2.094

FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS

SPRING 2008

Homework 6 - Solution

Problem 1 (20 points):

Let's define $\,^{t}\hat{R} = \dfrac{^{t}R}{2kL}$, $\,^{t}\hat{F} = \dfrac{^{t}F}{2kL}$ and $\,^{t}U = \dfrac{^{t}\Delta}{L}$.

Since $\,^{t}\hat{F} = \,^{t}\hat{R}$ at equilibrium,

$$^{t}\hat{F} = \,^{t}\hat{R} = \left\{ -1 + \frac{1}{\left[1 - 2\,^{t}U \sin 15° + \,^{t}U^{2} \right]^{1/2}} \right\} (\sin 15° - \,^{t}U)$$

The tangent stiffness can be calculated by

$$^{t+\Delta t}K^{(i-1)} = \frac{^{t+\Delta t}\hat{F}\left(^{t+\Delta t}U^{(i-1)} + \varepsilon \right) - \,^{t+\Delta t}\hat{F}\left(^{t+\Delta t}U^{(i-1)} \right)}{\varepsilon}$$

with a sufficiently small ε.

Then the Newton-Raphson iteration is performed

$$^{t+\Delta t}K^{(i-1)}\Delta U^{(i)} = \,^{t+\Delta t}\hat{R} - \,^{t+\Delta t}\hat{F}^{(i-1)}$$

$$^{t+\Delta t}U^{(i)} = \,^{t+\Delta t}U^{(i-1)} + \Delta U^{(i)}$$

with the initial conditions

$$^{t+\Delta t}U^{(0)} = \,^{t}U , \quad ^{t+\Delta t}K^{(0)} = \,^{t}K , \quad ^{t+\Delta t}\hat{F}^{(0)} = \,^{t}\hat{F}$$

The calculated values in each step are listed in Table 1. The number of iterations and the calculated values can be different from those in Table 1 depending on which value of ε you used for the calculation of the tangent stiffness.

Note that this problem has multiple solutions as shown in the graph in the textbook. Therefore if your stiffness is not correct, then you may get another solution which is far from the initial equilibrium state. Make sure that the displacement you calculated is the nearest one from the initial equilibrium state.

Table 1. Calculated values in each iteration ($\varepsilon = 0.0001$)

Step	$^{t+\Delta t}U$	$^{t+\Delta t}\hat{F}$	$^{t+\Delta t}K$	$^{t+\Delta t}\hat{R} - {}^{t+\Delta t}\hat{F}$
0	3.702230E-002	2.000000E-003	4.149292E-002	1.000000E-003
1	6.112280E-002	2.817306E-003	2.649827E-002	1.826936E-004
2	6.801735E-002	2.986220E-003	2.246174E-002	1.377952E-005
3	6.863081E-002	2.999909E-003	(Not necessary)	9.081258E-008

Problem 2 (10 points):

(a)

$$h_1 = \frac{1}{4}\left(1 + {}^0x_1\right)\left(1 + {}^0x_2\right), \ h_2 = \frac{1}{4}\left(1 - {}^0x_1\right)\left(1 + {}^0x_2\right), \ h_1 = \frac{1}{4}\left(1 - {}^0x_1\right)\left(1 - {}^0x_2\right), \ h_1 = \frac{1}{4}\left(1 + {}^0x_1\right)\left(1 - {}^0x_2\right)$$

Define

$$^0\hat{\underline{x}}^T = \begin{bmatrix} {}^0x^1 & {}^0x^2 & {}^0x^3 & {}^0x^4 & {}^0y^1 & {}^0y^2 & {}^0y^3 & {}^0y^4 \end{bmatrix}$$

$$^t\hat{\underline{x}}^T = \begin{bmatrix} {}^tx^1 & {}^tx^2 & {}^tx^3 & {}^tx^4 & {}^ty^1 & {}^ty^2 & {}^ty^3 & {}^ty^4 \end{bmatrix}$$

Then, using ${}^t\underline{u} = {}^t\underline{x} - {}^0\underline{x}$,

$$^t\underline{u} = \begin{bmatrix} {}^tu \\ {}^tv \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix}\left({}^t\hat{\underline{x}} - {}^0\hat{\underline{x}}\right)$$

$$= \begin{bmatrix} \dfrac{\sqrt{3}}{2}\left(1 + {}^0x_2\right) \\ -\dfrac{1}{2}\left(1 + {}^0x_2\right) \end{bmatrix}$$

(b)

$$
{}^{t}\underline{x} = {}^{0}\underline{x} + {}^{t}\underline{u} = \begin{bmatrix} {}^{0}x_{1} + \dfrac{\sqrt{3}}{2}\left(1 + {}^{0}x_{2}\right) \\ {}^{0}x_{2} - \dfrac{1}{2}\left(1 + {}^{0}x_{2}\right) \end{bmatrix}
$$

$$
{}^{t}_{0}\underline{X} = \begin{bmatrix} \dfrac{\partial {}^{t}x_{1}}{\partial {}^{0}x_{1}} & \dfrac{\partial {}^{t}x_{1}}{\partial {}^{0}x_{2}} \\ \dfrac{\partial {}^{t}x_{2}}{\partial {}^{0}x_{1}} & \dfrac{\partial {}^{t}x_{2}}{\partial {}^{0}x_{2}} \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{\sqrt{3}}{2} \\ 0 & \dfrac{1}{2} \end{bmatrix}
$$

$$
{}^{t}_{0}\underline{C} = {}^{t}_{0}\underline{X}^{T}\,{}^{t}_{0}\underline{X} = \begin{bmatrix} 1 & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & 1 \end{bmatrix}
$$

$$
{}^{t}\rho = \dfrac{{}^{0}\rho}{\det {}^{t}_{0}\underline{X}} = \dfrac{0.05}{1/2} = 0.1
$$

2.094 Finite Element Analysis of Solids and Fluids II
Spring 2011