College of Surgeons

from the Author

ON

THE INFLUENCE OF SIGNS

IN

MATHEMATICAL REASONING.

From the Transactions *of the* Cambridge Philosophical Society.

By CHARLES BABBAGE, Esq. M. A. Trin. Coll.

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XX. *On the Influence of Signs in Mathematical Reasoning.*

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It can scarcely excite our surprise that the earlier geometers, engaged in successfully employing the most powerful instrument of discovery which human thought has yet contrived, and seduced by the splendour of the view their science had opened to them, should press with earnestness to enlarge its boundaries by new applications, rather than exert their genius in explaining the causes which have combined to advance it to such unrivalled eminence. On the discovery of those branches which have so completely altered the face of the science, the use of the new acquisitions was too inviting to allow time for any very scrupulous enquiry into the principles on which they were founded: satisfied with the accuracy of the results at which they arrived, the desire of multiplying them naturally prevented any return on their steps for the purpose of applying themselves to the less promising task of establishing on secure foundations, principles of whose truth they felt confident.

These efforts to extend the reach rather than fix the basis of the new calculus, were undoubtedly to be admired at the period to which we refer: an acquaintance with its extensive bearings ought justly to have no inconsiderable influence on the form in which its elements should be delivered; hence the lapse of nearly a century has been required to fix permanently the foundations on which the calculus of Newton and of Leibnitz shall rest.

Time which has at length developed the various bearings of the differential calculus, has also accumulated a mass of materials of a very heterogeneous nature, comprehending fragments of unfinished theories, contrivances adapted to peculiar purposes, views perhaps sufficiently general, enveloped in notation sufficiently obscure, a multitude of methods leading to one result, and bounded by the same difficulties, and what is worse than all, a profusion of notations (when we regard the whole science) which threaten, if not duly corrected, to multiply our difficulties instead of promoting our progress.

As a remedy to the inconveniences which must inevitably result from the continued accumulation of new materials, as well as from the various dress in which the old may be exhibited, nothing appears so likely to succeed as a revision of the language in which all the results of the science are expressed, and the establishment of general principles which shall curtail its exuberance, and regulate that which has hitherto been considered as arbitrary — the contrivance of a notation to express new relations. Previous however to this, some observations on the nature of that assistance which signs lend to our reasoning faculties, and on the causes which give such certainty to the conclusions of analysis, may render our future enquiries more intelligible.

The nature of the quantities with which the mathematical sciences are conversant, is undoubtedly one of the first of those causes: in Geometry it has been well remarked* that its foundations rest on definition, and if this do not altogether hold in algebraical enquiries, at least the meaning of the symbols employed

* Elements of the Philosophy of the Human Mind, Vol. II. p. 150.

must be regulated by definition; and here arises one of the great differences which characterise this science, the definitions themselves being exceedingly simple, comprising but few ideas, whilst in other sciences they are usually much more complicated. In Geometry, definition is the beginning of any enquiry; in metaphysical science, it is frequently the result of one: thus that a triangle is a figure formed by three sides, is a convention on which many of Euclid's propositions rest, and from this, as a point of departure, numerous deductions are made: on the other hand, our idea, and consequently our definition of beauty, is only the result of considerable thought and enquiry.

In the language of analysis, it is very rare that any symbol possesses more than one meaning; in ordinary language, it is as rare to find a word having but one signification: nor is this the only difference; when an algebraical symbol has more than one meaning, they are always well defined and distinct, and should there exist several signs for the same operation, the only difference is in their external form, not the slightest in their meaning; whilst in common language, the meanings of words shade away into each other, and it is frequently difficult, even on mature consideration, to assign the precise limits of the signification of words which are nearly synonymous.

Now if this be the case when the words themselves are the especial objects of our thoughts, how open must all reasoning be to inadvertencies when the mind is compelled to occupy itself at once on the various meanings of the signs it uses and on the train of consequences which it endeavours to deduce by them.

The multitude of significations which attach to many of the words that compose our ordinary language, is a disadvantage which is completely removed from that of analysis. In our reasoning concerning any objects even of a moderately complicated nature, we are obliged to make use of the words attached to those objects, which consequently recall to the mind the variety of particulars of which they consist, some with more, others with less vividness according to our previous habits of thought; from this cause it sometimes happens that the real ground on which our reasoning depends, is with difficulty kept in view by a laborious effort of the attention, and is in many instances very indistinctly perceived.

In the use of algebraic signs this inconvenience entirely vanishes; we can always so arrange them, that that quality on which the whole force of our reasoning turns shall be visible to the eye, whilst the numerous others which contribute to form the expression we are considering, although thrown into the back ground, are still by no means excluded. This species of insulation of the property whose consequences we wish to trace, enables the mind to apply that attention, which must otherwise be exerted in keeping it in view, to the more immediate purpose of tracing its connection with other properties that are the objects of our research. As an example of these ideas, I would mention the word government, upon which we may reason in many different directions, either as it secures domestic liberty, or protects from foreign attacks, as it discourages vice or promotes commerce: in these and in numerous other courses, our reasoning may be pursued, and the word government will constantly recur without the possibility of avoiding it but by the most tedious circumlocution, or of restricting the view in which it is regarded but by the most unwearied efforts of attention. The word function in analysis possesses a still more extensive signification than that which has been just mentioned:

$$\text{The sign } \psi\left(\overline{x}, \overline{\tfrac{1}{x}}\right)$$

signifies any symmetrical combination of the two quantities x

and $\frac{1}{x}$. That combination is in this mode of expressing it left arbitrary and undefined. The same function ψ may at the same time be a function of

$$\frac{x^2+1}{x}, \quad \frac{x}{x^2+1}, \quad \frac{x^4+x^2+1}{x^2},$$

or of a thousand other quantities; all which circumstances although deducible from the original expression, are not presented to the eye, because in the consequences which it is proposed to deduce, they are entirely immaterial.

If two circumstances in the nature of the function are jointly the ground on which any of its properties depend, they may be separated from the rest and made prominent by several methods.

Thus the index n subjoined to a function

$$\psi(\overline{x}, \overline{y})_n$$

may be defined to mean that it is homogeneous with respect to x and y and of the dimensions n: these two circumstances are the causes on which the truth of the following property depends.

xt being substituted for y

$$\psi(\overline{x}, \overline{xt})_n = x^n \psi(\overline{t}, \overline{1})_n.$$

In all our attempts at mathematical generalization, it is of great importance to discover and distinguish these immediate causes of successful operations; in almost all cases they lead us at once to the highest point of generality, and very frequently contribute in no inconsiderable degree to simplify the processes of the investigation. This advantage so peculiar to algebraic signs, has been remarked by M. Degerando, from whose writings I have derived much satisfaction by observing the support which many of those views that I had taken previous to my acquaintance with them, received from the reflections of that distinguished philosopher. " La troisième raison," observes M. Degerando, " est dans la propriété qu' à l'algèbre de ne saisir, dans les idées des quantités, que certains rapports généraux, de ne présentir ainsi à notre esprit que les considérations qui lui sont vraiment utiles dans les recherches auxquelles il se livre. De là il arrive que notre attention se trouve débarassée d'un grand nombre d'idées accessoires, qui etrangères au but de ses méditations, n'auroient servi qu' à la distraire*."

The quantity of meaning compressed into small space by algebraic signs, is another circumstance that facilitates the reasonings we are accustomed to carry on by their aid. The assumption of lines and figures to represent quantity and magnitude, was the method employed by the ancient geometers to present to the eye some picture by which the course of their reasonings might be traced: it was however necessary to fill up this outline by a tedious description, which in some† instances even of no peculiar difficulty became nearly unintelligible, simply from its extreme length: the invention of algebra almost entirely removed this inconvenience, and presented to the eye a picture perfect in all its parts, disclosing at a glance, not merely the conclusion

* Des Signes et l'art de Penser, p. 214. tom. II.

† The difficulty which many students experience in understanding the propositions relating to ratios as delivered in the fifth book of Euclid, arises entirely from this cause, and the facility of comprehending their algebraic demonstrations forms a striking contrast with the prolixity of the geometrical proofs.

A still better illustration of this fact is noticed by Lagrange and Delambre, in their report to the French Institute on the translation of the works of Archimedes by M. Peyrard.

It occurs in the ninth proposition of the 2nd book on the equilibrium of planes, on which they observe, " La demonstration d'Archimede a trois énormes colonnes in-folio, et n'est rien moin que lumineuse." Eutochius commence sa note " en disant que le theorême est fort peu clair, et il promet de l'expliquer de son mieux. Il emploie quatre colonnes du même format et d'un charactère plus serré sans reussir d'avantage; au lieu que quatre lignes d'algebra suffisent a M. Peyrard pour mettre le verité du theorême dans le plus grand jour." Ouvrages d'Archimede traduites par M. Peyrard, p. 415. tom. II.

in which it terminated, but every stage of its progress. At first it appeared probable that this triumph of signs over words would have limits to its extent: a time it might be feared would arrive, when oppressed by the multitude of its productions, the language of signs would sink under the obscurity produced by its own multiplication: had these expectations been realized, still its utility would have been extensive, and mankind, whilst they felt grateful for the many stages it had advanced them, must have sought some more powerful auxiliary for their ulterior progress. Fortunately however such anticipations have proved unfounded; in whatever department of analysis the number of symbols has encreased to a troublesome extent, contrivances have soon occurred for diminishing it without any sacrifice of perspicuity: the inconvenience has always been temporary, the advantage permanent.

In later times the generalization and contraction introduced by the use of signs, seems even to have outstepped the discoveries which have resulted from them; and reasoning from the past course of science to its future advances, we may fairly presume that our power of condensing symbols will at least keep pace with the demands of the science.

Examples of the power of a well contrived notation to condense into small space, a meaning which would in ordinary language require several lines or even pages, can hardly have escaped the notice of most of my readers: in the calculus of functions this condensati is carried toon a far greater extent than in any other branch of analysis, and yet instead of creating any obscurity, the expressions are far more readily understood than if they were written at length: the instance I shall choose as an example is the equation

$$\psi^{\overline{9,9}}(x, y) = \psi(x, y)\,*.$$

* Transactions of Cambridge Philosophical Society, Vol. I. p. 68.

To any person acquainted with the notation belonging to the calculus, it is instantly intelligible; yet if it were written out at length, the letters x and y would be each repeated 257 times, the letter ψ would be found 512 times, whilst the expression would also contain 257 commas and 512 pairs of parentheses; thus comprising in the whole 2307 symbols; and it may be added that it would require a much longer time to understand the meaning of the equation written out at length, than it would to find its general solution.

The power which we possess by the aid of symbols of compressing into small compass the several steps of a chain of reasoning, whilst it contributes greatly to abridge the time which our enquiries would otherwise occupy, in difficult cases influences the accuracy of our conclusions: for from the distance which is sometimes interposed between the beginning and the end of a chain of reasoning, although the separate parts are sufficiently clear, the whole is often obscure. This observation furnishes another ground for the preference of algebraical over geometrical reasoning, and is one which had not escaped the notice of Lagrange. "Chaque membre de phrase est claire et très intelligible a le considerer seul, mais le tout est si long qu'on a souvent oubliè le commencement quand on arrive a l'endroit ou le sens est complet."

The closer the succession between two ideas which the mind compares, provided those ideas are clearly perceived, the more accurate will be the judgement that results; and the rapidity of forming this judgement, which is a matter of great importance, inasmuch as the quantity of knowledge we can acquire in a great measure depends on it, will be proportionably encreased. M. Degerando has clearly stated this advantage in comparing the decimal arithmetic with that of the Romans. "La rapidité d'une operation intellectuelle est toujours en raison inverse des efforts qu'on demande a l'attention et a la memoire. Cette operation qui consiste a fixer les rapports de ses idées pour leur appliquer les même judgements s'executera donc autant plus promptement qu'il nous sera facile de nous rappeler et de remarquer ces rapports*."

The almost mechanical nature of many of the operations of Algebra, which certainly contributes greatly to its power, has been strangely misunderstood by some who have even regarded it as a defect. When a difficulty is divided into a number of separate ones†, each individual will in all probability be more easily solved than that from which they spring. In many cases several of these secondary ones are well known, and methods of overcoming them have already been contrived: it is not merely useless to re-consider each of these, but it would obviously distract the attention from those which are new: something very similar to this occurs in Geometry; every proposition that has been previously taught is considered as a known truth, and whenever it occurs in the course of an investigation, instead of repeating it, or even for a moment thinking on its demonstration, it is referred to as a known datum. It is this power of separating the difficulties of a question which gives peculiar force to analytical investigations, and by which the most complicated expressions are reduced to laws and comparative simplicity. One of the most elegant illustrations of this opinion I shall at present briefly allude to, as a more detailed account of it will be given in a subsequent essay. Among the papers left by the late Mr. Spence, is one on a method of solving certain equations of differences: elimination is the means by which he proposed to

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* Degerando sur les signes, Tom. II. p. 196.

† Of so much importance is this maxim, that it has been adopted by Des Cartes as one of his principles of philosophizing. "Diviser chacune des difficultés en autant des parcelles qu'il se pourrait et qu'il serait requises pour les resoudre." *Discours de la Methode.*

accomplish his object, but the results soon became so complicated that little expectation could be formed of succeeding by that means. In this difficulty Mr. Spence introduced into the equation to be solved an arbitrary quantity a, which is merely employed as a letter by whose powers the resulting series may be arranged: if the attempt is now made by continually eliminating, a series arises proceeding according to the powers of a, and equations are found for determining their coefficients: finally, the arbitrary quantity a having performed its office is made equal to unity, and the result is the solution of the equation. The success of this plan depends entirely on breaking into a number of separate parts a very complicated expression, each of these portions being separately reducible to known laws.

On resolving into their separate parts a vast variety of questions which have occurred, it has been found that the number of individual difficulties is by no means so large as had been originally supposed; many of very different kinds have been found to depend perhaps on the same integral, or on the solution of the same equation. In proportion to the number of questions which are reduced to these new difficulties, they themselves assume importance, and the celebrity which always attaches to those who remove obstacles regarded as insuperable by their predecessors, induces many to attempt the solution of these purely abstract questions. Perhaps these ultimate points of reference may not from their nature admit of a comparison with, or reduction to, existing transcendents: the labour and ingenuity employed in the attempt are not however thrown away; relations are discovered by which, from a certain number of particular cases numerically given, all others may be readily calculated, approximations are discovered for determining the cases which are required as data, and, finally, they are arranged in tables and accompanied by rules for their employment, by which, as far as results in pure numbers are required, all questions that are made to depend on them may be considered as solved.

The power which language gives us of generalizing our reasonings concerning individuals by the aid of general terms, is no where more eminent than in the mathematical sciences, nor is it carried to so great an extent in any other part of human knowledge. In the transition from Arithmetic to Algebra, when letters began to be substituted for numbers, the first step consisted rather in the circumstance of the possibility of operating on a quantity determined but unknown. Thus if it were proposed to discover such a number, that its square added to three should be equal to four times the number itself; we commence by supposing the number to be represented by x: now it is quite certain, as soon as the question is stated, that there can only exist two numbers fulfilling the condition; x therefore must in reality mean either of these two, and the rest of the process is

$$x^2 + 3 = 4x,$$
$$x^2 - 4x + 4 = 1,$$
$$x - 2 = \pm 1,$$
$$x = 3 \text{ or } 1,$$

To point out more clearly the force of this observation, we adopt the plan which Vieta introduced into Algebra, that of denoting known quantities by letters: instead of the numbers 3 and 4, let us use the letters a and b; then the process is as follows:

$$x^2 + a = bx,$$
$$x^2 - bx = -a,$$
$$x^2 - bx + \frac{b^2}{4} = \frac{b^2}{4} - a,$$
$$x = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a}:$$

here it is true that a and b meant 3 and 4, but as no part of the reasoning employed in any manner depended on their numerical value, the result must be independent of it, and is consequently true for all possible values. It may perhaps be contended that by the assumption of x for the number to be found, it was meant to represent number in the abstract, and that such was also the meaning of a and b; but there exists this difference, that it is not in our power to alter the value of x, but we may give to those of a and b any numerical magnitude we may please *.

The utility of the unknown quantities in Algebra, arises from their capability of being operated on without reference to the determined values for which they are placed, the advantage of employing letters for the known quantities, consists in their similarity to general terms in language, and the consequent extension of the reasoning from an individual case to a numerous species. The light in which this question has been regarded, is purely arithmetical, it may however be placed in another point of view, in which without any change in the quantities concerned, it is still more general in its nature; instead of restricting the equation

$$x^2 - bx = - a$$

to number, it may be considered as indicating that x is composed of a and b in such a manner, that when its value is substituted in that equation, all the terms shall mutually destroy each other. This signification, it is true, is not contained in the original question, but arises from the equation into which it is translated: the language of signs is far more general than that of arithmetic, a circumstance which is not perhaps sufficiently attended to in the application of it to questions of pure number. In one respect this generality is not so unexpected, for if a number is required satisfying a certain condition, and if it should happen

* There is in truth one restriction, namely, that a must always be less than $\frac{b^2}{4}$; but this will be removed when the question is viewed in an algebraical light, and does not in the least affect the argument.

that more numbers than one fulfil that condition, there is no reason why the answer should produce one of these numbers rather than another, it must therefore contain them all.

The reasoning which is carried on in Geometry is of a general nature, and applies to a species, although it is impossible that the picture presented to the eye can be any thing else than that of an individual; hence, it not unfrequently happens that some peculiarity in the figure which is actually employed, either leads us into erroneous conclusions, or when the results are correct, they are supposed to be limited by the individual nature of the figure we have employed. If a line is made use of to represent number, since some other line is the standard unit, it is impossible by such means to represent number in the abstract, but if number is denoted by a letter, there is nothing in the sign which at all indicates the magnitude of that which it represents: it is evident therefore that a property which might lead us into error in the first case, is removed from our view in the second. It will perhaps be objected that the standard unit need not be visible to the eye, since the force of the demonstration is in no way affected by its magnitude, this observation is perfectly correct, and if only one line be considered, and no unity of linear measure be stated, that line may represent length in general, and is to all purposes an arbitrary sign: but the moment any other line is introduced into the diagram, although the unit should not be mentioned, the generality of the former sign is diminished, a relation is instantly established, and whatever may be the unit of length, the ratio of those two lines is fixed and determinate.

The position of a line is another circumstance in Geometry which must always remain particular, and this brings with it that of the points formed by its intersection, as well as that of the angles formed by it with other lines, and the attention which the mind must exert to perceive that no part of the reasoning it is pursuing rests on any of these individualities, itself requires a considerable effort. The substances of these observations may be expressed in this conclusion. ***The reasonings employed in Geometry and in Algebra are both of them general, but the signs which we use in the former, are of an* * *individual nature, whilst those which are employed in the latter, are as abstract as any of the terms in which the reasoning is expressed.***

The signs used in Geometry, are frequently merely *individuals* of the *species* they represent; whilst those employed in Algebra having a connection purely arbitrary with the species for which they stand, do not force on the attention one individual in preference to any other.

An example of the limitation which geometrical considerations introduce, we shall select from a very well known author. In determining the relation between the rectangle under the parts of two lines intersecting each other and cutting a circle, Euclid considers separately the two cases of the point of intersection being situated within and without the circle, and he shows that in the two figures

the rectangle under CP and CQ, is in both cases equal to that under $CP_{,}$ and $CQ_{,}$: the case of one of the lines becoming a

* Halley's paper on the determination of the foci of lenses, would furnish a very apposite example of this principle, and probably few of my readers will fail to recollect instances where the same identical words of a proposition, and the same letters apply to two, three, or more different geometrical figures.

tangent is also a separate investigation. Now in the algebraic mode of treating these questions, the three cases are comprised in one formula*.

The indication of the extraction of roots by means of an appropriate sign, instead of actually performing the operation, is one of the circumstances which add generality to the conclusions of Algebra, and the same principle of indicating operations, instead of executing them, when employed with judgement, contributes frequently in no small degree to the perspicuity of the result, and sometimes enables us to read in the conclusion every stage which has been passed through in the progress towards it. Any general rules to direct us in the application of this principle will be difficult to form, because they ought in a great measure to depend on the objects we have in view: it may, however, be stated generally that it is improper to adhere to it, when by an opposite course any reduction or contraction can be made in the formula; thus generally speaking it would be better to write

$$y = \sqrt{(a-x)^2 + b^2},$$

than

$$y = \sqrt{(a+x)^2 - 4\,ax + b^2},$$

and on the other hand, wherever in the course of any reasoning the actual execution of operations would add to the length of the formula, it is preferable merely to indicate them.

Some of the advantages which arise from the use of letters to denote known quantities, have already been adverted to; but there are others of considerable value which may now be noticed, and which relate in a great measure to the higher departments of analysis. If a player bet a certain sum of money u, he may either win it and become possessed of $+u$, or he may lose it and possess $-u$. If we now suppose that he regulates the amount of his

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* Book III. Prop. 36. and 37.

second stake by the result of the first, and that he makes it $u-v$ if he had won the first, but $u+v$ if he had lost it; on this second bet he will either win or lose

$$u+v, \text{ or } u-v.$$

Supposing him to determine his third stake from his second, in the same manner as he fixed his second from his first, it is clear that according to the determination of his previous bets, he may stake on the third event either of the three sums

$$u+2v, \; u, \; u-2v.$$

And generally on the nth event, if he proceed according to this law, there are n different bets which he may make according to the order in which the previous ones were decided. Now in any question in which such a mode of play entered, it would be exceedingly tedious to consider separately all these cases, and to repeat the same or nearly the same reasoning for each individual case. This may be avoided by rendering the events indeterminate, for we then find his first profit may be denoted by

$$u(-1)^a,$$

in which the letter a represents any whole number whatever; if it is an even number he wins, and if an odd one he loses; the same artifice applied to his second stake gives for it

$$u-v(-1)^a,$$

as a is still undetermined, this will represent that stake truly, whichever event has happened on the first.

The result of this second stake may be represented by

$$\{u-v(-1)^a\}\,(-1)^b$$

whether it is lost or gained, and this is still kept undecided by means of the letter b.

The third stake will be

$$u-v(-1)^a-v(-1)^b,$$

and the profit or loss by that bet will be

$$\{u - v(-1)^a - v(-1)^b\}(-1)^c$$

a new undetermined letter being again introduced.

By these means his nth stake will be represented by

$$u - v(-1)^a - v(-1)^b - v(-1)^c - \ldots v(-1)^m;$$

and thus all the n cases become reduced into one expression.

The influence of this indeterminateness in contributing to the success of the solution of the following problem is very apparent.

Supposing a player to regulate his stakes in the manner just described, and that his chances of winning and losing are equal, what will he have profited after $p+q$ events, p of which have been favorable, and q have been unfavorable?

At a first view it might perhaps be considered that the data are insufficient for the solution, and that the order in which the events take place ought to be given: the result however of the investigation will prove that the profit or loss is entirely independent of that consideration.

It has already been shewn that his profit on each event will be as follows:

$$u(-1)^a,$$
$$u(-1)^b - v\{(-1)^a\}(-1)^b,$$
$$u(-1)^c - v\{(-1)^a + (-1)^b\}(-1)^c,$$
$$u(-1)^d - v\{(-1)^a + (-1)^b + (-1)^c\}(-1)^d,$$
$$\cdots\cdots\cdots\cdots$$
$$u(-1)^{(n)} - v\{(-1)^a + (-1)^b + \ldots .(-1)^m\}(-1)^{(n)}.$$

On examining the composition of that collection of terms which multiply v it will appear that the sum of all the above quantities is equal to

$$u\{(-1)^a + (-1)^b + \ldots (-1)^{(n)}\},$$

$-v \times$ {the sum of all the products two by two which can be formed out of the quantities $(-1)^a$, $(-1)^b, \ldots (-1)^{(n)}$}

and since p of the quantities a, b, c, d, n are even and q are odd, this last expression is equal to

$$(p-q)\,u - v \times \{\text{the coefficient of the third term of } (x-1)^p\,(x+1)^q\}$$

$$= (p-q)\,u - v\left\{\frac{p.\overline{p-1}}{2} - pq + \frac{q.\overline{q-1}}{2}\right\}$$

$$= (p-q)\,u - \frac{p^2 - p - 2pq + q^2 - q}{2}\,v$$

$$= (p-q)\,u - \frac{(p-q)^2 - (p+q)}{2}\,v.$$

If the order in which the events happened, or what corresponds to it, if the quantities $a, b, c, \ldots n$ had been given, the process we have gone through could not have been completed, for the perception of the nature of the quantity, which in the sum of all the profits multiplies v, depends entirely on preserving those quantities distinct and unconnected with each other; a relation quite impossible, had each been an individual number.

The remarkable influence of signs in the successful termination of this process of reasoning, claims our particular attention: abstract number, from its very nature, admits of amalgamation when subjected to the various operations expressed by algebraic signs: hence all trace of the mode in which it originally entered is completely lost; or if this inconvenience be studiously avoided, by merely indicating instead of executing the arithmetical operations, still the individual nature of the several numbers presents so many points to which the attention is attracted, that it would be almost impossible, even for the most attentive observer, to seize that general view in which they all agree. This influence is still more remarkable in investigations, where characteristics of operation occur, and when letters are used to the exclusion of number, the relations are not merely more apparent, but the results, although attained with difficulty, are more worthy of confidence: the reason of which, is to be found in this circumstance, that when letters only are employed, the functional characteristics convey no meaning except that on which the force of the reasoning depends; but, if numbers are used, they convey, besides this signification, a multitude of others, which distract the attention, although they are quite insignificant in producing the result.

This principle of preserving undetermined* until the conclusion, the quantities on which we are reasoning, seems to be the only one, which promises success in questions, where two parties successively make choice either of things or of situations: of this nature are many of those questions which relate to games dependent on skill. In these cases, the number of things, out of

* The profound remark of Lagrange, that the true secret of analysis, consists in the art of seizing the various degrees of indeterminateness, of which quantity is susceptible, has been so beautifully illustrated by M. Carnot, and so well accords with my own ideas on the subject, that I should be unwilling to present it to my readers in any language but his own.

J'ai ouï dire plusieurs fois à ce profond penseur, (M. Lagrange) que le véritable secret de l'analyse consistait dans l'art de saisir les divers degrés d'indetermination dont la quantité est susceptible; idée dont je fus toujours pénétré, et qui m'a fait regarder la méthode des indéterminées de Descartes comme le plus important des corollaires de la méthode d'exhaustion.

Un nombre abstrait est moins déterminé qu'un nombre concret, parceque celui-ci spécifie non seulement le combien du nombre, mais encore la qualité de l'object soumis au calcul; les quantités algébriques sont plus indéterminées que les nombres abstraits parce qu'elles ne spécifient pas même le combien: parmi ces dernières les variables sont plus indéterminées que les constantes, parce que celles-ci sont considérées comme fixes pendant un plus longue période de calcul; les quantités infinitésimales sont plus indéterminées que les simples variables, parce quelles demeurent encore susceptible de mutation, lors même qu'on est déjà convenu de considérer les autres comme fixes; enfin les variations sont plus indéterminées que les simples différentielles, parce que celles-ci sont assujéties à varier suivant une loi donnée, au lieu que la loi suivant laquelle on fait changer les autres est arbitraire. Rien ne termine cette échelle de divers degrés d'indétermination, et c'est précisément dans cet assemblage de quantités plus au moins definies, plus au moins arbitraires, qu'est le principe fécond de la methode générale des indéterminées, dont le calcul infinitésimal n'est véritablement qu'une heureuse application. *Carnot, Reflexions sur la Metaphysique du Calcul Infinitesimal,* 2d edit. p. 207.

which the choice is to be made, is always finite, and the second player can only make his selection out of that number diminished by unity. The first player, at his second stroke, can only choose out of the same number diminished by two, and so on. Now it is evident, that if the individual actually chosen at each step, were fixed permanently, the reasoning must diverge into an immense multitude of cases; whereas, if any one indifferently is chosen by the first, and again, any one indifferently out of the $k-1$ remaining ones by the second, and so on, the things chosen by the two parties would all be comprehended in two expressions.

It is this power of representing any one of $k-p$ things, where p have already been taken out of k, (subject only to the condition that the expression for it can never represent any one of those already chosen,) which enables us to delay the decision of the individuals actually selected until the conclusion; and thus by their means, to satisfy the other conditions of the problem. Several instances of such questions, will be noticed in a future paper: The only means that have hitherto been employed for this purpose, are the roots of unity, and the sines, and other similar functions of submultiples of π, and from the great length of the formulæ in which they occur, I am by no means sanguine in my expectation of much success in those enquiries, until some more condensed method of indicating and to a certain extent also of executing such operations, shall have been contrived. The principle in discussion, appears to me to be the only one, by which any general and complete solutions can be arrived at: more partial views may doubtless be taken, more adapted to the present state of symbolic language, and even these become extremely valuable, in such difficult enquiries, not merely from the advances they themselves introduce, but as the ground-work of generalization to more perfect methods.

The principle of representing any one quantity indifferently, out of a given number, has been employed on several occasions, and occurs in some mechanical questions. Lagrange has made use of it in a memoir relative to the theory of Sound *, and M. Poisson has employed it on a similar occasion †. The cause, on which its successful application depends, seems to be the power which it gives of uniting together a number of cases totally distinct, and of expressing them all by the same formula, the consequence of which is, that one single investigation frequently supersedes the necessity of a multitude.

Many examples of the successful application of this principle, are to be found in a paper on Circulating Functions ‡, in which it is applied to the solution of a peculiar class of equations of finite differences: other instances will come under our notice, in a future communication.

An examination of the various stages, by which, from certain data, we arrive at the solution of the questions to which they belong, ought to constitute a prominent feature, in any work devoted to the philosophical explanation of analytical language. It is a subject, the consideration of which is too frequently omitted altogether, and as a correct view of it tends materially to advance the science, and also by pointing out more clearly the nature of the difficulties it has to contend with, and also by guarding us against those openings by which errors are most frequently introduced, I shall enter into the question at some length.

In the application of analysis to the various questions which are submitted to it §, there are three distinct stages, each subject

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* Memoirs de Turin, vol. I. † Journal de l'Ecole Polytecnique Cah. 18.

‡ Phil. Trans. p. 144. 1818. J. F. W. Herschel, Esq.

§ Of course questions already in an algebraic form are not here alluded to, such as the integration of equations, &c.

to its particular rules, and each liable to its peculiar difficulties. The order in which these succeed each other, will prescribe that which will be followed in the remarks upon them.

I. The first stage consists in translating the proposed question into the language of analysis.

II. The second, comprehends the system of operations necessary to be performed, in order to resolve that analytical question into which the first stage had transformed the proposed one.

III. The third and last stage consists in retranslating the results of the analytical process into ordinary language.

I. In the first step, which consists in translating the proposed question into the language of analysis, much caution is requisite: for unless this is correctly done, it is quite manifest that the labour bestowed on the remaining stages is useless. It is at this point that the principles applicable to the question are employed for its solution; in all the succeeding parts they are kept totally out of sight: the termination of this stage is usually marked by the circumstance of the difficulty being reduced to one or more equations*, which is the case in most of those questions that arise in the application of analysis to physics: but this however does not always happen, since very many questions relating to chances are reduced to the finding of the coefficient of a certain term in the developement of a given function into a series, and although most of these may be reduced to equations of differences, yet it is not universally the case. It is of some importance to observe, that those errors most difficult to detect and remove,

* The term equation in this place must be understood to comprehend such expressions as $A-B >$ or $< C$.

usually occur in this first stage; and it cannot be too strongly recommended, that every part even of the most difficult problems should be fully translated into the language of analysis, before any attempts at simplification are made. In the first stage, it is scarcely possible to see clearly in what degree the results will be affected by a proposed omission; whilst in the second, any quantity which it is conjectured will have little influence on the result, although it adds greatly to the difficulty of calculation, may be kept separate, and the operations to which it is submitted, may be indicated rather than performed. In many of the applications of analysis, and particularly in its treatment of mechanical questions, the principles which regulate the first stage of the process are completely known, and little difficulty is experienced in translating them into the language of signs, the difficulties when they occur, usually taking their rise in the solutions of the equations thus produced. A similar remark is applicable to optical questions, and indeed to by far the greater part of those which occur in the mixed sciences.

II. The second stage in the solution of any problem, generally begins with the equations into which it has been translated, and terminates with their solution. The point at which it commences is not always so well defined as that at which it ends, and this is more particularly the case when the question relates to geometrical figures, where in some instances, the first and second stages are much intermixed.

The difficulties which now occur are purely analytical, and are generally such as have been treated of in works devoted to the subject. The solution of one or more algebraic equations is frequently the object to be obtained: differential equations, or equations of finite differences are another class of analytical expressions to which physical problems are often reduced; many of these can only be resolved approximately, but in proportion to the interest these questions have excited, the variety and accuracy of the approximations have been multiplied. This is strongly exemplified in the problem of the three bodies, as applied to determining the place of the Moon: the great importance of the question has caused the approximations to be pushed to such an extent, that they have arrived at a degree of inelegance and complexity, which would long since have caused them to be rejected from any other question on the exact solution of which less important interests depended. But on this second stage in the solution of a question, it is less necessary to add many observations; the operations which are concerned in it, and the modes of effecting them, being more fully treated of in works of instruction than either of the others.

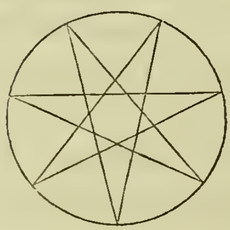
III. The last of the stages into which the resolution of a question has been distributed, has been more neglected than any other. It may perhaps appear singular that the answer to a question, which is of course the great object of research, should have been passed over without sufficient attention. It is not however of any errors in those results which are usually arrived at that I complain; but it is, that sufficient instruction is not given in elementary works, as to the full meaning of all the different circumstances which are contained in the result that analysis has presented. In those questions which lead to algebraic equations, it is not unfrequently the case, that some one or perhaps two roots are taken as the answer, whilst all the remaining ones are completely neglected. Now a question can never be said to be fully answered until every root of the equation to which it has conducted has been discovered, and its signification with reference to the data of that question been explained. It sometimes happens that superfluous roots have been introduced by the algebraic operations that were necessary to arrive at the final equation: these ought to be pointed out, and the step at which they were introduced should be noticed, and also whether they can admit of any translation into the language of the problem considered. Imaginary roots are very frequently introduced; they sometimes imply impossibility or contradiction amongst the data; their origin ought to be carefully traced, and such a course will frequently make us acquainted with the maxima and minima which belong to the question. It is still more necessary to attend to all the real roots whether positive or negative, and to explain the various circumstances in the solution to which they refer.

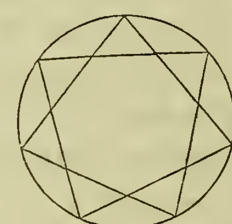
It is by no means uncommon with algebraical authors, when they have led their readers through a process which terminates in an equation, to select that root which gives the answer they require, without explaining the signification of the other roots that are equally comprised in it; and this incomplete mode of solution, which is censurable from revealing only a part of the truth, has in some instances caused the most interesting circumstances attending a question to be entirely overlooked. A singular example of this occurs in several authors who have sought analytically the side of a heptagon inscribed in a circle, or the radius of a circle which would circumscribe a regular heptagon whose side is given. In neither of these questions can the equation to which we are led, be reduced below the third degree, and the three roots of the cubic are always real: the largest of the positive roots gives the answer to the latter of these questions for the common heptagon of Euclid: but no reason is stated why this root should be considered as the true answer to the question in preference to either of the others. In the Analytical Institutions of M. Agnesi *, where the first problem is solved, no

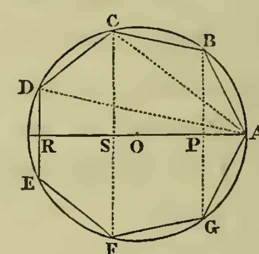
* Vol. I. p. 168. English Translation.

notice is taken of the fact that all the three roots are possible, nor am I aware of its being noticed by any author who has treated of this question: had it been observed and enquired into, the existence of three species of heptagons answering strictly to the definition, and the knowledge of the star-shaped polygons which were discovered by M. Poinsot, could not have remained so long unknown.

If $x = OP$, denote part of the diameter intercepted between the







centre, and a perpendicular from the extremity of the first side of the heptagon, then the usual trigonometrical formulæ give

$$x^6 - \frac{5}{4}x^4 + \frac{3}{8}x^2 - \frac{1}{64} = 0,$$

this contains six roots, of which the three positive are (abstracting the signs) equal to the three negative: it may be resolved into the two factors

$$\left(x^3 - \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{8}\right)\left(x^3 + \frac{1}{2}x^2 - \frac{1}{3}x - \frac{1}{8}\right) = 0,$$

the second of which is only the first with the signs of its roots changed. The three roots are real and are represented by

$$x = OP, \ x = OS, \text{ and } x = OR,$$

the first of these gives AB for the side of the heptagon, this is the same as that which has long been known; the other two roots give AC and AD, as the sides of the polygon, and by carrying them round the circle, the two star-shaped heptagons, (Figures 2 and 3,) are produced which have no re-entering angles, and which are in fact comprehended in Euclid's definition of regular polygons. The sum of the interior angles of the first of these heptagons is ten right angles, the sum of the interior angles of the second is six right angles, and that of the third species is two right angles. These new species of polygons were first noticed by M. Poinsot, in a highly interesting memoir on subjects connected with the *Geometry of Situation*, read before the Institute in 1809, and subsequently printed in the *Journal de l'Ecole Polytecnique*, 10ᵉ Cah.

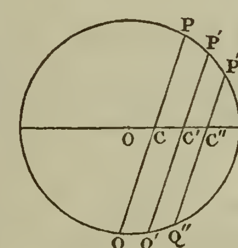
Another important business which belongs to this stage of the question, is to examine carefully what changes will ensue from supposing any peculiar relations amongst the data; or from any of the constant quantities becoming infinite or evanescent, such circumstances frequently introduce great simplicity, and when they refer to geometrical questions, are sometimes the means of making us acquainted with general properties, by which the construction of the problem is greatly facilitated.

A careful and laborious attention to all the possible modifications of a problem which might result from any relation amongst its data, was considered by the ancient Geometers as an indispensible part of its investigation, and the manner in which this was accomplished, was generally little else than a repetition of the whole process under the altered circumstances: when the data are numerous, the length of such a system of operations becomes intolerable, and if more rapid methods had not been contrived, Geometry must have become stationary from the accumulation of the details with which it was thus encumbered. Many instances of the extreme length to which a full investigation of comparatively a very simple problem will lead, occur in the treatises *De Sectione Rationis,* &c. The advantage of Algebraic language is in this respect very striking: all the data of the questions are embodied in the equation in which its solution terminates, and without repeating any part of the process by which that was produced, we can examine with ease all those modifications which any differences in the actual magnitude of the data can introduce into the question under consideration: and moreover the equation itself will suggest to us such relations amongst those quantities as will have the effect of lowering the number of its dimensions, or of rendering it the product of two or more factors.

It sometimes happens that by a peculiar relation amongst the data of a question, the number of solutions instead of being limited becomes infinite: thus, if the position of a line is determined by two points, when those points coincide, any line passing through the point in which they coalesce, will satisfy the conditions of the question which becomes to a certain extent indeterminate: this gives rise to a class of propositions in Geometry which are called porisms. When the data on which questions depend are numerous, it is by no means so easy to discover by Geometrical considerations that relation amongst them in which the question becomes indeterminate, as it is by an Algebraical inquiry where the solution is presented in its most condensed form: one consequence of this is, that such cases have frequently escaped the notice of those who have treated the problems to which they belong in a Geometrical manner. One celebrated and important oversight of this kind occurred in a problem which Newton solved in order to determine the orbit of a Comet.

Having four lines given in position, it was required to draw a fifth line which should be cut by the other four into segments, having a given ratio to each other. Of this question Wren, Wallis, and Newton had given solutions, but when Zanotti, Boscovich and other Astronomers made use of them, employing the observed places of a Comet, the results were found greatly erroneous. Boscovich inquired into the reason of this singular result, and having first assured himself of the accuracy of the solutions, he discovered that in a particular relation between the given lines the problem became indeterminate, and admitted of an infinite number of solutions, and that the case of a Comet approached extremely near to this, and consequently that any very small error in observations must produce an extremely large one in the result.

As an instance of the curious and elegant properties to which such an examination of the relations of the data contained in the final equation sometimes leads, I shall propose the following problem.



A circle whose radius is r *being given, and also three points in one of its diameters, at what angle must three parallel chords be drawn through these points so that the sum of the squares of two of them shall be equal to a given multiple* n *of the square of the remaining one?*

Let the distance of the three points in the diameter from the centre be

$$v, \; v_1 \text{ and } v_2,$$

and calling the angle which is sought θ, we have

$$CP = -v \cos\theta + \sqrt{r^2 - v^2 (\sin\theta)^2},$$

and

$$CQ = +v \cos\theta + \sqrt{r^2 - v^2 (\sin\theta)^2},$$

hence

$$PQ = 2\sqrt{r^2 - v^2 (\sin\theta)^2},$$

and similarly for the other two chords

$$P_1Q_1 = 2\sqrt{r^2 - v_1^2 (\sin\theta)^2},$$

$$\text{and } P_2Q_2 = 2\sqrt{r^2 - v_2^2 (\sin\theta)^2}.$$

These values of the chords being employed give

$$4\,(r^2 - v \sin\theta^2) + 4\,(r^2 - v_1^2 \sin\theta^2) = 4\,n\,(r^2 - v_2^2 \sin\theta^2)$$

At this step the first of the three stages which have been described terminates; the question is now translated into the language of Algebra, and must be treated according to its rules: the following reductions must then be made

$$r^2 - v^2 \sin\theta^2 + r^2 - v_1^2 \sin\theta^2 = nr^2 - nv_2^2 \sin\theta^2$$

$$(nv_2^2 - v_1^2 - v^2) \sin\theta^2 = nr^2 - 2\,r^2,$$

$$\sin\theta = \pm\, r \sqrt{\frac{n-2}{nv_2^2 - v_1^2 - v^2}},$$

The second stage is here concluded by the solution of the equation to which the first conducted us, and we have now to explain the meaning of its two roots, and the modifications which may arise from any peculiar relations amongst the data.

The two signs signify that the angle θ, may be measured either above the diameter or below it, as is apparent on inspecting the figure. As the result contains an even radical, we must enquire if in all cases a solution is possible, and if not, what are the conditions of possibility. For this purpose we observe that the numerator and denominator must be both positive or both negative, consequently

$$n - 2 > 0, \text{ and } nv_2^2 - v_1^2 - v^2 > 0,$$

or

$$n - 2 < 0, \text{ and } nv_2^2 - v_1^2 - v^2 < 0, \text{ and also } \sin\theta < 1,$$

are the two sets of conditions; in all other cases the question is impossible.

From this, however, must be excepted the case of the numerator and denominator simultaneously vanishing in consequence of the following relations taking place amongst the data,

$$n - 2 = 0, \text{ and } nv_2^2 - v_1^2 - v^2 = 0,$$

whence

$$n = 2, \text{ and } v_2 = \pm \sqrt{\frac{v_1^2 + v^2}{2}},$$

under which circumstance the value of sin θ, becomes really indeterminate, not depending even on the value of r the radius of the circle.

This indeterminate case suggests the following porism:

Any three points in a straight line being given, another point may be found about which as a centre if a circle with any radius be drawn, and if through the three given points, three chords be drawn in any direction, but parallel to each other; then the sum of the squares of two of them shall be always equal to the square of the third.

It is to be observed, that the origin of the lines denoted by $v, v_1 v_2$, may be changed by removing it to the distance a, then the latter of the two conditions which rendered the problem indefinite becomes

$$2(v_2 + a)^2 = (v + a)^2 + (v_1 + a)^2,$$

whence

$$a = \frac{2v_2^2 - v_1^2 - v^2}{2v + 2v_1 - 4v_2}.$$

Before I conclude my observations on this subject, which may perhaps be considered as a digression from that which the title prefixed to this Essay would seem to imply, I shall offer one more illustration of the division of a problem into the several stages which I have pointed out.

This examination of all the circumstances attending the equation containing the solution, is still more necessary when that equation is a differential one: if it be only capable of integration by means of transcendents or by approximating series, it sometimes happens that some relation amongst the data may be assumed, by which in the one case the transcendents shall disappear, and in the other, that the series shall terminate. Euler has taken advantage of the former of these circumstances, to discover curves whose indefinite quadrature should depend on a given species of transcendent, whilst the areas of particular portions of them are susceptible of an Algebraic expression*. The integration of the equation is not always sufficient for a complete analysis of a question, for in some cases besides the general integral, there exists another not included in it, which is known by the name of a particular solution; in order to be secure of not overlooking any such, it must be observed that a change in the magnitude of an index may cause the introduction of such a solution. When the complete integral as well as all the particular solutions are found, the interpretation of them according to the circumstances of the question is not always an easy task, nor are any general rules yet established to which we can refer for information. In the theory of curves the interpretation of particular solutions is sufficiently well known: they represent the curve which touches all those formed by the complete integral when its parameter varies. In mechanical questions a considerable degree of uncertainty prevails relative to these kind of solutions, as in some instances they seem to have no reference to the problem which gave rise to them, whilst in other cases its solution can only be fully represented by their assistance; some light has been thrown on this subject by M. Poisson† in a memoir in which he has explained the theory of particular solutions with great perspicuity.

Of whatever kind the equation to which our question conducts us, may be, it ought to be regarded, merely in an analytical point of view; and all its various roots or solutions, should

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* (Euler Acta Acad. Petrop.)

† Journal de l'Ecole Polytecnique, Cah. 13. p. 60.

be sought after; out of these, by means of some peculiarity in the problem, we must select that individual, which more immediately satisfies the particular view of it which we have taken, and the other solutions must be explained if possible, by means of the data, from which we commenced the process; or should that be impossible, their entrance must be traced to some generalization in signs, to which the language of the question was incapable of adapting itself. In the demonstration of the composition of forces, given in the Mecanique Cœleste, which has sometimes been unjustly censured on account of its analytical nature, this does not appear to have been completely attended to. In the enquiry, to which I refer, x one of the forces is assumed equal to $z\phi(\theta)$, where z is the resultant force, and $\phi(\theta)$ some function of the angle between it and the force x, which function it is required to determine; by changing x into y and θ into $\frac{\pi}{2}-\theta$, the two values of x and y are found to be

$$x=z\phi(\theta), \quad y=z\,\phi\left(\frac{\pi}{2}-\theta\right),$$

and the equation

$$x^2+y^2=z^2$$

is arrived at: this equation in fact amounts to

$$[\phi(\theta)]^2+\left[\phi\left(\frac{\pi}{2}-\theta\right)\right]^2=1,$$

which results* from it, by merely substituting for x and y, their values.

* This equation is one of that class whose general solution I have ascertained, and it may be exhibited in either of the following forms

$$\phi(\theta)=\sqrt{\frac{\phi_1(\theta)}{\phi_1(\theta)+\phi_1\left(\frac{\pi}{2}-\theta\right)}}$$

$$\phi(\theta)=\sqrt{\tfrac{1}{2}+\left(\frac{\pi}{2}-2\theta\right)\bar{\chi}\left(\theta,\frac{\pi}{2}-\theta\right)},$$

From this equation it appears to me, to be the direct course to deduce the form of ϕ: its general solution should first be shown, and then from the peculiar circumstances of the problem, that particular one which belongs to it should be pointed out. In the work, to which I refer, the particular form of ϕ has been deduced at once by properties peculiar to the problem, without any reference to the general solution of the equation. A similar objection may be made to other demonstrations of this celebrated theorem: the equation to which the investigation conducts, is usually solved in a manner not sufficiently general. This is the case in a work devoted to the analytical exposition of the elements of Geometry, pp. 53, 54;* the substitutions employed, although satisfying the conditions, not containing all possible solutions. M. Poisson has given an investigation of this theorem not quite so open to the objections just stated; by the introduction of two variables and the employment of one sign of function, the solution is necessarily more restricted in its extent. Equations of that class are frequently contradictory, although in the case referred to, a fortunate property leads directly to the solution. See Poisson, *Mecanique*, p. 14. I cannot conclude this slight criticism on a detached passage of the Mecanique Cœleste, without expressing that respect for its illustrious author, which is shared with all those, who are capable of appreciating the important additions he has made to mathematical science, or who have the happiness of being personally acquainted with him.

When any question leads to an algebraic equation, it is usual to resolve it generally, and then to point out amongst its roots that particular one which is sought; if the individual root re-

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where ϕ_1 is perfectly arbitrary, and $\overline{\chi}$ is any symmetrical function of θ and $\frac{\pi}{2} - \theta$.

* Precis d'une nouvelle methode pour reduire à de simples procedès Analytiques la demonstration des principaux theorêmes de Geometrie. Par. I. G. C. Paris, An. vi.

quired were discovered by some artifice without the solution of the equation, we could not feel assured that it alone fulfilled all the conditions, and we should, by arriving at an equation and rejecting its use, have the semblance of generality without its reality; nor do I perceive any reasons which should induce us to change our course, when we have to consider equations of a more comprehensive nature.

Any enumeration of the causes which contribute to give such extensive power to the employment of algebraic signs, would be justly considered incomplete, if no notice were bestowed on the symmetry that ought always to prevail, where the calculations in which it is employed are in any degree complicated.

In its least restricted signification, symmetry is applied to two things, which although sometimes connected, are yet, in many instances, totally independent: it either refers to a resemblance between the systems of characters assumed to represent the data of a question, or it implies a similarity of situation, between certain of the letters, which are found in an analytical formula. An attention to it in either of these senses, has a direct and very beneficial influence in relieving the memory from a considerable burthen. In the first case, its precepts would direct us to assume similar letters as the representatives of similar things: thus, if we have two series, and propose to find another, which consists of the product of the corresponding terms of the two former; if we assume for the first of two series

$$a+b+c+d+\ldots$$

the terms of the second ought to be

$$A+B+C+D+\ldots$$

or which is still more convenient

$$a'+b'+c'+d'+$$

and the series, whose sum we wish to find is then denoted by

$$aA+bB+cC+dD+..$$

or by

$$aa'+bb'+cc'+dd'+..$$

The assumption of

$$A_1+A_2+A_3+..$$
$$A_1'+A_2'+A_3'+..$$

would have been equally proper, and the result equally clear: but had we assumed for the two series

$$a+b+c+d+..$$
$$A_1+A_2+A_3+A_4+..$$

the resulting series

$$aA_1+bA_2+cA_3+dA_4+..$$

would have been devoid of that symmetry, which forms so prominent a character in the former cases.

The plan of accenting letters, in order to represent quantities which stand in similar relations, adds, when employed with discretion, much to the perspicuity of the formulæ in which it is used; but like many other innovations, whose tendency is on the whole decidedly beneficial, an attempt to extend it beyond its proper limits, has been productive of inconveniences as considerable as those which its introduction was proposed to remove. Indices in various positions have been substituted in many cases for the system of accentuation, and the admirers of this scheme, pursuing it with equal ardor, have not been more fortunate in avoiding the confusion, which a multitude of signs, differing but by the slightest shades, can scarcely fail of producing. The taste of the geometer is not less strongly tried by the choice of the letters in which he conducts his reasoning, than his skill and ingenuity are by the artifices he invents to surmount the difficulties opposed to him; in the one case, the elegance which a judicious selection produces, carries the reader pleasantly and almost imperceptibly through an abstruse calculation, whilst the latent cause, which gives facility to his progress, is rarely appreciated, because it is spread uniformly over the whole question: in the other, whose essence often consists in some happy substitution, which is always concentrated in some point, the effect is too remarkable to escape the least attentive enquirer, and its success too striking not to command his admiration.

Unlimited variety in the use of signs, is as much to be deprecated as too great an adherence to one class of them; the one conceals the appearance of relations that really exist, whilst the other, affecting to display them too clearly, fails from its want of distinctness. It is difficult to point out models of imitation, even amongst the most eminent; the same writer, who at one time might be safely trusted as a pattern of correct judgement, has indulged at another in the most unexampled innovation; completely setting at defiance many of those principles, whose authority I have endeavoured to establish. The fate, which has attended the greatest of these proposed reformations, though sanctioned by the illustrious name of Lagrange, is no slight testimony of the validity of those views, which it is the object of several of these Essays to advocate*. Having delivered a theory of the differential calculus, which rested on principles entirely new, he introduced it to the world, clothed in a language of his own creation. The value of the present was too great to allow of its utility being impeded by the garb with which it was encumbered; and the labor of acquiring the language was compensated by the truths which it revealed. Time, however, and experience, convinced even its immortal author, that the language of signs rests

* This relates more particularly to an Essay on the Principles of Notation, which is not yet published.

on principles, which cannot be neglected without danger, or violated with impunity;—the authority of the greatest geometer of the age, failed to make converts to the language he had invented, whilst the justice of the view he had taken, was admitted, and his explanations almost universally adopted. Whilst the language, in which the Theory and the Calculus of functions are conveyed, is pointed out as a warning, not to be neglected by the most successful, that of the Mechanique Analytique of the same author, may perhaps be held up to imitation, with fewer limitations than any other work of equal magnitude. In returning to the notation of Leibnitz, Lagrange has in this work, reduced the whole theory of mechanics to the dominion of pure analysis, and in the choice of his symbols has frequently displayed that happy selection, which so much facilitates the process of reading and comprehending analytical formulæ.

The value of symmetrical symbols is greater in proportion to the complexity of the operations, and the number of quantities, which are concerned; but unless their selection is attended to at the outset of our studies, it is not to be expected that a correct taste can be acquired, I would therefore recommend a degree of attention to this subject, which is not usually bestowed on it by elementary writers. Some instances I shall select from the simplest applications of Algebra to Geometry. The equation of a right line is usually written thus;

$$y = ax + b,$$

which is sufficiently convenient. M. Biot in his Geometrie Analytique * has employed this notation, as also has M. Hachette in his Introduction to the admirable work of Monge †: both authors in treating of lines referred to three co-ordinates have denoted it thus:

$$y = ax + \alpha,$$

* P. 30. † Application de l'Analyse a la Geometrie, 4me ed. p. 2.

the difference is apparently trivial, but the convenience or inconvenience of notation frequently depends on differences as trifling. It may be observed, that in the first equation, a denotes the tangent of an angle, and b an absolute line; two things which have no relation to each other, and which are therefore justly represented by dissimilar signs. In the second equation, the line and the angle, are both represented by the *same letter* of different alphabets; a circumstance, which will infallibly suggest some idea of a relation that does not exist. When two straight lines enter into the question, other reasons present themselves: they may be represented in any of these four ways;

$$\begin{array}{llll} y = a x + b & y = a x + \alpha & y = a x + b & y = a x + b \\ y = a'x + b' & y = b x + \beta & y = \alpha x + \beta & y = c x + d \end{array}$$

and if we seek the ordinate of their point of intersection it will be

$$y = \frac{a b' - a' b}{a - a'}, \quad y = \frac{a\beta - b\alpha}{a - b}, \quad y = \frac{a\beta - \alpha b}{a - \alpha}, \quad y = \frac{a d - c b}{a - c}:$$

the latter of these expressions is quite devoid of all symmetry in regard to its letters, and the larger the number of lines about which we reason, the more confused will such a mode of expressing them render the result. In the first and third mode, it is sufficient to remember that the letter a, under all its forms, represents the tangent of an angle, and that the letter b, in every form, always represents a particular ordinate: with this principle in our mind, we can see at a glance, however numerous the lines introduced, to what property of them each individual letter refers; whereas in the last method, we must, in order to discover the meaning of any letter, refer back for each individual one, to the original translation into algebraic language. The third plan will suffice, where only a few different lines are concerned, but its application is limited by the smallness of the number of different alphabets we can command. The second method may be defended on the ground, that the tangents are denoted by one class of letters;

namely, Italics, whilst the Greek letters are reserved for lines; perhaps it might still be improved by interchanging these significations of the two alphabets.

Before I pass on to the consideration of the second species of symmetry, I shall select from the Arithmetica Universalis, an example, in which the choice of the letters employed seems to have been made without any rule; and shall subjoin to it, the same problem expressed in a language consistent with the views I am illustrating. This course will render more apparent the advantages of attending even to the letters which we select to represent the quantities.

"The velocities of two moving bodies A and B being given, and also their distance, and the difference of the times of the commencement of their motion, to determine the point in which they will meet.

Let A have such a velocity that it will pass over the space c in the time f; and let B have such that it will pass over the space d in the time g, and let the interval between the two bodies be e, and that of the times when they begin to move be h.

Case I. Then if both move in the same direction, and if A is farther distant from the point of meeting, call that distance x, from this take away the interval e, and there will remain $x-e$ for the distance of B, from the same point. And since A passes over the space c in the time f, the time in which it will pass over the space x will be $\frac{fx}{c}$.

And so also, since B passes over the space d in the time g, the time in which it will pass over the space $x-e$, will be $\frac{gx-ge}{d}$. Now since the difference of these is supposed to be x, in order that they may become equal, add h to the smaller time; namely, to the time $\frac{fx}{c}$ (if B moved first) and it will become

$$\frac{fx}{c} + h = \frac{gx - ge}{d},$$

and by reduction

$$x = \frac{ceg + cdh}{cg - df} = \frac{ge + dh}{g - \frac{d}{c}f},$$

but if A began to move first, add h to the time $\frac{gx - ge}{d}$ and it will become

$$\frac{gx - ge}{d} + h = \frac{fx}{c},$$

and by reduction

$$x = \frac{cge - cdh}{cg - df}.$$

CASE II. If the bodies move in opposite directions, and if x is, as before, the distance of the body A from the point of meeting, then $e - x$ will be the distance of B from the same point, and $\frac{fx}{c}$ the time in which A will pass over the distance x; and $\frac{eg - gx}{d}$ will be the time in which B will perform the distance $e - x$. To the less of these times add the difference h, namely, to $\frac{fx}{c}$ if B first began to move, and thus we shall have

$$\frac{fx}{c} + h = \frac{eg - gx}{d},$$

and by reduction

$$x = \frac{cge - cdh}{cg + df}.$$

If A began to move first, add h to the time $\frac{eg - gx}{d}$, and it will become

$$\frac{eg - gx}{d} + h = \frac{fx}{c},$$

and by reduction

$$x = \frac{ceg + cdh}{cg + df}.$$

This same question with the following data may be solved in nearly the same way,

v = velocity per second of A,

v' = ... B,

s = the space one (A) is distant from the other B,

t = the time in seconds one (B) starts before the other.

Since the velocities are both positive, the bodies move in the same direction, and x being the distance of the point where they meet from the place of A, the number of seconds which A will take to pass over it, will be found from the ratio

$$v : 1'' :: x : \frac{x}{v},$$

the distance of the other body from the same point will be $x - s$, and the time it takes to arrive will be $\frac{x-s}{v'}$, but the other body began to move t seconds before B, therefore t added to the time of its motion must equal $\frac{x-s}{v'}$, or

$$\frac{x}{v} + t = \frac{x}{v'} - \frac{s}{v'}:$$

hence

$$t + \frac{s}{v'} = x\left(\frac{1}{v'} - \frac{1}{v}\right),$$

and

$$x = \frac{\frac{s}{v'} + t}{\frac{1}{v'} - \frac{1}{v}} = v\frac{s + tv'}{v - v'}.$$

If A move in an opposite direction, its velocity must be accounted negative; hence in that case v' becomes $-v'$, and

$$x = v\frac{s - tv'}{v + v'}.$$

If A begin to move after B, the time t must be made negative, and then these two cases become

$$x = v\frac{s - tv'}{v - v'},$$

and

$$x = v\frac{s - tv'}{v + v'},$$

the former of these referring to the case of the bodies moving in the same direction, and the latter to that of their direction being opposite.

There are some restrictions which ought to be noticed, if the velocities of the two bodies are equal, or $v' = v$ the first and third cases show that the bodies can never meet. To this there is however an exception, if $v' = v$, and also $s = tv'$, then $x = \frac{0}{0}$, an indefinite expression and x may have any value: the signification of this is that both bodies move in the same direction and with equal velocities, since $v' = v$, and that the hindermost *B*, which starts t seconds before the other, is situated at such a distance s from it, that it arrives at the point where the other is, exactly as it begins to move; this appears from the equation $s = tv'$, it is therefore obvious that the two bodies will be at the same point at every part of their progress and for every value of x. Whenever x is negative they can never arrive at the same point in the direction in which they move. If however we conceive that they had been moving at the same rate prior to the point of time at which we consider them, the negative value assigned to x marks a point through which they both passed at the same moment.

These two modes of translating the same question into Algebra, and of re-translating the result into ordinary language, give rise to several observations. The assumption of v to represent the velocity of one body per second, instead of the plan pursued in the first solution, was productive of two advantages: first, it substituted one letter instead of two; and secondly, it is so usual in all mechanical problems to make that letter denote velocity, that it is in such cases associated with it in the mind. The next assumption of v' for the velocity of the other body, possessed both these advantages, and tended to make the result more apparently symmetrical in case it was susceptible of that species of arrangement. The selection of the letters s and t, to represent space and time, was adopted with a similar view of making the signs recal the thing signified. In pure analysis there is but little room for taking advantage of this species of connection, but in all the mixed questions to which it is applied, it may be extensively employed. The general principle is, that either the initial or some prominent letter should be selected from the word which denotes the thing we wish to represent. The beneficial effect of this arrangement is felt a little in the first stage of the solution; it has no influence on the second; but in the last stage it saves considerable trouble by obviating the necessity of constantly referring back to enquire what particular letters represent.

In the first solution, there are in fact two distinct cases, in which the reasoning is repeated from the beginning to the end; and each of these cases has two sub-species: so that there are, in fact, four cases treated of in the Arithmetica Universalis. This defect has been remedied in the second solution, where it has been shown that the four cases are included in one formula, to which it is only necessary to give the proper interpretation, and every circumstance connected with the problem is brought to light. Two causes seem to have contributed to produce this separation into cases: in the first place, the extreme generality of the language of Algebra may not have been sufficiently noticed: the ancients were accustomed to divide their problems into cases, and the habit of treating these separately, may have produced the same cautious treatment of a question when resolved by methods of a far more general and comprehensive nature: such indeed would naturally be the case, until the degree of generalization introduced by the new method, was fully ascertained. In some instances, the action of another cause may be traced, and one that is more easily removed than that which arises from a want of confidence in the method employed: it may be referred to the imperfect manner, in which the very first elements of the application of Algebra to Geometry and to Mechanics, have been communicated to us: in order to explain clearly the result, it is quite indispensible that we should be perfectly familiar with the principles which regulate these: without them, we may frequently put the question into an algebraic form, because we view it only in one light; and from the nature of the language made use of, that one, whichever it may be, virtually embodies the whole; but when we require to retranslate our conclusions, as they contain every possible case, we must of course be able to translate each individual. To display in a more prominent point of view, the reason why there are so few failures in expressing the conditions of a problem in Algebra, and so many in giving the full account of all which the solution informs us, let us imagine a problem proposed that admits of ten cases; if we are capable of translating any one of these into an equation, such is the comprehensive nature of Algebraic language, that though they are all contained in that single expression, we may be ignorant of the manner of treating nine cases, and competent to manage one of them. If this happened, we might succeed in resolving the equation, and discovering all its roots, which would answer all the ten cases; yet it is scarcely probable, with such moderate knowledge, we should succeed in explaining in common language, the meaning of many of them.

The advantage of selecting in our signs, those which have some resemblance to, or which from some circumstance are associated in the mind with the thing signified, has scarcely been stated with sufficient force: the fatigue, from which such an arrangement saves the reader, is very advantageous to the more complete devotion of his attention to the subject examined; and the more complicated the subject, the more numerous the symbols and the less their arrangement is susceptible of symmetry, the more indispensible will such a system be found. This rule is by no means confined to the choice of the letters which represent quantity, but is meant to extend, when it is possible, to cases where new arbitrary signs are invented to denote operations.

In the formation of some of the most common algebraic signs, this maxim has been attended to; but although in many individual instances it has been admitted, it is still desirable that it should be recognised as a general principle. The sign of equality was obviously adopted from the circumstance of the same relation existing between its two parts, as that which it indicates between the two quantities which it separates, and the propriety of this selection has confirmed its use, although Girard employed $=$ to denote difference, and Descartes used $\propto$ to represent equality. In the two signs representing greater and less than

$$a > b \text{ and } b < a\ *,$$

the prevalence of the same motive of choice is equally apparent, for it is immediately seen, that these signs are so contrived, that the largest end is always placed next to the largest quantity, and

* That this principle operated in inducing Harriot, who first used these signs, to adopt them, I have now very little doubt: I may however, remark, that on my first initiation into Algebra, finding some difficulty in remembering their distinction, I formed the association above alluded to as a kind of artificial memory, a purpose, which it effectually answered.

consequently, the smallest end next to the smallest quantity. At a period when the more frequently occurring signs were not permanently settled, one method of denoting equality was by the following sign, 2|2, and the same author P. Herigone employed 3|2, and 2|3 to represent respectively, greater than, and less than. In each of these, the principle we have been contending for, has undoubtedly had its influence, and the signs themselves are objectionable, on entirely different grounds. The same writer made use of the common sign of equality to denote parallelism; a purpose for which it was then well adapted; since that time, however, its long continued use in another sense, has compelled geometers to change its position; and when it is proposed to state that two lines AB and CD are parallel, instead of putting $AB = CD$ as Herigone would have done, they merely change the position, and write $AB \parallel CB$; thus preserving the advantage, without infringing another rule, which ought never to be violated, that of avoiding the use of any sign in two senses.

In the doctrine of triangles, Lagrange has introduced a species of symmetry, which has been found productive of very advantageous results; it consists in denoting the three sides by the letters a, b, c, and the angles respectively opposite to them by the capitals A, B, C: by this arrangement, not merely the quantities themselves are indicated, but in some measure also their position, and the transition from any relation between one side and given data, to other sides and the corresponding data, are made with the greatest ease.

The more complicated the enquiries on which we enter, and the more numerous the quantities which it becomes necessary to represent symbolically, the more essentially necessary it will be found to assist the memory by contriving such signs as may immediately recal the thing which they are intended to represent. The notation which M. Carnot has contrived, for the purpose of illustrating his view of the application of Algebra to Geometry, possesses in an eminent degree the qualifications we are now considering; and although I cannot altogether agree with the conclusions, which it is the object of the author of that highly ingenious work, the "Geometry of Position," to establish, yet I am happy to acknowledge the instruction I have derived from the very original view which he has taken.

To denote a line, M. Carnot places a bar over the letters which represent it, thus, $\overline{AB}$: an arc of a circle or other curve, will naturally be represented by $\overset{\frown}{AB}$. To indicate the point where two lines meet, $\overline{AB}\cdot\overline{CD}$ is used, and similarly the points formed by the meeting of two arcs, or an arc and a circle, are denoted respectively,

$$\overset{\frown}{AB}\cdot\overset{\frown}{CD}, \text{ and } \overline{AB}\cdot\overset{\frown}{CD};$$

according to these principles, the line drawn from the point F, to the intersection of AB and CD, will be represented thus,

$$\overline{F\ \overline{AB}\cdot\overline{CD}},$$

and the two expressions,

$$\overline{\overline{AB}\cdot\overline{CD}\ \ \overline{FG}\cdot\overline{HK}}, \qquad \overline{\overline{AB}\cdot\overline{CD}\ \ \overline{FG}\cdot\overline{HK}}\cdot\overset{\frown}{LM}$$

denote the first, the line which joins the points, where AB, CD, and FG, HK respectively intersect each other: and the second, the point where that line cuts the arc LM,

$\pm$ is used to signify coincidence with, as

$$\overline{AB}\cdot\overline{CD} \pm \overset{\frown}{FG}\cdot\overset{\frown}{HK},$$

indicates, that the two points formed by the intersection of the lines AB, CD, and by the arcs FG and HK, coincide.

The angle ABC, is indicated thus, $\widehat{ABC}$, and if it is formed by the two lines AB and CD, it is thus expressed,

$$A\hat{B}C = \overline{AB} \wedge \overline{CD}.$$

The angle formed by the intersection of arcs, will therefore stand thus,

$$\overset{\frown}{AB} \wedge \overset{\frown}{CD}.$$

Surfaces are represented by the position of two lines above the letters, which indicate them, as

$$\overline{\overline{AB\ CD}},$$

and the angles which are formed by these, with other surfaces, or with lines, can be expressed on the principles already explained; as for example,

$$\overline{AB} \wedge \overline{CDEF},$$

is the angle formed by the line *AB*, with the surface *CDEF*. When single letters are employed to denote lines, as *a*, *b*, *c*, the angles formed by them are represented thus $a^\wedge b$, $a^\wedge c$, $b^\wedge c$.

The effect of these few abbreviations, which, when once explained, need no effort to fix them in the memory, is to render unnecessary the almost endless repetitions of the same words, and to convey the writer's meaning in the fewest terms. In the calculation of the values of annuities dependent on lives, if the question is in any degree complicated, the number of independent data is very considerable, and the letters which have been employed to denote them, have in many instances been selected, without the least regard to assisting the reader's memory. The symbols made use of are far too numerous to be extracted, and the immense advantages which attend a judicious selection, can only be appreciated by those, who have had occasion to study the subject, in the writings of Price and Morgan, and have afterwards perused those of Mr. Baily or Mr. Milne; the former gentleman has had the merit of explicitly stating the principle *, and the use, which he has continually made of it, has had the effect of giving great perspicuity to his formulæ.

In Astronomy, this principle has been adopted with much success, and signs ☉ and ☽ for the Sun and Moon constantly occur. It is rather singular, that a principle on which the earliest and most imperfect written language rested, should be found to add so essentially to the value of the most accurate and comprehensive: yet the language of hieroglyphics, is but the next stage in the progress of the art, to the mere picture of the event recorded; and the signs it employs, in most cases, closely resemble the things they express. In Algebra, although the principle has not been pushed to its extreme limits, the grounds of its observance are the same; the associations, are by its assistance, more easily and more permanently formed, and the memory most effectually assisted †.

I have entered into more detail respecting these causes, than the importance of the subject, may in the opinion of some appear

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* "When compound quantities are represented by more simple expressions, those characters ought to be preferred, which will most readily, and with least effort of memory, bring to our recollection the original quantity intended to be expressed." *Baily on Life Annuities,* Pref. p. 39.

† The benefit derived from a proper choice of signs, or from a judicious mode of presenting them, is not entirely confined to mathematical studies; wherever the multiplicity of particulars renders it of importance to assist the memory, or to give quickness to the apprehension of the terms employed, such a principle, if it can be adopted, will be found of considerable value. Amongst the authors who have availed themselves of this principle, in treating other subjects, than those, in which it is so eminently useful, the name of M. Cuvier, may be mentioned. In the preface to his work, Le Règne Animal, he remarks, "Partout les noms des divisions superieures sont en grandes majuscules; ceux des familles, des genres et de sous genres, en petit majuscules, correspondentes aux trois caractères employés dans le texte; ceux des espèces en Italiques; le nom Latin est a la suite du nom Français, mais entre deux parentheses.——Ainsi l'œil distinguera d'avance l'importance de chaque chose et l'ordre de chaque idée, et l'imprimeux aura secunder l'auteur de tous les artifices que son art peut preter à la mnémonique. *Cuvier, Le Règne Animal,* tom. I. Preface, p. 18.

to justify: the reasons which have induced me to do so, are to be found in the incomplete manner, in which these parts of the subject are treated by the generality of our elementary writers, a circumstance which impedes the subsequent progress of the student more than is perhaps usually allowed. To those who may hereafter employ themselves in supplying a considerable desideratum, in English Mathematics, by composing an introductory treatise on the application of Algebra to Geometry, I may be permitted to recommend a copious developement of its very first elements, not merely a detailed account of the changes of the situation or direction of lines, by changes in the signs, or magnitudes of quantity, or by the circumstances of the radicals, in which they may be involved: but by way of giving practical illustrations of the precepts so delivered, they should be accompanied by a series of examples, in which every circumstance, attending the section of two or more right lines, should be explained with scrupulous minuteness: the influence of such a course of reading, will be sensibly felt by the student as he proceeds; and the uninviting dryness of the results will be relieved by a judicious selection, which may render them valuable points of reference, in his future reading.* How much such a system is calculated to assist all our enquiries in theoretical mechanics will be allowed by those, who have had, in some measure, to form it for themselves, at the moment when the natural difficulties of the subject were sufficient to require their undivided attention. I am indeed inclined to refer a considerable portion of those difficulties, which are so much complained of by English students, well versed in the mathematical science of their own country, when they first open the works of the continental geometers, to

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* These observations were written prior to the publication of the Analytical Geometry of Mr. Lardner, and to the still more recent work of Mr. Hamilton. I am happy that one considerable impediment to the progress of the English student is at length removed.

the want of such a previous course of instruction, as that which I have now pointed out. I would however guard myself from being supposed to imagine, that this is by any means the sole obstacle; I mention it as one, which appears to me of some weight, and which might, without much difficulty, be removed.

The other, and more general, acceptation of the word symmetry, applies to the position, as well as the choice of the letters, employed in an enquiry: in this sense, it can scarcely exist, without a previous attention to that which has just been explained, for however regularly and analogously two series of quantities may be arranged, unless the signs, by which they are represented, are so constructed, as mutually to excite the idea of their correlatives, it is impossible, that the symmetry can be apparent to the eye. By employing the first species of symmetry, we assist the memory in remembering the ideas indicated by signs; by the use of the second, we enable it more easily to retain the form, in which our investigation has arranged those signs, as well as facilitate the processes, by which that final arrangement was accomplished. By the happy union of the two, our formulæ acquire that wonderful property of conveying to the mind, almost at a single glance, the most complicated relations of quantity, exciting a succession of ideas, with a rapidity and accuracy, which would baffle the powers of the most copious language.

The mode of expressing an angle of a triangle, in terms of the radii of three circles, which touch respectively each side, and the other two prolonged, will furnish the first example. Calling a, b, c those radii, and θ the angle opposite a, we have

$$\cot\frac{\theta}{2} = \sqrt{\frac{b}{a} + \frac{c}{a} + \frac{bc}{a^2}},$$

an unsymmetrical expression. This can be improved by a very trifling change, for it is equivalent to

$$\cot\frac{\theta}{2} = \frac{\sqrt{ab+ac+bc}}{a},$$

in which the numerator is instantly perceived to be the square root of the sum of the products of the radii, two by two. In as far as regards its form, it does not admit of any further improvement; but if this expression were to be employed in any enquiry into the properties of a triangle, it would be much more convenient to use the letters A, B, C, and a, b, c, to denote respectively the angles and sides, and to employ for the radii of these circles, the letters ρ', ρ'', ρ''', thus,

$$\cot\frac{A}{2} = \frac{\sqrt{\rho'\rho''+\rho'\rho'''+\rho''\rho'''}}{\rho'},$$

the convenience of so employing the letters A, B, C, a, b, c, has been already noticed, and the letters ρ', ρ'', ρ''', cannot fail, after a very short use, to recal the idea of radii, as well as fix the particular side, which each circle touches.

I have now enumerated what appear to me to be the principle causes which exert an influence on the success of mathematical reasoning, and have illustrated, with examples, those which were susceptible of it. They may be recapitulated in few words. The nature of the quantities which form the subject of the science, together with the distinctness of its definitions — the power of placing in a prominent light, the particular point on which the reasoning turns — the quantity of meaning condensed into small space—the possibility of separating difficulties, and of combining innumerable cases,—together with the symmetry, which may be made to pervade the reasoning, both in the choice, and in the position of the symbols, are the grounds of that pre-eminence, which has invariably been allowed to the accuracy of the conclusions deduced by mathematical reasoning.