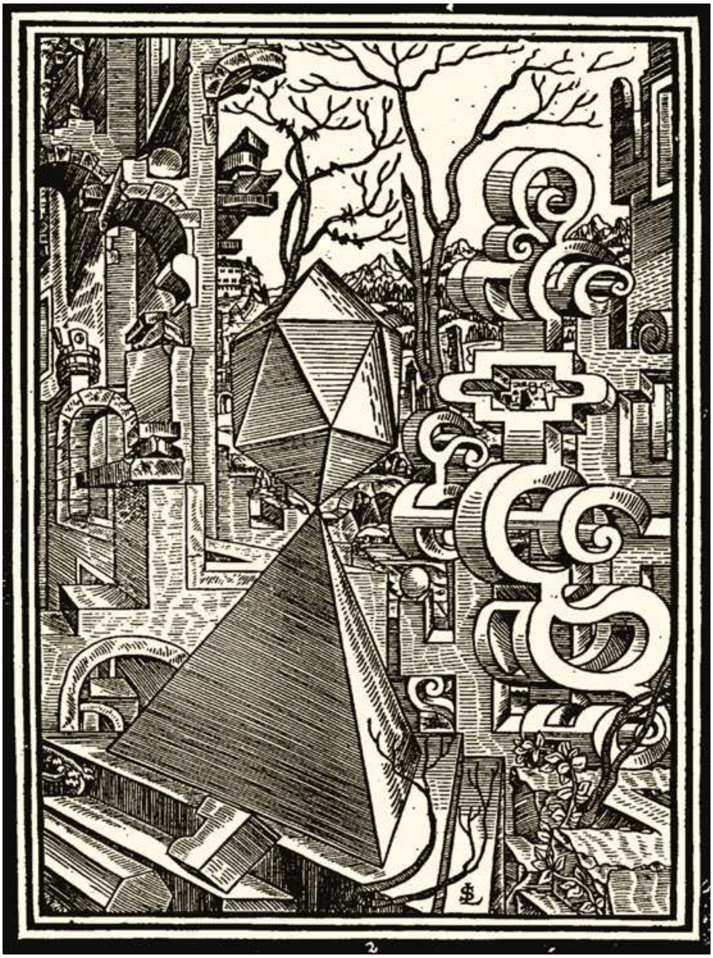
(e.g., see [Dehio 1895], [Fichtner 1984]), including frequent attempts to read ideas arbitrarily into it (Illus. 5.3.25). Pictures and architectural views are covered with complicated constructions afterwards.

However, proofs that the artists did indeed plan their work in this manner are rarely conclusive and the artists have seldom given us such clues themselves (also see [Conrad 1990, p. 82f.] for a knowledgeable critique of "the triangulating and squaring madness" in the architectural-historical literature). Nonetheless, there is a whole series of geometrically constructed picture drafts by Dürer (Illus. 5.3.26). The literature indicates that there was a competition to find proportions of the human body that could be mathematically grasped. This goes back until Vitruvius and was followed up on by Giotto, Leonardo da Vinci, Piero della Francesca, Alberti and others. The latter stated that he obtained his number ratios by means of multiply measuring different people. Dürer's 'Four Books on Human Proportion' (1528) also go beyond the traditional concern. It is the first time that geometrical transformations are used.

Dürer developed different heads based on a normal face by means of linear or non-linear net distortion (Illus. 5.3.28). He showed implicitly how many different faces we can produce based on just a few basic elements and/or by variation of a detail. May he also be the father of the method for producing facial composites at the end? We must mention that the construction of aesthetically beautiful and stylistically unified alphabets (stated in a modern fashion: typography) was used in the Renaissance and, hence, is theoretically close to the canon of proportion of the human body. Due to multiple reproductions, it is relatively known that Dürer constructed different alphabets with compass and straightedge (Illus. 5.3.27). However, he had forerunners in this case too, including Pacioli and Leonardo. Graphical design of letters and numbers has lost nothing of its fascination for artists up to the present time.



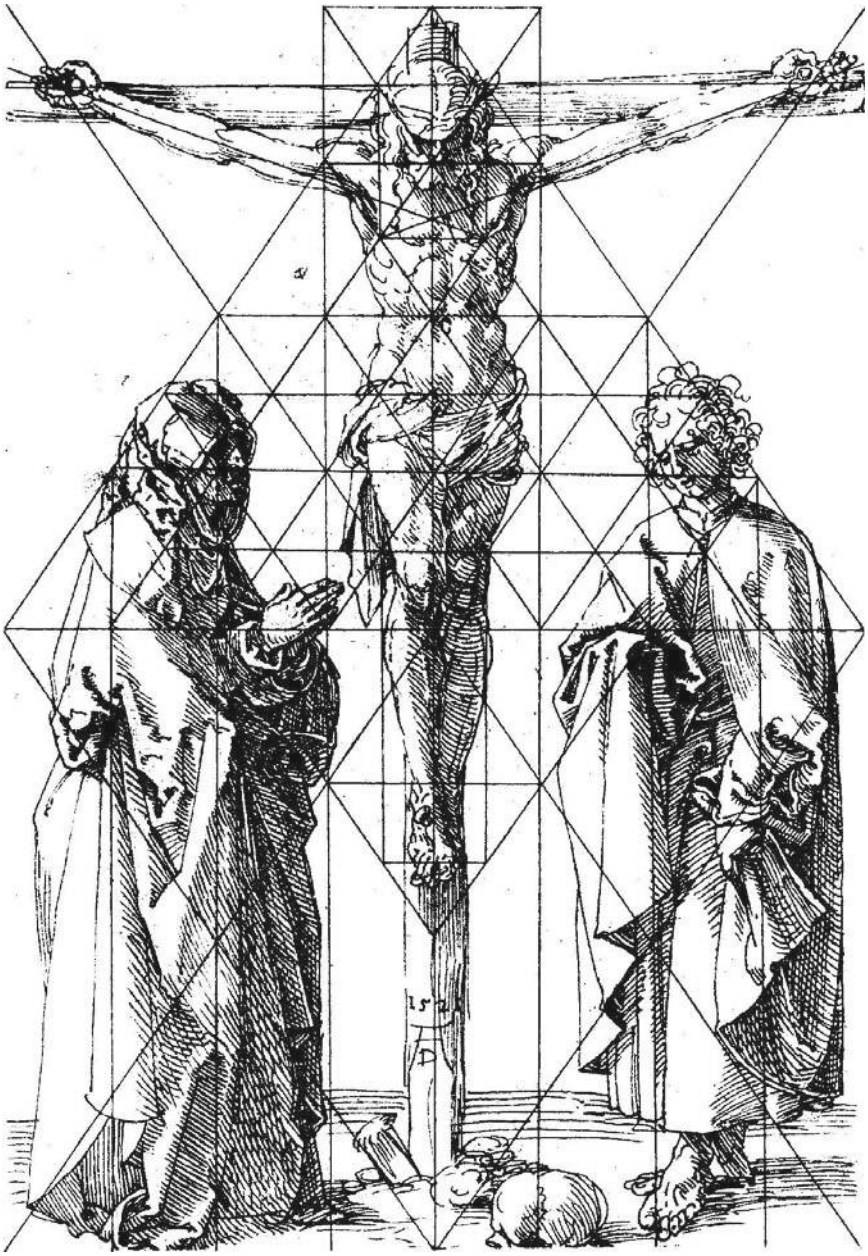
**Illus. 5.3.23** Wenzel Jamnitzer: *Perspectiva corporum regularium* [Nuremberg 1568, plate 23] Picture plates by Jost Ammann were engraved in wood based on drafts drawn by the author. The great dodecahedron is to be found on the left, middle row. Hence, another star polyhedron was known before Kepler.



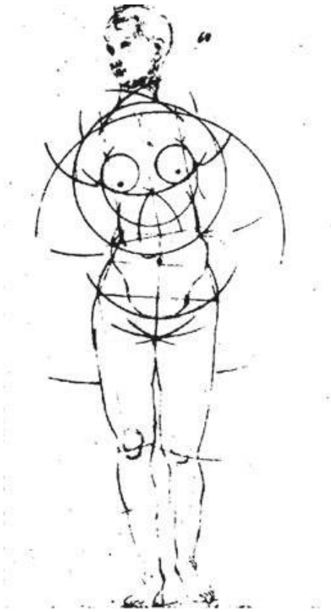
**Illus. 5.3.24** A geometrical still life of Mannerism
[Lorenz Stoer 1567]

### 5.3.7 Terminology

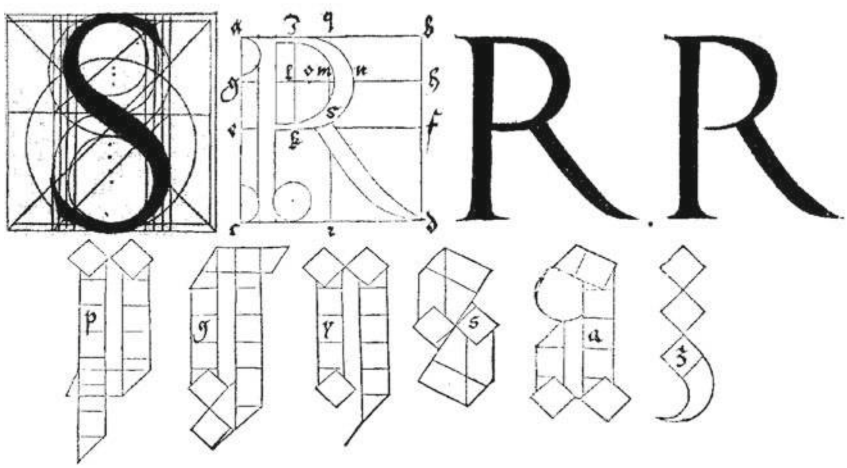
We will confine this section to the origins of a German terminology and, hence, find ourselves back at Dürer's 'Instruction'. It already starts with the German title "richtscheyt" (straightedge). However, the latinised "Lineal" (ruler) has won this "competition". Dürer's "brenlini" (burn/fire line) for parabola and "gabellini" (fork line) for hyperbola have not become accepted. (They were, in fact, merely suggestions for readers who did not know Latin and Greek, since Dürer also states terms taken from Greek.) A circle is a "zirckel lini" (compass line), a circular area is a "runde ebene" (round plane), a square a "gefierte ebene" (four plane) or "fierung" (four thing), ("vier"



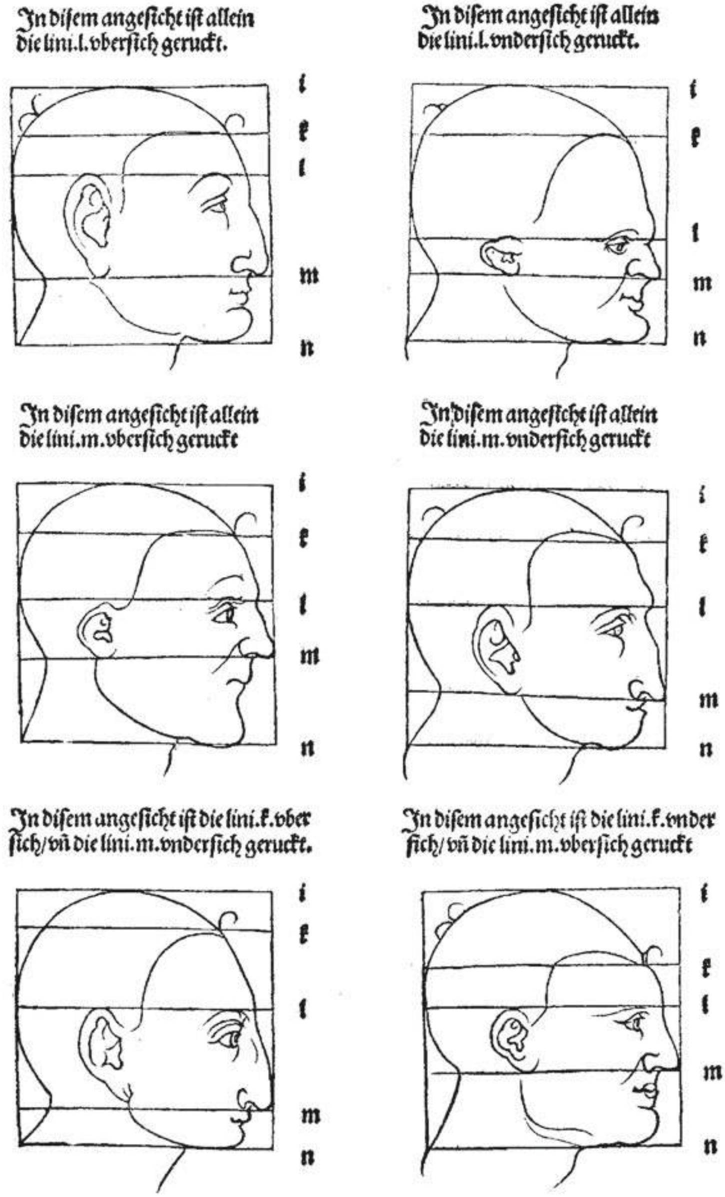

**Illus. 5.3.25** An "analysis" by Max Steck (Dürer, Crucifixion of Christ, 1521) The relation of the covering grid to the picture is not convincing here [Steck 1948, plate XXXV]



**Illus. 5.3.26** A geometrical auxiliary construction. Based on this study, Dürer himself has left his geometrical auxiliary construction standing [Steck 1948, plate X]



**Illus. 5.3.27** Some of Pacioli's and Dürer's examples of constructing letters. The first letter on the top left (S) is by Pacioli, the other ones are by Dürer [Underweysung 1525]

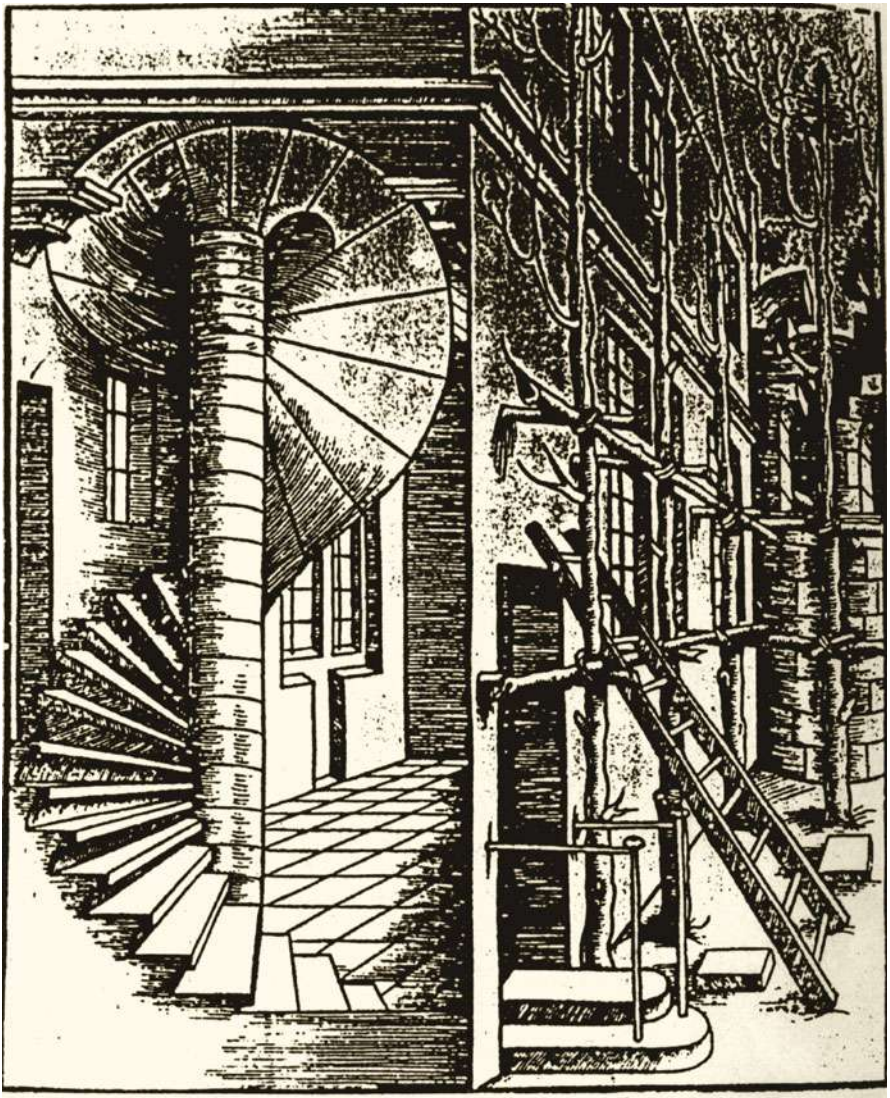



**Illus. 5.3.28** Dürer: 1528: Different methods of non-linear, but mathematically describable distortion of human heads [Dürer 1528]

(Dürer writes "fier") in German means "four" and Dürer used this word to make verbs (fieren, past part. gefiert) and abstract nouns (fierung)), a diagonal an "ortsstrich" (location line), a horizontal line a "zwerchlini" (athwart line), a parallel a "barlini" (bar line), More essential than understanding these clumsy individual words is that we nowadays understand Dürer's text and especially his descriptions of construction. It is crucial that the nature of the addressing technique developed by Euclid has been well preserved by Dürer.

Dürer's work had already been translated into Latin in 1532 and, thus, printed in Paris in several editions. Soon, several imitators surfaced in Germany, but they stood no chance of competing at Dürer's level: amongst others, Hieronymus Rodler (*Eyn schön nütlitz buchlin und underweisung* (A nice, useful booklet and instruction), Simmern 1531 (full of mistakes, see Illus. 5.3.29)), Heinrich Lautensack (*Des Circkels unnd Richtscheydts…underweisung* (Instruction of compass and straightedge), Frankfurt Main 1564), Hans Lencker (*Perspectiva literaria*, Nuremberg 1567, 1571) and Paul Pfinzing (*Ein schön kurtzer Extract der Geometriae und Perspectivae* (A nicely brief extract of geometry and perspective), Nuremberg 1599, 1616.

We will conclude this section by looking at Dürer's copperplate engraving "MELENCOLIA I" from 1514 (Illus. 5.3.30). According to general consensus, the angel lost in deep thoughts represents a symbolic self-portrait of Dürer: Melancholy – caused by slow and partially fruitless pondering about difficult mathematical problems. (For detailed discussions and interpretations see, amongst others, [Panofsky/Saxl 1923], [Steck 1948, p. 141], [Schröder 1980, p.64ff.], [Schuster 1991], [Schreiber 1999]. The displayed polyhedron will be further examined in Problems 5.3.10 and 5.3.11.)



**Illus. 5.3.29** Spiral staircase by H. Rodler (1531)
What is wrong with this spiral staircase? [H. Rodler: *Eyn schön nützlich büchlin und underweysung der kunst des messens mit dem Zirckel/Richtscheyt oder Linial* (A nice, useful booklet and instruction on the art of measuring with compass/straightedge or ruler), Simmern 1531]


**Illus. 5.3.30** A. Dürer: Melencolia I, copperplate engraving (1514)

## 5.4 Geometrical roots of infinitesimal mathematics

Modern mathematics, which originated in the $17^{th}$ century, stands on four pillars: notion of function, coordinate method, differential and integral calculus, of which at least the latter three originally belonged to geometry. The historical roots of these four core areas are to be found at different times in the past. It is clear that determining measured values (lengths, areas, volumes) for geometrical objects forms part of the oldest practical problems in mathematics. However, we can solve them by means of elementary geometry only for "linear objects": line segments, linearly bound plane areas and polyhedra. The first genuinely mathematical attempts to cross that boundary go, as we have seen, back to Democritus (not passed on in writing), Book XII of 'Elements' and Archimedes. In the Renaissance, in which everything was focussed on praxis, these parts of ancient mathematics are rehearsed with special interest and soon advanced. The edition of Archimedes' texts on spirals, circle measurements, squaring the parabola, conchoids and spheroids and floating bodies represented a crucial key for this (first, a Latin print, Basel, 1544, very effective, followed by Greek-Latin editions from Commandino, 1558, 1566). The attempts to replace the strict but tedious Archimedean proving methods for area and volume formulae with heuristic and generalisable considerations are characteristic for the Renaissance. The beginning of Kepler's text 'New Stereometry of Wine Barrels' (1615, herein shortened to 'Wine Barrels') is a typical example of this. In this text, he first compiled the outcomes known from Antiquity, but gave them new justifications: "A circumference has as many parts as points, namely infinitely many; each part can be seen as the basis of an isosceles triangle, the vertex of which lies in the centre of the circle." He concluded that the circular area is composed of the area of infinitely many triangles, the common height of which is the radius. The bases add up to the circumference and, hence, the area equals half the product of circumference and radius. Analogously, Kepler extracts from the pyramid volume formula that spherical volume equals a third of the product of radius and surface by imagining a sphere decomposed into infinitely many pyramids, the bases of which form the surface and with a common apex that is the centre of the sphere.

Whereas everything mentioned so far is thoroughly understandable given the historical context, it is strange that there was a theological reason for the emerging interest in problems of determining centres of gravity next to the needs of shipbuilding, which, for instance, encouraged Stevin to conduct his investigations. The medieval theologian and logician Jean Buridan argued that every geological process (including the fact that broadleaf trees shed all their leaves in autumn) shifts the centre of gravity of Earth, which then has to try hard over and over again to reach the centre of the universe. He continued to argue that this would lead to a constant tottering movement of Earth around this assumed centre. Having said that, the dogma of the

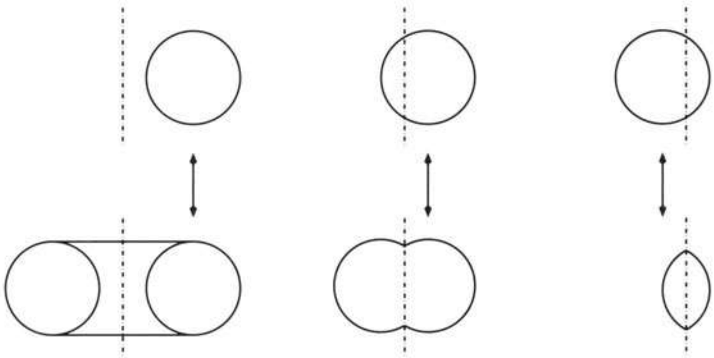

**Illus. 5.4.1** Bonaventura Cavalieri and the title of his book 1635

immobility of Earth was also accepted, causing a theological argument in which, amongst others, Bishop Nicole Oresme, who was so important to the history of mathematics, participated. Furthermore, for example, the Jesuit mathematician Paul Guldin still referred back to this argument in the $17^{th}$ century when engaging with gravity problems. He listed more metaphysical than mathematical arguments concerning the Guldinus theorem, named after him, in his main work in four volumes *Centrobaryca* (Vienna 1635-41).

Only Bonaventura Cavalieri attempted to prove the theorem based on infinitesimal methods (i.e., decomposition into "indivisibles") in 1635. We can also see Kepler as an essential forerunner, since he, in the first part of 'Wine Barrels', produced a large variety of solids of revolution based on circles and other conic section by means of simply shifting the axis of revolution away from the axis of symmetry. Theorems, such as Proposition XVIII, follow:

"The volume of every ring-shaped solid with circular or elliptical cross section equals a cylinder, the height of which equals the circumference specified by the revolution from the centre of the figure, and the base equals the cross section." To justify this, he says: "If we cut the ring into infinitely many and very thin discs by means of cuts from the centre, then the thickness of the disc will decrease and become a lot narrower towards centre A of the ring, whereas this thickness will increase symmetrically to it on the outside. Afterwards both parts together will be twice as thick as in the centre of the discs. This consideration would not apply if the discs with their parts inside and beyond the centre [of the revolving profile] were not symmetrical to each other."

Coordinate methods were used indirectly in the Antiquity whenever curves are described by their symptoms. The coordinate method is more explicitly worded in astronomy and geography. All in all, the Renaissance could not top



**Illus. 5.4.2** Several solids of revolution with shifted axis of revolution: torus, apple, lemon

these past achievements. However, G. B. Benedetti (little appreciated until now) can be seen as a forerunner of Descartes, since he was the first to compile the realisation of all four species and root extracting systematically by means of geometrical construction in his *Speculationum* from 1585. Based on this, he traced back the solution of any construction problem to the algebraic analysis of the wanted parts and their – so to speak – algorithmic construction based on given quantities.

The early development of differential calculus was heavily hindered at first by the fact that it is exceptionally possible to define tangents of non-degenerate conic sections elementarily (which had been the focus of theorems on tangents since Antiquity) because they are just those lines that have exactly one point in common with the conic section. This was sufficient for the requirements of geometrical optics, in which reflecting or refracting surfaces (which are created in the practically important cases by revolution of conic sections) are substituted locally by their tangential plane. The essential impulse to deal generally with the tangent problem was provided by questions of local extreme values of functions, which were hardly ever asked before the Renaissance, and if so, only in a hidden manner[15]. However, the crucial thought-provoking impulse, on which Fermat will follow up, came from the Renaissance and from Kepler.

The functional approach to thinking and the intention of finding maxima broke spontaneously through in Kepler's already mentioned 'Wine Barrels',

[15] Occasionally, we can find a hint in the literature pointing at isoperimetric observations of ancient geometers, particularly in one passage in Proclus's commentary on Euclid. The relevant passage in Proclus's work states, in badly worded form [Proclus/Morrow 1992, p.314]: "The square is demonstrably greater than all figures of same perimeter." We can only deduct that quadrilaterals alone are permitted here as rival figures by means of the context. Otherwise, this claim would naturally be wrong.


**Illus. 5.4.3** Monument of Brahe and Kepler, Prague [Photo: P. Schreiber]

which is basically dedicated to problems of calculating volumes of solids of revolution and already explicitly refers to Archimedes in the title (all quotations are translated into English based on the German translation of 'New Stereometry of Wine Barrels' in Ostwald's classics): Part One, Proposition XXIX: "If the lemon, the plums, the parabolic spindle and the double cone (creative, but precisely defined notions introduced by Kepler for his solids of revolution), which are all obtuse, have the same intersecting circle as well as the same circle around the centre of the solid, then the lemon is the *largest* solid..." (The emphases here and in the following are by Schreiber). Part Two, Proposition IV: "The cube is *the largest* amongst all parallelepipeda or columns, which are inscribed in one and the same sphere and stand on two opposite square bases." Corollary 2 for Proposition V: "This shows that a certain practical, geometrical meaning is inherent to the rule, according to which the Austrian cooper makes barrels... i.e., that it agrees with the rule of Proposition V and has the *largest* possible volume regardless of whether it deviates a little from completely fulfilling the rule. Other designs, which stretch across this side and beyond this side until the points are very close to G, hardly change the volume, since the volume is the largest possible for

AGC: *the* [volume] *adjacent on both sides to a greatest value only shows unnoticeable decrease at the beginning.*" There is not just functional thinking at work here, but for the first time the idea that under certain conditions, nowadays well defined, an extreme value can only occur if the changes of the functional value caused by arbitrarily small changes of the argument will be minimal. Of course, no proof is given, but based on geometrical intuition, Kepler worded the concept that will later be referred to as Fermat's theorem in differential calculus. Proposition XXVII re-highlights his new thought:
"If the two halves of an Austrian barrel are not completely similar (i.e., equal), but one barrel base is a little smaller and narrower than the other one, then the difference of the capacity of both halves is not noticeable, given that only the visor length is named However, this difference is always unnoticeable in those points in which a change from the smaller to the larger and here again to the smaller occurs according to a certain circle law."

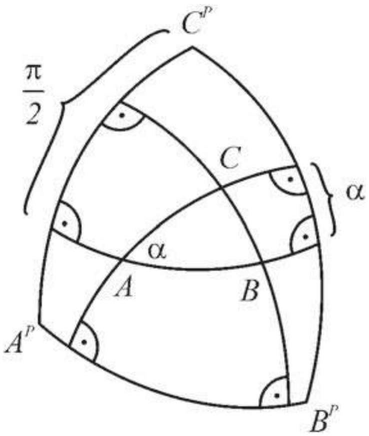
Further essential roots of initial functional thinking, which also belong to the Renaissance, concern the mathematical wording of laws of motion (Galilei, Kepler). However, they are not naturally geometrical, since time always occurs as an independent variable.


**Illus. 5.4.4** Detail of the title page of the book: Adam Risen, *Rechenbuch auff Linien und Ziphren in allerley Handthierung, Geschäfften und Kauffmanschafft*, 1574 (Calculation book on lines [i.e. abacus] and ciphers [i.e. calculation on paper] in all sorts of handlings, negotiations and merchants)

**Essential contents of geometry in the Renaissance**

| | |
|---|---|
| 1377–1446 | Filippo Brunelleschi invents “method of intersection” for perspective Pictures(according to Vasari) |
| 1404–1472 | Leone Battista Alberti: books on architecture and painting(with perspective) |
| Around 1420–1492 | Piero della Francesca:*De prospectiva pingendi*(On the perspective for painting) |
| 1436–1476 | Regiomontanus: establishes trigonometry in Europe as systematic theory, translation of classic mathematical and astronomical works |
| Around 1445–1517 | Luca Pacioli:*Divina proportione*(golden ratio; regular and semi-regular polyhedra) |
| Approx. 1450– | Mathes Roriczer: ‘Booklet on Pinnacle Correctitude’(1486) |
| Approx. 1500 | ‘Geometry [in] German’(1487), descriptions of decorations in Gothic architecture |
| 1471–1528 | Albrecht Dürer: Instruction(1525,1538),‘Four Books on Human Proportion’(1528) deal extensively with practical geometry, particularly perspective, multiplane method, curves, tesselations, regular polyhedra |
| 1473–1543 | Nicolaus Copernicus: *De revolutionibus orbium coelestium libri VI*(1543). heliocentric world system |
| 1502–1578 | Pedro Nunes: curves of constant course (loxodromes) in seafaring |
| 1508 – 1555 | Gemma Frisius: method of triangulation |
| 1508 – 1585 | Wenzel Jamnitzer: *Perspectiva corporum regularium* (1568) |
| 1512 – 1594 | Gerard Mercator: first world map as “Mercator projection” |
| 1514 – 1576 | Georg Rheticus: definitions of trigonometric functions in right-angled triangles |
| 1546 – 1601 | Tycho Brahe: astronomical measurements (foundation of Kepler’s work) |
| 1561 – 1613 | Bartholomaeus Pitiscus: *Trigonometriae sive dimensionae...* (1595), ten books on trigonometry |
| 1564 – 1642 | Galileo Galilei: establishes modern kinematics |
| 1571 – 1630 | Johannes Kepler: ‘New Stereometry for Wine Barrels’ (1615) (root of infinitesimal mathematics), ‘On the Six-cornered Snowflake’ (1611) and *Harmonice Mundi* (1619) contain numerous approaches to discrete and combinatory geometry, laws of planetary motion |

## 5.5 Problems to 5




**Illus. 5.5.1** Figure to Problem 5.2.1

**Problem 5.2.1:** Polar triangles on the sphere

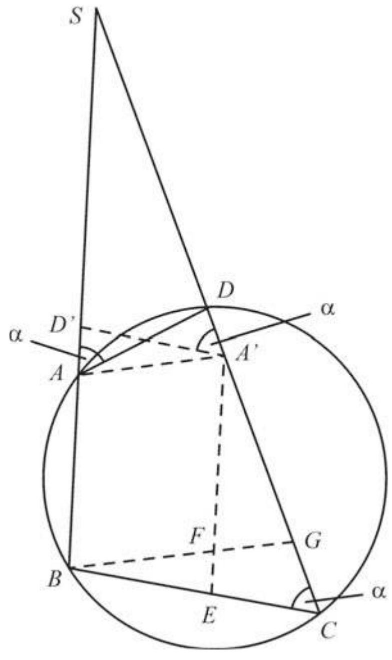
$A, B, C$ are corners of a triangle on a spherical surface. $C^P$ refers to the pole for the great circle through $A, B$, which lies on the same semi-sphere as $C$ regarding this great circle. Points $A^P$ and $B^P$ are defined analogously. $A^P B^P C^P$ is the polar triangle for $ABC$ (Illus. 5.5.1). Consider:

a) That the polar triangle of the polar triangle of $ABC$ is again $ABC$.
b) How the sides of the polar triangle measured in radians depend on the angles of the fundamental triangle and the angles of the polar triangle on the sides of the fundamental triangle.
c) How we can now determine the sides of the polar triangle based on the three given angles of a spherical triangle and the sides of the fundamental triangle based on its angles. (This corresponds to a derivation of the law of cosines for angles of spherical trigonometry by means of the law of cosines for sides.)

**Problem 5.2.2:** Vieta's construction of the inscribed quadrilateral

In the appendix of Vieta's Mesolabum from 1596, we encounter the following tempting construction problem: In order for a convex quadrilateral with the sides $a, b, c, d$ to exist, it seems to be necessary and also illustratively sufficient that each of these sides is shorter than the sum of the three remaining. Vieta showed that it is possible under these weak conditions to always give the quadrilateral a form that features a circumscribed circle, and that angle $\alpha$ , e.g., close to corner $C$ and necessary to construct this form based on given

$a, b, c, d$, can be constructed with compass and straightedge. We are dealing with a trivial special case, if two opposite sides are equal. (Then, the wanted figure is a symmetrical trapezoid.) The figure (Illus. 5.5.2) used by Vieta gives us a hint for the general case. Regarding this figure, $D'$ and $A'$ are so determined that $SD' = SD$ and $SA' = SA$. Furthermore, $BG$ is parallel to $AA'$ and $A'E$ is parallel to $AB$. Since a quadrilateral inscribed in a circle is characterised by the fact that the sum of each two opposite interior angles is 180 degrees ( this can be justified by means of the theorem of peripheral angles), $\alpha$ reoccurs as an external angle at $A$. To know $\alpha$, it now suffices to know all three sides of the triangle $A'EC$, whereby $A'F = AB = a, BC = b, CD = c$ and $DA = D'A' = BE = d$ are given and $EF$ (why?)$= DA'$. Thus, we need $x = EF$ and $y = GC$. (Solution: $x + y = c - a$ and $x : d = y : b$. Hence, we must divide the line segment $c - a$ in the ratio $d : b$.)Following this problem, the question naturally arises, if there is also always a convex inscribed $n-gon$ with given sides for $n > 4$ for n line segments $a_1, ..., a_n$ that fulfil the necessary inequations, and if we can construct it based on these sides with compass and straightedge. The answer to the first question is yes, the answer to the second one is no. The proof can be found in [Schreiber 1993]. Also see Problem 7.3.4.




**Illus. 5.5.2** Figure to Problem 5.2.2

**Problem 5.2.3:** Resection

Solve the problem of resection analytically instead of graphically, i.e., how do the Cartesian coordinates of the new point depend on the coordinates of the three given points? How do we express in the analytical formula that the new point must not be too close to the "dangerous circle"?

**Problem 5.2.4:** Archimedean and Catalan (Archimedean duals) polyhedra and polygons

A polyhedron is Archimedean semi-regular, if its areas are regular (but not necessarily pairwise congruent) polygons and its corners are pairwise congruent (consisting of the cyclic ordered amount of the areas meeting in one corner).

a) Prove that each Archimedean polyhedron has a circumscribed sphere, i.e., a centre, from which all corners have the same distance.

b) The Catalan polyhedron (Archimedean dual) $D^P$, which is reciprocal to an Archimedean polyhedron $P$ has the circumscribed sphere of $P$ as the inscribed sphere. Its areas in the corners of P are tangential to this sphere. Thus, it has its corners exactly on the axes from the centre of the sphere through the centres of the areas of $P$, and the edges of both polyhedra are injectively mapped. The equality of all areas of $D^P$ is due to the equality of all corners of the Archimedean polyhedron P and the possible inequality of the corners of $D^P$ results from the possible inequality of the areas of $P$.

   Prove that a polyhedron is absolutely regular, if it is Archimedean and Catalan, i.e., features both a circumscribed and an inscribed sphere.

c) In his *Harmonice mundi*, Kepler initiated a discussion on a two-dimensional analogon of the notion of Archimedean polyhedron. This is a convex polygon, the "corner figures" of which, made of one corner each and both adjacent edges (possibly of different length), are pairwise congruent. Continue this train of thought: for each of the infinitely many possible "corner figures", there is an Archimedean polygon that always has a circumscribed circle, in other words, is a cyclic polygon. This circumscribed circle is the inscribed circle of the respective reciprocal Catalan polygon (Archimedean dual). What characterises the latter ones?

**Problem 5.2.5:** Kepler's *Mysterium cosmographicum*

Recalculate Kepler's planet model, i.e., which radii feature the nested inscribed and circumscribed spheres if we equate the orbital radius of Earth with 1? Compare the obtained values with the ratio of the mean orbital radii stated by modern astronomical encyclopaedia. Would it be possible to include the planets discovered since then, Uranus, Pluto and Neptune, in Kepler's model by allowing suitable Archimedean and Catalan polyhedra (Archimedean duals)? (Archimedean polyhedra only have a circumscribed sphere, but no inscribed sphere; vice versa for Archimedean duals (Catalan solids).)

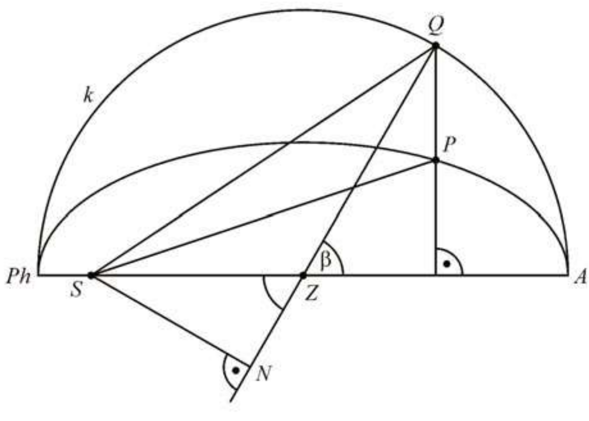
**Problem 5.2.6:** Kepler's Problem

$Z$ refers to the centre of circle $k$ with a diameter of $2a = AP_a$ ($A$ = aphelion = farthest point from sun, $P_a$ = perihelion = closest point to sun) of the orbit of planet $P$ (Illus. 5.5.3, also cf. Illus. 5.2.8). The sun is located at focus $S$ of the ellipse. Then, the orbital ellipse of $P$ is made of this circle $k$ by affine compression. Thereby, the planet location $P$ is the image of a "pseudo planet" $Q$, which moves on circle k in such a manner that $PQ$ is always perpendicular to the great axis $APh$ of the orbital ellipse.

a) State the reason for why the ratio, to be determined, of area $ASP$ to the total area of the semi-ellipse equals the ratio of area $ASQ$ to the area of the semi-circle. As a result, Kepler's initial problem is reduced to the elementary problem of determining the position of the pseudo planet $Q$ belonging to $P$ by means of the $2^{nd}$ of Kepler's law in dependence of time.
b) If we determine the position of $Q$(and, hence, of $P$) by means of angle $\beta = AZQ$, referred to as an eccentric anomaly, the arc segment $ASQ$ is calculated as a union of sector $AZQ$ and triangle $SZQ$ as $\frac{1}{2}(\beta a^2 + aesin\beta)$.(Height $SN$ of basis $ZQ = a$ of triangle $SZQ$ can be expressed by $\beta$ and eccentricity $e = SZ$ of the orbital ellipse.)
State reason for this.
c) By expressing $e$ by the numerical eccentricity $\epsilon = \frac{e}{a}$ of the ellipse, we finally obtain Kepler's equation, if the ratio $c$ of the area $ASP$ to the semi-ellipse is given:

$$\beta + \epsilon \cdot sin\beta = c \cdot \pi \ .$$

With this equation we can determine $\beta$. Recalculate this and consider that this yields again a uniform movement in zero approximation for very small eccentricities, such as approximately circular orbits around the sun.




**Illus. 5.5.3** Figure to Problem 5.2.6

**Problem 5.3.1:** Dürer's construction of the regular $5, 7$ and $10 - gon$

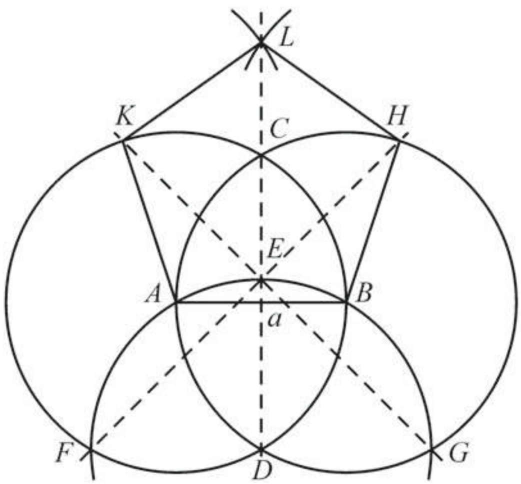
Check that the penta- and decagon edge are exact solutions, in respect to Dürer's construction of the penta-, hepta- and decagon edge for a given circle, as stated in the text (Illus. 5.3.11). (We obtain $\sqrt{\frac{1}{2}(5-\sqrt{5})}$ for the pentagon edge and $\frac{1}{2}(\sqrt{5}-1)$ for the decagon edge from both the given construction and the added figure at radius 1 of the circumscribed circle.)

Calculate the error made concerning the heptagon edge, for which the height of the isosceles triangle formed by the side of the regular hexagon is used.

**Problem 5.3.2:** Approximate construction of the regular pentagon with fixed compass span

The construction already described in Roriczer's 'Geometry [in] German' is as follows: $AB$ is the given edge a. The circles (all of radius a) around $A$ and $B$ intersect each other in $C$ and $D$. The circle around $D$ intersects the straight line $CD$ in $E$, the circle around $A$ in $F$ and the circle around $B$ in $G$. Then, the straight line $EF$ meets the circle around $B$ in the $C$-half-plane regarding $AB$ in $H$ and, symmetrically to this, the straight line through $EG$ meets the circle around $A$ in $K$. $H$ and $K$ are two further corners of the wanted pentagon after $A$, $B$. Hence, it is clear that the last corner $L$ lies on the straight line $DE$ and on the circle around $H$ or $K$, respectively. According to the construction, all five sides feature the length a of the used compass radius.

Calculate the size of the angles at $A$, $K$ and $L$ (Due to a reason of symmetry, the angle at $B$ equals the one at $A$ and the one at $H$ equals the one at $K$.) and verify that the angles at $A$, $B$ are approx. 108.37 degrees and the ones at $H, K$ are approx. 107.04 degrees. Hence the angle at $L$ is approx. 109.18 degrees.

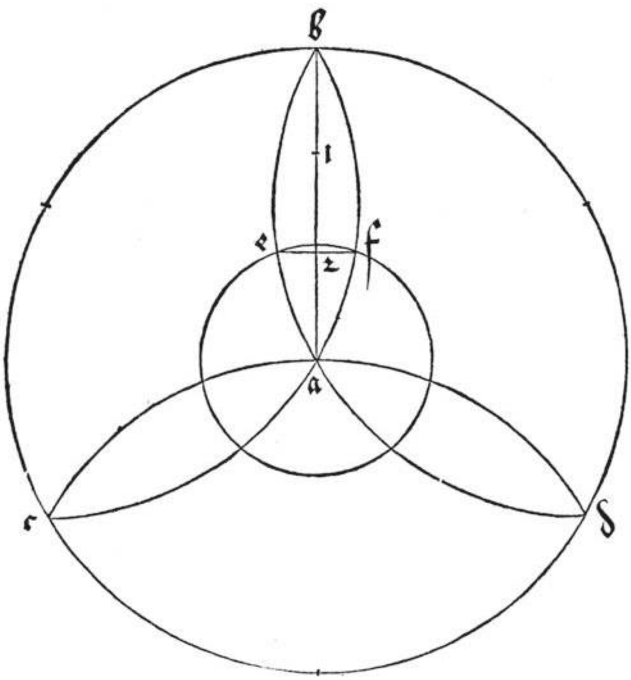



**Illus. 5.5.4** Figure to Problem 5.3.2

**Problem 5.3.3:** Dürer's construction of the nonagon

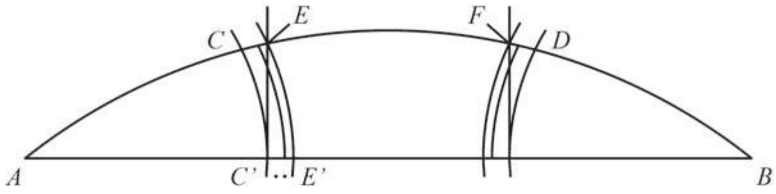
Dürer's peculiar approximate construction of a regular nonagon (originating from craftsmen praxis according to [Steck 1948, p. 49]) was described by him as follows (according to [Strauss 1977]). Of course, Dürer's original wording, clumsy from a modern point of view, is hard to translate. But it is remarkable that Dürer included letters that are used as marked between points to distinguish them from other letters, for example, .a..

"You can construct a nine-sided figure based on a triangle. Draw a large circle with center .a.. Then without changing the opening of the compass, draw three "fish-bladders" whose upper end on the periphery you will mark .b.. Mark the others .c. and .d.. Within the upper "fish-bladder" draw a vertical line ba and divide this line with two points 1 and 2 into three equal parts. Point 2 should be closest to .a.. Then draw a horizontal line through point 2 at right angles to the vertical line .ba. Where the horizontal line crosses the "fishbladder", mark points .e. and .f.. Then place one leg of a compass on center .a. and the other on point .e. and draw a circle through point .f.. Line .ef. will then represent one of nine sides which will compose a nonagon inside the smaller circle, as shown in the diagram below."(cf. Illus. 5.5.5):
The claim is made that angle .eaf. is approximately 40 degrees. Determine its exact value and judge its accuracy.




**Illus. 5.5.5** Figure to Problem 5.3.3

**Problem 5.3.4:** Dürer's approximate angle trisection




**Illus. 5.5.6** Figure to Problem 5.3.4

The construction (which is actually a trisection of a respective circular arc and, thus, can be improved in regards to its graphical accuracy by choosing a preferably long radius $r$) is as follows for an (w.l.o.g acute) angle: Divide the connecting line segment of the extremities $A, B$ of the circular arc into three equal parts and transfer both outer line segments by means of using the compass as in Illus. 5.5.6 and the middle line segment by means of lifting it perpendicularly onto the circular arc. The chords $AC, EF$ and $BD$ are now equal (and consequently, so are their arcs). We now need to divide the rests $CE$ and $FD$. It would be possible to apply the same construction to this and iterate it as often as we like. Dürer was satisfied with returning these residual arcs to the basic chord $AB$ by drawing circles around $A$ and/or $B$ with the compass. There, he divided them linearly into three and added the respective parts of two thirds to the outer partial arcs, but the two median thirds to the median arcs by means of repeatedly drawing circles around $A$ and/or $B$. But keep in mind: Since $E$ does not lie on a straight extension of $AC$, $C'E'$ is a little shorter than $CE$ and analogously on the other side! As calculated by F. Vogel in 1931, the mistake of the theoretical accuracy made here is less than 20 seconds of arc. No drawing could practically achieve this level of precision. Attempt to understand this calculus of errors.

**Problem 5.3.5:** Dürer's angle trisection according to Vahlen

In [Vahlen 1911], which also addresses Dürer's approximation, the described construction deviates a little, as follows: Divide chord $AB$ into three in $C', D'$ and construct $E, F$ as above. Take the arithmetic mean of the three chords $AE, EF, FB$ to be the chord of the third of the arc. Consider that this description yields the same result as Dürer's!

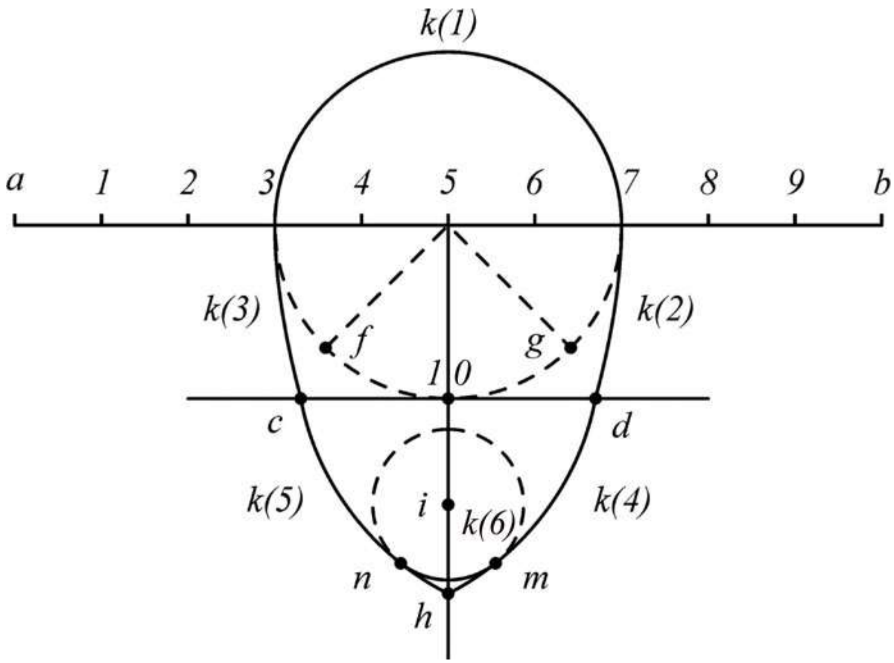
**Problem 5.3.6:** Dürer's approximate squaring of the circle

Dürer's approximate squaring of the circle is based on taking $\frac{5}{2}$ of the circle radius as the diagonal of the square of approximate equality of area. In which approximate value of $\pi$ does that result?

**Problem 5.3.7:** Dürer's "egg curve"

"Now I want to teach you how to draw a line, which is similar to a well-formed egg." Dürer's "splintered" egg curve (Illus. 5.5.7) is composed of the circular arcs $k(1)$ from point 3 until point 7 around point 5, $k(2)$ from point 7 until point $d$ around point $a$ (symmetrically to this, $k(3)$), $k(4)$ from $d$ until $m$ around $f$ (symmetrically to this, $k(5)$) and $k(6)$ between $m$ and $n$ around the centre $i$ of the line segment between 10 and $h$: "Position the compass with one foot in point $i$ and the other one in the circle $ch$ so that you reach it in the shortest way possible."

a) Justify why the curves in the points 3, 7, m and n are smooth, but have $a$ break in $c$ and $d$ (not visible with the naked eye)..
b) Calculate the angle between the one-sided tangents in $c$!
c) How would we have to adopt the construction of circle centre $f$ and/or $g$ to avoid all breaks?



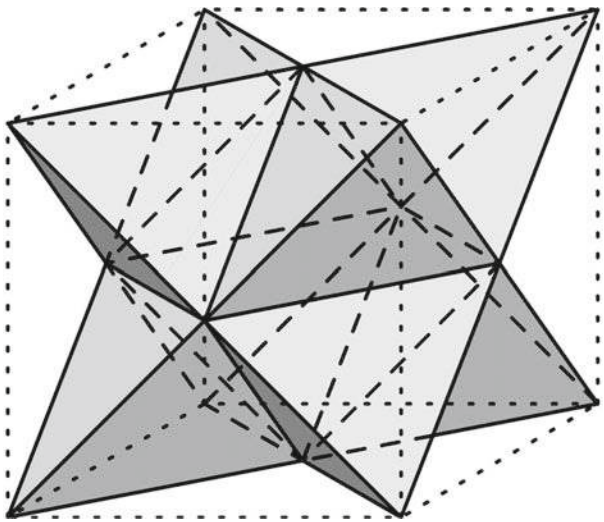

**Illus. 5.5.7** Figure to Problem 5.3.7

**Problem 5.3.8:** Symmetrical representations of floor patterns

a) Which symmetrical mappings do the floor pattern shown in Illus. 5.3.20 allow for (Pisa, end of $12^{th}$ century)? (In [Rosenthal/Schreiber 2004], the number of ornament groups is counted for the simplest case of woven ornaments, i.e., strip ornaments.)
b) Which additional mappings are made possible if we ignore the undercrossing of the ribbons and only see those as linear patterns?
c) Try to continue the pattern beyond the displayed detail.

**Problem 5.3.9:** Star polyhedra

a) What are the differences between most of Jamnitzer's drawn "regular", but not convex polyhedra (cf. Illus. 5.3.23) and his great dodecahedron (left middle) and Uccello's and Poinsot's polyhedra (Illus. 5.3.2; Illus. 7.9.2), which nowadays are accepted as real star polyhedra?
b) Why is Kepler's octahedral star (Illus. 5.5.8) not accepted as a star polyhedron from the modern perspective and/or how would we have to alter the definition of the notion of star polyhedra in order to be able to include Kepler's?
c) Which further star polyhedra would then be acceptable (cf. Problem 2.3.2 for the last question)?



**Illus. 5.5.8** Figure to Problem 5.3.9

**Problem 5.3.10:** Dürer's obtuse rhombohedron

The polyhedral solid in Dürer's "Melencolia I" (Illus. 5.3.30) is created if we stretch a cube in the direction of two diametral opposite corners to form a parallelepiped (rhombohedron) bound by rhombi and then cut off the two tops perpendicularly to this axis. Due to the distortion, the cube loses its property of having a circumscribed sphere. Cutting off the tops in an appropriate manner restores this property (why is this possible?). It is probable that this idea is the base of Dürer's much-discussed solid [Schreiber 1999].

a) Consider how this construction (i.e., cutting off the tops of the rhombohedron to restore the circumscribed sphere of the obtuse solid) can be realised by means of the two-plane method, i.e., a means that Dürer knew all too well.
b) The sphere in the foreground of the picture and the tools close by suggest that the angel is just thinking about how we can reverse this idea and cut out the relevant solid if the sphere is given. Try to solve this problem, too.

**Problem 5.3.11:** Dürer's obtuse rhombohedron again

A consequence of the intersection method is that it shows us how we can reconstruct the observer's standpoint, i.e., principal point, and eye distance, based on a picture of correct perspective, if the picture contains sufficient information, such as the case of Dürer's "Melencolia I", in which the horizon and the pictures of original right-angled objects must be given. We can then reconstruct the top and front views of the displayed polyhedra based on the observer's standpoint. Further instructions, including the result, can be taken from, for example, [Schröder 1980].

We want to invite the reader to execute this problem independently as far as possible and to check the hypothesis made in Problem 5.3.10 by means of the obtained top and front view of the polyhedron. Does Dürer's solid really have a circumscribed sphere? Furthermore, it is possible to extract the real shape of the side areas from the top and front view, to make the solid yourself, and to confirm that the smaller angle of the rhombi is 72 degree in Dürer's version, although this angle can actually be chosen arbitrarily in regard to the construction described above. Thus, a clear relation to the regular pentagon is established (and to the golden ratio, if we like).

**Problem 5.3.12:** A wrong net unfolding

Apart from the five regular and seven of the Archimedean polyhedra, Dürer also unfolds two non-Archimedean solids (in the modern sense) into a net in 'Instruction.' Both solids also feature a circumscribed sphere. He made a mistake when drawing the unfolding of one of these solids, which is created by cutting off the cube faces to form a dodecagon, whereby three isosceles and one equilateral triangle are created in each corner.

a) Why has this polyhedron a circumsphere?
b) The mistake of the net can be found by merely looking at Dürer's net drawing (Illus.5.5.9).What is obviously wrong?
c) Construct the solid in the oblique view or by means of the two-plane method and then calculate the correct ratio of both occurring edge lengths!

**Problem 5.4.1:** Guldin's theorem

Prove Guldin's theorems by means of elementary geometry:

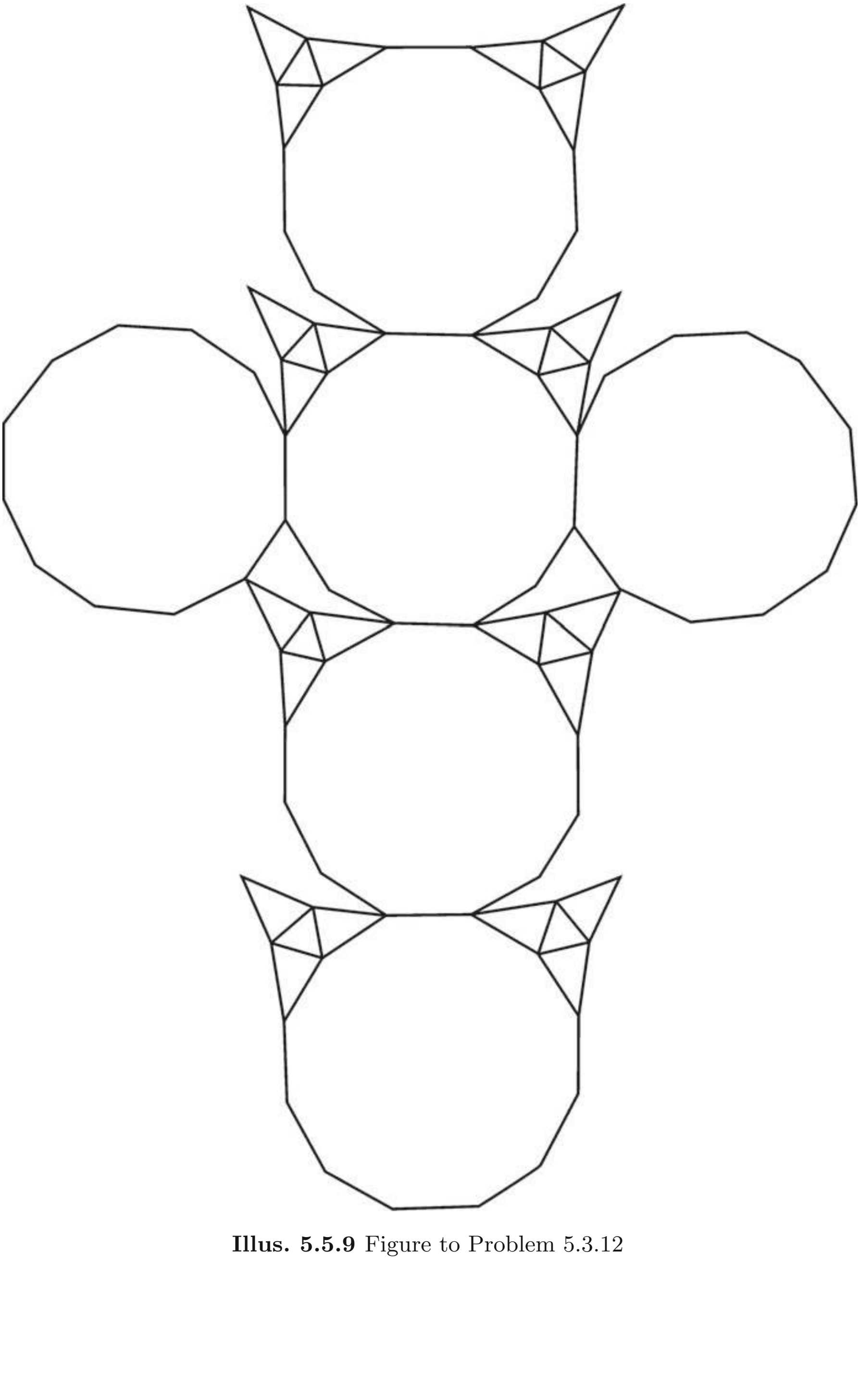
a) The volume of a solid of revolution is the product of a revolving area and the path of its gravity centre.
b) The area of a surface created by the revolution of a plane curve around an axis in its plane is the product of the length of the revolving profile curve and the path of its gravity centre.

in case a) for the volume of rings with triangular or rectangular profile, in case b) for the lateral surface of a conic frustum. (The general case results from this in accordance with the understanding of the $17^{th}$ century for both rules by means of decomposing the revolving area or curve in any amount of small parts.)

**Problem 5.4.2:** Kepler's rules for the connection between surface and volume

The relation between area and circumference of the circle and/or between volume and surface of the sphere, which, according to Kepler, are heuristic, can also be obtained via another heuristic approach: Imagine the circle composed of concentric circular rings of finite thickness $d$. If $d$ is small, the volume is approximately the product of $d$ and the circumference with the respective mean radius. Based on the summation of these part areas, this sum becomes the integral for a going towards zero.

Analogously, carry out this observation for the spherical surface and spherical volume.



**Illus. 5.5.9** Figure to Problem 5.3.12

# 6 The development of geometry in the 17$^{th}$ and 18$^{th}$ centuries




C.J. Scriba, P. Schreiber, *5000 Years of Geometry*,
DOI 10.1007/978-3-0348-0898-9_6

The known world is enriched in many directions by numerous geodesic, geographical expeditions as well as the invention and improvement of telescopes and microscopes. The architecture of Baroque (∼1600–1730), Rococo (∼1720–1780) and Classicism (∼1750–1830) is characterized by new shapes, ornaments and figures and/or a return to classic ancient architecture. Geometry made essential contributions to all of this.

| | |
|---|---|
| 1633 | Galilei is forced by the Inquisition to withdraw his commitment to Copernicus's world system |
| 1643–1715 | Louis XIV rules in France |
| 1644 | B. Pascal builds the first preserved mechanical computing machine (and receives a royal privilege to produce them in 1649) |
| 1646 | A. Kircher describes the "Laterna Magica" as the first (root of later film projection) |
| 1648 | Peace of Westphalia ends 30 Years' War |
| 1649–1658 | Commonwealth and Protectorate under Oliver Cromwell |
| 1660 | Restoration under Charles II (Stuart) |
| 1662 | (Official) founding of the Royal Society in London |
| 1666 | French Academy of Sciences is founded in Paris |
| 1666 | After a plague epidemic and the Great Fire of London: Rebuilding of St. Paul's Cathedral under supervision of Christopher Wren |
| 1666–84 | Canel du Midi built in France (connection Atlantic – Mediterranean) |
| 1672 | Leibniz invents staggered roll as element of mechanical computing machine |
| 1687 | Newton's *Philosophiae naturalis principia mathematica* |
| 1709 | E. W. v. Tschirnhaus and J. F. Böttger invent European white hard porcelain in Saxony |
| 1725 | Russian Academy of Sciences opens in St. Petersburg |
| 1729–96 | Catherine II (the Great) rules in Russia |
| 1733–43 | Great Russian North expedition under supervision of Vitus Bering |
| 1735–37 | Arc measurement expeditions of French Academy to South America and Lapland prove flattening of Earth |
| 1740–86 | Friedrich II (the Great) rules in Prussia |
| 1741 | Prussian Academy of Sciences re-founded in Berlin, Euler called to Berlin |
| 1756–1763 | Seven Years' War |
| 1768–1779 | James Cook's expeditions |
| 1769 | James Watt receives patent for steam engine |
| 1775–1783 | War of Independence in the USA |
| 1783 | England accepts independence of the USA |
| 1786 | Mechanical power loom by E. Cartwright |
| 1787 | Debut performance of opera 'Don Giovanni' by Mozart in Prague |
| 1789 | Civil revolution starts in Paris |
| 1794 | École Polytechnique founded in Paris |
| 1798 | Casanova dies at Castle of Dux in Bohemia |
| 1798 | Battle of the Nile: Nelson defeats the French fleet of Napoleon |
| 1799 | Napoleon I consul of France, de facto autocrat |

## 6.0 Preliminary remarks

From about 1630 until approximately 1800 (when mathematics yet again took a profound turn due to reasons we will discuss later on), those scholars who dealt with the development of mathematics in a manner that – from today's perspective – was significant for its historical development, were small in number and easy to identify. Generally speaking, they were in contact with one another. Of course, this contact was established by other means than are usual nowadays, in other words, not through conferences, journals and email, but through spreading their books and individually printed articles, and taking part in scientific talks at different, central locations that sometimes formed the basis for future academies. Later on, they stayed in contact by means of printed working titles and conference reports of those academies (which basically were the first scientific journals ever), and by writing letters and visiting each other. Friar Marin Mersenne, who lived in Paris, played a special role. From approximately 1623 until his death in 1648, he introduced almost all the important scholars of Europe to each other through correspondence and organising personal gatherings. Apart from a very few exceptions, all those scholars thought of themselves as philosophers of nature, meaning they engaged in mathematics in closest relation to philosophy, astronomy, geodesy, cartography, mechanics, optics, acoustics and other origins of physics and techniques that were growing step by step. At this time, mathematics was as embedded into the latest applied sciences as trigonometry was into astronomy until the beginning of the Renaissance. Within its own realm, mathematics had not yet specialised into subareas. Apart from the fact that some scholars, such as Newton and Hobbes, considered *geometry* to apply only within the scope of ancient geometry, and others, such as Descartes und Huygens, viewed the new infinitesimal methods very scep-

**Illus. 6.0.2** Thomas Hobbes, Marin Mersenne, Christiaan Huygens

tically, hardly any mathematician of that time would have understood our attempt to separate the new coordinate-orientated geometry from the infinitesimal mathematics rapidly developing at the time. Indeed, as already indicated, the basic problems of analysis are of geometric nature with hardly any exceptions. Geometry concerns the definition of the terms curve, area, solid, surface, tangent, tangential plane, evolute and evolvent, the calculation of curvature, arc length, surface area and volume. Of concern is determining the shape of certain curves, areas or solids due to geometric or physical requirements. For instance, looking at the problems of loxodromes, catenaries or brachistochrones (curves on which a mass point under the influence of gravity moves from point A to the below-positioned point B in the shortest time possible), equilibrium figures of revolving masses, etc., the first questions dealing with maximums or minimums of functions, were of geometric nature, as we have seen in Kepler's work.

## 6.1 The coordinate method – geometry and algebra

A coordinate system (in its most general sense) always accomplishes two things: First, it facilitates the algebraic treatment of geometric problems by allowing us to translate theorems and problems on geometric objects into equivalent theorems and problems via their coordinate formation and the – so to speak – calculative simulation of geometric processes. (Thereby, within the foundations of mathematics since the beginning of research, the possibility exists for demonstrating reliable models based on algebra and arithmetics for axiomatically characterised geometric structures.) Second, a coordinate system facilitates the illustration and optical representation of algebraic facts. This way, it not only crucially supports intuition, but also delivers insights into certain developmental stages of mathematics, which otherwise would constitute unreachable algebraic relations. Most importantly, the connection between possibly very abstract functional relations ( such as between economic, scientific or technical quantities) and the (two or three dimensional) graphic picture of the relevant function, which nowadays is taken for granted, is the "fruit" of this "inverse application" of the coordinate method. From today's point of view, this interlocking of geometric and algebraic methods is the premise and core for mathematics to be capable and effective. Consequently, describing the historic development that led to this interlocking will form a central chapter of the history of geometry.

The easiest and shortest answer to the question of the origin of the coordinate methodis well known. It is said to stem from René Descartes and Pierre de Fermat, who almost simultaneously and essentially independently of each other agreed in 1637 (the year in which Descartes' *La Géométrie* was published) that this exact year of 1637 should consequently be the natural border between the long prehistory of modern mathematics and its actual beginning. We will have to discuss three problems in this chapter:

1. A short summary of the prehistory, i.e., the base level upon which Descartes and Fermat built their accomplishments.
2. A preferably careful and, as we will see, difficult analysis of their contributions from today's perspective, as indicated above.
3. An overview (admittedly rather brief due to size) of the essential further steps and contributions that led to the full formation of the coordinate method through the end of the $18^{th}$ century.

Additionally, it is worth noting that the coordinate method has also been a necessary premise for the development of infinitesimal mathematics. Both approaches were pushed forward in close interaction, mostly by the same people, during the $17^{th}$ and $18^{th}$ centuries. Nonetheless, we attempt here to treat the development of analysis, as far as it must be touched on in regards to the history of geometry, separately. See also [Sonar 2011].

### 6.1.1 Prehistory

The relatively few special curves that were looked at as part of ancient mathematics, were either defined as plane sections of a simple spatial figure (like conic sections), or by means of a point-by-point construction (like the quadratrix) or an imagined mechanic procedure (like conchoids). In principle, based on this view a purely axiomatic-synthetic treatment for each of these curves would likewise be possible, just as Euclid delivered for the geometry of straight lines and circles. However, the ancient Greeks already used to transform each original curve definition into an equivalent constraint, which expresses that a point $P$ belongs to such a curve by means of an "algebraic" relation between certain variables (dependent on point $P$) and certain fixed (dependent on the assigning pieces of the curve) quantities (mostly line segments, but sometimes also areas, angles,... ), i.e., the "symptom" of the relevant curve. It is clear that the ancient Greeks' so-called geometrical algebra mainly served the purpose of working with such symptoms and solving problems or verifying theorems based on this premise. Furthermore, the sparse examples provided by the ancient world showed as a matter of experiential fact that a curve in a plane is normally characterised by a symptom with exactly two variables, whereas a symptom of a surface in space requires three variables. However, the ancient geometric algebra was restricted, since multiplication of quantities – stated in a modern fashion – was thought of as a geometrically realised Cartesian product, which is why:

a) Equations were burdened with the requirement for homogeneity: All summands must be of the same dimension. (Thus, if $a$, $b$, $c$, $x$ are line segments, then $ax^2 + bx + c = 0$ is meaningless, since $ax^2$ is a volume, $bx$ an area and $c$ a line segment.)

b) Dimensions could be spatial at most. Thus, $ax^2y = bxy^2$ is meaningless, even though it suffices the requirement for homogeneity. Premise b) was merely eased by the possibility of phrasing algebraic premises by means of proportions. Hence, it was possible to express the equation $ab = cd$, which is possibly meaningless due to dimensional reasons, with maximal volume quantities $a$, $b$, $c$, $d$ (whereby $a$, $d$ belong to the same dimension and $b$, $c$ also belong to the same dimension) by means of the meaningful proportion $a : d = c : b$. Additional premises are the result of:
c) Quantities being generally positive.
d) The lack of algebraic symbolism. Every algebraic equation and every algebraic conversion had to be justified by means of heavy geometrical reasoning.

Given all of these, we must stress that Pythagoras's theorem in Euclid's 'Elements' basically serves as an introduction and a "door opener" for the subsequent geometrical algebra of Book II (and also because of its position at the end of Book I). It is the first time in history that a theorem expresses a purely geometrical relation between three points, i.e., to form a right angle by means of an entirely algebraic relation between participating quantities, namely the pairwise distances between the points.

In regards to the status quo at the evening before the "invention of analytical geometry", we must add that the functional approach to geometric thinking may already have been thought of in Euclid's 'Porisms', although his followers struggled to understand it; likewise, it was touched on by the theory of form latitudes by Richard Swineshead and Nicole Oresme in the $13^{th}$ century and others, yet only on a purely qualitative level. In contrast, the thriving occidental natural sciences and techniques offered plenty of inspiration and examples for functional correlations. Hereby, Kepler, Galilei, and his students and followers Torricelli, Cavalieri and Viviani played their parts excellently. The practice of measuring geometric quantities by means of numbers is, of course, much older than Greek geometry and had been revived as part of the medieval Islamic geometry. Yet, it does not seem to have had a noteworthy influence on the development of the occidental coordinate geometry. However, the Cartesian principle regarding the graphic representation of functional correlations – especially concerning a time axis – had already based on the European musical notation developed around 1000, and the comparison with extra-European or contemporary musical notation shows that this type of representation is by no means mandatory [Schreiber 2003]. However, the advances of algebra during the European Renaissance have had a great influence. Fermat's essay on the coordinate method called *Ad locos planos et solidos isagoge* (written before 1636 and circulating in copies from 1636 onwards, but only printed posthumously in 1679) already hints at its dependence on Vieta's *In artem analyticem isagoge* (Introduction to the analytic art, i.e., 'letter algebra') due to the use of the Greek word "isagoge" (introduction) in the otherwise Latin text. Following 1637, Descartes repeatedly underlined the fact that he had neither heard of nor read Vieta before writing

**Illus. 6.1.1** René Descartes (portrait after Frans Hals, 1648); Pierre de Fermat (unknown painter, $17^{th}$ century)

down his *Géométrie.* Indeed, the notation introduced by him has its own take. However, it is evident that Descartes was influenced by his first mathematics tutor Isaac Beeckman, the German cossist Johannes Faulhaber, and his studies of Clavius. Finally, we must accentuate that Fermat and Descartes, like all mathematicians of this time, were completely familiar with the Greek approach of treating geometry by utilising algebra. At the beginning of the $17^{th}$ century, the time had come to redesign this concept entirely by using the advances that had already been achieved in algebra. Thus, we can already find parts of the ideas attributed to Fermat and Descartes some time earlier in the works of G. B. Benedetti (in *Diversarum speculationum* in 1585) and M. Ghetaldi (in *De resolutione et de compositione mathematica* in 1630).

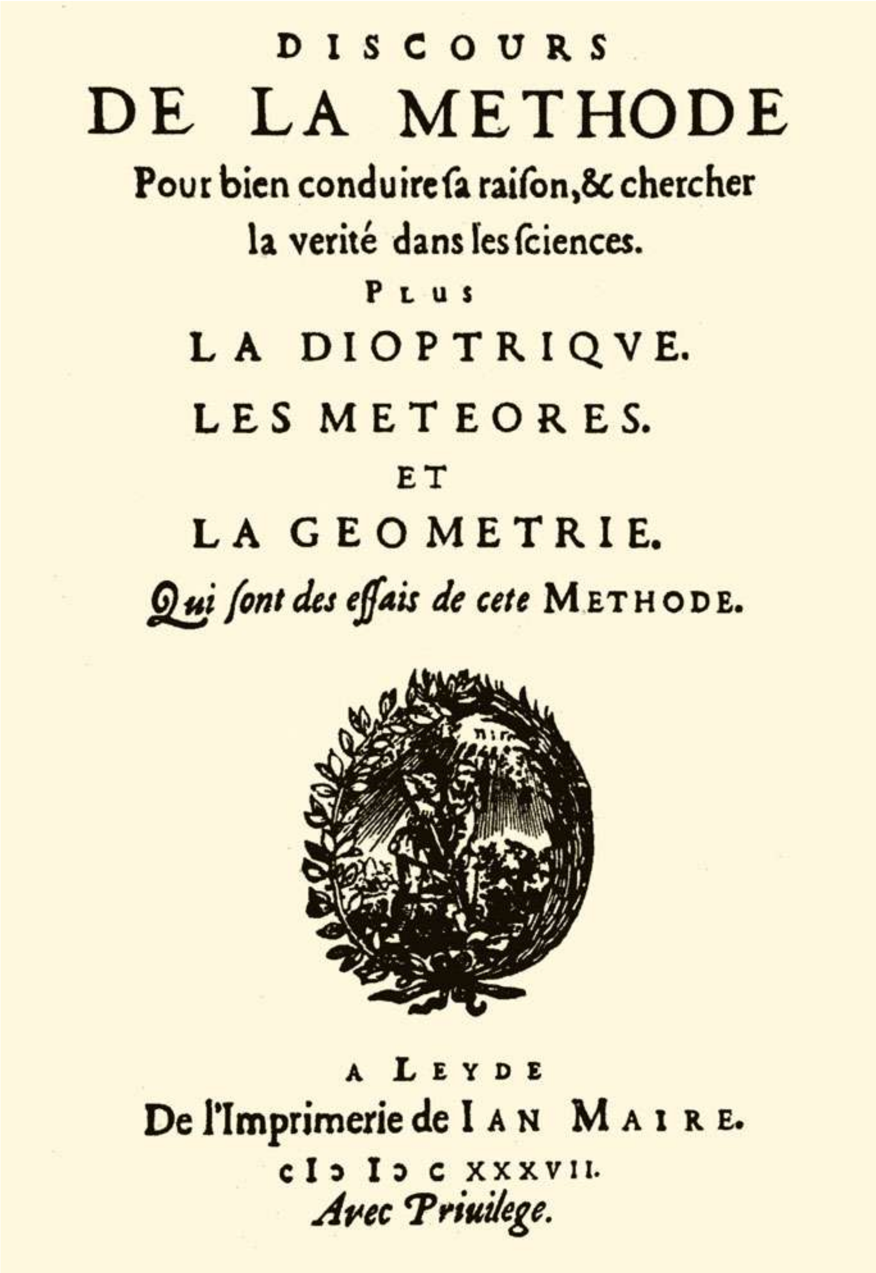
### 6.1.2 Fermat's and Descartes' accomplishments

First of all, we will take a closer look at Fermat. His contribution is more modest, yet fundamental and easier to pin down. According to his first innovation, both variable quantities (modern $x$, $y$) always refer to the coordinates of the variable point in regards to a mostly right-angled or at least an affine system of coordinates. In his short essay, the title of which has already been cited above, "loci plani" refer to plane locations, meaning symptoms or equations in which area quantities are equated. Thus, these are equations of straight lines, e.g., $ax + by = cd$ or special conic sections, such as $x^2 + y^2 = r^2$ or $ay = x^2$. "Loci solidi" are not spatial amounts of points, but plane curves described by an equation between volumes, such as $ax^2 + by^2 = cde$. Furthermore, Fermat's classification offers linear loci (all other curves), which he, however, did not deal with any further. He used Vieta's algebraic notation with capital vowels for variables and capital consonants for fixed quantities (parameter) and systematically examined all algebraic possibilities that resulted from this approach. Consequently, he remains obliged to adhere to ancient limitations of (a) homogeneity and (b) a maximum of three algebraic dimensions. The essence of his work lies within the fact that he started with equations. Thereby, he clarified the following:

1. Every meaningful algebraic equation between $x$ and $y$ describes (in regards to "Cartesian" or other coordinates) a quantity of points in a plane, which, assuming this equation, can be examined with respect to its geometric characteristics and independently of a possible mechanic or point-by-point creation. As a result, the set of all curves to be investigated is at once immensely expanded compared to the approach of the ancient world. Instead of single examinations, general examinations over entire, algebraically-described classes of curves could be conducted.
2. It is possible to classify curves based on algebra. The first result (of which Fermat was very proud, whereas the contents under (1.) had to be read between the lines) is as follows: the class of curves, geometrically defined as conic sections, is identical to the algebraically defined class of curves of second degree at most. Given that it is possible to obtain conic sections as pictures of a circle by means of central projections, it springs to mind to reduce the endless abundance of algebraic curves of a certain degree to one or finitely many "normal forms" via geometric transformation. (Admittedly, Fermat himself did not consider this great program; it was executed by Newton, Euler and others step by step later on.)

Neither of Fermat's results, illustrated above, can be deduced from Descartes' entire works. In general, his contributions are much more scattered and difficult to analyse, although much more effective in retrospect. His main effort probably lies within freeing algebra and, hence, geometry of the ancient limitations a) and b), as stated above, by means of a single ingenious, yet simple thought. By choosing a fixed line segment $e$ as a unit, it is possible to turn every rectangle $ab$ into a rectangle $ce$ of the same area and to use the line segment $c$ as a representation of the quantity $ab$. Since we can repeat this trick as often as we want to, it is possible to transform the product of any amount of linear quantities down to a linear quantity. Basically, all equations turn into homogeneous equations between linear quantities. There is another, weaker interpretation, whereby previously inhomogeneous equations between linear quantities, such as $ax^2 + bx + c = d$, are interpreted as a homogeneous equation $ax^2 + bex + ce^2 = de^2$ by means of an imagined topping up with relevant amounts of factors $e$. (This can be compared to the role of the sinus totus as part of the trigonometry of the Renaissance.) However, it is not possible to reconstruct if or to what degree Descartes was aware that the choice of a linear unit $e$ simultaneously creates an isomorphism between line segments and their measures. This is the basis of today's admission of the coordinate method, at least in its standard case. The dualism between, on one hand, "concrete" quantities, meaning quantities that are rational at most or can be described by root-terms or other construction rules, and, on the other hand, the continuously variable quantities, which can be imagined as straight lengths and, thus, depend on geometric intuition, was dragged along until the concept of a real number was defined set-theoretically-arithmetically by European mathematics (Dedekind and Cantor, 1872, Bachmann, 1892, Hilbert, 1900). Basically, this dualism was revived by the constructivist stream of mathematics, which was not especially evoked but aided by theoretical informatics. Descartes seems to have been stuck in the mind of the ancient world, which assumed that algebra dealt with geometrically represented quantities (according to him, always straight lengths) that could be described numerically or by construction only "as an exception".

We can already find Descartes' main contribution described in his first (and only posthumously printed work) *Regulae ad directionem ingenii* (Rules of guiding the mind). It may be even more clearly worded there than in *Géométrie*, in which his basic idea is pushed into the shadows due to an abundance of other concerns that were often only sketched or illustrated by examples. As known, the essay *La Géométrie* is one of three appendixes meant to serve as an elaborate application of his philosophical main work *Discours de la méthode*. Hereby, we only want to state that the principles demonstrated in *Discours de la méthode,* unfortunately, bore little influence on the clarity and systematics of *Géométrie*, even though they represented the foundation for the philosophy of rationalism and had a great impact on the general history of the mind. (However, Descartes has this in common with other distinguished mathematic philosophers, such as Leibniz or Lambert.) In Descartes' work, we find even less of the Cartesian coordinates (as referred to by Leibniz for the first time) than in Fermat's. Here, he defends the ancient view of describing the locus of a curve point by means of two suitably chosen straight lines, each time depending on the problem at hand. However, in 1637, he turned towards fixed quantities represented by $a, b, c, d, \ldots$ and variable quantities using the last letters of the alphabet, preferably $x, y, z$. Before that, he referred to parameters with capitals and to variables with lower case letters. He also hinted at a classification of curves, but followed another principle: For him, the simplest curves were those ones that can be generated by a single movement, like straight lines or circles. In 1619, he had already addressed a letter to Beeckman in which he talked about "generalised circles", with which it was possible to generate more such curves. Curves of class $n + 1$ are created by admitting already drawn curves of "class $n$" as means of construction. (This suggests an idea that will be looked at again by Jakob Steiner in 1832 in his *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander* (Systematic development of the dependence of geometrical forms to each other)). He intended to produce curves of second class as loci of intersections of straight lines, which each move uniformly on curves of first class. Thereby, he probably used model representation of producing the quadratrix already known in Antiquity, whereby a straight line moves uniformly parallel to an $x$-axis in direction of a perpendic-



DISCOURS
DE LA METHODE
Pour bien conduire ſa raiſon, & chercher
la verité dans les ſciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
*Qui ſont des eſſais de cete* METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
cIↃ IↃ c XXXVII.
*Avec Priuilege.*

**Illus. 6.1.2** Title page of *Discours de la méthode* [Descartes 1637] ular $y$-axis and the other one turns uniformly around the coordinate origin, i.e., along a circle. (Dürer's "shell curve" would be another good example, although probably not known by Descartes.) We can justify the problem of such a classification attempt by the fact that it is possible to produce several such motions synchronized with each other by means of a single motion and suitable coupling mechanisms. Descartes distinguished algebraic curves that can be solved geometrically by means of the stated approach from those that cannot be treated "rationally". However, he was mistaken in believing (and underestimated by far the difficulties of solving equations algebraically) that each polynomial equation could be solved with his methods. He showed us how he proceeded when graphically solving the equation $x^3 + ax = b$ in case of $a, b$ >0. In this case, we can find positive $p, q$ so that $a = p^2$, $b = p^2q$. Then, the solutions of $x^3 + p^2x - p^2q = 0$ are the solutions different to 0 of the equation $x^4 + p^2x^2 = p^2qx$, which is obtained when calculating the common solutions of

$$(x - q/2)^2 + y^2 = (q/2)^2 \tag{6.1.1}$$

and

$$x^2 = py. \tag{6.1.2}$$

The graphic representation of the situation in the first quadrant of an $x$-$y$-axes system shows that the circle (6.1.1) and the parabola (6.1.2) in the area of positive $x$-values have exactly one intersection, which "proves" a purely algebraic theorem by geometrical interpretation.

The notion of Cartesian ovals can also be found amongst the many inspirations that Descartes states in his *La Géométrie*. These are, generalising the ellipses, curves, which are defined by $n$ "foci" $P_1$... $P_n$ and the condition that $P$ is part of the curve if the sum of the intervals $PP_i$ is equal to a given constant. These curves were meant to play an important role just a little later in Fermat's problem of finding that point $P$ in a plane for which the sum of the intervals $PP_i$ is minimal.

A problem, the special cases of which go back to Pappus, is dealt with in great detail in *La Géométrie*, which perhaps is not appropriate for modern geometry: $n$ straight lines are given in the plane and a fixed angle $\alpha(g)$ for each of these straight lines $g$. Determine the locus of those points $P$ for which, given line segments $a(g)$ drawn from $P$ to $g$ for each of these straight lines so that they cut $g$ with angle $\alpha(g)$, it holds that the product of some of them stands in a given ratio to the product of the remaining ones.

Descartes could show with his method that the wanted points form a curve of $n^{th}$degree, if the product of $n$ intervals equals the product of $m \leq nm \leq n$ intervals, particularly a conic section for $n = m = 2$. He mistakenly concluded the unlimited ability and effectiveness of his methods from this. Whereas Descartes geometrically solved an algebraic problem in the case of the general cubic equation, Pappus's geometrical problem is solved here algebraically. However, nowhere does Descartes hint at this fundamental difference, despite his high methodological standards.

### 6.1.3 History of impact and reception

"Discours" was first published anonymously in 1637 in the Netherlands, where Descartes lived for quite some time. Mathematicians in contact with him very quickly recognised the trend-setting meaning of the appendix on geometry. Frans van Schooten published a Latin translation of this appendix in 1649 and, thus, made this text accessible to Italian, English, Dutch and German scholars for the first time. A second edition, which was extended and supplemented by him and some of his students (especially Johan de Witt and Jan Hudde) to form two volumes, was first published in 1659/61, then again in 1683 and 1695. This second edition already contains the three-dimensional coordinate method. Meanwhile, almost all significant mathematicians of that time had turned towards this topic: Leibniz was the first to use the words "abscissa" and "ordinate" (in a letter to Oldenburg on 08/27/1676), and "coordinates" (in *Acta eruditorum* in 1692). The brothers Jacob (I) and Johann (I) Bernoulli were the first to speak of "Cartesian coordinates". John Wallis dealt with conic sections purely analytically as algebraic curves in the plane in 1655 and, thereby, was also first to use negative coordinates. Above all, we must highlight Newton, who began engaging with coordinate geometry around 1665. The significance of this for the development of the coordinate method is often underestimated nowadays and/or pushed aside by his contributions to analysis and physics. We will try to systematise his contribution without considering the chronological order (which is already complex, since many aspects of his work were only published long after they were written):

1. Newton used plane and spatial Cartesian coordinates exactly in the now common manner; in other words, he accepted negative coordinates as fully equal.
2. Newton also used polar coordinates and taught how to convert both into each other. Hence, the notion of coordinate began to turn towards the now self-evident general meaning.
3. Since Newton always thought like a physicist, there is only one real independent variable for him, namely time $t$. According to him, a curve is the course of a point through time and, hence, is described primarily by the functions
$$x = x(t),\ \ y = y(t) \tag{6.1.3}$$
and, if spatially, by $z = z\ (t)$. The equation of a plane curve is created by either taking $t$ as proportional to $x$ or – generally – eliminating parameter $t$ in both equations (6.1.3).

Apart from the fact that the description of a curve by means of a parameter representation applies to a more general scope than the one by means of *one* equation, especially in three-dimensional or higher cases, we will, for good and all, include the fundamental remark here that the turn of coordinate geometry triggered by Fermat and Descartes towards algebraic manifolds (stated in a modern fashion) naturally caused a certain alienation of geometry from reality amongst many positive and fruitful aspects. The curves, areas, ... . occurring in reality only generally satisfy an algebraic relation between coordinates partially and approximately at the most. Furthermore, all order relations are lost when transferring a parameter representation (with which we can describe an originally mechanical creation of a curve very well and which can accommodate reality by means of restricting the parameter interval) to an equation of the curve so that an algebraic equation mostly describes a point set, which is too large in regards to the original problem. Moreover, it can reveal surprising additional parts compared to the original representation. (Also see the discussion on Dürer's conchoid in Problem 6.1.1.)

4. In 1667, Newton referred back to Fermat's problem of classification and listed 72 types of curves of third order in the plane. (He ignored six additional cases, although he was aware of them. These cases were only rediscovered in the $18^{th}$ century.) This classification was only published in 1704 as an appendix to Newton's *Optics*. This publication was his first one on a purely mathematical topic. However, it was already reprinted in 1710 in the second volume of John Harris's published *Lexicon Technicum*.
5. In contrast to Descartes and Fermat, Newton values the graphic representation of the subject matter. His classification of curves of third order is basically a picture atlas of possible forms, which we will look at in greater detail.

Newton based his work on the most general form of the polynomial of third degree in $x$ and $y$ and shows, first of all, that we can arrive at one of the four forms by means of suitably transforming the coordinate system (!):

$$\begin{array}{lr} \text{I.} & xy^2 + ey = ax^3 + bx^2 + cx + d, \\ \text{II.} & xy = ax^3 + bx^2 + cx + d, \\ \text{III.} & y^2 = ax^3 + bx^2 + cx + d, \\ \text{IV.} & y = ax^3 + bx^2 + cx + d. \end{array}$$

These four forms lead to the 72 listed types, if we continue transforming the coordinates. In case III, depending on whether the right-hand side has three different real zeros, a double and a single, a triple or just a single real zero, Newton arrived at the five forms shown in Illus. 6.1.3, since, in the case of the double zero $a_1$ and single zero $a_2$ , we must distinguish whether $a_1 < a_2$ or $a_2 < a_1$.

Strangely, Newton did not want to acknowledge all these aspects as geometrical. It may have been his conservative upbringing and educational background that made him shy away from understanding a greater number of aspects as geometrical, in contrast to the Greeks. However, the impulses he triggered caused further turbulent developments. An algebraic geometry began to form in Scotland at the hands of James Stirling and Colin MacLaurin, who were both active followers of Newton.

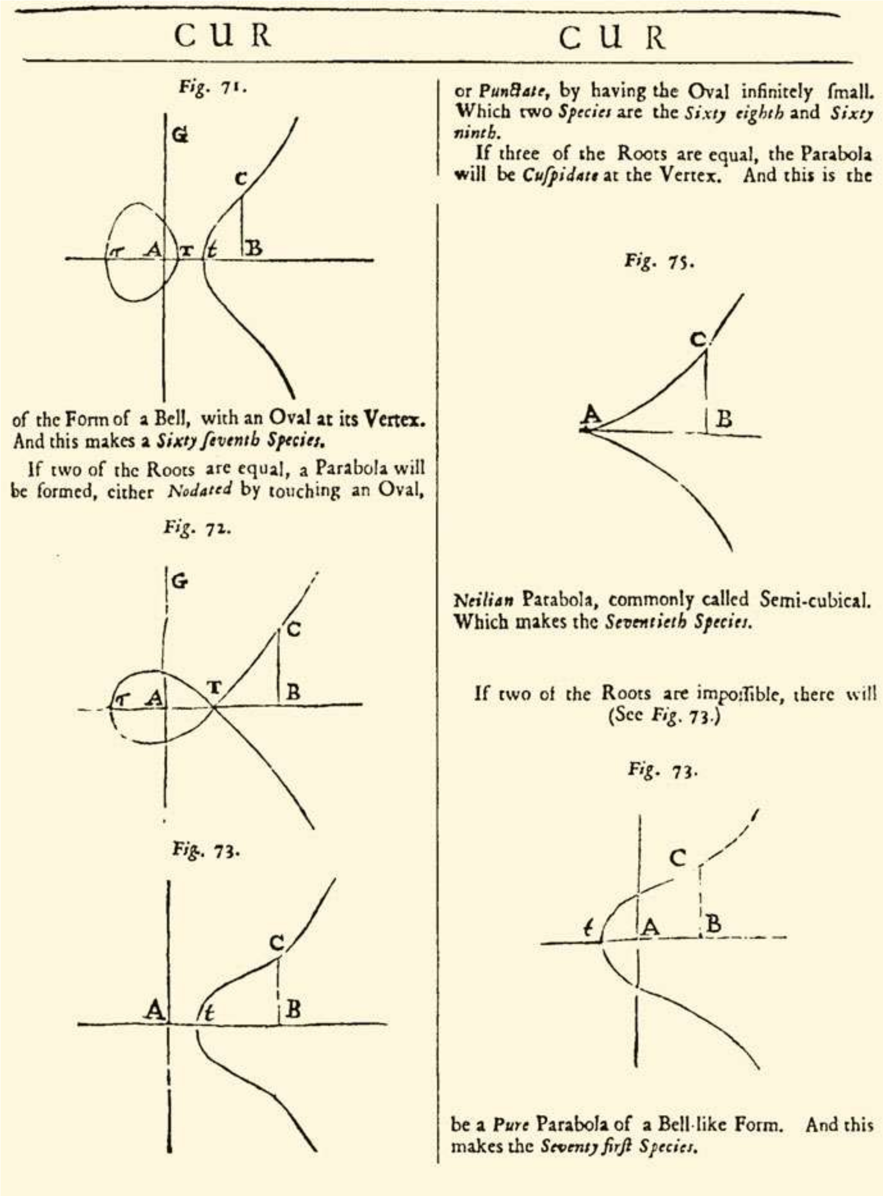

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*Fig.* 71.

of the Form of a Bell, with an Oval at its Vertex. And this makes a *Sixty ſeventh Species.*

If two of the Roots are equal, a Parabola will be formed, either *Nodated* by touching an Oval,

*Fig.* 72.

*Fig.* 73.

or *Punctate*, by having the Oval infinitely ſmall. Which two *Species* are the *Sixty eighth* and *Sixty ninth.*

If three of the Roots are equal, the Parabola will be *Cuſpidate* at the Vertex. And this is the

*Fig.* 75.

*Neilian* Parabola, commonly called Semi-cubical. Which makes the *Seventieth Species.*

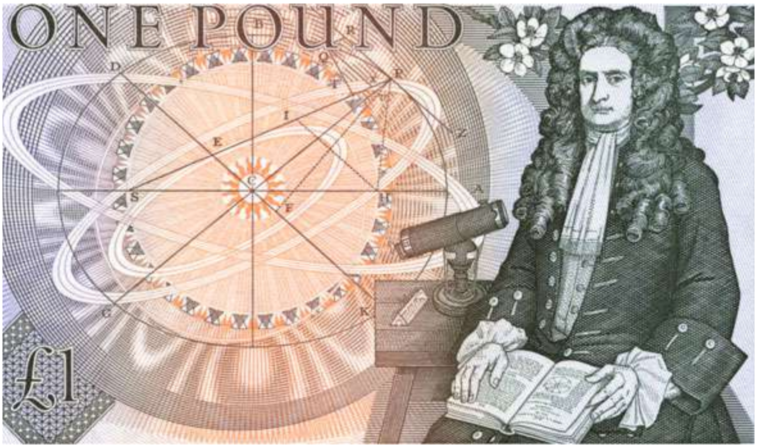
If two of the Roots are impoſſible, there will (See *Fig.* 73.)

*Fig.* 73.

be a *Pure* Parabola of a Bell-like Form. And this makes the *Seventy firſt Species.*



**Illus. 6.1.3** The five forms of cubic equations of type III [Newton: The Math. Works, Vol. II, p. 158]. Top left: three different real zeros. Middle left: the two greater zeros have become one (in figure : $T$). Bottom left: the two smaller zeros are pulled together into one point (not shown in the picture). Top right: the case of triple real zeros. Equation III becomes $y^2 = a(x - A)^3$ and the curve of the so-called Neilian semi-cubical parabola with the apex in $A$. Bottom right: there is only one real zero ($t$)




**Illus. 6.1.4** Isaac Newton honoured on one-pound-note

Stirling published the first book on plane algebraic curves in 1717. Therein, he found out that a curve of $n^{th}$ degree is generally given by $\frac{n(n+3)}{2}$ points by counting the coefficients of the relevant equation. MacLaurin's book followed up on this in 1720. For example, he mentioned that a curve of $n^{th}$ and a curve of $m^{th}$ order generally have $m \cdot n$ points in common. Then, he began to notice that his rule delivers 9 intersections for two different curves of third order. In contrast, Stirling's rule says that a curve of third order is uniquely determined by 9 points. Euler *(Introductio,* 1748, see below) and Gabriel Cramer *(Introduction á l'analyse des lignes courbes algébriques*, 1750), amongst others, deal with this apparent contradiction, which was later referred to as Cramer's paradox. Only J. Plücker could finally solve the problem of the "independent point systems". MacLaurin was first to draw a connection between the transformability of two curves into each other and the number of their singularities.

The second volume of Euler's *Introductio in analysin infinitorum,* published in 1748, represented a further important milestone. Whereas the development of analytic geometry was burdened by a nationalistically motivated, one-sided emphasis of either Descartes' or Newton's position, Euler succeeded in synthesizing all fruitful aspects of both schools of thought. Having compiled the arithmetic and algebraic means of analysis in the first volume of *Introductio*, he shared "all aspects of geometry worth knowing" in the second volume "since the analysis is ordinarily developed in such a way that its application to geometry is shown." [Euler a, p. V]. His subject matter is structured as follows:

- Theory of curved lines in general
- Equation of a curve
- Investigation and classification of conic sections based on their equations (without using differential calculus)

- Classification of curves of third order into 16 types and drawing up a relation to Newton's classification
- Classification of curves of fourth order following the same pattern (Euler lists 146 types)
- Purely algebraic treatment of tangents, normals, curvatures, inflexion points, peaks, multiple points, etc. ("Although all of these nowadays are ordinarily accomplished by means of differential calculus" [l.c. p. IX]. This hints at a style that Lagrange will take to the extreme.)
- Determination of curves with given properties.

Euler dealt with the following matters only in the appendix (this shows again the dominant role of plane geometry in the mind of the $18^{th}$ century):

- General theory of solids and their surfaces (of course, restricted to those aspects which can be described algebraically)
- Description "of one each" (!) surfaces by means of an equation between three variables
- Classification of surfaces according to the degree of their order and listing of six types of surfaces of second degree (Hereby, Euler introduces the names of quadric, hyperboloid of one and two sheets, parabolic hyperboloid, etc., which are still customary nowadays.)
- Description of a spatial curve as the intersection of two surfaces and their representation by an equation (Euler's ideas can be simplified a little as follows: the surfaces are given by the equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$. Resolve both according to $z : z = f(x, y), z = g(x, y)$. The equation $f(x, y) = g(x, y)$ yields all points of the $x$-$y$-plane for which there is a common $z$-value of both surfaces. Now, we parameterise this curve in the $x$-$y$-plane and represent $z$ as the function of the parameter $t$. It turns out that here again, as in many other cases, Euler generally thought and calculated like an intelligent older pupil without knowledge of modern mathematics.)
- Normal and tangential planes of the surfaces of second order (again purely algebraically without differential calculus).

If we add that Euler introduced the "Euler angles" in another work (on mechanics) to describe a point in space, basically introducing spatial polar coordinates, we can characterise the level of the coordinate method in the middle of the $18^{th}$ century as follows:

**Illus. 6.1.5** Isaac Newton (painting by G. Kneller 1702, National Portrait Gallery, London); Leonhard Euler (portrait by Emanuel Handman 1753, Kunstmuseum Basel)

1. Everything takes place in the two or three-dimensional Euclidean space. This space is grasped as physical space and not as space $R^n$ of n-tuple real numbers as was more and more customary later on. (Hence, nobody is consciously aware that relations between more than three physical quantities can be interpreted in a higher dimensional space analogous to the classic Euclidean case.)
2. Coordinates are Cartesian (in the modern sense), in special cases also oblique-affine or, as an exception, plane or spatial polar coordinates. Nobody is consciously aware that, for example, the spherical surface is an alternative geometrical structure, and that geographical coordinates there have a function analogous to the plane polar coordinates.)
3. Newton's view that geometry ends with conic sections continues to endure in the reservation of the term "analytic geometry", from the scope of which those aspects we take as algebraic geometry nowadays were increasingly excluded. *(Geometria analytica* first occurs in a title of a manuscript by Newton published posthumously in 1779. The well-known textbooks by S. F. Lacroix especially helped strengthen this term.)
4. Newton's discovery that functions relevant to the natural sciences back then can be defined as a power series and that we can largely reduce the analysis of polynomials of finite and infinite degree to algebra when waiving an exact logical foundation, caused, together with Descartes'

propagated alliance between algebra and geometry, the almost complete constriction of geometry to objects that can be addressed within this scope. There was no clear boundary between such problems that can really be dealt with algebraically, and those that require analysis. Both concepts were summarised under the notion "analytic".

5. All the numerous open questions of "elementary geometry" were pushed aside and the analytical method was not applied to elementary geometry, especially not to linear problems, as they are prompted nowadays at the beginning of every introduction to analytic geometry (and often at the end, too).

The overcoming of the restriction listed in 5 deserves a special mention. It started at the end of the $18^{th}$ century and was triggered by two different motives in two directions complementary to each other: Lagranges' analytic mechanics (1788) was the foundation for the transfer to $n$-dimensional Euclidean space, which was introduced here (still unconsciously) as state or phase space of mechanical systems. Once placed in this space, linear problems are, of course, immediately non-trivial and not to be dealt with by means of "synthetic" methods. Monge was concerned with the education of engineers. For him, everything took place in the three-dimensional "true physical space". He viewed descriptive geometry and coordinate geometry as two tools of equal rights and even of somewhat equal type, the task of which is to convert the problems of three-dimensional space into a structure that can be dealt with more easily; in one case, for the plane of projection, in the other case, for calculation. All standard problems of linear geometry of plane and space, coordinate transformation, transformation by reciprocal radii, problems of orientation, calculation of lengths, areas and volumes, are dealt with in the manner customary nowadays in textbooks on applying algebra and analysis to geometry, which circulated more and more from 1801 onwards. Somehow even the idea of the vector and the idea of viewing straight lines as basic elements of space instead of points were anticipated.

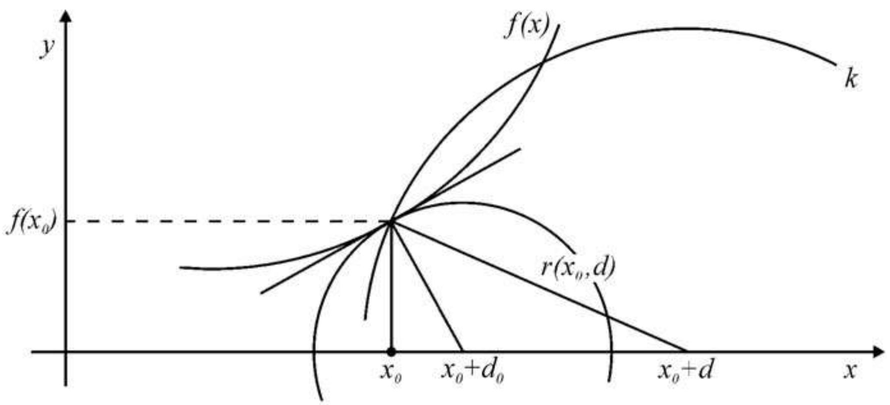
## 6.2 Geometry and analysis

We follow up here on Chapter 5.4 and stress that the following very tight representation of analysis is due to the extensive overlapping of the subject matter with several good books on the history of analysis (for example see [Sonar 2011]). However, a self-contained representation of the history of geometry requires us not to completely overlook the initially very close relations between geometry and analysis.

*Geometria indivisibilis* (1635) by Bonaventura Cavalieri forms a milestone of determining volumes. Indivisibles are "infinitely thin" parallel layers. We can imagine an area or a solid being cut into infinitely many of those layers. If the outcome of this was that, based on the comparison of two areas and/or solids standing on the same base, their parallel sections to the base in each height over the base had the same content, Cavalieri concluded that their volumes must be overall equal (Problem 6.2.1). The principle, which is rather fruitful in many individual cases, was named after Cavalieri and, as known, still plays an important role as a heuristic rule in didactics. Furthermore, it is, of course, inherent to each reduction of a higher dimensional integral to an integral reduced by one in dimension. Modern mathematics differs in this aspect from Cavalieri's only by the knowledge of "mathematical monsters" and the care taken as a result of this when specifying the area of validity. This principle likely goes back to Democritus, based on remarks by Archimedes and others. Thus, it surely stands in correlation to his ideas of the atomic composition of matter. Democritus's principle, if there was one, deals with material solids and their decomposition into layers of atomic thickness. Cavalieri did not refer to such an interpretation. For him and his followers (he also had opponents, such as Guldin and Huygens), it was pure heuristics, which was justified by the obtained results. This was a point of view that was characteristic for the entire era. Additionally, they were convinced that the ancient mathematicians had found their theorems with exactly such heuristic principles before converting their proofs into an unchallengeable form that was difficult to read and even more difficult to transfer onto new problems. In regard to this, Torricelli wrote: "I would not dare claim that geometry of indivisibles is a real new discovery. I would rather believe that the ancient geometers used it to discover the more difficult theorems, although they seem to have preferred a different approach in their proofs." We want to refer the reader to applied examples in the problem section, such as the standard problem of determining the semi-sphere volume according to Cavalieri. Furthermore, we want to point out how Roberval used this method in 1636 in order to determine the area under a cycloid arc (Problem 6.2.2). However, Roberval, who came from a simple farmer's family and acquired his education mainly autodidactically, discovered this method fully independently of Cavalieri and also used a different justification. His "little discs" are of finite but arbitrarily small selectable thickness and his arguments are much closer to those of modern integral calculus than Cavalieri's.

**Geometrical roots of differential calculus**

In contrast to determining volumes, differential calculus could hardly compete with the pre-accomplishments of Antiquity. The fact that the question of tangent directions was asked more generally than in regards to conic sections alone is due to the wealth of newly introduced and/or discovered curves. Nowadays, we are so focussed on the Leibniz-style access by means of the "characteristic triangle" of *dx, dy* and secant and the following passage to the limit, by which the secant becomes the tangent, that an encounter with a completely new approach must be puzzling. The idea, cultivated by Roberval and others, that the curve is created by the course of a moving point and the momentary speed is obtained by the result of a speed in $x-$direction and a speed in $y$-direction, leads to the characteristic triangle when executed. However, Descartes had a completely different idea, which, of course, was much clumsier concerning its calculation. His idea again connected geometrical considerations very closely with purely algebraic aspects and, thus, deserves our interest. He was concerned with determining the subnormal $d(x_0)$ (i.e., the projection of the normal to the curve point $(x_0,\ f(x_0))$ onto the $x$-axis. If $d(x_0) = d_0$ is known, we can determine the slope of the normal by means of elementary geometry and, thus, also the slope of the tangent. We describe Descartes' method (in the German translation of his *Géométrie,* p. 43) in modern language. Since Descartes belonged to the declared opponents of indivisibles and similar methods on the basis of "infinitely small quantities", we want to remark that his method seems to have made no use of such considerations. However, the truth is that there is a limit process, which lies in the assumption that the usual two intersections of the circles merge with the curve to a touching point, when $d$ approaches the wanted value.




**Illus. 6.2.1** The determination of the normal according to Descartes

We fix a value $x_0$ of the independent variable and an increase $d$ (Illus. 6.2.1) and draw circle $k$ around the point $x_0 + d$ of the $x$-axis through the curve point $(x_0,\ f(x_0))$. This circle will cut the curve twice for any $d$. We obtain the $x-$coordinates of both intersections as solutions of the equation system

$$(x-(x_0+d))^2+y^2=r(x_0,d)^2 \quad \textit{(equation of circle k)}, \tag{6.2.1}$$

$$y=f(x) \quad \textit{(equation of the curve)}. \tag{6.2.2}$$

In other words, after elimination of $y$, applying $[f(x_0)]^2+d^2$ for $r^2$ and erasing $d^2$ on both sides we obtain

$$x^2-2x(x_0+d)+{x_0}^2+2dx_0+[f(x)]^2-[f(x_0)]^2=0. \tag{6.2.3}$$

The wanted subnormal $d_0$ is characterised under all $d$-values by the fact that the usual two $x$-solutions of equations (6.2.3) given any $d$ collapse to a double solution $x_0$. Hence, we make the approach

$$x^2-2x(x_0+d)+{x_0}^2+2dx_0+[f(x)]^2-[f(x_0)]^2=(x-x_0)^2\cdot R(x) \tag{6.2.4}$$

with a rest factor $R(x)$, the degree of which depends on function $f$ taken to be a polynomial. (For instance, for a polynomial $f$ of third degree there is a polynomial of sixth degree on the left. Hence, $R$ must be of fourth degree.) By comparing coefficients, we obtain an equation system for the unknown coefficients of $R$ and finally also $d = d_0$ for the course of its solution. We invite the reader to carry this out for the case $f(x) = x^3$ and then to determine the slope of the normal (in dependence of $x_0$) and finally the slope of the tangent. Afterwards, the reader will have a completely new feeling of gratitude for Leibniz's differential calculus. However, we want to pinpoint: the notion of tangent can be explained in a completely different but familiar manner; not as a boundary position of secants, but as the perpendicular to the normal, which on its side is defined by means of touching a circle without using limit processes.

The concept of circles touching any curve proved to be key to advancing curve geometry. Based on what has been said so far, the reader can probably imagine that the definition of the circle of curvature in a curve point, whereby the reciprocal value of its radius yields the measure of the local curvature, was also accomplished in different ways by the followers and/or opponents of infinitesimal methods. The followers of the infinitesimal approach (initially Newton and Jacob I Bernoulli, who referred to the known formula for the curvature as the function of the first and second derivative of the function as "theorema aureum" (golden theorem)) determined the intersection of two adjacent curve normals and then let one converge against the other, whereby the intersection tends against the centre of the circle of curvature. Of course, this was a nightmare for the followers of the Cartesian method. They defined the circle of curvature as a circle that adapts best to the curve amongst all circles that touch the curve in the respective point, so that basically no circle fits better in the "angle of contingence" between curve and circle of curvature (cf. 5.1). However, it turned out that this version, free of limit processes, indeed complicates the actual calculation tremendously.

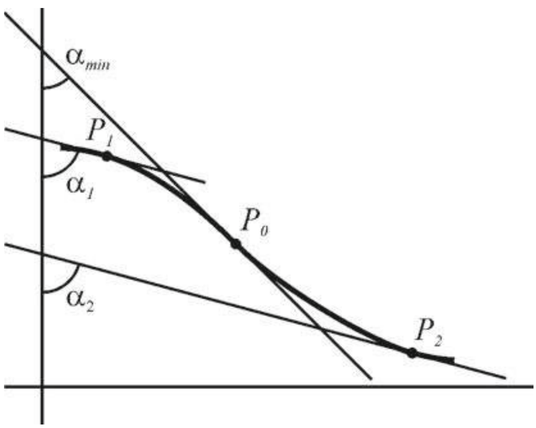
The formation of the notion of torsion of spatial curves as a measure for "local non-planeness", and the notion of "geodesic" as the shortest connection of its points on a curved surface in space, also belong to the history of curve geometry. After preparation from Jacob I and Johann I Bernoulli around 1700, as well as Henri Pitot in 1724, the first combining work on spatial curves was written by the 17-year-old Alexis Claude Clairaut in 1729, who also stood out later as a practical geodesist by participating in an arc measurement expedition to Lapland organised by the French Academy of Sciences (1736/37).

The basic problem of tangent direction was not so closely linked to determining extreme function values from the beginning, as suggested by modern "school analysis". It also took on completely different roles in natural scientific/technical questions of the $17^{th}$ and $18^{th}$ centuries, such as in ballistics, geometrical optics, the practical shape of ship bodies, wind and water wheels, and when constructing the pendulum clock. Christiaan Huygens was led to the general notion of evolute and evolvent of a plane curve by the question of how we could steer a clock pendulum so that it does not move in a circular arc but in a cycloid arc (in order to ensure the exact independence of an oscillation period from amplitude of oscillation at finite expansion). Nonetheless, his discovery did not contribute to the actual technical development of clocks in the end. However, he took on an important role for mathematical curve theory. We owe it primarily to Pierre de Fermat that the question of tangent in many cases turns into a question of the locus of horizontal tangents. He wrote his *Methodus ad disquirendam maxima et minima* between 1638 and 1646 in different stages, which were circulated (amongst others, by Mersenne), but only printed after Fermat's death. This is indicated by the fact that he constantly looked back to objections concerning earlier parts of his writings and tried to invalidate them. Fermat solved numerous examples of geometrical extreme value problems by determining those values of variables for which the tangent of the relevant function will be parallel to the $x$-axis and/or the local variable speed of the function value becomes zero. His method of determining these positions amounts mathematically to the calculation of the differential quotient of this function. We will demonstrate this through his first, still very simple example. (The result herein was already known in Antiquity and can be determined very easily by means of elementary geometry. However, it is exactly the fact that it can be so easily checked that serves Fermat in justifying his method.)

A given line segment of length $a$ is meant to be deconstructed into two parts $x$ and $a - x$ so that a rectangle of these line segments, the product $x(a-x)$, is as large as possible. If $x + e$ is an adjacent value to the assumed optimum value, it delivers a rectangle of size $(x+e)\ (a - x - e)$. The difference of both function values is $e(a-2x)-e^2$; hence, if $e$ is very small, almost $e(a–2x)$. (Fermat did not use the likely argument of different orders of magnitudes from $e$ and $e^2$.) Nowhere does this difference seem to grow smaller than for $x = a/2$. That means: amongst all rectangles of given perimeter, the square has the largest area. With this first example alone, Fermat had already shouted out: “You could not possible state a more general method.”

Given this method fostered by Fermat through many further examples, it is understandable that the dispute about priority between the followers of Leibniz and those of Newton was declared meaningless in France. By speaking of curvatures only qualitatively as convex or concave (without defining these terms), Fermat also recognised that the inflexion points of a curve can be interpreted and calculated as such points in which the angle between the tangent and a fixed direction, such as the $y$-axis, takes on an extreme value (Illus. 6.2.2). If we compare the tangent in inflexion point $P_0$, in which the curve changes from concave to convex characteristics, with the tangents in the points $P_1$ and $P_2$ located on the right and left, we find that the angle $\alpha_{min}$ formed between the tangent belonging to $P_0$ and the y-axis is smaller than angle $\alpha_1$ and/or $\alpha_2$. Keep in mind that this is an observation completely free of calculi and follows up on an illustrative geometrical concept of convex and concave.

The derivation of “Fermat’s principle” is the end of Fermat’s ‘Treatise on maxima and minima’ [Fermat b]. This principle is proof that the law of refraction follows from an extreme value principle, namely that it yields the *temporarily* shortest (and/or sometimes longest) optical path if the light in every permeable medium has a constant speed characteristic for this medium and, thus, in the densest medium, the lowest. This is accompanied by a harsh critique on Descartes, who had attempted to derive the law of refraction by




**Illus. 6.2.2** Inflexion point as locus of an extreme tangent slope according to Fermat

means of the opposite, physically incorrect assumption of speeds. Apart from mutual efforts concerning the coordinate method, this is one of many signs to come of the overall very tense, competitive attitude between Fermat and Descartes. We could almost say that France accommodated the dispute between Descartes and Fermat as an equivalent to the argument, unfortunately tainted by nationalism, between the followers of Leibniz and Newton.

It was significant for the history of geometry that the first extreme value problems solved successfully were mostly of geometrical nature. Some of them are still vigorously taught as part of analytic exercises at school and as part of basic studies. Nevertheless, we must add that Fermat, completely exalted by his successful method, put an extreme value problem that later proved to be almost unsolvable by differential calculus even in its fully developed form. This was the problem already mentioned of determining the point with minimal distance sum for $n$ given points, a problem that seems to have attracted mathematicians again and again up to the present day due to its theoretical difficulty (in the general case) and its significance for praxis (see Problems 6.2.3, 7.3.6, section 7.9 and Problems 7.9.1 and 7.9.2).

Looking at the history more closely, we see that differential and integral calculus had to establish themselves in strong competition against philosophically justified objections and, as a result of this, alternative, purely geometric or geometric-algebraic methods. However, we must simultaneously accept that calculisation of infinitesimal methods conducted by Leibniz, Newton, the Bernoullis and Wallis in different approaches symbolised the beginning of the separation of geometry and analysis. By changing from working with geometrical figures and geometrical arguments to working with formulae, the performance rate was increased dramatically, but also alienated itself more and more from the subject matter standing behind these formulae. Although the end result of such a calculation can be interpreted again as geometric, the intermediate steps can no longer be so interpreted at all times.

**Cartography**

There will be no discussion here of the relatively rapid subsequent expansion of analysis to include functions of several variables, and we will go over the development of actual differential geometry within the appropriate context in section 7.4. This section is meant to look at a special geometrical application. Whereas only a few special cartographic mappings were introduced in the Renaissance and examined by means of elementary geometry within the scope of possibility back then, Johann Heinrich Lambert was the first to ask the general question of the notion of cartographic mapping of the spherical surface or greater parts of it in the plane in his *Anmerkungen und Zusätze zur Entwerfung der Land- und Himmelscharten* (Notes and Comments on the Composition of Terrestrial and Celestial Maps). Furthermore, he raised the question of their mathematical description and the characterisation of properties, such as preservation of area or angle, by means of partial differential equations. Additionally, he introduced a number of new net drafts, of which most are still used nowadays under the name Lambert's projection preserving area and/or angle (...). Lambert, who acquired his education fully autodidactically, differs from the other significant mathematicians of the $18^{th}$ century, in that he did not handle the calculi of analysis as well. (He compensated for this by approaching problems unconventionally and justifying a number of fruitful new sub-disciplines and problem areas for mathematics.) Hence, it was up to Euler in 1777 and Lagrange in 1779 to state the general solutions to the cartographical questions asked by Lambert (whereby Lagrange already used complex numbers without further explanation), whereas Gauss, from 1816 onwards, only transferred the notion of preservation of angle and area onto the mappings between any two surfaces [Gauß' Werke (Gauss' works), vol. 8, p. 370ff, vol. 4, p. 189-216]. He later proved exactly what had been known intuitively for a long time, that preservation of area and angle in a mapping of spherical surface in the plane (and more generally, in a non-isometric mapping) cannot co-exist. Lagrange's work on cartographic mappings was, similar to that of Lambert and Euler, written in a broad, almost popular scientific manner, with extensive verbal introduction to the problem and very concrete examples of application. This was a style that disappeared very rapidly in the $19^{th}$ century. The mathematical language is still one of "differentials" grasped as "infinitely small increases". Gauss initially wrote in this manner too, but had already changed to the modern writing style for (partial) derivation around 1816.

## 6.3 En route to descriptive and projective geometry

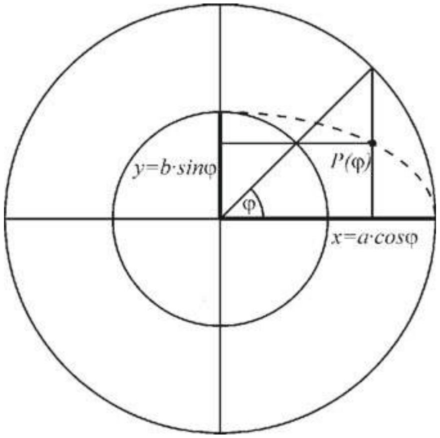
A retrospective conclusion of the Renaissance shows again that a certain type of work share was established in this time: When the "scholarly" mathematicians did not deal with the re-discovery and re-development of ancient knowledge, the advances achieved by them were preferably connected to the problems and questions of astronomy (including trigonometry and optics), whereas the impulse to develop a practical "ordinary geometry" came mainly from the class of artists, craftsmen and engineers. It would be wrong to interpret this simply as a separation into "pure" and "applied" mathematics. Astronomy also had a great practical significance for calendar calculation, seafaring, geodesy and cartography. Questions of descriptive geometry (perspective, multiplane method and others) began increasingly to develop as the most important part of "everyday geometry" in the $17^{th}$ and $18^{th}$ centuries. They were gradually passed on from the practitioners to the scientists, mainly in France. This took place especially in France, where a militarily and economically strong national state had been created under the absolutist government, which ruled uninterrupted until 1789. This state created good conditions for exchanges between theorists and practical scientists through the founding of academies and universities of different natures in Paris. Further-

**Illus. 6.3.1** Two paintings of Blaise Pascal

more, Mersenne's circle and his intellectual atmosphere benefitted the contact between pure scientists, like Descartes, Pascal and Roberval, on one side, and architects, stronghold master builders and military engineers, like Girard Desargues and Alain Mallet, crafting drawers and engravers, like Abraham Bosse, and geometrically engaged artists, like Laurent de La Hire, on the other side. The son of the latter, Philippe de La Hire, whose name is primarily connected nowadays to the so-called two-circle method for the pointwise construction of ellipses (Illus. 6.3.2), has up to now played a little appreciated role as a link between theorists and practical scientists of new geometry. He succeeded Roberval at the College Royale in Paris in 1682 and became professor at the Academie Royale d'Architecture in 1687. His teaching and numerous publications covered all areas of physics and techniques back then apart from actual mathematics; even other natural sciences, in part. His circle of friends included men such as Roberval and Bosse. Similar conditions would develop decades later in other European centres, such as Berlin, London or Petersburg, but never achieve this quality.

Let us start with the architect and engineer Desargues. His first publication in 1636 concerns central perspective. However, architects' job descriptions also included the construction of sundials, and if the sun moves along an orbit, the shadow of the peak of the gnomon describes a conic section on the plane of the sundial.

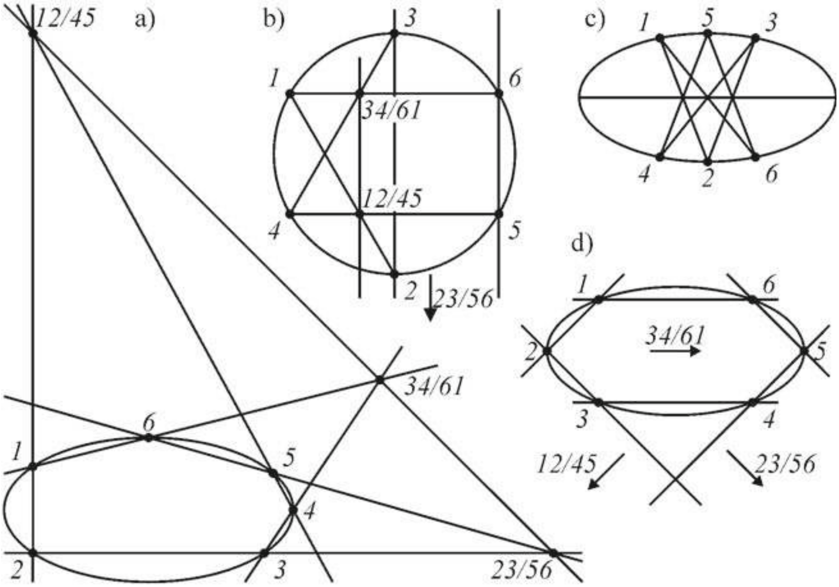
The studies of men like Fermat or Descartes, which were triggered based on Antiquity, encountered practical needs. In 1639, Desargues published his *Brouillon project...* (English: First draft of describing the events when a cone meets a plane; modern English translation in [Field/Gray 1987]). This text, of which only 50 copies were printed and which was long forgotten and lost, only




**Illus. 6.3.2** Two-circle construction of the ellipses according to de La Hire

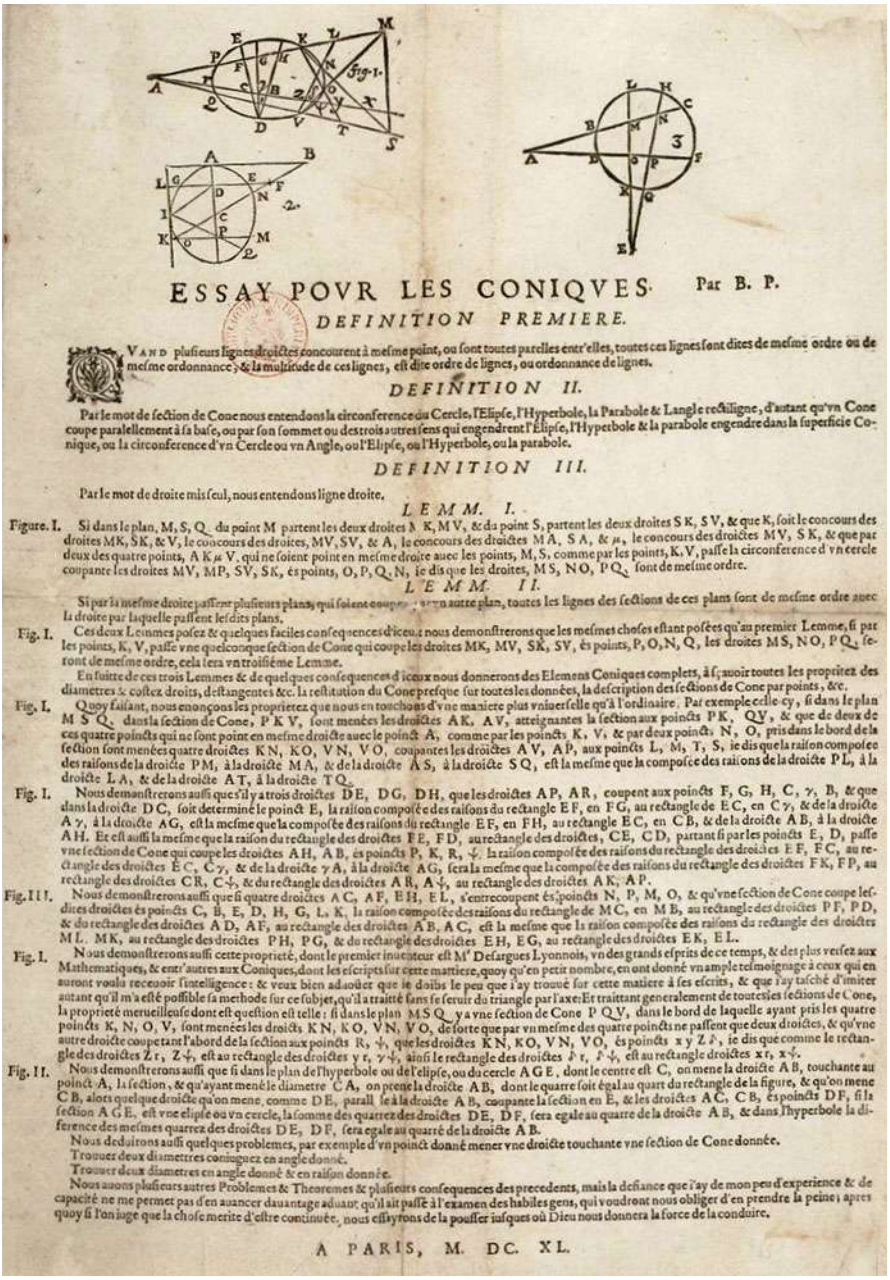
to be rediscovered[1] by M. Chasles in 1845, is nowadays accepted as the actual "birth certificate" of projective geometry. Desargues started by pinpointing that in his work all straight lines and planes are infinitely extended in every direction (in contrast to Euclid's view, which had been the dominant one until then and which is based on line segments and their potential extension). He recognised the analogy between the pencil of lines through a point and a flock of all straight lines parallel to each other (in Desargues' work, both types of pencils together are called "ordonnance", roughly "rule"), concluded correctly and assigned such flocks to an "infinitely distant" intersection (his common name for infinitely distant and finite points as centres of pencils of lines is "butte", roughly "target") and later concluded, amongst other things, that a circular cylinder is the special case of a circular cone with an infinitely distant apex. He realises the transfer from the simple cone to a double cone, which was highly important for the modern geometry of conic sections, and concentrated on deriving as many properties as possible common to all conic sections by producing them as a central perspective image of a circle. However, he first introduced proper and improper pencils of planes analogously to the proper and improper pencils of lines, studied the between and separation relation for three or four collinear points and found out that the latter is preserved when centrally projecting. He then examined the special case of harmonically separating point pairs, discovered the notion of a complete quadrilateral, constructed the fourth harmonic point for three given ones, and advanced to the theory of polarity of point and straight line in conic sections.

[1] The copy found by Chasles was a copy made by Philippe de La Hire in 1679. The efforts of copying this text manually do not just indicate how much it must have been appreciated, but also that La Hire could not get a printed copy of this rare text. An original only appeared in 1950, on which [Field/Gray 1987] based their translation.




**Illus. 6.3.3** Pascal's theorem, a) general case, b)-d) different special cases

Desargues' thoughts were appreciated by Fermat and Descartes and bore a strong influence on the sixteen-year-old Pascal back then. They inspired him to write his first treatise on conic sections, which already contains Pascal's theorem without proof. This short *Essay pour les coniques* (English translation in [Field/Gray 1987]) only circulated in a few copies in 1640. Modern understanding of conic sections is rounded therein by the fact that a pair of crossing straight lines (possibly in the infinite, thus, parallel) is viewed as a conic section, which reveals Pappus's theorem, already known in Antiquity, as a special case of Pascal's theorem. Pascal's original figure served the illustration of several claims and was, thus, overloaded with many unnecessary points and straight lines (Illus. 6.3.4). Illus. 6.3.3shows all basic aspects (with other, now more common names). This theorem in modern fashion is as follows: If 1, ... ., 6 are any points of a conic section and if 12, 23, etc., refer to the respective connecting line, 12/45, etc., refer to the respective (possibly improper) intersection of these straight lines, then 12/45, 23/56 and 34/61 are located on a (possibly improper) straight line (in the latter case, the three straight line pairs are pairwise parallel, as shown in, for example, Illus. 6.3.3d). Illus. 6.3.3b)-d) show some of many elementarily justifiable special cases and also illustrate that the order of the points on the conic section is not important. Since the collinearities are preserved when centrally projected, we can suppose that Pascal found his theorem by means of the systematic transformation of an elementary initial case on a circle by means of centrally projecting them onto other configurations. His 'Essay', which explicitly refers to Desargues and uses his notions, rather has



ESSAY POVR LES CONIQVES. Par B. P.

DEFINITION PREMIERE.

QVAND plusieurs lignes droictes concourent à mesme point, ou sont toutes paralleles entr'elles, toutes ces lignes sont dites de mesme ordre ou de mesme ordonnance, & la multitude de ces lignes, est dite ordre de lignes, ou ordonnance de lignes.

DEFINITION II.

Par le mot de section de Cone nous entendons la circonference du Cercle, l'Elipse, l'Hyperbole, la Parabole & Langle rectiligne, d'autant qu'vn Cone coupe paralellement à sa base, ou par son sommet ou des trois autres sens qui engendrent l'Elipse, l'Hyperbole & la parabole engendre dans la superficie Conique, ou la circonference d'vn Cercle ou vn Angle, ou l'Elipse, ou l'Hyperbole, ou la parabole.

DEFINITION III.

Par le mot de droite mis seul, nous entendons ligne droite.

LEMM. I.

Figure. I. Si dans le plan, M, S, Q, du point M partent les deux droites MK, MV, & du point S, partent les deux droites SK, SV, & que K, soit le concours des droites MK, SK, & V, le concours des droites, MV, SV, & A, le concours des droictes MA, SA, & μ, le concours des droictes MV, SK, & que par deux des quatre points, A K μ V, qui ne soient point en mesme droite auec les points, M, S, comme par les points, K, V, passe la circonference d'vn cercle coupante les droites MV, MP, SV, SK, és points, O, P, Q, N, ie dis que les droites, MS, NO, PQ, sont de mesme ordre.

LEMM. II.

Si par la mesme droite passent plusieurs plans, qui soient coupez par vn autre plan, toutes les lignes des sections de ces plans sont de mesme ordre auec la droite par laquelle passent lesdits plans.

Fig. I. Ces deux Lemmes posez & quelques faciles consequences d'iceux nous demonstrerons que les mesmes choses estant posées qu'au premier Lemme, si par les points, K, V, passe vne quelconque section de Cone qui coupe les droites MK, MV, SK, SV, és points, P, O, N, Q, les droites MS, NO, PQ, seront de mesme ordre, cela sera vn troisiéme Lemme.

En suitte de ces trois Lemmes & de quelques consequences d'iceux nous donnerons des Elemens Coniques complets, à sçauoir toutes les proprietez des diametres & costez droits, des tangentes &c. la restitution du Cone presque sur toutes les données, la description des sections de Cone par points, &c.

Fig. I. Quoy faisant, nous enonçons les proprietez que nous en touchons d'vne maniere plus vniuerselle qu'à l'ordinaire. Par exemple celle-cy, si dans le plan MSQ, dans la section de Cone, PKV, sont menées les droictes AK, AV, atteignantes la section aux poincts PK, QV, & que de deux de ces quatre poincts qui ne sont point en mesme droicte auec le poinct A, comme par les poincts K, V, & par deux poincts N, O, pris dans le bord de la section sont menées quatre droictes KN, KO, VN, VO, coupantes les droictes AV, AP, aux poincts L, M, T, S, ie dis que la raison composée des raisons de la droicte PM, à la droicte MA, & de la droicte AS, à la droicte SQ, est la mesme que la composée des raisons de la droicte PL, à la droicte LA, & de la droicte AT, à la droicte TQ.

Fig. I. Nous demonstrerons aussi que s'il y a trois droictes DE, DG, DH, que les droictes AP, AR, coupent aux poincts F, G, H, C, γ, B, & que dans la droicte DC, soit determiné le poinct E, la raison composée des raisons du rectangle EF, en FG, au rectangle de EC, en Cγ, & de la droicte Aγ, à la droicte AG, est la mesme que la composée des raisons du rectangle EF, en FH, au rectangle EC, en CB, & de la droicte AB, à la droicte AH. Et est aussi la mesme que la raison du rectangle des droictes FE, FD, au rectangle des droictes, CE, CD, partant si par les poincts E, D, passe vne section de Cone qui coupe les droictes AH, AB, és poincts P, K, R, ♆, la raison composée des raisons du rectangle des droictes EF, FC, au rectangle des droictes EC, Cγ, & de la droicte γA, à la droicte AG, sera la mesme que la composée des raisons du rectangle des droictes FK, FP, au rectangle des droictes CR, C♆, & du rectangle des droictes AR, A♆, au rectangle des droictes AK, AP.

Fig. III. Nous demonstrerons aussi que si quatre droictes AC, AF, EH, EL, s'entrecoupent és poincts N, P, M, O, & qu'vne section de Cone coupe lesdites droictes és poincts C, B, E, D, H, G, L, K, la raison composée des raisons du rectangle de MC, en MB, au rectangle des droictes PF, PD, & du rectangle des droictes AD, AF, au rectangle des droictes AB, AC, est la mesme que la raison composée des raisons du rectangle des droictes ML, MK, au rectangle des droictes PH, PG, & du rectangle des droictes EH, EG, au rectangle des droictes EK, EL.

Fig. I. Nous demonstrerons aussi cette proprieté, dont le premier inuenteur est Mr Desargues Lyonnois, vn des grands esprits de ce temps, & des plus versez aux Mathematiques, & entr'autres aux Coniques, dont les escripts sur cette mattiere, quoy qu'en petit nombre, en ont donné vn ample tesmoignage à ceux qui en auront voulu receuoir l'intelligence: & veux bien adoüer que ie doibs le peu que i'ay trouué sur cette matiere à ses escrits, & que i'ay tasché d'imiter autant qu'il m'a esté possible sa methode sur ce subjet, qu'il a traitté sans se seruir du triangle par l'axe: Et traittant generalement de toutes les sections de Cone, la proprieté merueilleuse dont est question est telle: si dans le plan MSQ, y a vne section de Cone PQV, dans le bord de laquelle ayant pris les quatre poincts K, N, O, V, sont menées les droicts KN, KO, VN, VO, de sorte que par vn mesme des quatre poincts ne passent que deux droictes, & qu'vne autre droicte coupe tant l'abord de la section aux poincts R, ♆, que les droictes KN, KO, VN, VO, és poincts x y Z δ, ie dis que comme le rectangle des droictes Zr, Z♆, est au rectangle des droictes yr, γ♆, ainsi le rectangle des droictes δr, δ♆, est au rectangle droictes xr, x♆.

Fig. II. Nous demonstrerons aussi que si dans le plan de l'hyperbole ou de l'elipse, ou du cercle AGE, dont le centre est C, on mene la droicte AB, touchante au poinct A, la section, & qu'ayant mené le diametre CA, on prene la droicte AB, dont le quarre soit égal au quart du rectangle de la figure, & qu'on mene CB, alors quelque droicte qu'on mene, comme DE, parallele à la droicte AB, coupante la section en E, & les droictes AC, CB, és poincts DF, si la section AGE, est vne elipse ou vn cercle, la somme des quarrez des droictes DE, DF, sera egale au quarre de la droicte AB, & dans l'hyperbole la difference des mesmes quarrez des droictes DE, DF, sera egale au quarré de la droicte AB.

Nous deduirons aussi quelques problemes, par exemple d'vn poinct donné mener vne droicte touchante vne section de Cone donnée.

Trouuer deux diametres coniuguez en angle donné.

Trouuer deux diametres en angle donné & en raison donnée.

Nous auons plusieurs autres Problemes & Theoremes & plusieurs consequences des precedents, mais la defiance que i'ay de mon peu d'experience & de capacité ne me permet pas d'en auancer dauantage aduant qu'il ait passé à l'examen des habiles gens, qui voudront nous obliger d'en prendre la peine; apres quoy si l'on iuge que la chose merite d'estre continuée, nous essayrons de la pousser iusques où Dieu nous donnera la force de la conduire.

A PARIS, M. DC. XL.

**Illus. 6.3.4** First text page of Pascal's *Essay pour les coniques* (1640). The figure on the top left contains the one belonging to Pascal's theorem as partial figure (see Illus. 6.3.3)

[Bibliothèque nationale de France]

the character of an announcement, but indicates that many problems concerning conic sections are solvable, such as pointwise construction given five points or construction of the tangent given one point by means of the found theorem based on linear construction (i.e., by cutting, linking or parallels). An extensive treatise on this topic, on which Pascal subsequently worked for many years, was, unfortunately, never published and is nowadays accepted as lost. Mersenne confirmed that Pascal drew more than 400 conclusions from his theorem. Leibniz also saw the text and reported on some of its content. Desargues' thoughts were appreciated by Fermat and Descartes and bore a strong influence on the sixteen-year-old Pascal back then. They inspired him to write his first treatise on conic sections, which already contains Pascal's theorem without proof. This short *Essay pour les coniques* (English translation in [Field/Gray 1987]) only circulated in a few copies in 1640. Modern understanding of conic sections is rounded therein by the fact that a pair of crossing straight lines (possibly in the infinite, thus, parallel) is viewed as a conic section, which reveals Pappus's theorem, already known in Antiquity, as a special case of Pascal's theorem. Pascal's original figure served the illustration of several claims and was, thus, overloaded with many unnecessary points and straight lines (Illus. 6.3.4). Illus. 6.3.3shows all basic aspects (with other, now more common names). This theorem in modern fashion is as follows: If 1, ... ., 6 are any points of a conic section and if 12, 23, etc., refer to the respective connecting line, 12/45, etc., refer to the respective (possibly improper) intersection of these straight lines, then 12/45, 23/56 and 34/61 are located on a (possibly improper) straight line (in the latter case, the three straight line pairs are pairwise parallel, as shown in, for example, Illus. 6.3.3d). Illus. 6.3.3b)-d) show some of many elementarily justifiable special cases and also illustrate that the order of the points on the conic section is not important. Since the collinearities are preserved when centrally projected, we can suppose that Pascal found his theorem by means of the systematic transformation of an elementary initial case on a circle by means of centrally projecting them onto other configurations. His 'Essay', which explicitly refers to Desargues and uses his notions, rather has the character of an announcement, but indicates that many problems concerning conic sections are solvable, such as pointwise construction given five points or construction of the tangent given one point by means of the found theorem based on linear construction (i.e., by cutting, linking or parallels). An extensive treatise on this topic, on which Pascal subsequently worked for many years, was, unfortunately, never published and is nowadays accepted as lost. Mersenne confirmed that Pascal drew more than 400 conclusions from his theorem. Leibniz also saw the text and reported on some of its content.

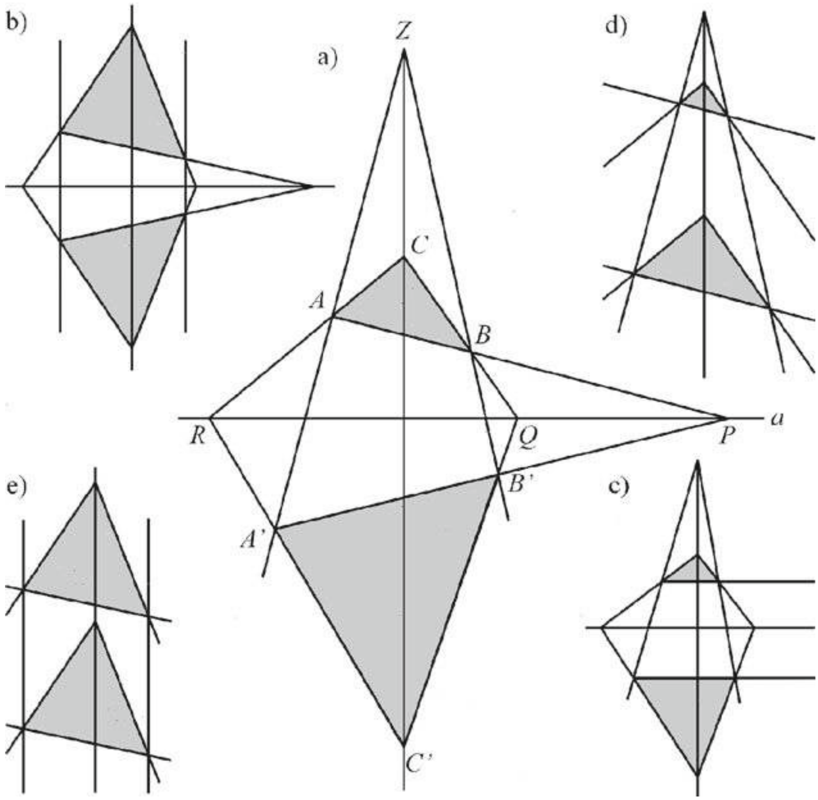
Hence, whereas Desargues via Pascal had an extraordinarily fruitful impact on the "science of geometry", his texts (apart from those papers on shadow constructions, stereotomy and sundial constructions from 1640) were heavily rejected by his professional peers. The thoroughly unusual way of thinking and a wealth of newly introduced notions and terms may have contributed to this rejection. However, it seems that the resistance of an established profession against "uncomfortable" changes and innovations was mainly to blame.

As a result, Desargues, practitioner himself, only found one follower amongst all practical scientists, though very faithful and active: the already mentioned illustrator Abraham Bosse, who initiated the creation of generally understandable representations of Desargues' texts on sundials and stereotomy in 1643 and an extended edition of *Perspective* in 1648, the appendix of which was the first to publish Desargues' theorem on equivalence of central and axial perspectivity (Illus. 6.3.5). Desargues himself, embittered by his peers' reactions, did not publish anything else after 1640.

**From perspective to multiplane methods**

Advances concerning mathematising perspective were also made in other countries in the $18^{th}$ century. Thereby, there often was no sharp boundary between actual central perspective, its borderline case of parallel projection, and the approach to multiplane methods. Illusionistic paintings enjoyed great esteem in the Baroque era (Illus. 6.3.6). Relief perspective was also frequently used in order to simulate spatial depth (Illus. 6.3.7, Problem 6.3.2). A treatise on perspective by Brook Taylor (mostly known in analysis) was published in London in 1715, released in several editions and also translated into French. A popular version by Joshua Kirby (London 1754) has become immortal, mainly due to its title page, which was designed by the famous, contemporary graphic designer William Hogarth (Illus. 6.3.8). It features numerous mistakes that can be committed if one does not study this exquisite book before actually drawing. It truly is an early piece of shock advertising, common nowadays, and on top of that a highly valued example of art of "contradictory perspective", which is so popular at present! Mathematically speaking, it seems that Taylor offered first and foremost the representation of straight lines and planes based on their traces, in other words, their intersections with the projecting planes. Furthermore, he had already studied the reconstruction of the observer's viewpoint based on a correctly constructed picture.

Lambert [Lambert/Steck 1943] engaged repeatedly with perspective in Germany. His *Freye Perspective* was published in 1759 following an early small text (1752), first printed in 1943, already describing the perspectograph that he invented, with which we can mechanically transform a top view into its central perspective view (Problem 6.3.1). The title shows that we are dealing with the idea of drawing a central perspective picture directly without using a top and/or front view. In Lambert's work, the idea of grasping the plane of projection as a model of space – which, of course, corresponded to the case of central perspective not injectively mapped – by giving oneself the picture of a spatial Cartesian trihedral becomes obvious. Afterwards, every construction that we imagine to be executed in space transforms itself into an assigned construction in the plane of projection. As we will see, exactly this thought will be perfected by Monge by combining it with the multiplane method. Since, in Lambert's viewpoint, for instance the operation of drawing a parallel to a given straight line through a given point is realised in the plane




**Illus. 6.3.5** Desargues' theorem. a) General case: triangles $ABC$ and $A'B'C'$ have a perspectivity centre $Z$ if and only if the sections $P, Q, R$ of the assigned sides are located on a common straight line, i.e., perspectivity axis $a$. Thereby, b) centre $Z$ can be infinitely distant, c) one of the intersections on the axis can drift into the infinite, i.e., the relevant triangle sides become parallel to the axis and, hence, also to each other. d) If a second pair of triangle sides becomes parallel at the central perspective position, axis $a$ yields two infinitely distant points. Thus, the third intersection must also be infinitely distant. e) shows that the centre can also be infinitely distant at the same time. "Desargues' figure" or "configuration" a) can be interpreted in several ways, e.g., $P$ is the perspective centre for triangles *AA'R* and *BB'Q*. In that case, straight line *ZCC'* takes on the function of the perspectivity axis. We now see that b) and c) represent the same special case. Imagine one of the figures turned by 90 degrees

of projection as the connection of the image point of the given point and the vanishing point of the given straight line, his text opens up further access to projective geometry:

**Illus. 6.3.6** Illusive vault of the church di Badia in Arezzo, painted onto a plane ceiling by Andrea Pozzo (1642–1700), a master of illusionistic perspective painting. Pozzo painted several similar ceilings and also wrote a book on perspective [Photo: A. Schreiber]

**Illus. 6.3.7** Wall painting (1767) in the Baroque church Saint-Roch in Paris. The frame in relief perspective simulates great depth. In reality, the niche is barely 25 cm deep [Photo: P. Schreiber]


**Illus. 6.3.8** Title page by William Hogarth for J. Kirby's *Dr. Brook Taylor's Perspective Made Easy* (1754). "Whoever makes a design without knowledge of perspective will be liable for such absurdities, as shown in this frontispiece." (picture caption)

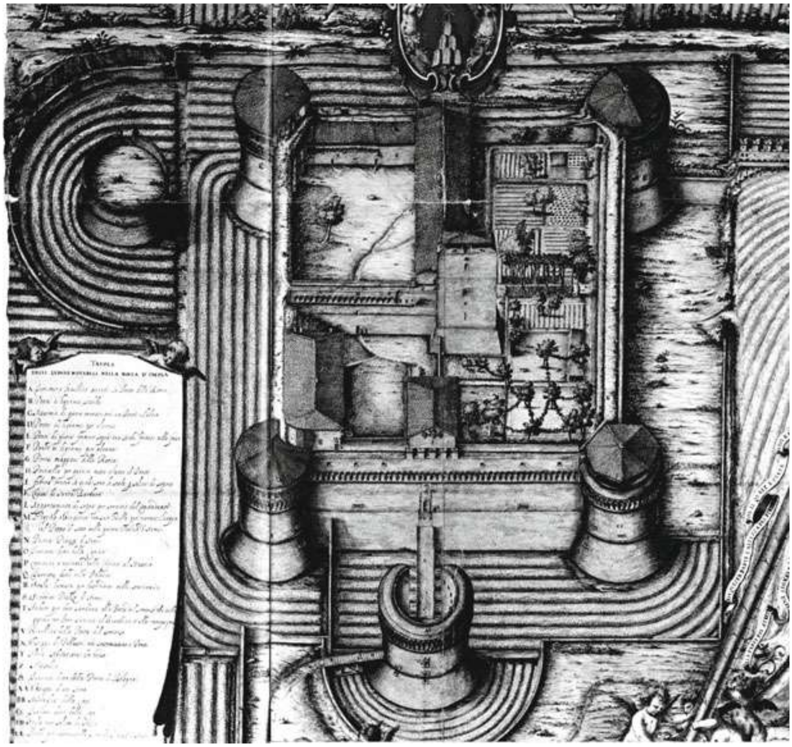
Within central perspective projections, most parallel straight lines feature an intersection too, i.e., the merely imagined infinitely distant points transform themselves into finitely located points. Moreover, drawing parallels seems like a natural borderline case of connecting points.

Lambert's text is at least partially the result of his stay in France in 1758 and was published in German and French in the same year. A second German edition was published in a highly extended version in 1774. Along the way, apart from his academy membership in Berlin, Lambert had also become a Prussian senior building officer in 1770. This office was subsequently held by famous architects like David Gilly and Karl Friedrich Schinkel. Such "part-time jobs", which also included being a building surveyor and a practical architect, was characteristic for mathematicians of the $18^{th}$ century. Stronghold and civil architecture will be regular subjects for mathematics professors at universities until the 1830s and will occupy a large number of pages in several volumes of mathematical textbooks, as written by, for example, Christian Wolff, Abraham Gotthelf Kästner and Wenceslaus Karsten.

So-called cavalier perspective had established itself as the main representation in the building trade since around 1600. In modern descriptive geometry, everything is based on the concept of a perpendicular upright perspective, which is, thus, true to scale and followed by shortened pictures of depression lines, mostly inclined by 45 degrees. In contrast, everything back then was based on a top view true to scale, above which the images of perpendicular lines rose (hence, the French term "elevations géométrales" for this type of representation, Illus. 6.3.9). The term "cavalier...." has nothing to do with the general meaning of the word "cavalier". Cavaliers back then were overhanging parts (bastions) of a stronghold. The development of artillery had led to a revolution in stronghold architecture. From the views of sovereigns and commanders, the planning of a stronghold so that the entire surrounding area could be seen and reached by ordnance, whilst making sure that nobody could see into the stronghold from any location, had developed into one of the most important tasks of applied geometry. Accordingly, the leading specialists of stronghold constructions, such as the French admiral Vauban, enjoyed great esteem. He could afford to compose a memorandum about the drawbacks and deficits in France under the almighty and impeccable King Louis XIV; this made him fall into disgrace with his king, but his memorandum came to be accepted as an important event in the history of France.

A strictly organised training regime of military engineers developed under his control, generating men like Lazare Carnot and Gaspard Monge at the end of the $18^{th}$ century. Christian Willenberg, who founded the present Czech Technical University in Prague in 1717, had enjoyed such an education and brought with him the typical French culture of military engineering training to the countries of the Habsburg Monarchy, where descriptive geometry still playes a greater role than at universities in other countries.

The authorities of stronghold constructions were opponents of the spreading central perspective, since they could not deduct the real measures and



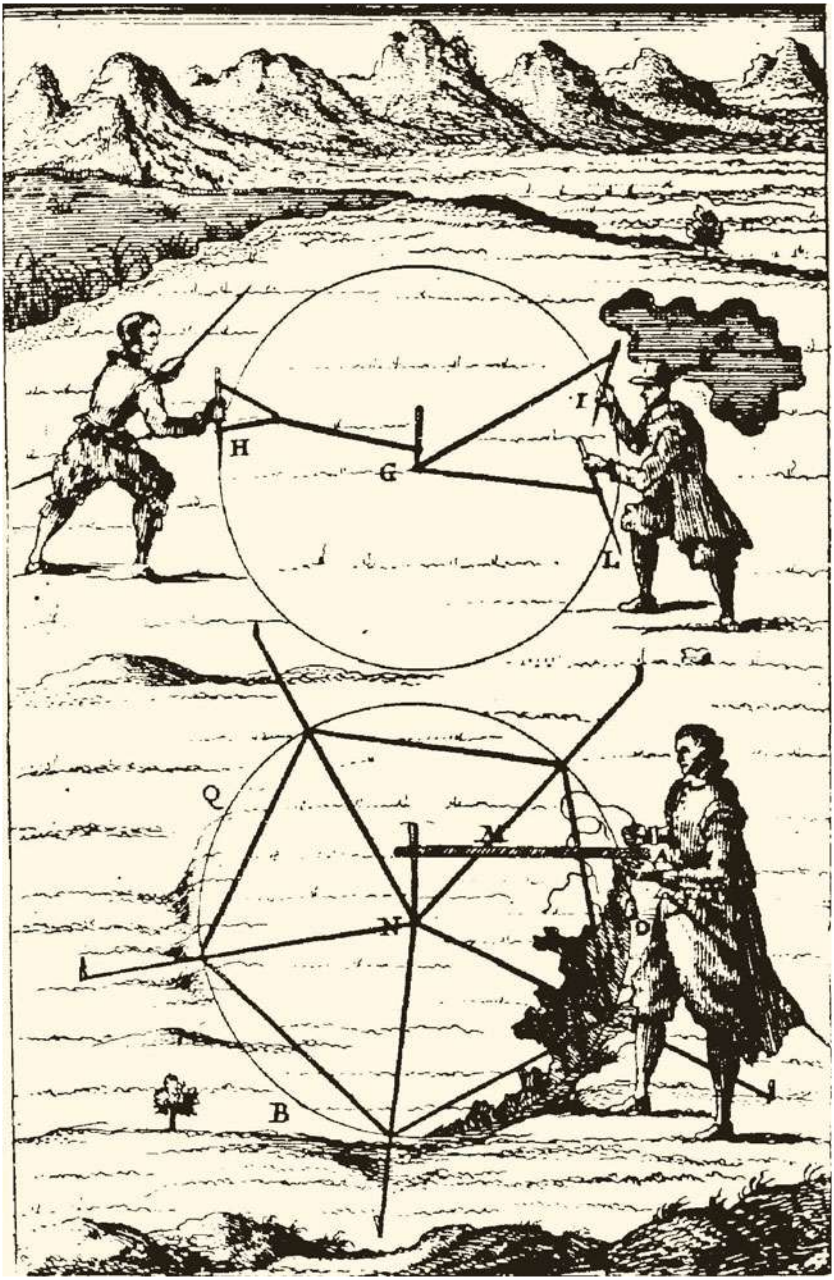
**Illus. 6.3.9** Fortress Rocca (Italy) in cavalier perspective ($17^{th}$ century) ([Leonardo: Forscher, Künstler, Magier] by permission from EMB-Service for Publisher, Adligenswil)

distances as easily from perspective pictures as from cavalier perspectives (nowadays generalised as axonometric single-plane projection). This argument would gain increasing significance with the rise of engine building, and would first push central perspective to the edge of descriptive geometry before banishing it to the art academies. There too, it was not very appreciated and was not practised much longer when the artists turned away from the paradigm of preferably naturalistic drawings and graphics at the end of the $19^{th}$ century. However, other approaches to constructive geometry also developed in the shadow of war constructions, such as constructing in an area laced with obstacles, the purely linear construction (by means of bearing, since a more large-scale work with the compass is naturally not possible) and first approaches to estimating and minimising the unavoidable error concerning practical works in the field. A book by the afore-mentioned A. Mallet from 1672 titled *Les travaux de Mars* (The Works of Mars; Illus. 6.3.10) is a rich source of such practical geometry. France and, in part, the needs of war were also the origin of a new direction in cartography: the description of the field by means of level curves and depression lines, which was later referred to as topographic single-plane projection and also consulted as an auxiliary device of pure mathematics to graphically represent functions of a complex argument respectively of two variables. Since measuring the depths of water by sound could be realised much more easily than determining heights in the field concerning a fixed zero level (approx. sea level), it is not surprising that maps with depth contour lines (isobaths) occur earlier than those with height contour lines (isohypsies). The first of the latter maps (1771 by Du Carla) was the map of an "imaginary island" – thus, the primary concern was just the principle itself. Jean Louis Dupain-Triel submitted the first map of France with contour lines to the French Academy of Sciences in Paris in 1791. There were already isobathic maps in 1697 from the Maas mouth (Pierre Ancelin), 1733 from Merwede (Samuel Cruquius) and 1737 from the English Channel (Philippe Buache) [Wiener 1884, I, p. 25 Kupčik 1980, p. 212f].

The knowledge of Dürer's remarkable contribution to the multiplane method (assigned normal views) soon spread across France. His 'Instruction' had already been published in a Latin translation in Paris in 1532 and reprinted several times after that. Another crucial source was known as stereotomy. When building vaults, window embrasures, spiral staircases or the like with natural stone, the cutting and working of which requires effort and is expensive, the form of each individual stone had to be determined exactly beforehand. Therefore the necessary geometrical knowledge was passed on mainly orally in the "Bauhütten" for centuries.

When written representations appeared, for example, in France by Philibert de l'Orme in 1576 or Derand in 1643, they were mostly limited to construction drawings, including auxiliary lines, with the advice that every knowledgeable person could deduct the method from this and spacious written explanations would only complicate the matter [Wiener 1884]. (Compare this with modern DIY instructions to assemble furniture!) The work in three volumes by the military engineer Amédée François Frézier published in 1737-39, which even mentions stereotomy in the title, achieved remarkable advances. The first volume is only dedicated to theory; the other two are of a practical nature. Frézier based his ideas on clear and unambiguous definitions, phrased general rules, and proved all his claims. He dealt extensively with curved surfaces in space and, following the needs of stereotomy, preferred such surfaces that can be generated by mechanical grinding processes, hence, stated in a modern fashion – ruled surfaces: a straight line is moved so that it is inserted along two curves at possibly different speed levels. He also solves the problems very generally for constructing the curve of intersection, for spreading a developable surface in the plane, and for determining the angle of intersection of two surfaces.

The historical accomplishment from Gaspard Monge is based on this prehistory. Coming from a very simple background, his talent was discovered early due to fortunate circumstances. He went to the military engineering school at Mézières, which, however, only offered officer education and training to aristocrats at that time. Talented individuals of lower class could join the so-called "plaster class", which led to an occupation as supervisor




**Illus. 6.3.10** Geometrical construction in the field
[A. Mallet: Les travaux de Mars (1672)]

at stronghold constructions or technical drawer and/or producer of models. Since Monge again sparked attention due to his original and well-considered solutions of geometrical-technical problems, he was taken out of class and promoted to teach his aristocratic classmates, although they did not pay him any respect whatsoever. This experience made him an enthusiastic follower of the French revolution. He was one of the initiators of the Polytechnic School founded in Paris in 1794 and was temporarily director. If we look at this very unemotionally, his main accomplishment in descriptive and/or constructive geometry was securing the leading educational place for boarding pupils of the Polytechnic School, who were treated like cadets. Furthermore, he conceived of excellent teaching programs and also gave excellent lectures himself, wrote a respective textbook and inspired a whole generation of French geometers due to his personal charisma. This occurred mainly in the 19th century and will be looked at individually later.

Monge's purely scientific significance for descriptive geometry lies, above all, within the fact that he was the first to say aloud as clearly as possible in his time that the responsibility of descriptive geometry is twofold:

"First, it should deliver methods to *map* all spatial figures that have all three dimensions, i.e., length, width and height onto a paper that only has two dimensions, i.e., length and width, given that these figures can be strictly defined.

Second, it should teach the method for *recognising* the shape of the spatial figures and for deriving all theorems that follow from the figure and the mutual position of the spatial figures, all based on an accurate drawing." ([Monge 1798], translation into English from German of the first sentences of his textbook, emphases added)

We can phrase the opinion expressed here even more precisely: we are not simply dealing with a plane mapping of spatial objects, but are concerned with creating a two-dimensional *model* of the three-dimensional space in which problems that actually apply to space can be alternatively solved based on their plane representatives. Thereby, the problems can address constructions in the narrow sense, decision processes and proving theorems. This thought approaches descriptive geometry methodologically to what we nowadays would refer to as coding, i.e., coding to make objects more manageable for algorithms, in other words, in an analogy to number calculation by means of number naming systems, to manipulate functions (calculating zeros, differential and integral calculus, ...) by means of formulae that represent these functions, and to edit geometrical questions by means of the coordinates of the relevant object. Monge must have been especially aware of the last mentioned analogy, i.e., the great methodological proximity between the coordinate method and the methods of descriptive geometry, since he also developed the coordinate method to serve as the tools for constructive problems of three-dimensional space and taught engineers how to use both methods parallel like their "left and right hand".

**Illus. 6.3.11** Gaspard Monge, painting by J. G. Elzidor Naigeon (Museée de l'Histoire de France, Versailles)

The fact that we are not insinuating that Monge had anything to do with this interpretation of his first sentences will be reflected by a few further sentences of his *Géométrie descriptive:*

"First problem: A point $P$ is given by means of its two projections $P', P''$ and a straight line $g$ is given by its two projections $g', g''$. Construct the projections of straight line $h$, which passes through point $P$ and is parallel to straight line $g$." [l.c., p. 23] If we translate this into the language of the end of the $20^{th}$ century, this means: In space we have an operation, that for each single point $P$ and one straight line $g$ assigns the parallel through $P$ to $g$. Point $P$ is now given in the plane (two-plane) model of space by its coding

$P', P''$ and the straight line by its coding $g', g''$. Describe an algorithm that will fabricate code $h', h''$ of the outcome.

"Compare descriptive geometry with algebra... Descriptive geometry acts in this point exactly as algebra, neither of which had a general method to convert a word problem into equations.

...and it is only possible to get beginners used to grasping these relations accurately and writing them in equations by using widely different examples. However, just as there are methods in analysis to further deal with equations and to deduct the values of the unknowns with their help after a problem has been converted into them, there are also general methods in descriptive geometry to construct everything, which results from the shape and position of the latter, once the solids have been projected (i.e. after the coding has been executed).

It is not without intention here that we compare descriptive geometry with algebra; both branches of mathematics are most closely related to each other. There is no construction in descriptive geometry, which cannot be transferred to analysis; and vice versa, concerning problems that do not contain more than three unknowns, every analytic operation can be grasped as a description of a geometrical operation. It is desirable that these two branches of mathematics would be looked after together..." [l.c. p. 17f.]

Monge was aware that the "empty" plane becomes a model of the entire space, if we mark a fixed straight line, i.e., the "axis", by interpreting it as the straight line of intersection of two planes perpendicular to each other that afterwards are folded onto one another in the plane of projection. Thus, he differed from his predecessors since the top view is not located on one semi-plane and the front view on another one regarding a base line, but each point in the plane of projection has a double function as the top view image of a point and simultaneously as the front view image of a generally different point. Of course, it soon became obvious that this fine program would encounter difficulties. Already, a straight line cannot always be determined by means of the pair of top and front view images. Hence, Monge utilised additional planes of projection and the so-called traces (intersections of straight lines and planes with the planes of projection) in order to be able to code a greater amount of spatial objects in the plane. Thereby, he used many features that his predecessors had already developed by means of the concrete case. However, Monge was primarily concerned with generality.

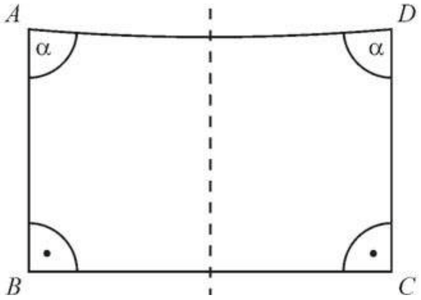
When Monge's *Leçons de géométrie descriptive* was first circulated in written form after his lectures in 1795 (first public version in 1798), his work had already fully matured, since Monge had tested his methods for many educational years. He was not allowed to publish any of that before 1789 since his methods were thought of as militarily important and, thus, secret. (Strangely, this secret was not only forbidden from being passed on to foreign countries, but also to other competing French military engineering schools like the ones at Metz and Besançon!) Hence, it seems that some of his lecture notes may have gotten into the hands of the competition. One of Monge's first students in Paris, S. F. Lacroix (mentioned in another context because of his influential textbooks on coordinate geometry, but who also taught in Besançon from 1788 until 1793, worked at the universities in Paris afterwards, and was friends with Monge), published his own textbook on descriptive geometry already in 1795, differing only slightly from Monge's. Lacroix explained in the foreword that he had already had access to drawings and notes in Besançon years ago via some of Monge's former students. He had ordered these drawings and notes himself into a textbook framework before Monge's lectures were printed. Other reports (e.g., Dupin's Monge biography, 1819) basically confirm this.

## 6.4 Competing for the parallel postulate

The question of the provability of the $5^{th}$ Euclidean postulate and/or its substitutability by an "evident" assumption, which had already been discussed by several authors in Antiquity, was basically revived in Europe by Christoph Clavius's edition of Euclid from 1574 (cf. Chap. 5.1). Clavius did not just talk about the efforts concerning this matter of ancient mathematicians, but also dealt with the problem himself and proved the $5^{th}$ postulate under the assumption that a line, which had a constant distance to another line, is also a straight line[2].

In 1641, Guldin used the same pre-condition as Clavius for his attempt at proof. It is not by coincidence that it was often the Jesuits who focussed on this question, since Euclid played a significant role in the development of their educational system into a school that encouraged sharp-witted debate. Afterwards, many philosophers dealt with this problem up to the end of the $18^{th}$ century, though it was rarely the leading mathematicians (as already mentioned, elementary geometry had become old-fashioned). In cases in which they did take part, their "contributions" remained far below their usual level of excellence, e.g., Euler [Belyi 1968] or Wallis. The latter was under certain external pressure. He was in charge of a chair founded by Sir Henry Savile, an office predecessor at Oxford. Savile had connected the founding of the chair with the condition that each one in charge of the chair would have to address the two afore-mentioned "weak links" of 'Elements', which he referred to as "naevi" (birthmarks). (Apart from the parallel problem, there is also the notion of ratios in Euclid's work, which at that time was unclear and is to be found especially in the $6^{th}$ definition of Book V and the $5^{th}$ definition of Book VI.) Hence, Wallis also held public lectures on both these topics (printed in the second volume of his mathematical works). The quintessence

[2] Taking a look at the spherical surface could have taught him the lesson that this train of thought is not self-evident: the curves, which are of constant distance there to the geodesic great circles, are circles but no great circles. Unfortunately, the extensive analogies between plane and spherical geometry were rarely consciously perceived.



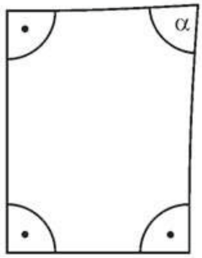

**Illus. 6.4.1** Saccheri's quadrilateral

of his attempt at proof from 1663 lies within replacing the $5^{th}$ postulate with the requirement that there are similar triangles (here meaning: agreeing in all angles) of any size for each triangle. At least, Wallis hinted at a critical distance to the general opinion of the time that axioms are characterised as theorems neither capable of proof nor requiring it. He writes: "However, I do not criticise the fact at all that Euclid did not provide proof. Rather, I would not even complain, if he had stipulated even more unproven conditions; for instance, if he (like Archimedes) had claimed that the straight line is to be the shortest one amongst all lines between the same extremities. In this case, he would not have needed to pre-deliver nineteen theorems before proving that two triangle sides together are greater than the third, and other things, which are actually self-explanatory." [Wallis 1693, p. 674].

Girolamo Saccheri, also a Jesuit and mathematics professor in Pavia, published an extensive and relatively high-class text in 1733 that refers to Savile's challenge in its title, *Euclides ab omnio naevo vindicatus* (The Euclid Free of Any Defect). Herein, he introduced the figure now referred to as the Saccheri quadrilateral (Illus. 6.4.1), whereby $AB = CD$ and the angles at $B$ and $C$ are right. This figure was linked to three possible hypotheses, i.e., angle $\alpha$ in the figure could be acute, obtuse or right.

The latter case is equivalent to the parallel postulate (we recognise the correlation to the requirement that the curves of constant distance from a straight line be straight lines themselves.). The second case also applies to spherical surface, but verifiably not to the plane. Saccheri made two failed attempts to show that the hypothesis of the acute angle is contradictive. However, he had thereby already discovered a series of theorems of the late so-called absolute geometry (those theorems that apply equally to Euclidean and non-Euclidean geometry and, thus, are independent of using or dismissing the parallel postulate).

It is to Friedrich Engel and Paul Stäckel that we owe the great accomplishment of having brought back into the general consciousness the extensive pre-history of non-Euclidean geometry, which had finally been accepted after almost having been forgotten by the end of the $19^{th}$ century, and that the important early texts, apart from Wallis, and particularly from Saccheri




**Illus. 6.4.2** Lambert's quadrilateral

and Lambert, were rediscovered and made available in German translation. Their bibliography of texts on the parallel problem covers over one hundred entries from between 1557 and 1800 (distributed very unevenly in time and accumulated towards the end of the stated period), amongst them the above-mentioned, but not,for example, Euler's texts, which were still unknown back then. The doctoral thesis by Georg Simon Klügel from 1763, inspired by A. G. Kästner, on the history of the parallel postulate was a great preparatory work for this undertaking. At that time in 1766, the most important work by Lambert had not even been written yet (it was published posthumously in 1786), whereas Lambert knew Klügel's doctoral thesis and was probably inspired by it to engage with the problem himself.

Lambert introduced a figure that was similar to Saccheri's, but easier to handle for the matter at hand, namely the one shown in Illus. 6.4.2. It was also shown that the assumption that $\alpha$ is obtuse is not compatible with the remainder of the axioms of Euclidean geometry, but applies to the sphere. Furthermore, he indicated that the assumption that $\alpha$ is a right angle is equivalent to the parallel postulate. Lambert also tried in several ways to contradict his "hypothesis of the acute angle", but remained unsatisfied with all his results in contrast to his predecessors. He went so far with the conclusion of the hypothesis of the acute angle that he recognised that the area of a triangle must then be proportional to the difference between the sum of angles and 180 degrees and compared this to the fact that the area of a triangle on a sphere is equal to the difference of the sum of angles against 180 degrees multiplied by the radius square. Then he wrote: "I might almost draw the conclusion that the third hypothesis occurs in imaginary spherical surfaces. [i.e., if we take radius $r$ of the sphere to be purely imaginary so that $r^2$ is negative and the symmetrical difference of the sum of angles turns into a defect]. At least, there always has to be some reason why it [the hypotheses of the acute angle] cannot be overruled in plane areas as easily as it was with the second [hypothesis of the obtuse angle]." (English translation of German text by [Engel/Stäckel 1895, p. 203])

Due to his remarks on the applicability of the hypothesis of the acute angle on a spherical surface of imaginary radius, Lambert had indeed approached non-Euclidean geometry more than Bolyai and Lobachevsky, who are now accepted as the founders of non-Euclidean geometry. However, in contrast to Lambert, they did not offer anything else other than largely extended edited conclusions based on the negation of the parallel postulate and their inner certainty that this theory did not yield any logical contradictions, but only collided with everyday experience or rather with the acquired prejudices. In contrast, Lambert would have had a *model* for this theory if he had just felt confident enough with his subject matter. It seems, though, he was not, since by no means did he conclude his efforts to find a contradiction at the cited location and left everything unpublished while alive. Johann III Bernoulli published this manuscript in 1786 as the first of a series from Lambert's legacy in the *Magazin für reine und angewandte Mathematik* (Magazine for Pure and Applied Mathematics), which only existed for a short period of time since it seems that it did not provoke any interest and was completely forgotten over the course of time. The publisher of the magazine, C. F. Hindenburg, made a comment regarding this matter: "I have ruled differently, based on versatile experience, that what is claimed, i.e., that the proof of Euclid's axiom can easily go so far that that which remains not just appears apparently correct, but also seems to be made up for and that the proof could be supplemented by this, namely: that what still needs to be proved seems to be *almost nothing* at first; but this apparent little something, if we want to correct it with all our strength, is always the *main issue* if we look more closely; usually it presupposed the theorem or one equal to it [i.e. logically equivalent]..." [Engel/Stäckel 1895, p. 143].

Hindenburg's commentary hints at psychic torment, which, for the participants, was linked to the ongoing competition to solve this problem and for which there was nothing comparable in mathematics before 1800. Such situations occur more and more often later on. For instance, think of Cantor's competing in vain for the continuum problem, the long fight for the four colour theorem (see section 7.8) or the still unresolved P versus NP problem in computational complexity theory[3]. In order to anchor this impression, we quote a section from a letter. The letter was written by Farkas Bolyai to his son Janos in 1820, having discovered that he was also dealing with the parallel problem just as his father had. "You must not investigate parallels in this manner. I know this way until the end – I also have measured this endless night, every light, every joy in my life has been erased by it – I summon you by God! Leave the theory of parallels in peace – You should feel the same repulsion from this as from a sloppy handling, it could cost you all your motivation, health, tranquillity and entire happiness in life. – ... " (Translation by J. Schreiber from German [Bolyai/Stäckel 1913, p. 76]). (This was indeed

[3] It concerns this question whether an algorithmically solvable problem that can be solved by a non-deterministically working system, which can follow many possibilities simultaneously and continue to branch out further and further within a polynomial time limit (in dependence of the format of input) can also always be solved by a strictly sequentially working system in polynomial time.

the destiny that awaited Janos Bolyai. By the way, father Bolyai's exuberant style is due to the fact that, although actually a mathematics teacher, he was also a treasured writer.) To complete the chapter on the fruitless competition for the parallel problem, we must mention that Adrien-Marie Legendre attempted to prove the $5^{th}$ postulate in his textbook *Eléments de géométrie,* first published in 1794. He realised the mistake of his attempt at proof shortly after publishing and fabricated a new proof for the next edition that was repeated over and over again until the $12^{th}$ edition in 1823. Thereby, Legendre found different correlations, which are still important nowadays, e.g., the two theorems named after him:

1. Under the pre-condition of the Euclidean axioms different to the $5^{th}$ postulate, the sum of angles of any triangle is always smaller than or equal to 180 degrees. (Nowadays, we know that we also need the Archimedean axiom for this.)

2. If the sum of angles in any triangle is 180 degrees, so it will be in every triangle.

Furthermore, for example, he could prove without the parallel postulate that we can construct triangles of the same sum of angles for which the sum of both base angles is smaller than $\epsilon$, for each triangle and any small positive error bound $\epsilon$. These triangles seem to be arbitrarily flat. It is indeed difficult to imagine that the third angle does not need to converge against 180 degrees, in contrast to what Legendre had believed and had tried to prove ([Legendre 1844, p. 16], see Problem 6.4.2). This took place in an environment in which the leading French mathematicians had assumed the point of view that it was uncalled for to keep trying to prove theorems that were generally accepted by everyone and the truth of which were granted by experience. Clairaut wrote in his *Eléments de Géométrie* from 1741: "Extensive examinations of things, which our healthy common sense already decides beforehand, are redundant and only serve to obscure the truth and to scare the reader off." (English translation of German text in [Engel/Stäckel 1895, p. 153] by J. Schreiber). Hence, there was a French anti-Euclid movement, which has continued through Dieudonné and the other members of Nicolas Bourbaki, thus, up to the present day!

Although the parallel problem appears to be either meaningless or a playground for outsiders, laymen and crazy people from the view of many of the most productive mathematicians of the $17^{th}$ and $18^{th}$ centuries working in accordance with the spirit of their time, we see in the painful competition of the participants, if we probe deeply, that something fundamentally new wanted to reveal itself, i.e., a deeper insight into the nature of mathematical proofs, which only led to the realisation of the formal character of mathematics (even then still incomprehensible and not fully understood by many mathematicians) at the end of the $19^{th}$ century, the clarifications of the notions axiom system, theory, structure, model, conclusion, independence, etc. We will again quote Lambert, who saw further here than his contemporaries as a logician and methodologist. If they had only read and understood! "And since Euclid's postulate and remaining axioms have been expressed now in words, we can and should demand that we can never call upon the matter itself but present the proof symbolically – if possible. In this respect, Euclid's postulates are equal as just as many equations, which we have already been faced with and from which we must derive *x, y, z & c* without looking back at the matter itself." [Engel/Stäckel 1895, p. 162].

The final solution of the parallel problem in the $19^{th}$ century was (despite several attempts by Lobachevsky) completely meaningless for the applications of mathematics. However, it made the greatest contribution to recognising the true nature of mathematics. This meant simultaneously the end of the concept that mathematics is a natural science and, above all, that geometry was the theory of the real and only possible physical space. We want to return again to the analogy of the painful competition for the continuum problem, the four colour theorem and the P versus NP problem: Why was the first painful? As a result of it, mathematicians had to say their goodbyes to the much-loved concept that real numbers, the apparently so familiar original matter of classic mathematics, could each be described completely unambiguously by formal means. Why was the second painful? Mathematicians had to accept that there are problems that cannot principally be solved in an "elegant" manner, in other words, in a manner that was straightforward for common sense and doable in human time, although they were solvable. This bade farewell to the long practised concept that, in the end, truths are always simple and nice once one has found the right path to them. Why is the P versus NP problem painful? We do not know yet, but one day it will open the door to a fundamental new cognition, and then we will know.

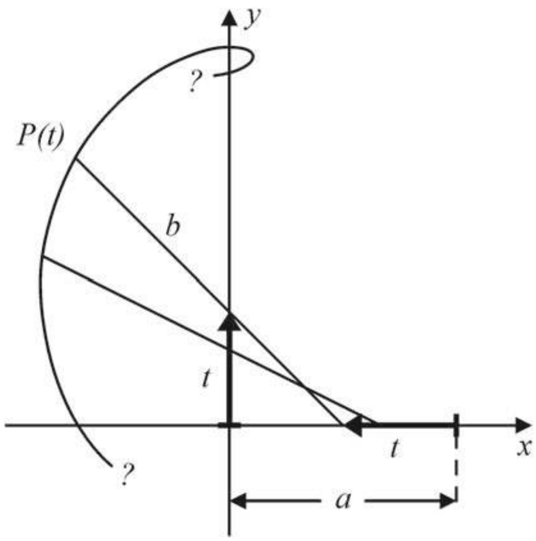
**Essential contents of geometry in the $17^{th}$ and $18^{th}$ centuries**

| | |
|---|---|
| 1591–1661 | Gerard Deargues: founding of projective geometry |
| 1598–1647 | Bonaventura Cavalieri: *Geometria indivisibilibus...* (1635), principle of calculating areas and/or volumes by deconstructing in "infinitely thin" parallel sections |
| 1596–1650 | René Descartes: *La Géométrie* (1637), founding of coordinate geometry: classification of curves according to their generation by motion |
| 1602–1675 | Gilles Personne Roberval: method of arbitrarily small discs to calculate area and/or volume |
| 1607–1665 | Pierre de Fermat: *Ad locos planos et solidos isagoge* (1636), introduction to coordinate method: description of plane point sets (curves) by means of algebraic equations (in two variables), algebraic classification of conic sections, Fermat's principle |
| 1613–1672 | Jan de Witt: Extension of the coordinate method to include the three-dimensional case |
| 1623–1662 | Blaise Pascal: *Essay pour les coniques* (1640) continuation of projective geometry, Pascal's theorem |
| 1643–1727 | Isaac Newton: parameter representation and classification of curves |
| 1646–1716 | Gottfried Wilhelm Leibniz: curve geometry and area calculations with infinitesimal methods |
| 1667–1733 | Giralomo Saccheri: *Euclides ob omnio naevo vindicatus* (1733): first approaches to non-Euclidean geometry |
| 1707–1783 | Leonhard Euler: *Introductio ad analysin infinitorum* vol. II (1748) (application of analysis to geometry), treatises on spherical trigonometry and map projections |
| 1713–1765 | Alexis Claude Clairaut: Examinations of spatial curves, moon and planet orbits |
| 1718–1777 | Johann Heinrich Lambert: Approach to non-Euclidean geometry, texts on perspective, access to projective geometry |
| 1746–1818 | Gaspard Monge: *Géometrie descriptive* (1798), development and fostering of descriptive and constructive geometry |

## 6.5 Problems to 6

**Problem 6.1.1:** Dürer's conchoid ("shell curve")

1. Describe Dürer's conchoid by means of a parameter representation. The parameter chosen by Dürer himself suggests itself (naturally, since he constructs the curve pointwise, only for equidistant, discrete values), we will name $t$. Furthermore, $a$ is the positive distance of the starting point of t from the origin on the $x$-axis, $b$ the length of the "pole producing" the curve (Illus. 6.5.1). Angle $\alpha$ between pole and $x$-axis suggests itself as an auxiliary parameter (which is expressed subsequently by $t$).



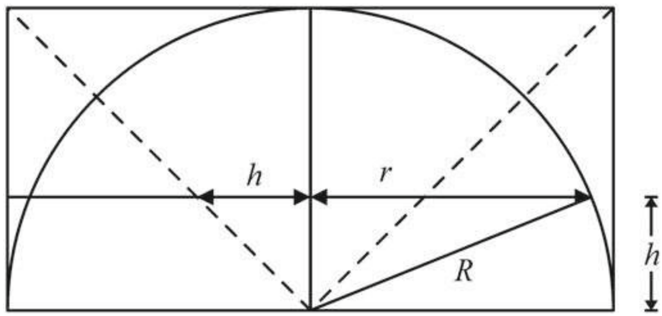

**Illus. 6.5.1** Figure to Problem 6.1.1

1. Show without calculation by elementary considerations that

   a. The maximum of the $y$-values of the conchoid is $b$ and exactly assumed for $t = a$
   b. The curve asymptotically approaches the straight line $y = b/2 \cdot \sqrt{2}$ for $t \to \infty$ (equal in meaning to $x \to -\infty$) and the straight line $y = -b/2 \cdot \sqrt{2}$ for $t \to -\infty$ (equal in meaning to $x \to \infty$), whereby it remains equal in both cases above.
   c. Determine those curve points in which the conchoid has tangents, parallel to one axis, in dependence of $a, b$ by means of the derivation $x'(t)$ and $y'(t)$.

Tip: Dürer's conchoid (shell curve) is also addressed on p. 58 in [Brieskorn/Knörrer 1981]. There, you can also find an answer to the further question of how the curve changes in dependence of the variable ratio of line segments $a$ and $b$. This ratio as second independent variable $v$ yields a "shell area" above the $x$-$v$-plane, from which another "shell curve" is cut out for every constant $v$.

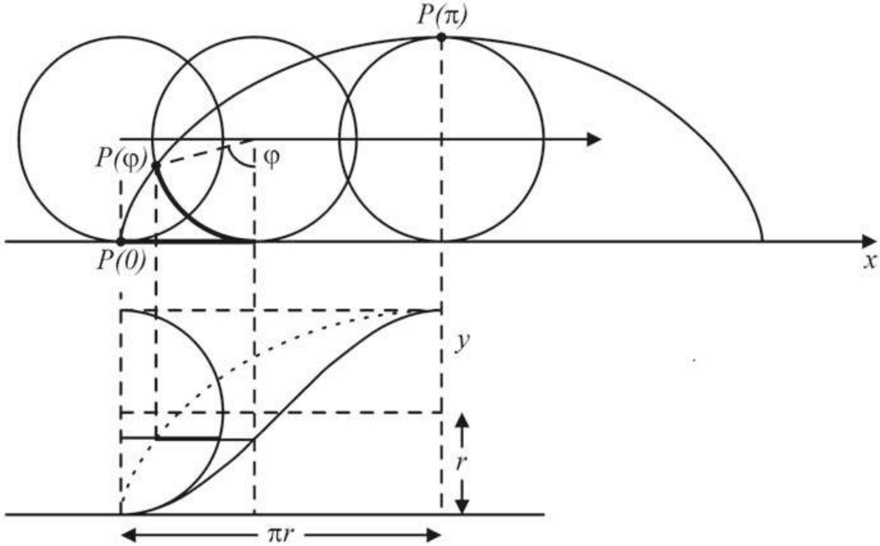
**Problem 6.2.1:** Sphere volume according to Democritus/Cavalieri
Imagine a semi-sphere with the circular area facing down and embedded in a circular cylinder with equal base, the height of which equals radius $R$ (Illus. 6.5.2). A cut made in any height $h(0 < h < R)$ through the difference-solid cylinder – semi-sphere parallel to the base yields a circular ring with outer radius $R$ and inner radius $r$. Recalculate that its area equals the area of the circle of intersection, with a circular cone that is put into the cylinder with its apex facing down at this height. Thus, from Cavalieri's principle follows that this cone and the difference-solid are equal in volume. If, consequently, the volume of the cone (1/3 base · height) is known, we obtain the semi-sphere volume 2/3 base · height, i.e., $\frac{2}{3}\pi r^3$.




**Illus. 6.5.2** Figure to Problem 6.2.1

**Problem 6.2.2:** Area under a cycloid arc

1. Describe the curve by means of a parameter representation, whereby the marked roll angle $\varphi$ shall serve as parameter.
2. Roberval imagined the motion, as suggested in the lower part of the picture, i.e., originating from a pure sine oscillation around the horizontal orbit of the centre and an oscillation of the orbital point to the back and/or put together in the front after passing the highest point. (This method to explain orbital curves by means of overlapping different motions is characteristic for Roberval.) Since the oscillation towards the base straight line is $r \sin\varphi$ for every parameter value, it is, according to Cavalieri's (and/or Roberval's) principle, the club-shaped correction, which




**Illus. 6.5.3** Figure to Problem 6.2.2 shows the development of a common cycloid by rolling a circle off onto a straight line

has to be attached to the area beneath the sine curve equal to the area of the semi-circle. However, the area beneath the sine curve is $\pi \cdot r^2$ due to symmetrical reasons (the piece missing under the straight line $y = r$ is congruent to the piece overlapping on the right). Consequently, the area below the cycloid arc between start point and apex is $\frac{3}{2}\pi r^2$ and the area below the entire arc $(0 < \varphi < 2\pi)$ is the double there, again due to symmetrical reasons.

3. Solve the same problem by means of integral calculus and compare the results. Roberval's approach is "heuristic", but reveals the reason for the surprisingly smooth result.

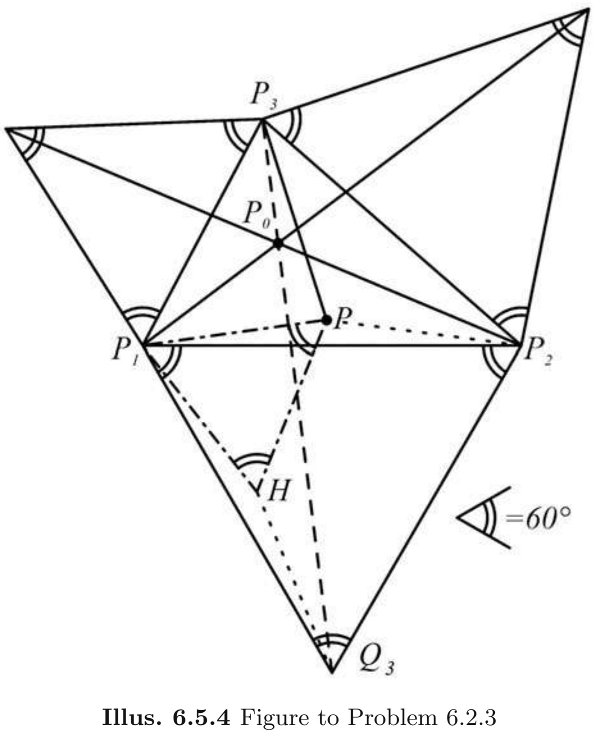
**Problem 6.2.3:** Fermat's problem, Torricelli point

We study Fermat's problem to determine point $P_0$ for $n$ given points $P_i$ $(i = 1, ..., n)$, for which the sum $f$ of the distances $P_0 P_i$ becomes minimal, first for any $n$ and in the $k$-dimensional Euclidean space.

a) Represent the distance sum $f$, which is to be minimised, as a function of $k$ coordinates $x_1 ... \, x_k$ of a variable point $P$ and form the $k$ partial derivatives. The necessary conditions for an extreme value yield that the wanted point $P_0$ can, at best, be located where either all these derivations disappear or at least one of them does not exist. The latter applies if $P$ equals one of the points $P_i$. The first condition leads to the fact that the vector sum of the unit vectors directed from $P$ toward point $P_i$ must be zero.

b) The existence of the wanted minimum point results from the fact that $(k$ - $1)$–dimensional subsets of the points $P$, for which the distance sum has a constant value $s$, tend with increasing $s$ to a $(k-1)$– dimensional sphere, the centre of which is located amidst the given finite point set. Illustrate this for the case $k = 2$. (In this case, the curves of constant distance sum $s$ become more and more similar to the circle for great $s$.) Since the $k$-dimensional "almost-sphere" of all $P$ with the distance sum $\leq s$ is closed and restricted, the continuous function $f$ must have an absolute maximum there (which is assumed everywhere on the edge) and, thus, inside an absolute minimum.

c) The necessary condition obtained in a) states for the plane $(k = 2)$ for $n = 2$ that the wanted point can only be located on the line segment $P_1P_2$ (of course, every point on the line segment fulfils the condition); for $n = 4$ that the wanted point can only be located on the intersection of the diagonals, if there is one, i.e., if the four given points yield a convex quadrilateral and/or that it must be the one point of the four that is located in the inside of the triangle formed by the other three. For the case $n = 3$, on which Fermat had only focused at first, the condition found in a) states that the three unit vectors must yield an isosceles triangle, i.e., For $n > 4$ and/or in case of $k > 2$ already for $n > 3$, the closed polygon




**Illus. 6.5.4** Figure to Problem 6.2.3

of $n$ unit vectors still has too many "degrees of freedom" in order for the necessary conditions delivered by differential calculus to be able to yield any useful information about the location of the wanted point. (However, we consider it approximately useful for practical purposes by gradually progressing from an estimated starting point towards the greatest decline of the distance sum.)

d) Take the case of $n = 3$, which in the plane is not as trivial as $n = 2$, 4, but still solvable by means of elementary geometry. Cavalieri and Torricelli independently found fine solutions for this original problem shortly after the problem had been released (by Mersenne). Part of this was the explanation that the "Torricelli point" (so called ever since then) of the triangle $P_1P_2P_3$ is only located inside the triangle if all angles of the triangle are, at most, 120 degrees. Otherwise, especially in the case of collinearity, that point of the three in which the possibly even degenerated triangle is too flat is the Torricelli point itself. Prove this! Illus. 6.5.4 shows the solution of Fermat's problem, which can be justified and constructively realised most easily. For any point $P$ inside triangle $P_1P_2P_3$, traverse $P_3PHQ_3$ has distance sum $PP_3+\ PP_1+\ PP_2$ as length. This distance sum is always larger than the direct line segment $P_3Q_3$ or is only equal if $P$ is located on this line segment. Give reasons for this (particularly: why is $HQ_3 = PP_2$ ?) and conclude from this that $P_0$ is the Torricelli point. How would one consequently have to construct it?

e) Here is a tip for how to find the necessary condition (*) from c) without differential calculus: if an ellipse with the foci $P_1, P_2$ intersects a circle around $P_3$, none of the intersections can be the minimum point, since we enter the circle area along the ellipse and, thereby, decrease the distance of $P_3$, whereas the distance sum towards $P_1$ and $P_2$ remains constant. Hence, the centre can only be located where such an ellipse and such a circle touch each other. Since this consideration also applies to all other divisions of the three points in a circle centre and two elliptical foci, it follows that all three angles are equal. Concerning the further history of Fermat's initiated problem, which is still exciting and newsworthy, and the extensive literature on this topic, see [Schreiber 1986]. Only in 1988 was it proven that the Torricelli point for more than four points in the plane cannot be constructed generally by means of compass and straightedge [Bajaj 1988]. [Mehlhos 2000] presents a much simpler proof for this and the analogous statement in case of $k = 3$, $n = 4$. See Problem 7.3.6.

f) Lightly modifying the idea sketched above indicates that the Torricelli point of four points in space, which yields a tetrahedron that is not too flat, can only be located where an ellipsoid revolving around two of the points (as foci of the revolving ellipse) and an ellipsoid revolving around the other two points touch each other. As shown by Mehlhos, the coordinates of this point cannot generally be represented by quadratic radicals anymore, i.e., the coordinates of the Torricelli point cannot be constructed with compass and straightedge based on the given points.

g) Prove that the Torricelli point of $n$ points, which have a (mirror or central) symmetry, must be located on the symmetry axis and/or in space on the symmetry plane and/or on the centre of the symmetrical mapping.

**Problem 6.2.4:** Construction of the circles of principal curvature of an ellipse

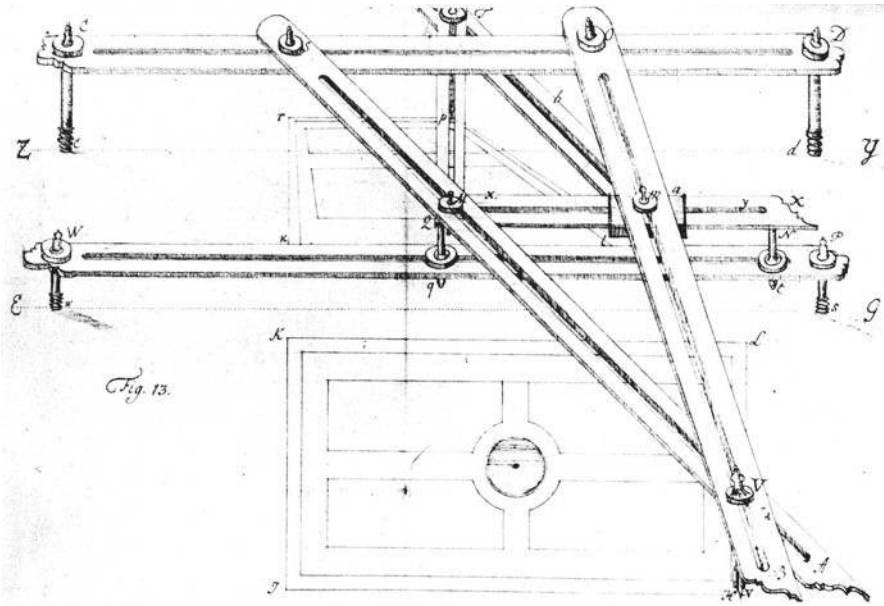
a) Based on the general formula for curvature radii of a plane curve given by a parameter representation, derive these radii for an ellipse in the vertices.
b) Consider how we can practically construct the circles of principal curvature from the given position of the vertices with compass and straight edge. (Concerning ellipses with an eccentricity not too large, a marginal adjustment between these circular arcs by eye normally suffices in order to draw the ellipse sufficiently exactly for most purposes, such as in descriptive geometry.)

***Problem 6.2.5:** Flatting of Earth

The two equations obtained in 6.2.4 for the radii of principal curvature of an ellipse in dependence of the semi-axes $a, b$ can easily be resolved according to $a$ and $b$. The flattening of the Earth ellipsoid determined during the arc measurement expedition of the French Academy of Sciences in 1735/37 to South America (Lacondamine, Bouguer) and Lapland (Maupertuis, Celcius) was based on this. By geodetically measuring the arc for a difference of latitudes of, for example, one degree on an arc belonging to a meridian at both the equator and the North Pole, we obtain (by adapting Eratostenes' method) the radii of the circles of principal curvatures. Since the flattening is not very strong and the curvature only changes slowly close to the vertex, we can take values measured in the greatest possible proximity to the North Pole and/or the equator as good approximations for the exact principal curvature. Subject the conducted calculation to an error analysis as already done by Celsius in his time in order to stand up to unjustified objections from the Cartesians, who, in favour of the Cartesian theory of the extended ellipsoid of revolution, doubted the results of the expeditions.

***Problem 6.3.1:** Lambert's perspectograph

a) Illus. 6.5.5 shows Lambert's drawing of the perspectograph he invented in 1752. Turn this "technical drawing" into a representation of the geometrical principle and use it to explain how it works.
b) How could we use a detail featured in Lambert's drawing that ensures the equality of two variable line segments, always perpendicular to each other, to complete Dürer's apparatus for drawing the conchoid?



**Illus. 6.5.5** Figure to Problem 6.3.1: Lambert's drawing of the perspectograph

***Problem 6.3.2:** Relief perspective and Desargues' theorem

Illus. 5.3.8 shows the principle of relief perspective. Having selected a frontal plane and a parallel vanishing plane as well as a visual point $A$ in the half-space averted from the vanishing plane regarding the frontal plane, the half-space behind the frontal plane is mapped injectively onto the layer between frontal and vanishing plane, so that the image point $f(P)$ of each point $P$ of the original half-space is located on the visual line from $A$ to $P$ and, hence, provokes the same impression in the eye as $P$. Thereby, every bundle of parallel straight lines $g$ has a common vanishing point $F_g$ in the vanishing plane, in which the images of the parallel straight lines meet. By arbitrarily choosing a vanishing point $F_g$ on the vanishing plane and/or equivalently by arbitrarily choosing a point $P_0$ on the frontal plane we can, given point $P$, construct image point $f(P)$ between the frontal and the vanishing plane or, given image point $Q$, the original point $P$ with $f(P) = Q$. Consider how the proof that each constructed point is independent of this selection must necessarily lead to Desargues' theorem (applicable in the general spatial case) whereby the infinitely distant straight line of intersection of front and vanishing plane occurs as axis of perspective and image point $Q = f(P)$ as centre of perspective. From the viewpoint of projective geometry, nothing would change if we chose a straight line located in the finite as axis of perspective, i.e., we could also map the half-space behind the frontal plane onto a wedge-shaped relief between front and vanishing plane.

Since the practical application of relief perspective was flourishing at Desargues' time, it is possible (but, unfortunately, cannot be proven) that Desargues was led to his theorem exactly by the idea sketched above.

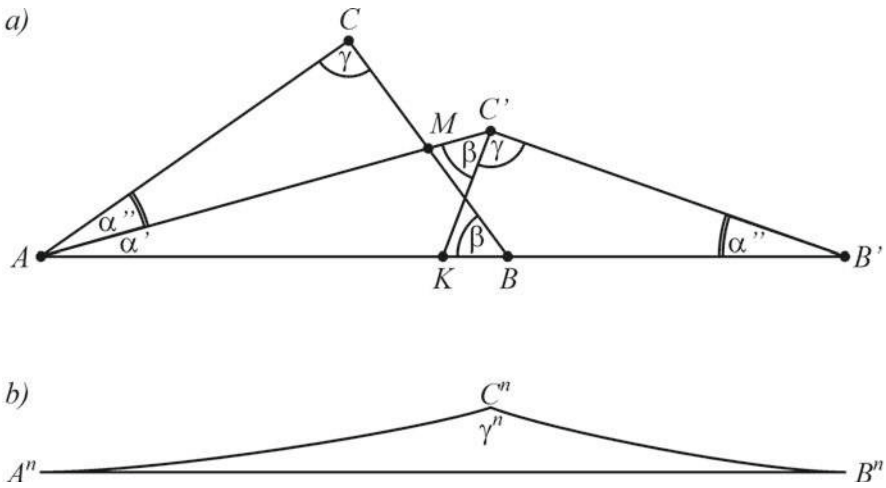
**Problem 6.4.1:** Defect of triangles as area measure

Under the assumption that the sum of angles of triangles is smaller than or at most equal to 180 degrees, the defect of a triangle is the difference of 180 degrees – sum of angles of the triangle. Congruent triangles have, of course, the same defect, based on this definition. Prove the following:

a) If triangle $D$ is deconstructed into two part triangles $D_1, D_2$ by a transversal (from a corner to a point at the opposite side), then defect $(D)$ = defect $(D_1)$+ defect $(D_2)$.
b) If there are any triangles with a positive defect at all, then there are arbitrarily large triangles and arbitrarily small triangles with positive defect.
c) Since, consequently, every triangle contains a part triangle with positive defect, every triangle has, according to a), a positive defect. Due to the invariance of the defect during motion and the additivity (a), the defect can serve as a measure of area. We just have to standardise it, i.e., determine area 1 for a freely selectable triangle.

***Problem 6.4.2:** Legendre's pseudo-proof for the angle sum of 180° (according to Legendre 1823)

Let $ABC$ be any triangle and, thereby, without restricting the generality $BC \leq AC \leq AB$.



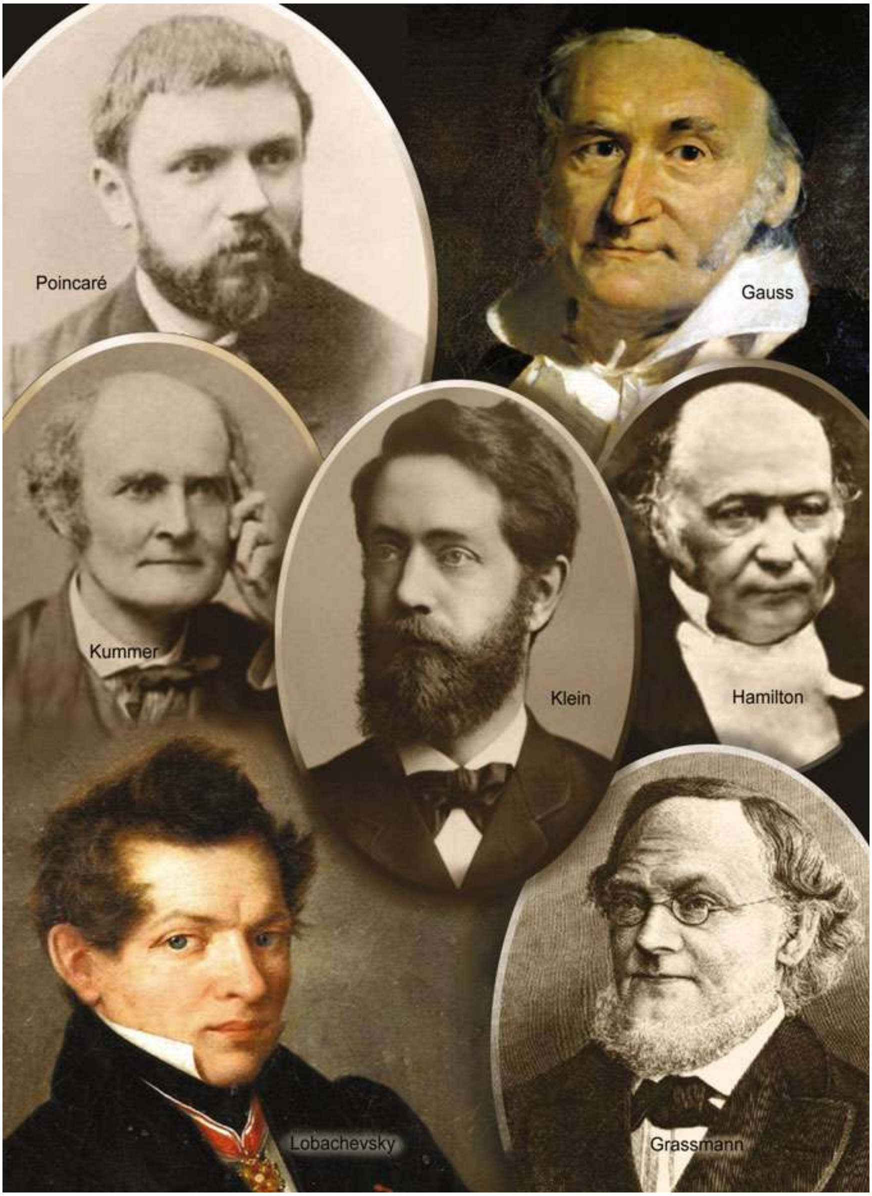

**Illus. 6.5.6** Figure to Problem 6.4.2

According to Euclid's 'Elements', I.18, we can conclude $\alpha \leq \beta \leq \gamma$, based on this and without relying on the $5^{th}$ postulate. Construct $M$ as the centre of $BC, C'$ on the axis $AM$ so that $AC' = AB, K$ on $AB$ so that $AK = AM$ and $B'$ as the duplication of $AK$ beyond $K$ (Illus. 6.5.6a). The reader is asked to prove by only using the theorems of triangle congruence (which are independent of the parallel postulate) that the angles drawn in the illustration and named the same are indeed equal. As a result, the sum of angles of triangle $AB'C'$ equals the sum of angles of triangle $ABC$. Furthermore, since $B^{'}C^{'} = AC \leq AB = AC'$, as a result of I. 18 applied to the triangle $AB'C'$, $\alpha' \leq \alpha''$ and consequently, $\alpha' \leq \frac{1}{2}\alpha$. By repeating this construction sufficiently enough, we obtain the triangles $A^n\,B^n\,C^n$, so that the following applies to the respective angles:

$$\alpha^n + \beta^n = \alpha^{n-1} \leq \left(\frac{1}{2}\right)^{n-1} \cdot \alpha.$$

Legendre believed to be able to conclude from this that point $C^n$ should converge against the straight line $AB$ if the sum of both base angles converges against zero and, thus, angle $\gamma^n$ should converge against 180 degrees. Since the sum of angles remains constant during the entire procedure, the sum of angles would already have to be 180 degrees in the initial triangle. However, in reality the triangles $A^n\,B^n\,C^n$ already adopt the shape shown in Illus. 6.5.6b in non-Euclidean geometry, whereby their initial defect is preserved, i.e., if the sum of both base angles is sufficiently small, angle $\gamma$ tends against the difference of 180 degrees and the defect of the initial triangle.

# 7 New paths of geometry in the 19$^{th}$ century






C.J. Scriba, P. Schreiber, *5000 Years of Geometry*,
DOI 10.1007/978-3-0348-0898-9_7

The $19^{th}$ century was a great century for geometry, discoveries and inventions in natural sciences and techniques, but also marked by colonialism and social tensions.

| | |
|---|---|
| 1810 | Gergonne's Annales de Mathematiques: first modern math. journal |
| 1813 | Battle of Leipzig ends Napoleon's domination in Europe |
| 1815 | Battle of Waterloo, Congress of Vienna |
| 1822 | Ch. Babbage finalises first program-controlled computing machine for tabling polynomials of the third degree |
| 1822 | J. N. Niepce invents photography |
| 1825 | First train in England (Stephenson) |
| 1826 | First multi-colour print (A. Senefelder) |
| 1830 | July Revolution in France |
| 1833 | Electromagnetic telegraph by Gauss and Weber |
| 1837-1901 | Reign of Queen Victoria, Expansion to the Empire |
| 1838 | F. W. Bessel, F. G. W. Struve and T. Henderson measure almost simultaneously the first fixed star parallaxes |
| 1846-1848 | War between Mexico and USA: Texas, California, Nevada, Utah, Arizona and New Mexico become States of the USA |
| 1851 | First world exhibition in London |
| 1854-1856 | Crimean War |
| 1854 | Electric bulb (H. Goebel) |
| 1861-1865 | Secession War in USA |
| 1862 | Society of Czech mathematicians and physicists: first national union of mathematicians |
| 1863 | London underground |
| 1869 | Suez Channel opens |
| 1870/71 | War between Germany and France |
| 1871 | First volume of *Jahrbuch über die Fortschritte der Mathematik* (First Annual Book on the advances of Mathematics) is published |
| 1874 | Cantor's first publication on set theory |
| 1875 | International metre convention |
| 1876 | Queen Victoria becomes Empress of India |
| 1879 | Carbon filament lamp (Th. A. Edison) |
| 1880 | First use of electric lighting |
| 1884 | International agreement on prime meridian at Greenwich |
| 1895 | Röntgen discovers X-rays |
| 1896 | Becquerel discovers radioactivity |
| 1897 | First International Congress of Mathematicians (Zurich) |
| 1898 | War between Spain and USA<br>Puerto Rico and the Philippines ceded to USA |

## 7.0 Preliminary remarks

At the turn from the $18^{th}$ to the $19^{th}$ century, both the character of mathematics and its external conditions changed fundamentally. The industrialisation beginning in 1770 was the general background for technological development. However, political turmoil in Europe caused by the French Revolution and the following Napoleonic Wars, which conveyed civil ideas to almost every corner of Europe, also contributed greatly to the changes. Next to the local and national academies, institutions of higher technical education were established in many countries for research and the teaching of mathematics. The philosophical faculties of the classical universities, which for centuries had only served as pre-study institutions for students of theology, medicine and law, took on a new function as educational bodies for teachers of higher schools of general education. This led to plenty of new professorial vacancies, especially in mathematics, over the course of the $19^{th}$ century to the formation of the status of "Privatdozent" (unsalaried university lecturer) as well as the founding of institutes and seminars. Examination regulations for teaching candidates were passed in Prussia in 1866, explicitly demanding that prospective mathematics teachers be put into the position of working independently within the main areas of geometry, analysis and mechanics. Many magazines were established, often in connection to local and, towards the end of the century, national associations and societies of mathematicians. Due to publications in the respective national languages of each country, Latin was quickly superseded as the international language of science. However, since this was not accompanied by an increase in the teaching of foreign languages at schools of general education (apart from a widespread knowledge of French), and due to some very strong nationalistic feelings, a (more or less silent) agreement on *one* well-established modern foreign language (as is completely normal for us nowadays) was politically impossible. The rapid increase in the number of productive mathematicians interacted with the suddenly occurrence of language barriers in such a manner that there were clusters of repeated discoveries, parallel developments difficult to follow, and, hence, arguments of priority. An international understanding of the progress of science was maintained until around 1870 by many book translations published shortly after the original release and by just a few linguistically gifted and interested mathematicians. For instance, the Irishman Hamilton easily read English, French, German and several Oriental languages. Gauss began learning Russian in old age. The *Jahrbuch über die Fortschritte der Mathematik* (Annual Book on the Advances in Mathematics) was first published in 1871 (its first year under review was 1868) as the first mathematical body of new books and seminar papers. It seems that the international effects of this German publication have not yet been examined.

Whereas mathematicians had an overview of and edited mathematics as a whole (without ever reflecting on it as a unit), including its most important applied areas, until the end of the $18^{th}$ century, a specialisation within mathematics developed very quickly in the $19^{th}$ century that was only fully comprehended by a few excellent scientists, such as Gauss, Cauchy, Jacobi, Jordan and Poincaré. According to the general consensus, geometry, apart from analysis, was *the* main area of mathematics during the entire $19^{th}$ century, and thus it branched out strongly. Some of its sub-disciplines were heavily cross-linked with other branches of mathematics and also with areas of application outside mathematics; differential geometry, for example, with analysis, geodesy, physics, the general (ultimately even philosophical) space problem, but also still with descriptive geometry and its applications in construction and machine building.

Until the $19^{th}$ century and beyond, elementary mathematical teaching consisted of calculating and, depending on country and type of school, more or less pursued study of Euclid's 'Elements'. In 1773, the rather important mathematics professor J. A. Segner from Halle (Germany) still stressed the role of geometry in accordance with Euclid for the development of logical thinking as well as oral and writing skills in classic languages as part of his foreword to a Latin school edition of 'Elements' (see [Schreiber 1987a, p. 124]). At the University of Edinburgh in the middle of the $19^{th}$ century, lack of awareness of the works of Bolyai and Lobachevsky on non-Euclidean geometry led to far reaching conclusions still being made based on the negation of the parallel postulate as sheer exercises for students to draw strictly logical conclusions [Kelland 1843] [Kelland 1864]. B. Bolzano also started his *Betrachtungen über einige Gegenstände der Elementargeometrie* (Inspection of Some Objects of Elementary Geometry) (Prague, 1804) with the sentences: "It is not unknown that mathematics, next to the widespread use which is granted by its applications to practical life, can also deliver a second hardly minor use, although not recognized at a first glance, by exercising and training the mind, by beneficially promoting a sound way of thinking; a use which the state primarily intends when demanding every academic to study sciences." The mathematical "studium generale", which was still mandatory for all first-year students at the universities of the Habsburg Monarchy around 1840, proved a great torment to the majority of those lucky students. For further information, we refer to reader to the passage taken from the memoirs of the German-Bohemian author Alfred Meissner to be found in the Appendix.

In the meantime, the emerging "Reformpädagogik" (progressive education) had discovered "geometrische Anschauung" (geometrical observation) as an important means of furtherance of education. Without going into detail, we want to refer to the texts by J. H. Pestalozzi, J. F. Herbart, F. Fröbel and A. Diesterweg. However, whereas Fröbel demanded the observation of simple geometrical solids and taking those into one's hands, Diesterweg went so far as to have his classrooms darkened in order to heighten imagination. Such pedagogical differences of opinion also affected researching geometers, such as Jakob Steiner, who, as one of Pestalozzi's students and followers, was a particularly argumentative representative of the style within geometry now called “synthetic” in contrast to “analytic”, or H. G. Grassmann and B. Riemann, who had been heavily influenced by Herbart. Whereas “analytics” emphasised the generality of the results achieved by algebraic and differential calculus and the elegance of derivation, “synthetics” criticised the accompanying neglect of the concrete case and geometrical observation. Gauss, who was more of an analytic, wrote in a discussion of the third French edition of Monge’s *Géométrie descriptive*:

“Indeed the investigations... [on spatial geometry] were dealt with exquisitely in modern times by means of analysis and, thus, simultaneously withdrawn from geometry, which only uses immediate observation. We also cannot deny that the advantages of an analytic treatment in contrast to a geometrical one, its briefness, simplicity, its uniform way, and especially its generality, usually reveal themselves even more resolutely depending on how difficult and complex the examinations are. However, in the meantime it becomes more and more important to cultivate the geometrical method continuously...Due to these reasons, it is a pleasure to see that in recent decades some French geometers have started cultivating this part of geometry [addressing] the ratios of points and lines that are not located in a plane, of different planes against each other ... as a special discipline under the name of géométrie descriptive. Above all, we must praise the following work on this science for its great clarity and precision of its recitation, a well-ordered transition from easy to difficult and the wealth of new views and successful demonstrations and, hence, recommend studying it as strengthening food for the mind, which is why without a doubt we can contribute a plenitude to the revival and maintenance of the real geometrical spirit sometimes missed in new mathematics.” (Translated into English by J. Schreiber based on German original in [Gauß a, vol. IV, p. 359f])

Moreover, we want to mention the unmanageably large number of geometrical textbooks for craftsmen and technicians of very different kinds (see text by F. Wolff in Appendix), but also the origin of the great model collections of different universities, which nowadays again enjoy an increasing appreciation, despite having been condemned to gathering dust for a long time in the central decades of the $20^{th}$ century [Fischer 1986], [Böhm 1991]. Their origin lies in France, where the geometrical materials of observation created since Monge’s time belong to the protected national relics at the museums in Paris, those that have survived the ages, anyway. Thus, looking at the broader context, geometry of the $19^{th}$ century offers a split picture between, on one hand, picture-less printed pages always difficult to read, far from any concrete view, and full of formulae, and on the other, a continuously polished culture of representing and modelling.

Whereas the mathematical contents were “generally understandable” until the end of the $18^{th}$ century, any attempt to lead the history of mathematics or one of its branches beyond this point harboured a risk of either presuming too much expert knowledge of the readers or going off course in the description of the developing mathematical contents in such a manner that the historical view was lost. Additionally, some developments that were marginal from the viewpoint back then led to important new theories and applications in the $20^{th}$ century, whereas many aspects that were at the centre of attention in the $19^{th}$ century had become old-fashioned in the meantime, broken off from mathematics as technical special knowledge, or become meaningless due to intra-mathematical or technical advances.

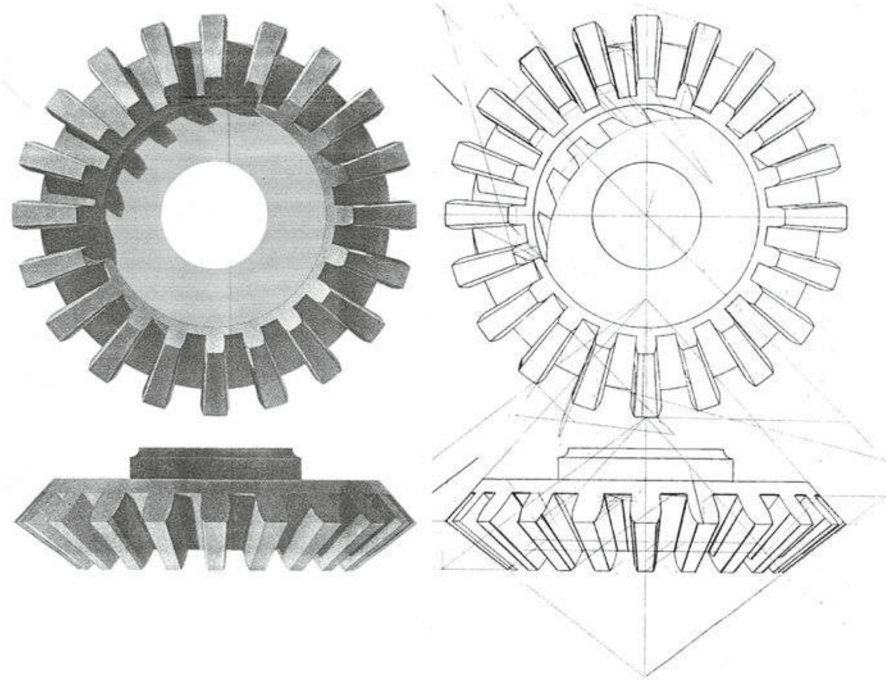
## 7.1 Descriptive and applied geometry

Paris at the beginning of the $19^{th}$ century seems to have been one of the few places where analytic and synthetic geometry cooperated under Monge's overwhelming influence, at least initially. Thereby, we should keep in mind that Monge only taught sporadically after 1789 due to the large number of offices he held and frequent extended absences (Italy in 1796/97, Egypt in 1798/99), not to mention his intensive engagement with chemistry apart from mathematics. Nonetheless, his rich contributions to both analytic and synthetic geometry were usually swiftly incorporated into his constantly improved and newly edited textbooks, many aspects of which were devised and added by his students. Monge's lifetime achievement harbours many pearls, which nowadays belong to "geometrical folklore" (in other words, nobody can remember where they come from), such as the notion of a director circle of a conic section, which allowed for the viewing of all conic sections in a perspective completely different from the algebraic or projective viewpoint (Problem 7.1.1). His mathematical productivity decreased rapidly after 1805. M. Chasles, himself a graduate of École Polytechnique, wrote in 1837: "It still continues to be told in the story of the polytechnic school that Monge understood to an unheard-of extent how to clarify the assembled shapes of the expansion in space and to sensualise its general relations and hidden properties by sole means of his hands, the motions of which followed his will wonderfully and were always accompanied by the speaker's true eloquence, precision, wealth and depth of ideas." ([Chasles 1837, p. 209], translated from French original into English)

A long list of names came out of the school founded by Monge: J.-B. Meusnier, J. N. P. Hachette and L. Carnot (the oldest students from the time in Mézières), E. Bobillier, Ch. Brianchon, B. Brisson, Ch. Dupin, S. F. Lacroix, G. Lamé, Th. Olivier, L. Poinsot, J. V. Poncelet, just to name the most important and gifted geometers of this time, which was also rich in noted names in respect to other fields of mathematics. J. Gergonne, who founded France's first public mathematical magazine *Annales des mathématiques pures et appliquées* after the journal of École Polytechnique, also called himself Monge's student, even though he had not studied in Paris. This magazine quickly became the main body of the style established by Monge, who did not disdain analytical methods, but for whom visual ability and practicality remained the core. Apart from the already multiply mentioned École Polytechnique founded in 1794 and the University the École Normal Superieure (specialised in educating teachers), the École des Ponts et Chaussées (bridges and streets), the École des Arts et Metiers (similar to higher vocational schools) and the École des Mines (something like a mining academy) all played a role in fostering geometry in Paris. Most significant French mathematics professors taught at several of these institutions at the same time after 1789, whereby they were partially supported and represented by “adjuncts” (assistant professors) and assistants. Monge, Carnot and Poncelet especially could only devote a portion of their time to mathematics due to their military and/or public offices.

Brisson and Hachette published several edited and extended editions of Monge’s main textbooks (apart from ‘Descriptive Geometry’ and ‘Application of Analysis to Geometry’, published under different titles). Brisson added appendixes on central perspective and shadow constructions to *Géométrie descriptive*. The latter did not primarily serve artistic purposes back then (although Monge had already referred to the use of descriptive geometry for artists at suitable occasions in his lectures), but was meant to improve comprehensibility and/or suggestiveness of technical drawings (Illus. 7.1.1). Olivier established the construction of geometrical models. Hachette rendered great services to the application of descriptive geometry regarding the technical drawings of machine building. *Géométrie perspective* (1828) featured alternatives to the method of assigned normal views preferred by Monge.

It was written by engineer B. E. Cousinery, who, on one hand, related more closely to central perspective and, on the other hand, represented the basic figures of space more symbolically, e.g., a point $P$ is represented by an orientated circle in the plane of projection, the centre of which is the foot of the perpendicular of $P$, the radius of which “codes” the distance of $P$ to the plane of projection and its sense-class “codes” the half-plane (and/or the algebraic sign) of this distance. (This approach was developed and called “cyclographics” by W. Fiedler in Germany in 1882.) The distance circle in central perspective occurs first in this context as a special case in Cousinery’s works. Thereby, the location of the visual point regarding the plane of projection is represented by a circle, the centre of which is the main point and the radius of which is the visual distance. In another description derived from central perspective, Cousinery represented planes by their straight line of intersection with the plane of projection and its “horizon” regarding a visual point $A$ (i.e., the straight line of intersection of the parallel plane placed through $A$ with the plane of projection) located outside the plane of projection, and proceeded analogously with the straight lines of space. Such ideas played no permanent role in applications, although they temporarily occupied a wide area in the textbooks on descriptive geometry, such as by Müller-Kruppa and Krames. However, they prepared the concept of the later, so-called “transfer principle”, the nature of which was to prove geometrical theorems by means of reinterpretation of the participating objects (see, for example, Problem 7.1.3).



**Illus. 7.1.1** Conic gear in top and front view with exactly constructed shadow. [Wolff 1840] (Descriptive geometry and its applications: Guide to teaching at the Royal Institute for craftsmen education, Berlin)

### Expansion of descriptive geometry

The mighty storm of geometry began to weaken in France in the middle of the $19^{th}$ century and the majority of significant French mathematicians turned towards other areas of mathematics, mainly due to the fact that the multitude of individual results and applications could not be tamed by categorising new theories and that Monge's strong tradition had hindered the mathematicians from turning towards issues of non-Euclidean geometry or inner differential geometry. M. Chasles, who had made important contributions to projective geometry in particular, wrote at the end of his famous *Aperçu historique...* from 1837: "In old geometry, the truths stood isolated, new ones were difficult to conceive of or to create and not every geometer who wanted could become an inventor...At present, everyone can absorb some random truth and subject it to different general principles, so that it is possible to replicate the number of new truths almost infinitely." (l.c., p. 263).

The theory of descriptive geometry spread quickly across Europe. For instance, the first Russian technical university was founded as an institute for engineers for traffic routes (construction of streets, bridges, etc.) in St. Petersburg in 1809 and the classes on descriptive geometry were absorbed by Faber and Potier, two of Monge's students, "imported" from France. Faber wrote the first crucial textbook for Russia in French in 1816, translated into Russian by Sevastyanov in the same year. (Later, Sevastyanov wrote textbooks himself and is accepted as a type of "Russian Monge" in Russia nowadays.) The first German textbook on descriptive geometry was composed by senior construction director F. Weinbrenner from Karlsruhe (1810). A book by Guido Schreiber, much richer in content, followed in 1828 in close connection to Monge. Further extensive textbooks were written by, amongst others, Ludwig Burmester (also on illumination geometry and kinematics), Karl Wilhelm Pohlke, and Christian Wiener (1884). The latter features a description of the historical development rich in detail. Further centres of descriptive and applied geometry were formed in Austria/Hungary (Vienna, Graz, Prague, Brno and others) and Italy. However, Monge's hope that descriptive geometry would become not just an essential aid and means of understanding for techniques but also a permanently fruitful area of mathematics did not come true. It fell very quickly into the hands of specialists who hardly had any contact with other areas of mathematics and, thus, had neither opportunity nor ability to address unresolved problems under new and general aspects. The fact that descriptive geometry was dropped from mathematics – too early as we now know – became apparent due to the fact that it was often an independent teaching or examination subject next to mathematics and had many peculiar notions and terms that were not in harmony with the remainder of mathematics.

Apart from the multiplane method and central perspective, descriptive geometry in the $19^{th}$ century covered the different forms of parallel projection (also called axonometry), the reconstruction of the spatial original image from two central projections (later referred to as photogrammetry), relief perspective and illumination geometry. The origins of photogrammetry can already be found in Lambert's works. Monge had also addressed them in his lectures in anticipation of practical application when mapping out the field from the air. (Remember that the first manned balloon flights took place in France from 1783 onwards.) Illumination geometry originated in the first half of the $18^{th}$ century (P. Bougner, Lambert, then Monge). Its central notions are isophote (curve of same luminance strength on a curved surface at given central or parallel illumination) and isopheng (curve of apparently same brightness, additionally dependent on an assumed observer's standpoint). Whereas illumination geometry was first meant to help design pictures more realistically, it was later also used in different applications of physics and techniques, such as for the optimal illumination of rooms or the even drying of varnished surfaces. We refer the reader to [Bohne 1989] for a more detailed history of this field.

Interesting approaches to a mindset that then developed mainly in entirely different areas of mathematics (such as theory of errors, interval mathematics, stability) can be found in applications of perspective in painting and theatre decoration, beginning in a textbook by De la Gournerie (1859). For instance, there was an investigation into which area the observer's eye could be altered whilst ensuring that the distortions when viewing a perspective would not exceed a given level. It seems that De la Gournerie was also the first to pinpoint that the correct interpretation of a picture of central perspective depends on additional information about the illustrated objects (e.g., that we already know that certain line segments are equally long, certain angles are 90 degrees, certain areas are horizontal or vertical, certain objects belong together). Such traces - usually ignored and thought of as insignificant by the classical historiography of mathematics - gain new importance and interest from the view of modern scientific issues (picture recognition, artificial intelligence).

Mathematically speaking, Pohlke's theorem or "principle theorem of axonometry" was a highlight. It is named after Karl Wilhelm Pohlke, who discovered it in 1853, but only published it in 1860. Pohlke was at first "Privatdozent" (unsalaried University lecturer), and then full professor from 1860 onwards, for descriptive geometry and perspective at the Building Academy and the Art Academy of Berlin. It is little known that a special case of the theorem (for orthogonal axonometry) had already arisen from a work by Julius Weisbach, professor at the Mountain Academy Freiberg (Germany), published at a remote location in 1844. This theorem has attracted the attention of mathematicians many times since then (amongst them, prominent ones, such as H. A. Schwarz in 1864), partially due to its theoretical and practical meaning, partially due to its difficulty in being proven by rather elementary[1] means. The big gap between mathematics and descriptive geometry is also made clear by the fact that the theorem, which is highly interesting and indispensible from a theoretical point of view, is not even mentioned in most textbooks on descriptive geometry, let alone proven. It states that we can indicate a spatial orthonormal basis and a direction of projection for each real two-dimensional tripod $OE_1E_2E_3$ (i.e., the vectors $OE_i$ span the plane of projection), so that the given plane tripod is the parallel projection of the spatial one. Thus, this theorem delivers the justification for the procedure common for axonometric sketches of selecting the pictures of three pairwise perpendicular cube edges arbitrarily or "intuitively". Regardless of how we do this, Pohlke's theorem states that there is always one line of sight from which the cube looks exactly as described.

### Further branches of applied geometry

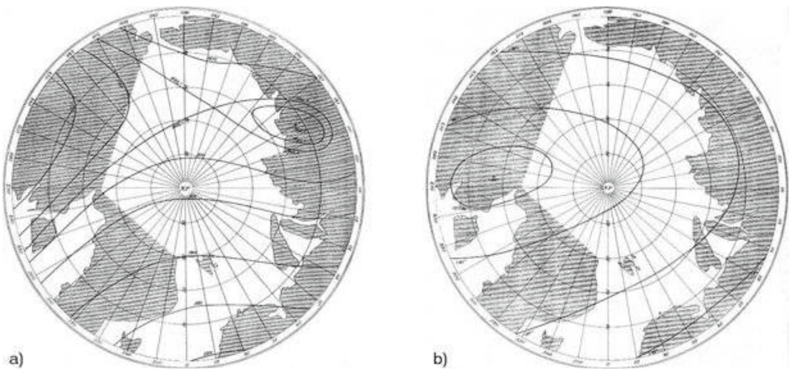
Apart from descriptive geometry, cartography and geodesy, as well as parts of optics and mechanics, also belong to the pronounced applied fields of geometry, the development of which was characteristic for the $19^{th}$ century. Optics ultimately distanced itself far from its classical relation to geometry due to its physical advances and now predominantly uses analysis as its mathematical aid.

[1] [Salenius 1978] gave a very illustrative newer proof (see Problem 7.1.2).

The discovery and technical utilisation of double refraction, polarisation and diffraction attracted enormous interest in crystallography and related questions of regular spatial configuration (more about this in section 7.9). Geodesy and cartography lost their elementary geometrical character and were related more and more to differential geometry. (Some of this will be mentioned in section 7.4.) The theory of maps true to angle (referred to as "conform" since Gauss) and area was approached and made equal to its modern concept. Apart from further net drafts (also by Mollweide, Gauß-Krüger, Tissot and Hammer), new topics for cartography were also made accessible. From a geometrical perspective, the atlas of the magnetic fields of Earth published by Gauss and Weber in 1840 may be especially impressive, since it was the first time that components of a complicated spatial vector field were represented graphically (Illus. 7.1.2). The close link of the detailed geodesic and cartographic exploitation of Earth to astronomical and mathematical aids led to the filling of the first chairs for geodesy and even partially for geography, established at German universities at the end of the century, initially with mathematicians (e.g., Hermann Wagner for geography, 1880, Göttingen; Ernst Hammer for geodesy, 1884, TH Stuttgart; Siegmund Günther for geography, 1886, TH Munich).

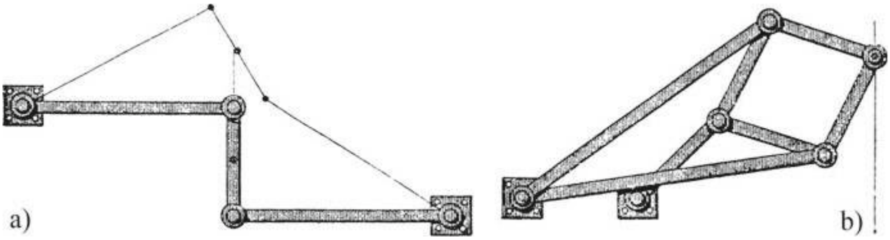
The notion of vector (more in section 7.6), the theory of composition of motions (more in section 7.7), graphic statics (above all, L. Cremona in Italy) and the theory of mechanisms developed within the scope of mechanics.

Watt's mechanism only delivers an approximately straight motion within certain limits. Peaucellier's mechanism represents the first historically exact solution to the problem. It was shown later that there are also exact solutions from five (instead of seven used by Peaucellier) parts and that an exact solution with less than the five adjustable parts is impossible [Kempe 1877, p. 9, 12].



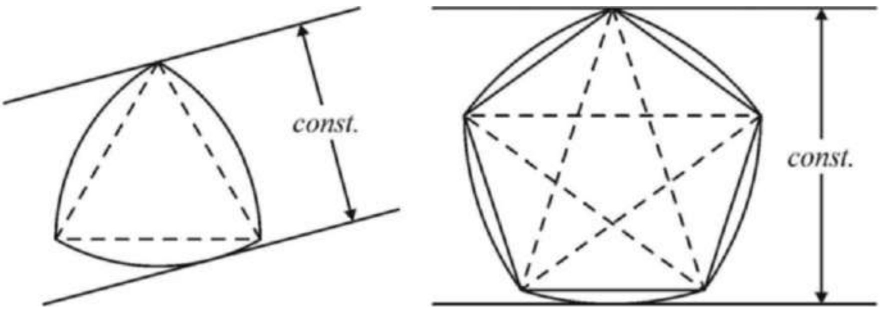

**Illus. 7.1.2** Maps for the calculated values of inclination.

*Atlas des Erdmagnetismus* (Atlas of Earth Magnetism) published by C. F. Gauss and W. Weber. (Leipzig-London-Paris-Stockholm-Milan-St. Petersburg 1840; cf. Gauß Werke (Gauss' works), vol. XII)




**Illus. 7.1.3** Linear guiding by a) Watt (1784), b) Peaucellier (1864)

The latter started with individual questions and their solutions, such as the linear guiding mechanism invented by James Watt in connection with the steam engine in 1784 (Illus. 7.1.3 and Problem 7.1.4). What had been invented in the Renaissance for the purpose of drawing special curves by means of mechanisms now merged with the new needs of machine building to move construction components along prescribed courses for the composition of approaches to theories, which were also developed by Poncelet in France, Chebyshev in Russia and J. J. Sylvester in Great Britain. A fine, popular scientific introduction of this kind of "kinematics", which reflects much of the 'Zeitgeist', is [Kempe 1877]. Some engineering scientists were heavily criticised by their professional peers because of their attempts largely to mathematise practical mechanics. We want to mention Franz Reuleaux as a representative who was also known in pure geometry for his discovery that, apart from the circle, there are numerous, further plane figures of constant width, particularly the simple "Reuleaux triangle" (Illus. 7.1.4), which, however, Euler had already published in 1778 in the Acta of Academy of St. Petersburg. [Fischer 1986], for instance, shows spatial analogies.



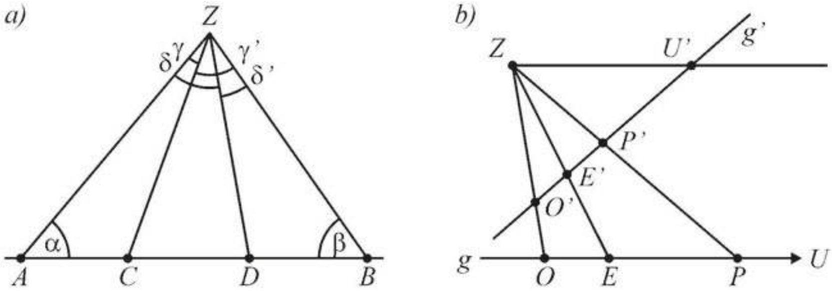

**Illus. 7.1.4** Reuleaux triangle and a further orbiform (figure of constant width)

## 7.2 Projective and synthetic geometry

Carnot's first book *De la corrélation des figures de géométrie* was published in 1801. An extended edition from 1803 under the title *Géométrie de position* is accepted as his geometrical main work. Since we can translate the title as 'geometry of position', which corresponds to the title of Christian v. Staudt's foundation of projective geometry discussed later on, his contribution is often associated primarily with the origin of projective geometry, which is not entirely accurate.

However, Carnot had another wish, which could be roughly described as the attempt to merge or reconcile the illustrative geometrical way of thinking with the algebraic calculus of the coordinate method. Above all, he tried to interpret the negative or even complex numbers that occur when dealing analytically with geometrical facts. For this purpose, he imagined figures in motion, which are constantly transformed into each other. (He himself highlighted the theoretical proximity to Euclid's 'porisms'.) Two figures stand in a relation, if one of them is transformed into the other one by a continuous change (which, of course, is not exactly defined by Carnot). For instance, if we move a straight line that intersects a circle away from the circle, both intersections merge first to one point, which is algebraically reflected in the double solution of the equation for the intersections, and, subsequently, both solutions of this equation turn complex. Then, Carnot also attempted in this case to interpret the zeros as "codings" of the opposite position of circle and straight line to each other. Thus, he gradually arrived at the geometrical interpretation of negative and sometimes even complex numbers. Such questions may have been triggered by Monge's approach, according to his student's credentials, of repeatedly stressing in his lectures early on the notion of radical axis (power line) of two circles as a phenomenon that first loses its original illustrative meaning as a straight line through the two intersections of two circles if these circles do not intersect each other anymore, but that, nonetheless, a purely geometrical interpretation is also possible for this case based on the correct interpretation of the equation of the radical axis (Problem 7.2.1). Carnot polemicised against negative quantities and wanted to replace them by algebraically (nowadays, we might rather say propositionally logically) working with notions, such as sense of description, sense-class, orientation, inner – outer, etc. However, this turns out to be rather tedious when executed, and, in the end, did not have any historical effect. To summarise, we could say that Carnot's ideas (in the shape of his collinear relations and the coordinates to determine those) only very indirectly influenced the development of projective geometry, but made way for the general concept of geometrical maps, the invariants in a map and the group-theoretical structure of map families.

a) According to the law of sine of plane trigonometry, it holds that $CA : CZ = \sin\gamma : \sin\alpha$, $CZ : CB = \sin\beta : \sin\gamma'$, hence (1) $CA : CB = \sin\gamma \sin$




**Illus. 7.2.1** Invariance of cross-ratio

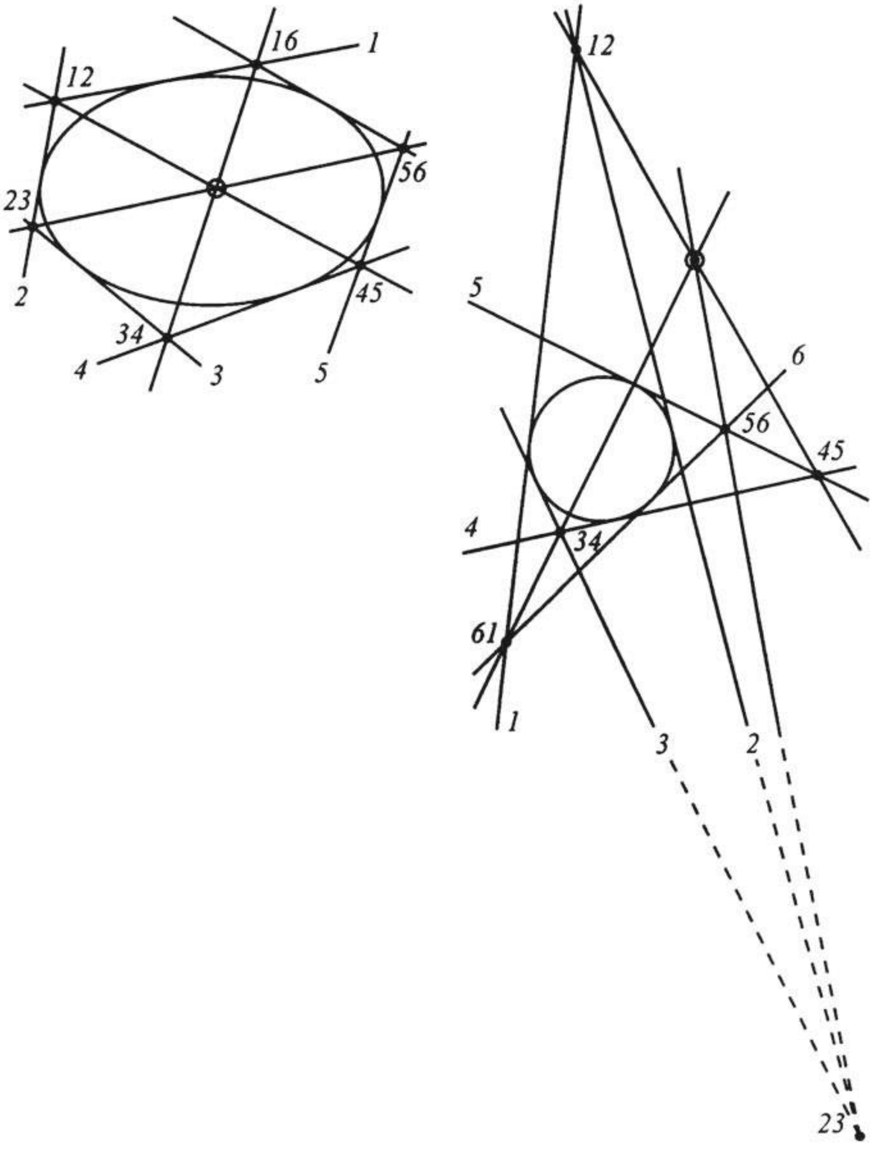
$\beta : \sin\alpha \sin\gamma'$, analogously (2) $DA : DB = \sin\delta \sin\beta : \sin\alpha \sin\delta'$. Thereby, we have to imagine line segments $CA, CB, DA, DB$ and consequently, the respective sine values as orientated or signed so that $CA, CB$ have the same algebraic sign, if they are equally orientated, i.e., $C$ is located outside of line segment $AB$. Thus, the affine ratio $(A, B; C) = CA : CB$ is positive if $C$ is outside, but negative if $C$ is within $AB$. It is zero for $C = A$, undefined for $C = B$. The cross-ratio $(A, B; C, D)$ of the four points $A, B, C, D$ (which lie on a common straight line) is defined as $(A, B; C) : (A, B; D) = CA \cdot DB : CB \cdot DA$, because of (1) and (2) consequently equal to (3) $\sin\gamma \sin\delta' : \sin\gamma' \sin\delta$. Formula (3) means that this cross-ratio is actually a property of the mutual position of the four straight lines passing through $Z$ to each other, independent of the straight line passing through $A, B, C, D$. Hence, if we project two straight lines onto each other from a centre $Z$, the cross-ratio of each four image points remains the same as that of their original image points.

b) Straight line $g$ is mapped onto $g'$ from centre $Z$ so that the infinitely distant point $U$ of $g$ is transformed, thereby, into the finitely located point $U'$. Regarding points $O$ (origin of coordinates) and $E$, point $P$ has a coordinate on $g$, which we can write as an affine-ratio $(P, E; O)$. Since $(P, E; U)$, if defined at all, can only have value 1 ($(P, E; Q)$ tends towards 1 if $Q$ tends towards U and independently of the direction of $Q$), we can also write the coordinate from $P$ regarding $O, E$ (and $U$) as a cross-ratio $(P, E; O, U)$. Due to the invariance of the cross-ratio, $P'$ on $g'$ has the same coordinate regarding the three basic points $O', E', U'$ as $P$ regarding $O, E, U$. The proof of the invariance of the cross-ratio stated concerning Illus. 7.2.1 is strange to projective geometry, since it uses notions of line segment length, ratio of lengths and sine of an angle, which make no sense in projective geometry as they are not invariant in projective mappings (or, according to the modern view, since they cannot be defined by the notions of point, straight line and incidence). V. Staudt's opinion differs from that of his predecessors, since he, so to speak, reverses the circumstances described above by gradually constructively obtaining the coordinates of the points of $g$ regarding the fixed points $O, E, U$ by means of continuous doubling and halving (in the projective sense) from the line segment already contained in $OE$ and, thus, obtains the notion of cross-ratio of four points reserved and "metric-free" based on the so-introduced coordinates (although he does not call them that).

The actual fathers of projective geometry (apart from Desargues, whose remarkable pre-accomplishments were only revived by Poncelet himself) were Poncelet and Gergonne. Poncelet's thoughts basically matured during his wartime captivity in Russia for two years (1812-1814) and he only published them in their original form under the title *Cahiers de Saratov)* in 'Collected Works' in 1862, specifically to re-establish his priority rights over Gergonne and Plücker. Indeed, all three developed the essential notions of projective geometry independently of each other, but in almost the same manner: the introduction of infinitely distant points, straight lines and planes, central projection, only then unrestrictedly executable (including its special case, if the centre is infinitely distant) and its composition as the universal method for transforming figures into each other whilst maintaining collinearity (hence, he used projective mapping instead of the continuous change suggested by Carnot!), cross-ratio of four points as the essential invariant of projective mapping (Illus. 7.2.1), the duality principles for plane and space, the theory of pole and polar in curves and surfaces of second degree, whereby the conic section is grasped explicitly as the set of its tangents for the first time. Above all, Poncelet was influenced by the works of his peer Charles Brianchon, who published the dual theorem to Pascal's theorem named after him in 1806 (Illus. 7.2.2), introduced the methods for obtaining theorems of plane geometry by means of central projection of spatial matter of facts in 1816 and was the first to compose the parts of plane geometry systematically only based on the notions of point, straight line and incidence whilst stressing the practical significance of the purely linear construction in his main work *Application de la théorie des transversales* in 1818. After Poncelet's mainwork, *Traité des propriétés projectives des figures*, which looked at Brianchon's results in a broader scope, had been published in 1822 (and again in a version extended to two volumes in 1864/65), Brianchon turned his interest (as his teacher Monge had done before him) to chemistry.

As already mentioned, Poncelet was also involved with the development of an illustrative geometrical mechanics preferably free of analysis. (See [Ziegler 1985] regarding this notion and its history.) His very intuitive style appealing to visualisation resulted in the fact that many of his accomplishments were only thought of as forerunners and ascribed to later authors (e.g., the Poncelet/Steiner theorem to be discussed in section 7.3). Cauchy, who was a professed opponent of the infinite within mathematics, especially criticised his naïve use of the infinitely distant as well as the infinitely small changes of a figure. As a member of the academy and an influential reviewer, Cauchy could delay the publishing of Poncelet's works considerably.

It seems that most countries were aware of France's leading role in mathematics, particularly in geometry, in the first decades of the $19^{th}$ century. Most French "classics" by, for example, Monge, Carnot, Poncelet, Chasles,




**Illus. 7.2.2** Brianchon's theorem.
If 1,..., 6 are tangents of a conic section, $ij$ the intersection of $i$ and $j$ and $ij/kl$ the connecting straight line of $ij$ and $kl$, then 12/45, 23/56, 34/61 go through one point

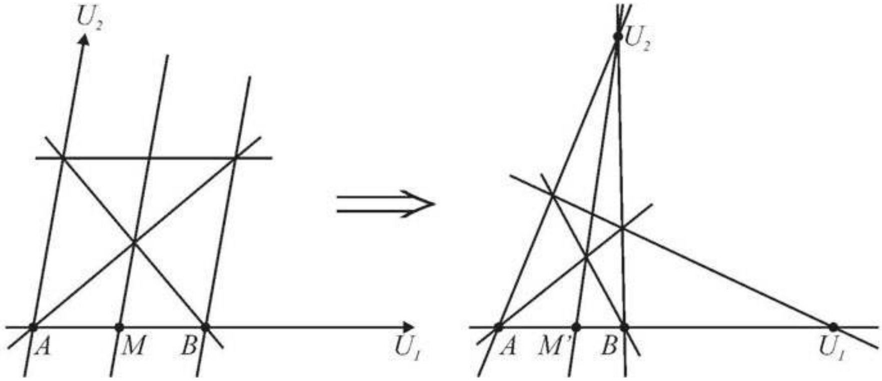
but also Lacroix, Legendre and Cauchy, were translated into other languages shortly after being published. We do not know of any cases in the reverse. Chasles complained on several occasions that he could not read the profound new works by German mathematicians. Projective geometry received its final shape in Germany through A. F. Möbius, J. Steiner, J. Plücker and Ch. v. Staudt. Möbius introduced homogenous coordinates in his *Barycentrischer Calcul* in 1827. This was based on questions of mechanics (we refer the reader again to [Ziegler 1985] for further details.): if three non-collinear points of the plane and/or four points in space in general position are plated with weights, the centre of gravity of this system assumes a position depending on this plating. By also allowing negative weights, Möbius demonstrated that every point in the plane and/or space in this manner can be the centre of gravity. Weights of the given points served then as coordinates of the respective centre of gravity. Since this fact does not change regardless of proportional change of all weights, the coordinates are homogenous, i.e., $\lambda a, \lambda b, \lambda c, ...$ describes the same point for any $\lambda > 0$ as $a, b, c, ... (a, b, c, ...$ real weights). We now see that the straight lines in the plane and/or planes in space are described by homogeneous linear equations and that we obtain the most general mappings preserving collinearity by assigning the basic points of a coordinate system to basic points of another coordinate system and continuously assigning coordinate-wise (see Problem 7.2.3). Möbius' work was highly acclaimed in Germany by Gauss, Jacobi, Dirichlet and others. Of course, nobody other than Cauchy himself reviewed this work in France.

Just one year after *Barycentrischer Calcul*, Julius Plücker's *Analytisch-geometrische Entwicklungen* (Analytic-Geometrical Developments) was published. The essential ideas are more or less the same, although Plücker had studied in Paris, but stressed the algebraic-analytical standpoint of projective geometry much more strongly than Möbius had. Plücker also introduced homogeneous coordinates, although in a different manner, and then conceived of the plane or spatial duality principle and the introduction of infinitely distant entities, since it is possible to exchange the role of the point coordinate and coefficient of the equation of straight line and/or plane. Whilst doing so, Plücker could also grasp straight lines in space rather elegantly by means of homogeneous coordinates: having fixed two planes and a projective coordinate system in each one, we can grasp every straight line in space by its intersections with these two planes and every one of these again by its three coordinates in the relevant plane. Dually to this, we can locate two points in space and turn the bundle of planes carried by each of these points into coordinates by means of each three coordinates. Each spatial straight line is now described as the section of a plane of one bundle and a plane of the other bundle. Plücker followed up on this in the 60s and composed *Neue Geometrie des Raumes, gegründet auf die Betrachtung der Geraden als Raumelement* (New Geometry of Space, Based on the Observation of the Straight Line as an Element of Space, 1868), which remained unfinished due to his death, but delivers an example of a "space" which has four dimensions, even though it consists of very visual and elementary objects (see Problem 7.2.4). Plücker, who turned more and more towards algebraic geometry, was first to phrase the idea very clearly of grasping the coefficients of a describing equation as coordinates of the respective object for algebraic figures of higher kind, so that the equation of a curve, surface,... assumes the nature of translating the incidence between point and figure into a relation between the coordinates of both these objects. It seems that the nearby step of expressing areas in the plane and/or solids in space by means of inequations in the relevant coordinates was first conceived of by Cauchy (1847; more in section 7.6.). Both Möbius and Plücker only occasionally devoted their attention to projective geometry. Möbius held a chair for astronomy in Leipzig and engaged with physical problems. Plücker also dedicated his attention mainly to experimental physics at times.

### V. Staudt's contribution

Basically, all contributions to projective geometry have been, so to speak, impure until now. The length of a line segment has always been assumed as given and nobody hesitated to use angle sizes freely (see the classic proof of invariance of the cross-ratio in central projection, Illus. 7.2.1). Ch. v. Staudt with his *Geometrie der Lage* (Geometry of Position, 1847) took the greatest step towards an axiomatic view of projective geometry free of metrics. The book presents itself in the foreword as an inspiration for geometry classes for advanced secondary schools that are out of the ordinary. Naturally, this fundamental book starts with a series of pre-observations on rays, angles, surfaces, solids, etc., which feature no hint at all that the first consequent projective geometry will soon follow based on just a few propositions and according to Euclid's model. Regardless of its pedagogic aim, the book is completely free from illustrations and examples or applications, and features a wealth of new terms. Some of them like "bunch" for simple infinite bundles (e.g., of all straight lines in a plane through a fixed point or of all planes in space through a fixed straight line) and/or "bundle" for twofold infinite bundles (e.g., of all straight lines in space through a fixed point or of all planes in space through a fixed point) have permanently established themselves. Others, such as "uniform figure" for the set of all points of a straight line or for a bundle of straight lines, make reading more tedious for the modern reader.

Ch. v. Staudt justifies the introduction of infinitely distant elements with the fact that point and "direction" accomplish the same for determining a straight line as two different points, and that analogous point and "position" (which means something like the direction of a plane in space) achieve the same as a point and a straight line that does not intersect it. Then, he justifies the dual principles (he calls them "reciprocity") for the plane and/or space (therein differing noticeably from Poncelet, Möbius and Plücker) with the duality of the propositions, which he found for the structure expanded by infinitely distant elements (these are implicitly the axioms of projective geometry),

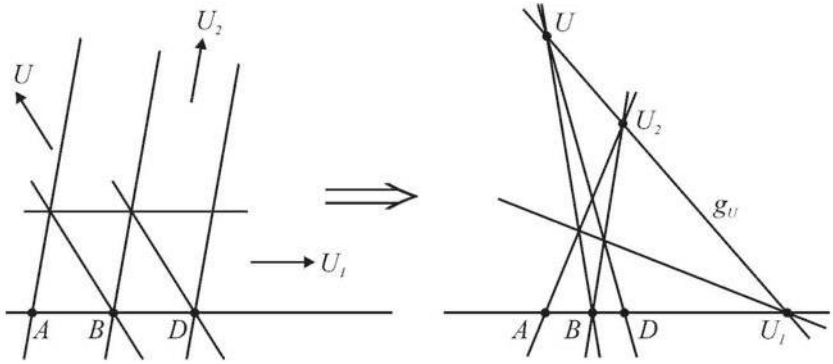



**Illus. 7.2.3** Harmonic quadruple of points $A, B, M, U$ in affine and projective view. Although definable by means of incidence geometry, $M$ is also the centre of line segment $AB$ from a metrical point of view. By replacing infinitely distant point $U$ by any point of straight line $AB$ (and, simultaneously, special auxiliary point $U_2$ by any point), we obtain the projective centre $M'$ of line segment $AB$ relating to $U_1$.

and, from then on, phrases every matter of fact in two columns parallel to both variants dual to each other. It seems he tried all of this with his students since, in his foreword, he explicitly states, "but that the law of reciprocity inspire every student open to geometry more than any individual theorem". Ch. v. Staudt now introduces harmonic pairs of points, which is nothing else but the relation distorted by central projection between the four points $A, B$, centre $M$ of $AB$, infinitely distant point $U$ and straight line $AB$ (Illus. 7.2.3). Furthermore, he shows the invariance of this relation in central projection and arrives first of all at projective coordinates on a straight line by means of continued projective halving and doubling of a unit $OE$ regarding an infinitely distant point $U$.

Afterwards, he arrives at the projective net in the plane and/or space (Illus. 7.2.4). He did all of this without explicitly speaking of coordinates. According to v. Staudt, a projective mapping is a mapping that transforms harmonic point pairs into harmonic point pairs. Central projections are such mappings, and every projective map can be represented by the composition of central projections. This is continued in a purely synthetic and coordinate-free style until the "degrees of freedom" of a projective mapping are clarified and conic sections are dealt with projectively. This book was appraised by v. Staudt's peers and established his fame.

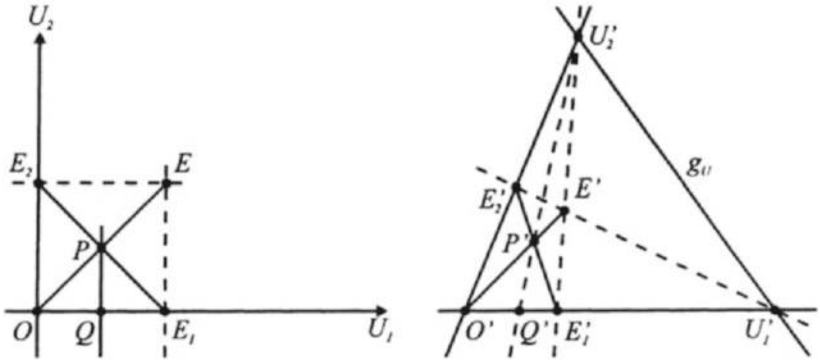
V. Staudt mainly dealt with the issue of how to obtain "imaginary elements" from the purely synthetical standpoint in three *Beiträge zur Geometrie der Lage* (Contributions to Geometry of Position), which were published between 1856 and 1860. He based this on the idea that an orientation on a projective straight line, which is completed to a closed curve by the infinitely distant




**Illus. 7.2.4** Projective coordinates on a straight line.
To determine the numeric value of the coordinates on an axis featuring $O, E$ and $U$ according to v. Staudt, the unit line segment $OE$ regarding the chosen $U$-point is added again and again in the projective sense by repeatedly doubling in both directions, and this integer point net is arbitrarily refined subsequently by continuously halving the part line segments. Illus. 7.2.4 shows how the construction of doubling $D$ from $AB$ beyond $B$ is transferred from the affine to the projective case by means of any improper auxiliary line passing through $U$. In the same manner, Illus. 7.2.3 shows one of the possible purely linear constructions of centre $M$ of any line segment $AB$, first affine and then generalised for the projective case. The coordinate definition of any point of axis $OEU$ should now result from the consideration of the limit cases. However, this presumes, like in the affine case, conclusions on the order, and is dealt with only at the level of illustrative evidence in v. Staudt's work

point, is given by three given points $O, E, U$ in order. A fourth point $X$ of this straight line is positive according to our interpretation, if we can obtain it moving from $O$ to $U$ via $E$. It is negative if we obtain it by moving from $O$ to $U$in the opposite direction. However, if the straight line now is directed in another sense (Durchlaufungssinn) from the beginning on (Durchlaufungssinn is the term that v. Staudt uses in this context), point $X$ can neither be obtained by moving in the positive nor the negative sense, although it lies on the straight line, since we can never reach $E$ in the positive sense, and thus, by no means can it go beyond. However, this concept is hidden in the motion of involution[2] in the work by v. Staudt and most authors following him,

[2] In projective geometry, involution refers to every non-identical projective map $f$, which agrees with the inverse map, i.e., points that are not fixed, pairwise inverted with each other. If $f_{A,B}$ for two different points $A, B$ of a straight line $g$ is the mapping of $g$ onto itself that assigns every point $P$ the fourth harmonic point to $A, B, P$, then $f_{A,B}$ is an involution, which has exactly the points $A, B$ as fixed points. As the analytic calculation with homogenous coordinates shows, there are also involutions without fixed points resp. the two solutions of the fixed point equation can become conjugate complex numbers. This was the approach to explaining "points with complex coordinates" geometrically. See Problem 7.2.5.




**Illus. 7.2.5** Projective coordinates in the plane.
From the view of plane projective geometry (it is analogous in space), a coordinate system is given by four points $O, U_1, U_2, E$. $O$ is the origin, $U_1$ the "improper" point of the $x$-axis so that $O$ and $U_1$ determine the $x$-axis together. Analogously, $O$ and $U_2$ determine the $y$-axis. By fixing point $E$ as the one that will have the coordinates (1, 1), we also determine the 1-point on the $x$-axis and the 1-point on the $y$-axis. An affine parallel coordinate system is a special case, in which $U_1$ and $U_2$ are "really" infinitely distant. A projective coordinate system is created vice versa, even by means of a Cartesian system, by putting the infinitely distant straight line $g_U$ to the finite

whereby the complex elements correspond to elliptical involutions. The first treatment of cyclic order (cf. p. 358) on the projective straight line closed by the infinitely distant point is remarkable. The intellectual proximity to Carnot's efforts is just as clear in v. Staudt's writing as the independent reorganisation of contents already given. Nevertheless, the complete lack of references to other literature and authors tends to alienate the modern reader.

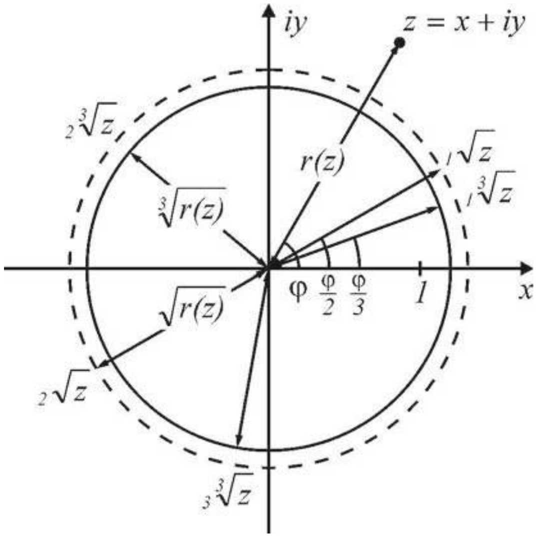
Projective geometry, which was a flourishing area from 1820 to roughly a century later, developed in two directions regarding the mentioned contributions. On one hand, the generalisation of the contributions of projective geometry from its very beginning to comprehension and mastery of conic sections was appropriate as an entrance for "algebraic geometry" (which is not looked at any further in this book, since we think of it as algebra rather than geometry), since polynomials with an arbitrary number of variables are always homogenous by being transformed into homogenous coordinates and, thus, make linear algebra more applicable, along with other reasons. On the other hand, it played several important roles in the investigations beginning around 1870 concerning the logical and methodological basics of geometry. We will further look at this in sections 7.7 and 8.1.

## 7.3 Theory of geometrical construction

As we have seen, geometrical construction problems and their solutions formed the origin and core of geometry. A theory that concerns the notion of problem, its solution and the methods of solving and/or proof of irresolvability, perhaps also the complexity of solutions, or examines the precision of approximations generally and theoretically, can develop in two directions: on one hand, within the scope of mathematical logic (which could not, of course, happen before the end of the $19^{th}$ century), and on the other hand, by transferring geometrical questions to the language of algebra. Modest approaches to this can already be found in Antiquity. For example, the proof of the irrationality of $\sqrt{2}$, as indicated by Aristotle and in 'Elements' (X, 1 15), could be so interpreted that the diagonal of a given square cannot be transferred towards one side by means of a purely linear operation. Other theorems from Euclid's Book X also say something about the impossibility of geometrical constructions wrapped up algebraically. However, such an algebraic theory of geometrical constructions could only become fruitful after constituting the coordinate method and making algebraic means available that allow proving the irresolvability of certain algebraic problems with given algebraic means. This was basically accomplished by Gauss, and the actual theory of geometrical constructions started with him. Nonetheless, we want to speak briefly of some preceding steps. Whenever we subsequently speak of theory, we always mean that we are not dealing with individual construction problems but with an entire class of problems or with the notion of problem and its solution as such.

In 1593, Vieta crowned his *Supplementum geometriae* with the proof that all problems that lead to equations of third or fourth degree become solvable if we, apart from compass and straightedge, allow instruments with which we can extract the third root of given quantities and trisect every angle. This outcome is actually understandable from the viewpoint of the complex numbers, since, according to Ferrari, an equation of fourth degree can be reduced to an equation of third degree and, according to Cardano, an equation of third degree can be solved by rational operations and additionally second and third roots. However, since we also need to make general use of complex intermediate results when dealing with real given quantities and real zeros, the plane of construction would have to be used temporarily as a complex plane, whereby square roots can consequently be constructed with compass and straightedge by means of the root of the radius coordinate of the relevant number and by halving its angle coordinate, whereas we need to be able to extract third roots from the radius and to trisect the angle coordinate regarding third roots. Did Vieta know of the geometrical interpretation of complex algebra in Gauss's "complex number plane" (Illus. 7.3.1)? We might just be able to conclude this based on his calculations.

Another direction for dealing theoretically with constructions is devoted to examining how changes (expecially the restricions) in the permitted instru-




**Illus. 7.3.1** Extracting roots in Gauss's complex number plane

ments and/or construction steps or the available area of construction affect the solvability of problems. Pappus, later medieval Islamic geometers like Abū'l-Wāfā, artists of the Renaissance like Leonardo da Vinci, and finally Cardano and Tartaglia had engaged with construction problems that can be solved with a compass with a fixed opening span and a straightedge [Hallerberg 1959]. The fixed compass can be replaced by a circle template, which is appreciated by each drawer whose circle turned out to be a spiral due to the compass being too loose or an enlarged hole in the drawing paper. In 1653, this resulted in a text by G. B. Benedetti, in which all problems from Books I – IV and VI of Euclid's 'Elements' were solved by the fixed compass and straightedge mentioned above. A booklet with the odd title *Compendium Euclidis Curiosi* was published in the Dutch language in Amsterdam in 1673. Therein, the anonymous author (Georg Mohr from Denmark, as is now known) revealed the same results. He wrote in the introduction that he had heard of the relevant text by "Joan Baptista" (apparently he was referring to Giovanni Battista Benedetti), but could not get access to it and subsequently had obtained these results himself after initial disbelief. There was even an English translation of this 'curious Euclid' in 1677 (excerpts in Problem 7.3.2). Both texts were lost for a long time and were only generally accessible again from 1982 onwards [Mohr 1673]. Moreover, the Danish geometer Johannes Hjelmselv had rediscovered in 1927 the forgotten treatise *Euclides Danicus* (the Danish Euclid) by the same author and published it [Mohr 1672]. Therein, Mohr made it plausible that all *points* constructible with compass and straightedge by means of given figures can be constructed with the compass alone.

In forgivable ignorance of this predecessor, Lorenzo Mascheroni devoted his book *Geometria del compasso* (Geometry of the Compass), printed in 1797, to the same topic. It is interesting that his intention was motivated by his opinion that constructions with the compass are more precise regarding their practical execution than the ones with the straightedge, whereas Lambert, Brianchon, Poncelet and others, in contrast, favoured the sole linear construction, or at least restricting the compass as much as possible, since only linear constructions can be realised in greater distances in the field (by means of taking a bearing and aligning). Jacob Steiner concluded this set of problems in 1833 with his theorem that a single drawn circle or just a piece of it, including (the indispensible) centre, would suffice in order to reduce all compass and straightedge constructions to purely linear ones (which basically was already shown by Poncelet in 1822). The development of the logical foundations of mathematics has put all these theorems in a new perspective under different aspects [Schreiber 1975], [Schreiber 1984]. However, we do not want to neglect the curious fact that Napoleon Bonaparte, who, as known, was interested in mathematics, got to know Mascheroni himself during his campaign in Italy just after he had finalised his book, but before it had been published in print. Filled with pride at having obtained his head start, he returned to Paris and was ever so happy to give the famous French mathematicians a "really easy" construction problem, which they, of course, could not solve ad hoc (see Problem 7.3.1).

**Algebraisation of the theory of geometrical construction**

So far, so good concerning the pre-history. The decision of the eighteen-year-old Gauss to dedicate his life back then to mathematics and not to ancient languages, as he had considered before, is, as known, due to his discovery that the regular 17-gon can be constructed with compass and straightedge. His diary also begins with this entry from 30/03/1796. Further entries indicate that he came back to questions of circle division again and again, reduced to purely algebraic questions by using complex numbers, for the equation $x^n = 1$ (and/or after separating the trivial factor $x - 1$ from $x^n - 1$) for which $n$ the equation $1 + x + ...x^{n-1} = 0$ can be solved by quadratic radicals. Gauss's final answer was: this is exactly the case, if $n$ has the form

$$2^m \cdot p_1 \cdot p_2 \cdot ... \cdot p_k \tag{7.3.1}$$

Thereby, $m \geq 0,\ k \geq 0$ and $p_1, ..., p_k$ are pairwise different prime numbers of the form $2^i + 1$ (for example, 3, 5, 17). A considerable part of his early work *Disquisitiones arithmeticae* published in 1801 is dedicated to the question as to which conditions of a polynomial equation must be fulfilled in order to be solvable by quadratic radicals. We also find there the essential theorems named after Gauss, with whose help most of the classical problems can be proven to be irresolvable by means of compass and straightedge: an irreducible polynomial with rational coefficients within the realm of rational numbers is at most then solvable by means of quadratic radicals, if its degree is a power of two. The question of the reducibility of a polynomial with rational coefficients within the range of rational numbers can be reduced to the question of the reducibility of an integer polynomial of same degree within the range of integer coefficients (and the latter can be decided by means of a finite case-by-case analysis). Two of the classical construction problems (cf. section 2.2.2), doubling the cube and angle trisection, can immediately be proven to be irresolvable with these algebraic means. Doubling the cube leads to the equation $x^3 - 2 = 0$ for the wanted edge $x$. The polynomial is irreducible and its degree is not a power of two (Problem 7.3.3). If angle trisection would be generally possible with compass and straightedge, then such is also the case in which the given angle is 60 degrees, and, thus, is actually constructible, i.e., an angle of 20 degrees, a regular 18-gon would also be constructible. However, 18 does not have the form stated in (7.6.1). Given these conclusions from Gauss's work, it seems it is due to the afore-mentioned communication difficulties of the $19^{th}$ century that the French mathematician Pierre Wantzel, who died young, proved these outcomes again between 1837 and 1845, and is claimed as the originator of these theorems in different books. The third famous ancient problem, squaring the circle, leads to an investigation under the new circumstances if the number $\pi$ is the root of a polynomial equation solvable by quadratic radicals. Lambert showed the irrationality of $\pi$ in 1767. His text *Vorläufige Kenntnisse für die, so die Quadratur und Rektifikation des Cirkuls suchen* (Provisional Knowledge for Those that Seek Squaring and Rectification of the Circle) was targeted at a broad audience. Therein, he invited the large number of amateurs who strove towards solving this problem with completely insufficient prerequisites (such amateurs still existed in the $20^{th}$ century!) rather to do something useful within their reach, like calculating prime number tables. This text is still a pleasure to read nowadays. After all, Charles Hermite could follow up on Lambert's pre-accomplishments when proving the transcendence of $e$ in 1873. Ferdinand Lindemann finally generalised the idea of proof in 1882, i.e., he proved the transcendence of $\pi$, meaning that $\pi$ cannot at all be the solution of any algebraic equation with rational coefficients. Thus, squaring the circle and the closely related rectification of the circle are not just irresolvable by means of compass and straightedge, but also by any other means and/or aids (such as insertion or a section with given conic sections), the algebraic analysis of which yields that we can only construct points with their help, the coordinates of said points depending algebraically on the coordinates of the given points. (We still recommend [Vahlen 1911] for mathematical details and a well-formed proof of transcendence of $e$ and $\pi$.)

Gauss's investigations had increased the qualitative level of the manner in which geometrical construction problems were addressed, but also turned them somehow into a branch of algebra as they have subsequently been grasped by many authors since then, with the negative side effect of a certain “de-geometrisation”. We will look at the correction of this development later on in the $20^{th}$ century. Nonetheless, the $19^{th}$ century also reveals certain approaches to another view on geometrical constructions. For instance, Carnot's *Géométrie de position*, discussed in section 7.2, is to be understood to a high extent as an attempt at a theory of geometrical construction in a completely different sense, and Gergonne's *Annales* are a rich source of individual contributions to an almost completely algebra-free treatment of special construction problems. August Adler, an educational expert from what constituted Austria back then, presented a new translucent and general proof for Mohr's/Mascheroni's theorem in 1890 by using the transformation by reciprocal radii in order to transform the entire figure made of given and wanted components as well as auxiliary lines into a figure that does not contain any straight lines at all. In 1902, he showed that both a ruler to be used in a certain manner with two parallel edges and a set square, the edges of which meet in the arbitrarily fixed angle, have the same level of performance as circle and straightedge. His compiled book on geometrical constructions, which is not just a pure collection of solutions and solution recipes (like, for example, [Petersen 1879]), was published in 1906. Approximately at the same time, Paul Zühlke wrote a summarising description on constructing with obstacles (limited instruments, limited drawing area). Despite the date, we mention this here, because intellectually it still belongs to the $19^{th}$ century.

### Geometrography

Pointing to the $20^{th}$ century, although actually originating in the $19^{th}$ century, a peculiar theory was formed, established under the term 'geometrography', by the Frenchman Emile Lemoine, who was very influential due to his activity in scientific organisations. Using the example of geometrical construction algorithms, the first attempt at complexity theory was made after Steiner had coined the oft-quoted saying of "solving only with the tongue" at the end of his text from 1833, mentioned above. He meant that the fashionable reduction of new problems to already solved ones and/or the pure sketching of an approach to a solution completely neglected the feeling for the actual effort of solving or the practical feasibility. Lemoine counted the steps of a construction according to an assessment suggested by him (and, separately, those steps that influence the practical precision, according to him) and referred to those solutions to a problem that revealed the lowest amount of steps as geometrographic. A considerable number of interested parties and combatants, especially teachers, hoped for a revival of the stiff geometry lessons by stressing a "sporty aspect". Surprising simplifications were found for a series of classical problems, for example, the number of steps needed to solve Apollonius's problem was reduced from around 500 (per Vieta and also Gergonne) to 150, above all by crafty multiple use of auxiliary lines. After a short-lived golden age, however, geometrography was quickly cast into oblivion from 1906 onwards. It revealed three deficits, which are particularly informative for modern computational complexity theory:

1. The century-old deficit of classic mathematics in respect to considering the notion of algorithm led to Lemoine and all his combatants thinking only within the category of a fixed step sequence without branching or cycling. Thus, they viewed the complexity measure as a pure counting of possibly weighted steps.
2. Lemoine and his followers could never agree on a common assessment of the steps. Rather, a considerable portion of the written literature is dedicated to polemical disputes, whereby,for example, it was meant to be proven with the stopwatch how much faster it would be to execute this step over that one.
3. There was no method for proving optimality or the approximate optimality of a solution. Rather, the term "geometrographic" was a kind of challenge cup, which, on the following day, was given to a solution faster by two steps. (Think of the modern use of the word "efficient"!)

However, apart from these three points, the literature on geometrography features many remarkable ideas, which nowadays must be considered to belong to an astonishing pre-history of computer science within classic mathematics, as should geometrical constructions in general.

## 7.4 Differential geometry

The $17^{th}$ and $18^{th}$ centuries accomplished a great deal of groundwork for addressing "objects of curvature" by means of infinitesimal mathematics, which we will now look at briefly. Whereas Clairaut's afore-mentioned book on spatial curves (1731) is limited to those notions and properties that can be treated by means of the first derivation of coordinates according to the curve parameter, Monge approached the notion of curvature of spatial curves more geometrically from 1771 onwards. The tangent determines a perpendicular normal plane in every curve point. Two "infinitesimally adjacent" normal planes generally intersect each other at a straight line, the normal plane of which through the given point is the osculating plane of the curve in the given point. Changing the inclination of this osculating plane is the torsion. (However, the term 'torsion' was only used from the $19^{th}$ century onwards.) The three unit vectors directed towards the curve tangent, towards the curve normal, and towards the normal of the osculating plane (referred to as binormal) form the "accompanying tripod" of the curve. J. F. Frenet, J. A. Serret, P.-O. Bonnet and J. Bertrand completed the theory of spatial curves around 1850 by proving, amongst other things, that such a curve is determined independently from its position regarding the coordinate system and/or motions by curvature $k(s)$ and torsion $w(s)$ as functions of arc length $s$. This result became the model for the coordinate-invariant characterisation of higher manifolds of curvature.

The systematic differential geometrical study of curved surfaces in a space basically began with Euler's textbook on differential calculus in two volumes (1755) and continued in the form of individual investigations from 1760-67. Euler looked at curves of intersection of all planes that pass through the normal to the surface in a given point $P$ of the surface and the surface depending on which side of the surface lies the centre of curvature. Each of the intersection curves (as plane curves) features a curvature in the sence of a plane curve and the angle $\varphi$ of its tangent in the tangential plane of the surface in P. Euler assigns the radius of curvature $R(\varphi)$ to each of these curves, in addition to that signed depending on which side of the surface lies the centre of curvature. He found out that this function, which, radically plotted in the tangential plane, adopts maximum $R_1$ and minimum $R_2$ in two directions perpendicular to each other, the so-called principal curvatures, and that the radius of curvature $R(\varphi)$ in direction of $\varphi$ suffices the equation

$$1/R(\varphi) = cos^2\varphi/R_1 + sin^2\varphi/R_2. \tag{7.4.1}$$

Meusnier in 1776 and Monge in 1784 followed up on this, and Dupin introduced the "indicatrix" named after him in 1813. After Euler, who, when exchanging ideas with Johann Bernoulli, had already developed the notion of geodesic curve as one that locally represents the shortest link of its respective points within the surface, Monge studied those curves in the plane that run either in the direction of the larger or smaller principal curvature in every point and, thus, intersect themselves everywhere perpendicularly. Hence, these geodesics are created as a solution to a variation problem. (However, the term "geodesic" was only made customary much later, namely by Liouville.) Lagrange was first to study the analogous problem of fitting a minimal surface into a given boundary curve (later called Plateau's problem) in 1760. He found the following necessary condition for minimality for the case that the surface can be represented in the form $z = f(x, y)$:

$$\left(1 + {f_y}^2\right) f_{xx} + f_x f_y f_{xy} + \left(1 + {f_x}^2\right) f_{yy} = 0. \tag{7.4.2}$$

Meusnier showed in 1776 (published in 1785) that this condition is equivalent to the condition $R_1 = -R_2$ in every point of the minimal surface. This, for instance, leads to the fact that a surface that features a positive "Gaussian" curvature $R_1 \cdot R_2$ in one point (and due to the continuity in its surrounding) cannot be minimal. Furthermore, he found the first two types of non-trivial minimal surfaces, namely the catenoids created by revolving a catenary, and the helicoids. It only became clear much later that the surfaces characterised by Lagrange's and/or Meusnier's conditions only represent "local minima" in the set of all surfaces fit into a given boundary, which corresponds to its physical generation by means of soapsuds (Plateau around 1850!).

**Beginning of inner differential geometry**

Up to this point, curved surfaces had always been seen as objects in three-dimensional space. However, their role as part of geodesy had sooner or later to reveal the question of notions and properties, which could be determined by measuring or experimenting within the surface without relating to its spatial embedding. It has been one of C. F. Gauss's greatest accomplishments to establish this "inner geometry", which is fundamental for modern physics and is the only possibility, speaking physically, for curved spaces of a higher dimension than two (since a surrounding space is then not imaginable for us or at least is not accessible), and to clarify how it differs from the viewpoint customary until then.[3]

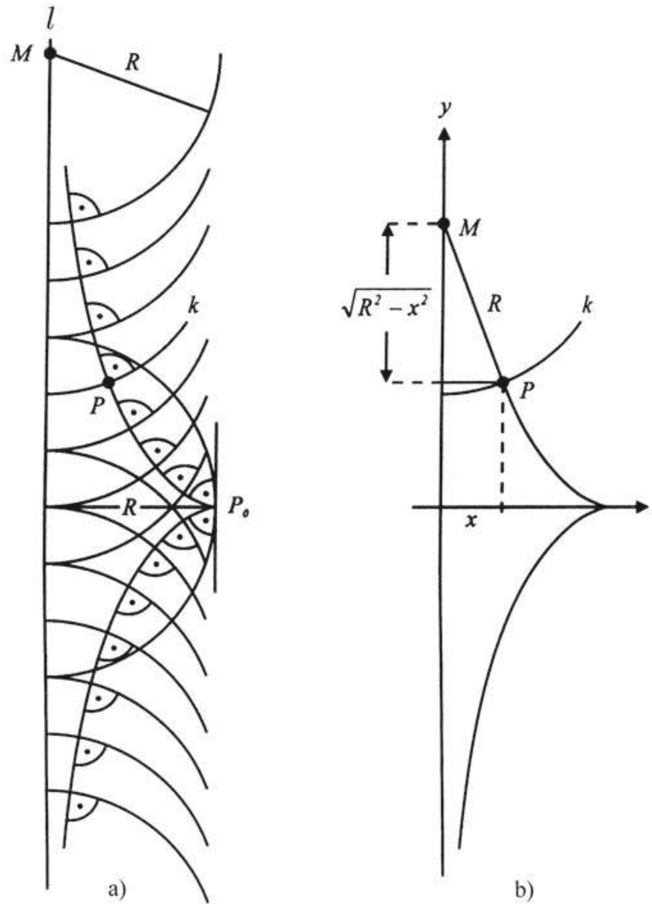
In 1818, Gauss was told to continue the arc measurement conducted by his friend, the astronomer H. C. Schumacher in Denmark, with the aim to measure exactly the difference of latitude between Altona and Göttingen, a task that he conducted himself through immense efforts between 1818 and 1827. This undertaking was followed by the measuring of the entire kingdom of Hannover, whereby Gauss was in charge of evaluating the results [Biermann 1990, p. 19ff]. Similar undertakings were finalised or partially worked on in other European countries around this time. These assignments caused a more intensive turn towards the differential geometry of curved surfaces. As indicated by certain letter excerpts, Gauss had been planning an extensive work on higher geodesy not later than 1822. However, it was only published in treatises in two parts in 1844-47 (reprinted in Ostwalds Klassikern) and was preceded by an incomplete manuscript from 1825 found in his estate [Gauß a, vol. 8, 408-422]. Having worked in 1825 with the representation of surfaces in the form of $F(x, y, z) = 0$ or even $z = f(x, y)$, he later switched to using representations by means of two parameters, which, until then, had only been used occasionally.

Over the course of this work, he must have become so aware of the significance of inner geometry that his first publication, *Allgemeine Untersuchungen über gekrümmte Flächen* (General Investigation on Curved Surfaces) finalised in 1827 and reviewed at a glance by Göttingische Gelehrte Anzeigen (Göttingen's Scholarly notifications), published in 1828 in Latin, German translation in Ostwalds Klassikern, reprinted in Teubner-Archiv vol. 1), purposely omits all that does not agree with this standpoint. Due to the large number of sources, it is easy to follow the development of ideas. Gauss transferred the definition of total curvature ("amplitude") of a curve segment, which originated from the Monge school, to surface segments: indicate the normal of

[3] Here, we have cause to mention a peculiar elementary geometric analogon: since Antiquity, spherical geometry has always been treated based on the idea that a spherical surface exists in three-dimensional space, so-to-speak, as a practically important part of spatial geometry. It seems that, up to 1980, nobody has thought of establishing elementary spherical geometry analogously to plane geometry completely axiomatically as inner geometry without any relation to the surrounding space [Schreiber 1984, chap. 2.3].

length one in every point, transfer its end point onto a fixed unit sphere by means of parallel translation and take the content of the created surface segment of the sphere as the measure of the total curvature of the surface segment. If the surface is, for example, developable, the assigned subset on the unit sphere is only one-dimensional, and the total curvature is, thus, zero. By dividing the total curvature of a small surface segment around point P by the area of this surface segment and conducting a limit process, we obtain a numeric measure for the local curvature in point $P$ (nowadays referred to as Gaussian curvature). His treatise from 1828 is entirely devoted to the question of how to determine the notion of total (and, thus, also local) curvature by means of measurements executed within the surface. As shown by Gauss, this is possible by means of triangulation and the difference between the sum of angles in the geodesic triangles and 180 degrees thereby obtained. The further outcome that he himself referred to as "theorema egregium" (something like "extraordinary theorem") states that this notion of curvature is invariant not just when shifting the surface in space, but also when arbitrarily bending (not distorting) the surface, since it can be defined by measuring within the surface, and that it can be calculated by formulae that only feature the coefficients $E, F, G$ (nowadays referred to as first fundamental quantities $g_{ij}$) of the differential arc element $\sqrt{Edp^2 + 2Fdpdq + Gdq^2}$ as functions of the surface parameters $p, q$. Gauss then shows that his local curvature is identical to the signed product of both principal curvatures studied since Euler. He generalised the notion of development, which, until then, had only been used for surfaces that could be isometrically mapped in the plane, to apply to the development of a surface onto another one (equal in meaning to the existence of a mapping, which is isometric in regards to the inner geometry of both surfaces), and showed that surface segments, which can be successively developed in this respect, have the same Gaussian curvature in the respectively assigned points. (Ferdinand Minding succeeded in reversing this concept in 1839. In other words, we are dealing with a successive local development of surfaces, given a map that leaves the Gaussian curvature pointwise invariant.)

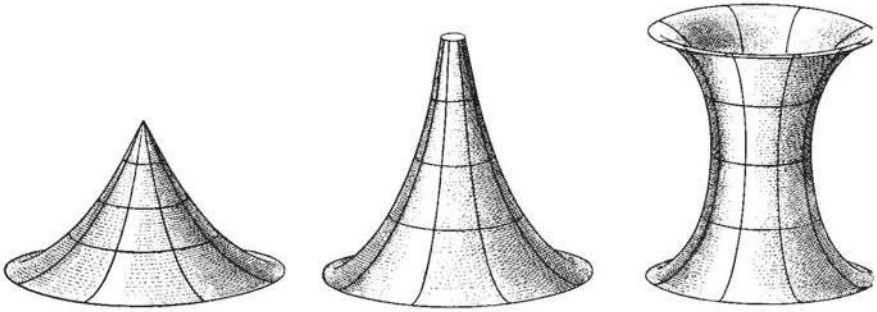
Further outcomes in Gauss's *Disquisitiones generales circa superficies curvas* (that is the original title) from 1828 concern analogies between plane geometry and inner geometry of curved surfaces: a circle around $P$ in terms of inner geometry is created by indicating geodesics of the same length from $P$ in every direction. It then intersects all its radii in a right angle. A distance line is created by marking a "geodesic perpendicular" of constant length on a geodesic in every point towards a given side. The so-created distance curve intersects all its perpendiculars again perpendicularly. All this serves the generalisation of the geographic grid to geodesic grids on any surfaces. Gauss also extended Legendre's theorem concerning spherical triangles, which is important for higher geodesy, to include geodesic triangles of any curved surfaces. If we compare a plane and a geodesic triangle of same side length, the angles of both triangles differ, apart from the quantities of fourth order, by amounts, the sum of which is the deviation of the sum of angles of the geodesic triangle




**Illus. 7.4.1** A tractrix with directrix $l$

a) A tractrix with directrix $l$ is defined by the fact that, for each curve point $P$, the section of the tangent of the tractrix has the constant value $R$ in point $P$ between $P$ and its intersection with $l$. According to this definition, it is – in contrast to many other "dynamically" defined curves – not exactly pointwise constructible. However, since it intersects all circles $k$ of radius $R$ with centre $M$ on $l$ perpendicularly, we can draw it quite well approximately as the orthogonal trajectory of the bundle of circles by starting with its cusp $P_0$, which is to be given anywhere in distance $R$ of the directrix.

b) If we choose $l$ as $y$-axis and the relevant perpendicular through $P_0$ as $x$-axis of a coordinate system, we obtain $y' = \pm\sqrt{R^2 - x^2}/x$ for the slope of the tangent in curve point $P(x, y)$, whereby the $+$ applies to the lower branch and the $-$ to the upper branch of the curve symmetrical to the $x$-axis. As a result, we obtain the following curve equation for the upper branch:
$y = Rln\left(R + \sqrt{R^2 - x^2}/x\right) - \sqrt{R^2 - x^2} \quad (0 < x \leq R)$



**Illus. 7.4.2** Surfaces of constant negative curvature. In the middle, the pseudo-sphere created by revolving the tractrix

of 180 degrees and the distribution of which depends on the local curvatures in the corners of the geodesic triangle. In the case of constant positive (that was Legendre's theorem) or negative curvature, the angles of the spherical or pseudo-spherical triangle are larger or respectively smaller than the angles of the plane triangle with equal sides by one third each of the excess or respective defect of the triangle. Here, Gauss's consideration touches not only on practical geodesy, but also on his interest in non-Euclidean geometry, nowadays referred to as hyperbolic or Lobachevskian geometry. Not later than 1827, Gauss must have been aware of the fact that they agree locally with the inner geometry of a pseudo-sphere, i.e., a surface of constant negative curvature, and, thus, are no phantasm. The fact that there are such surfaces and that, for example, such a surface (Illus. 7.4.2) is created by revolving the tractrix (Illus. 7.4.1) first introduced by Newton, was already known at the end of the $18^{th}$ century (Problem 7.4.1).

### Transition to $n$-dimensional differential geometry

Gauss must have known that his inner geometry only needed an appropriately generalised notion of "multiply extended quantities" in order to include higher dimensions and lead us to a completely new concept of space. That is why he pushed through that Riemann on occasion of his habilitation in Göttingen in 1854 had to speak about the third topic submitted by him instead of the first, as was, in fact, customary. So Riemann was forced to speak "On the Hypotheses that Geometry Is Based On" [Riemann 1876], [Klein 1928] (Lectures on Non-Euclidean Geometry).

Riemann fulfilled Gauss's expectations completely in this oral presentation imposed upon him, although basically it had a more non-mathematical character in a very informal manner almost without formulae. As examples for continuously varying, multiply extended quantities, he listed colours (this was followed up on later by Helmholtz and Ostwald) apart from the "locations for sensual objects" (i.e., physical space). However, he did not cite states of mechanical systems, which would have been an obvious idea. His verbal description of how to obtain an $(n+1)^{th}$ extended system based on a $n^{th}$ extended system by changing one parameter (for example, a parallelogram by means of a line segment and a cuboid by means of a rectangle) reminds us very much of Grassmann's *Ausdehnungslehre* (Theory of Extension) already published in 1844. However, it is unclear if Riemann knew this work, which at that time had hardly been circulated. Having expanded the fundamental metric quantities introduced by Gauss to include this general case, he focused on spaces of constant curvature and, thereby, on the special case of a curvature of zero, and clarified – without using these terms – that constant curvature is the necessary condition for homogeneity and isotropy of a space, in other words, for uniformity of all points and all directions, and for the unrestricted mobility of "solids". He then discussed the difference between infinity and unboundedness of a space and noticed that, given constant positive curvature, the space has to be necessarily finite. The last passage indicates his prophetic sharp eye:

"However, now the empirical notions in which the measure-relationships of space are established, the notion of firm solid and ray of light, seem to lose their validity in the infinitely small; thus, it is easy to imagine that the ratios of space do not suffice the pre-conditions of geometry in the infinitely small, and we would indeed have to assume this as soon as it is possible to explain appearances in a simpler manner with these means. The question of the validity of pre-conditions of geometry in the infinitely small is connected to the question of the inner reason of ratios of space. Concerning this question, which can be thought of as part of a theory of space,... Thus, that which is real and what space is based on must form a discrete manifold, or we must find the reason of ratios of measure outside the binding powers affecting those." (translated from German [Riemann 1876, p.267f.])

If Riemann was justifiably honoured as the intellectual father of an orientation, which in the end led to general theory of relativity, the last sentences reveal him to be the forerunner of a even more actual turn of physics that questions all physical concepts inspired by the paradigms of classic analysis and differential geometry.

Gauss's treatise on inner geometry had already been multiply reprinted in the $19^{th}$ century and also translated into French, which, in the end, led to differential geometry and its inner standpoint in particular advancing to form a central area of geometry in the $19^{th}$ century. The Gaussian definition of the main notions established itself, although other measures of curvature were occasionally suggested (for example, in 1831, by the French mathematician Sophie Germain, who was in contact with Gauss, and in 1889, by F. Casorati). Riemann's habilitation speech was only taken from his estate and published after his death by R. Dedekind in 1868. The German (Minding , Enneper, Lipschitz, Christoffel, Weingarten), the French (Dupin, Bonnet, Bertrand, Liouville, Bour, Darboux), and the Italian mathematicians (Brioschi, Betti, Dini, Bianchi, Codazzi, Mainardi, Beltrami, Casorati) were greatly involved in advancing the direction of geometry initiated by Gauss and Riemann in a predominantly calculus-like manner. Christoffel generally founded the so-called absolute, i.e., inner differential geometry free from coordinates of the surrounding space, and created in this context the elements of the later tensor calculus. However, other classical sets of problems more closely related to geodesy were also continuously fostered, such as C. G. J. Jacobi explicitly determining the equations of the geodesic on the ellipsoid with three axes in 1844.

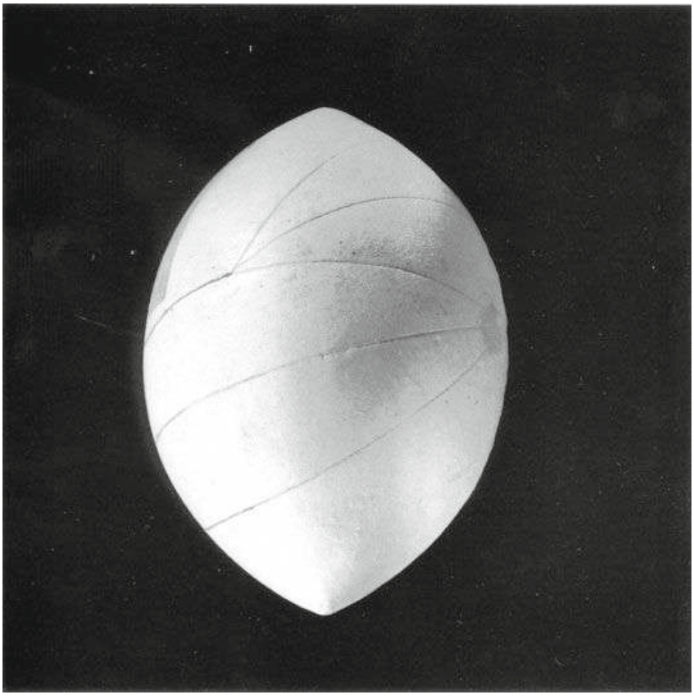
Many investigations in the second half of the $19^{th}$ century were devoted to surfaces of constant negative curvature, partially due to the correlation between constant curvature and free motion created by Riemann and partially because of the interest in non-Euclidean geometry, which had increased after Gauss's death. Minding systematically studied surfaces of revolution of constant negative curvature and classified their local characteristics as elliptic, hyperbolic or parabolic in 1839 (Illus. 7.4.2), whereby only the tractrix can be considered as the profile of the parabolic case.

However, in 1865, Dini realised that the surfaces with the profiles found by Minding do not necessarily have to be surfaces of revolution, but that surfaces of constant negative curvature are also created by screwing the profiles with an arbitrarily selectable pitch around the relevant axis. Bour had already proven in 1857 that every helicoid is developable onto a surface of revolution. The works by Ferdinand Joachimsthal (1846), Alfred Enneper (1868), and their students, as well as by Theodor Kuen (1884), discovered extraordinarily shaped surfaces (Illus. 7.4.3) and produced an unheard-of wealth of possible shapes of surfaces for the case of constant negative curvature.

**Illus. 7.4.3** Kuen's surface.
[Gerd Fischer: *Mathematische Modelle* (Mathematical Models), Vieweg Verlag, Braunschweig/Wiesbaden 1986]

Due to efforts made to provide an overview thereof and to be certain that the amount of surfaces found was complete, developments took place that would later show their main effects outside of differential geometry. In 1860, the French Academy of Sciences hosted a competition to find methods with which it was possible by means of a given surface to produce further surfaces, which were developable onto the given one. The first prize was awarded to Edmond Bour in 1862, who died shortly after turning 33. The methods found by Bour, Ribaucour, Bäcklund, and others to produce other surfaces with the same features of curvature based on one surface by means of geometrical transformation (consider the historical relation to the method to list all curves of third order applied by Newton and the following development!) finally led to the notion of contact transformation and its application to the theory of solving differential equations via works by Albert Bäcklund, Luigi Bianchi, Sophus Lie, and others. The seeds of group theory were also hidden therein. For instance, Lie discovered in 1883 that Bianchi's transformations regarding Lie's transformations are exactly the conjugated ones of Bäcklund's transformations (stated in a modern fashion). Delfino Codazzi was awarded the second prize by the French Academy of Sciences for the problem set in 1860 for phrasing the conditions that two given quadratic forms must fulfil in order to be the first and second fundamental form of a surface. Afterwards, G. Mainardi pinpointed that he had already published these equations in an Italian journal in 1857. It was found out that Mainardi-Codazzi's equations had already been featured in Gauss's manuscript from 1825 mentioned above and already published by Gauss's estate, and that K. Peterson had already used them in Dorpat (Tartu, Eesti) in his dissertation in 1853.

The generous use of differential calculus was of advantage for "local thinking" in geometry, and some not exactly precise claims of the $18^{th}$ and early $19^{th}$ century can be traced back to the wrong assumption that all that is locally possible can also be "continued" without problems. For example, Meusnier believed himself able to prove in 1776 that spherical surfaces and their parts are the only surfaces of constant positive curvature. Other surfaces of constant positive curvature were only found towards the end of the $19^{th}$ century. However, these feature singularities (Illus. 7.4.4). In 1890, Felix Klein phrased the question of those two- or three-dimensional manifolds of constant curvature in which it is possible to mark a geodesic of length $r$ from every point in every direction. This question was later generalised as the spherical space problem of Clifford-Klein to apply to any dimension. Hence, Euclidean, Lobachevskian or spherical geometry applies locally in such spaces. For instance, for the case of curvature 0 and dimension 2, we must also consider every curved cylinder surface of infinite length apart from the plane. Only Heinrich Liebmann proved around 1900 that complete spheres are the only surfaces of constant positive curvature free of boundaries and singularities. David Hilbert showed in 1901 that a surface of constant negative curvature cannot exist in three-dimensional space without boundaries or singularities, in other words, that the flaw of the "pseudo-sphere" (as used by Beltrami)



**Illus. 7.4.4** Surface of constant positive curvature.
This surface is created by cutting open a spherical surface along a meridian and sliding the cutting edges into each other. However, as a result, peaks are created, i.e., points without tangential plane, at the poles. [Gerd Fischer: *Mathematische Modelle* (Mathematical Models), Vieweg Verlag, Braunschweig/Wiesbaden 1986]

of only being a local model of non-Euclidean geometry was unavoidable if we want to realise it as inner geometry of a surface in $\mathbb{R}^3$.

For newer historical descriptions that go more into technical details, we refer the reader to [Reich 1973], [Scholz 1980], [Fischer 1986, Volume of commentaries, chap. 3] and the article on differential geometry in [Dieudonné 1985], as well as the commentary by Böhm and Reichardt in Teubner-Archiv, vol. 1 (Gauss, Riemann, Minkowski). However, this does not make the extensive historical and bibliographical details by Wangerin in his remarks on the first edition of Gauss's theory of surfaces redundant.

## 7.5 Non-Euclidean geometry

We immediately follow up on 6.4 by remarking that there is plenty of accessible detailed literature on this important portion of the history of geometry (amongst others, [Engel/Stäckel 1895], [Sommerville 1911], [Bonola 1911], [Bolyai/Stäckel 1913], [Sjöstedt 1968], [Reichardt 1985], [Trudeau 1987]). Thus, we will restrict this section to a relatively brief description of the course of time and content, but discuss some aspects little mentioned until now and follow the "broad effects" of the events.

Multiple passages from Gauss's letters show that he had already begun to engage with the parallel problem in 1792, along with the young Hungarian Wolfgang (Hungarian: Farkas) v. Bolyai. They studied in Göttingen at the same time and their interest in that problem was an essential catalyst for their friendship, which thrived in an irregular correspondence over the years. W. v. Bolyai's own contribution lies within the fact that he could deduct the uniqueness of the parallels based on the condition that a circle passes through each of the three points that do not lie on a mutual straight line. His textbook *Tentamen*, written in Latin and with the intention of being used for teaching mathematics at grammar school, was published in Hungary in 1832. At this time, it was not uncommon – partially as the result of Legendre's widespread books on geometry – to look at the parallel problem in such textbooks by means of, if nothing else, pedagogically well-balanced marginalia or footnotes. For instance, in 1834, J. A. Grunert, back then still professor at a grammar school in Brandenburg (Germany), wrote in his textbook *Lehrbuch der ebenen Geometrie für die mittlern Classen höherer Lehranstalten* (Textbook on Plane Geometry for the Middle Classes of Higher Educational Institutions):

"As known, the mathematicians have found the theory of parallels difficult since Euclid's time, and it cannot be our intention here to look at and explain this subject matter more closely, but rather to try to lift these difficulties themselves, since we would need an independent detailed treatise for this, which does not exactly belong in an elementary book like this one presented to you here."

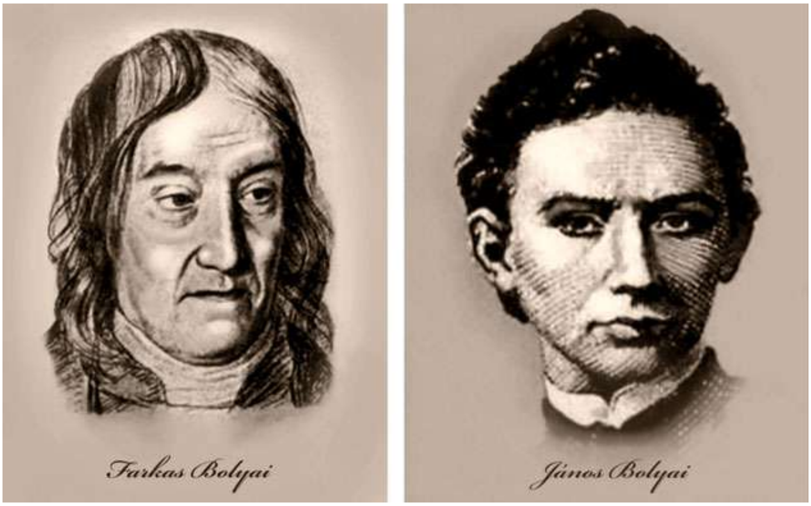
However, he could not help but present his own "view on this subject matter in a few words" (l. c. p. 51). To sum up, this view basically amounts to the fact that Euclid's other axioms and postulates are local – so to speak – and also states some aspects about the conditions for the congruence of triangles. However, the gist of the $5^{th}$ postulate states a "global" property of the plane and, hence, we cannot expect to be able to do without it (i.e., to prove it by means of the other axioms).

The text by Wolfgang's son Johann v. Bolyai was attached in the *Appendix* of the *Tentamen*. As had some before him, Johann encountered correlations by means of his own attempts to derive a contradiction based on the negation of the parallel postulate, which contradicted the naive concept, but seemed more and more meaningful to him the more he looked at it. Above all, he found that the created alternative for Euclidean geometry so depends on a constant that the ratios arbitrarily approach the Euclidean ratios given that the constant is sufficiently small. The basic tendency of his appendix lies within developing geometry, first of all, without using the parallel postulate or its negation as far as possible (following his wording, this part of geometry was later referred to as absolute geometry) and to develop the consequences of the negation of the parallel postulate so in dependence of this constant that we encounter the Euclidean case as a boundary case of an infinite bundle of possibilities. He stressed repeatedly that we could not expect more from mathematics here and determining the constant would be up to physical measurements, whereby the case of Euclidicity generally cannot be verified, since the derivation may be so insignificant that it cannot be determined within the scope of the possible accuracy of measurement. Next to many other individual results, he also showed that squaring the circle is possible with compass and straightedge in the case of non-Euclidicity. After this brief description of the content of the appendix, the title of his text is clear: *Raumlehre, unabhängig von der (a priori nie entschieden werdenden) Wahr- oder Falschheit des berüchtigten XI. Euklidischen Axioms*[4] *: für den Fall einer Falschheit derselben geometrische Quadratur des Kreises* (Theory of Space Independent of Truth or Falsity (never decided a priori) of the Infamous XI. Euclidean Axiom: For the Case of Falsity of the Same Geometrically Squaring the Circle). (A German version of the entire text composed by him in 1832 can be found in [Bolyai/Stäckel 1913], [Reichardt 1985] and others.)

We have already mentioned how vehemently the father warned his son beforehand not to get too deeply involved in this problem. (Further such letter passages can be found in [Bolyai/Stäckel 1913], [Reichardt 1985, p. 56 ff.] and others). These passages also reflect that the publication of the appendix in this given form was only finalised after longer discussions between father and son. Nonetheless, the elder Bolyai sent his son's work to his old friend Gauss and it is completely clear that both Bolyais not only counted on a strong affirmation, but also expected above all that the already very famous Gauss would publically approve and defend the new theory. Praise and approval followed, however in a manner that must have deeply hurt and disappointed the already psychically weak J. v. Bolyai:

"Now some things regarding your son's work. If I start by saying "that I must not praise such", you will be speechless for a moment. But I cannot do otherwise; to praise it would mean praising me, since the entire content of the text, the path your son chose and the results to which it led him already occur almost completely in my own meditation, which I have been engaging with for almost 30-35 years at parts. Indeed, I am extremely surprised at this. My intention was not to release any of my own work during my lifetime, of which I have hardly written down anything, by the way. Most people are not

[4] $5^{th}$ postulate




**Illus. 7.5.1** Wolfgang (Farkas) Bolyai and his son Johann (Janos). The "portrait" of Jonas Bolyai does not show Janos. It is a pseudo-portrait from a Romanian stamp that appeared 1960 on the occasion of the $100^{th}$ anniversary of his death. Because no real painting of Janos was found a portrait of a young man (probably member of the House of Habsburg) was used for the stamp, printed also on Hungarian stamps and in many books (see [Schreiber 2005c])

of the right mind for this, which matters here... Thus, I am very surprised that I can save these efforts [to work out everything in writing] and I am delighted that it is the son of my old friend who beat me to it in such a strange manner..." (Translated from the quote in [Reichardt 1985, p. 59f.]; some pages on suggestions of how to improve the content follow.)

Indeed, Gauss had occasionally mentioned his view on the parallel problem in letters to good friends and praised Johann v. Bolyai's work, but avoided any public statement and refused to refer to his statements made in letters in an almost abrupt manner. Having found out about the investigations of the legal expert F. K. Schweikart and his nephew F. A. Taurinus (partially printed in [Engel/Stäckel 1895]), he wrote to the latter in 1824: "I do not fear of a man who has revealed himself to me as a thinking mathematical mind that he could misinterpret that given herein [a short description of Gauss's results and views]: however, in any case, you must view this as a private message, which you must not, in any case, use in a public manner or in a manner that leads to publication." (Translated from German into English based on [Reichardt 1985, p. 39])

Completely independent of Lambert, Gauss, Bolyai, Schweikart and Taurinus, N. I. Lobachevsky had already given a first public talk on non-Euclidean geometry in distant Kazan in 1826. Several publications on this subject followed from 1829-40. 'Geometrical Investigations on the Theory of Parallel

**Illus. 7.5.2** Carl-Friedrich Gauss and Nikolai Ivanovich Lobachevsky

Lines' was even translated into German in 1840 (engl. translation by Halsted 1891). Lobachevsky had a completely different social rank as a meritorious vice chancellor at the University of Kazan for years and, above all, was in a completely different state of mind than the officer J. v. Bolyai, who had retired early and been surrounded by scandals. He did not give up on defending the theory of parallels, which he had believed to be true during his entire life, because of failures or non-approval. The main difference between this and Bolyai's theory was that, in Lobachevsky's work, the idea of constant (of curvature, stated in a modern fashion) and the deviation from Euclidicity, which cannot be perceived experimentally in a space of little curvature, did not play a dominant role. Of course, Gauss followed these works with the greatest of interest and was the reason that Lobachevsky became appointed corresponding member of the "Göttinger Gelehrten Gesellschaft" (Scholar Society of Göttingen). It is also true that he began learning Russian in his old age. However, it is just a myth that he did so to be able to read those works by Lobachevsky that were written in Russian.

Although, as mentioned, the parallel problem was discussed on different levels by many people at that time, even in textbooks, the publications by Bolyai, Lobachevsky and Schweikart did not have any effect. This only changed after Gauss's opinion on this subject became generally known after his death, first in 1856 due to a remark by his first biographer, Sartorius von Waltershausen, and even more greatly due to the publishing of the Gauss-Schuhmacher correspondence (6 volumes, 1860-65). One consequence was that Lobachevsky's 'Pangeometry' was published in France in 1856 and in Italy in 1867 and his main work (1840) in a French translation in 1866 (with an extensive commentary by the translator J. Hoüel [Sjöstedt 1968], who, as a result, had to be accepted as the pioneer of non-Euclidean geometry in France). Bolyai's

*Appendix* was only translated into Italian in 1868. The article 'Über den neuesten Stand der Frage von der Theorie der Parallelen' (On the Latest Updates on the Questions of the Theory of Parallels), published in *Archiv der Mathematik und Physik* (Archive of Mathematics and Physics) in 1867 and written by the already mentioned Grunert, draws a good picture of the time. Grunert had been appointed mathematics professor at the University of Greifswald by then and was founder and publisher of 'Archiv', a journal mainly targeted at teachers and spread across Europe. There he writes, amongst other things:

"...it seems now that uniting the views of the new geometers, partially very important voices [!], to make a decision about this [namely about the sum of angles in a triangle] in believing that the a priori theoretical investigation of the above (Legendre's theorems listed in 6.4) had reached its end, and nothing else is left than asking experience. Hence, geometry is at least in *one* point a science of experience!!" (Grunert Archiv 1867, p. 319).

We have now arrived at the crucial question of the philosophical interpretation of all the efforts (and further ones not looked at here) mentioned so far. The highly regarded philosopher Immanuel Kant from Königsberg had multiply decreed, mainly in his '*Critique of Pure Reason*' (1781), that Euclidean geometry is just as necessary a subject matter as the notion of natural numbers, a priori, given before all experience, so-to-speak the empty shelf, in which we then put the experiences. Which role such an authoritatively spoken opinion played back then can perhaps be better understood if we remember that mathematics was a subject studied within the scope of the philosophical faculties and that, naturally, all prospective mathematics teachers at the higher schools of general education (from which the professors of universities and institutions of higher technical educational were recruited) had to engage with a significant amount of philosophy as part of their studies. As a result, many of them had a priori philosophical tendencies for their entire life.

It is clear in the works of all authors of non-Euclidean geometry (the only exception is Lambert!) that they could only view the question under the aspect of possible doubt over the Euclidicity of physical space. Mathematics was a natural science; geometry was proto-physics, so-to-speak, the theory of empty but actually existing space. Schweikart named his geometry "Astralgeometrie", since he thought that their possible applicability could only be seen on a cosmic scale. J. v. Bolyai defended the same opinion. Lobachevsky spoke of "imaginary" geometry, since we can only "imagine" it as long as we have not proven its "truth". Gauss wrote to Taurinus in the often quoted letter: "All my efforts at finding a contradiction, an inconsequence in this non-Euclidean geometry have been fruitless and the only thing that resists our mind is that there would have to be a certain special linear magnitude (although unknown to us) in space, if it was true [!]" [Reichardt 1985, p. 38]. A passage in a letter from Gauss to his friend Schumacher from Nov. 28, 1846 is also highly remarkable: "It [the German edition of Lobachevsky's theory of parallel lines] features the basics of such a geometry, which had to take place [!] and strictly could take place, if the Euclidean is not the true one... You know that I have been convinced of this for fifty-four years already (with a certain later extension, which I will not mention here);..." (translated from German after [Reichardt 1985, p. 77]). Apart from repeatedly stressing non-Euclidean geometry as merely an alternative for the structure of physical space, it seems very plausible (and this is in contrast to Reichardt's assumption following the quoted passage) that Gauss already hinted at his insight gained in 1827 as part of his remark added in brackets that plane non-Euclidean geometry is the inner geometry of surfaces of constant negative curvature. Next to the repeatedly stated and surely applicable view that most mathematicians would not understand him and that "the wasps would unnecessarily fly around his ears", as he had once expressed, as well as Gauss's known aversion to all the turbulence of external life, a reason for his careful lifelong reservation was that he was missing the spatial analogies for curved surfaces. This would explain the unusual reaction that he showed towards Riemann's habilitation speech in 1854. According to witnesses, Riemann had simultaneously answered two open questions with which Gauss had fought a long time himself: how could be structured a space of higher dimension than three, in which we could imagine a three-dimensional curved space embedded? How could we generalise inner geometry of surfaces to inner geometry of curved spaces of higher dimension? It seems that imagining a curved space without a surrounding space of higher dimension, in which the curvature could be located, laid outside all psychological possibilities, like recognising that physical space can, or even must, in general, be inhomogeneous. In the quote, Gauss speaks of "either" (Euclidean) "or" (non-Euclidean in the classical sense of Lobachevsky-Bolyai). The modern extensive meaning of the words 'space' and 'non-Euclidean' is the result of much later developments.

### The first models of non-Euclidean geometry

In 1868, E. Beltrami published his famous and oft-quoted article *Saggio...* (English: Attempt at an Interpretation of Non-Euclidean Geometry) in the Italian language at the age of 33, explicitly referring to Gauss and Lobachevsky (without, however, mentioning Bolyai; cf. the year of publication of the *Appendix* in Italian, as stated above). Its essential content is proof that non-Euclidean geometry is the inner geometry of the surface of constant negative Gaussian curvature created by revolving the tractrix (i.e., a very special surface). It seems that Beltrami was first to use the term "pseudosphere" in this context. Beltrami's article finally made non-Euclidean geometry legitimate in the eyes of the mathematical public, since now it had been interpreted in the real world. In 1868, Beltrami still had doubts that something analogous would be possible for spatial non-Euclidean geometry. Having read Riemann's habilitation speech printed in 1868, he expanded his investigations to the spatial case. Meanwhile, Felix Klein had found a general model of non-Euclidean geometry within Euclidean geometry in 1871 that works completely analogously for the plane as for the spatial case. Based on the manner with which A. Cayley had explained measuring lengths and angles in projective space by means of an imaginary surface of second order (cf. 7.7), Klein now showed that it was possible to obtain models analogously for non-Euclidean space on the inside of a non-degenerated surface of second degree, whereby the intersection with any plane yields a model of non-Euclidean plane bound towards the outside by a conic section. Klein also introduced the term 'hyperbolic', since then customary for non-Euclidean in the sense of Lobachevsky-Bolyai, 'elliptic' for the geometry locally applicable to the sphere with sum of angles $> 180$ degrees, and 'parabolic' for the Euclidean case in this article. Thus, he had once and for all eliminated concerns regarding the possibility of a three-dimensional hyperbolic geometry, as had been explicitly stated by Bolyai and Beltrami, and presumably also mentioned by Gauss and others. Furthermore, he had wiped away the flaw of the pseudosphere of not delivering a global model due to the unavoidable singularities in three-dimensional space. Influential textbooks affirmed the position of non-Euclidean geometry, including those by J. Frischauf (1876) and W. Killing (1885), as did the proof stated by R. Lipschitz that the laws of mechanics are maintained in hyperbolic space, and the multiple contributions by H. v. Helmholtz for non-Euclidean geometry, partially by means of clever popular-scientific talks (for an example, see the talk [Helmholtz 1870], which is still a pleasure to read nowadays, excerpts in appendix A.9, p. 574).

Below, some critical remarks concerning the history:

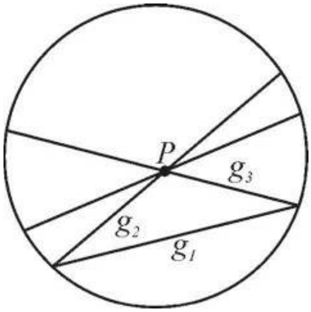
1. Both the basics of Klein's and Poincaré's (plane) model, which we will look at later on, are comprehensible. If we compare the original descriptions by Beltrami, Klein, Poincaré and others from the $19^{th}$ century, we are confused because of the wealth of complicated formulae and computations, which seem to be secondary from a modern perspective. Apart from the afore-mentioned differential-geometrical tendency fashionable in the $19^{th}$ century (unfortunately, trends and authorities still play an important role nowadays regarding the level of attention caused by a mathematical accomplishment), another main reason lay within the fact that, before the modern axiomatic foundation of Euclidean geometry first established by D. Hilbert in 1899, it was not clear at all which theorems in a "model" (this notion did not yet exist back then!) had to be proven as valid. After all, Euclid's axioms and postulates form a rather incomplete basis for axiomatic systems in the modern sense. Thus, all efforts were directed at finding the analoga of the formulae of Euclidean trigonometry and of measure determinations (lengths, angles, areas, volumes) in the non-Euclidean geometries (here including the spherical one), and then proving them as applicable in the found "Versinnlichungen" (sensualisation) (expression from Klein!). In contrast to checking a complete axiomatic system, this is an 'open' and rather incomplete program.
2. As repeatedly emphasised, before the rise of mathematical logics with its clear notions of formal language (or at least a sharply bound system

of notions), axiomatic systems, interpretation, model, etc., briefly, before the transformation of mathematics from a natural science to a science of structure non-Euclidian geometry could only be looked at through the aspect of its possible validity in the physical sense. The models found by Beltrami and Klein were also not grasped by the authors themselves as models in the modern sense, but only very vaguely as proof of logical consistency. Rather, visualisation or sensualisation served to strengthen the conviction that such a geometry, which until then had contradicted the previous visualisation, was possible in the physical sense.

3. The historical significance of Beltrami's publication from 1868 shall not be challenged. Nonetheless, it would have been utterly possible to deduct this result far earlier from the works by Gauss and others on inner geometry of surfaces of constant negative curvature. Furthermore, we find the basic concept of the model by Klein implicitly in Beltrami's work. Beltrami had the general intention (also in other publications) of representing curved surfaces in the plane in such a way that the geodesics thereby are transformed into straight lines. He also did this with the pseudo-sphere. If we look at his representation, we basically see Klein's plane model (Illus. 7.5.3). Having heard of Riemann's habilitation speech, Beltrami extended his model to a spatial one inside a sphere in the same year (1868, published in 1869). It remains Klein's accomplishment to have connected this model with Cayley's measure determination and, thus, to have included it in a general context. In contrast, substituting the interior of a circle for the interior of any non-degenerated conic section is rather tedious in its didactical aspects, if we are only focussed on a preferably elementary proof of consistency. In 1871, Klein referred to Beltrami's work from 1868, although it seems that he was unaware of the article from 1869, since he wrote rather unclearly: "Beltrami, to whom we owe the respective sensualisation of hyperbolic geometry, has proven that something analogous is not possible for space." [Nachrichten von der Kgl. Gesellschaft der Wissenschaften Göttingen (News from the Royal Society of Sciences, Göttingen) 1871, p. 626].

Hardly any other intra-mathematical advancement has ever provoked as much public attention as non-Euclidean geometry. And the mathematicians' "camp" remained split after Beltrami's and Klein's publications.

Next to disputatious propagandists (amongst them, for example, the radical-liberal Briton W. K. Clifford, and mathematics professor Kurd Laßwitz from Gotha, also known as an early writer of science fiction stories), there were militant opponents, such as J. Bertrand in France and I. C. V. Hoffmann (founder of the Journal for Mathematical and Natural Scientific Teaching) in Germany, some of whom took the position they did due to a conservative attitude (G. Frege, C. L. Dodgson, alias Lewis Carroll). Philosophers also felt affected and threatened in their very own area of expertise, such as, amongst others, R. H. Lotze and E. Dühring. Professor F. K. Zöllner in Leipzig, also known



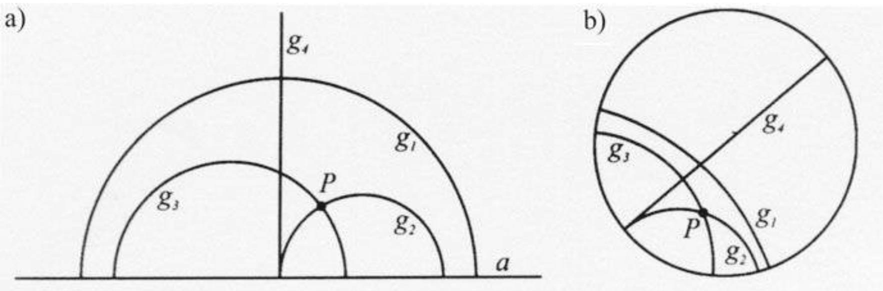

**Illus. 7.5.3** Beltrami-Klein Model for the plane case
The "world" only consists of the points inside the circle. Between the limiting parallels (also horoparallels) $g_2, g_3$ to $g_1$, there are an infinite number of further straight lines through $P$ that do not reach $g_1$. The inside of a sphere yields an analogous model of spatial hyperbolic geometry.

for his extraordinary astrophysical achievements, concluded the existence of a surrounding four-dimensional space from the now possible curvature of three-dimensional space. He further concluded that intelligent four-dimensional beings could live in this four-dimensional space. His spiritualistic attempts to make contact with these beings provoked an immense furore around 1880 [Wirtz 1887]. Finally, Helmholtz's appealing thought experiment of putting oneself in the position of two-dimensional beings living on a curved surface inspired a whole genre of literature [Helmholtz 1870] [Sjöstedt 1968] (see Appendix). For instance, *Flatland* (1884) by the British educational expert E. A. Abbott (no mathematician but a theologian; see A.10) and "Bolland" (1957) by the Dutchman D. Burger were translated into several languages and have remained successful up to the present day.

In 1881, Henry Poincaré found a new model of plane non-Euclidean geometry in correlation with investigations on applying conformal mappings of complex number fields to solve certain differential equations without first being aware of this geometrical interpretation. Through the medium of Felix Klein, he wrote a report summarising this for the journal *Mathematische Annalen* in 1882. Only through the thought exchange with Klein did gradual awareness of the model character of Poincaré's construction rise. Following the work *Sur les hypothéses fondamentales de la géométrie* published in 1887, Poincaré, with a notional clarity unmatched until then, represented his found model in his scientific philosophical book *La science et l'hypothése* first published in 1902 and translated into German in the same year (by F. Lindemann and his wife): $a$ is any straight line of the Euclidean plane, $H$ one of both open half-planes bounded by $a$. In the non-Euclidean sense, points are all points of $H$and straight lines are semi-circles in $H$, the centres of which are located in $a$, as well as all half-lines in $H$ standing perpendicularly on $a$ (which can be grasped as degenerated cases of the semi-circles named above for the case of a radius tending towards the infinite). Since every non-Euclidean congruence mapping can be produced by composition of (3 at most) straight line reflections, it suffices to say what reflections are in hyperbolic terms, namely inversions on the relevant semi-circles or half-lines, consequently conformal mappings (true to angle). All this can be done analogously for the three-dimensional case, whereby a plane takes the place of $a$, a half-space takes the place of $H$ and the non-Euclidean planes are represented by hemispheres and/or half-planes. When describing this model, Poincaré speaks of "some kind of dictionary" for the first time, with the help of which every proposition of plane geometry (regardless of whether it applies to Euclidean or non-Euclidean geometry, or to neither) can be "translated" into the corresponding proposition of this model. Finally, we can also check the validity of his basic propositions of hyperbolic geometry. Apart from the fact that this publication naturally preceded Hilbert's 'Foundations', the formal (this word seems to have a negative aftertaste) or, more appropriately, syntactic standpoint clearly breaks through: We have a system of notions. If any interpretation of these notions is given now, we can check if certain theorems become true, i.e., if it represents a model for these theorems.

Poincaré and many after him were fascinated with the property of the model that every hyperbolic angle is represented by a Euclidean angle of the same size (which does not apply to Beltrami's and Klein's models). Furthermore, he saw the opportunity to apply hyperbolic geometry independently of the question of whether it is valid in physical space by reverse-interpreting its theorems as theorems about conformal mappings.

Of course, we can represent the entire model by conformal mappings onto the inside of a circular disc. However, it would still differ from the Beltrami-Klein model, since all straight lines are represented as straight chords there




**Illus. 7.5.4** Poincaré's model. a) in a Euclidean half-plane, b) inside a circle Model b) is created by a conformal mapping of a). It still differs fundamentally from the Beltrami-Klein model, since the straight lines inside are not represented by chords, but by perpendicular circular arcs on the boundary circle. There are also spatial analoga for Poincaré's model in a semi-space bound by a plane or inside a sphere.

(Illus. 7.5.4). Only much later were some other properties of Poincaré's model realised, also distinguishing it from Klein's model: constructions of hyperbolic geometry, which can be done with compass and straightedge there, can be simulated by construction with compass and straightedge in the Euclidean sense here ([Schreiber 1984], [Schreiber 1996a]]; Problem 7.5.1).

The last outcome of the $19^{th}$ century worth mentioning concerning hyperbolic geometry was also only published in print in 1901. Analogous to spherical geometry, a triangle is also uniquely determined by its three angles in hyperbolic geometry. As known, the relevant construction problem regarding spherical geometry had its solution coming for a long time, until Vieta and Snellius found the polar principles, which reduce this problem to the problem of constructing a triangle based on its three sides. Lobachevsky and Bolyai had failed to solve the analogous problem for hyperbolic geometry. Now, H. Liebmann also found a kind of polar or duality principle for hyperbolic geometry, with the help of which we could reduce a construction based on three angles to the construction of a "polar" triangle based on its three sides. However, it only works directly for right-angled triangles, and no doubt made the construction possible for the general case, but only very complicated. (How lucky that we apparently never have to do it! See [Liebmann 1901], [Schreiber 1984].)

Until now, we have restricted the notion of non-Euclidean geometry to Bolyai-Lobachevskian hyperbolic geometry. However, hand in hand with its increasing acceptance and parallel to the development of inner differential geometry, awareness also grew that the hyperbolic space form is just one of many possible non-Euclidean forms. Of course, spherical geometry also defended its ancient right to a sensible alternative in a certain way. Nonetheless, it seems that geometers of the waning $19^{th}$ century were more interested in the example created when only looking at a hemispherical surface and identifying each pair of diametrically opposite boundary points. This has been referred to as the elliptical variant ever since then. On one hand, we obtain a geometry that corresponds locally to the spherical one with a difference between angle sum and 180 degrees, which is proportional to the area. However, each two geodesics corresponding to the straight lines always have only one intersection. On the other hand, this geometry is isomorphic to the geometry of the bundle of straight lines carried by the centre of the sphere.

In clear reference to the title of Riemann's habilitation speech after its first publication in 1866, H. Helmholtz subjected Riemann's space forms to the additional condition of free mobility of fixed solids in two published presentations: *Über die thatsächlichen Grundlagen der Geometrie* (On the Actual Foundation of Geometry, 1866) and *Über die Tatsachen die der Geometrie zum Grunde liegen* (On the Facts on which Geometry is Based, 1868) from the physicist's viewpoint. Based on this, he further deducted that only spaces of constant curvature could then be considered. Simple examples, like an infinitely long cylindrical surface, which has the constant Gaussian curvature of zero, already show that Helmholtz's condition makes globally different structures possible for every one of the three cases of curvature greater, smaller or equal to zero. These investigations were especially uplifted when Clifford discovered a finite surface of the constant curvature of zero in 1873, the global topological structure of which looks like a ring (see, for example, [Klein 1928], Chap. VIII and IX). This topic was called the space problem of Clifford-Klein since 1890, which means listing all such possibilities and proving their completeness. (There is also occasional reference to the names of Riemann, Helmholtz and Lie.) Indeed, it seemed that this question was of cosmic relevance before the rise of the theory of relativity.

## 7.6 Notion of vector and *n*-dimensional geometry

From our modern perspective, every mathematical problem featuring $n$ variables, takes place in an $n$-dimensional space. For instance, the names for higher powers of the unknown, introduced by Diophantus and conserved by Vieta until modern algebra was established, show that algebra often benefitted more or less consciously from analogous ideas between the case of $n \leq 3$ and the higher "dimensional" case: $\delta$ (dynamis) for $x^2, \kappa$(kubos) for $x^3, \delta\delta$ for $x^4$, $\delta\kappa$ for $x^5$, $\kappa\kappa$ for $x^6$, ... It seems particularly that the analogies between producing a line segment by linearly moving a point, producing a rectangle by linearly moving a line segment perpendicularly to it and producing a cuboid by linearly moving a rectangle perpendicularly to it inspired the mathematician's imagination early on. We find traces of speculation about this in both medieval Islamic and European mathematics. Abū-l'Wāfā imagined something like an $n$-dimensional analogon for Pythagoras's theorem around 970. With this, he wanted to transform the sum of more than three squares into one square by means of geometrical construction. Michael Stifel, in his edition of *Coß* by Christoph Rudolff in 1552, expressed his regret that it is not "allowed" in geometry to look at things that have no "shape", like, for example, the motion of a point perpendicularly to all three edges of a cube. As long as geometry was accepted as proto-physics, meaning as the theory of true physical space and, hence, as a "natural science", we could hardly expect a fundamental turn in this respect. The greatest possibility until then, namely interpreting time as a true fourth dimension, is only found explicitly in the article 'Dimension' by d'Alembert in the French Encyclopédie (1764).

We would remind the reader of Plücker's four-dimensional space of all straight lines of $\mathbb{R}^3$ (1846) and his approaches to grasp algebraic (or geometric?) objects, for the characterisation of which we need more coefficients, as elements of a higher dimensional space (1868). Cauchy's ideas, which matured around the same time, were aimed in a similar direction. He basically wrote in an academic note on 'Analytische Örter' (Analytic Loci, 1847): If a function depends on two or three variables, we can look at its domain of definition as an amount of points in the plane and/or space regarding Cartesian coordinates.

Let us now assume that the amount of variables is greater than three. Then, every value system determines for these variables what we want to call an 'analytical point' and of what these values are the coordinates. If these value systems are subject to certain conditions expressed by inequations, these analytical points, the coordinates of which suffice for these conditions, form an 'analytical locus'. This 'locus' is bounded by 'analytical closers', which are expressed by equations, created if we substitute the signs "<" and/or ">" with "=" in the inequations describing the locus. We want to refer to a system of analytical points as an 'analytical straight line', the entire coordinates of which are linear functions of one of them. Finally, the 'distance of two analytical points' is the root of the sum of the squares of the coordinate differences of these points. Looking at analytical points and loci enriches the means to solve many difficult questions, especially those which refer to the theory of polynomials. (Free translation based on [Cauchy, Works, 1. Ser., vol. X, p. 292]).

The analogies occur most clearly between the illustratively imaginable and the higher dimensional when speaking of linear notions and problems. Hence, the actual transition to $n$-dimensional geometry is most closely connected to the development of linear algebra and the origins of the notion of vector. The word 'vector' does not occur in mathematics before the middle of the $19^{th}$ century, perhaps first in W. R. Hamilton's work. The vectors themselves occur first in mechanics as velocities and forces, whereby both the manner of naming and the physical interpretation connected to the notion of force remained nebulous for centuries. The addition of forces according to the parallelogram rule occurred around 1600 in Stevin's and Snellius's works for special cases, the one of velocities in Roberval's work from 1635 onwards. Pierre Varignon played a role (even if possibly exaggerated earlier by Bossut, Lagrange and others) when teaching composition of forces that is still hardly mentioned in the overall description of the mathematical and physical history of the $20^{th}$ century. He used the parallelogram of forces fully generally and the whole mutual compensation of $n$ forces, the vector sum of which yields the zero vector. We can also determine Fermat's point for $n$ points with the device invented by Varignon to determine experimentally the point for $n$ forces at which they are balanced. Monge's statics reached a relatively concluded form (*Traité élémentaire de statique*, 1788) and, above all, in the works of his student L. Poinsot (*Éléments de statique*, 1803, many edited versions and translations). In 1832, the Italian mathematician G. Bellavitis was first to express the fact that translations also act like vectors in regards to composition. (This astonishingly late recognition is surely related to the difficult birth of the notion of mapping in geometry. We will look at this again in 7.7.)

At the beginning of the $19^{th}$ century, the physical notion of vector received another great impulse due to the examination of electric and magnetic fields (amongst others, the discovery of electromagnetism by Hans Christian Oersted in 1820). Now, the magnetism of Earth was understood as a spatial vector field, the state of which is characterised not just by the direction within the tangential plane of Earth (declination), but additionally by the inclination (inclination towards this plane) and intensity. Gauss, his colleague from Göttingen, the physicist Wilhelm Weber, and the Norwegian Ch. Hansteen (Oersted's student) especially rendered outstanding services to the worldwide exploration and evaluation of Earth's magnetism. Notions such as rotation and divergence emerged in this context. However, this led to great advances in vector analysis on the basis of coordinate-wise calculations before there was clear vector algebra.

**Vector algebra, complex numbers and quaternions**

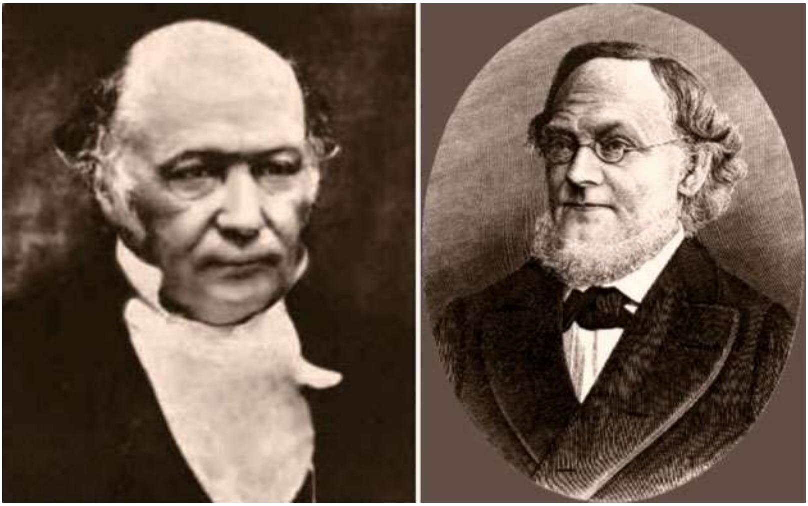
The geometrical interpretation of complex numbers played a strange role in the development of vector algebra. The Norwegian geodesist Caspar Wessel is nowadays accepted as its first (historically ineffective) founder (treatise 1797, published in Danish in 1799, for English translation see [Lützen/Branner 1999]). He had nothing else in his mind than an algebraic calculus for composing translations and rotations, the first expressed by the coordinate-wise addition (hence, the two-dimensional special case of vector addition), the latter by connecting the direction cosines, which corresponds to the addition theorems of sine and cosine and the multiplicative connection of real and imaginary parts of a complex number. Wessel also had a forerunner, although he did not know him, in the teacher Heinrich Kuehn from Danzig, who had published a Latin work on the geometrical visualisation of imaginary quantities in the proceedings of the Russian Academy of Sciences in St. Petersburg in 1751/52. The amateur mathematician J. R. Argand anonymously self-published a further work on this subject matter in Paris in 1806, hardly known at first. It was his explicit wish to state a geometrical interpretation of calculating with complex numbers that had been very obscure until then. His results included, amongst others, de Moivre's formula and observations regarding the fundamental theorem of algebra. This work became so well known after 1813 due to articles in Gergonne's *Annales* and the subsequent review by Cauchy and other significant French mathematicians that it bore a crucial influence. W. R. Hamilton studied it and let himself be guided by it when inventing quaternions (1843) for the construction of an algebraic calculus analogous to the complex numbers for the transformation of spatial vectors. Such a transformation consists of a scalar dilatation and/or compression (referred to as "tensor" by Hamilton) by a factor $a$, which acts multiplicatively during composition, and the revolution around the origin of coordinates in the new orientation, which depends on three parameters and acts like the outer product of vectors during composition. Thus, the quaternions form (stated in a modern fashion) a four-dimensional vector space, the elements of which consist of a one-dimensional "scalar part" $ae$ and a three-dimensional "vector part" $bi + cj + dk$ with the units $e, i, j, k$. Thereby, addition takes place component-wise and multiplication according to the rules below:

$$ee = e,\ ei = ie = i,\ ej = je = j,\ ek = ke = k,\ ii = jj = kk = -e,$$
$$jk = -kj = i,\ ki = -ik = j,\ ij = -ji = k.$$

If we multiply the vector part of two quaternions according to these rules, we obtain a quaternion, the scalar part of which corresponds to the inner product of these vectors apart from the algebraic sign and the vector part of which corresponds to its vector product. An extensive description of Hamilton's own considerations can be found in, for example, [Coolidge 1940, p. 257ff.]. It was only made clear much later that he had found the first and also final extension of complex numbers for the price of no more commutative multiplication. Whereas quaternions have maintained a restricted meaning for algebra from the contemporary view, they have proven to be a complete detour for arriving at actual vector algebra, because they cannot also be generalised for higher dimensions. However, Hamilton was already a very famous and influential mathematician around 1850, so a strong group of followers for his quaternions calculus was formed.

### Grassmann's theory of linear extension

In the meantime, the grammar school professor H. G. Grassmann published his *Lineale Ausdehnungslehre* (Theory of Linear Extension) in Stettin (now Szczecin in Poland) in 1844. This work hardly received any attention back then, since, on one hand, it was written in a rather "philosophical" style that was free of formulae and unusual for his contemporaries, and, on the other hand, it was, to a great extent, leaps and bounds ahead of the imagination of the mathematicians of the time. Nowadays, Grassmann is accepted worldwide as one of the most significant mathematicians of the $19^{th}$ century. He was not just the real founder of geometry of $n$-dimensional space, but also. within this context. triggered impulses of algebra, which only fully came to blossom in the $20^{th}$ century. (Moreover, he accomplished many things in the areas of physics, linguistics and folklore, for which he received much more praise during his lifetime than for his mathematical work. Cf. [Grassmann 1911], [Crowe 1967], [Zaddach 1994], [Schubring 1996]). The attempt to explain how Grassmann arrived at his theory of linear extension, seemingly without any predecessors or prehistory, only leads us back to his exam paper from 1839 on the 'Theory of Low and High Tide', in which he makes ample use of the contemporary knowledge of physical vector analysis, and refers to his father Justus Grassmann, also a mathematics teacher in Stettin, who played a role in developing crystallography. Grassmann the elder had already stated in a textbook in 1824 that parallelograms and/or parallelepipeds should be grasped in a specific sense as geometrical product formations. When he turned towards crystallography influenced by the dynamic natural philosophy dominating in Germany back then, he based his idea on the hypothesis of forming forces and, thus, described orientation and area of boundary surfaces by means of



**Illus. 7.6.1** William Rowan Hamilton and Hermann Günther Grassmann

length and orientation of their (stated in a modern fashion) normal vector. (For details, see [Scholz 1989, p. 48ff.].)

Grassmann the younger immediately based his concept on the notion of extensive quantities of any (finite) level, which, in the case of first level, corresponds to a vector in $n$-dimensional space (for Grassmann: area of $n^{th}$ level). To make it understandable and to the point, we will subsequently use the term 'vector' to refer to Grassmann's "extensive quantity". Grassmann himself, however, never used this term. He may have rejected it, as physics described vectors as objects in space that are characterised by direction and length, whereas Grassmann focussed on any dimensional case from the beginning on and, above all, had a very affine grasp of it all, whereby the notions 'length' and 'direction' first have no meaning at all. $n$-dimensionality of space is indicated by the existence of a base (for Grassmann: "system of units") of $n$ vectors. On one hand, these are linearly independent and, on the other hand, produce every vector by means of linear combination. The central basic idea now lies within forming successive higher dimensional extensive quantities by means of an "outer" product formation of vectors (in the simplest case, the parallelogram spanned by two vectors ("bivector") and/or the parallelepiped spanned by three vectors), which, on top of that, are signed so that its outer product is not just linear but also alternating. In contrast to Hamilton's vector or cross product, which was just so gladly adopted by the physicists and for which the position of a surface is represented by the orientation of a perpendicular vector and its size by the length of this vector, Grassmann's "bivector" remains a two-dimensional entity and is also an element of another new vector space of dimension $\binom{n}{2}$.

After 1870, when a party of "Grassmannians" had formed, the Hamiltonians were temporarily referred to as "monovectorians" and the Grassmannians as "bivectorians". This dispute, however, took place mainly amongst physicists, who had started to use vector analysis long before vector algebra. The adoption of Hamilton's, Cayley's and Grassmann's ideas in physics was mainly triggered by J. W. Gibbs in the USA (1881) and O. Heaviside in England. The first textbook of this kind in Germany was *Einführung in the Maxwellsche Theorie* (Introduction to Maxwellian Theory) by A. Föppl, 1894. We will not look at this any further. (See contribution by K. Reich in [Schubring 1996].)

Grassmann's 'Theory of Linear Extension' from 1844 contains almost all basic notions and theorems of *affine* vector algebra, although in a language difficult to read back then; for instance, linear dependence and independence, exchange theorem, invariance of dimension, dimension formula for subspace mutually created by two subspaces, etc. The reaction to this was disappointing for him: there simply was none. Möbius, who would have been most able to put himself into Grassmann's world, declined to review the 'Theory of Linear Extension'. Upon his friends' advice, Grassmann published a completely re-edited, much more mathematical version of his theory in 1862. Only after this one had gained ground was a new edition of the version from 1844 (a tremendous portion of the publication of which was turned into pulp in 1864, owing to low sales) published in 1877. Whereas Grassmann mentioned only hastily in the introduction in 1844 that he had also discovered another product with interesting algebraic properties, apart from the outer one (which we nowadays refer to as "inner" or "scalar product"), he devoted more attention to this in an awarded text published in 1847 and titled *Geometrische Analyse geknüpft an die von Leibniz erfundene Charakeristik* (Geometrical Analysis Linked to the Characteristics Invented by Leibniz). This text, submitted to the Jablonowskische Gesellschaft zu Leipzig (Jablonowskich Society of Leipzig), was evaluated by professor Möbius in Leipzig [Möbius, vol. 1, 615-33]. Möbius then wrote: "Studying the present treatise by Mr Grassmann, and especially the last part of this, may be connected with some difficulties regardless of the author's unmistakable striving for clarity, which result from the author trying to account for his new geometrical analysis in a manner that is far from the previously usual course of mathematical considerations... Since his new analysis seems to deserve much attention due to its simplicity, with which we can conduct geometrical investigations, I have attempted in the following to establish it in a manner apt for the mind of geometry and, thus, as I hope, in a more easily graspable manner and to show how those apparent quantities can be looked at as shortened terms of real quantities."

The 'Theory of Linear Extension' from 1862 differs from that of 1844 distinctively, in amongst other ways, by including metric notions. The inner product of vectors occurs there as a special case of a much more general product, which can be applied to extensive quantities of any level (multivectors). This is introduced by pre-supposing a (stated in a modern fashion) orthonormalised basis of the entire space. If $A, B$ are such multivectors, then

Grassmann defines,for example, the fact that they stand perpendicularly to each other by making this product $[A/B]$ disappear, despite $A \neq 0,\ B \neq 0$, the numeric value of a multivector $A$ by $\sqrt{[A/A]^2}$ and the cosine of the angle between $A$ and $B$ by dividing their product $[A/B]$ by the product of their numeric values. We owe it to the Cauchy-Bunjakowski-Schwarz inequation that the latter is meaningful – for a number to be a cosine value, its absolute value cannot be higher than one. We will now adumbrate the history of this inequation, since it exemplarily demonstrates in which manner fundamental notions and approaches to thinking of modern mathematics result from concrete, yet coincidental reasons and via many detours, but also how geometry turns into algebra or analysis and vice versa at certain points of its development, and that nowadays such assignments are often completely pointless.

Lagrange, in whose work we find plenty of implicit $n$-dimensional geometry wrapped up in algebra, wrote down the following identity (easy to prove by complete induction) in 1773 (modern notation by us):

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \sum_{i<j} (a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \tag{7.6.1}$$

[Lagrange 1923/1924, vol. 3, p. 662f.].

Cauchy added an appendix (note II) to his *Analyse algébrique* in 1821. Therein, he dealt with inequations for the first time without any relation to certain applications. (We have already mentioned that he later also used inequations to describe $n$-dimensional subsets of $n$-dimensional Euclidean space nowadays known as semi-algebraic, i.e., surfaces in $R^2$, solids in $R^3$, ...) There, we find "Cauchy's inequality" as Theorem XVI in the following manner:

$$\sum\nolimits_{i=1}^{n} a_i b_i < \sqrt{\sum\nolimits_{i=1}^{n} a_i^2}\sqrt{\sum\nolimits_{i=1}^{n} b_i^2} \tag{7.6.2}$$

with the following addition: if the ratios $a_i/b_i$ are not all equal (otherwise we have equality due to (7.6.1)) as conclusion from (7.6.1). (However, Lagrange is nowhere cited as a source of (7.6.1)). Cauchy's inequality is born, curiously enough, without any relation to geometry and only by suppressing the information of (7.6.1) regarding the difference between both sides of (7.6.2). In 1829, Cauchy again returned to a special case of this inequality and this time there was a hint of geometry in there, since he applied it to the complex number field for the case of $n = 2$ in order to show that the triangle inequality applies there [Cauchy, Oeuvres, ser. II, tome IV, p. 573-609]. In 1859, the mathematics professor V. J. Bunyakovsky from St. Petersburg, who had been Cauchy's student in Paris, applied this inequality to equidistant function values in order to account in a very intuitive manner for the analogous inequality for the case that (stated in a modern fashion) the vectors are real functions

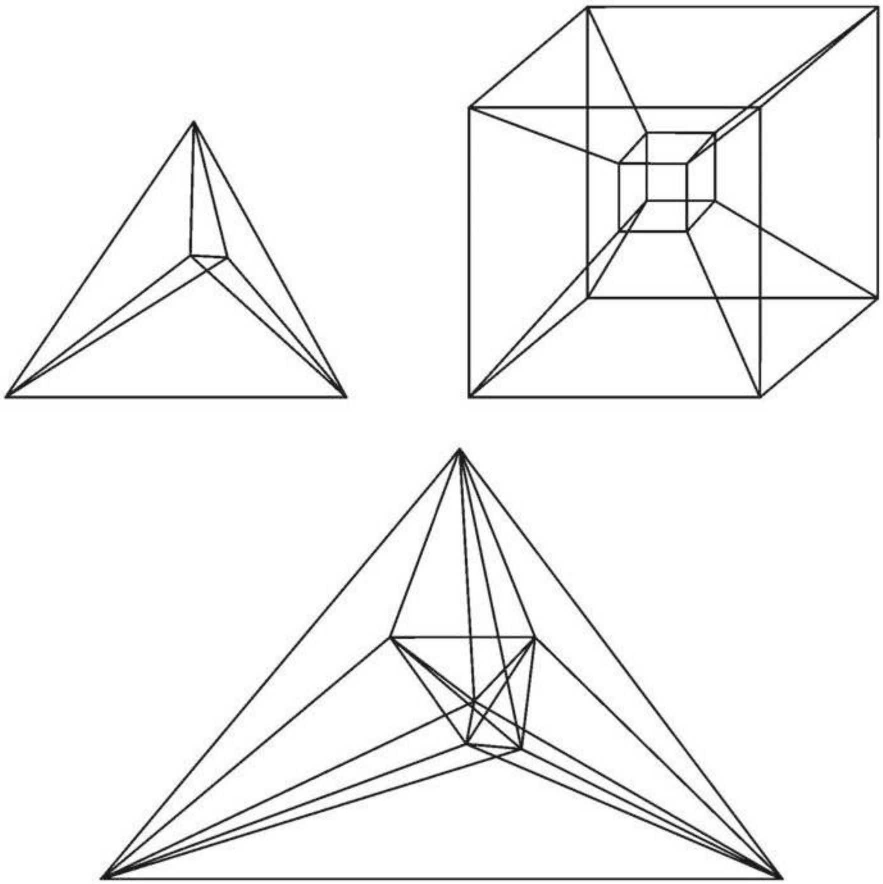
$f, g$ and that their inner product is formed as the integral of $fg$. In 1875, H. A. Schwarz phrased the corresponding analytical inequality for a special case he needed in his famous work on minimal surfaces, probably without knowing of Bunyakovsky . When F. Engel commented on both 'Theories of Linear Extension' in the edition of Grassmann's works he published in 1894/96, he noticed a footnote concerning Grassmann's definition of the angle between two multivectors, as mentioned above: "It is easy to show that the value of the term for $\cos AB$ lies between the boundaries -1 and + 1...". His following proof is based on introducing a vector basis and applying Cauchy's inequality (7.6.2) to the coordinate vectors. We first find in the book *Raum, Zeit, Materie* (Space, Time, Matter), written by Hermann Weyl in 1918, how much more elegantly we can obtain the inequality free of coordinates for any (also infinitely dimensional) vector spaces based on the axiomatic characterisation of the inner product (as bilinear, symmetric and positively definite), in order to apply it in reverse to the case that the vector space is a space of real $n$-tuples.

Let us go back to Grassmann. His groundbreaking ideas were, as stated, acknowledged very slowly. Hamilton himself was one of the first to comment on Grassmann's work. He wrote in 1853: "It is proper to state here that a species of non-commutative multiplication for inclined lines occurs in a very original and remarkable work by Prof. H. Grassmann with whom I did not meet till after years had elapsed from the invention and communication of the quaternions... Notwithstanding these and perhaps some other coincidences of view, Prof. Grassmann's system and mine appear to be perfectly distinct and independent of each other, in their conceptions, methods, and results..." (Quoted from [Zaddach 1994, p. 15]. The interested reader will find many more details about Grassmann, Hamilton and the history of their discoveries there.) In Germany, it was Hermann Hankel who was first interested in Grassmann's works and made them better known. However, it seems that the greatest impact was brought about by Giuseppe Peano with his book *Calcolo geometrico secondo* (i.e., following here, according to the role model) *l'Ausdehungslehre di H. Grassmann* published in 1888. We will later look at Peano's important role in fundamentally modernising the understanding of mathematics. We only want to state here that Peano's version essentially bases his work on the axiomatically grasped notion of vector space as we know it today.

### Further sources of $n$-dimensional geometry

We have reached an area where it becomes increasingly difficult to separate geometry from algebra, and particularly from linear algebra. For instance, there are determinants. They first occur implicitly in Leibniz's work (1693), and then are referred to by G. Cramer (1750), but only initially as formal terms in the solution of linear equation systems. The term 'determinant' was introduced by Gauss in *Disquisitiones arithmeticae* (1801). It seems that we owe the fact that this also turned into a piece of geometry to Cauchy, who had engaged with determinants since 1815 (whereby he introduced the effective notation as quadratic schema) and recognised the characters of determinants as multilinear and alternating functions of their rows or columns. Hence, the algebraic tool was already ready for recognising the significance of the determinants as orientated volume of the $n$-dimensional parallel solid spanned by $n$ vectors, as soon as just this solid itself existed as a notion, and also the realisation of the correlation between the determinants and the scalar triple product of vectors in the case of $n = 3$. The further organisation of linear algebra was mainly taken care of by the British mathematicians A. Cayley and J. J. Sylvester, who worked closely together. For instance, we can see how difficult it was for the participants back then to distinguish between algebra and geometry (this still applies to many mathematicians nowadays!) when looking at Cayley's work from 1843 with the promising title *Analytic Geometry in n Dimensions*, which basically does not feature anything but a theory for solving linear equation systems for the general case, clearly a sensationally modern idea given the circumstances back then (in the general case the rank of the system could be different from the number of variables and from the number of equations). If it was not like this, Cayley, instead of Grassmann, would have to be accepted as the father of $n$-dimensional geometry.

In correlation to the solution to a problem of probability calculus, W. K. Clifford determined the volume of an $n$-dimensional simplex and an $n$-dimensional sphere by means of complete induction over the dimension in 1866. Having concluded the two and three-dimensional case of his problem in the introduction, he wrote without any further ado: "Now consider the analogous case in geometry of $n$ dimensions. Corresponding to a closed area and a closed volume we have something which I shall call a *confine* [i.e., the enclosed]. Corresponding to a triangle and to a tetrahedron there is a confine with $n + 1$ corners or vertices which I shall call a *prime confine* as being the simplest form of confine. A prime confine has also $n + 1$ faces, each of which is, not a plane, but a prime confine of $n - 1$ dimensions." [Clifford 1866, p. 2]. The Swiss Ludwig Schläfli accomplished a further piece of real geometry of $\mathbb{R}^n$ (which could never be interpreted as linear algebra in any manner). His work, which could only be published in excerpts during his lifetime, supposedly due to its large size (but possibly because the publisher in charge did not understand the content) and was only published in full in 1901, six years after his death, has a title that does not tell the modern reader about its content any better than it did the readers back then, just like so many fundamental accomplishments of the $19^{th}$ century: *Theorie der vielfachen Kontinuität* (Theory of Multiple Continuity). Therein, Schläfli offers, amongst other things, an extension of the notions of polyhedra, corner, edge, lateral surface,..., convexity, regularity, etc., to include the $n$-dimensional case. He also introduces the 'Schläfli symbols', which are still used today to mark regular polytopes. His investigation climaxed in the proof that there are exactly six regular polytopes in $\mathbb{R}^4$ (Illus. 7.6.2), but that there are only three of such polytopes in all higher dimensions, namely



**Illus. 7.6.2** 4-dimensional regular simplex, 4-dimensional cube and the dual 4-dimensional solid bounded by 16 tetrahedra

the self-dual $n$-dimensional simplex (in generalisation of the tetrahedron), the $n$-dimensional cube and its dual solid. He conducted the analogous investigation for the regular decomposition of the surface of the $n$-dimensional sphere. Furthermore, he stated volume formulae for such entities and the classification of the motions in $\mathbb{R}^n$. In this context, we must also name Victor Schlegel, professor at the Higher School of Machine Building in Hagen, who had dealt with the higher dimensional, especially regular, polytopes and the possibility of tessellation of $\mathbb{R}^n$ with them in presentations and publications since 1883. His name has almost been forgotten, even in the specialised literature on polyhedra (e.g., [Cromwell 1997]). Only the "Schlegel diagrams", special projections of polytopes onto the Euclidean plane, remind us of him.

Another pioneer of true $n$-dimensional geometry was E. Betti in Italy, who, in 1871, befriended by Riemann and inspired by his works, continued the attempt, left unfinished due to Riemann's early death, to examine the (stated in a modern fashion) topological equivalence of $k$-dimensional manifolds in $n$-dimensional space. We will look at the consequences for the developing topology in 7.8.

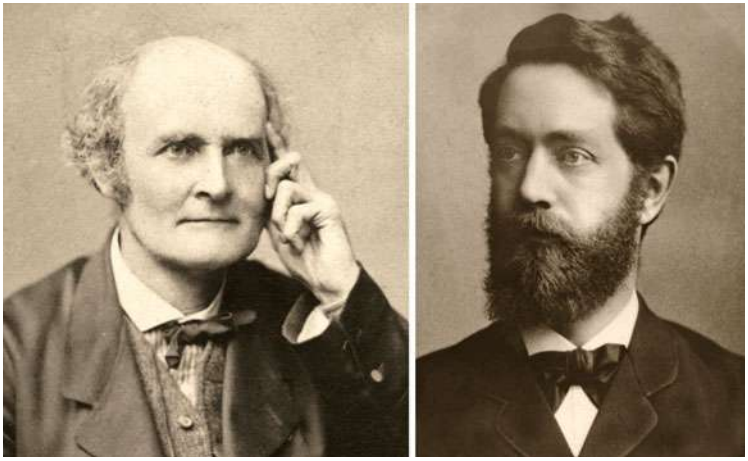
## 7.7 Transformation groups

For a long time, geometrical mappings were only looked at implicitly and by applied disciplines like optics, perspective, cartography or mechanics. The first three areas offer relatively little motive for thinking about the composition or reversion of mappings. Furthermore, the domain there is almost always different from the original domain. The notion of motion or rigid mapping differs in mechanics. There, however, reflections did not play any role, since they cannot be realised on material solids in space. Hence, it is understandable that the first implicit group-theoretical considerations concerned the area of motions (or congruence mappings) of Euclidean space onto itself, nowadays referred to as proper (i.e., orientation-preserving). The first finding possibly worth mentioning was that such a mapping can be represented as a screwing, i.e., as the composition of a revolution around a straight line and a translation towards this straight line, whereby one of these parts can be omitted so that translations and revolutions are kept as special cases. (Of course, there was a long way to go until the identical mapping was acknowledged as a mapping!) This is also only explicitly stated in a treatise by Chasles on the motion of solids in 1830. The classification and composition of spatial motions was further elaborated by O. Rodrigues in 1840 and L. Poinsot in 1851. Möbius made an attempt in 1838 to account for the chaining of motions synthetically free of reflections, which he had intended to be a contribution to mechanics. At the same time, we must point out that the analytical treatment of such questions, which dominated at this time, with traditional coordinate systems compared to other calculi, but especially the now customary representation of motions as a product of reflections, let the simple geometrical subject matter appear back then as being rather complicated. An alternative analytical treatment followed up on Hamilton's quaternions: in 1873, Clifford described the most general spatial motion of a solid by means of a biquaternion invented by him for this purpose. E. Study took a similar approach in his *Geometrie der Dynamen* (Geometry of the Screw Theory) in 1903. At the very least, the notion of a rigid mapping including the relevant implicit group-theoretical observations was so well established in 1866 that H. v. Helmholtz could use it to subject Riemann's manifolds, which could be considered as physically possible forms of space, to the additional condition of free mobility of rigid solids (cf. final part of section 7.5).

Carnot's blurry notion of (continuous) deformation of a figure in a "related one", which Poncelet narrowed down to the notion of projective mapping by means of composition of central projections, is a completely different source for the late triumph of the notion of transformation group within the entire geometry (cf. section 7.2). However, the proper father of the idea of mappings and group theory in geometry is Möbius. His *Barycentrischer Calcul* from 1827 was already inspired by the concept of geometrical relation in its general form. The second part of this book is titled *Von den Verwandtschaften der Figuren und den daraus entspringenden Classen geometrischer Aufgaben* (On the Relations of Figures and the Resulting Classes of Geometrical Problems). Projective mappings, the affinities resulting from fixing an improper straight line, the equiform transformations, the similarities as special cases of affine mappings and the "equality" as the special case of similarity are described on 150 printed pages. He determined the degree of freedom for each of these types of mappings, which we have when determining such a mapping, and afterwards classified notions that maintain their meaning under the listed types of mappings. (An appendix to [Staudt 1847] contains similar, but much briefer considerations without any comparison to Möbius or other forerunners.) Möbius always returned to this principle in later works, for instance, in 1834, when studying the not necessarily linear mappings (which, however, preserve area ratios), in 1846, when dealing with the motions of the spherical surface as such, and in 1852, when stereographically mapping those onto the plane, whereby the corresponding circle-preserving mappings are produced by composition of reflections at the circle (1853, 1855),this is why geometry of this group of mappings was later referred to as Möbius, but above all, in 1863, in his *Theorie der elementaren Verwandtschaft* (Theory of Elementary Relations), in which he attempted to describe those mappings that are now called topologic (also see section 7.8).

A. Cayley founded the matrix calculus during a thought exchange with J. J. Sylvester from 1845 onwards. As known, within the scope of linearity, this calculus permits us to treat all relations between the different types of mappings, their group-theoretical properties and their invariants purely algebraically and, on top of this, to generalise this without any problems for $n$-dimensional space. Of course, the following methodological key idea does not play any role here. Geometrical mappings are also objects of geometry. A matrix is the coordinate formation of such an object regarding a coordinate system that is at least projective. The advantage of homogeneous, or inhomogeneous, but linear coordinate systems (in contrast to, for example, polar coordinates) also lies within the fact that we can transform composition and reversion of linear mappings as well as determination of special properties and invariants into algebraically well manageable procedures. At the same time, the triumph of linear algebra found herein and the closely related algebraic invariance theory led to certain parts of geometry, especially those discussed in Sections 7.8 and 7.9, being pushed out of the picture.

One of the greatest successes of the Cayley-Sylvester contribution to the coordinate method was Cayley's discovery in 1859 that a Euclidean metric in a projective space can be generated by calling for the invariance of an exceptional imaginary formation of second degree (i.e., in the plane: of a



**Illus. 7.7.1** Arthur Cayley (unknown, portrait in London by Barrand & Jerrard) and Felix Klein

conic section). However, this outcome, preserved in Cayley's *Sixth Memoir upon Quantics*, has a prehistory: based on one of Poncelet's ideas, E. Laguerre had shown in 1853 that two straight lines being mutually perpendicular can be explained by the fact that these harmonically separate an exceptional pair of straight lines of the same bundle. Thereby, the other pair fixed for every bundle is determined by its two straight lines running through both special imaginary points of the infinitely distant straight line, which every circle has got in common with this straight line in the complex. It is no surprise that such discoveries fostered the tendency to base analytical, largely algebraicised geometry on complex instead of real coordinate values.

By replacing the complex formation with a real one of second degree, F. Klein had gained the metric of hyperbolic geometry in the interior of this formation in analogy to Cayley's approach. Hence, on one hand, he had accounted for the plane model already found by Beltrami in a novel manner useable in all dimensions, and, on the other hand, shown that Euclidean and non-Euclidean geometry are two very closely related theories, which both result from the view of the general Cayley-Sylvester invariance theory by subjecting projective mappings to a certain simple invariance requirement.

### Erlangen Program

From here on, there is only one small step to the famous "Erlangen Program", which Klein submitted in 1872 when he began working at Erlangen. However, this also has a prehistory. Due to a study trip to Paris together with his friend S. Lie in 1870, the twenty-one-year-old Klein had encountered the new group theory, which had only recently been published by C. Jordan. Jordan had been the first to unite the theory of finite permutation groups coming from algebra and the theory of geometrical transformations, although this did not yield the abstract notion of group in the modern sense, but the discrete subgroups of Euclidean motion, which Jordan had also studied, originating from crystallography, a bridging function between finite groups and the continuous groups of mappings of classic geometry. It is strange that Klein himself acknowledged the influence of Jordan's *Traité des substitutions* (1870), but not of the preceding *Mémoire sur les groups de mouvements* (1869), which had been much more relevant for the Erlangen Program: "Camille Jordan had a great impression on me, whose traité des substitutions et des equations algébriques has just been published and occurred to us as a mysterious book." [Klein 1921, Ges. Math. Abhandl. Vol. 1, p. 51]. Klein also made no use of the terminus group in the original version of the "Erlangen Program", bringing it up only in later editions. Jordan had first used explicitly the notion of a 'group of motions' in the afore-mentioned work from 1869 (whereby, however, he also silently presupposed the existence of the inverse) and classified 174 types of subgroups, which included both continuous and discrete, but were incomplete in different aspects. Jordan received a crucial impulse from theoretical crystallography, which had already prospered to a great degree, especially through the works of A. Bravais (around 1850). We want to refer the reader to [Scholz 1989] for the very complicated history of crystallography; furthermore, to [Wußing 1969] for the development of group theory, to [Tobies 1981] for Klein's biography, and to the newly annotated edition of [Klein 1872, Ostwalds Klassiker] for the history of development and effect of the Erlangen Program.

The "Erlangen Program" did not just become greatly popular because it approached the problem, maturing around 1870, of restoring order to the diversity of geometrical trends and opinions of its time. After Klein, *one* geometry (actually: a geometrical theory) is given by the fact that for any basic set $M$ (Klein had originally only thought of $n$-dimensional manifolds in Riemann's sense) a group $G$ of unique mappings of $M$ onto itself is fixed. The theory $(M, G)$ deals with the invariants $I(G)$ of this group. Thereby, Klein's idea of invariants was highly inspired by the contemporary state of algebraic invariance theory. If $G_1$ is a subgroup of $G_2$ given a fixed "space" $M$, then all invariants of $G_2$ are invariants of $G_1$, in other words, the smaller the group, the richer the theory. The unconstrained manner in which this specified the classification contemplated by Möbius and other forerunners is clear. It is also obvious that Klein's discovery from 1871 regarding definability of Euclidean metrics in projective geometry stands as a high point of his program and legitimizes projective geometry as the mother discipline of all classic geometries (i.e. excluding topology) and that Möbius's elementary relations fit in as well. Apart from many other consequences (not the least being didactical), the so-achieved marriage between geometry and group theory also gradually resulted in the inverse mappings of geometrical transformations finally becoming explicit. Until then, all named authors had silently presupposed and/or used the uniqueness of the mappings they had studied, but argued only about the closure of types of mappings regarding composition.

Only the inclusion of inverse mappings forced mathematicians to acknowledge the identical mapping as the logical consequence from both other closure requirements for a group.

Some geometers proud of their traditions may still be shocked nowadays to find out that the Erlangen Program ever so slightly missed its aim from the perspective of mathematical logic. From the present point of view, a theory is defined by a system of notions (i.e., if we want to be absolutely exact, by an aptly chosen formalised language) and an axiomatic system created by means of these notions (in this language). To each of such system of notions and to each of its interpretations (which may be a model for the axioms phrased in this system of notions) belongs the group of automorphisms of this interpretation. As easy as it is to show this through logic, every such automorphism also leaves all notions invariant that can be defined by means of these basic notions, apart from the notions of the relevant system by which it is defined. The theory given in a chosen system of notions actually deals with all notions that are *definable* by this system of basic notions (in a syntactically specified meaning). According to the above, these definable notions are invariants of the relevant group of automorphisms. However, in many cases the group of automorphisms also leaves notions invariant that are not definable and, hence, do not constitute the real subject matter of the theory, since we cannot talk about them in the chosen language. Thus, the area of objects characterised by Klein as the subject matter of the theory can be too large.[5] At a time when notions such as formalised language were not at all provided, and others such as an axiomatic system, provability, and definability only in a vague and intuitive sense, Klein's program meant substituting the not yet graspable definability based on the basic notions with the often more extensive property of invariance regarding the group of automorphisms. The fact that the groups presupposed by Klein as given are in reality always given as groups of automorphisms of a system of notions was also not considered back then.

Whereas Klein's works were dedicated to the geometrical transformation groups in the narrower sense, S. Lie, who had come across group theory together with Klein in Paris, developed his very own theory of contact transformations, whereby the transformed element pairs are given by a point of the observed space and a hyperplane indexed by this point (i.e., straight line in the two-dimensional, plane in the three-dimensional case, etc.), interpreted as a tangential manifold of a curved manifold $M$. Lie created one of the most important tools of present analysis by reversing the process to obtain the

[5] Perhaps the simplest example, although not very geometrical, is the theory of real numbers. Here, the identical map is the only automorphism, which, consequently, leaves every single real number invariant. However, simply due to reasons of cardinality, not every single real number could constitute a possible subject matter of the theory of real numbers. Above all, the language, by which we mean the way the theory and its axiomatic system are worded, generally and originally only featured the individual names 0 and 1, from which we can naturally derive countably many further individual names of reals by means of definition.

differential equation of the bundle of tangential manifolds and the group of those transformations, which leave these invariant, by means of analytically describing $M$. In contrast to Klein, by basing his idea on a differential equation, he determined the group of transformations that leave this one invariant, in order to obtain an overview of the solutions of the equation by means of geometrical ideas about a belonging manifold $M$. Thus, he founded a new role for geometry as a school of thought in other areas of mathematics. Similar to Riemann, his insights were at first rather intuitive and difficult for his contemporaries to comprehend, being used for calculating with coordinates. We owe it mainly to Engel's selfless commitment that Lie's idea was finally generally accepted and used.

### Groups and notion of symmetry

The notion of a transformation group is closely related to the notion of symmetry, which nowadays is so very important for mathematics and physics. Symmetry was only indirectly a mathematical notion from Antiquity until the $18^{th}$ century, and by no means had its present meaning. Rather, symmetry meant repeating certain proportions in many or all parts of, for example, a building or artwork. Crystallography, which turned away from empiricism in the $19^{th}$ century, also contributed considerably to the word 'symmetry' finally adopting its present meaning. At the same time, crystal symmetry next to optics (and against the needs of mechanics back then) constituted a reason for contemplating reflections from a physical point of view. R.-J. Haüy, one of the first pioneers of crystallography in France, had already spoken of symmetry in this context in 1815. J. F. Hessel, professor of mineralogy, technology and natural history in Marburg (Germany), who had translated Haüy's work into German in 1819, presented his own "crystallonometry", first published in Gehler's *Physikalischem Wörterbuch* (Physical Dictionary) in 1830, with a purely geometrical introduction to the possible forms of spatial symmetry, as well as a classification of polyhedra based on this and possibilities for filling space with them, whereby he, due to reasons of systematics, was first to consider everything mathematically possible, but not realisable in nature within the scope of knowledge back then. He says "...taking as a basis new general theories of pure theory of shapes (Gestaltenkunde)..." in the complete title of his extensive article, also separately printed in 1831 (Reprint in Ostwalds Klassiker, vol. 88/89, 1879). He distinguished here between congruence of the same sense (in Hessel, "equality of image") and congruence of *not* the same sense (in Hessel, "equality of counter-image"). Axes of revolution were classified according to the number of possible revolutions. Afterwards, the composition of revolutions around different axes was examined implicitly group-theoretically. Hessel's investigations, which climaxed in listing 32 crystal classes, received little attention during his lifetime. A. Bravais, physics professor at École Polytechnique at this time, could, after Hessel, begin anew within the scope of crystallography through purely geometrical examination of possible symmetries of polyhedra in 1849. Thereby, he distinguished point

**Illus. 7.7.2** August Ferdinand Möbius (engraving by Adolf Neumann, presumably before 1864); Richard Dedekind (unknown, about 1870)

reflections (central symmetries), axial or rotational symmetries and plane reflections in space. However, he was not aware of the existence of improper orthogonal mappings (combination of rotation and reflection). He called a polyhedron symmetrical if it at least featured one of the three types of symmetry.

Thus, while the first studies on symmetry were conducted within the context of crystallography from a modern perspective and not originated by actual mathematicians, this notion only attracted the mathematicians' interest from the middle of the $19^{th}$ century onwards. In 1849, Möbius began a work titled *Ueber das Gesetz der Symmetrie der Krystalle und die Anwendung dieses Gesetzes auf die Eintheilung der Krystalle in Systeme* (Regarding the Law of Symmetry of Crystals and the Application of This Law to the Classification of Crystals into Systems). He wrote: "A figure shall be called symmetrical (in its broadest meaning), if we can equate it to an equal or similar figure in more than one way as equal or similar." This is a correct, and yet baffling definition. Another work by Möbius followed in 1851 titled *Ueber symmetrische Figuren* (On Symmetrical Figures). Therein, he first demonstrated his knowledge of Bravais's work mentioned above and then put forward a new definition: "§1. As every quantity is equal to itself, so every figure is equal and similar to itself. However, there are figures that are equal and similar to themselves in more than one way, and such figures shall be called symmetrical." Having, amongst other things, begun to describe symmetries by tables of values of the corresponding classifications, he noticed a footnote, which said: "The degree of symmetry of a figure will be more certainly determined by the number that states in how many different ways the figure can be equal and similar to itself." Möbius finally reached the conclusion in a brief note from 1855 that the notion of symmetry (as he had grasped it) can be expanded to include (stated in a modern fashion) more general mappings than just the isometric ones: "This extended notion says that a figure stands to itself in the namely relation in more than one way." The highly general proposed concept increasingly dominated Möbius's way of thinking. Nonetheless, his projected great work remained unpublished during his lifetime. It was only reconstructed due to the publication of his Collected Works in 1886, by virtue of his estate and the efforts of Klein as the publisher, who acknowledged only having gotten to know Möbius's lifework thoroughly through this occasion. In a comment in the last edition of his "Erlangen Program" in 1921, Klein wrote: "By the way, I would like to refer to... Möbius's works (which I myself have only grasped according to their inner correlation having been allowed to assist with the complete edition of his works arranged by the Saxon Society of Sciences from 1885–1887). Möbius did not know the general notion of group and also many of the geometrical transformations that are consulted in the Erlangen Program for illustrative purposes. However, guided by a feeling of certainty, he established them in his consecutive geometrical works in a manner corresponding to the basic idea of the program." (translated from German; [Klein 1872, Ostwalds Klassiker vol. 253, p. 84]).

## 7.8 Beginnings of topology

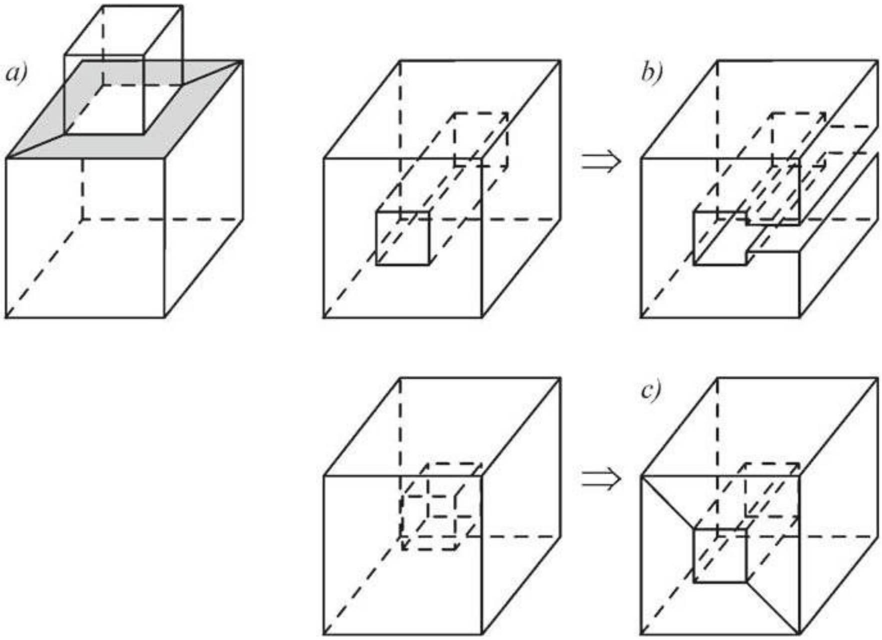
Although Leibniz's vision of "Analysis situs" ('Analysis of Position') had spooked through mathematical literature since the end of the $17^{th}$ century, references to it popping up every now and then without any relation to content, "we do not have a lot more than nothing after one and a half hundred years", Gauss stated in 1833 [Gauß a, vol. V, p. 605]. When topology began truly developing shortly after, it did so in two directions, the mutual relations of which have been rather blurry until today, although we do have names for them by now: general or set-theoretic topology and combinatorial (soon to be renamed, 'algebraic') topology. It is perhaps easiest, though not precisely exact, to characterise this relation as follows: general topology deals with continuity, thereby, often with the "local", the microcosm. Algebraic topology, in contrast, addresses discretised objects and, thereby, mostly "global" properties. Both collaborate in applications, but differ fundamentally in their methods.

As the name indicates, set-theoretic topology cannot actually exist without set theory, i.e., not before Cantor's and Dedekind's foundational theory and, thus, not before 1872. Conversely, set theory was born as an approach to set-theoretic topology. It first addressed notions such as inner, outer, frontier point of a set (in $\mathbb{R}^n$), open and closed set, connectivity, (metric) completeness, continuous mapping, before turning, almost against Cantor's initial intentions, towards the hierarchy of infinite cardinal numbers, well-ordering, the continuity problem and similar issues. In 1878, Cantor discovered that a line segment can be continuously mapped onto a triangular area. As a result, Jordan, Peano and others began competing for a sound notion of curve. As soon became obvious, without its embarrassingly exact definition and examination, even illustratively plausible propositions, such as the property of a curve for transforming into a curve in topological mappings (i.e., bijective and continuous in both directions), or Jordan's curve theorem, could not be proven. The latter states that a simple closed curve in the Euclidean plane dissects the plane into an inner and an outer area. A first attempt, which, as it turned out later, remained incomplete despite Jordan's well-known meticulousness in regard to proof, can be found in Jordan's *Cours d'Analyse* from around 1880.

Approaches towards a set-theoretic topology can already be found in Möbius's *Theorie der elementaren Verwandtschaft* (Theory of Elementary Relation) in 1863. It seems that Möbius understood elementary relation to refer to what we nowadays call "topological mapping". However, he was unable exactly to define it at this time. This work is full of fundamental propositions on topological equivalence of one, two or three-dimensional sets in Euclidean space. In hindsight, these often were not based on sound reasoning, and thus were sometimes partially wrong or at least carelessly worded. For instance, Möbius had a basically valid intuitive idea of the notions of inner point and frontier point and made it plausible that these notions stay invariant in elementary relations. However, he continued to conclude that "of two elementary related plane surfaces, one of the same number of closed curves must be bounded like the other one", and furthermore, "this condition is not just necessary but also sufficient for the elementary relation of two plane surfaces." In an undated fragment *Allgemeine Sätze über Räume* (General Propositions About Spaces) from Dedekind's estate [Gauß Werke (works), vol. 2, p. 353-355], it is "proven", amongst other things, that the boundary of a solid cannot be a solid after the correct metric definition of the notions 'inner', 'outer', and 'frontier point' by means of surroundings and by means of a "solid" as (stated in a modern fashion) a closed casing of a non-empty open subset of $\mathbb{R}^3$.

Combinatorial topology first dealt with the (expressible in whole numbers) relation between numbers of corners, faces, edges, overcrossing, holes, etc., of a one, two or three-dimensional geometrical entity and the resulting possibilities and impossibilities. Accordingly, it actually began in the despised Book XV of 'Elements' with the described relations between regular polyhedra and their corners, edges, etc. The next step is the famous problem called 'Seven Bridges of Königsberg', which was presented to Euler in 1735 by Mayor Ehler of Danzig and the solution to which was published by Euler in 1736. It seems there is no better proof for the categorical turn of the mathematicians' attitude towards such questions between the middle of the $18^{th}$ century and present time than the letter that Euler wrote to Ehler in 1736 regarding the 'Seven Bridges of Königsberg':

"As you see, My Noble Lord, this solution in its character hardly bears any relation to mathematics [!!], and I do not understand why mathematicians are especially expected to deliver it than any other person, since the solution is exclusively based on reason and does not require consulting any principles of mathematics to find it. Thus, I do not know how it happens that even




**Illus. 7.8.1** Solids with cavity, perforation or ring-shaped boundary faces
a) The solid consisting of two cubes standing on top of each other has first the shaded ring-shaped one under its boundaries. Check that Euler's formula does not apply in this case, but the one by l'Huillier does. The transition between both formulae takes place by cutting the ring in the part surfaces by means of two further edges so that now all boundary surfaces are coherent and no surface has any edge in common with itself. b) We analogously obtain the version of the formula valid for a solid with a perforation by cutting the solid open along a surface between the outer boundary and the boundary of the perforation. c) We also accomplish the occurrence of an inner cavity analogously. However, Listing criticised l'Huilliers's methods as having failed if the perforations are branched out in a sufficiently complicated manner and interlaced with each other, cf. Illus. 7.8.4 and 7.8.5

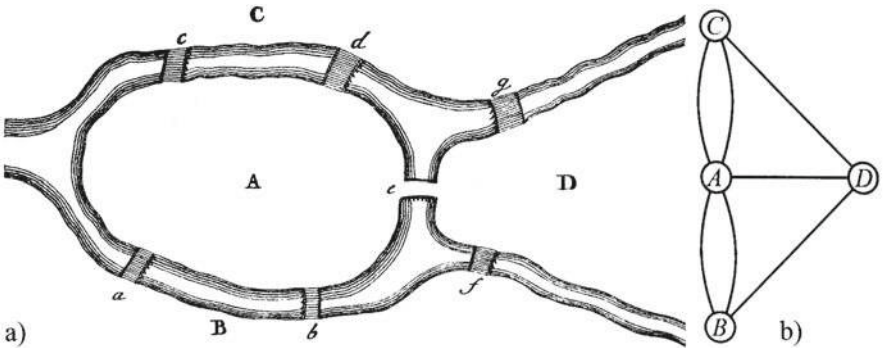
questions that bear very slight relations to mathematics are solved more quickly by mathematicians than by others. Instead, you, My Noble Lord, grant this question a place in geometry of position. However, regarding this new discipline, I admit that I do not know which kinds of problems Leibniz and Wolff want to see related to this." (Translated from [König 1936, p. 312]. The reader will also find a German translation of Euler's Latin treatise on the problem and further related details therein.)

A further, equally famous contribution by Euler regarding the so-called "Analysis situs" concerns the Euler characteristic named after him (1750): If $e$ refers to the number of corners, $k$ to the number of edges and $f$ to the number of the faces of a polyhedron (which suffices for conditions that still

need to be specified), then $e - k + f = 2$. A duplicate of a lost manuscript by Descartes made by Leibniz and only rediscovered in 1860 states that he had already known this formula. When Euler published it, he inspired several mathematicians to engage further with this concept. It slowly became clear that, in reality, it is a proposition regarding the dissection of the spherical surface or a surface topologically equivalent (particularly of the Euclidean plane closed by an infinitely distant point) into areas by any curves, whereby their validity does not depend on the convexity of the surfaces or the form of the curves, but rather on the "topological" character of the surface and their parts. However, as the name states, it was originally thought of for (semi) regular polyhedra and, thus, for straight edges and plane boundary faces. After Legendre had already noticed in 1794 that this formula fails in certain cases, the Swiss mathematician l'Huillier corrected the formula in 1812:
$e-k+f = 2(1+h-p)+c$, whereby $h$ refers to the number of closed cavities of the solid, $p$ to the number of perforations of the solid and $c$ to the number of ring-shaped boundary surfaces. (The latter is demonstrated by the examples shown in Illus. 7.8.1). Cauchy extended this formula in another direction by permitting inner dividing walls, which subdivide the solid like a cellular tissue. Moreover, he came to the correct conclusion and subsequently used it to look at the case first addressed by Euler, in which the entire space is dissected by a closed subdivided surface into an inner and outer solid. All these efforts would be followed up on in 1862 by J. B. Listing, the true father of combinatorial topology.

### Beginnings of graph theory

First of all, we want to note that the branch later called graph theory soon became independent of the later topology. This branch only deals with one-dimensional entities consisting of knots (or corners) and edges (in the directed case "arcs"). Due to its size and independence compared to geometry and topology, we will only look at its beginnings, when it was still closely connected to topology. The problem of the 'Seven Bridges of Königsberg' is followed by the puzzle invented by W. R. Hamilton in 1858, which requires us first to locate a so-called Hamiltonian cycle on the corner-edge-scaffolding of a dodecahedron, i.e., a closed edge path that passes through each knot exactly once. Although this problem seems to be closely related to the 'Seven Bridges of Königsberg', only notions of complexity theory, which originated around 1956, could specify that the decision concerning the existence of a Hamiltonian cycle of a given graph is principally more difficult than the almost trivial decision concerning the existence of an Eulerian path (open or closed) based on the Eulerian consideration. We will repeatedly see how even the most important mathematicians struggled when initially dealing with combinatorial-geometrical observations. In 1736, Euler devoted only brief and completely irrelevant remarks to the question of how to find a Eulerian path in a graph that fulfils the necessary conditions, demonstrating that he was not especially interested in that question.

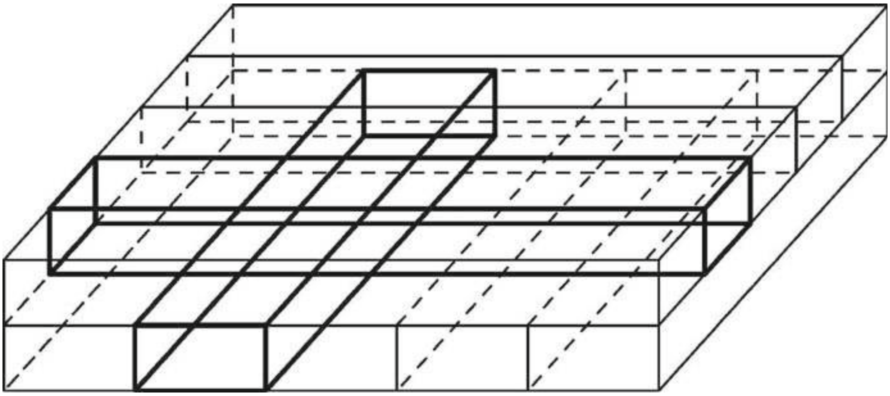



**Illus. 7.8.2** Seven Bridges of Königsberg
a) Drawing of the actual situation in Euler's work from 1736,
b) Transition to the graph, which represents the essential aspects of this situation.
to a) [König, Dénes: *Theorie der endlichen und unendlichen Graphen* (Theory of Finite and Infinite Graphs), Teubner Archiv zur Mathematik, vol. 6 ©1986 BSB B. G. Teubner Verlagsgesellschaft, Leipzig]

In 1871, the young mathematician Carl Hierholzer from Karlsruhe (who died shortly after) stated an algorithm for finding a Eulerian path, apparently without knowing Euler's work at all[6]. The first attempts by mathematicians to establish algorithms for searching labyrinths also form part of the beginnings of graph theory (Ch. Wiener 1873, G. Tarry 1895). However, a problem that only proved to be graph-theoretic after a certain transformation turned out to have the greatest consequences:

In 1852, Frederick Guthrie presented his mathematics professor A. de Morgan in London a question from his brother Francis Guthrie, as to whether it would be possible to colour every map with only four colours so that each two countries with common borders along an edge would have different colours. This program became known amongst mathematicians through de Morgan. It was only solved in 1976 after many errors and failed attempts, and then also in a rather controversial manner (with the help of a computer). Indeed, the original question is graph-theoretic, since we can transfer it to the "dual" graph in the case of a plane map, the knots of which correspond to the countries and are linked by an edge, if the corresponding countries have common borders. Then, the problem turns into a question of whether certain knot colourings of undirected graphs are possible. (The detailed history of the four colour theorem can be found in [Bigalke 1988] and [Fritsch 1994].) However, topo-

[6] The presentation given by Hierholzer on this subject was reconstructed and published after his death by Ch. Wiener and J. Lüroth [Math. Annalen, vol. 6, p. 30-32]. Therein, Wiener referred to the fact that Listing had already demonstrated this in the *Vorstudien zur Topologie* (Pre-studies of Topology) in 1847 in a footnote. However, checking this text passage shows that Listing had also failed to discuss how to locate a Eulerian path, but rather discussed the minimal number of Eulerian paths necessary to run completely through any graph.



**Illus. 7.8.3** Regarding the triviality of the four colour theorem in $\mathbb{R}^3$
If every single "bar" of $n$ such lying at the bottom is connected to exactly one of the $n$ bars lying across on top forming a solid, then each one of these $n$ solids along a surface borders with all others, so that we need $n$ colours to legitimately colour these solids. Thereby, $n$ is arbitrary.

logically speaking, the question remains as soon as we ask it more generally, i.e., either for maps on another surface than a plane or for the analogous colouring of three or higher dimensional "maps". Frederick Guthrie himself recognised that the latter case was trivial. He showed around 1880 that we can find $n$ polyhedra for every number $n$ in $\mathbb{R}^3$ that join each other pairwise along certain surfaces (see Illus. 7.8.3). The first mentioned generalisation (for maps on closed surfaces of higher gender) was finally answered after P. J. Heawood's groundwork and partial results for certain surfaces in the 1960s.

### The difficult birth of combinatorial topology

Let us return to the general development of combinatorial topology. First, it was mainly Gauss who repeatedly indicated this gap in contemporary geometry and also occasionally engaged with such pertinent questions. For instance, there are several studies from his estate on the correlation between total amplitude (something like sum of all changes of direction) of a closed curve and the number of overcrossings [Gauß a, vol. VIII, p. 271-286] as well as an integral formula, which yields the number of "enlacements" of two spatial curves intertwined with one another in dependence of the parameter representation of these two curves [Gauß a, vol. V, p. 605]. However, above all it seems that he influenced one of his students in this respect, namely Johann Benedict Listing, who worked as a physics professor in Göttingen from 1839 after several stopovers. Apart from the great versatility of his other accomplishments and interests, Listing is, next to Möbius, the true father of topology in many respects. Hence, it is ever so unclear as to why he has been so little dealt with in the literature of the history of science.

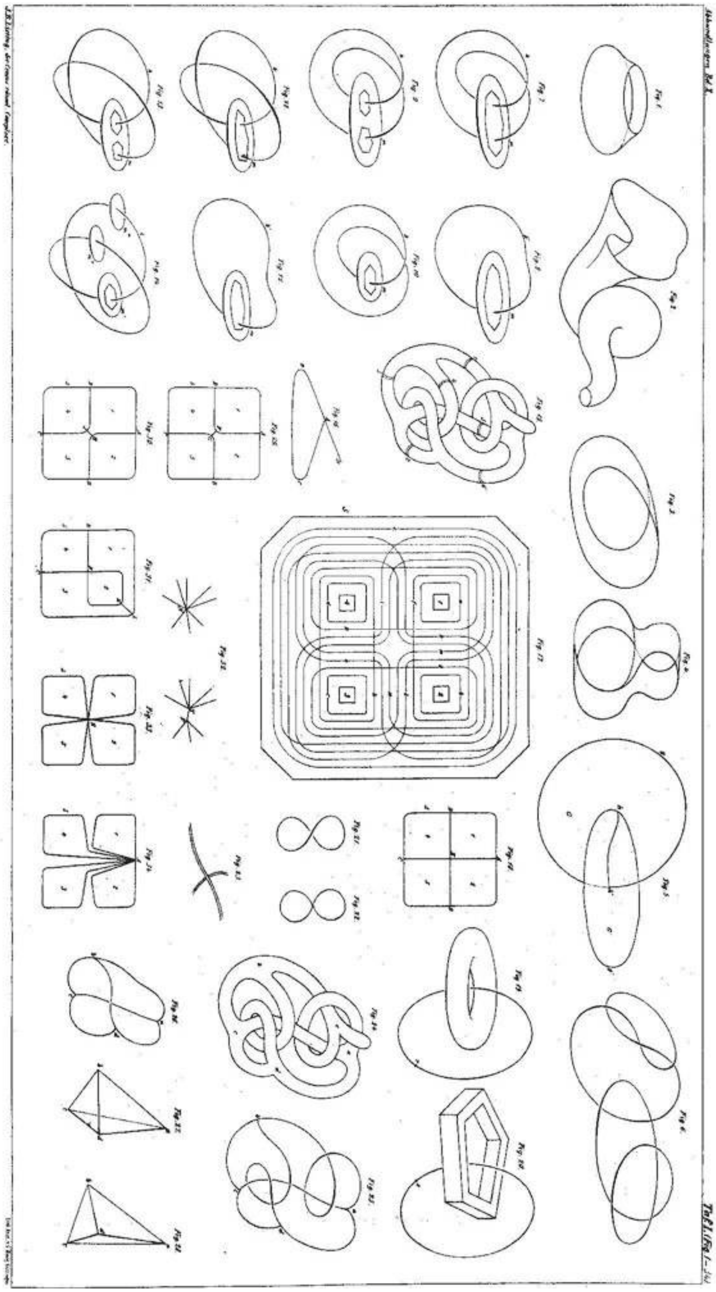
Listing had already coined the word 'topology' first in a letter from 1836. In his *Vorstudien zur Topologie* (Pre-studies of Topology) [Listing 1847], he states the following motive for the new term: Carnot's *Géométrie de position* is to have given the "geometria situs" a different meaning than Leibniz had originally intended. He also wrote:

"Having been made aware of the importance of the matter by the greatest geometer at the present time on several occasions, I have attempted for a rather long time now in the analysis of individual relevant cases, which are prompted by the natural sciences and their applications. By daring, even before these observations can claim a strict scientific form and method, to communicate some aspects as pre-studies of the new science, my intention can only be directed at rising awareness of the opportunity and the significance of this science by propaedeutic rudiments, examples and materials."

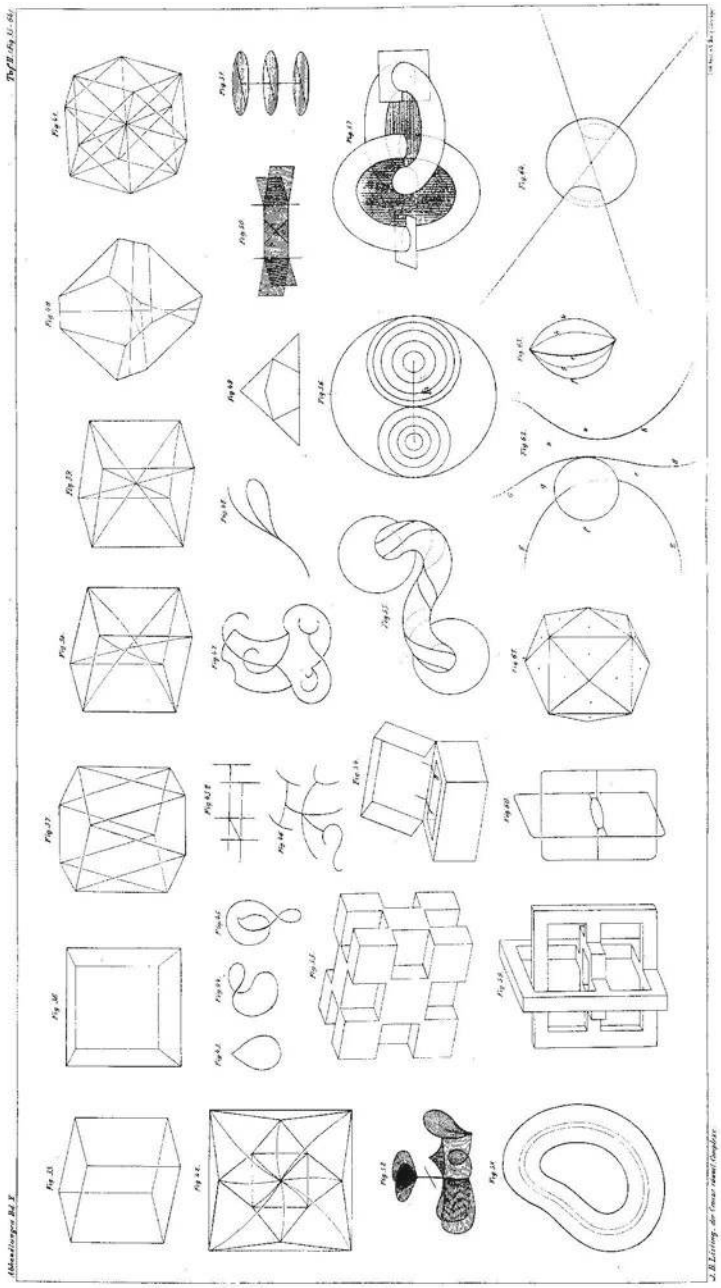
Nonetheless, the main content of 'Pre-studies' actually has hardly anything to do with topology in its present sense and only indirectly in smaller ways. For instance, stated in a modern fashion, it concerns the group-theoretic examination of reflections on three planes pairwise mutually perpendicular, with which Listing wanted to describe the mutual position of two solids in space, but also (in accordance with his physical interests) the possible mutual positions of preimage and image, which are produced by optical tools including mirrors or different types of telescope. Furthermore, he dealt with (in his opinion) the most general form of spiral-shaped curves and their possible mutual positions to each other, extensively discussed the notion of right and left hand thread and their occurrence and meaning in nature and techniques, and finally advanced to considerations which would count as starting points for knot and braid theory. Of course, these are again immediately followed by attempts to apply this to natural science, in this case to the real and seemingly mutual enlacement of orbits of planetoids. Although still worth a read, the book truly just shows how difficult it was back then to keep topological and non-topological notions and matters separate.

Listing then exceeded all his contemporaries in regards to topology because of his *Census räumlicher Complexe* (Census of Spatial Complexes), published in the treatises of the Royal Society of Sciences in Göttingen [Listing 1862]. Thus, the 'Census' is hidden in a local periodical in a pile of treatises on medical, natural scientific, philosophical and historical topics. We can only speculate to what extent this prevented a stronger reception by the international community of mathematicians. Hence, Listing followed up on the set of problems discussed above following Euler's characteristic. Thereby, he neglected all constraints used more or less explicitly by his predecessors, especially the straightness of all edges and surfaces distinct for polyhedral geometry.

Just one look at the relevant tables of illustrations (Illus. 7.8.4, 7.8.5) shows to what degree Listing's "topological fantasy" had developed by 1862 from his having engaged with such questions ever since his student days in Göttingen. Fig. 3 in Table 1 represents the so-called Möbius strip and next to it a



**Illus. 7.8.4** Listing's first table of illustrations
[Listing 1862: *Der Census räumlicher Complexe* (Census of Spatial Complexes). Treatises of the Kgl. Gesellsch. d. Wiss. zu Göttingen (Royal Society of Sciences at Göttingen), Vol. X, Table I]

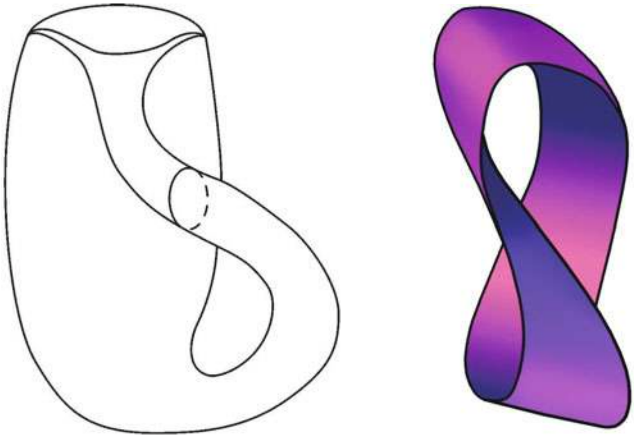


**Illus. 7.8.5** Listing's second table of illustrations [Listing 1862 l.c., Table II]

second example of a surface that had never before been considered by any pioneer of topology. The term "census" in the title (rather confusing for us, since in modern parlance it tends to mean "stock taking", "balance sheet", and, in particular, "population count") refers to Listing having established a much more general balance, though analogous to the Euler-l'Huillier effort, for the numbers of the relevant points, lines, surfaces (which do not necessarily restrict anything or are constrained by anything anymore) and solids of the respective "constituents" of a "complex", for which the total sum always yields zero. The 86 page work is very rich in content and has an index of newly introduced notions attached, featuring words such as 'amplexum' (the surrounding space extended into the infinite), 'diacrisis' (a constant number depending on the nature of the complex), 'periphractic' (closed all around), 'diatresis' (cancellation or annulment of periphraxis), etc. Amongst the many remarkable ideas in "Census" we find the closure of space by means of a single infinitely distant point for the spatial case, which will later be called an Alexandroff compactification (in contrast to the dominant idea back then of projective closure). Due to the stereographic projection of the plane onto the spherical surface, the analogous case for the plane had a longstanding tradition. Moreover, we find mappings of a spatial complex onto a plane "diagram", i.e., first starting points to distinguish between homeomorphism and homotopy. (Two objects are called homeomorph if they can be mapped onto each other topologically. Two objects embedded into a surrounding space are called homotopic if there is a topologic mapping of the surrounding space onto itself, converting one object into the other one. Regarding $\mathbb{R}^3$ the two closed curves in Fig. 6 of Table I of Listing are homeomorph but not homotopic.) I want to put forth the hypothesis that if Listing's tables had been more generally accessible, just looking at them would have advanced on an earlier date several combinatorial or topological discoveries. For example, it is only a small step from Table II, Fig. 53 to discovering infinitely extended, metrically regular polyhedral surfaces (J. F. Petrie and H. S. M. Coxeter 1926, see [Cromwell 1997, p. 79f.]), something most mathematicians still do not know about now.

### One-sided surfaces

A work by Möbius still relevant to topology was published just a few years later in 1867. Inspired by a work from 1769/70 on the content of plane polygons, which overlap with themselves, and apparently also by the renewed examination of star polyhedra in France (see section 7.9), he engaged with the analogous question in space and arrived at (open and closed) one-sided surfaces via a notion that will later be called a simplicial complex. He illustrated the nature of these surfaces by means of the strip named after him. We want to mention here that v. Staudt had claimed the following at the beginning of his *Geometrie der Lage* (Geometry of Position) from 1847: "Every surface has two sides." [Staudt 1847]. Möbius had applied in vain for the prize for substantial improvements of polyhedral theory awarded by the



**Illus. 7.8.6** Klein bottle; Möbius strip

French Academy of Sciences in 1862 with his two afore-mentioned works on elementary relation and the content of polyhedra. Further drafts concerning this set of problems were reconstructed from his estate [Möbius, vol. II, p. 513-60], which guarantee Möbius a coequal place next to Listing.

Several trends crossed and influenced each other during further development. The representation of algebraic surfaces in $\mathrm{R}^3$ offered a lot of inspiration. In 1876, Klein clarified the difference between one-sidedness of a surface (depending on the surrounding space) and non-orientability as an inner property independent of the surrounding space. Bear in mind the analogy to outer and inner differential geometry and the forming bridge of ideas concerning the difference between homology and homotopy! In 1882, he discovered the "bottle" named after him (Illus. 7.8.6) as a closed (i. e. without boundaries) analogy to the Möbius strip. In this respect, we must also mention that the topological structure of the projective plane as a non-orientable surface was discovered around 1874 during a thought exchange between Schläfli and Klein. Nonetheless, in 1871, Betti took up Riemann's plans, which were unable to be finalised due to Riemann's early death, for generalising the notion of a Riemann surface for higher dimensions, as it had been created first for the needs of complex analysis of a complex variable. (There is a relevant fragment from Riemann's estate that was published by H. Weber and R. Dedekind as part of Riemann's Collected Works in 1876.) In 1866, C. Jordan's investigations into embedding spatial curves in curved surfaces also made a significant contribution to forming the notion of homotopy.

In 1895, Poincaré concluded, to some extent, what Listing had begun for complexes of dimension $\leq 3$ and Riemann and Betti had pursued for the higher dimensional case.

The term combinatorial topology was bindingly used from that time onwards. Whereas the topological equivalence of two seemingly different manifolds can always be demonstrated by a more or less illustrative description of a bijective and continuous deformation of the one into the other, the actual problem of topology lies within proving that such a transformation is impossible, such as the topological difference of two given manifolds. Poincaré introduced the notion of an $n$-dimensional topological complex, for which the occurring "constituents" (in Listing's language) can only be $n$-dimensional polytopes and their $k$-dimensional boundary polyhedra (for all $k \leq n$), and invoked that every $n$-dimensional manifold could be approximated by such complexes with arbitrary precision. This, of course, was recognised as too special by the topology of the $20^{th}$ century. Brouwer will further generalise Poincaré's theory in 1912 by only looking at "simplicial" complexes consisting of "simplexes", i.e., the $k$-dimensional analoga of line segment, triangle and tetrahedron. Poincaré added a group structure to the set of $k$-dimensional chains (approx. "paths") in a complex, defined the homology groups as factor groups of this group, and, thus, obtained the notions of Betti group, Betti number and torsion number. These numbers can be calculated for complexes and are invariants in topological mappings. Hence, we can prove the topological difference of two manifolds by means of the non-agreement of such a number. (We recommend to those readers completely unfamiliar with combinatorial topology the very illustrative introductions by [Seifert/Threlfall 1934], the article in Vol. V of the Encyclopaedia of Elementary Mathematics, and [Fomenko 1994] as a modern read. None of these presuppose any pre-existing knowledge of the matter at hand.)

### Beginnings of knot theory

We will now describe one of those rare detours that the history of science occasionally took. The famous physicist W. Thomson (Lord Kelvin), around 1865, had spread the hypothesis that atoms may have to be thought of as some kind of swirls in "ether" and the molecular bond as a topologically irresolvable enlacement of several such swirls – a theory that was even temporarily supported by J. C. Maxwell. It delivered the motive for some notable mathematicians (particularly P. Tait and T. P. Kirkman) to turn towards a combinatorial-topological classification and listing of enlacements of several curves with one another. Gauss and Listing had already shown interest in this. There is even a note in this respect from 1771 by A. T. Vandermonde that was repeatedly cited simply because of its early date (and because Gauss knew and referred to it)[7]. Thereby, it turned out that the number of possibilities in dependence of the number of the legitimate overcrossings grows too rapidly in order to go beyond the modest beginnings of listing. However, notions and techniques were developed for reduction, which only blossomed

[7] The relevant text by Vandermonde starts as follows (based on a free translation of the French original): No matter how complicated the enlacements of a system of threads in space may be, we can, of course, principally describe it analytically. However, those who have to deal with interlaces, nets or other knots practically are interested in the positional ratios rather than the metric ones [Mém. Acad. Paris 1771, p. 566-574].

as part of the independent knot and braid theory in the $20^{th}$ century (M. Dehn, J. Alexander, K. Reidemeister, E. Artin and others). For further details, see [Turner/van de Griend 1995], especially Part IV, and [Epple 1999]. As history took its course, Thomson's idea re-awoke the physicists' interest in a modified manner as part of string theory.

## 7.9 Further, especially non-classical, directions

Even such a finely structured history of the geometry of the $19^{th}$ century as the one presented here can only communicate an insufficient and incomplete picture of the actual events. To finish, we will report some details, certain effects of which endured far into the $20^{th}$ century, cast further light onto the relation between research and application, and partially illustrate the wealth of parallel developments indicated in the preliminary remarks. The need to rely on chance findings when reconstructing the mathematics of the $19^{th}$ century has endured until the present day, owing to the fact that the first reviewing publication *(Jahrbuch über die Fortschritte der Mathematik* – Annual Report on the Success of Mathematics) only came into existence in 1868 and did not cover everything that appears now interesting to us, not to mention that the literature was classified according to aspects relevant at the time, which may not be so now. The articles in *Encyclopädie der Mathematischen Wissenschaften* (Encyclopaedia of Mathematical Sciences; published from 1913 onwards) offer better access to the literature.

A version of Fermat's problem of minimal sum of distance had already been introduced in the first year of "Annales" by Gergonne in 1810: "Find a connecting system of line segments with minimal total length for $n$ points of the plane." [Annales tome 1, p. 292]. Page 381ff. of the same year features some incomplete comments on the solution, particularly the fundamental statement that the additional knot points (now known as Steiner points), which may have to be added, must have a valence of 3, stated in a modern fashion, and that the edges originating from them must pairwise form an angle of 120 degrees. This follows from the solution of Fermat's problem for three points and also expresses the close relation between both problems (see Problem 7.9.1). Questions and partial solutions were stated anonymously and may even have been composed by the publisher Gergonne himself. We have selected this example, since the literature on the problem, generally though unreasonably referred to as the "Steiner" or "Steiner-Weber" problem, has increased a thousandfold in recent years. We also find historical papers and historical remarks in purely mathematical articles amongst this literature. However, each of these still contains only part of the true story [Schreiber 1986], [Wesolowsky 1993]. Furthermore, we have chosen this story because it demonstrates exemplarily the literary dilemma of the time without any reviewing before 1868. G. Lamé and B.-P.-E. Clapeyron, who were amongst the borrowed French scientists temporarily working in St. Petersburg at the beginning of the $19^{th}$ century, wrote an article about the same problem in the journal of the earlier mentioned Institute of Transportation in Petersburg in 1827, wherein they reached the same conclusions (and did not advance either for good reasons). Gauss certainly did not know of either article when exchanging letters on this subject with his friend, the astronomer Schumacher. Schumacher had asked for clarification of an apparent paradox concerning the ordinary Fermat problem of four points (Problem 7.9.1). This shows how far the interest in this question had already spread back then. Following the solution to this question, Gauss wrote to Schumacher: "If we are speaking of the shortest connecting system and not of the strictly mathematical [Fermat] problem of a 4-gon, then we must distinguish several individual cases. Thereby, we create a quite interesting mathematical problem, which is not strange to me. Rather, I have considered it when taking the train between Harburg, Bremen, Hannover, and Braunschweig, and have thought myself that this could be quite a decent bonus question for our students at the next opportunity." [Gauß a, vol. X, 1; p. 459-468]. Probably as a consequence of publishing this letter exchange, a dissertation by Karl Bopp was defended in Göttingen in 1879, exhausting the problem for four points. Steiner's relation to this set of problems is limited to a talk given at the Prussian Academy of Science in Berlin in 1837 on Fermat's problem for $n$ points. In the meantime, E. Fasbender had shown in 1846 that Fermat's problem could also be grasped as a problem of maximum for the triangle. The confusion surrounding the names, origin and history of this set of problems was completed by the fact that the question was grasped rather independently of the mathematics by economists and economic geographers towards the end of the $19^{th}$ century (amongst others A. Föppl in Schweizerische Bauzeitung (Swiss Building newspaper) 1884, A. Weber 1909).

### A geometrical balance problem

Another geometrical extremal problem with a strange history first occurred, as far as is known, in 1803 in a mathematical journal characteristic for Great Britain back then, mainly targeted at educated and interested amateurs. It dealt with determining the point $P$ for three (or more) straight lines $a, b, c, ...$, for which the sum of squares of the distances becomes minimal to $a, b, c, ...$ The problem originated in nautical science and geodesy: three astronomical localisations deliver three locally approximate straight "locations", which only pass through the correct common point in the ideal case. Thus, we have to determine the most probable position of the observation location according to the method of the smallest squares. L'Huillier (1809), Steiner (1828), E. W. Grebe (1847, after whom the relevant point was temporarily named), E. Catalan (1852) and at least eight further authors discovered this problem and its solution independently of each other before E. Lemoine released long lists of interesting properties of the Grebe point at the annual conference of the French Society for the Advancement of Science in 1873 and 1885. Under the wealth of "special points and lines of the triangle", this point has been known as the "Lemoine point" since then, and if we look up this name nowadays we find a lot of information, but usually not the minimum property of the point anymore [Weiße/Schreiber 1989] (see Problem 7.9.3).

### Equivalence by dissection

We remind the reader that Euclid's proof for Pythagoras's theorem was actually a proof for the finite equivalence by dissection of the square of the hypotenuse and both squares of the cathetes, even though we first have to carve out the actual creation of the relevant dissection of the square of the hypotenuse and the following assembly to form the square of the cathetes based on a series of given theorems. Later proofs of this theorem make much more immediate and illustrative use of the fact taken for granted that the equivalence of area of (plane or spatial) figures follows from their equivalence by dissection. Speaking of Pythagoras's theorem, there is an astonishingly long and extensive tradition of such dissections and closely related completion proofs[8]. C. Cramer gathered 93 such proofs in a book published in 1837 and again in 1880 J. Wipper stated 46 partially different proofs. (The exact details, as well as a series of further texts from the $19^{th}$ and beginning of the $20^{th}$ century, on Pythagoras's theorem also feature in the fourth edition of [Lietzmann 1911] published in 1930. Even later editions do not refer to this topic, since we can expect that these texts had become unattainable for most readers.) W. v. Bolyai's *Tentamen* (1832), of which Gauss approvingly emphasised how easy it was to notice the author's striving towards thoroughness and completeness, as well as the carving out of fundamental problems everywhere, was first to present the proof that each two figures of the plane equal of area and bounded by straight lines are equal by dissection, meaning that equality of area, if it exists, can always be proven by dissection, i.e., by means of the traditional methods that especially thrived during the Islamic Middle Ages (see Problem 7.9.4). Just one year later, a German amateur, the Prussian lieutenant P. Gerwien, published two articles, in which he – unaware, of course, of Bolyai's book (which was a Latin textbook for grammar schools in Hungary) – proved the same subject matter not just for the Euclidean plane, but also for the spherical surface with analogous means. The set of problems, although seemingly singular and not fitting in with the great theories of the $19^{th}$ century, again shows multiple aspects:

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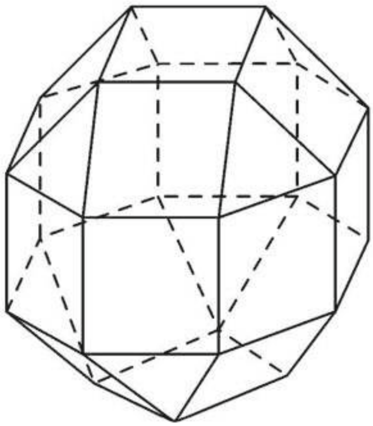
[8] Two plane (or spatial) figures bounded either by straight lines or straight planes are called equal of completion, if we can complete them to congruent figures by adding each the same amount of figures, of which each one of the first added is congruent to an assigned one of the second added. The classic example is the "Chinese" proof of Pythagoras's theorem (cf. Chap. 3). It is easy to demonstrate: If figures of equal area are always equal of dissection in a space, then they are also equal of completion. The reversed scenario already does not apply in $\mathbb{R}^3$.

1. These great theories by no means exhaust the essential geometrical events of the $19^{th}$ century.
2. People whose names are hardly known these days – amongst them typical amateurs – have also made significant contributions.
3. Some things that were thought of as marginal back then have become the roots of essential mathematical accomplishments of the $20^{th}$ century.

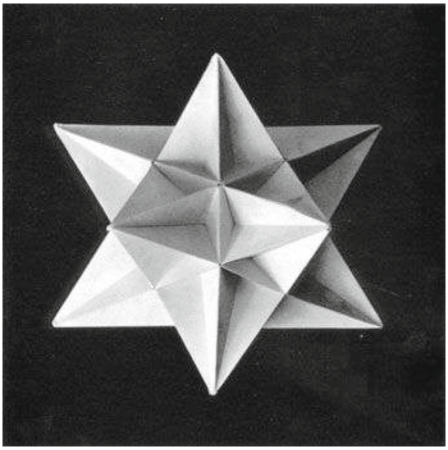
Of course, Bolyai's and Gerwien's outcome immediately opened up the question as to whether there is an analogous theorem that applies in space. In 1844, Ch. L. Gerling, one of Gauss's many pen pals, showed Gauss his proof for the fact that each two polyhedra that are the mirror image of each other are equal of dissection [Gauß a, vol. VIII, 240-246]. In 1896, the Briton M. J. M. Hill found three special types of tetrahedra, which are equal by dissection to a cuboid [Proc. London Math. Soc. 27, p. 39-53]. The now obvious difficulty for solving the problem generally led to D. Hilbert making it the third position of his list of 23 important unsolved problems as part of this famous talk at the second International Mathematics Conference in 1900. This resulted in M. Dehn proving the following in his habilitation text in 1901: cubes and regular tetrahedra equal of volume are not equal by dissection. The applied method, which was multiply improved and refined later, is so similar to the method introduced by Listing in 'Census' and advanced by Poincaré to the tool of combinatorial topology that we can suspect a connection regarding the history of ideas. We want to mention here that H. Brandes showed the following in his dissertation (written under Hilbert's indirect influence) in 1908: a proof by dissection for Pythagoras's theorem in the general case is impossible with less than seven triangles. Hence, a new direction in research had opened up and beaten the question of geometric extremity demonstrated by the examples above.

The boom of polyhedral geometry in the $19^{th}$ century has already been mentioned in two respects: in topology and graph theory and in crystallography and/or its role when forming the geometrical notion of group and symmetry. However, these are only two aspects of the literature on polyhedra which blossomed abundantly during the entire $19^{th}$ century. It is hardly possible to observe the phenomenon of mutual non-acknowledgement and multiple discoveries better in any other area of mathematics. Even an old theory, such as the one of semi-regular polyhedra, was in need of a re-awakening and addition. It started in 1807 with Meyer Hirsch, followed by Lidonne in 1808, l'Huillier in 1812, Gergonne in 1818, and many others until Badoureau in 1878. It concerns the proof that the list of regular and Archimedean polyhedra is complete, whereby for the first time the prisms and anti-prisms with regular base, which Kepler had rejected, were included. However, despite so many attempts, an Archimedean polyhedron unknown until then was only discovered in 1930 and multiple times after that (Illus. 7.9.1). L'Huillier was first to refer to the three regular tessellations of the plane as boundary cases of Platonic solids. Gergonne also looked at the Archimedean semi-regular

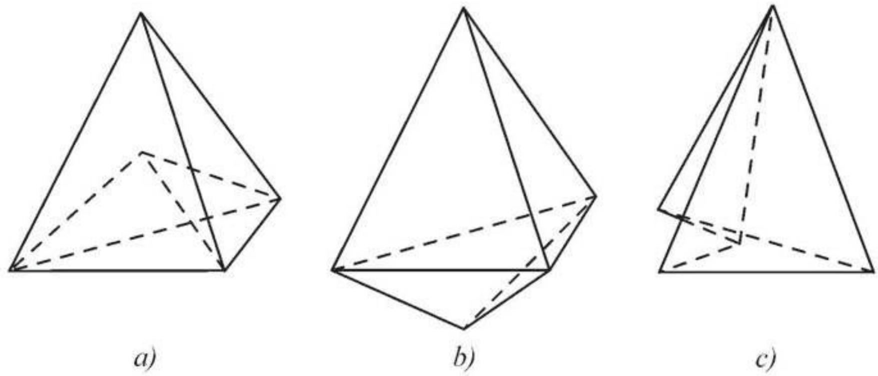


**Illus. 7.9.1** The Archimedean polyhedron discovered independently by Miller in 1930 and Aschkinuse in 1957

This polyhedron suffices for the classic definition, which states that it is bounded by regular polygons and that the surfaces joining in any corner are pairwise congruent corner figures. However, the group of its isometries is not transitive, i.e., there is no isometry for each two of its corners, which maps the one onto the other one and, thereby, the entire polyhedron onto itself. This Miller-Aschkinuse polyhedron is the only one of its kind and very instructively shows a deficit of the classic definition of Archimedean semi-regularity, since we intuitively refer to the pairwise equivalence of all corners. However, it does not suffice for this to look at its "nearest surroundings".



**Illus. 7.9.2** The fourth star polyhedron found by Poinsot and called great icosahedron [Gerd Fischer: Mathematische Modelle, Vieweg-Verlag, Braunschweig/Wiesbaden 1986]




**Illus. 7.9.3** Non-convex polyhedra
The non-convex polyhedron shown in a) has the same net as the one shown in b). This example (with a minimal number of faces) shows that the condition of convexity in Cauchy's theorem about the determination of a convex polyhedron by means of its net is necessary. If all boundary faces of a polyhedron have to be convex themselves, then a) is a non-convex polyhedron with a minimal number of faces. If non-convex faces can also occur, then c) shows the possibility of a non-convex polyhedron with only five surfaces

plane tessellations, but missed some cases. Only Badoureau delivered a complete list. The semi-regular solids reciprocal to the Archimedean ones were addressed by E. Catalan in 1865 and are usually named after him. Whereas combinatorial duality of tessellations and polyhedra has been known for a while, Catalan made it clear that duality in metric terms is provided by a sphere, which simultaneously is the circumscribed sphere of a polyhedron and the inscribed sphere of the reciprocal (he says, 'conjugated') polyhedron. In particular, the circumscribed sphere of an Archimedean polyhedron is the inscribed sphere of the dual "Catalan" polyhedron. Another substantial discovery within the realm of regular solids was made by L. Poinsot, who, apparently without knowing Uccello's and/or Kepler's priority regarding two of these solids, found the four polyhedra nowadays accepted as the regular star polyhedra (Illus. 7.9.2). Star polygons, star polyhedra and, above all, their correct determination of area and volume were one of the driving forces of the developing topology for contemplating the phenomena under which circumstances a curve dissects the surface on which it is located into two parts or that a surface in space can be one-sided, etc. [Günther 1876].

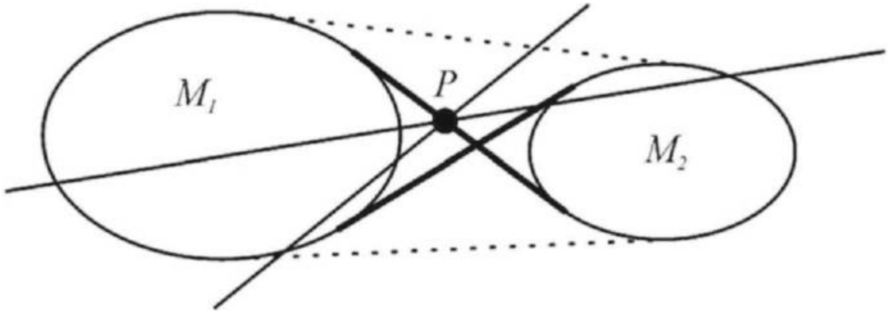
Whereas everything written here up to now refers to regular polyhedra at least to some extent, Cauchy proved in his first work from 1813 that a system of corners, edges and surfaces, which can be realised at all as a polyhedron in space, can only be realised in one way under the additional condition of convexity (Simple contra-examples for different realisations at given non-convexity are shown in Illus. 7.9.3).

This will be followed up on by E. Steinitz when showing that every "abstract" polyhedron, i.e., one given as an incidence instruction for a certain finite set of corners, edges and surfaces, that suffices for certain necessary conditions, can be convexly realised ([Steinitz 1930], published posthumously). However, J. Steiner [Steiner 1882, Vol. I, p. 154] had already asked the much more far-reaching question in 1832 as to whether such a polyhedron might not be realised then with a circumscribed sphere. This question could only be negated by a contra-example (also by Steinitz) in 1928. The great interest that polyhedral geometry attracted during the $19^{th}$ century is also expressed by the fact that the French Academy of Sciences in Paris asked the bonus question "to complete the geometric theory of polyhedra in any essential point". The bonus prize was not awarded, although Catalan's works and those by Möbius mentioned earlier were submitted.

[Brückner 1900], [Brückner 1906] features a smothering wealth of results regarding the theory of general and special polyhedra, meaning, for example, semi-regular or star polyhedra. Moreover, [Brückner 1900] contains historic and bibliographic information very rich in detail. [Cromwell 1997] is, of course, incomparably more up-to-date and also historically insightful. However, it suppresses many factual and historical details.

### Geometric probability

As a peculiar and nowadays highly important combination of probability calculus and geometry, the theory of geometric probability began developing in the $19^{th}$ century after having a famous forerunner from the $18^{th}$ century. This theory can be interpreted without any stochastic background and simply as an extension of measuring capacity of sets of points to sets of other geometrical objects. The first mentioned impulse was provided by G. L. L. Buffon's needle problem. He was one of those universal scholars almost typical for the $18^{th}$ century, engaged in, amongst other things, geology and mineralogy, cosmology, botany, and, within mathematics, above all, probability calculus and its statistical applications. The historic significance of the needle problem (see Problem 7.9.5) lies within the fact that, for the first time, a question of probability was proposed within the realm of even, innumerably many possible events. Thus, the inevitable geometrical question of measures for the relevant sample space superseded classic combinatorial counting of all possible and all favourable cases. The needle problem is to determine the probability that a needle of given length $l$ intersects with one of the lines on a horizontal plane with equidistant parallel lines (distance $d > l$) when randomly thrown. Buffon had already presented this problem at the French Academy of Sciences in Paris in 1733 and published it in 1777 within the scope of his 36 volumes of general natural history. It was probably inspired by an old French gamble, whereby we have to throw a coin or another flat object onto a floorboard so that we hit a gap in the floor. P. S. Laplace pointed out in 1812 that we could approximately determine $\pi$ through experiment, since $\pi$ is part of Buffon's result. This could be done by replacing probability with relative frequency,




**Illus. 7.9.4** Crofton's theorem
Exactly the straight lines through a point $P$ of the thickly marked rope piece around $M_1, M_2$, which do not pass through one of the punctured rope piece, intersect $M_1$ and $M_2$. Hence, if the length of a curve is the measure for the set of the mutually intersecting straight lines, Crofton's theorem follows

which settles after a sufficiently large number of needle throws. This would turn out to be meaningful over the course of the $19^{th}$ century, since it became obvious that measures in space of all straight lines or line segments seemingly can be defined in different ways. The fine agreement of Buffon's experiments with the approximation values for $\pi$ obtained with different methods delivers, above all, an argument for the rationality of the measure chosen in this case.

In 1841, Cauchy solved the problem for the more general case occurring as part of the mentioned gamble, whereby, instead of an infinitely thin needle, we throw a convex disc of any given form in one direction. Thereby, he was first to discover that the measure of the set of all straight lines $m$, which intersect a plane convex set, is given by the perimeter of this set.

J. Steiner and several British mathematicians had also dealt with generalisations of such geometrical content measurement since the middle of the $19^{th}$ century. The Briton M. W. Crofton concluded his "Crofton theorem" from Cauchy's outcome in 1867. Accordingly, the measure of the set of all straight lines that encounter two separately located convex plane figures $M_1, M_2$, equals the difference between the length of the curve crossed over and surrounding both figures and the perimeter of the mutual convex envelope (Illus. 7.9.4). J J. Sylvester generalised this theorem in 1890 for more than two convex figures. In the meantime, the first works and a book by E. Czuber in 1884 on "geometrical probabilities" had also been published in the German-speaking countries. However, given his own numerous contributions, W. Blaschke would coin the name 'integral geometry' for this new and increasingly important area in 1935 [Blaschke 1937].

In 1888, J. Bertrand shocked the mathematicians engaging with such questions with a "paradox" ([Bertrand 1888, p. 4f]): We are asked to calculate the probability for a random chord of a circle being longer than the side of the inscribed equilateral triangle. (We can also ask for the measure of the set of these chords and/or the straight lines created by extending them in the ratio to the measure of all straight lines that intersect the circle.) Determining measures for geometric objects is always based on describing these objects by means of a suitable number $k$ of independent parameters and measuring the capacity of such sets in $\mathbb{R}^k$, which thereby correspond to the original sets of objects. Bertrand selected three options from the large number of possibilities to determine the straight lines of the plane by means of two real parameters and showed that we arrive at three different answers for the posed question (see Problem 7.9.6). Hence, in this respect, Laplace's proposal offers a possibility for justifying the chosen measure for the case of Buffon's problem. E. Cartan and H. Poincaré stated a general theoretical justification for the choice of each appropriate parameterisation of the objects in 1896: the chosen measure must be invariant towards shifts of the relevant set, i.e., the motions of basic space must correspond to volume preserving coordinate transformations in the space $\mathbb{R}^k$ of the coordinates.

## Essential contents of geometry in the $19^{th}$ century

*Advancement of descriptive geometry*: multiplane method, central perspective, axonometry, photogrammetry, relief perspective, illumination geometry (G. Monge and his students)
*Projective geometry*: invariance of the cross-ratio (M. Chasles), infinitely distant points, straight lines, planes (J. V. Poncelet, J. Gergonne), duality principle (J. Plücker), homogeneous coordinates and Barycentric calculus (A. F. Möbius), "geometry of position", bundles, pencils, projective coordinates (v. Staudt)
*Differential geometry*: curvature and torsion of spatial curves (G. Monge), theory of spatial curves (Frenet, Serret, Bonnet, Bertrand), theory of surfaces of curvature in space (starting points by Euler; Meusnier, Monge, Lagrange; Dupin's indicatrix), foundation of "inner geometry" (C. F. Gauss), spaces of constant curvature are homogeneous and isotropic (Riemann), advancement of calculus (Lipschitz, Christoffel, Weingarten, Betti, Dini, Bianchi, Codazzi, Mainardi, Beltrami, Casorati)
*Theory of geometric construction*: *Disquisitiones arithmeticae* (C. F. Gauss, 1801): general theory of circle division, algebraic methods for proving the impossibility of doubling the cube and angle trisection with compass and straightedge
*Non-Euclidean geometry*: proof of existence of "non-Euclidean" geometries when negating the Euclidean parallel postulate – end of a thousand-year-old argument (W. v. Bolyai, N. J. Lobachevsky, B. Riemann), models of non-Euclidean (hyperbolic) geometry (F. Klein, H. Poincaré, E. Beltrami)
*Notion of vector and n-dimensional geometry*: magnetic and electric "vector fields", rotation and divergence (Oersted, Gauss, Weber), calculating with complex numbers as vectors (Weber, Argand), quaternions (Hamilton), 'theory of linear extension' (H. G. Grassmann: $n$-tuples of numbers as coordinates of a point, units, basis, outer product, linear dependence/independence of vectors, invariance of dimension, multivectors), determinants as multi-linear and alternating functions of row/column vectors, later as orientated volume of the $n$-dimensional parallel solid (Cauchy), Cauchy/Schwarz/Bunyakovsky inequality, *Analytic geometry in n dimensions* (A. Cayley, 1843), $n$-dimensional simplex (Clifford), polyhedra, convexity, regularity in $n$-dimensional spaces, regular polytopes (L. Schläfli)
*Transformation groups*: groups of motions (congruence mappings), projective maps, Möbius transformation amongst others (Chasles, Poinsot, Carnot, Möbius), matrix calculus (A. Cayley, J. J. Sylvester), Erlangen Program (F. Klein), Symmetry groups and crystallography (Haüy, Hessel)
*Beginnings of topology*: set-theoretic topology, topologic maps, Jordan curve theorem (Cantor, Dedekind, Jordan), combinatorial topology, graph theory, polyhedra, four colour theorem, one-sided surfaces, notion of homology (Gauss, Listing, Möbius, Jordan, Riemann, Poincaré)
*Misc*: Fermat's problem for $n$ points, Grebe-Lemoine point, equality by dissection, star polyhedra, geometric probability
*Basics and formation of geometry*: turn towards new perceptions (M. Pasch, G. Peano, H. Wiener, F. Klein), *Grundlagen der Geometrie* (Foundations of Geometry; D. Hilbert, 1899)

## 7.10 Problems to 7

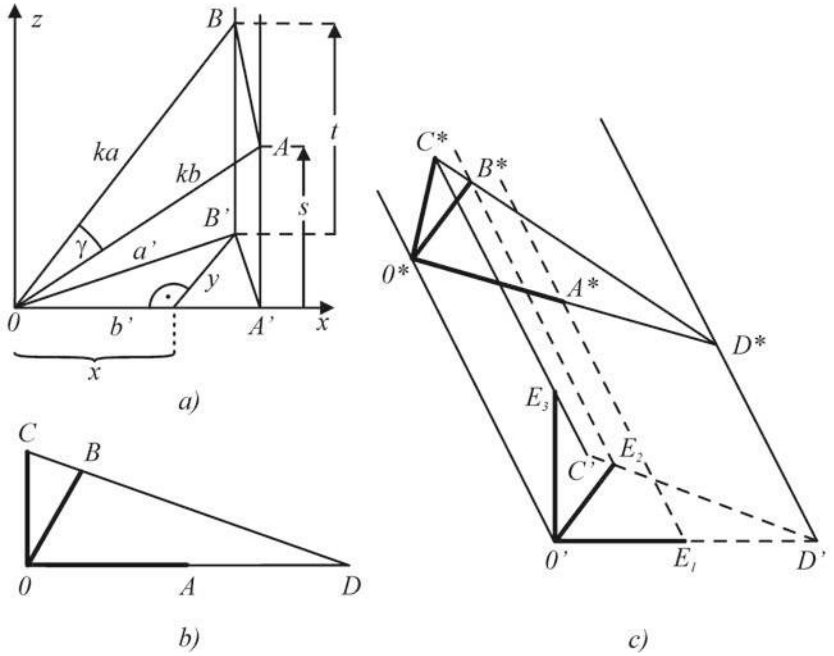
**Problem 7.1.1*:** Director circle of conic section according to Monge

Prove the following: if the apex of a right angle glides along a circle $k$ and one of its legs, thereby, always through a fixed point $F$, then the other leg runs through all tangents of a certain conic section $K$. Thereby, $K$ is an ellipse, if $F$ lies within $k$, or, respectively, a hyperbola, if $F$ lies outside $k$. If circle $k$ degenerates to a straight line $g$, then the produced conic section is a parabola. (The latter was already known by J. H. Lambert.) $k$ is called the director circle of the produced conic section (this term, however, is also occasionally used for other circles connected to the conic section) or director line, respectively. $F$ is always one of its foci.

**Problem 7.1.2*:** Proof of Pohlke's theorem according to [Salenius 1978]

We need a lemma, which is also of independent interest and is best proven by means of calculation: If a triangle $T$ is given, then every triangle can be mapped onto a triangle similar to $T$ by means of perpendicular parallel projection. (Equivalent wording: we can produce any triangular shape by means of plane sections of a given prism with three edges.) To prove this, position triangle $OA'B'$, onto which we are meant to map perpendicularly, as shown in Fig. 7.10.1 a) in the $x$-$y$-plane. Then, it is fully described by $b' = OA'$ and the coordinates $x, y$ of $B'$, whereby ${a'}^2 = x^2 + y^2$. The given triangular shape, which is meant to be mapped onto $OA'B'$, is given by angle $\gamma$ and the ratio $a : b$ of the adjacent sides. In order for a triangle $OAB$ of this shape, as shown in the illustration, to fit into the prism with profile $OA'B'$ above, we have to determine the parameters $k > 0, s$($z$-coordinate of $A$) and $t$($z$-coordinate of $B$) so that (1) $k^2a^2 = t^2 + {a'}^2$, (2) $k^2b^2 = {b'}^2 + s^2$ and (3) the inner product of vectors $OA, OB$ has to be $k^2ab\cos\gamma$, while also yielding $xb + st$, i.e. $xb + st = k^2ab\cos\gamma$ by means of the coordinates of these vectors. Show that this system of equations can always be solved under the given conditions regarding $b', a, b, \gamma, x, y$.

Now, let us assume that $O, A, B, C$ are not collinear points of a plane (Fig. b). We only provide the sketch regarding the proof for Pohlke's theorem for the general case that straight lines $OA$ and $BC$ intersect each other in one point $D$. The remaining exceptions are all trivial and are left up to the reader. Furthermore, $O', E_1, E_2, E_3$ shall form a Cartesian tripod (Fig. c). We determine point $D'$ on straight line $O'E_1$ so that the affine ratio of points $O', E_1, D'$ is the same as the one of points $O, A, D$. Then we determine point $C'$ on straight line $D'E_2$ so that the affine ratio of points $D', E_2, C'$ is the same as the one of points $D, B, C$. The projection direction $C'E_3$ determines a (generally oblique) prism with three edges together with base $O'D'C'$, into which we can fit a triangle $O^*D^*C^*$ similar to $ODC$ according to the lemma. Given this similarity, $A$ and $A^*, B$ and $B^*$ correspond to each other. Concerning the mentioned projection direction, $O'$ is now also mapped onto $O^*$, $D'$ onto $D^*$ and $E_3$ onto $C^*$. Since the affine ratios stay invariant, $E_1$ also converts into $A^*$ and $E_2$ into $B^*$.




**Illus. 7.10.1** Figures to Problem 7.1.2

**Problem 7.1.3:** Analogon of Pohlke's theorem for central perspective
If three non-collinear points $F_1, F_2, F_3$ are given in an image plane as vanishing points of three directions pairwise mutually perpendicular, then for each two of these points, a geometrical locus is given for the position of the belonging visual point $A$ by means of the "Thales hemisphere" above the line segment between these two points. If there is a point $A$ that fulfils these three conditions, it is located in the intersection of these three hemispheres. Since the section of each two of these hemispheres can be projected as a straight connection of the intersections of the base circles of these two hemispheres when projecting perpendicularly onto the image plane, these three line segments pass through the principal point as a projection of the visual point. Consider the following for this scenario:

1. What are these three line segments in regards to the triangle of the three vanishing points?
2. Which theorem of plane triangular geometry is obtained as a result of the projection of the spatial subject matter? (transfer principle)
3. Which conditions must the triangle of the three vanishing points fulfil in order to really feature a visual point, of which they appear as vanishing points of three pairwise perpendicular directions?

**Problem 7.1.4*:** Watt's and Peaucellier's straight line motion

1. Analyse which curve is produced by the tracing points of Watt's resp. Peaucellier's mechanisms shown in Illus. 7.1.3, and prove that Peaucellier's mechanism solves the problem exactly.
2. How do we have to dimension the parts of Watt's mechanism in order to produce a preferably approximately straight curve segment?

**Problem 7.2.1:** Power of a point with respect to a circle (Monge, Carnot)

The power $(P, k)$ of a point $P$ with respect to a circle $k$ with centre $M$ and radius $r$ is defined as $PM^2 - r^2$, thus greater than, equal to or smaller than zero depending on whether $P$ is located outside, inside or on the circle itself. For two non-concentric circles $k_1, k_2$ the set *power* $(k_1, k_2)$ of all points $P$, which have the same power concerning both circles, is a straight line perpendicular to the connection of the centres of both circles. This can be traced back elementarily as follows: first, it seems there is exactly one point $P_0$ on line segment $s$ between both centres that belongs to *power* $(k_1, k_2)$, since the power concerning both circles increases for each case monotonously and continuously with the distance to the centre. Every point on the perpendicular erected on $s$ in $P_0$ also belongs to *power* $(k_1, k_2)$. If in reverse $P$ belongs to *power* $(k_1, k_2)$, so every point of the perpendicular is also dropped from $P$ onto $s$.

If $k_1$ and $k_2$ intersect, the connection of the intersections is the power line (radical axis) of both circles, since both intersections concerning both circles have a power of zero. If they do not intersect, add an auxiliary circle $k_0$, which intersects both, determine *power* $(k_0, k_1)$ and *power* $(k_0, k_2)$ and their intersection – it is the power point of the three circles and it has the same power regarding all three circles – and drop the perpendicular onto connection $s$ of the midpoints of $k_1$ and $k_2$.

1. Prove that for each straight line through $P$ that intersects or touches $k$, *power* $(P, k)$ equals the product of the directed (i.e., algebraically signed) distances from $P$ to both intersections with $k$ (or equal to the square of the distance to the tangential osculation point), accordingly that this does not depend on choosing the chord [Euklid c].
2. If we subtract the normed equations of two non-concentric circles, we obtain a straight line equation. Show that this is always the equation of the power line of both circles and derive all properties of this straight line listed above based on its equation.
3. The notion of power facilitates elegant solutions for many construction problems. For instance, one of the ten cases of Apollonius's problem requires us to construct all circles through $P$ that touch both straight lines, for two straight lines and one point $P$ that is not located on either. Solve this construction problem by means of power.

**Problem 7.2.2:** Purely linear projective constructions

1. An alternative method to the one shown in Illus. 7.2.3 for halving line segments projectively speaking regarding a given infinitely distant point is given by the fact that the diagonals of a parallelogram halve each other. Elaborate this to a solution.
2. Generally speaking, every construction that can be accomplished in the Euclidean plane solely by means of linearly connecting, intersecting and drawing parallels can be "translated" into a construction, which can be made solely by intersecting and connecting by replacing the infinitely distant straight line by a finitely located one. This can also be interpreted as alternatively working in a central perspective image of the construction plane. (In this regard, the purely linear constructions have already been addressed in the last chapter of Lambert's *Freyer Perspective* (Free Perspective).) Provide concrete examples for the statements above.

**Problem 7.2.3*:** Homogeneous coordinates

A very illustrative access to plane homogeneous coordinates is created if we first look at the bijective map between a projectively closed (i.e., extended by infinitely distant points and a infinitely distant straight line) plane $e$ of the Euclidean space and the bundle of all straight lines through coordinate origin $O$ of a Cartesian (or more general affine) coordinate system in space that is not located on $e$. To every $P \in e$, straight line $OP$ corresponds. The straight lines through $O$ parallel to $e$ correspond to the infinitely distant points of $e$. (This bundle of straight lines is somehow a "better" model of the projective plane, since the infinitely distant objects are not especially distinguished from the outset. The objects parallel to $e$ are only distinguished as improper by choosing plane $e$.) Points of $e$ are now collinear if the corresponding straight lines of the bundle are located in one plane, i.e., the planes through $O$ correspond to the straight lines of projective plane $e$. Regarding the chosen spatial coordinate system, these planes correspond injectively to the equations of the form $ax + by + cz = 0$, whereby $a, b, c$ are not all 0 at the same time and are only determined except of a common factor different to zero. Having said that, every point different from $O$ (whose coordinates $x, y, z$ are not all zero at the same time), determines a straight line of the bundle and, hence, also a point of $e$. We now want to grasp $a, b, c$ as (determined for up to one factor) coordinates of the intersection lines $g$ in $e$, and $x, y, z$ as (determined for up to one factor) coordinates of the respective point $P$ in $e$. $P \in e$ is exactly then true, if $ax + by + cz = 0$. (Here, we see the duality principle in the form established by Plücker: we cannot see whether $a, b, c$ are supposed to be the coordinates of the straight lines and $x, y, z$ the coordinates for the points or vice versa.)

Now consider how choosing three non-collinear points in $e$ is equal in meaning to fixing three in $O$ originating axial directions of a coordinate system in the surrounding space and how the fourth point in general position in $e$ serves to fix a point in space with the coordinates 1, 1, 1, which is why (how?) a unit is determined on each of the three axes.

If we distinguish the straight line in $e$ with the equation $cz = 0$ as improper and/or have arranged our spatial coordinate system so that exactly this straight line "really" is the improper straight line of $e$, then the improper points located on it are characterised by $z = 0$; the proper ones, consequently, by $z \neq 0$. Regarding these proper points, we can go over to the new, "inhomogeneous" $x', y'$ by $x' = x : z, y' = y : z$. Then, the homogeneous equations of straight lines $ax + by + cz = 0$ turn into the known equations with two variables and absolute term.

**Problem 7.2.4*:** Coordinatisation of straight lines in space

Having introduced a Cartesian coordinate system, we can also coordinatise the space of all straight lines of $\mathbb{R}^3$ in an elementary (non-projective) manner by first assigning to every straight line $g$ the foot $F(g)$ of the perpendicular dropped from the coordinate origin $O$ onto $g$. If, conversely, we fix any point $P$ (described by three coordinates), then the straight lines $g$ through $P$, for which $P = F(g)$, can only vary in the plane through $P$ perpendicular to straight line $OP$, and consequently can be described by one further parameter. Thus, we obtain the four-dimensionality of this manifold in a completely different manner. Contemplate, whether and, if applicable, how we can include the straight lines of the projective closure of space by means of homogenising the used coordinates. Compare this approach to Plücker's.

**Problem 7.2.5:** Involutions of a projective straight line and points with complex coordinates

A projective map of a straight line onto itself is described by a fractional-linear transformation regarding the usual (inhomogeneous) coordinate $x$:

$$f(x) = \tfrac{ax+b}{cx+d} \text{ } (a, c \text{ not simultaneously zero}).$$

Whilst transforming into homogenous coordinates $x_0, x_1$, the transformation has the following system of linear equations for the coordinates $y_0, y_1$ of the image points:

$$y_0 = cx_1 + dx_0, \quad y_1 = ax_1 + bx_0.$$

Confirm how the characterising property $f(f(P)) = P$ of an involution is expressed by the coefficients $a, b, c, d$ and then show that the condition $f(P) = P$ always leads to a quadratic equation given this restriction, whereby in the case of real coefficients $a, b, c, d$ both fixed points are either real and the involution can be grasped as a one-dimensional transformation by means of reciprocal radii or both fixed points have mutually conjugated complex coordinates. Conclusion: points with complex coordinates are the "ideal" fixed points of involutions without real fixed points.

**Problem 7.3.1*:** "Napoleon's problem" (according to Mascheroni)

One of the first problems from Macheroni's *Geometria del compasso* requires us to divide a circle periphery in four equal parts (or equal in value: to construct the corners of a right-angled isosceles triangle) solely by means of the compass. (Solution tip: If given radius $r$, mark a line segment of length $r\sqrt{2}$ on the circumscribed circle, which we can find, according to Pythagoras, by constructing a line segment of length $r\sqrt{3}$ beforehand.) Elaborate this to a preferably "geometrographic" solution.

**Problem 7.3.2:** Construction à la Georg Mohr with compass by means of fixed span width (circle template) and straightedge

How do we find the following with the subsequent means (solution tips in brackets)?

1. The centre of a line segment (simultaneously the centre of each concentric shorter line segment),
2. An equilateral triangle for the given base (obviously, we only need an angle of 60 degrees at the extremities),
3. The perpendicular from a given point onto a given straight line $g$ (simultaneously the perpendicular for all straight lines parallel to $g$),
4. The parallel through the given point for the given straight line,
5.* Marking any given line segment on any axis.

Concerning 5., a little theory may be helpful, which was not provided by Mohr, but by Jacob Steiner in 1833: moving a line segment can be put together by parallel translation and revolution in the desired direction. The first is a purely linear construction. Thus, at most, we need the compass to construct parallels (see 4.). The latter can be reduced to the revolution of a line segment, the length of which corresponds to the provided circle/compass, by means of homothety (which also is purely linear). Since this revolving process can be conducted at any location due to the possibility of executing parallel translations, we do not necessarily need a circle template (although it often shortens the method). Principally, *a single* already drawn circle with centre suffices – for the given, and, as shown by Steiner, also for any other problem resolvable with compass and straightedge. We refer those who are now sufficiently curious to [Steiner 1833] or, if this is unattainable, to [Schreiber 1975].

**Problem 7.3.3:** Impossibility of doubling the cube with compass and straightedge

Show by means of case-by-case analysis that $x^3 - 2$ cannot be written as $(ax^2 + b)(cx + d)$ with integers $a, b, c, d$. According to Gauss's general theorems, it follows that doubling the cube by means of compass and straightedge is impossible.

**Problem 7.3.4*:** Non-constructability of the inscribed pentagon by means of its five given sides

Supplementing the construction of the inscribed quadrilateral by means of its four sides according to Vieta, show that an inscribed pentagon generally cannot be constructed by means of its sides with compass and straightedge. As always in such cases (cf. the explanations regarding angle trisection), one counter-example will suffice. We recommend the case $a_1 = a_3 = 3$ (length units), $a_2 = a_4 = 4, a_5 = 5$ and using Ptolemy's theorem, according to which the product of the diagonals equals the sum of the products of the opposite sides in the inscribed quadrilateral. We refer those who cannot find the solution with these hints to [Schreiber 1993], which also proves the existence of the wanted inscribed pentagon.

**Problem 7.3.5*:** Equivalence of construction problems in space

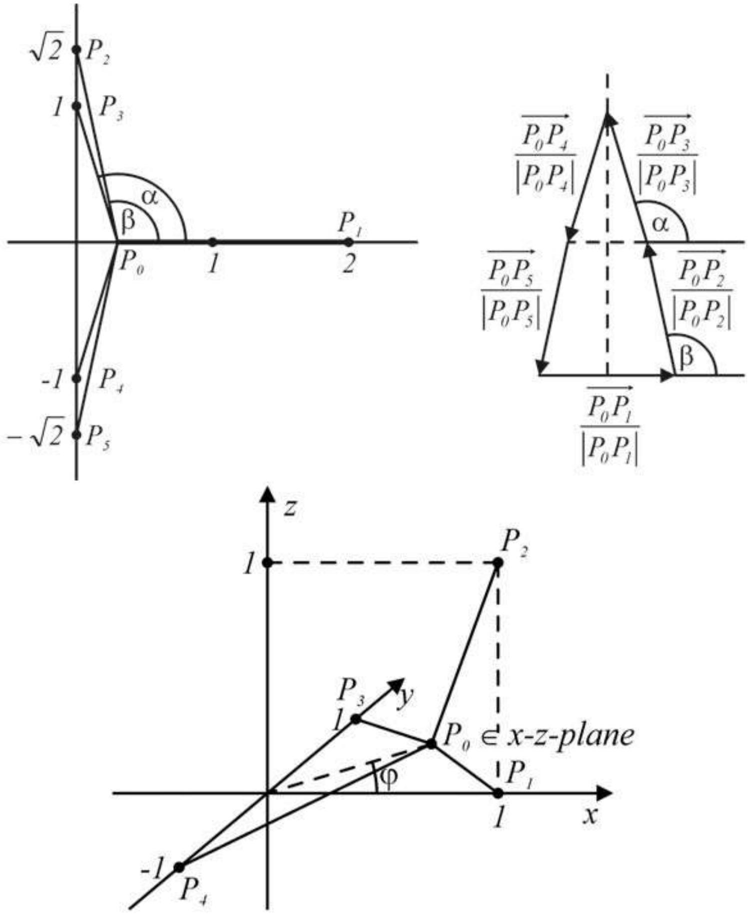
Show that the following three characterisations of the construction problems that can be solved in space are equivalent:

1. A point can be constructed by means of given points iff (if and only if) we can construct its three coordinates with compass and straightedge by means of the coordinates of the given points in an auxiliary plane regarding a given spatial-Cartesian coordinate system.
2. It can be constructed by means of given points iff, regarding a given position of the projection plane, its top and front view pair can be constructed by means of the top and front view pairs of the given points, meaning if the plane image problem that results from the spatial problem caused by Monge's two-plane method can be solved.
3. It can be constructed iff we can obtain it by means of the given points by using a planeal (i.e., an ideal instrument which produces the plane through each three non-collinear points in space) and a spherical compass (which produces the sphere of given radius around a given point) and locating the intersections of the so-constructed planes and spheres (additionally to compass and straightedge).

**Problem 7.3.6*:** Torricelli point of 5 points of the plane and 4 points in space respectively

The question of whether the Torricelli point can be constructed for more than 4 points of the plane or, respectively, more than 3 points in space with compass and straightedge has had no answer until recently, although Gauss had already stated the following hypothesis in a letter to his friend Schumacher in 1836 indicating that this will lead to "higher equations" [Gauß a, vol. X, p. 465]. As far as is known, this was first confirmed in [Bajaj 1988], although in a very tedious manner. Easily verifiable counter-examples for the plane and space are given in [Mehlhos 2000]:

1. Regarding a Cartesian coordinate system, take the five points with the coordinates $(0, \pm 1)$, $\left(0, \pm\sqrt{2}\right)$, $(2, 0)$. According to Problem 6.2.3, the Torri-




**Illus. 7.10.2** Figures to Problem 7.3.6

celli point $P_0$ has to be located on the $x$-axis due to reasons of symmetry. Thus, it suffices to find its $x$-coordinate $x_0$ or angle $\beta$ (Illus. 7.10.2) or, respectively, $y = \cos\beta$ or, respectively, to prove that one of these quantities cannot be constructed with compass and straightedge. If we define the auxiliary angle $\alpha$ as in the illustration and use what follows from the necessary condition (vector sum of the unit vectors drawn from $P_0$ to $P_1, ..., P_5$ is the zero vector): $2\cos\alpha + 2\cos\beta = -1$, then we obtain the following equation for the wanted quantity:

$$4y^4 + 4y^3 - 3y^2 + 4y + 1 = 0.$$

The irreducibility of this equation can either be proven directly or by looking at the belonging cubic resolvent.

2. Regarding a spatial Cartesian coordinate system, we choose the four points (0, -1, 0), (0, 1, 0), (1, 0, 0) and (1, 0, 1). Then, the Torricelli point $P_0$ has to be located on the $x$-$z$-axis due to reasons of symmetry. If $x$ refers to the cosine of the angle between the $x$-axis and the axis of the coordinate origin to $P_0$, we obtain the following equation for $x$:

$$(*)\ 8x^4 - 4x^3 - 7x^2 + 2x + 1 = (x-1)(8x^3 + 4x^2 - 3x - 1) = 0.$$

Thereby, $x = 1$ does not deliver a solution to the problem and the cubic rest polynomial is yet again proven to be irreducible. Follow this through and contemplate why this problem can be solved with compass and straightedge iff we can construct $x$ by these means. (In order to be able to appreciate the details of this fine solution, we ask the reader to contemplate, for example, the following: if $x = \sin\varphi$, only some algebraic signs change on the left side of the equation, but they have the effect that we can no longer split such a simple factor from the polynomial.)

**Problem 7.4.1:** Tractrix and Gaussian curvature of the pseudo-sphere

Based on the production of the tractrix demonstrated in Illus. 7.4.1, extract the parameter representation for the curve and for the surface created by the revolution around the asymptote and verify that the Gaussian curvature is negative and constant.

**Problem 7.5.1:** Constructions in Poincaré's model of the plane hyperbolic geometry

Solve the following construction problems in Poincaré's model with (Euclidean) compass and straightedge:

1. Straight connection of two points $P_1, P_2$ (Thereby, the case that the Euclidean connecting line is perpendicular to $a$ is trivial.)
2. Marking out a line segment on a point (Take into account that reflections are generally realised by transformation by means of reciprocal radii at the relevant circle.)

3.* Construction of a circle with given centre and radius (Use the solution to 2. as a "sub-program"!)

4.* Prove that trisecting any angle cannot also be done with compass and straightedge in hyperbolic geometry by using the preservation of angles of Poincaré's model.

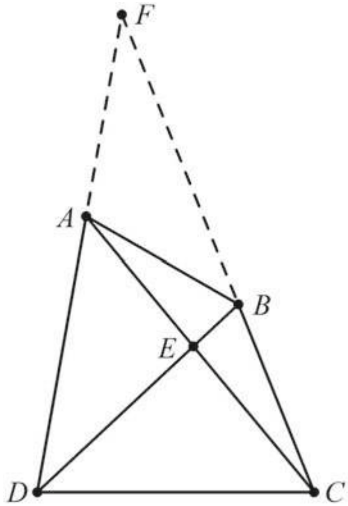
(Solutions in [Schreiber 1984], chap. 2.6)

**Problem 7.5.2:** The sum of angles of triangles is always smaller than 180 degrees in hyperbolic geometry. Consider that this deviation of the sum of angles from the Euclidean value ("defect" $\delta$) is additive, i.e., if a triangle (or, more generally, a polygonal figure) is decomposed into part triangles, the defect of the whole figure equals the sum of defects of the part triangles. Thus, we can use this defect as a measure of content.

* Squares and Pythagoras's theorem do not exist in hyperbolic geometry. However, we can use quadrilaterals with four sides of equal length and four equal acute angles, which can be easily constructed, as "pseudo-squares". If such a pseudo-square has an angle of $90° - \delta/4$($\delta$ defect of the pseudo-square) in every corner, a square of $n$-tuple content has the corner angle $90° - n\delta/4$. However, the construction of such a square based on what is given is already very tedious in the simplest case of $n = 2$, since it is unavoidably linked to the more difficult problem of constructing a triangle based on the given angles. Bearing this aspect in mind, look at the Euclidean triplication of the square by Abū'l-Wāfā (Illus 3.4.9). Would its application to a pseudo-square be correct in hyperbolic geometry? Attempt to sketch this with slightly curved square sides. The resulting "sling star" has exactly the triple content. But what happens when we change it to a pseudo-square?

**Problem 7.9.1:** Solution to Schumacher's paradox

The astronomer H. C. Schumacher wrote to his friend Gauss in 1836: "I have recently come across a paradox, which I am frank enough to present to you. I cannot yet sufficiently explain it. It is known that if we look for a point in a quadrilateral $ABCD$, of which the sum of the lines drawn at the angle points (i.e. $A, B, C, D$) shall be minimal, the wanted point is the intersection point E of the diagonals. If we now let points $A, B$ of lines $DA, CB$ approach more and more until they finally collapse in $F$ (Illus. 7.10.3), then $E$ also collapses




**Illus. 7.10.3** Figure to Problem 7.9.1

in $F$ at the same time and the quadrilateral turns into triangle $DFC$ and we would have point $F$ as the one, of which the sum of lines drawn to the angle points $F, C, D$ of the triangle shall be minimal. However and as known [!!], this is only true, if the angle [at] $F \geq 120°$." How can this "paradox" be solved? Those, who cannot solve it can read Gauss's answer in his Collected Works, vol. X, 1, p. 459f or in [Schreiber 1986].

**Problem 7.9.2:** Valence of Steiner points

1. Prove that a "Steiner point" $S$ always has a valence of three in the Euclidean case (in the plane or space of any higher dimension) and that the line segments from $S$ to the three points connected to $S$ have to enclose the angle of 120° pairwise.
2. How can we consequently construct Steiner trees for four points that span a convex quadrilateral in the plane?
3. Comprehend the problem in space by means of the top and front view method (cf. Problem 7.3.6b). It generally cannot be solved there by means of compass and straightedge, as recently proven [Mehlhos 2000].

**Problem 7.9.3:** Regarding the Grebe-Lemoine point

If $P$ is any point inside triangle $ABC$ and if $d_a, d_b, d_c$ are its distances from $a$ or $b$ or $c$, respectively, dissecting this triangle into the part triangles $ABP, BCP, CAP$ yields the following for the total area $\mathbf{A}$ of $ABC$:

$$\mathbf{A} = \frac{1}{2}(ad_a + bd_b + cd_c).$$

The problem of determining $P$ so that ${d_a}^2+{d_b}^2+{d_c}^2$ has a minimum given this side condition yields the following necessary condition by means of differential calculus:

$$a : d_a = b : d_b = c : d_c. \tag{*}$$

1. Check this and contemplate that (*) is also sufficient, since, on one hand, there must be a point $P$ in the triangle for which the minimum of the function is assumed (why?), and on the other hand, that a point can be uniquely determined by (*).
2. The construction of wanted point $K$ according to Grebe (1847) is as follows due to (*): enlarge the given triangle $ABC$ to $A'B'C'$ by drawing outside the triangle the parallel $A'B'$ in distance of the length of $c$ to side $c$, the parallel $B'C'$ in distance of the length of $a$ to $a$, etc. The fact that the straight lines $AA'$, $BB'$, $CC'$ intersect in one point $K$ already follows from Desargues' theorem. Prove that $K$ is the wanted minimum point.

3.* A synmedian of a triangle is created by reflecting a median on the corresponding bisectrix. First prove that the three synmedians also pass through a common point and then that this point fulfils condition (*).

**Problem 7.9.4:** Equality by dissection of rectangles of equal area

Show that any two rectangles of equal area are equal by dissection. (A proof can be found in e.g. [EdEM] (Encyclopaedia of Elementary Mathematics, vol. V, p. 150.) Why does this proof make use of the Archimedean axiom?

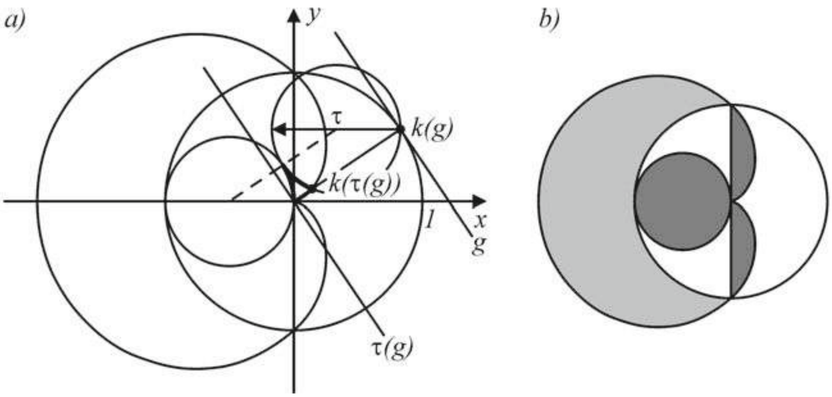
**Problem 7.9.5:** Buffon's needle problem

The distance of the parallel lines shall be $d$, the length of the thrown needle $l < d$. Characterise its location by means of distance $x$ of its right extremity of the straight line closest to the left of the given line grid and by angle $\alpha\left(-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}\right)$, which it forms with the perpendicular dropped onto this straight line. Then, a rectangle with the sides $d$ and $\pi$ is assigned to the set of all possible locations of the needle in the $x$-$\alpha$-plane. Determine the subset that corresponds to the "favourable" positions of the needle, and verify that the ratio of both capacities is $2l : d\pi$.

**Problem 7.9.6:** Bertrand's paradox

1. For version 1, the position of any straight line $g$ is characterised by its distance $r$ from centre $O$ of the circle and by angle $\alpha$ between the perpendicular dropped from $O$ onto $g$ and an arbitrarily distinguished initial direction. For version 2, it is described by angles $\alpha, \beta$ between this initial direction and the connections from $O$ to the intersections of $g$ with the circle. For version 3, it is described by the Cartesian coordinates $x, y$ of the foot of the perpendicular dropped from $O$ onto $g$. Check that the probability that the circle cuts out a chord from the straight line, which is longer than the side of the inscribed equilateral triangle, would be $1/2$ for the first version, $1/3$ for the second one and $1/4$ for the third one. Of course, we could find out for this case which of the three theoretical approaches is the right one by means of a statistical experiment. (Thereby, of course, we must only evaluate those throws of the straight line onto the circle or, respectively, the circle onto the straight line, for which there is any intersecting at all.) Nevertheless, we can also establish the right version by means of the Cartan-Poincaré criterion:
2.* For version 1, the set of all straight lines of the plane is injectively mapped onto the set of all pairs $(r, \alpha)$ with $r \geq 0$ and $0 \leq \alpha < 2\pi$, which we, for practical reasons, imagine aligned on a cylindrical surface of a perimeter of $2\pi$, rising from the floor $r = 0$ into infinite height. Since the measure of a set of straight lines should not depend on its position in the plane, we must now show that all[9] congruence mappings of the original plane

[9] In order to make the problem easier, consider that any congruent mapping of a plane can be produced by translations, revolution around the coordinate origin and reflection on any single selectable straight line. Hence, we only need to indicate for these special mappings that they correspond to mappings true to area on the cylindrical surface.




**Illus. 7.10.4** Figures to Problem 7.9.6

(in which the straight lines at hand are located) correspond to mappings true to area of the cylindrical surface onto itself. However, these cannot all be congruence mappings of the cylindrical surface onto itself, since no shifts towards the apothem ($\alpha$ constant) are possible due to its one-sided narrowness, whereas translations in the original plane change $r$ in general, but leave fixed the $\alpha$-coordinate of straight lines or change it by 180 degrees. It is easy to see by means of Cavalieri's principle that the mappings belonging to the translations of the cylindrical surface onto itself still remain true to area.

2. In order to show that the other versions for measuring the capacity of sets of straight lines suggested by Bertrand are not translation-invariant, a single counter-example for each suffices, i.e., a congruence mapping and a set $M$ of straight lines, the coordinate image $K(M)$ of which is transformed into a set not true to area for $K(M)$ in this mapping. In the case of version 3, for example, the unit circular disc is transformed into the set shown in Illus. 7.10.4 by means of the translation $\tau = (-1, 0)$. Thereby, its right half is considerably decreased and the left one is increased. Thereby, the perpendicular diameter described by $x = 0$ remains pointwise fixed. First, find a hypothesis for the image on the left and right half of the unit circle by pointwise construction.

4.* Show (preferably by means of elementary geometrical considerations, it is also possible by analytic means, but this is much more complicated) that the created heart-shaped curve is the conchoid of a circle produced by the unit line segment and the inserted circle

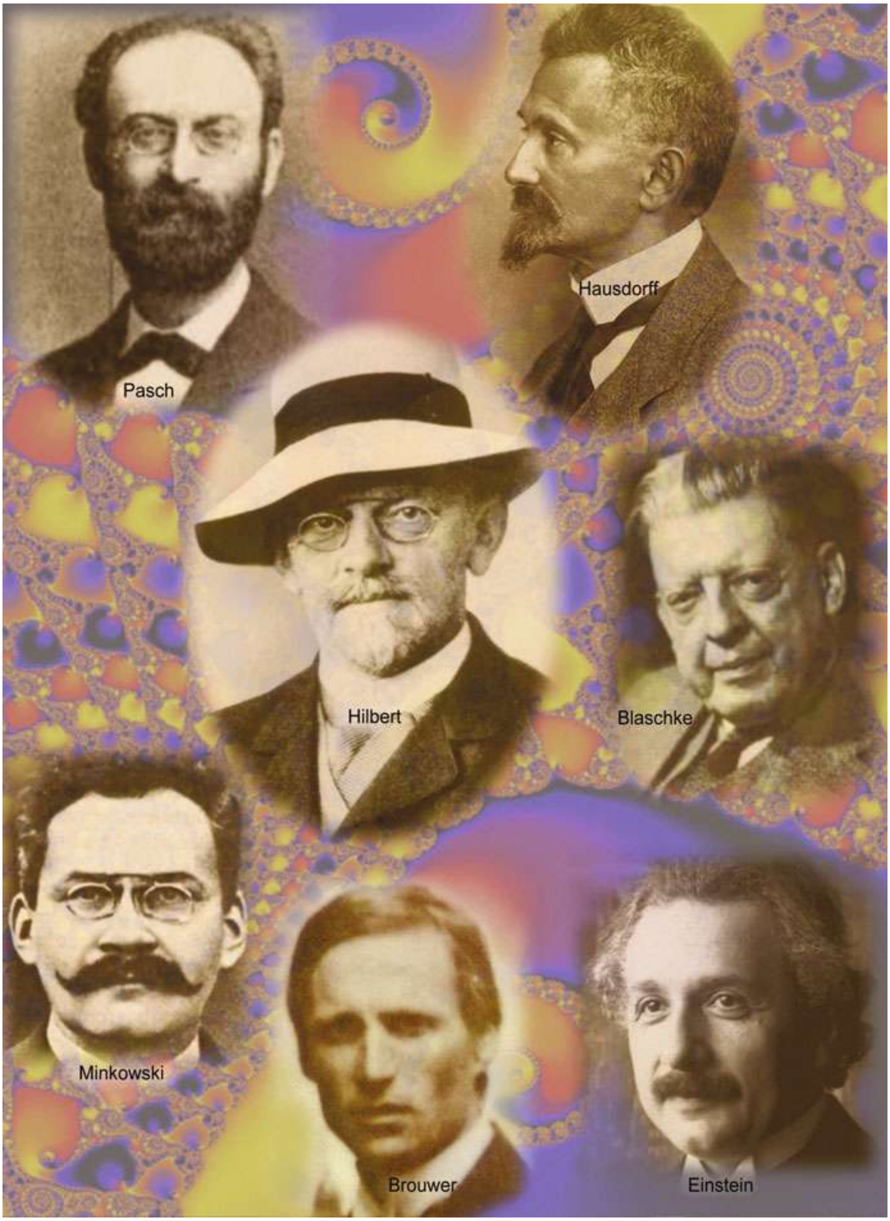
$$\left(x + \tfrac{1}{2}\right)^2 + y^2 = \tfrac{1}{4} \qquad (*)$$

and that it is identical to the cardioid, which is produced by unrolling an inserted variable circle on the fixed circle (*) (cf. Illus. 7.10.4).

**Problem 7.9.7:** Measuring sets of straight lines by means of arc lengths
As first shown by Cauchy in 1841, the measure of the set of all straight lines that intersect a limited plane convex set $M$ equals the length of the boundary curve of this set.

1. The reader shall verify this for the two special cases that set $M$ is a circle or a paraxial square.
2. Why is the theorem for non-convex sets not correct?

# 8 Geometry in the $20^{th}$ century





C.J. Scriba, P. Schreiber, *5000 Years of Geometry*,
DOI 10.1007/978-3-0348-0898-9_8

| | |
|---|---|
| 1895–1910 | Art Nouveau introduces new types of ornamentation |
| 1900 | Max Planck founds quantum theory |
| 1905 | Einstein publishes his theory of special relativity |
| 1905/07 | Civil Revolution in Russia |
| From 1908 | Cubism in painting(Picasso), abstract art(Kandinsky and others) |
| 1914 | Panama channel opens |
| 1914–1918 | First World War |
| 1916 | A.Einstein: general theory of relativity |
| 1917 | October Revolution in Russia |
| 1918 | End of German Empire, Habsburg Monarchy, Osman Realm and Empire of Tsar |
| From 1919 | Bauhaus founded in Weimar, modern architecture, constructivism in architecture |
| 1920 | Public broadcasting in the USA |
| 1933 | Hitler is Reich Chancellor in Germany |
| 1936 | Public television broadcasts Olympic Games in Berlin |
| 1938 | Nuclear fission of the Uranium core |
| 1939–1945 | Second World War |
| 1941 | First program-controlled electro-magnetic computer(K. Zuse) |
| 1945 | Use of atomic bombs against Japan (Hiroshima, Nagasaki) |
| 1947 | India and Pakistan gain independence, start of decolonisation |
| 1949 | Proclamation of Federal Republic of Germany and German Democratic Republic |
| 1957 | First artificial satellite(Sputnik) |
| 1957 | Treaty of Rome starts the European Union |
| 1961 | First manned space shuttle |
| From 1965 | Computer with integrated circuit |
| 1969 | First men land on the moon |
| From 1975 | Computer with microprocessor |
| 1979 | Ayatollah Khomeini founds theocrasy in Iran |
| 1989/90 | Collapse of communistic regimes in Eastern Europe, reunion of Germany, decline of Soviet Union |
| From 1991 | Boom of internet after the introduction of the World Wide Web(WWW) |
| 1979–1990 | Margaret Thatcher Prime Minister of the UK |
| 1991–2006 | Decay of Yugoslavia |
| 1997 | Sheep Dolly cloned as first mammal |
| 2001 | World Trade Center in New York destroyed by terrorists |
| 2001–2009 | G. W. Bush President of the USA |
| 2003 | USA and allies occupy Iraq; Space Shuttle Columbia disaster |
| 2004 | Space probe “Mars Express” starts cartography of Mars surface |
| 2004 | Approx. 230 000 people die from tsunami in Indian Ocean |
| 2008 | Burst of housing bubble and insolvency of Investment Bank Lehman Brothers cause global economical and financial crisis |
| 2009 | Barack Obama elected for President of the USA |
| 2011 | “Arabian Spring”: Revolts overthrow presidents of Tunisia, Egypt, Lybia, Yemen and starts civil war in Syria |
| 2012 | CERN Research Centre successful in hunting Higgs-Boson-Particle |
| 2014 | Islamic forces of ISIS try to found a theocracy in Iraq and Syria |

## 8.0 Preliminary remarks

If it was difficult for the $19^{th}$ century to fit the wealth of geometrical tendencies into a limited number of sub-chapters, it will be hardly possible to do so analogously for the $20^{th}$ century and would not serve the purpose of this book. The classification of mathematical disciplines currently valid (since 1991; MSC Math. Subject Classification) in the mathematical reviewing journals "Mathematical Reviews" and "Zentralblatt für Mathematik" pinpoints fourteen sub-areas under the main group (H) 51 (Geometry), three further sub-areas under H52 (Convex and Discrete Geometry) , and three sub-areas under H53 (Differential Geometry). Furthermore, there are fifteen sub-areas for H 14 (Algebraic Geometry), and eight sub-areas each for H 54 (General Topology) and H 55 (Algebraic topology). Under the main group 68 (Computer Sciences), there are, amongst others, the three sub-areas of Computer Graphics/Computational Geometry (which have little in common), Computer Aided Design, and Image Processing. If we also acknowledge that analytic geometry, now paired once again with descriptive geometry, forms one sub-group, that some geometrical works are reviewed under didactics, physics, logics and basics, model theory or recreational mathematics, and that some of the subjects added with so much care are basically empty of substance while others overflow, we begin to understand the complexity of mathematics in our time and a certain helplessness on the part of the authors of this classification in swimming against the stream of geometrical research and activity. In contrast, we will attempt here to put forth a picture of the geometry of the last century by looking at some of the most important extra-mathematical fields of application and also inspiration in relation to geometry. Sections 8.1 and 8.2 are exceptions, as they look at inner-mathematical development, which had already begun towards the end of the $19^{th}$ century, but only fully matured in the $20^{th}$ century.

Before starting with the history of geometry in the $20^{th}$ century, we should mention first that geometry now refers to two very different things due to the emancipation of geometry from the three-dimensional and Euclidean case, gradually concluded towards the end of the $19^{th}$ century. On one hand, (and the word 'geometry' is often understood as such by mathematicians nowadays, unless explicitly explained otherwise) geometry refers to the science of "spaces" as a whole, their different forms and possibilities for describing them and their mutual relations. On the other hand, geometry still and to an increasing extent means investigating the individual geometrical objects and/or a class of related objects in a space, which, of course, need not always be the classic $\mathbb{R}^2$ or $\mathbb{R}^3$.

After the turbulent and multiply branched development of the $19^{th}$ century, still in effect until around 1920, one segment of geometry had taken such a route towards abstraction that no consensus existed for a long time as to its inclusion as a part of geometry. Another, more illustrative segment had mostly been integrated into different engineering sciences. Geometry has only played a small role also in the mathematical curriculum of school education in many countries during the middle decades of the last century. The trend only began to take a clear turn in the last twenty to thirty years. The profound needs of computer sciences and other areas of application surely must have borne a great influence on these renewed turns towards geometry in illustrative terms. But also independently of this, knowledge has spread amongst mathematicians that geometrical problems that exist in the two or three-dimensional Euclidean space can be very difficult and are by no means unworthy of a contemporary mathematician's attention. In other words, a mathematician must not feel ashamed anymore, as was common at certain times, to engage with issues that can also be realised by laymen as geometric in nature.

The ups and downs of the development may become more comprehensible if we look at the background that mathematics as a whole went through a similarly turbulent history in the past 140 years: with the establishment of set theory and logics in the 1970s, caused last but not least by non-Euclidean and n-dimensional geometry, mathematics rapidly turned from a quasi-natural science into its present structure-theoretic existence. A phase of self-reflection, of deepening the philosophical, methodological and logical basics began, but with it the discovery of new possibilities, resulting from the fact that objects suddenly, or at least not immediately, did not require a corresponding object in the material world. This development, at least seemingly detached from reality to a certain extent, reached its extreme in the statement ascribed to one of the most prominent mathematicians of this time: "Mathematics is nothing more than a luxury in which modern civilisation indulges." However, the great political and economic turmoil of the recent past (in which computer sciences again take part) have made 'civilisation much' less willing than it was several decades ago to finance something that is not profitable. At the same time, a mathematician's job is a mass profession to a degree unheard of until recently with blurred boundaries to computer sciences, economics and other mass professions, reflected in the content and orientation of academic studies. Thus, mathematicians were exposed to external pressure to engage with application-orientated questions much more than during the first decades of the $20^{th}$ century, statistically speaking, and the results of pure research reflect the actual role of geometry for mathematics and the world less than they did before.

Having said that, we do not believe the reader will make any wrong conclusions based on the following statistical details. *The Jahrbuch über die Fortschritte der Mathematik* (Yearbook on the Advances of Mathematics) published a report around the year 1868, structured in 12 chapters, of which two were assigned to geometry: analytic and synthetic. Differential geometry was partially hidden in analytic geometry, partially in the chapters on differential and integral calculus, mathematical physics and geodesy. Both main geometrical chapters took up 153 of 396 pages in the reviews, i.e. about

39 %[27]. Of course, this says nothing about the extent of the reviewed works or the significance of the content. In 1884, geometry only took up 283 of 1097 pages, or about 26 %, although the chapters were structured in the same manner. Having updated the structure, in 1900 there were 109 of 909 pages concerning the topic 'Pure, elementary and synthetic geometry' and 'Analytic geometry' (including further parts of differential geometry), all in all about 21 %. In 1910, there were 99 resp. 115 of a total of 1054 pages with the same structure, or about 20 %. The last published semi-volume of 'Jahrbuch' (yearbook) for 1942 devoted 190 of 657 pages to geometry, or about 28 %. However, geometry had been re-structured as follows:

- Foundations, non-Euclidean geometry
- Elementary geometry
- Analytic and projective geometry
- Algebraic geometry
- Vector and tensor calculus
- Differential geometry, particularly Lie groups
- Rieman manifolds, transformation
- Topologic differential geometry, convex objects, integral geometry
- Kinematics
- Applied geometry

Topology had been assigned its own main section, which also contained graph theory as a sub-area. Applied geometry was further structured as follows:

- Descriptive geometry
- Photogrammetry
- Geodesic measuring
- Localisation, cartography, nautical science
- Geometric optics

The 'Jahrbuch' faltered in its attempt to further develop a presentation of one properly-structured year of achievements. Meanwhile, the 'Zentralblatt' had come into existence, initially structuring geometry as follows:

- General
- Elementary geometry
- Descriptive geometry
- Analytic, projective and non-Euclidean geometry

[27] Here and in the following sections, we refer not to the number of reviews and, hence, works counted, but the number of pages in the reviewing institution, since we assume that the length of each review can be taken as a rough measure for the extent and significance of the content of the text.

- Algebraic curves and surfaces (We see an attempt here to separate the "geometrical part" of algebraic geometry and the more algebraic tendency)
- Differential geometry, Riemannian geometry, tensors
- Topology (also see set theory; here, we see an attempt to split topology into a more set-theoretic and a more geometric, i.e., mainly combinatorial-algebraic, part.)

Here, geometry took up 59 of 432 pages, or only about 14 %. However, all areas apart from descriptive geometry summarised in 'Jahrbuch' from 1942 as "applied" fell prey to the greater modernity and even more so to the present classifications. 'Zentralblatt' dedicated 26 % of itself to geometry in 1942 (hence, the 28 % stated above for 'Jahrbuch' for this time gains more relevance). The formal part of geometry decreased drastically after the Second World War: Zentralblatt Vol. 30 (1949) 18 %, Vol. 45 (1955) 17 %, Vol. 60 (1957) 15 %. We will consult samples from *Mathematical Reviews* founded in 1940 for further development: 1960, 12 %; 1980 and 1990, each around 6 %; slowly increasing from 1991 onwards. However, it looks even more dismal if we limit statistics to geometry in narrow terms, i.e., to the main groups 51 and 52 (see above): 1960, around 1.5 %; 1970 until 1990, less than 1 % on average; 1996, 1.4 %. However, this is all put into perspective by the literature on "computer-relevant" geometry, which has been overflowing for years in order to summarise everything that belongs here under a neutral name literature, which, to a great degree, is either not included in the mathematical reviewing organs at all or in a different section.

Let us now return to the turn to the 20th century. When Hilbert presented his famous 23 problems at the 2nd International Mathematics Conference in 1900, he was just engaging intensively with the foundations of geometry himself. Hence, it did not come as a surprise that at least seven of these problems are geometrical in nature. They are:

3. The question as to whether a theorem analogous to Bolyai's and Gerwien's also applies to space.

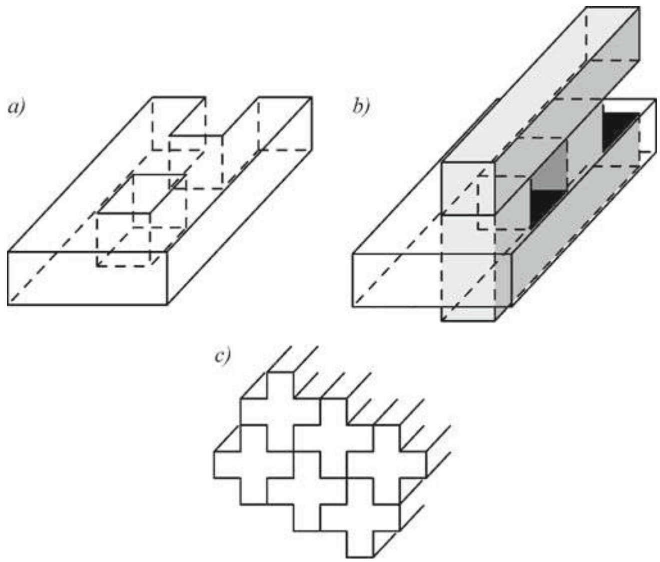
4. The question of geometrical theories that are similarly adjacent to Euclidean geometry as are the Lobachevskian or the spherical geometries, meaning they only differ in respect to a few propositions, especially the question of more general geometries, in which triangle inequality still applies.

5. The question of the dispensability of conditions of differentiability for continuous transformation groups.

15. The task of strictly accounting for Schubert's calculus of enumeration (which belongs to algebraic geometry).

16. Complete clarification of the topological nature of algebraic curves and surfaces.

18. The question as to whether there are polyhedra with which we can fill space without any gaps, but only so that these polyhedra are not a fundamental domain of a discrete group of motion; furthermore, if there is only a finite number of discrete groups of motion with a fundamental domain in $n$-dimensional space.




**Illus. 8.0.2** Spatial building stone for tessellation by Reinhardt
The hole in Reinhardt's stone (a) can only be closed by taking a second stone turned by 90 degrees and shifted by a quarter of the length in longitudinal direction and uniting it with the first one to form a solid (b), which is then obviously a fundamental domain. (Use it to form infinitely long strings of cross-shaped cross-sections and pave the plane lying across as indicated in (c).)

The $6^{th}$ problem is also related to geometry. It demands the axiomatic configuration of sub-areas of physics according to the model of formal-axiomatic Euclidean geometry as concluded by Hilbert shortly beforehand. Detailed descriptions of the status of the solution to these problems from the view of the year 1969 can be found in [**?**] [Browder 1976]. The solution to the third problem by M. Dehn in 1901 has already been mentioned in section 7.9. K. Reinhardt stated the first spatial tile in 1928, with the help of which Hilbert's $18^{th}$ question could be negated. The idea, hereby, was to dissect a suitable fundamental domain $F$ of a discrete group into two (or more) mutually congruent sections $T_1, T_2, ...$ such that no motion that transforms $T_1$ into $T_2$ leaves the $F$-tessellation invariant (Illus. 8.0.2). H. Heesch obtained an analogous result for the plane in 1932 (Illus. 8.0.4). However, whereas Dehn's result basically concludes the third problem, apart from later simplifications and/or deepening of proofs, the set of problems (in broadest terms) of the irregular tessellations respectively pavements with a finite number of types of paving stones, has established an extensive, new geometrical area of research in the past decades. Apart from its purely mathematical and aesthetical attractiveness (see Illus. 8.0.3), this new area also has some important natural scientific-technical applications (more in sections 8.3, 8.4 and 8.5).

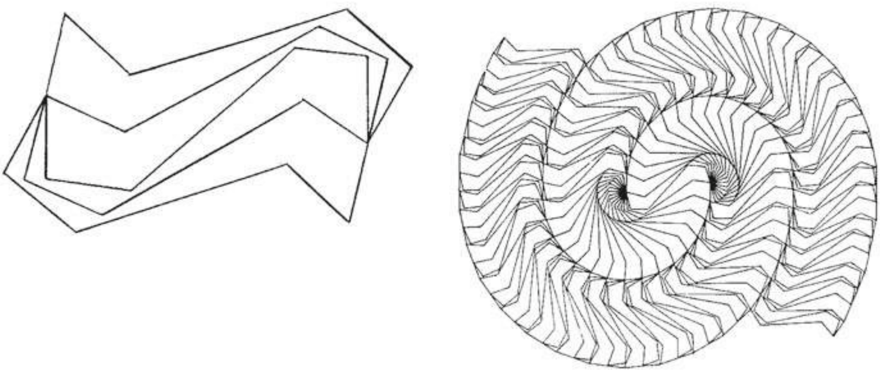
The last mentioned set of questions signals a fundamental change in contrast to the $19^{th}$ century, both in respect to the inner-mathematical "fashion" and the demands asked of mathematics by praxis. Problems that nowadays are referred to all too readily as "discrete" and that only received little attention during the $19^{th}$ century, despite having positioned themselves outside the main orientations of geometry, are not only equal in significance nowadays, but have been moved almost to the centre of interest. Those things that please a mathematician's eye are also different than they were one hundred years ago. Whereas then it was the models of analytic functions (the famous "plaster models"), of algebraic surfaces or of surfaces of constant curvature, also of semi-regular or star polyhedra, nowadays it is fractional formations, such as the Mandelbrot set, sophisticated non-periodic tessellations or graphs with special properties.

It is daring and presumptuous, but also tempting to list some unsolved or unsatisfactorily solved problems and main foci of prospective geometrical research according to Hilbert's role model. We will attempt to do so here:

1. A geometry of the discretised plane dissected into pixels demands a formal-axiomatic foundation, which is not trivial, but helpful for solving problems of computational geometry.

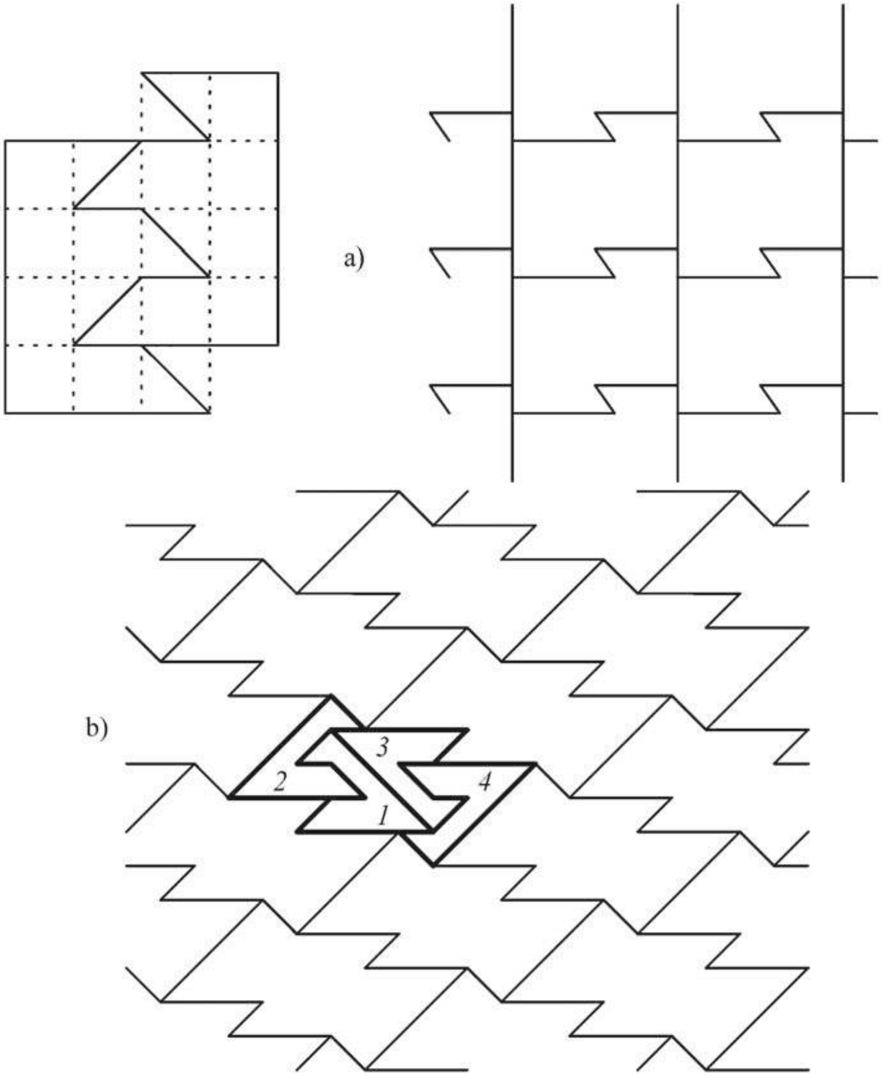
2. The cooperation of optical information of a two-dimensional image of a spatial scene with the non-optical knowledge either given additionally to and/or in advance of this scene demands a theoretic comprehensive analysis.

3. The algorithmic description of construction processes in three-dimensional space can still not compete in more demanding cases with the clarity of geometrical construction algorithms in the plane.



**Illus. 8.0.3** Polygonal tile by Voderberg

In 1936, Heinz Voderberg, one of Reinhardt's students, found this form, which can also be used to form an aesthetically pleasing double spiral, as the solution to the problem of two congruent polygonal tiles completely surrounding one or two others of the same type.




**Illus. 8.0.4** Building stones for tessellation in the plane

The original building stone a) by Heesch (1933) and a simpler example b) with only seven instead of ten corners; both cases indicate that two or, respectively, four assembled samples each yield a fundamental domain. However, thereby both "witch profiles" a) can only be interlocked by a glide reflection, which never leaves the whole tessellation invariant. In b), parts 2, or respectively, 3 are created by means of glide reflection and/or revolving part 1, and can be interlocked without any gaps only in this manner. However, these illustrations do not achieve congruence of the respective other part with one stone of the entire tessellation.

4. The P versus NP problem of computational complexity theory seems to be one of the most difficult unsolved mathematical problems at present. It concerns the question as to whether every algorithm, which given divided or respectively non-deterministic work of an input of a quantity $n$ works in "polynomial time", i.e., in a number of steps, that polynomially depends on $n$, can be replaced by an algorithm that also works in polynomial time, but strictly sequentially. The still unproven assumed answer is that this generally is not true. There already is a number of techniques that represent different levels of complexity of algorithms by geometric algorithms. So, could we perhaps solve the P versus NP problem by taking geometrical considerations into account, i.e., finding an example of a geometrical problem that can be solved non-deterministically in polynomial time, of which, however, we can prove by means of specific geometrical methods that it cannot be solved deterministically (sequentially) in polynomial time?

A fine, up-to-date introduction to newer problems and outcomes of some subareas of geometry is provided by the overview in [Giering/Hoschek 1994].

## 8.1 Foundations of geometry

The turn of mathematics from a quasi-natural science to a science of structure, which began around 1870, caused a flood of literature on logical-methodological and/or philosophical questions of mathematics, which had never been discussed before with such intensity. Geometry played a considerable role in the reasons for this development: the attempts, first successfully accomplished in the $19^{th}$ century, to prove the basic irresolvability of certain construction problems with compass and straightedge[28], as well as the non-provability of the parallel axiom, were mathematical outcomes of a completely new type. They inspired mathematicians to contemplate mathematics as a whole, the nature of proving, the nature of being of mathematical objects, the truth of mathematical propositions, and to make all of this the subject of mathematical investigations. The works on $n$-dimensional and non-Euclidean geometry (the latter to be understood in broader terms) did not just enable us to doubt the Euclidicity of physical space, but also split the mathematical notion of space from the physical one. Similarly, the gradual logical foundation of different number domains, which was linked to knowing alternative number domains like finite, non-Archimedean ordered or non-continuous domains of quantities, caused the notion of number or quantity to take over from the physically inspired notion of measure and number.

A simplified picture of history (which can be quite appropriate for some purposes) links the beginning of geometrical foundational research with Hilbert

[28] Especially the classical problems of doubling the cube, angle trisection and squaring the circle

and his *Grundlagen der Geometrie* (Foundations of Geometry), which was first published in a smaller version in 1899 as a "Festschrift" for unveiling the Gauss-Weber monument in Göttingen. [Toepell 1986] was first to make the pre-history of this book generally accessible, as far as it concerns Hilbert himself. We know now that the 'Grundlagen' were preceded by some of Hilbert's lectures, as well as a summer course regarding further training for teachers on this topic. Hilbert's notes on these events were published in extracts in [Toepell 1986] so that we can gain a detailed insight into the gradual maturing process of Hilbert's thoughts. After all, he had basically addressed completely different areas of mathematics before 1899. Apart from a letter on straight lines as the shortest connection between two points (which was added to the $2^{nd}$ edition of 'Grundlagen' as Appendix I) published in *Mathematische Annalen* in 1895, he had published nothing before 1899 on geometry or even merely its logical basics. Before thoroughly discussing the content and historical significance of Hilbert's book, we should also acknowledge that there had already been an ongoing discussion on the foundations of geometry at this time. This discussion covered the following problems amongst others:

– What is true physical space like and how is it related to geometrical theory?

– What justifies the selection and assumption of geometrical axioms and/or which criteria do we have to follow thereby?

– What role do certain axioms play therein, particularly those that determine the structure of the domain of quantities with which the distances are measured, i.e., above all the Archimedean axiom and continuity and/or completeness.

– On which assumption, which had not been reflected on until then, are the measuring processes in physical space based?

– How complete may a proof be? Which are its smallest steps? How far "down" do we have to go when defining basic notions and/or wording axioms?

– Prominent participants in these discussions were, amongst others, H. Hankel, O. Stolz, F. Lindemann, W. Killing, G. Frege, H. v. Helmholtz and F. Klein in Germany and the Italians G. Peano, A. Padoa, M. Pieri and G. Veronese.

Additionally, there were many people nowadays unknown, whose comments have been forgotten. Even a pronounced analyst like K. Weierstraß could not deny the spirit of the time and gave lectures on 'the principles of geometry'. Insight into the controversy, which is often pointless from a modern view, but extremely interesting from a historical perspective, can be found in the chapters "Geschichte und Philosophie" in 'Jahrbuch über die Fortschritte' (History and Philosophy in Yearbook on the Advances). A good overview is also featured in Klein's report [Math. Annalen, Vol. 50 (1898), p. 583-600] for the first award ceremony of the Lobachevsky prize, sponsored by University of Kasan for works on the foundations of geometry; for a modern viewpoint, see [Dieudonné 1985, chap. 13].

Above all, there are three men who must be mentioned regarding the more detailed pre-history of Hilbert's 'Grundlagen': Moritz Pasch, Giuseppe Peano and Hermann Wiener. Pasch published his book *Vorlesungen über neuere Geometrie* (Lectures on Newer Geometry) in 1882. The main aim of this book was the initiation of a program that Klein had inspired many times since 1871: the foundation of projective geometry (in other words: the constructive introduction of the projective plane or, respectively, projective space) independent of the parallel axiom. The projective closure was meant to be accomplished by means of adjunction of improper points on the basis of a "local absolute geometry", for which the question of uniqueness of parallels remains open at first. Hence, we must determine by means of local propositions (for which Pasch needed congruence and was upset with himself that he could not do a better job of achieving it) when line segments belong to a common bundle, even if their extensions do not intersect each other in the accessible part. Then, we must adjoin the defined bundles as improper objects to the accessible part of the plane. The usual projective closure features therein as that special case that is created when the available area in space constitutes the entire Euclidean space from the outset. If, in contrast, the available area of space is hyperbolic, adjunction delivers a much greater set of ideal points that form a projective space together with the proper points, as Klein's model shows. The relevant geometrical subject matter had, of course, already played a practical role when making constructions in restricted parts of the plane, where inaccessible auxiliary points are represented by accessible replacement objects. However, in Klein's program [Math. Ann. Vol. 4 (1871), p. 624; Vol. 6 (1873), p. 131], these techniques are assigned a new role as the logical safeguard of projective geometry, which yields models for both Euclidean and non-Euclidean geometry according to the Erlangen Program.

**Illus. 8.1.1** David Hilbert and Moritz Pasch

This wish met with Pasch's ability to penetrate problems logically, a skill strongly pronounced for his time (as reflected in his book on the basics of analysis and his talent, praised by his contemporaries, for phrasing propositions of any kind) and his clearly emphasised empirical approach. Hence, he rejected the notion of infinitely extended straight lines and returned to Euclid's line segments as the basic notion with which we can connect two points uniquely and successively extend. Accordingly, the incidence between points and line segments is the crucial basic notion of purely linear geometry. Hence, Pasch arrived at the three-digit relation of betweenness as the term for point $C$ being located on the line segment with the extremities $A$ and $B$. Hilbert would follow up on this, but chose the relation of betweenness instead of the incidence of point and line segment as the basic notion and some of the "basic propositions" conceived by Pasch (which, in contrast to Hilbert, can always be finitely experienced, according to Pasch) as axioms; amongst them, the one he named "Pasch's axiom". It is worded by Pasch as follows: "If three points $A, B, C$ are pairwise connected by the straight line segments $AB, AC, BC$ in a plane area, and if the straight line segment $DE$ is drawn through a point within the line segment $AB$ in the same plane area, then the straight line segment $DE$ or an extension of the same either passes through a point of line segment $AC$ or through a point of line segment $BC$." (l.c. p. 21)

Bear in mind that Pasch did not speak of a plane but of a plane area (imagined as bounded, but extendable) and of the idea that line segment $DE$ or an extension of $DE$ fulfils the known condition. He is a consistent empiricist (just like Helmholtz and, to a certain degree, Klein). Thus, his geometry deals with the space of "observation" or "experience" despite its (relatively) logical rigour. This distinguishes him crucially from Hilbert, who started his 'Grundlagen' from 1899 with the frequently quoted lines: "Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters $A, B, C, ...$ ; those of the second, we will call straight lines and designate them by the letters $a, b, c, ...$ And those of the third system, we will call planes and designate them by the Greek letters $\alpha, \beta, \gamma, ...$ The points are called the elements of linear geometry; the points and straight lines, the elements of plane geometry; and the points, lines, and planes, the elements of the geometry of space or the elements of space. We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as 'are situated', 'between', 'parallel', 'congruent', 'continuous', etc. The complete and exact description of these relations follows as a consequence of the axioms of geometry." What Hilbert meant with this wording, namely the complete loosening of the formal system 'geometry' by a fixed and possibly illustrative or physical interpretation of the occurring notions and theorems, is even clearer in the also frequently quoted report by O. Blumenthal, according to which Hilbert states: "At all times, we must be able to say 'tables', 'chairs' and 'beer mugs' instead of 'points', 'straight lines' and 'planes'." (cf. [Blumenthal 1922, p. 68] or [Toepell 1986, p. 42]) or in Hilbert's letter to Frege cited in Toepell l.c.

Hermann Wiener (son of the descriptive geometer Ch. Wiener) was the founder of this view regarding the axiomatic treatment of geometry. He had given a talk titled "über Grundlagen und Aufbau der Geometrie" (On the Foundations and Composition of Geometry) at the annual conference of the *Gesellschaft deutscher Naturforscher und Ärzte* (Society of German Natural Researchers and Medical Doctors) in Halle in 1891. Unfortunately, only a brief summary has been published in [Jahresbericht (annual report) DMV 1, 45-48], which, however, strongly influenced Hilbert in favour of the mentioned trend. On his way home from this conference, whilst reflecting on impressions Wiener's talk had left behind, he is said to have made the well-known statement about tables, chairs and beer mugs. The way Wiener spoke during this talk was indeed novel, if not revolutionary: "Let us presuppose two types of elements and two operations by assuming that connecting each two of these elements of the same type yields an element of the other type." (He was referring to the projective plane incidence structure made of points and straight lines with the operations of cutting and connecting.) Now, Hilbert attempted the synthesis of the novel, abstract structure-theoretic perspective as it had mainly been articulated in England up to then as part of the developing symbolic algebra and logics, with Pasch's groundwork, in which Pasch was not exactly aware of this view, and was even hostile towards it. In 1893, he wrote in a letter to Klein: "I have not even managed to hold my $3^{rd}$ lecture on non-Euclidean geometry [due to lack of audience]. However, I am elaborating it for myself and think the best way to gain an understanding for the geometers' argument concerning the axioms is to look into Pasch's clever book. We also owe it to Pasch that the necessity of the axioms regarding the notion 'between' has been recognised. The question of the smallest system of postulates (axioms), which I demand answer to by a system of units so that the same can assist me in describing the geometrical appearances of the external world [referring to the external shape of entities] seems not to have been completely answered until present day." (translated based on [Toepell 1986, p. 46f]). This demonstrates Pasch's influence on Hilbert and also indicates that Hilbert was not well-read within the realm of the foundations of geometry, which he had been addressing only casually and not for very long. Otherwise he would have known that Desargues and v. Staudt had already debated the notion of 'between' and given his relations to the one for the closed projective straight line, and could instead have found a four-digit separation relation. We must also mention Peano's axiomatic foundation of Euclidean geometry, which had already come into existence in 1894. However, Peano could only influence his students and followers with his work since it had been published only in Italian and in a heavily formalised manner as was characteristic for him.

To sum up, we want to pinpoint that Hilbert's 'Grundlagen' did by no means constitute the beginning of a new development, but rather mark a certain climax. Its charisma and effect, which were much greater compared to all its predecessors, must have been due to a certain degree of the fame that Hilbert had already gained in other areas of mathematics. Having said that, it must also have been due to the fact that he presented a much broader spectrum of interesting and fruitful questions regarding the axiomatic foundation of elementary geometry than all his forerunners. In contrast, it did not really matter that many of these questions were solved by others, particularly by his own students and/or that his own solution proposals were not always the best ones and his insight into problems was occasionally limited. This will be demonstrated by means of a few examples.

Hilbert's wording of some axioms is incomplete. For instance, regarding axiom II.1 he wrote in 1899: "If $A, B, C$ are points of a straight line and $B$ lies between $A$ and $C$, then $B$ lies also between $C$ and $A$." From the $7^{th}$ edition onwards (1930, in the meantime, Hilbert had thoroughly engaged with formal logics and published the book *Grundzüge der theoretischen Logik* (Outlines of Theoretical Logics) together with his student W. Ackermann), II.1 is: "If a point $B$ lies between a point $A$ and a point $C$, then $A, B, C$ are three different points of a straight line and B then also lies between $C$ and $A$." This is what he had meant from the beginning, but it could not be strictly deducted from the original wording.

In 1899 he only phrased the Archimedean axiom under the headline "Group of axioms V: axiom of continuity". In the second edition from 1903, there is also an axiom V2 of "completeness" apart from the Archimedean axiom (now V1) under group of axioms V: "To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus extended shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible to extension, if we regard the five groups of axioms as valid." This, of course, is a proposition about an axiomatic system, and thus belongs to a completely different language level than all other axioms. Axiom V2 did not go through any more changes, including in Hilbert's last edition from 1930, although Hilbert had already worded an intention expressed by V2 much better in the mentioned letter to Klein (later Appendix I), namely by means of the geometrically worded axiom regarding the existence of the upper and/or lower limit of bounded monotonic sequences: "If $A_1, A_2, A_3, ...$ are an unbounded sequence of points of a straight line $a$ and $B$ is a further point on a of the nature that generally $A_i$ lies between $A_h$ and $B$, as soon as index $h$ is smaller than $i$, then there is a point $C$, which has the following properties: all points of the infinite sequence $A_2, A_3, A_4, ...$ lie between $A_1$ and $C$ and every other point $C'$, to which this also applies, lies between $C$ and $B$" (translation based on $2^{nd}$ edition of 'Grundlagen', p. 84). Hilbert had also not considered that the Archimedean axiom can be concluded from the axiom of continuity phrased by Dedekind, although it had already been known from the theory of real numbers [O. Stolz, Math. Annalen, Vol. 31 (1888)]. The notion of categoricity of an axiomatic system (apart from isomorphy, there is only a single model of this axiomatic system), which played an especially important role for theories á la Euclidean geometry and which Hilbert had intended to refer to with his failed axiom V2, does not occur at all in his work. It was formed in 1903 by the American O. Veblen, who also made other significant contributions to the foundation of geometry.

The Archimedean axiom constitutes one of many issues for which (until now) striving towards preferably independent axiomatic systems is not exactly compatible with striving towards a preferably deep insight into the role of single propositions. Veronese (1891 in his 'Outlines of Geometry'[29], O. Stolz 1894, and others had attracted attention to the strange role of the Archimedean axiom when conducting different geometrical proofs (for example, Legendre's theorems on sum of angles, equivalence of equality by dissection, equality by completion and equality of area)). Since the Archimedean axiom enforces the commutativity of multiplication in a purely arithmetical manner and the latter, as shown by Hilbert, is equivalent to Pappus's theorem[30], he could provide a model of non-Archimedean geometry by means of a non-Archimedean ordered coordinate domain and a model for which Pappus's theorem is false, by means of a non-commutative coordinate division ring. All of these are fruits of a completely novel relation between axiomatic-synthetic and analytic geometry, which Hilbert introduced in his 'Grundlagen'. Whereas geometrical intuition of line segments and of the measuring process had until then mainly served as justification for the still unspecified notion of real numbers, Hilbert founded the consistency of geometrical axiomatic systems based on models formed of pairs and/or triples of appropriate numbers and, thereby, stated almost automatically which properties of the coordinate domain are responsible for the validity of which geometric axioms. However, he chose the reverse path for the calculus of segments by defining an algebraic structure (addition, multiplication, ordering process) by means of geometrical construction (Descartes had already done so!) and basing their structural properties on geometrical theorems. Thereby, he was the first person that did not shy away from considering theorems by, for example, Desargues and Pappus that play a major role in this context as possible axioms regardless of their complicated structure, and unveiled their central position for the structure of the geometrical subject matter. Axioms are no longer characterised by their "simplicity" or "immediate evidence" and/or as "experienced facts", but freely selectable and should be marked by their position within the logical structure. We owe it to Hilbert that the respective coordinate system was unveiled as a result of the immense wealth of wonderful mutual analoga between certain geometrical the-

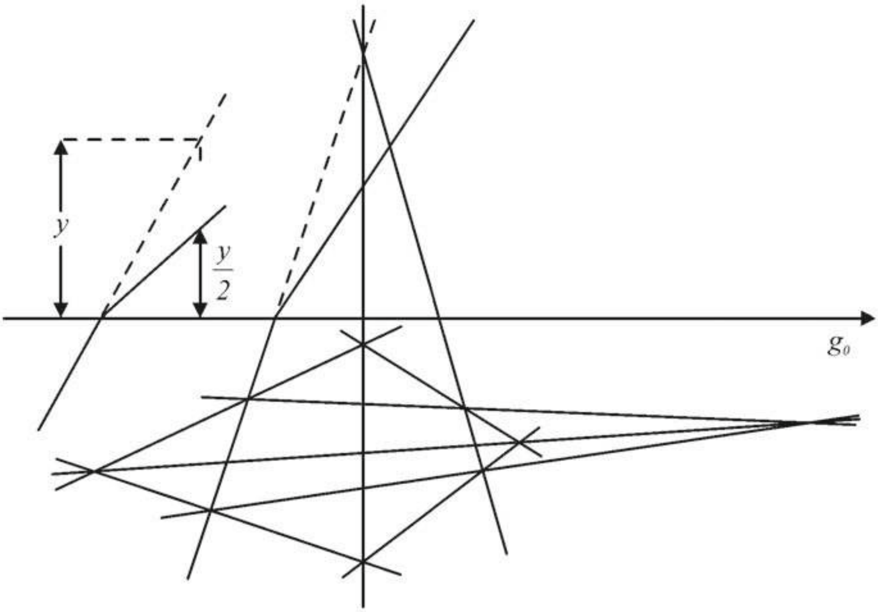
[29] Unfortunately, this flawed book was the only one from the Italian contribution to the 'foundation' of geometry that was soon (1894) translated into German.

[30] Hilbert and all of his contemporaries always spoke of Pascal's theorem within this context. However, we only need the special case ascribed to Pappus for all foundation-theoretic investigations, whereby the conic section degenerates to a double line.

orems and certain algebraic properties. A gap that existed for a long time in this respect was closed in 1970, when L. W. Szczerba and W. Szmielew pinpointed the equivalence of Pasch's axioms and the monotony of multiplication in the coordinate domain and proved the independence of Pasch's axiom by means of a suitably constructed "pathological" coordinate domain (Lit. see [Schwabhäuser/Szmielew/Tarski 1983]).

Hilbert's model of a non-Desarguesian plane (i.e., a plane to which the remaining axioms of incidence apply, but not Desargues' theorem) was rather complicated. The American Moulton had already found another model in 1902, one so simple that it can be explained to students easily (Illus. 8.1.2, Problem 8.2.2). Hilbert only adopted it in the $7^{th}$ edition from 1930 and annotated it as "a little simpler" in a footnote. G. Hessenberg proved in 1905 that Desargues' theorem follows from Pappus's theorem, which was also only included in 1930. It had become clear in the meantime that every finite field (or any other finite algebraic domain with corresponding properties) could also be used for constructing finite models of geometrical axiomatic systems, opening up another new field of study (see, for example, [Karteszi 1976]). As a forerunner, we must mention here the minimal projective plane with 7 points and 7 straight lines already published by G. Fano in 1892. O. Veblen, as well as his students and fellows, delivered excellent representations of geometrical axiomatics in the USA (amongst others, a *Projective Geometry* in two volumes together with J. W. Young in 1910/18). J. Hjelmslev started an axiomatics on the basis of reflections in 1907 (Fr. Bachmann conceived of something of a concluding monograph in this respect in 1959). Thereby, the points $P$ of a space are represented by point reflections (i.e., that point $R$ for which $P$ is the centre of line segment $QR$, corresponding to point $Q$), the reflections on straight lines correspond to these straight lines, planes analogous, etc., and the entire axiomatics refers to propositions on the composition of such reflections. One of the advantages resulting from this is a natural access to absolute geometry, i.e., that subject matter that Euclidean and Lobachevskian geometry have in common. For instance, it applies to absolute geometry that the composition of two revolutions in the plane is always a revolution again (but around a possibly improper centre). Only if we add the parallel axiom do revolutions around improper centres become translations, which in this context are the result of degeneration cases of revolutions in a very natural manner. Another benefit lies within the fact that "metric" reflections can be generalised as oblique reflections and, thus, gain access to affine geometry, which pays off for so-called affine differential geometry.

In Chapter IV of Hilbert's 'Grundlagen', the notion of area of polygonal figures was also first *defined*, and, herewith, acknowledged the (psychological) obstacle of needing to prove the independence of the product of base and height of a triangle of the chosen base, for which we need proportions. Furthermore, the mathematically difficult proof had to be accomplished so that the area of a polygonal figure defined as the sum of areas (i.e., half the product of bases and heights) of all partial triangles does not depend on the chosen dissection into triangles. Hilbert also did not mention here the groundwork




**Illus. 8.1.2** Moulton's non-Desarguesian plane

It consists of all points of a Euclidean plane in which any straight line $g_0$ and a direction of this straight line is chosen (hence, also a "lower", namely the one on the right regarding the direction, and an "upper" half-plane). All straight lines parallel to $g_0$ and all that intersect $g_0$ "from top left to bottom right" or perpendicularly serve as straight lines of the model. The straight lines running from "bottom left to top right" are refracted when passing through $g_0$ in a fixed refraction ratio (in the illustration, 2 : 1) in the same manner that we know from optics for the passage through the separation layer between two optical media. The reader is asked first to contemplate that all basic axioms of plane Euclidean geometry (existence of non-collinear points, unique possibility for connecting points, existence and uniqueness of parallels) apply to this model and how we could conduct the respective operations constructively. If we now position a Desarguesian figure, for example, in the manner shown (of course, there are other possibilities) so that all essential points, except for the perspective centre, are located in the lower half-plane and we have to refract exactly one of the three projection rays, then the triangles maintain their perspective axis, but have no more perspective centre. Beyond this concrete purpose, this example serves as a fine inspiration for constructing non-standard models in order to clarify logical non-dependencies within the realm of geometrical theorems.

of Bolyai, Gerwien and others. As already mentioned, it is probable that Hilbert's literary knowledge within the field of geometry was rather limited. However, he managed to close almost all outstanding gaps in his ingenious conceptual framework with his students and the students of his students. We again refer the reader to Dehn's solution to the $3^{rd}$ Problem and to the 1908 dissertation in which H. Brandes, a student of Hilbert's student F. Bernstein, proved that the equality by dissection of the square of the hypotenuse and the sum of the squares of the cathetes cannot generally be realized with less than 7 partial triangles.

We must now turn our attention to a mathematician who remained in Hilbert's shadow for no good reason. Friedrich Schur had already entered the discussion on the foundations of geometry with the publication of several works beginning in 1891. It is easy to see his external significance for Hilbert, since he is one of the most quoted authors in Toepell's book on the origins of Hilbert's 'Grundlagen'. However, his view on the subject matter was only published in the form of a summarising book in 1909. There are important differences from Hilbert: Schur used the notion of motion instead of the notion of congruence as a basic notion, just like Killing, Peano and Pieri had done before him. From a purely logical standpoint, both approaches are equally valid, since we can define each of these notions by means of the other one and then translate all axioms and theorems into theorems equal in meaning via the other notion. Nonetheless, Schur's approach had three advantages: on one hand, the axiomatic characterisation of motions as a sub-group of affine mappings with specific bounded degrees of freedom fit in better with the group-theoretic classification of geometries according to Klein. On the other hand, the congruence of line segments and, more generally, of any sets or figures can naturally only be verified by attempts to make those congruent (such an experiment would be measuring a line segment with a scaled ruler or measuring tape), i.e., executing motions. Conversely, the attempt to prove a mapping as a motion, i.e., as rigid, leads to us having to prove the congruence of a certain system of pairs of line segments, which, again, is only possible by means of motions. Third, the manner of how congruence is defined by means of motion can be generalised for any other group of mappings and fosters fruitful analogous thinking. Nevertheless, a consistent axiomatisation in Schur's sense requires us to grasp motions as a new kind of arbitrary basic object (like Hilbert's tables and beer mugs) and to characterise their nature as bijective mappings of the set of points onto itself as well as the application of a motion to a point as a binary operation between undefined objects by axioms (see Problem 8.1.2). Obviously, Schur was nowhere near this.

Hilbert had introduced the geometrical instrument called the 'gauge', with which we can just construct a minimal model of Hilbert's axioms (excluding completeness) by means of the four axiomatic non-coplanar points taken as existent and which is equal in value to the closure of the coordinate field regarding the operation that forms the line segment $\sqrt{a^2 + b^2}$ by means of the line segments $a, b$. In contrast, Schur introduced an elementary axiom that more rigorously affirms the closure of the coordinate domain regarding the operation $\sqrt{a}$ for $a > 0$ than the gauge and simultaneously assures that the necessary condition for two circles to intersect – both radii and the distance of the centres must suffice for the three triangle inequalities – is also sufficient. Accordingly, this axiom by Schur, together with the other elementary axioms, forms the appropriate scope for a real elementary geometry of compass and straightedge constructions, whereas the axiom of continuity features the real plane $\mathbb{R}^2$ as its only model, but is too demanding for the purposes of elementary geometry and also burdens geometry with all problems of real numbers (such as uncountable sets). Schur referred to the book by Veronese, who indeed had already thoroughly engaged with questions of closure of the coordinate domain regarding different operations in 1891 and, in this context, had also discussed the question of the existence of the intersections of circles [Veronese 1891, German transl. 1894, Book II, 1. Chap. 16]. However, this very extensive book is nearly indigestible. Klein wrote in the report mentioned above: "I find it very difficult to follow the way the author thinks even just a little bit" [l.c. p. 596]. Hilbert, Peano and, decades later, Freudenthal also commented negatively [Toepell 1986, p. 56].

It is a historical curiosity that the logician G. Frege heavily criticised the formal part of Hilbert's 'Grundlagen', but did not consider the deficits essential from a modern viewpoint [Jahresber. DMV 12 (1903), 319-324 and 368-375], and that Th. Vahlen (the author of the book on geometrical construction cited in section 7.3) published *Abstrakte Geometrie* (Abstract Geometry) in 1905, which dwelled on formal subtleties, contained many errors and basically did not provide anything new. The crushing critique by M. Dehn [Jahresber. DMV 14 (1905), 535-537, Vahlen's response 591-595], which Vahlen's had brought upon himself, made him an eternal enemy of such structure-emphasizing mathematical work, an applied mathematician in a sense already archaic back then, and, finally, one of the few prominent Nazis (Dehn was Jewish) and exponents of 'illustrative German mathematics' amongst German mathematics professors.

Many investigations into different versions of axiomatic composition by means of other notions or other axiomatic systems, which more or less directly followed up on Hilbert's 'Grundlagen', brought new geometrical knowledge as well as clarity for the mutual relations of notions and theorems that Hilbert was unable to achieve during his lifetime, and a set of theories "adjacent" to Euclidean geometry due to alterations of such new single axioms (see, for example, [Bernays 1959]). A. Padoa, one of Peano's students, conceived a method in 1900 with which we can prove the non-definability of a notion $b$ by a given system of notions $B$: since all notions definable by $B$ remain invariant for each automorphism belonging to $B$, it suffices to state a $B$-automorphism that does not leave $b$ invariant in a model. Thus, we could prove, amongst other things, that congruence cannot be defined by the relation of betweenness and that the relation of betweenness cannot be defined by incidence relations (whereas reverse definitions are possible), so that there is a real hierarchy amongst Hilbert's basic notions. Within this context, it also became clear that the parallel axiom, positioned towards the end by Hilbert due to traditional reasons, belongs with affine incidence geometry, meaning at the beginning of the now common hierarchical composition of geometry of incidence, order and motion. Mathematical logics, which matured in the 1930s, made clear the fundamental difference between elementary theory (i.e., also the notion of elementary geometry, which until then had been used rather intuitively) and the non-elementary axioms, such as the Archimedean or the axiom of continuity.

Nonetheless, most geometers in this field of study were not aware that their work basically only referred to the level of manipulation with formalised language. As soon as systems of notions are proven to be mutually definable, the mutual translation of axioms and theorems becomes routine. In contrast, changes to axiomatic systems, which are of a completely different nature, go back to J. Hjelmslev. His 'Geometry of Reality' (1916) and 'Natural Geometry' (1928) criticised the traditional idealisation, for example, that arbitrarily close points should be uniquely connectable. This concept has begun to harmonize with the development of interval mathematics since the 1970s (see, for example, [Schreiber 1984]).

A. Tarski's result from 1940 that an appropriately defined elementary partial theory of full Euclidean geometry is decidable heralded a new quality regarding the investigations on the foundations of Euclidean (and also hyperbolic and other "classic" geometrical) theories. These theories were now indeed only grasped as sets of character strings, which had to be examined according to the criteria of enumerability, finite axiomatisability, decidability, definability, or complexity. [Schwabhäuser/Szmielew/Tarski 1983] offer a good insight into this topic. The contributions by Engeler, Schreiber, Seeland and others since 1967 have taken another direction, which treats geometrical constructions as model cases of algorithms in any axiomatically characterised structure. [Schreiber 1984] also includes non-Euclidean geometries, interval-mathematical and other aspects of practical geometry.

## 8.2 Total abstraction?

As we will see now, the $19^{th}$ century produced almost all essential notions and questions on which basic mathematics would rapidly reach abstract heights in the $20^{th}$ century, a fact nobody can comprehend anymore without having thoroughly studied them. (Here, comprehension does not just refer to the technical details, but also to content and intended goal.) Within the scope of this development, the part of mathematics, which was accepted as geometrical in the $19^{th}$ century, at least to the extent that it generalised notions and subject matter of ordinary Euclidean space in an obvious manner or looked at them under new aspects, dissolved into a mathematical style that transferred originally geometric notions onto completely non-geometrical objects and problems. The notion of vector space is still linked to the $\mathbb{R}^n$ or $K^n$ (whereby $K$ is another coordinate field, especially the one of the complex numbers), i.e., with the existence of a finite base, at the end of the $19^{th}$ century, even after the first axiomatic presentation by Peano. The step of ignoring this last restriction and at least subjecting coordinate vectors of infinite length, i.e., countable sequencies of numbers $(a_i)$, to term-wise addition and scalar multiplication, whereby the domain of sequencies $(a_i)$ has to be restricted to such with a convergent sum of squares, if we want to obtain an inner product, i.e., measurement of lengths or angles, was not made for the needs of geometry or pure linear algebra (an obvious conclusion from a modern perspective). Having said that, the mathematicians had been aware since Buniakovsky and Schwarz of the fact that the definite integral of $f \cdot g$ acts like an inner product for functions $f, g$ with sufficient conditions. This includes the fact that a linear structure is given that suffices for the same laws as the usual vector addition and scalar multiplication, by the operations $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ for a set of functions with a common domain of definition. Generally speaking, there was a multitude of attempts to transfer algebraic operations formally onto problems of analysis over the course of the $19^{th}$ century. This included attempts to write a differential or integral "operator" (of course, this notion did not exist at the time) in the form $(E-A)$ (with identical operation $E$) and then to dissolve the inverse operator $(E-A)^{-1}$ into the infinite series $E + A + A^2 + ...$ Such techniques played a central role from around 1895 onwards in works by V. Volterra, I. Fredholm and Hilbert on linear integral equations. Hilbert's student E. Schmidt clearly established the 'geometrical core' of these techniques in his dissertation from 1907 and subsequent works. Furthermore, he created what is now referred to as Hilbert space as a not necessary finitely dimensional vector space with countable base and positively definite inner product, and showed how we can always produce an orthonormal base in such a space and, by means of this base, an isomorphism on the standard space of the sequences with convergent sum of squares already mentioned above. One chapter of his work from 1908 is directly titled "Geometrie in einem Funktionenraum" (Geometry in a Function Space). However, the space of the functions integrable within a given domain of definition was at first not complete in terms of convergence of any Cauchy sequence. Yet, F. Riesz and E. Fischer had shown at approximately the same time how we can complete it by using Lebesgue's notion of integral. After all, this is also a fruit of the new approach to geometrical thinking in analysis, which means that functions, operators, sets, ... were increasingly taken to be "points of a space". The now common type of definition and axiomatic treatment of Hilbert spaces and their generalisation to Hermite spaces (in case of a complex-valued scalar product) was only used from 1929 onwards and first by J. v. Neumann.

Let us turn towards another source of this new geometrical thinking in analysis. Since the beginning of systematic error analysis, a 'pessimistic' version, intended to determine the maximally possible deviation between exact object and approximate object, had competed against a 'realistic' version, which looks at something like the statistical average of deviation. If objects are functions (they could also be, for example, series of measured values, since a series is nothing else but a function defined on the basis of the set of natural numbers), the realistic version leads to a value that results from suitable integration or the addition of differences between exact and approximated values. Gauss had made great efforts always to state new reasons for why we should take the sum of squares of differences and/or, in the case of continuous functions, the integral of the square of difference as the measure of deviation. Only about a century later did it become clear that this leads exactly to the metric of Hilbert spaces. In the case of functions, the pessimistic version, which originated from Chebyshev's theory of mechanisms, looked at the maximum of difference as the measure of deviation. The following became gradually clear over the course of the $19^{th}$ century: a factual pre-condition for the existence of this maximum is that the common domain of definition is bounded and closed (or something equivalent) and that functions are continuous there. Once more, we are faced with a vector space of functions as the basic set, very similar to the one of a Hilbert space, but with an entirely different metric, which cannot be founded upon an inner product this time (and, thus, neither yields an angular measure), but simply assigns a norm or length $||x||$ with certain basic properties[31] to every vector $x$ so that the distance of two vectors $x, y$ can be measured by $||x - y||$.

The axiomatic generalisation of this special norm to the notion of (possibly additionally metrically complete) normed vector space was due to different approaches (amongst others, H. Lebesgue in 1910 and 1913) in St. Banach's dissertation from 1922, after whom the complete normed spaces are named nowadays. However, it is very strange that H. Minkowski had already conceived of the notion of normed vector space for the finite dimensional case when pursuing number-theoretic problems in 1896. He found that we only need to state the set of the vectors the norm of which is 1 (today: standard body), and that this can be any bounded convex and, regarding the coordinate origin, centrally symmetrical subset of $\mathbb{R}^n$ in order to determine a distance in $\mathbb{R}^n$ compatible with the vector structure. (Thereby, convexity means validity of the triangle inequality; central symmetry means: $||x|| = || - x||$.) If the standard body is the unit circle or, respectively, its higher dimensional generalisation, we obtain the metric of the Euclidean or Hilbert space. If it is bounded by the hyperplanes parallel to the coordinate axes (i.e., in the two-dimensional case, a paraxial square of side length 2), we obtain the (Chebyshev) maximum norm. If it is a square, the corners of which are the

[31] The axioms characterising a norm are: $(1)||x|| \geq 0, (2)||x|| = 0, iff$ $x = 0, (3)||\lambda x|| = |\lambda|||x||, (4)||x + y|| \leq ||x|| + ||y||$.

four points with the coordinates 1, -1 on the axes, or the corresponding $n$-dimensional generalisation, we obtain the sum of the absolute values of the coordinates as norm. It seems that back then nobody considered the fact that they were dealing with the last mentioned metric, if they – like in a big city with streets crossing at right angle – can only take paths that are composed of line segments parallel to the axes – which is why this metric is handily called a Manhattan metric – and that such a metric would in the future play a practical role when designing electric circuits, even in the "trivial" 2-dimensional case. In 1935, J. v. Neumann and P. Jordan showed that the so-called parallelogram law

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2) \tag{8.2.1}$$

(geometrically: the sum of squares of the diagonals of a parallelogram equals the sum of squares of the four sides), the validity of which can easily be checked for norms defined by means of the inner product, is also sufficient for being able to derive a given norm from an inner product (Problem 8.2.1).

We have repeatedly made reference to "metric" in recent pages with a silent assumption that the reader is familiar with this notion or at least has a valid intuitive notion thereof. Generally speaking, a metric space is a pair of a set $M$ and a distance function $d$ defined in $M$ so that for any $x, y, z \in M$:

1. $d(x, y)$ is real and non-negative
2. $d(x, y) = 0$ exactly then, when $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$.

At the same time, this notion is very illustrative and very abstract. It was introduced by M. Fréchet in its just defined general form in 1906, also due to functional-analytic motives. Nonetheless, it seems that the name "metric space" was only coined by F. Hausdorff in 1914. The self-evident observation that many different metrics are possible in one and the same basic set is linked to the notion of metric space. Hereby, geometry seemingly needed the help of analysis, although much earlier the fact had been clear that there will always be an inner metric and a metric regarding the surrounding space (chordal distance) on, for example, curved surfaces, and especially on a spherical surface. Of course, Gauss and all differential geometers following him knew that, but it did not seem to be worth pinpointing or contemplating any further.

All fundamental notions of analysis are connected to the notion of metric: the "$\epsilon$ - neighbourhood" and, thus, all notions of inner, outer, frontier point, convergence, continuity, open and closed set, compactness, connectivity, etc., but also the notion of Cauchy sequence and metric completeness. From this viewpoint, many to some extent old theorems of analysis take on a new meaning. For instance, in 1838, Ch. Gudermann had already introduced the notion of uniform convergence of sequences of functions, which was then generally made known by his most important student K. Weierstraß towards the end of the $19^{th}$ century. The classic theorem that any sequence of continuous functions uniformly convergent on a compact domain $M$ converge towards a continuous limit function is now understood to mean nothing other than the metric completeness of "the space of functions continuous on $M$" regarding the maximum norm of this vector space. Thus, the notion of metric is the most important key for transferring geometrical concepts to other areas of mathematics. If we now talk of transferring notions, such as approximate solution or a measure of deviation to areas, in which mathematics has to establish itself first, the crucial question is again mostly one of suitable metrics in the relevant area. Thereby, the inner-mathematical observation that there is never just one inherent, completely determined metric in any basic set can prevent errors and false approaches. However, for proper geometry explicitly invoking the notion of metric means the development of a new and very fruitful sub-discipline called "general metric geometry" (often somewhat misleadingly equated with convex geometry). A summarisation entitled *Die innere Geometrie der metrischen Räume* (The Inner Geometry of Metric Spaces) by W. Rinow was published in 1961.

Based on the notion of metric space, it is possible to derive metrics from originally given metrics in spaces $R_1, R_2, ...$ for mostly abstract spaces, consisting of cartesian products of $R_1, R_2, ....$ For instance, we can immediately generalize the manner for obtaining the "Pythagorean" metric of the Hilbert space or the metrics of the Banach-Minkowski spaces by means of the natural distance of two real numbers to techniques for defining metrics with different properties in Cartesian products of sets with given metrics. The set of all compact subsets of a given space turns into a metric "hyperspace", according to Hausdorff. He showed simultaneously how we can obtain a symmetrical notion of distance from an initially unsymmetrical one (as, for example, occurs in a terrain, when the way downhill may be "shorter" than the way uphill). (See Problem 8.2.2 for details and elementary applications to the theory of errors of geometrical operations.)

When writing above that the fundamental notions of analysis are based on the notion of a metric, we deliberately stated a half-truth. Although the relevant metric is indispensible for measuring and calculating, as well as for the entire numeric realm of analysis, most fundamental theorems can also be obtained by means of even more abstract pre-conditions, as long as they are topological in nature. We remind the reader that the first notions and theorems of general topology, such as inner and/or frontier point, are based on a metric. Boundedness and closure of a domain are important for the validity of several such theorems. Continuity of functions is based on the notion of an epsilon-neighbourhood. Many small steps were necessary to split those notions and theorems for which a metric is really necessary gradually from those for which a metric only serves to define topological notions. For instance, E. Heine showed in 1872 that the theorem on the continuity of limit functions cited above is based on subject matter now referred to as the Heine-Borel covering


**Illus. 8.2.1** Luitzen Egbertus Jan Brouwer and Felix Hausdorff

theorem. This theorem states that we can choose a finite subset, which also already covers $M$, from every covering with epsilon-neighbourhoods, if the domain $M$ is bounded and closed. As was later recognized, the seemingly necessary metric property of $M$ to be bounded and closed is reduced to the equivalent topological property of compactness, which can be defined without use of a metric just by the validity of the Heine-Borel theorem.

As the quintessence of such a series of insights, F. Riesz (1908), H. Lebesgue and L. E. J. Brouwer (based on other notions) and, finally, F. Hausdorff, in his book *Grundzüge der Mengenlehre* (Outlines of Set Theory) from 1914 [Hausdorff 1914] introduced the abstract notion of a topological space. Hausdorff defined this notion as a basic set in which, as the only geometrical structure, only a system of subsets is distinguished as "open", sufficing for some simple axioms, for example, that the common part of two and/or the union of any number of open sets is again open. It was discovered that we can base (in part, after tightening the axioms to some extent) the entire general or set-theoretic topology and, thus, also further parts of analysis on such an abstract base of notions. It was recognized that many different metrics can lead to the same topological structure of a "space" in a fixed basic set, but also that different topologies are possible there (Problem 8.2.3). Based on this foundation, the general set-theoretic topology quickly developed into a blossoming and independent field of study, which, to a certain degree, split from geometry and/or has more in common with analysis than with proper geometry. This was expressed externally by the fact that one of the first mathematical journals not to be dedicated to all mathematics, but to certain special areas, such as *Fundamenta Mathematicae*, founded in Poland in 1920, focused on the fields of set theory, topology and (higher) analysis, and that one of the first international conferences (Moskow 1935), which was devoted to only a sub-area of mathematics, also addressed topology.

From the wealth of newly arisen questions, we will pick only two: how can we define the dimension of such an abstract topological space so that it, on one hand, remains invariant when topologically mapping[32] it onto another space, and, on the other hand, so that it agrees with the natural dimension in such spaces in which there is already a "natural dimension" based on the number of necessary coordinates due to given coordinatisation? This problem was finally solved by K. Menger and P. Uryson in 1922 after the groundwork by L. E. J. Brouwer and others. When and how can we introduce a metric to a topological space that yields the given topology? This question was also answered by Uryson in 1924/25 with further contributions by J. Nagata and J. M. Smirnov in 1950/51. Uryson established the (astonishingly weak) conditions under which a topological space is even homeomorphic to a subset of the standard Hilbert space. The practical significance of this question is also clear at present: in a new domain of discourse (think of, for example, economy or psychology!), it is easier to say intuitively what is proximal and what is not than to state a concrete measure of distance. Thus, we often have a topology first and look for a matching measurement of distance afterwards. However, given the spirit of the time back then, it is very probable that the authors mentioned were motivated to take up their investigations by a purely theoretical interest[33].

A number of so-called fixed point theorems form part of the fundamental gains of the general "notion of space". These theorems can all be so interpreted that the convergence of certain approximate methods of analysis or numeric mathematics is essentially based on interpreting the objects to which these methods are meant to be applied as points of a suitable space and the methods as suitable geometrical mappings of this space onto themselves, and then using the topological properties of this space, such as metric completeness or compactness. The first and most famous of these theorems was the one by Brouwer (1912): every continuous map of a set homeomorphic to an $n$-dimensional full sphere in itself has at least one fixed point. The theorem by Banach (1922) is also relevant: every contractive map $f$ of a complete metric space in itself has exactly one fixed point. Thereby, $f$ is called contractive if there is a constant $c < 1$, so that the following applies to the distance of any two points $x, y$ and their image points: $d(f(x), f(y)) \leq cd(x, y)$. Brouwer's fixed point theorem was generalised by J. P. Schauder to convex and compact subsets of any Banach space (Problem 8.2.4). It contains Brouwer's fixed point theorem as a real special case, but we need this one exactly to prove Schauder's theorem. (Both proofs can be found in, for example, [Naas/Tutschke 1986].) The Stone-Čech compactification (1937) also repre-

[32] A map between two topological spaces is called topologic if it is bijective and transfers open sets to open ones in both directions (equal in meaning: $f$ and $f^{-1}$ are continuous).

[33] The author of these lines considers himself to be qualified to pass such a judgement, as he was an assistant at an institute at the beginning of the 1960s that, back then, could have been called a centre for set-theoretic topology.

sents a significant generalisation of classic geometrical methods: within the general set of problems to "close" geometrical spaces by adding ideal (infinitely distant) elements, the closure by a single infinite point, as is assumed in Möbius's geometry, plays a minimum role. Not only the extension of a hyperbolic space by all infinitely ideal points, as suggested by Klein's embedding of the hyperbolic into the projective space, but also the projective closure of an affine plane are so-called 'middle cases', which always result in embedding a non-compact set in a compact one, from the topological point of view. The Stone-Čech compactification now yields the maximum result of such an embedding process, for which 'as many points as possible' are adjoined.

Nowadays, it has almost been forgotten/repressed that the question of measures (length, area, volume,...) was one of a geometrical nature for millennia (and one of the most important of geometrical questions at that). Measure theory is an independent field of mathematics today and may be closer to stochastics than to geometry due to its most essential applications. However, the path to modern measure theory still belongs to geometry. Cauchy had defined the notion that we now refer to as definite integral as the common limit value of upper and lower sums in his *Cours d' Analyse* in 1821 and, thus, given the relation between antiderivation and area, which until then had only been intuitive, a clear foundation. He proved the existence of the definite integral for continuous functions in 1823. Riemann had casually generalised this to the notion of Riemann integrability in his habilitation text in a short section titled "über den Begriff eines bestimmten Integrals and den Umfang seiner Gültigkeit" (On the Notion of a Definite Integral and the Extent of Its Validity) in 1854. The relevant notion of content for any set (which, hence, need not have the form $\{(x,y)|x \in D_f \text{ and } 0 \leq y \leq f(x)\}$ of an "ordinate set" regarding a suitable function f) was created by Peano and Jordan around the same time in 1887, and also made clear that the "outer content" can really be greater than the "inner content" for a bounded set[34]. It seems that both were aware that the needs of analysis at this time already demanded a notion exceeding the Riemann-Jordan-Peano content, namely a totally (meaning countably) additive notion of measure. First attempts to define totally additive measures, for which in particular to every countable set is automatically assigned the measure zero, go back to A. Harnack (1881), O. Stolz (1884), G. Cantor (1884) and E. Borel (1894). (For details, see [Dieudonné 1985] Chaps. 6.3 and 6.9.) These efforts were temporarily abandoned in 1902 due to the creation of the notion of measure and integral by H. Lebesgue. In 1905, Lebesgue also asked the question as to whether or not the domain of those sets that can be assigned a totally additive measure can be further extended. In the same year, G. Vitali (using the axiom of choice) constructed a counterexample of a bounded set $V$, of which countably many

[34] Think of a carpet with fringes. The smallest covering is bigger than the biggest exhaustion.

congruent and pairwise disjoint samples are contained in a certain bounded interval (which is why the measure of $V$ would have to be zero, if it was definable), and cover a smaller interval without gaps (which is why the measure of $V$ cannot be zero). After the problem of a totally additive measure function defined for all bounded sets had been negated by this, Hausdorff showed in an addendum for his 'Grundzüge' in 1914 that we cannot even define a finitely additive content for any bounded set in the three-dimensional case and, thus, even less in one of higher dimension. He dissected the spherical surface and, hence, also the full sphere in three congruent and pairwise disjointed sets $A, B, C$ and a rest with necessarily vanishing content, again by means of the axiom of choice, so that $A$ is also congruent to the union of $B$ and $C$. We can draw two possible conclusions from this paradox: either it is a further argument (next to many others) against the axiom of choice, or the classic geometry founded upon real numbers is a very bad model for physical reality due to reasons completely different than the simply 'global' ones, which will be discussed in this course. Banach showed in 1923, also by means of the axiom of choice, that there are universally defined finitely additive content functions in the one or two-dimensional case, but uncountable equivalent ones, of which we cannot distinguish any as preferable in constructive manner. (For more details on the history of measure problems, see [Schreiber 1996b].)

Finally, there is one of Hausdorff's accomplishments, which belongs to measure theory in its broadest terms: in 1919, he introduced an "outer measure" for any sets by means of which we can assign them a dimension, which, generally, is no longer an integer. This Hausdorff dimension became very meaningful during the age of intensive study of "fractal sets", since it reflects the state of such sets, like the one of the v. Koch curve (see, for example, [Mangoldt-Knopp, Vol. 2, no. 145]), between one and two dimensionality rather well.

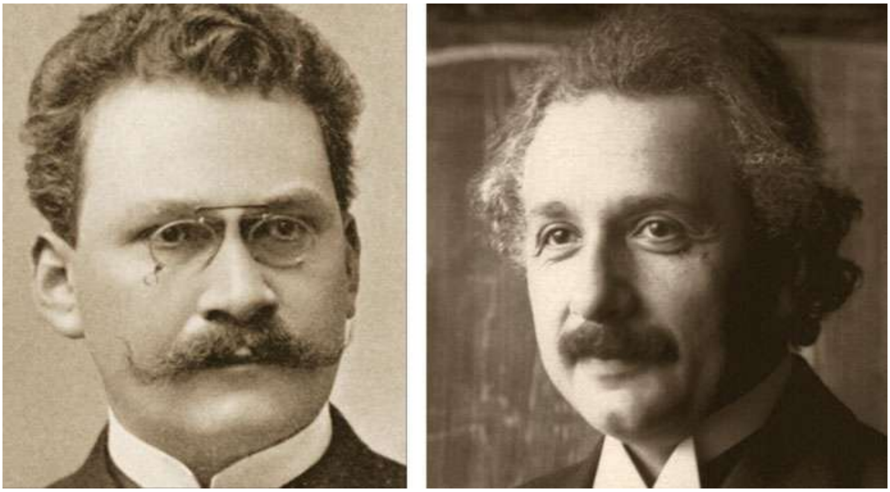
Generally speaking, the theories that we have rudimentarily introduced here have developed originally from geometrical notions and problems and are characteristic for the mathematics of the $20^{th}$ century. Next to their intramathematical and purely epistemological meaning, they represent a great source of notions and methods for the most different and most recent applications, which often serve the applied fields of mathematics, as well as techniques, natural sciences and humanities, in an unpredictable manner. However, we could not possibly miss the fact that the golden age of those directions, which are relatively poor in algorithms but rather dealt with establishing often profound subject matter, is long over.

## 8.3 Geometry and natural sciences

First of all, there is physics here. It is generally known what role modern geometry, and especially its forerunner in the $19^{th}$ century, played in establishing both the special and then the general theory of relativity, as well as for the cosmological hypotheses that followed. One of the main tasks of this section is to make the reader look at completely different applications of geometry in physics, as well as chemistry, biology and geosciences. Nonetheless, let us first look at what the reader expects.

The results of the end of the $19^{th}$ century concerning hyperbolic, Riemann and multi-dimensional geometry had led to an intellectual climate in the circles interested in mathematics, natural sciences and also philosophy, in which almost every statement about physics regarding non-Euclidicity or non-three-dimensionality of physical space would have been accepted by a majority without much further questioning. However, that which enforced the experientially ensured invariance of speed of light compared to reference systems moved against each other (Michelson 1881, improved together with Morley 1887) was somehow still an unprepared change of traditional concepts, which did not just have to include time in geometry in a manner unimaginable up to that point, but also to equip the newly created four-dimensional space with a kind of geometry, a notion unheard-of until then. Einstein had been a weak mathematician, at least at the beginning of his career (also according to his own opinion). When he published the work titled 'On the Electrodynamics of Moving Bodies' in 1905, soon to be known as the special theory of relativity, it was less geometrically thorough than philosophically daring, since something that had already been known in terms of formulae since 1895 due to H. A. Lorentz and mathematically elaborated by Poincaré as the group of Lorentz transformations was here claimed to be physical reality. Einstein wrote: "Examples of a similar kind as well as the failed attempts to state a movement of Earth relative to the 'medium of light' lead to the assumption that properties of appearances do not just correspond to the notion of absolute idleness in mechanics, but also in electrodynamics... We will advance this assumption ( the content of which shall be called the principle of relativity in the following) to precondition and furthermore introduce the seemingly incompatible precondition that light always travels in empty space with a fixed speed $V$ independent of the state of motion of the emitted body..." (l.c. p. 891).

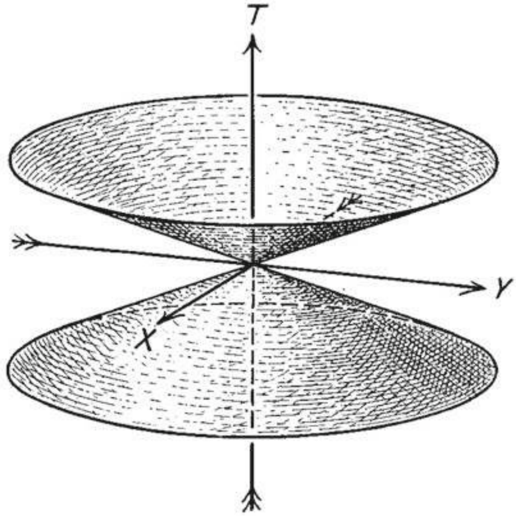
We owe it to the mathematician H. Minkowski that Einstein's theory was soon equipped with a clear geometrical foundation despite its complete lack of mathematical preparation. (Unfortunately, Minkowski died young shortly after.) In 1908, he demonstrated that we would simply have to eliminate the precondition applying to the already established theory of finitely dimensional Euclidean (or Hilbert) spaces that the inner product defining the metric is positively definite, in order to obtain a mathematical model cor-



**Illus. 8.3.1** Hermann Minkowski and Albert Einstein

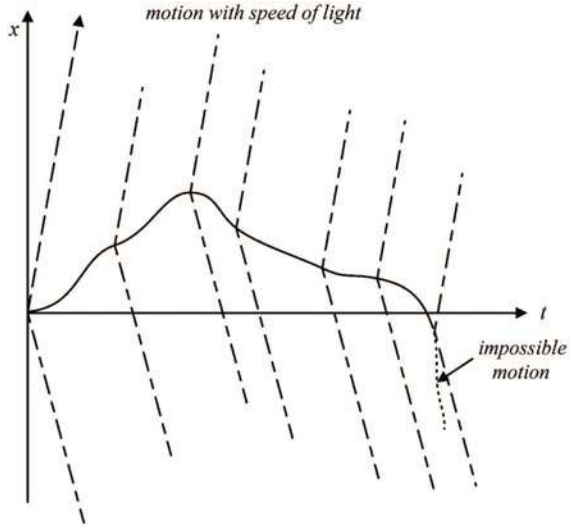
responding to Einstein's space-time. Then, the norm of a vector can also be purely imaginary or zero without having to be the zero vector itself. In the case of four-dimensional space-time, these latter vectors form the "light cone", if we add them to any event (point of space-time). The light cone separates the events reachable from this event from the unreachable ones, and, conversely, those that could have had an influence from those that are independent. Hyperboloids replace spheres as locations of constant distance from a given point in this geometry. In general, it was discovered that many elements of projective and hyperbolic geometry play a role, although the space itself, including its lower dimensional sections, is neither projective nor hyperbolic. Due to reasons of illustration and didactics, we often only look at the subspace spanned by the time axis and one or two space coordinates (Illus. 8.3.2).

Nowadays, this pseudo-Euclidean or Minkowski geometry (not to be confused with that of the Banach-Minkowski spaces) has been thought through and so thoroughly adapted that we can discuss it with older students without any problems [Liebscher 1991]. Hence, the great quake turned into a small one, especially since the ratios in the realm of lower speeds remain naturally classical with any approximation. It is worth pointing out that it was just pseudo-Euclidean Minkowski geometry that additionally motivated us to, analogously, think through the geometrical ratios in non-relativistic space-time, which is by no means Euclidean in terms of $\mathbb{R}^4$. Thus, the Galilei-Newton geometry was created as the boundary case of speed of light, being assumed to be infinitely fast based on pseudo-Euclidean geometry. It seems Klein was first to demonstrate and explain it in this way in his lectures on non-Euclidean geometry ([Klein 1928], I. M. Yaglom 1966, also see [EdEM Vol. V].) In this case, the light cone degenerates to a double-hyperplane (i.e., in the 'didactical' case of one or two space axes sketched above to a straight




**Illus. 8.3.2** Three-dimensional space-time with light cone
The more a space-time-point $P$ is spatially distant from the space-time-point at the apex of the double cone, the more time the information regarding an event in $P$ or its effects on the point of the apex of the cone will need in case of a past event $P$ and/or the later the information/effect regarding an event at the apex of the cone will arrive in the future $P$. The surface of the double cone bounds the set of events that can in any way stand in a causal relation with the events at the apex of the cone, and its apex angle is greater the higher the maximal propagation speed. All of this would also apply to a restriction of propagation, for example, by old-fashioned techniques of telecommunication and/or traffic, and even more so to the universal restriction of all propagation by speed of light. [F. Klein: *Vorlesungen über nicht-euklidische Geometrie* (Lectures on non-Euclidean Geometry), Springer. Berlin 1928, p. 30]

line or plane $t_0 = \text{const.}$), which effectively separates future events ($t > t_0$) from past ones ($t < t_0$). The history of a particle of matter that changes its location over the course of time corresponds (both in the Einstein-Minkowski and in the Galilei-Newton space-time) to a curve that leaves the past cone and enters the future cone at all times. Consequently, it lies over the t-axes for purely geometrical reasons (Illus. 8.3.3). However, the direction of the time axis in space is distinguished in the Galilei-case. Co-ordinate transformations must map it onto itself, i.e., can only shift it in a translative manner (which corresponds to a transition to another 0-time) and, hence, map the space onto itself in an orthogonal manner. Compared to the Minkowski-world, the light cone must be mapped onto one with equal asymptotes (i.e., its section with the infinitely distant must remain invariant), which leaves sufficient space for transformations, mapping a time axis onto another one: time loses its absolute character.




**Illus. 8.3.3** World line in the $x$-t-co-ordinate system

The special theory of relativity was followed by the general theory (first announced by Einstein in 1915, with complete publication in 1916, and a version easier to comprehend and targeted at a broader audience by Einstein himself in 1917). In 1918, the fundamental book from a mathematical viewpoint, *Raum, Zeit, Materie* (Space, Time, Matter) by H. Weyl, was published, also adapting the entirety of basic mathematical knowledge into a new and modern manner (for example, it was first to derive the Cauchy-Buniakovsky-Schwarz inequality in the now common manner (cf. 7.6)). By 1923, five editions of this book had been released, one for every year it had been in print.

The general theory of relativity states that four-dimensional space-time has such a curvature varying from location to location and determined by the distribution of masses that the orbits of moved particles turn into the geodesic under the influence of gravitation in this space. The first experimential confirmation of this theory consisted of proving the (very long-term) perihelion precession of the orbit of Mercury and the (very slight) deflection of the light rays in the gravity field of the sun, only observable during a solar eclipse. Therefore, the mathematical model is now the general Riemann geometry of four-dimensional space-time. Hence, the geometry of space is also Riemannian at all times. (To illustrate this, contemplate that the section of a curved surface with a plane generally yields a curve.) The space-time of the special theory of relativity relates to the general theory as the linear tangential manifold of a curved manifold to this manifold. Hence, it reflects the ratios locally with sufficient approximation if the curvature is not too large (i.e., in physical terms, sufficiently distant from large masses). From a purely geometrical perspective, the curvature of space-time could make it possible that the curved future cone of an event overlaps with the past cone of the same event and, thus, closed world lines are possible, meaning things could "travel into the past" after all. However, further reaching physical considerations contradict this idea. Moreover, only elementary particles could survive such a journey. We refer the reader to the up-to-date, generally comprehensible and very geometrically illustrated depiction in [Hawking 1988].

However, it turns out that, along the way, the extended investigations of the late $19^{th}$ century regarding space forms of constant curvature and their correlation to free motion lose several aspects of their physical relevance. True space cannot have a constant curvature, since matter is not distributed homogeneously in it. Having said that, if we take such a distribution to be the first approximation, the space forms of constant curvature describe more or less what is possible in general terms. Due to physical reasons, a statistically homogeneous distribution of masses seems unimaginable in an infinitely extended space, since it would yield an infinitely large total mass and, thus, an infinitely large gravity potential. Therefore, almost all contemporary cosmological hypotheses favour unlimited, but at all times spatially finite (and lately also finite in terms of time) models. Without intending to enter an actual physical discussion or to pick on mathematical details, we have seen how many ways the purely geometrical theories of the $19^{th}$ century have laid the ground for relativistic physics and cosmology of the $20^{th}$ century.

The true revolution in physics was not triggered by relativity theory, but by quantum physics, which began developing in 1905 and which was forced upon physicists by experimential outcomes and against their intuition. In contrast to everything sketched above, it is impossible to illustrate quantum physics, even its basic assumptions. In other words, it cannot be explained by didactically edited geometrical models. Hence, it is even stranger that geometry (or analysis?), of course, in its most abstract forms, has also provided us with one of the necessary mathematical means dressed as the notion of the (infinitely dimensional) Hilbert space. Nevertheless, it is not the elements of this Hilbert space that indicate something illustratively imaginable. Rather, linear operators with certain properties correspond to the "observable quantities" of classical physics in this space, and the physical laws are expressed by relations between these operators. The notion of symmetry, which also comes from geometry, plays a dominant role in this kind of physics. Since this is a book on history, we may mention that very intelligent scientists, who were also successful in other areas, struggled greatly with accepting a theory of physics that gives up classical logic and the principles of determinism and causality and the mathematical tools of which are no longer classical analysis, vector and differential calculus. Einstein was also not happy about this development and looked for an alternative throughout his life. Of all the strange attempts to establish an alternative, we should mention *Rechnender Raum* (Calculating Space) [Zuse 1969] by computer pioneer K. Zuse, based on the concept of the cellular field, which J. v. Neumann had left behind after his death in 1957: an infinitely extended two or three-dimensional flock of regularly and mutually aligned, equal elementary automata that exchange synchronized signals with their respective neighbours and, thereby, change their statuses according to a uniform instruction within the scope of a finite set of possible statuses. Zuse proposed imagining physical space as such a space, which is assembled by cells that cannot be further dissected and in which matter moves due to conglomerates of statuses of adjacent cells transmitting from cell to cell in a synchronized manner (similar to the computer game 'Life'). Then, space and time are discrete in the microcosm and the speed of propagation of the signals distinct to the system is the highest achievable speed. In particular, if a moving object $O$ with maximum speed emits light particles on all sides, then the macroscopically observed speed towards the motion of $O$ is as fast as in the opposite direction. Nonetheless, Zuse's hypothesis can easily be disproved (Problem 8.3.1).

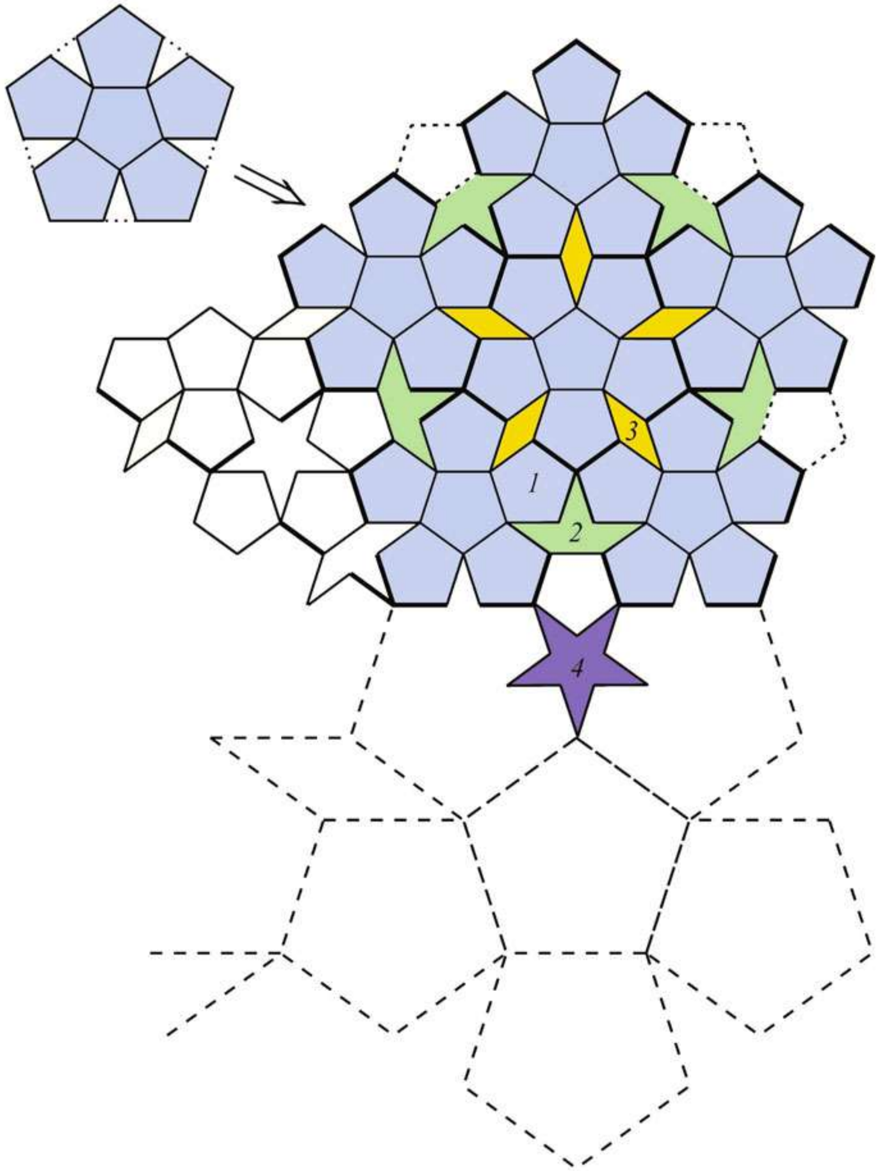
Zuse's curious proposal yields a bridge to a completely different branch of modern physics: condensed matter physics. According to [Schreier 1988], it has been "an established sub-discipline of physics since the 1940s" and "is nowadays accepted as the main area of physics, which at the same time is closely connected with practical applications. Depending on how large we take the area to be, the amount of works of the total number of physical publications is between 25 and 30 percent in the last thirty to forty years" (l.c. p. 363). Since the subject of condensed matter physics mainly takes place in an order of magnitude in which matter is already presupposed to be discrete, but the intra-atomic structure (which, according to the present knowledge, cannot be illustrated at all) does not play an essential role, it has also become an ideal field of application for discrete geometry, as classical geometry was for classical physics. Processes at boundary layers between two different or differently orientated regular orders of points, concluding the producing crystal structure based on adjacent diffraction images and, recently, the theory of quasi-crystal, which is closely linked to non-periodic tessellations, are typical fields of contact between physics and geometry nowadays.

The pure geometrical play instinct has yet again created a forerunner for an unpredictable natural scientific development in a remarkable manner. R. M. Robinson (1971) and R. Penrose (1973) found the first systems of plane tiles ("prototiles"), with which we can plaster the plane only non-periodically (Illus. 8.3.4). Although it was only four prototiles at the beginning, Penrose and Ammann could already confirm systems of only two prototiles in 1974. Essentially, they are based on the possibilities that had already been studied by Dürer and Kepler. Concerning the terminology, to be non-periodic is a property of a certain tessellation. It means that there are no shortest translations that map tessellations onto themselves, in two (or, in space, three) linearly independent directions. Of course, there are many systems of tiles that can be used to make both periodic and non-periodic tessellations. This is already possible when using only pairwise congruent squares or cubes. A system of prototiles is called non-periodic if we can use it to plaster the plane or space, but only in a non-periodic manner.

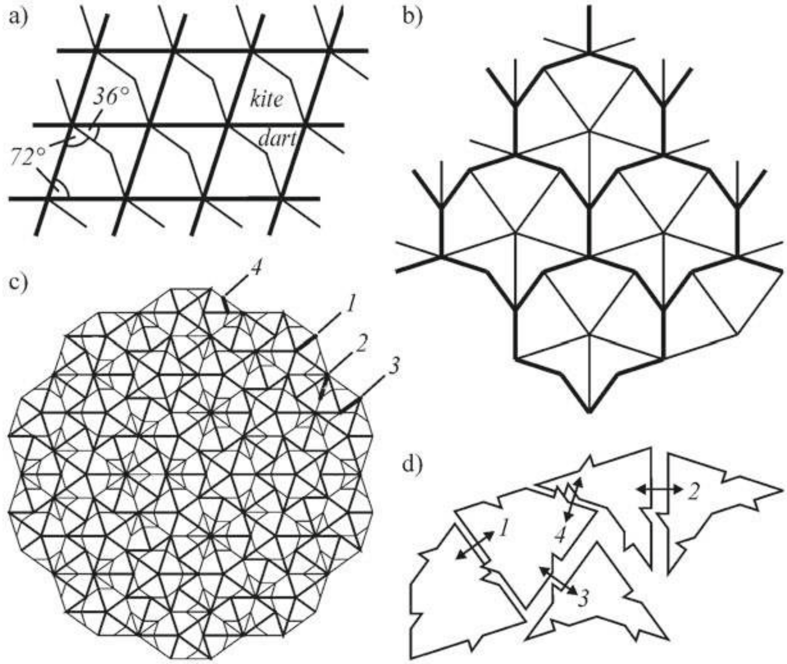
Every non-periodic tessellation system prompts us to ask for proof that we can indeed continue the tessellation in all directions. The most important method used up to now is based on the ability to assemble a new system of tiles similar to those originally given, but enlarged by a certain scale to supertiles, which can be repeated any number of times (so-called inflation). However, if we, like Berger in 1966, simulate the work of a Turing machine by a row-wise tessellation, it suffices to prove that the respective machine program will never stop when applying it to the initial row. As a result, a hardly predictable, interesting link between discrete geometry, on the one hand, and algorithm and complexity theory, on the other hand, has developed. As Illus. 8.3.5c shows, and looking at the bigger picture, such non-periodic tessellations can show apparent symmetries, for example, the return of local penta- or decagon shapes, which are impossible in terms of classical crystallography due to purely geometrical reasons. In 1984, it became known that a certain aluminium-manganese alloy macroscopically shows a diffraction image with icosahedron symmetry under special physical conditions [Senechal 1995, 1.1]. Numerous such phenomena have been discovered since then, so we must at least partially attempt to explain the strange fact that this had not been discovered much earlier with the established dogmas of classical crystallography and the well-known saying "it cannot be what may not be". The material-scientific research and its technical applications have experienced a great ascension ever since then, just as the purely geometrical investigations inspired by this had.

In 1985/86, Danzer, Schmitt, Levine and Steinhardt also found real three-dimensional non-periodic tessellations. In fact, Penrose's tiles, by now almost classic, also permit periodic tessellations. This has to be prevented by either altering the edges so that certain matches become impossible in terms of shape (keeping in mind that this implies hindering the principle of inflation), or using the matching rules, as generally known from dominoes, according to which only equally marked or equally coloured edges can touch each other. The latest physical findings hint at the idea that such matching rules correspond well to the physical conditions for forming quasi-crystals. Moreover, fractal prototiles have also been considered recently, since self-similarity is given as the natural consequence of combining form adjustment (instead of rule) and the principle of inflation.

First considerations on the correlation between geometrical shapes, their creation, function and purpose in nature were put forward by Leonardo da Vinci and J. Kepler. In approximately the mid-$19^{th}$ century, the physiologist Karl Vierordt from Tübingen advocated the extensive application of mathematical concepts in medicine and biology. The geometrical aspects of biology were examined and demonstrated systematically and extensively at the beginning of the $20^{th}$ century [Cook 1914], [Thompson 1917]. The latter book especially, written by a Scottish zoologist, has been revived with the increasing interest in the application of mathematical methods to biology since approx. 1950. This has been reflected by new editions and translations. Whereas mainly




**Illus. 8.3.4** The first non-periodic tessellation by Penrose




**Illus. 8.3.5** The tessellation system "kites and darts" by Penrose
This tessellation can be used for both periodic (a, b) and non-periodic (c). By altering the edges in the manner indicated by (d) (or, equivalently, "prohibiting" certain contacts), we can prevent periodic versions from being included

classical mathematical means were initially used for descriptions, for example, differential equations that roughly explained the formation of certain spiral shapes or the manner with which biological supply chains branch out, discrete geometry played a constantly increasing role in the $20^{th}$ century. One of the latest books pointing in that direction is [Meinhardt 1998]: algorithmically describable growing processes often lead to approximately fractal formations and, thus, to a better understanding of the manifolds of shapes within flora and fauna. For instance, the formalism of cellular fields is used to simulate the spread and mutual influence of populations.

A completely different application of geometry in biology is based on the fact discovered by the biologist Tammes in 1930 that pollen have a certain number of contact membranes on their surface in a regular structure, which, however, differs from type to type. Thereby, polyhedral theory led to a classification of pollen and to new insights into how they function as well as the relations between different genera [Fejes Tóth 1964, p. 214ff.]. Studying viruses has recently revealed similar connections to geometry of regular polyhedra. Geometry had been involved in the creation of a theoretical forerunner in this and related fields of application since the 1930s with investigations of (densest) packings and (thinnest) overlappings and related extreme questions "in the plane, on the sphere and in space" (as per the subheading of the book

**Illus. 8.3.6** Fractals approximately describe growing processes
l.: Computer designed fractal shape of a tree [Solkoll], r.: Fractal shape of Romanesco broccoli [Photo: John Sullivan 2004]

of well-established relevance by [Fejes Tóth 1953]). Such questions have become important for understanding physical, chemical and micro-biological phenomena, whereby we, for example, look at spheres from a finite number of different diameters, which have to stand in given neighbourhood relations. For instance, if we ask about the smallest convex set that can enclose $n$ congruent spheres, we find tube-like structures from $n$ spheres for small numbers that suddenly collapse into a more compact optimal structure at a certain $n$. The analogous question for higher dimensions leads to one limit number $n(d)$ characterised by every dimension $d$ (Problem 8.3.2).

Even if some classically educated geometers first looked upon such investigations very sceptically or mockingly, it is clear that, within the field of discrete geometry, not only has an excellent harmony been established between interesting and difficult mathematical problems on one hand and the versatile applications on the other, but also, an unpredictably broad spectrum of methods and outcomes from distant branches of mathematics has become necessary in order to answer seemingly simple questions; for instance, the question of the distribution of $n$ points on a spherical surface for which the minimum distance of two points turns out to be as large as possible. For some values of $n$ $(4, 6, 12)$, the question is easily answered by the existence of regular polyhedra with triangular areas and a respective number of corners, but for $n = 8$, the alignment corresponding to the corners of the cube is already by no means optimal. As the reader can easily recalculate, at the very least, the corners of the anti-prism with a square base yield a better solution on its circumscribed sphere. The difficulty in proving that this solution cannot be improved any further has been a significant problem for years [Fejes Tóth 1953, p. 162ff.]. It is remarkable and symptomatic of the attractiveness of the topic that this question and its analog for some further $n$ were solved by two mathematicians with different specialisations in 1931: van der Waerden, one of the most famous algebraists of the $20^{th}$ century, and K. Schütte, a well-known logician.

If quantum and condensed matter physics, as well as the more recent developments of chemistry, biology and biochemistry, leave the impression that discrete geometry (and in general, discrete mathematics) have stolen the lead position from classical geometry of what is continuous and smooth in present natural science, the inclusion of other areas shows a more balanced picture. For instance, nowadays geodesy includes surveying and monitoring the gravity field of Earth, having begun with the surveying and monitoring of Earth's magnetic field in Gauss's time. Thereby, classical geometrical methods are applied. In soliton theory[35], certain discrete problems of condensed matter theory can be described with the same type of differential equation as solitary water waves (already recognised as a phenomenon since 1834). In this theory, with its strongly growing significance, a bridge between continuous and discrete phenomena has started to become apparent, which then also contributes to explaining the wave-particle duality. Generally speaking, we will only regain practical control concerning discrete phenomena, for which the number of participating objects exceeds a certain quantity, as in statistics, by (like the reverse case of numeric analysis) approximating the discrete by the continuous.

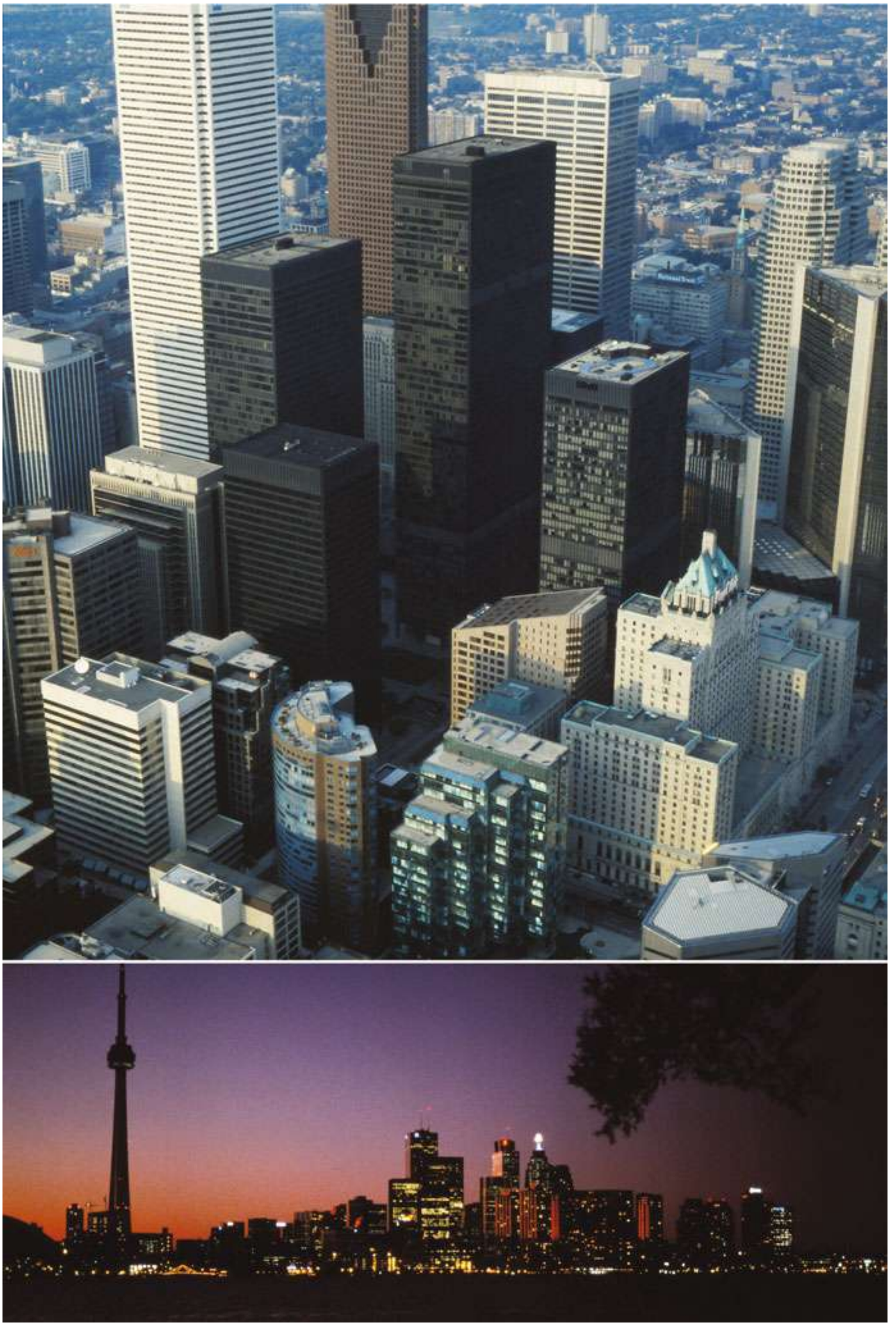
Finally, we want to pinpoint that, even in geography, an area that stands with one foot in the natural sciences and the other in economics, and in which there was special (partially objectively, but partially also subjectively founded) resistance against mathematisation, the source of novel problems and inspirations of a geometrical nature does not run dry: if it was geodesic and cartographic questions that were handed over from geography to geometry a long time ago, followed by the mathematical means of geophysical problems and even later the statistical ones, nowadays we also have to deal with optimising locations and transport networks, defining adequate measures for the degree of fissure or the accessibility of an area, the penetrability of toxic substances or biological populations, exactly determining catchment basins, the density of information of maps, and graphically representing complicated relations (overview and literature in [Schreiber 1989]).

[35] A soliton equation describes the emission of a certain type of "standing wave", which behaves like "particles" during, for example, mutual encounters (for further details, see, for example, [Meinel/Neugebauer/Steudel 1991]).

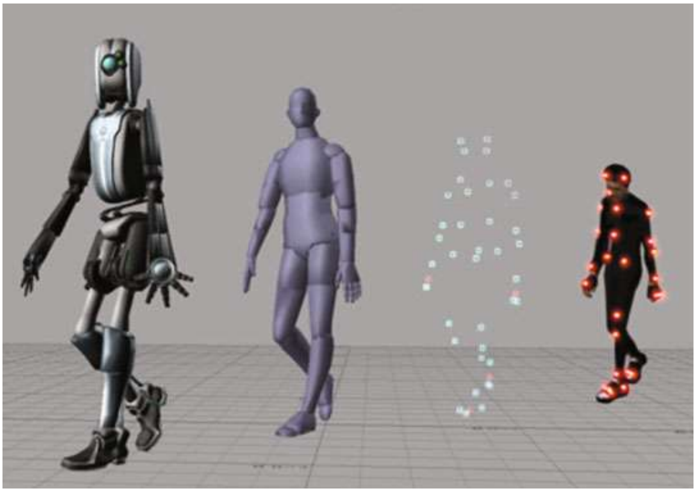
## 8.4 Geometry and techniques

F. Klein stated in the introduction to a lecture called ‘Application of Differential and Integral Calculus’, which was published in 1902 and again in 1907: “There is a profound gap in the modern mathematical literature that you must all have come across: the theorists’ interests and their trains of thoughts differ extraordinarily from those methods to which we actually help ourselves when executing applications. Not only does the individual’s scientific education suffer from this, but also the prestige of science itself” (l.c. p. 1). In this book, he continuously speaks of “precision-” and “approximation”-mathematics. His viewpoint of the relation between these two sides of mathematics, which was rather revolutionary for a geometer back then, shall be explained for the case of geometry by means of the following example: the theorems of Euclidean geometry are logical conclusions from the axioms, which are idealisations of practical relations. For instance, the axiom “through any two points, there is exactly one line” is a ‘shortened version’ of the empirical subject matter, “if the diameter of two plane areas $a, b$ turn sufficiently small, the set of the possible straight connections between a point $A \in a$ and a point $B \in b$ tends against a uniquely determined limiting position”. The task of precision-mathematics is to derive logical conclusions from the shortened versions. The task of approximation mathematics is to re-interpret these conclusions as propositions about reality and to supply those with error estimates (which mostly need the help of analysis). Thus, the theorem “the heights of a triangle intersect at one point” really means: “if $ABC$ is a triangle and the straight lines passing $A, B, C$ closely enough are sufficiently perpendicular to the relevant opposite sides, then their pairwise points of intersection slide together arbitrarily closely.” This theorem (as analogous to every other one) of theoretical mathematics contains the demand for solving the problem of how the diameter of the area in which the points of intersection of the heights could lie depends on the presupposed maximal deviation of the points $A, B, C$ and the deviation of the heights from the perpendicular position.

Consider Klein’s position in front of the historical background that he was first to advocate drastically for the re-approximation of pure and applied mathematics, that he was responsible for establishing the first chair of Germany for applied mathematics in Göttingen in 1904, and that this applied mathematics, which actually only developed as an independent field in the $20^{th}$ century, also included graphical methods for solving numeric problems until the 1940s. Precision techniques, which are at an unattainable level for amateurs nowadays, had also developed parallel to this since the end of the $19^{th}$ century. The first worldwide chair of this kind at a technical university was founded in Dresden in 1929, and appointed to G. Berndt, a pioneer in this field. Nowadays, gear techniques (gears, worm wheels, antifriction bearings, forced operations, etc.) use a sophisticated combination of differential-geometrical and numeric methods, in order to fulfil these higher demands [Giering/Hoschek 1994].



**Illus. 8.4.1** Skyscrapers in Toronto
The centres of many towns became more and more dominated by skyscrapers. The CN-Tower (Toronto) was classified as one of the Seven Wonders of the Modern World (American Society of Civil Engineers 1995)
[Photo: H. Wesemüller-Kock 1991]



**Illus. 8.4.2** Motion of human body (motion capture)

Our everyday environment consists of objects most often created by mankind. All these reveal geometrical aspects in the most versatile manner. Hence, this section can only be concerned with pinpointing some developmental trends that distinguish the $20^{th}$ century in this respect from the already industrially-dominated $19^{th}$ century. Furthermore, we will show by means of some examples how results of 'pure' geometry have practical effects in a sometimes surprising manner and how geometry unlocks fields of applications that we rarely think of within this context.

A still blossoming area, which is close to both classical geometry (its origins go back all the way to the Renaissance) and mechanics, is geometrical kinematics. W. Blaschke, one of the most versatile geometers of the $20^{th}$ century, referred to it as 'the geometers' paradise' in [Blaschke/Müller 1956]. The core questions are: which orbits do certain points of a line segment, area or solid describe, if other points of these objects move along given orbits or, respectively, on prescribed areas; how can we conversely enforce certain motions; which space do certain motions need and how can we optimise it? Think of the geometrical question in connection with the so-called rotary piston engine (also called a Wankel engine). Theorems of Holditch's type state something about the fact that a certain volume (area) independent of the special form of the motion is covered during certain motions of a line segment (area). The concept refers to the first of such theorems found by H. Holditch in 1858: a chord of suitable length moves inside a convex curve so that each of its two extremities passes through the boundary curve exactly once. In that case, a fixed point of this chord, the distances of which from both extremities are $x, y$, describes a curve inside the given boundary curve so that the ring area between both curves has the capacity of $\pi xy$ (Problem 8.4.1).

Biomedical engineering is one of the most unusual applications of kinematics. Amongst other things, it deals with exact descriptions of the forms and motions of the human body in order to construct prostheses, which reproduce this mobility as perfectly as possible. In 1904, O. Fischer wrote extensive reports on "physiological mechanics" as part of the communications of the DMV (Deutsche Mathematiker-Vereinigung, German Mathematical Society) and in Volume IV of *Enzyklopädie der mathematischen Wissenschaften* (Encyclopaedia of Mathematical Sciences), which were already based on an astonishingly rich literature back then. A small book on this topic was also published in the Czech language in 1952 and was probably inspired by the fact that F. Kadeřávek, one of the two authors and professor of descriptive geometry at the Czech Technical University in Prague, was the son of one of the manufacturers, who engaged with fabricating prostheses at one of the surgical university clinics in Prague. [Giering/Hoschek 1994, p. 191-194] offer an up-to-date depiction.

A large field of application for differential geometry concerns curvature of surfaces: the surfaces of gear parts moving against each other should be capable of being developed onto each other as exactly as possible. In order to manufacture ship, plane and car bodies, we need to find optimal dissections of the entire surface into parts that have a total curvature as small as possible and, thus, can be made from plane material with only slight deformations. A related question concerns the wrapping of curved pipelines and similar objects with plane ribbons so that these cannot shift. Optics constitutes a further field of application of differential geometry. Thereby, we are dealing with, for example, calculating panorama and "frog eye" objectives, which yield non-linearly distorted panorama images with certain properties, as well as with the distortion of these images [Drs 1981]. Such optics are used,for example, in medical endoscopy, but also for the inner control of technical systems (cauldrons, pipelines, etc.).

Following up on Cauchy's theorem [Cauchy 1813, cf. section 7.9]) that a polyhedron is uniquely determined apart from its location in space by its lateral surfaces and the rule of linking them under the naturally necessary additional condition of convexity, a problem had already started to develop shyly over the course of the $19^{th}$ century that addresses the possible difference of spatial positioning of a given polyhedral surface or its edge structure. Concerning the latter case, the applicability in the statics of framework constructions is, at least, obvious. Hence, it feels even stranger for us nowadays that the motives were originally of a rather theoretical nature: Euclid's axiom that what can coincide is equal, in combination with the notion of coincidence, which he did not define, made philosopher A. Schopenhauer wonder in 1844 about the fact that mathematicians accepted this without any questioning, but spent all their time contemplating the much clearer parallel axiom [Cromwell 1997, p. 221]. Nowadays, we say that a polyhedral surface is 'wobbly' or 'infinitesimally flexible', if it has two adjacent realisations so that we can transform one into the other by means of slight deformation. (Concerning pure bar framework, only the edges must not cross during this transformation; concerning wobbly polyhedra, the surfaces must not intersect in the meantime.) We experience it every day that bar frameworks respectively polyhedra with sufficiently many edges respectively surfaces are generally not very stable. Hence, almost from the beginning on, we were concerned with finding counter-examples with a preferably small number of surfaces or edges and recalculating the deformations that are necessary for transforming one position into the other one. Some simple wobbly bar frameworks were depicted by the French engineer R. Bricard in 1897. It seems that it was first mentioned in [Brückner 1900] that closed chains of at least six (not necessarily regular) tetrahedra, for which each two opposite edges serve as connection 'hinges', are rotatable within themselves (cf. also section 8.6 and the literature referring to caleidocyles given there). Simple wobbly polyhedra had already been demonstrated by G. T. Bennett in 1912, and furthermore by W. Wunderlich in 1965 (Problem 8.4.2), M. Goldberg in 1978, and others. However, there were doubts until recently that there are simply connected and overall real three-dimensional polyhedra that are *continuously* flexible, meaning that they can pass through an infinite family of positions without deformation and self-overlapping of surfaces in the meantime. In 1977, R. Conelly found the first surprising counter-example when he was still a student. His idea was to insert little indents in suitable infinitesimal flexible polyhedra exactly at those places that feature a small amount of self-overlapping, in order to make a smooth passage possible. In the meantime, it had already become known that almost all simply connected polyhedra are stable in the "space of polyhedra" in terms of a suitable measure (Gluck 1975). See [Cromwell 1997] for further details and literature.

All that which makes a tempting but rather playful impression here has a surprising practical consequence. Since radar had been invented (technical application in the military field only since approx. 1935, expanded for civil purposes after 1945) and especially since the rise of the use of lasers (approx. from 1970), it has been possible to measure distances much more exactly than angles. For instance, when determining the distance from the Earth to the moon, nowadays the deviation will only be about 4in. Thus, classical triangulation was rapidly superseded by trilateration in geodesy. (Every reader can imagine what this change of terms means.) The wobbly respectively continuously deformable polyhedra now show that, considering the analogy to the 'dangerous circles' of triangulation (cf. section 5.2), there are constellations in trilateration for which either their mutual position cannot be uniquely determined at all except by means of the pairwise distances of the points or very small errors in length measurement can lead to big errors in localisation. B. Wegner demonstrated the invariance of wobbliness properties for projective maps in 1984. (See [Giering/Hoschek 1994, p. 177-183] for details and further literature.)

The definition and investigation of a meaningful notion of measure for sets of geometrical objects, based on which we can establish propositions of probability or statistics, such as the ones above concerning the rigidness of polyhedra, is addressed by integral geometry or the geometrical measure theory already mentioned in section 7.9. However, whereas this theory in its classic era (since approx. 1930: Bieberbach, Santaló, Maak) engaged with arbitrary mutual positions of well-defined geometrical objects and with propositions of the form “almost all or, respectively, almost no objects of a certain type have a certain property”, stochastic geometry has emerged from under the pressure of practical needs since approx. 1965. It seems that this term occurred first in the title of the omnibus volume [Harding/Kendall 1974] published in 1974. It concerns statistic propositions on irregular orderings of a generally large number of objects. A specialised field of stochastic geometry, also around since approx. the mid-1960s, is stereology, which draws statistical conclusions on spatial distributions based on random plane sections. Stochastic geometry features a wealth of applications in biology, medicine, mineralogy and material testing, whereby we have to derive propositions on the global distribution of certain features and shapes from random samples or tissue sections. We recommend [Stoyan/Mecke 1983] as an introductory read.

Geometrical questions of a completely different nature result from optimally exploiting tissue (or metal) sheets to cut textiles or car body parts; suitably packing non-cuboid-shaped objects into cuboid-shaped boxes; optimally selecting locations for measuring or transmitting stations with given coverage, which together supply a certain territory, the coverage of which should overlap as little as possible and exceed the total area not at all or as little as possible; optimally selecting locations and designing traffic or distribution networks. There are already thousands of books concerning just the so-called Steiner-Weber problem and its generalisations introduced in section 7.9 [Cieslik 1998].

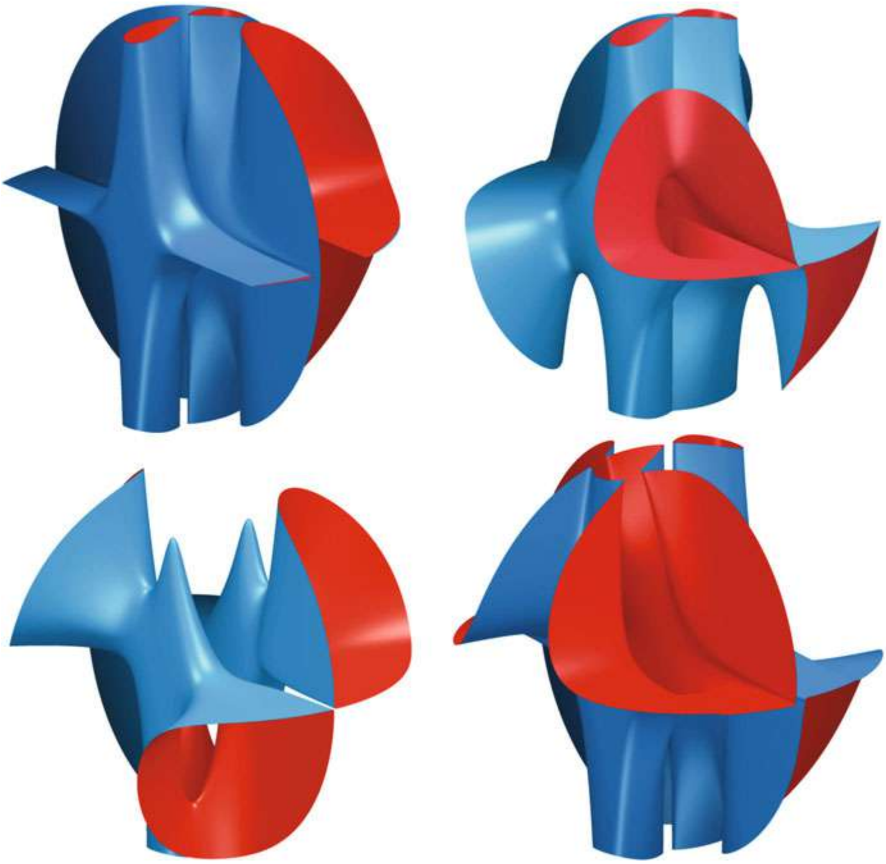
However, whereas all fields of application mentioned so far are only accessible in detail for the respective specialists, the ordinary environment offers opportunities over and over again for contemplating geometry and its effects: rectangular plasters or tiles can be relatively easily separated from their bond or tilt if the underground sinks. This is due to the fact that they have an axis of revolution to which two lateral surfaces are orthogonal, but the perpendiculars only have to incline a little, respectively. In contrast, the zigzag-shaped stones, which we can spot everywhere these days, can only be separated from their bond completely perpendicularly to the tread.

## 8.5 Geometry and computer sciences

Practical geometry without a computer is hard to imagine nowadays. Even visual artists have long discovered (since approx. 1963) the computer as a possible tool. Thus, it is even more astonishing that none of the original computer pioneers apparently expected this development. Despite (or maybe because of) a constantly rising flood of literature, the entire field of computer geometry is distinguished by a lack of boundaries and structure, which is why some notional explanations shall be provided here at the beginning of this section.

Although computer geometry tends to unite more theoretically and practically orientated fields, it is an area of application in respect to its relation to geometry. Hence, it centres on solving problems by means of algorithmic processes. An initial rough classification results from structuring these problems as follows:

A. Those for which we are meant to produce a non-geometrical output based on a geometrical input, e.g., determining measured values of geometrical objects, extracting features, sorting of objects according to certain criteria (also recognising handwritings), ...
B. Those for which we are meant to produce a geometrical output based on a non-geometrical input, e.g., drawing a function graph based on a table of values or a description of the function in formulae, including showing the object in its temporal change given by a table or formula, producing an object based on a verbal respectively formal description (this includes all interactive CAD systems), ...As tools serve all further parts of classical descriptive and coordinate geometry, including the revived illumination geometry, which equips computer images with impressive realism. “Splining” has also drastically advanced due to the needs of CAD, i.e., assembling curves and surfaces by means of parts that have common tangents or tangential planes at the joints. The underlying mathematics is so classical that we could say that this area originated in the $18^{th}$ century.
C. Those for which we are meant to produce a geometrical output based on a geometrical input, e.g., transforming an object given by assigned views into a perspective view, rotating an object into another position for the viewer, executing certain procedures to relieve ourselves from having to recognize or classify the original object “digital image enhancement”),... The basic operations of digital image enhancement often go back to theoretical concepts from around 1900 (Minkowski, Voronoi, Thue, ...). For instance, we pass over from any plane set of points $A$ to unite all circles with a fixed radius $r$ around any point of $A$ that ‘inflates’ the set and makes isolated points visible in the first place; or we reduce the set to this ‘essential part’ by keeping only those points that have an entire surrounding of a suitable chosen radius in the original set. Combining and multiply repeating such procedures yield surprising outcomes.



**Illus. 8.5.1** Algebraic surfaces constructed with computer
The relation between formulae and forms can be experienced by SURFER software. In an interactive way one can visualize real algebraic geometry. Based on formula $y^4 + x^2z^2 = y^2$ (called 'Helix'), surfaces like these examples (generated by H. Wesemüller-Kock) can be created

The objects produced or edited in computer geometry are always represented materially (the object containing the image representation being the main case), hence Euclidean and two- or three-dimensionally. However, the latter does not refer to so-called "three-dimensional" computer graphics, which, nonetheless, only yield two-dimensional images, but to the fact that the program-controlled working machines need not necessarily be a computer in narrow terms, communicating with its surroundings by means of scanners, plotters, printers or screens, but that we should also include, for example, program-controlled machine tools and industrial robots. There have been three-dimensional printers since approx. 2000 that, depending on the program, can produce any complicated spatial object by means of powder and binding agents.

Three further classifications, which can be superimposed on the one above, concern the questions as to whether the participating objects are (on the input or output side) as follows:

I. two or three-dimensional,
II. “black and white” or multi-colour or respectively feature several materials/textures,
III. sharply defined or diffused, fussy, ... All in all, this yields 80 different process types. We are already now at the point where little imagination and expertise are needed to contemplate at least one meaningful already existing or desirable example for each of these.

The classical disciplines, such as descriptive, analytic, algebraic and differential geometry, belong to the instruments of applied computer geometry (see, for example [Faux/Pratt 1979]). However, this is not just the continuation of classical applied geometry with a new, much higher performing instrument. This is due to the following: first, this new instrument makes many applications possible that nobody would have dared think of fifty years ago. Second, it includes objects (such as X-rays or satellite photos) that cannot be described by means of the notional system of classical geometry. Third, it gives old questions concerning geometrical problems a new quality:

- The different nature of the objects of many problems demands the development of matching notional systems, problem-orientated programming languages, as well as new geometrical theories, for example, different closure operations, integral geometry, stochastic geometry, collaborations of optical and non-optical information to interpret an image correctly.
- Classical coordinate geometry turns into a small special case of the question as to how we can describe geometrical objects by means of finite data structures and, on top of that, in such a manner that these “codings” ‘support’ the intended algorithms.
- Classical theory of irresolvability proofs for certain problems (such as with compass and straightedge) and the modest starting points of geometrography for comparing the effort of different solutions of the same problem (cf. section 7.3) have advanced to complexity theory for geometrical algorithms. The now internationally common term “computational geometry” for studying the complexity of geometrical algorithms was introduced by M. I. Shamos in 1975. Good introductions are [Preparata/Shamos 1985, 1985], [Edelsbrunner 1987] and [de Berg 1997]. The first of these books is distinguished by a historical view on the prehistory, which is not so self-evident for modern mathematical literature. It covers all fundamental aspects from Euclid via Mohr and Gauss until Lemoine, described in the preceding chapters of our book regarding geometrical constructions. Computational geometry focuses on problem classes, which hardly

played a role in classical constructive geometry, but now attract more interest due to the combination of (sometimes also questionable) practical importance and high complexity. Amongst those are:

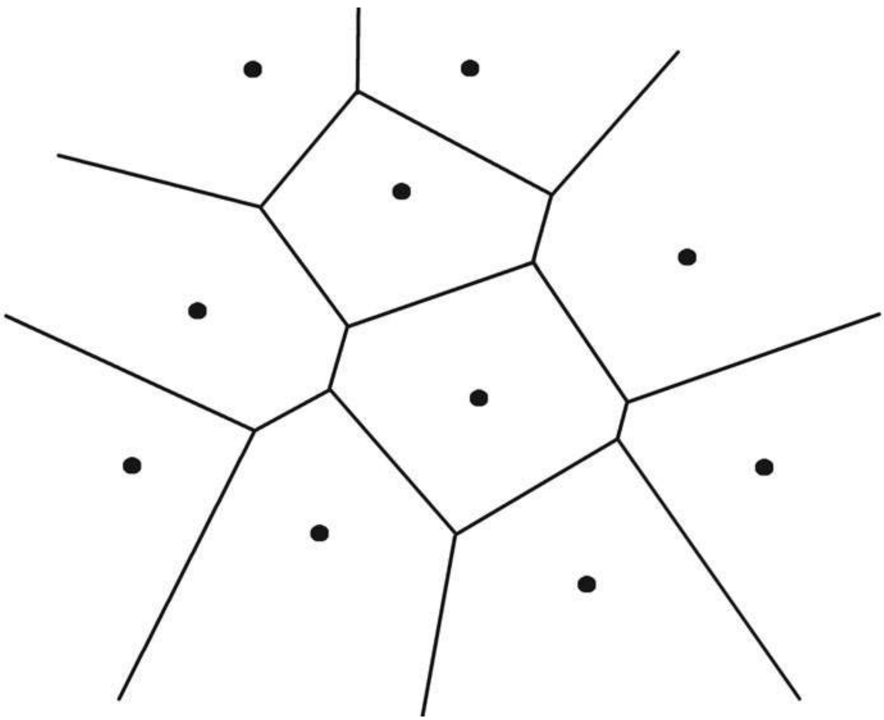
- Determining convex closures of a finite number of given points,
- Determining the common part of several polyhedra,
- Determining the visibility of line segments and areas for a given observer's viewpoint,
- Determining the nearest neighbour, the two points in a finite set of points $M$, which are either closest to or farthest from each other, generally the so-called Voronoi diagram of a finite set of points $M$, which divides the plane into Voronoi (or Dirichlet) cells, assigning every point of the finite set the set of those points that have the shortest distance to it from all other points of $M$ (Illus. 8.5.2).
- The group of guardian problems for which we must find number and location of a minimal set of points that can see every part of a given set (equal in meaning to those which can illuminate the entire set).
- Determining shortest connection systems (Steiner-Weber problem, cf. section 7.9)

On one hand, the computer caused a massive turn towards geometry of the Euclidean plane and the ordinary Euclidean space, whereas, on the other hand, it led to novel, different complications from those discussed in section 8.2 concerning the question of the boundary between geometry and non-geometry. For instance, the problems of image interpretation are closely connected to information theory and artificial intelligence, and many questions of designing programming languages and efficient algorithms that play a central role in computer geometry are really questions that go far beyond geometrical application. The geometrical construction algorithm as an informative model case of algorithms in any not necessarily discrete structures had been addressed by E. Engeler by 1967 and including further aspects also by [Schreiber 1975, Schreiber 1984] (see there for literature). Amongst these typical sets of problems are:

- many-sortedness of geometrical structures (relation to "algebraic specification" in any programming language),
- the equivocation of typical geometrical operations, as well as the necessity for using random 'auxiliary points', which led to questions of non-determinism in a natural manner,
- questions of mutual simulatability of different operational systems,
- propositions of irresolvability respectively complexity based on geometrical ratios and transfer principles. We will cite two theorems as examples for the latter: although every problem solvable with compass and straightedge can be solved in terms of the earlier described meaning according to the Mohr-Mascheroni theorem solely by means of the compass,

solutions with a constantly limited number of steps are generally transformed into solutions with cyclic subprograms: [Schreiber 1975], generalisation [Schreiber 1984]. Furthermore, not even in hyperbolic geometry can any angle be trisected with compass and straightedge (use of Poincaré's model) [Schreiber 1984].

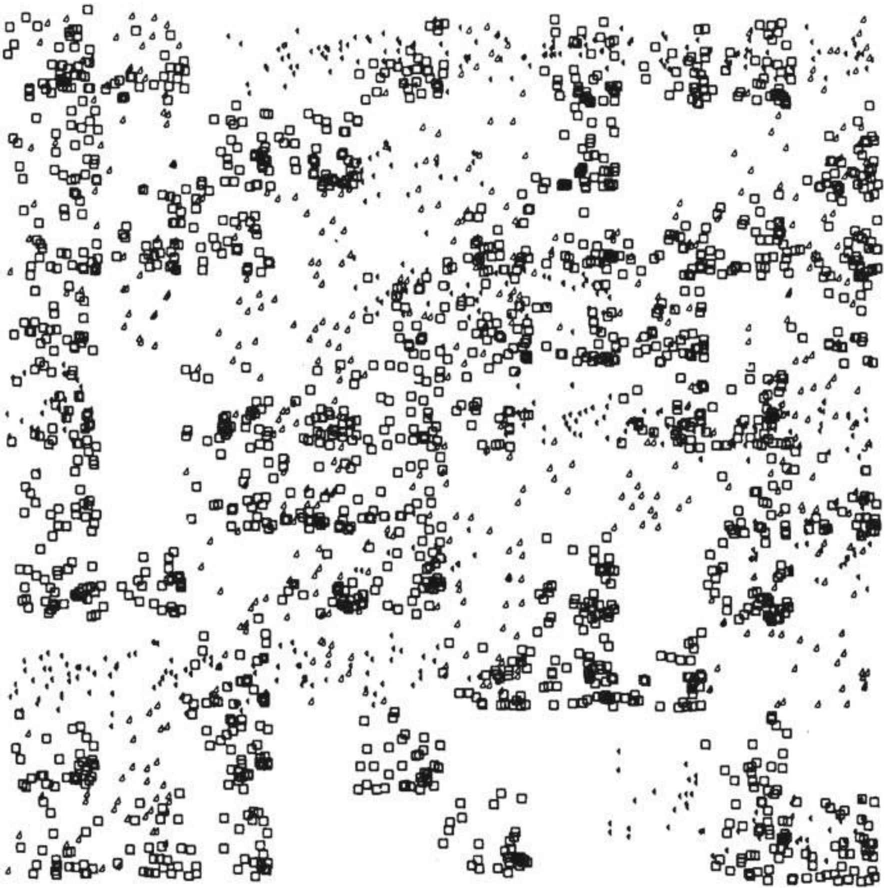
After these preliminary remarks, we will attempt to sketch the historical development. Of course, computer geometry is only possible with a given periphery. The first input devices have been used to evaluate satellite photos automatically in the USA since the 1960s. These "grey value analysers" by IBM transformed photos in which the eye could hardly recognize anything into bit strings, which were then numerically processed. The first plotters (used for computer-aided design) were launched a little later, for example, in Germany in 1964 by the Zuse company. The mainframe computer ILIAC, especially equipped for graphic data processing, began being used in the USA in 1963. In the same year, the first dialogue-orientated graphic program system called SCETCH-PAD was developed. However, there had already been initial efforts to develop graphics-orientated program systems for didactical purposes in the USA in the 1960s. Hence, MIT created LOGO as a dialect of the language LISP and the program-controlled drawing turtle was born.



**Illus. 8.5.2** Voronoi diagram for ten randomly chosen points

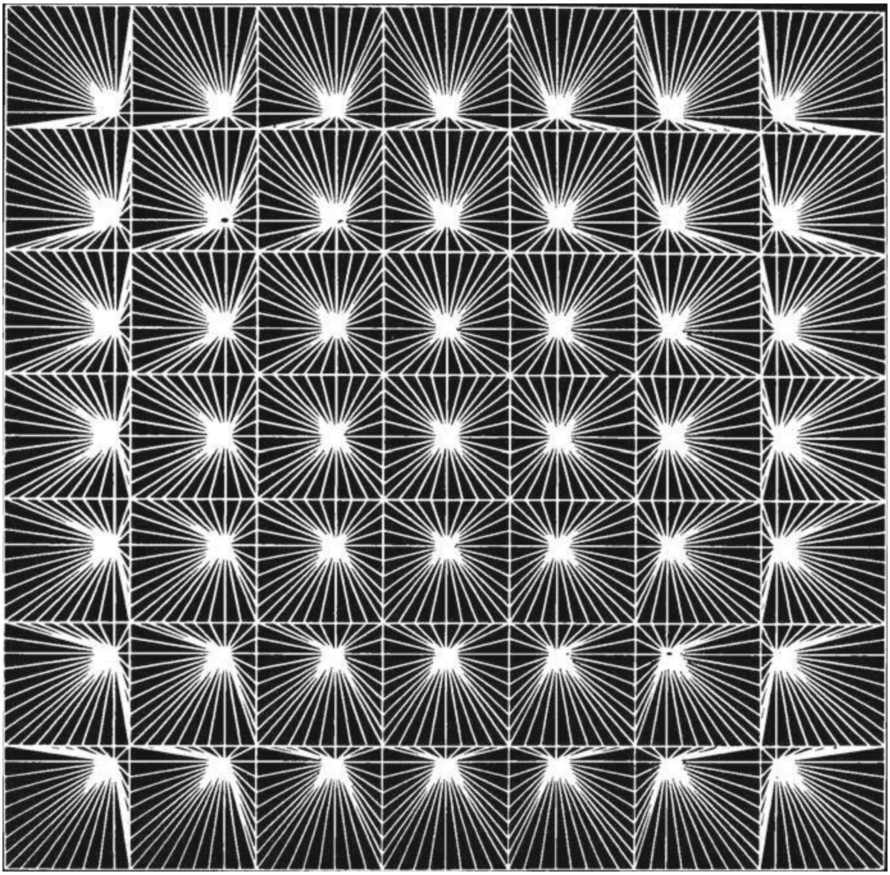
In 1967, the American ACM (Association for Computing Machinery) founded a SIG (Special Interest Group) for computer graphics and the first specialized journal *Computer Graphics*. The term CAD (computer-aided design) was formed soon after, in honour of which a first international specialized conference (Southampton, GB) had already been held under this name in 1969, accompanied by the founding of a journal under the same name. The terms of picture or image processing have been used since 1969/70.

The first experiments using them for artistic purposes also commenced in the mid-1960s (for example, G. Nees in Stuttgart, F. Nake in Karlsruhe, Illus. 8.5.3). The Computer Art Society was founded in London in 1969. In 1970, five stamps that reflected computer graphics (after drafts by the Technical University of Eindhoven) were released in the Netherlands. In many

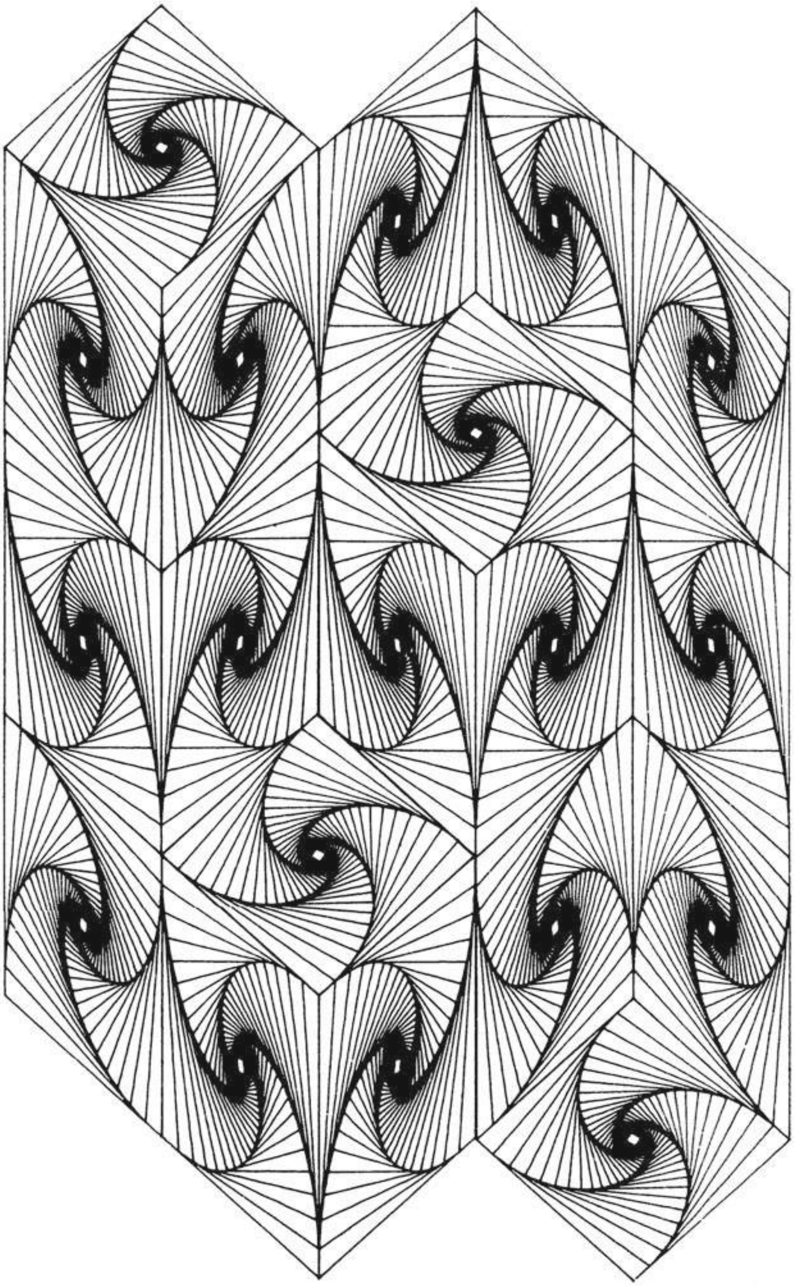


**Illus. 8.5.3** Fields with character distribution twice superimposed
Computer graphics by [Frieder Nake, no. 5, 13/09/1965]

cases, the computer was just used as a 'hard-working sketch artist' for the generation of complicated mathematically defined patterns (Illus. 8.5.5). We could first lay eyes on the "Mandelbrot set" in 1967. Simpler fractal formations can also be easily produced nowadays, leading to versatile applications [Herfort/Klotz 1997]. Within the realm of artistic design, we often either run through and order a combinatorial system of possibilities accordingly (Illus. 8.5.4) or a statistic distribution function is given and a globally legitimate pattern, though random in its details, is produced by including a random generator. An extensive factual and historical depiction of the topic of computer art is given by [Steller 1992].



**Illus. 8.5.4** Computer graphics according to the combinatorial principle: Systematically running through and ordering cases [Horst Bartnig: computer graphics 2, Weiß auf schwarzem Grund (White on black ground), 1979/1980, screen printing, 35x35, spectrum issue 9, Akademie-Verlag der DDR, 1981]



**Illus. 8.5.5** The computer as a hard-working artist © Helmut Schwigon, Computergraphic from: [Gert Prokop: Das todsichere Ding (The Dead Sure Thing), Verlag Das Neue Berlin. Berlin, 1986]


**Illus. 8.5.6** Computer graphics by Technical University of Eindhoven [Dutch stamps from 1970]

A fundamental turn that affected computer sciences as a whole was the 'marriage' between computer and screen in the 1960s and the launch of the personal computer (PC) around 10 years later (1979 Apple, 1982 IBM-PC). Computer geometry has become a mass phenomenon since then, whereby, speaking purely quantitatively, the overwhelming majority of all graphic computer applications are targeted at the entertainment industry. Amongst the huge amount of such products, there are some that are able to awaken geometrical interest. One of those is the now classic "Life", which was invented by the mathematician J. H. Conway in 1970 and is nothing other than a version of the idea of the cellular automaton passed on to us by J. v. Neumann. However, it is distinguished by an excellent combination of simplicity and wealth of diversity. All the versions of Blockout belong here, too. Computer geometry in narrower terms has become possible for everybody due to integrated mathematical program packages, such as MAPLE (1980) and MATHEMATICA (1988). Simple drawing and painting programs are part of the basic software of every home PC. There have been calculators with display, capable of achieving simple graphics, since approx. the 1990s.

## 8.6 Geometry and art

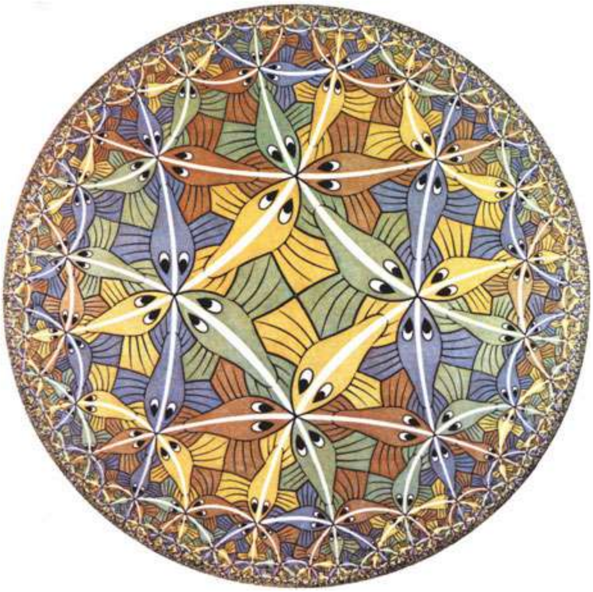
We have seen that past needs and inspiration from architecture, design and the visual arts have played an important role for the development of geometry. We have mentioned along the way that the building trade was covered by the mathematics professor's teaching and that practical work as an architect or building expert was a typical source of their additional income. We want to add in this respect that seminars on descriptive and elementary geometry were sometimes also held by 'academic drawing teachers' at the same time. These individuals were typical, at least for German universities, along with dancing and fencing teachers. On first look, it seems as if these close relations were permanently halted due to the exclusion of descriptive geometry from 'proper mathematics' and the artists' decreasing interest in perspective.

However, there were also a great number of artists interested in mathematics in the $20^{th}$ century, amongst them, some that were very well known, at least within mathematical circles, but also many lesser known. Some of them only absorbed/used mathematical forms and notions. However, others have provided geometry with so much fruitful inspiration through their work that we are justified in ranking them among mathematicians like Dürer. We will next look more closely at the work of some of these artists. Afterwards, we will discuss some profound analogies between mathematics and visual arts. Computer art has already been discussed in section 8.5 and need not be revisited. We will also bypass artworks the relation of which to mathematics is not primarily geometrical (for example, representations of numbers, formulae, laws and pictures that were inspired by self-reference and other topics of logics) as well as the entire complex of portraits, busts and monuments of mathematicians.

We refer the reader to [Schreiber 1999][Schreiber 2012] for a complete overview with extensive literature and to [Guderian 1990], [MUMOK 2008], [Maur 1997] and [Lauter/Weigand 2007] for extensive examples.

Those who hear the phrase 'geometry and art in the $20^{th}$ century' probably think first of M. C. Escher. Indeed, this Dutch graphic artist was an exception at best comparable to Dürer, both of them a type seemingly only bestowed upon the world in large time lags. Escher was an average student at best, according to both objective facts and his personal evaluation, even in mathematics. This only shows that ordinary mathematics instruction is little suited for the discovery and fostering of exceptional talents and that the common idea of what mathematics is, also held by many mathematicians, needs urgently to be corrected, as had already been hinted at in the introduction to this book as 'unconscious mathematics'. Escher was distinguished from many other artists who turned towards geometrical topics in the $20^{th}$ century by a great variety of addressed topics and aspects, as well as the strong inspiration that his works have given to actual mathematical questions. His topics can be roughly classified as follows:

- Ornaments, tessellations, surface packings (inspired by multiple visits to the Alhambra and, historically speaking, at the beginning of his turn towards geometrical topics from around 1936 onwards). Having got to know the Poincaré model of the hyperbolic plane, Escher designed multiple regular tessellations in the Poincaré model inside a circular disc (Illus. 8.6.1). [Herfort/Klotz 1997] have analysed this in great detail from a mathematical standpoint; also see [Henderson 1983].
- Problems of two-dimensional images of the three-dimensional.
- Beyond this, further questions connected to the interpretations of images.
- Topology, polyhedra, one-sided surfaces, ...
- Anamorphisms (especially if the object is distorted by curved mirrors).



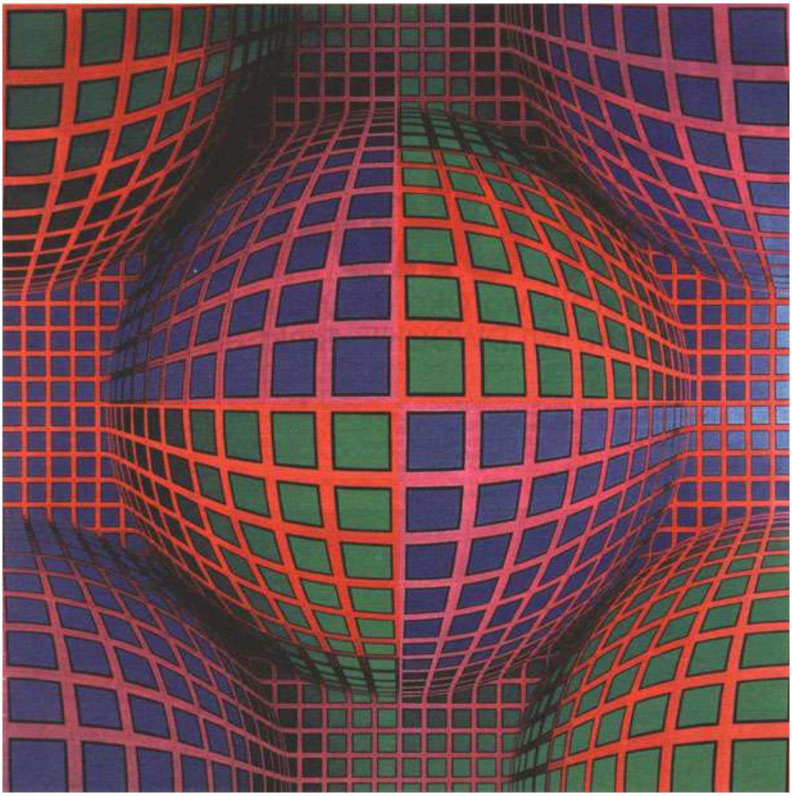
**Illus. 8.6.1** M. C. Escher's "Circle Limit III" [© 2014 The M.C. Escher Company-The Netherlands. All rights reserved, www.mcescher.com]

We will discuss all these topics factually when introducing further artists of relevance. We only want to add here that Escher was ignored by the official art scene for a long time and was heavily critizised later on.

(Bear in mind here the analogy to the rejection that Paolo Uccello had already received from his artistic peers because of his overly geometrical style!) Mathematicians and physicists were the first to show how excited and enthused they were with Escher's works. Some, especially H. S. M. Coxeter, initiated close personal relationships with him. The Dutch mathematician B. Ernst (a pseudonym for J. A. F. Rijk) wrote a successful book about him [Ernst 1978]. Eventually, art criticism also had to accept that there was greater public interest in and enthusiasm for Escher's work than the art experts would have ever thought. Some of his works are so widespread nowadays and, hence, so well known that we could compare them in this respect to the "Mona Lisa" or the "Sistine Chapel". For Escher's entire catalog, also see [Locher 1994] and [Coxeter, Emmer, Penrose, Teuber 1987]. The Swiss Sandro Del Prete can be compared to Escher in terms of originality, but his topics were not as versatile. The Swede Oscar Reutersvärd also created 'contradictory' figures. After an Austrian stamp displayed the 'Escher cube' in 1981 at a mathematics conference, some of the impossible figures conceived of by Reutersvärd were then used as designs for permanent stamps in Sweden from 1982 onwards (Illus 8.6.2). This deserves an honourable mention. Through actions such as this, many were introduced to a kind of geometry that, unfortunately, they did not encounter at school.


**Illus. 8.6.2** Escher cube and contradictive forms by Reutersvärd as motives for stamps

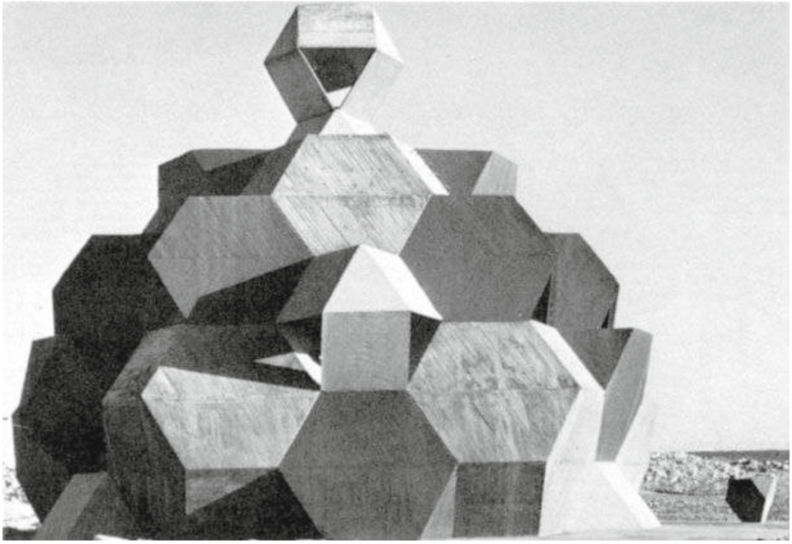
During the $20^{th}$ century, at least three art forms were created that, all in all, are closely related to mathematics. Escher, S. Dali, M. Ernst, Man Ray and R. Magritte were Surrealists (G. Apollinaire from 1917, theoretically founded by André Breton in 1924). All these artists thought of mathematics, in the way amateurs perceive it by formulae and illustrations, as well as models of mathematical objects made of glass, wire, thread and cast, as a fantastic irreal parallel world, similar to the dream worlds they had created. This served as the source of much inspiration and numerous motives. 'Concrete art' (a notion conceived by Theo van Doesborg in 1930) deals with the (often spatial) technically perfect design of aesthetic objects that denote or represent nothing other than themselves. Thus, simple geometrical shapes, such as cubes, spheres and their parts, regular and semi-regular polyhedra and mosaics, grids and Möbius strips, were designed with joy. Max Bill was an outstanding representative of this. Amongst other things, he created a series of dissections of the sphere into two mutually congruent halves for the University of Karlsruhe (1966), a series of plastics for Jerusalem (1973) that have different sections of the cube as their theme, as well as (already from 1935) numerous versions of the Möbius strip, which he initially believed to have in principle discovered himself. In one of his last works, *Kontinuität* (Continuity, 1986, in folklore terms 'Colossus of Frankfurt', see Illus. 8.6.4), he returned to this topic, whereby the question as to whether we are really dealing with a one-sided surface was left unanswered on purpose. We list "Op Art" as the third art form relevant to geometry. It works systematically with large-scale regular patterns, translucent surfaces shifted against each other, very different materials and illuminations, etc., in order to create optical effects, in particular simulating space through plane patterns (Illus. 8.6.3). The Hungarian V. Vasarély, who mainly lived in France, is one of the primary representatives of this. Further geometrically interesting works originate from W. Leblanc, B. Riley, F. Morellet, S. Le Witt, J. R. Soto and others. Finally, the fact that fractal sets were made visible by high-performance computers, especially the famous Mandelbrot set by B. Mandelbrot, led to many other similar patterns as, for example, textile and advertising designs, which, however, could often not be described by a mathematical law.



**Illus. 8.6.3** Motive of Op Art
Producing vivid impressions of plasticity caused by special patterns is one of the basic themes of Op Art [Tiré de Vasarely no 3, publié par les Editions du Griffon, Neuchatel (Suisse) 1984]

After 1950, following a period characterized by a lack of imagination, architecture again turned increasingly towards interesting geometrical figures. Next to striking roof shapes, for the designs of which technical aspects played as great a role as the aesthetic, and interesting surface designs, there are also a few remarkable experiments of floor planning based on polyhedra that are not right-angled. To give the reader an example, there is the synagogue of the Israeli officer academy, which was built 1968/70 by A. and N. Neumann and Z. Hecker according to the principle of regular space packing with tetrahedra and octahedra. However, the corners are obtuse so that all in all obtuse tetrahedra, obtuse octahedra and cuboctahedra occur as part solids (Illus. 8.6.5). Furthermore, there is the housing estate Ramot in Jerusalem, which was built according to plans by Z. Hecker with dodecahedra between 1972 and 1985. Escher also drew an architecture based on a tetrahedra-octahedra space packing ('Flatworms', 1959). However, it is one of his weakest works,

**Illus. 8.6.4** Max Bill: Continuity (1986)
The approx. 6'4" tall monumental plastic in front of the Deutsche Bank building in Frankfurt (Main) shows a multiply folded Möbius strip. See the article by G. Fischer in DMV communications 4-1999, p. 24f. [Photo: H. Wesemüller-Kock]



**Illus. 8.6.5** Synagogue of the Israeli officer academy [Photo: P. Schreiber]

artistically speaking. Striking and innovative design of products and advertising plays so important and constantly increasing a role nowadays, given the modern conditions of global competition in all areas, that, in the applications of geometry, a new field of application has opened up that is definitely to be taken seriously both scientifically and economically.

But now, let us turn to a more ordered look at the outcomes of inspiration that resulted from art (in the terms adumbrated above) of the $20^{th}$ century and from which the science of geometry benefited.

1. The great rise of the complex of ornaments and tessellations in the $20^{th}$ century, of which the volume [Grünbaum/Shephard 1987], basically a classic nowadays, bears witness. Here, some quotes from the preface:

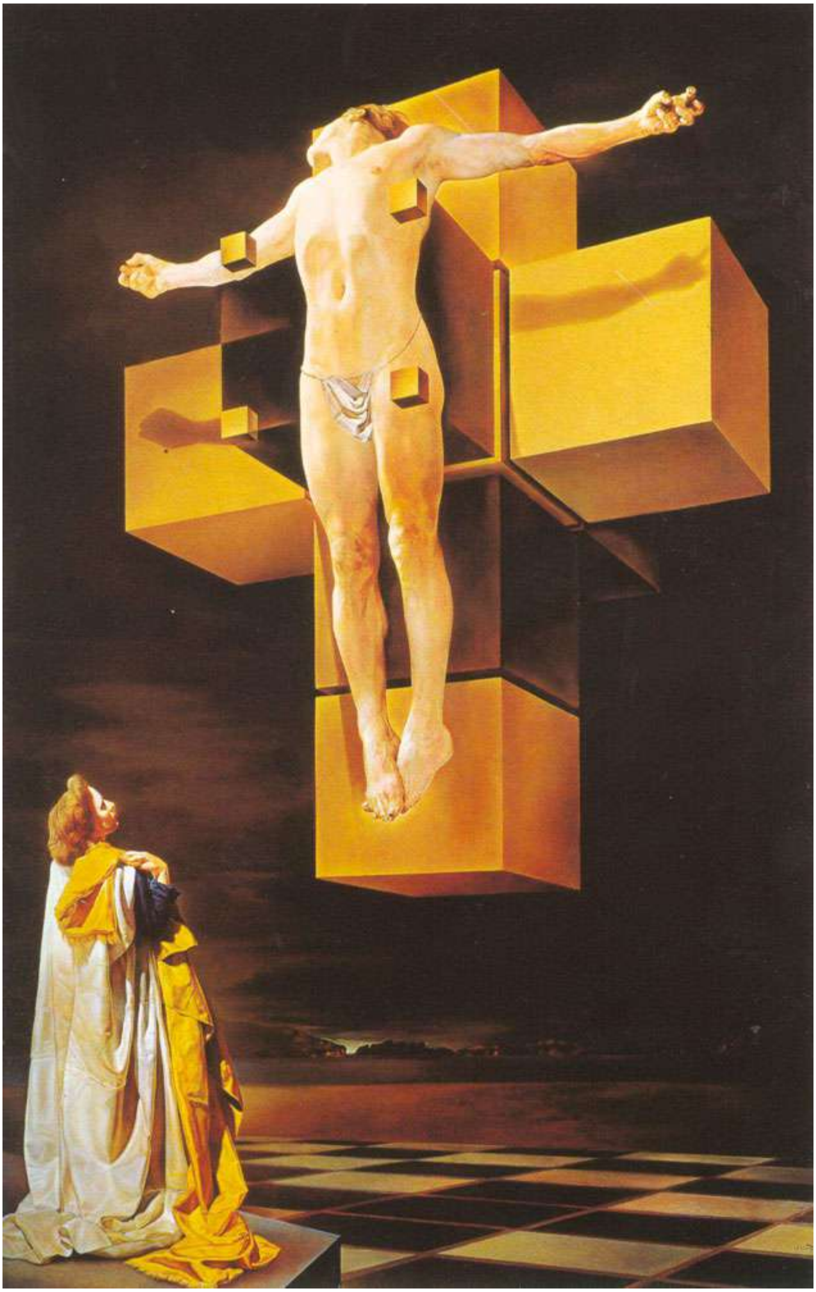
   "Perhaps our biggest surprise when we started collecting material for the present work was that so little about tilings and patterns is known. We thought, naively as it turned out, that the two millenia of development of plane geometry would leave little room for new ideas. Not only were we unable to find anywhere a meaningful definition of a pattern, but we also discovered that some of the most exciting developments in this area (such as the phenomenon of aperiodicity for tilings) are not more than twenty years old... we are rejecting the current fashion that geometry must be abstract if it is to be regarded as advanced mathematics, and that dispensed entirely with diagrams. To consider geometry without drawings as a worthy goal... seems to us as silly as to extol the virtues of soundless music (suggesting, of course, that the sign of true musical maturity is to appreciate it by merely looking at the printed score!) While assembling the material we realized that the field abounds with challenging but tractable problems, and that many previous publications contain serious errors."

The physical-technical significance of non-periodic tessellations in connection with quasi-crystals has been addressed in section 8.3. We want to stress complimentarily here that a second highly interesting and promising connection between problems of algorithms and complexity theory on one hand, and its geometrical reinterpretation on the other hand has developed since the first outcomes of the simulatability of Turing machines and similar models of recursive computation by means of tessellations (R. Berger 1966) after the combinatorially difficult graph-theoretic problems.

2. Broadly speaking, descriptive geometry takes place everywhere in which a two or three-dimensional Euclidean model of another (mathematically) defined structure is established, first in order to foster illustrative aspects, but then to realize algorithms of the material representation of their subject matter [Schreiber 2002]. We want to re-stress that operations (and tests) can only be *executed* if the objects or, respectively, their representative code objects are physically given. When physically representing them, we basically need to distinguish between two types:

   a) The geometrical form is not important. For instance, these are realized sequences of characters as series of physical statuses, and also sequences of letter characters, as long as we can neglect the aspect that their form matters again when identifying them.
   b) The geometrical form carries the crucial information. In this case, it has to be two or three-dimensional and (approximately) Euclidean.

   We are dealing with descriptive geometry in a sense still very close to Monge when, for example, describing four-dimensional regular polyhedra by means of their three-dimensional projection or development (see Illus. 7.6.1 and 8.6.6) and when modelling the plane hyperbolic or spherical geometry in the plane. (The latter is the subject of classical cartography.) We already go a little beyond this scope if cartography passes over to the graphic representation of meteorological, political, economical, ecological or traffic conditions. Actually, we are also facing descriptive geometry if an abstract network of relations (such as the old problem of the wolf, goat and cabbage) is grasped by a drawn graph. A profound analogy between descriptive geometry in mathematical terms and visual arts can be found within the fact that, in most cases, visual arts similarly strive ( excluding so-called concrete art, meaning the creation of the object for its own sake) to produce a two or three-dimensional image of something that, according to its nature, is either abstract, does not have the same dimension, is too far away in terms of space or time, or is too big or too small to be immediately perceivable by our senses. Art and mathematics can learn a lot from each other in regards to both the possible themes and the techniques of representation.



**Illus. 8.6.6** Salvador Dali: Corpus hypercubus (1954)
"The crucifix is replaced here by the two-dimensional map of the three-dimensional development of the four-dimensional hypercube. The key to understanding is the development of the three-dimensional cube on the floor." [Gilles Néret: Salvador Dali. Benedict Taschen Verlag, Cologne 1995 © Nicolas Descharnes]

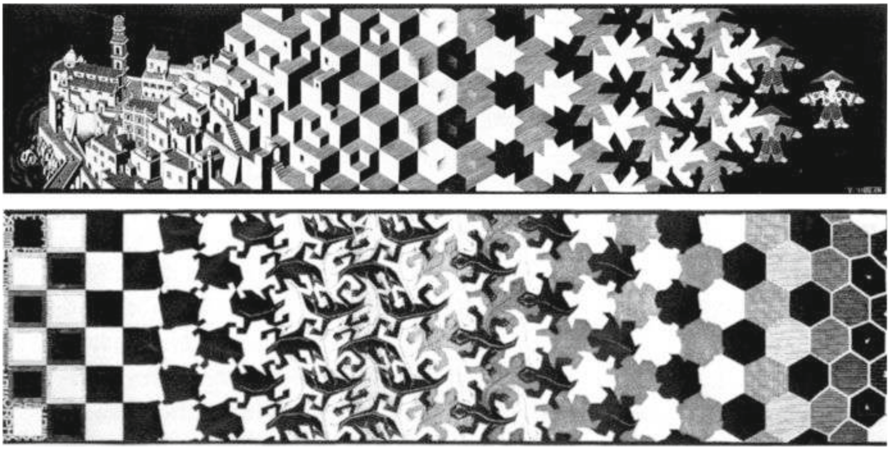
3. Image interpretation, special case perspective (detailed in [Ernst 1986]): Since the respective inventions by Escher, R. Penrose and the Swedish graphic artist O. Reutersvärd, the so-called contradictive pictures of perspective or impossible figures are very fashionable. We found out, subsequently, that their tradition goes far back into the past, requiring that we distinguish the following concerning the older pictures:

   a) if they have been created due to the artist's plain inability to master the laws of perspective,
   b) if the artist was more interested in other aspects, such as making important details more visible (often applicable to old Egyptian pictures or medieval 'technical drawings') or indicating a person's significance by their size (sacral pictures),
   c) if they had already played or experimented with perspective in humorous intentions in earlier times.

   The customary terminology is very bad. There are no such things as contradictive pictures of perspective: every picture has an infinite number of original pictures in space, since we can randomly shift every point of the image along its visual line without changing anything of the picture. (Standard method for creating tricks on film before they began to be realized with computers.) The new wave of products of 'art of contradictive perspective' has reminded us again that we cannot conclude anything from the three-dimensional original image based on the two-dimensional perspective picture without any additional information. Rather, different artists and mathematicians have enjoyed building objects or scenes of well-known pictures so that they reflect the given picture (only) from a certain viewpoint. The interaction between the viewed picture and additional information on what is seen (without which we could not orientate ourselves in our environment) has become a central field of research for artificial intelligence in the age of robots. However, this is still in its nascent stages, at least in regards to grasping the theoretical problems.

   Again, there is a profound analogy between the actual geometrical problem and the reception of artworks. The message that an artwork conveys is also usually grasped by the combination of what is visible and what we know about the artist, his time, his views and intentions, and possibly about the picture itself.

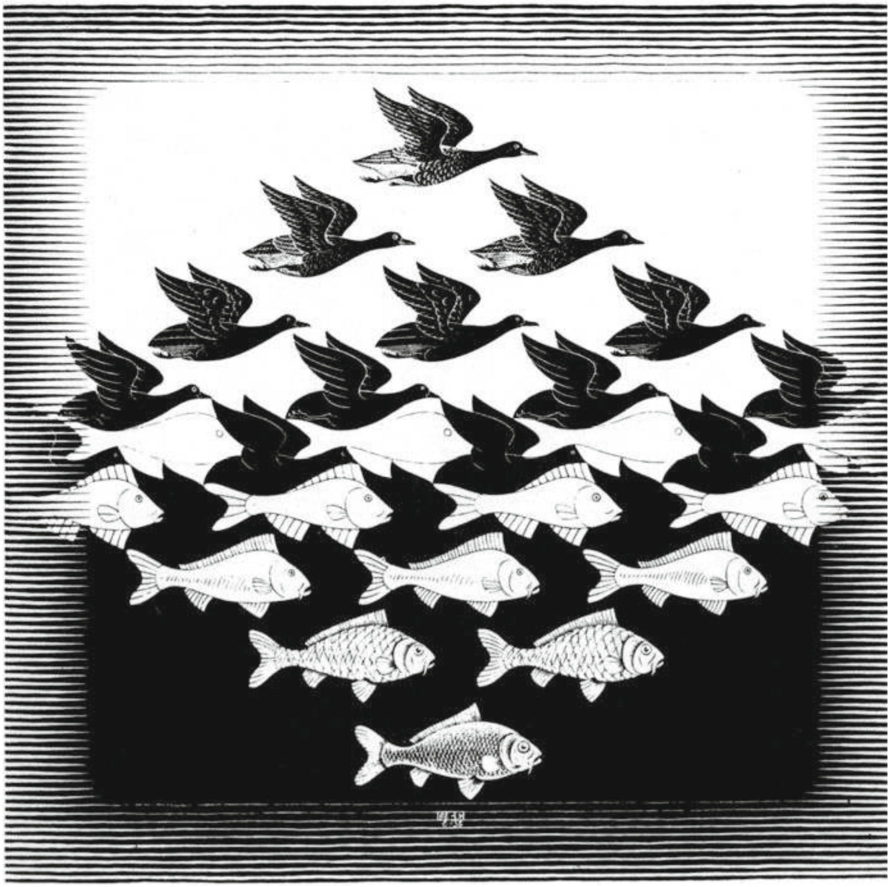
4. Image interpretation, broadly speaking. How is it that we interpret the image of a known object correctly even if it is strongly interrupted, alienated, with rough screens across it or simplified in another manner? May it depend on the Hausdorff metric discussed in section 8.2 Escher made us think with his pictures, in which a form is gradually transformed into another one. In Metamorphosis I (1937, Illus. 8.6.7), we can still uniquely



**Illus. 8.6.7** M. C. Escher's "Metamorphoses I and II" [© 2014 The M.C. Escher Company-The Netherlands. All rights reserved, www.mcescher.com]

determine where the cubes stop being cubes. But what about Air and Water I (1938, Illus. 8.6.8)? From what layer onwards can we determine fish as fish and birds as birds? Could we also do this without the respective environment?

Dali asked a completely different question with his picture "Gala Contemplating the Mediterranean Sea which in a distance of Twenty Meters becomes a Portrait of Abraham Lincoln" (1976, Illus. 8.6.9). It concerns, just as in similar works, a new type of anamorphosis for which there is a second one hidden behind the information in the foreground, as conveyed by the picture. This can only be grasped if we find the correct viewpoint (in the literal or in the figurative meaning).



**Illus. 8.6.8** M. C. Escher's "Air and Water I"
Hence, this picture inspires us to contemplate the relation between the notions of picture puzzle and anamorphosis on one hand, and the notion of (secret) code on the other hand. 

**Illus. 8.6.9** Salvador Dali: "Gala Contemplating the Mediterranean Sea..." (1976) Dali painted this picture thinking of the digitalisation of Lincoln's portrait by the American cyberneticist Leon D. Harmon. [Gilles Néret: Salvador Dali. Benedict Taschen Verlag, Cologne 1995, © Nicolas Descharnes]

**Illus. 8.6.10** One of Janas paintings (aged 11)
Children discover the aesthetic appeal of legitimate geometrical objects over and over again [Jana Schreiber]

## 8.7 Instead of an afterword: geometry and game(s)

Let us again return to our initially worded and multiply highlighted thesis of the existence of an 'unconscious mathematics'. Perhaps even before the material and cultic motives, the innate human instinct for play stands next to our tendency to engage with mathematics and in particular with geometry. Everyday life is full of examples of people who, due to their experiences in school, are averse to mathematics or at least under the impression that they are completely talentless at same, and yet are capable of engaging in astonishing trains of thoughts as soon as their interest has been awoken and their play instinct inspired. Let us remind ourselves of the remarkably early 'invention' of regular star-polyhedra by Uccello (around 1425) and Jamnitzer (1568), that Miller was first to discover a new Archimedean polyhedron after more than two thousand years when doing handicrafts, and that Reutersvärd invented the contradictive-in-perspective "tribar" during a class in 1934 during which he was so bored that he doodled lots of drawings in his notebook. The Dane Piet Hein invented the soma-cube in a possibly similar mood during a lecture on quantum physics by W. Heisenberg. Packing problems are closely related to this, indicating both practical and playful aspects. The knights tour problem, the eight queens puzzle and many further problems originally of a geometrical nature have originated from board games. Games that concern separating interlaced rings and loops or freeing objects seemingly bonded by ropes by means of topological tricks had already been addressed by Cardano in 1550. They became a fundamental component of puzzle books and so-called table magic. Numerous types of 'witch knots' made of wooden parts are also related to this. They may have originated from the ancient Scandinavian and Slavic techniques of woodwork without connecting metal pieces. We do not know how old the Chinese tile puzzle 'tangram' is (cf. Illus. 3.1.9). It first occurred in Europe at the beginning of the $19^{th}$ century and has been the subject of extensive literature and many interesting mathematical questions since then (such as the number of different convex figures that we can assemble with the 7 pieces). The Hungarian architect Ernö Rubik did not just become famous with his cube (1977) and a series of subsequent geometrical games, but also rich. Other inventors of games and puzzles also became famous, such as Sam Lloyd, as well as some mathematicians whose full-time job was basically to collect, edit and publish recreational mathematics. We could not possibly count the amount of books on this topic. Some older ones are sought-after rarities. As introductory reads, we recommend [Thiele 1984, Thiele 1988], the classics in several editions [Ball 1892], [Ahrens 1918], the books and articles by M. Gardner and the conference report [Guy/Woodrow 1994].

Herewith, we intend to express the following: Nobody should be talked into believing that geometry is boring or that he/she is untalented. Nowhere else than in the so-called experimential geometry, with or without a computer, is there a greater chance of discovering something new, even nowadays and without extensive knowledge of 'higher mathematics', which could even be useful, but above all interesting.

**Essential contents of geometry in the $20^{th}$ century**

| | |
|---|---|
| *Foundations of geometry*: | transformation from a quasi-natural science to a structure science, notion of space separated from physical meaning, axiomatic foundation of geometry (D. Hilbert); methodological and logical investigations (Schwabhäuser, Tarski and others). |
| *Abstraction*: | Change to infinitely dimensional “spaces”, application of geometrical notions to other objects, e.g., metric, inner product, norm, orthogonal base, Hilbert spaces, normed vector space, metric spaces, topological spaces, filter theory, fixed point theorems (D. Hilbert, E. Schmidt, E. Fischer, F. Riesz, S. Banach, M. Frechet, F. Hausdorff, H. Lebesgue, L. Brouwer, P. S. Alexandrow, P. Uryson), introduction to different notions of content and measure (Peano, Jordan, Borel, Lebesgue, Hausdorff). |
| *Applications in the natural sciences*: | special and general theory of relativity (A. Einstein), three or four-dimensional spaces as space-time models, Minkowski geometry (H. Minkowski, H. Weyl), relations to quantum physics, condensed matter physics, non-periodic tessellations (R.M. Robinson, R. Penrose). |
| *Applications in techniques*: | geometrical kinematics, robot constructions, manufacturing of prostheses, differential geometry for curved surface parts to assemble cars, planes and similar; stochastic geometry in biology, medicine, material testing; dense or optimal packings for transporting goods, choosing standpoints, traffic networks and timetables. |
| *Applications in computer science*: | computer geometry to represent two or three-dimensional objects graphically (CAD, 3D progr.), to recognise patterns, for mathematical program packages and computer games. |
| *Geometry within art*: | irreal representations (Escher, Reutersvärd), Op Art (Vasarely), plastics (M. Bill), Anamorphisms (S. Dali), tessellations and ornaments (M. C. Escher). |
| *Solution to an old problem*: | proof of the four colour theorem based on considerations by H. Heesch by means of computers (K. Appel, W. Haken) |
| *Proof of an old problem:* | Proof of Fermat’s Last Theorem by A. Wiles |

## 8.8 Problems to 8

**Problem 8.1.1:** Moulton's plane (see Illus. 8.1.2)

a) Look for different positions of the Desarguesian figure in this model, so that the theorem is not fulfilled respectively.
b) Consider where we encounter difficulties when we want to define translations in this model or to prove that the translations form a commutative group.
c)* The projective closure of the Moulton plane is, of course, possible, since it is only based on the incidence axioms valid here. What happens to the infinitely distant points when changing from the Euclidean standard plane to the Moulton plane? Attempt to construct a non-Desarguesian plane with only a finite number of points.

**Problem 8.1.2*:** Axiomatisation of Euclid's plane geometry according to F. Schur

In order to phrase this correctly according to the modern standards of logic, we assume two types of variables: points $P, P_i$, motions $m, m_j$, the relation of betweenness $B(P_1, P_2, P_3)$, an "application operation" $A$ and an already given axiomatisation of the plane affine geometry by means of the notions point and betweenness $B$. These suffice to define straight lines and incidence of point and straight-line. $A(m, P) = P_1$ means: Applying $m$ to $P$ results in $P_1$. The meaning of $A$ and $m$ is still open from an axiomatic viewpoint. For now, we require the following:

1. For all $m$ and all $P$ there is exactly one $P_1$, so that $A(m, P) = P_1$.
This justifies the definition $m(P) =_{def} A(m, P)$ and $A$ will then not be used anymore.

2. If $m_1(P) = m_2(P)$ for all $P$, then $m_1 = m_2$;
(i.e., a motion is entirely determined by what it does with the points, analogous to how a set is determined by its elements. 2. is an "extensionality axiom" for mappings).

3. For all $m$ and all $P$ there is exactly one $P_1$, so that $m(P_1) = P$.
The axioms 1-3 express those aspects that are communicated outside formal language in Schur's work, within the chosen language. We leave it up to the reader to phrase Schur's original axioms, namely:

4. The motions form a subgroup of the affine mappings;
(i.e., they are closed concerning composition and inversion, and leave the relation of betweenness invariant).

5. There is exactly one motion $m$, which transfers one figure into the other one, for each two figures consisting of a point, the axis starting from this point and the half-plane hanging on this axis.

6. For each two different points, there is a motion that swaps both.

7. For each angle, there is a motion that exchanges both arms.

**Problem 8.2.1:** Norm and inner product

How can we define the inner product based on the norm in a normed vector space if the parallelogram law holds for the norm? (Use the bilinearity!) Attempt to prove the basic properties (bilinearity, symmetry and positive definiteness) for the so-defined inner product based on the preconditions of the norm. For what do we need the precondition of the parallelogram law?

**Problem 8.2.2:** Hausdorff metric

$A, B, C$ shall be bounded and closed subsets of a metric space (For illustrative purposes, think of, for example, sets in the Euclidean plane.) For any point $p, gd(p, A)$ shall be the minimum of all distances $d(p, a)(a \in A)$. From which precondition does its existence result? Further, $gd(B, A)$ shall be the maximum of all $gd(b, A)$ $(b \in B)$.

a) Show that the following applies to this directed (asymmetrical) distance of sets:

   1) $gd(B, A) \geq 0, gd(B, A) = 0$, if $B \subseteq A$.
   2) $gd(A, C) \leq gd(B, A) + gd(B, C)$.
   3) If $f$ is a motion (i.e., a map preserving distance) of the space onto itself, then $gd(B, A) = gd(f(B), f(A))$.

   In order to take this notion in illustratively, consider the following: if $b$ is a point-shaped hare that can freely move on set $B$, and $a$ is a point-shaped dog that can freely move on $A$, $gd(B, A)$ is realised as the distance of two points $b_0, a_0$, which assume dog and hare, if the hare wants to be as far away from the dog as possible in any case and then the dog approaches the hare as much as possible. Based on this idea, find $gd$ for simple plane sets. When is $a_0$ and when is $b_0$ uniquely determinable?

b) Contemplate by means of a counter-example that, in general, $gd(A, B) \neq gd(B, A)$. Hausdorff's symmetrisation consists of transferring from $gd$ to $d(A, B) = max[gd(A, B), gd(B, A)]$. Show that this turns into a metric that is additionally invariant in terms of 3), i.e., the distance of two sets only depends on their mutual relative position.
c) Now show that the Hausdorff distance of two circles of the Euclidean plane with radii $r_1, r_2$ and the distance $r_0$ between the centres equals $r_0 + ||r_1 - r_2||$. (The analogous case applies to the full sphere of dimension n in every $\mathbb{R}^n$, since the distance between hare and dog is to be realized on the connecting line of the sphere centres in all dimensions.)
d) If we now coordinatize the set of all circles by the respective coordinates of its centre and its radius, the Hausdorff distance proves to be the product metric in this case, which we obtain in the Cartesian product from the plane (as location of all centres) and $\mathbb{R}^+$ as the location of all possible radii, if we choose the Euclidean distance, respectively, in the first and second factor.

e) Contemplate what the $\epsilon$-neighbourhood of a fixed circle $k_0$ looks like concerning the metric described above in the set of all circles of the plane for a given $\epsilon$. In terms of interval mathematics, this is the set of all circles, for which the sum of the deviations of the centres and the radius of those of circle $k_0$ is smaller than the given $\epsilon$. Those fill in a torus concentric with $k_0$ [Schreiber 1984, p. 215ff].

**Problem 8.2.3:** Equivalence of all norms in $\mathbb{R}^n$

Why do all Banach-Minkowsky norms yield the same topology in $\mathbb{R}^n$?

**Problem 8.2.4:** Convexity and Schauder's fixed point theorem

Show by means of a simple, plane counterexample that the precondition of convexity is necessary for Schauder's fixed point theorem.

**Problem 8.3.1:** Contradiction of Zuse's "calculating space"

For readers who know the game 'Life' or the notion of cellular field at all: assume that single signals that are emitted linearly from cell to cell correspond to photons and this speed of propagation shall be the speed of light, which cannot be beaten by any conglomerate of signals. Why is Zuse's proposal not compatible with physical reality?

**Problem 8.3.2:** Sphere-packings

Imagine six congruent spheres put together once in a straight line, once in two rows of three spheres each and once in an octahedral structure so that everyone touches four other ones.

a) Determine the volume of the convex closure for the three cases and verify that the octahedral structure is, surprisingly, not the densest one.
b) Examine the surfaces of the convex closures for the three cases. The latter explain why a small number of approximately sphere-shaped objects are mostly packed in a linear or plane structure.

**Problem 8.4.1:** Holditch's theorem

Verify Holditch's theorem mentioned in section 8.4 for the following elementary special cases:

a) The chord of length $a + b$ glides inside circle $k$ of a diameter of $d > a + b$ around once. Then point $P$ of the chord, which divides it into section $a$ and section $b$, describes a circle apparently concentric to $k$. Compare the content of the ring bounded by both circles with the proposition of Holditch's theorem.
b) The outer boundary curve shall now be a rectangle with sides $x, y > a+b$. What kind of curve does point $P$ describe now if the chord is only moved around the boundary once? It seems that it then piecewise agrees with the edges of the rectangle, but what happens in the corners? If we already know the curved arcs that are created there, we obtain the formula for the area of ...as the special case of Holditch's theorem.

c) What changes in contrast to b), if we take any (sufficiently large) convex polygon instead of the rectangle as the outer boundary curve? The reader only needs to approximate now a completely random (sufficiently large) convex boundary by a polygon in order to obtain Holditch's theorem.

**Problem 8.4.2*:** Wunderlich's wobble octahedron

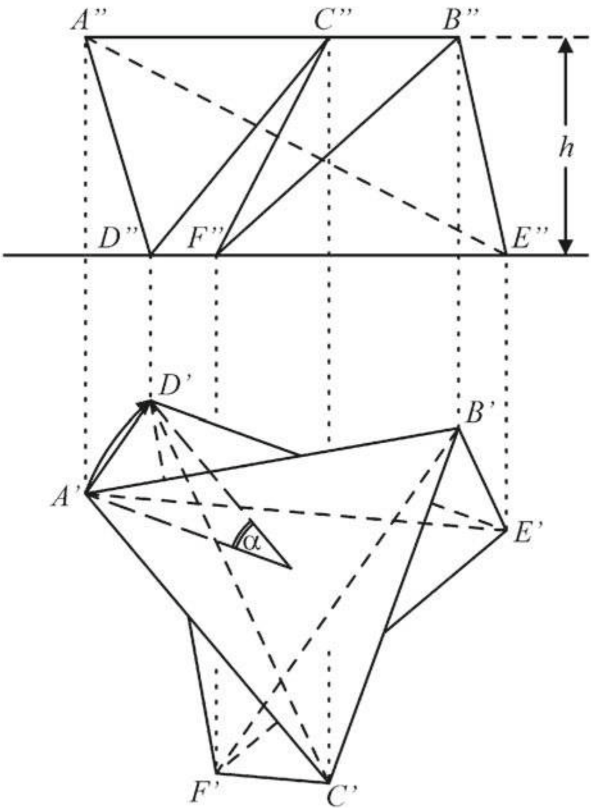
We start off with a straight prism, the top surface $ABC$ of which forms an isosceles triangle with the opposite points $DEF$ of the base. Height shall be $h$ and radius of the circumcircle of base and top surface $r$. (For the purpose of the following calculation, take $r = 1$.) If we rotate the base with respect to the top by angle $\alpha < 60°$, the four corners of a lateral surface (e.g., $ABED$) do not stay coplanar. However, by inserting diagonal $AE$ we get two triangles. Analogously insert diagonals $BF$ and $CD$ (Illus. 8.8.1; every space point $X$ is represented by its top view $X'$ and its front view $X''$ in the two-plane method.). A non-convex solid is created, bounded by 8 triangles that are analogously linked to the regular octahedron combinatorially. It is easy to see that for $\alpha = 60°$, the three added diagonals meet in the centre of the solid, i.e., in height $h/2$ over the midpoint of the base triangle. For $\alpha > 60°$, the triangles would mutually penetrate each other. If we imagine the rotation around the central axis given constant $h$, length $d$ of the diagonals and the lateral edges $s = AD(= BE = CF)$ change continuously. But we want to fix $s$, which is why the solid with increasing $\alpha$ becomes flatter and flatter. For given $r$ and $s$ and variable $\alpha$, also apart from $h$, length $d$ of the three diagonals becomes a function of $\alpha$. The problem is to establish $d(\alpha)$ explicitly. Then, we can deduct from the formula that $d\ (60° - \alpha) = d(\alpha)$ and $d$ grows monotonously between 0° and 30°. Therefore, each $d$-value between $d(0°)$ and $dmax = d(30°)$ occurs at exactly two positions $30° \pm \epsilon$. If we additionally determine $d$ so that the relevant angle values are located sufficiently close right and left of the maximum position, it means that $d$ only changes a little bit in the intermediate interval (according to Kepler!!).

Hence, we can push the solid from one position into another with only slight deformation. This example illustrates quite well what we are supposed to understand by 'wobble' or, respectively, 'infinitesimal flexibility'. Readers are invited to craft the solid themselves with suitable self-calculated values for $r, s$ and $d$ and to confirm the belonging values experimentially.

**Problem 8.7.1:** Witch knots

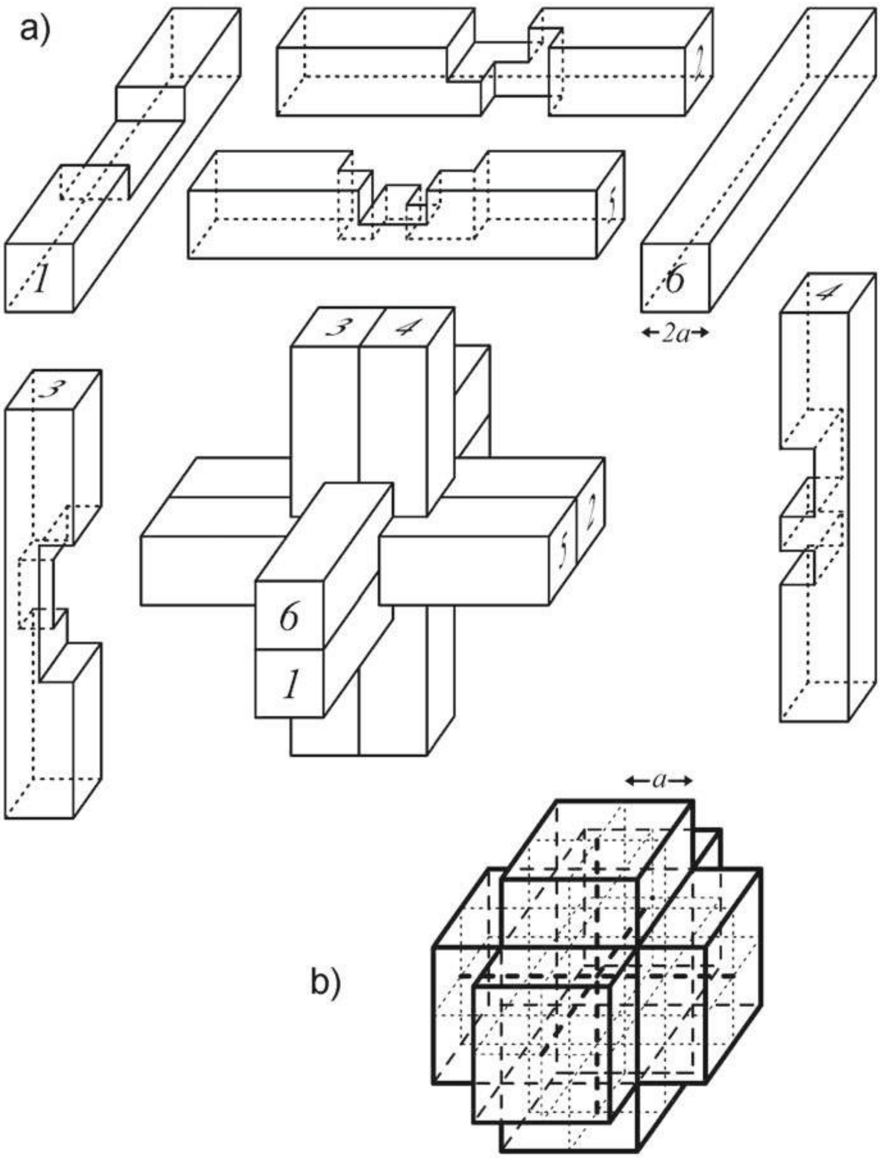
Illus. 8.8.2a shows a 'commercial' witch knot and its individual components. They have been numbered in order of assembly. However, this is only one of an immense number of possibilities for designing the 6 square wood pieces so that in the area where they cross (Illus. 8.8.2)

a) the knot does not collapse on its own,
b) there is no gap inside,
c) it can be constructed and deconstructed again, whereby exactly one of the wood pieces, the "key" no. 6, remains without notch so that we can pull it out and, as a result, are able to move the other wood pieces.




**Illus. 8.8.1** Wunderlich's wobble octahedron

C. A. Cross and W. H. Cutler have determined by means of computers from 1977-79 that there are 113 577 such possibilities [Thiele 1988, p. 54 and 198]. Our problem is more modest: find at least one version that differs from the one shown here so sharply that it also cannot result from rotations and/or reflections thereof. To draw the components individually, as in Illus. 8.8.2a, is a good exercise in descriptive geometry. If 2a refers to the lateral length of the square cross section of the wood pieces and if we imagine the core (Illus. 8.8.2b) of the knot in cubes of edge length a dissected, then they form the corners of a graph very regular in space. Our problem can also be phrased like this: assign the numbers $1, \ldots, 6$ to the corners of this graph (i.e., the a-cubes) so that the conditions corresponding to the ones stated in a) to c) above are fulfilled.




**Illus. 8.8.2** Figure for Problem 8.7.1

# Appendix: Selection of original texts

## A.1 Plato: "The Republic"

Following his well known "Allegory of the Cave" Plato developed the educational program for the prospective people responsible for the state in the $7^{th}$ book of "The Republic" – by letting Socrates explain his ideas in a dialogue with some partners. It is meant to enable them to recognise the unchangeable being and the truth based on this behind all perishable appearances of becoming and decaying. He argues that studying the four mathematical sciences is indispensible for this. Here, some extracts from the conditions Socrates lists:

"And our guardian is both warrior and philosopher?
Certainly.

Then this is a kind of knowledge which legislation may fitly prescribe; and we must endeavour to persuade those who are prescribe to be the principal men of our State to go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only; nor again, like merchants or retail-traders, with a view to buying or selling, but for the sake of their military use, and of the soul herself; and because this will be the easiest way for her to pass from becoming to truth and being.

That is excellent, he said. (...)

Let this then be made one of our subjects of education. And next, shall we enquire whether the kindred science also concerns us?

You mean geometry?

Exactly so.

Clearly, he said, we are concerned with that part of geometry which relates to war; for in pitching a camp, or taking up a position, or closing or extending the lines of an army, or any other military manoeuvre, whether in actual battle or on a march, it will make all the difference whether a general is or is not a geometrician.

Yes, I said, but for that purpose a very little of either geometry or calculation will be enough; the question relates rather to the greater and more advanced part of geometry — whether that tends in any degree to make more easy the vision of the idea of good; and thither, as I was saying, all things tend which compel the soul to turn her gaze towards that place, where is the full perfection of being, which she ought, by all means, to behold.

True, he said.

Then if geometry compels us to view being, it concerns us; if becoming only, it does not concern us? (...)

And suppose we make astronomy the third — what do you say?



C.J. Scriba, P. Schreiber, *5000 Years of Geometry*,
DOI 10.1007/978-3-0348-0898-9

I am strongly inclined to it, he said; the observation of the seasons and of months and years is as essential to the general as it is to the farmer or sailor.

I am amused, I said, at your fear of the world, which makes you guard against the appearance of insisting upon useless studies; and I quite admit the difficulty of believing that in every man there is an eye of the soul which, when by other pursuits lost and dimmed, is by these purified and re-illumined; and is more precious far than ten thousand bodily eyes, for by it alone is truth seen. (...)

Then, I said, in astronomy, as in geometry, we should employ problems, and let the heavens alone if we would approach the subject in the right way and so make the natural gift of reason to be of any real use. (...)

The second, I said, would seem relatively to the ears to be what the first is to the eyes; for I conceive that as the eyes are designed to look up at the stars, so are the ears to hear harmonious motions; and these are sister sciences — as the Pythagoreans say, and we, Glaucon, agree with them?

Yes, he replied. (...)

Now, when all these studies reach the point of inter-communion and connection with one another, and come to be considered in their mutual affinities, then, I think, but not till then, will the pursuit of them have a value for our objects; otherwise there is no profit in them."

[Plato: The Republic (On shadows and realities in education), Plain Label Books: translated by Benjamin Jowett, 1930, p. 431 – 445]

### A.2 Archimedes: Introduction to treatise "On Spirals"

Differently to the other scholars, who worked at the Musaeum in Alexandria and could converse with their colleagues there about their research, Archimedes worked very isolated in his hometown Syracuse on the island of Sicily. In contrast to, e.g., Euclid, who did not start his "Elements" with any kind of explanatory introduction at all, Archimedes stated details on the motives, which had inspired him to his investigations, in accompanying letters to some of his texts, which he sent to the mathematicians in Alexandria. After his friend Conon had died, he addressed several letters to Dositheus, since he had heard that he had also been Conon's friend and, furthermore, was a experienced mathematician.

"Archimedes to Dositheus greeting.

Of most of the theorems which I sent to Conon, and of which you ask me from time to time to send you the proofs, the demonstrations are already before you in the books brought to you by Heracleides; and some more are also contained in that which I now send you. Do not be surprised at my taking a considerable time before publishing these proofs. This has been owing to my desire to communicate them first to persons engaged in mathematical studies and anxious to investigate them. In fact, how many theorems in geometry which have seemed at first impracticable are in time successfully worked out! Now Conon died before he had sufficient time to investigate the theorems referred to; otherwise he would have discovered and made manifest all these things, and would have enriched geometry by many other discoveries besides. For I know well that it was no common ability that he brought to bear on mathematics, and that his industry was extraordinary. But, though many years have elapsed since Conon's death, I do not find that any one of the problems has been stirred by a single person. I wish now to put them in review one by one, particularly as it happens that there are two included among them which are impossible of realisation [and which may serve as a warning] how those who claim to discover everything but generate no proofs of the same may be confuted as having actually pretended to discover the impossible. What are the problems I mean, and what are those of which you have already received the proofs, and those of which the proofs are contained in this book respectively, I think it proper to specify.

The first of the problems was, given a sphere, to find a plane area equal to the surface of the sphere; and this was first made manifest on the publication of the book concerning the sphere, for, when it is once proved that the surface of any sphere is four times the greatest circle in the sphere, it is clear, that it is possible to find a plane area equal to the surface of the sphere. [Six more problems concerning the sphere follow.]

Of all the propositions just enumerated Heracleides brought you the proofs. The proposition stated next after these was wrong (...). If a sphere be cut into unequal parts by a plane at right angles to any diameter in the sphere, the greater segment of the surface will have to the less the same ratio as the greater segment of the sphere has to the less a ratio less than the duplicate ratio of that which the greater surface has to the less, but greater than the sesquialterate of that ratio.

After these came the following proposition about the spiral, which are as it were another sort of problem having nothing in common with the foregoing; and I have written out the proofs of them for you in this book. They are as follows. If a straight line of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line revolves, a point moves at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane. I say then that the area bounded by the spiral and the straight line which had returned to the position from which it started is a third part of the circle described with the fixed point as centre and with radius the length traversed by the point along the straight line during the one revolution... ."

[Archimedes: The works of Archimedes. Translated and edited by T. L. Heath, (1897), published by Cambridge University Press, p.151 – 154]

## A.3 Pope Gregory the Great mentions the art of land surveying

One of the last mentions of the Roman art of land surveying is found in a letter by Pope Gregory the Great from July 597:

Gregory to John, the Bishop of Syracuse:

"To prevent disputes about secular matters estranging the hearts of the faithful, great care must be taken that a dispute may be settled as easily as possible. We have learnt from Caesarius, Abbot of the Monastery of St. Peter at Baiae [at the Golf of Naples], that a serious dispute has arisen between him and John, Abbot of the Monastery of St. Lucia at Syracuse, about certain lands. To prevent this being prolonged, we have decided that it must be settled by a surveyor's ruling. We have therefore written to Fantinus the lawyer to send John the surveyor, who has set out from Rome for Palermo, to your brotherhood. We accordingly urge you to go with him to the area in dispute, and by presence of both parties on the spot to make an end to a dispute kept up by both sides despite [?] a limitation of forty years. Whatever is decided, your brotherhood should see that it scrupulously maintained, so that henceforth no dispute about the matter may reach us."

In another letter of this time You may read on a surveyor: "He is a judge, at any rate of his own art; his law-court is deserted fields; you might think him crazy, see him walk along tortuous paths. If he is looking for evidence among rough woodland and thickets, he doesn't walk like you or me, he chooses his own way."

[O. A. W. Dilke: The Roman Land Surveyors. Newton Abbot 1971, p. 46]

## A.4 The old Chinese "Zhou Bi Suan Jing"

The oldest conserved Chinese mathematical-astronomical work, obviously composed of different versions, starts with a dialogue between Chou Kung (Duke of Chou) and his scholarly dialogue partner Shang Kao. Their conversation concerns the properties of the right-angled triangle, the gnomon, the circle and the square, as well as measuring heights and distances. The statement in sentence (3) that geometry has its origins in measuring is very informative. Needham saw an indication here that the Chinese arithmetic-algebraic way of thinking had already been expressed in earliest times [Needham 1959, p. 23-34]. Also spend attention to the emphasis of the algebraic aspect of the proof of Pythagoras' theorem! Furthermore, the assignments of geometrical objects to the cosmos are rather striking. (The numbering of the paragraphs was introduced by later publishers.)

"(1) Of old, Chou Kung addressed Shang Kao, saying, 'I have heard that the Grand Prefect (Shang Kao) is versed in the art of numbering. May I venture to enquire how Fu Hsi anciently established the degrees of the celestial sphere? There are no steps by which one may ascend the heavens, and the earth is not measurable with a foot-rule. I should like to ask you what was the origin of these numbers?'

(2) Shang Kao replied, 'The art of numbering proceeds from the circle (*yuan*) and the square (*fang)*. The circle is derived from the square and the square from the rectangle (lit. T-square or carpenter's square; *chu.*)

(3) The rectangle originates from (the fact that) $9 \cdot 9 = 81$ (i.e., the multiplication table or the properties of numbers as such).

(4) Thus, let us cut a rectangle (diagonally), and make the width (*hou*) 3 (units) wide, and the length (*ku) 4* units) long. The diagonal (*ching*) between the (two) corners will then be 5 (units) long. Now after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a (square) plate. Thus the (four) outer half-rectangles of width 3, length 4, and diagonal 5, together make (*té chéng*) two rectangles (of area 24); then (when this is subtracted from the square plate of area 49) the remainder (*chang*) is of area 25. This (process) is called "pilling up the rectangles" (*chi chü*).

(5) The methods used by Yü the Great in governing the world were derived from these numbers.

(6) Chu Kung exclaimed, 'Great indeed is the art of numbering. I would like to ask about the Tao of the use of the right-angled triangle.'

(7) Shang Kao replied, 'The plane right-angled triangle (laid on the ground) serves to lay out (works) straight and square (by the aid of) cords. The recumbent right-angled triangle serves to observes hights. The reversed right-angled triangle serves to fathom depths. The flat right-angled triangle is used for ascertaining distances.

(8) By the revolution of a right-angled triangle (compasses) a circle may be formed. By uniting right-angled triangles squares (and oblongs) are formed.

(9) The square pertains to earth, the circle belongs to heaven, heaven being round and the earth square. The numbers of the square being the standard, the (dimension of the) circle are (deduced) from those of the square.

(10) Heaven is like a conical sun-hat. Heaven's colours are blue and black, earth's colours are yellow and red. A circular plate is employed to represent heaven, formed according to the celestial numbers; above, like an outer garment, it is blue and black, beneath, like an inner one, it is red and yellow. Thus is represented the figure of heaven and earth.

(11) He who understands the earth is a wise man, and he who understands the heavens is a sage, knowledge is derived from the straight line. The straight line is derived from the right angle. And the combination of the right angle with numbers is what guides and rules the ten thousend things.'

(12) Chu Kung exclaimed 'Excellent indeed!' ..."

[J. Needham: Science and Civilisation in China, vol. 3: Mathematics and the Sciences of the Heavens and the Earth. Cambridge 1959, p. 22-23]

## A.5 Cassiodorus Senator: *Institutiones*

Cassiodorus wrote his "*Institutiones*" in the monastery Vivarium founded by him approx. between 531 and 562. They were directed at the monks at the monastery, who are said to have introduced the theological sciences and also the indispensible foundations of the worldly sciences. As justification for studying the latter, he stated:

"However, in the second book we will briefly speak of the liberal arts and sciences. A mistake is less likely in this area if we miss concerning the firmly standing faith. However, what is found about such things in the Holy Bible will be better understood due to the recognition given in advance. It is certain for once that the foundations of the worldly sciences lie at the beginning of the theological sciences just like seeds: the teachers of the worldly sciences have adapted them very cleverly in their own principles later. Perhaps we have been able to prove this at an appropriate place when interpreting the Psalter."

[translated from German, K. S. Frank: Frühes Mönchtum im Abendland. Vol. 1: Lebensformen. Zurich and Munich 1975, p. 206-207.]

## A.6 Preface of Albrecht Dürer addressing W. Pirckheimer

Extracts from [A. Dürer: Preface to "Four books on human proportions" 1528]

"However, nobody is forced to follow my teaching as if it were entirely perfect, since human nature has not yet so decreased that someone else might not invent something better. It would have to be a very thin mind that does not dare invent something more, but is on the old path, only follows others and does not allow himself to think further. It is obvious that the German painters are not little skilled with their hands and using colours, although they have lacked in the art of measuring, also perspective and similar... But without correct proportion, no picture can be perfect... In order for this instruction to also be better understood, I have published before a book of measurement, namely concerning lines, planes, solids etc., without which this instruction may not be thoroughly understood... Nobody should reject it since he will soon understand all these things, because what is very easy cannot be very artificial. But what is artificial that will need diligence, effort and work."

## A.7 Alfred Meißner (1822 – 1885): Geschichte meines Lebens (History of my Life, 1884)

Reprinted under the title [Meißner: "Ich traf auch Heine in Paris" (I also met Heine in Paris), book publisher: Der Morgen, Berlin 1973]

(About studying at University of Prague 1837)

"'In the so called "first philosophical year"erything revolved around the professor of mathematics, Ladislaus Jandera. Nobody could know in advance if he would satisfy this fruitful enthusiast and if he did not satisfy him, one was lost; because with a bad mark in mathematics it was impossible to rise. We two – Hartmann[1] and I – think nature has denied us all aptitude in this area. We felt that we stayed behind; but instead of doubling our efforts in this field, we slowed down and were full of bad premonitions regarding the final result.

The terrifying Ladislaus Jandera was a very small old man, a figure like taken from a fairy tale by E. T. A. Hoffmann. He was a Premonstratensian[2], but did not wear a monk's robe, instead high mighty boots, a civil skirt, and because he handled so much chalk a blue-white one like a miller. A terrible eager for the holy sciences had been engraved in his face, the hard, angled face of a gnome. If he had climbed up the teacher's desk, which mostly happened with a storming auditorium, he had the habit of crossing his arms across his chest like Napoleon and to dominate the audience with wild looks until everything was silent. In front of him was the so called 'Me-mo-ria-le', the chalk in his hand, a short white stick under his arm, with which he used to demonstrate and often started hammering onto the board as if possessed. He started his speech with a yelling voice which passed through every storm and dissected every word into its single syllables. 'Cla-ri-ty' was his motto, and 'Now e-very cook must un-der-stand this!', his last word after every longer debate, with which he paid himself the greatest tribute, according to him. Unfortunately, I must confess that I very often did not grasp what every cook was supposed to.

One time, when we laid hands onto a collection of old copper engravings from the time of the French Revolution, we both made the discovery at the same time that Professor Ladislaus Jandera bore the greatest similarity to Robespierre. It was exactly the same head, only much older, the same forehead, the same mouth. But I also cannot compare this man to nobody less than the virtuous delegate of Arras. Jandera also was the personified virtue,

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[1] Moritz Hartmann (1821-1872), German-Bohemian poet, author, liberal journalist and politician; poetry collection *Kelch und Schwert* (Goblet and sword; 1845), hist. novel *Der Krieg um den Wald* (The war over the forest; 1850) and others.

[2] Premonstratensian: cath. cleric order, founded in 1120. Chaplains teaching mathematics at universities had a long tradition going back to the Middle Ages. It was customary until far into the $19^{th}$ century in some countries.

the justice and the incorruptibility itself, but terrifying, because he did not accept anything else than his own principles and the individuals were nothing in front of his eyes. Those, who did not pass him and his memorials, had 'de-mon-stra-ted their use-less-ness for the scien-ti-fic pro-fes-sion'. Thousands of young people had already had to change their career, because they did not suffice his standards. It did not even bother him in his sleep. The personal use of all other professors could never ever have put him off an explicit 'No'. One nice sunny morning, Professor Ladislaus Jandera stood in front of his board and explained us the theorem from teaching the ellipse, that the plane of the curve AQMQ stands perpendicularly on the triangle KLM and that any section, which is parallel to the base LBMD, must have its centre in the axis KC in any case, if we put a plane KLM perpendicularly against the plane RS through the axis KC of a cone.

Meanwhile, Hartmann still sat happy with his bank and read a book, which I had gotten hold of yesterday and given to him today. Grabbes' *Faust and Don Juan*. Grabbe was one of our favourite poets. A 'Quod erat demonstrandum!' slammed into the air like a rooster's cry awoke the reader from his dreams. The terrifying nutcracker face up there had finished his line of argument. 'And now', the terrible continued in a seemingly mild tone – his sharp eye may have followed the inattentive for the longest time – 'and now (slowly turning the pages in his catalogue, until he had found it), my dear Mortiz Hartmann, come up to me and show your colleagues that you have understood me. Hartmann, Moritz, come up!'

That was a shock that also made me, the friend, lose my joy, vision and hearing! One would have to adhere to his call at all times. The storm of expectation was already stirring amongst the immense number of students. However, there was only a short appearance. Hartmann had gone up, drawn some lines onto the board and murmured some words. Then he had taken the first opportunity to escape and had dived into the sea of heads with a fire-red face, whereas up there the evil goblin had thrown his hands up moaning about so much ignorance."

## A.8 Preamble by F. Wolff

[F. Wolff: Descriptive geometry and its applications.Guide for teaching at Royal Trade-Institute, second part, Berlin 1840] Extracts from the preface:

"The Mister-Really-Secret-Senior-Government-Councillor *Beuth* was first to draw my attention to descriptive geometry and its importance, and later generously approved the means on behalf of a high ministry of finance, which were drawn on when creating the work: hence, I must be grateful in two respects. I also owe gratitude to Mister Senior-County-Building-Director *Schinkel* for the great favour to advise us concerning several papers belonging to this department... Since the industry has decisively separated from what is given, and, based on science, has experienced a more intellectual and daring boom, we have been careful to provide the younger with a scientific education. In particular, the Prussian government has spent the greatest means to set up a department in the trade-institute, which in regards to external facilities and healthy thoughts, which it is originally based on, resembles the most excellent features. Because of the application alone, the technician conducts scientific studies, but only the application has meaning, which he practises independently. The independent application demands two things: material knowledge and educating the mind. First of all, we must state the most useful for the moment from the sciences, but also touch on the least striking, as far as there is enough time. What is an unapparent little detail nowadays, may be of great importance tomorrow... The technician must not mention that he is done with what he hears in the lectures, can read in books, or with the applications, which he made himself. He should absorb all that, but then strive further unbiased and independently... He should not have himself bewitched by the gossip about theory and praxis. According to many people's opinion, this is nothing else than a convenient means to suspect what is inconvenient, and to talk sugar-coated what they approve of... I have confirmed that geometrical drawings are drawings of purely geometrical projections, to which we add illumination and shadow, in order to obtain a more visual representation, which simplifies understanding, but does not have the purpose of illusion as perspective... Two things are part of geometrical drawings: the skills to draw lines, to wash out and similar and the science of construction... The still missing sections of this volume shall address perspective in respect to the teachings of newer geometry, the stereometry and other technical applications, finally the methods to produce drawn objects in reality. Concerning my own status as a private teacher, the circumstance that I taught some classes at the trade-institute annually and also on descriptive geometry during some weeks cannot be reason to spend significant amounts of time and effort after an approximate estimate, which demands the execution of that plan despite much groundwork. And if a year-long effort has only led to one position, which does not even allow us to deny the effort spent on books, much less to move otherwise, as it would be necessary and the private teacher has no other opportunity than to live in suppressing conditions, as long as he is hale, and to be a beggar, when he is old, it seems he is only now sufficiently been warned to give his additional occupations a direction, which will be more fruitful."

## A.9 Hermann von Helmholtz: "The Origin and Meaning of Geometrical Axioms"

Speech given in Heidelberg in 1870.

"The fact that a science like geometry can exist and can be developed as it has been has always attracted the closest attention among those who are interested in questions relating to the bases of the theory of cognition. Of all branches of human knowledge, there is none which, like, it, has sprung as a completely armed Minerva from the head of Jupiter; none before whose death-dealing Aegis doubt and inconsistency have so little dared to raise their eyes. It escapes the tedious and troublesome task of collecting experimental facts, which is the province of the natural sciences in the strict sense of the word; the sole form of its scientific method is deduction. Conclusion is deduced from conclusion, and yet no one of common sense doubts that these geometrical principles must find their practical application in the real world about us. Land surveying as well as architecture, the construction of machinery no less than mathematical physics, are continually calculating relations of space of the most varied kind by geometrical principles; they expect that the success of their constructions and experiments shall agree with these calculations; and no case is known in which this expectation has been falsified, provided the calculations were made correctly and with sufficient data.

Indeed, the fact that geometry exists, and is capable of all this, has always been used as a prominent example in the discussion on that question, which forms, as it were, the centre of all antitheses of philosophical systems, that there can be a cognition of principles destitute of any bases drawn from experience. In the answer to Kant's celebrated questions, 'How are synthetical principles *a priori* possible?' geometrical axioms are certainly those examples which appear to show most decisively that synthetical principles are *a priori* possible at all. The circumstance that such principles exist, and force themselves on our conviction, is regarded as a proof that space is an *a priori* mode of all external perception. He appears thereby to postulate, for this *a priori* form, not only the character of a purely formal scheme of itself quite unsubstantial, into..."

[Ewald, William B., ed., 1996. *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, 2 vols. Oxford Uni. Press. 1876, *The Origin and Meaning of Geometrical Axioms*, 663–88 , Extracts (p. 665-667 in vol. 2).]

### A.10 E. A. Abbott: “Flatland”

[E. A. Abbott: Flatland. A Romance of many dimensions. 1884] Extracts (The somehow peculiar writing style adheres to the original)

“To the Inhabitants of Space in General And H. C. in Particular. This Work is Dedicated By a Humble Native of Flatland In the Hope that Even as he was Initiated into the Mysteries Of Three Dimensions Having been previously conversant With Only Two So the Citizens of that Celestial Region May aspire yet higher and higher To the Secrets of Four Five or even Six Dimensions Thereby contributing To the Enlargement 580 of the Imagination And the possible Development Of the most rare and excellent Gift of Modesty Among the Superior Races of Solid Humanity.

Imagine a vast sheet of paper on which straight Lines, Triangles, Squares, Pentagons, Hexagons, and other figures, instead of remaining fixed in their places, move freely about, on or in the surface, but without the power of rising above or sinking below it... Our Women are Straight Lines. Our Soldiers and Lowest Classes of Workmen are Triangles with two equal sides, each about eleven inches long, and a base or third side so short (often not exceeding half an inch) that they form at their vertices a very sharp and formidable angle. Indeed when their bases are of the most degraded type, (not more than the eighth part of an inch in size), they can hardly be distinguished from Straight Lines or Women; so extremely pointed are their vertices...Our Middle Class consists of Equilateral or Equal-sided Triangles. Our Professional Men and Gentlemen are Squares (to which class I myself belong) and Five- Sided Figures or Pentagons. Next above these come the Nobility, of whom there are several degrees, beginning at Six-Sided Figures or Hexagons, and from thence rising in the number of their sides till they receive the honourable title of Polygonal, or many-sided. Finally when the number of the sides becomes so numerous, and the sides themselves so small, that the figure cannot be distinguished from a circle, he is included in the Circular or Priestley order, and this is the highest class of all. It is a law of Nature with us that a male child shall have one more side than his father, so that each generation shall rise (as a rule) one step in the scale of development and nobility. But this rule applies not always to the Tradesmen, and still less often of the Soldiers, and to the Workmen; who indeed can hardly be said to deserve the name of human Figures, since they have not all their sides equal... Chapter 13: How I had a Vision of Lineland: ...I saw before me a vast multitude of small Straight Lines (which I naturally assumed to be women) interspersed with other Beings still smaller and of the nature of lustrous points — all moving to and fro in one and the same Straight Line, and, as nearly I could judge, with the same velocity... Approaching one of the largest of what I thought to be Woman, I accosted her, but received no answer. A second and a third appeal on my part were equally ineffectual. Losing patience, I brought my mouth into a position full in front of her mouth so as to intercept her motion, and loudly repeated my question ‘Woman, what signifies this concourse, and this strange and confused chirping, and this monotonous motion to and fro in one and the same Straight Line?' 'I am no Woman', replaid the small Line, 'I am the Monarch of the world . . ."

We believe that these extracts sufficiently illustrate in what genius manner social conditions were satirically criticised here together with the knowledge of geometrical as up-to-date as it was at that time. The reference to the method of illustration multiply used by Helmholtz cannot be missed.

## A.11 Th. Storm[3]: "Der Schimmelreiter" (The Rider on the White Horse)

Extract from the short story, translated by J. Schreiber.

"Well, he said, in the middle of the last century, or rather, to be more exact, before and after the middle of that century, there was a dikemaster here who knew more about dikes and sluices than peasants and landowners usually do. But I suppose it was nevertheless not quite enough, for he had read little of what learned specialists had written about it; his knowledge, though he began in childhood, he had thought out all by himself. I dare say you have heard, sir, that the Frisians are good at arithmetic, and perhaps you have heard tell of our Hans Mommsen from Fahretoft, who was a peasant and yet could make chronometers, telescopes, and organs. Well, the father of this man who later became dikemaster was made out of this same stuff–to be sure, only a little. He had a few fens, where he planted turnips and beans and kept a cow grazing; once in a while in the fall and spring he also surveyed land, and in winter, when the northwest wind blew outside and shook his shutters, he sat in his room to scratch and prick with his instruments. The boy usually would sit by and look away from his primer or Bible to watch his father measure and calculate, and would thrust his hand into his blond hair. And one evening he asked the old man why something that he had written down had to be just so and could not be something different, and stated his own opinion about it. But his father, who did not know how to answer this, shook his head and said: 'That I cannot tell you; anyway it is so, and you are mistaken. If you want to know more, search for a book to-morrow in a box in our attic; someone whose name is Euclid has written it; that will tell you.' The next day the boy had run up to the attic and soon had found the book, for there were not many books in the house anyway, but his father laughed when he laid it in front of him on the table. It was a Dutch Euclid, and Dutch, although it was half German, neither of them understood. 'Yes, yes', he said, 'this book belonged to my father; he understood it; is there no German Euclid up there?' The boy, who spoke little, looked at his father quietly and said only: 'May I keep it? There isn't any German one.' And when the old man nodded, he showed him a second half-torn lit-

[3] Theodor Storm (1817 – 1888) has been one of the most famous German writers of his time. The story on The Rider on the White Horse is based on real events.

tle book. “That too?” he asked again. “Take them both!” said Tede Haien; “they won’t be of much use of you.” But the second book was a little Dutch grammar, and as the winter was not over for a long while, by the time the gooseberries bloomed again in the garden it had helped the boy so far that he could almost entirely understand his Euclid, which at that time was much in vogue. I know perfectly well, sir, the story teller interrupted himself, that this same incident is also told of Hans Mommsen, but before his birth our people here have told the same of Hauke Haien–that was the name of the boy. You know well enough that as soon as a greater man has come, everything is heaped on him that his predecessor has done before him, either seriously or in fun. When the old man saw that the boy had no sense for cows or sheep and scarcely noticed when the beans were in bloom, which is the joy of every marshman, and when he considered that his little place might be kept up by a farmer and a boy, but not by a half-scholar and a hired man, inasmuch as he himself had not been over-prosperous, he sent his big boy to the dike, where he had to cart earth from Easter until martinmas. “That will cure him of his Euclid”, he said to himself. And the boy carted; but his Euclid he always had with him in his pocket, and when the workmen ate their breakfast or lunch, he sat on his upturned wheelbarrow with the book in his hand.”

[Th. Storm: Der Schimmelreiter. Berlin 1888]

## A.12 K. Fladt: Euclid

Extract, translated by J. Schreiber.

“Therefore, there was no lack of attempts to represent geometry genetically. However, the Euclidean manner of representation has enjoyed such a high prestige for centuries that it was almost accepted as the matter itself that we believed that the meaning of mathematics exhausted itself in its logical consistency. Hence, mathematics should just be a formal subject at school, i.e. there to educate the mind. A further consequence of this misunderstanding was that schoolbooks for young pupils were created based on ‘Elements’, the textbook of the students at Alexandria. And since it was not every student’s cup of tea to distinguish form from core, mathematics was thought to be difficult and it was believed that a special talent was necessary. All the tribute, which extensive circles of our scholarly people paid to the adept juniors of mathematical secret knowledge and are still paying, but also the secret horror, which they feel towards mathematics, goes back to the effects of the Euclidean ‘Elements’ at the end.”

[K. Fladt: Euklid, Berlin 1927]

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