HANDBOOK OF THE HISTORY OF GENERAL TOPOLOGY

VOLUME 2

Edited by
C.E. Aull and R. Lowen

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History of Topology

Volume 2

The titles published in this series are listed at the end of this volume.

Handbook of the History of General Topology Volume 2

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Introduction

This account of the History of General Topology has grown out of the special session on this topic at the American Mathematical Society meeting in San Antonio, Texas, 1993. It was there that the idea grew to publish a book on the historical development of General Topology. Moreover it was felt that it was important to undertake this project while topologists who knew some of the early researchers were still active.

Since the first paper by Fréchet, "*Généralisation d'un théorème de Weierstrass*", C.R.Acad. Sci. 139, 1904, 848–849, and Hausdorff's classic book, "*Grundzüge der Mengenlehre*", Leipzig, 1914, there have been numerous developments in a multitude of directions and there have been many interactions with a great number of other mathematical fields. We have tried to cover as many of these as possible. Most contributions concern either individual topologists, specific schools, specific periods, specific topics or a combination of these.

The first volume, which was published in 1997, contains the following articles:

Felix Hausdorff (1868–1942) (G. Preuß)
Frederic Riesz' Contributions to the Foundations of General Topology (W.J. Thron)
The Contributions of L. Vietoris and H. Tietze to the Foundations of General Topology (H. Reitberger)
Some Aspects of the Work and Influence of R.L. Moore (B. Fitzpatrick Jr.)
The Works of Bronisław Knaster (1893–1980) in Continuum Theory (J.J. Charatonik)
Witold Hurewicz – Life and Work (K. Borsuk, transl. by K. Kuperberg, A. Kuperberg)
The Early Work of F.B. Jones (M.E. Rudin)
The Beginning of Topology in the United States and the Moore School (F.B. Jones)
Some Topologists of the 1940s (A.H. Stone)
Miroslav Katětov (1918–1995) (Petr Simon)
Origins of Dimension Theory (M. Katětov, P. Simon)
General Topology, in Particular Dimension Theory, in The Netherlands: the Decisive Influence of Brouwer's Intuitionism (T. Koetsier, J. van Mill)
The Flowering of General Topology in Japan (J. Nagata)
Rings of Continuous Functions in the 1950s (M. Henriksen)
Categorical Topology – its Origins, as exemplified by the Unfolding of the Theory of Topological Reflections and Coreflections before 1971 (H. Herrlich, G.E. Strecker)
History of Sequential Convergence Spaces (R. Frič)
Interaction between General Topology and Functional Analysis (E. Kreyszig)

The present second volume contains articles covering the work of:

W. Sierpiński (R. Engelking)
K. Kuratowski (R. Engelking)
S. Mazurkiewicz (R. Pol)
R.H. Bing (M. Starbird)

Furthermore there are articles covering:

Uniform, Proximinal and Nearness Concepts in Topology (H.L. Bentley, H. Herrlich, M. Hušek)
Hausdorff Compactifications (R.E. Chandler, G. Faulkner)
Continua Theory (J.J. Charatonik)
Generalized Metrizable Spaces (R.E. Hodel)
Minimal Hausdorff Spaces and Maximally Connected Spaces (J.R. Porter, R.M. Stephenson Jr.)
Orderable Spaces (S. Purisch)
Developable Spaces (S.D. Shore)
The Alexandroff-Sorgenfrey Line (D.E. Cameron)

And finally there is a short paper concerning:

History of Mathematics (D.E. Cameron)

We decided to publish this work in volumes of 300–400 pages each, as papers became available. Waiting for all contributions to be completed before proceeding with the publication would indeed have involved an unacceptable delay for many authors. At the point of writing of this introduction, sufficient material for two more volumes has either been written or is in preparation. Nevertheless, at this moment, there are still some significant topologists, schools, periods and subareas of the field that we are seeking authors to write about.

In addition to the articles contained in the first two volumes, the following articles will appear in the next volumes.

A. Arhangel'skiĭ, *Some Observations on the History of General Topology*
C.E. Aull, *Toward an Outline of the History of General Topology*
P.J. Collins, *The Work of Hugh Dowker and its Legacy*
W. Heath, *History of Metrization*
R.E. Hodel, *History of Cardinal Functions*
P.T. Johnstone, *History of Pointless Topology*
B. Karl, *On the Early History of Topology*
A. Lelek, *Dilemma in Topology (and in Science): Bizarre versus Common*
E. Lowen, R. Lowen, *Supercategories of TOP and the Inevitable Emergence of Topological Constructs*
S. Marsedic, J. Segal, *History of Shape Theory*
J. Mioduszewski, *Polish Topology Between the Two World Wars*
S. Nowak, S. Spiez, H. Torunczyk, *Karol Borsuk – his life and contributions to topology and geometry*

P. Nyikos, *History of the Normal Moore Space Problem*
G. Reed, *History of Counterexamples*
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Ju M. Smirnov, *The Development of Topology in Moscow*
M.G. Tkačenko, *Topological Features of Topological Groups*
J.E. West, *History of Infinite Dimensional Topology*

Most of the authors for this work either were contacted personally by one of the editors or were recommended by experts in the field. The first drafts of papers were sent to readers and their suggestions were forwarded to the authors. We expect that there will be some disagreement among some authors, but we also consider this to be healthy. We hope that this work will encourage, not only further study in the history of the subject, but also further mathematical research in the field.

We would like to thank all colleagues who willingly contributed to what we hope will become a standard reference work on the History of General Topology. In view of the fact that most contributors would consider themselves primarily mathematicians rather than historians of mathematics, we are especially grateful for their efforts.

Finally, we would like to thank D. Vaughan for his extensive TEXnical help in turning a varied set of manuscripts into a uniform entity, and Kluwer Academic Publishers for their professional support in the publication of this book.

C.E. Aull, R. Lowen
The editors

WACŁAW SIERPIŃSKI (1882–1969)
HIS LIFE AND WORK IN TOPOLOGY

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Contents

C. E. Aull and R. Lowen (eds.), Handbook of the History of General Topology, Volume 2, 399–414.

1. Biographical sketch

Wacław Sierpiński was born in Warsaw on March 14, 1882, in the family of a well-known physician. Poland was then divided between Austria, Prussia and Russia. Warsaw belonged to the Russian part, and there was only a Russian university there (the only two Polish universities were in the Austrian part: in Kraków and Lwów). In 1900 Sierpiński entered the Department of Physics and Mathematics of the university in Warsaw, where his mathematical talent was appreciated and developed by an outstanding number theorist G. Voronoi. He finished university in 1904 and started to work as a high-school teacher. In 1905 he took part in an important school strike, protesting against the tsarist régime's attitude towards Poles, and then left Warsaw for Kraków. There he entered the old Jagiellonian University, and after one year obtained the PhD degree, presenting as his thesis a paper on number theory written earlier in Warsaw.

In 1908 he started to teach at the university in Lwów. Somewhat earlier he discovered, to his great astonishment, that the position of a point on the plane can be fully described by *one* real number. He wrote about this to one of his friends then studying in Göttingen, and for an answer got a cable with only one word: CANTOR. Further explanations followed, and, after a short stay in Göttingen, Sierpiński's interests switched from number theory to set theory. Sierpiński's course in set theory taught in Lwów in 1909 was among the first in the world. The notes for this course became the basis for his *Outline of Set Theory* published (in Polish) in 1912. About that time Sierpiński invited two younger Polish mathematicians, Zygmunt Janiszewski and Stefan Mazurkiewicz (both born in 1888), to Lwów. The first one had just obtained his doctorate in Paris, and the second had received his PhD degree from Sierpiński in 1913, the subject of his thesis being a study of the multiplicity of points in square-filling curves. The three of them, working together and sharing their interests in set theory and topology, formed a nucleus of what was to become a new scientific school.

The outbreak of World War I in 1914 caught Sierpiński on Russian territory, and, since he was by then an Austrian citizen, he was interned. From Vyatka, the first place of his internment, he was transferred to Moscow on the request of Russian mathematicians. There he met I. Egoroff, N. Lusin and M. Souslin, and got interested in real functions and analytic sets. At that period he started his important common research with Lusin in the field of analytic sets that eventually led both of them to the discovery of projective sets. In February 1918 he returned to Lwów, and in the Fall of that year he moved to Warsaw, by then the capital of Poland, brought back to life.

At Warsaw University he found Janiszewski and Mazurkiewicz. The three professors decided to concentrate the mathematical research at the university on set theory and its various applications – new domains where it would be relatively easy to catch up with world mathematics. Janiszewski came up with the idea to create, in Warsaw, an international journal devoted to these fields, and *Fundamenta Mathematicae* was born. After the untimely death of Janiszewski in 1920, Sierpiński and Mazurkiewicz took over the editorial tasks. Sierpiński was also most active as a teacher. His "higher mathematical seminar" attracted many brilliant mathematicians, among others Kazimierz Kuratowski (who finished under Sierpiński his PhD thesis on continua irreducible between two points begun under Janiszewski), Bronisław Knaster, Alfred Tarski and Edward Marczewski. Their most important papers were printed in *Fundamenta*; Sierpiński alone published 263 articles there. In 1931 a book series *Monografie Matematyczne* consisting of mathematical monographs written by Polish authors was launched. Sierpiński's first contribution was his famous treatise on the continuum hypothesis written in French and entitled *Hypothèse du continu*. The existence of a new mathematical school became evident, and it was called the Warsaw School of Mathematics. Sierpiński was its main founder and its chief representative in the period between the two World Wars. In the same period he also conducted intense organizational work in learned societies. His achievements have been duly acknowledged: the most famous universities conferred honorary doctorates onto him, and he became a member of several academies.

The admirable development of Mathematics in Poland was cut short by the German invasion in 1939. During World War II about one half of all Polish mathematicians engaged in research were killed by the Nazis who systematically tried to destroy all science and culture in the occupied country. During the war Sierpiński worked as a petty municipal clerk; he did not interrupt his research, though, and was also active in organizing the underground Warsaw University.

After the war for a short time he taught in Kraków and in 1946 resumed his tasks at Warsaw University, then quickly rising from the ruins. He was again remarkably active: teaching, researching, publishing. *Fundamenta Mathematicae* and *Monografie Matematyczne* started to reappear and what was left of the Warsaw School of Mathematics set to work. Sierpiński, endowed with highest scientific honours and dignities, retired in 1960. In the final period of his scientific activity he returned to number theory, the calling of his youth. He died on October 21, 1969.

When I entered Warsaw University in 1952, Sierpiński was for us, young students, a living legend. I vividly remember his robust silhouette, bowed down by age, gliding along the corridors of the institute. He was formally dressed, silent, and a bit lost in reality, absent in a way. In the Fall of 1955 he announced a special course on ordinal numbers, and I signed in. I must confess that my hopes were deceived; he lectured as if the class were empty. Evidently, I expected too much from this 73 years old and already tired mathematical giant. Some of us were happier; Andrzej Schinzel, a number theorist, my coeval and the last of Sierpiński's students that became eminent mathematicians, writes: "Mere teaching, transferring of knowledge, was for Sierpiński only a duty; as all his duties, he fulfilled it as best he could. The true contact with his students began when they started to work on unsolved problems. Then the distance vanished, professor shared his ideas, helped writing papers, rejoiced in students' success." We feel a whiff of the air from the first period of the Warsaw School of Mathematics.

2. Work in topology

Topology – for its wealth of concrete examples and possibility of visualisation – belonged to these domains that specially attracted Wacław Sierpiński. He worked in it from 1910, when he published his note *A new way of proving the Bolzano–Weierstrass Theorem* [1910], until the middle of the 1950s. His most important results, classics of general topology, were obtained between 1915 and 1920; they all relate to curves and continua. About the same time Sierpiński became interested in various kinds of "highly disconnected" subsets of Euclidean spaces, studied them and compared their properties; the results of this analysis now belong to the theory of dimension. Somewhat later – until the beginning of the 1930s – he sporadically used to investigate more general topological properties, such as compactness, the Lindelöf property, separability and complete metrizability. Between 1917 and the end of the 1930s Sierpiński obtained his best results in descriptive set theory.

Topological content can also be detected in many of Sierpiński's theorems and examples that formally belong to other domains, such as the theory of real functions or measure theory. It is impossible to strictly partition and classify the discoveries of such a universal and independent mind as Sierpiński's who, when absorbed in a question about the real line or the plane, probably never asked himself if he was doing topology or something else, but just sought a solution and applied all methods he could devise.

Sierpiński was always interested in the effectiveness of proofs and constructions, and particularly in the possibility of eliminating the axiom of choice from mathematical arguments: hence a number of papers, among them topological ones, with new proofs of results established earlier by others or by himself. Let us note that the notion of an *effective* argument, that appears in many papers of Sierpiński and of his contemporaries, does not seem to have a quite precise meaning. It is sometimes explained by such phrases as "indicate a *procedure*" or "*specify* a set/function" [italics mine, RE]. Its main ingredients seem to be avoiding the axiom of choice and defining mathematical objects in a "constructive" way, e.g. defining sequences of sets by recursion. Thus the construction of the Cantor set is effective, and that of the Vitali nonmeasurable set is noneffective.

Limiting ourselves to the subjects traditionally seen as a part of topology, we shall now succinctly discuss the most important of Sierpiński's achievements in this area.

2.1. CURVES AND CONTINUA

When Sierpiński started his work in topology, two non-equivalent definitions of plane curves were in use: cantorian curves (plane continua with an empty interior) and jordanian curves (plane continuous images of the interval $[0, 1]$). Both these notions at the beginning seemed quite sound, but were eventually rejected as conflicting with the intuitive notion of curve: it appeared, surprisingly, that the square is a jordanian curve (G. Peano 1890) and that some of the cantorian curves contain no arc, i.e. no subset homeomorphic to the interval $[0, 1]$ (Janiszewski 1912). Let us add here that "Peano curves" – continuous mappings of the interval onto the square – intrigued Sierpiński for a long time. In [1912] he gave his own construction of such mapping, and in [1937] he showed that from any such mapping one can easily obtain a continuous mapping of the interval onto the Hilbert cube; the latter mapping has been applied in [1945a].

Among the most important of all papers Sierpiński devoted to curves are [1915] and [1916], with the intention of continuing the attacks on the definitions. Sierpiński wanted to show that even some curves that are both cantorian and jordanian exhibit paradoxical properties and he constructed two famous examples: an example of a curve (both cantorian and jordanian) each point of which is a ramification point, i.e. a common end-point of three arcs, otherwise disjoint, and an example of a curve (both cantorian and jordanian) which topologically contains every cantorian curve. The pictures of these two curves, or rather of their approximations (see Figure 1), are widely known as Sierpiński's gasket (left) and Sierpiński's carpet (right).

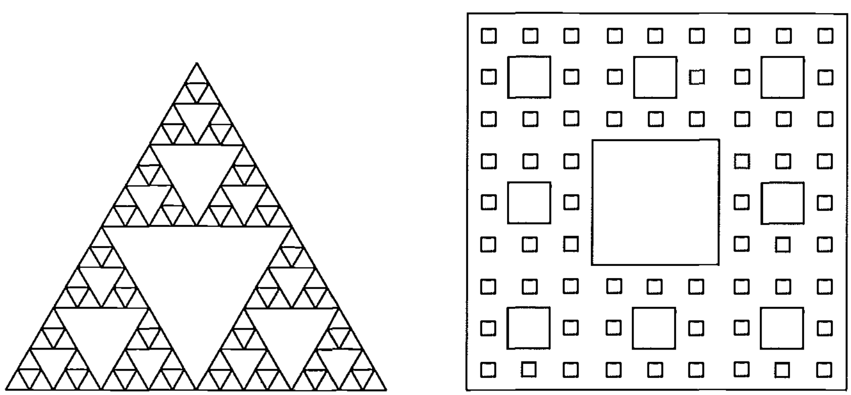

Figure 1

To turn the vertices of the triangle into ramification points, Sierpiński pasted together six copies of his curve placed in a hexagon. Let us add that, knowing of Sierpiński's gasket, Mazurkiewicz proved that every point of Sierpiński's carpet is a ramification point of infinite order. On this occasion Sierpiński's carpet was defined and studied for the first time; it is not clear if it was devised by Sierpiński or by Mazurkiewicz. The latter never published his result which was announced by Sierpiński in the 1916 Polish version of [1915] and in [1916].

Further development of topology showed that there is no better way of defining plane curves than by joining the conditions of Cantor and Jordan; the curves which satisfy these two conditions are, as we call them now, the locally connected plane curves. The two examples of Sierpiński which seemed so strange to his contemporaries and to himself were later studied in detail and led to two important topological notions, viz., the notion of the order of a point, and the notion of a universal space. Recently, growing interest in fractals endowed them with new significance and made them quite popular.

Pursuing his study of curves, Sierpiński in [1917] topologically characterized the arc as a continuum X that has the property that for each point $x \in X$ distinct from two fixed points $a, b \in X$ the space X can be represented as the union of two closed subsets A and B containing, respectively, the points a and b, and having only the point x in common. For historical accuracy we should add that a Polish version of [1917] was published in 1916, and that the same result was obtained by S. Straszewicz in 1918 and by R.L. Moore in 1920; for arcs lying in the plane, a similar characterization was established by N.J. Lennes in 1911.

Sierpiński's main contribution to continua theory consists of two important theorems (published in [1918a] and [1920c], respectively) that now can be found in most topology textbooks dealing with continua. The first theorem says that no continuum can be decomposed into countably many pairwise disjoint non-empty closed subsets, and the second one characterizes continuous images of the interval $[0, 1]$ (i.e. locally connected continua) as the continua which for each $\varepsilon > 0$ can be represented as the union of finitely many subcontinua with diameter less than ε (the latter was announced in the Russian version of [1916]).

2.2. HIGHLY DISCONNECTED SETS AND DIMENSION THEORY

Most likely, Sierpiński's interest in "highly disconnected sets" was awakened by Mazurkiewicz's 1913 decomposition of the plane into two punctiform sets (i.e. sets that contain no nontrivial continua; the notion was introduced by Janiszewski). Mazurkiewicz defined his two sets using the axiom of choice, and then, at the end of his paper, indicated an effective procedure that yields such sets. Sierpiński devoted his papers [1913], [1920] and [1922a] to similar questions. In the first paper he gave another effective construction of such a decomposition. In the second one he proved that the complement of any punctiform set lying in the plane is connected (this was generalized in [1921a] to punctiform sets in arbitrary Euclidean spaces) and – in a footnote – showed how the plane can be "non-effectively" decomposed into two sets each of which contains no perfect subset and *a fortiori* no nontrivial continuum. This important footnote introduced the so-called Bernstein sets to topology, where they continue to be an important tool for constructing various examples (in 1908 F. Bernstein defined a subset of the real line that has the cardinality of the continuum and contains no perfect subset).

For Sierpiński such a blunt way of solving the problem, as sketched in the footnote, was highly unsatisfactory: in the construction of Bernstein sets the axiom of choice and transfinite induction were used. He treasured concrete, effective constructions and was interested in decompositions into sets of the smallest possible Borel class. Such decompositions are discussed in [1922a], written together with Kuratowski, where the plane is decomposed into two punctiform sets, one of them being the intersection of a G_δ-set and an F_σ-set and the other one the union of a G_δ-set and an F_σ-set, and where it is shown that, as far as the Borel classification is concerned, this is the best of possible examples. This last paper is perhaps more important for the tool it uses than for the results it contains. It is among the very first papers where a peculiar topological space is obtained as the graph of a suitably defined function; this construction has been used more than once since.

A conspicuous place among "highly disconnected sets" belongs to countable sets, and they were also studied by Sierpiński in two important papers. In [1920a] he proved that every dense-in-itself countable metric space is homeomorphic to the space of rational numbers (announcement in a 1915 Polish paper), while [1920b], a paper written with Mazurkiewicz, contains a complete classification of countable compact metric spaces via their topological characterization as some sets of countable ordinal numbers.

In the already mentioned paper [1921a], Sierpiński thoroughly studied some classes of spaces larger than the class of countable spaces but smaller than that of the punctiform ones. He considered there the class of hereditarily disconnected spaces (i.e. spaces that contain no nontrivial connected subspaces), defined in 1914 by F. Hausdorff, and introduced two new kinds of high disconnectedness that are now called total disconnectedness and zero-dimensionality. Recall that a space is totally disconnected if for every pair of distinct points there exists an open-and-closed set that contains only one of these points, and that a space is zero-dimensional if for every point and each closed set not containing the point there exists an open-and-closed set containing the point and disjoint from the set. The relations between the various classes of highly disconnected spaces are exhibited in Figure 2.

Sierpiński provided us with ingenious examples of a totally disconnected space that is not zero-dimensional and of a hereditarily disconnected space that is not totally disconnected; the existence of punctiform spaces that are not hereditarily disconnected followed from his [1920] result on the connectedness of complements of punctiform sets lying in the plane.

The class of zero-dimensional spaces constitutes the lowest level in the classification of spaces according to their dimension. This last notion was introduced independently by P.S. Urysohn in 1922 and K. Menger in 1923, and is studied

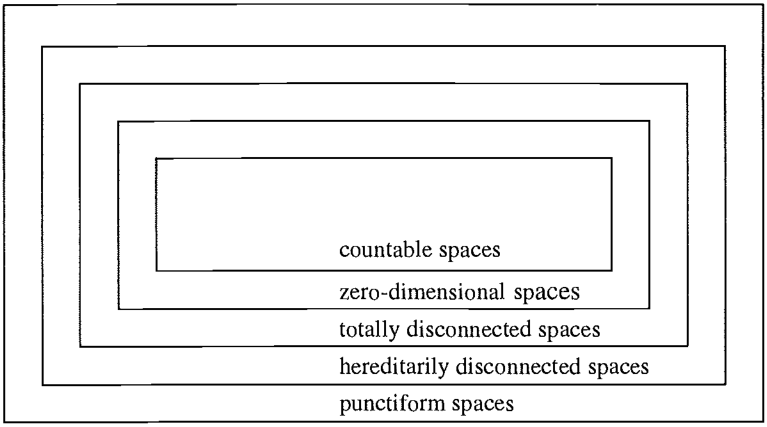

Figure 2

in one of the most interesting branches of topology: dimension theory. Sierpiński anticipated the creation of dimension theory also in his paper [1922], where he proved that every subset of Euclidean n-space that has an empty interior is homeomorphic to a nowhere dense set in that space. The proof of his result yields a nowhere dense set whose closure is contained in the set of points with at most $n - 1$ rational coordinates. Thus, in fact, Sierpiński characterized the n-dimensional subsets of Euclidean n-space as the sets with a non-empty interior; Urysohn and Menger obtained this important characterization somewhat later. One should also mention that Sierpiński's example of a hereditarily disconnected space that is not totally disconnected given in [1921a] was the first example of a weakly one-dimensional space, i.e. of a space that is globally one-dimensional but whose points of one-dimensionality constitute a zero-dimensional set (weakly n-dimensional spaces for $n = 2, 3, \ldots$ were defined later by Mazurkiewicz). Another result of Sierpiński that belongs to dimension theory is the simple but nice and useful theorem from [1928] asserting that every nonempty closed subset of a zero-dimensional separable metric space is a retract of that space.

The series of five papers [1926], [1929], [1929b], [1929c] and [1932] devoted to "types of dimensions" has nothing to do with dimension in the present understanding of this word. The notion of the type of dimensions was introduced in 1910 by M. Fréchet: he declared two spaces X and Y to be of the same type of dimensions, if X is homeomorphic to a subspace of Y and, conversely, Y is homeomorphic to a subspace of X (the terms "homoïe" and "topological rank" are also used). This notion long ago ceased to attract topologists' attention, and would not deserve being mentioned here, were it not for the lesson the five Sierpiński papers teach us: clever examples, even if constructed to illustrate second-rate notions, often remain valuable. In the first of these papers Kuratowski and Sierpiński gave such a clever example in which the Bernstein sets reappeared, this time in the main body of the paper. They also presented an interesting method for constructing examples (developed in [1932]) which was later used for various purposes, and which is known as the technique of killing homeomorphims.

2.3. GENERAL TOPOLOGICAL PROPERTIES

At the beginning of this century several non-equivalent concepts of an abstract space with a topological structure were introduced. Some of them, such as the notions of a metric space (Fréchet 1906), a Hausdorff space (Hausdorff 1914) and a topological space (Kuratowski 1922) proved very convenient; others were much less useful. Among the latter there were Fréchet's $\mathcal{L}$-spaces, based on the notion of the limit of a sequence. This notion appealed, however, to Polish topologists who devoted a number of papers to it. In [1921b] Kuratowski and Sierpiński characterized those $\mathcal{L}$-spaces in which the Borel–Lebesgue theorem and the "generalized Borel–Lebesgue theorem", i.e. the Lindelöf theorem, hold. Their arguments, translated to the language of topological spaces, yield the characterizations of compact spaces and hereditarily Lindelöf spaces in terms of complete accumulation points and condensation points, respectively. In [1921c], another paper couched in the language of $\mathcal{L}$-spaces, Sierpiński in fact established the characterizations of hereditarily separable and hereditrarily Lindelöf spaces in terms of the stabilization of increasing and decreasing transfinite sequences of closed subsets. There he also started a study of relations between hereditary separability and the hereditary Lindelöf property by providing examples of Hausdorff spaces that have only one of these two properties. His modification of the usual topology of the real line taking into account a well-order in the set of real numbers, applied in the example of a hereditarily separable non-Lindelöf space, has been often exploited later.

Sierpiński's predilection for effectiveness in constructing mathematical objects found expression in his study of sets with relatively simple borelian structure. In [1921d], together with Kuratowski, he characterized differences of two closed sets as those sets that are locally closed. And in two later papers [1924b] and [1930a] he studied G_δ-sets in Euclidean spaces; his results, stated in the current terminology, consist of a topological characterization of completely metrizable separable spaces and of a proof of the invariance of complete metrizability under open mappings in the realm of separable metric spaces (this result was later extended to arbitrary metric spaces by Hausdorff).

We should also cite here two quite late papers of Sierpiński: [1945a] and [1945b] devoted to universal metric spaces. In the first one he gave an elementary proof of the theorem on the metrical universality of the space of continuous real-valued functions on the interval $[0, 1]$ for the class of separable metric spaces (established by S. Banach and S. Mazur). In the second one he proved, among other things, that from the generalized continuum hypothesis it follows that for every uncountable cardinal number $\mathfrak{m}$ there exists a universal space for the class of metric spaces of cardinality $\mathfrak{m}$. The results of these two papers were announced in two short notes presented by Sierpiński's friend M. Picone in April 1940 to the Torino Academy of Sciences and were published the same year in the *Atti* of the Academy. Sierpiński's claim that "complete proofs of the above results will appear in volume 33 of *Fundamenta Mathematicae*," that concluded the announcements, sounded a bit weird: Poland was then under German occupation and the Nazis had already started their work of destruction, universities were closed and no scientific journals were being published. Sierpiński's intuition was unfailing, though, and the claim proved prophetic.

2.4. DESCRIPTIVE SET THEORY

We already mentioned Sierpiński's common research with Lusin on analytic sets. They published two joint papers on this subject, [1918] and [1923]. In the first of these, they gave a simplified proof of the Souslin characterization of Borel sets, stating that a set E of real numbers is a Borel set if and only if E itself and its complement are analytic sets, i.e. if E is both an A and a CA set. In their proof they established an important fact: every CA set can be represented as the union of $\aleph_1$ Borel sets, and they introduced an important tool: the Lusin–Sierpiński index. In a note they detail that the original – never published – proof of Souslin was based on "geometrical considerations" and that Souslin already considered an index, defined in a "geometrical way". They also proved in [1918] that the family of Lebesgue measurable sets is closed with respect to the (A)-operation.

In [1923] they gave another proof of the Souslin characterization theorem, proved that analytic sets have the Baire property (this result was announced earlier by Lusin), and showed that every analytic set can be represented as the union of $\aleph_1$ Borel sets. The theorem on the uniformization of Borel sets by CA sets is also due to both of them; although Sierpiński presents the contents of his paper [1930] as an alternative proof of an already published result by Lusin, the latter, on announcing the result, says that the theorem was proved simultaneously, independently and in different ways by Sierpiński and himself (see C.R. Acad. Paris 190(1930), p. 351, cf. Mathematica (Cluj) 4(1930), p. 59). Among Sierpiński's contributions to the theory one should also quote the separation theorem for Borel sets of multiplicative class $\alpha > 0$ (any pair of disjoint set of multiplicative class $\alpha > 0$ can be separated by an ambiguous set of class α) established in [1924a], and generalized in [1934]. In [1926a] Sierpiński still gave another proof of the Souslin characterization of Borel sets. In the last paper – as in many others – he introduced important new methods to the theory of analytic sets. The significance of Sierpiński's work in the theory of analytic sets makes some historians of mathematics to consider him, along with Souslin and Lusin, a founder of the theory.

There have been some discussions about the priority of Lusin and Sierpiński in defining projective sets. In this context it seems appropriate to quote from the opening words of Kuratowski's communication *Sur les ensembles projectifs* (*On projective sets*) presented at the First International Topology Conference in Moscow in 1935 (see Mat. Sbornik 1 (43) (1936), p. 713): "The investigation of the notions of analytically representable functions and Borel sets led in a natural way to the notion of projective sets." In the mid 1920s the notion of projective sets hovered in the air, it was *natural* to define and consider them. No wonder that Lusin and Sierpiński introduced them independently. In 1924–25, at Moscow University, Lusin gave a course on projective sets, and between May and August 1925 published a series of five notes on the subject in the Paris Comptes Rendus; he also introduced the name "projective set". Sierpiński in [1925a], which appeared before May (see Lusin's statement in Fund. Math. 10 (1927), p. 94), defined classes $\mathcal{K}_1, \mathcal{K}_2, \ldots$, where $\mathcal{K}_6$ is identical with the class of PCA sets, $\mathcal{K}_7$ with the class of CPCA sets, and so on (this identity is recorded by Sierpiński at the end of [1928]). In his paper Sierpiński observed that the classes $\mathcal{K}_n$ are all distinct and that PCA sets can be represented as the union of $\aleph_1$ Borel sets. In [1928a] and [1929a] Sierpiński proved that all classes of projective sets are closed with respect to countable intersections and unions. Sierpiński used to call projective sets "les ensembles projectifs de M. Lusin" because this expression was at the same time precise and polite. On p. 1 of his 1950 booklet *Les ensembles projectifs et analytiques* (*Projective and analytic sets*) he says that "the theory of projective sets [was] created by Nicolas Lusin in 1924", the date of Lusin's Moscow University courses, but on p. 15 he quotes his [1925a]. It seems that he remembered well that, in printed form, he and Lusin introduced projective sets independently and simultaneously, but – knowing perfectly well that these sets had to appear anyway – he was not particularly keen on recalling this fact.

We should also mention here that Sierpiński's methods and results from [1921], [1924], and [1945], although developed within the framework of descriptive set theory (more precisely: the theory of functions of the first Baire class), recently found applications in capacity theory and functional analysis.

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Wacław Sierpiński published over 700 research papers and 31 books and booklets devoted to various branches of mathematics. His *Introduction to General Topology*, published in Polish in 1928 and translated to English in 1934, as well as his *General Topology*, published in 1952, were quite often cited in their time. Among at least a dozen of his students and collaborators that obtained their doctorate from him there are two names famous in topology: Mazurkiewicz (1913) and Kuratowski (1921).

The reader of Sierpiński's topological papers is struck by the ease with which the author finds problems to be solved. Some of them might seem trifling today, but they witness of a very characteristic and personal style of doing mathematics. Any "natural" mathematical problem concerning sets in Euclidean spaces was interesting for Sierpiński, if only it was difficult enough and the arguments providing the answer were clever. He was much more interested in answering such questions, often posed by himself, than in building up theories. To quote a few examples: Is the set of all distances between points of a plane G_δ-set a Borel set? When can an F_σ-set of real numbers be represented as a countable union of *disjoint* closed sets? Given two uncountable zero-dimensional analytic sets of real numbers, can one of them always be mapped continuously onto another? The answers are given in [1925], [1929d] and [1929e].

It seems that simple curiosity was one of Sierpiński's main characteristics as a mathematician, and also his main stimulus. Far in his advanced age he retained a freshness and spontaneity of thought that prompted him to study any attractive question that he devised or was asked. One could see that mathematics was his element and his pleasure.

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THE WORKS OF STEFAN MAZURKIEWICZ IN TOPOLOGY

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Contents

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1. Introduction

Stefan Mazurkiewicz was born in Warsaw in 1888, a son of a well-known barrister. He attended high schools in Warsaw and Cracow. At that time, Poles were deprived of their own state, and the only Polish universities were in Lvov and Cracow, then a part of Austro-Hungarian Monarchy. Mazurkiewicz studied mathematics at Jagiellonian University in Cracow (1906–7), München (1907–10), Göttingen (1910–12) and Lwow (1912).

His main contributions to mathematics concern the topological structure of subspaces of the Hilbert space. At the time Mazurkiewicz was starting his research, the subject already had firm foundations, established in works by Cantor, Baire, Borel, Brouwer, Lebesgue, Schoenflies, Zoretti, and others: basic notions were clearly defined, standards for precision set, the topological character of the dimension of Euclidean spaces proved, and subtle examples exhibiting complexity of the matters constructed. It took, however, a prolonged effort of many great successors to raise and develop to the form we recognize now as "classical" the three basic parts of the subject: continuum theory, dimension theory, and descriptive set theory. Mazurkiewicz played an eminent role in this process.

The choice of his first research subject was strongly influenced by Zygmunt Janiszewski, his coeval, who was working on his doctoral thesis about irreducible continua, approved in 1911 in Paris by Poincaré, Borel and Lebesgue.

Mazurkiewicz obtained a doctoral degree at Lwow University in 1913, supervised by Professor W. Sierpiński. This thesis [M2], [M3], [M4], devoted largely to the structure of continuous images of the interval, contained what is now referred to as "classical Hahn–Mazurkiewicz and Mazurkiewicz–Moore theorems".

In 1919 Mazurkiewicz obtained the title of Docent from Jagiellonian University, presenting a habilitation thesis [M10] about the topological structure of G_δ-sets in Euclidean spaces. A topological characterization of the irrationals given in the thesis is now a standard one.

In November 1915, Warsaw University was resurrected, as a result of dramatic changes brought by WW I, and Mazurkiewicz was appointed an "extraordinary professor" at the university. He was associated with Warsaw University throughout the rest of his life, several times elected Dean of the Department of Mathematics and Sciences, the last vice-rector of the university in the interwar period, a lecturer at the Underground Warsaw University (1942–44) after the German invasion destroyed the Polish state in 1939.

After WW I, together with Janiszewski and Sierpiński, Mazurkiewicz organized mathematical life in Warsaw, the capital of Poland, brought back to existence as an independent state. Mazurkiewicz and Sierpiński were the first editors of Fundamenta Mathematicae (Janiszewski, who strongly influenced the creation of the journal, did not live to see the first issue, passing away untimely in January 1920, a victim of the epidemic of influenza). Fundamenta Mathematicae marked an emergence of a new active mathematical center, which had strong impact on research in topology, set theory and logic in the interwar period. Mazurkiewicz was one of the main driving forces behind this phenomenon – the Warsaw Mathematical School.

Mazurkiewicz died in June 1945, soon after WW II in Europe was over. As Kuratowski wrote [10], "expelled from Warsaw, he had no possibility of curing the illness with which he was stricken during the war, and he ended his life as yet another of countless victims of the war".

He left over 140 publications in mathematics. The list, a collection of some of his most outstanding papers, and an overview by Kuratowski, can be found in *Stefan Mazurkiewicz, Travaux de Topologie et ses Applications*, Warszawa 1969.

Among the best-known accomplishments of Mazurkiewicz are the following theorems: each analytic set is a continuous one-to-one image of a coanalytic set [M22]; there exist totally disconnected absolute G_δ-sets of arbitrarily large dimension [M24]; no $(n-2)$-dimensional set can cut the n-cube [M27]; among continua in the square, hereditarily indecomposable ones are typical [M34]; almost every continuous function on the unit interval is nowhere differentiable [M42]; any open map between separable sets in complete spaces extends to an open map over G_δ-sets [M47]; the set of differentiable functions on the interval is coanalytic, but not analytic [M57]. One should also mention a proof of the Brouwer's fixed point theorem, found jointly with Knaster and Kuratowski, which became classical [M29].

Topology was the main area of Mazurkiewicz's scientific activity, but his mathematical interests were much wider, including real and complex analysis, probability theory and hydrodynamics. In particular, his work on foundations of probability theory, although apparently without essential influence on the topic, is of some historical interest.

Kuratowski's reminiscences [10] and Janiszewski's letters describe Mazurkiewicz as a brilliant and inspiring lecturer, if not always systematic, and as a true and sharp intellectual, with rather conservative views. Searching for biographical information about Mazurkiewicz, I came across his booklet, printed in Warsaw in 1927, containing several carefully crafted short sonets; here are a few lines:

> In vain Sisyphean labor bands you to the floor –
> Uselessly piercing with a hack the granite seal
> You start your quest each sunrise with fresh zeal
> Seeking a golden handful torn out the ore.

From the early twenties, a great deal of Mazurkiewicz's papers in topology were devoted to specific problems stated by other mathematicians. One gets the impression that on many occasions, Mazurkiewicz was driven by a pure intellectual challenge, and sometimes, while even not much interested in broader aspects of new seminal ideas, he would be inventive in his thrust toward a solution.

Maybe, an important element of Mazurkiewicz's legacy is that such an attitude, assumed by a great master in the field guided by strong instincts for depth and elegance, can create an impressive body of first rate mathematics with lasting value.

2. Continuum Theory

The first publication of Mazurkiewicz [M1] contains a proof of Janiszewski's theorem on the existence of irreducible continua which avoids transfinite induction. The reasoning was in fact a special case of what is called now Brouwer's irreducibility principle, formulated in Proc. Kon. Akad. Wett., 1911, where the paper of Mazurkiewicz is quoted.

Three years later, in two notes [M3], [M4] presented to Comp. Rend. Varsovie in March and November by Sierpiński, Mazurkiewicz obtained results concerning continuous images of the interval I, which became classical. In the first note he shows that a continuum C can be parametrized continuously on I if, and only if, each pair of points p, q can be joined in C by a simple arc $C(p,q)$, with diameters of the arcs $C(p,q)$ tending to zero, whenever the distance between p and q converges to zero. The proof of necessity is a direct analysis of a given parametrization of C, and the proof of sufficiency starts from an arbitrary Peano curve filling up the cube I^n containing C. In the second note, simple arcs $C(p,q)$ are replaced in this characterization by arbitrary continua. Put together, the results show that locally connected continua are arcwise connected and coincide with continuous images of the interval.

More precisely, the local connectedness aspects are discussed in a later note [M8] in terms of "continua of condensation" introduced by Janiszewski, and the "oscillation of a continuum at a point". These notes were written in Polish, and a systematic exposition of the results in French appeared in [M15]. The paper contains also the following interesting construction, to which Urysohn later devoted a separate section in the memoir [13], Ch. II, §9: to each metric space G a new metric is associated, with the distance between points p, q being the infimum of diameters of continua in G containing p and q; Urysohn denoted the resulting metric space by G^*.

Independently, in 1914, H. Hahn characterized continuous images of the interval as locally connected continua (cf. Hahn, Atti di Congresso Internationale dei Matematici, Bologne 1928), and R.L. Moore published in 1915 results which, properly interpreted, yield the arcwise connectedness of locally connected continua.

Mazurkiewicz carried out further analysis of locally connected continua in [M18] and [M20], prompted by some problems raised by Knaster and Kuratowski. In the first paper, it is shown that any locally connected continuum contains at least two points which do not separate it, and if such a continuum is separated by any two of its points, it must be a simple closed curve. Independently, stronger results, without local connectedness, were obtained by R.L. Moore in 1920. However, an essential aspect of Mazurkiewicz's paper is a refined analysis of locally connected continua which do not contain any simple closed curve. It turned out that such continua, called now dendrites, form an important class of curves (a conjecture of Mazurkiewicz, stated in the paper, that every dendrite can be embedded in the plane, was later confirmed by Ważewski in 1924 and Menger in 1926).

The second paper contains a theorem that the only locally connected homogeneous continua in the plane are simple closed curves. A problem formulated in this paper, if there are any other homogeneous continua in the plane, remained open until R.H. Bing demonstrated in 1948 that the pseudoarc is homogeneous.

We shall concentrate now on some results of Mazurkiewicz about indecomposable continua, i.e. continua which are not the union of any two proper subcontinua. Brouwer's famous common boundary of three plane regions, defined in 1910, was an indecomposable continuum, and Janiszewski's thesis, mentioned in §1, originated a systematic study of this extremely important class of continua. Let us recall that the composant of a continuum C determined by a point p is the union of all proper subcontinua of C containing p.

Mazurkiewicz proved in [M13] that each composant of any indecomposable continuum C is of first Baire category in C, and in [M23], that there is always a Cantor set in C intersecting each composant in at most one point. The proof of this second assertion deserves special attention. Taking from Mazurkiewicz's reasoning what really matters, one arrives at the theorem that for any F_σ-equivalence relation with first category equivalence classes, there is a Cantor set independent with respect to this relation. Results of this kind, in various settings, proved very important later.

Two years later, in [M28] and [M30], Mazurkiewicz published subtle results concerning indecomposable continua C in the plane. He showed that the unions of all composants in C which contain an accessible point (i.e. an end of an arc in the plane with no other points in common with C) is of first Baire category in C, and that only countably many composants of C can contain more that one accessible point.

The next year brings one of the most important accomplishments of Mazurkiewicz [M34], the theorem that in the hyperspace of the square (i.e. the space of compact subsets with the Hausdorff distance) hereditarily indecomposable continua form a set residual in the space of all continua. A continuum is hereditarily indecomposable if all its subcontinua are indecomposable. The first such continuum, the pseudoarc, was created by Knaster in 1922. Hereditarily indecomposable continua are extremely crooked and hard to grasp intuitively, and it was indeed a startling discovery that from the point of view of the Baire property, these are typical continua in the square.

In a broad conceptual sense, a grounding for such an approach was laid in earlier papers of Banach, Saks and Steinhaus. Investigating linear operations on function spaces, they linked the classical "principles of condensation of singularities" (cf. [7]) with the Baire Category Theorem. Mazurkiewicz's brilliant application of this idea in topology opened, however, entirely new perspectives. It was one of the turning points in forming what is called now the Baire Category Method, a trademark of mathematics related to Lwow and Warsaw in the thirties. Mazurkiewicz himself used the method on several other occasions, and we shall see a few such instances in §5.

Mazurkiewicz's last publication on indecomposable continua is [M55]. Mazurkiewicz solves here a problem of P.S. Alexandroff, demonstrating that each continuum of dimension $\geq$ 2 contains an indecomposable continuum (much later, in 1951, R.H. Bing considerably strengthened this result, proving that any continuum can be separated by a compactum, all non-trivial components of which are hereditarily indecomposable).

A few years earlier, Borsuk and Mazurkiewicz [M40] initiated a new important topic in continuum theory, showing that the hyperspace 2^K of any continuum K is arcwise connected. In a subsequent paper [M46], Mazurkiewicz added new information, proving that 2^K is a continuous image of the Cantor fan.

We shall end this section with a few more remarkable results of Mazurkiewicz concerning continua: a continuum C can be mapped continuously onto the Cantor fan if, and only if, the space C^* (defined earlier in this section) is non-separable, and every continuum is a continuous image of a curve (the second result was proved independently by W. Hurewicz) [M31]; examples of locally connected planar continua of uncountable order at each point, without any uncountable collection of disjoint subcontinua (solving a problem of Urysohn) [M33], [M35]; a characterization of dendrites which admit a continuous map with finite fibers onto the interval (answering a question of E. Čech) [M44].

3. Dimension theory

The dimension theory of arbitrary subspaces of the Hilbert space was launched in the early twenties by Menger and Urysohn (Brouwer's paper, published in 1913, contained a clear notion of dimension, but had little impact on the development of the subject). Nevertheless, several earlier papers of Mazurkiewicz provided important material for these creators, and after the theory emerged, the results were reinterpreted from the new point of view. To this category belongs Mazurkiewicz's second publication [M2], a chapter from his doctoral thesis. In this paper, he splits the plane into two punctiform sets (i.e. sets containing no continuum). But the principal value of the paper is that it brings into the subject a new method, which we shall briefly recall.

Mazurkiewicz fixes two points p, q in the plane, a Cantor set K in the segment $\overline{pq}$, and considers a one-to-one correspondence $C \to L(C)$ between the collection $\mathcal{C}$ of all continua in the plane containing p and q, and the lines perpendicular to $\overline{pq}$ through points in K. Then, for each $C \in \mathcal{C}$, he selects a point $x_C \in C \cap L(C)$. The set $\{x_C : C \in \mathcal{C}\}$ projects in a one-to-one way onto K, and intersects each continuum from p to q. The required decomposition is then obtained readily through a condensation procedure. The construction appeals to the Axiom of Choice, and Mazurkiewicz conciously adresses the effectiveness problem in the appendix. He indicates a specific procedure of selection of the points x_C, and hidden in the procedure, but not explicitely stated, is a parametrization of the collection $\mathcal{C}$ on the Cantor set. This, indeed, was a cluster of very friutful ideas. Still lacking was a precise notion of continuous parametrization of a collection of continua, and measurable selection theorems. The missing elements would be delivered by Mazurkiewicz much later, in his celebrated paper [M24], but then he would not return to his original ideas, leaving it to his successors to reveal their full potential, cf. [5], §6.2 and Problems 6.2. A–C.

In [M14], Mazurkiewicz constructs a punctiform G_δ-set in the plane which cannot be embedded in the real line. He had already noticed in his habilitation thesis [M10] that the punctiform sets from his decomposition of the plane cannot be embedded in the real line, but, as he put it "I did not succeed in completing a construction of such G_δ-sets". With hindsight, it is clear that such examples can be obtained by pursuing further his original ideas from [M2]. Instead, Mazurkiewicz embarks on a very clever, but special, geometrical construction. Interestingly, the graph of any derivative discontinuous on every nontrivial interval is a punctiform and connected G_δ-set, hence not embedable in the real line. Mazurkiewicz was interested in such derivatives (he used one in his paper [M9]) and W.H. Young clearly stated in 1911 relevant analytic properties of derivatives. Yet, it seems that the first to make the point were Knaster and Kuratowski, in Rend. Cir. Mat. Palermo 1925.

Another interesting paper of Mazurkiewicz from the prehistory of dimension theory is [M6]. Inspired by the tiling principle of Lebesgue, Mazurkiewicz defines the dimension of a compactum as the minimal order minus one of continuous parametrizations of the compactum on compact subsets of the Cantor set (he was aware of a gap in Lebesgue's original proof, but overlooked, as many other contemporaries did, the important paper of Brouwer from 1913, with a complete proof and a notion of dimension). This was a penetrating idea, but had to wait until Hurewicz in 1926 included it in the main stream of dimension theory.

The first publication of Mazurkiewicz addressing dimension theory in full bloom was [M24]. This brilliant paper, one of his most outstanding contributions, provides a construction of totally disconnected absolute G_δ-sets of arbitrarily large dimension, solving fundamental problems κ and λ of Urysohn [12]. The construction introduced some new seminal ideas, which we shall sketch below.

Mazurkiewicz begins with a selection theorem. Given a compact space K, he considers a continuous parametrization $p : C \longrightarrow K$ of the compactum on the Cantor set in the unit interval, and selects from each compact $A \subset K$ the point $p(\min(p^{-1}(A)))$. Then he proves that the selection procedure applied to any upper semi-continuous decomposition of K yields a G_δ-selector for the decomposition (this important fact appeared in [3], IX, §6 as Exercise 9, which became a standard source of references).

In the next step, Mazurkiewicz applies the classical idea of "universal functions" in a new setting, opening again a road toward fruitful applications. He considers the hyperspace of the product $C \times I^n$ of the Cantor set and the n-cube, and maps C continuously onto the collection of all increasing sequences of compacta in $C \times I^n$, $S(t) = (S_1(t), S_2(t), \ldots)$, $t \in C$. Letting D_i be the set of all t for which $S_i(t)$ is the first element of the sequence $S(t)$ intersecting $\{t\} \times I^n$, he defines a σ-compact "universal" set $M = \{(t,x) : t \in D_i, (t,x) \in S_i(t)\}$. Then he discloses the following remarkable property of the projection $\pi : M \longrightarrow C$: if $S \subset M$ hits every fiber $\pi^{-1}(t)$, then any G_δ-set in $C \times I^n$ containing S contains also some $\{t\} \times I^n$. Combined with a theorem of Tumarkin that k-dimensional sets extend to k-dimensional G_δ-sets in the ambient space, this shows that each selector S for π must be n-dimensional. Finally, Mazurkiewicz applies his selection theorem and the countable sum theorem to obtain absolute G_δ totally disconnected sets of dimension n.

In their fundamental works on dimension theory Menger and Urysohn devoted much attention to the sets $E^{(n)}$ of points at which E has dimension at least n, cf. [12], [8]. If $\dim E = n$, then $\dim E^{(n)} \geq n-1$, and if the lower bound is assumed, E is called weakly n-dimensional. Even before the theory was created, Sierpiński constructed a G_δ-set in the plane, which, reinterpreted, provided an example of a weakly 1-dimensional set. However, the problem of the existence of weakly n-dimensional sets for $n > 1$ remained open, until another beautiful paper of Mazurkiewicz [M27] provided such completely metrizable spaces for all n. Mazurkiewicz presented this result at the International Congress of Mathematicians in Bologne, in September 1928. A very important element of the construction is a theorem, which became one of the best known results of Mazurkiewicz: no open connected set G in the n-cube can be cut by any $(n-2)$-dimensional set A, i.e. each two points in $G \setminus A$ can be joined in $G \setminus A$ by a continuum.

Let us now give a brief overview of some other noteworthy results of Mazurkiewicz in dimension theory. Solving problems of Menger concerning weakly n-dimensional spaces, Mazurkiewicz constructs in [M25] a non weakly 1-dimensional set in the plane which is a union of two closed weakly 1-dimensional subsets, and in [M26], he defines a set E in the plane with $(E^{(1)})^{(1)} \neq \emptyset$, but $((E^{(1)})^{(1)})^{(1)} = \emptyset$ (in fact, as he demonstrates, the iteration of this process may stop at any countable ordinal).

Irreducible continua, the first research object of Mazurkiewicz, appear again in [M37]. He proves there that if $f : K \longrightarrow I$ is a continuous monotone map (i.e. with connected fibers) of an irreducible continuum K onto the interval, then the dimension of K is the maximum of the dimensions of the fibers $f^{-1}(t)$.

The papers [M48] and [M50] deal with n-dimensional components of n-dimensional compacta K, i.e. the maximal n-dimensional Cantor manifolds in K. Mazurkiewicz proves that any two n-dimensional components intersect in a set with dimension $\leq n-2$, and, solving a problem of P.S. Alexandroff, that the collection of n-dimensional components is either countable or of cardinality continuum. There he also solves an analogous problem of Alexandroff about the modular dimension.

We close this section with recalling a joint paper by Knaster, Kuratowski and Mazurkiewicz [M29], containing proofs of Brouwer's fixed point theorem and Lebesgue's tiling principle, which entered textbooks on the subject. Numerous applications beyond topology of "KKM-maps" defined implicitly in the paper are discussed in the monograph [4], §5.

4. Descriptive set theory

Classical descriptive set theory was concerned with the structure of Borel sets in complete separable metric spaces, analytic (A-) sets, i.e. continuous images of the Borel sets (following Souslin's discovery in 1916 that the projection of a Borel set in the plane may not be Borel), coanalytic (CA-) sets, i.e. the complements of analytic sets, and the sets on higher levels of projective hierarchy PCA, CPCA, ..., obtained by the successive iteration of the two operations.

Mazurkiewicz's first publication on the subject [M7] contains a theorem that if A and B are homeomorphic subsets of Euclidean spaces, and A is a G_δ-set, then so is B (the argument requires only the surrounding spaces to be complete metrizable). This demonstrated that being a G_δ-set in a complete space is an absolute property and encouraged Alexandroff and Sierpiński to seek an internal characterization of this property. Five years later, Mazurkiewicz [M17] proved a similar result for $F_{\sigma\delta}$-sets, raising the problem of absoluteness of higher Borel classes. This was solved in 1924 by M.M. Lavrentieff, who derived the invariance of Borel classes ≥ 1 from his famous theorem about the extensions of homeomorphisms over G_δ-sets. In turn (answering a question of Sierpiński, quoted by Aronszajn in Fund. Math. 17 (1931), p. 120) Mazurkiewicz [M47] generalized the separable case of Lavrentiev theorem to the effect that any open map $f : A \longrightarrow B$ between separable subsets of complete metrizable spaces X, Y can be extended to an open map $f^* : A^* \longrightarrow B^*$ over G_δ-sets $A^* \subset X$, $B^* \subset Y$. The separability assumption is indispensable here, but as was proved by I.A. Vainstein in 1947, an analogous result for closed maps is true without any restrictions.

We should now return to Mazurkiewicz's habilitation thesis [M10], which we have already considered in §3. It contains a thoro analysis of punctiform G_δ-sets in Euclidean spaces in terms of sequences of disjoint covers consisting of open regions. One of the results, Theorem IV in Ch. III, provides a topological characterization of the irrationals, which is now regarded as classical.

One of the most valuable contributions of Mazurkiewicz to descriptive set theory is his result [M22] that each A-set is a continuous one-to-one image of a CA-set. Twelve years later Kondô published a definitive result in this direction, which implies that this holds true for all PCA-sets. But at the time Mazurkiewicz's theorem appeared (solving a problem of Sierpiński) it was a significant event in the subject. Lusin [11] presented this theorem in a separate section, introducing it as a "result of great importance". Of great significance for the subject are also Mazurkiewicz's papers [M41], [M57] and [M58], devoted to an exact evaluation of the descriptive complexity of three naturally defined sets.

A general method of finding an upper bound for the complexity was already clearly explained in papers by Kuratowski and Tarski. The problems concerning the lower bounds are, however, more specific, and as a rule, much harder. Mazurkiewicz's three papers set a pattern of general strategy for numerous further results in this topic, which is still of considerable interest. Let us illustrate the approach by examining the main idea of the first paper. Mazurkiewicz proves there that, in the hyperspace of the cube, locally connected continua form a set of exactly class $F_{\sigma\delta}$. To this end he associates to an arbitrary $F_{\sigma\delta}$-set A in the interval I a compactum $K(A)$ in I^3 and an open map $f : K(A) \longrightarrow I$ such that $f^{-1}(t)$ is an arc if $t \in A$, and a non-locally connected continuum if $t \in I \setminus A$. It follows that the descriptive complexity of the set of locally connected continua in the hyperspace of $K(A)$ is the same as the complexity of A. A similar strategy, but in a quite different setting, is used in a very ingenious way in the next two papers to show that the continuous differentiable functions on I form a CA-set, but not an A-set, and the continuous functions on I^2 with at least one y_0 such that the partial derivative on the variable x exists at all (x, y_0) form a PCA-set, but not a CA-set.

Some elements of such an approach can be traced back to a joint paper with Sierpiński [M21], where (answering a question of Banach) the authors show that $A \subset I$ is analytic if and only if there is a Borel set $B \subset I \times I$ such that the lines $x = t$ intersect B in an uncountable set exactly when $t \in A$.

Let us finish this section with a brief discussion of two papers [M12] and [M16] involving the Cantor–Bendixon process. The first one, a joint work with Sierpiński, establishes a topological equivalence between compact countable sets and the ordinals $\omega^\alpha \cdot n + 1$ with countable α and natural n, and demonstrates the existence of "continuum many" topological types among scattered subsets of the real line. The second paper follows a discovery of Sierpiński of a Baire-1 function (i.e. a function of the first Baire class) which is not the difference of two u.s.c. (upper semicontinuous) functions. Mazurkiewicz provides a new method of constructing such functions and shows that each bounded Baire-1 function can be uniformly approximated by differences of u.s.c. functions. This topic seemed isolated until recently, when some important links with functional analysis were established. In particular, the results of Sierpiński and Mazurkiewicz were rediscovered in this new setting, cf. Haydon, Odell, Rosenthal, "On certain classes of Baire-1 functions with applications to Banach space theory", Springer Lecture Notes 1470 (1991). Remarkably, the "index of oscillation", a countable ordinal associated to a Baire-1 function through a Cantor–Bendixon process, which proved useful in recent investigations, was used implicitly by Mazurkiewicz in his construction.

5. Miscellaneous results

Mazurkiewicz was among pioneers in applying the Baire Category Theorem to various existence problems in topology and analysis. His most important contribution to the topic [M34] was already discussed in §2. But probably the best known is his theorem (answering a question of Steinhaus) that the continuous functions on the unit interval without finite one-sided derivative at each point form a residual set in the function space [M42]. (Later, Saks proved that the continuous functions without finite or infinite one-sided derivative at each point, i.e. Besicovitch functions, form a set of first category.) Two more remarkable applications of the Baire Category Method to function spaces are published in [M39] and [M43]. The first provides an answer to a question of Lusin concerning complex power series, and in the second one it is demonstrated that a certain delicate property of continuous functions, considered by Hardy and Littlewood, is in fact generic.

Very intersting applications are given in [M49] and [M52], where the Baire Category Theorem is used in the hyperspace of the unit interval to investigate some special Cantor sets playing a vital role in the theory of Fourier Series. In the second paper, Mazurkiewicz proves the existence of a Cantor set of logarithmic capacity null, which is not a countable union of compact H-sets, introduced by Raichman. This was related to the problem of whether each compact set of uniqueness, i.e. U-set, is a countable union of compact H-sets. Recently, Debs and Saint-Raymond proved that for any Borel collection $\mathcal{B}$ of compact U-sets, closed with respect to compact subsets, there is a compact U-set which is not a countable union of elements of $\mathcal{B}$ and, as Mazurkiewicz already noticed in his first paper, H-sets form a $G_{\delta\sigma}$-set in the hyperspace (see [9]).

Another beautiful result of Mazurkiewicz in this area is that for any curve K, a typical continuous function from K into the plane maps K onto a continuum homeomorphic with the Sierpiński's Universal Curve [M59].

Changing the subject, let us stop at Mazurkiewicz's paper [M32]. It contains a theorem that if a locally connected continuum K in the complex plane C has empty interior and does not cut C, then every continuous map from K to C can be approximated uniformly by complex polynomials. This extended some earlier results of Hartogs and Rosenthal; in a subsequent paper Math. Ann. 104 (1932) the authors quoted an unpublished work of Lavrentieff, who apparently obtained results similar to those of Mazurkiewicz. Finally, in 1935 Lavrentieff published a memoir containg a definitive result in this direction, without the local connectedness assumption, which became classical, cf. [6], Ch. II, 8.7. It seems that Mazurkiewicz's intermediate result was overlooked.

One should also recall a result of Mazurkiewicz [M38] (answering a question of Banach) that in the dual of a separable Banach space, endowed with the weak*-topology, there may be a linear subspace whose sequential closure is not sequentially closed. Banach [1] devoted to this topic a separate section (Annexe).

Another noteworthy theorem of Mazurkiewicz, proved in [M44], asserts that for any sequence of continuous functions on the interval, there is a Cantor set K and a subsequence of this sequence uniformly converging on K. Recently, R. Laver, J. London Math. Soc. 29 (1984), 385–396, considerably strengthened this theorem.

There is also a contribution of Mazurkiewicz to the theory of retracts. In a joint paper with K. Borsuk [M53] a compact absolute retract is constructed which is not a countable union of compact absolute retracts of arbitrarily small diameter. This phenomenon, in sharp contrast with properties of polytopes, was called in Borsuk's monograph [2], Ch. VI, §4 the "singularity of Mazurkiewicz".

Starting from his early publications, Mazurkiewicz showed a profound interest in subtle constructions of real functions. A remarkable display of his ingenuity in this field is a construction in [M36] of a continuous function from the unit interval into itself which preserves sets of Lebesgue measure zero, but loses this property upon adding any non-trivial linear function.

In some of Mazurkiewicz's papers there is an intriguing interplay between set theory, geometry and topology. Such is his construction [M5] of a set in the plane intersecting each line in exactly two points, which raised effectiveness problems of considerable interest.

Mazurkiewicz was also among the first topologists to examine properties of closed unbounded sets separating the plane [M19], as Urysohn pointed out in the introduction to his memoir [12] (footnote (1) on p. 34).

We finish this article with a glance at Mazurkiewicz's papers on "primends". The topic deals with adding to a separable metrizable space a "boundary" in a way reflecting its connectivity properties. It goes back to Carathéodory's work in 1913, related to conformal maps onto the unit disc, where he adjoined to simple connected regions in the plane some special points at infinity – the "Primenden".

Mazurkiewicz interpreted the process of adding the primends as a result of an appropriate Cantor–Meray–Hausdorff completion procedure. In [M56] he considers simply connected regions in the plane. First, he introduces a new metric in G, to the effect that, roughly speaking, the distance between points p, q is small if there is a closed connected set H separating G between p, and a fixed point in G with small Euclidean diameter, such that p and q are in the same component of $G \setminus H$. Then he shows that the completion of G with respect to this metric yields the Carathéodory's primends. A few years earlier B. Kaufmann considered the primends in a more general situation – for open sets in Euclidean spaces – and Mazurkiewicz hints at a possibility to include also Kauffmann's results in his setting. However, the exposition of these results would be published only in [M60], in the first issue of Fundamenta Math. after WW II. This was the last publication of Mazurkiewicz on topology, completed in March 1940, in Warsaw, then under German occupation. Mazurkiewicz concentrates in this paper on manifolds G, equipped with the metric introduced in [M15] (i.e. the spaces G^* we mentioned in §2), and combines the Cantor–Meray–Hausdorff completion procedure with some identifications reflecting the path structure of G.

H. Freudenthal presented in Fund. Math. 39 (1952), 189–210 a more general approach to the subject, based on different ideas. The last section of this paper contains an enlightening exposition of Mazurkiewicz's results on primends.

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Cited papers of Mazurkiewicz

Abbreviations

BAP: *Bulletin de l'Academie Polonaise des Sciences*
CRV: *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III*
FM: *Fundamenta Mathematicae*
PMF: *Prace Matematyczno Fizyczne (Works on Mathematics and Physics)*
SM: *Studia Mathematica*

[M1] Compt. Rend. Paris 151, pp. 296–298.
[M2] BAP, pp. 46–55.
[M3] CRV 6, pp. 305–311 (in Polish, abstract in French).
[M4] CRV 6, pp.941–945 (in Polish, abstract in French).
[M5] CRV 7, pp. 382–384 (in Polish, abstract in French).
[M6] PMF 26, pp. 113–120 (in Polish, abstract in French).
[M7] BAP, pp. 490–494.
[M8] CRV 9, pp. 428–442 (in Polish, abstract in French).
[M9] PMF 27, pp. 87–91 (in Polish, abstract in French).

[M10] Wektor 6, pp. 129–185 (in Polish, abstract in French).
[M11] BAP, pp. 44–46.
[M12] FM 1, pp. 17–27.
[M13] FM 1, pp. 35–39.
[M14] FM 1, pp. 61–81.
[M15] FM 1, pp. 166–209.
[M16] FM 2, pp. 28–36.
[M17] FM 2, pp. 104–111.
[M18] FM 2, pp. 119–130.
[M19] FM 3, pp. 20–25.
[M20] FM 5, pp. 137–146.
[M21] FM 6, pp. 161–169.
[M22] FM 10, pp. 172–174.
[M23] FM 10, pp. 305–310.
[M24] FM 10, 311–319.
[M25] FM 12, pp. 111–117.
[M26] CRV 22, pp. 51–58.
[M27] FM 13, pp. 210–217.
[M28] FM 14, pp. 107–115.
[M29] FM 14, pp. 132–137.
[M30] FM 14, pp. 271–276.
[M31] Comptes Rendus du I Congrès des mathématiciens des payes slaves, Warszawa 1929, Warszawa 1930, pp. 66–71.
[M32] CRV 23, pp. 136–142.
[M33] FM 15, pp. 222–227.
[M34] FM 16, pp. 151–159.
[M35] FM 16, pp.337–347.
[M36] FM 16, pp. 348–352.
[M37] Monash. für Math. 37, pp. 163–168.
[M38] SM 2, pp. 68–71.
[M39] Compt. Rend. Paris 192, pp. 1525–1527.
[M40] CRV 24, pp. 149–152.
[M41] FM 17, pp.273–274.
[M42] SM 3, pp. 92–94.
[M43] SM 3, pp. 114–118.
[M44] FM 18, pp. 88–98.
[M45] FM 18, pp. 114–117.
[M46] FM 18, pp. 171–177.
[M47] FM 19, pp. 198–204.
[M48] FM 19, pp. 243–246 .
[M49] BAP pp. 18–20.

[M50] FM 20, pp. 47–51.
[M51] FM 20, pp. 98–99.
[M52] FM 21, pp. 59–65.
[M53] Compt. Rend. Paris 199, pp. 110–112.
[M54] FM 24, pp. 118–134.
[M55] FM 25, pp. 327–328.
[M56] FM 26, pp. 272–279.
[M57] FM 27, pp. 244–249.
[M58] FM 28, pp. 7–10.
[M59] FM 31, pp. 247–258.
[M60] FM 33, pp. 177–228.

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KAZIMIERZ KURATOWSKI (1896–1980)
HIS LIFE AND WORK IN TOPOLOGY

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Contents

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1. Biographical sketch

Kazimierz Kuratowski was born in Warsaw on February 2, 1896, in the family of an eminent lawyer. His father, caring for patriotic education of his children, sent him to a Polish school, although – in then Russian Warsaw – graduation from such a school did not grant any privileges. In 1913 the young Kuratowski went to Moscow to pass official, Russian, graduation; having finished a Polish school, he had very little chance to pass such a graduation in Warsaw. Since after the 1905 school strike Polish youths boycotted the Russian university in Warsaw, Kuratowski went abroad to study. He passed the school year 1913–14 at the university in Glasgow, studying engineering.

Due to the events of World War I, Polish Warsaw University reopened in 1915, and Kuratowski was among its first students. This time he chose mathematics, his main interest. From his beginnings as a student until 1918 when Poland regained independence, he was very active in patriotic and political youth organizations. He also indulged in his interests in human sciences, and attended classes of many celebrities. Among his professors of mathematics there were two twenty-seven year old eminent topologists: Zygmunt Janiszewski and Stefan Mazurkiewicz. At the end of the war, in 1918, Wacław Sierpiński also became professor of Warsaw University. The three mathematicians, and the logician Jan Łukasiewicz, influenced young Kuratowski strongly and durably. His relations with Janiszewski were particularly close. This most talented young professor studied philosophy and mathematics in Zurich, Munich, Göttingen and Paris, where in 1911 he presented his PhD thesis *On continua irreducible between two points* to the Committee consisting of Émile Borel, Henri Lebesgue and Henri Poincaré. Janiszewski was the prime force of the Warsaw School of Mathematics: he proposed to concentrate the mathematical research in Warsaw on set theory and its applications and to found *Fundamenta Mathematicae*, an international journal devoted to these subjects. He died untimely in 1920, and the PhD thesis of Kuratowski began under him was finished under Sierpiński in January 1921. By the end of that year Kuratowski became a lecturer at Warsaw University and a close collaborator of Sierpiński and Mazurkiewicz on the editorial tasks of *Fundamenta Mathematicae*. He formally joined the editorial board of the journal in 1928, and since 1945 till his death acted as the editor-in-chief (at first together with Sierpiński). He was also conducting common research with Sierpiński and Mazurkiewicz, and wrote joint papers with each of them. At that time his closest collaborator and friend was Bronisław Knaster; they entered the university at the same time, worked together on connectedness and continua, and published their important results in joint papers.

As all the chairs of mathematics at Warsaw University were taken, in 1927 Kuratowski accepted a chair at the Polytechnic Institute in Lwów. There he spent almost seven years. Lwów was then a strong mathematical center where research was concentrated mainly on functional analysis. Stefan Banach and Hugo Steinhaus were the leaders and the editors of *Studia Mathematica*, the journal of the Lwów School of Mathematics. Kuratowski wrote two joint papers with Banach during this period. In the first one they solved the famous measure problem by proving (using the continuum hypothesis) that there exists no σ-additive real-valued measure defined for all sets of real numbers and vanishing on points. Further results were obtained one year later by Stanisław Ulam, "discovered" by Kuratowski among his Lwów students. Ulam was the first of Kuratowski's PhD students, and a close friend.

In 1932 a series of mathematical monographs by Polish authors was launched (*Monografie Matematyczne*). It was Kuratowski's idea that Polish mathematics should have its own series of monographs, just as it had its own journals *Fundamenta* and *Studia*. Kuratowski's first contribution to this famous series was volume I of his *Topologie*, published in 1933.

In 1934 Kuratowski returned to Warsaw, where a chair at the University was offered to him. At the First International Topology Conference in Moscow, in 1935, he met with Solomon Lefschetz who invited him to the United States. Next year Kuratowski sailed to New York and visited Princeton, Harvard, and Duke Universities. At the Institute of Advanced Studies he met John von Neumann and wrote a paper with him.

In the 1920s and 1930s Kuratowski obtained his greatest results in topology and gained international fame. Since the beginning of the 1930s he worked on his important treatise *Topologie*, the first volume of which appeared in 1933. In connection with this undertaking, a part of his research at that time consisted in reworking several parts of topology, such as descriptive set theory, dimension theory, etc. In doing this, owing to the new methods he devised, he always succeeded in generalizing the range of applicability of the theorems and in simplifying the proofs. His *Topologie* contains a number of results and ideas that were not published separately. Volume II of this monumental work was prepared for publication in 1939. Before it went to the printer, World War II broke out, and the scholarly activity of Kuratowski suffered a five year interruption.

The German occupation of Poland (1939–1945) brought many victims, also among mathematicians, and has been disastrous for Polish culture and science. The Nazi antisemitism forced Kuratowski to hide for a time under an assumed name. Several times he managed to escape death by a hair's breadth. Nevertheless, he lectured at the underground Warsaw University and collaborated with the Polish resistance, in spite of the fact that such activities involved the loss of life. In addition, he suffered from tuberculosis.

In the after-war period Kuratowski was very active in raising Polish education and science from its ruins. His dedication and organizational talents proved particularly useful. He took part in the rebuilding of Warsaw University, as well as in the restoring of scientific publications and international scientific relations.

Due to his initiative and efforts the State Mathematical Institute (later the Mathematical Institute of the Polish Academy of Sciences) was organized, and was opened in 1949. Kuratowski was its director from 1949 till 1967. For a long period he served as vice-president of the Polish Academy of Sciences and for one term as vice-president of the International Mathematical Union. He very much liked scientific travels, and travelled to almost all European countries, to the United States, to China and to India. Everywhere, honours were conferred on him. He became a member of six foreign Academies and received honorary doctoral degrees from four universities.

He was also very active in research and teaching. In the years 1945–80 he published 61 papers, volume II of *Topologie* (the manuscript has been saved in Switzerland), *Set Theory* (a monograph co-authored by Andrzej Mostowski), and two university textbooks. Until his retirement in 1965, he was teaching at Warsaw University and was admired and cherished by generations of students. Till the end of his life he had an office at the Mathematical Institute of the Polish Academy of Sciences, where all of us could talk to him and seek his advice. He died suddenly on June 18, 1980.

My acquaintance with Professor Kuratowski started in the Spring of 1954 in his Topology class. I remember him well from this time: a bit slow and composed in movements, of middle height, stoutish, formally dressed with perfect taste, smiling – he at once conquered the students. He lectured exceptionally well, and used the blackboard masterly. In this introductory class he kept strictly to the syllabus. At the oral examination he introduced a friendly atmosphere; his exquisite manners created the impression that he was really interested in a conversation on the rudiments of topology. Later I attended his special courses on the topology of the plane and on dimension theory. There, by his numerous digressions, he introduced us to the world of topology. He invited some of his students to join the research topology seminar that he then conducted jointly with Professor Karol Borsuk. I was one of them. Quite consciously he put us to the test, and the test was hard. For two years I understood practically nothing from what was going on. Professor Kuratowski, very busy then, proposed to me a topic for my Masters thesis and did not pay particular attention to my progress. When I gave him my manuscript, he kept it for some time, and then invited me to his home for a talk. He surprised me a lot by handing to me a French manuscript in his elegant, small, slightly slanting handwriting: this was a part of my paper, rewritten and translated by him, and prepared for publication. He asked me to read and approve the text. Somewhat later I wrote my PhD thesis under him, and I was already able to translate it by myself. Professor Kuratowski was rather reserved with his students, and not only with them. He kept his distance and hid behind his good breeding. He took true interest, though, in people in his charge, and was always ready to help in many ways, not only scientific. He observed the highest standards of scholarly propriety, but rarely talked about them, and never explicitly. He had some axioms he repeated from time to time: every mathematical paper should be written several times; the success of his pupils is the greatest pleasure for a mathematician.

His particular art of living has been admired by all who met him. He enjoyed many aspects of life: mathematics, music, books, travels, social life, politics. Everywhere he was in his place; at the Sorbonne when an honorary doctorate was being conferred to him, and at an improvised party in a hotel room. In all that he did there was an exceptional elegance and some peculiar harmony.

It has been my good fortune to have such a teacher.

2. Work in topology

Kazimierz Kuratowski started his research with two polemical papers published in the leading Polish philosophical journal *Przegląd Filozoficzny* in 1917 and 1919. He there discussed conditions for the correctness of definitions and the question of the independence of axioms. This start under the sign of logic is a symbolic expression of one of the main and permanent features of his intellect: the need for clarity and precision. In times when the basic set-theoretic and topological notions were still taking shape, Kuratowski wanted to know exactly what he was talking about. That is why, in the beginning of the 1920s, he took such interest in the notions of finite set and order, in the role of ordinal numbers in mathematical reasoning, and in the axioms for topological spaces. For him this kind of analysis meant, in the first place, building up the necessary foundations for specific and concrete topological research he was already conducting.

He took an interest in topology under the influence of Zygmunt Janiszewski and Stefan Mazurkiewicz; among his early collaborators were Wacław Sierpiński and Bronisław Knaster. Topology remained his main interest till the end of his life, although he occasionally wrote also on set theory and measure theory. His research extended over the whole realm of set-theoretic topology. He obtained many fundamental results in this domain, and – by his personal exquisite style – influenced many topologists.

We shall now discuss, in accordance with his changing interests, Kuratowski's main achievements in topology.

2.1. CONTINUA

By a continuum we shall understand a metrizable compact and connected space. Such spaces first attracted Kuratowski's attention. In his earliest topological paper [1920] published in volume I of *Fundamenta Mathematicae*, he characterized locally connected continua as continua where the components of open subspaces are open; thus he made the first step towards the well-known characterization of local connectedness, established one year later by H. Hahn who proved that the result holds for arbitrary spaces, not just continua. In the same volume of *Fun-*

damenta, Kuratowski and Janiszewski jointly published [1920a], a paper devoted to indecomposable continua (i.e. continua that cannot be represented as the union of two proper subcontinua). The first continuum of this kind was constructed in 1910 by L.E.J. Brouwer, and was the common boundary of three plane regions. The example, shattering simplistic intuitions, attracted attention of contemporary topologists. Kuratowski and Janiszewski's paper contains several characterizations of indecomposable continua and a study of their composants.

Kuratowski's PhD thesis [1922] and [1922a], also published in *Fundamenta*, was devoted to continua irreducible between two points; further results on this subject were published five years later in [1927]. In these papers Kuratowski completely described the structure of irreducible continua, refining older analyses of Z. Janiszewski, L. Vietoris, H. Hahn and W.A. Wilson, and establishing results that quickly became classical in this domain. Before recalling the most important of them, we should stress that his results on irreducible continua, published in [1922a], were presented by Kuratowski as the principal part of his thesis. The first part, [1922], where he formulated and discussed his famous closure operator axioms for topological spaces, was merely an introduction where the realm of further investigations was properly delineated. Later, this introduction became more widely known than the main body, and sometimes was even taken for Kuratowski's thesis.

Returning to [1922a] and [1927], let us recall that a continuum X is irreducible between its points x and y if no proper subcontinuum of X contains both x and y. Thus, the closed interval $[0, 1]$ is irreducible between its endpoints. Another classical example (see Figure 1) is the graph of the function $y = \sin(1/x)$, $0 < x \leq 1$, together with its limit segment $x = 0$, $-1 \leq y \leq 1$; this continuum is irreducible between the points $(1, \sin 1)$ and $(0, 1)$.

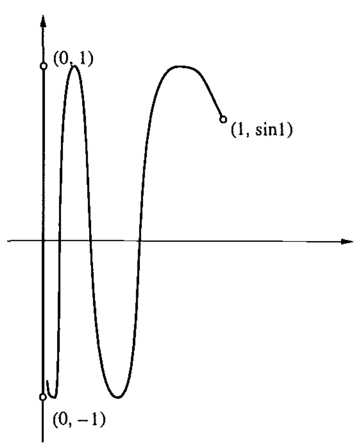

Figure 1

On the whole, irreducible continua are much more complicated than the two just named; in particular, as proved by Mazurkiewicz, each indecomposable continuum is irreducible between a pair of its points. Now, the most spectacular result on irreducible continua obtained by Kuratowski (in collaboration with Knaster, see [1927], p. 248) states that each irreducible continuum is built up from uniquely determined subcontinua, the so-called layers, the whole family of which is ordered in a natural way exactly in the same manner as the individual points on the interval $[0, 1]$. In this context one should also quote the related paper [1938a] where a technique of parametrization of monotone families of closed sets was developed and applied to the study of irreducible continua as well as to the study of the family of closed separators of a connected space.

The series of papers [1926], [1929] and [1929c] deals with locally connected continua; it contains an interesting analysis of relations between certain topological properties that appeared in earlier theorems on the sphere S^2 and the square I^2. In [1926] Kuratowski proved that the so called Phragmén–Brouwer theorem (established for $X = S^2$ by Brouwer in 1910) holds in a locally connected continuum X if and only if X is unicoherent, i.e. if for every decomposition of X into the union $A \cup B$ of two subcontinua A and B, the intersection $A \cap B$ is connected. He also proved there that a plane locally connected continuum X separates the plane if and only if X is not unicoherent. The handy notion of unicoherence – introduced by Kuratowski (and, independently, by Vietoris) – has been thoroughly studied in the 1930s, after its reappearence in [1929] and [1929c]. In the first of those papers, Kuratowski proved the equivalence of the following two properties of a locally connected continuum X:

(J_1) If A and B are closed subsets of X none of which separates X between points x and y, and if the intersection $A \cap B$ is connected, then the union $A \cup B$ does not separate X between x and y.

(J_2) If A and B are subcontinua of X and if the intersection $A \cap B$ is not connected, then the union $A \cup B$ separates X.

For $X = S^2$ these are Janiszewski's first and second theorems established in 1913, and this is why Kuratowski calls a locally connected continuum satisfying (J_2) a Janiszewski space (note that, as shown by Kuratowski in [1929b], Janiszewski spaces have dimension at most 2, and in $\mathbb{R}^2$ coincide with dendrites). In [1929] Kuratowski showed that every Janiszewski space X is unicoherent and, if it cannot be separated by a one-point set, then every simple closed curve $C \subset X$ separates X into two regions of which it is the common boundary (for $X = S^2$ this is the famous Jordan theorem). The latter result has been used in the proof of the main theorem of [1929], the beautiful topological characterization of S^2 as a locally connected continuum X that satisfies (J_2) and cannot be separated by a one-point set (a related characterization of S^2 was obtained in 1930 by L. Zippin). Besides its simplicity, the characterization of S^2 discovered by Kuratowski has the advantage of quickly leading to some interesting and important results on the sphere, such as the renowned Moore theorem stating that the quotient space determined by an upper semi-continuous decomposition of S^2 into continua that do not separate the sphere is homeomorphic to S^2.

In [1929c], the last paper of the series, Kuratowski proved that if the Brouwer fixed point theorem holds in a locally connected continuum X (i.e. for every continuous mapping f of X into itself there exists a point $x \in X$ such that $f(x) = x$), then X is unicoherent. This result permitted the Jordan theorem to be swiftly deduced from the Brouwer fixed point theorem for the square I^2, via Janiszewski's first theorem.

The paper [1930b], co-authored by G.T. Whyburn, is also devoted to locally connected continua; it develops the theory of cyclic elements and specifies its numerous applications.

The series of three papers we just discussed is intimately related with the next topic.

2.2. TOPOLOGY OF THE PLANE AND OF EUCLIDEAN n-SPACE

Kuratowski also obtained conspicuous results in the topology of the plane. In [1924] and [1928] he studied irreducible cuts of the plane, i.e. closed subsets E of $\mathbb{R}^2$ which are minimal with respect to the property that in the complement $\mathbb{R}^2 \setminus E$ there exist two points that cannot be joined by a continuum disjoint from E (since E is closed, the latter property is equivalent to the disconnectedness of the complement $\mathbb{R}^2 \setminus E$). In [1924] he observed that each irreducible cut of the plane is connected, and that E is an irreducible cut if and only if E is the common boundary of all components of its complement. He also proved there that if an irreducible cut is a locally connected continuum, then it is a simple closed curve. His main result in this paper is that every irreducible cut E of the plane that is the common boundary of $n \geq 3$ regions either is an indecomposable continuum or is the union of two indecomposable continua. This result was strengthened in [1928] by relaxing the assumptions on E: it suffices to assume that E is a closed connected set in the plane whose complement has $n \geq 3$ components at least 3 of which have E as their boundary. The principal subject of [1928] is a study of the structure of continua that are common boundaries of two plane regions; Kuratowski showed that such continua either are "monostratic" or have a natural "cyclic structure", i.e. are built up from layers naturally ordered in the same way as the individual points of the circle.

One of Kuratowski's most widely known results, his planar graph theorem established in [1930], also belongs to the topology of the plane. The theorem states that a graph (or, more generally, a local dendrite, i.e. a locally connected continuum that contains only finitely many simple closed curves) is embeddable in the plane if and only if it contains no subspace homeomorphic to either of the two graphs represented in Figure 2. The original proof, based on some properties of curves lying in the plane, was purely topological. Since then quite a number of proofs of this fundamental result of graph theory, based on various ideas, have been published. For historical accuracy, we should add that – as stated in a note on page 272 of [1930] (the only contemporary testimony, based on a private letter from P.S. Alexandroff) – "an analogous result" was obtained earlier by L. Pontrjagin but was never published. Further research on the embeddability of curves in the plane was carried out by S. Claytor, who extended Kuratowski's result by showing that a locally connected continuum is embeddable in the plane if and only if it contains no subspace homeomorphic to any of four test spaces: the two graphs represented in Figure 2 and two specific non-planar curves described by Kuratowski in [1930].

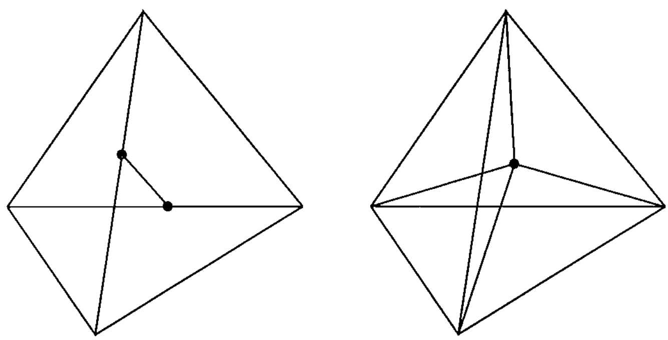

Figure 2

Besides his important contributions to the topology of the plane discussed above, Kazimierz Kuratowski had significant influence on the methodology of this domain. As we already observed, by combining the results from [1929] and [1929c] one obtains a clean topological proof of the Jordan theorem, sharply contrasting with earlier arguments where polygonal lines have been profusely employed (as in Brouwer's 1910 proof). Kuratowski always stressed the importance of "eliminating non-topological methods" from the topological study of the plane. He meant by this using only topological notions (thus, no polygonal lines) and reducing algebraic apparatus to a minimum. He used to say that this was important both from the methodological and from the aesthetic point of view. This opinion prevailed in Warsaw during the 1930s, and Kuratowski, together with his colleagues and students, strived to devise new tools for the topology of the plane. In [1928a] he developed a "topological" method for studying cuts of the plane. The study of mappings of a space to the circle S^1, originated by Borsuk and applied to the topology of the plane by Eilenberg, however, proved more useful. The latter, in his 1936 PhD thesis, obtained fundamental theorems of the topology of the plane by considering mappings of plane sets to the circle.

More exactly, he proved that some topological properties of a subspace X of the plane can be expressed in terms of the quotient group of the group of all continuous mappings of X to S^1 by its subgroup consisting of all mappings homotopic to a constant mapping. Kuratowski applied this new method in two papers published in 1938 and 1939, and directly after World War II presented further results in [1945]. Among other things, he there established interesting topological counterparts of classical theorems of Runge, Weierstrass and Rouché from the theory of analytic functions, and found a new setting for the duality between a subset of a plane and its complement (it should be noted that for technical reasons he was considering mappings to the complex plane without the point 0, rather than to the circle). He continued his research in this direction in three further papers published in *Fundamenta Mathematicae* between 1947 and 1954. As a result of all these endeavours, the topology of the plane became a relatively simple and homogenous theory, as presented in Kuratowski's *Topology*.

Passing to Kuratowski's contributions to the topology of Euclidean n-space, we shall first mention the simple and elegant proof of the Brouwer fixed point theorem given in [1929a], a paper co-authored by Knaster and Mazurkiewicz, where the theorem is deduced from Sperner's lemma. The method used in this proof has recently found applications in various fields of mathematics, including functional analysis and mathematical economy.

Later in his life, Kuratowski turned back to the topology of Euclidean spaces. In 1958–59 he adapted the method of mappings to the circle, used in the topology of the plane, to the case of arbitrary subsets of Euclidean n-space by using in this context the cohomotopic multiplication defined by Borsuk in 1936. He thus extended Borsuk's results obtained in 1950 for compact subsets of $\mathbb{R}^n$. As an outcome of his research (see [1958], [1958a] and [1959]), in the second volume of his *Topology* (starting with the 1961 French edition), he presented a new set-theoretic approach to problems of separation of $\mathbb{R}^n$ by its subsets. The most valuable part of the theory is that concerning the existence of an isomorphism – discovered by Kuratowski – between the group of components of the space of continuous mappings of a set $X \subset \mathbb{R}^n \subset S^n$ to the space $\mathbb{R}^n$ without the point 0, and a group of measures assigned to the complement $S^n \setminus X$.

2.3. DIMENSION THEORY

In the 1930s Kuratowski's main interests switched from continua to dimension theory and descriptive set theory. Dimension theory was then a new branch of topology. It was originated and developed in 1922–24 by K. Menger and P. Urysohn for the class of compact metric spaces and later generalized by W. Hurewicz and L. Tumarkin to the class of separable metric spaces. The formulation of an exact definition of dimension, and the discovery of the basic properties of this important concept, so wonderfully agreeing with earlier intuitions, was deemed a major achievement, and many topologists became interested in the new theory. Ten papers by Kuratowski devoted to the notion of dimension constitute an important contribution to the theory. In his study of dimension properties Kuratowski, following an idea of Hurewicz, often applied function spaces and the category method, and his best results were obtained in this way. In [1932a] he applied the category method to prove (and refine) an important theorem of Hurewicz on parametric representations of n-dimensional compact spaces on zero-dimensional sets. This theorem (generalized later to larger classes of spaces) says, as restated by Kuratowski, that for every compact metric space X with $\dim X = n \geq 0$ there exists a continuos mapping $f : A \to X$ defined on a closed subset A of the Cantor set such that $f(A) = X$ and for every $x \in X$ the fiber $f^{-1}(x)$ consists of at most $n + 1$ points; if, moreover, X has no isolated points, then one can assume that A is the whole Cantor set. The category method allowed Kuratowski to strengthen some of the basic theorems in dimension theory and to simplify their proofs. Thus in [1937a] he proved that if the dimension of a separable metric space does not exceed n, then the space of its continuous mappings to the $(2n + 1)$-cube contains a dense G_δ-set consisting of homeomorphic embeddings, and in a similar way he generalized Hurewicz's theorem stating that for every separable metric space X and countable family $F_1, F_2, \ldots$ of its closed subspaces there exists a compact space Y containing X as a dense subspace such that for $i = 1, 2, \ldots$ the dimension of the closure of F_i in Y is equal to the dimension of F_i. More precisely, he proved that in the space of continuous mappings of X to the Hilbert cube the set of all mappings f such that $f|F_i$ is an embedding and $\dim \mathrm{cl} f(F_i) = \dim F_i$ for $i = 1, 2, \ldots$ contains a dense G_δ-set. In [1938], using the same method, he obtained another interesting variant of the embedding theorem: he proved that every separable metric space can be compactified in such a way that the dimension at all its points is preserved.

In the proofs of theorems from [1937a] and [1938], besides the category method, another ingenious device was used: that of a κ-mapping, a simple and handy way to map spaces to nerves of their coverings. This class of mappings was introduced independently in 1933 by Hurewicz and by Kuratowski, who in [1933b] applied κ-mappings to prove in a straightforward way the fundamental theorem of Alexandroff on transforming spaces to polyhedra, the theorem that built a bridge between general and algebraic topology. Since that time κ-mappings have remained among the basic topological tools.

Kuratowski's clever generalization of the classical Tietze–Urysohn extension theorem formally also belongs to dimension theory, but its applications reach far into other branches of topology. The generalization, obtained in [1935], states that for every continuous mappings f from a closed subset A of a separable metric space X to Euclidean n-space (or the countably infinite Cartesian power of the real line) there exists a continuous extension F defined on X and taking values in the set $f(A)$ enlarged by the union of countably many simplexes whose dimension does not exceed the dimension of the complement $X \setminus A$. His extension theorem permitted Kuratowski, in [1935a], to expand the theory of local connectedness in dimension n from the realm of compact metric spaces (where it was developed by S. Lefschetz) to the realm of separable metric spaces. In the same paper Kuratowski introduced integral connectedness in dimension n and investigated the relationship between two notions and the theory of retracts.

2.4. DESCRIPTIVE SET THEORY

Kazimierz Kuratowski devoted about twenty papers to descriptive set theory. His best known and probably most important contribution is the discovery of the relations between the descriptive classification of sets and logical operations. This discovery, presented in [1931] and [1931a], eventually brought radical changes in the language and methods of descriptive set theory. It strongly influenced A. Mostowski, and then J.W. Addison, who established links between recursive properties of sets of natural numbers and the classical descriptive hierarchy, setting foundations for modern descriptive set theory. In the first of the two papers, co-authored by A. Tarski, the link between the existential quantifier and the projection of the Cartesian product onto one of its axes is exhibited, and is applied to showing that by performing the five basic logical operations on the propositional functions that define projective sets one still obtains definitions of projective sets (more exactly, that the family of propositional functions of n real variables defining projective sets in Euclidean n-space is closed under five basic logical operations). In the second paper, a method of evaluating the borelian and projective classes of various sets encountered in topology and in the theory of real functions is presented. The method is based on a simple but shrewd observation that the projection parallel to a compact axis maps closed sets to closed sets. Applied to the spaces of functions and spaces of sets, the method permits one to evaluate in a *mechanical way* – as Kuratowski stresses in the introduction to his paper – the class of a set while its definition is written formally in the language of logic. Kuratowski's method has since been applied in many instances (in particular in his two papers written in 1932 and 1933, and co-authored, respectively, by E. Szpilrajn-Marczewski and S. Banach), and it is still in use.

In the years 1936 and 1937 Kuratowski published a series of seven papers that can be considered as a sequel to [1931] and [1931a]. In [1936a] he showed that – under quite general assumptions – transfinite induction applied in the realm of projective sets does not lead outside this realm, and he evaluated the projective class of sets thus obtained. As an application, he proved that the famous 1905 Lebesgue set, universal with respect to analytically representable functions, is projective of class 3. He thus solved one of the famous problems of Lusin and showed that Lusin's intuitions about the degree of complexity of this set were wrong. A more precise evaluation of the class of the Lebesgue set was established in a paper co-authored by J. von Neumann [1937]. Further papers in the series deal with the "geometrization of countable ordinals" and its applications. Kuratowski, following Lebesgue, assigned points of the Cantor set to countable order types, then translated properties, relations, etc. pertaining to these order types to properties, relations, etc. pertaining to real numbers, and thus introduced the notion of projectivity to the domain of countable order types. This procedure has various applications in descriptive set theory. It can also be applied in logic: Kuratowski used it in [1937b] to prove Tarski's theorem that the notion of a well-ordering is not first order definable; in his proof he used an infinitary language in a precursory way.

Another important contribution of Kuratowski to descriptive set theory consists of the introduction and study of the notion of generalized homeomorphism. It allowed Kuratowski in [1934] to establish a Borel isomorphism theorem asserting that the structure of Borel and projective sets is the same in all uncountable complete separable metric spaces, in particular in Euclidean spaces and in the space of irrationals. In this context we should mention Kuratowski's very useful extension theorems proved in [1933] and [1933a].

An analysis of separation theorems in descriptive set theory led Kuratowski in [1936] to introduce the concept of reduction property. There he established reduction theorems for borelian sets of additive classes and for the classes of CA and PCA sets, and deduced the corresponding separation theorems. This neat and natural approach to separation theorems has since been followed by many authors.

Among Kuratowski's contributions to descriptive set theory one should also mention his exposition of the theory, in consecutive editions of the famous monograph *Topologie*, starting with the first one published in 1933. In his book he unified and simplified the classical theory, providing the standard reference source that has been used by several generations of mathematicians.

2.5. GENERAL TOPOLOGICAL PROPERTIES

We already mentioned Kuratowski's axioms for topological spaces presented in [1922]. At that time, beside the convenient but restrictive notion of metric space, several notions of "an abstract space" were in use. Some of them were based on the notion of "a limit point" (limit of a sequence in Fréchet's $\mathcal{L}$-spaces, and accumulation point of a set in Riesz's setting), others were based on the notion of "a neighbourhood" (culminating in 1914 in Hausdorff's "topological spaces" equivalent to our Hausdorff spaces). It was rather clear how to pass from the limit point approach to the neighbourhood approach, and vice versa, but the translations were not exact. Some reflection was necessary to realize that the notion of limit point was not proper for an axiomatization, the proper notion being that of the closure of a set: the set enlarged by all its accumulation points (the latter was introduced by Baire in 1906). Closure was often used in Janiszewski's papers, beginning with his 1911 PhD thesis. As Professor Kuratowski emphasised, Janiszewski was the first topologist to understand the predominance of the closure over the set of accumulation points. Following his line, the young Kuratowski, a friend and the most talented student of Janiszewski, devised the famous axiomatics in terms of the closure operator, axiomatics that allows an exact translation to the open set (or neighbourhood) approach.

Probably the most important among Kuratowski's early papers devoted to general topological properties is [1921a], written jointly with Knaster. Besides a thorough analysis of the notion of connectedness, the paper contains some examples that proved to be very useful and almost infinitely modifiable. The first of these (Example α) is the celebrated Knaster–Kuratowski fan – regarded by Alexandroff as "one of the most brilliant examples in point-set topology." In [1921a] it served as an example of a biconnected space, i.e. an infinite connected space that admits no decomposition into two connected and disjoint proper subsets. Not long ago, a modification has been used by E. Pol and R. Pol to solve an important problem in dimension theory, viz., to show that the covering dimension is not monotone in the realm of hereditarily normal spaces. The idea at the basis of Examples β and γ – a refined diagonal construction used at the same time by Mazurkiewicz – keeps reappearing in topology and leads to most interesting examples; it has been applied in R. Pol's construction of a weakly infinite-dimensional compact metrizable space that is not countable-dimensional.

Three papers of Kuratowski written at that time with Sierpiński, [1921], [1921b] and [1926a], are discussed in the companion article (see pp. 400–414) devoted to the older of the two topologists. We should, however, return to the technique of killing homeomorphisms, introduced in [1926a], and mention that it was recently applied by J. van Mill (to give an example of an infinite-dimensional separable normed space that is not homeomorphic to its product with the real line) and by W. Marciszewski (to give an example of separable normed space that is not homeomorphic to a prehilbert space). The method used in the fourth joint paper with Sierpiński, [1922b] – defining a peculiar topological space as the graph of a suitable function – was again successfully applied in [1925] (co-authored by Knaster), where a biconnected G_δ-set in the plane is constructed, and in [1932b], where an ingenious example of a weakly one-dimensional space is given. This last example has quite recently been used by G. Delistathis and S. Watson to solve an old problem in dimension theory, viz., to establish the existence of a regular space with a countable network and different dimensions dim and Ind.

We should also cite here Kuratowski's "measure of non-compactness" $\alpha(X)$ which found applications in functional analysis. This is a numerical invariant, defined in [1930a] for bounded metric spaces, which has the property that

$\alpha(X) = 0$ for a complete metric space X if and only if X is compact.

Kuratowski was a true master in simplifying proofs and in finding simple definitions for various mathematical objects. Let us recall here two such definitions using the distance $\rho(x, A)$ from a point x to a subset A of a metric space (X, ρ):

Assigning to every bounded subset A of X the function $f_A \in C^*(X)$ defined by letting $f_A(x) = \rho(x, A) - \rho(x, a)$, where $a \in X$ is fixed, yields an isometric embedding of the space of all non-empty bounded closed subsets of X with the Hausdorff metric to the space of all bounded continuous real-valued functions on X (this was proved in the 1948 French edition of *Topology* I, and in [1935b] for the embedding of X to $C^*(X)$).

For a closed subspace M of a metric space X and a continuous mapping $f : M \to L$ onto a subspace L of the Hilbert cube $I^{\aleph_0}$, the formula $F(x) = (f^*(x), \rho(x, M), \rho(x, M) \cdot f_x)$, where $f^* : X \to I^{\aleph_0}$ is a continuous extension of f and $f_x \in C^*(X)$ is defined as in the last paragraph, yields a continuous mapping $F : X \to I^{\aleph_0} \times R \times C^*(X)$ such that $F|X \setminus M$ is a homeomorphism of $X \setminus M$ onto $Y \setminus L$ ([1938b]; if f is a homeomorphism, then $F : X \to F(X)$ is a homeomorphism as well). This is a separable case of a 1938 theorem of Hausdorff, proved in a much more involved way. As observed later by Arens (who applied the paracompactness of metric spaces), the same formula with $I^{\aleph_0}$ replaced by $C^*(L)$ works in the case of an arbitrary metric space L, i.e. yields Hausdorff's theorem in its full generality).

2.6. HYPERSPACES AND SELECTORS. FUNCTION SPACES

Kuratowski's interest in spaces of closed subsets (hyperspaces) originated in relation with his method of evaluating the borelian class of sets. In [1932] he proved, among other things, that the families of all arcs and of all simple closed curves in a compact metric space X are $F_{\sigma\delta}$-sets in the space of all nonempty closed subsets of X with the Hausdorff metric. He returned to hyperspaces later in his life. In [1956] he described a method of defining complete metrics on some families of compact subsets of a completely metrizable space, e.g. on the family of all compact subsets that are locally connected in dimensions $\leq n$ or on the family of all locally connected continua. And in [1973] he established a clever and useful theorem on independent sets (a similar result was obtained somewhat earlier by J. Mycielski): if X is a complete metric space and $R \subset X^n$ is a closed and nowhere dense n-ary relation in X, then the family of all sets $F \subset X$ independent in R constitutes a dense G_δ-set in the space of all nonempty closed subsets of X (recall that F is independent in R if for every $(x_1, x_2, \ldots, x_n) \in F^n$ with $x_i \neq x_j$ for $i \neq j$ we have $(x_1, x_2, \ldots, x_n) \notin R$).

From 1965 till the end of his life, Kuratowski was mainly interested in a particular aspect of hyperspaces, viz., in set-valued mappings (or multifunctions) and in selectors. His interest in multifunctions arose much earlier: to obtain his results from [1932], he there defined the notions of lower and upper semi-continuity for set-valued mappings and studied their properties. Since then, these notions have proved crucial in the theory of selectors, and, accordingly, in [1963] Kuratowski extended his definitions of semi-continuity for set-valued mappings to the context of topological spaces and the Vietoris topology. The series of his 13 papers devoted to multifunctions and selectors opens with the often cited paper [1965], written jointly with C. Ryll-Nardzewski. The power of the main theorem of that paper consists in its properly chosen generality: it is formulated in such a way that it could be applied in topology, descriptive set theory and measure theory. Thus, one considers a field $\boldsymbol{L}$ of subsets of a set X (i.e. a family of subsets of X closed under the operations of taking finite unions and intersections, as well as complements) and denotes by $\boldsymbol{S}$ the family of all countable unions of members of $\boldsymbol{L}$. Now, if F is a function assigning to each point $x \in X$ a closed subset $F(x)$ of a separable complete metric space Y and if $\{x : F(x) \cap G \neq \emptyset\} \in \boldsymbol{S}$ whenever $G \subset Y$ is open, then there exists a selector $f : X \to Y$ such that $f^{-1}(G) \in \boldsymbol{S}$ whenever $G \subset Y$ is open. In particular, for every lower or upper semicontinuous function F assigning to each point x of a metric space X a closed subset $F(x)$ of a separable complete metric space Y there exists a selector $f : X \to Y$ such that $f^{-1}(G)$ is an F_σ-set, i.e. a selector of the first Baire class.

We shall leave aside Kuratowski's theorems on selectors that belong to descriptive set theory or measure theory, and concentrate on purely topological results. One series of such results pertains to choice functions, i.e. functions choosing individual points from subsets of a given space. In [1965] it is shown that if Y is a separable and complete metric space, then there always exists a choice function of the first Baire class for the family of all nonempty closed subsets of Y (if $\dim Y = 0$, there exists a continuous choice function). The existence of a continuous choice function is quite exceptional: as shown in [1970] (co-authored by S.B. Nadler and G.S. Young), if for all two-element and one-element subsets of a metric continuum Y (of a locally compact separable metric space Y) there exists a continuous choice function, then Y is an arc (Y is homeomorphic to a subset of the real line). Another series of results pertains to selectors from partitions of spaces into nonempty disjoint closed sets. Here the most important result is a general theorem on selectors from partitions (in the spirit of the main result from [1965]) obtained jointly with A. Maitra in [1974].

Function spaces were extensively exploited in Kuratowski's study of dimension. As we remember, their completeness allowed the application of the category method and yielded most interesting results. On this occasion Kuratowski discovered many properties of function spaces which were useful for his study, but are too technical to be discussed here.

In the late papers [1968], [1969] and [1971] he resumed the subject, applying his results on semicontinuous set-valued functions. In the first of these papers he proved that the family of all monotone mappings (i.e. continuous mappings with connected fibers) of a compact metric space X to a complete metric space Y is completely metrizable, and thus is a G_δ-set in the space of all continuous mappings. In the second one he proved, with his co-author R.C. Lacher, that under the additional assumption that Y is compact and locally connected the family of all monotone continuous mappings from X to Y is a closed set in the space of all continuous mappings (the latter result was obtained for topological spaces and the compact-open topology). And in the last of the three papers he showed that under natural assumptions the class of light mappings and the class of non-alterating mappings are G_δ-sets in the space of all continuous mappings.

We should mention here also the paper [1955], devoted to the space of "partial functions", i.e. functions from (varying) closed subsets of a compact metric space X to a metric space Y. For such functions there exists a natural notion of convergence, and Kuratowski showed that this convergence stems from a metric; moreover, he proved in [1956] that when Y is complete, then the space of partial functions is completely metrizable. The space of partial functions proved useful in the theory of differential equations.

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Kazimierz Kuratowski published 172 research papers, two monographs *Topology* and *Set Theory*, the latter co-authored by A. Mostowski, and two textbooks (*Introduction to Set Theory and Topology* and *Introduction to Calculus*), both translated into many languages. He has also written some thirty papers on mathematical life in Poland, a book of remembrances and reflections *A half Century of Polish Mathematics*, and a more personal book *Notes for an Autobiography* (available only in Polish).

The list of mathematicians that obtained their doctorate under him opens with the name of S. Ulam, continues with S. Eilenberg (who was also Borsuk's student), R. Sikorski, S. Mrówka, J. Jaroń, R. Engelking, M. Całczyńska-Karłowicz, J. Krasinkiewicz, and closes with J. Kaniewski. The list of those who were influenced by his style of doing and presenting mathematics would be much longer.

His unforgettable style was precise, gracefully concise, simple, and most elegant, not to say charming. The precision and conciseness have been related to his inner need for clarity and his excellent feeling for logic. The simplicity was obtained by a deep analysis and discovering the true core of the problem. The elegance was his personal secret, related to his belief in analogy between mathematics and music. As many of those who admired him, I tried to grasp the secret; it proved impossible. I succeeded though in discovering a part of it. It seemed that for Professor Kuratowski lecturing was a mere pleasure, and that his excellent talks were dictated by inspiration and natural ease. There was a saying in Warsaw during my student years that his lectures, simply copied from the blackboard, would produce a flawless book. This was absolutely true, but – as I learned later – the book has been written, it was his set of notes. All that he had to say was perfectly prepared and most often written down.

Those who never met him can get an idea of Kuratowski's personal and scientific style and manner by reading his *Topology*. The two volumes, written in French, and rewritten for each of the two subsequent French editions, were translated "revised and augmented" into English in 1966 and 1968. This masterly and highly original synthesis of set-theoretic topology – the life-work of a great mind – remains a masterpiece of mathematical literature.

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R.H. BING'S HUMAN AND MATHEMATICAL VITALITY

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It was a dark and stormy night. So R.H. Bing volunteered to drive some stranded mathematicians from the fogged-in Madison airport to Chicago. Freezing rain pelted the windscreen and iced the roadway as Bing drove on – concentrating deeply on the mathematical theorem he was explaining. Soon the windshield was fogged from the energetic explanation. The passengers too had beaded brows, but their sweat arose from fear. As the mathematical description got brighter, the visibility got dimmer. Finally, the conferees felt a trace of hope for their survival when Bing reached forward – apparently to wipe off the moisture from the windshield. Their hope turned to horror when, instead, Bing drew a figure with his finger on the foggy pane and continued his proof – embellishing the illustration with arrows and helpful labels as needed for the demonstration.

Two of Bing's mathematical colleagues, Armentrout and Burgess, independently told us versions of this memorable evening. Those of us who knew Bing well avoided raising mathematical questions when he was driving.

In this paper, we will give a sense of the personal life and style of R.H. Bing. This personal view makes this paper something of a diversion from the strictly mathematical and intellectual aspects of the history of general topology. Perhaps the reader will excuse that detour from the main road and enjoy this paper for what it is – a celebration of a life well led, a life whose joy came partly from significant contributions to topology and partly from an overflowing *joie de vivre*.

R.H. Bing loved to work on problems in topology, perhaps because he was so successful at solving them. He was a mathematician of international renown including authorship of seminal research papers in several branches of topology. His work garnered for him honors such as membership in the National Academy of Sciences and presidencies of both the Mathematical Association of America and the American Mathematical Society. But we who knew him will remember

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him most for his zest for life which infected everyone around him with a contagious enthusiasm and good humor.

R.H. Bing started and ended in Texas. He was born on October 10, 1914, in Oakwood, Texas, and there he learned the best of the distinctively Texas outlook and values. What he learned in Oakwood guided him clearly throughout his life. He had a strong Texas drawl which became more pronounced proportionate to his distance from Texas; and he spoke a little louder than was absolutely necessary for hearing alone. He might be called boisterous with the youthful vigor and playful curiosity that he exuded throughout his life. He was outgoing and friendly and continually found ways to make what he did fun. You could hear him from down the hall laughing with his T.A.'s while grading calculus exams or doing other work that deadens most people. He did not sleep well and when he woke at 4 a.m., he would get up and work. He especially enjoyed working on things requiring loud hammering at that hour on the grounds that if you are going to be up at 4, the family should know about it. He practiced the traditional Texas value of exercising independent judgment, both in general matters and in matters mathematical. He treated people kindly and gently – unless he knew them, in which case it was more apt to be kindly and boisterously.

Both of Bing's parents were involved in education. His mother was a primary teacher and his father was the superintendent of the Oakwood School District. Bing's father died when R.H. was five, so Bing most remembered his mother's impact on his character and interests. Bing attributed his love for mathematics to his mother's influence. He recalled that she taught him to do mental arithmetic quickly and accurately and to enjoy competition both physical and mental.

After high school, Bing enrolled in Southwest Texas State Teachers College in San Marcos (now Southwest Texas State University) and received his B.A. degree in 1935 after two and a half years there. Later in life, Bing was named as the second Distinguished Alumnus of Southwest Texas State University. The first person so honored was Lyndon Baines Johnson. Bing's college education had prepared him as a high school mathematics teacher. He also was a high jumper on the track team and could jump his own height – which was over six feet.

Bing's final academic position was as the Mildred Caldwell and Blaine Perkins Kerr Centennial Professor in Mathematics at The University of Texas at Austin, but his first academic appointment was as teacher at Palestine High School in Palestine, Texas. There his duties included coaching the football and track teams, teaching mathematics classes, and teaching a variety of other classes, one of which was typing. His method of touch typing involved anchoring his position over the keys by keeping some constant pressure on his little fingers. This habit was hard to break, apparently, because later he said that when he used an electric typewriter or computer keyboard (neither of which he did often) he tended to produce large numbers of extraneous "a's".

Nowadays one frequently hears complaints about a school system which gives the football coach the added assignment of teaching a mathematics class. One wonders if those football boosters of a bygone day in Palestine complained to the local school board about a real mathematics teacher coaching the football team.

In an effort to improve public school education in the '30's, the Texas Legislature had approved a policy whereby a teacher with a Master's degree would receive more pay than a teacher with a Bachelor's degree. So, many teachers saved (and scrimped) during the nine- month session and went to summer school during the three summer months in an effort to upgrade their talents and their salaries. Bing was among them.

R.H. had begun public school teaching in 1935, and by taking summer school courses at The University of Texas at Austin, he had earned a Master of Education degree in 1938. During one summer, Bing took a course under the late Professor R.L. Moore. Moore was inclined to deprecate the efforts of an older student such as Bing was, so Bing had to prove himself. But he was equal to the task.

Bing continued to take some summer courses while teaching in the high schools. In 1942 Moore was able to get Bing a teaching position at The University which allowed him to continue graduate study to work towards a doctorate, and to try his hand at research. [This practice of allowing a person of instructor rank or higher to work towards an advanced degree was allowed in those days.]

An unofficial rating scheme sometimes used by R.L. Moore and his colleagues went something like this: You could expect a student with Brown's talents and abilities every year; you could expect a student with Lewis's talents and abilities once every four years; but a student with Smith's talents and abilities came along only once in twelve years. Bing's talents and abilities threw him in the twelve- year class, or in an even higher class, since he was one of the most distinguished mathematicians ever to have received his degree from The University of Texas at Austin. Several of Moore's later graduate students have written that in the days after Bing, Moore used to judge his students by comparing them with Bing – probably not to their advantage.

Bing received his Ph.D. in 1945 – writing his dissertation on "Planar Webs" [B3]. Planar webs are topological objects now relegated to the arcana of historical topological obscurity. The results from his dissertation appeared in one of his earliest papers in the Transactions of the American Mathematical Society [B7]. He told us that the Transactions had sent him fifty reprints at the time and if we were interested we could have some because he still had forty-nine or so left.

But Bing did not have long to wait for recognition of his mathematical talent. He received his Ph.D. degree in May 1945, and in June 1945, he proved a famous, long-standing unsolved problem of the day known as the Kline Sphere Characterization Problem [B9]. This conjecture states that a metric continuum in which every simple closed curve separates but for which no pair of points separates the space is homeomorphic to the 2-sphere.

When word spread that an unknown young mathematician had settled this old conjecture, some people were skeptical. Moore had not checked Bing's proof since it was his policy to cease to review the work of his students after they finished their degrees. Moore believed that such review might tend to show a lack of confidence in their ability to check the work themselves. So when a famous professor wired Moore asking whether any first-class mathematician had checked the proof, Moore replied, "Yes, Bing had."

Primarily because of the renown among mathematicians generated by his having solved a famous conjecture, Bing was offered positions at Princeton University and at the University of Wisconsin, Madison. Moore naturally wrote letters of recommendation. One comment he made was that, although the Kline Sphere Characterization Problem was a much better known topic than that of planar webs, Moore felt that it was Bing's work on planar webs that demonstrated that Bing had the mathematical strength to be an outstanding mathematician.

One of the leading topologists of the time was at Princeton, but Bing did not wish to follow in anyone's footsteps, so in 1947 he accepted a position at Wisconsin. He remained at Wisconsin for 26 years except for leaves: one at the University of Virginia (1949-50), three at the Institute for Advanced Study in Princeton (1957-58, 1962-63, 1967), one at The University of Texas at Austin (1971-72), and brief teaching appointments elsewhere. He returned to The University of Texas at Austin in 1973; but it was during his tenure at the University of Wisconsin, Madison that his most important mathematical work was done and his prominent position in the mathematical community established.

Bing's early mathematical work primarily concerned topics in general topology and continua theory. He proved theorems about continua that are surprising and still central to the field. Among these results is Bing's characterization of the pseudo arc as a homogeneous indecomposable, chainable continuum [B14]. The result that the pseudo arc is homogeneous contradicted most people's intuition about the pseudo arc and directly contradicted a published, but erroneous, "proof" to the contrary. Bing continued to do some work in continua theory throughout his career; including directing a Ph.D. dissertation in the subject at UT in 1977.

Around 1950, one of the great unsolved problems in general topology was the problem of giving a topological characterization of the metrizability of spaces. In 1951, Bing gave such a characterization in his paper *Metrization of topological spaces* in the Canadian Journal of Mathematics [B20]. Nagata and Smirnov proved similar, independent results at about the same time, so now the result is referred to as the Bing–Nagata–Smirnov Metrization Theorem. That 1951 paper of Bing has probably been referred to in more papers than any other of his papers, even though he later was identified with an altogether different branch of topology.

Bing's *Metrization of Topological Spaces* paper enfolds in a manner consistent with an important strategy he practiced in doing mathematics. He always explored the limits of any theorem he proposed to prove or understand. Consequently, he would habitually construct counterexamples to demonstrate the necessity of each hypothesis of a theorem. In this paper, Bing proved theorems numbered 1 to 14 interspersed with examples labeled A through H. The impact of this paper came both from his theorems and from his counterexamples.

Bing's metrization theorems describe spaces with bases formed from countable collections of coverings or spaces where open covers have refinements consisting of countable collections of sets. He defined and discussed screenable spaces, strongly screenable spaces, and perfectly screenable spaces – terms that have been largely replaced by the "σ-" style of terminology.

He proved in this paper that regular spaces are metrizable if and only if they are perfectly screenable. (The term *perfectly screenable* means that the space has a σ-discrete basis.)

His metrization theorems hinged strongly on his understanding of a strong form of normality and certainly one of the legacies of this paper is his definition of and initial exploration of *collectionwise normality.*

This paper contains the theorem that a Moore space is metrizable if and only if it is collectionwise normal.

After identifying this important property of collectionwise normality, he explored its limits by constructing an example of a normal space that is not collectionwise normal. Bing was known for his imaginative naming of spaces and concepts, but this example enjoys its enduring fame under the mundane moniker of "Example G".

Immediately following his description of Example G, Bing included the following paragraph that formed the basis of countless hours of future mathematicians' labors:

> "One might wonder if Example G could be modified so as to obtain a normal developable space which is not metrizable. A developable space could be obtained by introducing more neighbourhoods into the space [Example G]. However a difficulty might arise in introducing enough neighbourhoods to make the resulting space developable but not enough to make it collectionwise normal."

Nowadays if you refer to "Bing-type topology," you are referring to a certain style of geometric analysis of Euclidean 3-space that came to be associated with Bing because of the fundamental work he did in the area and the distinctive style with which he approached it. The first paper Bing wrote in this area appeared in 1952 [B25] and contains one of his best-known results. The result in this paper describes a method of shrinking geometric objects in unexpected ways. When Bing first worked on the question considered in this 1952 paper, he naturally did not know whether it was true or false. He claimed that he worked two hours trying to prove it was true, then two hours trying to prove it was false. When he originally worked on this problem, he used collections of rubber bands tangled together in a certain fashion to help him visualize the problem. The mathematics that Bing did is abstract, but he claimed to get ideas about these abstruse problems from everyday objects. A final note about this problem involves a paper which Bing wrote in 1984 containing one of his last results [B16]. If one shrinks the rubber bands in the manner described in Bing's 1952 paper, each rubber band becomes small in diameter, but very long. It became interesting to know whether one could do a similar shrinking without lengthening the bands – in other words, could you do the same thing with string as Bing had proved could be done with rubber. Bing's original procedure had been studied by numerous graduate students and research mathematicians for more than 30 years, and yet no one had been able to significantly improve Bing's shrinking method. It was left for Bing himself to prove that "Shrinking without lengthening" (the title of this final paper) is possible.

Bing's results in topology grew in number and quality. He proved several landmark theorems and then raised lots of related questions. Because of his habit of raising questions, many other mathematicians and students were able to prove good theorems in the areas of mathematics which he pioneered. He emphasized the importance of raising questions in one's papers and encouraged his students and colleagues to do so. He felt that mathematicians who read a paper are often more interested in what remains unknown than they are interested in what has been proved.

The period from 1950 until the mid-60's was Bing's most productive period of research. He published about 116 papers in his lifetime – most during this period at the University of Wisconsin, Madison.

His research success brought him honors, awards, and responsibilities. He was quickly promoted through the ranks at the University of Wisconsin, becoming a Rudoph E. Langer Research Professor there in 1964. He was a Visiting Lecturer of the Mathematical Association of America (1952-53, 1961-62) and the Hedrick Lecturer for the Mathematical Association of America (1961). He was chairman of the Wisconsin Mathematics Department from 1958 to 1960, but administrative work was not his favorite. He was President of the Mathematical Association of America (1963-64). In 1965, he was elected to membership in the National Academy of Sciences. He was Chairman of the Conference Board of Mathematical Sciences (1966-67) and a U.S. Delegate to the International Mathematical Union (1966, 1978). He was on the President's Committee on the National Medal of Science (1966-67, 1974-76), Chairman of the Division of Mathematics of the National Research Council (1967-69), Member of the National Science Board (1968-75), Chairman of the Mathematics Section of the National Academy of Sciences (1970-73), on the Council of the National Academy of Sciences (1977-80), and on the Governing Board of the National Research Council (1977-80). He was a Colloquium Lecturer of the American Mathematical

Society in 1970. In 1974 he received the Distinguished Service to Mathematics Award from the Mathematical Association of America. He was President of the American Mathematical Society in 1977-78. He retired from The University of Texas at Austin in 1985 as the Mildred Caldwell and Blaine Perkins Kerr Centennial Professor in Mathematics. He received many other honors and served in many other responsible positions throughout his career. He lectured in more than 200 colleges and universities in 49 states and in 17 foreign countries.

Bing believed that mathematics should be fun. He was opposed to the idea of forcing students to endure mathematical lectures which they did not understand or enjoy. He liked to work mathematics out for himself and thought that students should be given the opportunity to work problems and prove theorems for themselves. During his years in Wisconsin, Bing directed a very effective training program for future topologists. The first year graduate topology class which he often taught there would sometimes number 40 or more students. He directed the dissertations of 35 students and influenced many others during participation in seminars and research discussions.

Bing enjoyed teaching and felt that experiments in teaching were usually successful – not because the new method was necessarily better, but because doing an experiment showed an interest in the students which they appreciated and responded to. Here are a couple of the experiments he tried while teaching at UT. Bing thought that a person who could solve a problem quickly deserved more credit than a person who solved it slowly. He would say that an employer would rather have an employee who could solve two problems in as much time as it took for someone else to solve one. So in some of his undergraduate classes he introduced "speed points." For a fifty minute test, he gave an extra point for each minute before the fifty minutes elapsed that the test was submitted. He noticed that often the people who did the work the quickest also were the most accurate. Speed points were somewhat popular and sometimes he would let the class vote on whether speed points would be used on a test.

Another experiment in test giving was not popular. One day Bing had prepared a calculus test that he realized was too long. Instead of deleting some questions, however, he decided to go ahead and give the test, but as he phrased it, "Let everyone dance to the tune of their own drummer." That is, each person could do as many, or as few, of the problems as he or she wished and would be graded on the accuracy of the problems submitted. The class was quite angry when the highest score was obtained by a person who had attempted only one problem.

In the 1971-72 school year, Bing accepted an offer to visit the Department of Mathematics at The University of Texas at Austin. In 1973, the mathematics department, under Leonard Gillman's chairmanship, persuaded Bing to accept a permanent position at UT. When he arrived in 1973, Bing was the highest paid professor in the state of Texas. He soon showed that he was worth the money.

Bing believed that part of the fun of life was to take on a variety of challenges. When he accepted the position at UT, he came with the idea of building UT's mathematics department into one of the top 10 state university mathematics departments in the country. While he was at Texas from 1973 until his death in 1986, he helped to improve the research standing of the department by recruiting new faculty and by helping to change the attitudes and orientation of the existing faculty. Raising research standards was the watchword of that period and is the guiding principle for the mathematics department now. Bing was chairman of the department from 1975 to 1977 but used his international prominence in recruiting efforts throughout his stay at UT. The UT Department of Mathematics was considered one of the most improved departments over the period of Bing's tenure at UT. The 1983 report of the Conference Board of Associated Research Councils listed Texas as the second most improved mathematics department in research standing during the period 1977-1982, ranking it number 14 among state university mathematics departments at that time.

Bing accomplished much during his life and left us with many ideas, personal and mathematical, to consider and enjoy. He left topologists a treasure-trove of theorems and techniques and left the UT Department of Mathematics with a goal and thirteen years of good progress toward it. He was a man of strong character and integrity who liked to understand things for himself. For example, he never claimed to understand a theorem unless he personally knew a proof of it. He made decisions based on his own experience – relying on his independent judgment of a person or a cause whenever possible, rather than averaging the opinions of others. He was a kind man and respected people for their own merits rather than measuring them on a single scale.

R.H. Bing died on April 28, 1986. He suffered from cancer and heart troubles during his last years; but he never complained about his health problems nor did he allow discomfort to dampen his enthusiasm and good spirits. He was an exemplary person. His friends, his family, and his students have been enriched beyond bound by his character, his wisdom, and his unfailing good cheer and continue to be enriched by his memory.

R.H. Bing's PhD Students

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FROM DEVELOPMENTS TO DEVELOPABLE SPACES

The Evolution of a Topological Idea

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Contents

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Abstract

This investigation highlights a thread of mathematical activity that begins with E.H. Moore's introduction of the mathematical notion of a "development for an abstract class" in 1910, before general topology was established as a field of mathematics, and suggests an evolution of ideas that leads to the present-day topological concept of "developable space", as it was defined by R.H. Bing in 1951.

In addition, we are guided by the following principle:

> *to reconstruct the viewpoint of a writer, one must consider the backdrop of a classical past that remains in view in contrast with the powerful present that informs and judges this past.*
>
> John McCleary [1989]

Thus, rather than paraphrase or "report" the evolution of concepts, as I interpret it, I want, first, to create the historical context in which an activity took place; then, to allow the researcher himself to communicate his contribution by quoting the work; and finally, to add my reflection and observations about the work. In this way I hope "to reconstruct the viewpoint of the writer" and present that work in "the backdrop of a classical past" and still not judge the work in the light of "the powerful present".

Introduction

The topological concept of developability has emerged in the last half of this century as a rich and almost ubiquitous one.

A topological space $(X, \mathcal{T})$ is *developable* iff it has either of the following equivalent properties:

(i) There is a sequence $\{\mathcal{G}_n\}$ of open covers of X such that, if $U_n(x) = \bigcup\{G \in \mathcal{G}_n \mid x \in G\}$ for every x and every n, then for every point p in X, if $x_n \in U_n(p)$ for every n, then the sequence $\langle x_n \rangle$ converges to p;

(ii) there is a distance $\delta(p, q)$ between any two points p and q in X that is nonnegative and real–valued such that

  (a) every convergence sequence is Cauchy and,

  (b) $\{U_n(p) | n \in \mathbf{N}\}$ is a countable neighborhood base for each p in $(X, \mathcal{T})$, where $U_n(x) = \{y \in X \mid \delta(x, y) < 1/2^n\}$.

Thus, the concept of developability is essentially either a first countability property in which the topology is determined by the convergent sequences, or it is a geometric property of a distance whose spheres determine the topology. In either case the essence of the developability concept strikes close to the seminal notions that Maurice Fréchet had in mind in 1906 when he initiated a study of *classe* ($\mathcal{L}$) and *classe* ($\mathcal{E}$) in the theory of "abstract sets".[1] How these ideas planted the seeds that were to lead to the field of topology as we know it today is part of the evolution this paper seeks to unfold.

Moore spaces are regular Hausdorff developable spaces. An insider's view on the importance of Moore spaces and the role they played in generating mathematical activity in the first half of this century is related in the chapters by Ben Fitzpatrick [1997] and Burton Jones [1997] in volume one of *Handbook of the History of General Topology*. Of particular interest to us is the Normal Moore Space Conjecture[2] that emerges in the work of Jones [1937], a student of R.L. Moore. At the same time in Alexandroff and Niemytzki [1938] the Russian, V.W. Niemytzki, is refining some earlier work of E.W. Chittenden on distances and finding striking characterizations that relate to developable spaces. All of this sets the stage for Bing's paper [1951], which raises significant questions, introduces tantalizing ideas and presents interesting examples – and gives birth to a generation of set-theoretic topologists.

For further details and additional observations that highlights the importance of developability in the recent history of general topology, see R.E. Hodel's chapter, "A History of Generalized Metrizable Spaces" in this volume of *Handbook of the History of General Topology.*

1. 1900–1910: Primordial Beginnings

David Hilbert had just published his *Grundlagen der Geometrie.* Georg Cantor and Richard Dedekind had just made significant advances in placing analysis on a more firmly-based set-theoretic foundation. The mathematical community as a whole was beginning to think of classes of objects which were "completely unconditioned"; that is, a class might be considered as an *abstract class* of elements without making any specific reference to, or assumptions about, the nature of those elements. E.H. Moore had just established the University of Chicago as the leading center for mathematical research and graduate study in the United States.

In this context mathematical activity often centered around finding independent, consistent axioms for *abstract* groups and fields, for a *linear continuum* or for *ordinary* complex algebra. Eliakim Hastings Moore, Leonard Eugene Dickson, Oswald Veblen, Edward V. Huntington and others are filling the pages of the *Transactions of the American Mathematical Society* with sets of axioms, revisions of sets of axioms and critiques of the aesthetics of revised versions of sets of

[1] As we will see, the *classe* ($\mathcal{L}$) is defined by making certain assumptions about the convergence of sequences, whereas the *classe* ($\mathcal{E}$) is defined by the use of a distance.

[2] The importance of this conjecture is discussed briefly in section 5 of this work; see also Mary Ellen Rudin's chapter (Rudin [1997]), "The Early Work of F.B. Jones," in volume one of *Handbook of the History of General Topology.*

axioms.[3] Often at issue is the appropriate selection of axioms from which the fundamental results of a theory can be proved. In some cases an author is attempting to be "categorical", while others are seeking the cleanest, most aesthetically appealing axiomatic foundation on which to base a theory.

Speaking generally of E.H. Moore, Gilbert A. Bliss, one of Moore's first students, recalls that "If there were two characteristics of his research which could be distinguished above others, I should say that they would be rigor and generality."[4] Furthermore, as Bliss characterizes the mathematical climate of the time at the University of Chicago, "The appearance of Hilbert's book on the foundations of geometry in 1899 attracted the attention of Moore and his students to postulational methods, including the earlier work of Pasch and Peano as well as that of Hilbert."[5]

E.H. Moore himself mentions briefly his motives for generalization and points specifically to the influence of Fréchet.

> *The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.*[6]
>
> As apart from the determination and extension of notions and theories in analogy with simpler notions and theories, there is the extension by direct generalization. The Cantor movement is in this direction. Finite generalization, from the case $n = 1$ to the case $n = n$, occurs throughout Analysis, as, for instance, in the theory of functions of several independent variables. The theory of functions of a denumerable infinity of variables is another step in this direction [which Moore refers in a footnote as undertaken by D. Hilbert in [1906] and [1909].]
>
> We notice a more general theory dating from the year 1906. Recognizing the fundamental role played by the notion *limit-element* (number, point, function, curve, etc.) in the various special doctrines, Fréchet has given, with extensive applications, an abstract generalization of a considerable part of Cantor's theory of classes of points and of the theory of continuous functions on classes of points. Fréchet considers a general class $\mathcal{P}$ of elements p with the notion *limit* defined for sequences of elements. The nature of the elements p is not specified; the notion *limit* is not explicitly defined; it is postulated as defined subject to specified conditions. For particular applications explicit definitions satisfying the conditions are given. More theorems are obtained by defining *limit* as usual in terms of the notion *distance (écart, voisinage)* postulated as defined, subject to specified conditions, for the pairs of elements.†[7]

[3] See, for example, Volume 6, published in 1905.
[4] Bliss, *The scientific work of Eliakim Hastings Moore*, p. 501.
[5] Bliss, *Ibid.*, p. 503.
[6] E.H. Moore, *Introduction to a Form of General Analysis*, p. 1.
[7] Moore, *Ibid.*, pp. 3–4.

Here † refers the reader to the footnote:

> F. Rietz in his paper: *Stetigkeitsbegriff und abstracte Mengenlehre*, read before the section on Analysis of the Rome Congress (Atti, etc., vol 2 (1909), pp. 18–24), indicates a more general theory involving, instead of Fréchet's *limit* for sequences of elements, the notions: *element of condensation (Verdichtungsstelle)*; *connection (Verkettung)*, postulated as defined for subclasses of the class $\mathcal{M}$; for pairs of subclasses of the class $\mathcal{M}$.

It is with this "backdrop of a classical past" that we examine Moore's ideas about developments.

1.1. E.H. MOORE'S NOTION OF DEVELOPMENT FOR AN ABSTRACT CLASS

Our story begins in 1910 when the American Mathematical Society publishes E.H. Moore's "text", *Introduction to a Form of General Analysis*, which is based on a series of lectures he gave in September of 1906 at Yale University as part of the AMS Colloquium series of lectures. The review (Birkhoff [1911]) of this publication in the *Bulletin of the American Mathematical Society* by G.D. Birkhoff, another of Moore's first students at the University of Chicago, summarizes Moore's point of view and his intention for the work.

> It is obvious to those who have been following recent mathematical progress that, since the researches of Hill, Volterra, and Fredholm in the direction of extended linear systems of equations, mathematics has been in the way of a great development. That attitude of mind which conceives of the function as a generalized point, of the method of successive approximation as a Taylor's expansion in a function variable, of the calculus of variations as a limiting form of the ordinary algebraic problem of maxima and minima is now crystallizing into a new branch of mathematics under the leadership of Pincherle, Hadamard, Hilbert, Moore, and others. For this field, Professor Moore proposes the term "General Analysis", defined (page 9) as "the theory of systems of classes of functions, functional operations, etc., involving at least one general variable on a general range."[8] He has fixed attention on the most abstract aspect of this field by considering functions of an absolutely general variable. The nearest approach to a similar investigation is due to Fréchet (Paris thesis, [1906]), who restricts himself to variables for which the notion of a limiting value is valid.
>
> In the General Analysis we consider a class $\mathcal{M}$ of a real single valued function ϕ_p of the variable p; important illustrative cases are:
>
> I. $p = 1, 2, \ldots, n$; $\phi_p \equiv (\phi_1, \phi_2, \ldots, \phi_n)$, where $\phi_1, \phi_2, \ldots, \phi_n$ are arbitrary real quantities.

[8]The "range" of a function for E.H. Moore is the "domain" in current terminology.

II. $p = 1, 2, \ldots, n, \ldots$; $\phi_p \equiv (\phi_1, \phi_2, \ldots, \phi_n, \ldots)$, where $\phi_1, \phi_2, \ldots$ are restricted in that $\lim_{p=\infty} \phi_p = 0$.

III. The same as II, except that the convergence restriction is now that $\sum_{p=1}^{\infty} {\phi_p}^2$ converges.

IV. $0 \leq p \leq 1$; ϕ_p is any real continuous function of p.

The class $\mathcal{M}$ is linear (L) if the sum of every two functions of $\mathcal{M}$ or the product of one such function by any real constant is in $\mathcal{M}$.

The class $\mathcal{M}$ is closed (C) if the limit ϕ_p of a convergent sequence α_p, $\beta_p, \ldots$ of functions of $\mathcal{M}$ is itself a function of $\mathcal{M}$ whenever there exists a function λ_p of $\mathcal{M}$ such that the difference between the successive members of the sequence and the limit ϕ_p becomes and remains uniformly not greater in absolute value than $\epsilon\lambda_p$, where ϵ is an arbitrary small positive quantity. ...

The class $\mathcal{M}$ possesses the dominance property D if for every finite or infinite sequence $\alpha_p, \beta_p, \ldots$ in $\mathcal{M}$ there exists a dominating function λ_p of $\mathcal{M}$ such that each particular element of the sequence does not exceed in absolute value a suitable constant multiple of λ_p for any p.

Finally if the absolute value of any function in the class $\mathcal{M}$ is a function of $\mathcal{M}$, that class is said to be absolute (A).[9]

In this memoir Moore studies in some detail certain closure and dominance properties of classes $\mathcal{M}$ of real-valued functions with respect to the properties like A, D, L and C. In the course of obtaining a generalized setting for an abstract set, Moore introduces the notion of a development, which he points out is intended to generalize the notion of a partition as it is used in defining the Riemann integral.

Moore introduces his idea with an example of a development for $\mathcal{P}'$, which he says "denotes the interval: $0 \leq p' \leq 1$, of the real number system."[10] In Moore's notation this particular development is given as follows:

For the system:

$$((ml)) \;\; (l \leq m;\; m = 1, 2, 3, \ldots;\; l = 0, 1, 2, \ldots),$$

of indices ml we introduce the corresponding systems:

$$((r'^{\,ml}));\;\; \Delta' \equiv ((\mathcal{P}'^{\,ml})),$$

of rational numbers $r'^{\,ml}$:

$$r'^{\,ml} \equiv \frac{l}{m};$$

and of subintervals $\mathcal{P}'^{\,ml}$:

$$\mathcal{P}'^{\,ml} \equiv \left[p'^{\,ml}\right],\;\; r'^{\,m\,l-1} \leq p'^{\,ml} \leq r'^{\,m\,l+1},$$

[9]Birkhoff, *The New Haven Colloquium Lectures*, pp. 415–416.
[10]E.H. Moore, *Introduction to a Form of General Analysis*, p. 114.

where as an additional understanding:

$$r'^{\,m\ -1} \equiv 0; \ \ r'^{\,m\ m+1} \equiv 1.$$

The system:

$$\Delta' \equiv ((\mathcal{P}'^{\,ml})),$$

is a *development* of the class $\mathcal{P}'$, the *stage* m *of the development* being the system:

$$\Delta'^{\,m} \ \equiv \ (\mathcal{P}'^{\,ml}),$$

of $m+1$ intervals $\mathcal{P}'^{\,ml}(0 \leq l \leq m)$.[11]

In addition, Moore draws attention to some intrinsic characteristics of this development which he intends to generalize and thereby utilize in the theoretical framework that he will establish.[12]

The relation $K_2' \ \ldots$ is a property of pairs p_1', p_2' of elements of $\mathcal{P}'$ with integers m, having reference to the development Δ' of $\mathcal{P}'$. We write

$$K'_{p_1' p_2' m},$$

to denote that p_1', p_2' are two elements belonging to the same interval (say $\mathcal{P}'^{\,m_0 l_0}$) of some stage $m_0 \geq m$ of the development Δ'; ...

One understands how questions of infinitesimal nature on $\mathcal{P}'$ may be treated in terms of the development Δ' of $\mathcal{P}'$, the expression:

$$\exists\, d_e \ni \ A\,(p_1' - p_2') \leq d_e,$$

being replaced by

$$\exists\, m_e \ \ni \ K'_{p_1' p_2' m_e}.$$

With this example as a guide, Moore proceeds to introduce his general theory and his ideas about the essential nature of a development for an abstract class:

A *development* Δ *of a class* $\mathcal{P}$ *of elements* is a sequence $\{\Delta^m\}$ of systems Δ^m of subclasses of $\mathcal{P}$, each system Δ^m consisting of a finite number, say l_m, of subclasses of $\mathcal{P}$. The system Δ^m is *stage* m *of the development* Δ. ...

There is no question of order but there may be repetitions amongst the l_m classes of stage m, and it is to provide for this possibility of repetition that we say *system* Δ^m instead of *class* Δ^m.

With this understanding, we introduce an *index* l^m:

$$l^m = 1; 2; \ \ldots \ ; l_m,$$

[11]E.H. Moore, *Introduction to a Form of General Analysis*, p. 115.
[12]Moore, *Ibid.*, p. 116.

of stage m, and omit the superscript m of the index l^m when, in conjunction with the index m of the stage, it occurs in the bipartite index ml. We denote the l_m subclasses of stage m respectively by the notations:

$$\mathcal{P}^{ml}, \ [p^{ml}] \qquad\qquad (l = 1, 2, \ldots, l_m).$$

Then the stage m of the development Δ is the system:

$$\Delta^m \equiv \left(\mathcal{P}^{ml}\right),$$

and the development Δ is the system:

$$\Delta \equiv \left(\left(\mathcal{P}^{ml}\right)\right),$$

of subclasses $\mathcal{P}^{ml}$ of the class $\mathcal{P}$.[13] ...

As to an element p the indices l of existent classes of stage m are designated:

$$g^p;\ h^p,$$

according as the class $\mathcal{P}^{ml}$ contains or does not contain the element p. Then the class $\mathcal{P}^{pm}$:

$$\mathcal{P}^{pm} \equiv \bigcup \left(\mathcal{P}^{mg^p} \mid g^p\right),$$

is the least common superclass of the classes of stage m which contain p, the class of all elements each of which is by stage m of the development *directly connected* with the element p, in that they belong to the same class of stage m. The class $\mathcal{P}_{pm}$:

$$\mathcal{P}_{pm} \equiv \bigcup \left(\mathcal{P}^{pm_0} \mid m_0 \geq m\right),$$

is the least common superclass of the classes $\mathcal{P}^{pm_0}$ $(m_0 \geq m)$, the class of all elements each of which is by some stage $m_0 \geq m$ of the development directly connected with the element p.[14]

Moore goes on to point out that any "development Δ gives rise to a relation [for which] $K_{p_1p_2m}$ *denotes two elements* p_1, p_2 (*not necessarily distinct*) *belonging to the same class of some stage* $m_0 \geq m$."[15] This kind of binary relation had been noted on several other occasions by Moore. Specifically, he mentions[16] that "we had the metrical instances:

$$K_{pm} \cdot \equiv . \, p > m; \quad K_{p_1p_2m} \cdot \equiv . \, A(p_1 - p_2) \leq \frac{1}{m}. \ \dagger"$$

[13]E.H. Moore, *Introduction to a Form of General Analysis*, p. 136.
[14]Moore, *Ibid.*, pp. 137–138.
[15]Moore, *Ibid.*, p. 138.
[16]Moore, *Ibid.*, p. 126.

Here † refers to the footnote:

More generally, if δ is a nowhere negative function ... of the general nature of *distance*, e.g. Fréchet's *voisinage* or *écart*, an instance is

$$K_{p_1p_2m} \,.\equiv.\, \delta_{p_1p_2} \leq \frac{1}{m}.$$

A relation K_2 gives rise to numerous relations K_1, e.g. for every element p_0 of $\mathcal{P}$ there is a relation K_1, viz.,

$$K_{pm} \,.\equiv.\, K_{pp_0m}.$$

REFLECTION AND OBSERVATIONS ON THIS WORK OF E.H. MOORE.

One might suggest that the word *development* itself derives from the analogous use at this time of *developing a function* as an infinite or finite sum of functions of some sort. Thus, the analog for sets is to *develop* an abstract set as an infinite sum of subsets. In a parallel usage of the word Moore introduces the idea of developing a (real-valued) function in a function space as an infinite sum of functions from a *developmental system* of functions that are intrinsically associated with the development itself.

To place Moore's work in perspective, let us use current set-theoretic notation and suppose that for integers m and l, $\mathcal{P}_{m,l}$ is a subset of $\mathcal{P}$, then

$$\Delta^m = \{\mathcal{P}_{m,l} \mid l = 1, 2, \ldots, l_m\} \text{ is } \textit{stage } m \text{ of the development } \Delta, \text{ and}$$

$$\Delta = \{\Delta_m \mid m = 1, 2, \ldots\} \text{ is a countable set of sets of subsets of } \mathcal{P}.$$

Further, if

$$st(p, \Delta^m) \equiv \bigcup\{A \in \Delta^m \mid p \in A\} \quad \text{(that is, the } \textit{star of p in } \Delta^m \text{),}$$

then

$$\mathcal{P}^{pm} = st\,(p, \Delta^m)\,; \text{ and } \mathcal{P}_{pm} = \bigcup\{st(p, \Delta^{m_0}) \mid m_0 \geq m\}.$$

Although the stages themselves need not be "covers" as a general characteristic, Moore's examples have the property of being such. Elements that would occur in sets at the stage m in a development were said to have *been developed* at this stage.

Moore's use of developments in the foundations of general analysis suggests that the sets in Δ^m are intended to be the sets of *uniform diameter*, say $1/m$, since the idea originates with the example of a partition of the unit interval into subintervals of uniform length.

We note also that Moore's work reflects his underlying passion of being an engaging teacher. The ideas he wishes to introduce are first motivated with familiar examples or known facts; his notation is intended to be clear and suggestive in an attempt to be unambiguous and efficient; and, finally, he draws his readers into his discussions and invites their participation in the mathematical evolution of the ideas that are to serve as the centerpiece for his work.

On the other hand, let us note that Moore was not attempting to define some sort of "generalized space" or to create a generalized setting in which to study limits or continuous functions. According to Bliss, who spent more than twenty years as a colleague, the aim of Moore's work was to establish a setting for doing analysis in which "the special theories of linear equations which he desired to unify could be regarded as special instances of a general theory of linear functional equations in which the functions $\mu(p)$ involved are defined on an entirely unrestricted range $\mathcal{P}$ of elements p." But, perhaps most significantly, as Bliss explains, "the methods of Moore in the theory of linear functional equations are epoch-making in that they shift the attention from the properties of individual functions to the properties of classes of functions, and from the form of the operator J to its properties, thus attaining far-reaching generality."[17]

Despite all of this, Moore's research and seminars touched on ideas that stimulated his students, already steeped in what was to be called Fréchet's *Calcul Fonctionnel*, to make significant contributions to advance the evolution of *topological* ideas.

2. 1910–1915: Refining the Neighborhood Concept

The evolution of the notion of developability is not an isolated thread, nor is its creation independent of other mathematical research. Its origin is likely embedded in the mathematical activity at the turn of the century. Certainly the contemporary work of prominent mathematicians of that time, for example, is a significant factor in understanding the rudimentary ideas that will lead to its creation. In retrospect generalization and axiomatization seem to be key elements in motivating much of the research in the first decade of the twentieth century.

Moreover, since developability can be viewed as a property about neighborhoods of a point, the evolution of the "neighborhood" concept plays a role in the evolution we wish to trace. The idea of neighborhood is clearly present, for example, in the work of those who wish to examine Cantor's point set theory for the real line and to generalize it to abstract classes in which the elements are "completely unconditioned". Other research that employs the concept of "neighborhood of a point" or the notion of "interior element of a class" in a *more general space* is relevant to our considerations and will enrich our understanding of why the notion of developability evolves.

[17]Bliss, *The scientific work of Eliakim Hastings Moore*, pp. 507–508.

2.1. THE PRECURSORY CONTRIBUTIONS OF HILBERT AND VEBLEN

We turn first to the work of David Hilbert. By 1903 the second edition of his "Foundations of Geometry" is published along with several papers that include a further axiomatization of what he calls an "abstract plane". We appeal to Angus E. Taylor's appraisal based on his extensive research:

> Perhaps the first occurrence of the notion of neighborhoods in the context of an entirely axiomatic set theory is in the work of Hilbert. On two occasions in 1902 he used the neighborhood notion in discussing the foundations of geometry. ... The paper [reference to Hilbert [1902b]] and a footnote in [reference to Hilbert [1902b]] are included as Appendix IV in the second edition of Hilbert's book on the Foundations of Geometry. In this appendix a plane is, for Hilbert, a collection of objects called points; with each point is associated a family of subsets of this plane, called neighborhoods of the given point. There are six axioms, two of which relate the "abstract plane" to the "number plane" of coordinate point-pairs (x, y):
>
> (1) A point belongs to each of its neighborhoods.
>
> (2) If B is a point in a neighborhood U of the point A, then U is also a neighborhood of B.
>
> (3) If U and V are neighborhoods of A, there is another neighborhood of A that is contained in both U and V.
>
> (4) If A and B are any two points, there is a neighborhood of A that contains B.
>
> (5) For each neighborhood there is at least one mapping of its points, one-to-one onto the points (x, y) of some Jordan region (the interior of a simple closed curve) in the number plane.
>
> (6) Given a point A, a neighborhood U of A, and a Jordan region G that is the image of U, then any Jordan region H that lies in G and contains the image of A is also the image of some neighborhood of A. If a neighborhood of A has two different Jordan regions as images, the resulting induced one-to-one correspondence between these images is bicontinuous.
>
> As can be seen the first four axioms are abstract. Hilbert's axiom system was not designed for the purpose of pursuing general point set topology in the abstract. Rather, Hilbert was intent upon founding plane geometry ... solely on the foregoing axioms together with a group of three axioms about a group of continuous one-to-one transformations of points in the number plane. ...
>
> To what extent Hilbert's use of the concepts of neighborhoods influenced the subsequent development of abstract general topology by means of axioms

> about neighborhoods is, I think, likely to remain speculative unless more firm evidence is found.[18]

To place other work of Hilbert at this time in perspective we note the following remarks of Michael Bernkopf in his extensive study of the evolution of the concept of a function space.

> With the appearance of Fredholm's work, David Hilbert (1862–1943) became interested in the study of integral equations, and in 1904 published his first of six papers on the subject. ... However, before investigating Hilbert's work, it will be instructive to examine the point of view maintained by him throughout the several papers. One sees that he did not set out to abstract from the finite dimensional to the infinite dimensional solely for the sake of abstraction itself. In fact, he never refers explicitly to "spaces" or "function spaces" in any of his work. Hilbert's chief aim is to get on with the job of solving analytic problems. ... In short he is not interested in, nor does he even mention, function spaces *per se*, but seems only to be concerned with developing a method to handle the particular problems he has in mind.[19]

Another paper that is published in the 1905 volume of the *Transactions of the American Mathematical Society* will have a significant influence on the work of researchers we wish to follow. In this paper Oswald Veblen defines a linear continuum as "a set of elements $\{P\}$ which we may call points, subject to a relation $<$ which we may read *precedes*, governed by the following conditions ..."[20] Veblen goes on to list postulates of order, which define a total ordering for $\{P\}$; a postulate of closure, which defines completeness using Dedekind cuts; a "pseudo-archimedean" postulate; a postulate of density, which places a point between any two given points; and a postulate of uniformity, as follows:

> *Definition.* An element P_2 such that $P_1 < P_2 < P_3$ is said to lie between P_1 and P_3. If P_1 and P_3 have an element between them the set of all such elements is called the *segment* P_1P_3. Thus the segment P_1P_3 does not contain either P_1 or P_3 which are called its *end-points*. A segment plus its end-points is called an interval. ...
>
> *Postulate of uniformity. For every element P of a set $\{P\}$ ... and for every ν ($\nu = 1, 2, 3, \ldots$) there exists a segment $\sigma_{\nu P}$ such that the set of segments $[\sigma_{\nu P}]$ has the following properties* † :
>
> 1. *For every fixed P, $\sigma_{\nu P}$ contains $\sigma_{\nu+1\ P}$.*
> 2. *For a fixed P, $\mathcal{P}$ lies on every $\sigma_{\nu P}$ and is the only such element.*

[18]Taylor, *A study of Maurice Fréchet: II. Mainly about his work in general topology, 1909–1928*, pp. 297–298.

[19]Bernkopf, *The development of function spaces with particular reference to their origins in integral equation theory*, pp. 9–10.

[20]Veblen, *Definition in terms of order alone in the linear continuum and in well-ordered sets*, p. 165.

3. *For every segment τ, there exists a ν, ν_τ, such that for no P does $\sigma_{\nu_\tau P}$ contain τ.*[21]

Here † refers to the footnote:

> A metrical special case of a set $[\sigma_{\nu P}]$ is obtained by letting $\sigma_{\nu P}$ be a segment of length $1/\nu$ whose middle point is P. It is evident that as ν increases the segment $\sigma_{\nu P}$ closes down on the points P in the *uniform* manner described in number 3 above. The theorem of uniform continuity involves consideration of a set like $[\sigma_{\nu P}]$.

Young Veblen, a then recent graduate of the University of Chicago and a student of E.H. Moore, proceeds to make a few observations that relate to the point set theory of Cantor.

> From [these] conditions ... follow the theorems that every bounded set has a least upper bound and greatest lower bound,* that every bounded set has at least one limiting element, and also the important.
>
> *Heine–Borel Theorem. If every element of an interval P_1P_2 belongs to at least one segment σ of a set of segments $[\sigma]$, then there exists a finite subset of $[\sigma]$, $\sigma_1, \sigma_2, \ldots, \sigma_n$, such that every element of P_1P_2 belongs to at least one of the segments $\sigma_1, \sigma_2, \ldots, \sigma_n$.*[22]

Here * refers to the footnote: "A *limiting element* of a set $[x]$ is an element A such that every segment which contains A contains an element of $[x]$ different from A."

In particular let us note, since its usage will occur in further discussions, that an *interval* in this era refers to what would later be called a closed interval with two endpoints; further, if A and B are the endpoints, then the interval would be denoted as (A, B). Significantly, a limiting point of a *set* is defined as an element whose *deleted* segments intersect the set.

2.2. THE EARLY CONTRIBUTIONS OF FRÉCHET AND RIESZ

The idea of finding a generalization for Cantor's fundamental notion of "limiting element of a point set" will attract the attention of both Maurice Fréchet and Frederic Riesz.

Maurice Fréchet [1906] chooses to assume certain axioms concerning an undefined "limit concept" that assigns limits to sequences in a structure he calls *classe* $(\mathcal{L})$. T.H. Hildebrandt, in a 1912 publication of his thesis work, provides a succinct overview of Fréchet's *Calcul Fonctionnel*:

> Fréchet's work may be divided into two parts: (1) a theory of continuous functions on an abstract set, and (2) a generalization of the theory of linear point sets. The first of these was no doubt suggested by the analogies between

[21] Veblen, *Definition in terms of order alone in the linear continuum and in well-ordered sets*, p. 166.

[22] Veblen, *Ibid.*, pp. 166–167.

> theorems on continuous functions of a single variable, of n variables, of lines, of curves, etc. The element of generality enters in the consideration of a class or set Q of elements q, which are not specifically defined. For the class there is postulated the existence of a notion of limit of a sequence of elements; satisfying a number of conditions which are properties of the limit of a sequence of real numbers. In terms of limit, it is possible to define a sequentially continuous function, and hence to construct a theory of sequentially continuous functions. To attain the second end, there is postulated for the class Q the existence of a *voisinage* or distance function δ of pairs of elements, there being a value δ for every pair of elements of the class. This distance function δ is subject to a number of conditions, generalizations of properties of its real variable analogue, the absolute value of the difference between two numbers. In terms of such a δ, a limit is definable, and a theory of sets, concerning derived, closed, etc., sets, is obtainable.[23]

In Fréchet's *classe* $(\mathcal{L})$ a set (*ensemble*) is *closed* if it contains all of its limiting elements; it is *compact* if every infinite set of its elements has at least one limiting element; it is *extremal* if it is both compact and closed; and an element is *interior to* a set if it is not the limiting element of a sequence of distinct elements which do not belong to the set.

Fréchet also introduces the notions of *compactness*, *separability*, and *completeness* although the current-day meanings of these concepts have been refined. His work seeks to prove the theorems generally associated with point sets, most particularly those concerning compactness. Properties of continuous (real-valued) functions are a focal point, which connects this work to the work of E.H. Moore. Fréchet also introduces some notions of distance, specifically those of *voisinage* and *écart*, and uses them to define limiting elements.

Contemporaneously Frederic Riesz [1907] suggests that any generalized notion of "space" needs to free itself of the underlying notion of distance. Rather, he proposes a geometric notion based on the concept of neighborhood. Riesz may have been influenced by Hilbert's use of "neighborhood" [1902] and by Veblen's use of "segment" [1905]. In any case the extensive research of Angus E. Taylor presents a compelling account of the influence of Riesz.

> Riesz's paper of 1905 is on multiple order-types (mehrfache Ordnungstypen), an extension of the theory of simple order-types that is bound up with Cantor's theory of ordinal numbers. Cantor's theory of sets, including the concept of a limit point, is employed in the theory of order-types. Riesz explains his desire to examine the unexplored field of multiple order-types as a natural result of the then recent upsurge of attention to point set theory in the plane and higher dimensions. But, he says, it is necessary to dispense with such concepts as distance and Jordan's écart, which do not fall within the group

[23] Hildebrandt, *A contribution to the foundations of Fréchet's calcul fonctionnel*, p. 237.

of basic notions that occur in the theory of order-types. One must, Riesz says, carry over into this theory the concept of neighborhood in the general setting that was then current in the investigations of point set theory. There is no reference to Fréchet or his work in this paper; nor is this paper a contribution to abstract point set topology. The reference to the role of neighborhoods and the desire to avoid the use of distance do nevertheless represent an early indication of the way Riesz's ideas were taking shape.

Riesz's long paper of 1907 in German on the origins of the concept of space is a translation of a paper originally published in Hungarian in two installments in 1906 and 1907. ... This paper is not, in the main, about general point set topology, either concrete or abstract. It is a quasi-philosophical paper in which Riesz attempts to construct a mathematical model for the geometry of space as needed or used in physics. In so doing he formulates a notion of what he called a mathematical continuum. This notion is, in fact, that of an abstract space with a rudimentary topology defined axiomatically. ...

Riesz's mathematical continuum is a class (he refers to it as a Mannigfaltigkeit) of elements in which there is a rule, subject to four axioms, that specifies, for each element A and each set t, one and only one of the relationships: either A is isolated from t or is a limit element (Verdichtungsstelle) of t. It is clear from the paper that "Verdichtungsstelle" should *not* be translated as "condensation point" in the special sense given to that term by Lindelöf. The set of all limit elements (if any) of t is denoted by t' and is called the derived set (Ableitung) of t. To say that A is isolated from t means merely that A is not an element of t'.

In stating the four axioms given by Riesz I shall use symbolism more than he did. The axioms are :

(1) If t is a finite set, every element is isolated from t. (This is the same as saying that the derived set of a finite set is empty.)

(2) If t is a subset of u, t' is a subset of u'.

(3) If t is the union of u and v, t' is contained in the union of u' and v'. (This, together with (2), implies that t' is the union of u' and v'. Riesz did not use the set-theoretical term "union".)

(4) If A is in t' and $B \neq A$, there exists a subset u of t such that A is in u' but B is not.

Riesz then introduces concepts as follows: A set u is called a neighborhood (Umgebung) of A if A is in u but is isolated from the complement of u. Riesz points out that the elements common to a finite number of neighborhoods of A (that is their intersection) is a neighborhood of A. An element is called an interior element of a set t if t is a neighborhood of A. A set is called open if all its elements are interior elements. ...

Riesz does not, in this context, define the concept of a closed set. He does, however, introduce the symbol for the union of t and t', namely $\{t, t'\}$. Riesz points out in a footnote that $(t')'$ is not necessarily contained in t'. Then, forgetting that he has not formally defined the property of being closed, he says that the fact that in ordinary set theory the derived set is closed must be done without (literally, missed or regretted) in the general theory of continua.

Riesz proves that if A is in t', every neighborhood of A contains infinitely many elements of t. He asserts (but does not prove) that the inverse is true, meaning that if A and t are such that every neighborhood of A contains infinitely many elements of t, then A is in t'.[24]

REFLECTION AND OBSERVATIONS ON THIS WORK OF FRÉCHET AND RIESZ.

In any case both Fréchet and Riesz considered their work as "first attempts" toward a fruitful theory, and their first papers seemed to have been written independent of one another. Both considered their works new and original and both fully believed that a richer theory would require additional axioms and perhaps some refinements. Indeed, Fréchet makes the point explicitly on the second page of his thesis, having just introduced a context for the research that he will undertake:

> Le présent travail est une première tentative pour établir systématiquement quelques principes fondamentaux du Calcul Fonctionnel et les appliquer ensuite à certains exemples concrets.

According to Taylor, "Riesz expressed the view that to go further with a theory one would need additional axioms. Of the first three axioms alone he wrote that they are so broadly conceived that one cannot build much more on them alone."[25]

Significantly, Riesz will make no further contributions to this line of inquiry. Fréchet, on the other hand, will continue to revisit these issues, to revise and refine his definitions and then to raise questions that will stimulate an increasing corps of researchers to enrich the theory that has its roots in his 1906 thesis.

Interestingly, and likely due to the influence of E.H. Moore, the work of Fréchet and Riesz catches the eye of the Americans. We note in particular the contributions of E.R. Hedrick as well as those of T.H. Hildebrandt and Ralph E. Root, both of whom write a thesis with Moore at the University of Chicago in 1910. Following what appears to be "standard practice" for E.H. Moore's students at that time both Hildebrandt and Root present their work at regional meetings of the American Mathematical Society and later publish their work in prominent journals.

[24]Taylor, *A Study of Maurice Fréchet: I. His early work on point set theory and the theory of functionals*, p. 267–269.

[25]Taylor, *Ibid.*, p. 270.

2.3. THE CONTRIBUTION OF HEDRICK

Earle Raymond Hedrick finished his Ph.D. work at Göttingen in 1901 and spent some time in Paris, where Fréchet was enrolled at the Ecole Normale Supérieure, before taking a position at the University of Missouri.

Hedrick is familiar with Fréchet's thesis and is one of the first to attempt to rid Fréchet's theory of distance. Based on his knowledge of the recent work of Fréchet and his reading of Shoenflies's *Punktmannigfaltigkeiten* in Bericht der Deutsch Mathematiker-Vereingung, Ergängung, published in 1908, Hedrick chooses to undertake his investigation in a certain *classe* ($\mathcal{L}$) that is (in hindsight) "purely topological". The trail of abstracts for Hedrick's talk before regional meetings of the American Mathematical Society, first, in Columbia, Missouri on November 27, 1909, and then in Lincoln, Nebraska, on November 26, 1910, is interesting in that it reveals Hedrick's intention for his work and how he relates his research to that of Fréchet's *Calcul Fonctionnel.*

> "On a property of assemblages whose derivatives are closed"
>
> Fréchet proved, in his thesis, that the Heine–Borel theorem holds for countable families in any compact assemblage in which "limit" is defined by means of "distance" in the sense in which those terms are there used. Professor Hedrick points out in the present paper that the same theorem is true for any compact, closed assemblage whose derivative is closed.[26]

> "On assemblages with closed derivatives"
>
> At the last meeting of the section, Professor Hedrick proved that a few important theorems of the theory of point sets hold true for any compact set of objects for which the first derived set is closed. In the present paper the hypothesis is added that any element of the set is enclosable in the interior of a family of diminishing assemblages in a uniform manner. With this hypothesis practically all of the fundamental theorems of point sets are shown to hold for general sets of objects; the list of theorems includes the general Heine–Borel theorem, the Cantor–Bendixon theorem, the Cauchy fundamental theorem, the oscillation theorem, and many others. No distance notion is used; those theorems which usually involve distance are reworded to avoid that concept.[27]

Hedrick proceeds to establish many of what were considered the important theorems in point set theory of that time as well as to make some observations about the "Hilbert space", which had been discussed in the 1910 work of Fréchet. Hedrick's introduction sets the context and explains the purpose for his work.

> Fréchet, in his thesis, defines a general class of assemblages (L) of objects of any sort, with which some definition of limiting element of a sequence is associated.

[26]Bulletin of the American Mathematical Society 16 (1910), 230.
[27]Bulletin of the American Mathematical Society 17 (1911), 229.

The most interesting type of such assemblages is that which Fréchet calls *compact*. Among such assemblages one very important class are those which have the property that the *first derived set* (E') *of every subset* (E) *of* (D) *is closed.*

In this paper, some of the properties of any fundamental assemblage (D) which has both the properties described above will be discussed. We shall call (D) the fundamental domain.[28]

Using the limiting element ideas of Fréchet and the associated definitions, Hedrick proceeds to establish some standard results and to focus on the important example that had been attributed to Hilbert.

Theorem 1. If an element A is interior to a set (E), all but a finite number of elements of any sequence whose limiting element is A are interior to (E). ...

The Heine–Borel Theorem. The theorem just proved enables us to show that the Heine–Borel theorem holds for a fundamental domain (D) which has the properties of section 1.

Theorem 2. Let $(E_1), (E_2), \ldots, (E_n), \ldots$ be a countable set of assemblages, such that every element of a certain closed assemblage (K) is interior to at least one of the assemblages (E_i); then every element of (K) is interior to one of a finite number of the assemblages (E_i). ...

Examples: The Hilbert Space. As an instance of a fundamental domain to which the preceding theorems apply we may cite, of course, any limited region of an ordinary space of one dimension (or in fact of n dimensions), with the usual definition of limiting point. In fact, any assemblage of points in n dimensions which is itself closed may be chosen as the (D). Such a domain evidently has the property that the first derived assemblage of any subset (E) is closed. A more characteristic example, in which the definition of distance has nothing to do with the definition of limiting object is the Hilbert space of an infinite number of dimensions; the objects used are the points $(x_1, x_2, \ldots, x_n, \ldots)$ with the restriction $\sum x_1^2 \leq 1$ [*sic*]. The definition of distance between two points is

$$\left[\sum \left(x_1^{(1)} - x_2^{(2)}\right)^2\right]^{1/2} \quad [\textit{sic}];$$

and the points $(x^{(1)}), (x^{(2)}), \ldots, (x^{(n)}), \ldots$ approach the point (x) as a limit if $\lim_{n=\infty} x_i^{(n)} = x_i$ for every value of i. This space evidently possesses the properties of §1; yet the set of points $(x^{(n)})$ which *approach* (x) may all lie *at unit distance* from (x), as in the example $x_i^{(n)} = 0$ if $n \neq i, x_n^{(n)} = 1; x_i = 0$.

[28]Hedrick, *On properties of a domain for which any dervied set is closed*, p. 285.

Fréchet has given a definition of distance (voisinage) for which the preceding definition of limit point makes the distance between $(x^{(n)})$ and (x) approach zero. The preceding work shows that it is not always necessary to set up a definition of distance for the Hilbert space; for other domains of objects to do so might be very difficult or even impossible. . . .

That the several theorems proved above hold true without any restrictions on the definition of distance, or any connection between the definition and the definition of limiting element, is especially remarkable because the proofs of analogous theorems for real numbers apparently make essential use of the distance notion. That such is the case, however, is demonstrated here by the entire absence of any definition of distance.

It is also clear that the property emphasized in section 1 – that the derivative of any set be closed – may be used to characterize a type of domains (with assigned definitions of limiting elements) which have in common some fundamental properties.

Another important theorem which follows from these same assumptions occurs incidentially in the next section.

Enclosable Domains. In the theory of linear point sets the following statement may be taken as an axiom: If the interval (A_i, B_i) is interior to the interval (A_{i-1}, B_{i-1}) and if the distance $\overline{A_iB_i}$ approaches zero, there is a single point common to all these intervals.

We shall now assume the analogous property: *Corresponding to any element A of (D) there exists a countable set of closed assemblages $Q_i(A)$ which contain the single common interior element A, and which have the following properties:*

(a) *$Q_i(A)$ is interior to $Q_{i-1}(A)$.*

(b) *Corresponding to any integer m an integer n exists, such that, for any point B whatever, any Q_n which contains B is interior to $Q_m(B)$.*

When the domain D with its definition of limiting element satisfies this requirement, we say that D has the enclosable *property.*[29]

Hedrick then makes it clear that his work may have been suggested by the earlier work of Veblen as he makes the following remark in a footnote:

My attention is called to the similarity between this hypothesis and that given by Veblen for point sets, these Transactions vol. 6 (1905), p. 168. The notion of uniformity of enclosure is essential in both cases. It is suggested that the word "enclosable" might read "uniformly enclosable"; the former is retained here, however, for simplicity.[30]

[29] Hedrick, *On properties of a domain for which any dervied set is closed*, pp. 286–289.
[30] Hedrick, *Ibid.*, p. 289.

Hedrick proceeds to develop a theory for point sets based on his assumptions for the enclosable domains. The following excerpts reflect the tone and intent of his work.

> *Lemma 1. If (D) contains more than one point, then (D) is perfect. ...*
>
> *Lemma 4. If B is interior to (E), a number m exists such that any Q_m which contains B is interior to (E). ...*
>
> *Theorem 6. If any sequence Q'_i of assemblages of the types Q_i are such that Q'_i is interior to Q'_{i-1}, there is one and only one point B interior to all Q'_i and also Lim $A_i = B$, if A_i is any element interior to Q'_i. ...*
>
> *Theorem 7. (Cauchy Fundamental Theorem) The necessary and sufficient condition that the sequence A_j have a single limiting element is that, corresponding to any arbitrarily assigned value of i, a number n exists such that, for all values of p all the elements A_{n+p} are interior to a single assemblage of the type Q_i. ...*
>
> *Theorem 15. The domain (D) itself is the first derived assemblage of a certain countable set of its own elements; that is (D) is "separable"* [at which point Hedrick refers to page 23 of Fréchet's thesis].
>
> *Theorem 16. Let (E) be an assemblage, each of whose elements is interior to at least one of a family (H) of assemblages I, where (H) is not necessarily countable; then the necessary and sufficient condition that every point of (E) is interior to at least one of a finite number of the assemblages I is that (E) be closed. ...*
>
> We have now proved a set of theorems analogous to the principal fundamental theorems of the theory of bounded sets of points; for these the only assumptions made were those of §1 and §6. The hypothesis that the first derived assemblage of every set is closed is therefore seen to be a fundamental hypothesis of far-reaching character. The fundamental domains which possess that property include many important examples besides the original bounded sets of points, and have characteristics in common which make that class of domains one of great importance.
>
> It should also be noticed that the enclosable property of section 6 is possessed by all ordinary domains under any of the usual definitions of distance, if $Q_i(A)$ be taken to be the spheroid about A as center, with radius $1/i$.

After publication of his paper, Hedrick apparently sends a reprint to Fréchet, suggesting that "it closely followed his thesis."[31] The ensuing correspondence leads Fréchet to write from Poitiers on January 3, 1912, in order to connect Hedrick's results to prior work of his. In particular, Fréchet defines a *classe* (Δ) that is somewhat more general than Hedrick's fundamental domain (D) and

[31] See Taylor [1985] for more complete details of this correspondence.

shows that any *classe* (Δ) is *classe* $(\mathcal{V})$ *normale*, a class that is carefully studied in Fréchet's thesis; therefore, much of Hedrick's results is subsumed in Fréchet's thesis results.

> Ceci étant, je vais démontrer d'abord que: *toute classe* (Δ) *est une class V normale. Il en résultera que les théorèmes que vous démontrez pour la classe* (Δ) *peuvent être considérés comme déjà démontrés dans ma Thèse. ... Par contre les théorèmes où vous ne faites pas usage de l'*"enclosable property" *restent essentiellement nouveaux et constituent une importante généralisation.*[32]

In order to carry out his proof Fréchet must show the existence of a *voisinage* that carries the same convergence structure as the *classe* (Δ); his proof uses the following ideas for constructing a *voisinage*:

Assume $Q_1(A) = D$ for every A. Given distinct points A and B, let β be the last index of $Q_i(B)$ containing A and α be the last index of $Q_i(A)$ containing B and define a *voisinage* by

$$(A, B) = 1/\min\{\alpha, \beta\}.$$

Fréchet concludes by showing that this is a *voisinage* with the required sequential convergence structure. Conversely, Fréchet notes:

> *Réciproquement toute classe V normale et compacte est une classe* (Δ)*; c'est même à votre sens, une classe* (D) *ayant exactement l'"enclosable property."* ...[33]
>
> Ainsi *nous voyons que la famille des classes* (D) *ayant l'"enclosable property" forme une partie seulement des classes* (V) *normales, à savoir, exactement l'ensemble de celles-ci qui sont compactes.*[34]

REFLECTION AND OBSERVATIONS ON THIS WORK OF HEDRICK

In order to prove many of the important theorems of Fréchet, Hedrick intentionally establishes a context that is distinctly "topological". The general idea of introducing subsets of a point that "close down" on a point conveys the sense of neighborhoods that will inspire those who follow to refine and clarify the more appropriate concepts that are important in the evolution of the notion of topological space and in the evolution of the notion of a development; this will become apparent when we discuss the work of Root [1914] and Chittenden [1927].

Hedrick's work also created a context that led Fréchet to construct a distance function (in this case a *voisinage*) that provides the same "topological"

[32]Fréchet, *Sur les classes V normales*, p. 320.
[33]Fréchet, *Ibid.*, p. 323.
[34]Fréchet, *Ibid.*, p. 324.

information. Fréchet's result, therefore, is the first in the spirit of an explicit metrization theorem. One might easily suggest that this is the first step in defining the "metrization problem" which plays a huge role in motivating the evolution of topological ideas.

2.4. THE CONTRIBUTION OF HILDEBRANDT

In this prenatal stage of topology a central issue, based on the fundamental work of Fréchet and Riesz, is to identify an appropriate "primitive" concept on which to base a theory. There seems to be some consensus in the belief that the theory itself needs to include a notion of limit point so that one can follow a Cantor–Fréchet course in defining the notions of derived set, closed set and perfect set, as well as a notion of convergence for a sequence. From these the notions of sequentially continuous function or even that of continuous function in general and its relationship to uniform continuity is a context to be defined and explored.

Let us begin by setting the stage for the work that is to follow. Fréchet's thesis is published in 1906, is widely read and in many cases has stimulated additional papers by prominent mathematicians (e.g. Hans Hahn, Arthur Shoenflies, etc.[35]). Fréchet himself publishes a sort of addendum in 1910. Riesz has published his work in 1906 and 1907 and then presented a summary at the International Congress in Rome in 1908. This work is well known and of great interest to the research group, led by E.H. Moore at the University of Chicago. Both Hildebrandt and Root in their 1910 thesis attempt to move away from "distance" in order to establish a "topological context" such that

(1) the *classe* $(\mathcal{L})$ theory of Fréchet can be recovered;
(2) the derived set axioms of Riesz are satisfied; and
(3) the primitive notions are not depend on distance.[36]

One might speculate that Hildebrandt's work pursues the direction suggested by Fréchet while Root's research is more influenced by the work of Riesz.

The motivation for a very young Hildebrandt seems to come from his mentor's research in foundations of *General Analysis*, the recently published thesis of Fréchet [1906] and the seminars[37] conducted by Moore at the University of Chicago in either 1907 or 1908. He will consider a system $(\mathcal{Q}; K)$ in which the fundamental notion generalizes the relation $K_{q_1q_2m}$ that had been introduced by Moore. Hildebrandt's objective is to consider a list of properties for this relation and to extract the essential ones needed to establish the individual theorems of Fréchet. His own words set the stage for his work.

> The present work concerns itself with the Fréchet point of view. It had its inception in an attempt to replace the distance function δ of Fréchet by a

[35]See Taylor [1982].
[36]Hedrick, *On properties of a domain for which any dervied set is closed*, pp. 289–294.
[37]This is reported by C.E. Aull in [1981] as a result of his conversations with Hildebrandt.

weaker condition of the class Q. The fact that in most instances the δ appears in connection with an inequality of the type

$$\delta_{q_1 q_2} \leq \frac{1}{m}$$

suggested the adoption of the second K-relation of Moore, $K_{q_1 q_2 m}$, in place of the δ. By stating, in the case of every theorem, the precise condition of K sufficient to carry the argument, and extending this idea to the case in which the class Q is subjected only to the condition of the existence of a limit, a two-fold result was obtained:

(a) *that an unconditioned limit suffices for the theorems on sequentially continuous functions obtained by Fréchet, and*

(b) *that it is possible to obtain the theory of sets of elements with a distance δ, subjected to weaker conditions than those imposed by Fréchet.*[38]

In this investigation Hildebrandt will consider a given class Q in the context of three possible systems – $(Q; K)$, $(Q; L)$, and $(Q; \delta)$, where K denotes a binary relation suggested by the work of Moore, L denotes an assignment of limits for sequences as suggested by Fréchet and δ denotes a distance. By identifying the properties of interest in each of these theories and subsequently establishing their interrelationships, Hildebrandt lays the foundation for comparing various options for generalization. Notably, Hildebrandt simplifies and extends much of Fréchet's theory.

For the structure L, Hildebrandt considers six properties, which include the three properties assumed by Fréchet for his *classe* $(\mathcal{L})$.

Following notation suggested by Moore, Hildebrandt will denote a structure that has properties, say (1), (2) and (6), by L^{126}; this is, in fact, Fréchet's *classe* $(\mathcal{L})$. Similarly, Hildebrandt considers eight properties of a relation K and corresponding eight properties of a distance δ and uses the associated "Moore notational scheme". Hildebrandt makes no claims to having created new and original work. His intent is to present three potential axiomatizations that can generalize the work of Fréchet. As a student of Moore, his intent is to consider axiomatic systems in which the axioms can be shown to be independent and then to consider the fewest possible assumptions for proving the fundamental theorems of point set theory. He sums up much of his contribution in the following:

> The theorems derived are in the main the theorems of Fréchet. However, instead of permanently conditioning the L and the K in the systems $(Q; L)$ and $(Q; K)$, we have preferred to indicate in each case the precise properties of L or K, sufficient to carry the argument. In this way it appears that it is not necessary to condition the L for the theorems on continuous functions.

[38] Hildebrandt, *Loc. cit.*, pp. 238–239.

Further, that a K-relation having the properties (1), (3), (6), (7) of §4, is sufficient for all the theorems, and in some cases even weaker conditions on the K will do. It will be noticed that the symmetry property (2) and the property (4) do not occur. The former is really a matter of convenience. It is avoided by the use of properties K^{167}, which combination we have seen is weaker than K^{125}, the combination it replaces. Property (4) serves to avoid the separate consideration of the limit of a sequence which consists of a finite number of elements only. Its presence as a condition restricts the generality of the theorems. We have therefore preferred to gain in generality at the expense of convenience, replacing the property (2) by a weaker combination, and taking up a more detailed discussion, if necessary, instead of using the property (4).[39]

REFLECTION AND OBSERVATIONS ON THIS WORK OF T.H. HILDEBRANDT.

Hildebrandt's intention is to identify essential properties of Moore's (uniformity) relation K and associated properties of Fréchet's limit relation L and distance function δ. By considering the systems $(\mathfrak{Q}; K)$, $(\mathfrak{Q}; L)$ and $(\mathfrak{Q}; \delta)$ and then making a careful and exhaustive investigation into their interrelationships Hildebrandt lays out the potential directions for the further evolution of these ideas. Note that Hildebrandt is the first to consider distances that are nonsymmetric and that like Hedrick his intention is to develop a general theory that is not based on a notion of distance.

2.5. THE CONTRIBUTION OF ROOT

A step to focus exclusively on the notion of neighborhood is taken by Ralph E. Root in his 1910 thesis at the University of Chicago. Root announces his approach at a Chicago meeting of the American Mathematical Society in April, 1911. After a year with Hedrick at the University of Missouri, Root takes a position at the United States Naval Academy, where he writes in February of 1914, "The investigations leading to the present paper were completed in June, 1911, and the manuscript left the hand of the author April, 1912."[40]

Root explains the intent of his work:

The paper has its origin in the thought that in most of the definitions of limit that are employed in current mathematics a notion analogous to that of "neighborhood" or "vicinity" of an element is fundamental. In the domain of general analysis various ways of determining a neighborhood of an element have been employed, notably the notion of *voisinage* used by M. Fréchet, and the relations K_1 and K_2 used by E.H. Moore, either as undefined or as

[39] Hildebrandt, *Ibid.*, p. 267.
[40] Root, *Iterated limits in general analysis*, p. 133

defined in terms of a "development" of the class of elements constituting the fundamental domain.

A definite class of elements being assumed, the notion of "neighborhood" of an element is essentially that of a subclass having a special relation to the element. In taking this relation as undefined and at the basis of our system of postulates we occupy a position immediate, as regards generality, between the extreme position of those who take the notion of "limit" itself as undefined, and that of those who define "limit" by means of other relations which give rise to notions analogous to that of "neighborhood".[41]

In terms of our relation R we define limit for a sequence of elements so as to fulfill the conditions of the Fréchet limit, and our definition of limiting element of a subclass fulfils the Riesz postulates. We show that in a system satisfying our postulates the definitions of continuity by means of "sequence" and by means of "vicinity" are equivalent. On the other hand our system is less special than the system of Fréchet in which he uses the notion of écart, since a relation R fulfilling our condition may be defined for any class for which an écart exists. The gain in generality over the Fréchet treatment is seen in the possibility of determining the relation R by means of order relations in which distance and magnitude play no part. An R may be defined for any class having linear order, or for the composite of any finite number of such classes.[42]

Root's systems include a class $\mathcal{P}$ of elements, a class $\mathcal{U}$ of ideal elements and a dyadic relation R between subclasses of $\mathcal{P}$ and individual elements of $\mathcal{P}$ or $\mathcal{U}$. When Root's relation R is interpreted as neighborhood, one obtains the following postulates:

(I) If A is a neighborhood of p, then p is in A.

(II) Every neighborhood of an ideal element contains a point of $\mathcal{P}$.

(III) For any p there is a sequence A_n of neighborhoods of p such that, if B is a neighborhood of p, then there is k such that A_n is contained in B for all $n \geq k$.

(IV) For every neighborhood A of p there is a neighborhood B of p such that each element of B has a neighborhood that is contained in A.

(V) Distinct elements of $\mathcal{P}$ have disjoint neighborhoods.

Root's year with Hedrick produces a deeper study of neighborhoods, an overview of which is given in his abstract for his presentation to American Mathematical Society in Chicago on March 21 of 1913.

While the conditions used in the paper by Dr. Root are fulfilled by any simply ordered class, provision is made for multiple interpretation of its postulates and theorems. The work pertains to a class of elements q of arbitrary

[41]Root, *Iterated limits in general analysis*, p. 79
[42]Root, *Iterated limits of functions on an abstract range*, p. 539

> character with a relation B of the type of "betweenness". ... The relation B replaces betweenness in the definition of segment, and an element q is said to be a limiting element of any set that has an element distinct from q in every segment containing q. Certain fundamental theorems, including the proposition that every derived class is closed, result from the type of the system, without postulates. ... An additional postulate leads to a form of the Heine-Borel theorem, and to theorems on bounds of functions on a compact class. A sequence of elements is said to have the limit q if every segment containing q contains all but a finite number of elements of the sequence. On the basis of the postulates this definition of limit fulfils the conditions specified by Fréchet, thus securing, by means of the sequential definition of limiting element of class, a considerable body of theorems developed by Fréchet, Hedrick, and Hildebrandt. There arise, then, two parallel theories, of about equal extent, one based on the neighborhood definition of limiting element, the other on the sequential definition. That these two theories are not equivalent is shown by an example of a simply ordered class fulfilling all the postulates of the paper and possessing a limiting element that is not the limit of any sequence of elements of the class.[43]

In the manuscript that was to follow, being submitted from the University of Missouri in April of 1913, Root not only sets the stage for his research, but he also points to the works of others that influenced his work and his point of view.

> The importance of the role of the order relation in the theory of the linear continuum has been recognized since the appearance, in 1872, of Dedekind's *Stetigkeit und irrationale Zahlen*, and it received added importance at the hands of Cantor in 1895. A later contribution to the theory was made by O. Veblen in 1905, and a systematic account of the theory of the continuum as a type of order was given by E.V. Huntington the same year. The work of these writers is directed toward a complete characterization of the linear continuum in terms of order alone. A set or class of elements, otherwise undefined, is assumed to fulfil conditions, stated in terms of order, sufficiently restrictive to admit of effective use of the class in the role of range of the continuous independent variable.
>
> Meanwhile certain classes have been recognized as being, in effect, the range of an independent variable, while not fulfilling all the conditions of the linear continuum. The desirability of utilizing analogies that exist between theories that pertain to these classes and the theory of the continuum is obvious. To this end it is important to generalize, as far as may be, the notions of point sets so as to be applicable to such classes. ...
>
> The object of the present paper is to use in this field of generalization some of the simple properties of ordered classes. A triadic relation of the type

[43]Root, Bulletin of the American Mathematical Society 19(1913) pp. 445–446

of "betweenness" serves in the definition of segment, which, with certain mild postulates that are fulfilled by any simply ordered class, is effective in the development of a theory of limits and continuity.[44]

Root characterizes the work that he produces as a "neighborhood theory", since it is based on a definition of limiting element that is given in terms of "segment", a type of neighborhood. As neighborhoods Root's first three postulates take the following form:

Postulate I. Elements are contained in their neighborhoods and every element has a neighborhood.

Postulate II. Any two neighborhoods of an element have in common a third neighborhood of that element.

Postulate III. Any two distinct elements have neighborhoods with no element in common.

After developing the "desired theorems" for point sets, based on these three axioms alone, Root turns his attention to a comparison of the "neighborhood theory" with the parallel "sequential theory."

In the theory of point sets a limiting point of a set may be defined as a point every neighborhood of which contains a point of the set distinct from itself, or as a point which is the limit of a sequence of distinct points of the set. The notion "limit of a sequence" being defined in terms of "neighborhood", the two definitions are known to be equivalent. In the domain of general analysis, where elements of arbitrary character are considered, it may happen that, even though the notions involved in both definitions of limiting element are well defined, and even though either definition may lead to a theory of very considerable extent, the two theories need not be equivalent.

The definition of limiting element we have employed in the foregoing sections is an instance of a "neighborhood" definition, and the theory developed is in terms of "segment", which is an instance of the general notion of neighborhood. The theory might be characterized as a "neighborhood theory" for the system $(\mathcal{Q}, B)$. But since the notion "limit of a sequence" is defined (Def. 7), we might adopt a "sequential" definition of limiting element and develop a "sequential theory" for a system of this type. Fortunately, however, this theory becomes a special case of the general theory receiving consideration in the papers of M. Fréchet, E.R. Hedrick, and T.H. Hildebrandt. Our definition of limit of a sequence satisfies all conditions used by Fréchet and Hildebrandt in the purely sequential parts of their papers.

Hildebrandt shows that for many of the theorems none of these conditions are necessary. It is obviously true, also, that in this sequential theory the

[44]Root, *Limits in terms of order, with example of limiting element not approachable by a sequence*, pp. 51–52.

> theorem "every derived class is closed", which in the neighborhood theory must be stated, "if every element of a sequence whose limit is q is a limit of a sequence of distinct elements of $\mathcal{P}$, q is a limit of a sequence of distinct elements of $\mathcal{P}$." This proposition, with the condition on "limit" postulated by Fréchet, forms the basis for the first part of the paper by Hedrick.
>
> It appears, then, that a considerable body of "sequential theory" is available for the system $(\mathcal{Q}, B)$ subject to our three postulates. We show by the following two examples that for such a system theorems of like form in the sequential and neighborhood theories need not be equivalent theorems.[45]

Root's first example is the set of all sequences of real numbers with a sequence being between two others iff each term of that sequence is strictly between the terms of the two. Letting q be the constant sequence 0 and $\mathcal{P}$ be the set of sequences of positive real numbers, Root shows that q is a (neighborhood) limiting element of $\mathcal{P}$. However, for any sequence $\{p_n\}$ in $\mathcal{P}$, if q_1 has all negative terms and $q_2(n) = p_n(n)$, Root notes that q is between q_1 and q_2, but the segment given by these elements contains no members of the sequence $\{p_n\}$; that is, q is not the limit of a sequence in $\mathcal{P}$. Root's second example is the linearly ordered real line with the open unit interval replaced by the set of all denumerable subsets of a non-denumerable well ordered set and for which a linear ordering is then defined.

Having noted then that the "sequential theory" and the "neighborhood theory" are distinct, Root goes on to identify conditions under which the theories would be the same.

> We shall say that a subclass $\mathcal{P}$ of $\mathcal{Q}$ has the sequential property if every subclass of $\mathcal{P}$ that has a limiting element q contains a sequence of distinct elements whose limit is q. This property is definite for any subclass of a class $\mathcal{Q}$ for which the two notions, "limiting element of a subclass" and "limit of a sequence" are defined. In particular, if the idea of neighborhood of an element is available for a class $\mathcal{Q}$, then definitions of the form of definitions 2 and 7 of section 2 afford a definite sequential property.
>
> Relative to a system of the type $(\mathcal{Q}, B)$ we consider the following postulates.
>
> *Postulate IV. For every q there exists a sequence $\{\sigma_n\}$ of segments containing q such that, for every segment σ containing q, there is an n_σ such that, if $n > n_\sigma$, σ_n is a subclass of σ.*
>
> *Postulate IV′. For every q there exists a sequence $\{\sigma_n\}$ of segments containing q and having no other common element.*[46]

In the spirit of his mentor, E.H. Moore, Root discusses interrelationships and independence of the postulates as well as their dependence on having the neigh-

[45]Root, *Limits in terms of order, with example of limiting element not approachable by a sequence*, pp. 67–68.

[46]Root, *Ibid.*, p. 69.

borhoods defined by a linear order. For example, Root notes that "Without the assumption of linear order, it is easily seen that Postulate IV implies I and II, and that III and IV together imply IV′."[47]

Root attempts to show that his work, although written in the context in which an order in used to define the notion of segment, is also a theory about neighborhoods. He suggests that this work naturally evolves from the prior work of Fréchet, Hedrick and Hildebrandt.

> Turning now to a consideration of the sequential property in relation to the general notion of "neighborhood," of which the notion segment is obviously a special case, we observe that this property is present in systems of much more general character than a system $(\mathcal{Q}, B)$ fulfilling Postulates III and IV. ... These include cases of neighborhood defined in terms of *voisinage* as used by M. Fréchet, and in terms of suitably conditioned K relations of the type used by E.H. Moore and T.H. Hildebrandt.
>
> In his paper on properties of a domain for which any derived set is closed, E.R. Hedrick reverses the process we have used and defines "neighborhood" in terms of "limit". Assuming a limit relation fulfilling the conditions prescribed by Fréchet, and using a sequential definition of limiting element, he postulates that the fundamental domain is compact and that the derived class of every subclass is closed. He later assumes the "enclosable property", which affords a set of subclasses subject to conditions that validate their use in the sense of neighborhood. The type of neighborhood thus available may be used to set up a new definition of limiting element, in which case the fundamental domain is found to have the sequential property, so that the sequential theory he secures for the domain thus conditioned is equivalent to a neighborhood theory based, in the usual manner, on the type of neighborhood his assumptions afford. In fact, a domain with the enclosable property leads, in a very obvious manner, to a system ... fulfilling all five of the postulates we have used for such a system.[48]

REFLECTION AND OBSERVATIONS ON THIS WORK OF RALPH E. ROOT.

Taking into account the work of Hildebrandt and Hedrick, Root writes a culminating chapter for the American contribution to the evolving neighborhood theory for abstract sets. The emergence of the work of Chittenden in 1917 on the equivalence of *écart* and *voisinage* will focus the attention of American research on issues surrounding distance, while the formulation of a school of topologists in Poland and Russia would spearhead a movement toward fundamental results and terminology in *topological spaces* that they originate and consequently deflect

[47] Root, *Limits in terms of order, with example of limiting element not approachable by a sequence*, p. 69.

[48] Root, *Ibid.*, pp. 70–71.

research away from the derived set ideas of the Fréchet school and their study of *abstract set theory.*

Root seems careful in documenting sources for his work. Therefore, it seems unlikely that he was aware of the work of Hausdorff.

2.6. THE CONTRIBUTION OF HAUSDORFF

Hausdorff's book (Hausdorff, 1914) appears in 1914 in Germany with a war in progress. The term "metric space" and "topological space" are introduced in a form that is particularly lucid and instructive. How Fréchet may have influenced Hausdorff in this writing is unclear and whether Root and Hausdorff are aware of each other's work is equally obscure. In any case a central issue for research is the primitive concept on which to base a theory.

Again we rely on Angus Taylor's careful research to shed light on attribution and to add perspective to our study.

> I turn now to Chapter 7 of Hausdorff's book. Hausdorff begins with general remarks about the success of Mengenlehre in clarifying and sharpening the fundamental principles of geometry by its application to point set theory. He makes some general comments about alternative ways of laying the foundations of point set theory. He speaks of using distance to define the notion of convergent sequences and their limits, or of using distance to define neighborhoods and then building up the whole theory from neighborhoods. ... Next he says that the choice between using distance, sequential limits, or neighborhoods as basic notions is to some extent a matter of taste. He opts for neighborhoods as being more general than the use of distance, and as preferable to sequential limits, which bring in denumerability, whereas neighborhoods do not. However, he says, in order to provide the reader with a concrete picture, he will begin with the special neighborhoods defined by means of distance.
>
> Hausdorff then proceeds to define a metric space as a class of elements (points) with distance between x and y denoted by $\overline{xy}$ and subject to three axioms: (1) $\overline{yx} = \overline{xy}$, (2) $\overline{xy} = 0$ if and only if $x = y$, (3) $\overline{xy} + \overline{yz} \geq \overline{xz}$. The neighborhoods of x in a metric space are defined to be spheres with the center x and without the "surface;" that is, sets of points y such that $\overline{xy} < \varrho$, where ϱ can be any positive number. Hausdorff next states that spherical neighborhoods have properties of which only a few will be used. He indicates that, in accordance with his decision to make neighborhoods fundamental, he will abstain from using distance and will make use solely of certain properties of neighborhoods, thus treating the properties as axioms.
>
> Finally, on page 213, Hausdorff comes to his definition of what he calls a topological space – a class of elements (points) to each of which correspond certain sets from the class, called neighborhoods. There are four axioms:

(A) To each point corresponds at least one neighborhood U_x, and U_x contains x.

(B) If U_x and V_x are neighborhoods of x, then there is a neighborhood of x, W_x, which is a subset of U_x and V_x.

(C) If y is in U_x, then there is a neighborhood U_y of y such that U_y is a subset of U_x.

(D) For two distinct points x, y there exist two neighborhoods U_x and U_y with no point in common.[49]

Hausdorff proceeds to introduce the concepts of interior point and limit point, the notions of closed and open sets and to prove their fundamental properties. The usual theorems of that time, including the variations on the Heine-Borel theorem, are carefully discussed.

Concerning attribution, Taylor makes the following assessment:

> On the broader question of influence that might have led Hausdorff to choose to develop his point set topology on the basis of the neighborhood concept, I can only speculate. I think he probably was influenced by Hilbert and F. Riesz. Careful and industrious scholar that he was, Hausdorff would surely have seen Hilbert's work on the Foundations of Geometry and would, likewise, have seen the paper ... that was read [by Riesz] at the International Congress of Mathematicians in Rome in 1908. In that paper there are footnotes referring to work of Hilbert and Riesz although not to [the paper Riesz [1905]]. This last paper was on a subject that lay close to Hausdorff's particular interests (as evidenced by some of his publications on ordered sets and order types). It is highly likely that Hausdorff saw this paper. In it Riesz stressed his view that one should get away from distance and use the notion of neighborhood. ...
>
> The notes that Hausdorff included in his book were not comprehensive enough to indicate the general source of his ideas; therefore I do not attach much significance to the lack of references to the foregoing works of Hilbert and Riesz. He *does* refer to Fréchet occasionally, but not as often as if he were providing thorough scholarly documentation. For example, he does not give Fréchet credit for the notion of a metric space.[50]

In any case it seems clear that by 1915 the idea of neighborhood has evolved as one that merited serious consideration in the study of point set theory. In later years Fréchet [1917,1918] himself will add his perspective to the salient role that neighborhoods might play as he introduces and modifies his *new classe* $(\mathcal{V})$, where the fundamental structure $\mathcal{V}$ finally takes on the characteristics of a "*voisinage*". Following Fréchet but true to their roots, E.W. Chittenden and A.D.

[49]Taylor, *Loc. cit.*, pp. 301–302.
[50]Taylor, *Ibid.*, pp. 303–304.

Pitcher, both students of E.H. Moore, place the work of Fréchet and Hausdorff in the context of E.H. Moore's developments. However, these contributions crop up in another strand of mathematical evolution – namely, in the evolving role of distances that seems to have attracted the primary interests of the Americans. Thus, before this saga in the evolution of developability can continue, we must step back and take a look at the questions Fréchet raised in his thesis about the kinds of distance that might be identified as a fundamental construct and how these might be used to prove the standard theorems in abstract point set theory.

3. 1916–1923: The Influence of "The Metrization Problem"

In his thesis Fréchet introduces two classes of abstract sets for which a specific kind of distance is used to define the notion of a limit – the *classe* $(\mathcal{V})$ and the *classe* $(\mathcal{E})$. Fréchet himself in both [1906] and [1910] makes some remarks about the role that a distance might play in his theory, but he focuses exclusively on this in Fréchet [1918], entitled "Relation entre les notions de limite et distance," which is published in *Transactions of the American Mathematical Society*. There he makes it clear how his thinking evolved and why he wants to introduce a distance.

> Un certain nombre de propriétés des ensembles linéaires peuvent être étendues aux ensembles abstraits, c'est à dire aux ensembles dont les éléments sont de nature quelconque ou inconnue. . . .
>
> Mais si l'on veut aller plus loin, on est amené à introduire la notion de limite ou même la notion de distance tout en s'efforçant de les présenter sous une forme qui ne fasse pas intervenir la nature des éléments de les classe d'éléments considérée.
>
> En ce qui concerne la notion de limite, on peut se contenter d'admettre qu'il existe une règle d'ailleurs quelconque ou inconnue permettant de décider si une suite infinie d'éléments $A_1, A_2, \ldots, A_n, \ldots$ de la classe considérée converge ou non vers un élément déterminé A. On se bornera à supposer que ce critère est tel que 1° si les éléments de la suite sont identiques, la suite converge et converge vers A_1; 2° si une suite $\{A_n\}$ converge vers A, toute suite infinie extraite de la suite $\{A_n\}$ converge aussi vers A.
>
> *De telles classes d'ensembles*, que j'ai appelées classes $(\mathcal{L})$, *ne sont pas moins générales qu'une classe abstraite quelconque*. . . .
>
> Mais si l'on veut généraliser des théorèmes plus précis de la théorie des fonctions, une définition aussi indéterminée de la limite ne suffit plus. J'ai donc été amené, dans ma Thèse, à étudier l'effet de restrictions de plus en plus grandes apportées à l'idée de limite, ces restrictions restant toutefois telles que toutes les définitions classiques de la limite dans les ensembles les plus importants (ensembles de points, de courbes, de fonctions continues, etc.) y satisfont par avance.

> La notion qui intervient de la façon la plus essentielle pour faciliter l'étude de la limite dans la théorie des ensembles linéaires est *la notion de distance*. Réduite à ses caractéristiques abstraites, on peut la décrire ainsi:
>
> Dans une *classe* $(\mathcal{L})$, on *peut définir la distance* si à tout couple A, B d'éléments de la classe correspond un nombre $(A, B) = (B, A) \geq 0$ jouissant des propriétés suivantes:
>
> 1° (A, B) n'est nul que si A, B coïncident;
>
> 2° si, A restant fixe, la suite $\{B_n\}$ converge vers A, (A, B_n) doit converger vers zéro et réciproquement;
>
> 3° on a, quels que soient les éléments A, B, C de la classe,
>
> $$(B, C) \leq (A, B) + (A, C).$$[51]
>
> J'appellerai un tel nombre *la distance de A à B* et je dirai d'une classe $(\mathcal{L})$ où on peut définir une distance que *c'est une classe* $(\mathcal{D})$.‡ [52]

Here ‡ refers to the footnote: "Je modifie donc ici et dans la suite les dénominations adoptées dans ma Thèse. Ce que j'y ai appelé écart et voisinage je l'appelerai ici distance et écart uniformément régulier."

In his thesis Fréchet defines an abstract set with the properties given here as a *classe* $(\mathcal{E})$ and a distance with the given properties as an *écart*; a *voisinage* in his thesis is a distance with the property 3° replaced with the property:

> Il existe une fonction positive bien déterminée $f(\varepsilon)$ tendant vers zéro avec ε, telle que les inégalités $(A, B) \leq \varepsilon, (B, C) \leq \varepsilon$ entrainent $(A, C) \leq f(\varepsilon)$, quels que soient les éléments A, B, C.[53]

A *classe* $(\mathcal{V})$ is a *classe* $(\mathcal{L})$ in which there is a *voisinage* that is compatible with the limit as in property 2°.

It is clear that the *classe* $(\mathcal{V})$ includes the *classe* $(\mathcal{E})$ and, perhaps because it has a defining property that is founded in the limit notion, Fréchet proves all the results of his thesis, with the exception of one proposition, for the *classe* $(\mathcal{V})$. He conjectures that the exceptional result, which was proved only for *classe* $(\mathcal{E})$, is in fact valid for *classe* $(\mathcal{V})$; this was later verified by Hahn in [1908]. Each of the examples of a distance in Fréchet's thesis, which comprised the pages 34 through 75, is in fact an *écart*, and Fréchet makes no attempt to distinguish the two classes. One has the impression that Fréchet fully believes that the classes are identical.

[51] Note that this appears as $(A, B) \leq (A, C) + (C, B)$ in Fréchet's 1906 thesis.

[52] Fréchet, *Relation entre les notions de limite et distance*, pp. 53–54.

[53] Fréchet's Thesis [1906], p. 18.

In early 1907 Fréchet will send a copy of his thesis to Arthur Schoenflies at the University of Königsberg; the documentation of their correspondence is detailed in Taylor [1982: pp. 264–266]. Schoenflies will include Fréchet's work in his report[54] in the "Bericht erstattet der Deutschen Mathematiker-Vereinung" in 1908. It is in response to this report and to update the results of his thesis that Fréchet writes [1910]. Here he mentions Hahn's results and comments that this further confirms his belief that the *classe* $(\mathcal{V})$ and *classe* $(\mathcal{E})$ are identical; he even suggests that Hahn's work will, in fact, provide the mechanism for verifying this conjecture.

At this point world affairs will intervene. In 1910 Fréchet is appointed to a position at the University in Poitiers, but in 1914 he is enlisted as an officer into the French Army, serving primarily as an interpreter, although he still maintains his position at the University of Poitiers. In any case Fréchet's next installment to the evolution we wish to trace occurs in the paper we have just quoted and which appeared in 1918 in the *Transactions of the American Mathematical Society.*

Meanwhile the Americans have been making some contributions of their own.

3.1. CHITTENDEN'S FIRST METRIZATION THEOREM

E.R. Hedrick will spend his entire academic life doing analysis; R.L. Moore, another student of E.H. Moore and now at the University of Pennsylvania, is working on a "geometric" characterization of the Euclidean plane (Moore [1915, 1916, 1919]); T.H. Hildebrandt takes a position at the University of Michigan doing functional analysis; and Ralph Root, after his year with Hedrick, ends up at the U.S. Naval Academy and does not write another research paper.

E.W. Chittenden seems to emerge as the only American to work with Fréchet on the generalized theory of point sets. Although he also collaborates with A.D. Pitcher on extending the theory of developments to a body of research that is aimed at advancing E.H. Moore's general analysis, Chittenden himself seems primarily motivated by work that is closely associated with Fréchet. His work is essentially "topological".

Edward Wilson Chittenden completed his A.B. degree at the University of Missouri in 1909 and remained an extra year to write a master's thesis under the direction of E.R. Hedrick. One surmises that Hedrick encouraged him to continue his work at the University of Chicago where he finished his Ph.D. work in 1912 under the supervision of E.H. Moore. Chittenden began his career as an instructor at the University of Illinois in Urbana;[55] in 1918 he moved on to the University of

[54] Hedrick mentions in his paper [1911] that it is this report that has provided some direction for his paper [1911].

[55] An interesting aside is that Fréchet had planned to spend the academic year 1914–1915 at Urbana, except that his plans were interrupted by the war; additional details may be found in Taylor [1985: p. 283].

Iowa where he quickly became a full professor and remained until his retirement in 1954.

In 1916 Chittenden establishes a major breakthrough in Fréchet's *Calcul Fonctionnel* by verifying Fréchet's conjecture that for any abstract set of the *classe* $(\mathcal{V})$ there is an *écart* with the same convergent sequences, and hence, the set is also *classe* $(\mathcal{E})$. Note that Chittenden is using the terminology from Fréchet's thesis; thus, this result in today's terminology shows that *any abstract set of classe* $(\mathcal{V})$ *is a metric space.*

Chittenden will announce his result at the April meeting of the American Mathematical Society in Chicago; his full proof appears a year later in the *Transactions*. Chittenden's abstract explains how this work originated and suggests the idea for his proof.

> "On the equivalence of écart and voisinage"
>
> Hahn has shown that if a voisinage $V(a,b)$ is defined for a class $\mathcal{Q}$ containing at least two elements, then for every element q of $\mathcal{Q}$ there exists a continuous function μ^q, vanishing only at q and assuming values between zero and one. Dr. Chittenden modifies the existence proof of Hahn and secures a type of uniformity among the functions μ^q. Defining (a,b) as the least upper bound of $|\mu^q(a) - \mu^q(b)|$ for all elements q of $\mathcal{Q}$, it appears that (a,b) is an écart and is defined for every pair of elements of $\mathcal{Q}$. By means of the uniformity mentioned it is shown that $L_n(a_n,a) = 0$ implies $L_nV(a_n,a) = 0$ and that $L_nV(a_n,a) = 0$ implies $L_n(a_n,a) = 0$. Hence voisinage is equivalent to écart. This result was anticipated by Fréchet on the ground that no theorem was known for an écart which had not been established for a voisinage. This paper will be offered to the *Transactions* for publication.[56]

Apparently, Chittenden had a productive summer in 1916. His paper in the *Transactions* is dated September 25, 1916, and a second paper, "On the equivalence of relations $K_{q_1q_2m}$," published in the *American Journal of Mathematics*, is dated October 14, 1916. Both papers arrive from Urbana and have a common strategy in that the objective is to find a construction for a distance that will have particular convergence properties; this is work that clearly has the flavor of Fréchet's prior research in distance construction.

The second paper addresses issues of nonsymmetric distances that were raised by Hildebrandt in his 1912 paper, also published in the *American Journal of Mathematics*. In the cases considered by Hildebrandt, Chittenden shows that symmetric distances always exist to provide the same convergence structure so that the importance of nonsymmetric distances is still questionable.

In the meantime Fréchet has taken this theory to another level.

[56]Bulletin of the American Mathematical Society 22(1916), p. 430.

3.2. FRÉCHET POSES "THE METRIZATION PROBLEM"

One might easily suggest that the research we are about to examine evolves naturally from Fréchet's earlier work [1913], which was written in response to Hedrick [1911]. In that instance it is assumed that a convergence structure is given (that is, an assignment of limits to sequences) in the context of Fréchet's *classe* $(\mathcal{L})$ and that the question being resolved is whether this same convergence can be obtained using some sort of distance.

In the present context Fréchet defines a new class with certain *neighborhood* properties, which he will now call *classe* $(\mathcal{V})$, and a more general notion of distance, which he will now call an *écart*; further, he raises the issue of identifying necessary and sufficient conditions for the convergence in a *classe* $(\mathcal{V})$ to be equivalent to that of a metric, which he now calls *classe* $(\mathcal{D})$. This change in terminology is characteristic of Fréchet, who is continually revising his theory as he enriches and deepens its fundamental concepts.

Fréchet quickly realizes the limitation of even this *classe* $(\mathcal{V})$ which allows for only countably many neighborhoods at each point. Hence, a newer, and final, *classe* $(\mathcal{V})$ is defined so that the neighborhoods at a point need not be countable. One wonders if Fréchet is influenced by Root's work that had been published in 1914.

Fréchet himself explains the research context and the problem he wishes to explore.

> Le problème qui se pose n'est donc pas de savoir à quelle condition une classe abstraite est une classe $(\mathcal{D})$, mais à quelles conditions simples et indépendantes une classe $(\mathcal{L})$ où les suites convergentes et leurs limites sont *déjà* définies est-elle une classe $(\mathcal{D})$ et comment on y peut calculer la distance (A, B).
>
> *Ainsi, l'objet de ce travail est le suivant: sachant qu'on peut toujours de bien des façons définir dans une classe d'éléments abstraits les suites convergentes et leurs limites, il s'agit de déterminer à quelles conditions supplémentaires il faut assujettir ce choix pour que l'on puisse définir sur cette classe une distance telle que la convergence définie d'avance ne soit pas modifiée quand on la définit au moyen de cette distance.*[57]

In the theory of *classe* $(\mathcal{L})$ Fréchet has decided to alter his previous definitions for *classe* $(\mathcal{V})$ and *classe* $(\mathcal{E})$ and to introduce several new classes. A *classe* (S) is a *classe* $(\mathcal{L})$ in which all derived sets are closed. A *classe* $(\mathcal{L})$ for which there is a symmetric distance with properties 1° and 2°, but not necessarily property 3° (i.e. a metric without the triangle inequality) will be called "une *classe* $(\mathcal{E})$, ou encore qu'on a pu y définir un *écart*, à savoir le nombre (A, B), qui sera l'écart des éléments A et B. Si cet écart satisfait en même temps à la condition 3°, ce

[57] Fréchet, *Relations entre les notions de limite et de distance*, pp. 54–55.

sera en même temps une 'distance'."[58] Thus, the *classe* $(\mathcal{D})$ now denotes the class of metric spaces along with its associated sequential convergence structure.

Fréchet notes that any *classe* $(\mathcal{D})$ is both *classe* $(\mathcal{E})$ and *classe* $(\mathcal{S})$, but that a given *classe* $(\mathcal{E})$ need not be a *classe* $(\mathcal{D})$; and, for a given *classe* $(\mathcal{S})$ there need not even be an écart with the same limits. In order to study the question of when a *classe* $(\mathcal{L})$ is a *classe* $(\mathcal{E})$, Fréchet will introduce a new *classe* $(\mathcal{V})$, but in various footnotes and addendums writes that he will soon abandon this class in favor of a new *classe* $(\mathcal{V})$, primarily to avoid the restriction to enumerable neighborhood systems.

> Nous pouvons en outre *décomposer en plusieurs autres conditions indépendantes la condition pour une classe* $(\mathcal{L})$ *d'être une classe* $(\mathcal{E})$.
>
> Si une classe est $(\mathcal{E})$, la condition nécessaire et suffisante pour qu'une suite d'éléments $\{A_n\}$ converge vers l'élément A est que pour q donné on puisse trouver p tel que
> $$(A_n, A) < \frac{1}{q}$$
> pour $n > p$. Autrement dit si l'on appelle Σ_q l'ensemble des éléments B tels que $(A, B) < 1/q$, il faut et il suffit que quel que soit q, la suite des A_n soit contenue dans Σ_q à partir d'un certain rang variable avec q.
>
> Remarquons d'ailleurs que Σ_{q+1} fait partie de Σ_q et que $\Sigma_1, \Sigma_2, \ldots$ ont un seul élément commun savoir A.
>
> On est alors amené à la définition suivante. Soit une classe $(\mathcal{L})$; on dira que c'est *une classe* $(\mathcal{V})$ ou encore qu'on peut y définir *un voisinage** lorsque la condition suivante est réalisée:
>
> Pour tout élément A, il existe une suite d'ensembles $U_1, U_2, \ldots, U_q, \ldots$ telle que la condition nécessaire et suffisante pour qu'une suite $\{A_n\}$ converge vers A est que quel que soit q, la suite $\{A_n\}$ fasse partie de l'ensemble U_q à partir d'un certain rang (variable en général avec q).[59]
>
> [Here * refers to the footnote: "J'emploie ici la notation $(\mathcal{V})$ et le mot de voisinage dans un sens plus général que celui que avais adopté dans ma Thèse" – at which point Fréchet refers the reader to the following supplementary note that arrived too late for inclusion in the main body of the paper: "Dans une note récente aux Comptes-Rendus de l'Académie des Sciences (Septembre, 1917) je généralise une nouvelle fois en abandonnant la condition que les ensembles U forment une suite dénombrable."]

Evidently the *classe* $(\mathcal{V})$ is intermediate between the *classe* $(\mathcal{E})$ and the most general *classe* $(\mathcal{L})$. Fréchet raises the questions of what properties might be added to those defining the *classe* $(\mathcal{L})$ in order to characterize the more restrictive classes. In particular, for *classe* $(\mathcal{V})$ Fréchet characterizes those sets for which an écart

[58] Fréchet, *Relations entre les notions de limite et de distance*, p. 55.
[59] Fréchet, *Ibid.*, pp. 56–57.

exists; namely, a necessary and sufficient condition that a *classe* $(\mathcal{V})$ be a *classe* $(\mathcal{E})$ is that

> l'on puisse attacher à chaque élément A une suite d'entiers non décroissants et qui tendent vers l'infini
>
> $$1 = r_1^A \leq r_2^A \leq \cdots \leq r_n^A \leq \cdots,$$
>
> choisis de sorte que si l'on se donne arbitrairement un entier N et un élément A de la classe, on puisse déterminer un entier m pour lequel l'ensemble $T^{(B)}_{r^B_N}$ contient nécessairement l'élément A, si B appartient à $T^{(A)}_m$. [60]

Furthermore, Fréchet obtains necessary and suficient conditions in order (1) that a set of *classe* $(\mathcal{L})$ be *classe* $(\mathcal{V})$ and (2) that a set of *classe* $(\mathcal{E})$ be *classe* $(\mathcal{S})$. Fréchet suggests several properties of an écart that others will show to be of importance, but he makes no attempt to explore their significance.

Notably, in the context of his *classe* $(\mathcal{L})$ Fréchet poses the "metrization problem" in a quite explicit and emphatic fashion along with an acknowledgement that he is unable to give a complete solution in this paper:

> Pour élucider complètement les relations entre les notions de limite et de distance, il reste à résoudre le problème suivant:
>
> Nous avons vu que pour qu'une classe $(\mathcal{L})$ soit une classe $(\mathcal{D})$, il faut qu'elle soit $(\mathcal{S})$ et $(\mathcal{E})$. Ces deux conditions nécessaires, qui sont indépendantes, *sont elles suffisantes*? Et dans le cas de la négative, *quelle est la nature de la condition supplémentaire*?
>
> Ce problème n'est pas résolu dans le présent mémoire.[61]

REFLECTION AND OBSERVATIONS ON THIS WORK OF FRÉCHET.

As in his response (Fréchet,1913) to Hedrick, Fréchet's strategy in this situation is to explicitly construct a distance. In this case a *classe* $(\mathcal{V})$ is assumed and an *écart* is constructed as follows:

$$(A, B) = (B, A) = \frac{1}{q_A^B} + \frac{1}{q_B^A}$$

where for any elements A and B

$$q_B^A = \max\left\{n \mid B \text{ is in } T^{(A)}_{r_n^A}\right\}$$

[60]Fréchet, *Relations entre les notions de limite et de distance*, p. 61–62.
[61]Fréchet, *Ibid.*, p. 63.

and the neighborhoods $\left\{T_n^{(A)}\right\}$ are assumed to be decreasing as the index increases.

Furthermore, Fréchet discusses some properties of distance functions (A, B) that include the notion of a *uniformly regular écart*, as follows:

(1) For every δ and every A there is ν such that, if $(A, B) < \nu$, then there is $\omega = \omega(A, B, \delta, \nu)$ such that, if $(B, C) < \omega$, then $(A, C) < \delta$.
(2) For every δ and every A there is $\omega = \omega(A, \delta)$ such that, if $(A, B) < \omega$ and $(B, C) < \omega$, then $(A, C) < \delta$.
(3) For every δ there is $\omega = \omega(\delta)$ such that, if $(A, B) < \omega$ and $(B, C) < \omega$, then $(A, C) < \delta$.

Actually, (1) is a necessary and sufficient condition that a *classe* $(\mathcal{E})$ be a *classe* $(\mathcal{S})$.

Property (2) will reappear as Pitcher-Chittenden's condition (3), which for symmetrics is the property of being *coherent*. A nonsymmetric distance with this property will be called a *γ–distance* (or a *Δ–metric*); this distance is important in the study of quasimetric spaces. Later work by Niemytzki [1927] will show, in fact, that sets with such a symmetric distance are indeed metric spaces.

Condition (3) defines the *classe* $(\mathcal{E}_r)$ of sets with a uniformly regular écart, which Pitcher–Chittenden will call condition (4) and show is equivalent to being a *classe* $(\mathcal{L})$ with a *voisinage* (and hence, equivalent to being a metric space).

In retrospect one must acknowledge that Fréchet was unparalleled during this era in his understanding of the intrinsic properties of distance, especially as this relates to ideas of continuity and convergence. He continues to add additional perspective to the important class of metric spaces, to pose the fundamental questions that must be answered and to suggest properties that need to be considered; but again, he would leave the important details of these investigations for other researchers to complete.

3.3. THE JOINT WORK OF CHITTENDEN AND PITCHER

Immediately following Fréchet's paper in the *Transactions of the American Mathematical Society* is "On the foundations of the calcul fonctionnel of Fréchet". This is A.D. Pitcher and E.W. Chittenden's version of a study of distance functions and their associated convergence structure with particular emphasis on the significance of compactness in this setting.

Pitcher and Chittenden begin with their view of Fréchet's *Calcul Fonctionnel* and the contributions their paper will add to this theory.

> In his thesis, Fréchet gave a very beautiful generalization of the theory of point sets and of the theory of real valued functions of a real variable. His functions are real valued but the range of the independent variable is an abstract class Q of elements q. He secures his results principally through the medium of a properly conditioned distance function δ, a generalization of

the distance between two points, which associates with each pair q_1q_2 of elements a real non-negative number $\delta(q_1q_2)$. He is thus enabled to secure the important theorems of point set theory and of real function theory, especially those relating to continuous functions and their properties. This theory of Fréchet has excited considerable interest and has received much attention from mathematicians. Various contributions to its foundations and to its content have been made.

In the present paper we follow the example of Fréchet in assuming once for all that $\delta(qq) = 0$ and that $\delta(q_1q_2) = \delta(q_2q_1)$. ... In the first part of the paper we give very simple conditions on systems $(\mathcal{Q};\delta)$ which are sufficient for many purposes and which, in the case of compact sets, we show to be equivalent, so far as limit of a sequence is concerned, to the *voisinage* and thus to the *écart* of Fréchet [where the authors refer to the recently published *On the equivalence of écart and voisinage*]. The remainder of the paper is devoted to the theory of functions on sets $\mathcal{Q}$ of systems $(\mathcal{Q};\delta)$. ...

We follow Fréchet in saying that q is a limit of a sequence q_n, $\mathrm{L}_n q_n = q$, when and only when $\mathrm{L}_n\delta(q_nq) = 0$. We shall be interested in the following properties of δ, or of systems $(\mathcal{Q};\delta)$.

(1) If $\delta(q_1q_2) = 0$ then $q_1 = q_2$.

(2) If $\mathrm{L}_n q_{1n} = q$ and $\mathrm{L}_n\delta(q_{1n}q_{2n}) = 0$, then $\mathrm{L}_n q_{2n} = q$.

(3) If $\mathrm{L}_n q_{1n} = q = \mathrm{L}_n q_{2n}$, then $\mathrm{L}_n\delta(q_{1n}q_{2n}) = 0$.

(4) If $\mathrm{L}_n\delta(q_{1n}q_{2n}) = 0$ and $\mathrm{L}_n\delta(q_{2n}q_{3n}) = 0$, then $\mathrm{L}_n\delta(q_{1n}q3n) = 0$.

(5) There is a function $\phi(e)$ such that $\mathrm{L}_{e=0}\phi(e) = 0$ and such that,

$$\text{if } \delta(q_1q_2) \le e \text{ and } \delta(q_2q_3) \le e, \text{ then } \delta(q_1q_3) \le \phi(e).$$

(6) $\delta(q_1q_2) + \delta(q_2q_3) \ge \delta(q_1q_3)$

It will be seen at once that (2), (3), and (4) are important properties which are implied by (5) and (6). We will show that (2), (3), and (4) play a fundamental role.

The notation δ^n denotes the fact that δ has the property (n). Thus δ^{13} denotes a δ which possesses the property (1) and (3). The *voisinage* of Fréchet is a δ^{15} and the *écart* a δ^{16}.

Our terminology is that of Fréchet[62] except that we do not wish to imply that the limit of a sequence is unique. Thus a set $\overline{\mathcal{Q}}$ of the set $\mathcal{Q}$ of system

[62]Note that the authors continue to use the terminology from Fréchet's thesis rather than the revised terminology in the paper that has immediately preceded theirs in the *Transactions*. Since Chittenden had read and suggested some changes in Fréchet's manuscript early in 1917 (see Taylor [1985: pp. 307–308]), one surmises that Pitcher was the lead writer for this paper or that the paper was already in publication format before Fréchet's apparently untyped manuscript reached the editors.

Note also that the authors adhere to the notation of E.H. Moore and Hildebrandt.

$(\mathcal{Q};\delta)$ is said to be *compact* in case every sequence of distinct elements of $\overline{\mathcal{Q}}$ gives rise to at least one limiting element. $\overline{\mathcal{Q}}$ is *closed* in case every limiting element of $\overline{\mathcal{Q}}$ is of $\overline{\mathcal{Q}}$. It should be noted that, in a system $(\mathcal{Q};\delta)$ where $\delta(q_1q_2)$ may be zero for q_1 and q_2 distinct, an element q may be the limit of a sequence composed of a single element, other than q itself, repeated infinitely often. If $\overline{\mathcal{Q}}$ is compact and closed then $\overline{\mathcal{Q}}$ is said to be *extremal*. For the case of non-unique limits it is desirable to take special note of classes $\overline{\mathcal{Q}}$ which may not be closed but which are such that every sequence $\{q_n\}$ of $\overline{\mathcal{Q}}$, which has a limit, has a limit in $\overline{\mathcal{Q}}$. Such sets are said to be *self-closed*. If a set $\overline{\mathcal{Q}}$ is compact and self-closed it is said to be *self-compact*. The property (2) will prove to be of fundamental importance and, for lack of a better term we venture to call a system $(\mathcal{Q};\delta^2)$ a *coherent* system.* A system $(\mathcal{Q};\delta)$ is said to be *limited* if there is a positive number h such that for every q_1q_2, $\delta(q_1q_2)\le h$.

[Here * refers to the footnote: The importance of coherent systems is apparent from the fact that in such systems the derived class of every class is closed, cf. E.R. Hedrick, these Transactions, vol. 12(1911), p. 285.]

We may have two systems $(\mathcal{Q};\delta)$ and $(\mathcal{Q};\overline{\delta})$, the set $\mathcal{Q}$ being the same in each case but the two distance functions, δ and $\overline{\delta}$, not the same. If we wish to indicate that q is a limit of a sequence $\{q_n\}$ we write, in the first case, $\mathrm{L}_nq_n = q$, and in the second case $\overline{\mathrm{L}}_nq_n = q$. Two systems $(\mathcal{Q};\delta)$ and $(\mathcal{Q};\overline{\delta})$ are said to be L-equivalent in case $\mathrm{L}_nq_n = q$ implies $\overline{\mathrm{L}}_nq_n = q$ and conversely.[63]

The following theorem reflects the kind of theorem that has motivated much of Chittenden's work as exemplified by his metrization theorem.

Theorem 7. If $(\mathcal{Q},\delta^1)$ is a coherent system then there is an L*-equivalent, limited system $(\mathcal{Q};\overline{\delta}^{13})$ such that on every compact set $\overline{\mathcal{Q}}$ of $\mathcal{Q}$, $\overline{\delta}^{13}$ is a voisinage, i.e. the system $(\overline{\mathcal{Q}},\overline{\delta}^{13})$ is a system $(\overline{\mathcal{Q}},\overline{\delta}^{15})$.*

This theorem follows from Theorem 6 and the fact that the property (1) is undisturbed by the transformation used in establishing the above theorems.† [64]

Here † refers to the footnote: "From Theorem 7 and the work of Chittenden (loc. cit.) it follows that in compact sets coherence and *écart* are *infinitesimally* equivalent."

The remainder of this paper conveys a more "topological bent", suggesting an emerging feature in the work of these researchers, even though the underlying setting is generated by a distance so that sequential convergence is still providing the context for the research.

A system $(\mathcal{Q},\delta)$ is said to be *biextremal* in case it is true that when two sequences $\{q_{1n}\},\{q_{2n}\}$ are such that $\mathrm{L}_n\delta\,(q_{1n}q_{2n}) = 0$ then there is a pair of

[63] Pitcher and Chittenden, *On the foundations of the calcul fonctionel of Fréchet*, pp. 66–68.

sequences $\{q_{1n_k}\}$, $\{q_{2n_k}\}$ (subsequences of $\{q_{1n}\}$, $\{q_{2n}\}$ respectively) which have a common limit.

Theorem 10. Every biextremal system (Q, δ) *is also extremal.* ...

Theorem 12. If a function μ *is continuous on the set* Q *of a biextremal, connected system* (Q, δ)*, then* μ *is uniformly continuous, bounded, assumes its bounds and every value between these bounds.* ...

Thus every continuous function on a set Q of a biextremal connected system (Q, δ) possesses the important properties which we usually associate with a continuous function on a closed interval or region. However in a particular case there may be very few continuous functions. Indeed it may happen that only constant functions are continuous. For example let Q be any set of elements and for every q_1, q_2 let $\delta(q_1 q_2) = 0$. The system thus defined is biextremal and connected. Certainly a proper set of continuous functions on a set Q should, in general, contain functions other than the constant functions. ...

We have already seen that every biextremal system (Q, δ) is extremal and that every biextremal, L–unique system (Q, δ) is coherent. It is also true that every coherent, extremal system (Q, δ) is biextremal. ...

Theorem 21. For L*-unique systems* (Q, δ) *biextremal is equivalent to the two properties extremal and coherent.* ...

The following properties are interesting and important.

(a) every sequence $\{q_n\}$ contains a sequence $\{q_{nk}\}$ which has a limit;

(b) if a sequence $\{q_n\}$ has a limit and $\mathrm{L}_n\delta(q_{1n}, q_{2n}) = 0$, then $\{q_{1n}\}$ and $\{q_{2n}\}$ have subsequences $\{q_{1n_k}\}$, $\{q_{2n_k}\}$ with a common limit;

(c) If q is a limit of $\{q_{1n}\}$ which has a limit in common with $\{q_{2n}\}$ then q is a limit of $\{q_{2n}\}$.

Theorem 22. The property biextremal is equivalent to properties (a) and (b) and the property coherent is equivalent to properties (b) and (c).[65]

Note that, until now, a prime underlying strategy in the work of Fréchet and Chittenden has been to find a distance that comes equipped with an inherent convergence structure to match an axiomatically defined sequential limit structure, which is usually given in the context of some kind of *classe* $(\mathcal{L})$. Now observe, in the work that is to follow, a focus on more general considerations that have the characteristics of "neighborhood". This work evolves in Chittenden's collaboration with Pitcher and is likely influenced by their formative years at the University of Chicago.

Arthur D. Pitcher completed his Ph.D. work with E.H. Moore along with Hildebrandt and Root in 1910. Chittenden's thesis follows two years later in 1912 and seeks primarily to extend Moore's work in general analysis to allow

[65]Pitcher and Chittenden, *On the foundations of the calcul fonctionel of Fréchet*, pp. 73–78.

for *infinite* developments. The stream of abstracts for presentations before various regional meetings of the American Mathematical Society suggests that both Chittenden and Pitcher were engaged in serious research to develop further the ideas that Moore had initiated in his *Introduction to a Form of General Analysis*. This work culminates with the paper "On the theory of developments of an abstract class in relation to the calcul fonctionnel," whose leading author is likely Chittenden. In it Chittenden and Pitcher attempt to unify the work of Fréchet, Hausdorff and R.L. Moore by basing their theory on a fundamental concept that extends E.H. Moore's original definition of development. Significantly, the research has moved away from focusing exclusively on distance in the spirit of Fréchet's thesis work and has taken on a "topological" flavor.

This change in perspective is indicative of the work of others of this era. Most importantly, Fréchet has recognized the important role that neighborhoods might play in the development of abstract classes. The outline for this work, which will highlight his *classe* $(\mathcal{H})$[66] as a tribute to Hedrick, is presented in Fréchet [1918]. This is Fréchet's initial response to the work of Hausdorff. Characteristically, Fréchet will argue for the optimum in generality and, hence, considers the ideas (1) that neighborhoods need not be "open" and (2) that the neighborhoods of a point need not contain the point. In light of this work one might suggest that Chittenden and Pitcher are merely following the lead of Fréchet.

Chittenden and Pitcher provide a context for their work and describe what they expect to achieve:

> In his *Introduction to a Form of General Analysis* E.H.Moore has called attention to the great importance of developments Δ in analysis, and has used them in a general theory which includes the theories of continuous functions and convergent series. The authors of the present paper have made further studies of the properties of developments in relation to the theory of Moore.
>
> It is the purpose of the present paper to develop the theory of developments along lines inaugurated by Fréchet and developed by him and other investigators. The methods of analysis suggested by this theory have led the authors to results in the Calcul Fonctionnel, some of which have been published previously [referring to the Pitcher and Chittenden paper, *On the foundations of the calcul fonctionnel of Fréchet*].
>
> The theory of developments Δ is placed, in the present paper, into close relation with the theories of systems $(\mathcal{L})$ and $(\mathcal{D})$ of Fréchet and the topological space of Hausdorff.
>
> We develop the general theory in terms of five completely independent properties of a development Δ which together suffice to make the developed class $\mathcal{P}$ a compact metric space.

[66]This class is that which will become known as the T_1-spaces.

> The theory is applied to determine necessary and sufficient conditions that a topological space be a compact metric space. A further application is made to spaces s satisfying axiom systems Σ_1 or Σ_2 introduced by R.L Moore as bases for a theory of plane curves in non-metrical analysis situs. It is shown that each such space s is the sum of an enumerable set of compact metric sets. A development Δ of s is defined such that the associated distance function δ is equivalent to an écart with respect to limit of a sequence.[67]

Next the authors highlight Moore's notion of development in its relationship to Fréchet's recent work in his new *classe* $(\mathcal{V})$. Note that they quickly set the stage to include the work of Fréchet and Hildebrandt.

> In general a development Δ of an abstract class $\mathcal{P}$ is a sequence of systems Δ^m, called stages, each system Δ^m being composed of subclasses $\mathcal{P}^{ml}$ of $\mathcal{P}$, the index l having for each integer m a range L^m. In the present paper we restrict ourselves to the cases in which the classes L_m are composed of integers in the natural order and the classes $\mathcal{P}^{ml}$ are existent classes for every value of the composite index ml in the system $((ml))$ of indices for the development Δ. We make here the fundamental hypothesis that we are given a class $\mathcal{P}$ and a development Δ of $\mathcal{P}$.
>
> M. Fréchet has recently introduced the following considerations: a class of elements is a class $(\mathcal{V})$ if to every element p of the class there is assigned a family of sets V_p called neighborhoods § of p [where § refers to the footnote: Comptes Rendus, vol. 165 (1917), p. 359. Fréchet uses the term "voisinages".].
>
> Denote by V_p^m the class of all elements of $\mathcal{P}$ which belong to a class $\mathcal{P}^{m'l}$ $(m' \geq m)$ containing p. This class will be called *the neighborhood of rank* m of p defined by the development Δ. Then $\mathcal{P}$ is a class $(\mathcal{V})$. ...
>
> In terms of the neighborhood V_p^m we define a distance function $\delta(p,q)$ as follows: If q belongs to V_p^m and not to V_p^{m+1} then $\delta(p,q) = 1/m$; if q does not belong to V_p^m for any m, $\delta(p,q) = 1$; if q belongs to V_p^m for every m, then $\delta(p,q) = 0$; for every p, $\delta(p,p) = 0$.*
>
> [Here * refers to the footnote: This definition is a special case of a definition of distance in terms of a K relation given by T.H. Hildebrandt, *A contribution to the foundations of Fréchet's calcul fonctionnel*, American Journal of Mathematics, vol. 34 (1912), p.248.]
>
> For the purposes of this paper we shall define limiting element as follows: *an element p is a limiting element of a class* $\mathcal{Q}$ *if and only if every neighborhood of p contains an infinity of elements of* $\mathcal{Q}$. †
>
> [Here † refers to the footnote: This definition is more restrictive than that of Fréchet (loc. cit.) and becomes equivalent to his only when every two

[67] Chittenden and Pitcher, *On the theory of developments of an abstract class in relation to the calcul fonctionnel*, pp. 213–214.

neighborhoods of p have a neighborhood of p in common, p has an infinity of distinct neighborhoods, and the neighborhoods of p have no common element, except possibly p. Only the first of these conditions is satisfied by the neighborhoods $\mathcal{B}_p^m$ in general.]

By an obvious application of the Zermelo axiom of choice we obtain the following important proposition:

If p is a limiting element of a class $\mathcal{Q}$, $\mathcal{Q}$ contains a sequence $\{q_n\}$ of elements q_n, no two alike, such that $L_n\delta(q_n,p) = 0$, that is, p is a limit of the sequence $\{q_n\}$.

The notion of fundamental sequence is helpful in studying developments. A fundamental sequence F is a sequence F $\equiv$ $\{\mathcal{P}^{m_n l_n}\}$ of classes of Δ subject to the conditions: $m_n \geq m_{n-1}$; $\mathrm{L}_n m_n = \infty$; *the classes* $\mathcal{P}^{m_1 l_1}$, $\mathcal{P}^{m_2 l_2}, \ldots, \mathcal{P}^{m_n l_n}$ *have at least one common element for every value of* n.

A fundamental sequence F of the development Δ is *closed* if there is an element common to all the classes $\mathcal{P}^{m_n l_n}$.[68]

After establishing a few properties of fundamental sequences, Chittenden and Pitcher introduce the definitions they wish to consider.

The development Δ is:

finite, if for every integer m the number of classes $\mathcal{P}^{ml}$ is finite;
complete, if for every m and element p there is a class $\mathcal{P}^{ml}$ containing p;
closed, if every fundamental sequence is closed. ...

Theorem 4. If the development Δ is finite, complete, and closed, the class $\mathcal{P}$ is compact and separable. ...

An element p is *interior* to a set $\mathcal{D}$ if $\mathcal{D}$ contains a neighborhood of p. A class $\mathcal{D}$ consisting entirely of interior elements will be called a domain.†

[Here † refers to the footnote: Hausdorff, loc. cit., p. 215, calls attention to the importance of this concept. He used the term "Gebiet" in this connection. R.L. Moore, loc. cit., p.36, following the usage of Weierstrass assumes that a domain is also a connected set.]

Theorem 8. If the development Δ is finite, complete, and closed and $\mathcal{Q}$ is a closed set then any family ($\mathcal{D}$) of domains whose sum contains $\mathcal{Q}$, contains a finite subfamily $\mathcal{D}_1, \ldots, \mathcal{D}_n$ with the same property.[69]

Now, Chittenden and Pitcher introduce their idea of a *regular* development and continue the research they had begun in their just-published joint paper.

The development Δ is *regular* if for every pair of elements (p_1, p_2) contained in a class $\mathcal{P}^{ml}$ $(m > 1)$ there is a class $\mathcal{P}^{m-1,\, l}$ which also contains the pair (p_1, p_2).

[68] Chittenden and Pitcher, *On the theory of developments of an abstract class in relation to the calcul fonctionnel*, pp. 214–215.
[69] Chittenden and Pitcher, *Ibid.*, pp. 216–217.

Two sequences $S_1 \equiv \{p_{1n}\}$, $S_2 \equiv \{p_{2n}\}$ are connected in case

$$L_n \delta(p_{1n}, p_{2n}) = 0.$$

A necessary and sufficient condition that two sequences S_1, S_2 *be connected is that for every value of* n *there is a class* $\mathcal{P}^{m_n l_n}$ *containing* p_{1n} *and* p_{2n}, *where* $L_n m_n = \infty$. ...

Theorem 13. If the development Δ *is finite, complete, closed, and regular the system* $(\mathcal{P}; \Delta)$ *is biextremal.** ...

[Here * refers to the footnote: Cf. Pitcher-Chittenden, loc. cit., Section 4. A system is *biextremal* in case there exists for every pair of connected sequences S_1, S_3 a pair S'_1, S'_2 [*sic*] of connected subsequences of S_1, S_3, respectively, which have a common limit.]

The theory of coherent systems $(\mathcal{P}, \delta)$ was developed in the previous paper. A system $(\mathcal{P}, \delta)$ is *coherent* if whenever two sequences S_1, S_2 are *connected* (cf.§7) then every limit of S_1 is a limit of S_2. The following theorem is a consequence of Theorem 4 above and Theorem 6 of the previous paper.

Theorem 15. If the system $(\mathcal{P}, \delta)$ *derived from the development* Δ *of the given class* $\mathcal{P}$ *is coherent, the derived class of every class is closed; and if furthermore* Δ *is finite, complete, and closed,* δ *is equivalent with respect to limit to a distance function* $\overline{\delta}^{2345}$. ‡ [70]

Here ‡ refers to the footnote: "The properties 2, 3, 4, 5 of a distance function δ are defined in Section 1. of the previous paper. If $\overline{\delta}$ has the further property δ^1; $\delta(p, q) = 0$ implies $p = q$, then $\overline{\delta}$ is a voisinage as defined in the thesis of Fréchet (loc. cit.)."

Not to be overlooked is the following construction that the authors include and apply to obtain their most significant contributions in this paper.

Denote by $\mathfrak{R}_p^m$ the class of all elements q such that p and q are contained in some class $\mathcal{P}^{m'l'}$ for every $m' \leq m$. Evidently $\mathfrak{R}_p^m$ contains $\mathfrak{R}_p^{m+1}$.

Theorem 18. If the development Δ *is finite, complete, and closed, and if limit of a sequence is unique, then every class* $\mathfrak{R}_p^m$ *contains an infinite subclass of every class* $\mathcal{Q}$ *which has* p *for a limiting element.* ...

Theorem 19. Under the hypothesis of Theorem 18 *the class* $\mathcal{P}$ *admits a definition of a distance function* ρ, *which is equivalent to* δ *with respect to limit of a sequence, such that the system* $(\mathcal{P}; \rho)$ *is coherent.* ...

Theorem 20. Under the hypothesis of Theorem 18 *the distance function* ρ *is equivalent (with respect to limit of a sequence) to an écart.*

Cf. Theorem 19 above and Theorem 7 of the previous paper. As an immediate consequence of the preceding theorem we have:

[70] Chittenden and Pitcher, *On the theory of developments of an abstract class in relation to the calcul fonctionnel*, pp. 219–220.

Theorem 21. If the development Δ is finite, complete, and closed, and if limit of a sequence is unique the class $\mathcal{P}$ is a compact metric space. † [71]

Here † refers to the footnote: "Cf. Hausdorff, loc. cit.,p. 211 et seq., for the consequences of this theorem."

Interestingly, the axiomatic influence of E.H. Moore remains apparent in the work of Chittenden and Pitcher as they include the following:

> We have developed the theory of developments Δ in terms of five properties: (1) finite; (2) complete; (3) closed; (4) every fundamental sequence has at most one element in its core; (5) if two closed fundamental sequences are connected with a sequence $\{p_n\}$ their cores have a common element. We shall show that these five properties are completely independent in the sense of E.H. Moore. The proof of complete independence requires the exhibition of $2^5 = 32$ examples, representing all possible combinations of the five properties and their negatives.[72]

Perhaps the most important mainstream contribution of Chittenden and Pitcher comes here where they connect their work with that of Hausdorff.

> F. Hausdorff has introduced the concept of *topological* space. Such a space consists of a class $\mathcal{C}$ of points x which correspond to subsets $\mathcal{U}_x$ of $\mathcal{C}$ called regions[73] and are subject to the following conditions:
>
> (A) Each point x implies at least one region $\mathcal{U}_x$; every region $\mathcal{U}_x$ contains the point x;
>
> (B) If $\mathcal{U}_x$, $\mathcal{V}_x$ are two regions belonging to the same point x, then there is a region $\mathcal{W}_x$ contained in both $\mathcal{U}_x$ and $\mathcal{V}_x$;
>
> (C) If a point y is contained in $\mathcal{U}_x$ there is a region $\mathcal{U}_y$ which is a subset of $\mathcal{U}_x$;
>
> (D) For every two distinct points x, y there are two regions $\mathcal{U}_x, \mathcal{U}_y$ without a common point.
>
> Let $\delta(x,y)$ be a symmetric distance function defined on a class $\mathcal{C}$. We define regions $\mathcal{U}_x$ as follows: for every positive number a the class of all points y such that $\delta(x,y) < a$ is a region $\mathcal{U}_x$ of x. With this definition of region *a necessary and sufficient condition that* C *be a topological* space is that:
>
> (1) *If $\delta(x,y) < a$ there exists a positive number b such that whenever $\delta(y,z) < b$ then $\delta(x,z) < a$.*

[71] Chittenden and Pitcher, *On the theory of developments of an abstract class in relation to the calcul fonctionnel*, pp. 221–222.

[72] Chittenden and Pitcher, *Ibid.*, p. 228.

[73] Chittenden, in a footnote, remarks that Hausdorff uses the term "Umgebung".
We note Chittenden's use of "region" that was prevalent in the work of R.L. Moore at that time, rather than the use of "neighborhood" that had occurred in the work of Root.

(2) *If x and y are any two points there is a positive number a such that $\delta(x, z) < a$ implies $\delta(y, z) > a$.*

From the above one easily deduces the conditions under which a development Δ determines a topological space.

The properties (1), (2) of δ stated above do not imply the property of coherence as is shown by the following example. . . .

The topological space defined by a symmetric distance function is not general but satisfies the first enumerability axiom of Hausdorff in the sense that the system $(\mathcal{U}_x)$ is equivalent to a system $(\mathcal{V}_x)$ satisfying this axiom.

We shall now state the conditions under which a topological space becomes equivalent to a metric space. In place of axiom (A) we shall employ the stronger axiom:

(A_ω) There exists for each point x a fixed sequence $\{\mathcal{U}_x^m\}$ of regions of x such that $\mathcal{U}_x^m$ contains x and $\mathcal{U}_x^{m+1}$.§

[where § refers to the footnote: This is virtually the assumption of a type of development.]

(G) If $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots, \mathcal{U}_m \ldots$ is any sequence of regions such that $\mathcal{U}_m$ is of rank m (at least) and there is a point x common to the classes $\mathcal{U}_m^0$ † then there is for every region $\mathcal{U}_x$ of x an m_0 such that for every $m \geq m_0$, $\mathcal{U}_m$ is contained in $\mathcal{U}_x$.

[where † refers to the footnote: $\mathcal{U}_m^0$ is the class of all α–points of $\mathcal{U}_m$. An α–point (Hausdorff) is a point x such that every $\mathcal{U}_x$ contains a point of $\mathcal{U}_m$.]

(H) If $(\mathcal{U})$ is any family of domains such that $\mathcal{C}$ is the aggregate of classes $\mathcal{U}$ then there is a finite subfamily $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots, \mathcal{U}_n$ of $(\mathcal{U})$ whose aggregate is $\mathcal{C}$.

From axioms (A_ω), (G), it follows that *for every region $\mathcal{U}_x$ there is an m such that $\mathcal{U}_x^m$ belongs to $\mathcal{U}_x$.* Every region $\mathcal{U}_x$ is a domain. It follows readily from axiom (H) that $\mathcal{C}$, and every closed subset of $\mathcal{C}$, is compact and admits the Heine–Borel theorem.

From axioms (A_ω) and (H) we have for every positive integer m a finite set of regions;

$$\mathcal{U}^{m1}, \mathcal{U}^{m2}, \mathcal{U}^{m3}, \ldots, \mathcal{U}^{ml_m};$$

such that

$$\mathcal{C} = \sum_{l=1}^{l=lm} \mathcal{U}^{ml}$$

The development $\Delta = ((\mathcal{P}^{ml}))$, where $\mathcal{P}^{ml}$ is the set of α–points of $\mathcal{U}^{ml}$ is a finite complete development of $\mathcal{C}$. We shall show that Δ is closed and equivalent to the system $(\mathcal{U}_x)$ with respect to limit of a sequence.

> [A proof follows appealing to Theorem 21.]
>
> *Theorem 31. A necessary and sufficient condition that a topological space be a compact metric space is that the axioms* (A_ω), (G), (H) *be satisfied.*
>
> The extension of this theory to non-compact spaces will be the topic of a separate discussion.[74]

Chittenden and Pitcher conclude this paper by connecting their work to that of R.L. Moore.

> We shall apply the results of the preceding article to space S satisfying the axiom systems Σ_1, Σ_2 of R.L. Moore [referring to Moore [1916]]. In the systems Σ_1, Σ_2 the undefined elements are points p and classes R of points called regions. A region $\mathfrak{R}$ will be a region $\mathfrak{R}_p$ in the sense of §14 if $\mathfrak{R}$ contains p. Then S is easily shown to be a topological space. We shall establish the further result
>
> *Theorem 32. A space S satisfying axiom systems Σ_1 or Σ_2 is a topological space satisfying the additional axioms* (A_ω), (G). *Any limited closed subset of S satisfies axiom* (H). ...
>
> On the basis of the complete axiom systems Σ_1, Σ_2 we define an infinite development Δ of S for which the corresponding distance function is equivalent to an écart on every limited subset of S, while limit relative to Δ is equivalent to limit relative to the system of regions R. ...
>
> It follows ... that every closed limited subset of S is a compact metric space and that S is therefore the sum of an enumerable infinity of compact metric spaces. As the proof that S is in fact a metric space does not involve a direct application of developments Δ it is reserved for a separate paper. We summarize the results of this section in the following theorem:
>
> *Theorem 34. A space S satisfying systems Σ_1 or Σ_2 admits a regular, complete, closed development Δ whose reduction relative to any limited closed set Q is finite, complete, and closed. Furthermore, if Q is perfect, Q is a set of type "(E) normale" of Fréchet. Limit of a sequence is the same whether defined in terms of the development Δ or of regions $\mathfrak{R}$.*[75]

REFLECTION AND OBSERVATIONS ON THIS WORK OF CHITTENDEN AND PITCHER.

The body of work in this emerging field is beginning to take shape as Chittenden and Pitcher connect the work of Fréchet, Hausdorff and R.L. Moore.

In using their construction for V_p^m, namely

$$V_p^m = \bigcup \left\{ \mathrm{st}\left(p, \Delta^{m'}\right) \mid m' \geq m \right\},$$

[74] Chittenden and Pitcher, *Ibid.*, pp. 228–230.
[75] Chittenden and Pitcher, *Ibid.*, pp. 230–233.

along with its associated distance,

$$\rho(p,q) = \begin{cases} 1/m, & \text{if } q \text{ belongs to } V_p^m \text{ and does not belong to } V_p^{m+1}; \\ 0, & \text{if } q \text{ belongs to } V_p^m \text{ for every } m, \end{cases}$$

Chittenden and Pitcher employ a *now standard* construction for neighborhoods and distance functions; note that this construction was given first by E.H. Moore in his 1910 memoir.

On the other hand, by defining

$$\Re_p^m = \left\{ q \in X \mid \text{for each } k \leq m, p \text{ and } q \text{ are in some subclass of } \Delta^k \right\},$$

Chittenden and Pitcher are advancing a new idea, although this notion is implicit in the work of Hildebrandt. Of importance are the properties that

$$\Re_p^{m+1} \subseteq \Re_p^m \text{ and } q \in \Re_p^m \text{ iff } p \in \Re_q^m.$$

Also note their focus on the importance of having *decreasing neighborhoods* in order to produce *a sequence of covers* for establishing constructions that continue to refine the notion of a "uniformity." Clearly the precursory idea of *refinement* is being born in this construction as well as in their notion of a "regular" development.

In retrospect this paper presents a state-of-the-art perspective of the "topological world" of 1919 from the viewpoint of the Americans.

3.4. ALEXANDROFF AND URYSOHN'S METRIZATION THEOREM

Depending heavily on the 1917 theorem of Chittenden, Alexandroff and Urysohn solve the metrization problem for *classe* ($\mathcal{L}$), which they note also solves the problem for the topological spaces of Hausdorff. Like Chittenden and Pitcher, this work takes into account the work of major researchers of the day, especially Fréchet and Hausdorff.

To place this work in perspective let us note the events that have transpired since the work of Chittenden and Pitcher.

- Fréchet has published (Fréchet, 1921) and *Esquisse d'une théorie des ensembles abstraits* in 1922, works that will become the foundation for a book representing the culmination of his work in the theory of abstract sets.
- The Polish School has been born with a narrow focus on set theory, topology and the foundations of mathematics, which for them included mathematical logic. They had begun publishing *Fundamenta Mathematica* in 1920 with a publication policy of considering only the works in the narrow interests the Polish School. Kuratowski has published his defining properties for the topological notion of *closure*.

- Alexandroff and Urysohn are attracted to the work of Hausdorff and Fréchet. Since both are fluent in both German and French, they establish active correspondence with Hausdorff in Bonn and Fréchet in Strasbourg. Their academic work takes place in Moscow, but they summer in Europe where they visit both Hausdorff and Fréchet and where much of their research originates.

Importantly, in this work Alexandroff and Urysohn refine at least two concepts of fundamental importance for this story. First, they elucidate the notion of a point being determined in a topological context by a *countable* neighborhood system; this clarifies the seminal work of Fréchet, Hedrick, Root and Hausdorff. Secondly, they introduce the concept of *chaîne complète* which for topological spaces is equivalent to development. With this understanding Alexandroff and Urysohn prove that a Hausdorff space is metrizable iff it has a development such that each stage is *inscribed* in the immediately preceding one. Thus, the notion of a development is implicitly introduced in this paper; however, its specific characteristics are confounded by the "inscribed" property. That is, Alexandroff and Urysohn do not isolate and study the properties of "spaces" that have a complete chain of open covers; rather they focus only on spaces for which a complete chain with the inscribed property is present.

With a heightened sense of attribution, Alexandroff and Urysohn set the stage for their results. To fully appreciate the publication of this work, one might examine the correspondence between the Russians and Fréchet, as well as the correspondence between Lebesgue, an editor for *Comptes Rendus*, and Fréchet. Much of this is documented in the personal letters to Fréchet.[76] After several drafts the introduction to this landmark paper of Alexandroff and Urysohn appears as follows:

> C'est M. Fréchet qui a le premier formulé explicitement le problème d'indiquer les conditions pour qu'une classe $(\mathcal{L})$ soit une classe $(\mathcal{D})$, c'est-a-dire pour qu'on puisse déterminer dans une classe $(\mathcal{L})$ une distance telle que les relations limites auxquelles elle donne naissance soient identiques à celles qui étaient d'avance. Ce problème auquel plusieurs auteurs (M. Hedrick, Fréchet, Chittenden, Moore,[77] Vietoris, Urysohn, Alexandroff) ont déjà apporté des contributions importantes en résolvant dans des cas particuliers, est équivalent au problème suivant: *quelles sont les conditions pour qu'un espace topologique soit un espace métrique*? En effet, tout espace métrique peut être regardé comme un espace topologique et comme une classe $(\mathcal{L})$ [même

[76] See Taylor [1985: pp. 319–328], where the sensitivity of Fréchet to the issues of attribution are revealed. According to Taylor, none of Fréchet's letters to Alexandroff or Urysohn survive.

[77] This refers to R.L. Moore's work which is reported in the work of Chittenden and Pitcher.

comme une classe $(\mathcal{S})$] et l'on peut indiquer facilement les conditions pour qu'un espace topologique soit une classe $(\mathcal{L})$ et *vice versa*.[78]

Without wasting words, Alexandroff and Urysohn quickly establish the necessary definitions, state their theorem and provide a proof.

Définitions. – I. Soit $\{V_n\}$ une *suite* de *domaines* dont chacun contient le point ξ; nous dirons qu'elle *détermine* ce point dans l'espace topologique donné E si chaque domaine G contenant ξ contient au moins un domaine V_n.

II. Nous dirons qu'un système Π de domaines *recouvre* l'espace E si chaque point ξ de E appartient à un au moins des domaines du système Π.

III. Soient Π_1 et Π_2 deux systèmes de domaines dont chacun recouvre E; nous dirons que Π_2 est *inscrit* dans Π_1 si tout couple V_2 et W_2 de domaines de Π_2 ayant des points communs, correspond un domaine V_1 de Π_1 qui contient tous les deux.

IV. Soit $\{\Pi_1, \Pi_2, \ldots, \Pi_n, \ldots\}$ une suite de systèmes recouvrant l'espace; nous dirons que c'est une *chaîne compléte* si la condition suivante est vérifiée: *soient ξ un point quelconque de E et $V_1, V_2, \ldots, V_n, \ldots$ des domaines contenant ξ et appartenant respectivement à $\Pi_1, \Pi_2, \ldots, \Pi_n, \ldots$; dans ce cas la suite $\{V_n\}$ détermine le point ξ dans E.*

V. Une chaîne complète $\{\Pi_1, \Pi_2, \ldots, \Pi_n, \ldots\}$ sers dite *régulière* si, pour tout n, Π_{n+1} est inscrit dans Π_n.

Théoreme – Pour qu'un espace topologique puisse être considéré comme un espace métrique, il faut et il suffit qu'il y existe une chaîne complète régulière.

Démonstration. – Pour voir la nécessité de cette condition, il suffit d'appeler Π_n le système formé de toutes les *sphères* de rayon 2^{-n} [une sphère de centre x et de rayon ϵ étant, par définition, l'ensemble des points y tels que $\rho(x, y) < \epsilon$].

Montrons maintenant qu'un espace topologique E admettant une chaîne complète $\{\Pi_1, \Pi_2, \ldots, \Pi_n, \ldots\}$ est une classe $(\mathcal{E})$. x et y étant deux points arbitraires de E, nous définissons leur écart $\rho(x, y)$ comme il suit: s'il n'existe aucun domaine de Π, qui les contient tous les deux, nous posons $\rho(x, y) = 1$; dans le cas contraire, soit n le premier entier tel qu'aucun domaine de Π_{n+1} ne contienne ces deux points simultanément: nous poserons alors $\rho(x, y) = 2^{-n}$. Il s'agit de démontrer que cet écart ne modifie pas les relations limites, c'est-a-dire que:

1° A tout point ξ de E et à tout nombres $\epsilon > 0$, il correspond un domaine G contenant ξ dont tous les points x satisfont à l'inégalité $\rho(\xi, x) < \epsilon$. En effet, soit n le premier entier tel que $2^{-n} < \epsilon$, et soient $V_1, V_2, \ldots, V_n$

[78] Alexandroff and Urysohn, *Une condition nécessaire et suffisante pour qu'une classe $(\mathcal{L})$ soit une classe $(\mathcal{D})$*, p. 1274.

des domaines appartenant respectivement à Π_1, Π_2, ..., Π_n et contenant chacun le point ξ; il suffit de désigner par G leur partie commune;

2° A tout point ξ de E et à tout domaine G contenant ξ, il correspond un $\epsilon > 0$ tel que tout point x satisfaisant à $\rho(\xi, x) < \epsilon$ est situé dans G. En effet, il existerait dans le cas contrire des points x étrangers à G tels que $\rho(\xi, x)$ serait arbitrairement petit; il y aurait donc, dans chaque Π_n, au moins un domaine V_n contenant ξ sans être contenu dans G: ce qui est en contradiction avec la définition des chaînes complètes.

Or si la chaîne complète est régulière, les deux inégalité $\rho(x, y) < 2^{-n}$ et $\rho(y, z) < 2^{-n}$ entraînent évidemment $\rho(x, y) < 2^{-(n-1)}$, c'est-a-dire que l'écart est régulier; donc, d'après le theorème de M. Chittenden, E est un espace métrique. C.Q.F.D.[79]

Undoubtedly, the Russians add a note as a concession to their referee, who is Fréchet.

Note supplementaire – M. Fréchet a eu l'obligeance de nous communiquer que la condition pour qu'une classe $(\mathcal{L})$ soit $(\mathcal{D})$ peut être énoncée d'une manière bien simple que celle qu'on obtient en se servant des espaces topologiques. En effet, notre théorème relatif à ces espaces (de même que la démostration ci-dessus) s'applique aussi *directement* aux classes $(\mathcal{S})$, vérifiant la condition 3° (voir la note n° 4) et même, plus généralement, aux classes $(\mathcal{H})$.[80]

REFLECTION AND OBSERVATIONS ON THIS WORK OF ALEXANDROFF AND URYSOHN.

In a succinct and clarifying fashion Alexandroff and Urysohn have pulled together necessary ideas to provide a "neighborhood characterization" for the metrizability of a topological space. In the course of developing their proof they pinpoint a number of ideas that certainly are fundamental in any topological study. We note at least the following:

(1) The notion of *local base* at a point in a topological setting;
(2) the notion of a *open cover* and the usefulness of a *sequence of open covers* that leads to
(3) their notion of a *complete chain*, from which the notion of development can be derived; and finally,
(4) their notion of a cover being *inscribed* in another, which one might perceive as precursory to the notion of *refinement*.

[79] Alexandroff and Urysohn, *Une condition nécessaire et suffisante pour qu'une classe* $(\mathcal{L})$ *soit une classe* $(\mathcal{D})$, pp. 1275–1276.
[80] Alexandroff and Urysohn, *Ibid.*, p. 1276.

There is little doubt about the importance of this paper in the history of general topology. Much of general topology, as we know it today, remains intact as it was born in the work of Alexandroff and Urysohn. This particular work begins a divergence away from the specific results of Fréchet in abstract sets (and his various classes) and leads eventually to a theory that almost ignores the approach of Fréchet. Note that the main theorem is stated in the context of *topological spaces* rather than in the *classe* $(\mathcal{L})$ or *classe* $(\mathcal{V})$ setting of Fréchet's research. Since Fréchet was the referee, this says something about the convictions of Alexandroff and Urysohn concerning the appropriate setting on which to base further research in general topology.

Still let us recognize that it is the intriguing questions of Fréchet that have motivated the young Russians. It is the recently published *Esquisse* (Fréchet [1922]) and the extensive publication Fréchet [1921] that has captived these researchers and inspired them. The rich exchange of personal correspondence provides clear evidence that, although the test of time would obliterate the original work of Fréchet, it is Fréchet, the one that Alexandroff always referred to as *Mon cher Maître* in personal correspondence,[81] who provided essential ideas that stoked the fires in these research efforts of Paul Alexandroff and Paul Urysohn.

4. 1923–1951: Refining and Naming of Developable Spaces

The 1920's usher in and establish point set topology as a recognized field of mathematics. The creation of the Polish School of Mathematics with Sierpiński and Kuratowski as two of its leaders, along with the developing research of the Russians headed by Alexandroff and Urysohn and their students, are creating the foundation for "modern" general topology and publishing prodigiously, mostly in the *Mathematische Annalen* and *Fundamenta Mathematica*. In the United States R.L. Moore establishes himself as the leader of his brand of *analysis situs* at the University of Texas and continues to work primarily on geometrically conceived problems associated with the plane. But, taking center stage for our story is Chittenden, still at the University of Iowa, who will add a major contribution to the evolution of the development concept.

4.1. CHITTENDEN'S OUTLINE FOR GENERAL TOPOLOGY

Following his Colloquium presentation at the summer meeting of the American Mathematical Society, held in Columbus, Ohio, in September of 1926, Chittenden writes what might be characterized as a first outline for a foundational development of general topology. In this work Chittenden uses the metrization problem as a theme for recapitulating the important topological facts of the day and making some additional contributions of his own. Although still tightly drawn to Fréchet's

[81] See Taylor [1985].

"derived set" approach for defining the topological setting, Chittenden manages to discuss the prominent separation properties (attributed to Tietze, Vietoris and others), to prove Urysohn's Lemma for arbitrary spaces (rather than just Hausdorff spaces), to note the recent results of Urysohn, Tychonoff and Alexandroff (on separation, compactness and metrization) and to show that R.L. Moore has also made his contributions.

In 1926 *a standard text* that would provide an introduction to the study of topological spaces has not yet been written. Chittenden's opening section establishes a setting that Chittenden might have considered well known.

> 1. *Topological Space.* In the theory of abstract sets we assume that we are given an arbitrary aggregate $\mathcal{P}$ and a relation between subsets of $\mathcal{P}$ which corresponds to the relation between a set and its derived set in the classical theory of sets of points.† That is, the mathematical concept abstract set in its current sense includes the notion limit point or point of accumulation. The introduction of limit point permits the definition of continuous 1–1 correspondence or homeomorphy. The study of such correspondences, particularly of invariants under homeomorphic transformations, constitutes the science of topology or analysis situs.‡ It seems proper therefore to speak of an abstract set as a topological space.§ Throughout this paper, the term *topological space* or *abstract set* refers to any system of the form $(\mathcal{P}, K)$ composed of an aggregate $\mathcal{P}$ and a relation of the form EKE' between the subsets E, E' of $\mathcal{P}$ which is subject to the condition, for every subset E of the aggregate $\mathcal{P}$ there is a unique set E' in the relation K to E. That is, the relation K defines a single-valued set-valued function on the class $\mathcal{U}$ of all subsets of the aggregate.||
>
> [Here footnotes occur as:
>
> † See M. Fréchet, *Esquisse d'une théorie des ensembles abstraits*, Sir Asutosh Mookerjee's Commemoration volumes, II, p.360, The Baptist Mission Press, Calcutta, 1922; *Sur les ensembles abstraits*, Annales de l'École Normale, vol. 38 (1921), p. 341ff.
>
> ‡ See H. Tietze, *Beiträge zur allgemeinen Topologie*, I., Mathematische Annalen, vol. 88 (1923), p. 290.
>
> § This terminology is suggested by Fréchet. See Comptes Rendus, vol. 180 (1925), p. 419.
>
> || These functions are studied in detail in an unpublished article by the writer.]
>
> 2. *The Metrization Problem.* The problem is to state in term of the concepts point, and point of accumulation the conditions that a topological space be metric.
>
> A metric space [referring in a footnote to Hausdorff [1914] and adding "The definition of distance is due to Fréchet" with a reference to Fréchet [1906]] is any topological space in which the points of accumulation are

defined or definable in terms of a function (p, q) called the distance between the points p and q and satisfying the following conditions.

(0) *The distance (p, q) is a definite real number for every pair of points p, q.*

(1) *Two points are coincident if and only if their distance is zero.*

(2) *For any three points p, q, r,*

$$(p, q) \leq (p, r) + (q, r).$$

It follows readily from these conditions that the distance (p, q) is non-negative, and that it is symmetric in p and q, $(p, q) = (q, p)$. In a metric space a point p is a point of accumulation of a set E provided its distance from a variable point of E which is distinct from p has a lower bound zero. ...

3. *Hausdorff Spaces.* A remarkable and important class of topological spaces has been defined by F. Hausdorff. In a Hausdorff space the points of accumulation are defined in terms of a family of neighborhoods U conditioned by the following four postulates: [following which the usual properties are given]. ...

The topological spaces of Hausdorff are evidently included among the classes $(\mathcal{V})$ of Fréchet ... In a class $(\mathcal{V})$ a point p is a point of accumulation of a set E in case every neighborhood of p contains a point of E distinct from p.

It is easy to see that every metric space is a Hausdorff space.

[Here Chittenden gives necessary and sufficient conditions that a topological space be a Hausdorff space by listing four properties of derived sets.]

To complete the solution of the metrization problem we need only add the conditions that a Hausdorff space be metric.

4. *Existence of Non–Constant Continuous Functions.* The metrization problem is included in another problem proposed by Fréchet in correspondence with Paul Urysohn and with me. It is evident that the distance (p, q) of two points p and q is a continuous function of its arguments, and is not constant in a space of two or more points. Thus the metrization problem is related to the more general problem, under what conditions does a topological space admit the existence of a non-constant continuous function.† The topological conditions for the existence of such functions in a Hausdorff space have been discovered by Urysohn. I have recently succeeded in formulating these conditions for topological spaces in general.[82]

[Here Chittenden gives his proof of Urysohn's Lemma for *any normal* space, while † refers to the following footnote:

The definition of continuous function for general topological space is given by Fréchet in the following form: *A point p of space is interior to a set*

[82]Chittenden, *On the metrization problem and related problems in the theory of abstract sets*, pp. 13–17.

I if it belongs to I and is not a point of accumulation of any subset of the complement of I, $P - I$. A function $f = f(p)$ is continuous at a point p if the oscillation of the function f on the sets I to which p is interior has the lower bound zero. Esquisse, p. 363.]

Chittenden continues his treatise on the foundations of general topological spaces.

5. *Perfectly Separable Spaces.* Among the spaces which were considered by Hausdorff are those whose points of accumulation are definable in terms of an enumerable family of neighborhoods. Such spaces are said to satisfy the second axiom of enumerability. Tychonoff and Vedenissof [in [1926]] have called them separable spaces. In a letter to me Fréchet calls attention to the fact that the word separable is already in use in a more general sense and suggests the term perfectly separable. The following important and remarkable theorem was discovered by Urysohn.

Theorem. A necessary and sufficient condition that a perfectly separable Hausdorff space be metric is that it be normal.[83]

Here Chittenden introduces the usual separation axioms with appropriate references and follows that with a proof of Tychonoff's generalization of Urysohn's theorem to regular spaces.[84] Now Chittenden will acknowledge the work of R.L. Moore.

8. *An Axiom of R.L. Moore.* The importance of the regular and perfectly separable, therefore metric, spaces in the analysis of continua is indicated by the fact that nine years before the publication of the discoveries of Urysohn, R.L. Moore assumed these properties in the first of a system of axioms for the foundations of plane analysis situs. [Here Moore [1916] is referenced with a note that "The hypothesis of regularity was also made (apparently independently) by L. Vietoris, Monatshefte, vol. 31 (1921), p.176."] This axiom is furthermore of particular interest historically since it yields when slightly modified a necessary and sufficient condition that a topological space be metric and separable. The modified axiom of Moore may be stated as follows.

Axiom (R.L. Moore). *We are given a space $\mathcal{P}$ in which point of accumulation is defined in terms of a family of classes of points called regions. Among the regions there exists a fundamental enumerable sequence*

$$R_1, R_2, R_3, \ldots, R_n, \ldots$$

with the following properties: (0) for every region R there is an integer n such that R_n is a subset of R; (1) for every point p and every integer n there is an integer n' greater than n such that $R_{n'}$ contains p; (2) if p and q are distinct

[83]Chittenden, *Ibid.*, p. 18.

[84]Here Chittenden references both Tychonoff [1926] and Urysohn [1925].

points of a region R there is an integer m such that if n is greater than m and R_n contains p, then R_n is a subset of $R - q$.

It is easy to show that in a space satisfying this axiom the regions are open sets, and that the space is a regular and perfectly separable Hausdorff space, therefore metric and separable. The converse of this proposition is also true and is established by the following chain of propositions. [An argument follows.]

Theorem. A necessary and sufficient condition that a topological space be metric and separable is that it satisfy the axiom of R.L. Moore.‡ [85]

[Here ‡ refers to the footnote: The fact that Axiom 1 is a sufficient condition for metrizability was inferred by R.L. Moore from the theorem of Tychonoff in §7 above.]

Turning to the metrization of compact spaces, a class of spaces he has studied and made significant contributions to its theory, Chittenden notes:

Theorem. A necessary and sufficient condition that an infinite compact Hausdorff space be metrizable is that it be perfectly separable. ...

Theorem. A necessary and sufficient condition that a locally compact Hausdorff space be metrizable is that the space be perfectly separable or else the sum of a set (of arbitrary cardinal number) of disjoined domains which are perfectly separable subspaces of the given space. ...

Corollary. A necessary and sufficient condition that a connected and locally compact space be metrizable is that it be perfectly separable.[86]

Next, Chittenden considers the role that the study of distances has contributed to the foundations of general topology.

12. *The Equivalence of Distance and Uniformly Regular Écarts.* Attempts have been made by Fréchet, E.R. Hedrick, A.D. Pitcher and the writer to obtain effective generalizations of the theory of metric spaces. That is, to impose hypotheses which yield substantially the same group of theorems about point sets and are less restrictive. It has however been established in each case that the conditions proposed imply that the resulting space is equivalent to a metric space

[Chittenden then gives a short history for the evolution of uniformly regular écarts which culminates with the following theorem and a new proof using his work on families of equally continuous functions.]

Theorem. The concepts uniformly regular écart and distance are equivalent. ...

[85]Chittenden, *On the metrization problem and related problems in the theory of abstract sets*, pp. 22–23.

[86]Chittenden, *Ibid.*, pp. 24–25.

13. *Coherent Spaces.* Another attempt to generalize effectively the theory of metric spaces was made by A.D. Pitcher and E.W. Chittenden. They considered an écart (p, q) in which

$(2'')$ if $L(p, p_n) = 0$ and $L(p_n, q_n) = 0$, then $L(p, q_n) = 0$.

A space in which the points of accumulation are definable in terms of a symmetric écart satifying condition $(2'')$ is said to be coherent. It has recently been shown by Niemytski [in [1927]] that if the écart satisfies the further condition: $(p, q) = 0$ is equivalent to $p = q$, then the space is metric.[87]

Finally and most importantly, Chittenden isolates the property of a space having a development, although he uses the terminology "regular development". In this setting Chittenden presents three sets of necessary and sufficient conditions for a Hausdorff space to be metrizable.

14. *Spaces Defined by Developments.* The metrization problem for spaces whose points of accumulation are defined in terms of a development Δ was studied by Pitcher and Chittenden. A development Δ is an arbitrary system of subclasses $\mathcal{P}^{ml}$ $(m = 1, 2, 3, \ldots,\ l = 1, 2, \ldots, l_m)$ of a fundamental class $\mathcal{P}$ in which the classes $\mathcal{P}^{ml}$ for a fixed index m form a stage Δ^m of the development. The index l_m may have a finite or infinite range. If its range is finite for all values of m the development is said to be finite, otherwise infinite.

A point p is a point of accumulation of a set of point E relative to a development Δ provided there is a sequence of indices $m_1 < m_2 < m_3 \cdots$ and a corresponding sequence of classes $\mathcal{P}^{m_n l_n}$ each of which contains an element of the set $E - p$. The reader is referred to the original article for the details of this investigation. The metrization problem was solved for compact spaces. The paper contains a set of necessary and sufficient conditions that a compact Hausdorfff space be metric.

15. *The General Metrization Problem.* The general problem, to determine the topological conditions for the metrization of a topological space, was first explicitly stated and solved by Alexandroff and Urysohn. It is however of interest to note that E.R. Hedrick in continuing the search begun by Fréchet for a generalization of metric space discovered that a number of important theorems stated by Fréchet for classes $(\mathcal{V})$ normales could be proved in any class $(\mathcal{L})$ in which derived sets are closed, providing the given space admits a property called the *enclosable* property. While Fréchet soon proved that the space thus defined by Hedrick was in fact a "classe $(\mathcal{V})$ normale" (therefore metric), it is important to observe that a slight modification of the conditions imposed by Hedrick constitute a set of necessary and sufficient conditions for the metrization of an abstract set.

[87] Chittenden, *On the metrization problem and related problems in the theory of abstract sets*, pp. 26–30.

We shall give three solutions of the general metrization problem; the first contains a modified form of the enclosable property of Hedrick, the second is due to Alexandroff and Urysohn, and the third is based upon the notion of coherence introduced by Pitcher and Chittenden. These three sets of conditions are alike in requiring the existence of a type of development Δ of fundamental importance which it is proposed to call *regular*.

Each stage of a *regular* development of a topological space is a family Δ^m of open sets V^m which covers the space $\mathcal{P}$. The development proceeds by consecutive stages, that is, each set V^{m+1} of the $(m+1)$st stage is a subset of a set of the mth stage. Furthermore, if $V^1, V^2, V^3, \ldots, V^m, \ldots$ is any infinite sequence of open sets one from each stage of the development and if there is a point p which is common to the sets of the sequence, then that point is *determined* by the sequence. That is, if V is any neighborhood of the point p, then for some value of the integer m, we have $V_m = V$. The sets V^m of the mth stage of the development are said to be of *rank* m.

The following additional definitions will be needed. Two points p, q are *developed* of stage m provided there is a set of rank m which contains them both. In a regular development two points which are developed of rank m are developed of any lower rank.

Two sequences p_m, q_m are connected by a regular development provided the points p_m, q_m are developed of stage m $(m = 1, 2, 3, \ldots)$.

A regular development Δ will be said to be *coherent* provided the connection of sequences is transitive. That is, a sequence $\{p_m\}$ is connected with a sequence $\{q_m\}$ whenever there is a sequence $\{r_m\}$ such that $\{p_m\}$ is connected with V_m [Chittenden probably means $\{r_m\}$] and $\{r_m\}$ is connected with $\{q_m\}$.

The $(m+1)$st stage of a development is said to be *inscribed* in the mth if every pair of sets of rank $m+1$ which have a common point is contained in a set of rank m.

Theorem. Let $\mathcal{P}$ be a Hausdorff space admitting a regular development Δ in open sets V^m. Each of the following three conditions is a necessary and sufficient condition that $\mathcal{P}$ be equivalent to a metric space.

I. (Hedrick) *For any positive integer m there is an integer n such that for any point p there is a set V_m of rank m which includes all sets of rank n which contains p.*

II. (Alexandroff and Urysohn) *For each value of the integer m the $(m+1)$st stage of the development is inscribed in the m-th.*

III. *The development Δ is coherent.*[88]

[Then follows Chittenden's proof for this theorem.]

[88]Chittenden, *Ibid.*, pp. 30–32.

REFLECTION AND OBSERVATIONS ON THIS WORK OF CHITTENDEN

One can not overlook the value in a paper, presented at a national meeting of the American Mathematical Society, that provides such a succinct overview of what needs to be known about the current status of this emerging field. Most impressive is what Chittenden adds to the intriguing problem of metrization – a problem situation in which he is recognized internationally at the time as a leading figure. One can not fail to note the "name-dropping" that constitutes an almost "Who's who?" in the field at this time.

The first metrization result is based on neighborhood considerations, which Chittenden attributes to Hedrick. Essentially, this states that a space is metrizable if, and only if, it has a development such that for each stage there is a later stage that *star-refines* it. In essence one has the theorem:

> A topological space $(X, \mathcal{T})$ is metrizable if and only if there is a sequence of open covers $\mathcal{G}_n$ such that, for each n, $\mathcal{G}_{n+1}$ star-refines $\mathcal{G}_n$.

Such a sequence would later be called a *normal development* for $(X, \mathcal{T})$.

Notably, Chittenden has made implicit use of a "star" concept, which will become a fundamental construct for defining developability. Note that this idea of "starring" occurs prominently in the 1910 work of E.H. Moore and in the 1919 paper of Chittenden and Pitcher so that it seems quite reasonable to expect that a refinement might evolve in the work of Chittenden. Thus, it is Chittenden in 1927 who introduces the concept of a development as it applies to a topological space; however, it is Bing who will point out the fruitful properties of such spaces, again in the context of the metrization problem, and create an interest in studying them.

The second set of metrization conditions is simply a restatement of the conditions that are given in the 1923 metrization theorem of Alexandroff and Urysohn.

The third condition, given last and perhaps attesting to the modesty of Chittenden, is a "coherent" condition that closely relates to Chittenden's own work on uniformly regular écarts.

We note several publications of significance to our study that close out the events of the 1920's.

In a brief abstract in the *Bulletin of the American Mathematical Society* R.L. Moore announces a modification of his axioms. In particular, we see that in the spirit of Alexandroff and Urysohn, and now Chittenden, Moore will replace the fundamental *countable sequence of regions* for his space S with a *countable collection of covers*.

> [Referring to his 1916 paper, *Foundations of plane analysis situs*, Moore announces:] Axiom 2, Theorem 4, and Axiom $1'$ stated below hold true in all these spaces, and form a basis for a considerable body of theorems, in particular, Theorems 1–10, and 15, of the paper cited above.
>
> Axiom $1'$: There exists a countable sequence $G_1, G_2, G_3, \ldots$ such that

(a) for each n, G_n is a collection of regions covering space,

(b) if P_1 and P_2 are distinct points of a region R, there exists an integer δ such that if $n > \delta$ and K_n is a region containing P_1 and belongs to G_n, then K_n' [the closure of K_n] is a subset of $R \setminus P_2$,

(c) if $R_1, R_2, R_3, \ldots$ is a sequence of regions such that, for each n, R_n belongs to G_n and such that, for each n, $R_1, R_2, \ldots, R_n$ have a point in common, then there exists a point common to all the point sets R_1', R_2', $R_3', \ldots$[89]

Note that Moore announced this modification at the Annual Meeting of the American Mathematical Society in December of 1926, just after Chittenden had delivered his Colloquium address.

A second major event to occur in the late 1920's is the publication in 1928 of the long-awaited Fréchet memoir, *Les espaces abstraits et leurs théorie considérée comme introduction à l'analyse générale,* by the prominent publishing firm of Gauthier–Villars in Paris. The work itself was heralded by a sequence of article that had appeared, beginning in 1927, in *Fundamenta Mathematica* and in the *American Journal of Mathematics.*

The outpouring of research papers that were to follow, written by a host of American researchers, suggest that the book was well received and heavily referenced. One of those to appear in 1929 is a long paper in *Transactions of the American Mathematical Society* by Chittenden, who in a fashion similar to that of his important survey with Pitcher of 1919, writes his version of a state-of-the-art reprisal of the general theory of topological space, again using the derived set as the essential theme. The influence of this paper is less apparent, suggesting perhaps the rapid advances in the development of the field had passed him by. No additional papers of significance would be penned by Chittenden.

4.2. CONTRIBUTIONS OF THE 1930'S

Research publications in the 1930's suggest that the theory of general topological spaces has attracted the interest of the mathematical community as a plethora of monographs appear to replace Hausdorff's *Mengenlehre.* Particularly notable are the following:

- R.L. Moore's *Foundations of Point Set Theory* [AMS Colloquium Publication, 1932];
- Casimir Kuratowski's *Topologie I, Espaces Métrisables, Espaces Complets* [Warsaw–Lwow; 1933];
- Waclaw Sierpiński's *Introduction to General Topology* [University of Toronto Press, 1934];

[89]R.L. Moore, *Abstract sets and the foundations of analysis situs,* p. 141.

- Paul Alexandroff and Heinz Hopf's *Topologie I* [Springer, Berlin; 1935]; and
- Eduard Čech's *Bodové množiny I* (Point Sets I) [1936].

Most particularly, let us note that the year 1937 is a banner year for this story as two significant papers are published in the *Bulletin of the American Mathematical Society*.

Although his results were announced much earlier (in 1933, before he had finished his doctoral work with R.L. Moore at the University of Texas), F. Burton Jones will not publish his most important paper until 1937. In this paper he names Moore spaces and poses a question that will become known as *The Normal Moore Space Problem*.[90] That is, Jones notes that a Moore space need not be normal whereas a metrizable space is always normal (even completely normal) and conjectures that it is this separation property alone that distinguishes metric spaces from Moore spaces.

Jones quickly introduces the focus of his paper and establishes his preliminary results:

> Urysohn has shown that any completely separable, normal topological space is metric. It is the principal object of this paper to establish a similar result for certain separable spaces.
>
> *Theorem 1. Every subset of power c of a separable normal Fréchet space–$\mathcal{L}$ (or –$\mathcal{H}$) has a limit point.*
>
> [Here a short proof is given.]
>
> The above argument with slight changes establishes the following three theorems.
>
> *Theorem 2. Every subset of power c of a separable completely normal Fréchet space–$\mathcal{L}$ (or –$\mathcal{H}$) has a limit point of itself.*
>
> *Theorem 3. If $2^{\aleph_1} > 2^{\aleph_0}$, every uncountable subset of a separable normal Fréchet space–$\mathcal{L}$ (or –$\mathcal{H}$) has a limit point.*
>
> *Theorem 4. If $2^{\aleph_1} > 2^{\aleph_0}$, every uncountable subset of a separable completely normal Fréchet space–$\mathcal{L}$ (or –$\mathcal{H}$) has a limit point of itself.*
>
> A space–$\mathcal{L}$ may, however, be separable and normal but contain an uncountable point set *not* containing a limit point of itself. This is shown by the following example.[91]
>
> [Here a lemma is proved and an example is given.]

In just two pages Jones presents an incredible collection of mathematical thought that will stimulate at least a generation of point set topologists.

[90] See Nyikos [1979] for an insightful account of this problem and its role in the history of point set topology.

[91] Jones, *Concerning normal and completely normal spaces*, pp. 671–672.

In order to make an application of Theorem 4, two lemmas will be established. Throughout the rest of this paper M denotes a space satisfying Axiom 0 and parts 1, 2, and 3 of Axiom 1 of R.L. Moore's *Foundation of Point Set Theory* and is referred to as a *Moore space* M. . . .

Lemma B. In order that a Moore space M should have the Lindelöf property it is necessary and sufficient that every uncountable subset of M should have a limit point. . . .

Lemma C. If every uncountable subset of a Moore space M has a limit point, M is a completely separable metric space. . . .

[Note that Jones uses the term *completely separable* for Hausdorff's *second countability* property.]

Theorem 5. If $2^{\aleph_1} > 2^{\aleph_0}$, *then every separable normal Moore space M is completely separable and metric.*

This follows from Theorem 4 and Lemma C.

The author has tried for some time without success to prove that $2^{\aleph_1} > 2^{\aleph_0}$. But although Theorem 5 is unsatisfactory in this respect, it does raise a question of some interest: *Is every normal Moore space M metric*? This question is as yet unsettled. However, if the answer is *yes*, then it should be possible to establish directly certain results for normal Moore spaces M which are known to hold in metric spaces but which are known not to hold in all Moore spaces M. The author has established a number of such theorems but it seems likely that only one of them may be of use in settling the question itself.

Theorem 6. A normal Moore space M is completely normal.[92]

One might argue that it is these remarks that provide the impetus for Bing's research that is to follow – and even that it is this problem alone that has spawned the field of set-theoretic topology that will occupy perhaps the most talented topologists in the last half of the twentieth century. Thus, one wonders why these results were *not* included in the dissertation work of Jones and why he waited so long to publish such impressive results. According to Jones, this is what happened.

When I was a student at Texas, Moore apparently didn't consider work in abstract spaces very highly. Typical of this attitude is the fact that he never published or even stated his metrization theorem except in a hidden sort of way. It may be that he had tried to solve Fréchet's problem and Chittenden beat him to it. But I rather doubt this. I think he just felt there was more substance and beauty in other less abstract kinds of problems – continua,

[92]Jones, *Concerning normal and completely normal spaces*, pp. 675–676.

> decompositions, etc. So when I proved that if $2^{\aleph_0} < 2^{\aleph_1}$ then every separable, normal, Moore space was metrizable, I wasn't asked to present the proof in class but rather to the Mathematics Club (as it was called). And when I wanted to use it together with other things for a thesis, Moore didn't like the idea at all but chose instead a couple of embedding theorems I had gotten. With hindsight he certainly was right. Since I thought that one ought to be able to prove that $2^{\aleph_0} < 2^{\aleph_1}$, I might still be there!
>
> As you might guess, I got into the problem by accident. I think Roberts in one of his papers that I had to look up in another connection had stated that certain Moore spaces were metrizable. Now I can't find this paper but at the time I couldn't see why and eventually showed that they were not. They couldn't be metric because they weren't normal. So what if they were?
>
> The first proof which I got for the separable case was about eight pages long. I didn't save a copy of it but I remember that it had real functions in it and that the key to the proof was the theorem that the set of points of continuity of a function from $[0,1]$ to the real numbers formed a G_δ-set. I have forgotten how it went. Anyway, Lubben didn't like it and insisted that there should be a better way. After working on it for some time I did find a better way, a way so short (3/4 of a page) that it made the whole thing look trivial. This short argument made it clear why $2^{\aleph_0} < 2^{\aleph_1}$ was needed. And it also made me feel that the general case (for every normal Moore space whether separable or not) couldn't be so much harder. Certainly one might need something a bit stronger than $2^{\aleph_0} < 2^{\aleph_1}$, e.g. the generalized continuum hypothesis. (Remember this was in 1932 or 1933 and I had never heard of Gödel.)[93]

Certainly less influential, but nonetheless stimulating in its effect on huge numbers of other researchers, is the often quoted, *and only*, paper of Aline H. Frink. This work, entitled "Distance functions and the metrization problem", recapitulates and puts into perspective the various aspects of distance function research, especially as this relates to issues of metrization in general topology. Notably she improves and elegantly proves the important results of Niemytzki and Chittenden with respect to the metrization problem and presents them in a fashion that one might argue paved the way to a solution of the general metrization problem. Notably, Frink considers the role that might be played by nonsymmetric distance functions; in particular the notion that will become known as a *weak base* is implicit is this work.

Clearly the considerable research in general topology is now gaining momentum, but alas, the events of the world would again intervene as a second world war breaks out to occupy significant minds for almost a decade.

[93]Jones, *Metrization, non-metrization and a bit of history*, pp. 1–2.

4.3. BING'S DEVELOPABLE SPACES

After World War II Jones returns to the University of Texas, where R.H. Bing has just finished his degree as a student of R.L. Moore and is revisiting Chittenden's 1927 paper and struggling with Jones' normal Moore space conjecture.

This work culminates in 1951 in a paper published in a new journal, *The Canadian Journal of Mathematics*. In it Bing names developability and uses it to solve the metrization problem – thus "finishing up" Chittenden's approach to the metrization problem. In addition, Bing introduces the notion of collectionwise normal and notes that, as a partial solution to the normal Moore space conjecture, this property is necessary and sufficient for a Moore space to be metric. Thus, Bing emerges as a leading figure in American topology by making major contributions to two of the most prominent problems in point set topology of that time.

Without fanfare, Bing introduces the context and focus for his paper.

> A single valued function $D(x, y)$ is a *metric* for a topological space provided that for points x, y, z of the space:
>
> 1. $D(x, y) \geq 0$, the equality holding if and only if $x = y$.
> 2. $D(x, y) = D(y, x)$ (*symmetry*),
> 3. $D(x, y) + D(y, z) \geq D(x, z)$ (*triangle inequality*),
> 4. x belongs to the closure of the set M if and only if $D(x, m)$ (m element of M) is not bounded from 0 (preserves limit points).
>
> A function $D(x, y)$ is a metric for a point set R of a topological space S if it is a metric for R when R is considered as a subspace of S. A topological space or point set that can be assigned a metric is called metrizable.
>
> If a topological space has a metric, this metric may be useful in studying the space. Determining which topological spaces can be assigned metrics leads to interesting and important problems. For example, see [reference to Chittenden [1927]].
>
> A regular topological space is metrizable if it has a countable basis. However, it is not necessary that a space be separable in order to be metrizable. Theorem 3 gives a necessary and suficient condition that a space be metrizable by using a condition more general than perfect separability.
>
> Alexandroff and Urysohn showed [in [1923]] that a necessary and sufficient condition that a topological space be metrizable is that there exist a sequence of open coverings $\mathcal{G}_1, \mathcal{G}_2, \ldots$ such that
>
> (a) $\mathcal{G}_{i+1}$ is a refinement of $\mathcal{G}_i$,
>
> (b) the sum of each pair of intersecting elements of $\mathcal{G}_{i+1}$ is a subset of an element of $\mathcal{G}_i$, and
>
> (c) for each point p and each open set D containing p there is an integer n such that every element of $\mathcal{G}_n$ containing p is a subset of D.

We call a sequence of open coverings satisfying condition (c) a development. A developable space is a topological space that has a development. In section 2 we study conditions under which developable spaces can be assigned metrics.

The results of this paper hold in a topological space as defined by Whyburn in [G.T. Whyburn's book, *Analytic Topology*] or in a Hausdorff space.[94]

In a style reminiscent of the 1923 paper of Alexandroff and Urysohn Bing proceeds in a no–nonsense fashion to introduce his definitions and prove his theorems.

1. *Screenable spaces.* We use the following definitions:

Discrete. A collection of point sets is *discrete* if the closures of these points sets are mutually exclusive and any subcollection of these closures has a closed sum.

Screenable. A space is *screenable* if for each open covering H of the space, there is a sequence $H_1, H_2, \ldots$ such that H_i is a collection of mutually exclusive domains and $\sum H_i$ is a covering of the space which is a refinement of H. A space is *strongly screenable* if there exists such H_i's which are discrete collections.

Perfectly screenable. A space is *perfectly screenable* if there exists a sequence $G_1, G_2, \ldots$ such that G_i is a discrete collection of domains and for each domain D and each point p in$D$ there is an integer $n(p, D)$ such that $G_{n(p,D)}$ contains a domain which lies in D and contains p.

Collectionwise normal. A space is *collectionwise normal* if for each discrete collection X of point sets, there is a collection Y of mutually exclusive domains covering X^* such that no element of Y intersects two elements of X. We use X^* to denote the sum of the elements of X.

The following result follows from the definitions of perfectly screenable and strongly screenable.

Theorem 1. A perfectly screenable space is strongly screenable. ...

The proof of the following theorem may be compared with one given by Tychonoff to show that any regular perfectly separable topological space is normal.

Theorem 2. A regular strongly screenable space is collectionwise normal. ...

The following theorem may be compared with the result of Urysohn which states that a normal perfectly separable topological space is metrizable.

Theorem 3. A necessary and sufficient condition that a regular topological space be metrizable is that it be perfectly screenable. ...

[94]Bing, *Metrization of topological spaces*, p. 175.

[A proof follows making use of Stone's result (Stone [1948]) in the proof of necessity. In his proof of sufficiency Bing constructs a metric using the given sequence of covers.]
In the following modification of Theorem 3 we dispense with the supposition that the elements of H_i are mutually exclusive. E.E. Floyd suggested that such a modification might be possible.

Theorem 4. A regular topological space is metrizable if and only if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ such that

(a) *$\mathcal{G}_i$ is a collection of open subsets of S such that the sum of the closures of any subcollection if $\mathcal{G}$ is closed and*
(b) *if p is a point and Δ is an open set containing p there is an integer $n(p, D)$ such that an element of $\mathcal{G}_{n(p,D)}$ contains p and each element of $\mathcal{G}_{n(p,D)}$ containing p lies in D. ...*

2. *Developable spaces.* For a developable topological space there exist a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ such that

(a) $\mathcal{G}_i$ is a covering of the space with open sets,
(b) $\mathcal{G}_{i+1}$ is a refinement of $\mathcal{G}_i$,
(c) for each domain and each point p in D there is an integer $n(p, D)$ such that each element of $\mathcal{G}_{n(p,D)}$ which contains p lies in D. ...

Regular developable topological spaces have been studied extensively because a Moore space is such a space. ...

Theorem 5. A separable screenable developable space is perfectly screenable. ...

Theorem 6. A strongly screenable developable space is perfectly screenable. ...

Theorem 7. A regular developable space (Moore space) is metrizable if it is strongly screenable. ...

Theorem 8. A screenable Moore space is metrizable if it is normal. ...

Theorem 10. A Moore space is metrizable if it is collectionwise normal.[95]

Bing finishes up his paper with a thorough study of collectionwise normality in its relationship to full normality and normality by establishing all possible relationships and giving counterexamples.

REFLECTION AND OBSERVATIONS ON THIS WORK OF BING

In a footnote Bing remarks that "A topological space is regular if for each open set D and each point p in D there is an open set containing p whose closure lies in

[95]Bing, *Metrization of topological spaces*, pp. 176–182.

D" – suggesting perhaps the intuitive utility of the R.L. Moore version of the concept over the now-standard separation property of a point and a closed set by disjoint open sets. Also, note Bing's definition of a discrete family in contrast to the usual definition (i.e. each point has a neighborhood intersecting at most one member of the family). One senses in Bing's writing the ancestral ties to E.H. Moore and R.L. Moore that produces a strongly intuitive and natural flow of mathematical ideas that engaging teachers develop to inspire their students.

This intuitive undercurrent of Bing's thinking is expressed by Bing himself in the following, which gives some insight into the kind of mathematics that attracted his interest.

> Burton Jones is the first mathematician with whom I collaborated after receiving my Ph.D. Although R.D. Anderson, E.E. Moise, C.E. Burgess, Mary Ellen Estill (Rudin), Eldon Dyer, Billy Jo Ball were at Texas at this time, they were students and R.L. Moore insisted that his students develop independent work habits. Hence, after receiving my Ph.D., I did not discuss research with these students. However, Jones had received the Ph.D. several years earlier and was just returning from Cambridge where he had done work on underwater sound related to war work. I did not feel restrained in discussing mathematical research with him.
>
> An unsolved problem of considerable interest to me was the question Jones had asked in 1937 [J_1] – is a normal Moore space metrizable? To understand how we worked at unsolved problems, it is necessary to know our modus operandi. Our first approach in attacking a problem was to look for a counterexample. If no one of our vast store of examples worked, we would try modifying known examples to discover a counterexample. It was my gut-reaction (and still is) that there is a real counterexample to the normal Moore space conjecture but it may be more complicated than anything we have examined. I soon learned that Jones had examples in his repetoire that were missing from mine.[96]

Here Bing goes on to discuss Jones' "tin-can-space". Then he adds:

> Not all questions are solved with counterexamples. One of the best known metrization results is the one stating that a second axiom regular Hausdorff space is metrizable. Here the proof consists of constructing a distance function. I recall my disappointment when I saw the traditional argument. While it gave a distance function, the distance function constructed had little resemblance to ordinary distance. If a straight line interval were metrized by such a technique, there would be scant hope that the resulting metrized arc could be isometrically embedded in any Euclidean space. It embeds isometrically in a Hilbert cube.

[96]Bing, *Metrization problems*, p. 1.

> I resolved to try for a more realistic distance function if I had the opportunity.
>
> [Here Bing discusses some of the work that preceded his work on his metrization theorem. Then, he goes on to discuss the sufficiency proof in which he will construct a metric. Here he assumes that the regular Hausdorff space H has a basis which is the countable union of discrete, open collections $\mathcal{G}_n$.]
>
> To get a first approximation to a distance, convolute H so that part of it not covered by $\mathcal{G}_1$ lies in a horizontal plane and points in elements of $\mathcal{G}_1$ lie on hills of height less than or equal to $1/2$. If p and q are two points,
>
> $$d_1(p,q) = \text{height } p \text{ if } q \text{ is on the bottom level,}$$
>
> $$d_1(p,q) = |\text{height } p - \text{height } q| \text{ if } p, q \text{ are on the same hill,}$$
>
> $$d_1(p,q) = (\text{height } p + \text{height } q) \text{ if } p \text{ and } q \text{ are on the different hills.}$$
>
> Next turn to $\mathcal{G}_2$ but get d_2 by deciding that the heights of the hills are less than $1/4$. The procedure is continued and the final distance is defined as $d(p,q) = \sum d_i(p,q)$. Note that instead of embedding H in a Hilbert cube as was done in the case where H was second countable, it was embedded in the cartesian product of cones.
>
> It is not our purpose here to show that a collectionwise normal Moore space is metrizable but we remark that in getting a metric for it, we may use the "hills approach".[97]

Furthermore, Bing's paper is loaded with instructive examples that further highlight the role that might be played by additional set-theoretic axioms that would inspire others. In contrast Bing's own interests will shift from problems of "abstract" general topology to more concrete "geometric" problems in the topology of Euclidean 3-space. Interestingly, however, this paper will spawn new interest in the role of set theory and one might claim that this is the birth of a new era in topology that might be called "set-theoretic" topology. But, it will be Mary Ellen Rudin and others at the University of Wisconsin along with their students who will be blazing this trail.

This era in the development in point set topology comes to a close with the publication in 1955 of John L. Kelley's book, *General Topology*. For Americans this will become the national *standard text* in an era of incredible growth for graduate education in the United States. There is little doubt that Hausdorff's *Mengenlehre* and Fréchet's *Espaces Abstraits* have been replaced and a new era of research in general topology has just begun.

[97] Bing, *Metrization problems*, pp. 7–8.

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A HISTORY OF GENERALIZED METRIZABLE SPACES

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Contents

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Introduction

Metrizable spaces, together with compact spaces, occupy a central position in general topology. Both classes of spaces have been generalized in a wide variety of ways. For example, locally compact and countably compact spaces are two well-known and especially useful generalizations of compact spaces. This paper is a history of the search for generalizations of metrizable spaces.

There are a number of reasons why generalized metrizable spaces are worthy of study. Perhaps the most important reason is that such classes increase our understanding of the metrizable spaces. But in addition, topologists are continually seeking broader classes of spaces for which especially important results hold. Two examples that immediately come to mind are: (1) find spaces more general than metrizable spaces for which a satisfactory dimension theory can be developed; (2) find spaces more general than metrizable spaces for which the Dugundji Extension Theorem holds.

From one point of view, any topological property that generalizes metrizability is a generalized metrizable space. But this interpretation is too broad, and I prefer a more restricted point of view: a generalized metrizable space is one that is useful in characterizing metrizability and at the same time retains many of the pleasant properties that we have come to cherish when working with metrizable spaces.

Unfortunately, this more restricted point of view does not lead to a precise notion of a generalized metrizable space. Simply put, there is no general agreement among topologists on how to answer the question *What is a generalized metrizable space*? Burke and Lutzer, in their 1976 survey paper, gave a list of conditions that could serve as a definition; but they also emphasized the difficulty in finding a list that everyone is willing to accept. Not surprisingly, eight years later Gruenhage [1984] began his very nice survey paper with the observation that we still do not have a precise definition of what constitutes a generalized metrizable space.

Rather than pursuing this problem further, I will instead give an overview of the classes of spaces that I do plan to discuss.

$$
\begin{tikzcd}
\text{Compact spaces} \arrow[d] \arrow[dr] & & \text{Metrizable space} \arrow[dll] \arrow[dl] \arrow[d] \\
\text{Cover and separation} & \begin{array}{c}\text{Unification}\\ \text{(with compactness)}\end{array} & \text{Uniformity}
\end{tikzcd}
$$

Cover and separation: paracompact, metacompact, subparacompact, meta-Lindelöf, θ-refinable, collectionwise-normal.

Unification with compactness: p-space, M-space, $w\Delta$-space, Σ-space, β-space.

Uniformity: Developable, base of countable order, point-countable base, G_δ-diagonal, point-countable separating open cover, M_1-space, stratifiable, Nagata space, semi-metrizable, symmetrizable, quasi-metrizable, σ-space, γ-space, semi-stratifiable, θ-base, $\delta\theta$-base, monotonically normal. (All of these properties, save monotone normality, can be characterized in terms of a countable collection.)

Metrizable spaces have many useful and important properties; for example:

1. every subspace of a metrizable space is metrizable (*hereditary*);
2. a countable product of metrizable spaces is metrizable (*countably productive*);
3. in a metrizable space, compact = countably compact (*coincidence of compactness properties*);
4. in a metrizable space, CCC = her. CCC = separable = her. separable = Lindelöf = her. Lindelöf = ω_1-compact = her. ω_1-compact = countable base (*coincidence of countability properties*).

As I trace the history of generalized metrizable spaces, I will use this list of properties as benchmarks, or as points of comparison with metrizable spaces. (A property not enjoyed by metrizable spaces is preservation by closed maps; this gives rise to the *Lasnev spaces* [1965].)

Most of the properties $\mathbf{P}$ that I will discuss not only generalize metrizability but also satisfy at least one of the following two conditions:

I. *Countably compact* $+ \mathbf{P} \Rightarrow$ *countable base*. If X is countably compact and has property $\mathbf{P}$, then X has a countable base (and hence is compact and metrizable).

II. *Countably productive with paracompactness*. Let $\{X_n : n \in \mathbb{N}\}$ be a countable collection of paracompact spaces, each of which has property $\mathbf{P}$. Then ΠX_n is paracompact and has property $\mathbf{P}$.

All of the properties listed under the heading *Uniformity*, save monotone normality and base of countable order, satisfy I. (Every *compact* space with a base of countable order does have a countable base). All of the properties listed under the heading *Unification*, save β-spaces, satisfy II. A number of spaces satisfy both: for example, σ-spaces, developable spaces, and Nagata spaces. On the other hand, the following classes satisfy neither: first-countable spaces; perfect spaces; paracompact spaces.

All spaces are T_1 and regular; $\mathbb{N}$ denotes the set of natural numbers. A number of excellent survey papers on generalized metrizable spaces are listed in the references under the heading "Survey Papers". The papers by Gruenhage [1984] and Nagata [1989] were especially useful.

1. Developable, semi-metrizable, and quasi-metrizable spaces (early history)

I will take a somewhat modern approach and give a quick overview of the fundamentals of metrizable spaces. Given a set X, a *distance function on* X is a function d from $X \times X$ into $[0, \infty)$ such that $d(p,p) = 0$ for all $p \in X$. For each $p \in X$ and $\varepsilon > 0, B_d(p,\varepsilon)$ is the subset of X defined by $B_d(p,\varepsilon) = \{x : x \in X \text{ and } d(p,x) < \varepsilon\}$. Note that $d(p,p) = 0$ if and only if $p \in B_d(p,\varepsilon)$ for all $\varepsilon > 0$ (thus the requirement $d(p,p) = 0$). A distance function d on X induces a topology T_d on X in the following way:

$$U \in T_d \iff \text{for all } p \in U, \text{ there exists } \varepsilon > 0 \text{ such that } B_d(p,\varepsilon) \subseteq U.$$

In modern terminology, the collection $\{\mathcal{B}_p : p \in X\}$, where each $\mathcal{B}_p = \{B_d(p,\varepsilon) : \varepsilon > 0\}$, is a *weak base* for the topology T_d (see Arhangel'skiĭ [1966]).

Now consider the following three axioms for a distance function d on X:

1. if $d(x,y) = 0$, then $x = y$ (*coincidence*);
2. $d(x,y) = d(y,x)$ (*symmetry*);
3. $d(x,z) \leq d(x,y) + d(y,z)$ (*triangle inequality*).

It is easy to check that d satisfies (1) if and only if the topology T_d induced by d is a T_1-topology. Since we are assuming that all spaces are T_1, this amounts to the assumption that that all distance functions satisfy (1). With this convention, the emphasis will henceforth be on properties (2) and (3) of a distance function. Recall that the set $B_d(p,\varepsilon)$ is always open whenever d satisfies (3). I will now define three important classes of spaces in terms of a distance function.

Definition. Let $\langle X, T\rangle$ be a topological space.

(a) X is *metrizable* if there is a distance function d on X that satisfies (2) and (3) and such that $T = T_d$;
(b) X is *quasi-metrizable* if there is a distance function d on X that satisfies (3) and such that $T = T_d$;
(c) X is *semi-metrizable* if there is a distance function d on X that satisfies (2), $T = T_d$, and in addition $p \in B_d(p,\varepsilon)^\circ$ for all $p \in X$ and all $\varepsilon > 0$.

The metric and semi-metric spaces were introduced by Fréchet in his thesis in 1906. The first systematic study of quasi-metric spaces is due to Niemytzki [1931] and Wilson [1931a]. Some members of the Russian school of topology use Δ-*metrizable* for *quasi-metrizable.*

A space X is *developable* if there is a countable sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X such that the following condition is satisfied: given $p \in U$ with U open, there exists $n \in \mathbb{N}$ such that $st(p,\mathcal{G}_n) \subseteq U$; the sequence of open covers $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is called a *development for* X. (Recall that $st(p,\mathcal{G}) = \bigcup\{G : p \in G \text{ and } G \in \mathcal{G}\}$.) Every metrizable space is developable and every developable space is semi-metrizable.

Developable spaces have always played a central role in metrization theory. Shore has a paper in this volume on the history of developable spaces, and Heath's paper on metrization emphasizes their importance in metrization theory. For these reasons, I do not plan to devote as much attention to this important class of spaces as they properly deserve.

The fundamental treatise on developable spaces is the now classical book *Foundations of point set theory* by R.L. Moore [1932]. One especially important result from Moore's book is the following:

Theorem 1.1 *Let X be a developable space.*

1. *If X is countably compact, then X has a countable base.*
2. *If X is hereditarily separable, then X has a countable base.*

The Niemytzki plane (or tangent disc space) is developable, semi-metrizable, and quasi-metrizable, yet is not normal and hence not metrizable. Stated in terms of distance functions, it is possible for a topological space $\langle X, T\rangle$ to have

- a distance function d on X that is symmetric and such that $T = T_d$;
- a distance function ρ on X that satisfies the triangle inequality and such that $T = T_\rho$;
- no metric on X that induces the topology T.

Anyone interested in generalizing the notion of a metric space should naturally look at the two axioms (2) and (3) for a metric, omit one, and arrive at the quasi-metrizable and the semi-metrizable spaces. This is, in fact, exactly what happened, and three early results in this direction are the following.

Theorem 1.2 (Niemytzki [1927], Wilson [1931b]) *A space X is metrizable if and only if there is a semi-metric d on X with the property: if $d(y_n, p) \to 0$ and $d(x_n, y_n) \to 0$, then $d(p, x_n) \to 0$.*

Theorem 1.3 (Niemytzki [1931]) *Let X be countably compact. If X is either quasi-metrizable or semi-metrizable, then X has a countable base (and hence is compact and metrizable).*

Theorem 1.4 (Alexandroff–Niemytzki [1938]) *Let X be semi-metrizable with semi-metric d. If d satisfies the following additional condition, then X is developable: if $d(x_n, p) \to 0$ and $d(y_n, p) \to 0$, then $d(y_n, x_n) \to 0$.*

Despite these early results, I believe it is fair to say that relatively little progress was made in the study of semi-metrizable and quasi-metrizable spaces during the 1930's and 1940's.

The modern era of semi-metrizable spaces began in the early 1950's with the important work of F.B. Jones and two of his Ph.D. students, Heath and McAuley. Jones stated in a brief survey paper [1955]: "Starting in 1950 I gave my topology classes as problems to be worked out in regular semi-metric topological spaces the theorems in the first chapter of Moore's book (with a few omissions and additions). All of these turn out to be theorems for semi-metric spaces with few exceptions."

Both McAuley [1955] and Heath [1966] wrote survey papers in which they emphasized that many properties that hold for metric, even developable spaces, continue to hold for semi-metrizable spaces; for example, the following are equivalent in a semi-metrizable space: ω_1-compact, hereditarily Lindelöf, hereditarily separable.

During this same period, Morton Brown [1955] asked two interesting questions:

1. Does every semi-metric space have a semi-metric under which all spheres are open?
2. What "topological" property can be added to a semi-metric space to get a developable space?

Heath [1961] gave a negative solution to the first problem by constructing an appropriate example, and somewhat later he obtained the following deep result related to (2).

Theorem 1.5 (Heath [1965b]) *Every semi-metrizable space with a point-countable base is developable.*

This theorem is one of the outstanding achievements in the theory of generalized metrizable spaces. Nevertheless, I do not believe that Heath regarded it as an entirely satisfactory solution of (2), since a developable space need not have a point-countable base (for example, a non-metrizable, separable developable space, such as the Niemytzki plane, cannot have a point-countable base).

I will return to Brown's second question later. But today, the question does not seem so difficult, and it is interesting to speculate on why it was such a problem in 1955. One difficulty, of course, is the fact that topologists had not yet "discovered" the classes such as p-spaces and $w\Delta$-spaces that make the question an easy one to answer. But it seems that another difficulty is that the relationship between semi-metrizable spaces and covering properties weaker than paracompactness was not well understood. I will elaborate on this point later in section 3.

2. Nine metrization theorems: 1923–1957

In this section I will survey nine metrization theorems that have played an important role in the development of generalized metrizable spaces. The first result is the Alexandroff–Urysohn solution to the problem of finding a topological characterization of metrizability. (Recall that a development $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is *regular* if the following holds for all $n \in \mathbb{N}$: if $U, V \in \mathcal{G}_{n+1}$ and $U \cap V \neq \emptyset$, then there exists $W \in \mathcal{G}_n$ such that $U \cup V \subseteq W$.)

Theorem 2.1 (Alexandroff–Urysohn [1923]) *A space X is metrizable if and only if it has a regular development.*

A little after Alexandroff and Urysohn obtained their metrization theorem, Urysohn proved the following fundamental result for spaces with a countable base. (Urysohn assumed normality; Tychonoff [1926] subsequently showed that every regular space with a countable base is normal.)

Theorem 2.2 (Urysohn [1925]) *Every space with a countable base is metrizable.*

Related to Urysohn's theorem is the following result by Jones (compare with Theorem 1.1).

Theorem 2.3 (Jones [1937]) *Every ω_1-compact developable space has a countable base and hence is metrizable. In particular, a developable space is metrizable if it is countably compact, Lindelöf, or hereditarily separable.*

At about the same time that Jones obtained this result, Frink published a paper on metrization that is important for at least two reasons: first, she greatly simplified the procedure for constructing a metric (or a quasi-metric) from a suitable collection of open covers; second, her result emphasized the important role of neighborhood systems in characterizing metrizability.

Theorem 2.4 (Frink [1937]) *A space X is metrizable if and only if each $p \in X$ has a sequence $\{V_n(p) : n \in \mathbb{N}\}$ of neighborhoods such that the following hold:*

1. *$\{V_n(p) : n \in \mathbb{N}\}$ is a local base for p;*
2. *for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that if $V_k(p) \cap V_k(q) \neq \emptyset$, then $V_k(q) \subseteq V_n(p)$.*

Šneider, in 1945, proved a rather unexpected but quite influential theorem on the metrizability of compact spaces. A space X is said to have a *G_δ-diagonal* if its diagonal $\Delta = \{(x, x) : x \in X\}$ is a G_δ-subset of $X \times X$. What could this property possibly have to do with metrizability?

Theorem 2.5 (Šneider [1945]) *A compact space with a G_δ-diagonal has a countable base (and hence is metrizable).*

It is indeed fortunate that the converse of Urysohn's metrization theorem (countable base $\Rightarrow$ metrizable) is false. This gives rise to the metrization problem: find a characterization of metrizability from which Urysohn's theorem follows as an easy corollary. This problem was solved in the early 1950's independently by three mathematicians:

Theorem 2.6 (Bing [1951]) *A space X is metrizable if and only if it has a σ-discrete base.*

Theorem 2.7 (Nagata–Smirnov [1950], [1951]) *A space X is metrizable if and only if it has a σ-locally finite base.*

Urysohn's metrization theorem does indeed follow easily from either Bing's theorem or the Nagata–Smirnov theorem. One would thus expect these results to mark the end of research in metrization theory. But, as Alexandroff pointed out in a survey paper [1964]: "As so often happens in mathematics, following the complete solution of the problem new points of view have opened up and they have given rise to new interesting research." Alexandroff then discussed some metrization theorems obtained by the Soviet school of topology. In retrospect, Alexandroff's remark certainly understated the situation; in the early 1960's, metrization theory and the theory of generalized metrizable spaces entered its golden age.

Bing's 1951 paper on the solution to the metrization problem also contained another theorem that has played an important role in the theory of generalized metrizable spaces. Bing introduced a strong type of normality condition that he called *collectionwise normality*; he then proved:

Theorem 2.8 (Bing [1951]) *Collectionwise-normal + developable = metrizable.*

Bing's theorem was influential for a number of reasons. First, it introduced a new and important separation axiom into topology. Second, it reinforced the central role of developable spaces in the study of metrization. Finally, it is perhaps the first metrization theorem of the *factorization type*. Prior to 1950, metrization theorems were stated in terms of strong base conditions. But Bing's theorem factored metrizability into a base condition (developability) and a separation axiom. Bing's theorem ultimately lead to a shift in the emphasis from factoring metrizability to that of factoring developability. Indeed, as we proceed, the reader will see a number of theorems of the form

$$\begin{matrix}\text{paracompact}\\ \text{(or cw normal)}\end{matrix} + \begin{matrix}\text{unification}\\ \text{property}\end{matrix} + \begin{matrix}\text{uniformity}\\ \text{property}\end{matrix} = \text{metrizable}$$

or more generally of the form

$$\theta\text{-refinable} + \begin{matrix}\text{unification}\\ \text{property}\end{matrix} + \begin{matrix}\text{uniformity}\\ \text{property}\end{matrix} = \text{developable.}$$

Theorem 2.9 (Nagata [1957]) *A space X is metrizable if and only if for each $p \in X$, there exist two sequences $\{U_n(p) : n \in \mathbb{N}\}$ and $\{V_n(p) : n \in \mathbb{N}\}$ of neighborhoods of p such that the following hold:*

1. *$\{U_n(p) : n \in \mathbb{N}\}$ is a local base for p;*
2. *if $q \notin U_n(p)$, then $V_n(p) \cap V_n(q) = \emptyset$;*
3. *if $q \in V_n(p)$, then $V_n(q) \subseteq U_n(p)$.*

Nagata's 1957 paper is important for a number of reasons. First of all, we see for the first time the idea of proving a metrization theorem of a very general nature and then deriving from it many others, thereby giving a unified approach to metrization theory. Nagata used Theorem 2.9 to give direct proofs of the Nagata–Smirnov theorem, Morita's metrization theorem in terms of a sequence of locally finite closed covers [1955], Frink's metrization theorem [1937], and others. (This idea is taken to its logical conclusion by the author in the paper [1975]; a long list of base conditions, each stated in terms of an infinite cardinal κ, are proved equivalent; this is metrization theory "without normality".)

There is another reason for the importance of Nagata's paper, namely the discovery of new and useful classes of generalized metrizable spaces. Note that Nagata's theorem factors metrizability into two properties, each defined in terms of systems of neighborhoods of a point. I will discuss this idea later in section 5 (stratifiable and Nagata spaces) and in section 11 (γ-spaces and Nagata first-countable spaces).

3. The role of covering properties in generalized metrizable spaces

The early 1950's saw the beginning of a long and fruitful period of research in metrization theory and the theory of generalized metrizable spaces. There can be little doubt that covering properties played a major role in this development. Indeed, it is often difficult to separate these two areas of research, and progress in one almost immediately led to progress in the other. The first step was taken as early as 1944, when Dieudonné introduced paracompactness as a covering property that *obviously* generalizes compactness and, with some effort, also generalizes separable metrizable spaces. But the fact that paracompactness also generalizes metrizability was not so obvious, and in 1948, A.H. Stone proved one of the deepest and most important theorems we have about metric spaces: every metric space is paracompact. It would be difficult to overestimate the important role of this theorem in metrization theory and the theory of generalized metrizable spaces. As a reminder, by 1951 Nagata, Smirnov, and Bing had independently solved the metrization problem by characterizing metrizability in terms of a σ-locally finite or a σ-discrete base; this breakthrough would not have been possible without Stone's result.

Another important influence on the theory of generalized metrizable spaces was the sequence of three papers of Michael [1953], [1957], [1959] that gave characterizations of paracompactness in terms of locally finite closed refinements, σ-discrete open refinements, closure-preserving open (or closed) refinements, σ-closure preserving open refinements, cushioned refinements, and σ-cushioned open refinements. I will return to this point later when I discuss the M_i-spaces ($i = 1, 2, 3$) of Ceder. But for now, I will give an overview of the covering properties that repeatedly occur in the study of generalized metrizable spaces.

$$\begin{tikzcd}
 & & \text{paracompact} \arrow[dl] \arrow[dr] & \\
 & \text{metacompact} \arrow[dl] \arrow[dr] & & \text{subparacompact} \arrow[dl] \\
\text{meta-Lindelöf} & & \theta\text{-refinable} &
\end{tikzcd}$$

A space X is *subparacompact* if every open cover has a σ-discrete closed refinement. In his 1951 paper, Bing proved that every developable space is subparacompact; this result was a key step in his factorization of metrizability into developablility and collectionwise normality. Bing's result on subparacompactness was an important breakthrough: a covering property weaker than paracompactness was used in an essential way.

Bing did not give a name to this covering property. This was done later by McAuley [1958], who introduced the name *F_σ-screenable* and generalized Bing's theorem by proving that every semi-metrizable space is subparacompact. McAuley also observed that

$$\text{collectionwise-normal} + \text{subparacompact} = \text{paracompact}.$$

In addition, every ω_1-compact, subparacompact space is Lindelöf and every countably compact, subparacompact space is compact.

Burke [1969] did for subparacompact spaces what Michael did for paracompact spaces; he gave a number of characterizations of the subparacompact spaces, including the following: every open cover has a σ-locally finite closed refinement; every open cover has a σ-closure preserving closed refinement. At that time Burke also suggested the more informative name *subparacompact.*

Metacompact spaces were introduced by Arens and Dugundji in 1950 (called by them *weakly paracompact* spaces). By definition, a space X is *metacompact* if every open cover has a point-finite open refinement $\mathcal{G}$ ($\mathcal{G}$ is *point-finite* if every $p \in X$ belongs to at most a finite number of elements of $\mathcal{G}$). Arens and Dugundji proved that every countably compact, metacompact space is compact. Somewhat later, Michael [1955] and Nagami [1955] independently proved the following result (which greatly enhanced the importance of metacompactness):

$$\text{collectionwise-normal} + \text{metacompact} = \text{paracompact}.$$

There is an obvious generalization of the metacompact spaces: a space X is *meta-Lindelöf* if every open cover has a point-countable open refinement. Aquaro [1966] generalized the result of Arens and Dugundji by proving that every ω_1-compact, meta-Lindelöf space is Lindelöf. Thus, the meta-Lindelöf spaces, like the subparacompact and the metacompact spaces, fall in the category of "What can you add to a countably compact space to get a compact space?" Also note that every separable, meta-Lindelöf space is Lindelöf.

It is probably fair to say that subparacompactness was invented as a tool for the analysis of developability (see Bing [1951]). The same is surely true of the next covering property that I will discuss. This covering property is extremely important and was introduced by Wicke and Worrell in 1965.

Definition. A space X is *θ-refinable* if every open cover has a open refinement $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$, where each $\mathcal{G}_n$ covers X, and for each $p \in X$, there exists $n \in \mathbb{N}$ such that $\mathrm{ord}(p, \mathcal{G}_n)$ is finite. (Note: $\mathrm{ord}(p, \mathcal{G}_n)$ is the cardinality of the set $\{G \in \mathcal{G}_n : p \in G\}$.)

Every metacompact space and every subparacompact space is θ-refinable. I will return to this covering property later in section 6 (base of countable order spaces); for now I will simply record the following results of Wicke and Worrell:

$$\theta\text{-refinable} + \omega_1\text{-compact} = \text{Lindelöf};$$

$$\theta\text{-refinable} + \text{countably compact} = \text{compact};$$

$$\theta\text{-refinable} + \text{collectionwise-normal} = \text{paracompact}.$$

In 1978, Junnila obtained a number of characterizations of θ-refinability, and at the same time he suggested the more descriptive name "submetacompact". However, in this paper I will continue with the original terminology. For more on covering properties, the reader is referred to the survey article of Burke [1984].

4. Unification with compactness: p-, M-, $w\Delta$-, Σ-, and β-spaces

Compact spaces and metrizable spaces are so central to topology that it is desirable to find properties that are common to both classes. Indeed, Arhangel'skiĭ [1963a] began his fundamental memoir on p-spaces as follows: "Metric spaces on the one hand and locally bicompact spaces on the other are two classes of spaces which have long occupied an important place both in set-theoretic topology and in its applications. At the same time it is clear that in the whole family of spaces studied in topology these classes take antipodal places; their qualitative difference hits the eye even when comparing definitions." Arhangel'skiĭ then went on to say that "Locally bicompact spaces and metric spaces, in spite of all their dissimilarity, have quite a number of common properties; for example, closedness relative to the taking of a finite product, of a perfect image, of a closed subspace; the addition theorem on weight is true both for these and for other spaces. This series of examples can be augmented, but already we can note that these common properties seemingly stem from different causes and, at any rate, require different explanations. Therefore it is very tempting to try and find a class of spaces which would contain all metric spaces and all locally compact spaces and in which the facts stated would acquire a common meaning. Such a class is proposed in the present paper." I will return to Arhangel'skiĭ's p-spaces later in this section.

There is an old but important result of Čech [1937] that is a precursor of the problem of unifying compactness and metrizability. The Baire category theorem, one of the most important and useful results in topology and analysis, gives conditions under which a countable intersection of open, dense sets is dense (and hence non-empty). It is well known that the Baire category theorem holds for completely metrizable spaces and for locally compact spaces. Čech introduced a class of spaces, now called the Čech-complete spaces, that generalizes both complete metrizability and local compactness and for which the Baire category theorem holds. Čech's original definition was given in terms of an embedding in the Stone-Čech compactifiction. Later, Frolík [1960b] and Arhangel'skiĭ [1961] independently gave an internal characterization of this important class of spaces.

Theorem 4.1 (Frolík, Arhangel'skiĭ) *A space X is Čech-complete if and only if there is a countable sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X such that the following holds: if $\mathcal{F}$ is a collection of closed sets with the finite intersection property, and if for all $n \in \mathbb{N}$ there exists $F \in \mathcal{F}$ and $G \in \mathcal{G}_n$ such that $F \subseteq G$, then $\cap\mathcal{F} \neq \emptyset$.*

Let us focus the problem of unifying compactness and metrizability by looking at product spaces. As is well known, any product of compact spaces is compact, and moreover any countable product of metrizable spaces is metrizable. But the situation for paracompact spaces is not so nice: the Sorgenfrey line is a paracompact space whose product with itself is not even normal. This suggests the following problem.

Countable product problem. Find a topological property $\mathbf{P}$ such that

1. every compact space and every metrizable space has property $\mathbf{P}$;
2. if $\{X_n : n \in \mathbb{N}\}$ is a countable collection of paracompact spaces, each of which has property $\mathbf{P}$, then ΠX_n is paracompact and has property $\mathbf{P}$.

Frolík [1960a] found a neat way to solve the countable product problem: solve the following pre-image problem.

Pre-image problem. Find a topological property $\mathbf{P}$ such that the following holds:

X is paracompact and has property $\mathbf{P}$	$\Longleftrightarrow$	there is a metrizable space M and a continuous, perfect map f from X onto M.

(Note: f is *perfect* if it is closed and $f^{-1}(y)$ is compact for all $y \in M$.)

Let us see how the pre-image problem solves part (2) of the countable product problem. Let $\mathbf{P}$ be a property that solves the pre-image problem. Let

$\{X_n : n \in \mathbb{N}\}$ be a collection of paracompact spaces, each of which has property $\mathbf{P}$. For each $n \in \mathbb{N}$, there exists a metrizable space M_n and a continuous, perfect map f_n from X_n onto M_n. Now ΠM_n is metrizable, and the product map from ΠX_n onto ΠM_n is continuous and perfect, so ΠX_n is paracompact and has property $\mathbf{P}$ as required.

We can summarize these important and pioneering results of Frolík as follows.

Theorem 4.2 (Frolík [1960a]) *A space X is a paracompact Čech-complete space if and only if there is a completely metrizable space M and a continuous, perfect map f from X onto M.*

Corollary. *A countable product of paracompact Čech-complete spaces is paracompact and Čech-complete.*

A very general version of the pre-image problem was emphasized by Alexandroff [1961] in a talk at the 1961 Prague Symposium. In his address, Alexandroff set forth the following problem (also see Arhangel'skiĭ's paper [1966]):

Pre-image problem (general version) Let $\mathcal{B}$ be a "nice" class of spaces and let $\mathcal{C}$ be a "nice" class of mappings. Find the class $\mathcal{A}$ of spaces that can be mapped to a space in class $\mathcal{B}$ by a map of type $\mathcal{C}$.

We will be concerned with the following version of Alexandroff's problem:

$\mathcal{B} =$ class of metrizable spaces;

$\mathcal{C} =$ class of continuous, perfect (or quasi-perfect) maps.

In his Prague talk, Alexandroff emphasized the important contributions of Frolík to the pre-image problem.

Shortly after these results of Frolík, Arhangel'skiĭ and Morita independently made very important contributions to the countable product problem. First let us look at Morita's idea. (Note: the cover $\mathcal{G}$ *star-refines* the cover $\mathcal{H}$ if $\{st(p, \mathcal{G}) : p \in X\}$ refines $\mathcal{H}$.)

Definition (Morita) A space X is an *M-space* if there is a countable sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X such that the following holds:

1. for all $n \in \mathbb{N}$, $\mathcal{G}_{n+1}$ is a star-refinement of $\mathcal{G}_n$;
2. if $x_n \in st(p, \mathcal{G}_n)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

The class of M-spaces unifies the metrizable spaces and the countably compact spaces; Morita proved the following pre-image and countable product theorems ($f : X \to Y$ is *quasi-perfect* if f is closed and $f^{-1}(y)$ is countably compact for all $y \in Y$).

Theorem 4.3 (Morita [1964]) *A space X is an M-space if and only if there is a metrizable space Y and a continuous, quasi-perfect map f from X onto Y.*

Corollary. *A countable product of paracompact M-spaces is a paracompact M-space.*

Now I will discuss Arhangel'skiĭ's p-spaces [1963a]. This class of spaces was obviously inspired by the Čech-complete spaces, and the original definition was given in terms of an embedding in the Stone–Čech compactification. Rather than give Arhangel'skiĭ's original definition, I will instead give an internal characterization due to Burke [1970].

Theorem 4.4 (Burke [1970]) *A completely regular space X is a p-space if and only if there is a countable sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X such that for all $p \in X$, if $p \in G_n$ and $G_n \in \mathcal{G}_n$ for all $n \in \mathbb{N}$, then*

1. $C_p = \cap\{G_n^- : n \in \mathbb{N}\}$ *is compact;*
2. $\{\cap_{k \leq n} G_k^- : n \in \mathbb{N}\}$ *is a "base" for C_p in the sense that if U is open and $C_p \subseteq U$, then there exist $n \in \mathbb{N}$ such that $\cap_{k \leq n} G_k^- \subseteq U$.*

It is not difficult to see that the class of p-spaces unifies the metrizable spaces and the locally compact spaces. In addition, Arhangel'skiĭ proved the following pre-image and countable product theorems.

Theorem 4.5 (Arhangel'skiĭ [1963a]) *A space X is a paracompact p-space if and only if there is metrizable space Y and a continuous, perfect map f from X onto Y.*

Corollary. *A countable product of paracompact p-spaces is a paracompact p-space.*

Borges quickly picked up on Morita's class of M-spaces, and in 1968 he introduced the following definition (equivalent form).

Definition (Borges) A space X is a *$w\Delta$-space* if there is a countable sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X such that the following holds: if $x_n \in st(p, \mathcal{G}_n)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

In other words, if we omit the star-refinement property in the definition of an M-space, we obtain the $w\Delta$-spaces. [Aside: the w stands for *weak*. Borges also introduced a stronger class of spaces called the Δ-spaces; in retrospect, this class is not as useful as the $w\Delta$-spaces.] The $w\Delta$-spaces have many of the same properties as the M-spaces, but in addition the $w\Delta$-spaces generalize the developable spaces (unlike the M-spaces). For this reason they occur frequently in factorization theorems. We also emphasize that Burke [1970] proved the following very useful and simplifying theorem:

$$p\text{-space} = w\Delta\text{-space} \quad (\text{assuming } \theta\text{-refinability}).$$

From this it follows that for paracompact spaces, p-spaces, $w\Delta$-spaces, and M-spaces are the same. Henceforth, when working with θ-refinable spaces, I will consistently use $w\Delta$-space rather than p-space.

One of the most interesting and important contributions to the countable product problem is due to Nagami. In 1969, Nagami introduced the class of Σ-spaces, defined as follows.

Definition (Nagami [1969]) A space X is a *Σ-space* if there is a countable sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of locally finite closed covers of X such that the following holds for all $p \in X$: if $x_n \in c(p, \mathcal{F}_n)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point. Note: $c(p, \mathcal{F}) = \bigcap\{F : F \in \mathcal{F} \text{ and } p \in F\}$.

Nagami proved that if there is a continuous, quasi-perfect map from X onto Y, then X is a Σ-space if and only if Y is a Σ-space; from this it follows that every M-space is a Σ-space. Moreover, every subparacompact $w\Delta$-space is a Σ-space.

Σ-spaces generalize the M-spaces and hence both the countably compact spaces and the metrizable spaces. In fact, Σ-spaces are a rather natural generalization of the σ-spaces of Okuyama (discussed later in section 8). The class of Σ-spaces has many nice invariance properties, of which the following is particularly important.

Theorem 4.6 (Nagami [1969]) *A countable product of paracompact Σ-spaces is a paracompact Σ-space.*

There is another class of spaces that I want to mention here, namely the β-spaces (see Hodel [1971]). This class of spaces is quite broad: semi-metrizable, $w\Delta$-spaces, and Σ-spaces are all β-spaces. I will define this class later in section 7, and discuss its motivation in section 10 in connection with the semi-stratifiable spaces. But I want to emphasize here that this class of spaces, while generalizing both the metrizable spaces and the countably compact spaces, does not solve the countable product problem. In 1970, Berney constructed (under CH) an example of a Lindelöf semi-metrizable space X such that $X \times X$ is not paracompact.

To summarize our discussion thus far, it seems that there are at least three reasons for the remarkable activity in generalized metrizable spaces that began around 1950:

1. the interest in semi-metrizable spaces by Jones and his students, especially Heath (and his students!);
2. the introduction of paracompactness by Dieudonné in 1944, Stone's proof that every metrizable space is paracompact, the influential papers of Michael characterizing paracompactness, generalizations of paracompactness by Bing, Wicke–Worrell, and others;
3. the successful search for topological properties that generalize compactness and metrizability and are countably productive with paracompact spaces.

5. M_1-, stratifiable, and Nagata spaces

During the early 1960's, two Ph.D. students of E. Michael introduced and studied a number of important classes of generalized metrizable spaces. Ceder [1961] introduced three classes of spaces that he called the M_1-spaces, the M_2-spaces, and the M_3-spaces. Each of these classes is defined in terms of a base condition that is a variation of a σ-locally finite base; these new base conditions are obviously inspired by various characterizations of paracompactness due to Michael. (Much later, Gruenhage [1976] and Junnila [1978b] independently solved a longstanding problem by showing that $M_2 = M_3$, so I will concentrate on the M_1- and the M_3-spaces.) By definition, a space X is an M_1-*space* if it has a σ-closure-preserving base. This is a natural generalization of a σ-locally finite base. But the M_1-spaces have certain technical problems; for example, it is still unknown if every subspace (even closed subspace) of an M_1-space is an M_1-space.

For this reason, Ceder introduced another class of spaces that he called the M_3-spaces and are defined as follows: a space X is an M_3-*space* if it has a σ-cushioned pair-base. Every M_1-space is an M_3-space; it is not known whether the converse holds. Indeed, this may be the most important unsolved problem in the theory of generalized metrizable spaces. Every subspace of an M_3-space is an M_3-space.

M_1-spaces and M_3-spaces have nice base properties; as a result, both classes, like the metrizable spaces, are countably productive, imply paracompactness, and are perfect. By Bing's factorization theorem, a developable M_3-space is metrizable; for an M_3-space, the following are equivalent: CCC, ω_1-compact, her. Lindelöf, her. separable. Moreover, every countably compact M_3-space has a countable base.

In 1966, Borges, another student of Michael, published a paper in which he obtained a number of results about the M_3-spaces and renamed them the *stratifiable* spaces. In particular, Borges obtained a characterization similar to the following.

Theorem 5.1 *A space X is stratifiable if and only if for each closed subset H of X, there is a countable sequence $\{U(n,H) : n \in \mathbb{N}\}$ of open sets such that*

1. $H = \cap\{U(n,H)^- : n \in \mathbb{N}\}$;
2. *if K is closed with $H \subseteq K$, then $U(n,H) \subseteq U(n,K)$ for all $n \in \mathbb{N}$.*

Borges also showed that the Dugundji extension property holds for stratifiable spaces.

In his 1961 paper, Ceder also introduced the class of Nagata spaces; by definition, a space X is a *Nagata* space if it satisfies conditions (1) and (2) of Nagata's Theorem 2.9. Later I will give a characterization of the Nagata spaces due to Heath that is often easier to work with. Ceder proved that the Nagata spaces are precisely the first-countable stratifiable spaces and that every Nagata space is semi-metrizable.

I will return to the stratifiable spaces in section 10 when I discuss the semi-stratifiable spaces. Also, I will mention in section 9 on G_δ-diagonals yet another important contribution from Ceder's 1961 paper.

Aside: Burke, Engelking, and Lutzer [1975] proved that a space with a σ-hereditarily closure-preserving base is metrizable.

6. Base of countable order, θ-base, and quasi-developable spaces

The work of Ceder and Borges on stratifiable spaces emphasized spaces with base conditions that are similar to a σ-locally finite base; such spaces are automatically paracompact. At about the same time that these advancements were taking place, Wicke and Worrell [1965] published an important paper in which they made significant contributions that are, so to speak, at the other end of the spectrum. Namely, they gave a penetrating analysis of developability and introduced a number of important properties in the theory of generalized metrizable spaces. The starting point of this discussion is the following pair of metrization theorems (the first is Bing's theorem, the second is due to Arhangel'skiĭ [1963b]):

(1) collectionwise-normal $+$ developable $=$ metrizable;
(2) paracompact $+$ base of countable order $=$ metrizable.

Spaces with a base of countable order (often written BCO) were first introduced by Arhangel'skiĭ [1963b]; the characterization below is due to Wicke and Worrell.

Theorem 6.1 (Wicke–Worrell [1965]) *A space X has a base of countable order if and only if there is a sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of bases for X such that the following holds for all $p \in X$: if $p \in B_n \in \mathcal{B}_n$ and $B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$, then $\{B_n : n \in \mathbb{N}\}$ is a local base for p.*

Wicke and Worrell observed that if we omit the decreasing condition $B_{n+1} \subseteq B_n$ in Theorem 6.1, we obtain a characterization of the developable spaces. They also showed that spaces with a base of countable order are hereditary, and moreover have the following remarkable feature, called the *local implies global property*: if $\mathcal{G}$ is an open cover of X, each element of which has a base of countable order, then X itself has a base of countable order.

Wicke and Worrell also gave a factorization of developability into a base condition and a covering property:

(3) base of countable order $+$ θ-refinable $=$ developable.

As noted earlier, they also introduced the important covering property θ-refinability and proved

(4) collectionwise-normal $+$ θ-refinable $=$ paracompact.

Note that Bing's result (1) follows from (2), (3), and (4).

By the *local implies global* property, every locally metrizable space has a base of countable order; in particular, ω_1 (with the order topology) is a countably compact space with a base of countable order that is not compact. This example is certainly compelling evidence that base of countable order theory factors out all trace of a covering property! Although a countably compact space with a base of countable order need not have a countable base, every compact space with a base of countable order does have a countable base.

In their 1965 paper, Wicke and Worrell introduced another useful base condition as follows. A collection $\mathcal{B} = \cup\mathcal{B}_n$ of open subsets of X is a *θ-base* for X if for each $p \in X$ and each open set U with $p \in U$, there exists $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$ such that $p \in B \subseteq U$ and $\text{ord}(p, \mathcal{B}_n)$ is finite. Wicke and Worrell proved:

(5) perfect $+$ θ-base $=$ developable.

In the late 1960's, and independent of this work by Wicke and Worrell, Bennett began a study of the *quasi-developable spaces* (a quasi-development for X is like a development except that each $\mathcal{G}_n$ need not cover X). Bennett [1971] observed that such spaces are hereditary and countably productive; he also proved that

(6) perfect $+$ quasi-developable $=$ developable,

and that the following are equivalent for a quasi-developable space X: her. ω_1-compact, her. Lindelöf, her. separable. A little later, Bennett and Lutzer [1972] simplified the theory of these two classes by showing that

(7) θ-base $=$ quasi-developable.

Bennett and Berney [1974] then collaborated to prove that

(8) β-space $+$ quasi-developable $\Rightarrow$ base of countable order.

From this result, Bennett and Berney obtained the following generalization of an earlier theorem of Bennett [1969] and a new factorization of metrizability:

(9) θ-refinable $+$ β-space $+$ quasi-developable $=$ developable;
(10) paracompact $+$ β-space $+$ quasi-developable $=$ metrizable.

7. g-functions: a unified approach

In this section I will discuss an important unifying idea in the theory of generalized metrizable spaces. The motivation for a unified approach is nicely put by Arhangel'skiĭ [1966]: "Careful analysis and broad classification of entities that at first glance appear to be dissimilar are the essence of point-set topology and form its main task."

This unifying technique is motivated as follows. Let X be a space with metric d, let $p \in X$, and let $\langle x_n \rangle$ be a sequence in X. If $x_n \in B_d(p, 1/2^n)$ for all $n \in \mathbb{N}$, then $x_n \to p$. It is also true that if $p \in B_d(x_n, 1/2^n)$ for all $n \in \mathbb{N}$, then $x_n \to p$. Even more is true; for example, if $B_d(p, 1/2^n) \cap B_d(x_n, 1/2^n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x_n \to p$. These observations suggest that we consider these various conditions in a more general setting, thereby allowing us to isolate and analyze the various components of metrizability.

Definition (idea due to Heath [1962]) Let $\langle X, T\rangle$ be a topological space and let $g : \mathbb{N} \times X \to T$. Such a function is called a *g-function for X* if it satisfies the following two conditions for all $p \in X$:

1. $p \in g(n,p)$ for all $n \in \mathbb{N}$;
2. $g(n+1,p) \subseteq g(n,p)$ for all $n \in \mathbb{N}$.

Now consider the following conditions on a g-function.

(DEV) If $p, x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $x_n \to p$.
($w\Delta$) If $p, x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $\langle x_n\rangle$ has a cluster point.
(DIA) If $p, q \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $p = q$.
(σ) If $p \in g(n, y_n)$ and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then $x_n \to p$.
(SS) If $p \in g(n, x_n)$ for all $n \in \mathbb{N}$, then $x_n \to p$.
(β) If $p \in g(n, x_n)$ for all $n \in \mathbb{N}$, then $\langle x_n\rangle$ has a cluster point.
(NA) If $g(n,p) \cap g(n, x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x_n \to p$.
(ST) If $p \in R$ (open), then there exists $n \in \mathbb{N}$ and a neighborhood W of p such that if $W \cap g(n,x) \neq \emptyset$, then $x \in R$.
(γ) If $y_n \in g(n,p)$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $x_n \to p$.
(FC) If $x_n \in g(n,p)$ for all $n \in \mathbb{N}$, then $x_n \to p$.

Heath [1962] first used this idea to characterize the semi-metrizable spaces in terms of a g-function that satisfies (SS) and (FC); in other words, as the first-countable semi-stratifiable spaces. In this same paper he also characterized the developable spaces in terms of a g-function that satisfies (DEV). Later ([1965a], [1970]) he characterized the Nagata spaces in terms of a g-function that satisfies (NA) and the stratifiable spaces in terms of a g-function that satisfies (ST). Hodel [1971], [1972] used this functional approach to introduce new classes of spaces, including the β-space (a g-function that satisfies (β)) and the γ-spaces (a g-function that satisfies (γ)). Advantages of this unified approach are:

1. emphasize the similarities and the differences between various classes of generalized metrizable spaces;
2. discover new classes of spaces and theorems of the factorization type.

I will illustrate with an example. Developable spaces, $w\Delta$-spaces, and spaces with a G_δ-diagonal can be characterized in terms of a g-function that satisfies (DEV), ($w\Delta$), and (DIA) respectively. Thus we see at a glance the relationship between the three classes. Moreover, the following conjecture is suggested:

$$w\Delta\text{-space} + G_\delta\text{-diagonal} \Rightarrow \text{developable}. \qquad (???)$$

The basis of the conjecture is this: ($w\Delta$) guarantees that $\langle x_n \rangle$ has a cluster point, say q; (DIA) shows that $q = p$. This does not quite work; one also needs θ-refinability (see (2) in section 9). The required counterexample was not found until 1988 by Alster, Burke, and Davis.

8. Nets and σ-spaces

The notion of a net was first introduced into topology by Arhangel'skiĭ [1959]. A *net* for a space X is a collection $\mathcal{V}$ of subsets of X with the following property: given $p \in U$ with U open, there exists $V \in \mathcal{V}$ such that $p \in V \subseteq U$. In other words, a net is like a base except that its elements need not be open sets. Arhangel'skiĭ proved the beautiful theorem that a compact (even p-space) with a countable net has a countable base. This result then enabled Arhangel'skiĭ to give a very elegant solution to the following problem (solved earlier by Smirnov [1956] using an entirely different method): Let X be a compact space that is the union of a countable number of spaces, each of which has a countable base. Does X itself have a countable base?

In 1967, Okuyama combined the idea of a net with the Nagata–Smirnov metrization theorem to obtain the class of σ-spaces. By definition, a space X is a *σ-space* if it has a σ-locally finite net. The class of σ-spaces has many nice properties: hereditary; countably productive; subparacompact; perfect; G_δ-diagonal; Σ-space; preserved by a continuous, closed map; preserved under a countable union of closed sets. Moreover, in a σ-space, the following are equivalent: ω_1-compact, Lindelöf, countable net. A σ-space need not be first-countable or paracompact. Moreover, there is a first-countable, paracompact σ-space that is not metrizable (see Heath [1966]). On the other hand, Burke–Stoltenberg [1969] and Coban [1969] independently proved

(1) σ-space $+$ $w\Delta$-space $=$ developable.

Nagata and Siweic [1968] characterized σ-spaces in terms of a σ-closure preserving net and a σ-discrete net. Heath [1969] proved the important and difficult result that every stratifiable space is a σ-space. These results are all incorporated in the following fundamental theorem on σ-spaces due to Heath and Hodel [1973].

Theorem 8.1 (Heath–Hodel [1973]) *The following are equivalent for a space X:*

1. *X has a σ-discrete net;*
2. *X has a σ-locally finite net (i.e. X is a σ-space);*
3. *X has a σ-closure preserving net;*
4. *X has a g-function satisfying: if $p \in g(n, y_n)$ and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then $x_n \to p$.*

9. G_δ-diagonal, point-countable base, point-countable separating open cover

In 1945, Šneider proved that every compact space with a G_δ-diagonal has a countable base (and hence is metrizable). The number of far-reaching generalizations of this result is a major success story in the theory of generalized metrizable spaces. An important first step was taken by Ceder [1961], who gave an internal characterization of spaces with a G_δ-diagonal that allows for an easy comparison with developability.

Theorem 9.1 (Ceder [1961]) *A space X has a G_δ-diagonal if and only if there is a countable sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X such that for all $p, q \in X$ with $p \neq q$, there exists $n \in \mathbb{N}$ such that $q \notin st(p, \mathcal{G}_n)$.*

A sequence of open covers of X that satisfies the condition in Theorem 9.1 is called a *G_δ-diagonal sequence for X*. Ceder [1961] then generalized Šneider's result by showing that every paracompact, Čech-complete space with a G_δ-diagonal is completely metrizable. As Ceder noted in his paper, many of these ideas go back to Nagata [1950] (see, for example, Nagata's discussion of the α-countability axiom and his Theorem 3). Later, Borges [1966] and Okuyama [1964] independently proved the following more general result:

(1) paracompact $+$ $w\Delta$-space $+$ G_δ-diagonal $=$ metrizable.

In turn, this result was extended to developable spaces independently by Hodel [1971] and Kullman [1971]:

(2) θ-refinable $+$ $w\Delta$-space $+$ G_δ-diagonal $=$ developable.

Nagami, in his paper on Σ-spaces [1969], proved that every paracompact Σ-space with a G_δ-diagonal is a σ-space; Shiraki [1971] then proved the more general result:

(3) subparacompact $+$ Σ-space $+$ G_δ-diagonal $=$ σ-space.

But the deepest, most remarkable, and most interesting generalization of Šneider's theorem is the following result of Chaber.

Theorem 9.2 (Chaber [1976]) *Every countably compact space with a G_δ-diagonal is compact and has a countable base.*

Chaber's theorem tells us that the G_δ-diagonal property is a uniformity property in the following sense: when combined with countable compactness, it implies the existence of a countable base. Chaber's theorem has the following consequences (the first generalizes (3) above):

(4) Σ-space $+$ G_δ-diagonal $=$ σ-space.
(5) M-space $+$ G_δ-diagonal $=$ metrizable.

Now I will introduce another idea that is related to the previous discussion of spaces with a G_δ-diagonal. A σ-locally finite base is point-countable, hence every metrizable space has a point-countable base. In 1962, Miščenko proved the following "companion" to Šneider's theorem:

Theorem 9.3 (Miščenko [1962]) *A compact space with a point-countable base has a countable base (and hence is metrizable).*

Miščenko also gave an example of a paracompact space with a point-countable space that is not metrizable. Somewhat later, Filippov [1968] obtained the following "companion" to the Borges–Okuyama result (1):

(6) paracompact $+$ $w\Delta$-space $+$ point-countable base $=$ metrizable.

These results by Miščenko and Filippov should be compared with the work of Heath [1965b] on semi-metrizable spaces:

(7) semi-metrizable $+$ point-countable base $\Rightarrow$ developable;
(8) cw-normal $+$ semi-metrizable $+$ point-countable base $=$ metrizable.

To prove (6), Filippov isolated the following combinatorial result that is the essence of Miščenko's argument. (This is a very useful and interesting result – an application of topology to combinatorial set theory!)

Miščenko's Lemma. *Let X be a set and let $\mathcal{S}$ be a point-countable collection of subsets of X. Then the number of finite minimal covers of X by elements of $\mathcal{S}$ is at most countable.*

Nagata [1969] found a clever way to unify the work of Filippov and Borges–Okuyama (that is, (6) and (1)). Suppose X has a G_δ-diagonal, and let $\{\mathcal{G}_n : n \in \mathbb{N}\}$ be a G_δ-diagonal sequence for X; suppose further that X is meta-Lindelöf. For each $n \in \mathbb{N}$, let $\mathcal{S}_n$ be a point-countable open refinement of $\mathcal{G}_n$, and let $\mathcal{S} = \bigcup_{n\in\mathbb{N}}\mathcal{S}_n$. Then $\mathcal{S}$ has the following properties: (1) $\mathcal{S}$ is point-countable; (2) given $p \neq q$, there exists $S \in \mathcal{S}$ such that $p \in S$ and $q \notin S$. A collection $\mathcal{S}$ of open sets satisfying (1) and (2) is called a *point-countable separating open cover of X* (abbreviated to PCSOC). Nagata used Miščenko's lemma to prove the following result:

(9) paracompact $+$ $w\Delta$-space $+$ PCSOC $=$ metrizable.

This result obviously generalizes Filippov's theorem, and since every paracompact space is meta-Lindelöf, it also generalizes the Borges–Okuyama theorem.

The notion of a point-countable separating open cover entered topology from another direction. Arhangel'skiĭ and Proizvolov [1966] generalized Miščenko's Theorem 9.3 by proving

(10) countably compact $+$ PCSOC $\Rightarrow$ countable base.

This result, which should be compared with Chaber's Theorem 9.2, shows that the property of having a point-countable separating open cover is a uniformity property in the following sense: when combined with countable compactness, it implies the existence of a countable base.

Nagata's theorem (9) was generalized by Hodel [1971] as follows:

(11) θ-refinable $+$ $w\Delta$-space $+$ PCSOC $\Rightarrow$ developable;

and Skiraki [1971] proved the following σ-space version:

(12) Σ-space $+$ PCSOC $\Rightarrow$ σ-space.

Skiraki [1971] also proved:

(13) M-space $+$ PCSOC $=$ metrizable.

Moreover, (12), when combined with the result of Heath (7), gives

(14) Σ-space $+$ point-countable base $\Rightarrow$ developable,
(15) cw normal $+$ Σ-space $+$ point-countable base $=$ metrizable.

In 1974, Aull introduced the following common generalization of point-countable base and θ-base: a collection $\mathcal{B} = \cup\mathcal{B}_n$ of open subsets of X is a *$\delta\theta$-base* for X if for each $p \in X$ and each open set U with $p \in U$, there exists $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$ such that $p \in B \subseteq U$ and $\mathrm{ord}(p, \mathcal{B}_n)$ is countable. Every developable space has a θ-base and hence a $\delta\theta$-base. Aull proved the following nice extension of Heath's theorem:

(16) semi-stratifiable $+$ $\delta\theta$-base $=$ developable.

Again, Chaber [1977] obtained far-reaching generalizations of these results with the following theorem.

Theorem 9.4 (Chaber [1977]) *Let X be a θ-refinable β-space.*

1. *If X has a point-countable separating open cover, then X has a G_δ-diagonal.*
2. *If X has a $\delta\theta$-base, then X is developable.*

Chaber's theorem gives a unified approach to the results of Hodel (11) and Shiraki (12), as well as a generalization of the results of Heath (7) and Aull (16). Finally, the deep results of Chaber in [1976] and [1977] give the following result:

(17) countably compact $+$ $\delta\theta$-base $\Rightarrow$ countable base.

In particular, a countably compact space with a θ-base (equivalently, a countably compact quasi-developable space) has a countable base.

10. Semi-metrizable, symmetrizable, and semi-stratifiable spaces

I have already discussed the early study of semi-metrizable spaces by Niemytzki and Wilson and the interest generated in this class of spaces by Jones, Heath, and McAuley. Semi-metrizable spaces can be generalized in two interesting and non-equivalent ways: symmetrizable spaces; semi-stratifiable spaces. Moreover, the following hold (Arhangel'skiĭ [1966] and Heath [1962] respectively):

(1) symmetrizable + first-countable = semi-metrizable;
(2) semi-stratifiable + first-countable = semi-metrizable.

So, instead of dwelling on the semi-metrizable spaces, I will instead concentrate on the two new classes, keeping in mind that the semi-metrizable spaces are obtained from either by adding first-countability. First I will consider the symmetrizable spaces, introduced by Arhangel'skiĭ [1965], [1966].

Definition. A space $\langle X, T\rangle$ is *symmetrizable* if there is a distance function d on X with these properties:

1. $d(p,q) = d(q,p)$ for all $p, q \in X$;
2. the topology T_d induced on X by d is T.

Note: If we also require that $p \in B_d(p,\epsilon)^\circ$ for all $p \in X$ and all $\epsilon > 0$, we obtain the semi-metrizable spaces.

Arhangel'skiĭ [1966] asserted that every symmetrizable first-countable (even Fréchet) space is semi-metrizable; he also proved ([1966], [1965] respectively)

(3) countably compact + symmetrizable space $\Rightarrow$ countable base,
(4) collectionwise-normal + p-space + symmetrizable space = metrizable.

This second result is a remarkable metrization theorem: we are given a list of properties that together imply the metrizability of a space, yet it is not even clear that each point of the space is a G_δ! In fact, this is the hardest step of the proof; once we know that every point is a G_δ, the proof then proceeds as follows:

(a) p-space + every point a G_δ $\Rightarrow$ first-countable;
(b) first-countable + symmetrizable $\Rightarrow$ semi-metrizable;
(c) semi-metrizable + p-space $\Rightarrow$ developable;
(d) developable + collectionwise-normal $\Rightarrow$ metrizable.

I would like to emphasize that Arhangel'skiĭ's 1966 paper *Mappings and Spaces* played a very important role in the theory of symmetrizable spaces. Part (c) above was proved independently by Burke–Stoltenberg [1969] and Coban [1969]. In 1967, Nedev generalized (3) by proving that

(5) ω_1-compact + symmetrizable space $\Rightarrow$ hereditarily Lindelöf.

Symmetrizable spaces have some drawbacks: the class is not hereditary; under CH, there is a Lindelöf semi-metrizable space X such that $X \times X$ is not normal and hence not paracompact (see Berney [1970]); there is a symmetrizable space that is not subparacompact and not perfect (Davis, Gruenhage, and Nyikos [1978]). Harley and Stephenson [1976] introduced F-spaces as a generalization of symmetrizable spaces and proved a number of interesting results for this broader class of spaces.

Now I will discuss the semi-stratifiable spaces. This class of spaces was introduced independently by Creede [1970] and Kofner [1969] as a generalization of the stratifiable spaces. The semi-stratifiable spaces also generalize the σ-spaces; like the σ-spaces, this class of spaces has many nice properties:

– every subspace of a semi-stratifiable space is semi-stratifiable;
– a countable product of semi-stratifiable spaces is semi-stratifiable;
– every semi-stratifiable space is subparacompact, perfect, and has a G_δ-diagonal;
– every countably compact, semi-stratifiable space has a countable base.

Moreover, in a semi-stratifiable space, the following are equivalent: ω_1-compact, her. separable, her. Lindelöf. On the other hand, the product of two paracompact semi-stratifiable spaces need not be paracompact (see the example of Berney discussed above.)

The semi-stratifiable spaces are motivated by the following observation: every closed set in a metric space is a G_δ, and moreover the proof shows that one closes down on closed sets in a uniform way.

Definition. A space X is *semi-stratifiable* if for each closed set H of X, there is a countable sequence $\{U(n,H) : n \in \mathbb{N}\}$ of open sets such that

1. $H = \bigcap\{U(n,H) : n \in \mathbb{N}\}$;
2. if K is closed with $H \subseteq K$, then $U(n,H) \subseteq U(n,K)$ for all $n \in \mathbb{N}$.

A semi-metrizable space with semi-metric d is semi-stratifiable as follows: $U(n,H) = \bigcup\{B_d(p,1/2^n)^\circ : p \in H\}$. If we replace (1) with the following stronger condition

(1)′ $H = \bigcap\{U(n,H)^- : n \in \mathbb{N}\}$,

we obtain the stratifiable spaces.

As I noted earlier in section 2, Heath [1965b] gave a partial solution to Brown's problem by showing that every semi-metrizable space with a point-countable base is developable. Creede [1970] gave what is perhaps the first "satisfactory" solution:

(6) $w\Delta$-space + semi-stratifiable = developable.

Semi-stratifiable spaces can also be characterized in terms of a g-function as follows: if $p \in g(n,x_n)$ for all $n \in \mathbb{N}$, then $x_n \to p$. This characterization suggested the definition of a β-space, and the following is due to Hodel [1971]:

(7) θ-refinable + β-space + G_δ-diagonal = semi-stratifiable.

What do you need to add to a semi-stratifiable space to get a stratifiable space? Heath [1966] gave an example of a paracompact semi-metrizable space that is not stratifiable, so the obvious choice, paracompactness, does not work. What does work is the class of monotonically normal spaces, introduced and systematically studied by Heath, Lutzer, and Zenor [1973]. The basic idea is to capture the idea of normality in a uniform way.

Definition. A space X is *monotonically normal* if for each pair $\langle H, K\rangle$ of disjoint closed sets of X, there is an open set $W(H,K)$ such that

1. $H \subseteq W(H,K) \subseteq W(H,K)^- \subseteq X - K$;
2. if $H \subseteq H'$ and $K' \subseteq K$, then $W(H,K) \subseteq W(H',K')$.

This class of spaces gives further insight into the relationship between stratifiable and semi-stratifiable spaces as follows:

(8) monotonically normal $+$ semi-stratifiable $=$ stratifiable.

The class of monotonically spaces has certain nice features: preserved by closed, continuous maps; hereditary; collectionwise-normal. Moreover, Heath, Lutzer, and Zenor proved that

(9) monotonically normal $+$ p-space $+$ G_δ-diagonal $=$ metrizable.

But product theory for this class of spaces is unsatisfactory. For example, the Sorgenfrey line S is monotonically normal and hereditarily Lindelöf, but $S \times S$ is not normal. Also, a compact monotonically normal space need not have a countable base.

In 1982, Palenz obtained a number of important results on the relationship between monotone normality and paracompactness; among other results she proved that

(10) monotonically normal $+$ G_δ-diagonal $\Rightarrow$ paracompact.

Note that (9) above follows from (10) and the Borges–Okuyama metrization theorem.

11. Quasi-metrizable and γ-spaces

The spaces that I will discuss in this section are related as follows:

$$\begin{gathered}\text{non-archimedean quasi-metrizable (= }\sigma\text{-interior preserving base)}\\ \downarrow\\ \text{quasi-metrizable}\\ \downarrow\\ \gamma\text{-space (= Nagata first-countable)}\end{gathered}$$

moreover, none of the arrows can be reversed.

I have already discussed the early work by Niemytzki and Wilson on quasi-metrizable spaces (see section 1). The quasi-metrizable spaces have several positive features; most noteworthy are

– every subspace of a quasi-metrizable space is quasi-metrizable;
– a countable product of quasi-metrizable spaces is quasi-metrizable;
– every countably compact quasi-metrizable space has a countable base.

On the other hand, the Sorgenfrey line S is a quasi-metrizable, hereditarily Lindelöf space such that $S \times S$ is not normal and hence not paracompact. Van Douwen and Wicke [1977] have given a remarkable example of a separable, ω_1-compact, quasi-metrizable space that is not perfect, not subparacompact, and not meta-Lindelöf. So we see that the quasi-metrizable spaces fail to have many of the properties that we usually associate with metrizable spaces.

A characterization of the quasi-metrizable spaces in terms of a g-function was given quite early by Ribeiro [1943]:

Theorem 11.1 (Ribeiro) *A space X is quasi-metrizable if and only if there is a g-function for X that satisfies these two conditions for all $p \in X$:*

1. *$\{g(n,p) : n \in \mathbb{N}\}$ is a local base for p;*
2. *for all $q \in X$ and $n \in \mathbb{N}$, if $q \in g(n+1,p)$, then $g(n+1,q) \subseteq g(n,p)$.*

But the search for other characterizations and for factorizations of quasi-metrizability has been somewhat frustrating. The following sufficient condition for quasi-metrizability is due independently to Norman [1967] and Sion–Zelmer [1967]:

(1) σ-point finite base $\Rightarrow$ quasi-metrizable.

An easy consequence of (1) is that every metacompact developable space is quasi-metrizable. On the other hand, Heath [1972] has given an example of a developable space that is not quasi-metrizable.

The converse of (1) is false; for example, $S \times S$ (S = Sorgenfrey line) is quasi-metrizable but fails to have a σ-point finite base. The attempt to modify (1) so that we obtain a characterization of quasi-metrizability leads to the following ideas.

Definitions. A distance function d on X is *non-archimedean* if for all $x, y, z \in X$,

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}.$$

Note that a non-archimedean distance function is a quasi-metric. A space $\langle X, T\rangle$ is *non-archimedean quasi-metrizable* if there is a non-archimedean distance function d on X such that $T = T_d$. A collection $\mathcal{B}$ of open subsets of a space X is said to be *interior-preserving* if for each $p \in X$, $\bigcap\{B \in \mathcal{B} : p \in B\}$ is an open set.

Fletcher–Lindgren [1972], Nedev [1967b], and Kofner [1973] independently obtained the following result:

(2) non-archimedean quasi-metrizable $=$ σ-interior preserving base.

This theorem compares very nicely with the Nagata–Smirnov theorem: metrizable $=$ σ-locally finite base. But (2) does not give a characterization of the quasi-metrizable spaces; Kofner [1973] constructed a space, now called the *Kofner plane*, that is quasi-metrizable but is not non-archimedean quasi-metrizable.

Let us consider the following theorem in connection with the problem of characterizing the quasi-metrizable spaces.

Theorem 11.2 (Fletcher–Lindgren [1974]) *The following are equivalent for a g-function on a space X:*

1. *if $y_n \in g(n,p)$ and $x_n \in g(n,y_n)$ for all $n \in \mathbb{N}$, then $x_n \to p$.*
2. *if $x_n \in g(n,y_n)$ for all $n \in \mathbb{N}$, and p is a cluster point of $\langle y_n \rangle$, then p is a cluster point of $\langle x_n \rangle$.*
3. *if $K \subseteq U$ with K compact and U open, then there exists $n \in \mathbb{N}$ such that $\bigcup\{g(n,p) : p \in K\} \subseteq U$.*

Spaces satisfying 1. are called γ-*spaces* and were introduced by Hodel [1972] in connection with the problem of generalizing Nagata's general metrization theorem (see Theorem 2.9). At about the same time, Sabella [1973] introduced the class of spaces satisfying 2.; Sabella called a g-function satisfying 2. a *coconvergent open neighborhood assignment.* Condition 3. has been studied by Martin [1976].

It is obvious that every γ-space is first-countable. Call a space X *Nagata first-countable* if each $p \in X$ has two sequences $\{U_n(p) : n \in \mathbb{N}\}$ and $\{V_n(p) : n \in \mathbb{N}\}$ of neighborhoods such that the following hold for all $p \in X$:

1. $\{U_n(p) : n \in \mathbb{N}\}$ is a local base for p;
2. for all $n \in \mathbb{N}$ and $q \in X$, if $q \in V_n(p)$, then $V_n(q) \subseteq U_n(p)$.

These are conditions 1. and 3. of Theorem 2.9 (recall that Nagata spaces are defined in terms of 1. and 2. of Theorem 2.9). In a survey paper [1974] on metrization theorems that I gave at the VPI&SU Topology Conference in 1973, I noted that every Nagata first-countable space is a γ-space. Somewhat later, Fletcher and Lindgren [1974] proved:

(3) γ-space $=$ Nagata first-countable.

The situation comes down to this. Every quasi-metrizable space is a γ-space; the γ-spaces can be characterized in a number of interesting ways. But are they the same? Before answering this question, I will state a result that gives a clearer picture of the relationship between the three classes of spaces under discussion. Let X have a g-function such that for all $p \in X$, $\{g(n,p) : n \in \mathbb{N}\}$ is a local base for p. Now consider the following additional conditions on g:

(NA) if $q \in g(n,p)$, then $g(n,q) \subseteq g(n,p)$;
(QM) if $q \in g(n+1,p)$, then $g(n+1,q) \subseteq g(n,p)$;
(γ) for each $p \in X$ and each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that if $q \in g(m,p)$, then $g(m,q) \subseteq g(n,p)$.

Lindgren and Nyikos [1976] observed the following:

(4) X is non-archimedean quasi-metrizable $\Leftrightarrow$ g satisfies (NA);

(5) X is quasi-metrizable $\Leftrightarrow$ g satisfies (QM);

(6) X is a γ-space $\Leftrightarrow$ g satisfies (γ).

Alas, not every γ-space is quasi-metrizable. The first example is due to Fox [1982]; a regular example was later constructed by Fox and Kofner [1985].

As already noted, the Niemytzki plane is both quasi-metrizable and semi-metrizable, yet is not metrizable. But we can say that any space with both of these properties is developable. This follows from the following theorem of Hodel [1972]:

(7) β-space + γ-space $\Rightarrow$ developable.

An easy consequence of (7) is the result that

(8) symmetrizable + γ-space $\Rightarrow$ developable.

Moreover, Junnila proved (see Fletcher–Lindgren [1982]) that

(9) developable + γ-space $\Rightarrow$ quasi-metrizable.

If we combine (7) and (9), we obtain the result that for β-spaces, γ-space = quasi-metrizable. For a further discussion of quasi-metrizable spaces, the reader is referred to the survey paper by Kofner [1980] and the monograph of Fletcher–Lindgren [1982].

12. Concluding remarks

In this paper I have tried to capture the exciting and central developments that took place in the theory of generalized metrizable spaces during the time period 1950–1980. I have called this paper "A History" rather than "The History" for two reasons. First, another author might see the development of the field in a quite different way from the one presented here. But more important perhaps is the fact that the study of generalized metrizable spaces is by no means complete; rather, it continues to grow with many new and important results appearing every year. For a discussion of some of these recent developments, the reader is referred to the surveys by Gruenhage [1992] and Nagata [1989].

Survey Papers

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THE HISTORICAL DEVELOPMENT OF UNIFORM, PROXIMAL, AND NEARNESS CONCEPTS IN TOPOLOGY

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Contents

"I've never seen anything like this on television!"
Peter Sellers in "Being There" by **J. Kosinski.**

[1]" 'Men work together,' I told him from the heart, 'Whether they work together or apart.' "
The Tuft of Flowers, Robert Frost.

C. E. Aull and R. Lowen (eds.), Handbook of the History of General Topology, Volume 2, 577–629.

For formulating results or definitions from older papers and books we shall mostly use modern terms and notation to avoid misunderstanding.

1. Uniform and Proximity Structures

We shall briefly visit the last century and observe how uniform concepts appeared there, namely the completion of rational numbers and the uniform continuity of functions.[2] A third common uniform concept, the uniform convergence of functions, had no influence on the development of uniform structures. In the second section we shall look at the development of uniform concepts before 1937, i.e. before A. Weil initiated the theory of uniform spaces as we know it today. Then we shall follow the development of uniform spaces up to the time when J. Isbell "finished" another period by publishing his famous book – that period will be covered in the penultimate subsection of this section. We shall close the section by describing the development of proximity spaces.

1.1. UNIFORM CONCEPTS BEFORE 1900

"There is plenty of room for new trees and new flowers of all kinds."

J. Kosinski[3]

Historical notes concerning uniform concepts usually start with the well known Bolzano–Cauchy property of convergence as the first such notion. That property appeared in Bolzano [1817] and Cauchy [1821]. Bolzano needed his property of convergence of reals for proving the existence of infima of bounded sets of reals, which he applied to get a root of a continuous function between variables with opposite signs of their values. It may be interesting to quote Bolzano's original assertion ([1817, p. 35]; the members of the sequence $\overset{n}{F}x$ are partial sums of a series): Wenn eine Reihe von Größen

$$\overset{1}{Fx}, \overset{2}{Fx}, \overset{3}{Fx}, \ldots, \overset{n}{Fx}, \ldots, \overset{n+r}{F}x, \ldots$$

von der Beschaffenheit ist, daß der Unterschied zwischen ihrem $n$ten Gliede $\overset{n}{F}x$ und jedem späteren $\overset{n+r}{F}x$, sei dieses von jenem auch noch so weit entfernt, kleiner als jede gegebene Größe verbleibt, wenn man n groß genug angenommen hat: so gibt es jedesmahl eine gewisse **beständige Größe**, und zwar nur **eine** der sich die Glieder

[2]See also the discussion of the history of uniform concepts in topology which appears in the paper by Alexandroff and Fedorchuk [1978].

[3]From the movie *Being There*, starring Peter Sellers.

immer mehr nähern, und der sie so nahe kommen können, als man nur will, wenn man die Reihe weit genug fortsetzt.

One cannot say that what Bolzano and Cauchy did had a "uniform" nature since their condition was applied only to continuity. Bolzano used his condition to prove the "Bolzano–Weierstrass" theorem. He realized later that the proof is not rigorous because real numbers were not precisely given. So, during 1830–1835 he wrote a theory of real numbers that remained unpublished at that time (published and commented upon by K. Rychlík in [1957], [1962]). His procedure is cumbersome and not quite precise from the present point of view but at that time a theory of sets and of infinite series was not known. Bolzano's idea was right: to describe real numbers as equivalence classes of sequences of rational numbers (Bolzano's language is, however, more complicated). Using that approach, he established the arithmetic of the real numbers, and he proved among other things that the set of real numbers is dense-in-itself, and the Bolzano–Cauchy and Bolzano–Weierstrass theorems.

Probably, Ch. Méray was the first who rigorously used a "uniform" procedure to define the irrational numbers in [1869]. Three years later he published his approach in the book [1872]. Parts of Méray's paper [1869] can be found in Dugac [1982] and Dugac [1984]. The procedure described there is the, nowadays, standard one (except, of course, for the notation and terms: sequences were called *variantes*, fundamental (Cauchy) sequences were called *convergent*, and irrational numbers were called *incommensurables*). Méray proves the Bolzano–Weierstrass theorem (completeness of the real numbers).

G. Cantor is better known for the same procedure of defining irrationals in [1872]. He was motivated by his approach to the uniqueness of trigonometric series on $(0, 2\pi)$ if it is required on a subset P only. He realized that to deal with accumulation points of P he needs a solid introduction of real numbers and of their properties. (Even more than 15 years later some mathematicians did not understand such reasons, see [1889].) Cantor does not prove that his real numbers are complete but he says that to every rational number one can assign an equivalence class of fundamental sequences (he calls them Zahlengrößen) but not conversely; to every fundamental sequence of "Zahlengrößen" one can assign a "Zahlengröße". In spite of that fact, he continues the process to get "Zahlengröße der λ^{ter} Art" and says that "der Zahlenbegriff, soweit es hier entwickelt ist, den Keim zu einer in sich notwendigen und absolut unendlichen Erweiterung in sich trägt". It was E. Heine who explicitly proves in the same year that the procedure stops after the first step. (Heine thanks Cantor for explaining the procedure of defining irrationals – both were in Halle, Germany, at that time.)

That approach of constructing the reals was a qualitatively new approach and 60 years later it was one of the main properties leading to uniform structures. We should add that Dedekind's theory of constructing the real numbers as an order-completion of the rational numbers was also published in 1872. K. Weierstrass lectured on his theory of real numbers at the beginning of the 60's (see Dugac [1973], Kossak [1872]); he defined irrational numbers as (sums of) series of rational numbers.

The concept of uniform continuity originates at almost the same time as the completion of the rationals. It was defined precisely in 1870 by E. Heine who needed it for the convergence of trigonometric series. Heine describes uniform continuity (gleichmässige Continuität) for functions of two variables. For a proof of convergence of a trigonometric series he considered the series

$$\frac{1}{2}a_0 + r(a_1 \cos x + b_1 \sin x) + r^2(a_2 \cos x + b_2 \sin x) + \cdots$$

for x belonging to some interval and $r \in [0, 1]$. So, the two variables were x, r, and Heine noticed that to get convergence for $r = 1$ (from Abel's theorem) one must assume a stronger condition than continuity which may be called uniform continuity because "sie sich gleichmässig über alle Punkte und alle Richtungen erstreckt". He defines it in the following way: "A function f of two variables x, y is called uniformly continuous in its domain if for any given small positive number ε there exist two positive numbers h_1, k_1 such that the difference $f(x+h, y+k) - f(x, y)$ remains less than ε whenever h and k, resp., do not exceed h_1, k_1, and that must be true, for given ε and found h_1, k_1 for all the points $(x, y), (x + h, y + k)$ which belong to the domain including its boundary." In Heine's paper [1872], one can already find the present form of the definition of uniform continuity of a function defined on a closed interval; Theorem 6 asserts that every continuous function on a closed bounded interval is uniformly continuous. He used a procedure that is standard even today: starting with 0, he finds successively largest points x_n such that $|f(x_{n+1}) - f(x_n)| \leq 3\varepsilon$; after finitely many steps he must finish otherwise $\{x_n\}$ has a supremum x, and there is an x_n with $|f(x) - f(x_n)| \leq 2\varepsilon$.

U. Dini in his book [1878] defines uniform continuity for functions of one variable and proves the following result on page 49: *If f is a continuous function on $[a, b]$, then for every $\varepsilon > 0$ there exists a finite cover of $[a, b]$ by intervals such that the oscillation of f on each of these intervals is less than ε.* That formulation was used again by M. Fréchet in 1928 for mappings on metric spaces (see Section 1.2).

In 1897, T. Brodén uses, probably for the first time, completion as a uniform concept. He proves an extension theorem for uniformly continuous functions. He considered a countable dense set X in a closed interval, say $[0, 1]$, and a continuous function f on X. He proves that *a necessary and sufficient condition for f to have a continuous extension on $[0, 1]$, is that f be uniformly continuous on X.* He calls uniform continuity "gleichförmige Stetigkeit": for small enough $|\delta|$ and for any x, $|f(x + \delta) - f(x)|$ is smaller than a given arbitrarily small positive number ε. The proof is given by means of series; for a convenient Cauchy sequence $\{x_n\}$ the limit

$$\lim f(x_n) = f(x_1) + (f(x_2) - f(x_1)) + (f(x_3) - f(x_2)) + \cdots$$

exists (the series converges) and gives rise to a continuous function provided f is uniformly continuous. This was probably the first usage of uniform continuity on a noncompact set, and an important usage for the extension of functions.

As we mentioned at the beginning, we shall not describe the development of uniform convergence; see for instance I. Grattan-Guiness [1986] and P. Dugac [1973]. It proceeds from A.L. Cauchy (1821) via N.H. Abel (1826), E.H. Dirksen (1829) and C. Gunderman (1838) to K. Weierstrass (1842), Bjórling (1846), G.G. Stokes (1847) and P.L. Seidel (1848) back to A.L. Cauchy (1853).

1.2. UNIFORM CONCEPTS TILL 1937

"To generalize is to be an idiot. To particularize is the alone distinction of merit. General knowledges are those knowledges which idiots possess."

William Blake[4]

The concepts that lead to uniform structures were *uniform continuity*, *completeness* and *metrization*. All of these notions appeared in various generality, various settings and with various characterizations in the years 1904–1936. It was André Weil who, in a sense, completed the work of his predecessors and contemporaries after realizing the mutual connections and relations of corresponding results and concepts. We shall trace now how those results and concepts developed.

The beginning of this century was so full of ideas concerning various types of properties of the reals or of functions that it was natural to expect an abstract approach to appear soon. Probably there were other mathematicians who studied quite general structures, but two of them are known today for having suggested reasonable structures for further development: M. Fréchet and F. Riesz. Perhaps we could add here E.H. Moore [1906], [1909], who also tried to introduce a general abstract look at the basics of analysis; he worked with properties that are valid uniformly or relatively uniformly (i.e. instead of ε for all x one uses $\varepsilon f(x)$ for some f), which generalized, e.g. the uniform convergence of functions; probably his ideas had no influence on the development of uniform structures. Since Riesz' contribution concerns proximity spaces, we shall postpone the discussion of his ideas to Section 1.5.

[4] Cited after the book *Unbegreifliches Geheimnis* by Erwin Chargaff.

Maurice Fréchet was the first to suggest an abstract approach in a finished form in his thesis published as a 74-page paper in Rendiconti Circ. Matem. Palermo in 1906. After presenting his results in 1904–1905 in the Academy of Science in Paris, he published a short exposition of a general notion of distance function in C.R. in 1905. Fréchet defined several classes of abstract spaces that were convenient for generalizing properties used in various special situations: convergence spaces (he called them L-spaces), metric spaces (E-spaces, E from écart; the term "metrische Räume" was first used by Hausdorff in 1914), and a kind of generalized metric space which he called V-space:

A *V-space* is a set X and a nonnegative real-valued function d on $X \times X$ such that the following three axioms hold (d was not used, the distance was denoted by (x, y)):

1. for all $x, y \in X$ $d(x, y) = 0$ iff $x = y$
2. for all $x, y \in X$ $d(x, y) = d(y, x)$
3. there exists a nonnegative real-function f defined on the reals with $\lim_{t \to 0} f(t) = 0$ such that for all $x, y, z \in X$ and all $r \geq 0$,

$$\text{if } d(x, y) \leq r \text{ and } d(y, z) \leq r \text{ then } d(x, z) \leq f(r)\,.$$

The number $d(x, y)$ was called "voisinage" of x and of y. We should note that Fréchet allowed his écart to have infinite values (see, e.g., the example 38 on page 24 for sup-distance of two unbounded functions).

It was quite natural for Fréchet to define for his V-spaces some uniform concepts: the uniform continuity of functions into the reals, and completeness (he was aware that he cannot define these concepts for L-spaces). The former concept was not difficult to transfer from the situations used before, whereas the latter concept could not be transferred automatically from such earlier results in the classical settings. Fréchet does not use the term *complete space*, but uses the term *space admitting a generalization of the Cauchy theorem*; the term complete spaces (vollständige Räume) is again due to Hausdorff, 1914. Fréchet proves for instance that *every continuous function on a compact subset of a V-space is uniformly continuous* and, for separable spaces without isolated points, *a complete V-space is compact iff it is totally bounded*; total boundedness is defined by means of finiteness of r-nets. Total boundedness thus came naturally into consideration as another uniform concept.

Separability and the condition on isolated points for the above last quoted result was found to be unnecessary in Fréchet's paper published in the same journal four years later. Some other new facts about complete metric spaces are proved in that latter paper, e.g. *a complete perfect infinite V-space is uncountable.* The last result is a generalization of Cantor's result for the reals – F. Hausdorff in [1914] generalized it even further to G_δ-subsets of complete metric spaces.

Since all the topological results valid for metric spaces could be proved also for V-spaces, Fréchet conjectured (see [1910, pp. 22-23]) that for every V-space (X, d) one can find a metric having the same convergent sequences as the generalized metric d. He might have had in mind that Hahn's construction of a nonconstant continuous function applied to V-spaces might possibly lead to a proof. It was E.W. Chittenden in 1917 who, using Hahn's procedure, proved the conjecture to be true (cf. Aull [1981]). The proof that is used nowadays comes from G. Birkhoff [1936] and A.H. Frink [1937]. Hahn's procedure is natural and was used later by other authors, too.

Fréchet did not mention the uniform continuity of mappings between his V-spaces, nor did he define completions. The completion of metric spaces was described in the basic book of general topology, in Felix Hausdorff's *Grundzüge der Mengenlehre* from 1914. Hausdorff's procedure is the "classical" one, adding equivalence classes of nonconvergent fundamental sequences to the given space. The next step, the uniformly continuous extension to closures of uniformly continuous mappings, had not yet been found. But Hausdorff comes very close to that result with his theorem: *If f is a uniformly continuous mapping into a complete metric space, defined on a subspace A of a metric space, then there exists a continuous extension of f defined on the closure of A.* – see page 368.

Brodén's extension result, mentioned at the end of Section 1.1, was not generalized and, probably, neither Hausdorff nor Fréchet were aware of it. Neither was Bouligand in 1923 who brought attention to uniform extensions. He proved there that *every uniformly continuous real-valued function on the set of polynomial functions* (regarded as a subset of the space C of continuous functions on a closed bounded interval and endowed with the sup-metric) *can be extended to a uniformly continuous function on C*; that the extension is uniformly continuous is proved but not stated explicitly. Reading that paper, Fréchet realized immediately its meaning and proved in [1924] that *every uniformly continuous real-valued function defined on a subspace A of a metric space can be extended to a uniformly continuous function defined on the closure of A.* The possibility to replace the reals by another complete space had not been proved yet.

After publishing several more papers on abstract structures, Fréchet summarized the results in his book *Les Espaces Abstraits* published in 1928 in the series edited by E. Borel. The notation he uses there differs from his notation of 1906 and he changes some approaches. For some classes of spaces, convergence of sequences is the primary structure, and the other structure (as a metric) is the secondary one. For instance he defines a "complete" metric space as a metric space admitting a topologically equivalent complete metric, i.e. what was later called a topologically complete space. Fréchet notes that E.W. Chittenden in [1924] first found a space that is not topologically complete (Fréchet's problem from [1921]), and mentions that his notion differs from the completeness of Hausdorff. In the same year, P.S. Alexandroff [1924], for separable spaces, and F. Hausdorff

[1924], for all metric spaces proved that every G_δ-subspace of a topologically complete space is topologically complete.

We shall now mention two places in Fréchet's book where an indication of uniformization is shown. The first one is a kind of uniformization of countable neighborhood bases in a metrization theorem appearing on page 217: *A topological space satisfying the first axiom of countability is semimetrizable* (triangle inequality not needed) *iff there exist countable bases* $\{V_x^n\}$ *of* $x \in X$ *and a mapping* $\phi : \mathbb{N} \to \mathbb{N}$ *such that* $x \in V_y^n$ *provided* $y \in V_x^{\phi(n)}$. The second place concerns a possibility to define uniform continuity on more general structures than on those where a distance function is defined. Fréchet writes on page 237: "M.M. Alexandroff et Urysohn se sont rencontrés avec l'auteur pour penser qu'on éviterait ainsi l'introduction de la distance dans la définition de la continuité uniforme." On the next page he says that such a definition is possible for topological spaces that are compact: *A function* $f : X \to \mathbb{R}$ *is uniformly continuous iff for every* $\varepsilon > 0$ *there exists a finite open cover of* X *such that the oscillation of* f *on each member of the cover is less than* ε. In connection with this result see also Dini's result mentioned in Section 1.1.

Some kind of uniformization (from the present point of view) was included in other metrization theorems, already in the 20's. We have in mind the metrization theorems using systems of covers (e.g. P.S. Alexandroff and P. Urysohn [1923], E.W. Chittenden [1927]): there exists a countable system $\mathcal{V}_n$ of covers such that $\mathcal{V}_{n+1}$ refines $\mathcal{V}_n$, the system generates the topology (stars of points form a neighborhood base) and every $\mathcal{V}_n$ has a kind of star–refinement in the system.

Nobody made use of these last mentioned hints to get closer to a general uniform structure in the next few years. The force that drove some mathematicians at that time to the right concept of uniformity was a possibility to deal with completeness. The problem was to go behind metric spaces, i.e. to find structures where completeness could be reasonably defined and studied. The only thing that was clear was that the Bolzano–Cauchy property needed to be used. However, this approach restricted mathematicians to sequences, even though a general type of convergence was published by E.H. Moore and H.L. Smith [1922].

The first successful study of completeness and completion without using a distance function was carried out in spaces endowed with both a topological and an algebraic structure. D. van Dantzig noticed in [1933] that in a topological group one can define Cauchy sequences $\{x_n\}$ without using a distance: $x_n {x_m}^{-1}$ converges to the neutral element (in both n, m). He uses sequences; in fact he uses only first countable topologies on groups (i.e. metrizable, as was proved later by Birkhoff [1936] and Kakutani [1936]); van Dantzig proves the basic classical results for his completeness: a complete subgroup is closed, a closed subgroup of a complete group is complete, a sequentially compact group is complete, a locally compact group is complete, a metric group is complete as a group iff it is complete as a metric space. He proves that *a group completion exists iff whenever* $\{x_n\}$ *converges to the neutral element and* $\{y_n\}$ *is Cauchy, then the sequence* ${y_n}^{-1}x_ny_n$ *converges to the neutral element.* In the next sections and the next paper, he studies completeness and completions for rings, linear spaces, algebras, and other algebraic structures.

Still, completeness was restricted to using countable sequences. Even von Neumann, who first generalized completeness for "noncountable" situations, did not use nets in [1935]. He noticed that to define completeness, one needs a kind of uniformity in order to define Cauchy sequences. A topological space does not have such a property and so "it is improbable that a reasonable notion of completeness could be defined in it". But the situation is different in topological linear spaces, where Cauchy sequences can be defined; von Neumann does not quote van Dantzig and he repeats basic definitions and properties. His attempt to define completeness for general topological linear spaces L is the following one: *L is complete if the closure of every totally bounded subset of L is countably compact* (he uses the term compact set, but defines it by the property that each of its infinite subsets has an accumulation point in the set), i.e. L is sequentially complete and, thus, he did not go out of sequences; nevertheless, he showed another approach to completeness. Von Neumann proved that for topological linear spaces satisfying the first countability axiom, his completeness coincides with the completeness defined by the convergence of every Cauchy sequence.

Postponing A. Weil for the moment, the final step for defining completeness in topological groups or topological linear spaces was carried out by Garrett Birkhoff [1937a] (submitted in April 1936, so he was not aware of A. Weil's paper [1936]). He describes how to use Moore-Smith convergence of nets in topological spaces and, quite naturally, he defines Cauchy nets in topological groups. First, he extends van Dantzig's results to general topological groups and topological linear spaces (an important fact for Birkhoff was that complete spaces are absolutely closed – he and his predecessors worked in spaces with unique limits). Then he shows that von Neumann's definition of completeness for topological linear spaces is weaker than his (Birkhoff's) definition (the example that it is strictly weaker was communicated to him by von Neumann: Hilbert space in its weak topology).

In his next paper in the same issue, [1937b], Birkhoff suggests a general idea as to what completeness should be; he uses an external view. Perhaps one can consider that attempt as a general description of hulls. It is close to extremal reflections, but nobody at that time connected an existence of such a hull with the extensions of mappings.

André Weil publishes his first paper on uniform spaces in 1936. For a topological space X he defines a class $\mathcal{P}$ of covers to be *regular* if

– $\{\mathrm{Star}_{\mathcal{V}} x : \mathcal{V} \in \mathcal{P}\}$ is a local base at x in X;

- for every $\mathcal{V}, \mathcal{V}' \in \mathcal{P}$ there exists some $\mathcal{V}'' \in \mathcal{P}$ such that $\mathrm{Star}_{\mathcal{V}''}x \subset \mathrm{Star}_{\mathcal{V}}x \cap \mathrm{Star}_{\mathcal{V}'}x$ for every $x \in X$;
- for each $\mathcal{V} \in \mathcal{P}$ there exists some $\mathcal{V}' \in \mathcal{P}$ such that $\mathrm{Star}_{\mathcal{V}'}(\mathrm{Star}_{\mathcal{V}'}x) \subset \mathrm{Star}_{\mathcal{V}}x$ for every $x \in X$.

Weil notes that if one has a class $\mathcal{P}$ of covers satisfying the last two properties, then the system of stars as described in the first property defines a topology on X with respect to which the given system is regular. Since he works with Hausdorff spaces, he needs one more property to achieve the desired separation:

for $x \neq y$ there exists $\mathcal{V} \in \mathcal{P}$ such that no $V \in \mathcal{V}$ contains both x and y;

Weil asserts that using a method that was communicated to him by Pontryagin, it is possible to show that X is completely regular.

Then he defines the concept of a Cauchy family $\mathcal{F}$ in X endowed with the regular class $\mathcal{P}$: it is a centered family such that for every $\mathcal{V} \in \mathcal{P}$ there exists $F \in \mathcal{F}$ such that $F \subset \mathrm{Star}_{\mathcal{V}}x$ for every $x \in F$. Using Cauchy families, completeness can be defined, and completions can be constructed by adding equivalence classes of some Cauchy families.

Total boundedness (which Weil called *relative compactness*) is then defined and Weil asserts that a complete space is compact iff it is totally bounded. Also he asserts that on a compact space all the regular classes of covers are equivalent (equivalence means that the corresponding covers of stars of points are mutually refined).

As one can see, the definition is still not the right one. First, the structure is primarily connected with a topological space. That is not so important; worse is the fact that one must speak about equivalent structures. Graves [1937] (received in May 1936) formulates different axioms that are equivalent to Weil's, but he does not remove the main imperfection: he also deals with equivalent structures. Nevertheless, he does remove one formal difficulty from Weil's approach, namely, the fact that one deals not with the original covers but with the covers of the stars of points. Graves' definition of a structure on a set X is the following: he takes a directed ordered set S and for each $s \in S$ he has a cover $\mathcal{E}_s$ of X such that

- if $s > s'$ then $\mathcal{E}_s \subset \mathcal{E}_{s'}$;
- for each $s \in S$ there exists an $s' \in S$ such that $\{\mathrm{Star}_{\mathcal{E}_{s'}}x : x \in X\}$ refines $\mathcal{E}_s$.

He adds the separation axiom and also an axiom stating that members of the covers are open sets. The first axiom above is unpleasant because the covers must be big (they must contain with each member many smaller open sets).

His other approach using uniformization of systems of neighborhoods of points seems somewhat more convenient. Graves' structure on a set X is a mapping assigning to each $(s, x) \in S \times X$ (S is again a directed set) a set $V_s(x) \subset X$ containing x such that

- if $x \neq y$ then $y \notin V_s(x)$ for some s;
- if $s > s'$ then $V_s(x) \subset V_{s'}(x)$ for every $x \in X$;
- for each $s \in S$ there exists $s' \in S$ such that if $y \in V_{s'}(x)$ then $V_{s'}(y) \subset V_s(x)$.

Cauchy nets and their equivalence are defined (it is mentioned that only Cauchy nets with domain S may be considered). Complete spaces are assigned to the structures (he does not use the term completion nor does he embed the original space into the constructed complete space). For his structures defined by means of covers he uses a different description of completeness, namely by means of systems $\{M_t : t \in T\}$ of closed sets, T being a directed set, such that $M_t \subset M_{t'}$ if $t > t'$, and every defining cover of the structure must contain a member which is larger than some M_t; completeness then means that such families must have nonempty intersections.

Graves describes the equivalence between his two structures and Weil's structure and says that the three corresponding notions of completeness coincide. At the end of the paper he defines his "neighborhood" structure on a power X^Y (it is the product structure), provided X has such a structure and asserts that X^Y is complete provided X is complete. The same statement is shown to be true for (what we call today) the power endowed with the uniformity of uniform convergence, and for some subspaces of the powers, e.g. for the spaces of totally bounded functions, or continuous functions, or uniformly continuous functions.

Graves' paper was published in 1937, in the same year as the basic booklet by André Weil. The main aspect which makes A. Weil quoted as a founder of uniform spaces is the fact that, in his booklet, he proved the key basic results and found the basic relations, as Fréchet did in 1906 for metric spaces and Hausdorff in 1914 for topological spaces. By his theory, he convinced mathematicians that the approach is the right one.

We have omitted discussing another way that leads from metric spaces to uniform spaces, namely generalized metric spaces. We shall mention only Kurepa's contributions. In 1934, he defined *espaces pseudo-distanciés* and he returned to this concept in 1936. His definition corresponds to Fréchet's definition of V-spaces, the main difference being that Kurepa used a kind of general set for the range of the distance function. But Kurepa made no attempt to develop some notions characteristic to uniform structures, e.g. uniformly continuous mappings or completions. Also, his notation is difficult to deal with. It is true that a special case of his concept is equivalent to uniform spaces but no systematic theory was given. For a survey by Kurepa see [1976]. Similar situation occured in other papers dealing with generalized metrics at that time. Connections between uniformities and generalized metrics were established, e.g., by G.K. Kalisch [1946], A. Appert [1947] and [1950], L.W. Cohen and C. Goffman [1950] and J. Colmez [1949].

1.3. FROM A.WEIL TO J.R. ISBELL

> "The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact."
>
> **A.N. Whitehead**[5]

Weil's booklet has 37 pages of text. It is interesting that he changed his approach from the paper of 1936: instead of systems of covers he works with a uniform system of neighborhoods of points as Graves did (Weil was probably not aware of Graves' work), or with uniform neighborhoods of the diagonal. He mentions and uses in examples the fact that one can use a system of covers for the characterization of uniform spaces but does not specify any axioms. Next we sketch his definitions and results.

A *uniform system of neighborhoods* on a set X is an indexed collection $\{V_\alpha(p); \alpha \in A, p \in X\}$ satisfying three axioms that correspond to those of Graves (the only difference is that Weil does not assume A to be ordered, so the second axiom of Graves is changed to *for each* α, β *there exists* γ *such that* $V_\gamma(p) \subset V_\alpha(p) \cap V_\beta(p)$ *for every* p).

This system of axioms is trivially equivalent to the usual system of axioms for uniform neighborhoods $\{V_\alpha\}$ of the diagonal. Weil establishes that fact immediately after stating the definition: $V_\alpha = \{(p,q) : q \in V_\alpha(p)\}$). The system V_α satisfies the following axioms:

- $\bigcap V_\alpha = \Delta$,
- for each α, β there exists γ such that $V_\gamma \subset V_\alpha \cap V_\beta$,
- for each α there exists β such that $V_\beta V_\beta^{-1} \subset V_\alpha$.

A uniform space is a set together with a uniform structure, which is a class of equivalent systems of neighborhoods of the diagonal satisfying the above properties.

Of course, both systems exhibit the lack mentioned in the previous subsection, namely, that it is not a filter and one must speak about equivalent systems (Weil shows that every system of neighborhoods of the diagonal is equivalent to a system of symmetric sets in $X \times X$); he remarks in a footnote that the theory can be simplified by use of H. Cartan's filters that were introduced during the final

[5]From *Science and the Modern World.*

preparation of the booklet. One should note that the approach via neighborhoods of points does have an advantage in that the topology induced by the uniformity is easily defined.

Weil gives two basic examples: metric spaces and topological groups (the right uniformity). As for constructions, Weil defines subspaces and arbitrary products of uniform spaces and says that the topology of the product of uniform spaces is the product of the induced topologies. He says that the proof of the fact that induced topologies are completely regular is identical to Pontrjagin's proof that the topologies of topological groups are completely regular (a letter sent to Weil in 1936 – the same idea as Kakutani used in [1936]). In fact, this idea goes back to Hahn [1908] as for construction of functions, and to Chittenden [1917] and Alexandroff and Urysohn [1923] as for using that in metrization (the proof used today which omits using dyadic numbers is due to G. Birkhoff [1936] and A.H. Frink [1937]). Weil establishes that every uniform space can be uniformly embedded into a product of metric spaces and he proves the uniform metrization theorem: *In order that a uniform space E, defined by means of a family V_α of neighborhoods of Δ in E^2, be isomorphic (in the sense of uniform structures) to a metric space, it is necessary and sufficient that either the family V_α is countable or equivalent to a countable family.* As mentioned above, the proof is more complicated than that following Birkhoff or Frink.

According to Weil, *Cauchy families* on a uniform space X are bases of filters $\mathcal{C}$ such that for every α there is some $C \in \mathcal{C}$ such that $C \times C \subset V_\alpha$. After defining equivalent Cauchy families and completeness, a completion is constructed as the set of equivalent Cauchy families. The uniqueness of completions is deduced from the following extension theorem: *Let X, Y be uniform spaces, Y complete, $A \subset X$, and $f : A \to Y$. For f to have a continuous extension to the closure of A it suffices and is necessary that f preserves Cauchy families.* That is clearly satisfied when f is uniformly continuous – then the extension of f is also uniformly continuous; of course, the extension is unique. So, Weil proves more than is usually attributed to him in the literature.

Weil then proves the basic results about relations between uniformities and compact spaces: *All uniformities on a compact space are equivalent. Continuous mappings on compact spaces into a uniform space are uniformly continuous. A uniform space is compact iff it is complete and totally bounded.*

The part dealing with local compactness is rather important. He studies uniformly locally compact spaces and shows that *a connected, locally compact, topological space X has a uniformity that is uniformly locally compact iff X is σ-compact.* Such uniformities can be described by means of *locally finite* (countable) covers. He suggests to use locally finite covers to generalize Alexandroff–Kuroš procedure (they used nerves of *finite* covers).

Several pages are devoted to completions of topological groups. Conditions are found when the group operations can be prolonged to the completion of the right uniformity; for instance if the right uniformity coincides with the left uniformity, or if the inverse operation is locally uniformly continuous (e.g. when the space is locally compact or locally totally bounded). The two-sided uniformity is mentioned, too; it is interesting that the other "two-sided" uniformity (the intersection of the right and left uniformities) first appeared in Dierolf and Roelcke's book [1981].

John W. Tukey published his book *Convergence and Uniformity in Topology* in 1940. He defines a uniformity on a topological space to be a system of (open) covers that is a filter with respect to star-refinement. So it is not necessary to speak about equivalent systems of covers, but a topological space is primarily present. In notes, Tukey says that his uniformities are equivalent with the ones by Weil.

Since he works with normal covers, he knows about existence (and description) of topologically fine uniformities (all normal covers) and the finest totally bounded uniformity on a topological space (all finite normal covers); those two uniformities do not commute with subspaces. He is not sure whether the collection of countable normal covers forms a uniformity (but if it is, it induces the same topology). He knows that all finite interior covers form a uniformity on a normal space (this fact was proved independently by Morita in [1940]).

He proves that *compact spaces have unique uniformities; a continuous mapping on a topologically fine space (or the finest totally bounded uniformity on a topological space) into a uniform space (or a totally bounded space, resp.) is uniformly continuous; a uniformity is metrizable iff it has a countable base* (here he uses the procedure due to Birkhoff and Frink); *there are unique completions; the completion of a metrizable or of a totally bounded uniformity has the same property.* His approach to completeness is via nets and ultranets (but not described in a nice way – Graves' approach was more natural and is used today).

Some authors say that Tukey's approach via covers is better than the procedure described by Weil. This is really difficult to say. As we saw, Weil also defined uniform spaces via covers and, moreover, the interrelation between those two approaches is so clear that it really depends on the person what definition is preferable for a proof or explanation. Of course, there are situations, where one of those approaches (or another) has an advantage.

In between Weil's and Tukey's work, Jean Dieudonné published three papers on uniform spaces. He was interested in Weil's problem to find conditions for a topological space to admit a complete uniformity (Weil [1937, p. 38]). In his papers [1939a], [1939b], [1939c], Diedonné showed that there is a non-normal space admitting a complete uniformity (the Niemytzki example) and that, conversely, there is a normal space not admitting a complete uniformity (the space of countable ordinals: using an Aronszajn idea, Dieudonné proves that there is exactly one uniformity on that space, namely the restriction of the unique uniformity of the one-point compactification of the space). *He characterizes topological spaces admitting a complete uniformity as those homeomorphic to subspaces of products of (complete) metric spaces.*

We should also mention another book published in 1940 that provides a more systematic study of uniform spaces in the style we know it today: *Topologie Général I,II* by Nicolas Bourbaki.

There were not too many papers on uniform spaces in the 40's but some of them are important, for instance, Mibu's and Loomis' work on Haar measures in uniform spaces ([1947], [1949] resp.) and Hu's definition of boundedness in uniform spaces ([1947]). Pierre Samuel [1948], investigated extensions by means of ultrafilters. His approach is analogous to that of F. Riesz mentioned at the beginning of Section 1.5, and also to the extension method using "ends" in the papers by P.S. Alexandroff [1939] and H. Freudenthal [1942]. For a uniform space X, an envelope of a filter is formed by all uniform neighborhoods of members of the filter; two filters are equivalent if they have identical envelopes. Now, defining $\widehat{X}$ to be the quotient set of all ultrafilters on X along the equivalence, endowed with the usual hull-topology, one gets a compact space containing X topologically. The completion of X is also topologically contained in $\widehat{X}$. The restriction of the unique uniformity on $\widehat{X}$ to X is totally bounded and coarser than the original uniformity. Samuel also shows that a topological space X is normal iff the finite open covers form a uniformity on X (see Tukey and Morita above), that *there exists a coarsest uniformity on X iff X is locally compact* (the covers are composed of a complement of a compact set and of finitely many open sets with compact closures), and that a countably compact space has all compatible uniformities totally bounded (the converse holds if the space is normal). R. Doss [1947] proved that *a completely regular Hausdorff space is pseudocompact* (i.e. every continuous function on it reaches its upper bound) *iff every compatible uniformity is totally bounded.*

Around 1950, several papers appeared concerning the lattice of uniformities on a given completely regular T_1-topological space X. First, it was R. Doss [1949] who proved that *there is a unique uniformity on X iff there is a unique totally bounded uniformity on X iff, from any two functionally separated closed subsets of X, one of them must be compact.* Consequently, such a space X must be locally compact (compact if metrizable). Doss used the following assertion: *Every filter in X without accumulation points is a Cauchy filter for a uniformity on X.* A similar procedure was used by T. Shirota [1950] where he again described locally compact spaces as those X admitting the coarsest uniformity (i.e. topologically coarse uniformity); there exists a coarsest complete uniformity iff X is compact. In fact, Shirota studied more general situations, namely for uniformities having a base of covers of cardinalities not bigger than a given cardinality (usually the weight of X).

In 1950–1952, Shirota studied complete uniformities on X. He showed that *all countable normal covers on X form a uniformity* (denoted by eX) and that *eX is complete iff X is realcompact.* He proved his famous result that *eX is complete if X has a complete uniformity and wX is weakly accessible from $\aleph_0$ in Tarski's sense.* This gives as a corollary for paracompact spaces a special case of Katětov's result from [1951a] that *a paracompact space X is realcompact if every closed discrete subspace of X is realcompact, i.e. it has no two-valued measure.*

Up to 1950, probably nobody was interested in the extension of uniformly continuous functions from subspaces (except the mentioned extension onto closures of mappings into complete spaces). The known procedures of extension of continuous real-valued functions from subspaces of metric spaces onto the whole spaces, as given by H. Tietze, H. Bohr, F. Hausdorff, F. Riesz and probably others, work also for uniform continuity (see Hušek [1992]). This fact was probably not noticed. It was Miroslav Katětov who published in [1951b] his result on the extension of uniformly continuous functions: *Every bounded uniformly continuous real-valued function f defined on a subspace Q of a uniform space P can be extended to a uniformly continuous function defined on P.* He says "This theorem seems to be essentially known without having been explicitly stated as yet." The extension is achieved by extending the sets $\{x \in Q : f(x) < t\}$, t rational, to outside of Q in a convenient manner, using a special lemma for Birkhoff's interpolation property.

As in the topological case (the Hausdorff result on extension of metrics) it took several years until John Isbell generalized the result to extension of pseudometrics in [1959]: *Every bounded uniformly continuous pseudometric defined on a subspace of a uniform space can be extended to a uniformly continuous pseudometric on the whole space.* It was shown later that unlike the topological case, Isbell's result can be deduced from Katětov's (see Čech [1966, p. 474]), but the procedure is not much easier than the direct proof.

Extending mappings uniformly continuously into more general spaces is, of course, more difficult. The problem is in obtaining an inner characterization of ranges that enable such extensions. Aronszajn and Panitchpakdi [1956] found a characterization of metric extensors for a special case of uniformly continuous mappings, namely for extending Lipschitz continuous mappings; they called a metric space (X, d) "hyperconvex" if any system of balls has a nonempty intersection provided for every two balls the sum of their radii is larger or equal to the distance of their centers. The result then is *a metric space X is hyperconvex iff every Lipschitz mapping from a subspace of a metric space into X can be extended to a Lipschitz mapping on the whole space.* J. Isbell [1959] then characterized hyperconvex spaces in planes. Not too much was done later in the nonmetric situation. In 1982, the authors of this contribution were interested in metric spaces allowing extension of uniformly continuous mappings into them.

Their result says: *a metric space X is a uniform retract of a bounded hyperconvex metric space iff every uniformly continuous mapping from a subspace of a metric space into X can be extended to a uniformly continuous mapping on the whole space*, see Hušek [1992]. It follows from Isbell's paper [1959] that the phrase "*a uniform retract of*" cannot be omitted.

At the end of this subsection let us mention two other characterizations of uniform spaces, namely by means of systems of pseudometrics and by means of nets. Both characterizations have disadvantages in that the corresponding axioms are not too friendly. Probably the first one describing uniformities by means of pseudometrics was J. Dieudonné [1939c], then N. Bourbaki [1948]; in both cases the axioms are not explicitly stated. L. Gillman and M. Jerison [1960] used that approach via systems of pseudometrics for uniform spaces.

As for nets, one cannot use single nets as in topological spaces but an equivalence relation on nets; equivalent nets $\{x_\alpha\}, \{y_\alpha\}$ are called adjacent nets if the net $\{(x_\alpha, y_\alpha)\}$ converges to the diagonal, i.e. it is eventually in every uniform neighborhood of the diagonal. An equivalence on nets must satisfy two more axioms to characterize uniformities, see V.A. Efremovič and A.S. Švarc [1953], V.S. Krishnan [1953], J. Nagata [1954].

1.4. UNIFORM SPACES IN THE 60'S AND 70'S

" 'Well, I'se thinkin', sar, I must ha' digressed,' said the [accused] when questioned by the magistrate what he was doing with a sack in Deacon Abraham's poultry-yard at twelve o'clock at night."

Jerome K. Jerome[6]

A revival of an investigation of uniform structures started around 1960. One of the main persons of that time figuring in new developments around uniformities was John R. Isbell. His contribution till 1964 is included in his famous book [1964]. This book was for a long time a leading source not only for results on uniform spaces but also for ideas and open problems. One of the main new features brought by Isbell is a systematic categorical look at uniform structures. We cannot introduce here many results with details and shall concentrate on only some of them. We shall completely omit uniform dimension theory and related geometrical results.

In the 70's two groups appeared that had an influence on subsequent developments in uniform spaces. One was around Wesleyan University in Middletown, Connecticut and the other group was formed around Zdeněk Frolík in Prague.

[6] From *Three men on the Bummel.*

Although geographically both groups were far from each other, their cooperation was rather tight. The contribution of the Middletown group was mainly in the investigation of uniform spaces using primarily tools from analysis and descriptive set theory (Baire sets, measurable covers) and initiating new constructions of interesting classes of uniform spaces like metric-fine spaces. In Prague, both approaches were developed further (descriptive set theory, coz_--spaces) and, mainly by J. Pelant, combinatorial methods in the investigation of uniform covers were brought into consideration (by means of that approach, he was able to solve several of Isbell's problems).

One of the new features at the beginning of the 50's was the use of categorical methods. That meant a new look at some situations. Completion becomes a special reflection (a maximal one in a certain sense), there are many other reflections and also, unlike the situation in topological spaces, many interesting coreflections, e.g. (topologically) fine or proximally fine uniformities. How do these special functors behave relatively to each other? What properties do they preserve? There were a lot of unsolved questions. J. Isbell started to solve such questions already in the 50's. Probably the best known paper on uniformities from that period is his joint paper with S. Ginsburg [1959] containing many results on reflections and coreflections in uniform spaces and their behaviour (for instance several results on commutation of basic reflections with completion: for the locally fine reflection on any space, for the Euclidean reflection on locally fine spaces having no uniformly discrete subspaces of measurable cardinality – that is a generalization of Shirota's theorem).

Another categorical notion is that of injective objects, i.e. spaces into which prescribed types of uniformly continuous mappings from subspaces can be extended onto the whole spaces as uniformly continuous mappings. By Katětov's theorem, closed bounded real intervals are injective with respect to all uniformly continuous mappings into them. Isbell deduced from that result that *uniform powers of closed bounded real intervals are injective* (and, thus, *every uniform space can be embedded into an injective space*); also, *the space of uniformly continuous functions from a metric space into, say,* $[0, 1]$, *endowed with the uniformity of uniform convergence, is injective.* The problem whether one may omit "metric" in the last result is still an open problem.

Except for the completion, the so called cardinal reflections are probably the most important modifications of uniform spaces. The totally bounded reflection (a base formed by finite normal open covers) of fine spaces appeared already in Tukey [1940]. The separable reflection (all countable normal covers) of fine spaces was mentioned also by Tukey but he was not sure whether all the countable normal open covers of a given topology form a base for a uniformity (the so called separable modification). It was proved to be so by Shirota [1952]. Both totally bounded and separable reflections of any uniformity were described by

Ginsburg and Isbell (all finite or countable, resp., uniform covers). It was not clear for a while how it is for higher cardinals (except for covers with nonmeasurable members which was described also by Ginsburg and Isbell). A. Kucia proved in [1973], that under the Generalized Continuum Hypothesis, all uniform covers of cardinalities less than a given cardinal form a base of a uniformity. G. Vidossich [1969] proved the same statement in ZFC for uniformities having a base of σ-point-finite covers. Finally, J. Pelant [1975a], [1977a], showed that there is a model of ZFC and a uniform space such that its uniform covers of cardinalities at most ω_1 does not form a base for a uniformity. He also showed in the same issue, [1975b], that there exists a complete space (with covers composed of non-measurable sets) such that the separable modification is not complete (this cannot happen if the space has a base of point-finite covers as shown by G. Reynolds and M.D. Rice [1978] – this paper contains also interesting results on commuting completions and other reflections). That solved a problem of Ginsburg and Isbell asking whether the separable modification commutes with completion (clearly, the totally bounded modification does not commute with completion). We should add that Ginsburg and Isbell investigated also finite-dimensional reflections and star-finite reflections.

Pelant's examples also exhibit (in ZFC) a uniform space not having a base of σ-point-finite covers (thus not uniformly locally finite covers, or σ-uniformly discrete covers), or by Reynolds and Rice [1978] a uniform space not having a base composed of point-finite coverings. This was a problem posed by A.H. Stone [1960]. He was interested in uniform spaces having such bases since the spaces have then nice properties. Such an example was also constructed by Ščepin in the same year, [1975]. The difference is that Ščepin's example is rather large, $l_\infty(\beth_\omega)$, Pelant's example is $l_\infty(2^{\omega_0})$ (in [1976], Pelant improved it for $l_\infty(\omega_1)$ – the new result, for $l_\infty(\omega)$, has not yet been published by him).

Unlike in topological spaces, uniform properties are global but in some cases one may deal with locally defined properties. For instance the question when a function uniformly continuous on every member of a uniform cover (i.e. uniformly locally uniformly continuous) is uniformly continuous everywhere leads to the so called locally fine spaces examined by Ginsburg and Isbell [1959] and by Isbell [1964]: they are the uniform spaces where every cover being uniformly locally uniform, is uniform. Locally fine spaces form a coreflective subcategory (the coreflection can be obtained by a transfinite process adding uniformly locally uniform covers – the so called derivatives) containing all subspaces of fine spaces. It was again J. Pelant who solved Ginsburg and Isbell's problem whether, conversely, every locally fine space is a subspace of a fine space; in [1987] he showed that the answer is yes, and also that the derivatives are not always uniformities, [1975c], [1977b].

Another nice result of Isbell's was a characterization of supercompleteness. A uniform space is said to be supercomplete if its hyperspace (endowed with its Hausdorff uniformity) is complete. F. Hausdorff defined his metric D on the set Y of bounded closed nonvoid subsets of a metric space (X,d) in his book [1914, p. 124]. H. Hahn proved in [1932] that *if* (X,d) *is complete then* (Y,D) *is complete.* For nonmetric situations, the same result was proved by Bourbaki [1940] for compact X (then Y is the hyperspace endowed with the Hausdorff uniformity). A.P. Robertson and W. Robertson [1961] generalized Bourbaki's result for a noncompact uniform space X, when one takes for Y the set of all nonvoid compact subsets of X (that one cannot take all the closed subsets was shown by J.L. Kelley [1958]. J.R. Isbell [1962] proved that *a uniform space is supercomplete iff it is paracompact and its locally fine modification is fine.*

The investigation of uniformly continuous mappings brought several interesting results. We should start with several papers by J.E. Fenstad published in 1963 in Math. Zeitschrift; probably the most important of his series is the first one [1963] concerning a Stone–Weierstrass type theorem for uniform spaces (since the set of uniformly continuous real-valued functions on a given uniform space is not an algebra, he deals with lattice–groups of uniformly continuous functions and his assumptions for such a lattice–group to be dense in the space of all uniformly continuous functions are rather complicated).

It is not difficult to show that *every uniformly continuous mapping into a metric space, defined on a subspace X of a product of uniform spaces, depends on countably many coordinates.* This was noticed by Vidossich [1970] (Miščenko [1966] for X to be the whole product). For proximity spaces and proximally continuous mappings, a similar theorem holds in a weaker form only (Hušek [1975]): *A proximally continuous mapping, defined on a subspace of a product of κ many uniform spaces, into a uniform space having a uniform base with cardinal less than cof κ, depends on less than κ coordinates*; in case the domain of the mapping is the whole product, the result is true for products of more than κ many spaces. The motivation for proving such factorization theorems was an investigation of products of proximally fine spaces (again a problem of Isbell [1965]); M. Hušek could improve the results of J. Isbell, V.Z. Poljakov, V. Kurková to get: *A product of proximally fine spaces is proximally fine iff every finite subproduct is proximally fine*, so that *every product of uniform spaces having monotone bases is proximally fine, every injective uniform space is proximally fine, a product of proximally fine spaces is proximally fine provided all but one of the spaces are totally bounded*; the last mentioned result needs another theorem: *If X, Y, Z are uniform spaces, Y is totally bounded, and $f : X \times Y \to Z$ is proximally continuous and separately uniformly continuous, then f is uniformly continuous.* A first example of two proximally fine spaces the product of which is not proximally fine, is constructed there, too (in fact, *every uniform space is a quotient of a product of a uniformly discrete space and a fine space* – see Hušek [1979]. Using other factorization results, in [1978] M. Hušek and M.D. Rice could generalize to infinite products Isbell's result that a finite product of quotient mappings between uniform spaces is quotient (Isbell [1964, p. 53]) and, using that result, construct many productive coreflective subcategories of uniform spaces (a big difference from topological spaces). One such class is the class of uniformly sequential uniform spaces under the condition that no uniformly sequential cardinal exists (true in some models of set theory).

G. Tashjian proved in 1974 (published in [1977]) that *a mapping into a metric space, defined on a product X of sets, depends on countably many coordinates iff it is $\Sigma(X)$-measurable*, where $\Sigma(X)$ is the collection of all subsets of X depending on countably many coordinates. As corollaries, she obtained many interesting results; to formulate them one needs some definitions and, so, we quote the following result only: *Any Baire mapping from a product of complete metric spaces to a metric space is of bounded class*; it is a generalization of D. Preiss' result from [1974] that every completely additive disjoint system of Baire sets in a topological space is of bounded class.

Another useful result was proved by D. Preiss and J. Vilímovský [1980]. They characterized the situation when for given real-valued functions $f \geq g$ on a uniform space X, there exists a uniformly continuous function h with $f \geq h \geq g$. The necessary and sufficient condition is *for each $\delta > 0$ there exists a uniform cover $\mathcal{V}$ of X such that for all $n \in \omega, r, s \in \mathbb{R}$ with $s - r > (n+1)\delta$, the sets $f^{-1}((-\infty, s]), g^{-1}([s, \infty))$ are* $\mathrm{Star}^n\mathcal{V}$*-far*. For bounded functions f, g one may put $n = 1$ in the condition. Extension theorems for uniformly continuous functions, Katětov, Michael and other "in-between" theorems, can be obtained then as corollaries.

We shall finish this subsection by going back to the categorical examination of uniform spaces. At the beginning of the 70's, it was mainly Z. Frolík, A.W. Hager, M. Hušek, and M.D. Rice who started a new investigation of uniform spaces from a categorical point of view. The motivation came from the investigation of algebras of functions (Hager) and descriptive set theory (Frolík, Rice). J. Vilímovský [1973] and M. Hušek [1976] examined a categorical background of that approach. Hager's ideas are described in his papers [1972], [1974]; he defines metric-fine spaces as such uniform spaces X that every uniformly continuous mappings on X into a metric space M is uniformly continuous also into the fine uniformity on M. Z. Frolík later characterized the metric-fine spaces as coz_--spaces, i.e. *the class of metric-fine spaces is the coreflective class of uniform spaces determined by the supremum of all functors on uniform spaces that are finer than identity and preserve the families of cozero sets* (see [1975]); another of Frolík's characterization of metric-fine spaces X is that *every l_∞-partition of unity on X is l_1-uniformly continuous*. These approaches are related to methods of descriptive set theory and we cannot go into details here; we remark that one can find about 14 papers by Frolík in four issues of *Seminar Uniform Spaces* published in 1974–1977.

1.5. PROXIMITY SPACES

> "My name is Alice, but —" "It's a stupid name enough!" Humpty Dumpty interrupted impatiently. "What does it mean?" "Must a name mean something?" Alice asked doubtfully. "Of course it must," Humpty Dumpty said with a short laugh.
>
> **Lewis Carroll**

Although the theory of proximity spaces may be considered as a part of the theory of uniform spaces, it has its own features and generalizations. For that reason we shall devote this subsection to the development of proximity spaces.

Probably the first occasion when a structure on a set was mentioned that is defined as a relation between subsets of the set, was F. Riesz' lecture during the mathematical congress in Rome in 1908, that was based on his paper published in Hungarian and German in [1906]. He suggested to study two kinds of abstract structures. As the title of his lecture states, the basic notion is continuity. To define such a notion, one must have a concept of limit or accumulation points (Grenzelement, Verdichtungstelle). He calls "mathematisches Kontinuum" what now may be called a generalized topological space, namely a set X for which the concept of accumulation points of subsets is defined satisfying several axioms.

An important example is given showing that Fréchet's "countable" approach does not suffice. Every infinite subset of a space has an accumulation point iff every decreasing countable system of subsets has a common accumulation point; if any ordered decreasing system of subsets has a common accumulation point, then every infinite set has an accumulation point, but not conversely. Today, we would say that countable compactness does not implies compactness; in fact, this was probably the first occurence of general compactness).

Then F. Riesz notes that the spaces $\mathbb{R} \setminus \{0\}, \mathbb{R} \setminus [0, 1]$ are homeomorphic, but from a metric point of view the first space is connected, the other is not. To avoid such situations (another one concerns Jordan regions) one must introduce another kind of structure, namely chaining (Verkettung): for every pair of subsets of X we know whether it is chained or unchained (in agreement with later notation we shall denote that relation by ApB or $A\text{non}pB$). Three axioms corresponding to those for accumulation points must be satisfied for subsets and points of X:

- if ApB, and C contains one of the sets A, B and D contains the remaining set, then CpD,
- if $Ap(B \cup C)$ then either ApB or ApC,
- if $x \neq y$ then $\{x\}\text{non}p\{y\}$.

A basic example is that of metric: ApB if the distance of A, B is zero. Every chaining determines accumulation points: $x \in A'$ if $\{x\}pA$. The fourth axiom for chaining connected with the underlying topology is the following:

if $A' \cap B' \neq \emptyset$ then ApB.

A generalized topological space may have more than one chaining structure (the example above) and one of them is a natural one (loseste in German):

ApB if either one of the sets contains an accumulation point of the other or if they have a common accumulation point.

The very interesting end of the lecture describes how by adding some points to the space with chaining one gets a "loseste" structure containing the original one as a subspace. F. Riesz says that the importance of such procedure lies in the fact that *properties of the original space with chaining can be described by topological concepts of the larger space*. He adds to the original space X *ideal points* that are systems $\mathcal{S}$ of subsets of X satisfying:

- if $A \in \mathcal{S}$ and $A \subset B$ then $B \in \mathcal{S}$,
- if $(A \cup B) \in \mathcal{S}$ then either $A \in \mathcal{S}$ or $B \in \mathcal{S}$,
- if $A, B \in \mathcal{S}$ then ApB,
- $\mathcal{S}$ is maximal with respect to previous properties,
- $\mathcal{S}$ has no common accumulation point.

The systems just described are now sometimes called clusters. In the case ApB iff $A \cap B \neq \emptyset$ one gets free ultrafilters. Riesz's procedure is very close to what Alexandroff, Freudenthal and Samuel did 30 or 40 years later (see Section 1.3).

So, F. Riesz anticipated what was studied in detail much later. He suggested in his lecture that mathematicians should study such abstract approaches. Unlike his other suggestions, this one was not taken up and almost forgotten. It was V.A. Efremovič who defined proximity spaces in his lecture *Geometry of infinite proximity* at a conference in Moscow in 1936. He waited more than 10 years to come back to that idea. But then he and his colleagues from Moscow (J.M. Smirnov, N.S. Ramm, A.S. Švarc) developed most of the basic results about those structures.

In [1921] Fréchet studied a relation "enchaînés" satisfying Riesz' axioms; he uses that for treating connectivity, continua, composants. Similarly, A.D. Wallace [1941], [1942], used a general structure on X defined by means of a "separation" relation on subsets of X (he called the structured space *s–space*). His aim was to describe in a general way a connectivity. He wanted to know which sets are separated and which are not; that was a primary concept from which he could derive a topology and connectivity. His separation is symmetric, disjunctive, hereditary, additive, the void set is separated from every set and ($A|B$ means that A, B are separated):

- if $x \neq y$ then $\{x\}|\{y\}$;
- if $\{x\}|A$ then $\{x\}|\{y : \text{not } \{y\}|A\}$;
- if for each $x \in A$ and $y \in B$ we have $\{x\}|B$ and $\{y\}|A$ then $A|B$.

The last but one axiom was needed to get a topology (closure of a set must be closed). The last axiom was sufficient for Wallace' purpose; it shows that his relation is, as was Fréchet's, determined by closures. So this was not the right notion as we know it now, but he could define some basic notions like, what we call now, proximal neighborhoods of sets, and rewrite the axioms for separation as axioms for proximal neighborhoods of sets. He also defines a kind of proximally continuous mapping f (the term he used was *s–continuity*): $A|B$ iff $f^{-1}(A)|f^{-1}(B)$; that coincides with the present proximal continuity for the case considered by Wallace, namely for compact metric spaces, where it means continuity. A similar approach was independently carried out by S.B. Krishna-Murti [1940] and P. Szymanski [1941]. A category theoretic study of Wallace's separation spaces has been recently carried out by Nakagawa [1976].

A kind of separation properties between open and closed sets was considered also by H. Freudenthal [1942]. So, some topology must be given and then a relation of proximal neighborhoods $G \Subset H$ for open sets is given satisfying the usual axioms (described later by Efremovič) except, of course, symmetry (a weaker axiom is here: $G \Subset H$ implies $X \setminus \bar{H} \Subset X \setminus \bar{G}$). Freudenthal says that $G \Subset H$ means "G und $X \setminus \bar{H}$ besitzen einen Abstand". Taking maximal filters of open sets of X having the property that each of its member is a proximal neighborhood of another member, he gets a compactification of X. The Freudenthal procedure generalizes P.S. Alexandroff's approach of constructing the Wallman and Čech–Stone compactifications.

We shall now briefly go through some papers of the Moscow proximity school from 1951–1954. Many of the results proved there appeared later again, due to the fact that some people could not read Russian. We shall not mention those repeated results although they were probably proved independently.

The motivation of Efremovič was again geometrical. He studied manifolds and found out that, e.g. a paraboloid, Euclidean plane, Lobačevsky plane are homeomorphic but no bijective map preserves "infinitesimality" of subsets, [1949]. In [1951a], [1951b], he introduced the axioms for proximity spaces almost exactly as they are used now; he suggested to call the new structures plesiospaces, from the Greek word $\pi\lambda\eta\sigma\iota o\nu$ – close, proximal. Then he described topologies induced by proximities (completely regular topologies) and proved some basic results: *If two sets are far in a proximity space X, then they are functionally separated by a proximally continuous function* (the method is the one used by Urysohn in the proof of his famous lemma; as a consequence, it is stated that the same proof can be used to show that topological groups are completely regular); *a mapping between metric spaces is uniformly continuous iff it is proximally continuous* (see also [1952]).

The last mentioned result implies that *every metric uniformity is the finest one inducing its proximity* – such a uniformity is called proximally fine. It was reproved by Smirnov, Ramm and Švarc. The last named authors used for their proof a countable system of neighborhoods of a diagonal instead of a metric, which allowed them to generalize the result: *Every uniformity having a monotone base is proximally fine*; they could deduce from their procedure that *Every proximity generated by linear adjacent nets is induced by a proximally fine uniformity* – the proof used is the same as that used presently. The paper contains also a characterization of spaces having a metrizable fine proximity (equivalently, a metrizable fine uniformity): the subset of nonisolated points is compact metrizable. Another result of Ramm and Švarc is the following: *A proximity space X is induced by a unique uniformity iff there is no infinite proximally discrete subset.* Other parts of their paper reprove some results of Smirnov without using compactifications.

J.M. Smirnov, [1952a], [1952b], first proved that (in modern terminology) there is a categorical bijection between all proximities on a given completely regular space X and all compactifications of X. For constructing compactifications he uses Alexandroff's and Freudenthal's maximal centered regular collections (ends). In fact, in proving the extendibility of proximally continuous mappings onto compactifications, Smirnov proves independently the known Taimanov result on extendibility of continuous maps that was published in the same issue. From those results he deduced that *every separated proximity space can be proximally embedded into a Tychonov cube*, that *every proximity is induced by a totally bounded uniformity and this uniformity is the coarsest one inducing the proximity.* A description of completely regular spaces X having a unique proximity is given (two closed noncompact subsets of X cannot be functionally separated) and a negative answer is provided to a question of P.S. Alexandroff, whether only compact spaces have unique proximities: the space of countable ordinals. Using Urysohn's extension theorem, Smirnov proves that *every proximally continuous and bounded real-valued function on a subspace of a proximity space can be extended to a proximally continuous function on the whole space* (originally it was not noticed that this result is equivalent to Katětov's result on extendability of bounded uniformly continuous maps). He then considers Ramm's idea to use proximal covers (now we would say: uniform covers of uniformities inducing the proximity). He shows that proximal covers are directed with respect to refinement iff the proximity is induced by a finest uniformity (a proximally fine uniformity) and reproves Ramm's result that metric uniformities are proximally fine (he shows that every proximal cover has a Lebesgue number). An open problem is explicitly stated, whether every proximity is induced by a proximally fine uniformity.

The first example answering this problem in the negative was given by M. Katětov (announcement in [1957], details in [1959]). He noticed that a sum of two uniformly continuous functions is uniformly continuous, while there is a proximity space where a sum of two proximally continuous functions is not proximally continuous (the proximity space is $\mathbb{N} \times \mathbb{N}$ endowed with the proximity generated by the functions of the form

$$f(x,y) = \sum_{i=1}^{k} g_i(x)h_i(y)\,,$$

where $k \in \mathbb{N}, g_i, h_i$ are bounded functions on $\mathbb{N}$). An easier approach for constructing such an example was found by J. Isbell [1964, p. 34], where he showed that if X is a proximity space induced by a totally bounded uniform space Y and by a non-totally bounded uniform space Z, then both $Y \times Z$, $Z \times Y$ induce the proximity on $X \times X$ but the coarsest uniformity finer than both $Y \times Z$, $Z \times Y$ does not induce the proximity on $X \times X$ – the proof is very easy if one uses neighborhoods of the diagonal (they are proximal neighborhoods of the diagonal). We should remark that Katětov's result was reproved independently by several other authors, e.g. by C.H. Dowker [1962].

In his next papers [1953a], [1953b], [1954], [1955], J.M. Smirnov considered proximal completeness, i.e. completeness with respect to all proximal covers (thus, it coincides with the completeness of the proximally fine uniformity if it exists). For instance, he defines a proximity space to be totally bounded if its proximal completion is compact (i.e. the proximity is induced by a unique uniformity), which holds iff every real-valued function on the space preserving Cauchy filters (with respect to all proximal covers) is bounded – it does not suffice to use proximally continuous functions. In [1955], Smirnov suggests to use generalized uniformities in the study of proximity spaces in order to get a proximally fine uniform structure for any uniformity. That was done, e.g., by E.M. Alfsen and O. Njåstad [1963].

2. Generalized Uniform Structures

> "Topologists who study completely regular spaces often accept uniform spaces as reasonable entities since the completely regular spaces are precisely those topological spaces which arise as the underlying space of some uniform space. Having thus admitted uniform spaces into the universe of discourse, it is a disappointment to learn that the set of all open covers of a completely regular space may fail to be a base for any uniform structure: the open covers do form a base if and only if the space is paracompact. One is therefore forced either to avoid considering the set of all open covers as any acceptable type of structure or to admit structures which are more general than uniformities."
>
> **Bentley, Herrlich, Ori [1989]**

We begin with a broad view of the enormously diversified field of those uniform-type structures which are more general than the uniform structures "proper" treated in Section 1. We then narrow our focus to symmetric structures; these have received considerable attention by topologists during the period since about 1960, perhaps because their relationship to topological spaces is tighter than in the nonsymmetric case. Following this thread, which begins mainly with Katětov's merotopic spaces in 1963, we look briefly at Cauchy spaces and then turn our main attention to nearness spaces; it is nearness spaces which have the most perfect relationship with (symmetric) topological spaces. We end with a rather lengthy outline of the vast literature in this area.

2.1. PREAMBLE

> "Den Gegenstand der allgemeinen Topologie bilden schon länst nicht nur topologische Räume, sondern vielmehr verschiedene Typen von Räumen, ihre Zusammenhänge und mannigfaltige Prozesse der Erzeugung von Räumen."
>
> **M. Katětov [1963]**

A prominent feature of modern mathematics has been the exploration of general axiom systems, the motivating factor being a desire to find convenient settings for the various kinds of mathematical systems. While several specialities within mathematics have been represented by these general investigations, it is a fact that general topologists have carried out the boldest and most varied experimentation in this search for interesting and suitable axiom systems. Moreover, this investigation of axiomatics has been a feature of general topology since its genesis around the beginning of the twentieth century. After the concept of a topological space reached a reasonably settled stage with the axioms of Hausdorff and Kuratowski, a few decades passed during which most general topologists concerned themselves with exploring relationships between various special kinds of topological spaces and with constructions involving these. Even though topological spaces held center stage, during the 30's and 40's uniform spaces arose as a setting where uniformly continuous maps could be defined (see Section 1 of this article). Besides the concept of uniform spaces a vast number of uniform-like structures got created by suitable axiomatizations of a variety of **basic concepts**, such as

- *Entourages* (= uniform neighborhoods of the diagonal of $X \times X$); see Appert [1946], Appert and Fan [1951], Jaffard [1952], Aotani [1953], Tamari [1954], Mamuzić [1954], [1956], Krishnan [1955a], [1955b], and Hušek [1964], [1965] – to mention just those who started such investigations.
- *Uniform covers*; see mainly Nachbin [1948], Morita [1951], and Suzuki [1951].

– *Uniform systems of neighborhoods of points*; see Cohen [1937], [1939] and Inagaki [1943].

Whereas most of the above mentioned papers consisted mainly in sorting out appropriate axioms and providing straightforward consequences, deeper results started to emerge in the late fifties. Let us mention, without details, the work of E.F. Steiner, A.K. Steiner, Thampuran, Leader, Seiber, Pervin, Mock, Matzinger, Hušek, Lodato, Goetz, Haddad, Ålfsen, Njåstad, Švedov, and Harris. A particularly fruitful branch, obtained by dropping the symmetry axiom in the approach via entourages, led to *quasi-uniform* spaces; see the book by Fletcher and Lindgren [1982] and the recent surveys by Künzi [1993], [1995]. The alternative, still using entourages but dropping the triangle axiom while retaining the symmetry axiom, was named semi-uniform spaces and was extensively developed in the book of E. Čech [1966]. Other concepts such as *set-valued maps* (Konishi [1952]) and *maps of subsets* (Gómez [1948], [1953]) got experimented with but didn't get pursued. As opposed to the last mentioned, the creation of the following two concepts turned out to be highly fruitful and to invigorate the theory of generalized uniform structures considerably:

– *Micromeric collections*; see Katětov [1963] and [1965], based on a characterization of uniform spaces by Sandberg [1960] and Mordkovič [1965].
– *Nearness collections*; see Herrlich [1974a], based on a generalization of proximity relations to finite collections by Ivanova and Ivanov [1959] and Terwilliger [1965].

In fact, the concepts of nearness collections, micromeric collections, and uniform covers turned out to be just different facets of the same fundamental idea – an idea most suitable for the development of a theory of *symmetric generalized uniform structures*. The history of this theory will be outlined in some more detail in the following sections. Before we come to this let us end this preamble by mentioning two related developments whose detailed history cannot be given here.

First, the generalizations of *uniformities, topologies,* and *metrics* naturally led to concepts which simultaneously generalize several of the above three concepts and thus provide a unifying view. Besides *merotopic spaces*, *nearness spaces* and *Cauchy spaces*, which will be discussed in the following sections, let us mention Császár's *syntopogeneous spaces* [1963], (see also Deák's [1993]), A.K. Steiner and E.F. Steiner's *binding spaces* [1972], Doitchinov's *supertopological spaces* [1964], Leseberg's *nearform spaces* [1989], Deák's *bimerotopic spaces* [1990], and Lowen's *approach spaces* [1989]. See also the article in this handbook by Lowen and Lowen-Colebunders, *Topological categories and supercategories of Top*.

Secondly, the plethora of structures created by topologists would have lead to obfuscation rather than enlightenment were it not for the unifying touch of category theory. The latter provides a suitable language that helps to compare, to order, to systematize. In particular:

(1) Katětov [1963] started to develop a *general theory of topological structures* that allows not only to recognize the many pleasant features that all topological categories share but also to compare the merits and deficiencies of various topological categories. This theory was developed by the Prague school (Frolík, Hedrlín, Hušek, Pultr, Trnková), by Ehresmann [1966], Antoine [1966], Brümmer [1971], [1984], Wyler [1971], Hoffmann [1972], Bentley [1973], Herrlich [1974c], Nel [1975], Schwarz [1983], Herrlich's *Universal topology* [1984], and others. For more recent studies see, e.g., Bentley, Herrlich, and Lowen's *Improving constructions in topology* [1991] or Herrlich, Lowen-Colebunders and Schwarz' *Improving Top: PrTop and PsTop* [1991]. Again, reference is made to the article in this handbook by Lowen and Lowen-Colebunders, *Topological categories and supercategories of Top*.

(2) Herrlich's [1968] lecture notes *Topologische Reflexionen und Coreflexionen* provided a stimulus to many topologists for a systematic investigation of subcategories of **Top** (and related categories, such as **Haus, Unif**, etc.). A history of this theory and its topological roots is provided by Herrlich and Strecker's *Topological reflections and coreflections* [1997] in this handbook; see also Herrlich and Hušek's *Categorical topology* [1992]. A recent list of problems is provided by Herrlich and Hušek's *Some open categorical problems in Top* [1993].

2.2. MEROTOPIC SPACES AND FILTER-MEROTOPIC SPACES

> "Es erscheint daher intuitiv angemessen, den Begriff einer allgemeinen Stetigkeitsstruktur auf den Begriff von beliebig kleinen Objekten zurück zuführen. In diesem Sinne kann also die allgemeine Topologie als eine allgemeine Lehre von Unendlich-kleinen aufgefasst werden."
>
> **M. Katětov [1963]**

A filter in a uniform space is called a *Cauchy filter* provided it contains arbitrarily small members. Sandberg [1960] and Mordkovič [1965] realized that, by extending this concept to arbitrary collections $\mathcal{A}$ of subsets of a uniform space X as follows

$\mathcal{A}$ *contains arbitrarily small members* iff for every uniform cover $\mathcal{U}$ of X there exist $A \in \mathcal{A}$ and $U \in \mathcal{U}$ with $A \subset U$, they were able to recover the given uniformity from the set of all collections that contain arbitrarily small members. This observation enabled Sandberg to present descriptions of uniform spaces (and Mordkovič to present descriptions of proximity spaces) by axiomatizing the concept of collections of subsets that contain arbitrarily small members.

Katětov [1963], [1965] took a bold step forward in defining *merotopic spaces*[7] by prescribing a weakened form of axioms for collections containing arbitrarily small members, called *micromeric*, as follows:

(Mer0) $\emptyset$ is not micromeric.
(Mer1) If $\mathcal{A} \subset PX$ contains a member A with card $A \leq 1$, then $\mathcal{A}$ is micromeric.
(Mer2) If $\mathcal{A}$ corefines $\mathcal{B}$ (i.e. for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ with $B \subset A$) and $\mathcal{A}$ is micromeric, then so is $\mathcal{B}$.
(Mer3) If $\mathcal{A} \cup \mathcal{B}$ is micromeric, then $\mathcal{A}$ or $\mathcal{B}$ is micromeric.

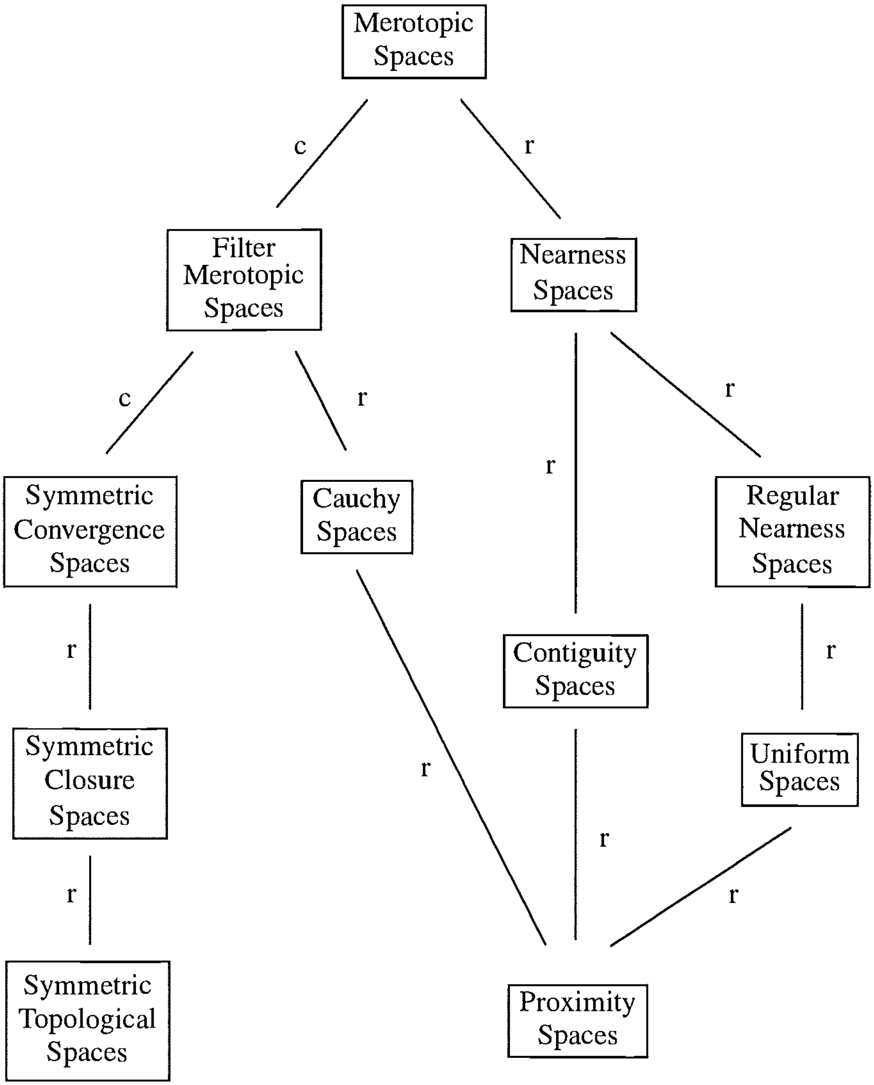
A function between merotopic spaces is *uniformly continuous* provided that it preserves micromeric collections; thus arises the category **Mer** of merotopic spaces and uniformly continuous maps. The main reason that Katětov introduced merotopic spaces was the observation that **Mer** contains a large cartesian closed topological subcategory, the category of filter merotopic spaces, i.e. those merotopic spaces with a base consisting of filters. The category **Mer** turned out in the years to come to be the most natural *symmetric* generalization of the category **Unif** of uniform spaces, forming an umbrella for various useful categories in symmetric topology[8] as the diagram overleaf indicates. (In the diagram r, respectively c, stands for a bireflective, respectively bicoreflective, embedding.)

Some of the larger categories are obtained from some of the smaller ones by canonical constructions, e.g.:

(1) Merotopic spaces are precisely the quotients of the zerodimensional uniform spaces in **Mer** (Katětov [1976]).
(2) Filter merotopic spaces are precisely the subspaces of symmetric convergence spaces in **Mer** (Robertson [1975]).
(3) Symmetric convergence spaces are precisely the quotients of symmetric topological spaces in **Mer** (Robertson [1975]).

[7]Isbell [1964] encountered the equivalent concept in terms of uniform covers in his study of locally fine uniform spaces, but didn't follow up on its potentials. Even earlier, Dowker [1952] used these structures in establishing the isomorphism of the Čech and Vietoris homology and cohomology groups.

[8]Symmetric topological spaces have also been called "weakly regular spaces" (Šanin [1943]) and "R_0-spaces" (Davis [1961]); they are those topological spaces which satisfy $x \in \mathrm{cl}\{y\}$ iff $y \in \mathrm{cl}\{x\}$.



Moreover, the category **Fil** of filter-merotopic spaces is particularly nice from a categorical point of view:

(0) **Fil** is a topological construct (Katětov [1965]).
(1) **Fil** is cartesian closed (Katětov [1965]).
(2) **Fil** is a quasitopos hence a topological universe (Bentley, Herrlich, Robertson [1976]).

(3) In **Fil** arbitrary products of quotient maps are quotient maps (Bentley, Herrlich, Robertson [1976]).
(4) In **Fil** finite products commute with direct limits (Bentley, Herrlich, Robertson [1976]).

For more recent results and historical remarks on subcategories of **Fil** see in particular Bentley, Herrlich, Lowen-Colebunders [1990].

2.3. CAUCHY SPACES

"Pudding and pie,"
Said Jane, "O, my!"
"Which would you rather?"
Said her father.
"Both," cried Jane,
Quite bold and plain.

Anonymous (ca. 1907)

"The fun with **Chy** [the category of Cauchy spaces] was greater, once it was embedded in **Mer** [the category of merotopic spaces]!"

Eva Lowen-Colebunders

In order to be able to study uniform continuity and completeness in spaces more general than completely regular spaces, Cook and Fischer [1967] introduced *uniform convergence structures*. They proved that every compact separated Choquet convergence space is induced by a uniform convergence structure.

They also defined a uniform convergence structure on the set of all uniformly continuous maps from one uniform convergence space to another. Later Wyler [1974] modified Cook and Fischer's axioms slightly and Lee [1976] showed that the category of uniform convergence spaces, as modified by Wyler, is cartesian closed.

Keller [1968] found necessary and sufficient conditions on a collection of filters on a set X to be the collection of all Cauchy filters of some uniform convergence space with underlying set X, thus creating the category **Chy** of Cauchy spaces. Bentley, Herrlich, and Lowen-Colebunders [1987] showed that **Chy** is cartesian closed. That **Chy** is (concretely isomorphic to) a bireflective subcategory of Katětov's category **Fil** was established by the same authors somewhat later [1990]. Thus, Cauchy structures can be viewed quite naturally either as generalized uniform structures or as generalized convergence structures – in other words: they are both. More on Cauchy spaces and their history can be found in Lowen-Colebunders [1989].

2.4. NEARNESS SPACES

> "The idea of nearness is one of those rare items in Mathematics — a concept which is simultaneously intuitive and rigorous. It is so intuitive that to quote Lagrange it is possible to 'make it clear to the first person one meets on the street' ... Its simplicity and depth provide a powerful tool in research in Topology and Functional Analysis."
>
> **Som Naimpally [1976]**

As Smirnov has shown (see Section 1) there is a close connection between compact Hausdorff extensions of completely regular topological spaces and completions of proximity spaces. Attempts to obtain a similar result concerning strict T_1-spaces via a suitable weakening of the axioms for proximity spaces failed. However, Ivanova and Ivanov [1959] and Terwilliger [1965] succeeded by replacing the concept of *nearness of a pair of sets* by that of *nearness of a finite collection of sets*, thus creating the concept of contiguity spaces. Going one step further, Herrlich [1974a] introduced *nearness spaces* by axiomatizing the concept of *nearness of arbitrary collections of subsets* of X as follows:

(N1) If $\cap\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is near,
(N2) If $\mathcal{A}$ corefines $\mathcal{B}$ and $\mathcal{B}$ is near, then so is $\mathcal{A}$,
(N3) If $\{ A \cup B \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B} \}$ is near, then so is $\mathcal{A}$ or $\mathcal{B}$,
(N4) If $\{ \mathrm{cl}\, A \mid A \in \mathcal{A} \}$ is near, then so is $\mathcal{A}$, where $x \in \mathrm{cl}\, A$ iff $\{\{x\}, A\}$ is near,
(N5) $\emptyset$ is near, $\mathrm{P}X$ is not near.

He defined the concept of completeness for nearness spaces, constructed completions, and showed that every strict T_1-extension of a topological space can be obtained as the completion of a suitable nearness space. Whereas a merotopic space has an underlying symmetric closure space, a nearness space has an underlying symmetric topological space. Moreover, the categories $\mathbf{Top}_{\mathrm{S}}$ of symmetric topological spaces and continuous maps and **Unif** of uniform spaces and uniformly continuous maps can both be regarded as full (!) bicoreflective, respectively bireflective, subcategories of the category **Near** of nearness spaces and nearness preserving maps, their intersection being – in topological terms – precisely the paracompact spaces. That **Near** itself can be regarded as a bireflective full subcategory of **Mer** and that for merotopic spaces each of the concepts

- micromeric collections
- nearness collections
- uniform covers

can be recovered from either of the others by means of the following equivalences (where X denotes the underlying set)

$$\begin{aligned}
\mathcal{A} \text{ is near} &\iff \{\, B \subset X \mid B \text{ meets every } A \in \mathcal{A}\,\} \text{ is micromeric}\\
\mathcal{A} \text{ is micromeric} &\iff \{\, B \subset X \mid B \text{ meets every } A \in \mathcal{A}\,\} \text{ is near}\\
\mathcal{A} \text{ is a uniform cover} &\iff \{\, X \setminus A \mid A \in \mathcal{A}\} \text{ is not near}\\
\mathcal{A} \text{ is near} &\iff \{\, X \setminus A \mid A \in \mathcal{A}\} \text{ is not a uniform cover}
\end{aligned}$$

is shown in Herrlich [1974d]. Via these "translations" the following hold

$$\mathbf{Unif} \subset \mathbf{Near} \subset \mathbf{Mer}$$

each of the categories being a full bireflective subcategory of the following ones. Moreover, regular nearness spaces fit in as follows:

$$\mathbf{Unif} \subset \mathbf{Reg} \subset \mathbf{Near}$$

In the uniform covers version they had been introduced much earlier by Morita [1951] as regular T-uniform spaces. Morita demonstrated that every regular extension of a topological space can be obtained as a completion of a suitable regular T-uniform space. Morita's regular T-uniform spaces were studied by Steiner and Steiner [1973a], [1973b] under the name "semi-uniform spaces", and in a somewhat modified form by Harris [1971]. Morita studied these spaces again in 1989. For a recent study comparing Morita's extensions with nearness completions, see Bentley and Herrlich [preprint].

Since not only **Unif** but also the category $\mathbf{Top}_\mathrm{S}$ of symmetric topological spaces can be considered as a full subcategory of **Near**, nearness spaces can be regarded not only as a generalization of uniform spaces but also of (symmetric) topological spaces. This observation has led to some surprising consequences concerning the stability of topological properties under fundamental categorical constructions. Whereas products and subspaces in **Unif** are formed exactly as in **Near** (since **Unif** is bireflective in **Near**), they are formed differently in $\mathbf{Top}_\mathrm{S}$: namely by forming them in **Near** as a first step and then pushing the resulting objects back into $\mathbf{Top}_\mathrm{S}$ (as the underlying topological spaces, i.e. via the topological coreflections). As shown by Herrlich [1976], [1977] it is the second step that destroys many desirable topological properties. E.g. the following theorems fail in $\mathbf{Top}_\mathrm{S}$ but hold in **Near**:

(1) Products of paracompact spaces are paracompact.
(2) Subspaces of paracompact spaces are paracompact.
(3) $\dim(X \times Y) \le \dim X + \dim Y$ for nonempty paracompact spaces.

Here, the terms "paracompact" and "dim" have to be understood as the canonical extensions of the topological notions to the nearness setting. In particular, the *paracompact nearness spaces* are precisely the *uniform spaces*, and the above results are trivial ((1), (2)) or well known (3). It might be worth noting that in the nearness setting the paracompact topological spaces are precisely those nearness spaces that are simultaneously topological and uniform, shortly,

$$\text{paracompact} = \text{topological} \cap \text{uniform}.$$

These ideas have been developed further in Herrlich [1977] and Bentley, Herrlich, Lowen [1991]. Herrlich [1982] surveys the results and open problems on nearness spaces up to 1981.

2.5. FURTHER DEVELOPMENTS ABOUT CATEGORIES OF MEROTOPIC SPACES

"So away they went with a hop and a bound,
And they hopped the world three times round;
And who so happy — O, who,
As the Duck and the Kangaroo?"

Edward Lear

By now there is a rich literature on merotopic, Cauchy, and nearness spaces. For comprehensive treatments see the books: Preuß [1988b], Herrlich [1988], and Lowen-Colebunders [1989]. In the following we list special developments and give associated references.

(1) Completeness and completions

(a) For uniform convergence spaces:
Wyler [1970], Reed [1971], Müller [1976], S. Gähler, W. Gähler, and Kneis [1976], Kneis [1978].

(b) For Cauchy spaces:
Gazik and Kent [1974], Kent and Richardson [1974], [1984], Frič and Kent [1978], [1979], [1982], Lowen-Colebunders [1982], [1984].

(c) For nearness spaces:
Herrlich [1974a], [1974d], Carlson [1975], [1981c], Bentley and Herrlich [1978a], [1979a], [1979b], Brunet [1986].

(2) Extensions

(a) Strict T_1-extensions of topological spaces via completions of suitable nearness structures:
Herrlich [1974a], [1974d], Bentley [1975], [1977], [1991], Naimpally and Whitfield [1975], Bentley and Herrlich [1976], Reed [1976], [1978], [1993], Ward [1978], Carlson [1979], [1980b], [1981a],

[1983a], [1983c], [1984a], [1985], [1987], [1989], Dean [1983], Dong and Wang [1988], Bentley and Hunsaker [1992]. For a recent survey, see Bentley [1992].

(b) Extensions of closure spaces via merotopies:
Chattopadhyay, Njåstad, and Thron [1983], Chattopadhyay [1988], Chattopadhyay and Guin [1994].

(c) Extensions of merotopies:
Császár and Deák [1990], [1991a], [1991b], [1992].

(d) Extensions of (uniformly) continuous maps:
Wooten [1973], Herrlich [1974b], Chattopadhyay and Njåstad [1983a], [1983b], Di-Concilio and Naimpally [1987], Hunsaker and Naimpally [1987].

(3) Properties of nearness / merotopic spaces

(a) Separation properties:
Herrlich [1974a], [1974d], Bentley and Herrlich [1976], Bentley [1977], [1991], Carlson [1978], [1994], Heldermann [1978], [1979a], [1979b], [1980], Brandenburg [1978a], [1978b], [1980], [1988], Baboolal, Bentley, Ori [1985], Bentley, Herrlich, and Ori [1989], Bentley and Lowen-Colebunders [1991].

(b) Generalized compactness notions:
Bentley and Herrlich [1976], Carlson [1984a], [1991], Bentley, Hastings, and Ori [1984].

(c) Connectedness notions:
Bentley [1976], Preuß [1982], [1989], Baboolal, Bentley, and Ori [1985], Baboolal and Ori [preprints].

(d) Dimension, homology, and cohomology:
Bentley [1974], [1982], [1983], Czarcinski [1975], Pust [1977] (and 2 unpublished preprints circa 1978), Preuß [1983a], [1983a], [1983b], [1988a], [1988b], [1991], [1992], Zacharias [1992], McKee [1994].

(e) Orderability:
Herrlich [1981], Hušek [1982].

(f) Metrizability and linearly ordered basis:
Reichel [1976], Kong [1992].

(g) Cardinal invariants, countable bases, and developable topological spaces:
Simon [1971], Steiner and Steiner [1973a], Brandenburg [1978a], [1978b], [1980], Carlson [1980a].

(h) Homogeneity:
Hicks and McKee [1991].

(i) Fixed point theorems:
Chattopadhyay and Njåstad [1986].

(4) Standard constructions in **Mer** and **Near**

(a) Subspaces, quotients, products, and sums:
Katětov [1965], [1976], Herrlich [1974a], [1974d], Robertson [1975], Bentley, Herrlich, Robertson [1976], Ward [1976], Wattel [1977].

(b) Reflective and coreflective subcategories:
Hušek [1976], Heldermann [1982], Bentley and Lowen-Colebunders [1992], Preuß [1993].

(c) Function spaces, cartesian closedness, monoidal closed structures, and stability of quotients under products and pullbacks:
Katětov [1965], Robertson [1975], Bentley, Herrlich, and Robertson [1976], Hong, Nel, and Rho [1978], Schwarz [1982], Brandenburg and Hušek [1982], Greve [1982], Rhineghost [1984], Bentley and Herrlich [1985], Činčura [1985], Herrlich, Lowen-Colebunders, and Schwarz [1991], Naimpally and Tikoo [1990].

(d) Hyperspaces:
Naimpally [1991].

(e) Endomorphism-monoids:
McKee [1989a], [1989b], Katětov [1993].

(5) Relations between various kinds of structures

(a) Comparison of constructions in **Top** and **Near** and the epireflective hull of $\mathbf{Top}_S$ in **Near**:
Herrlich [1974a], [1974d], [1976], [1977], [1983], Hastings [1975], [1981], [1982], Heldermann [1978], [1979a], [1979b], [1980], Bentley, Herrlich, and Lowen-Colebunders [1990].

(b) Merotopies/nearnesses via convergences, topologies, quasi-uniformities, proximities, etc.:
Katětov [1967], Hunsaker and Sharma [1974], Gagrat and Thron [1975], Schwarz [1979], Carlson [1983b], [1984b], Brümmer [1984], Császár [1988], Preuß [1995].

(c) Merotopies/nearnesses via lattices:
Hu [1982], [1983], Weiss [1984], [1988].

(d) Supercategories of **Mer** and/or **Near**:
Vorster [1977], Hastings [1982], Leseberg [1989], [1990].

(e) Nearness algebras:
Backhouse [1985], Gompa [1992].

(f) Nearness frames:
Frith [1987], Kim [1994], Hong and Kim [1995] (and a preprint), Dube [1995a], [1995b], Banaschewski and Pultr [1996].

(g) Motivation:
Cameron, Hocking, and Naimpally [1974], Naimpally [1974], [1976], Herrlich [1976], [1977], [1983], Bentley and Herrlich [1978b], Gauld [1978], Carlson [1980c], Bentley, Herrlich, Lowen-Colebunders [1990], Kong [1992].

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"Is that all?" Alice timidly asked. "That's all," said Humpty Dumpty. "Good bye."

Lewis Carroll[9]

[9]From *Through the Looking Glass.*

HAUSDORFF COMPACTIFICATIONS: A RETROSPECTIVE

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Contents

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Introduction: Compactness

[1]It was far from clear in the early years of general topology how to abstract the proper concepts from $\mathbb{R}$ or $\mathbb{I}$, first to metric spaces and later, to topological spaces. Today it seems that topology of the early 20th century was a virtual battleground of competing ideas, with various notions vying for attention. To quote Engelking [1989], pp. 132–133 on the topic of compactness:

> When general topology was in its infancy, defining new classes of spaces often consisted in taking a property of the closed interval $\mathbb{I}$ or of the real line $\mathbb{R}$ and considering the class of all spaces that have this property; classes of separable, compact, complete and connected spaces were defined following this pattern. At first this method was used to define some classes of metric spaces, later definitions were extended to topological spaces. Sometimes properties equivalent in the class of metric spaces, when extended to topological spaces, led to different classes of topological spaces ... and it was not immediately clear which class was the proper generalization. This happened with compactness, and for some time there was doubt whether the proper extension of the class of compact metric spaces is the class of compact spaces, the class of countable compact spaces, or the class of sequentially compact spaces ... By now, it is quite clear that it is the class of compact spaces; this class behaves best with respect to operations on topological spaces, is most often met in applications and leads to the most interesting problems.

Two major results of the late 19th – early 20th century are at the heart of our present definition of compactness: the Bolzano–Weierstraß Theorem and the Borel–Lebesgue Lemma. Cantor had defined the notions of accumulation point, closed set, open set, etc., for the real line and then for Euclidean spaces. During the 1870's the idea of uniform continuity attained (more or less) its present form through the efforts of Heine, Weierstraß, and others and the Bolzano–Weierstraß theorem had emerged as one of the major tools in dealing with limit behavior of real and complex valued functions. To summarize Engelking's discussion of the evolution of the notion of compactness in general topology: Alexandroff–Urysohn [1929] (announced in Alexandroff–Urysohn [1923], [1924]) pulled together related ideas which had been put forward by Vietoris, Riesz, Janiszewski, Kuratowski, Sierpiński, Saks, and others, provided a deep analysis of the basic ideas and ended up with an effective definition. Alexandroff [1960] has said "In those far off times when Urysohn and I constructed the theory of bicompact spaces in our memoir, the Bolzano–Weierstraß theorem seemed so much

[1]Several persons offered suggestions and advice in the preparation of this paper and they have our appreciation. We want to express our special thanks to Donald J. Hansen for his invaluable assistance.

more important than the Borel–Lebesgue lemma that we considered the definition in terms of complete accumulation points to be the basic definition of bicompactness." (There is a fascinating account of an interview with Alexandroff and Tychonov in Cameron [1985] which discusses the genesis of Alexandroff–Urysohn [1929]. We will follow Cameron's lead in transliterating the two names Александров, Тихонов: normally the Cyrillic 'в' which ends both names would be transliterated as 'v'; according to Cameron, Alexandroff preferred 'ff'. However, the papers of Tychonov which we reference all appeared in *Mathematische Annalen.* Thus when we refer to one of them we will use the transliteration which appeared there: 'Tychonoff'.)

If X is a topological space with $A \subseteq X$ then $x \in X$ is a *complete accumulation point* of A provided for each neighborhood U of x (in X) we have $|U \cap A| = |A|$. ($|K|$ is the *cardinal number* of K.) Alexandroff–Urysohn [1929] proved that the following three conditions are equivalent for a topological space R:

- (A) Every infinite set contained in R possesses at least one complete accumulation point.
- (B) Every well-ordered, infinite, decreasing sequence of closed subsets of R has at least one point belonging to all sets in the sequence. (Note: in this context, *sequence* meant a family indexed by an infinite ordinal.)
- (C) From every infinite family of open sets covering R one can extract a finite family of open sets having the same property.

Property (A) was, of course, inspired by the Bolzano–Weierstraß Theorem while the Borel–Lebesgue Lemma was the antecedent for Property (C).

Relative to the development of our understanding of the real number system Bourbaki [1989], p. 164 has said: "... by far the most important acquisition was the theorem of Borel–Lebesgue ..." The original theorem, for countable open covers of a closed, bounded interval of $\mathbb{R}$ had been given by Borel [1894]. He then placed the result in arbitrary Euclidean spaces (Borel [1903]). Curiously, the theorem is incorrectly stated here: "*Soit, dans l'espace à n dimensions, une infinité dénombrable d'ensembles fermés (c'est-à-dire tels que chacun contienne son dérivé)* $E_1, E_2, \ldots, E_n, \ldots$, *et une* <u>*ensemble quelconque E*</u> *tel que tout point de E soit intérieur à l'un des* E_i. *On peut, dès lors, choisir parmi les* E_i *un nombre limité d'ensembles tels que tout point de E soit intérieur à l'un d'eux.*" The underline is ours: no restriction is placed on the set E! Also note that the covering sets are the interiors of closed sets. Lebesgue [1904] (pp. 104–105) removed the necessity that the cover be countable.

It is interesting to note which of the properties of Alexandroff–Urysohn [1929] was used as the definition for compactness in several important and influential works which appeared in the 15 or so years following its publication:

– Tychonoff [1930]: Property (A).
– Sierpiński [1934]: Property (A).
– Alexandroff–Hopf [1935]: Property (C).
– Čech [1937]: Property (A). (States all three, with (B) modified.)
– Stone [1937]: Property (A).
– Pontrjagin [1939]: Property (C).
– Lubben [1941]: Property (B). (Uses linearly-ordered families.)

All of these were certainly aware of the equivalence of the three properties and several indicate this. The choice of a particular property as the definition for (bi)compactness merely reflected personal inclination or training. Later generations of topologists were brought up using Property (C) (with (B) modified to be its contrapositive); complete accumulation points are now the stuff of exercises. Today then we say that a topological space X is *compact* if it is Hausdorff (T_2) and every open cover of X has a finite subcover. It should be noted that the term *bicompact* has been commonly used in Eastern Europe for this concept, where the name *compact* was used for that property which in the West is called *countably compact.*[2]

It is certainly not universal to require T_2; however, in the present context nothing is lost since the theory of compactifications which we espouse is for *Tychonov spaces*, i.e. completely regular Hausdorff spaces. Spaces satisfying only the covering condition are sometimes termed *quasi-compact.*

Perhaps *the* decisive factor weighing the balance in favor of compactness (over e.g. countable compactness or sequential compactness) was the fact that it is productive for arbitrary families:

Theorem 1. *The Cartesian product $\prod_{\alpha\in A} X_\alpha$ (where each $X_\alpha \neq \emptyset$) is compact if and only if each X_α is compact.*

This remarkable result, arguably the most important theorem in general topology, was first explicitly stated in Tychonoff [1935a] although it can be deduced from the theorem Tychonoff [1930] that all cubes $\mathbb{I}^p$ are compact.

[2]The following anecdote, sent to us by Melvin Henriksen, is an interesting sidelight to this discussion: At a conference in Madison in 1955, Raymond Wilder told me [Henriksen] of an international conference in topology held in Ann Arbor in 1925. An attempt was made there to standardize topological terminology because all the leading lights of that new field were there. According to Wilder, the meeting on this subject broke up in an hour or so – one of the many bones of contention being the meaning of the word "compact".

R. Engelking has related in a private communication "They (Alexandroff–Urysohn [1929]) consider there, among various notions related to compactness, (modulo their terminology and some inaccuracies related to non-regularity of cardinals considered) $\mathfrak{m}$-INITIALLY compact spaces, i.e. such that every open cover of cardinality at most $\mathfrak{m}$ has a finite subcover and $\mathfrak{m}$-FINALLY compact spaces, i.e. such that every open cover has a subcover of cardinality less than $\mathfrak{m}$. Bicompact came from BOTH initially and finally compact."

1. Compactifications: Early Efforts

The foundations of the modern theory of Hausdorff compactifications rests on six papers (in addition to the previously discussed Alexandroff–Urysohn [1929]): Tychonoff [1930], Čech [1937], Stone [1937], Cartan [1937], Wallman [1938], and Lubben [1941]. We will discuss each of these (as well as some ancillary works) in some detail in this section.

Most references cite Carathéodory [1913] as the first to construct compactifications (of open sets in $\mathbb{R}^2$) with his theory of Ends and we have discovered nothing earlier of a general nature. Nineteenth Century mathematicians from at least Cauchy forward "knew" that a continuous, real-valued function on a closed and bounded domain attained its maximum and minimum values, long before the real number system was understood well enough (through the efforts of Cantor, Dedekind, and others) to allow a rigorous proof. Thus it was a natural problem, once a topological concept generalizing "closed and bounded" had emerged, to try to "extend" a non-compact space to a compact one. The first general method for abstract spaces was the one-point compactification of Alexandroff [1924]: For the locally compact T_2 space X let $\omega X = X \cup \{\infty\}$ where ∞ is some point not in X. Sets open in X are open in ωX as are sets of the form $\omega X \setminus K$ where K is compact in X. The resulting topological space ωX will be compact, T_2, and will contain a dense copy of X (so long as X is not compact).

Alexandroff's ωX would perhaps have remained merely a curiosity were it not for Tychonoff [1930]. Certainly this is *the* fundamental paper in compactification theory (in fact, one would be hard-pressed to identify a more significant paper in all of topology). We need a concept first considered in Urysohn [1925]: X is a *completely regular* (or *Tychonov* or $T_{3\frac{1}{2}}$) space provided it is T_1 and for each closed set $A \subseteq X$ and each $x \in X \setminus A$ there is a continuous $f : X \to \mathbb{I} = [0, 1]$ with $f(x) = 0$ and $f(y) = 1$ for each $y \in A$.

In this paper Tychonov

- defines what we know today as the *Tychonov* or the product topology for arbitrary families of copies of $\mathbb{I}$,
- proves that such a product is compact,
- shows that the spaces which can be embedded in some such cube $\mathbb{I}^\tau$ are precisely the completely regular spaces.

The theorem which we today call *the* Tychonov Theorem (that the product of compact spaces is compact) was first explicitly stated in Tychonoff [1935a]. Several authors identify Tychonoff [1935] as the proper reference. In both the 1930 paper and the 1935 paper, special cases of the theorem are stated. We believe the 1935a paper has the most unequivocal statement. In the middle of page 772 we have: *Das Produkt von bikompakten Räumen ist wieder bikompakt. Diesen Satz beweist man wörtlich so wie die Bikompaktheit des Produkts von Strecken.*[7] ("The product of bicompact spaces is again bicompact. This theorem can be proved literally as was the bicompactness of the product of intervals." Footnote 7 refers the reader to §2 of Tychonoff [1930]. In the paragraph preceding these sentences he gives an explicit statement of the product topology for an arbitrary family of arbitrary spaces. Ironically, in the 1930 paper he does not identify the product result as a *theorem*, although §2 is entitled "Beweis der Bikompaktheit von R_τ".)

Did Tychonov actually construct the "Stone–Čech" compactification of an arbitrary completely regular space as some have said? (For example, see W. Rudin [1991], p. 411.) Literally, he did not. He proved that a completely regular space can be embedded in a cube *of the same weight*. Applied to $\mathbb{N}$, for example, this would embed $\mathbb{N}$ into the second countable cube $\mathbb{I}^{\aleph_0}$. Of course, $\beta\mathbb{N}$ is not second countable. Tychonov restricted the number of factors in the product so as to obtain a cube of the same weight. By using exactly the same technique but employing the "maximal" number of factors Čech [1937] obtained the compactification which has the maximal extension property. Cameron [1985] has a substantial discussion of the parts that Tychonov and Čech each contributed to the discovery of the "Stone–Čech" compactification (as well as to the "Tychonov" product theorem: as we have seen Tychonoff [1935a] does contain an explicit definition of the product topology as well as a statement of the general product theorem; Čech [1937] contains perhaps the first published proof of the latter).

Cameron [1985] and others have raised the question of the first reference to the expression "Stone–Čech compactification". Hewitt [1943] contains the following: "A celebrated theorem, due originally to Tychonov and since extended by Stone and Cech . . . states that every completely regular space can be imbedded as a dense subset in a suitable bicompact Hausdorff space." As Cameron indicates, Hewitt [1948] contains the expression "theorems of Stone and Čech" at least twice. The review of Hewitt's paper Dieudonné [1949] refers to "la compactification bien connue de Stone–Čech . . . " This is the earliest reference we have been able to find. It is instructive and interesting (given the Dieudonné connection) to compare two different editions of Bourbaki's *Topologie Général, Chapitre IX, Utilisation des Nombres Réels en Topologie Général*. The first edition, published in 1948, refers to "*la plus grande extension compacte*" (p. 14, Ex. 7) while the second edition ("Revue et Augmentée"), published in 1958, in the same exercise (now on p. 22) has "le *compactifié de Stone–Čech* . . . "

Cameron also wonders why the expression "Stone–Čech" rather than the usually preferred alphabetical "Čech–Stone". It is probable that in the United States the expression derived from Hewitt's usage. Why would Hewitt express it this way? It seems to us there are two plausible reasons: he may have been honoring his mentor (Stone directed Hewitt's Ph.D. thesis at Harvard), or he may simply have been giving historical precedent. Čech's paper was received by the *Annals* on Feb. 3, 1937. Unfortunately, the *Transactions* does not indicate when Stone's paper was received but it must have been well before Feb. 3, given the monumental refereeing task it represented and the fact that it appeared in the May, 1937 issue.

Another trivia question might concern Čech's initial usage of the now standard notation "β". In his perceptive and amusing review of Weir [1975], Comfort [1976] addresses this issue, at the same time answering the parallel question of Hewitt's use of "v" for the realcompact analogue. He argues persuasively that the source of "β" is simply "its affinity with the word 'bicompact', or in order to contrast with the notation αX (used commonly, when X is locally compact but not compact, to denote the one-point compactification of Alexandroff)."

As we will see, 1937 was a very good year for topology. First, Stone [1937] and Čech [1937] developed the compactification which now bears their name. As indicated, Čech extended Tychonov's idea of embedding the completely regular space X in a cube. In this wonderful paper he establishes most of the basic properties of βX. To quote some of the more important passages, keeping Čech's notation intact:

> If S is a completely regular space, let $\beta(S)$ designate any topological space having the following four properties: (1) *$\beta(S)$ is a bicompact Hausdorff space*, (2) $S \subset \beta(S)$, (3) S is dense in $\beta(S)$ (i.e. the closure of S in the space $\beta(S)$ is the whole space $\beta(S)$), (4) *every bounded continuous function f in the domain S may be extended to the domain $\beta(S)$* (i.e. there exists a continuous function ϕ in the domain $\beta(S)$ such that $\phi(x) = f(x)$ for every $x \in S$).
>
> *The space $\beta(S)$ exists for every completely regular S.*
>
> *Let S be a completely regular space. Let B be a space having properties* (1)–(3) of $\beta(S)$ (but not necessarily property (4)). *Then there exists a continuous mapping h of $\beta(S)$ onto B such that* (i) $h(x) = x$ *for each* $x \in S$, (ii) $h[\beta(S) - S] = B - S$.
>
> Let T be a closed subset of S; let $\overline{T}$ denote the closure of T in the space $\beta(S)$. Then $\overline{T} = \beta(T)$ *if and only if every bounded continuous function in the domain T admits of a continuous extension to the domain S.*

Of these "classic" papers it is about the only one which can be read now with virtually no translation of notation, definitions, *etc.* If it were written today, more than 50 years later, we would remove the parentheses: βS instead of $\beta(S)$. This paper not only provided one foundation block for the whole study of compactifications, it also initiated one of the most vital areas of research in all of general topology: the study of $\beta\mathbb{N}$ and of $\beta\mathbb{N}\backslash\mathbb{N}$ (see §6 below). He mentioned several problems he was unable to solve: To quote again: "If I denotes the space of integer numbers, then I think it is impossible to determine effectively (in the sense of Sierpiński) a point of $\beta(I) - I$." And again: "It is an important problem to determine the cardinal number $\mathfrak{m}$ of $\beta(I)$. All I know about it is that

$2^{\aleph_0} \leq \mathfrak{m} \leq 2^{2^{\aleph_0}}$." He was of course correct regarding points of $\beta(I) - I$ (today $\beta\mathbb{N}\backslash\mathbb{N}$): Sierpiński [1938] showed that the existence of a non-trivial {0,1}-valued additive function defined on the subsets of $\mathbb{N}$ implies the existence of a non-Lebesgue measurable function $\phi : \mathbb{R} \to \mathbb{I}$. Now a point in $\beta\mathbb{N}\backslash\mathbb{N}$ may be associated with a free ultrafilter $\mathcal{F}$ on $\mathbb{N}$ (i.e. $\cap\mathcal{F} = \emptyset$). We can then define a non-trivial additive function $\alpha : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ by setting $\alpha(A) = 0$ *if and only if* $A \in \mathcal{F}$. If $x \in \mathbb{R}$ has dyadic representation $x = n + \sum \frac{c_n}{2^n}$ (c_n is 0 or 1) then defining $\phi(x) = \alpha(\{n|c_n = 0\})$ produces a non-Lebesgue measurable function. (This argument from Semadeni [1964].)

The question regarding the cardinality of $\beta\mathbb{N}$ was resolved by Pospíšil [1937]: "Let T denote an infinite isolated (= discrete) space of cardinal number $\mathfrak{h}$; then the cardinal number of $\beta(T)$ is exp exp $\mathfrak{h}$ (= $2^{2^{\mathfrak{h}}}$)." This easily follows from Čech's next-to-last property quoted above if we have some compactification of T with cardinality $2^{2^{\mathfrak{h}}}$. Pospíšil constructs one by embedding T densely in a space S of cardinality $2^{2^{\mathfrak{h}}}$ and then embedding S into a cube (à la Tychonov). This gives a compactification with cardinality at least $2^{2^{\mathfrak{h}}}$. Finally, he observes (as did Čech) that the cardinality of any Hausdorff space cannot exceed $2^{2^{\mathfrak{d}}}$, where $\mathfrak{d}$ is the cardinality of any of its dense subsets (first proved in Pospíšil [1937a]).

An entirely different approach was taken by Stone [1937]. Beginning with an arbitrary Boolean ring A he imposes a topology on $\mathfrak{C}$, the set of all prime ideals in A. When this technique is applied to a specific Boolean ring, namely that of subsets of $\mathfrak{R}$ of the form $f^{-1}((a, b))$ where $f \in C^*(\mathfrak{R})$, we get the space $\mathfrak{Q}$ referred to below. Of primary importance to us are his Theorems 79 and 88:

Theorem 79. The space $\mathfrak{Q}$ of Theorem 78 is an immediate, and hence strict, H-extension of the given CR-space $\mathfrak{R}$. Every function in $\mathfrak{M}$ can be extended from $\mathfrak{R}$ to $\mathfrak{Q}$ so as to be continuous, and hence bounded, in $\mathfrak{Q}$. If $f \in \mathfrak{M}$ and f^* is its extension to $\mathfrak{Q}$, then the correspondence $f \to f^*$ is an analytical isomorphism between the function-rings for $\mathfrak{R}$ and $\mathfrak{Q}$. In case $\mathfrak{R}$ is a bicompact H-space, $\mathfrak{Q}$ coincides with $\mathfrak{R}$.

Theorem 88. Let $\mathfrak{R}$ be a CR-space; let $\mathfrak{Q}$ be the bicompact strict H-extension of $\mathfrak{R}$ constructed in Theorems 78 and 79; let $\mathfrak{T}$ be a CR-space which is a continuous image of $\mathfrak{R}$ by virtue of a correspondence $\mathfrak{t} = \tau(\mathfrak{r})$; and let $\mathfrak{S}$ be any bicompact immediate or strict H-extension of $\mathfrak{T}$. Then there exists a continuous univocal correspondence $\mathfrak{s} = \sigma(\mathfrak{q})$ from $\mathfrak{Q}$ to $\mathfrak{S}$ which coincides in $\mathfrak{R}$ with $\tau(\mathfrak{r})$. In particular, every bicompact immediate or strict H-extension of $\mathfrak{R}$ is a continuous image of $\mathfrak{Q}$.

To translate to modern terminology: $\mathfrak{Q}$ is an *immediate extension* of $\mathfrak{R}$ means that $\mathfrak{R}$ is dense in $\mathfrak{Q}$. *H* means Hausdorff, *CR* means completely regular, and $\mathfrak{M}$ denotes $C^*(\mathfrak{R})$. Thus Stone's space $\mathfrak{Q}$ is what we today call $\beta\mathfrak{R}$ and in Theorem 88 he has one of the defining characteristics of the Stone–Čech compactification, that any continuous $\tau : \mathfrak{R} \to \mathfrak{T}$ has a continuous extension $\sigma : \beta\mathfrak{R} \to \alpha\mathfrak{T}$, where $\alpha\mathfrak{T}$ is any compactification of $\mathfrak{T}$. The second part of Theorem 79 contains a result which Čech did not get, namely that the correspondence between f and f^* establishes an isomorphism between $C^*(\mathfrak{R})$ and $C^*(\beta\mathfrak{R})$.

Stone's method does not rely on the Tychonov product theorem, whereas Čech's technique is completely dependent on it. Thus we could have had βX even if the product theorem had never been proved; there is some logical dependency (cf. §8).

"Despite the services rendered in topology by the consideration of denumerable sequences, their use is not adapted to the study of general spaces." Continuing, Cartan [1937] presented to the Paris Academy of Sciences in October and November of 1937 two short accounts (expanded and detailed in Bourbaki [1940]) which provided an alternative to the "Moore–Smith" convergence theory (nets) applicable to general topological spaces. The ideas of filter and ultrafilter given here were modified later by Kohls [1957] to the notions of z-filter and z-ultrafilter which were utilized to great effect in the popular and influential Gillman and Jerison [1960].

Independently Wallman [1938] developed something very close to Cartan's ultrafilters with his idea of maximal subsets (of a distributive lattice) having the "finite intersection property", i.e. the infimum of any finite collection of elements is not 0. In fact, applied to the lattice $\mathcal{F}(X)$ of all closed subsets of X a maximal subset having the finite intersection property *would* be an c-ultrafilter (where the superset condition of filters is restricted to the family of closed supersets). Wallman defines a topology on the family of maximal subsets in such a way that it becomes a compact T_1 space if the original lattice is $\mathcal{F}(X)$ and X is a T_1 space. We will label this space $\omega_{\mathcal{F}}X$. Also, if we identify the point $x \in X$ with the maximal subset of $\mathcal{F}(X)$ consisting of all closed subsets of X which contain x, we have a way of embedding X into $\omega_{\mathcal{F}}X$; this produces a T_1 (quasi)compactification of X. When X is normal, $\omega_{\mathcal{F}}X$ is T_2 and, in fact, is βX.

By using certain families (which he called *normal bases*) of zero-sets of continuous real-valued functions on X instead of all of $\mathcal{F}(X)$, Frink [1964] was able to produce T_2 compactifications of a completely regular space. He proposed one of the celebrated problems of the subject: "By choosing different normal bases Z for a non-compact semi-normal space X, different Hausdorff compactifications of X may be obtained in the form of Wallman spaces $w(Z)$. It is natural to ask if every Hausdorff compactification may be obtained in this way. I have not yet been able to answer this question." We will discuss the resolution of this problem in §9.

The last of the so-called "early" works which we will discuss is Lubben [1941]. This lengthy work is very difficult to read today, primarily because of the eccentric mathematical style in which it is written:

> The theory of point elements and that of ends suggest our regarding a point not as a static entity but rather as that of a relation of the point to the remain-

der of space, that involves methods of approach. Secondly, a point need not necessarily be indivisible. We have a conception according to which our space S consists of a quantity of basic matter; this may be decomposed and put together in various ways, and the same applies for each portion of this matter. In particular, the points of S are subject to such operations. There exists a maximal portion, that of all the matter; there prove to be atomic portions.

Because of its idiosyncratic exposition we will paraphrase (hopefully correctly) rather than quote further from this paper. The contents were presented to the American Mathematical Society in three parts, on September 9, 1937, December 28, 1937, and September 6, 1938. The manuscript was received by the editors of the *Transactions* on July 18, 1938 (in final revision on September 9, 1940). The purpose behind all this pedantry is to make the point that Lubben deserves some credit for having also discovered βX, independently of Stone and Čech. See the appendix for more on this. Without making any claim concerning precedence in this matter, Lubben does point out that Stone [1937] appeared in May, while Cartan [1937], Čech [1937], and Wallman [1938] all appeared after his first presentation in September, 1937. The extent to which these papers influenced his writing of the final published version is, of course, impossible to determine. In the last section of the paper he observes that there is a distinguished compactification $\lambda(S)$ of the completely regular space S which has the property that any other compactification of S is a quotient of $\lambda(S)$. Of course, this is one of the many properties which characterize βS. His final theorem states that for a completely regular space S, $H(S)$ (the set of compactifications of S with equivalent members identified) is a complete lattice if and only if S is locally compact. (Curiously enough, he seems to define the order on $H(S)$ so that $\lambda(S)$ is the smallest element, the opposite of our present convention.)

It is for this final result that he is acknowledged today. He recognized that by identifying equivalent compactifications (in much the same way that we identify equivalent rational numbers) we can consider the *set* of compactifications of a completely regular space S; that an interesting partial order can be defined on this set; with this partial order the set becomes a complete upper semi-lattice; for it to be a complete lower semi-lattice it is necessary and sufficient that S be locally compact. He left one small window of opportunity for future researchers: when is the set of compactifications of a completely regular space S a lower semi-lattice? We will discuss this further in §3.

2. Modern Notation and Conventions

All spaces considered will be *completely regular* (and therefore implicitly Hausdorff). A *compactification* of X is a pair $(\alpha, \alpha X)$ where αX is compact, $\alpha : X \to \alpha X$ is a homeomorphism, and $\alpha(X)$ is dense in αX. We will frequently abuse this definition by identifying αX as the compactification and identifying X with $\alpha(X)$. $\alpha X \setminus \alpha(X)$ ("=" $\alpha X \setminus X$) is the *remainder* of X in αX. βX and ωX will always denote the Stone–Čech and the Alexandroff one-point compactifications, respectively, (ωX exists exactly when X is locally compact) and X^* will represent $\beta X \setminus X$. We say that $(\alpha, \alpha X) \geq (\gamma, \gamma X)$ provided there is a continuous $\pi_{\alpha\gamma} : \alpha X \to \gamma X$ with $\pi_{\alpha\gamma} \circ \alpha = \gamma$ (briefly, $\pi_{\alpha\gamma}: \alpha X \geq \gamma X$). Note that for any αX we always have $\beta X \geq \alpha X \geq \omega X$. If $(\alpha, \alpha X) \geq (\gamma, \gamma X)$ and $(\gamma, \gamma X) \geq (\alpha, \alpha X)$ then we say that $(\alpha, \alpha X)$ and $(\gamma, \gamma X)$ are *equivalent* and denote this by $(\alpha, \alpha X) \approx (\gamma, \gamma X)$. Generally, we do not distinguish between equivalent compactifications. If $f : X \to K$ has an extension to αX we will label it f_α; this would mean that $f = f_\alpha \circ \alpha$. For a given space X the cardinality of any compactification of X is at most $2^{2^{|X|}}$ by the aforementioned theorem of Pospíšil. This in turn limits the possible topologies for compactifications (there can be no more than exp exp exp exp $|X|$). Thus we may refer to the *set* $\mathcal{K}(X)$ of nonequivalent compactifications of X: each element of $\mathcal{K}(X)$ is a compactification of X, any compactification of X is equivalent to some member of $\mathcal{K}(X)$, and no two distinct elements of $\mathcal{K}(X)$ are equivalent.

$C(X)$ ($C^*(X)$) denotes the set of continuous (respectively, continuous and bounded) functions from X to $\mathbb{R}$. $X \subseteq Y$ is C-embedded (C^*-embedded) in Y provided every $f \in C(X)$ (respectively, every $f \in C^*(X)$) has a continuous extension to Y.

A space X is *realcompact* (Hewitt [1948]) if it is not C-embedded as a dense proper subset in any space Y. Note the analogy with compactness: X is compact *iff* it is not C^*-embedded as a dense proper subset in any space Y. X is a *P-space* (Gillman and Henriksen [1954]) if every prime ideal in $C(X)$ is maximal. A more "topological" condition which is equivalent is that every G_δ-subset of X should be open or alternatively, for each $x \in X$ and each $f \in C(X)$ there is a neighborhood U_f of x with $f|_{U_f}$ constant. X is an *F-space* (Gillman and Henriksen [1956]) provided every finitely generated ideal in $C(X)$ is principal.

3. The Structure of $\mathcal{K}(X)$

For any compactification αX let $\mathcal{F}(\alpha X) = \{\pi_{\beta\alpha}^{-1}(p) \mid p \in \alpha X \setminus X\}$. Magill [1968] called this the *β-family of* αX. Note that the non-degenerate elements of $\mathcal{F}(\alpha X)$ are compact subsets of $\beta X \setminus X$. The quotient space obtained from βX by collapsing each of these to a distinct point is equivalent to αX. Exploiting this, Magill was able to prove the following beautiful result: If X and Y are locally compact then $\mathcal{K}(X)$ is (lattice) isomorphic to $\mathcal{K}(Y)$ if and only if $\beta X \setminus X$ is homeomorphic to $\beta Y \setminus Y$. This result was generalized (in one direction) to non-locally compact spaces by Rayburn [1973]: Let X and Y be any two (completely regular) spaces. If there is a homeomorphism from $\mathrm{cl}_{\beta X}(\beta X \setminus X)$ onto $\mathrm{cl}_{\beta Y}(\beta Y \setminus Y)$ which carries $R(X)$ onto $R(Y)$, then $\mathcal{K}(X)$ is isomorphic to $\mathcal{K}(Y)$. ($R(X)$ is the set of points of X which do not have compact neighborhoods.)

As we indicated in §1 $\mathcal{K}(X)$ is always a complete upper semi-lattice. This can be seen as follows. For any nonempty $\{\alpha_i X\}_{i\in\Lambda} \subseteq \mathcal{K}(X)$ let $e : X \to \prod \alpha_i X$ be the evaluation map: $\langle e(x)\rangle_i = \alpha_i(x)$. Then e is an embedding of X into $\prod \alpha_i X$, a compact set. If we let $eX = \overline{e(X)}$ then (e, eX) is a compactification of X, $(e, eX) \geq (\alpha_i, \alpha_i X)$ for each $i \in \Lambda$, and if $(\gamma, \gamma X)$ also has this property, then $(\gamma, \gamma X) \geq (e, eX)$; i.e. $(e, eX) = \mathrm{Sup}\{\alpha_i X\}_{i\in\Lambda}$. We can obtain $\mathrm{Inf}\{\alpha_i X\}_{i\in\Lambda}$ by taking $\mathrm{Sup}\{\gamma X \in \mathcal{K}(X) \mid \alpha_i X \geq \gamma X \forall i \in \Lambda\}$ *provided this set is nonempty.* This will always be the case if X is locally compact: ωX is a member of it.

Note that the minimal element of $\mathcal{K}(X)$ (if it exists) must be ωX: if p, $q \in \alpha X \setminus X$ and $p \neq q$ then we may construct a compactification γX smaller than αX by taking the quotient space of αX obtained by identifying p and q ($\pi_{\alpha\gamma}$ is the quotient map). Thus, $\mathcal{K}(X)$ is a complete lattice if and only if X is locally compact (Lubben [1941] and Shirota [1950]). (*Complete lattice* = every nonempty subset has an infimum and a supremum; *lattice* = every pair of elements has an infimum and a supremum.)

If x has a countable neighborhood base in X then x has a countable neighborhood base in βX (Čech [1937]) so that if x has no compact neighborhood in X then we can obtain a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $\beta X \setminus X$ which converges to x. Form $\alpha_1 X$ by collapsing each pair $\{x_1, x_2\}$, $\{x_3, x_4\}$, $\{x_5, x_6\}$, ... to a distinct point and likewise form $\alpha_2 X$ by collapsing each pair $\{x_2, x_3\}$, $\{x_4, x_5\}$, $\{x_6, x_7\}$, ... to a distinct point. Then $\alpha_1 X \wedge \alpha_2 X$ does not exist (Shirota [1950] and Visliseni and Flaksmaĭer [1965]). This means that for the first countable space X, the only way that $\mathcal{K}(X)$ can be a lattice is if X is locally compact; e.g. $\mathcal{K}(\mathbb{Q})$ is *not* a lattice.

Can $\mathcal{K}(X)$ be a lattice if X is not locally compact? The answer to this question is "Yes, but ... " Magill [1979], in his review of one such answer, said "The problem of finding necessary and sufficient conditions on a completely regular Hausdorff space X for $\mathcal{K}(X)$ to be a lattice is likely to be a difficult one, and it is even possible that no nice solution exists". We will indicate several efforts to solve this problem. $\mathcal{K}(X)$ will be a lattice if:

- Tzung [1978]: $\beta X \setminus X$ is C^*-embedded in βX and for some $\alpha X \in \mathcal{K}(X)$, $\alpha X \setminus X$ is either realcompact or a P-space.
- Ünlü [1978]: Either $\beta X \setminus X$ is C^*-embedded in βX and $\beta X \setminus X$ is realcompact or $\beta X \setminus X$ is a P-space and $\mathrm{cl}_{\beta X}(\beta X \setminus X)$ is an F-space.

An interesting aside: Tzung and Ünlü almost simultaneously and completely independently obtained these strikingly similar results as part of their Ph.D. theses. Notice how several of their conditions generalize the notion of local compactness for X. In particular, if X is locally compact then (i) $\beta X \setminus X$ is compact so that it is C^*-embedded in βX, (*ii*) $\beta X \setminus X$ is realcompact, (*iii*) $\alpha X \setminus X$ is both realcompact and a P-space for some $\alpha X \in \mathcal{K}(X)$, specifically for $\alpha X = \omega X$.

We can construct a compactification of X by taking the quotient space of βX obtained by collapsing a family of disjoint compact subsets of $\beta X \setminus X$ to distinct points. This has been a favorite technique for creating new compactifications from old ones; all that must be done is to devise some condition(s) on the family of compact subsets which will guarantee that the resulting quotient is Hausdorff. It is easy to see that this will be the case if there are only finitely many subsets in the family.

For $\alpha X \in \mathcal{K}(X)$ define $\mathcal{F}(\alpha X) = \{\pi_{\beta\alpha}^{-1}(p) \mid p \in \alpha X \setminus X\}$ and let $\mathcal{F}^*(\alpha X) = \{F \in \mathcal{F}(\alpha X) \mid |F| > 1\}$. Note that αX is the quotient space obtained from βX by collapsing the elements of $\mathcal{F}(\alpha X)$ to distinct points. From the other direction, suppose $\mathcal{H} = \{F_1, F_2, \ldots, F_n\}$ is a finite collection of compact subsets of $\beta X \setminus X$. The quotient space obtained from βX by collapsing each element of $\mathcal{H}$ to a distinct point is an element of $\mathcal{K}(X)$ which we will denote by $\beta_{\mathcal{H}} X$. Note that for any given $\mathcal{H}$ we must have $\beta_{\cup\mathcal{H}} X \leq \beta_{\mathcal{H}} X$. Thus in Ünlü [1978], when he defines $\mathcal{K}(X)$ to be a *bounding lattice* provided for each $\alpha X \in \mathcal{K}(X)$ there is a compact set $F_\alpha \subseteq \beta X \setminus X$ with $\beta_{\{F_\alpha\}} X \leq \alpha X$, there is no restriction by requiring singleton families. For αX and γX in $\mathcal{K}(X)$, a bounding lattice, we have $\beta_{\{F_\alpha \cup F_\gamma\}} X$ is less than or equivalent to both αX and γX so that $\{\eta X \in \mathcal{K}(X) \mid \eta X \leq \alpha X \text{and } \eta X \leq \gamma X\}$ is non-empty. The supremum of this set is $\alpha X \wedge \gamma X$. Thus a bounding lattice is, in fact, a lattice (Ünlü [1978]). Note that if X is locally compact then $\mathcal{K}(X)$ is a bounding lattice since $\beta_{\{\beta X \setminus X\}} X = \omega X$. Ünlü then goes on to show that for X satisfying either of his conditions (stated above), $\mathcal{K}(X)$ is a bounding lattice.

The question of determining when $\mathcal{K}(X)$ is a lattice was listed in Chandler [1976] as "Major Problem 1." Although there have been several nice results since 1976, they are of an *ad hoc*, fragmented nature; we hope for a more inclusive result. It is still a very good problem.

4. Special Properties

The focus of this section is primarily on properties of remainders; however, the final topic involves an early question: do the product and Stone–Čech operations commute?

If X is locally compact, then we have seen that X has a one-point compactification ωX. Conversely, if X has a compactification αX with finite remainder, then, since $\alpha X \setminus X$ is finite, X is open in αX so that X is locally compact. Under what circumstances can a (necessarily locally compact) space X have a compactification with a finite or countable remainder apart from ωX? Magill, in a series of papers appearing in the late 1960's, basically wrapped up this question:

Theorem 2. (Magill [1965]) *For some $\alpha X \in \mathcal{K}(X)$, $|\alpha X \setminus X| = n$ iff X is locally compact and contains n non-empty, pairwise disjoint, open sets $\{G_i\}_{1 \le i \le n}$ such that $K = X \setminus \bigcup_{i=1}^{n} G_i$ is compact, but for each $i, 1 \le i \le n, K \cup G_i$ is not compact.*

Theorem 3. (Magill [1966]) *For a locally compact space X the following are equivalent:*

i) *$\beta X \setminus X$ has infinitely many components.*
ii) *There is an $\alpha X \in \mathcal{K}(X)$ with $\alpha X \setminus X$ infinite and totally disconnected.*
iii) *For some $\gamma X \in \mathcal{K}(X)$, $|\gamma X \setminus X| = \aleph_0$.*
iv) *For each $n \in \mathbb{N}$, X has a compactification with n points in its remainder.*

At the other end of the cardinal spectrum, we expect that most of the time $|X^*| = |\beta X \setminus X|$ is quite large (e.g. if X is discrete then $|X^*| = 2^{2^{|X|}}$ Pospíšil [1937]), although in view of the famous Exercise 9.K.5 of Gillman and Jerison [1960] (for any Y there is an X with $Y = X^*$), it is possible to get "peculiar" spaces having Stone–Čech remainders of arbitrary cardinality. Čech [1937] initiated a powerful technique for guaranteeing that X^* have a large cardinal: find a condition on X that requires X^* to contain a copy of $\beta\mathbb{N}$. For example, he has the following theorem:

Let the normal Riesz space S be not compact. Then the cardinal number of $\beta(S) - S$ is at least equal to the cardinal number of $\beta(I)$ (hence at least equal to $2^{\aleph_0}$).

(In modern terminology, *normal Riesz* = T_4, *compact* = countably compact, and $I = \mathbb{N}$. Also, recall that he only knew that "$2^{\aleph_0} \le \mathfrak{m} \le 2^{2^{\aleph_0}}$" where $\mathfrak{m} = |\beta\mathbb{N}|$; thus his lower bound estimate can be improved to $2^{2^{\aleph_0}} = 2^{\mathfrak{c}}$.) Chapter 9 of Gillman and Jerison [1960] is concerned with extensions of Čech's techniques. Their best result: "*If X is locally compact and realcompact, then every infinite closed set in $\beta X \setminus X$ contains a copy of $\beta\mathbb{N}$ (and so its cardinal is at least $2^{\mathfrak{c}}$).*"

For locally compact X, can every cardinal $\mathfrak{m}$ with $1 \le \mathfrak{m} \le |X^*|$ be realized as the cardinal number of some remainder of X? This question has an easy "No" answer: $\mathbb{H} = [0, \infty)$ has no compactification $\alpha\mathbb{H}$ with $|\alpha\mathbb{H} \setminus \mathbb{H}| = \mathfrak{m}$ for any $\mathfrak{m}$ with $1 < \mathfrak{m} < \mathfrak{c}$. The reason, as we will see below, is that $\mathbb{H}^*$ is connected. Now every remainder of $\mathbb{H}$ is a quotient of $\mathbb{H}^*$ and is therefore connected. But no connected, non-degenerate, completely regular space can have cardinality less than $\mathfrak{c}$.

A *continuum* is a compact connected space. A continuum is *indecomposable* if it is not the union of two proper subcontinua. We can actually say more about $\mathbb{H}^*$: it is a non-metric, indecomposable continuum. This occurs independently in the theses Bellamy [1968] and Woods [1968], another good example of coincidence in mathematics. Bellamy also shows that if we are given any metric continuum K and any non-degenerate subcontinuum L of $\mathbb{H}^*$ then K is a continuous image of L. The former result was generalized in Dickman [1972]: Let X be a non-compact, locally compact, locally connected metric space. Then a necessary and sufficient condition that X^* be an indecomposable continuum is that X have the *strong complementation property* (that is, whenever U is a connected open subset of X with $\overline{U}$ not compact, then $X \setminus U$ is compact).

For a point p in a continuum K, the set $C_p = \{x \in K \mid p \text{ and } x \text{ are contained in a proper subcontinuum of } K\}$ is called the *composant* of p in K. For example, in $\mathbb{I} = [0,1]$ there are three different composants: $C_0 = [0,1)$, $C_1 = (0,1]$, and for any p, $0 < p < 1$, $C_p = [0,1]$. This example shows that distinct composants do not generally have to be disjoint. This is not the case if a metric continuum is indecomposable. Theorem 138 of Moore [1962] gives us "No two composants of an indecomposable continuum have a point in common." This is followed by the result (Theorem 129) that every compact indecomposable continuum has uncountably many composants. Given the Bellamy–Woods result above, it is natural to wonder about the number of distinct composants in $\mathbb{H}^*$. M.E. Rudin [1970] showed that in ZFC + CH, $\mathbb{H}^*$ has $2^{\mathfrak{c}}$ distinct composants. That CH (or some axiom beyond ZFC) is necessary was shown in Blass [1987] where he proved that in ZFC + NCF, $\mathbb{H}^*$ has only one composant. (Near Coherence of Filters is the statement that for every pair $\mathcal{U}_1$, $\mathcal{U}_2$ of ultrafilters on $\mathbb{N}$ there is a finite-to-one function $f : \mathbb{N} \to \mathbb{N}$ with $f(\mathcal{U}_1) = f(\mathcal{U}_2)$.) Thus, the statement that $\mathbb{H}^*$ has $2^{\mathfrak{c}}$ distinct composants is independent in ZFC.

While on the subject of $\mathbb{H}^*$ we should mention one of its unsolved problems. Browner [1979], utilizing methods pioneered in Keesling [1977], proved that $\mathbb{R}^{n*}$ contains $2^{\mathfrak{c}}$ non-homeomorphic subcontinua for $n \geq 3$. Shortly afterward she improved this result to $n = 2$ (Browner [1980]). Is this the case for $\mathbb{H}^*$ as well? Refinement of the algebraic topological arguments which she used for $n = 2$ and $n \geq 3$ are not likely to be effective for $\mathbb{H}^*$. Hart [1992] observes that it is consistent in ZFC for $\mathbb{H}^*$ to have at least $\mathfrak{c}$ topologically distinct subcontinua.

An early result with many applications for remainders is from Magill [1966]: If X is locally compact and K is compact T_2 then K is a remainder of X in some compactification *iff* K is the continuous image of $\beta X \setminus X$. There are many results concerning connectivity of remainders other than the Stone–Čech. Many are of this type: if X is from a class $\mathcal{X}$ of spaces and K is from a class $\mathcal{K}$ of spaces then K is a remainder of X. Recall that a *Peano space* is any compact, connected, locally connected metric space. Using the theorem cited above, Magill then proves that any Peano space K is a remainder of any locally compact, non-compact metric space. This was a corollary of a preceding result to the effect that any Peano space is a remainder of any locally compact, normal space X which contains an infinite, closed, discrete subset. A crucial step here is to get a C^*-embedded copy of $\mathbb{N}$ in X. Gillman and Jerison [1960] have perhaps the best criterion guaranteeing this (in completely regular spaces): X contains a C^*-embedded copy of $\mathbb{N}$ *iff* X is not pseudocompact (Theorem 1.21). Comfort [1967], in his review of Magill's paper, observes that what has actually been proved is that any locally compact, non-pseudocompact space has any Peano space as a remainder. Chandler [1978] generalized this result by extending the class of remainders to include any *weak Peano space*, i.e. any compact space containing a dense continuous image of $\mathbb{R}$.

A slightly different direction was taken by Aarts–van Emde Boas [1967], proving that any metric continuum is a remainder of any locally compact (non-compact) metric space. Rogers [1971] brought the two directions together, showing that any metric continuum is a remainder of any locally compact, non-pseudocompact space.

The final topic of this section is concerned with the Stone–Čech compactification of a product of spaces. Theorem 14 of Hewitt [1948] states: "Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a non-void family of completely regular spaces. Then $\beta(\mathbf{P}_{\lambda\in\Lambda}X_\lambda)$ is homeomorphic to the space $\mathbf{P}_{\lambda\in\Lambda}\beta X_\lambda$." (This error was noted in the review Dieudonné [1949].) The very next paper in that issue of the *Transactions* was Samuel [1948], who has on p. 127 the statement: "The Čech structure of a product of uniformizable spaces is, in general, different from the product of the Čech structures of its factors." When is it true that $\beta\prod X_\alpha = \prod \beta X_\alpha$? This is really two questions, depending on what is meant by '='. One interpretation would mean homeomorphic as topological spaces; the other, homeomorphic as compactifications (that is, the homeomorphism restricted to $\prod X_\alpha$ must be the identity). Gillman and Jerison [1960] in Exercise 14Q show that $\beta\mathbb{R}\times\beta\mathbb{R}$ is homeomorphic to no subspace of $\beta(\mathbb{R}\times\mathbb{R})$. The second question was examined by Henriksen–Isbell [1957] who obtained partial results for a product of two spaces. It was finally resolved by Glicksberg [1959] who proved (his Theorem 1): "*Let $\{X_\alpha\}$ be a set of completely regular spaces, and suppose the set $\mathbf{P}_{\alpha\neq\alpha_0}X_\alpha$ is infinite for every α_0. Then a necessary and sufficient condition that $\beta(\mathbf{P}X_\alpha) = \mathbf{P}\beta(X_\alpha)$ is that $\mathbf{P}X_\alpha$ be pseudo-compact.*"

5. Singular Compactifications

Given $f : X \to Y$ where Y is compact Hausdorff, it is a natural question to ask if a compactification of X having remainder Y might be constructed using f to somehow "glue" Y to X. There were many *ad hoc* constructions of such compactifications before a definitive theory emerged; we mention several:

- Alexandroff–Urysohn [1929] The Two Circle Construction glues the circle with its usual compact topology to a concentric discrete circle (using radial projection) to form a compactification of the discrete space of cardinality $\mathfrak{c}$ having the circle (with its usual topology) as remainder.
- Magill [1966] For locally compact X, K is a remainder of X if and only if K is a continuous image of $\beta X\setminus X$.

- Loeb [1967] Let X be a noncompact, locally compact Hausdorff space and suppose Q is a family of mappings of X, with each $f \in Q$ having a compact space Y_f for range. If $e_Q : X \to \prod Y_f$ is the evaluation mapping then $\Delta = \cap\{cl[e(X \setminus K)] \mid K \subset X \text{ is compact }\}$ is a remainder of X.
- Engelking [1968] extended the Alexandroff–Urysohn result to arbitrary discrete spaces of cardinality $\mathfrak{k}$ with an arbitrary compact space having a dense subspace of cardinality $\mathfrak{k}$.
- Steiner–Steiner [1968] If X is locally compact, K is compact Hausdorff, and $f : X \to K$ is continuous with $f(X \setminus F)$ dense in K for each compact $F \subset X$ then there is such a "gluing" using the natural embedding $\omega \times f$ of X into $\omega X \times K$ (ωX is the Alexandroff 1-point compactification).
- Magill [1970] If X is locally compact, $f : X \to K$ is continuous, K is compact, and $f^{-1}(k)$ is not compact for each k in some dense subset of K then K is a remainder of X.

Of these, Steiner–Steiner [1968] was enthusiastically embraced as a method of concretely generating new compactifications which are, in some sense, tangible and "geometric" rather than the forbiddingly abstract and unintuitive Stone–Čech construction. We list a sampling of the many papers to use this method: Magill [1970], Rogers [1971], Chandler [1978], Chandler–Tzung [1978].

Eventually, these separate ideas were brought together into a comprehensive theory. In Whyburn [1953] there had been, perhaps, the first formal definition of the idea of gluing two spaces together using a function to define the topology. Whyburn called the new object the *unified space* of the mapping $f : X \to Y$: On $Z = X \mathring{+} Y$, the disjoint union of X and Y, define a topology by saying $Q \subset Z$ is open provided

i) $h^{-1}(Q \cap X)$ is open in X and $k^{-1}(Q \cap Y)$ is open in Y, where $h : X \to Z$ and $k : Y \to Z$ are the inclusion mappings.
ii) For any compact $K \subset k^{-1}(Q \cap Y)$, $f^{-1}(K) \cap [X \setminus h^{-1}(Q \cap X)]$ is compact.

Whyburn's original intent was to compactify the mapping rather than its domain; that is, to provide a compact extension of $f : X \to Y$. He observed several crucial properties of the construction which make it particularly useful from the point of view of compactifying X but he did not seem to be concerned with this problem:

- X and Y are homeomorphic to the appropriate subspaces of Z.
- The mapping $r : Z \to Y$ defined by $r(z) = f(z)$ if $z \in X$ and $r(z) = z$ if $z \in Y$ is a retract.
- If X is locally compact, Z is Hausdorff.
- If Y is compact, so is Z.

While in later papers Whyburn [1966], Cain [1966], and Cain [1969], the focus remained on compactifying the mapping, an idea was born which proved to be the key in pulling together all the loose ends mentioned above. The unifying concept was that of the *singular set* of a map $f : X \to K$ (Cain [1966]):

$$\mathcal{S}(f) = \{p \in K \mid \forall U \in \mathcal{N}(p)\ \exists F \in \mathcal{K}_K\ \ (p \in F \subseteq U \wedge f^{-1}(F) \notin \mathcal{K}_X)\}.$$

(Note: $\mathcal{N}(p)$, $\mathcal{K}_Z$ denote respectively the neighborhood system at p, the family of compact subsets of Z.) Note that, since we want the resulting unified space to be Hausdorff, we need X to be locally compact.

We say that $f : X \to K$ is *singular* provided $\mathcal{S}(f) = K$. In this case the unified space $X \mathring{+} K$ is the *singular compactification induced by* f. Thus to say that the compactification $(\alpha, \alpha X)$ is singular means that there is a mapping $f : X \to \alpha X \setminus X$ with $\mathcal{S}(f) = \alpha X \setminus X$ and with $\alpha X \approx X \mathring{+} (\alpha X \setminus X)$. Note that from Whyburn's second property above we then have that $\alpha X \setminus X$ is a retract of αX. Conversely, suppose that γX is a compactification of X and $r : \gamma X \to \gamma X \setminus X$ is a retraction. It then follows that $\mathcal{S}(r|_X) = \gamma X \setminus X$ (Faulkner [1988]). The question of when $\beta X \setminus X$ is a retract of βX has been explored rather extensively beginning with Comfort [1965]. (See also Bentley [1972] and van Douwen [1978].) Clearly, if X is connected and $\beta X \setminus X$ is not connected (e.g. $X = \mathbb{R}$) then βX is not a singular compactification.

Two Ph.D. theses explored the order structure of singular compactifications. Guglielmi [1985] proved that the infimum of two singular compactifications is a singular compactification; consequently the subset of $\mathcal{K}(X)$ consisting of the singular compactifications of the locally compact space X is a lower semilattice. André [1992] proved that the two-point compactification of $\mathbb{R}$ is *not* the supremum of any collection of singular compactifications of $\mathbb{R}$, answering a question of Faulkner [1990]. He also characterizes those compactifications which are the suprema of sets of singular compactifications and characterizes those locally compact X for which the supremum of the set of singular compactifications is singular.

6. $\beta\mathbb{N}$

We have seen in §1 that it was Čech [1937] who began the interest in $\beta\mathbb{N}$, although Engelking [1989], p. 179 has observed that Tychonoff [1935] actually contains a construction which produces $\beta\mathbb{N}$. Čech's initial question (What is the cardinality of $\beta\mathbb{N}$?) was quickly answered by Pospíšil [1938]; this was the genesis of what ultimately became one of the most vigorous topics in topology. $\mathbb{N}$ is essentially a set-theoretic object, not a topological one. This, taken together with the fact that $\beta\mathbb{N}$ is the Stone space of $2^{\mathbb{N}}$, places the study of $\beta\mathbb{N}$ squarely in the realm of set theory. Thus most of the important historical results have set-theoretic overtones, and one can barely touch the study of $\beta\mathbb{N}$, and with it, the study of all compactification, without being somewhat armed with a knowledge of set theory.

The beginning of these results, and in many ways the most elegant, was the non-homogeneity of $\beta\mathbb{N}\backslash\mathbb{N}$ by W. Rudin [1956]. Certainly $\beta\mathbb{N}$ is not homogeneous, but since any two fixed ultrafilters are so transparently identical, why should this differ so much for the free (nonprincipal) ultrafilters. Mary Ellen Rudin had the following response (personal communication) to our query regarding the background of this paper: "... the question certainly came up as a result of that Conference in Madison in 1955. Walter attended the seminar on rings of continuous functions run by Gillman and Henriksen, and homogeneity and compactifications and such things were basic to all discussions. But he doesn't think he got the problem from anyone else specifically – he just wondered about the simplest possible case, namely $\beta\mathbb{N}\backslash\mathbb{N}$, and came up with this solution."

A point x in a topological space X is a *P-point* provided every continuous $f : X \to \mathbb{R}$ is constant on some neighborhood of x, e.g. ω_1 in the ordinal space $\omega_1 + 1$. A space X is a *P-space* if every point in X is a P-point.

Using the Continuum Hypothesis (CH) W. Rudin was able to show that $\beta\mathbb{N}\backslash\mathbb{N}$ contained a dense set of P-points. Clearly any homeomorphism of $\beta\mathbb{N}\backslash\mathbb{N}$ to itself would take P-points to P-points so that we would know $\beta\mathbb{N}\backslash\mathbb{N}$ is not homogeneous if it contains non-P-points. For any continuous $f : \beta\mathbb{N}\backslash\mathbb{N} \to \mathbb{R}$ with infinite range there is a point $q \in \beta\mathbb{N}\backslash\mathbb{N}$ such that f is not constant on any neighborhood of q. Thus q is not a P-point. In fact any pseudocompact P-space must be finite.

At the Madison Conference Gillman presented a paper entitled *Some Special Spaces* in which the idea of P-points were discussed. Gillman–Henriksen [1954] contained the notion of a P-space, and therefore the idea of P-points was introduced. In part, the paper concerned the question of when the prime ideals of $C(X)$ are maximal. This happens exactly when X is a P-space. Since prime ideals are maximal in $C(X)$ when X is discrete, Gillman and Henriksen named these spaces P-spaces for *"pseudo-discrete"*. P-spaces can be very far from discrete however; they can be extraordinarily complicated. Since being a P-point is hereditary, a dense set of P-points in $\beta\mathbb{N}\backslash\mathbb{N}$ is a P-space. The structure of the P-sets in $\beta\mathbb{N}\backslash\mathbb{N}$ is still a very active area of research with many results, as should be expected, being consistency results. For example, Frankiewicz–Shelah–Zbierski [1989] announced that there is consistently no **ccc** P-set in $\beta\mathbb{N}\backslash\mathbb{N}$. **ccc** sets are in some sense small. On the other hand we have from Balcar–Frankiewicz–Mills [1980] that any closed subset of $\beta\mathbb{N}\backslash\mathbb{N}$ can be realized as a P-set in $\beta\mathbb{N}\backslash\mathbb{N}$. In particular, $\beta\mathbb{N}\backslash\mathbb{N}$ is homeomorphic to a closed nowhere dense P-set in $\beta\mathbb{N}\backslash\mathbb{N}$.

Following Rudin we are left with the intriguing question of whether or not CH is necessary for the proof that $\beta\mathbb{N}\backslash\mathbb{N}$ is non-homogeneous. Frolík [1967], using a cardinality argument (and not CH) on the types of points in $\mathbb{N}^*$ showed that for each point $p \in \mathbb{N}^*$ there were $2^{\mathfrak{c}}$ points in $\mathbb{N}^*$ which cannot be mapped onto p by any automorphism of $\mathbb{N}^*$. In fact, Frolík proved that $\beta X \setminus X$ is non-homogeneous for any non-pseudocompact space X, not simply for $\mathbb{N}$. Isiwata

[1957] had originally proved this assuming CH.

But these results were not, as van Douwen would say, "honest" proofs since they did not exhibit topologically distinct points. So focus changed from the question of homogeneity to the status of P-points without the Continuum Hypothesis. In $\beta\mathbb{N}\backslash\mathbb{N}$, P-points have many alternate combinatorial definitions. These emphasize the set-theoretic flavor of the problem and allow for more obvious generalizations. The most common definition is as follows:

A free ultrafilter p on $\mathbb{N}$ is a *P-point* provided for each countable subcollection $\{G_n\}$ of p, there is a $G \in p$ so that $|G \setminus G_n| < \infty$ for each n. Without loss of generality the collection $\{G_n\}$ can be assumed to be decreasing. The P-point p is said to be *selective* provided G can be chosen so that $|G \setminus G_n| \leq 1$. A point p of a topological space is a *weak-P-point* if it is not in the closure of any countable set not containing it. Clearly a P-point is a weak-P-point, but not conversely.

In the sequence of weakening the assumption of CH, Booth [1969] showed that under Martin's Axiom, $\beta\mathbb{N}\backslash\mathbb{N}$ has a selective P-point, while Kunen [1970] showed that it is consistent that there be no selective P-points in $\beta\mathbb{N}\backslash\mathbb{N}$.

Let $\mathcal{F}$ be the collection of all functions $f : \mathbb{N} \to \mathbb{N}$. For $f, g \in \mathcal{F}$ we say that $f \leq^* g$ if there is an $n \in \mathbb{N}$ so that for $m > n$, $f(m) \leq g(m)$. A subcollection $\mathcal{D}$ of $\mathcal{F}$ is said to be *dominating* if for each $f \in \mathcal{F}$ there is a $g \in \mathcal{D}$ such that $f \leq^* g$.

Ketonen [1976] showed that under a weakening of Martin's Axiom, which he called (H), namely that the cardinality of any dominating family must be $\mathfrak{c}$, $\beta\mathbb{N}\backslash\mathbb{N}$ has P-points.

Ultimately the descending sequence converged. Shelah proved (Wimmers [1982]) that it is consistent that P-points do not exist in $\beta\mathbb{N}\backslash\mathbb{N}$.

Although there are no P-points guaranteed by ZFC, there are still distinguished points. Kunen [1978] showed that there are $2^{\mathfrak{c}}$ weak P-points in $\beta\mathbb{N}\backslash\mathbb{N}$. In fact the fledgling has become a flock, van Mill [1982].

Probably we should pause to mention the Rudin–Frolík ordering and the Rudin–Keisler orderings on $\beta\mathbb{N}$. Certainly no current treatise on the structure of $\beta\mathbb{N}$ or $\beta\mathbb{N}\backslash\mathbb{N}$ could be complete without discussing this theory. However it is a new theory, still growing vigorously. So we have decided to leave it out of a discussion of the history of $\beta\mathbb{N}$. This is done with some trepidation and uncertainty, but be that as it may.

Finally there is one clearly important fact about the structure of the group of automorphisms of $\beta\mathbb{N}$ and $\beta\mathbb{N}\backslash\mathbb{N}$. Rudin [1956] noted that every automorphism of $\beta\mathbb{N}$ arises from a permutation of $\mathbb{N}$and is consequently very simple. In fact this would imply that there are only $\mathfrak{c}$ automorphisms of $\beta\mathbb{N}$. However, the automorphisms of $\beta\mathbb{N}\backslash\mathbb{N}$ could be much more complicated. He observed that under CH that $\beta\mathbb{N}\backslash\mathbb{N}$ has precisely $2^{\mathfrak{c}}$ automorphisms. In fact, any of the $2^{\mathfrak{c}}$ P-points could be mapped to any other. But the same bell rings again. Shelah has shown that there is a model for ZFC in which every automorphism of $\beta\mathbb{N}\backslash\mathbb{N}$ is simple (arises from a permutation of $\mathbb{N}$). So it is possible to have only $\mathfrak{c}$ automorphisms of $\beta\mathbb{N}\backslash\mathbb{N}$.

Now as with P-spaces, Henriksen–Gillman [1956] introduced the class of F-spaces. A space X is an *F-space* precisely when every finitely generated ideal in $C(X)$ is principal. There are many topological characterizations of F-spaces. In particular, a space is an F-space if every cozero set is C^*-embedded. Now $\beta\mathbb{N}\backslash\mathbb{N}$ is a perfect, compact, zero-dimensional F-space of weight $\mathfrak{c}$ in which every G_δ has nonempty interior. Such a space is known as a Parovičenko space. Parovičenko [1963] showed that under the assumption of CH, $\beta\mathbb{N}\backslash\mathbb{N}$ is the only such space. From this it follows that every compact Hausdorff space of weight at most $\mathfrak{c}$ is the continuous image of $\beta\mathbb{N}\backslash\mathbb{N}$ and that every Boolean algebra of cardinality at most $\mathfrak{c}$ can be embedded algebraically into the Boolean algebra $2^{\mathbb{N}}$ *mod fin.* Eric van Douwen proved, but did not publish, the fact that under the assumption of MA $+\neg$CH there exist Parovičenko spaces which are not homeomorphic to $\beta\mathbb{N}\backslash\mathbb{N}$. (The paper was published posthumously: van Douwen [1990]). van Douwen–van Mill [1978] proved that the Parovičenko characterization of $\beta\mathbb{N}\backslash\mathbb{N}$ implied CH. Thus CH is equivalent to the statement that every Parovičenko space is homeomorphic to ω^*. However, Baumgartner, Frankiewicz, and Zbierski [1990] have shown that there is a model for ZFC in which the two famous corollaries

- Any compact T_2 space of weight less than or equal to c is the image of ω^*.
- Any Boolean algebra of cardinality less than or equal to c embeds in $\mathcal{P}(\omega) \setminus \text{fin}$.

are weaker. These discoveries have inspired many results about the structure of $\beta\mathbb{N}\backslash\mathbb{N}$, all of which of course depend heavily on set theoretic assumptions.

There is one final paper which we wish to mention. It is not old enough to be fully appreciated as history, but it will probably be viewed as one of the most important papers in contemporary topology. In addition the paper answers a question about $\beta\mathbb{N}$ asked by Smirnov (Rudin [1975]). The paper is Šapirovskiĭ [1980] and it concerns the question of when a compact space can be mapped onto the Tychonov cube $\mathbb{I}^\tau$. The answer is the following: A compact space X can be mapped onto $\mathbb{I}^\tau$ if and only if there is a compact $F \subset X$ such that $\pi_\chi(x, F) \geq \tau$ for every $x \in F$. This is a beautiful result which parallels the Tychonov embedding theorem. It implies an answer to Smirnov's question as follows: CH is equivalent to the statement that every compact Hausdorff space contains either a copy of $\beta\mathbb{N}\backslash\mathbb{N}$ or a point of countable π-character. The most pleasing implication of this paper is that $\beta\mathbb{N}$ is in some sense "prime", for it follows from the above that if $\beta\mathbb{N}$ embeds into a product of fewer than the cofinality of $\mathfrak{c}$ compact spaces, then it must embed in one of the factors. Unfortunately we have lost Šapirovskiĭ early in life, to a disease that might have been cured had it been given proper timely treatment.

7. Alternative Constructions of βX

Since its first discovery in the 1930's, the Stone–Čech compactification has been constructed in many different ways. As we saw in §1, Čech's construction was a variation of Tychonov's technique of embedding in a cube. To be precise, for a completely regular space S Čech defined Φ to be the set of all continuous mappings of S into the unit interval $[0,1]$. The evaluation mapping e (§2) embeds S into the cube $\prod_{f\in\Phi}[0,1]_f$ and βS is then defined as the closure of $e(S)$. With this definition, it is quite easy to prove that every bounded real-valued function on S has a continuous extension to βS, one of the characteristic properties of the Stone–Čech compactification.

An extension of this is employed in Engelking [1989]. We have seen that for a completely regular space there is a set $\mathcal{K}(X)$ of compactifications of X having the properties that no two elements of $\mathcal{K}(X)$ are equivalent, that every compactification of X has an element of $\mathcal{K}(X)$ to which it is equivalent, and with respect to the order defined in §2 $\mathcal{K}(X)$ is a complete upper semi-lattice. Simply define βX to be the largest element of $\mathcal{K}(X)$. (Note that this technique is basically the one pioneered by Lubben [1941].) In effect, this method constructs βX by embedding X into the product of all the elements of $\mathcal{K}(X)$ using the individual evaluation maps which embed X into the elements of $\mathcal{K}(X)$.

Stone first defined $\mathcal{G}$ to be the set of open sets of the form $f^{-1}((a,b))$ where f is any element of $C^*(X)$ and $(a,b)\subseteq\mathbb{R}$. Then letting A be the basic Boolean ring generated by $\mathcal{G}$ he obtained βX as the set of real ideals in A (i.e. those ideals I for which A/I is isomorphic to $\mathbb{R}$) suitably topologized. Gelfand–Kolmogoroff [1939] streamlined this process somewhat by constructing βX as the maximal ideal space of $C^*(X)$. More surprisingly, they also obtained it as the maximal ideal space of $C(X)$. Wallman [1938] adopted the other half of Stone's construction, using maximal prefilters of closed sets in X as the underlying set for his compactification. The disadvantage of this method is that it does not produce a Hausdorff space unless X is normal; then it produces βX. Using ultrafilters of *zero sets* rather than closed sets will produce βX; this is the basic construction espoused in the popular and influential Gillman–Jerison [1960].

Weil [1937] defined the concept of a uniform space in order to generalize the idea of uniform convergence from metric spaces. Bourbaki [1940] (in the English translation, p. 217) says:

> Once one has the definition of a uniform space, there is no difficulty (especially if one also has the notion of a filter at one's disposal) in extending to these spaces almost the whole of the theory of metric spaces as given e.g. by Hausdorff (and similarly in extending, for example, to arbitrary compact spaces the results on compact metric spaces given in Alexandroff–Hopf's *Topologie*). This is what we have done in this chapter; in particular, the theorem on completion of uniform spaces ... is no more than a transposition,

without any essential modifications, of Cantor's construction of the real numbers.

For the completely regular space X, each finite family $\{f_1, f_2, \ldots, f_n\} \subseteq C^*(X)$ determines a pseudometric

$$\delta_{f_1,f_2,\ldots,f_n}(x,y) = \max\{|f_1(x) - f_1(y)|, \ldots, |f_n(x) - f_n(y)|\}.$$

If we let $\mathcal{P}^*$ denote the family of all such pseudometrics and define $\mathcal{B}^*$ to be the family of all subsets of $X \times X$ of the form

$$\{(x,y) : \delta(x,y) < 1/2^i\} \text{ for } \delta \in \mathcal{P}^* \text{ and } i \in \mathbb{N}$$

then $\mathcal{B}^*$ is a base for a uniformity $\mathcal{U}^*$. The completion of $(X, \mathcal{U}^*)$ is βX.

Proximity spaces were introduced in Efremovič [1951], axiomatizing a notion of 'closeness' for subsets of a topological space. A penetrating analysis by Smirnov [1952] then established a 1–1 correspondence between the compactifications of a completely regular space X and its proximity relations. For a compactification αX of X we may define a proximity on X by saying $A\delta B$ *iff* $\overline{A} \cap \overline{B} \neq \emptyset$ (closures in αX). Conversely, for a proximity δ on X, the completion of the uniformity induced by δ is a compactification and these two constructions are essentially "inverses" of one another; see §8.4 of Engelking [1989] for details. For a given (completely regular) space X we may define a proximity δ on X by saying $A\delta B$ provided there is no continuous, real-valued function f with $f(A) = \{0\}$ and $f(B) = \{1\}$. (That is, $A\delta B$ *iff* A and B are not functionally separated.) The compactification corresponding to this proximity is βX.

8. The Axiom of Choice

"It is a hopeless endeavor, doomed to failure, to attempt to prove either the Stone–Čech compactification theorem or the Tychonov product theorem without invoking some form of the axiom of choice, as is well known ..., the latter theorem implies the axiom of choice and the former implies one of its weaker forms." Thus begins Comfort [1968], a striking and informative paper we will return to below. Since its inception many questions in topology have been tied to problems in set theory. We have indicated in §1 that Čech early saw that the existence of points in $\beta\mathbb{N}\backslash\mathbb{N}$ was probably dependent on the Axiom of Choice (AC). Let us see precisely what this dependence is.

The original proof of the (special) Tychonov product theorem in Tychonoff [1930] as well as the proof in Čech [1937] of the general case made use of Zermelo's well-ordering principle. Conversely, Kelly [1950] proved that the product theorem (for arbitrary rather than T_2 spaces) implies AC. Since then there has been considerable fine-tuning of these results (see Rubin–Rubin [1985] pp. 167–177 for details) but basically the product theorem for arbitrary quasi-compact spaces is equivalent to AC. (Recall that *quasi-compact* spaces satisfy the covering condition but are not necessarily T_2.) Generally, the existence of βX for an arbitrary completely regular space X can be obtained (e.g. in the Čech formulation) from the theorem that the product of a family of closed bounded intervals is compact. This is certainly no stronger than the product theorem for families of compact T_2 spaces. This result, that the product of compact T_2 spaces is compact, was shown in Łoś–Ryll-Nardzewski [1954] to be equivalent to the Boolean Prime Ideal Theorem (Every Boolean algebra contains a proper prime ideal). This is weaker than AC (Halpern [1964]). Thus the existence of βX for an arbitrary completely regular space X is weaker than AC.

On the other hand, a development of υX (the Hewitt *realcompactification* of X) can avoid any use of AC. The point of Comfort [1968] is to offer an explanation for this rather perplexing breakdown for the otherwise almost perfect analogy between the development of βX and that of υX. One of the many possible definitions for the completely regular space X to be *realcompact* is that each real maximal ideal in $C(X)$ is fixed. (An ideal $I \subset C(X)$ is *real* provided $C(X)/I \cong \mathbb{R}$. I is *fixed* if there is an $x \in X$ for which $f(x) = 0$ for every $f \in I$.) Comfort then observes that by using an analogous definition for compact (one which is, with AC, equivalent to the usual covering definition), AC may be avoided in the construction of βX for an arbitrary completely regular X and in the proof of the Tychonov theorem. To avoid confusion he names his concept *compact**: the completely regular space X is *compact** provided every maximal ideal in $C^*(X)$ is fixed. (All maximal ideals in $C^*(X)$ are real.) Before there is a general stampede to change the definition of compact to that of compact* we should note that Comfort concludes with the observation that it is impossible (without AC) to prove that a real-valued continuous function with a compact* domain is bounded!

9. Wallman–Frink Compactifications

In addition to ωX and βX there are two particular compactifications which deserve special attention. One of these, the so-called *Freudenthal compactification* (Freudenthal [1942]) for *rim compact* spaces (i.e. those having a base of open sets with compact boundaries), is the subject of an excellent recent review, Dickman–McCoy [1988]; there is no need to discuss it further here.

Several persons had examined Wallman's method of compactification prior to 1964 (e.g. Šanin [1943], Samuel [1948], Fan–Gottesman [1952], Banaschewski [1962, 1963]) but it was Frink [1964] who gave this study special impetus and who provided one problem which for more than 10 years generated intense interest. A base $\mathcal{Z}$ for the closed sets of completely regular space X is a *normal base* provided:

i) it is *disjunctive*, that is, given a point $x \in X$ and a closed set F not containing x, there is an $A \in \mathcal{Z}$ with $x \in A \subseteq X \setminus F$;

ii) it is a *ring of sets*, that is, it is closed under finite intersections and unions;
iii) given two disjoint members A and B of $\mathcal{Z}$, there exist C, $D \in \mathcal{Z}$ with $A \subset (X \setminus C)$, $B \subset (X \setminus D)$, and $(X \setminus C) \cap (X \setminus D) = \emptyset$.

By giving the set $\omega_{\mathcal{Z}}X$ of $\mathcal{Z}$-ultrafilters the same topology as Wallman had and identifying $x \in X$ with the $\mathcal{Z}$-ultrafilter of all elements of $\mathcal{Z}$ which contain x, we obtain a compactification of X, called by Frink a *Wallman type* compactification. If we let $\mathcal{Z}$ be the set of all zero sets of real-valued, continuous functions on X, then $\omega_{\mathcal{Z}}X$ is the same as βX, as constructed in Gillman–Jerison [1960]. The big question asked by Frink was whether every compactification of X could be realized as $\omega_{\mathcal{Z}}X$ for an appropriate $\mathcal{Z}$.

The answer to this question was most eagerly sought during the next dozen years. Typical results usually identified a specific compactification or class of compactifications as being of Wallman type:

- Njåstad [1966]: compactifications with finite remainders; the Freudenthal compactification; the Fan–Gottesman compactification.
- Brooks [1967]: the Alexandroff one-point compactification.
- Steiner [1968]: any product of compact subsets of $\mathbb{R}$ as a compactification of any dense subset.
- Steiner–Steiner [1968a]: compactifications with countable remainders.
- Steiner–Steiner [1968b]: any product of compact metric spaces as a compactification of any dense subset.
- Aarts [1969]: every metric compactification of any of its dense subsets.
- Biles [1969]: any compact ordered space as a compactification of any dense subset.
- Bandt [1977]: any compact space X with $w(X) \leq \omega^+$ as a compactification of any dense subset.

(We have not attempted an exhaustive list here; only a selection of typical results.) In the early to mid 1970's a rather unusual theorem was obtained by several persons, independently of one another: If there is a non-Wallman type compactification then there is a non-Wallman type compactification of a discrete space. This first appeared in Šapiro [1974], a paper which was largely unknown in the West at the time. Following this, Bandt [1977] and Steiner–Steiner [1977] obtained the same result. Note: the Steiners give credit to Ünlü for a proof from which they were able to remove the restriction to locally compact spaces. Ünlü's paper, which had been accepted by the Proceedings of the AMS, was withdrawn from publication after his thesis advisor, J.R. Porter, saw a copy of Šapiro [1974].

Frink's question was finally resolved by Ul'yanov [1977]: "*Suppose* $2^\tau \geq \aleph_2$. *Then there exist a completely regular space* S *of cardinality* τ *and a connected Hausdorff bicompact extension* νS *of weight* 2^τ *which is not an extension of Wallman type.*" In addition he proves that "*the assertion that every Hausdorff bicompact extension of an arbitrary separable completely regular space is an extension* *of Wallman type is equivalent to the continuum hypothesis.*" He attributes to Šapiro the additional result that for any cardinal τ such that $2^\tau \geq \aleph_2$, there exists a family of cardinality 2^{2^τ} of nonhomeomorphic Hausdorff bicompact extensions of a discrete space of cardinality τ which are not extensions of Wallman type.

10. Secondary Sources

If we apply Hilbert's criterion (the importance of a scientific work is directly related to the number of earlier publications rendered superfluous by it), then the latest manifestation of Engelking's encyclopedic treatise, Engelking [1989], must be one of the paramount texts of all time. It is truly a gold mine of information on general topology and intersects much of what we have discussed here, either as text or exercises or historical notes. Another book on general topology, Willard [1970] (unfortunately now out of print), is not so ambitious, but in our opinion it has proved to be an excellent text for beginning classes on the subject. Arkhangel'skiĭ–Fedorchuk [1990], like Willard, does not contain a lot of material specifically devoted to compactification but looks as if it would be an excellent text for a beginning course using the so-called "Moore Method" of instruction. Chapters 1–5 are subtitled *The Basic Concepts and Constructions of General Topology*, while Chapters 6–12 are *The Fundamentals of Dimension Theory*. Arkhangel'skiĭ–Ponomarev [1983] is a basically an exhaustive set of problems (well over 1500 of them) and solutions, with each section of preceded by basic notation, conventions, and definitions. Chapter 4 is about compactifications and contains 184 problems on this topic.

Another collection of problems in general topology, mostly unsolved, is van Mill–Reed [1990]. The articles by Dow, Steprāns, and Hart and van Mill contain many problems associated with $\beta\mathbb{N}$ and with $\beta\mathbb{N}\backslash\mathbb{N}$.

It is perhaps a little misleading to include Gillman–Jerison [1960] in this section on Secondary Sources since for many of us it was *the* place where we first learned the material of its title. It is difficult to realize that it is now more than 35 years old; it seems timeless. Beautifully written and virtually error-free, we have used it many times as a text in a follow-up to the beginning course in general topology. The balance of exposition to exercises is masterful. Its immediate source was a seminar at Purdue during 1954–1955 organized by Melvin Henriksen to explore the interrelationships between the topology of a space and the algebra of its rings of continuous real-valued functions. This interplay had been the motivation for three important papers which were the genesis of this subject: Stone [1937], Gelfand–Kolmogoroff [1939], and Hewitt [1948]. Gillman [1990] and Henriksen [1997] have first-hand accounts of the events which culminated in this wonderful book.

For an infinite set X the set of *ultrafilters* on X can be viewed as the points of βX when X is given the discrete topology. For this reason it is appropriate to include Comfort–Negrepontis [1974] here. It contains a wealth of information, primarily on the set-theoretic aspects of these issues. Chapter 9 is concerned with the Rudin–Keisler order on ultrafilters (see §6), and the final chapter is devoted specifically to $\mathbb{N}$. (See also Comfort [1980].)

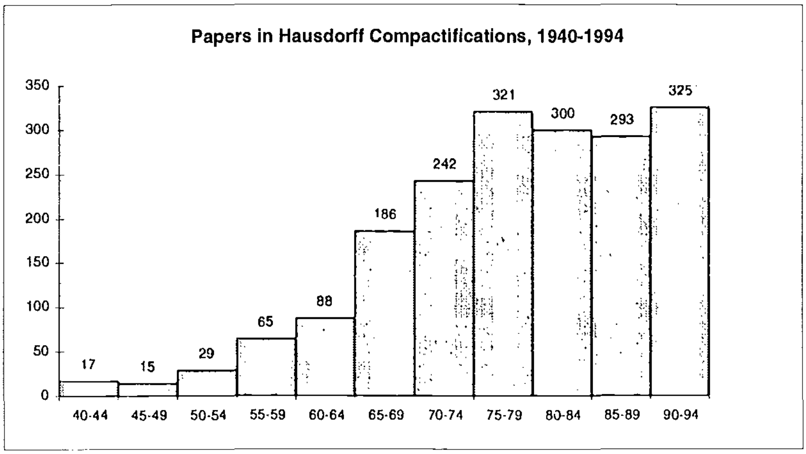
An *extension* of X is any space Y containing a dense subset homeomorphic to X. Thus a compactification of X is a particular example of an extension. The *absolute* of a compact space X is a compact space $A(X)$ together with a continuous irreducible map $\pi_X : A(X) \to X$ which "lifts" any onto mapping $f : X \to K$ when K is compact. π_X *irreducible* means that π_X maps $A(X)$ onto X but is not onto X if restricted to any proper subset of $A(X)$. π_X *lifts* $f : X \to K$ means that there is a function $f_\pi : A(X) \to K$ for which $f_\pi = f \circ \pi_X$. These two ideas form the central themes of Porter–Woods [1988]. Chapter 4 is primarily concerned with compactifications.

The title of Naber [1977] (*Set-Theoretic Topology with Emphasis on Problems from the Theory of Coverings, Zero Dimensionality and Cardinal Invariants*) provides a good description of its contents. Since $\beta\mathbb{N}$ has proved such an important source of examples for these topics, he has an extensive discussion devoted to it. Walker [1974] is restricted almost exclusively to the Stone–Čech compactification and represents practically everything that was known about this topic through the early 1970's. While his purpose does not seem to be a presentation of the historical development, there is an excellent "influence diagram" on page 27. Weir [1975] does for realcompact (= Hewitt–Nachbin) spaces what Walker does for the Stone–Čech compactification. His introductory chapter contains an effective historical development of the subject. Bolton [1978] is a "newspaper" style account, primarily concerned with four topics: Wallman type compactifications, Freudenthal compactifications, extension of mappings, and remainders and the lattice of compactifications. The purpose behind Chandler [1976] was to present a unified approach for obtaining all compactifications of a completely regular space using Tychonov's technique of embedding in cubes.

Most of the volumes mentioned in this section contain extensive bibliographies, several in excess of 20 pages.

11. Research Levels Since 1940

While not approaching the fecundity of Fixed Point Theory, Hausdorff Compactifications has been a vital part of General Topology almost from its inception. While we recognize that quantity does not equate to quality, it is interesting to see the sheer number of papers (and books) on this subject. A careful perusal of *Mathematical Reviews* gave the following data. There are surely papers we missed, and just as surely there are papers not reviewed in *MR*; with these two caveats we believe the chart to be reasonably accurate.



12. Appendix: On the Origin of "Stone–Čech"

As indicated in §1 the first use of the expression "Stone–Čech" is elusive. We queried Melvin Henriksen by e-mail and thought his interesting and informative response should be included here.

> Your question has been gnawing at me ever since I found it on my E-mail in Prague. I checked a number of sources in Warsaw and even asked Engelking. He defended Čech–Stone on an alphabetical basis, but could not answer your question. Kakutani used Čech compactification in his 1942 paper on abstract M-spaces. I encountered Hewitt's 1948 paper while writing my thesis at Wisconsin in 1951 and got a seminar going on it in early 1953 at Purdue. Gillman and I had started to write our paper on P-points in the winter of 1952. Stone wrote an expository paper in the ancestor of the Bulletin of the Polish Academy of Sciences in 1946 or 1947[3] in which he said, in essence, that he knew how to develop βX a la Čech all the long. His nose seemed out of joint when so many authors called it the Čech compactification. I feel sure that some use of it is made by Nakano in his works in the Tokyo Mathematical book series in the 1950's, but he always used his own terminology.
>
> To the best of my knowledge, this terminology is used first in Hewitt's 1948 paper and in Dieudonné's review of it, as you know. It appears in vol 2 of the Bourbaki *Topologie Générale*[4] but in neither case is any justification given. K.P. Hart has the same guesses as we do.
>
> Melvin Henriksen

[3]This is Stone [1949].
[4]See our remarks on the evolution of Bourbaki's usage in §1.

13. Appendix: Lubben's discovery of the "Stone–Čech" Compactification

There is evidence which suggests that R.G. Lubben knew of a "largest" compactification long before Stone [1937] or Čech [1937]. We wrote to F.B. Jones soliciting information. Below are copies of our letter and his response.

Dear Professor Jones:

I have read a draft copy of Ben Fitzpatrick's paper (on R.L. Moore) for the *Handbook of the History of General Topology*, edited by Charlie Aull. In it, he points out that R.G. Lubben has never been recognized for his independent discovery of the Stone–Čech compactification. A colleague and I are writing a history of compactifications for the same volume and had come to the same conclusion regarding Lubben's work that Fitzpatrick had. This was probably caused by the difficulty in reading Lubben's paper. I have tried quite diligently and it is very hard to penetrate. Fitzpatrick indicated that you had informed him that Lubben presented the content of his paper much earlier (1931–32) in a seminar at the University of Texas. Would you care to elaborate on this? I would like to include your comments in the paper for Aull, if this is OK with you.

⋮ ⟨ Irrelevant material deleted ⟩

I would like to thank you in advance for any thoughts you might have on the Lubben question.

My very best regards,
Richard Chandler

Dear Professor Chandler:

In the spring of 1931–32 I was a Senior chemistry major at the University of Texas (Austin) taking R.L. Moore's beginning topology. It met MWF at 4 in a classroom located in the south east corner of the third floor of Garrison Hall. At 3 PM on MWF his "seminar" met, and then quit at 4 PM. Most of my class would be waiting outside, sitting on the stairs up to the attic. We would hear some of what was going on in the seminar. It was at this time that Lubben (pronounced with a long u: L$\bar{\text{uuuu}}$ben) was presenting his results on compactifications and also results on amalgamations and decompositions. I didn't know very much about it. One didn't talk about what he was doing; nor did he work with others. It seemed OK to me at the time—I tended to be a "loner" myself—which seems very odd to me now. But this explains partly why Lubben's work is (or was) not so well known. In 1932–33 I was a part-time Instructor with my office in one corner of Lubben's office, so I heard a good deal more. And of course, his writing is hard to read.

Stone must have read some of Lubben's work because in his hour talk as retiring President of the Society he remarked that Lubben (he used a short "u") had anticipated some of his (Stone's) results. I heard Stone's talk (summer of '41?[5] I'm not sure) and I don't remember what the theorems were—but certainly not the compactification theorems. It might be that Stone didn't read Lubben himself but had one of his graduate students do so. Perhaps Hewitt (at the University of Oregon). At any rate, the referee required Lubben to cut out about the last third and connect the rest with the known literature. Lubben is so strong [willed] that this could have gone on for some time.[6] You could get a better idea of the time of writing by looking at the abstracts of papers submitted to the Society. It wouldn't be easy because the titles are frequently changed or even lost and I don't know that any of those attending Moore's "seminar" are still living. I think Basye might be alive but he is not listed in the latest directory. It may be that Lubben has referred to this seminar.[7] I have moved so many times during the recent past that I have given away so many books and reprints that I'm sure not to be of any real help.

I know that Lubben had done a lot more and still thought there was much more to do. For some reason I never got really interested in working in such "abstract" things although I remember doing an example or two for Lubben. I never got interested in less "abstract" things—like Bing's work in 3-space.

I hope this is of some use and interest and also hope you can read my writing. I might be able to help on some of it if you have a question and it's not too cold.

Regards,
Burton Jones

P.S. You certainly may quote me in your paper. — B.J.

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[5]This was more likely 1944; Stone was President of the American Mathematical Society in 1943–44.

[6]The time between first submission and final acceptance for Lubben [1941] was more than 2 years.

[7]Not that we could discover.

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MINIMAL HAUSDORFF SPACES – THEN AND NOW

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Contents

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The partially ordered collection of all Hausdorff topologies on a set is a complete upper semilattice; this semilattice always contains a maximum element (the discrete topology) but only contains a minimum element whenever the set is finite. When the set X is infinite, there are minimal elements in the semilattice. The minimal elements of the collection of all Hausdorff topologies on a set are called minimal Hausdorff topologies. A compact Hausdorff topology is minimal Hausdorff. In 1940, Katětov characterized those Hausdorff spaces which are minimal Hausdorff and produced an example of a minimal Hausdorff space which is not compact. The theory of the minimal elements of the semilattice of all Hausdorff topologies on a set is presented and developed in this paper.

Introduction

Let X be a set, $\text{top}(X) = \{\tau \subseteq \mathcal{P}(X) : \tau \text{ is a topology on } X\}$, and $\text{Haus}(X) = \{\tau \in \text{top}(X) : \tau \text{ is Hausdorff}\}$. The partially ordered set $(\text{top}(X), \subseteq)$ is a complete lattice with the discrete topology as the top element and the indiscrete topology as the bottom element. The subcollection $\text{Haus}(X)$ is a complete upper semilattice, and when X is infinite, there is no minimum element but there are minimal elements. In this paper, we present the development of the theory of the minimal elements of $\text{Haus}(X)$.

All spaces considered in this paper are assumed to be Hausdorff.

A space X is defined to be minimal Hausdorff, shortened to *minimal* in this paper, if the topology on X, denoted as $\tau(X)$, is a minimal element of the partially ordered set $(\text{Haus}(X), \subseteq)$.

A starting point for the theory of minimal spaces is a problem posed in 1924 by Alexandroff and Urysohn of whether a space can be densely embedded in some H-closed space (the semiregularization of an H-closed space is minimal).

Throughout this paper, the unit interval (= [0,1]) with the usual topology is denoted by $\mathbb{I}$, the space of rational numbers with the usual topology by $\mathbb{Q}$, the space of real numbers with the usual topology by $\mathbb{R}$, the set of positive integers by $\mathbb{N}$, and the set of all integers by $\mathbb{Z}$. The topology on a space X is denoted by $\tau(X)$; if σ is topology on the underlying set of X, then σ is called an *expansion* or *supertopology* (resp. *compression* or *subtopology*) of $\tau(X)$ if $\tau(X) \subseteq \sigma$ (resp. $\sigma \subseteq \tau(X)$).

A technique which is helpful in the theory of minimal topologies is that of semiregularization. A subset A of a space X is *regular open* if $\text{int}_X\text{cl}_X A = A$. The collection of all regular open subsets of a space X is denoted by $\mathcal{RO}(X)$ and is a complete Boolean algebra (see Porter and Woods [1987]); the topology on X generated by $\mathcal{RO}(X)$ is denoted by $\tau(s)$. The space $(X, \tau(s))$ is called the *semiregularization* of X and denoted by $X(s)$. A space is *semiregular* if $\tau(X) = \tau(X(s))$. It follows that the space $X(s)$ is semiregular. It is straightforward to verify the next result.

0.1 (a) A space X is Hausdorff iff $X(s)$ is Hausdorff.
(b) If X is semiregular, then $A \subseteq X$ is semiregular whenever $A \in \tau(X)$ or A is dense in X.
(c) A product of nonempty spaces is semiregular iff each coordinate space is semiregular.

Let X be a space, $\mathcal{U}$ be a maximal filter of dense subsets of X, and $\tau(\mathcal{U})$ the topology generated by $\tau(X) \cup \mathcal{U}$. It is an exercise (see Bourbaki [1941] and Porter and Woods [1987]) to show that $\tau(\mathcal{U})$ is a maximal element of $s(X) = \{\sigma : \sigma(s) = \tau(s)\}$; maximal elements of $s(X)$ are called *submaximal.* A topology σ is a maximal element of $s(X)$ iff there is a maximal filter on the collection of dense sets of X such that $\sigma(s) = \tau(X(s))$. A space X is submaximal iff every dense set is open.

1. Minimal Spaces

When X is a finite space, Haus(X) is a singleton, and, in particular, Haus(X) contains a minimum element. So, the interesting part of the theory of minimal spaces is the setting when X is an infinite space.

A characterization of minimal spaces starts with a necessary condition and a sufficient condition. First a definition is needed. A space X is *H-closed* if X is closed in every Hausdorff space in which X can be embedded.

1.1 (Parhomenko [1939], Katětov [1940], Bourbaki [1941])
A compact space is minimal, and a minimal space is H-closed.

Since there are compact topologies on every set X, it is immediate that Haus(X) always contains minimal elements. By 1.1, the class of minimal spaces lies between the class of compact spaces and the class of H-closed spaces. The next result provides a characterization of H-closed spaces in terms of open covers and is an indication of how close the concept of H-closed is to compactness.

1.2 (Alexandroff–Urysohn [1924])
(a) A space is H-closed iff every open cover has a finite subfamily whose union is dense.
(b) An H-closed space is compact iff it is regular.

The characterization by Katětov, provided in the next result, indicates that the study of the class of H-closed spaces is very close to and includes a study of the class of the minimal spaces.

1.3 (Katětov [1940])
(a) A space is minimal iff it is H-closed and semiregular.
(b) If X is an H-closed space, then $X(s)$ is minimal and $\{\sigma \in \text{Haus}(X) : \sigma \subseteq \tau(X)\}$ is a complete lattice which contains a unique minimal topology, namely $\tau(X(s))$.

A space is *Urysohn* if each distinct pair of points are contained in disjoint closed neighborhoods.

By 1.2(b), we have that a necessary and sufficient condition for an H-closed space to be compact is regularity. We now present the corresponding condition for minimal spaces.

1.4 (Katětov [1940], Bourbaki [1941])
A minimal space is compact iff it is Urysohn.

1.5 (Bourbaki [1941])
(a) A space is minimal iff every open filter on the space with an unique adherent point converges.
(b) A space is H-closed iff every open filter on the space has nonempty adherence.

1.6a Example (Alexandroff–Urysohn [1924])
Let $\mathbb{J}$ be the set $[0,1]$ with topology generated by $\tau(\mathbb{I}) \cup \{\mathbb{Q} \cap \mathbb{I}\}$. The space $\mathbb{J}$ is H-closed, Urysohn, and not compact.

1.6b Example (Urysohn [1925], Katětov [1940], Bourbaki [1941])
Start with a subset Y of $\mathbb{R}^2$. Let $Y = \{(\frac{1}{n}, \frac{1}{m}) : n, \pm m \in \mathbb{N}\} \cup \{(\frac{1}{n}, 0) : n \in \mathbb{N}\}$ and $X = Y \cup \{p^+, p^-\}$. A subset $U \subseteq X$ is defined to be open if
(a) $U \cap Y$ is open in Y (where Y is treated as a subspace of $\mathbb{R}^2$) and
(b) $p^+ \in U$ (resp. $p^- \in U$) implies for some $k \in \mathbf{N}$, $\{(\frac{1}{n}, \frac{1}{m}) : n \geq k, m \in \mathbb{N}\} \subseteq U$ (resp. $\{(\frac{1}{n}, \frac{-1}{m}) : n \geq k, m \in \mathbb{N}\} \subseteq U$).
The space X is minimal but not compact. Observe that $\{(\frac{1}{n}, 0) : n \in \mathbb{N}\}$ is a closed discrete subset of X and so is not H-closed. Also, note that the subset $U = \{(\frac{1}{n}, \frac{1}{m}) : n, m \in \mathbb{N}\}$ is an open subset of X, and $\mathrm{cl}_X U$ is not minimal. However, $\mathrm{cl}_X U$ is the continuous, open perfect image of X.

1.7 (Katětov [1940])
The continuous image of an H-closed space onto a space is H-closed.

Example 1.6b shows that the continuous open perfect image of a minimal Hausdorff space onto a space may not be minimal. Willard [1971] asked whether every H-closed space is the continuous image of a minimal space. In response to Willard's question, Vinson–Dickman and Friedler–Pettey answered the question with a strong affirmative as indicated in the next result.

1.8 (Vinson–Dickman [1977], Friedler–Pettey [1977])
An H-closed space is the continuous, open perfect image of some minimal space.

Both pairs of authors used the same technique, inverse limits, in establishing 1.8. We now present their technique.

Let $(D, \leq)$ be a directed set, $\{X_\alpha : \alpha \in D\}$ a family of spaces, and

$$\{f_\alpha^\beta : X_\beta \to X_\alpha : \alpha, \beta \in D, \alpha \leq \beta\}$$

a family of functions such that f_α^α is the identity function on X_α, and such that if $\alpha \leq \beta \leq \gamma$ then $f_\alpha^\gamma = f_\alpha^\beta \circ f_\beta^\gamma$. Then $\{(X_\alpha, f_\alpha^\beta)\}_D$ is called an *inverse system* and the functions $\{f_\alpha^\beta : \alpha, \beta \in D, \alpha \leq \beta\}$ are called *bonding* functions. The *inverse limit* space X_∞ is the subspace

$$\left\{g \in \prod\{X_\alpha : \alpha \in D\} : \text{ for } \alpha, \beta \in D,\ \alpha \leq \beta,\ f_\alpha^\beta(g(\beta)) = g(\alpha)\right\}.$$

For $\alpha \in D$, the restriction of the projection function $\prod\{X_\beta : \beta \in D\} \to X_\alpha$ to X_∞ is denoted by f_α. In particular, for $\alpha, \beta \in D$ such that $\alpha \leq \beta$, $f_\alpha = f_\alpha^\beta \circ f_\beta$.

1.8a The inverse limit of an inverse system of nonempty H-closed (resp. minimal) spaces with continuous open surjective bonding functions is a nonempty H-closed (resp. minimal) space.

Let X be a space and $\emptyset \neq U \in \tau(X)$. Let $X(U)$ denote the quotient space of the topological sum $X \oplus X (= X \times \{0,1\})$ where $(x,0)$ and $(x,1)$ are identified for $x \in X \backslash U$. Essentially $X(U)$ is X with two copies of U; in $X(U)$, $U \times \{0\}$ and $U \times \{1\}$ are regular open sets. Also, $X(U)$ is H-closed iff X is H-closed and the function $X(U) \to X : (x,i) \to x$, for $i = 0,1$, is continuous, open, perfect, and onto. By combining this construction with 1.8a, we obtain the desired inverse system.

1.8b Let X be an H-closed space and $\lambda = wX$ (the weight of X). There is an inverse system $\{(X_\alpha, f_\alpha^\beta)\}_\lambda$ with continuous, open, perfect surjective bonding functions such that X_∞ is minimal, $wX_\infty = wX$, and for each $\alpha \in \lambda$, f_α is a continuous open surjection.

1.8c (Friedler–Pettey [1977])
There is an inverse system of nonempty minimal spaces with bonding functions that are open embeddings and the inverse limit is not H-closed. Let X (denote it as X_0) be the minimal space described in 1.6b. For $n \in \mathbb{N}$, let

$$X_n = X_0 \setminus \{(\tfrac{1}{m}, \tfrac{1}{k}) : 1 \leq m \leq n,\ 1 \leq |k| \leq n\}.$$

For $n \leq m$, the bonding function f_n^m is the inclusion function – an open embedding. In this case, X_∞ is homeomorphic to

$$\bigcap\{X_n : n \in \mathbb{N}\} = \{(\tfrac{1}{n}, 0) : n \in \mathbb{N}\} \cup \{p^+, p^-\}$$

which is an infinite discrete space. Thus, X_∞ is not H-closed.

Suppose X and Y are spaces. A function $f : X \to Y$ is *θ-continuous* if for each $p \in X$ and open neighborhood U of $f(p)$, there is an open neighborhood V of p such that $f[\mathrm{cl}_X V] \subseteq \mathrm{cl}_Y U$. A function f is *irreducible* if f is onto and $f[A] \neq Y$ whenever A is a proper closed subset of X.

1.9 (Rudolf [1972] and Vermeer–Wattel [1981])
Let Y be a compact space, X a set, $f : Y \to X$ a compact, irreducible surjection. The set X with the topology generated by $\{X \backslash f[A] : Y \backslash A \in \tau(Y)\}$ is minimal and $f : Y \to X$ is perfect and θ-continuous.

1.10 (Katětov [1940])
If X is H-closed and $U \in \tau(X)$, $\mathrm{cl}_X U$ is H-closed.

1.11 (M. Stone [1937], Katětov [1940])
If every closed subspace of a space is H-closed, the space is compact.

1.12 (Chevally–Frink [1941])
A product of nonempty spaces is H-closed iff each coordinate space is H-closed.

1.13 (Obreanu [1950])
A product of nonempty spaces is minimal iff each coordinate space is minimal.

A major question, posed by Alexandroff–Urysohn in 1924, is which spaces can be densely embedded in an H-closed space?

1.14 (Tychonoff [1930])
A space can be embedded in an H-closed space.

Tychonoff realized that 1.14 did not answer the question by Alexandroff and Urysohn as the closure of subspace of an H-closed space is not necessarily H-closed. Some 60 years later, Porter showed that Tychonoff was on the right track and would have succeeded if he had continued.

1.15 (Porter [1992])
The closure of Tychonoff's embedding is H-closed.

In the late 1930's and early 1940's, many topologists, aware of Tychonoff's work, successfully solved the "denseness" portion of the 1924 H-closed embedding problem.

1.16 (Stone [1937], Katětov [1940], Fomin [1941], A.D. Alexandroff [1942], Šanin [1943])
Every space can be densely embedded in an H-closed space.

We now present Katětov's proof of this result. Let X be a space and

$$\kappa X = X \cup \{\mathcal{U} \text{ is an open ultrafilter on } X : a(\mathcal{U}) = \emptyset\}.$$

A set $U \subseteq \kappa X$ is defined to be open if $U \cap X \in \tau(X)$ and for $\mathcal{U} \in U \backslash X$, $U \cap X \in \mathcal{U}$. κX is an H-closed extension of X (called the *Katětov extension* of X) and X is open (and dense) in κX. Furthermore, κX has this property: if hX is an H-closed extension of X, then there is a continuous surjection $f : \kappa X \to hX$ such that $f(x) = x$ for $x \in X$; this was established by Porter–Thomas [1969] in response to a problem posed by Alexandroff [1960].

1.17 (Banaschewski [1961])
A Hausdorff space can be densely embedded in a minimal space iff the space is semiregular.

1.18 (Strecker–Wattel [1967], Porter–Thomas [1969])
A Hausdorff space can be embedded as a closed subspace of a minimal space.

1.19 Herrlich [1966] developed a technique for converting H-closed spaces into H-closed spaces with certain properties. Here is a description of the technique: Let D be a subspace of the space X. For $n \in \mathbb{N}$, let $n = \{0, 1, \ldots, n-1\}$ and $X(D^n) = (D \times n) \cup X \backslash D$. A set $U \subseteq X(D^n)$ is defined to be open if (a) $U \cap (D \times \{i\})$ is open in $D \times \{i\}$ for $i \in n$, and (b) for $x \in U \backslash D$, there is $V \in \tau(X)$ such that $x \in V$ and $(V \times n) \cup V \backslash D \subseteq U$.

(a) If X is H-closed, then $X(D^n)$ is H-closed.
(b) If $n \geq 2$ and $i \in n$, $D \times \{i\}$ is a regular–open subset of $X(D^n)$.
(c) If $n \geq 2$ and X is semiregular, then $X(D^n)$ is also semiregular.
(d) If $n \geq 2$ and X is minimal Hausdorff, then $X(D^n)$ is minimal Hausdorff.

The construction presented after 1.16 shows that every space can be embedded as a dense and open subspace of an H-closed space. By 1.17, a semiregular space can be embedded as a dense subspace of a minimal Hausdorff space. This leads to the problem of whether a semiregular space can be embedded as an open subspace or as an open and dense subspace of a minimal space. By using 1.19(b,d), it is immediate by 1.17, that if X is a semiregular space, X can be embedded as an open subspace of a minimal space and $X \oplus X$ can be embedded as an open and dense subspace of a minimal space; Vermeer established these two facts in 1979. Also, in 1979, Vermeer asked whether a semiregular space can be embedded as a dense and open subspace of a minimal space. This question was answered by Vermeer as follows:

1.20 (Vermeer [1983]) Let D be an infinite discrete space and $X = D \cup \{\infty\}$. A set $U \subseteq X$ is defined to be open if $U \subseteq D$ and $\infty \in U$ implies $D \backslash U$ is countable (i.e., the one–point Lindelöf extension of D).
(a) If $|D| > 2^{2^\omega}$, then X cannot be embedded as a dense and open subspace of a minimal space.
(b) If $|D| = 2^{2^\omega}$, then X can be embedded as a dense and open subspace of a minimal space.

Since it is consistent with ZFC to assume that $2^{2^{\omega_1}} > 2^{2^\omega}$ and consistent to assume that $2^{2^{\omega_1}} = 2^{2^\omega}$, it follows by 1.20 that the question asked by Vermeer in [1983] can not be answered in ZFC without additional set–theoretic assumptions.

1.21 Also, using 1.19, Herrlich [1966] has given an example of a minimal space which is of first category. Start with the unit interval $\mathbb{I}$ (usual topology) and let D be the dense subspace of rationals in $\mathbb{I}$. Then $\mathbb{I}(D^2)$ is minimal Hausdorff. Since

$\mathbb{I}\backslash D$ is a closed, nowhere dense in $\mathbb{I}(D^2)$ and $D \times 2$ is a countable set (recall that $2 = \{0, 1\}$), it follows that $\mathbb{I}(D^2)$ is the countable union of closed, nowhere dense subsets.

Mioduszewski [1971] has shown that an H-closed space is not the countable union of compact nowhere dense spaces; in the space described in 1.21, the closed, nowhere dense subset $\mathbb{I}\backslash D$ is not compact (nor H-closed). Dickman and Porter [1975] extended Mioduszewski's result by showing that an H-closed space is not the countable union of θ-closed, nowhere dense subsets (a subset is *θ-closed* if every point not in the subset has a closed neighborhood disjoint from the subset). Also, in 1971, Mioduszewski asked whether an H-closed space is the countable union of H-closed, nowhere dense subspaces. McCoy and Porter answered this question in the negative.

1.22 (McCoy–Porter [1977])
Let Q be the set of rational numbers in the unit interval $\mathbb{I}$ and $P = \mathbb{I}\backslash Q$. Let $X = P\times\{0\} \cup Q\times(\mathbb{Z}\backslash\{0\})$. A set $U \subseteq X$ is open if $(x, i) \in U$ implies there is $V \in \tau(\mathbb{I})$ such that $x \in V$ and
(a) if $i = 0$, $(V \cap P)\times\{0\} \cup (V \cap Q)\times(\mathbb{Z}\backslash\{0\}) \subseteq U$,
(b) if $i = 1$, there is $n \in \mathbb{N}$ such that

$$((V \cap Q)\times\{1\}) \cup \bigcup\{(V \cap Q)\times\{j\} : j \geq n\} \subseteq U,$$

(c) if $i = -1$, there is $n \in \mathbb{N}$ such that

$$((V \cap Q)\times\{-1\}) \cup \bigcup\{(V \cap Q)\times\{j\} : j \leq -n\} \subseteq U,$$

and (d) if $i \in \mathbb{Z}\backslash\{-1, 0, 1\}$, $(V \cap Q) \times \{i\} \subseteq U$.

The space X is minimal (H-closed and semiregular). Also, $T = P \times \{0\} \cup (Q \times \{-1, 1\})$ is minimal and a nowhere dense subset of X. Now, $X = T \cup \bigcup\{\{y\} : y \in X\backslash T\}$ is the countable union of minimal, nowhere dense subspaces.

Recall, by 1.10, a regular-closed subset of an H-closed space is H-closed. So, for a minimal space X, since the regular-open sets from a base for the open sets of X, it follows that the family of H-closed subsets of X form a base for the closed sets of X. The converse is also true, i.e., a space in which the family of H-closed sets form a base for the closed sets is minimal. By 1.11, a space in which every closed set is H-closed is compact.

A concept related to an H-closed set is that of H-set. A subset A of a space X is an *H-set* if, whenever $\mathcal{C}$ is a family of sets open in X and $\mathcal{C}$ covers A, there is a finite subset $\mathcal{F} \subseteq \mathcal{C}$ such that $A \subseteq \bigcup\{\text{cl}F : F \in \mathcal{F}\}$. Dickman and Porter [1984] proved that a space X is minimal iff the family of H-sets of X form a base for the closed subsets of X. Viglino [1969] showed that 1.11 is not true when "H-closed" is replaced by "H-set"; in fact, a space is called *C-compact* if every into a space is closed.

1.25 (Dickman–Zame [1969], Viglino [1971])
A C-compact space is functionally compact, and a functionally compact space is minimal.

The space described in 1.6b is minimal but not functionally compact.

1.26 (Goss–Viglino [1970] and Lim–Tan [1974])
Let $X = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ and $\{\mathbb{N}_i : i \in \mathbb{N}\}$ be a partition of infinite subsets of $\mathbb{N}$. For $i, j \in \mathbb{N}$, let $\mathbb{N}_i = \{n_{ik} : k \in \mathbb{N}\}$, $P_i = (\{i\} \times (\mathbb{N} \cup \{0\})) \cup (\mathbb{N}_i \times \mathbb{N})$, and $Q_j = ((\mathbb{N} \cup \{0\}) \times \{j\}) \cup (\mathbb{N} \times \mathbb{N}_j)$. A set $U \subseteq X$ is defined to be open if
(a) $(i, 0) \in U$ implies there are $m, n \in \mathbb{N}$ such that

$$\Big(\{i\} \times [m, \infty)\Big) \cup \Big((\mathbb{N}_i \backslash \{n_{ik} : 1 \leq k \leq m\}) \times [n, \infty) \Big) \subseteq U,$$

(b) $(0, j) \in U$ implies there are $m, n, r \in \mathbb{N}$ such that

$$\{1\} \times [n, \infty) \cup \Big[\Big([n, \infty) \times (\mathbb{N}_j \backslash \{n_{jk} : 1 \leq k \leq m\}) \Big) \setminus \bigcup \{P_i : 1 \leq i \leq r\} \Big] \subseteq U,$$

and (c) $(0, 0) \in U$ implies there is $n \in \mathbb{N}$ such that

$$(\mathbb{N} \times \mathbb{N}) \setminus \{P_i \cup Q_i : 1 \leq i \leq n\} \subseteq U.$$

The space X is functionally compact but is not C-compact.

2. Katětov Spaces

A space is *Katětov* if it has a minimal subtopology. In 1940, Katětov established that if X is H-closed, then $X(s)$ is minimal; i.e., an H-closed space is Katětov. For a set X, a natural question is whether Haus(X) is atomic, i.e., is every space Katětov? In 1961, Bourbaki mentioned that it was unknown if the space $\mathbb{Q}$ of rational numbers is Katětov. However, the next result was established.

2.1 (Bourbaki [1941])
The space $\mathbb{Q}$ has no compact subtopology.

2.2 (Herrlich [1965])
The space $\mathbb{Q}$ is not Katětov.

A straightforward consequence of Herrlich's proof is that a Katětov space is not the countable union of compact, nowhere dense sets. Actually, the result by

Herrlich is an immediate corollary to the following 1961 result by Bourbaki.

2.3 (Bourbaki [1961])
The set of isolated points of a countable, H-closed space is dense.

A useful tool in studying spaces is the absolute of a space. The *absolute* of a space X is a pair (EX, k_X) where EX is an extremally disconnected, Tychonoff space and $k_X : EX \to X$ is a perfect, irreducible θ–continuous surjection. If $\mathcal{RO}(X)$ is the complete Boolean algebra of regular open sets of X and $St(\mathcal{RO}(X))$ is the Stone space of $\mathcal{RO}(X)$, then EX is the dense subspace $\{\mathcal{U} \in St(\mathcal{RO}(X)) : a_X\mathcal{U} \neq \emptyset\}$ and $k_X : EX \to X$ is the function defined by $\{k_X(\mathcal{U})\} = a_X\mathcal{U}$. The absolute is uniquely determined in this sense: if F is an extremally disconnected Tychonoff space and $f : F \to X$ is a perfect, irreducible θ–continuous surjection, then there is a homeomorphism $h : F \to EX$ such that $k_X \circ h = f$. The absolute provides a connection between a space and a rather nice space, namely, an extremally disconnected, Tychonoff space; also, the link is nice as it is perfect, irreducible, θ–continuous, and onto. The strength of this link in the theory of H-closed spaces is illustrated by the result that a space X is H-closed iff EX is compact.

2.4 (Dow–Porter [1982] and Vermeer [1985])
A space is Katětov iff it is the remainder of some H-closed extension of a discrete space.

Proof: Let Y be a minimal space. We use a technique developed by Porter [1993] to obtain an extension of a discrete space with Y as the remainder. Let $S = Y \cup (EY \times \mathbb{N})$. For each $U \in \tau(Y)$, let $U^{\#} = U \cup \bigcup\{\{\alpha\} \times \mathbb{N} : U \in \alpha \in EY\}$. A set $W \subseteq S$ is open if $p \in Y \cap W$ implies there is $U \in \tau(Y)$ and a finite set $F \subseteq EY \times \mathbb{N}$ such that $p \in U^{\#}\backslash F \subseteq W$. Note that the points of $EY \times \mathbb{N}$ are isolated in S, S is minimal, S is an extension of $EY \times \mathbb{N}$, and Y is homeomorphic to $S\backslash(EY \times \mathbb{N})$. Let X be a Katětov space. Then there is a subtopology σ on X such that $Y = (X, \sigma)$ is minimal. So, there is an extension S of a discrete space $D \times \mathbb{N}$ such that S is minimal and Y is homeomorphic to $S\backslash(EY \times \mathbb{N})$. Consider the expansion topology μ on S generated by

$$\tau(S) \cup \{W \cup (EY \times \mathbb{N}) : W \in \tau(X)\}.$$

The space (S, μ) is H-closed, an extension of $EY \times \mathbb{N}$, and $(S, \mu) \setminus (EY \times \mathbb{N})$ is homeomorphic to X. Conversely, suppose H is an H-closed extension of a discrete space D. EH is a compact space and $k_H^{\leftarrow}[D]$ is a discrete space of EH. So, $EH\backslash k_H^{\leftarrow}[D]$ is a compact subspace such that $k_H|C : C \to H\backslash D$ is a perfect, irreducible surjection. By 1.9, $H\backslash D$ has a minimal subtopology.

2.5 (Porter–Vermeer [1985])
A space is Katětov iff it is the perfect image (not necessarily continuous) of an H-closed space.

The strength of this latter result, especially the lack of a continuity hypothesis, is demonstrated by the next corollary.

2.6 Corollary
Let Y be a H-closed extension of a space X and A a closed discrete subspace of X. If $|Y\backslash X| \leq |A|$, then X is Katětov.

Proof: Let $f : Y\backslash X \to A$ be any one-to-one function. The function g defined by $g(x) = x$ for $x \in X$ and $g(x) = f(x)$ for $x \in Y\backslash X$ is a perfect surjection from Y to X.

A surprising result and an indication of the delicate balance between the existence and nonexistence of a minimal subtopology is provided by the next example.

2.7 Example (Porter–Vermeer [1985])
A Katětov space X such that $X(s)$ ($= X$ with the topology generated by the regular open sets) is not Katětov. Recall that the Stone-Čech compactification $\beta\mathbb{Q}$ of $\mathbb{Q}$ has cardinality of 2^{2^ω}. Let $\beta^+\mathbb{Q}$ be $\beta\mathbb{Q}$ with this topology: $U \subseteq \beta\mathbb{Q}$ is open in $\beta^+\mathbb{Q}$ iff $U \cap \mathbb{Q}$ is open in $\mathbb{Q}$ and $p \in U\backslash\mathbb{Q}$ implies there is some $V \in \tau(\beta\mathbb{Q})$ such that $p \in V$ and $V \cap \mathbb{Q} \subseteq U$. Let $X = \mathbb{Q} \times \beta^+\mathbb{Q}$.

Note that $\beta^+\mathbb{Q}\backslash\mathbb{Q}$ is a closed, discrete subspace of $\beta^+\mathbb{Q}$; so, $\{0\} \times (\beta^+\mathbb{Q}\backslash\mathbb{Q})$ is a closed, discrete subspace of X of cardinality 2^{2^ω}. Now, $\beta\mathbb{Q} \times \beta^+\mathbb{Q}$ is an H-closed extension of X and $|(\beta\mathbb{Q} \times \beta^+\mathbb{Q})\backslash X| \leq |\{0\} \times (\beta^+\mathbb{Q})|$. By 1.24, X is a Katětov space. But $X(s) = \mathbb{Q}(s) \times (\beta^+\mathbb{Q})(s) = \mathbb{Q} \times \beta\mathbb{Q}$ is the countable union of compact spaces. By the comment after 2.2, $X(s)$ is not Katětov.

2.8 (Porter–Vermeer [1985])
(a) A regular, Lindelöf, Čech-complete space is Katětov.
(b) A complete metric space is Katětov.
(c) If the absolute EX of a space X is Katětov, so is X.
(d) Let Y be a compact, extremally disconnected space without isolated points and S a countable discrete subspace of Y. Then $Y\backslash S$ is not Katětov.
(e) A countable Katětov space is scattered.
(f) A space which is locally compact (resp. locally H-closed) at all except one point has a compact (resp. H-closed) subtopology.

The first two examples of 2.9 show that the converses of 2.8(c) and 2.8(e) are not true.

2.9a Example (Porter–Vermeer [1985])
A space X with a compact subtopology such that EX is not Katětov. Consider $E\mathbb{I}$ and for each $n \in \mathbb{N}$, choose a point $s_n \in k_{\mathbb{I}}^{\leftarrow}(\frac{1}{n})$. Then $S = \{s_n : n \in \mathbb{N}\}$ is a countable discrete subspace of the compact extremally disconnected Hausdorff space $E\mathbb{I}$, and $\mathrm{cl}_{E\mathbb{I}}S\backslash S$ is a compact, nowhere dense subset of $E\mathbb{I}\backslash S$. Let X be the quotient space of $E\mathbb{I}\backslash S$ where $\mathrm{cl}_{E\mathbb{I}}S\backslash S$ is identified to a point. The

quotient function from $E\mathbb{I}\backslash S$ to X is perfect and irreducible. Since $E\mathbb{I}\backslash S$ is extremally disconnected and Tychonoff, $E\mathbb{I}\backslash S$ and EX are homeomorphic by the uniqueness property of absolutes. By 2.8(d), EX is not Katětov. The space X is zero-dimensional and locally compact at all points of X except where $\mathrm{cl}_{E\mathbb{I}}S\backslash S$ is identified to a point. By 2.8(f), X has a compact subtopology.

2.9b Example (Porter–Tikoo [1989])
A scattered, countable space X which is not Katětov. Let $X = \mathbb{Q} \times (\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\})$. A subset $U \subseteq X$ is defined to be open if for each $(r, 0) \in U$, there are $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $(r - \varepsilon, r + \varepsilon) \times \{\frac{1}{n} : n \geq m\} \subseteq U$. Now, X is a countable space and points of $X\backslash(\mathbb{Q} \times \{0\})$ are isolated. Also, $\mathbb{Q} \times \{0\}$ is a closed, discrete subspace of X. So, X is scattered. A modification of the proof of 2.2 shows that X is not Katětov.

2.9c (Parhomenko [1939])
Consider the subspace $X = \bigcup\{\{\frac{1}{n}\} \times \mathbb{I} : n \in \mathbb{N}\} \cup \{(0,0), (0,1)\}$ of $\mathbb{I}^2$. The space X does not have a compact subtopology. Since $(\mathrm{cl}_{\mathbb{I}^2}X)\backslash X = \{0\} \times (0,1)$ is a σ-compact space, it follows by 2.8(a), X is Katětov.

A space is *completely Hausdorff* if each pair of points can be separated by a real-valued continuous function. A completely Hausdorff space X has the *Stone–Weierstrass property* if each subring K of $C^*(X)$ which separates points and contains all constant functions has the property that each $f \in C^*(X)$ is the uniform limit of a sequence of functions in K.

For a space X, $X(w)$ denotes the set X with the topology generated by $\{X\backslash Z(f) : f \in C^*(X)\}$.

2.10 Let X be a Hausdorff space. The following are equivalent:
(a) X is H-closed and Urysohn.
(b) (Katětov [1940]) $X(s)$ is compact.
(c) (Porter [1966]) X is H-closed and has the Stone–Weierstrass property.
(d) (Proizvolov [1965], Porter [1966]) X is H-closed and every pair of disjoint H-closed subsets can be separated by a real-valued continuous function.
(e) (Stephenson [1967]) X is H-closed and $X(s) = X(w)$.

2.11 (Dow–Porter [1982])
If X is an H-closed space, then $|X| \leq 2^{\chi(X)}$.

If X is a first countable, compact space, we know that $|X| \leq \omega$ or $|X| = 2^\omega$.

2.12 (Dow–Porter [1982] and Gruenhage [1993])
The statement that there is an uncountable, first countable H-closed space with size less than 2^ω is equiconsistent.

3. Problems

A special case of Katětov spaces is those spaces with compact subtopologies. This section starts with an old problem by Banach dating back to July 17, 1935.

3.1 Banach (reported in Banach [1947] and Maudlin [1981])
Identify those metrizable spaces with a compact metrizable subtopology.

Even among completely separable metrizable spaces, a simple characterization of those spaces with a compact metrizable subtopology is unknown. It is known that the space of irrationals (Sierpiński [1929]) and Banach spaces (Klee [1957]) have compact metrizable subtopologies. By 2.2, we know that the space of the rational numbers is not Katětov, and in particular it has no compact subtopology.

Katětov [1949] proved that a countable regular space has a compact metrizable subtopology iff the space is scattered. Kulpa and Turzański [1988] extended Katětov's results by showing that a scattered, hereditarily paracompact space has a compact subtopology. Also, Pytkeev [1976] has shown that a strongly zero-dimensional, complete metrizable space has a compact subtopology (a Tychonoff space X is a strongly zero-dimensional iff βX is zero-dimensional). The basic problem can be traced to P.S. Alexandroff.

3.2 (Attributed to Alexandroff in Smirnov [1968])
Identify those spaces (specifically, the second countable Tychonoff spaces) with a compact subtopology.

Smirnov [1968] has shown that the subspace

$$\begin{aligned} X = {} & \{(x,y,0) \in \mathbb{R}^3 : x^2+y^2=1\} \cup \\ & \{(x,y,\tfrac{1}{2n-1}) \in \mathbb{R}^3 : x^2+y^2 \le 1, n \in \mathbb{N}\} \cup \\ & \bigcup \left\{ \left\{(\sin(\tfrac{2k\pi}{2^n}), \cos(\tfrac{2k\pi}{2^n}), z) \in \mathbb{R}^3 : 0 \le k \le n, 0 \le z \le \tfrac{1}{2n-1}\right\} : n \in \mathbb{N} \right\} \end{aligned}$$

of $\mathbb{R}^3$ is a second countable, σ-compact, connected, locally connected, completely metrizable space with no compact subtopology. This answers negatively another problem (also attributed to Alexandroff in Smirnov [1968]) of whether a second countable, or whether a σ-compact, connected, locally connected, completely metrizable space has a compact subtopology.

An external characterization of Katětov spaces is provided in 2.4. An internal characterization of Katětov spaces should be helpful and is the next basic problem.

3.3 Find an internal characterization of Katětov spaces.

The results of 2.8 report some classes of spaces which are Katětov. However, there are two classes of spaces for which it is unknown if they are Katětov. These are listed in the next two questions.

3.4 (Porter–Vermeer [1985])
Is a Čech-complete, paracompact space Katětov?

3.5 (Porter–Vermeer [1985])
If X is a completely metrizable space, is EX Katětov?

By 2.4, we know that a space is Katětov iff it is the remainder of an H-closed extension of a discrete space. This result motivated Porter in 1988 to propose the next problem.

3.6 Characterize those spaces which are remainders of minimal extensions of discrete spaces.

Now, the remainder of an H-closed extension of a discrete space is an H-set. Vermeer asked this question which is related to 3.3.

3.7 (Vermeer [1985])
If a space X can be embedded as an H-set of some H-closed space, is X Katětov?

By 1.11, a space is compact if every closed set is H-closed. On the other hand, by 1.24, there are noncompact spaces in which every closed set is an H-set. Vermeer has asked these two related questions.

3.8 (Vermeer [1985])
(a) Is an H-closed space compact if all H-sets are H-closed?
(b) Is an H-closed space compact if all H-closed subspaces are minimal?

The strong results of 1.8 motivate the question of whether every H-closed space is the finite-to-one continuous image of a minimal space. Friedler and Pettey [1977] have shown there is an H-closed space X which is not the finite-to-one continuous image of a minimal space. The space X is the unit interval $\mathbb{I}$ with a topology finer than the usual topology $\tau(\mathbb{I})$. The topology of X is generated by $\tau(\mathbb{I}) \cup \{U \backslash H : U \in \tau(\mathbb{I})$ and H is a countable infinite subset of $\mathbb{I}\backslash\{0\}$ such that 0 is the only limit point of $H\}$. The space X is not first countable at the point 0.

Friedler and Pettey [1972] asked this next question.

3.9 Is every first countable H-closed space the finite-to-one continuous image of a minimal space?

The preservation of H-closedness in the inverse limit technique under continuous open surjective bonding functions is the key in establishing 1.8. By 1.8c, "surjective" cannot be dropped in establishing 1.8a. Vinson and Dickman [1970] asked if "open" can be removed.

3.10 Is the inverse limit of nonempty H-closed spaces with continuous surjective bonding functions both nonempty and H-closed?

By 1.18, a space can be embedded as a closed subspace of some minimal space. Dickman and Porter [1984] showed that a minimal space containing $\mathbb{Q}$ as a closed subspace cannot be embedded in any C-compact space. The embedding problem for functionally compact spaces remains unsolved even though it was posed twenty-five years ago by Dickman and Zame.

3.11 (Dickman–Zame [1969])
Characterize those spaces which can be embedded in a functionally compact space.

A problem proposed by E. van Douwen and noted in Kunen [1990] is whether there is a compact homogeneous space with cellularity greater than 2^ω? In a private conversation with J. Porter in 1991, J. Norden asked the following two questions.

3.12 (a) Is there a minimal, homogeneous space with cellularity greater than 2^ω?
(b) Is there an H-closed, homogeneous space with cellularity greater than 2^ω?

A space X is *minimal totally disconnected* if $\tau(X)$ is a minimal element in $\{\tau \in \mathrm{Haus}(X) : (X, \tau)$ is totally disconnected$\}$. Bdeir and Stephenson [1994] have shown that a minimal totally disconnected space is H-closed and have constructed an H-closed, totally disconnected space X such that $X(s)$ is not totally disconnected. They asked the next question.

3.13 (Bdeir–Stephenson [1994])
Is a minimal totally disconnected space necessarily minimal?

We conclude this paper with a question by Friedler; this is the only place in the paper where the "Hausdorff" assumption is dropped.

3.14 (Friedler [1988])
Is every compact T_1, countable, second countable space the continuous image of a minimal space?

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A HISTORY OF RESULTS ON ORDERABILITY AND SUBORDERABILITY

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Dedicated to Horst Herrlich on his 60th birthday

This is an early history of the theory of ordered sets and of results characterizing various classes of orderable and suborderable spaces until 1975. Many excellent results (such as most of D.J. Lutzer's work and the influence of the Souslin conjecture) which only give properties of these spaces are not covered. However, at the end of this article historical references are given for the more general theory of ordered spaces including set-theoretical developments.

A topological space is defined to be (linearly) orderable if there exists a linear order on the underlying set such that the open interval topology coincides with the original topology on the space. A space is suborderable if it can be embedded into an orderable space.

Order is a concept as old as the idea of number, and much of early mathematics was devoted to constructing and studying various subsets of the real line.

G. Cantor developed the theory of sets and introduced a rich new class of ordered sets – the cardinals (see [16], Introduction). In 1883 ([13]) Cantor postulated the existence of ω_0, developed the ordinal numbers, and defined addition and multiplication of these numbers. He derived the idea that any set can be well ordered and promised to return to this subject later. He also defined order type for the special case of well ordered sets and was the first to construct the famous Cantor set (example 10, pages 590–591 of [13]).

In 1895 ([14]) and 1897 ([15]) Cantor investigated more extensively than in his earlier papers the properties of cardinal and ordinal numbers and paid far more attention to logical form. He introduced in [14] the general definition of ordinal types and defined an arithmetic on them. Also in [14] it was shown that a countable linearly ordered set which is densely ordered with no end points is order-isomorphic to the rationals with the usual order. F. Hausdorff generalized this

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result in his 1914 book ([42], theorem II on page 181) by proving that two linearly ordered sets of regular cardinality α are order-isomorphic if in each of these sets every initial segment has cofinality α or its complement has coinitiality α.

The first orderability theorem that I am aware of was published in 1905 by O. Veblen (see page 89 of [104]). He proved that every metric continuum with exactly two non-cut points is homeomorphic to the unit interval. Veblen also combined the concepts of ordered set and topological space in defining a simple arc. For similar early results see [66], [54], [92], [38], [79], and theorem 6.3, page 56 of [109].

In 1909 R. Baire ([6], page 103) proved that the irrationals are homeomorphic to the Cartesian product $\mathbb{N}^{\mathbb{N}}$ by considering the convergence of sequences and using ideas from the 19th century that the irrationals in $(0, 1)$ are representable by the collection of infinite continued fractions, which in turn corresponds bijectively to the set of infinite sequences of positive integers. Also see section 31 of [5] (the first part of this two part paper) for earlier results related to this.

L.E.J. Brouwer ([12]) characterized the Cantor set in 1910 as a compact, perfect, totally disconnected metric space. In parts of chapters 7 and 8 in Hausdorff's 1914 book ([42]), ordered spaces are discussed including Brouwer's result, and on page 457 of the appendix, references are given to early work in arbitrary ordered spaces. (The reference for A. Haar and D. König was given as Journ. f. Math. 139 (1908), which is probably incorrect. Perhaps it was a mixup between [36] and either [34] or the announcement in [35]. The title of these papers was repeated in H. Hahn's [37]. Hausdorff gave no page numbers for the Haar and König article or the Hahn article.)

Also in 1910 M. Fréchet ([32]) proved that every countable metrizable space is homeomorphic to a closed subset of the rational numbers.

A punctiform is a space that contains no nondegenerate continua. Hence, every totally disconnected space is a punctiform, and a subset of the real line is a punctiform iff it is 0-dimensional. However, there are connected punctiforms (see [56]). S. Mazurkiewicz ([75], page 163) used continued fractions in 1917 to prove that a punctiform dense G_δ subset of a nondegenerate interval in the real line is homeomorphic to the set of irrational numbers in $(0, 1)$. He cited Baire's 1906 paper but was probably also aware of the $\mathbb{N}^{\mathbb{N}}$ characterization.

Fréchet's result was extended by W. Sierpiński ([93]), who showed in 1920 that every countable dense-in-itself metric space is homeomorphic to the rational numbers.

In the same year Mazurkiewicz and Sierpiński ([76]) proved that any compact, countable, metric space P is homeomorphic to a well ordered set. Moreover, they showed that if $P^{(\alpha)}$ is the last nonempty derivative of P and $P^{(\alpha)}$ consists of n points, then P is homeomorphic to a set of ordinal type $\omega^\alpha \times n + 1$.

In 1921 Sierpiński ([94]) showed that a separable metric punctiform is suborderable iff it is 0-dimensional. With this, Mazurkiewicz's characterization of the irrationals can be described as on page 370 in [29]: a G_δ set which is both dense and co-dense in a separable, 0-dimensional, completely metrizable space.

P. Alexandroff and P. Urysohn ([4]) used continued fractions in 1928 to characterize the irrational numbers as a topologically complete, separable, 0-dimensional, metric space that contains no nonempty compact open set. In 1937 this result was rediscovered by Hausdorff ([45]) using Baire's result of the homeomorphism between the irrationals and $\mathbb{N}^{\mathbb{N}}$ (which Hausdorff referred to as "der Bairesche Nullraum"). It is interesting to note that it was not unusual for Alexandroff and Hausdorff to independently prove the same result. For example they both verified the truth of the Continuum Hypothesis for the class of Borel sets on the real line ([1] and [43]), and they both proved that every nonempty compact metrizable space is the continuous image of the Cantor set ([3] and theorem V in section 35, page 197 in [44] as well as the announcement in [2]).

L.W. Cohen ([19]) stated in 1929 that a separable metric space X is suborderable iff

(a) each component of X is orderable,
(b) the set of cut points of each component of X is open, and
(c) X is *diskontinuierlich* (i.e. each point of X has a neighborhood base, each member of which is clopen relative to the complement of the point's component).

Although this theorem is true, in 1962 H. Herrlich pointed out that Cohen's proof is completely wrong and impossible to patch up. Unknowingly, J. de Groot ([24]) and M.E. Rudin ([89]) used Cohen's result before 1962. I am not sure if Cohen knew of a topological characterization of (a) in the above theorem, although definitions for arcs homeomorphic to intervals (open, closed, half open) in $\mathbb{R}^2$ had been known (e.g. [79]). In particular, did Cohen know the next result?

In 1936 A.J. Ward ([107]) proved that if a metric space X is separable, connected, and locally connected, and if $X - \{p\}$ consists of exactly two components for every element p in X, then X is homeomorphic to the real line. In 1970 S.P. Franklin and G.V. Krishnarao [31] showed that if in the above theorem 'metric' were changed to 'regular', then the resulting theorem would be true. They constructed a counterexample to demonstrate that 'metric' in the above theorem cannot be changed to 'Hausdorff'.

A space is weakly orderable if there exists a linear order on its elements such that the original topology on the space is finer than its open interval topology. In 1941 S.Eilenberg ([28]) proved that a connected space X is weakly orderable iff the subset of its square $X \times X$ obtained by deleting the diagonal is not connected. This condition is also necessary and sufficient for a connected, locally connected space to be orderable. He also showed that a nondegenerate, connected, weakly orderable space has precisely two compatible orders and that these orders are inverses of each other. Hence, this property concerning compatible orders also holds for connected orderable spaces. Approaching this problem from a different point of view, E. Michael ([77]) showed in 1951 that a connected Hausdorff space X is weakly orderable iff there is a continuous function (called a selection) $f : \mathcal{C}(X) \to X$, where $\mathcal{C}(X)$ is the space of all nonempty compact subsets of X with the Vietoris topology, and $f(A) \in A$ for each $A \in \mathcal{C}(X)$. The above results from [28] then imply that "weakly orderable" can be replaced by "orderable" if X is also locally connected.

I am unaware of other orderability results during the decade between [28] and [77]. However, other results were published on ordered sets and spaces throughout the 1940s especially concerning order types and trees (see [98] and [74]).

A space is non-archimedean (defined by Đ. Kurepa in [62] and [63] under the name "spaces with ramified bases") if it is T_1 and has a base any two members of which are either disjoint or comparable by inclusion. (The expression "non-archimedean" for these spaces comes from A.F. Monna [78].) It is interesting to note that in 1906 Baire ([5]) constructed a sequence of decompositions of the irrationals which he used in 1909 ([6]) as a non-archimedean basis for the space in proving his characterization of the irrationals. A non-archimedean space is suborderable. However, it is unclear where this result first appeared. In P. Papić's 1953 thesis ([83], see [84]) it is shown that a non-archimedean space is orderable iff it has a tree base $\mathcal{B}$ such that, for $\mathcal{B}' \subseteq \mathcal{B}$, if $\cap\mathcal{B}'$ is nonempty compact open then $\cap\mathcal{B}' \in \mathcal{B}'$. Papić ([84]) and his thesis advisor, Kurepa ([64]), also showed

1. if the union of all compact open subsets is closed in a non-archimedean space X, then X is orderable, and
2. a non-archimedean metrizable space is orderable.

(These papers were not widely known, and rediscovery in the literature of Kurepa's and Papić's results was a common occurrence.) In 1956 de Groot ([25]) proved that a metrizable space X is non-archimedean iff $\operatorname{Ind} X = 0$. (De Groot did not use the result with the erroneous proof by Cohen mentioned above as he did in [24].) This with (2) above yields the fact that a metrizable space X is orderable if $\operatorname{Ind} X = 0$, which was proved independently by H. Herrlich in 1965 and generalizes a result from 1962 by I.L. Lynn. However, it should be mentioned that neither Kurepa nor Papić were cited by de Groot, and so he was probably unaware of these orderability results.

A T_1 space is linearly uniformizable (called "espace pseudodistancié" in [61] by Kurepa) if its topology can be derived from a base for a uniformity which is linearly ordered by reverse inclusion. The non-metrizable ones are non-archimedean. Kurepa showed ([65]) that a non-metrizable linearly uniformizable space is orderable if it is dense-in-itself. Using an idea of Papić, Kurepa constructed for every uncountable regular cardinal κ a non-orderable linearly uniformizable space which has a compatible uniformity with a base of cofinality κ. The development (especially by Kurepa and Papić) of the theory of linearly uniformizable and non-archimedean spaces is the subject of Chapter 12 of the book [73] by Z.P. Mamuzić.

L. Gillman and M. Henriksen ([33]) proved in 1954 that the Hewitt realcompactification of an orderable space X that does not contain a closed discrete monotone transfinite sequence of measurable cardinality is suborderable. Moreover, the proof in ([33]) of this theorem is also valid if X were only suborderable.

A space X is defined to be generalized orderable if there is a linear order on its elements such that the original topology on X is finer than its open interval topology, and each of its points has a local base consisting of (possibly degenerate) intervals. E. Čech introduced the class of generalized orderable spaces and proved that it coincides with the class of suborderable spaces. I was unable to discover where the above definition and theorem originally appeared. They may be contained in the work on ordered spaces he did before World War II. They are, however, contained in Čech's book ([17] par. 9, pages 113–122) published in 1959. They are also contained in what started out as an English translation of Čech's 1959 book but ended up as a complete rewrite by Z. Frolík and M. Katětov ([18] theorem 17A.22, pages 285–286) published in 1966, four years after Čech's death. Later D.J. Lutzer ([68]) defined a topological space with a linear order to be a GO space if the conditions in Čech's definition were satisfied.

In 1961 H.J. Kowalsky ([60], theorem 15.5, page 109) proved that a connected T_1 space X is orderable iff X is locally connected T_1 and among every three distinct, connected, proper subsets of X, there are two which together do not cover the space X. Therefore, he obtained a better characterization of connected orderable spaces than did Eilenberg ([28]). Later (a footnote in [27] from 1968) D. Zaremba-Szczepkowicz observed that a connected space X is weakly orderable iff among every three distinct points of X, there is one which separates the other two. (Proofs of this were given by H. Kok in [58] and E. Wattel in [108].) Again [28] implies that "weakly orderable" can be replaced by "orderable" if X is also locally connected.

B. Banaschewski ([8]) generalized Eilenberg's result in 1961. A linear order on a set is dense if between any two distinct elements there is a third one. Every compatible order on a connected orderable space is a dense order. A uniform space X is precompact (or totally bounded) if for each entourage D of X there exists a finite subset S of X such that $D(S) = X$. Banaschewski proved that a precompact uniform space $(X, \mathcal{U})$ has a compatible dense order iff $\mathcal{U}$ contains a basis $\mathcal{V}$ with the following properties:

(a) if $V \in \mathcal{V}$ and if $V^{n+1} = V^n \circ V$ for each $n > 0$, then $\cup\{V^n : n > 0\} = X \times X$ and
(b) if $x, y \in X$ and there are two sequences $x = x_0, x_1, \ldots, x_n = y$ and $y = y_0, y_1, \ldots, y_n = x$ in X such that for some sets $V_0, \ldots, V_n$ in $\mathcal{V}$ we have $V_i(x_i) \cap V_{i+1}(x_{i+1})$ and $V_i(y_i) \cap V_{i+1}(y_{i+1})$ both nonempty for $0 < i \leq n-1$, then $V_j(x_j) \cap V_j(y_j)$ is nonempty for some j.

I.L. Lynn proved in 1962 ([70]) that every 0-dimensional, separable, metric space is orderable, which improves upon the 1921 theorem of Sierpiński ([93]) mentioned earlier. But as mentioned above this is implied by earlier results of de Groot, Kurepa and Papić. In 1964 Lynn ([72]) gave several complicated sufficient conditions for a subset of the real line, i.e. a suborderable separable metric space, to be orderable.

The 1962 doctoral dissertation of H. Herrlich ([46]) contains several important orderability theorems. He showed that a totally disconnected metric space is orderable iff it is strongly 0-dimensional (Ind 0) (also see [47], 1965). Hence, Lynn's theorem ([70]) is also a corollary of this theorem. But again as mentioned above this is implied by earlier results of de Groot, Kurepa and Papić since any totally disconnected suborderable space is strongly 0-dimensional. Herrlich characterized the connected orderable spaces as locally connected *randendlich* T_1 spaces, where a space is *randendlich* if every connected subspace has at most two non-cut points (also see [48], 1965). Thus, as Kowalsky did before him, Herrlich obtained a better characterization of connected orderable spaces than did Eilenberg.

As was stated earlier Herrlich discovered that L.W. Cohen's proof of his characterization of suborderable separable metric spaces is incorrect. However, in his thesis Herrlich proved this result using a completely different method than did Cohen. In 1968 ([49]) he substituted the condition "each component of X has a clopen neighborhood base" for the condition "X is *diskontinuierlich*" in this theorem. It is easy to show that these two conditions are equivalent if X is a T_2 space each of whose components has a finite number of boundary points.

The orderable subsets of the real line were characterized in 1965 by M.E. Rudin ([90]). This generalizes the results of Lynn in [72]. She defined a subset Q of a subspace X of the real line $\mathbb{R}$ to be the union of all nontrivial components of X all of whose endpoints belong to the closure of $(\mathrm{cl}X) - X$. Now taking closures to be in X, Rudin proved that X is orderable iff

(a) if Q is a proper closed subset of X, then $X - Q$ is not compact, and
(b) if I is an open interval of $\mathbb{R}$ and if p is an endpoint of I such that $\{p\} \cup (I \cap (X - Q))$ is compact and the intersection of the closures of $I \cap (X - Q)$ and $I \cap Q$ is $\{p\}$, then the component of X containing p, if any, is a singleton.

At the end of her paper, Rudin stated a theorem which characterizes the orderable subspaces of an ordered space. However, it is more difficult in general to show that a space satisfies the last condition of this theorem than to show that the space is orderable. (But about 20 years later I was able to simplify the statement of this last condition and found it useful in proving that certain spaces were not orderable. This condition is now called a friendship function.)

Lutzer showed in 1969 ([67]) that an orderable space is metrizable iff it has a G_δ diagonal. Hence, a non-metrizable suborderable space is not orderable if it has a G_δ diagonal (e.g. the Sorgenfrey line). This and results by Lynn ([72]) and Rudin ([90]) were the only nontrivial theorems that I know of up until that time that differentiate between suborderability and orderability.

In 1970 M. Venkataraman, M. Rajagopalan, and T. Soundararajan announced that a necessary condition for the Stone–Čech compactification of a Tychonoff space to be orderable is that it be normal and countably compact. They also gave several sufficient conditions. This later appeared in [106]. I proved in 1971 ([85]) that in the above condition if normal is changed to suborderable, then the resulting condition is both necessary and sufficient (also see [86]). In 1973 this result was obtained independently by J.H. Weston.

Also in [106] it was shown that (1) a totally disconnected topological group with a G_δ identity element is orderable iff it is metrizable and strongly 0-dimensional, and (2) a non-totally disconnected topological group is orderable iff it contains an open normal subgroup isomorphic to $\mathbb{R}$ as an additive group with the usual topology (and hence the entire group is metrizable).

Generalizing a result mentioned earlier by Mazurkiewicz and Sierpiński ([76]), J.W. Baker in 1972 ([7]) characterized the compact ordinal spaces. In the following let $X^{(\alpha)}$ denote the αth derived set of a space X. If λ is the least ordinal α such that $X^{(\alpha)}$ is finite and $n = \mathrm{card} X^{(\alpha)}$, then (λ, n) is called the characteristic of X. A space is scattered iff it contains no dense-in-itself subsets. Baker defined a space X to have property (D) if each x in X has a neighborhood base consisting of a decreasing, possibly transfinite, sequence $\{U_\alpha : \alpha < \lambda\}$ of clopen sets with the additional property that for each limit ordinal β with $\beta < \lambda$, $\bigcap\{U_\alpha : \alpha < \beta\} - U_\beta$ contains at most one point. Baker showed that a compact scattered Hausdorff space with characteristic (λ, n) is homeomorphic to $\omega^\lambda \times n + 1$.

In 1972 D. Jakel, M. Rajagopalan, and T. Soundararajan ([53]) characterized those spaces that are homeomorphic to the compact ordinal space $\omega_1 + 1$. Their characterization seems to be nicer than the above characterization of Baker when restricted to this case. They proved that a compact Hausdorff space X is homeomorphic to $\omega_1 + 1$ iff

(a) $\mathrm{card} X = \aleph_1$,
(b) X has a unique P-point x_0 (a P-point is a point for which a countable intersection of neighborhoods is a neighborhood),
(c) $X - \{x_0\}$ can be expressed as $\bigcup\{A_\alpha : \alpha \in \mathcal{A}\}$ where each A_α is a collection of compact sets such that if $\alpha, \beta \in \mathcal{A}$, then either $A_\alpha \subseteq A_\beta$ or $A_\beta \subseteq A_\alpha$, and
(d) each A_α in (c) is countable and, for each $\alpha \in \mathcal{A}$ the set $\bigcup\{A_{\alpha'} : \alpha' \in \mathcal{A};\ A_{\alpha'} \subset A_{\alpha'};\ A_{\alpha'} \neq A_{\alpha'}\}$ has at most one more point in its closure.

De Groot and P.S. Schnare ([26]) showed in 1972 that a compact T_1 space is orderable iff there is an open subbase $\mathcal{S}$ of the space which is a union a two nests of open sets such that every cover of the space by elements of $\mathcal{S}$ has a two element subcover. They used a similar subbasis notion to characterize those spaces that are products of compact orderable spaces.

In 1973 J. van Dalen and E. Wattel ([102]) gave complete topological characterizations of orderable spaces and of suborderable spaces. Using the subbasis idea in [26] they stated that a T_1 space is suborderable iff there exists an open subbase of the space which is a union a two nests. A T_1 space is orderable iff there exists a subbase $\mathcal{S}$ of the space which satisfies the following properties:

(a) $\mathcal{S}$ is the union of two nests, and
(b) if $S_0 \in \mathcal{S}$ and $S_0 = \cap\{S : S_0 \subseteq S \text{ and } S \in \mathcal{S} - \{S_0\}\}$, then $S_0 = \cup\{S : S \subseteq S_0 \text{ and } S \in \mathcal{S} - \{S_0\}\}$.

However, a subbase of a T_1 space X satisfying (a) induces a unique order up to inverse on X. Conversely, any order on X induces a unique maximal subbase $\mathcal{S}$ satisfying (a). E. Deák ([23]) independently discovered this result. Deák defined a concept called a direction of a set X, which consists of a collection of ordered pairs (G, F), $G \subseteq F \subseteq X$, satisfying certain properties which imply that the collection of all such G's and the collection of all such $X - F$'s form two nets satisfying conditions (a) and (b) above. As a historical footnote to these results, both van Dalen and Deák presented their solutions at the same conference in Keszthely, Hungary in 1972 (see [101] and [23]). It is still an open problem to find a topological characterization of orderable spaces, or even compact orderable spaces, that does not induce a particular order on a space.

The word "suborderable" was coined in my 1973 Ph. D. dissertation ([86]) as a translation of the term "*unterordnungsfähig*" which is found in Herrlich's dissertation. Solving a problem of Herrlich ([50]) the orderability and suborderability of metric spaces were characterized in [86]. (Also see [88].) It was shown in [86] that a metric space X is suborderable iff

(a) each component of X is orderable,
(b) the set of cut points of each component of X is open, and
(c) each closed subset of X which is a union of components has a clopen neighborhood base.

(Note that there had previously been several characterizations of connected orderable spaces.) In a space X let Q denote the union of all nondegenerate components each of whose non-cut points has no compact neighborhood. Modifying conditions in [90] it was also shown that a metric space X is orderable iff

(a) X is suborderable,
(b) $X - Q$ is not a proper compact open subset of X, and
(c) if W is a neighborhood of an element p of X and K is the component in X containing p such that $(W - K) - Q$ has compact closure and $\{p\}$ is the intersection of the closures of $(W - K) - Q$ and $(W - K) \cap Q$, then K is a singleton.

Also raised in [86] was the conjecture that orderability is equivalent to monotone normality for compact, totally disconnected, separable spaces. (I gave a counterexample in 1990.)

In 1975 P.J. Nyikos and H.-C. Reichel ([82]) extended the results on topological groups in [106] by proving that a non-metrizable topological group is orderable iff the identity element has a linearly ordered clopen local base. Hence, a topological group is orderable iff it is linearly uniformizable.

For additional historical information see [16], [20], [100], [29], [42], [51], [52], [68], [73], [74], [80], [81], [91], [97], [99] and [110]. Excellent bibliographies in the areas of ordered sets and ordered spaces are to be found in [100], [68], [74], [86] and [91]. There is also a 1980 survey article by D.J. Lutzer ([69]). Many set-theoretic aspects of ordered sets and spaces are covered in a 1984 survey by S. Todorčević ([98]). [9] and [10] are the proceedings of two workshops on topology and linear orderings held in 1980 and 1981.

A number of dissertations on ordered spaces are cited in the bibliography. Maarten Maurice was the advisor for all but one of them between 1972 and 1984. Contained in a memorial tribute to him is an article ([40]) about his contributions as the leader of a research group on ordered spaces at the Vrije Universiteit te Amsterdam.

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HISTORY OF CONTINUUM THEORY

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Contents

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By a *continuum* we usually mean a metric (or Hausdorff) compact connected space. The original definition of 1883, due to Georg Cantor, [126], p. 576, stated that a subset of a Euclidean space is a continuum provided it is perfect (i.e. closed and dense-in-itself, or – equivalently – coincides with its first derivative) and connected, i.e. if for every two of its points a and b and for each positive number ϵ there corresponds a finite system of points $a = p_0, p_1, \dots, p_n = b$ such that the distance between any two consecutive points of the system is less than ϵ. The equivalence of the two definitions for compact metric spaces is shown e.g. in Kuratowski's monograph [390], vol. 2, §47, I, Theorem 0, p. 167.

1. Introduction

Without any doubt the roots of the concept of a continuum lie in the notion of continuity, which goes back as far as the Greeks, who studied the linear continuum and tried to understand and clarify its nature. However, the history of this, as well as the contribution of medieval and later mathematicians (up to the XVIII century) is adequately covered in the literature, and is not the subject of this article.

In the second half of the XIX century, especially in the last decades, mathematicians started a slow (and difficult) progress in establishing the basic concepts of *Analysis Situs*, as topology was called then. These concepts were defined, studied, and slowly ordered in a system of results by Lorentz Leonard Lindelöf (1827–1908), Marie Ennemond Camille Jordan (1838–1922), Georg Cantor (1845–1918), Arthur Moritz Schoenflies (1853–1928), Jules Henri Poincaré (1854–1912), Giuseppe Peano (1858–1932), David Hilbert (1862–1943), Eliakim Hastings Moore (1862–1932), Felix Hausdorff (1868–1942), Félix Edouard Justin Émile Borel (1871–1956), Henri Léon Lebesgue (1875–1941), Maurice René Fréchet (1878–1973), Hans Hahn (1879–1934), Frigyes Riesz (1880–1956), Luitzen Egbert Jan Brouwer (1881–1966), Robert Lee Moore (1882–1974), Wacław Sierpiński (1882–1969), Solomon Lefschetz (1884–1972), Zygmunt Janiszewski (1888–1920), Stefan Mazurkiewicz (1888–1945), Eduard Čech (1893–1960), Bronisław Knaster (1893–1980), Heinz Hopf (1894–1971), Pavel Sergeevich Alexandroff (1896–1982), Kazimierz Kuratowski (1896–1980), Raymond Louis Wilder (1896–1982), Pavel Samuilovich Urysohn (1898–1924), and their successors.

2. Basic concepts

Topology, as a branch of mathematics with its own face, was born but in the 20th century, and it had its origin in the process of giving mathematical analysis a rigorous foundation ([447] contains extensive information about this; compare also Hermann Weyl's classical work [678]). Its basic apparatus was point set theory, created in the XIX century by G. Cantor, and its fundamental concepts were derived from geometry of the Euclidean n-space. Indeed, most objects studied in the very beginning period of topology were considered as subsets of the real line, the plane or – more generally – of the Euclidean n-space for an arbitrary integer n. The first class of abstract spaces to which several notions and results, discovered in the infancy of topology, were successfully generalized was the class of metric spaces (see [202], p. 256). The concept of a metric was defined by M. Fréchet in 1906 in his thesis ([224], p. 17; compare also [226], p. 54). The term "metric space" was introduced by F. Hausdorff in [278], p. 211. Also in [278] the notion of the Hausdorff space was established, however for a long time after this metric spaces were much more popular for researchers in continuum theory than topological ones, in particular those of Hausdorff. For the role of Hausdorff's book [278] in the development of topological concepts see [82] and Appendix 3 of [447].

One of the basic concepts in topology was connectedness. The present definition of this concept was introduced in 1893 by C. Jordan [327] for the class of compact subsets of the plane; generalization to abstract spaces is due to F. Riesz [565], N.J. Lennes [422] and F. Hausdorff [278]. A systematic study of connectedness was originated by F. Hausdorff [278] and by B. Knaster and K. Kuratowski [354]. The reader is referred to the article [705] by R.L. Wilder, where various aspects of evolution of this concept are pointed out, and early contributions to definition and basic properties of connectedness by B. Bolzano, G. Cantor, F. Hausdorff, C. Jordan, N.J. Lennes, F. Riesz, A. Schoenflies, W.H. and G.C. Young, and others, are discussed.

Another topological concept which is related to the notion of a continuum is that of compactness. The genesis of the notion of compactness is connected with the Borel theorem proved in 1895 ([86], p. 51), stating that every countable open cover of a closed interval has a finite subcover, and with the Lebesgue observation of 1905 ([399], p. 105) that the same holds for every open cover of a closed interval. In [87] Borel generalized this result, in Lebesgue's setting, to all bounded closed subsets of Euclidean spaces. For numerous references see [287]. The present definition is essentially due to P.S. Alexandroff and P.S. Urysohn [8]. For the infancy of compactness and for development of this notion see, e.g., historical and bibliographical notes in Engelking's monograph [202], pp. 132–133.

An especially important technique for obtaining interesting examples of continua is the use of nested intersections. A theoretical base for such constructions is a theorem which was established in the very early period of continuum theory, and which says that the intersection of a decreasing sequence of continua is a continuum. As it is stated in Zoretti's paper [726], p. 8, the result (in a slightly different form) was proved by Paul Painlevé (1863–1933) for continua in the plane (in one of his lectures at l'École Normale Supérieure in 1902). For a generalization see [386].

3. The Jordan Curve Theorem and the concept of a curve

One of the first problems of a purely topological nature, obviously related to connectedness, in particular to continuum theory, was the Jordan Curve Theorem, a statement saying that a simple closed curve in the plane cuts the plane into two regions and is their common boundary. The theorem was pointed out and discussed in 1887 by C. Jordan ([327], vol. III, pp. 587–594; also vol. I, §96–103). However, Jordan's proof was not a proof in the modern sense. The first rigorous proof of the result was given in 1905 by Oswald Veblen (1880–1960) [640]. In the subsequent twenty years the theorem was reproved, completed and generalized by a number of topologists (see a survey article [177] for details).

At the very beginning of its history, a continuum was understood as a connected, closed and bounded (sometimes not necessarily bounded) subset of a Euclidean space. But this definition had been formulated on the basis of a study of another (and that time much more important) object of mathematical investigation: the concept of a line or curve (compare G.T. Whyburn's article [699]).

In the second half of the XIX century a curve was frequently understood as the path (or locus) of a continuously moving point. Such a definition was formulated by C. Jordan in his book [327], and the term "Jordan curve" denoting a subset of the plane or of the the space which is the continuous image of a closed interval has been commonly accepted. However, it turned out that this definition is too general. In 1890 G. Peano showed [552] that the unit square can be obtained as a continuous image of a closed interval of reals. The original definition of Peano's function was arithmetical (compare also [136]). Its geometrical interpretation was found in 1900 by E.H. Moore [520] and (independently) by A. Schoenflies ([593], p. 122). Other functions "fulfilling the square" (i.e. mappings from $[0,1]$ onto $[0,1]^2$) have been constructed by H. Hahn, D. Hilbert, W. Sierpiński and others (see [264], [286], [555], [608]; see also a survey article [409]).

Peano's unexpected example shattered the intuitive notion of the dimension of a space as being the least number of continuous parameters needed to describe the space, and it precipitated a search for a rigorous definition of dimension. The first satisfactory definition was given in 1913 by L.E.J. Brouwer [119], who developed an idea of H. Lebesgue [401]. Another definition of dimension was formulated in the early twenties by P.S. Urysohn [634] and Karl Menger (1902–1985) [486]. Although almost the same definition of the (Menger–Urysohn) concept of dimension has been obtained independently, P.S. Alexandroff argued for Urysohn's priority ([5], pp. 25–35). See [200], pp. 6–9, for more details on the early period of the development of the notion of topological dimension.

Peano's result was an impulse for seeking a more adequate definition of a curve (and of a surface, too). G. Cantor defined a curve (in the plane) as a closed subset of the plane having empty interior (compare [492], Chapter 1, Section 10, p. 71). Having defined the dimension of a metric (or a topological) space, a curve was understood as a one-dimensional continuum. This definition agrees with Cantor's one for planar curves. In 1912 Z. Janiszewski [307] announced that there exists a curve containing no arc. Early theorems concerning basic properties of curves, presented in a form of a theory of these continua, were obtained in the twenties by Menger [487] and [492], and by Urysohn [637].

One of the important concepts related to the structure of curves is that of the order of a point in a curve (or, more general, in a continuum). It was considered in 1906 by W.H. Young and G.Ch. Young in [718], pp. 219–221, for planar curves in the sense of Cantor. Namely, Young and Young have defined a point p in a curve X to be of order k provided that there are in X exactly k continua every two of which have p as the only common point. Janiszewski, trying to give a more precise definition, used the concept of an irreducible continuum, introduced in 1909 by Ludovic Zoretti [727] as an attempt to characterize the segment $[0,1]$. A continuum is called irreducible (between two of its points) if no proper subcontinuum contains these points. The simplest example is an arc.

The first large paper devoted entirely to the study of irreducible continua was Janiszewski's thesis [306] published two years after Zoretti created the concept. Modifying the definition by Young and Young of a point of order k in a continuum, Janiszewski assumed ([306], Chapter 4, §1, p. 63) that the continua under consideration (which are mutually disjoint outside the point p) have to be irreducible and their union forms a neighborhood of p. He calls such points regular. In the particular case when the irreducible continua are arcs, the point p is said to be simple. Janiszewski's thesis [306] contains several theorems related to various structural properties of sets of these points. All these efforts certainly had an influence upon the final form of the definition of the order of a point in the Menger–Urysohn theory of curves. Namely, in its modern meaning, the concept has been understood as, roughly speaking, the minimum cardinality of boundaries of small open neighborhoods of the point. Points of order one are called end points, ones of order three or more, ramification points. If the order of each point of a curve is finite, then the curve is called regular; if the order is at most countable, the curve is defined to be rational. More generally, a space is said to be rational (or rim-finite, respectively), if it has a basis of open sets with countable (finite, respectively) boundaries. A curve X is said to be Suslinian provided each collection of pairwise disjoint subcurves of X is countable.

A number of theorems concerning interrelations between these concepts, as well as concerning the structure of sets of points satisfying various conditions expressed in terms of the order of a point are due to K. Menger and P.S. Urysohn, and can be found in [492] and [637]. A study of rational curves was the subject of Helene Reschovsky's thesis in 1930, [563]. Many results concerning the structure of regular, rational, and Suslinian curves are proved in a sequence of four papers by H. Cook, A. Lelek and L. Mohler in the early seventies devoted to the topology of curves [412], [413], [419] and [164]. A comprehensive exposition of this part of the theory of curves is given in Whyburn's book [695] (where it is shown that every hereditarily locally connected curve is rational), as well as in vol. 2 of Kuratowski's monograph [390]. We mention here a few results only.

Special attention was paid to the structure of the set of ramification points of a curve. As early as 1915 Sierpiński constructed [609] a curve every point of which was a ramification point (called later the Sierpiński triangular curve, or the Sierpiński gasket). In the final part of the Polish version of his paper Sierpiński wrote: "... one year ago Mr. Stefan Mazurkiewicz found an example of a curve every point of which is a ramification point of an infinite order ... The curve is constructed by Mr. Mazurkiewicz by dividing a square into 9 smaller squares (using lines parallel to the sides) and removing the interior of the central square, and repeating this procedure for every of the 8 remaining squares, etc. in infinitum." So, the idea of the construction of the famous universal planar curve (the Sierpiński carpet) had come from S. Mazurkiewicz. Next year the basic property of this curve (i.e. its universality) was proved [610]. Both curves are early examples of fractals or self-homeomorphic spaces, which have been extensively studied recently.

The Sierpiński carpet is composed exclusively of points of order the continuum. The set of such points in a continuum was studied by many authors. K. Menger ([487], p. 287), P. Urysohn ([637], p. 19) and W. Hurewicz ([297], p. 759) proved in the twenties that in every compact space the set of irregular points, as well as the set of irrational ones, is the union of nondegenerate continua. Other early results in this direction are due again to Mazurkiewicz [464], [466], [393] (the last paper is with Kuratowski). The structure of points of a finite order was studied by William Leake Ayres in [33] and by P. Urysohn in [637]. The latter has shown ([637], Chapter 6, the fundamental theorem, p. 105) that if all points of a curve C are of order at least n, where n is a natural number, then C contains a point of order at least $2n - 2$. Urysohn also constructed ([637], p. 109–123) examples of curves composed entirely of points of order ω, of order $\aleph_0$ and, for each natural n, of orders n and $2n - 2$.

4. Local connectedness; plane continua

Coming back to Jordan's definition and the result of Peano, the situation was clarified by H. Hahn and S. Mazurkiewicz who obtained (independently) about 1913 a result that characterized local connectedness. Recall that a space is said to be locally connected provided that each of its points has a local base composed of connected open sets. If the space is metric, this means that for every point of the space and for every number $\epsilon > 0$ there exists an open connected neighborhood of the point whose diameter is less than ϵ. First ideas related to this concept (compare Kuratowski's monograph [390], vol. 2, §49, footnote 1 on p. 227, and

Menger's book [492], p. 40) are traced to Pia Nalli [536], S. Mazurkiewicz [457] and H. Hahn [265]. Namely, it has been shown (for Mazurkiewicz see [457] and [460]; for Hahn see [265] and [266]) that a metric continuum is locally connected if and only if it is a continuous image of the unit closed interval $[0,1]$. In 1920 Sierpiński showed [614] that a continuum is locally connected if and only if it can be represented as the union of finitely many arbitrarily small subcontinua (property S). The two characterizations of locally connected continua are called the Hahn–Mazurkiewicz–Sierpiński theorem. In the same year Kuratowski characterized [375] locally connected continua as continua in which the components of open subsets are open, thus making the first step towards Hahn's well-known characterization of arbitrary locally connected spaces [268].

A very important circle of results concerning the structure of locally connected continua is related to arcwise connectedness. N.J. Lennes in 1911 proved [422] that if every point of a space, except two of them, separates the space between these two points, then there exists a one-to-one continuous transformation of the space onto the closed unit interval $[0,1]$ of reals. Later the conclusion of Lennes' theorem was taken as a definition of an arc.

A point of a continuum is called a cut point of the continuum provided that its complement is not connected. In 1920 R.L. Moore showed [522] that every nondegenerate continuum contains at least two non-cut points. Topological characterizations of an arc as a continuum containing exactly two non-cut points (or expressed in similar terms) were obtained in 1916–1920 by W. Sierpiński [611] and [612], S. Straszewicz [623] and R.L. Moore [522], who also characterized a simple closed curve as a continuum that is separated by any pair of its points. Some characterizations of an arc and a simple closed curve were also considered in 1911 by Z. Janiszewski in his thesis [306]. J.R. Kline showed [347] that a continuum which is separated by no connected subset is a simple closed curve. For other results in this direction see [223].

A concept which is defined using some disconnection properties is that of θ_n- and of θ-continuum. A continuum is said to be a θ_n-continuum (a θ-continuum) provided that the complement of each of its subcontinua has at most n (at most finitely many) components. These concept were introduced and studied by R.W. FitzGerald in 1974 in his thesis [214]. The structure of θ_n- and of θ-continua was investigated in connection with monotone decompositions (see, e.g., [236], [238], [650]).

In 1913 S. Mazurkiewicz ([457] and [460]) and in 1916 R.L. Moore [521] proved that a locally connected continuum is locally arcwise connected. Several proofs of arcwise connectedness of locally connected continua are given in Whyburn's book [695]. Arcwise connectedness of locally connected continua is related to the concept of the order of a point by Menger's n-arc theorem: if a locally connected continuum X contains a point p of order at least n in X (where n is a natural number), then there exist n arcs in X which are pairwise disjoint except at p ([489], p. 98; for generalizations see [492], Chapter 6, and [696]). The existence of arcs in rational curves was studied by J. Grispolakis and E.D. Tymchatyn in [248].

Locally connected continua became one of the main fields of interest of Gordon Thomas Whyburn (1904–1969). Define a continuum to be cyclic if every two its points are contained in a simple closed curve contained in the continuum. In the cyclic connectedness theorem, Whyburn proved ([682] and [686]) that a locally connected continuum is cyclic if and only if it has no cut points. Define a true cyclic element of the continuum to be a connected set consisting of more than one point, which is maximal with respect to having no cut points of itself. The following theorem (see [219], p. 61) summarizes the basic facts of cyclic element theory created by Whyburn during the thirties and presented in a concise form in his book [695]. Let a locally connected continuum X be given and let $p \in X$ be neither a cut point nor an end point. Then p is contained in a unique true cyclic element. A true cyclic element is a locally connected cyclic continuum. X has at most a countable number of true cyclic elements, and their diameters tend to zero. Any two of them intersect in at most a point and the point of intersection must be a cut point. Besides [695] and [219] the reader is referred to [473] and [475] for more information about the history and development of cyclic element theory. Some generalizations are contained in [476] (compare also [474]).

Special attention was paid to those locally connected continua that contain no simple closed curve, i.e. to dendrites. From among various results concerning these curves let us mention the construction of a universal dendrite, due to Tadeusz Ważewski (1896–1972). Recall that a space U is said to be universal for a class C of spaces provided that U is in C and each member of C can be embedded in U (i.e. is homeomorphic to a subspace of U). The (planar) universal dendrite was constructed in 1923 in [673] (see also [492], p. 318; an approximation of this curve is pictured on the cover of Nadler's book [533], where a modern description of Ważewski's dendrite is presented). Besides, Ważewski constructed in [673] for every natural number $n \geq 3$ a universal dendrite for the class of dendrites all points of which are of order at most n.

The existence of a universal space for all planar curves was known even earlier. As was mentioned above, already in 1916 W. Sierpiński had proved [610] that a curve which was later called the Sierpiński carpet is universal in the class of all planar curves. This curve has been characterized in 1958 by Whyburn [698]. Sierpiński's result was extended in 1926 by Menger, who constructed in the second part of [488] a universal curve (see also [490], Chapter 12, pp. 345–360; for a nice picture see [84], p. 502). R.D. Anderson gave in 1958 a topological characterization of Menger's universal curve in [19] (see also [20]; further important properties of the curve are in [456]). For each pair of nonnegative integers n and k, with $k > n$, Menger described [488] an n-dimensional continuum M_n^k in the Euclidean k-space E^k which is universal with respect to containing homeomorphic copies of every n-dimensional continuum which can be embedded in E^k. For any $k > 0$ the set M_0^k is the standard Cantor set, M_1^2 is the Sierpiński universal planar curve, and M_1^k (for each $k \geq 3$) is the Menger universal curve. These continua were characterized in 1984 by Mladen Bestvina [55].

In 1931 G. Nöbeling produced [540] a different space which he showed to be a universal space in the class of n-dimensional separable metrizable spaces. Other essential steps for higher dimensional continua were made by S. Lefschetz [405], H.G. Bothe [107], M.A. Shtan'ko [603] and others (see [200], p. 129, for more detailed information). In 1931 G. Nöbeling also proved [541] that in the family of all rim-finite spaces, rim-finite compact spaces and of all rim-finite continua there is no universal element. Using results of H. Reschovsky's [563] K. Menger showed ([492], p. 294) that in the family of all rational compact spaces and in the family of all rational continua there does not exist a universal element. The classes of all metric continua and of all locally connected metric continua have the Hilbert cube as universal continuum by the well-known Urysohn embedding theorem [635]. The existence of universal continua for various classes of not necessarily locally connected continua will be discussed later.

Phenomena related to the topology of the plane, in particular ones concerning the structure of continua disconnecting the plane, started to be studied in a very early period in the creation of topology (see, e.g., a very informative book by B. von Kerékjártó [345]). One of the earliest results in this area was the Brouwer–Phragmén theorem. In 1885 Edvard Phragmén proved [553] that the boundary of an open bounded subset of the plane contains a nondegenerate continuum. In 1910 L.E.J. Brouwer showed [115] that the boundary of every bounded component of the complement $\mathbb{R}^2 \setminus K$ of a continuum K in the plane $\mathbb{R}^2$ is itself a continuum (for a generalization to an n-dimensional space see P.S. Alexandroff [1]). And in 1921 it was shown by M. Torhorst [628] that if the continuum K is locally connected, then the mentioned boundary is locally connected as well, and moreover, it is rim-finite and contains no θ-curve. Compare also related results by B. von Kerékjártó [346], K. Kuratowski [384], R.L. Moore [521], G.T. Whyburn [683], [694], and R.L. Wilder [703].

Disconnecting the plane by continua was the subject of interest of Z. Janiszewski (compare [352]). In 1913 he proved in his habilitation thesis [308] that if the intersection of two planar continua neither of which disconnects the plane is connected, then their union also does not disconnect the plane (the first theorem of Janiszewski). A locally connected continuum X having the property that for every two of its subcontinua A and B with non-connected intersection there exist two points in X which are separated in X by the union $A \cup B$ was named (by Kuratowski, [390], vol. 2, p. 505) a Janiszewski space. The second theorem of Janiszewski said that the 2-dimensional sphere is a Janiszewski space. Leo Zippin in 1929 proved [724] that a locally connected continuum is a Janiszewski space if and only if each of its cyclic elements which does not reduce to a point is homeomorphic to the 2-dimensional sphere. A later contribution was made in 1922 by Anna M. Mullikin in her thesis [529]. Further generalizations and modifications of these two theorems of Janiszewski are contained in [195], [313], [355], [395], [528], [537], [624] and [625]. A strengthening of these theorems was proved in the middle of the forties by R.H. Bing in [57] and [58].

Kuratowski's investigations of continua disconnecting the plane and, in particular, his analysis of Janiszewski's theorems, led to his characterization of the 2-dimensional sphere [382] presented at the International Congress of Mathematicians in Bologna in 1928 [387]. He proved that every nondegenerate compact metric space X that satisfies the conditions:

(1) the space X is connected and locally connected,
(2) no one-point set disconnects X,
(3) if the intersection $A \cap B$ of two subcontinua A and B of X is not connected, then the union $A \cup B$ disconnects X,

is homeomorphic to the 2-dimensional sphere. For this and other achievements of Kuratowski in continuum theory see Engelking's article [201]. For various questions related to characterizations of the sphere see R.L. Moore's paper of 1916, [521], p. 131, Irmgard Gawehn's [232] and H. Whitney's [681]. Leo Zippin proved in 1930 [725] that the 2-sphere can be characterized as a locally connected continuum containing at least one simple closed curve and every simple closed curve is its irreducible separator. Other topological characterizations of the sphere were given by L. Zippin [724], R.L. Wilder [701], S. Claytor [157], and E.R. van Kampen [331]. This last characterization was incorporated in G.T. Whyburn's book [695]. J.R. Kline asked whether a nondegenerate locally connected continuum which is separated by each of its simple closed curves but by no pair of its points is homeomorphic with the surface of a sphere. Partial solutions were obtained by D.W. Hall [270] and [271]. An affirmative answer was given in 1946 by R.H. Bing [59], who presented in 1949 another proof, using brick partitioning, in [63]. Further progress was made by R.A. Slocum in [618]. For the history of the problem and for early references see [331]. See also Jones' article [320].

A continuum X is said to be unicoherent if, for every decomposition of X into the union of two subcontinua, the intersection of the subcontinua is connected. This condition was considered in 1926 by K. Kuratowski in [380], where it has no name and is denoted by α, and three years later in [383], where the connection between unicoherence, the validity of the Brouwer fixed-point theorem and the validity of Janiszewski's theorems in locally connected continua are thoroughly investigated. This paper called the topologists' attention to the notion of unicoherence and promoted further study of this notion, carried out in the thirties by K. Borsuk [90], [95], [96] and S. Eilenberg in his thesis [195].

The problem of the possibility of embedding a (one-dimensional) continuum in the plane was solved first for local dendrites (i.e. for continua each point of which has a neighborhood being a dendrite) in 1930 by K. Kuratowski, who proved [385] that a local dendrite is nonplanable if and only if it contains a curve homeomorphic to one of the following two, called primitive skew graphs: the curve K_1 which is the union of all six edges of a tetrahedron and of a segment joining two midpoints of a pair of disjoint edges, and the curve K_2 which is the union of all six edges of a tetrahedron and of four segments joining the center of the tetrahedron with its four vertices. The same characterization is valid if the class of local dendrites is replaced by one of graphs or surfaces (i.e. 2-dimensional manifolds) distinct from the sphere [385], or by the class of locally connected continua having no cut points ([156]; for a simpler proof see [518]).

Kuratowski's original proof, based on some properties of curves lying in the plane, was purely topological; since then quite a number of proofs of this fundamental result of graph theory, based on various ideas, have been published. Further topological investigations of this topic were carried out by S. Mazurkiewicz [469] and S. Claytor. To show that local acyclicity is essential in his result, Kuratowski constructed in [385] two locally connected nonplanable curves C_1 and C_2 which can be described as the union of a null-sequence of some similar (very simple) cyclic graphs, all lying in the plane and tending to a limit point p which is not accessible from the complement of the union in the plane, and of a segment perpendicular to the plane at p. In 1937 W. Shieffelin Claytor obtained [157] a complete characterization of nonplanable locally connected continua: a locally connected continuum can be embedded in a 2-sphere if and only if it contains no homeomorphic image of the primitive skew graphs K_1 and K_2 or the curves C_1 and C_2. For further references in this direction see, e.g., Kuratowski's monograph [390], vol. 2, p. 306, footnote (1)). However, as was proved in 1973, no theorem of this kind can be shown for all continua, not necessarily locally connected [140].

One of a few known obstacles to embedding a curve in the plane is that it contains a large family of pairwise disjoint triods. A continuum is defined to be triodic if it contains three continua such that the common part of all three of them is a nonempty proper subcontinuum of each of them and the common part of every two of them. In 1929 R.L. Moore proved [526] (see also [525]; for a more general result see [407]) that each uncountable family of triodic continua in the plane contains an uncountable subfamily every two members of which do intersect. For various concepts of triodic continua see R.H. Sorgenfrey's paper [619].

A metric d on a metric space X is said to be convex (in the sense of Menger [491], p. 81 and 82) provided that for each two distinct points x and y of X there exists a point z in X different from x and y which lies between x and y, i.e. such that $d(x,z) + d(z,y) = d(x,y)$. Convex metrics were studied by a number of authors (for some old results see, e.g., Aronszajn [31] and Wilson [711]). In 1928 K. Menger proved ([491], p. 98) that every continuum with a convex metric is locally connected, and asked if the inverse implication holds in the sense that every locally connected continuum is homeomorphic to one admitting a convex metric. This problem received much attention. There was a discussion of it in Blumenthal's book [83], and main partial results were obtained by Kuratowski and Whyburn [396], Beer [42], Harold [275] and Bing [61]. Finally R.H. Bing (1914–1986) [62] and simultaneously E.E. Moise [516] and [517] proved in 1949 that for every locally connected continuum there exists an equivalent convex metric (see also Bing's expository article [67] as well as Part 5 of M. Brown's article [120]). Therefore locally connected continua were characterized as ones having a convex metric.

A concept which is, in a way, related to local connectedness is one of aposyndesis. A continuum X is said to be aposyndetic provided that for each distinct points p and q of X there exists a subcontinuum of X that contains p in its interior and does not contain q. See [321] for Jones' description of his way to create the concept and [314], p. 546 for the explanation of the name. The set-function T is related to aposyndesis, which assigns to a point p of a continuum X the set of all points q of X such that each continuum containing q in its interior must contain p. The notion, which was introduced in 1941 by Floyd Burton Jones [314], has been recognized as an important tool in investigating structural and mapping properties of continua. For a survey of early results see [317]. Some of Jones' concepts and results on aposyndesis were generalized by H.S. Davis, D.P. Stadtlander and P.M. Swingle in [171] and [172]. A large bibliography on aposyndesis has been collected by E.E. Grace in [235].

A special method of investigating properties of a continuum, especially of a locally connected one, relies upon the study of mappings of the continuum into a sphere. The method was introduced in 1931 by Karol Borsuk (1905–1982) in [90], where a simple characterization of unicoherent locally connected continua is obtained in terms of mapping them into a circle (see also [96]). The method was later developed by K. Borsuk in cooperation with S. Eilenberg [103], by S. Eilenberg [195] and [196], and by K. Kuratowski [389]. Finally, it was well-organized into a self-consistent theory and was incorporated into several monographs (e.g. [390] and [695]).

In the early thirties Borsuk defined [89] the concept of a retract, an absolute retract and [92] an absolute neighborhood retract, and started to investigate basic properties of these concepts. We say that a metric space Y is an absolute retract, AR (an absolute neighborhood retract, ANR) if for every metric space X containing a homeomorphic copy Y' of Y as a closed subset there is a mapping, called a retraction, from X (from a neighborhood of Y') onto Y' whose restriction to Y' is the identity. Thus the property of being an AR or an ANR is stronger than local connectedness. Among early papers devoted to this topic are [93] and [97], where the concepts of local contractibility and of a deformation retract were introduced, [105], containing an example of an indecomposable ANR-space, and [98], where an acyclic polyhedron is constructed which is not the union of two acyclic polyhedra. This continuum, known in the literature as Borsuk's tube or the dunce hat, has proved a useful example in other branches of topology. It should be mentioned here that in 1964 Bing and Borsuk constructed [78] a 3-dimensional AR containing no disk. These and many other results obtained in this area were later developed in a large branch of topology, called the theory of retracts [102], [296]. The topic impinges upon infinite-dimensional manifolds, as displayed in Chapman's book [137]. For K. Borsuk's results in topology, in particular in continuum theory, see Nowak and Sieklucki's article [543].

5. Indecomposability

Locally connected continua are rather simple ones. Essential progress in continuum theory is related to the investigation of curiosities; the study of curiosities led to the discovery of regularities.

In 1904 A. Schoenflies started publishing a series of papers [594] which became an important step in the development of the theory of continua by introducing new concepts, bringing new results, and even making some faulty assertions. Relying heavily on intuition, Schoenflies claimed that there do not exist three regions in the plane with a common boundary. The claim was refuted in 1910 by L.E.J. Brouwer [114] who constructed continua which are the common boundary of three regions and showed that they are indecomposable, i.e. they are "closed curves, which cannot be divided into two proper subcurves". More precisely, a continuum is said to be decomposable provided it is the union of two its proper subcontinua. Otherwise it is called indecomposable. After several years these continua have been shown to be involved in many topological questions. In particular, for early papers on the applications of indecomposable continua to the theory of topological groups see L. Vietoris [644], D. van Dantzig [169] and A. van Heemert [284], where it is shown that any connected but not locally connected one-dimensional compact commutative topological group (solenoid) is an indecomposable continuum.

The discoverer of the first solenoid, L. Vietoris, mentioned in 1927 that it was indecomposable [644], but he gave no proof of this fact. A characterization of solenoids was the main subject of C.L. Hagopian's paper [258] of 1977. Other ones are due to Bing [72] and Krupski [370]. A very strange and unexpected example of a planar continuum was constructed in 1930 by G.T. Whyburn [685], and was later called Whyburn's curve. It was a continuum X every subcontinuum of which separated the plane. Moreover, X was the common boundary of two domains; every subcontinuum of X contained a homeomorphic copy of X; it contained no uncountable collection of mutually disjoint subcontinua, and thus no indecomposable subcontinuum. Obviously, it contained no arc. Another curve having all these attributes and being, moreover, a continuous image of the pseudo-arc, was constructed in 1962 by A. Lelek [410].

Besides Brouwer, indecomposable continua were announced also in 1910 by Arnauld Denjoy [176] and in 1917 by Kunizô Yoneyama ([716], p. 60), the latter describing the examples due to Wada (known as "lakes of Wada"). A further investigation of lakes of Wada was made in 1926 by P.S. Urysohn ([636], pp. 231–233). Urysohn's contribution to the lakes of Wada was an outline of a proof of the indecomposability of a continuum, based on his characterization of this concept ([636], p. 226). As he remarked in [636], p. 232, he did not know if Wada's construction always lead to an indecomposable continuum. The problem was solved in the negative in 1933 by R.L. Wilder who described ([702], pp. 275–278) an example of a locally connected continuum in the 3-space being the common boundary of three (or even countably many) domains. A stronger result was obtained in 1953 by M. Lubański [431] who constructed in the 3-space an ANR-set having the same property (compare also [102], p. 150–151). For further results in this direction see [626].

The common boundary problem was investigated in the twenties by Knaster [349] and Kuratowski [379] and [381]. Surely Brouwer and Urysohn knew it was possible to have planar continua being the common boundary of countably many regions, but Knaster was the first who published a specific description of such sets. In [349] he constructed a continuum which is the union of two indecomposable continua and which is the common boundary of infinitely many domains. Finally, the problem was solved in 1928 by Kuratowski who proved in [379] and [381]) that every plane continuum that is the common boundary of n open domains either is indecomposable or is the union of two indecomposable continua whenever $n \geq 3$, and when $n = 2$ it either is "monostratic" or has a natural "cyclic structure" in the sense that it is built up from layers naturally ordered in the same way as the individual points of the circle. More general results were obtained later, in 1951, by C.E. Burgess in his thesis [122].

The simplest indecomposable continuum is that of 1910 of Brouwer [114], who also indicated that his construction could be used to describe a common boundary of a finite number (greater than two) or even countably many domains. A simplification of Brouwer's example was made next year by Z. Janiszewski in his thesis ([306], p. 114), and finally B. Knaster gave in 1922 a nice description of this continuum in Kuratowski's paper ([378], I, pp. 209–210; compare [390], vol. 2, pp. 204–205). However, there was no proof of indecomposability with the example when it appeared in [378].

Recall that in 1909 L. Zoretti introduced [727] the concept of an irreducible continuum. Brouwer was later involved in the development of irreducible continua, again as a critic. In 1910 he pointed out [116] several errors in Zoretti's work, saying in particular that his own example of an indecomposable continuum was a counterexample to Zoretti's statement that the "exterior boundary of a domain" can be decomposed into two subcontinua having only two points in common. Zoretti took note of these comments [728] by pointing out that he had already published corrections. Some ten years later it turned out that both concepts, indecomposability and irreducibility, not only have common historical origin, but also are very closely related mathematically.

In the early years of their history, indecomposable continua were treated as curiosities, and were constructed to show inaccuracies in some statements. Beginning in the early twenties, they were studied more as entities in themselves, rather than just as pathological examples. The first paper devoted exclusively to studying properties of indecomposable continua, apart from that of 1917 of Yoneyama [716], was published in 1920 by S. Mazurkiewicz. Answering a question of Janiszewski, Knaster and Kuratowski he showed [459], using the Baire category theorem (known from René Baire's thesis of 1899, [36], p. 65, for the real line only; more generally, for metric complete spaces proved in 1914 by F. Hausdorff in [278]), that an indecomposable continuum in the Euclidean n-space has three points such that the continuum is irreducible between any two of them (see Kuratowski's comments on this and other results of Mazurkiewicz concerning continuum theory in [391]).

Parenthetically, this paper of Mazurkiewicz's [459] is the first one in which the term "indecomposable" appears. R.L. Moore credits Mazurkiewicz with being the originator of the term (see [524], p. 363). A more important paper on indecomposable continua, published in the same (first) volume of Fundamenta Mathematicae, was one by Janiszewski and Kuratowski [309]. It contains several necessary and sufficient conditions for a continuum to be indecomposable, as well as the concept of a composant, i.e. the union of all proper subcontinua of the whole continuum each of which contains a given point. Later, by many important applications, the concept was shown to be a fundamental one in indecomposable continua theory. It is proved there that composants of distinct points are either disjoint or coincide. Further, the following conditions are shown to be equivalent:

(1) a continuum X is indecomposable;
(2) for each point x of X there is a point y of X such that X is irreducible between x and y;
(3) there exists a point of X whose composant has empty interior;
(4) there are three points of X such that X is irreducible between any pair of them;
(5) there are two disjoint composants in X.

As a corollary one gets that a continuum is indecomposable if and only if it contains a point such that the continuum is irreducible between the point and each point of a dense subset. As an application of condition (4) above a method of constructing an indecomposable continuum is presented in [293], p. 142, as the intersection of a decreasing sequence of unions of chains of disks in the plane. In connection with (5) it is worth recalling that Mazurkiewicz later (1927) showed [463] that a metric indecomposable continuum has as many composants as there are real numbers. Metrizability of the continuum is essential in this result, since in 1978 David P. Bellamy showed [50] that there is an indecomposable Hausdorff continuum having only two composants (compare also [48]). Earlier, in 1968, Bellamy showed the existence of a non-metric indecomposable continuum in his thesis [44].

Another concept, due to Janiszewski, 1911, is that of a continuum of condensation [306]. Janiszewski used this name for a subcontinuum which is contained in the closure of its complement, and proved ([309], p. 210) that a continuum is indecomposable if and only if each of its proper subcontinua is a continuum of condensation. As a consequence it follows that an indecomposable continuum is not locally connected at any of its points. A further study of this concept is contained in the second part of Urysohn's memoir (Chapter 3 of [637]), where another concept, viz. that of a continuum of convergence, is also studied under the name of a continuum of a full condensation. Recall that a subcontinuum K of a continuum X is said to be a continuum of convergence of X provided that K is the limit of a sequence of mutually disjoint continua K_n contained in X which are also disjoint with K. Thus each continuum of convergence is a continuum of condensation. In these terms Urysohn obtained a characterization of hereditarily locally connected continua as such continua which contain no subcontinuum of convergence (Theorem 13 of Chapter 3 of [637]; simultaneously the same result was obtained by Kazimierz Zarankiewicz [722]).

An indecomposable continuum has a strong property related to its connectivity: no point disconnects it (see [354], p. 37). R.L. Moore even proved in 1926 ([524], p. 361) that for every Hausdorff indecomposable continuum X, for each of its proper subcontinuum K and for every subset L of K the complement $X \setminus L$ is connected. Moreover, even if the whole composant in an indecomposable continuum is removed, the resulting set is still connected ([390], vol. 2, p. 210).

Janiszewski and Kuratowski also established in [309] another result on the structure of indecomposable continua. To present it, recall concepts of the relative distance and of the oscillation, both due (1913 and 1916) to S. Mazurkiewicz ([457] and [458]; also [460]). Given two points x and y in a metric space M, the infimum of the diameters of connected subsets of M containing x and y is called the relative distance between x and y. The relative diameter of a set A in M is defined as the supremum of relative distances of pair of points in A. The oscillation of M at a point p of M means the infimum of relative diameters of all subsets A of M such that $p \in \operatorname{int} A$. The result says that for any point of an indecomposable continuum C in a metric space the oscillation of C at the point is a constant and equal to the diameter of C. The relative distance was also applied to study locally connected continua. Urysohn has shown [637] that if a given metric on a locally connected continuum X is replaced by relative one, then the resulting space is homeomorphic to X. This transformation, called by Whyburn the relative distance transformation, was in 1932 applied by him to study locally connected continua and planar regions ([687]; also [695], p. 154).

A semicontinuum means any set S every two points of which lie together in a continuum contained in S. Using this notion, Urysohn obtained in 1925 ([636], p. 226) one more (compare Mazurkiewicz's condition (4) above) relation between irreducibility and indecomposability. An irreducible continuum X between two points a and b is indecomposable if and only if it contains a semicontinuum S such that either a or b is in S and both S and its complement $X \setminus S$ are dense subsets of X. As mentioned earlier, Urysohn used this theorem to outline a proof of the indecomposability of the lakes of Wada.

Indecomposable continua, in the first stage of their theory, were considered as pathological examples which need special constructions to be shown, so (intuitively speaking) as rather rare objects in the family of all continua. However, as was shown by Mazurkiewicz, it is not so. First, in 1930, he has shown [465] that in the hyperspace $C(I^2)$ of all subcontinua of the unit square I^2 metrized by the Hausdorff metric ([278], Chapter 8, Section 6) the set of all continua which are not only indecomposable but even hereditarily indecomposable is a dense G_δ-set. Second, using results of P. Alexandroff [3], K. Borsuk [88] and S. Eilenberg [194] on essential mappings, he proved in 1935 that every compact metric space of dimension greater than one contains an indecomposable continuum, [471]. This result has been strengthened in 1951 by R.H. Bing who has established [64] a similar result for an even more singular type of continuum, namely for pseudo-arcs.

Another, and rather popular way of describing indecomposable continua (but which can be successfully applied to other spaces, too) involves inverse limits. P.S. Alexandroff introduced [2] the concept in 1929. The most frequently used definition was first stated in 1931 by S. Lefschetz [405] and studied by Hans Freudenthal (1905–1990) [227]. An exhaustive discussion of inverse systems was presented in 1952 by Eilenberg and Steenrod [198]. Compare also [128] and [530]. Coming back to indecomposable continua, D.P. Kuykendall proved in 1973 in his thesis [397] the following characterization. Let $\{X_n, f_{n,m}\}$ be an inverse sequence of nondegenerate metric continua X_n (equipped with a metric d_n) and of surjective bonding mappings $f_{n,m}$. Then the inverse limit continuum is indecomposable if and only if for each $\epsilon > 0$ and for each positive integer n there are a positive integer $m > n$ and three points of X_m such that if K is a subcontinuum of X_m containing two of them, then $d_n(x, f_{n,m}(K)) < \epsilon$ for each point $x \in X_n$. Other characterizations are in [600]. For an application of inverse limits of arcs or simple closed curves with some special bonding mappings to study indecomposable continua see [148] and [175].

It should be also remarked that indecomposable continua appear – in a natural way – in the investigation of problems related to homogeneity and to the fixed point property of planar continua (see Sections 8 and 11 below). Further, they appear even in a more general context, namely in dynamics of plane continua. In 1932 G.D. Birkhoff observed [80] that complicated dynamical properties of an annulus homeomorphism must necessarily lead to a complicated topological structure for certain invariant subsets. In 1934 M. Charpentier proved [153] that Birkhoff's "remarkable curve" – an invariant plane separating continuum – is indecomposable. For other places where indecomposable continua do appear, in particular in connection with M.L. Cartwright and J.E. Littlewood's investigation of solutions to the forced van der Pol equations ([131] and [132]) see M. Barge and R.M. Gillette's article [37]. In that paper remarkable results are proved (by means of the theory of prime ends due to Constantin Carathéodory (1873–1950), see [130]; compare also [591]) which show why and how indecomposable continua appear in the study of an orientation-preserving homeomorphism of the plane that leaves invariant a certain continuum. For yet a different approach to these and other results see [38] and compare [111]. Also recent studies made by J.T. Rogers, Jr. [585] on local Siegel disks indicate that their boundaries can be – in certain circumstances – indecomposable. Recall that in 1983 A. Douady and D. Sullivan asked [178] whether the boundary of a Siegel disk of a complex polynomial of degree greater than one is necessarily a simple closed curve, and that no answer to their question is known till now in general (compare Rogers' article [586]). A discussion of these and related results can be found in J.C. Mayer and L.G. Oversteegen's expository article [455].

For the history of indecomposable continua theory the reader is referred to the doctoral dissertation of F.L. Jones [324] from which many ideas of this article are taken. See also Kuratowski's article [392].

6. Irreducible continua; decompositions

Zoretti, considering that an irreducible continuum is a generalization of an arc, conjectured that any irreducible continuum could be given a linear ordering. Moreover, he published in 1909 a theorem which would provide the basis for this ordering [727]. When it was pointed out to him (by Brouwer) that his method failed for an irreducible continuum that is also indecomposable, he published [729] a new method based on a weaker theorem. Brouwer also observed that this theorem was false for an indecomposable continuum. The most that could be done in this case was to order the points of each composant separately ([117], pp. 144–145). Thus, Brouwer continued to play the role of critic in the development of indecomposable continua theory (compare [324], pp. 16 and 72).

The linear ordering question for an irreducible continuum is strongly related to its decompositions. Here a decomposition of a space X means a family $\mathcal{F}$ of closed and mutually disjoint subsets of X whose union is X. If members of the decomposition are connected, then the decomposition is said to be monotone. A decomposition $\mathcal{F}$ of a space X is said to be (a) linear if the decomposition space $X/\mathcal{F}$ (i.e. the space obtained from X by shrinking each member of $\mathcal{F}$ to a point, equipped with the quotient topology) can be linearly ordered; (b) upper (lower) semicontinuous (a concept due 1925 to R.L. Moore [523]; compare Chapter 5 of [528]) provided that for each open (closed) set $U \subset X$ the union of all members of $\mathcal{F}$ contained in U is open (closed). If a decomposition is both upper and lower semicontinuous, it is said to be continuous.

Linear monotone upper semicontinuous decompositions of irreducible continua were earlier studied in 1921 by H. Hahn [267], L. Vietoris in his thesis [641], and Wallace Alvin Wilson in a sequence of papers [708], [709] and [710] (1925 and 1926).

For each point p of an irreducible continuum X, Hahn defined in [267], p. 224, a "Primteil" $P(p)$ as a set (being a subcontinuum of X) composed of the point p itself and of all points x that can be joined with p, for each $\epsilon > 0$, by a chain of points $p, p_1, \ldots, p_n, x$ such that the distance of any two consecutive points of the chain is less than ϵ and that each point $p_1, \ldots, p_n$ belongs to a non-degenerate continuum of condensation. Wilson ([709], p. 536) and Kuratowski (Part 2 of [378], p. 226) criticized the Hahn decomposition of X into the sets $P(p)$ indicating examples which showed that the decomposition was not fine enough.

L. Vietoris considered in [641] only such continua X irreducible between some points a and b such that:

(v) X contains a connected set which is irreducible between these points.

Recall that a connected set S is said to be irreducible between points a and b if it contains these points and if each connected subset of S containing them is equal to S (the concept was introduced 1911 by N.J. Lennes in [422], p. 308). According to Vietoris ([641], p. 196) for every two points p and q of X we write $p \prec q$ if there exist two disjoint irreducible continua, one from a to p, and the other from q to b; the points p and q belong to the same "*Schichte*" $S(p)$ (as an element of the decomposition is called) if neither $p \prec q$ nor $q \prec p$. It is proved that, for irreducible continua X satisfying (v) the decomposition of X into $S(p)$ is monotone, upper semicontinuous and linear. However, if the irreducible continuum X does not satisfy (v), then Vietoris' method cannot be applied (compare Part 2 of [378], p. 265).

The initial concept for W.A. Wilson's approach is that of the oscillation of a continuum at a point, due to Mazurkiewicz in [457] and [458]. Let, as previously, a continuum X be irreducible between a and b. For each point p of X Wilson defines ([708], p. 433) an oscillatory set $C(p)$ as the intersection of all subcontinua of X that contain p in its interior. Denote by $S(a,p)$ the saturated semicontinuum of $X \setminus C(p)$ containing the point a and let $S(b,p)$ have a similar meaning for b. If neither $S(a,p)$ nor $S(b,p)$ is empty and $p \in (\operatorname{cl} S(a,p)) \cap (\operatorname{cl} S(b,p))$, then $C(p)$ is called a complete oscillatory set. Likewise, $C(a)$ is said to be complete if it does not contain b, and if $a \in \operatorname{cl} S(b,a)$. Under like conditions $C(b)$ is complete. Also if $C(p)$ is identical with $C(a)$ (or $C(b)$) and $C(a)$ (or $C(b)$) is complete, then $C(p)$ is complete ([709], p. 545 and [710], pp. 148 and 149). In [709] and [710] Wilson has proved a sequence of important properties of irreducible continua and developed the theory of complete oscillatory sets. Conclusions obtained by him can be summarized as follows. An irreducible continuum X can be decomposed into complete oscillatory sets if and only if

(w) X contains no indecomposable subcontinuum with nonempty interior.

If condition (w) is satisfied, then decomposition of X into complete oscillatory sets is monotone, upper semicontinuous and linear (it has the interval $[0,1]$ as its decomposition space). If not, Wilson's theory is not applicable, just as with Vietoris' case. This is nothing surprising, because as was shown in 1927 by B. Knaster [350] Vietoris' "Schichten" and Wilson's complete oscillatory sets coincide, and conditions (v) and (w) are equivalent. To show this, Knaster carried out a deeper study of connected sets irreducible between two points.

Both the above discussed decompositions, i.e. of Vietoris into the sets $S(p)$ and of Wilson into the sets $C(p)$ are applicable only to some kinds of irreducible continua, namely to ones satisfying the (equivalent) conditions (v) and (w). The problem of "linear ordering" for all irreducible continua was finally completely solved by K. Kuratowski in [378].

The second part of K. Kuratowski's thesis published in 1922 (i.e. Part 1 of [378]) is an extensive study of irreducible continua theory. The basic concept of Kuratowski's theory is one of a closed connected domain (introduced in 1921, under additional assumptions, by H. Lebesgue [402], p. 273), and investigated in the first part of his thesis [377] (where his famous four axioms of the closure operation were formulated). Given a space X, a subset of X is called a closed domain of X if it is the closure of its interior. Kuratowski considered in a continuum X irreducible from a to b a family $\mathcal{F}$ composed of the empty set and of all closed connected domains D in X containing the point a. He proved that $\mathcal{F}$ is a strictly monotone family, i.e. that if $D_1, D_2 \in \mathcal{F}$ and $D_1 \neq D_2$, then either $D_1 \subset \operatorname{int} D_2$ or $D_2 \subset \operatorname{int} D_1$. Further, the family $\mathcal{F}$ ordered by the relation $D_1 \subsetneq D_2$ has no gaps, i.e. when it is decomposed into two disjoint nonempty subfamilies $\mathcal{F}'$ and $\mathcal{F}''$ such that each element of $\mathcal{F}'$ is a subset of each element of $\mathcal{F}''$, then either $\mathcal{F}'$ has the last element or $\mathcal{F}''$ has the first one. Applying his results concerning the structure of monotone families of closed sets in separable metrizable spaces (developed 1938 in [388]) one can order the family $\mathcal{F}$ linearly, i.e. in such a manner that $\mathcal{F}$ is similar to a subset of the unit interval $[0,1]$.

Two members D_1 and D_2 of $\mathcal{F}$ form a jump if each member D of $\mathcal{F}$ with $D_1 \subset D \subset D_2$ equals either D_1 or D_2. This concept was used to show some further relations between irreducibility and indecomposability. Namely, Kuratowski proved (Part 1 of [378], pp. 210–212) that a nondegenerate indecomposable subcontinuum K of a continuum X irreducible between a and b is either a continuum of condensation of X or a closed connected domain. In the latter case, there is a member D of $\mathcal{F}$ such that D and $D \cup K$ form a jump. As a converse, it is shown that if members D_1 and D_2 of $\mathcal{F}$ form a jump, then $\mathrm{cl}\,(D_2 \setminus D_1)$ is either empty or an indecomposable continuum. Further, X is indecomposable if and only if $\mathcal{F} = \{\emptyset, X\}$.

The solution of the linear ordering question was published in 1927 as Part 2 of [378]. Studying again the family $\mathcal{F}$ of closed connected domains containing the point a in the continuum X irreducible between a and b, Kuratowski proved that it is possible to assign to each element D of $\mathcal{F}$ a number $y \in [0,1]$ so that the condition $y_1 < y_2$ is equivalent to the relation $D(y_1) \subsetneq D(y_2)$. Since $\mathcal{F}$ contains the first and the last element (namely the empty set and the whole X) and has no gaps, the set J of numbers y which correspond to members D of $\mathcal{F}$ is closed and may be assumed to contain 0 and 1. So there exists a mapping $f : X \to [0,1]$ such that for every $y < 1$ we have $f^{-1}([0,y]) = D(y)$ if $D(y)$ has an immediate successor in $\mathcal{F}$, and $f^{-1}([0,y]) = \bigcap\{D(z) : z > y\}$ if $D(y)$ has no immediate successor in $\mathcal{F}$.

To describe the decomposition concretely Kuratowski considered two cases. If the family $\mathcal{F}$ is uncountable, let P be the perfect kernel of J (thus $J \setminus P$ is countable) and let P^* be the set obtained from P by removing the end points of its contiguous intervals. Thus there exists a continuous nondecreasing function $\varphi : [0,1] \to [0,1]$ which is increasing on P^* and such that $\varphi(P) = [0,1]$ (the function φ was defined in a similar manner as the well known Cantor–Lebesgue "step-function" – see [127] and [400], p. 210). For $t \in [0,1]$ let $\gamma(t)$ and $\Gamma(t)$ be the first and the last y, respectively, such that $\varphi(y) = t$. In the second case, if the family $\mathcal{F}$ is finite or countable, we put $\varphi(x) = 0$ for all $x \in [0,1]$, $\gamma(0) = 0$ and $\Gamma(0) = 1$. Defining $g = \varphi \circ f : X \to [0,1]$ for both cases we have in the second one $g(x) = 0$ for all $x \in [0,1]$. For each point $p \in X$ Kuratowski defined its tranche (or layer) $T(p)$ as the inverse image of the function g, i.e. $T(p) = g^{-1}(g(p))$. It can be shown that, for each $t \in [0,1]$,

$$g^{-1}(t) = \bigcap\{D(z) : \Gamma(t) < z\} \cap \bigcap\{\mathrm{cl}\,(X \setminus D(u)) : u < \gamma(t)\}.$$

This equality was originally taken as the definition of the tranche (Part 2 of [378], p. 254).

Kuratowski showed that his decomposition $\mathcal{D}$ of an irreducible continuum X into tranches (the theory of which is a common result of Kuratowski and Knaster, see Part 2 of [378], footnote 1, p. 248) has the following properties:

(a) $\mathcal{D}$ is linear, upper semicontinuous and monotone,
(b) it is the finest possible among all decompositions like in (a), i.e. if $\mathcal{D}'$ is any decomposition satisfying (a), then each member of $\mathcal{D}'$ is the union of some members of $\mathcal{D}$.

In the case when the family $\mathcal{F}$ is finite or countable (and only in this case) the decomposition of X into tranches is trivial: the whole X is its only tranche.

The theory of upper semicontinuous decompositions of continua was (and still is) studied by a large number of topologists. A continuation of research recalled above can be found in a sequence of papers. We mention here only a few of them. For example in 1966 E.S. Thomas, Jr. gave [627] a large study of monotone decompositions of irreducible continua; in [445] and [500] interesting particular problems were discussed concerning such decompositions. Some of these results, originally proved for metric continua, have been extended to Hausdorff continua ([234], [446]). Besides decompositions of irreducible continua, ones of other continua were studied, as, e.g., decompositions of continua irreducible about a finite set ([589], [620], [646]), or having other special properties ([645], [647], [649]) or else satisfying particular conditions regarding the structure of either members of the decomposition or the (quotient) space of the decomposition. For example, following Knaster who in 1935 constructed [351] an example of an irreducible continuum all tranches of which are arcs, J.W. Hinrichsen studied in 1973–1982 ([288], [289], [290], [291]) the class of all continua K for which there exists an upper semicontinuous decomposition of an irreducible continuum with each member of the decomposition homeomorphic to K. In [477] and [478] decompositions were studied with an aposyndetic continuum as the decomposition space, and ones with a semi-locally connected decomposition space were the main objects of [215]. Monotone upper semicontinuous decompositions of continua having hereditarily arcwise connected decomposition space were studied in [141], and the results were extended to Hausdorff continua in [648]. For other results related to decompositions the reader is referred, e.g., to Z.M. Rakowski's thesis of 1980, [560], and the bibliography therein, and to E.J. Vought's survey article [651].

In 1935 Knaster [351] gave an example of a one-dimensional irreducible continuum that admits a continuous decomposition into tranches which are all nondegenerate. Continuous decompositions of irreducible continua were later extensively investigated ([515], [274], [186], [547]). In particular Oversteegen and Tymchatyn in 1983 showed [547] that if an irreducible continuum admits a continuous decomposition into tranches, and if all tranches are nondegenerate, then the continuum must contain a dense family of indecomposable tranches containing indecomposable subcontinua of arbitrarily small diameters. However, a continuum having the discussed property was constructed in 1987 by L. Mohler and L.G. Oversteegen [512] that contains no hereditarily indecomposable continuum.

Besides decompositions of continua, one can meet another object of interest in continuum theory that concerns decompositions, namely decompositions of other spaces, in particular Euclidean ones, into continua. In 1925 R.L. Moore proved [523] that if no element of a monotone upper semicontinuous decomposition $\mathcal{D}$ of the plane $\mathbb{R}^2$ separates $\mathbb{R}^2$, then the decomposition space $\mathbb{R}^2/\mathcal{D}$ is homeomorphic to the plane. In 1929 he considered in [527] a more general problem, without the assumption that elements of the decomposition do not separate the plane. The same year the first example of an upper semicontinuous decomposition of the plane into nonseparating subcontinua was presented by J.H. Roberts [566]; subsequently, in 1936 he gave an example in which, in addition, each element is a locally connected continuum, and proved that there is no upper semicontinuous decomposition of the plane into arcs [568]. The latter result was generalized in 1968 by S.L. Jones who showed [325] that there is no such decomposition of an arbitrary Euclidean space and that [326] there is no continuous decomposition of a Euclidean space into k-cells for $k \geq 1$. In 1952 R.D. Anderson provided [14] an example of a continuous decomposition of the plane into nonseparating subcontinua and showed [15] that there does not exist a continuous decomposition of the plane into nondegenerate locally connected nonseparating subcontinua. In 1955 Eldon Dyer (1929–1993) showed [187] that there does not exist a continuous decomposition of the plane into nonseparating subcontinua each of which is decomposable. In 1950 R.D. Anderson announced [13] that there exists a continuous decomposition of the plane such that each element of the decomposition is a pseudo-arc. In 1978 W. Lewis and J.J. Walsh constructed [428] such a decomposition. Examples of continuous decompositions of manifolds of dimension greater than two into nondegenerate subcontinua can be found in [658].

In general, the situation in higher dimensions, in particular in $\mathbb{R}^3$, is much more complicated than in the plane. In 1908 A. Schoenflies showed [595] that any homeomorphism between simple closed curves in the plane can be extended to a homeomorphism of the plane onto itself (in other words, there are no knots in the plane). The first examples of wild embeddings of the Cantor set, of an arc and of a 2-sphere in $\mathbb{R}^3$ appeared in the early twenties and were due to Louis Antoine (Antoine's necklace) ([22] and his thesis [23]), and to J.W. Alexander (Alexander horned sphere) [10] (compare also Bing's article [75]). Antoine has shown [24] that there is an arc (Antoine's arc) A in the 3-space $\mathbb{R}^3$ which is knotted in the sense that $\mathbb{R}^3 \setminus A$ is not homeomorphic with the complement of a point. A wild arc whose complement is homeomorphic to the complement of a point was first defined and investigated by R.L. Wilder in 1930 [701]; its properties were studied in 1948 (using other methods) by Ralph H. Fox and Emil Artin [222]. For details see, e.g., Moise's book [519] and Bing's book [77].

Coming back to decompositions, the fundamental problem was to find an analog of R.L. Moore's result on decompositions of the Euclidean 3-space $\mathbb{R}^3$ into continua such that the decomposition space is still $\mathbb{R}^3$. In 1936 G.T. Whyburn observed [689] that even with only one nondegenerate decomposition element, being an arc (namely Antoine's arc, [24]), the decomposition space might not be $\mathbb{R}^3$. Whyburn suggested that, perhaps, one should therefore study pointlike decompositions of $\mathbb{R}^3$, i.e. those in which the complement of each member of the decomposition is homeomorphic to the complement of a point. In 1957 R.H. Bing showed that Whyburn's suggestion did not work. He constructed [69] a decomposition of $\mathbb{R}^3$ into points and tame arcs such that the decomposition space (called later the dogbone space) is topologically different from $\mathbb{R}^3$. Two years later Bing proved [70] that the dogbone space is a factor of the Euclidean 4-dimensional space: the Cartesian product of the dogbone space and of the line $\mathbb{R}$ is homeomorphic to $\mathbb{R}^4$. In further papers Bing studied other properties of pointlike decompositions of $\mathbb{R}^3$, e.g. in [73] he described another example of a semicontinuous decomposition of the 3-space into straight arcs and singletons. His construction was modified in 1970 by Steve Armentrout, who defined such a decomposition and showed [27] that it has the two considered properties: the decomposition space is different from $\mathbb{R}^3$ and its product with the real line is homeomorphic to $\mathbb{R}^4$. See the introduction to [27] for more information on related results. Summarizing, Bing's results related to decompositions were of fundamental importance in the development of 3-dimensional topology, but they also contained seminal ideas that were at the heart of the great further accomplishments of higher dimensional topology of the late 1970's. See [120] for more detailed comments.

The results quoted above are related to decompositions of manifolds [170]. For a discussion of early results in this area see, e.g., the introduction to Robert J. Daverman's book [170]. During the 1950s R.H. Bing introduced and exploited several form of a remarkable condition now called his shrinkability criterion. In its most general form the criterion is expressed as follows. An upper semicontinuous decomposition $\mathcal{D}$ of a space X is shrinkable if and only if (shrinkability criterion) for each $\mathcal{D}$-saturated open cover $\mathcal{U}$ of X and each arbitrary open cover $\mathcal{V}$ of X there is a homeomorphism h of X into itself satisfying:

(1) for each $x \in X$ there exists $U \in \mathcal{U}$ such that $x, h(x) \in U$, and
(2) for each element D of $\mathcal{D}$ there exists $V \in \mathcal{V}$ such that $h(D) \subset V$.

Recall that a compact subset C of a space X is said to be (a) cellular, if X is an n-manifold and there exists a sequence $\{B_i\}$ of n-cells in X such that $B_{i+1} \subset \operatorname{int} B_i$ (for each $i \in \mathbb{N}$) and $C = \bigcap B_i$; (b) cell-like in X if, for each neighborhood U of C in X, the set C can be contracted to a point in U. By a cell-like decomposition of a metric space X we mean an upper semicontinuous decomposition of X whose elements are cell-like sets. Similarly, a mapping between locally compact metric spaces is said to be cell-like, provided that each point-inverse is a cell-like set.

A metric space X is said to have the disjoint disk property if for any two mappings f, g from the 2-dimensional disk B^2 into X and for each $\epsilon > 0$ there exist approximating mappings f' and g' again from B^2 into X, which are ϵ-near to f and g respectively, and $f(B^2) \cap g(B^2) = \emptyset$. The concept of the disjoint disks property was introduced into topology by R.H. Bing. He used a version of it to prove that his dogbone space was not a manifold ([69] and [74]). Its present form is due to J.W. Cannon [125].

Now let a cell-like decomposition $\mathcal{D}$ of an n-manifold M be given. If $n = 3$ and the elements of $\mathcal{D}$ are cellular, then $M/\mathcal{D}$ is homeomorphic to M if and only if $\mathcal{D}$ is shrinkable (R.H. Bing [68] and [69]; S. Armentrout [28]; compare also [25] and [26]). If $n \geq 5$ then $M/\mathcal{D}$ is homeomorphic to M if and only if $M/\mathcal{D}$ is finite-dimensional and has the disjoint disk property (R.D. Edwards [190]). In both cases the existence of a homeomorphism between M and $M/\mathcal{D}$ is obtained by showing that the quotient mapping of M onto $M/\mathcal{D}$ is a near homeomorphism. It should be stressed that Edwards' original manuscript [190] was never completed for publication. Instead, Edwards prepared an outline of the proof in his survey article [191]. Complete versions can be found in [398], [659] and [170].

If X and Y are either Q-manifolds (i.e. Hilbert-cube manifolds) or n-manifolds for $n \neq 3$, then a closed surjection is cell-like if and only if it is a near homeomorphism. This remarkable result was proved for the 2-sphere in 1925 by R.L. Moore [523] and for other 2-manifolds in 1938 by J.H. Roberts and N.E. Steenrod [571] and in 1948 by J.W.T. Youngs [719]. Its analogs for $n = 3$ (if the mapping is cellular) were shown in 1968 by S. Armentrout [25] and in 1972 by L.C. Siebenmann [604]; and in 1982 F.S. Quinn [559] proved the theorem for $n = 4$. For $n \geq 5$ the result was obtained in 1972 by L.C. Siebenmann [604]; it is also a corollary to Edwards' result quoted above. For Q-manifolds it was shown in 1976 by T.A. Chapman [137]. A comprehensive information on topological manifolds can be found in a survey article by D. Repovš [562].

7. Hereditary indecomposability; $\mathcal{P}$-like continua

Knaster and Kuratowski in 1921 asked [354] if there exists in the plane a continuum which was not only indecomposable itself, but also each of whose subcontinuum was indecomposable, too. Next year the answer was shown to be affirmative: the continuum has been constructed in 1922 in Knaster's thesis [348]. In today's terminology such a space is called a hereditarily indecomposable continuum, although no special name was given to it originally. Later Russians, following P.S. Urysohn [638], started to call it "Knaster's continuum". Mazurkiewicz had used the name "absolutely indecomposable continuum" in [465], but it was not accepted later. The construction constituted the major portion of Knaster's forty page thesis. F.L. Jones in [324] writes about Knaster: "He called his construction technique 'method of bands', and he credits Sierpiński with originating the concept in 1918 [348], p. 247. Essentially, the method of bands provides a way of constructing of a nested sequence of continua in the plane in which the 'nesting' is done in a special manner. By varying this manner slightly, Knaster first constructed a previously unknown example of an ordinary indecomposable continuum. Then by placing more restrictions on the nesting, he constructed the first hereditarily indecomposable continuum. Since each continuum in the nested sequence resembles a band, it is not hard to see where the name of the method probably originated."

However, Knaster's discovery of the hereditarily indecomposable continuum had not any immediate influence on further study of various attributes of the example which for the duration of the next quarter of a century was treated as one more curiosity in mathematics; neither Knaster's thesis nor several papers that appeared shortly after it and were related to the example contained deeper analysis of its properties. As an exception one can consider a characterization of hereditarily indecomposable continua due to Roberts and Dorroh who answered [570] a question of Whyburn [684] showing that a metric continuum X is hereditarily indecomposable if and only if no subcontinuum M of X contains an irreducible separator of M itself. Along the same lines W.R. Zame has proved [720] that a Hausdorff continuum is hereditarily indecomposable if and only if the difference of every two of its subcontinua is a connected set.

During the first thirty years of the development of continua theory, that is in the years 1910–1940, a great deal of results are due to Europeans, in particular to the Polish school of mathematics. The situation was drastically changed by the Second World War. The 33rd volume of Fundamenta Mathematicae (1945) contains a list of victims of the war. Some professors died, some emigrated from Europe to the United States, the rest remained without students, sometimes without universities, and therefore they were occupied with the reorganization of their own lives and of scientific life in general, rather than with working on new mathematical results. This is a part of the reason why after the Second World War most of the work in continuum theory seems to have been done by Americans, primarily by the first, second and third generation of Robert Lee Moore students. For an interesting account of Moore's famous teaching method see [700] and [630]. Compare also [704] and [706].

In as early as 1921 Mazurkiewicz posed a question [461] as to whether every planar continuum homeomorphic to each of its nondegenerate subcontinua is an arc. Later a nondegenerate continuum X which was homeomorphic to each of its nondegenerate continua was said to be hereditarily equivalent. In 1948 E.E. Moise in his thesis [513] constructed an example answering Mazurkiewicz's question in the negative. Moise called his example the pseudo-arc. More precisely, Moise described a general construction which produced a family of topologically equivalent planar continua having the property that they were hereditarily indecomposable and homeomorphic to each of their nondegenerate subcontinua. Some ten years later G.W. Henderson showed in his thesis [285] that the question has a positive answer provided that the continuum under consideration is decomposable.

To present Moise's construction let us recall the concept of a chain, introduced in 1916 by R.L. Moore [521], and used there to show arcwise connectedness of plane domains. A chain in a metric continuum X from a point p to a point

q is a finite collection $\mathcal{C}$ of open sets $C_1, \ldots, C_n$ (called links of the chain) such that $p \in C_1$, $q \in C_n$, and $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$. If $\operatorname{diam} C_i < \epsilon$ for each $i \in \{1, \ldots n\}$, then $\mathcal{C}$ is called an ϵ-chain. A chain $\mathcal{D}$ is said to refine a chain $\mathcal{C}$ if each link of $\mathcal{D}$ is a subset of some link of $\mathcal{C}$. A chain $\mathcal{D}$ is said to be crooked in $\mathcal{C}$ if it refines $\mathcal{C}$ and if, for every pair of links C_i and C_j of $\mathcal{C}$ such that $i + 2 < j$ and for every pair of links D_s and D_v of $\mathcal{D}$ with $C_i \cap D_s \neq \emptyset \neq C_j \cap D_v$ there are links D_t and D_u of $\mathcal{D}$ lying between the links D_s and D_v in the same order, i.e. $s < t < u < v$ or $s > t > u > v$, such that $D_t \subset C_{j-1}$ and $D_u \subset C_{i+1}$.

Now, let there be given in the plane two points p and q, and an infinite sequence $\{\mathcal{C}_m : m \in \mathbb{N}\}$ of $(1/m)$-chains of open disks, such that, for every $m \in \mathbb{N}$,

(a) the point p belongs to the first, and the point q to the last link of the chain $\mathcal{C}_m$,
(b) the chain $\mathcal{C}_{m+1}$ is crooked in the chain $\mathcal{C}_m$.

Given a chain $\mathcal{C}$, we denote by $\mathcal{C}^*$ the union of all elements of $\mathcal{C}$. Then a pseudo-arc P is defined by

$$P = \bigcap\{\operatorname{cl} \mathcal{C}_m^* : m \in \mathbb{N}\}$$

([513], p. 583). Moise proved the uniqueness of the pseudo-arc (i.e. that any two sets satisfying the definition of the pseudo-arc are homeomorphic) and that the pseudo-arc is hereditarily indecomposable. He commented that his method of construction of the pseudo-arc resembled Knaster's method of bands, and that his proof of hereditary indecomposability of P was quite similar to the corresponding proof in Knaster's thesis, so he suspected that the two continua might be homeomorphic ([513], p.581). Three years later, in 1951, Moise's conjecture was shown by Bing to be true.

A metric continuum is said to be chainable (or snake-like; Bing credits this term to Gustave Choquet, [65], p. 653) provided that for each $\epsilon > 0$ it can be covered by an ϵ-chain. The concept of chainability can be derived from Moore's paper [521] of 1916, and was investigated in the thirties. For instance, in 1930 J. H. Roberts showed [567] that each chainable plane continuum has uncountably many disjoint copies in the plane. Characterizations of hereditarily decomposable, of decomposable and of indecomposable chainable continua are in [40]. In 1951 Bing has shown [64] that each chainable continuum can be embedded in the plane, and that every two chainable and hereditarily indecomposable continua are homeomorphic. Thus the continuum described by Moise is homeomorphic to that described earlier by Knaster. In the same year Bing characterized [65] the pseudo-arc as a chainable continuum each point of which is an end point (here an end point of a chainable continuum means a point which belongs, for each $\epsilon > 0$, to the first link of an ϵ-chain covering the whole continuum). The concept of an end point was in 1966 generalized to one of a terminal continuum by J.B. Fugate [228] and used in 1978 by Sam Bernard Nadler, Jr. to characterize hereditarily indecomposable continua as those ones in which each subcontinuum is terminal

([532], (1.58), p. 109). In 1959 J.R. Isbell showed [304] that each metrizable chainable continuum is the inverse limit of an inverse sequence of arcs. That metrizability is an essential assumption in this result was shown by Sibe Mardešić [449]. For various properties of the pseudo-arc, including its characterizations, mapping properties, decompositions, and open questions related to it, see W. Lewis' expository article [426], with 110 items of references.

But not all hereditarily indecomposable continua are homeomorphic to the pseudo-arc, because there are as many non-homeomorphic planar hereditarily indecomposable continua as there are real numbers ([64], p. 50). It might be conjectured by now that all hereditarily indecomposable are one-dimensional. In 1942 John L. Kelley has proved in his thesis [343] that if there is a hereditarily indecomposable continuum of dimension greater than one, then there is one of infinite dimension. However, the major result in this direction was proved in 1951 by Bing: there are infinite dimensional hereditarily indecomposable continua in the Hilbert cube and n-dimensional hereditarily indecomposable continua in $(n+1)$-dimensional Euclidean space. More generally, each $(n+1)$-dimensional continuum contains an n-dimensional hereditarily indecomposable continuum ([66], p. 270).

Recall that a continuum is said to be circularly chainable if for each $\epsilon > 0$ it can be covered by a circular ϵ-chain, i.e. by an ϵ-chain whose first and last links intersect each other. In 1951 Bing described a planar non-chainable circularly chainable hereditarily indecomposable continuum, which has since became known as a pseudo-circle. His construction runs as follows ([64], p. 48). Let $\{\mathcal{C}_i : i \in \mathbb{N}\}$ be a sequence of circular $(1/i)$-chains in the plane such that

(a) each link of $\mathcal{C}_i$ is an open circular disk;
(b) the closure of each link of $\mathcal{C}_{i+1}$ is contained in a link of $\mathcal{C}_i$;
(c) the union $\mathcal{C}_i^*$ of all links of $\mathcal{C}_i$ is homeomorphic to the interior of an annulus;
(d) each complementary domain of $\mathcal{C}_{i+1}^*$ contains a complementary domain of $\mathcal{C}_i^*$;
(e) if $\mathcal{D}_i$ is a proper subchain of $\mathcal{C}_i$ and $\mathcal{D}_{i+1}$ is a proper subchain of $\mathcal{C}_{i+1}$ contained in $\mathcal{D}_i$, then $\mathcal{D}_{i+1}$ is crooked in $\mathcal{D}_i$.

Then the pseudo-circle was defined as the intersection $\bigcap\{\operatorname{cl}\mathcal{C}_i^* : i \in \mathbb{N}\}$. Bing proved that it separates the plane, and asked if all such continua are homeomorphic. The question was answered in the affirmative in 1969 by L. Fearnley [209] and [211]. A classification of hereditarily indecomposable circularly chainable continua is presented in [212].

Besides chainable and circularly chainable continua there are two other important classes of continua defined by means of the structure of coverings: weakly chainable and tree-like continua. Let us recall that a weak ϵ-chain in a metric continuum X is a finite collection $\mathcal{C}$ of open sets $C_1, \ldots, C_n$ called links of the weak chain such that $\operatorname{diam} C_i < \epsilon$ and $C_i \cap C_j \neq \emptyset$ if $|i-j| \leq 1$. A weak chain $\mathcal{D} = \{D_1, \ldots, D_m\}$ is said to refine a weak chain $\mathcal{C} = \{C_1, \ldots, C_n\}$ if each link D_i of $\mathcal{D}$ is a subset of some link C_{j_i} of $\mathcal{C}$ such that $|j_i - j_k| \leq 1$ if $|i - k| \leq 1$. A continuum X is weakly chainable provided that there exists an infinite sequence $\{\mathcal{C}_n : n \in \mathbb{N}\}$ of finite open covers of X such that each $\mathcal{C}_n$ is a weak $(1/n)$-chain and $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$ for each $n \in \mathbb{N}$. A. Lelek [410] in 1962 and (independently) L. Fearnley [206] in 1964 showed that a continuum is weakly chainable if and only if it is the continuous image of a pseudo-arc. It is also shown in [206] that this class of continua is identical with the class of all continuous images of chainable continua. A further study is contained in [207]. A characterization of the continuous images of all pseudo-circles is presented in [208]. As a conclusion one gets that every planar circularly chainable continuum, as well as every chainable continuum, is the continuous image of a pseudo-circle.

A continuum X is said to be tree-like provided that for each $\epsilon > 0$ there exists an ϵ-covering of X whose nerve is a tree (i.e. an acyclic one-dimensional polyhedron). Another (equivalent) definition runs as follows. A collection $\mathcal{C}$ of sets in a space X is said to be coherent (Moore [528], p. 46) if there do not exist two subcollections of $\mathcal{C}$ whose union is $\mathcal{C}$ such that each element of one subcollection is disjoint with each element of the other. A finite coherent collection of open sets is called a tree chain if no three elements of the collection have a point in common and no subcollection is a circular chain. A continuum is called tree-like provided that for each $\epsilon > 0$ there is a tree chain covering it such that each element of the tree chain is of diameter less than ϵ. This concept of 1951 is due to R.H. Bing [65] who proved that each planar continuum which does not contain a continuum which separates the plane is tree-like. Tree-like continua X were characterized in 1960 by Case and Chamberlin [134] as those curves for which each mapping from X onto a one-dimensional polyhedron is inessential.

A continuum is said to be hereditarily unicoherent if the intersection of any two of its subcontinua is connected. A hereditarily unicoherent continuum which is arcwise connected (hereditarily decomposable) is called a dendroid (a λ-dendroid). These notions are due to B. Knaster who initiated about 1960 a more systematic study of these continua (see [408] and [138]). In 1970 H. Cook proved that every λ-dendroid (in particular every dendroid) [161] as well as every hereditarily equivalent continuum [162] is tree-like. The main problems related to these continua are connected with their mapping properties. The concepts of chainable, circularly chainable and tree-like continua are special cases of $\mathcal{P}$-like continua, where $\mathcal{P}$ is a class of polyhedra. In 1929 P. S. Alexandroff introduced a class of mappings called ϵ-mappings. Given a number $\epsilon > 0$, a mapping $f : X \to Y$ between compact spaces X and Y is called an ϵ-mapping provided that $\operatorname{diam} f^{-1}(y) < \epsilon$ for each point y in Y. Let $\mathcal{P}$ be a class of polyhedra. A compactum X is said to be $\mathcal{P}$-like provided that for each $\epsilon > 0$ there is a polyhedron Y in $\mathcal{P}$ and an ϵ-mapping $f : X \to Y$ of X onto Y. It was shown by Sibe Mardešić and Jack Segal in 1963 [450] that if $\mathcal{P}$ is a class of connected polyhedra, then the class of $\mathcal{P}$-like compacta coincides with the class of inverse limits of inverse sequences of some members of $\mathcal{P}$ with surjective bonding mappings. Universal $\mathcal{P}$-like compacta were studied by M.C. McCord in [483].

In 1964 A. Lelek defined [411] the notion of the span of a metric continuum. To recall this, some auxiliary concepts are needed. If X and Y are continua, we let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ denote the first and second coordinate projections respectively. The surjective span of X, $\sigma^*(X)$, (respectively, the surjective semi-span, $\sigma_0^*(X)$) is defined [417] as the least upper bound of all real numbers ϵ for which there exists a subcontinuum $Z \subset X \times X$ such that $\pi_1(Z) = X = \pi_2(Z)$ (respectively, $\pi_1(Z) = X$) and $d(x, y) > \epsilon$ for each $(x, y) \in Z$, where d stands for the metric on X. The span of X, $\sigma(X)$, and the semi-span, $\sigma_0(X)$, are defined by

$$\sigma(X) = \sup\{\sigma^*(A) : A \text{ is a subcontinuum of } X\},$$

$$\sigma_0(X) = \sup\{\sigma_0^*(A) : A \text{ is a subcontinuum of } X\}.$$

Lelek developed a number of properties of these concepts; in particular, he proved in [411] that chainable continua have span zero, and asked [414] in 1971 if the converse is also true, i.e. if continua with span zero are chainable. Later it was proved (see [418] and [549]) that continua with span zero are atriodic and tree-like. A stronger result has been established by H. Kato, A. Koyama and E.D. Tymchatyn [336] who showed that the same conclusion holds for continua with surjective span zero. An atriodic tree-like continuum with positive span was constructed in 1972 by W.T. Ingram [300], and an uncountable collection of mutually disjoint plane continua with this property was presented by him two years later, [301]. In 1984 J.F. Davis showed [173] that span zero and semi-span zero are equivalent. Continua with span zero are characterized in [165] as those for which every indecomposable subcontinuum has semi-span zero. This generalized an earlier result of J.B. Fugate [229]. Other characterizations are in [549], where it is shown that continua with span zero are continuous images of the pseudo-arc. In [550] a result in the opposite direction is proved: if a continuum is a continuous image of the pseudo-arc and if all its proper subcontinua are pseudo-arcs, then the continuum itself is the pseudo-arc.

8. Homogeneity

A topological space is said to be homogeneous (this notion was introduced in 1920 by W. Sierpiński [613]) provided that for every two points of the space there is a homeomorphism of the space onto itself which maps one of the points to the other. In 1920 Knaster and Kuratowski asked [353] whether every nondegenerate homogeneous planar continuum is a simple closed curve. Four years later Mazurkiewicz proved [462] that the answer is yes provided the continuum is locally connected. This result was slightly generalized in 1949 by F.B. Jones who proved [315] that the only aposyndetic nondegenerate plane continuum is the simple closed curve. Homogeneous continua were the main subject of van Dantzig's paper [169], where it was shown that every n-adic solenoid is homogeneous (as a topological group; a definition of a solenoid in terms of inverse limits was given in 1937 by H. Freudenthal [227]), but since solenoids are not planar, these results were not related to the question of Knaster and Kuratowski.

In 1937 a false answer to the question was published by Z. Waraszkiewicz [665] who seemed to be able to delete the assumption of local connectedness from Mazurkiewicz's result. In 1949 F.B. Jones (using his concept of aposyndeticity of a continuum which is intermediate between local connectedness and decomposability) proved [311] that under slightly stronger hypothesis Waraszkiewicz's result is correct. Namely if the nondegenerate homogeneous planar continuum either is aposyndetic at all of its points or contains no cut points, then it is a simple closed curve. Two years later H.J. Cohen improved the result of Mazurkiewicz showing [158] that if a nondegenerate homogeneous plane continuum is either arcwise connected or contains a simple closed curve, then the continuum is itself a simple closed curve. Jones also suggested that Waraszkiewicz's error may have been to confuse the idea of a cut point of a continuum (whose complement is not connected) with that of a separating point (whose complement is not continuumwise connected) ([319], p. 66). The same opinion was given to the author by B. Knaster. Relying upon Waraszkiewicz's erroneous result, G. Choquet gave [154] a false classification of homogeneous planar compact sets. The two results (of Waraszkiewicz and of Choquet) were exhibited to be false when R.H. Bing in 1948 proved [60] that the pseudo-arc is homogeneous. Shortly thereafter E.E. Moise presented his own proof [514]. See F.B. Jones' article [323] on his impressions about these matters.

A few years later Issac Kapuano claimed [332] that the pseudo-arc is not homogeneous. However, an error was discovered in his work, so he published an attempt to correct it [333]. Mathematicians seemed more inclined to accept the results of Bing and Moise than those of Kapuano, but A.S. Esenin-Vol'pin, a reviewer of Referativnyĭ Zhurnal, wrote in 1955 that "in the light of this, the problem of Knaster and Kuratowski remains open" [203]. It is not surprising that the discussion greatly interested B. Knaster, the discoverer of the pseudo-arc, who asked in 1955 two of his students, Andrzej Lelek and Marek Rochowski, to verify Bing and Kapuano's arguments and clarify the situation. They did this hard work, the results of which were presented to Knaster in the form of a handwritten 60-page paper (in Polish; never published), and which concluded that Bing was right. Several years later Jerzy Mioduszewski, a member of Knaster's seminar group, gave in [504] and [505] his own proofs of the basic properties and characterizations of the pseudo-arc involving inverse limits and always oscillating functions.

In 1959 R.H. Bing proved [71] that each homogeneous nondegenerate chainable continuum is a pseudo-arc. Since each chainable continuum was known to be planable [65], the result was a step to a problem of finding all homogeneous planar continua. Earlier, in 1951, F.B. Jones proved [316] that each homogeneous planar continuum which does not separate the plane must be indecomposable, and in 1955 classified [319] homogeneous planar continua as those which (a) do not separate the plane (hence are indecomposable), (b) separate the plane and are decomposable, and (c) separate the plane and are indecomposable. At the time Jones gave his classification, a point and the pseudo-arc were the only known examples of type (a). A simple closed curve and an example discovered simultaneously in 1959 by Bing and Jones [79], called a circle of pseudo-arcs, were the only known examples of type (b). It was conjectured that the pseudo-circle was an example of type (c), but in 1969 L. Fearnley [210] and J.T. Rogers, Jr. in his thesis [574] showed that it is not homogeneous. F.B. Jones proved in 1955 [318] that each continuum of type (b) is a circle of continua of type (a). C.L. Hagopian proved in 1976 [257] and in 1984 [260] that continua of type (a) are hereditarily indecomposable. J.T. Rogers, Jr. proved [577] that there are no continua of type (c). C.E. Burgess proved in 1969 [124] that a nondegenerate circularly chainable planar continuum is homogeneous if and only if it is either a simple closed curve, a pseudo-arc, or a circle of pseudo-arcs. In 1960 Bing proved [72] that the circle is the only homogeneous planar continuum that contains an arc. A simpler proof was presented in 1975 by F.B. Jones [322]. In the same year the result was generalized by C.L. Hagopian who showed [256] that the theorem remains true if the property of containing an arc is replaced by one of containing a hereditarily decomposable continuum. In 1988 J.R. Prajs extended Bing's result proving ([557] and [558]) that homogeneous continua in Euclidean $(n+1)$-space which contain an n-cube are n-manifolds.

With regard to homogeneous continua out of the plane, let us recall that in 1931 O.H. Keller showed [344] that the Hilbert cube is homogeneous. Homogeneity of the Menger universal curve was established by R.D. Anderson in 1958 ([19] and [20]; compare also [456]). Concerning homogeneity of the Menger intermediate universal continua M_n^k it was shown in 1987 by W. Lewis [425] that they are never homogeneous for $n > 0$ and $k < 2n + 1$, while in 1984 M. Bestvina proved [55] homogeneity of these continua for $k = 2n + 1$ (whence the homogeneity for any $k \geq 2n + 1$ follows), and gave their characterization.

An important tool in the investigation of homogeneity of continua is the Effros property. Its roots were in a theorem proved in 1965 by E.G. Effros [192] on the action of the topological group of all homeomorphisms on a continuum X. Ten years later the theorem was employed by G.S. Ungar [632] and C.L. Hagopian ([253] and [255]) in the study of homogeneous continua. The result says that if a continuum X is homogeneous, then for each $\epsilon > 0$ and for each point x of X there is a $\delta > 0$ such that for every two points y and z of a

δ-neighborhood of x there exists a homeomorphism h of X onto itself satisfying $h(y) = z$ which is ϵ-near to the identity. An alternative proof of the Effros theorem was shown in 1987 by F.D. Ancel [12]. For some stronger results see [151].

There have been other partial solutions of the homogeneity problem, as well as other classifications of these spaces. An important one, proposed by J.T. Rogers, Jr. [580] in 1983 was based on the Jones aposyndetic decomposition theorem (see [318] and [579]) saying that each decomposable homogeneous continuum admits a continuous decomposition into indecomposable cell-like mutually homeomorphic homogeneous continua so that the resulting quotient space is an aposyndetic homogeneous continuum (for a general theory of decompositions of homogeneous continua see [578]). Namely six types of these continua were distinguished; types (1)–(3) for decomposable, and types (4)–(6) for indecomposable continua:

(1) locally connected;
(2) aposyndetic and not locally connected;
(3) decomposable and not aposyndetic;
(4) indecomposable and cyclic;
(5) acyclic and not tree-like;
(6) tree-like.

See [580] and [584] for a detailed discussion. If homogeneous curves are under consideration, type (1) continua are the simple closed curve and the Menger universal curve only ([19] and [20]). Homogeneous curves of type (2) are not planar. Examples were constructed in 1961 by J.H. Case who showed [133] that the Menger universal curve can be combined with a solenoidal construction to yield a new class of homogeneous non-locally connected curves containing an arc, and by J.T. Rogers, Jr. who observed in 1983 [581] that these continua are aposyndetic. Rogers' techniques can be applied to homogeneous continua of higher dimension as well. By combining the Menger universal curve with multiple solenoidal constructions, Minc and Rogers constructed in 1985 [502] other homogeneous curves modeled after those of Case.

Concerning type (3) curves it follows from Jones' aposyndetic decomposition theorem [318] improved by Rogers [579] that each type (3) curve admits a continuous decomposition into type (6) curves such that the quotient space is a type (1) or type (2) curve. The above mentioned circle of pseudo-arcs, constructed in 1959 by Bing and Jones [79] was the first known example of type (3) curve. A general result was shown in 1985 by W. Lewis [424] stating that for each curve (in particular for each homogeneous curve) X there is a (homogeneous) curve Y that admits a continuous decomposition into pseudo-arcs with quotient space X. The only known homogeneous curves of type (4) are solenoids and solenoids of pseudo-arcs (solenoids of pseudo-arcs were constructed by Rogers in 1977 [576]). Rogers obtained in 1987 a sequence of results [582] leading to a theorem that every acyclic indecomposable homogeneous curve is tree-like. Consequently, there is no continuum of type (5). The pseudo-arc is the only known homogeneous curve of type (6). Answering an old question of Bing, Krupski and Prajs [371] proved in 1990 that every type (6) curve has to be hereditarily indecomposable. A survey article (with 116 references) which presents an excellent outline of the history as well as the state of the art in the area of homogeneous continua was recently written by W. Lewis [427].

The concept of homogeneity has been generalized in many ways. In as early as 1930 D. van Dantzig considered n-homogeneous spaces, i.e. such spaces X that for every pair A, B of n-element subsets of X there exists a homeomorphism of X onto X which maps A onto B. If, in the above definition the subsets A and B are taken to be countable and dense (instead of being n-element ones), then one gets the concept of a countable dense homogeneous space, introduced and studied in 1972 by Ralph B. Bennett [53]. Connected manifolds without boundary are the simplest and the most natural examples of spaces which satisfy all of these homogeneity conditions. In 1958 R.D. Anderson proved in [19] that the Menger universal curve M is n-homogeneous for every natural n. The curve was used in 1980 by K. Kuperberg, W. Kuperberg and W.R.R. Transue [373] to show that the product of 2-homogeneous continua need not be 2-homogeneous. Using another result of R.D. Anderson [20] of homogeneity of curves, R.B. Bennett showed in 1972 [53] that M is countable dense homogeneous. In 1975 G.S. Ungar, solving a problem of C.E. Burgess [123], proved [632] that every 2-homogeneous metric continuum is locally connected. Another result of Ungar says [633] that for continua distinct from a simple closed curve, countable dense homogeneity is equivalent to n-homogeneity for each natural n. M. Bestvina showed [55] that for each natural n the Menger universal continua M_n^{2n+1} are countable dense homogeneous. Results on countable dense homogeneity are summarized in [216].

A space X is called $1/n$-homogeneous provided that the group of autohomeomorphisms $H(X)$ of X has exactly n orbits, i.e. if there are n subsets $A_1, \dots, A_n$ of X such that $X = A_1 \cup \cdots \cup A_n$ and, for any $x \in A_i$ and $y \in A_j$, there is a homeomorphism $h \in H(X)$ mapping x to y if and only if $i = j$. Józef Krasinkiewicz proved in 1969 in his MSc thesis [360] that the Sierpiński universal plane curve is $1/2$-homogeneous. Using Whyburn's characterization of the curve [698] one can list all $1/2$-homogeneous planar locally connected continua, and using Anderson's characterization of the Menger universal curve [19], [20] all the $1/2$-homogeneous locally connected curves can also be classified. In 1981 Hanna Patkowska classified [551] all $1/2$-homogeneous compact ANR-spaces of dimension at most two, and also gave a full classification of $1/2$-homogeneous polyhedra.

A continuum X is said to be bihomogeneous if for each pair of points x and y of X there exists a homeomorphism $h : X \to X$ with $h(x) = y$ and $h(y) = x$. Until recently every known homogeneous continuum was bihomogeneous, motivating the question of B. Knaster in 1922 of whether every homogeneous space was bihomogeneous [376]. A partial affirmative answer was given by K. Kuratowski [376] for totally disconnected spaces (i.e. spaces with one-point quasi-components, a concept introduced in 1921 by W. Sierpiński [615]) and for subspaces of a closed interval of reals. In 1990 however the question was answered in the negative by Krystyna Kuperberg. She constructed [372] a locally connected homogeneous non-bihomogeneous continuum of dimension 7.

Another method of generalizing homogeneity is to consider classes of mappings other than homeomorphisms. A continuum X is said to be homogeneous with respect of the class M of mappings provided that for every two points x and y of X there is a surjective mapping $f : X \to X$ such that $f \in M$ and $f(x) = y$. Several results on this topic have been obtained (see, e.g., [142], [145], [146], [147], [151], [334], [369]) but the whole area belongs rather to the future than to the history of continuum theory.

There are several concepts of continua which are opposite to homogeneous ones. In 1975 B.M. Scott studied [601] totally inhomogeneous spaces, i.e. ones such that the complements of every two distinct points are not homeomorphic. Earlier, in 1959, Johannes de Groot (1914–1972) studied [251] rigid spaces, i.e. spaces having a trivial autohomeomorphism group. He and R.J. Wille considered continua which are strongly rigid, i.e. continua X such that the only homeomorphism of X onto itself is the identity, and gave an example of a rigid but not strongly rigid universal planar curve [252]. In 1925 K. Zarankiewicz [721] asked whether, given any dendrite D, there always exists a proper subdendrite of D homeomorphic to D? Later a topological space which is homeomorphic with no proper subspace of itself was said to be incompressible [217]. Zarankiewicz's problem was answered in 1932 in the negative by E.W. Miller [499]. A stronger result was obtained in 1945 by A.S. Besicovitch [54] who constructed a totally heterogeneous dendrite, i.e. a dendrite such that no two of its open sets were homeomorphic. Spaces having this property were later named chaotic [539]. Interrelations between the considered concepts with references to the literature, and a study of chaotic curves are contained in [143].

Much more complicated examples of continua are known. In 1959 R.D. Anderson and G. Choquet, applying inverse limit techniques, constructed [21] three examples of planar continua no two of whose nondegenerate distinct subcontinua are homeomorphic. The first one does not contain uncountably many disjoint nondegenerate subcontinua (recall that continua having this property are called Suslinian, [413]; Suslinian continua are hereditarily decomposable and hence one-dimensional) and no subcontinuum of it separates the plane. The second curve is such that each of its nondegenerate subcontinua separates the plane (the property of Whyburn's curve [685]). The third example is a Suslinian curve no nondegenerate subcontinuum of which is embeddable in the plane.

In 1955 J. de Groot asked [250] if there exists a connected set which cannot be mapped onto any of its nondegenerate proper subsets, and R.D. Anderson asked [16] whether there exists a nondegenerate continuum admitting only the identity or a constant mapping onto itself; if so, whether there exists one, all of whose nondegenerate subcontinua have this property. R.L. Moore asked whether there is a hereditarily indecomposable continuum no two of whose nondegenerate subcontinua are homeomorphic. All these questions were answered in the affirmative by Howard Cook in 1967 who constructed one example of a curve having all these properties [160]. Further strong results in this direction were obtained by T. Maćkowiak in [440] and [443].

9. Mapping properties – families of continua

Various phenomena related to continuous mappings between continua were important and very interesting subjects of research for specialists in continuum theory. In the thirties several questions in this area were asked, and some of them have been answered. For example, in 1938 S. Mazurkiewicz proved [472] that in the space of all mappings f from a curve C into the plane, the set of mappings for which $f(C)$ is homeomorphic to the Sierpiński universal plane curve is residual (i.e. it is the complement of the union of a countable sequence of closed nowhere dense sets). Compare also [310] and [470]. K. Kuratowski asked in 1929 if every sequence of sets contains a member which is a continuous image of all other members of the sequence. As an answer, in 1932 Z. Waraszkiewicz constructed [661] an uncountable family of planar curves no member of which is a continuous image of another member of the family. Such families are called incomparable (with respect to continuous mappings). In the same year, using some invariants of continuity due to N. Aronszajn [30] he also constructed [662] a family of curves whose types of continuity filled a closed interval, that is, to each number $t \in [0, 1]$ he assigned a planar curve $P(t)$ so that a mapping from $P(t_1)$ onto $P(t_2)$ exists if and only if $t_2 < t_1$. In 1930 D. van Dantzig proved [169] that an m-adic solenoid is a continuous image of an n-adic solenoid if and only if m is a factor of a power of n. In 1967 H. Cook generalized this result in several directions [159] and proved that there is an uncountable incomparable collection of circle-like continua. Three years later an uncountable incomparable collection of pseudo-circles was constructed by J.T. Rogers, Jr., in [573] (Rogers extended the term of a pseudo-circle to all hereditarily indecomposable circularly chainable continua which are not chainable). Answering a question of Rogers [572], D.P. Bellamy exhibited [47] in 1971 an uncountable incomparable collection of chainable continua. Other results in this direction were obtained by T. Maćkowiak [440], [443] and Marwan M. Awartani [32].

A continuum K is a common model for the class $\mathcal{K}$ of continua provided that each member of $\mathcal{K}$ is a continuous image of K. According to the Hahn–Mazurkiewicz–Sierpiński theorem the closed unit interval $[0, 1]$ is a common model for all locally connected (metric) continua. In 1974 W. Kuperberg showed [374] that the cone over the Cantor set is a common model for uniformly pathwise connected continua. In 1962 J. Mioduszewski [504] and A. Lelek [410], and in 1964 L. Fearnley [206] showed that the pseudo-arc is a common model for all chainable continua, and J.T. Rogers, Jr., presented in 1970 a continuum called pseudo-solenoid which is a common model for circle-like (i.e. circularly chainable) continua [573]. In 1930 H. Hahn asked [269] if there exists a continuum, a common model for all continua. A negative answer to the question was presented in 1934 by Z. Waraszkiewicz who constructed [663] an uncountable family of planar curves (later called Waraszkiewicz's spirals) being irreducible continua, such that there is no common model for this family. In the same year Waraszkiewicz improved his result [664] showing that members of the family were incomparable with respect to continuity, and constructed another uncountable family of arcwise connected curves having both discussed properties (incomparability and no common model). However, even a quarter of a century after Waraszkiewicz's answer, Hahn's question reappeared in the literature as an unsolved problem [221].

Further essential progress in this topic was obtained in the seventies. Using Waraszkiewicz's result, D.P. Bellamy [46] showed in 1971 that there is no common model for indecomposable continua. Five years later Krasinkiewicz and Minc proved [365] that there is no common hereditarily decomposable model for planar fans (a fan means a dendroid having exactly one ramification point). In the seventies T. Ingram constructed uncountable families of planar atriodic indecomposable [302] and hereditarily indecomposable [303] continua without any common model, and R.L. Russo showed [590] that there is no common model for: planar tree-like continua, arcwise connected continua, planar indecomposable continua, planar λ-dendroids, (planar) curves, and aposyndetic continua. It is also proved in [590] that if $\mathcal{P}$ is a family of polyhedra such that $\mathcal{P}$-like continua have a common model, then either $\mathcal{P} = \{\text{arc}\}$, or $\mathcal{P} = \{\text{circle}\}$, or $\mathcal{P} = \{\text{arc, circle}\}$. Some of Russo's results were reproved in a simpler way in 1984 by T. Maćkowiak and E.D. Tymchatyn [444] using some ideas of D.P. Bellamy.

Another problem, closely related to the common model problem, is the following. Given a collection $\mathcal{K}$ of continua, characterize the collection of all continuous images of members of $\mathcal{K}$. A particular case was studied in the literature when the collection K consisted of only one specific continuum. The Hahn–Mazurkiewicz–Sierpiński theorem says that all continuous images of an arc are just locally connected (metric) continua. W. Kuperberg [374] gave in 1974 a characterization of continuous images of the Cantor fan as precisely the uniformly pathwise connected continua. Continuous images of the pseudo-arc were characterized in the early sixties by L. Fearnley [206] and A. Lelek [410]. L. Fearnley also proved [206] that the class of all continuous images of the pseudo-arc coincides with the class of all continuous images of all chainable continua and gave [208] a characterization of all continuous images of all pseudo-circles. Continuous images of circularly chainable continua were characterized in 1970 by J.T. Rogers Jr. in his thesis [573]. For some other results and questions see expository articles by D.P. Bellamy [51] and by J.T. Rogers, Jr. [572].

Let $\mathcal{K}$ be a class of continua. Recall that an element U of $\mathcal{K}$ is said to be universal for $\mathcal{K}$ provided that each member of $\mathcal{K}$ can be embedded in U. The existence of universal elements for the classes of all continua, all curves, all plane curves and some other classes was mentioned previously. A universal continuum for the class of chainable continua was constructed by Richard M. Schori in 1965 in his thesis [596]. Essential progress was made the next year by M.C. McCord who studied $\mathcal{P}$-like compacta in [483]. The class $\mathcal{P}$ of polyhedra is called amalgamable if for each finite sequence $(P_1, \ldots, P_n)$ of members of $\mathcal{P}$ and mappings $\phi_i : P_i \to Q$ (where $i \in \{1, \ldots, n\}$ and $Q \in \mathcal{P}$) there exist a member P of $\mathcal{P}$, embeddings $\mu_i : P_i \to P$ and a surjective mapping $\phi : P \to Q$ such that $\phi_i = \phi \circ \mu_i$ for each i. It was shown in 1966 [483] that if $\mathcal{P}$ is an amalgamable class of (connected) polyhedra, then there exists a universal element in the class of $\mathcal{P}$-like continua, and that the following classes of polyhedra are amalgamable: acyclic, contractible, of dimension at most k, acyclic and of dimension at most k, contractible and of dimension at most k, trees, and k-cells. Some negative results were also shown in [483], e.g. the nonexistence of any universal element for the class of closed connected triangulable n-manifolds. In 1931 G. Nöbeling showed [540] that there is no universal element in the classes of regular or of rational continua (i.e. of continua every point of which has arbitrarily small neighborhoods with a finite (resp. countable) boundary). But if the class of completely regular continua is considered (i.e. of continua every nondegenerate subcontinuum of which has a nonempty interior; such continua were studied by P.S. Urysohn in [637]), then a universal element exists and was constructed in 1980 by S.D. Iliadis in [299].

A dendroid (i.e. a hereditarily unicoherent and hereditarily decomposable continuum) X is said to be smooth ([150] and [495]) if there exists a point p in X such that the partial order $\leq_p$ with respect to p (defined by $x \leq_p y$ provided x is a point of the only arc from p to y) is closed (for some generalizations of the concept of smoothness see [435] and [432]). A structural characterization of smooth dendroids was given in 1988 by E.E. Grace and E.J. Vought [240]. A universal smooth dendroid was constructed in 1978 by J. Grispolakis and E.D. Tymchatyn [247]. For other constructions see [152] and [511]. J. Krasinkiewicz and P. Minc proved in 1976 [365] that there is no universal element for the following classes of (one-dimensional) continua: fans, dendroids, λ-dendroids, hereditarily decomposable, Suslinian, or for corresponding classes of planar members of the above mentioned ones. A modification of an argument from [365] led to nonexistence of any universal element for the class of hereditarily decomposable chainable continua [444]. There is no universal object for the class of smooth planar dendroids [436]. A universal hereditarily indecomposable continuum was constructed in 1985 by T. Maćkowiak [441].

10. Special mappings

All the (above discussed and other) problems related to (continuous) mappings between continua lead to a variety of other ones if the nature of the considered mappings is restricted in one way or another, e.g. to a particular class of mappings. Special mappings between continua were studied even in an early stage of continuum theory. Continuous mappings and homeomorphisms of abstract spaces were first considered in 1910 by M. Fréchet in [225]; in a narrower sense the notion of a homeomorphism was introduced earlier by H. Poincaré. The first exhaustive and systematic exposition of these classes of mappings was given in 1914 by F. Hausdorff in his book [278] (compare [202], p. 35).

A mapping is said to be k-to-1 provided that each point inverse has exactly k elements. O.G. Harrold, Jr., in 1939 showed [276] that every 2-to-1 function on the closed interval $[0,1]$ must be discontinuous. Since then numerous mathematicians have considered k-to-1 mappings. Much research had concentrated on which spaces can be the domain of such a map and for which k. In the beginning of the study the mappings were investigated when defined on linear graphs [277], [233]. In particular, in 1940 O.G. Harrold, Jr., showed [277] that there is no continuous k-to-1 from $[0,1]$ into $[0,1]$. The most interesting case is that of $k = 2$. For early results, especially for 2-manifolds, see [569] and [454]. In 1943 P. Civin showed [155] that there is no 2-to-1 mapping on an n-cell for $n \leq 3$. Mappings whose point inverses consist of at most two points (called simple mappings) were studied in 1958 by K. Borsuk and R. Molski [106] and by B. Knaster and A. Lelek [357]. The former showed that there is a simple mapping of an arc onto the Sierpiński universal plane curve, which in turn can be mapped in such a way onto the disk and onto the 2-dimensional sphere. A large study of 2-to-1 mappings was presented in 1961 by J. Mioduszewski [503], where these mappings were studied not only on graphs and other locally connected continua, but also on some irreducible and indecomposable ones. More recent results on k-to-1 mappings between continua were obtained by Nadler and Ward [535], by S. Miklos [497], and by Jo W. Heath, who studied the set of discontinuities of a k-to-1 function from some continua to others (in particular from or to dendrites) in a sequence of her papers [279], [280], [281], [282]. Wayne Lewis in 1983 gave ([423], [282]) an example of a chainable continuum that admits an exactly 2-to-1 mapping onto a continuum, and J.W. Heath constructed [283] k-to-1 mappings between tree-like continua.

In 1931 K. Borsuk introduced [89] a concept of a mapping called a retraction. A mapping f of a space X into itself is called a retraction if it satisfies the functional equation $f \circ f = f$. Retractions are a special kind of r-map, introduced in 1947 also by Borsuk in [99], i.e. mappings $f : X \to Y$ for which there exists a mapping $g : Y \to X$ such that the composition $f \circ g : Y \to Y$ is the identity map on Y. The theory of these mappings and their various invariants was developed during the past half of the century and is now well-known under the name of the theory of retracts. The reader is referred to [102] and [296] for details. Here we recall just two results, both related to large families of locally connected continua, and both due to Karol Sieklucki. In 1959 he constructed [605] an uncountable family of r-incomparable dendrites, and two years later a family of dendrites r-ordered similarly to a closed interval was presented in [606].

One of the most important classes of mappings, lying in a natural way between the class of all mappings and the class of homeomorphisms is the class of open mappings, defined as those that transform open subsets of the domain to open subsets of the range. This notion was introduced in 1913 (for mappings of the plane into itself) by H. Weyl [677] and in 1928 by S. Stoïlow [622] who additionally assumed that fibers of the mappings did not contain nondegenerate continua (lightness of the mapping). Open mappings of topological spaces were defined in 1931 by Natan Aronszajn [29] (see also W. Sierpiński [616]), and were studied by S. Eilenberg who proved [194] that a mapping of a compact space is open if and only if the decomposition of the domain into point-inverses is continuous. S. Eilenberg also initiated in 1935 a systematic study of some special open mappings, namely local homeomorphisms [193]. For some further results in this direction see [406] and [328]. In the end of the thirties G.T. Whyburn studied open mappings of compact spaces [690] and showed that Menger–Urysohn order of a point is never increased when the domain space is openly transformed, whence it follows that concepts of a curve of order less than n, of a regular curve and of a rational curve are invariants under open mappings. He also described the action of an open mapping on an arc, a simple closed curve, on a linear graph and on some surfaces [691], [692], [693]. These results were later incorporated into his book [695], where a systematic study of open mappings (called interior ones) was contained, and where special attention was paid to open mappings of locally connected continua. With regard to other classes of continua we recall only one, but very important result. In 1974 Ira Rosenholtz proved [588] that an open image of a chainable continuum is chainable.

The first example of a dimension-raising open light mapping was described in 1937 by Andrej Nikolaevich Kolmogorov (1903–1987) [359]. Ludmila Vsevolodovna Keldysh (1904–1976) [337] defined in 1954 an open light mapping from a one-dimensional continuum onto a square; a detailed description of her example can be found in [7]. In 1972 D.C. Wilson showed [707] that the Menger universal curve M_1^3 can be mapped onto every locally connected continuum under an open mapping whose point-inverses are all homeomorphic to the Cantor set. Recall that in 1956 R.D. Anderson announced [18] that M_1^3 can be mapped onto every locally connected continuum under an open mapping whose point-inverses are all homeomorphic to M_1^3. A proof of this result was given by D.C. Wilson in [707]. John J. Walsh gave in 1976 [656] sufficient conditions for the existence of light open mappings between piecewise linear manifolds. A wide spectrum of results and problems concerning open mappings of continua is contained in two survey articles by Louis F. McAuley [479] and [480] and in the proceedings of the 1970 conference devoted to open and to monotone mappings [481]. Finite-to-1 open mappings on chainable and on circularly chainable hereditarily decomposable continua were studied by Philip Bartick and Edwin Duda in [181] and [41], respectively.

Monotone mappings are defined as those having connected point-inverses. They were first studied in 1925 by R.L. Moore [523] in terms of upper semicontinuous decompositions. Namely an upper semicontinuous decomposition $\mathcal{D}$ of a compact space X is equivalent to a monotone mapping $f : X \to X/\mathcal{D}$ whose point inverses coincide with elements of the decomposition. The class of monotone mappings was introduced in 1934 by G.T. Whyburn in [688] and was studied in 1942 by Alexander Doniphan Wallace (1905–1985) in [654]. Whyburn's book [695] contains many divers characterizations and many other properties of monotone mappings (especially of locally connected continua), in particular the Whyburn factorization theorem saying that every mapping $f : X \to Y$ of a compact space X is the composition of two mappings, $f = f_2 \circ f_1$, where $f_1 : X \to X'$ is monotone, and $f_2 : X' \to Y$ is light; if f is open, then f_2 is also open (and light).

In 1934 Whyburn proved [688] that the properties of being an arc and a simple closed curve are invariant under monotone mappings. In 1951 R. H. Bing showed [65] that chainability of continua is preserved under monotone mappings. Monotone dimension-raising mappings of cubes were investigated in the fifties by L.V. Keldysh [338], [339], [340]. Monotone and open mappings defined on piecewise linear manifolds were studied by J.J. Walsh [655], [657]. A particular monotone mapping is an atomic one, a concept of 1956 due to R.D. Anderson [17] in connection with decompositions, and studied, e.g., in [478] and [199], defined as a mapping $f : X \to Y$ between continua such that for each subcontinuum K of X with nondegenerate image $f(K)$ the condition $K = f^{-1}(f(K))$ is satisfied. Let us also observe that cell-like mappings, as defined in the final part of Section 6 above, are monotone. The problem of increasing dimension by cell-like mappings is extensively discussed in a survey article [180]. Strong results in this area, concerning the existence of such mappings from lower-dimensional spaces onto infinite-dimensional ones were obtained in 1988 by A.N. Dranishnikov [179], who answered in the negative a question of P.S. Alexandroff [4] on the equality between the integral cohomological dimension and the Lebesgue covering dimension on metric compacta, and in 1993 by J. Dydak and J.J. Walsh who proved [185] in particular that for each natural number $n \geq 5$ there exists a cell-like mapping f from the Euclidean n-space $\mathbb{R}^n$ such that $\dim f(\mathbb{R}^n) = \infty$.

For more information on this topic the reader is referred to J.E. West's survey article [676].

There are a number of generalizations of the concept of a monotone mapping. One of the most important is the notion of a quasi-monotone mapping defined in 1940 by A.D. Wallace [652] as a mapping $f : X \to Y$ between continua X and Y such that for each subcontinuum Q of Y with nonempty interior the set $f^{-1}(Q)$ has finitely many components and f maps each of them onto Q. Several characterizations and various properties of these mappings when considered on locally connected continua are discussed in Whyburn's book [695]. In particular it is shown there that the degree of multicoherence of a continuum (introduced in 1936 by S. Eilenberg in [196]) is never increased under a quasi-monotone mapping. Images of θ_n-continua under quasi-monotone mappings were studied in [237] and [239]. Basic theorems and problems concerning monotone mappings were presented by Louis F. McAuley in a survey article [482].

In 1972 A. Lelek introduced [415] two classes of mappings called OM- and MO-mappings which were defined as compositions of monotone and open, and of open and monotone mappings, respectively, and showed that the former class coincides with the class of quasi-monotone mappings studied in 1950 by G.T. Whyburn [697]. Another common generalization of monotone and of open mappings were confluent mappings defined in 1964 in [139]. A mapping $f : X \to Y$ between continua X and Y is said to be confluent provided that for each subcontinuum Q of Y and for every component C of $f^{-1}(Q)$ we have $f(C) = Q$. It was shown in [139] that the concepts of dendrite, dendroid and λ-dendroid are invariants under these mappings. T.B. McLean proved in 1972 in his thesis [484] that confluent images of tree-like curves are tree-like, while a similar implication for arc-like (i.e. chainable) continua, asked in 1971 by A. Lelek [414], is still an open problem. For further results concerning these mappings see, e.g., [246], [416], [420], [561].

Confluent mappings have been generalized in many various ways. The most important are weak confluence, introduced in 1972 by A. Lelek [415], semi-confluence defined the next year by T. Maćkowiak [434] (see [174]) and pseudo-confluence due to A. Lelek and E.D. Tymchatyn [421] in 1975. Each of these classes of mappings has been studied extensively and proven to be useful in a variety of ways. For example, hereditary indecomposable continua are characterized as those continua Y such that every mapping from a continuum onto Y is confluent, [160] and [420]. All interrelations between the 24 classes of the above mentioned and other related mappings, their various properties, such as the composition property, the composition factor property, the product property, the product factor property, and the limit property, and their actions from or to spaces belonging to some 18 classes of continua have been studied (and the results have been collected in six tables) in the thesis (published in 1979, [438]) of Tadeusz Maćkowiak (1949–1986). See also a survey article [249] with an extensive bibliography. Weakly confluent mappings have been generalized to inductively weakly confluent ones in [453]. A particular case of weakly confluent mappings are refinable ones, i.e. mappings which can be approximated arbitrarily closely by surjective ϵ-mappings. The class of refinable mappings was introduced in 1978 by Jo Ford and J.W. Rogers, Jr. [220], and it found many applications in continuum theory. See a survey article [335].

Finally let us mention one more class of mappings, viz. expansive homeomorphisms. A homeomorphism f of a metric space (X, d) onto itself is said to be expansive provided that there exists an $\epsilon > 0$ such that for each pair of distinct points x and y of X there exists an integer n such that $d(f^n(x), f^n(y)) > \epsilon$. The study of expansive homeomorphisms started in the the mid-1900's, and it is extensively continued in the area of topological dynamics. Several important recent results concerning these mappings are recalled in Chapter 9 of the survey article [455] of J.C. Mayer and L.G. Oversteegen.

11. Fixed point theory

Besides homogeneity (which was discussed above), an area of continuing interest in continuum theory which is also related to mappings, is fixed point theory. The reader is referred to introductory chapters of monographs and handbooks, to survey articles, and to many other publications in the area (see, e.g., [56], [76], [121], [184], [204], [262], [263], [307], [660]) for general information related to history and bibliography of fixed point problems. We mention here only some basic facts related to continuum theory.

We say that a space X has the fixed point property (f.p.p.) if for every mapping $f : X \to X$ there is a point $p \in X$ such that $f(p) = p$. Brouwer's theorem on the f.p.p. for an n-cell is one of the oldest and best known results in topology. It was proved in 1909 for $n = 3$ by L.E.J. Brouwer [112]; an equivalent result was established five years earlier by P. Bohl [85]. It was Jacques Salomon Hadamard (1865–1963) who gave in 1910 (using the Kronecker index) the first proof for an arbitrary n. Around 1910 L.E.J. Brouwer discovered [112], [113], [118] the degree of a continuous mapping of one n-manifold into another, and used it to extend Poincaré's definition of the index [554] from two to n dimensions, and to prove fixed point theorems for the n-cell, n-sphere (for continuous mappings of degree different from $(-1)^n$) and the projective plane. In 1922 J.W. Alexander [9], and G.D. Birkhoff and O.D. Kellog [81] gave other proofs for the first two results, and in 1929 B. Knaster, K. Kuratowski and S. Mazurkiewicz presented [356] a short and elegant proof of the f.p.p. for an n-cell using Sperner's lemma [621]. Another major step in the history of fixed point theorems was the formula of S. Lefschetz discovered in 1926 for orientable n-manifolds without boundary [403] and extended the following year to manifolds with a boundary [404]. The

Brouwer theorem as well as the Lefschetz theorem were extended to set-valued mappings in 1941 by S. Kakutani [329] and in 1946 by S. Eilenberg and D. Montgomery [197], respectively.

In 1926 W. Scherrer proved [592] that every dendrite has the f.p.p. for homeomorphisms. In 1930 W.L. Ayres gave [34] several extensions of Scherrer's theorem to arbitrary locally connected continua, some of which were expressed in terms of cyclic element theory. He proved, in particular, that if a locally connected planar continuum does not separate the plane, then it has the f.p.p. for homeomorphisms. Further results in this direction were obtained by J.L. Kelley [341], [342] and by O.H. Hamilton (1899–1976) [272] who proved in 1938 that each homeomorphism of a λ-dendroid (i.e. hereditarily decomposable and hereditarily unicoherent metric continuum) into itself leaves some point fixed. In 1932 G. Nöbeling [542] and K. Borsuk [91] (using Whyburn's cyclic element theory) extended Scherrer's and some of Ayres' results to all continuous mappings. The extensions also follow from a theorem of H. Hopf [294] that for every closed covering of a unicoherent locally connected continuum X and for every mapping $f : X \to X$ there exists a member M of the covering such that $M \cap f(M) \neq \emptyset$. In 1941 A.D. Wallace showed [653] that the techniques introduced by Hopf could also be applied to show that a tree (i.e. a Hausdorff continuum in which each pair of distinct points is separated by a third point) has the f.p.p. for continuous mappings. Other proofs of this result were given in 1957 by L.E. Ward, Jr. [667] and by C.E. Capel and W.L. Strother [129] who used the order-theoretic characterization of trees due to Ward [666]. Hamilton's result was extended to the non-metric case for pseudo-monotone mappings [670].

Introducing local connectedness by a change of topology, G.S. Young proved [717] in 1946 that an arcwise connected Hausdorff continuum such that the union of any nested sequence of arcs is contained in an arc has the f.p.p. A similar argument (on nested sequences of arcs) was used in 1954 by K. Borsuk who proved [100] that every dendroid (i.e. an arcwise connected and hereditarily unicoherent metric continuum) has the f.p.p. As a corollary to Borsuk's result it follows that each contractible curve has the f.p.p. One-dimensionality is essential here: in 1953 Shin'ichi Kinoshita constructed a contractible acyclic 2-dimensional continuum without the f.p.p. and such that the cone over it also does not have the f.p.p. In 1967 Ronald J. Knill showed [358] that the cone over a circle with a spiral does not have the f.p.p. Results of Young and Borsuk were generalized in 1959 by L.E. Ward, Jr. [668]. Two years later he proved [669] that an arcwise connected metric continuum has the f.p.p. for upper semicontinuous continuum-valued mappings if and only if it is hereditarily unicoherent. Further essential progress was made in 1976 by Roman Mańka who showed in his thesis [448] that the λ-dendroids have the f.p.p. for a class of set-valued mappings. The result was generalized in 1981 by T. Maćkowiak [439] to a wider class of set-valued mappings and Hausdorff λ-dendroids.

Regarding continua that may contain indecomposable ones, in 1951 O.H. Hamilton proved [273] that each chainable continuum has the f.p.p. The result was extended in 1956 to the product of an arbitrary family of chainable continua by E. Dyer [188]. Fixed point theorems for the limit of an inverse limit system were obtained in 1959 by R.H. Rosen [587] and in 1962 by J. Mioduszewski and M. Rochowski [506], [507]. In 1951 R.H. Bing asked [65] (compare also [76]) whether any tree-like continuum had the property as well. Recall that, according to H. Cook [161], tree-like continua are known to form a wider class than one of λ-dendroids, which do have the f.p.p. In 1975 J.B. Fugate and L. Mohler showed [230] that if a tree-like continuum has finitely many arc components, then it has the f.p.p. The condition concerning the arc components had appeared to be essential, because in 1979 David P. Bellamy answered Bing's question in the negative [52] by constructing a nonplanar tree-like continuum which admits a fixed point free mapping. The continuum was obtained as a modification of a solenoid. Next, he used this example and an inverse limit technique of J.B. Fugate and L. Mohler [231] to construct a second tree-like continuum that admits a fixed point free homeomorphism. It is not known if this second example can be embedded in the plane. For related results see also [545] and [546]. M.M. Marsh proved [451], [452] a general result that implies f.p.p. for tree-like continua which can be represented as the inverse limits of triods with bonding mappings satisfying some special conditions. In 1982 L.G. Oversteegen and E.D. Tymchatyn, answering a question of C.L. Hagopian, showed [548] that every planar tree-like homogeneous continuum has the f.p.p. (since any such continuum has span zero). In 1993 L. Fearnley and D.G. Wright gave [213] a geometric realization of a Bellamy continuum describing a tree-like continuum T without the f.p.p. The continuum T consists of a Cantor set of arcs in the form of a fan together with an indecomposable chainable continuum.

An important part of continuum theory which is related to fixed point theory concerns covering and mapping properties of spheres. The Lusternik–Schnirelman–Borsuk theorem says that in any closed covering of an n-dimensional sphere S_n by $n + 1$ sets at least one of the sets must contain a pair of antipodal points (i.e. the n-sphere S_n cannot be decomposed into the union of n closed sets with diameters less than the diameter of S_n). This was discovered in 1930 by L. Lusternik and L. Schnirelman [433] and in 1933 by K. Borsuk. Borsuk proved [94] (using properties of the space of all continuous mappings of a compactum into the n-dimensional sphere) two its equivalent formulations:

1. Borsuk antipodal theorem: An antipodal-preserving mapping between n-spheres is essential (i.e. is not homotopic to a constant mapping of S_n onto a singleton).
2. Borsuk–Ulam theorem: Every continuous mapping of the n-sphere S_n into the n-dimensional Euclidean space sends at least one pair of antipodal points to the same point.

For a combinatorial proof of the equivalence see [242] and [184]. The first proof of this type was given in 1945 by A.W. Tucker [631] for $n = 3$, and was extended in 1952 to arbitrary n by Ky Fan [205]. Another combinatorial proof of the antipodal theorem was given in 1949 by M.A. Krasnoselskij and S. Krein [368]. For more details see [184].

As a special case of this theorem one gets a basic result in fixed point theory: the identity map on S_n is not homotopic to a constant mapping, which in turn is equivalent to Brouwer's fixed point theorem for an $(n + 1)$-ball, and to the nonexistence of a retraction of the $(n + 1)$-ball onto its boundary S_n. This last equivalence was observed in 1931 by K. Borsuk [88], [89]. For other proofs of the result see, e.g., [292] and [6]; compare also [184]. The Borsuk–Ulam theorem, conjectured by Stanisław Marcin Ulam (1909–1984) and proved by K. Borsuk, had many various consequences and generalizations (see, e.g., [11], [35], [108], [109], [110], [189], [205], [218], [241], [243], [295], [311], [312], [330], [366], [367], [429], [430], [508], [714], [715], [723]).

A special place in fixed point theory is occupied by fixed point problems connected with plane continua, and dually, one of the most interesting problems in topology of the plane is the following, still open, plane fixed-point problem:

(P) Does every nonseparating plane continuum have the f.p.p.?

The problem first appeared in 1930 in a paper by W.L. Ayres [34]. Ayres gave a partial positive answer to it under very restrictive conditions with respect to spaces as well as mappings: spaces were assumed to be locally connected continua, and the mapping were homeomorphisms. Ayres' result was generalized in 1932 by Borsuk [91]. He proved that every retract of a space having the f.p.p. also has the property. Since every locally connected nonseparating plane continuum is a retract of a disk, such continua have the f.p.p. In 1967 H. Bell [43], in 1968 K. Sieklucki [607] and in 1970 S.D. Iliadis [298] independently showed that every nonseparating plane continuum that admits a fixed point free mapping into itself contains an invariant indecomposable continuum in its boundary. This result was generalized by P. Minc [501] who established the f.p.p. for every nonseparating plane continuum in which each indecomposable subcontinuum of its boundary is contained in a weakly chainable continuum (i.e. in a continuous image of the pseudo-arc). In 1971 C.L. Hagopian used the Bell–Sieklucki result mentioned above to prove [254] that every arcwise connected nonseparating plane continuum has the f.p.p. In 1975 L. Mohler applied some measure-theoretic techniques (viz. the Markov-Kakutani theorem) to prove [510] that every homeomorphism of a uniquely arcwise connected continuum (i.e. a continuum that is arcwise connected and contains no simple closed curve) into itself has a fixed point. Note that this result concerns not necessarily planar continua. For planar continua a much stronger result was established the following year by C.L. Hagopian [259]: every uniquely arcwise connected plane continuum has the f.p.p. Later Hagopian even generalized his theorems proving that (1) every arc component preserving mapping of a nonseparating plane continuum has a fixed point [261], and that (2) if $\mathcal{D}$ is a decomposition of a plane continuum X such that elements of $\mathcal{D}$ are uniquely arcwise connected, then every mapping of X which preserves the elements of $\mathcal{D}$ has a fixed point [263].

Let us come back to Ayres' problem (P) and note that even a much weaker version is not answered. Namely we do not know if a homeomorphism of a nonseparating plane continuum into itself must have a fixed point. If Bellamy's second example of [52] (see above) can be embedded in the plane, the answer to this question, and to (P), is no. These and some other open questions in fixed point theory are discussed in Hagopian's expository article [262] from which information presented here is taken.

12. Hyperspaces

Hyperspace theory has its beginnings in the early 1900's with the work of F. Hausdorff and L. Vietoris. Given a topological space X the hyperspace 2^X of all closed subsets is equipped with the Vietoris topology, also called the exponential topology or finite topology, introduced in 1922 by L. Vietoris [642]. The hyperspace of all closed connected subsets of X is denoted by $C(X)$ and is considered as a subspace of 2^X. Vietoris proved the basic facts of the structure of 2^X related to continua: compactness of X implies that of 2^X (and vice versa, if X is a T_1-space, [642]); 2^X is connected if and only if X is ([643]). In the case when X is a metric space, the family of all bounded, nonempty closed subsets of X can be metrized by the Hausdorff metric 'dist' introduced in 1914 by F. Hausdorff [278]. A slightly different metric on this family if X was the plane was studied in 1905 by D. Pompéiu [556]. Two other metrics were introduced and investigated in 1955 by K. Borsuk [101]. Topologies on these and other families of subsets of a topological space X were studied in 1951 by E. Michael [493]. In particular it is shown in this paper that if X is compact, then the Vietoris topology coincides with one introduced by the Hausdorff metric dist.

The first results about hyperspaces of locally connected continua are due to L. Vietoris [643] and T. Ważewski [674] who proved in 1923 that the local connectedness of X is equivalent to that of 2^X and to that of $C(X)$. Of particular importance for the structure of hyperspaces of continua are results proved in 1931 by K. Borsuk and S. Mazurkiewicz in [104], where it is shown that, for a continuum X, the two hyperspaces 2^X and $C(X)$ are arcwise connected. Shortly thereafter Mazurkiewicz sharpened the first result showing [468] that 2^X is a continuous image of the Cantor fan (i.e. of the cone over the Cantor set), and that, if the continuum X is not locally connected, then the Cantor fan is a continuous image of 2^X [467] (for properties of mappings from or onto the Cantor fan see Bellamy's papers [45] and [49]). Arcwise connectedness of hyperspaces of Hausdorff continua with the Vietoris topology and of generalized arcs was proved in 1968 by M.M. McWaters [485]. In 1939 Menachem Wojdysławski (1918–1942/43?) proved [713] that a continuum X is locally connected if and only if 2^X and $C(X)$ are absolute retracts (thus also locally connected continua). He also proved [712] that the hyperspace 2^X for a locally connected continuum X is locally contractible and contractible in itself.

One of the important aspects of Wojdysławski's paper [712] was that the following question appeared there for the first time: if X is any locally connected continuum, then is 2^X homeomorphic to the Hilbert cube? Note that in 1931 S. Mazurkiewicz proved [467] that for any continuum X the hyperspace 2^X contains the Hilbert cube. After Wojdysławski's results of [713] some partial answers to the question were obtained in the late sixties by Neil Gray [244] and [245]. In 1972 James E. West proved [675] that for every dendrite D the Cartesian product of $C(D)$ and the Hilbert cube is homeomorphic to the Hilbert cube, and that, if the set of ramification points of D is dense in D, then $C(D)$ is homeomorphic to the Hilbert cube as well. Also in 1972 R.M. Schori and J.E. West gave [597] an affirmative answer for $X = [0, 1]$ (the full proof appeared in [599]; a generalization for graphs is in [598]). In 1974 D.W. Curtis and R.M. Schori answered the question in the affirmative for all locally connected continua X, proving in [167] and [168] that a continuum X is locally connected if and only if 2^X is homeomorphic to the Hilbert cube, and that if a locally connected continuum X contains no free arc, then also $C(X)$ is homeomorphic to the Hilbert cube. The proofs rest heavily on techniques from infinite dimensional topology. In 1980 Henryk Toruńczyk obtained [629] a characterization of the Hilbert cube and showed how to use it in proving some of the above mentioned results. For a detailed discussion see Nadler's book [532].

In 1942 one of the most important papers in hyperspace theory appeared: it was John L. Kelley's thesis [343]. Many previous results about hyperspaces were given a systematic treatment, a variety of topics in the theory were discussed and new results were obtained. In particular, Mazurkiewicz' result of [468] was extended by showing that not only 2^X but also $C(X)$ is a continuous image of the Cantor fan. It was the first paper in which hyperspaces of hereditarily indecomposable continua were studied. It was proved there that a continuum X is hereditarily indecomposable if and only if $C(X)$ is uniquely arcwise connected, and that a continuum X is decomposable if and only if $C(X) \setminus \{X\}$ is arcwise connected. A similar characterization with 2^X in place of X was obtained in 1976 by S.B. Nadler, Jr., [531]. Also J.L. Kelley was the first who used Whitney maps to investigate hyperspaces. Let a continuum X be given and let $\Lambda(X)$ stand for either 2^X or $C(X)$. By a Whitney map for $\Lambda(X)$ is meant any mapping $g : \Lambda(X) \to [0, \infty)$ such that $g(A) < g(B)$ for every $A, B \in \Lambda(X)$ with $A \subset B$ and $A \neq B$, and $g(x) = 0$ for each $x \in X$. In a context different from hyperspaces, a mapping satisfying these conditions was introduced in 1932 by Hassler Whitney (1907–1989) [679]. Another one was defined by him in [680].

But the simplest and the most natural Whitney map was constructed in 1978 by J. Krasinkiewicz [364]. Using Whitney maps, J.L. Kelley defined in [343] also another tool for studying hyperspaces, namely a special mapping from the closed unit interval $[0, 1]$ into the hyperspace, which he called a segment. Both Whitney maps and segments have become standard tools in hyperspace theory. Finally, Kelley's paper was the first one that gave applications of hyperspaces to other areas.

The next important steps in creating hyperspace theory were the achievements of Ernest Michael. In 1951 his paper [493] appeared, where various topologies on 2^X were considered, relations between numerous properties of the space X and hyperspaces 2^X and $C(X)$ were investigated, and mappings from or to hyperspaces, in particular selections, were studied. Also in the fifties he published his basic papers on selections [494], [495], [496]. Recall that if $\mathcal{A} \subset 2^X$, then a mapping $s : \mathcal{A} \to X$ is a selection for $\mathcal{A}$ provided that $s(A) \in A$ for each $A \in \mathcal{A}$. Given a continuum X, the most important cases of the problem of existence of a selection for $\mathcal{A}$ are when $\mathcal{A} = F_2(X)$ (i.e. the hyperspace composed of subsets of X having at most two points), or $\mathcal{A} = 2^X$, or $\mathcal{A} = C(X)$. It was shown by E. Michael in [493] that having a selection for $F_2(X)$ is equivalent to having one for 2^X. For 2^X the problem was completely solved in 1970 by K. Kuratowski, S.B. Nadler, Jr., and G.S. Young who proved [394] that a continuum X has a selection for 2^X if and only if X is an arc. For $C(X)$ the problem is still open. Partial results were obtained in 1970 by S.B. Nadler, Jr., and L.E. Ward, Jr., who proved [534] that a locally connected continuum X has a selection for $C(X)$ if and only if X is a dendrite, and that the existence of a selection for $C(X)$ implies that X is a dendroid. The inverse implication is not true in general, but it holds if the dendroid is smooth [671]. Other conditions related to the existence of a selection for $C(X)$ were studied in [144], [148], [437] and [442].

Convex structures introduced by E. Michael in [496] were studied by Doug W. Curtis, who considered them on the space of order arcs and used to investigate contractibility of hyperspaces [166]. Contractibility of hyperspaces was first studied by M. Wojdysławski [712], and next by J.L. Kelley [343], who used a special property, now called the property of Kelley, in connection with contractibility. A continuum X has the property of Kelley at a point $p \in X$ provided that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every continuum $A \subset X$ containing p and for every point $x \in X$ with $d(p, x) < \delta$ there exists a continuum $B \subset X$ containing x and such that $\operatorname{dist}(A, B) < \epsilon$. It is known that any continuum has the property of Kelley at each point of a dense G_δ-set [672]. A continuum is said to have the property of Kelley if it has the property at each of its points. Kelley proved [343] that if a continuum X has the property of Kelley, then 2^X and $C(X)$ are each contractible. S.B. Nadler, Jr. shown in 1978 that contractibility of hyperspaces is preserved under open mappings, (16.39) of [532]. A year before Roger W. Wardle proved [672] that hereditarily indecomposable continua, as well as homogeneous ones, have the property of Kelley. The latter result was extended in 1983 to continua which are homogeneous with respect to open mappings, [146], and cannot be extended to homogeneity with respect to confluent ones, [334]. Other results on contractibility of hyperspaces, in particular with an application of the property of Kelley, were obtained in [538] and [564].

In 1959 Jack Segal related hyperspaces and inverse limits by showing [602] that for the hyperspace $C(X)$ the hyperspace operation commutes with inverse limits. For 2^X this result was proved in 1968 by S. Sirota [617] in a general setting when X is a compact Hausdorff space. Compare also [183]. A number of applications of these results are given in Nadler's book [532].

Locating n-cells and Hilbert cubes in hyperspaces is an important ingredient in understanding the structure of hyperspaces. Generalizing a result of Kelley [343], J.T. Rogers, Jr. proved [575] that if a continuum X contains an n-od, then $C(X)$ contains an n-cell. Kelley showed [343] that for locally connected continua $\dim C(X)$ is finite (and $C(X)$ is a connected polyhedron) if and only if X is a linear graph. Kelley's proof was corrected by R. Duda who investigated [182] the polyhedral structure of $C(X)$ when X is any linear graph.

In 1992 Robert Cauty showed [135] that, in the hyperspace $C(M)$ of an arbitrary surface M the subspace $P(M)$ of all pseudo-arcs lying in M is homeomorphic to the product of M and the Hilbert space ℓ_2. In particular, it follows that $P(\mathbb{R}^2)$ is homeomorphic to ℓ_2, which answers a question of S.B. Nadler, Jr. ([532], (19.33), p. 618).

The structure of hyperspaces when the continuum X is not locally connected was studied already in Kelley's paper [343], especially in the case if X is indecomposable. In particular it was shown there that X is indecomposable if and only if $C(X) \setminus \{X\}$ is not arcwise connected, and that X is hereditarily indecomposable if and only if $C(X)$ is uniquely arcwise connected. In 1962 Jack Segal showed [602] the fixed point property for hyperspaces of chainable continua. A shorter proof and an extension of this fact to circle-like continua was given by J. Krasinkiewicz in [361], where the structure of these hyperspaces was studied. Whitney maps are applied to investigate hyperspaces of hereditarily indecomposable continua in [362] and [363]. For various properties expressed in terms of Whitney maps see Chapter 14 of Nadler's book [532]. This book contains a large list of references and a number of remarks related to the history of the theory of hyperspaces of continua.

13. Final remarks

The reader can find further information on continuum theory, in particular on sources of more recent results, in several survey articles. We have already mentioned above J.C. Mayer and L.G. Oversteegen's article [455]. The third part of Jan van Mill and George M. Reed's book [498] is devoted to continuum theory and it contains two articles [163] and [583] in which some open problems are discussed. But the reader certainly will find many interesting places in other parts of this book where results in the area of continuum theory are considered (e.g. [39]). Finally let us call the reader's attention to two large survey articles of A.A. Odintsov and V.V. Fedorchuk [544] containing extensive historical information and more than seven hundred items of references.

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WHY I STUDY THE HISTORY OF MATHEMATICS

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I was inspired to study the history of mathematics by my doctoral dissertation advisor Dr. Charles (Charlie) E. Aull. This is one of the many benefits which my mathematical father has given me (one of the others being the joy of researching both history and mathematics). I can remember that during my three years at Virginia Polytechnic Institute (and State University) history was always on Charlie's mind – which is probably one of the reasons that we used *Topological Structures* by Wolfgang Thron as our textbook (probably the other being that Thron is my mathematical grandfather). This provided my introduction into the history of general topology and began my acquaintanceship with the lives and works of Pavel Sergeevich Alexandroff[1] and Pavel Samuelovich Urysohn. Charlie told me that Alexandroff wanted to have the Memoirs on compact topological spaces translated into English. Acquiring a copy of the original French version, I did so – only to learn later that it was the 1971 Russian version with its voluminous footnotes which Alexandroff wanted translated. This became a project which necessitated my learning to read (and speak, albeit poorly) Russian.

This project became the basis for a proposal in the Eastern European Exchange program of the National Academy of Sciences and it took my family and me to Moscow January–June 1979 and myself October–December 1981.[2] While in Moscow translating the work, I spent many hours in the library at the Steklov Institute for Mathematics reading articles concerned with different individuals and organizations. One difference between our culture and that in the former USSR is the Soviet adoration of their personalities in all branches of

[1]This is the transliteration which Alexandroff himself preferred and is also the transliteration which appears on the notice of his death I received, which was in both Russian and English.

[2]In truth, I must thank my wife Nancy Calhoun Cameron for much of the impetus in this effort. Since her chemistry course in high school when she read Dostoyevsky's *Crime and Punishment* for extra credit, she had always wanted to visit Russia (the Soviet Union). While at Ohio University (Athens, Ohio) as part of a State of Ohio faculty exchange program for the 1976–77 academic year, I happened to see an announcement about the National Academy of Sciences program. After I mentioned it to her, she inspired me to tender my initial proposal, and for that (and many other things) I thank her.

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study. Almost every issue of *Russian Mathematical Surveys* (*Uspehki Matematicheskikh Nauk*) contains articles honoring someone's fiftieth, sixtieth, seventieth, etc., birthday or the centennial of the birth of some person. While the translations might not necessarily contain them all (I don't know for certain), they do contain most, thus providing access to these people.

The often personal (yet reserved) perspectives of these people's lives and personalities in many cases provide an insight into their mathematical reasoning and an understanding and respect for their abilities. In the Spring of 1979, I had the opportunity on several occasions to observe the following occurrence. I knew that Paul Alexandroff had worn thick glasses from his youth and in his advancing years he was effectively blind.[3] He would sit in the front row of the seminar room with his head bent down, eyes closed – to the casual observer he was asleep. Suddenly he would raise his head and speak, correcting a proof, reference, or asking a pointed question. He did not hesitate to rake some young mathematician (or older) over the coals – often embarrassing him but usually for the best. This view of Alexandroff provided me with a stronger respect for the mathematics which he did.

One of his earliest pieces of work was studying the cardinality of Borel sets, a problem which had been assigned to him by Nicholas Nicholaevich Luzin (1883–1950), one of the most fascinating personages in modern mathematics in my opinion. Alexandroff developed a method for manufacturing Borel sets which he explained to Luzin who said that the operation was a dead end and he should try some other approach. Fortunately Alexandroff was persistent and successfully continued with his study of the technique now known as the A-operation. Knowing this story about this mathematical tool gives it a different import.

One of the most famous theorems in the history of point-set topology is the Nagata–Bing–Smirnov metrization theorem ([5], [1] and [6]). Topological spaces evolved from the study of metrics and the generalization of them. One of the first problems investigated by many topologists (including Alexandroff and Urysohn) was the determination of necessary and sufficient conditions for a topological space to be a metrizable space. There are a couple of stories related to this which give the theorem a little more personality.

During my latter visit to the Soviet Union, I was invited to have dinner with Yu. M. Smirnov in his apartment at Moscow State University. We sat in his living area talking about his younger life and his topological breakthrough. Alexandroff was his advisor and had suggested the metrization problem to him. Studying the literature and what was already know about metrizable spaces, Smirnov concluded that one of the key elements was the recent definition of paracompact spaces by J. Dieudonné [4]. When he explained his reasoning to Alexandroff,

[3]This was fortunate because Alexandroff suffered from claustrophobia and his weekly seminar was held on the seventeenth floor of Moscow State University's main building and normal access was via an almost always crowded elevator.

he (like Luzin to him) said that the approach was a dead end. However, Smirnov successfully continued with his approach.

Another story which has been told at the annual spring topology conference several times recently is that Bing had arrived at his version of the metrization theorem while working on the Moore space problem and had put it aside because it was not the goal he was after. It was only upon hearing of the impending publication of the results of Nagata and Smirnov that his version was rushed into print. As famous as these three were, their first meeting occurred at the International Topology Conference held in Moscow June 1979. It was following a lecture by Nagata that I was fortunate enough to get the three of them to pose for a picture. To my knowledge, it is the only time the three of them were together and the picture is the only existing photograph [3].

While most of the stories I have just related are important to me personally, there are many such incidents in the history of mathematics. Such trivia (or trivial encounters depending upon your point of view) make the learning of mathematics more human. In the State of Ohio, secondary mathematics majors are required to take a course in the history of mathematics and I teach this course at The University of Akron. I try to make the mathematicians I discuss as real as I can by talking about their family lives and adding such anecdotes as above. To many students whom these people will be teaching in the future, mathematics is a mystical/incomprehensible/unintelligible science contrived by "gods with depraved minds". Hopefully, learning that these creators are human will make the learning a bit more tolerable and much more interesting. If it does, then my efforts in the history of mathematics will be all the more worthwhile.

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THE ALEXANDROFF–SORGENFREY LINE

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The example popularly known as "The Sorgenfrey Line" had its place in topological literature long before Sorgenfrey used it as an example to show that paracompactness is not productive. It existed as a subspace of an example P.S. Alexandroff called "The Double Arrow Space" and on one occasion he expressed the fact that he was a little peeved that Sorgenfrey has received all the credit. We will discuss the origins of both spaces and examine the controversy which existed mainly in the mind of P.S. Alexandroff. All translations in this paper are the author's.

The 'Memoirs on Compact Topological Spaces' of Paul Sergeevich Alexandroff and Paul Samuelvich Urysohn is one of the classics of mathematics. Written during the summer and following winter of 1922 and first presented to the Moscow Mathematical Society in June and October 1923, it was only published in 1929 in *Verhandelingen Der Koninklijke Akademie Van Wetenschappen Te Amsterdam* [2]. This publication in French has been followed by three publications in Russian: 1950 in *Works of the Steklov Institute* [3]; 1951 in *P.S. Urysohn: Works on topology and other fields of mathematics* [1]; and 1971 in an individual volume [4]. The article had first been accepted for publication in *Fundamenta Mathematicae* but was switched to the Dutch journal at the insistence of L.E.J. Brouwer when Alexandroff and Urysohn visited him during their ill-fated trip to Europe during the summer of 1924. The change of publication was not unreasonable in that the editor of *Fundamenta Mathematicae* chose to delay publication of the lengthy manuscript because of the recent publication of Urysohn's paper on Cantor Manifolds ('Memoire sur les Multiplicités Cantoriennes') which occupied volumes 7 and 8. Following Urysohn's tragic death in a swimming accident in Batz, France, preparation of the final manuscript was left in Alexandroff's hands. Sending manuscript and galley proofs back and forth between Moscow and the Netherlands caused much of the delay.

The manuscript (which can be used as a introductory text in topology) not only contains a wealth of knowledge about compact topological spaces but also an interesting variety of examples. Many of the examples are identified according

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to their creator: those originated by Alexandroff denoted A_i while those created by Urysohn are indicated by U_i.[1] There are some examples which bear no such indication and we must assume that they are examples of a cooperative effort or from other sources. An example which is important in the topic under consideration in this paper is one of the latter and also occurs in *Grundzüge der Mengenlehre* by Felix Hausdorff.[2] The example appears in Chapter II ('On the structure and cardinality of compact spaces'), Section 3 ('On the strengthening of the first axiom of countability in compact spaces').[3]

The purpose of the example was to construct a compact Hausdorff space which had an uncountable G_δ set which does not contain a nonempty perfect set. The example

> ... is the space TW where W is an ordered set of type q^2, where q is an interval of order type $0 < t < 1$. This space is very simple to realize geometrically. In the plane OXY take the unit square (Figure 6)[4], the points (x, y) of which are ordered in the following manner:
>
> $$(x, y) < (x', y') \ \text{ if } \ x < x' \text{ or } (\, x = x' \text{ and } y < y'\,).$$
>
> The intervals obtained as a result of this ordering form a base of our space R.[5]

The authors then offer the following description of the space which is accurately reflected in the accompanying figure (our Figure 1a).

> The same topology in the space R is obtained if for an arbitrary $n = 1, 2, 3, \ldots$ the neighborhoods $O_n(x, y)$ of an arbitrary point (x, y) $(0 \leq x \leq 1,$ $0 \leq y \leq 1)$ are defined in the following manner:
>
> – For $0 < y < 1$, a neighborhood $O_n(x, y)$ consists of all points (x, y'), $y(1 - y/n) < y' < y + (1 - y)/n$ ('vertical intervals', see Figure 6);

[1]This is a judgement made by the author.

[2]The author does not know whether or not this is the origin of the example.

[3]The origin of the term "first axiom of countability" is of interest. The term originated in Felix Hausdorff's *Grundzüge der Mengenlehre* [6] in 1914. Hausdorff stated that he wanted to investigate some cardinality properties of topological spaces. The first axiom studied he referred to as "the first axiom of countability" and the name stuck as did that of the second axiom of countabiliy which he studied.

[4]The reference to 'Figure 6' occurs in both the French and the 1950 and 1971 Russian versions. In the 1951 Russian version, it is 'Figure 52' since the figures in the two volumes are numbered consecutively. However, in the French version the figure is labeled 'Figure 5'. A comparison of manuscripts reveals that it was intended that Figure 2 in the 1950 and 1971 Russian version and Figure 48 in the 1951 Russian version be a diagram of the proof of Theorem 3 of various equivalences of compactness but this figure is omitted in the French version. The author assumes that this was a result of the difficulty in communications at the time.

[5]In the original French version, the second condition of this ordering was "if $x = x'$ and $y > y'$" which fits the description of the neighborhoods which follows. The inequality was changed in the Russian version but the figure and the descriptions were not. This example has taken the name of the lexicographic square.

- $O_n(x,0)$ consists of all points (x',y') for which $x < x' < x+(1-x)/n$, $0 < y' < 1$, and of all points (x,y') for which $0 \le y' < 1/n$;
- $O_n(x,1)$ consists of all points (x',y') for which $x - (1/n)x < x' < x$, $0 \le y' \le 1$ and of all points (x,y') for which $1 - 1/n < y' \le 1$.[6]

This description corresponds to that given in Figure 6 (Figure 5 in the French version), which we label as Figure 1a, which agrees with the order as given in the French version. Under the "French ordering" $(0,1)$ is the smallest point and $(1,0)$ is the largest, and intervals containing points are indeed those of the above description. However, in the "Russian ordering," $(0,0)$ is the smallest point and $(1,1)$ is the largest, which is eminently more gratifying. However, to correspond with this we would need to describe neighborhoods for the points $(x,0)$ and $(x,1)$ in the following manner and arrive at the accompanying Figure 1b.

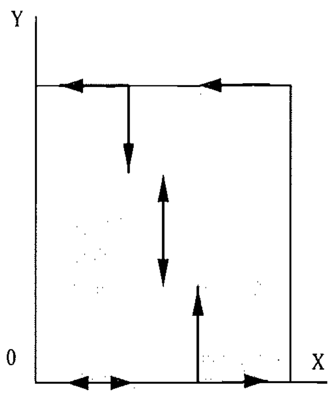

Figure 1a

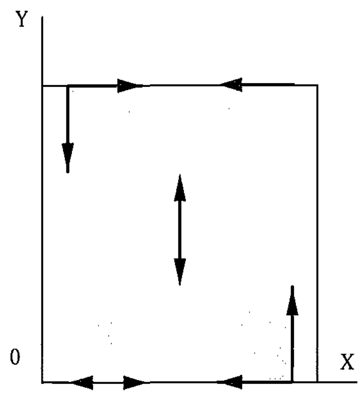

Figure 1b

$O_n(x,0)$ consists of all points (x',y') for which $x(1-x)/n < x' < x$, $0 < y' < 1$, and of all points (x,y') for which $0 \le y' < 1/n$;

$O_n(x,1)$ consists of all points (x',y') for which $x < x' < x + (1-x)/n$, $0 \le y' \le 1$ and of all points (x,y') for which $1 - 1/n < y' \le 1$.[7]

It is from this French version that the example of Alexandroff under consideration in this paper is derived. The example would be similar using the Russian ordering but that would have necessitated many changes which Alexandroff may not have wanted to undertake. The example which Alexandroff considers as the origin of the Sorgenfrey topology is given in Chapter 5 ('Metrizable compacta

[6]The description given is from the Russian 1971 version [the author's translation] but is the same in the other Russian versions. In the original French version, x and y coordinates of the point for which neighborhoods were being described were given as x_0 and y_0 respectively, and the notation $O_n(x,y)$ was not used. Also the last inequality is a strict "less than" which would leave the point itself out of the neighborhood.

[7]The G_δ set of the example is the set $\{(x,1/2) \mid 0 \le x \le 1\}$.

and locally compact spaces'), Section 1 ('Questions of metrizability and the second axiom of countability'). Labeled Example A_7 (it has become known as the 'Two Arrow' or 'Double Arrow' space) the example was used as a bicompact perfectly normal space which has the following six properties possessed by a second countable space but which is not second countable and not metrizable:

1. The space does not contain an uncountable set of pairwise disjoint open sets ("The Suslin property");
2. An uncountable monotone sequence of closed and open sets;
3. Every closed set is the union of a complete set and an at most countable set (Cantor-Bendikson Theorem);
4. Lindelöf; [8]
5. An everywhere dense set of cardinality $\aleph_0$;
6. Every system of open sets contains a countable subsystem with the same union (referred to as the Lindelöf property).

The space is defined as "being all points of two half open intervals as represented in Figure 7".

Definition of the neighborhoods: A neighborhood O_ξ of some point ξ of the upper half-open interval consists of all points of the half-open interval $[\xi; x)$ (where x is an arbitrary point lying to the right of ξ) and all the points of the interval $(\xi'; x')$ which is the projection of the interval $(\xi; x)$ on the lower half-open interval. An arbitrary neighborhood O_η of a point η on the lower half-open interval (Figure 7) consists of all points of the half-open interval $(y; \eta]$ (where y is an arbitrary point lying to the left of η) and of all the points of the interval $(y'; \eta')$ which is the projection of the interval $(y; \eta)$ on the upper half-open interval.

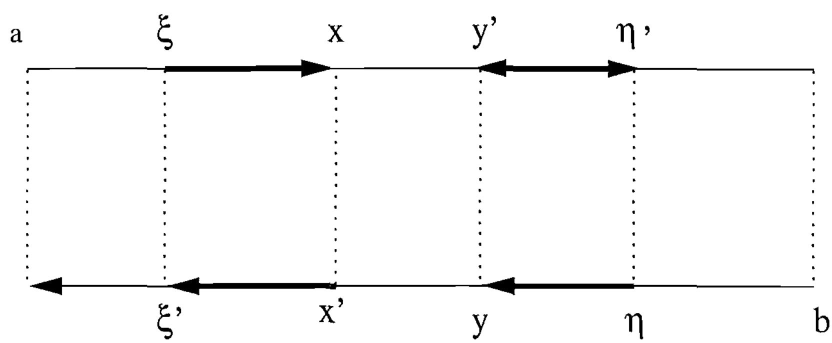

Figure 2

It is obvious that this space (depicted above in Figure 2) is a subspace of TW with the Russian ordering, or a mirror reflection of the French ordering. Conceivably, this was done in this manner so as not to reveal its origins. It is obvious

[8]Referred to as "final compactness" in the former Soviet Union.

that if we were to consider only one of the lines in its relative topology then we would have a subspace of what we know as the Sorgenfrey Line. The space which has become known as the Sorgenfrey line was first introduced into the literature in 1947 in a paper 'On the topological product of paracompact spaces' by R.H. Sorgenfrey [7]. The paper was written to show that paracompactness, introduced by J. Dieudonné in 1944 in a paper entitled 'Une généralization des espaces compacts' [5], is not preserved by products. During one of two visits to the former Soviet Union, it was mentioned to the author that it irritated Alexandroff that the example known as the Sorgenfrey Line had that name when in fact his example preceded the other by more than 25 years. In 1985 the author wrote Professor Sorgenfrey about this irritation and asked him about the origin of his example. His reply was as follows:

> Back in the mid-forties I was engaged in familiarizing myself with properties of spaces satisfying various separation axioms. (I was at that time almost completely unacquainted with the literature, having been trained in Texas under R.L. Moore and hence forbidden to read.) One problem that concerned me was whether the product of normal spaces was itself normal. After some time I came up with the example in the paper you cite – completely on my own I assure you. I did not publish it at that time however, because I was told by a senior colleague that an example already existed. (I believe that he had in mind the Tychonoff plank with the corner point deleted.) Some time later I become [sic] aware that Dieudonné had, in the 1944 paper, left open the question as to whether paracompactness was a productive property. After verifying that my example provided a counterexample, I submitted it for publication. I was at that time unaware of Aleksandroff [sic] and Urysohn's 'double-arrow space' and, I am afraid, remained so until your letter arrived this morning. I hope that Professor Aleksandroff [sic] can be convinced that my failure to give proper credit was entirely inadvertent.[9]

This is well understood by those people who know Professor Moore's method. It is equally understandable that even if Professor Sorgenfrey was familiar with the literature, he might not have undertaken to read the 100 page French version of the Memoirs which was all that existed at that time. This also explains why other people are equally unfamiliar with the examples and other contents of this work. The next line of Professor Sorgenfrey's letter opened another question in this topic: "Incidentally, I have no idea who first used the term 'Sorgenfrey plane (or space)'; it certainly wasn't I."[10]

[9]There are several different transliterations of Alexandroff's name. The one used in this paper is the one which Alexandroff preferred and is the one used on the announcement of his death on November 16, 1982. The one used by Professor Sorgenfrey (which the author may have inadvertently used in his letter) is another, as is 'Aleksandrov.'

[10]Preliminary research into the question raised by Professor Sorgenfrey indicates that the term "Sorgenfrey Plane" was first used in the index to the topology text by Pervin.

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THE FLOWERING OF GENERAL TOPOLOGY IN JAPAN – CORRECTION

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Theorem 7.5(1) of J. Nagata, *The Flowering of General Topology in Japan* (vol. 1 of this Handbook) is erroneous. Actually K. Nagami [1969c] proved the theorem in the case that Y was a paracompact σ-space. Recently the theorem has been proved for a Σ-space Y by K. Yamazaki.

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