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ORIGINAL ARTICLE

Soft Topology

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Abstract In this paper, the notions of topology on soft subsets and soft topology have been introduced. Some basic properties of these topologies are studied. Also the definition of continuity of soft mappings in both the topologies is given with their properties studied.

Keywords Soft mapping · Open soft set · Topology of soft subsets · Soft topology · *e*- open sets · Soft continuity · Homeomorphism

1. Introduction

In 1999, Molodtsov [10] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. Later other authors like Maji et al [8, 9] have further studied the theory of soft sets and used this theory to solve some decision-making problems. After that many authors have defined different structures on soft sets and shown several applications. In 2007, H.Aktas and N.Cagman [2] has developed soft groups. Later several other authors have developed many areas of soft set theory. In 2008, Feng et al [4] introduced soft semirings, Jun [5, 6] investigated soft BCK/BCI-algebras and its applications, Ali et al [3] and in 2009, Shabir and Irfan Ali [12] studied soft semigroups and soft ideals and idealistic soft semirings. In 2011 Kharal and Ahmed [7] has introduced the notion of mapping on soft classes. Recently in 2011, Shabir and Naz [11] introduced soft topological spaces. Also Aygunoglu and Aygun [1] have studied soft product topologies and soft compactness as well.

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In this paper, we have introduced two notions of topological structures in soft set settings, namely topology of soft subsets and soft topology. We have also studied some of their properties including continuity. Through important results we have pointed out the subtle difference between two concepts.

The organization of the rest of this paper is as follows: In Section 2, a few preliminaries, definitions and examples of soft subsets are given. In Section 3, a definition of the topology of soft subsets are given with some of its properties have been investigated. In Section 4, using the notion of soft mapping given in [7], we have studied some algebraic properties of soft mappings such as injectivity, surjectivity, bijectivity and composition of soft mappings and use them in studying their continuity properties under topology of soft subsets. In Section 5, a notion of soft topology has been introduced and its connection with soft mappings has been studied. Section 6 concludes the paper.

2. Preliminaries

In this section, a few definitions and properties regarding soft sets are given.

Definition 2.1 [9] *Let U be an initial universal set and E be a set of parameters. Let $P(U)$ denote the power set of U. A pair (F, A) is called a soft set over U iff F is a mapping given by $F : A \to P(U)$, where $A \subset E$.*

Example 1 As an illustration, consider the following example.

Suppose a soft set (F, A) describes attractiveness of the shirts which the authors are going to wear.

U = the set of all shirts under consideration = $\{x_1, x_2, x_3, x_4, x_5\}$.

A = { colorful, bright, cheap, warm } = $\{e_1, e_2, e_3, e_4\}$.

Let $F(e_1) = \{x_1, x_2\}, F(e_2) = \{x_1, x_2, x_3\}, F(e_3) = \{x_4\}, F(e_4) = \{x_2, x_5\}$. So, the soft set (F, A) is a subfamily of $\{F(e_i) : i = 1, 2, 3, 4\}$ of $P(U)$.

Definition 2.2 [9] *For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if* (i) *$A \subset B$,* (ii) *$\forall \epsilon \in A, F(\epsilon)$ is a subset of $G(\epsilon)$.*

Definition 2.3 [9] (Equality of two soft sets) *Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).*

Definition 2.4 (Complement of a soft set) *The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \to P(U)$ is a mapping given by*

$$F^c(\alpha) = U - F(\alpha), \forall \alpha \in A.$$

Definition 2.5 [9] (Null soft set) *A soft set (F, A) over U is said to be null soft set denoted by $\tilde{\Phi}$ if $\forall \epsilon \in A, F(\epsilon)$ =null set ϕ.*

Definition 2.6 [9] (Absolute soft set) *A soft set (F, A) over U is said to be absolute soft set denoted by $\tilde{A}$ if $\forall \epsilon \in A, F(\epsilon) = U$.*

Definition 2.7 [9] *Union of two soft sets* (F, A) *and* (G, B) *over a common universe* U *is the soft set* (H, C)*, where* $C = A \cup B$ *and* $\forall e \in C$,

$$H(e) = F(e), e \in A - B, = G(e), e \in B - A, = F(e) \cup G(e),$$

$e \in A \cap B$.

This is denoted by $(F, A)\tilde{\cup}(G, B)$.

Definition 2.8 [4] *Intersection of two soft sets* (F, A) *and* (G, B) *over a common universe* U *is the soft set* (H, C) *, where* $C = A \cap B$ *and* $\forall e \in C$,

$$H(e) = F(e) \cap G(e).$$

This is denoted by $(F, A)\tilde{\cap}(G, B)$.

3. Topology of Soft Subsets over the Universe (U, E)

In this section, a definition of topology of soft subsets over (U, E) is given. Here it is assumed that the parameter set is the same and is equal to E for simplicity in all soft subsets. Unless otherwise mentioned soft subset (F, E) will be denoted by simply F.

First we investigate some properties of soft subsets of (U, E).

Definition 3.1 *Let* $\{F_i : i \in J\}$ *be an indexed family of soft sets over* (U, E)*. Then* $\tilde{\bigcup_i} F_i$ *is a soft set* G *defined as follows:* $G(e) = \bigcup_i F_i(e) \forall e \in E$*. Similarly,* $\tilde{\bigcap_i} F_i$ *is again a soft set* H *defined as follows:* $H(e) = \bigcap_i F_i(e) \forall e \in E$.

Theorem 3.1 *The following holds in the case of soft subsets*:

(i) $F\tilde{\cap}G \subset F, G$.

(ii) $F\tilde{\cap}(\tilde{\bigcup_i} G_i) = \tilde{\bigcup_i}(F\tilde{\cap}G_i)$.

(iii) $F\tilde{\cup}(\tilde{\bigcap_i} G_i) = \tilde{\bigcap_i}(F\tilde{\cup}G_i)$.

(iv) $F \subset G \Rightarrow F^c \supset G^c$.

(v) $\tilde{\Phi}^c = \tilde{A}$.

(vi) $\tilde{A}^c = \tilde{\Phi}$.

(vii) $(F^c)^c = F$.

(viii) $(\tilde{\bigcap_i} F_i)^c = \tilde{\bigcup_i} F_i^c$.

(ix) $(\tilde{\bigcup_i} F_i)^c = \tilde{\bigcap_i} F_i^c$.

(ix) $F\tilde{\cap}F^c = \tilde{\Phi}$.

(x) $F\tilde{\cup}F^c = \tilde{A}$.

Proof The proof is straightforward.

Definition 3.2 *Let* τ *be a family of soft sets over* (U, E)*. Define* $\tau(e) = \{F(e) : F \in \tau\}$ *for* $e \in E$*. Then* τ *is said to be a topology of soft subsets over* (U, E) *if* $\tau(e)$ *is a crisp topology on* U $\forall e \in E$*. In this case,* $((U, E), \tau)$ *is said to be a topological space of* *soft subsets. If τ is a topology of soft subsets over (U, E), then the members of τ are called open soft sets and a soft set F over (U, E) is said to be closed soft set if $F^c \in \tau$.*

Note: It is clear that the intersection of two topologies of soft subsets over (U, E) is also a topology of soft subsets over (U, E) .

Theorem 3.2 *Let Ω be a family of all closed soft sets over (U, E) , then*

(i) $\tilde{\Phi}, \tilde{A} \in \Omega$.

(ii) $F_i \in \Omega, \Rightarrow \tilde{\bigcap_i} F_i \in \Omega$.

(iii)$F_1, F_2 \in \Omega \Rightarrow F_1 \tilde{\cup} F_2 \in \Omega$.

Proof Trivial.

Example 3.1 Let $U = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. Also let $F_1 \in P(U)^E$ be defined as follows:

$$F_1 = \{F_1(e_1) = \{x_1\}, F_1(e_2) = \{x_2, x_3\}\}.$$

Here $\tau(e_1) = \{\phi, U, \{x_1\}\}$ and $\tau(e_2) = \{\phi, U, \{x_2, x_3\}\}$ are crisp topologies on U.

Thus $\tau = \{\tilde{\Phi}, \tilde{A}, F_i\} \subset P(P(U)^E)$ is a topology of soft subsets over (U, E).

Remark 3.1 The collection $\tau_I = \{\tilde{\Phi}, \tilde{A}\}$ and τ_D =Collection of all soft sets of (U, E) are topologies of soft sets, which are called respectively indiscrete and discrete topologies of soft sets over (U, E). Further the collection $\tau_{(U,E)}$ of all topologies of soft sets over (U, E) forms a lattice with respect to 'set inclusion' relation of which τ_I and τ_D is the smallest and greatest elements respectively.

Definition 3.3 *Let F be a soft set over (U, E). Then the closure of F in $((U, E), \tau)$, denoted by $cl_\tau F$ and is defined by $(cl_\tau F)(e) = cl_{\tau(e)} F(e)$ for $e \in E$.*

Proposition 3.1 *Let $((U, E), \tau)$ be a topological space of soft subsets over (U, E). Then for $F \in P(U)^E$, the following holds:*

(i) $cl_\tau \tilde{\Phi} = \tilde{\Phi}$,

(ii) $cl_\tau \tilde{A} = \tilde{A}$,

(iii) $F \subset cl_\tau F$,

(iv) $cl_\tau F$ is a closed soft set,

(v) F is closed iff $F = cl_\tau F$,

(vi) $cl_\tau F$ is the smallest closed soft set containing F,

(vii) $cl_\tau(F \tilde{\cup} G) = cl_\tau F \tilde{\cup} cl_\tau G$,

(viii) $cl_\tau(cl_\tau F) = cl_\tau F$.

Proof (i) $(cl_\tau \tilde{\Phi})(e) = cl_{\tau(e)} \tilde{\Phi}(e) = cl_{\tau(e)} \phi = \phi \forall e \in E$. Therefore $cl_\tau \tilde{\Phi} = \tilde{\Phi}$.

(ii) $(cl_\tau \tilde{A})(e) = cl_{\tau(e)} \tilde{A}(e) = cl_{\tau(e)} U = U \forall e \in E$. Therefore $cl_\tau \tilde{A} = \tilde{A}$.

(iii) $(cl_\tau F)(e) = cl_{\tau(e)} F(e) \supset F(e) \forall e \in E$. Therefore $F \subset cl_\tau F$.

(iv) $(cl_\tau F)(e) = cl_{\tau(e)} F(e)$ is a closed soft set in $(U, \tau(e))$ for each $e \in E$. Therefore $(cl_\tau F)^c(e)$ is open in $(U, \tau(e))$ for each $e \in E$. Therefore $(cl_\tau F)^c \in \tau$ and hence $cl_\tau F$ is closed.

(v) Let F be closed. Therefore $F^c(e) \in \tau(e) \forall e \in E \Rightarrow F(e)$ is closed for each $e \in E$. Therefore $cl_{\tau(e)}F(e) = F(e) \forall e \in E$. Thus $(cl_\tau F)(e) = F(e) \forall e \in E$. Hence $cl_\tau F = F$.

Conversely, let $cl_\tau F = F$. Then F is closed.

(vi) Clearly, $cl_\tau F$ is a closed soft set containing F. Let G be a closed soft set containing F. $G \supset F \Rightarrow cl_\tau G \supset cl_\tau F$. Therefore $G \supset cl_\tau F$. Thus $cl_\tau F$ is the smallest closed soft set containing F.

(vii) and (viii) can similarly be proved.

4. Soft Mappings between Two Soft Universes

Throughout this paper, the pair (U, E), the universal set with a parameter set classifying the elements of U, will be called a soft universe. Kharal and Ahmed [7] have introduced the notion of mappings on soft classes (which is mentioned in this paper as soft mappings). They have studied several properties of these mappings including images and inverse images of soft sets under these mappings. Their idea of soft mappings is very important for us and therefore we have used this definition and some properties of soft mappings in the rest of this paper and also extended their theory as per our requirement.

Definition 4.1 [7] *Let $\tilde{f} : U_1 \to U_2$ and $\hat{f} : E_1 \to E_2$ be two mappings. Then the pair $f = (\tilde{f}, \hat{f})$ is said to be a soft mapping from $P(U_1)^{E_1}$ to $P(U_2)^{E_2}$ and the image $f(F)$ of any $F \in P(U_1)^{E_1}$ is defined as:*

$$f(F)(e') = \tilde{f}\Big(\bigcup_{e \in \hat{f}^{-1}(e')} F(e)\Big) \text{ if } \hat{f}^{-1}(e') \neq \phi$$
$$= \phi \text{ if } \hat{f}^{-1}(e') = \phi, \forall e' \in E_2.$$

Let M denote the set of all soft mappings from (U_1, E_1) to (U_2, E_2).

Definition 4.2 [7] *Let $f = (\tilde{f}, \hat{f})$ be a soft mapping from $P(U_1)^{E_1}$ to $P(U_2)^{E_2}$ and $F \in P(U_2)^{E_2}$. Then the inverse of f is again a soft mapping f^{-1} defined by*

$$f^{-1}(F)(e) = \tilde{f}^{-1}(F(\hat{f}(e)) \forall e \in E_1.$$

Definition 4.3 *A soft mapping $f = (\tilde{f}, \hat{f})$ is said to be injective if $\tilde{f}, \hat{f}$ are both injective. A soft mapping $f = (\tilde{f}, \hat{f})$ is said to be surjective if $\tilde{f}, \hat{f}$ are both surjective. A soft mapping $f = (\tilde{f}, \hat{f})$ is said to be bijective if $\tilde{f}, \hat{f}$ are both bijective.*

Definition 4.4 *Let $f = (\tilde{f}, \hat{f})$ be a soft mapping from $P(U_1)^{E_1}$ to $P(U_2)^{E_2}$ and $g = (\tilde{g}, \hat{g})$ be another soft mapping from $P(U_2)^{E_2}$ to $P(U_3)^{E_3}$. Then their composition, denoted by gof is again a soft mapping from $P(U_1)^{E_1}$ to $P(U_3)^{E_3}$, defined as $gof(F_1) = g(f(F_1)) \forall F_1 \in P(U_1)^{E_1}$.*

Example 4.1 Let (U_1, E_1) and (U_2, E_2) be two soft universes, where

$U_1 = \{x_1, x_2, x_3\}, E_1 = \{e_1, e_2\}, U_2 = \{y_1, y_2, y_3\}$ and $E_2 = \{f_1, f_2, f_3\}$.

We define two mappings $\tilde{f}, \hat{f}$ as follows:

$\tilde{f} : U_1 \to U_2$ be a function such that $\tilde{f}(x_1) = y_1, \tilde{f}(x_2) = y_2, \tilde{f}(x_3) = y_2$ and $\hat{f} : E_1 \to E_2$ is such that $\hat{f}(e_1) = f_1, \hat{f}(e_2) = f_3$.

Then $f = (\tilde{f}, \hat{f})$ is a soft mapping. For example if $F \in P(U_1)^{E_1}$ be as follows:

$$F = \{\{x_1\}, \{x_1, x_3\}\}.$$

Then

$$f(F)(f_1) = \tilde{f}(\bigcup_{e_i \in \hat{f}^{-1}(f_1)} F(e_i)) = \tilde{f}(\bigcup_{\{e_1\}} F(e_i)) = \tilde{f}(F(e_1)) = \tilde{f}(\{x_1\}) = \{y_1\} \subset U_2.$$

Similarly, $f(F)(f_2) = \tilde{f}(\bigcup_{e_i \in \hat{f}^{-1}(f_2)} F(e_i)) = \phi$ and

$$f(F)(f_3) = \tilde{f}(\bigcup_{e_i \in \hat{f}^{-1}(f_3)} F(e_i)) = \tilde{f}(F(e_2)) = \tilde{f}(\{x_1, x_3\}) = \{y_1, y_2\}.$$

Propositions 4.1 [7] *Let $f = (\tilde{f}, \hat{f})$ be a soft mapping from (U_1, E_1) to (U_2, E_2). Then the following holds*:

(i) *For two soft sets F, G over (U_1, E_1) such that $F \subset G \Rightarrow f(F) \subset f(G)$.*

(ii) $[f(F)]^c \subset f(F^c)$, *if $\tilde{f}, \hat{f}$ are surjective, where $F \in P(U_1)^{E_1}$.*

(iii) $f^{-1}(H^c) = [f^{-1}(H)]^c$, *where $H \in P(U_2)^{E_2}$.*

(iv) $H \subset K \Rightarrow f^{-1}(H) \subset f^{-1}(K)$, *where $H, K \in P(U_2)^{E_2}$.*

(v) $f(f^{-1}(F)) \subset F$, *equality holds if f is surjective.*

(vi) $F \subset f^{-1}(f(F))$, *equality holds if f is injective.*

(vii) $f^{-1}(\bigcup_i H_i) = \bigcup_i f^{-1}(H_i), H_i \in P(U_2)^{E_2}$.

(viii) $f^{-1}(\bigcap_i H_i) = \bigcap_i f^{-1}(H_i), H_i \in P(U_2)^{E_2}$.

(ix) $(fog)^{-1}(F) = g^{-1}(f^{-1}(F))$, where $g = (\tilde{g}, \hat{g})$ *is a soft mapping from* (U_2, E_2) to (U_3, E_3).

Proof (i) Let F and G be two soft sets over (U_1, E_1) such that $F \subset G$. Let $e' \in E_2$.

If $\hat{f}^{-1}(e') \neq \phi$, then $f(F)(e') = \tilde{f}(\bigcup_{e \in \hat{f}^{-1}(e')} F(e)) \subset \tilde{f}(\bigcup_{e \in \hat{f}^{-1}(e')} G(e)) = f(G)(e')$.

If $\hat{f}^{-1}(e') = \phi$, then $f(F)(e') = \phi = f(G)(e')$.

Thus $f(F)(e') \subset f(G)(e')$. Hence the result.

(ii) Let $F \in P(U_1)^{E_1}$ and $f = (\tilde{f}, \hat{f})$ be surjective.

Now

$$\begin{aligned} f(F^c)(e') &= \tilde{f}(\bigcup_{e \in \hat{f}^{-1}(e')} F^c(e)) \text{ (since } \hat{f} \text{ is surjective, } \hat{f}^{-1}(e') \neq \phi \forall e' \in E_2) \\ &= \tilde{f}(\bigcup_{e \in \hat{f}^{-1}(e')} (F(e))^c) = \bigcup_{e \in \hat{f}^{-1}(e')} \tilde{f}((F(e))^c) \supset \bigcap_{e \in \hat{f}^{-1}(e')} (\tilde{f}(F(e)))^c \\ &= (\bigcup_{e \in \hat{f}^{-1}(e')} \tilde{f}(F(e)))^c = (\tilde{f}(\bigcup_{e \in \hat{f}^{-1}(e')} F(e)))^c = (f(F)(e'))^c = (f(F))^c(e'). \end{aligned}$$

(iii)
$$\begin{aligned} f^{-1}(H^c)(e) &= \tilde{f}^{-1}(H^c(\hat{f}(e))) = \tilde{f}^{-1}((H(\hat{f}(e)))^c) \\ &= (\tilde{f}^{-1}(H(\hat{f}(e))))^c = (f^{-1}(H)(e))^c = (f^{-1}(H))^c(e). \end{aligned}$$

Hence the result.

(iv) Proof is straight forward.

(v) Let $\hat{f}^{-1}(e') \neq \phi$. Then

$$\begin{aligned} f(f^{-1}(F))(e') &= \tilde{f}(\bigcup_{e \in \hat{f}^{-1}(e')} f^{-1}(F)(e)) = \tilde{f}(\bigcup_{e \in \hat{f}^{-1}(e')} \tilde{f}^{-1}(F(\hat{f}(e)))) \\ &= \tilde{f}(\tilde{f}^{-1}(F(e'))) \subset F(e'). \end{aligned}$$

If $\hat{f}^{-1}(e') = \phi$, then the result is obvious, since the left hand side is then ϕ.

If further f is surjective, then $\hat{f}^{-1}(e') \neq \phi \forall e' \in E_2$ and $\tilde{f}(\tilde{f}^{-1}(F(e'))) = F(e')$. Therefore $f(f^{-1}(F)) = F$. Hence the result.

(vi) $$f^{-1}(f(F))(e) = \tilde{f}^{-1}(f(F)(\hat{f}(e))) = \tilde{f}^{-1}(\tilde{f}(\bigcup_{\bar{e}\in\hat{f}^{-1}(\hat{f}(e))} F(\bar{e}))) \supset \tilde{f}^{-1}(\tilde{f}(F(e))) \supset F(e).$$

Thus $f^{-1}(f(F)) \supset F$.

If f is injective, then $f^{-1}(f(F)) = F$ as $\hat{f}^{-1}(\hat{f}(e)) = \{e\}$ and $\tilde{f}^{-1}(\tilde{f}(F(e))) = F(e)$. Hence the result.

(vii) $$f^{-1}(\bigcup_i H_i)(e) = \tilde{f}^{-1}((\bigcup_i H_i)(\hat{f}(e))) = \tilde{f}^{-1}(\bigcup_i H_i(\hat{f}(e))) = \bigcup_i \tilde{f}^{-1}(H_i)(\hat{f}(e)).$$

Hence the result.

(viii) $$f^{-1}(\bigcap_i H_i)(e) = \tilde{f}^{-1}((\bigcap_i H_i)(\hat{f}(e))) = \tilde{f}^{-1}(\bigcap_i H_i(\hat{f}(e))) = \bigcap_i \tilde{f}^{-1}(H_i(\hat{f}(e))) = \bigcap_i f^{-1}(H_i)(e).$$

Hence the result.

(ix) $$f^{-1}og^{-1}(F)(e) = f^{-1}(g^{-1}(F))(e) = \tilde{f}^{-1}(g^{-1}(F)(\hat{f}(e))) = \tilde{f}^{-1}(\tilde{g}^{-1}(F(\hat{g}(\hat{f}(e))))) = (\tilde{g}o\tilde{f})^{-1}(F(\hat{g}(\hat{f}(e)))) = (gof)^{-1}(F)(e).$$

Hence the result comes out.

Definition 4.5 *Let $((U_1, E_1), \tau_1)$ and $((U_2, E_2), \tau_2)$ be two topological spaces of soft sets. Then a soft mapping $f = (\tilde{f}, \hat{f}) : ((U_1, E_1), \tau_1) \to ((U_2, E_2), \tau_2)$ is said to be continuous if $f^{-1}(F) \in \tau_1 \forall F \in \tau_2$.*

Theorem 4.1 *Let $((U_1, E_1), \tau_1)$ and $((U_2, E_2), \tau_2)$ be two topological spaces of soft sets. Then a soft mapping $f = (\tilde{f}, \hat{f}) : ((U_1, E_1), \tau_1) \to ((U_2, E_2), \tau_2)$ is continuous iff inverse of any closed soft set in τ_2 under f is a closed soft set in τ_1.*

Proof The proof is straightforward.

Theorem 4.2 *The soft mapping $f = (\tilde{f}, \hat{f}) : ((U_1, E_1), \tau_1) \to ((U_2, E_2), \tau_2)$ is continuous iff $f(cl_{\tau_1} F) \subset cl_{\tau_2} f(F) \forall F \in P(U_1)^{E_1}$.*

Proof The proof is straight forward.

Definition 4.6 *Let $((U_1, E_1), \tau_1)$ and $((U_2, E_2), \tau_2)$ be two topological spaces of soft sets. Then a soft mapping $f = (\tilde{f}, \hat{f}) : ((U_1, E_1), \tau_1) \to ((U_2, E_2), \tau_2)$ is said to be closed iff $f(F)$ is a closed soft set in τ_2 for every closed soft set F in τ_1.*

Definition 4.7 *A bijective soft mapping $f = (\tilde{f}, \hat{f}) : ((U_1, E_1), \tau_1) \to ((U_2, E_2), \tau_2)$ is said to be a homeomorphism if f is continuous and closed.*

Theorem 4.3 *A bijective soft mapping $f = (\tilde{f}, \hat{f}) : ((U_1, E_1), \tau_1) \to ((U_2, E_2), \tau_2)$ is a homeomorphism iff $f(cl_{\tau_1} F) = cl_{\tau_2} f(F) \forall F \in P(U_1)^{E_1}$.*

Proof The proof is straight forward.

5. Soft Topology

In this section, we give the definition of soft topology and study some of its properties.

Definition 5.1 *Let (U, E) be the universe. Let $\mathcal{T} : E \to P(P(U)^E)$ be a soft set over $(P(U)^E, E)$. Now $\mathcal{T}$ is said to be a soft topology over (U, E) if for each $e \in E, \mathcal{T}(e)$ is a topology of soft subsets over (U, E). In this case, $((U, E), \mathcal{T})$ is called a soft topological space over (U, E). Elements of $\mathcal{T}(e)$ are called e-open sets of $\mathcal{T}$.*

Note 1: It is to be noted that $\mathcal{T}$ is a soft topology over (U, E) iff $\mathcal{T}$ is a mapping from E to the collection $\tau_{(U,E)}$ (as defined in Remark 3.1).

Note 2: A soft set F on (U, E) is called e-closed of $\mathcal{T}$ if $F^c \in \mathcal{T}(e)$. Then clearly $\tilde{\Phi}$ and $\tilde{A}$ are e-open as well as e-closed, for all $e \in E$.

Example 5.1 Let $U = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2, e_3\}$. Let $F = \{\{x_1\}, \{x_1, x_2\}, \{x_3\}\}$ and $G = \{\{x_1\}, \{x_1\}, \{x_3\}\}$ be two soft sets over (U, E).

Further let $\mathcal{T} : E \to P(P(U)^E)$ be defined as follows:

$\mathcal{T}(e_1) = \{\tilde{\Phi}, \tilde{A}\}, \mathcal{T}(e_2) = \{\tilde{\Phi}, \tilde{A}, F\}, \mathcal{T}(e_3) = \{\tilde{\Phi}, \tilde{A}, F, G\}$.

Now $\mathcal{T}(e_1)(e_1) = \{\phi, U\}\ \forall e_i, i = 1, 2, 3$.

$\mathcal{T}(e_2)(e_1) = \{\phi, U, \{x_1\}\}, \mathcal{T}(e_2)(e_2) = \{\phi, U, \{x_1, x_2\}\}, \mathcal{T}(e_2)(e_3) = \{\phi, U, \{x_3\}\}$.

$\mathcal{T}(e_3)(e_1) = \{\phi, U, \{x_1\}\}, \mathcal{T}(e_3)(e_2) = \{\phi, U, \{x_1, x_2\}, \{x_1\}\}$,

$\mathcal{T}(e_3)(e_3) = \{\phi, U, \{x_3\}\}$.

Then $\mathcal{T}$ is a soft topology over (U, E).

Example 5.2 Let (U, E) be the universe. Let us define functions $\mathcal{T}_I, \mathcal{T}_D : E \to P(P(U)^E)$ as follows:

For each $e \in E, \mathcal{T}_I(e) = \tau_I, \mathcal{T}_D(e) = \tau_D$ (where τ_I, τ_D are as in Remark 2.7) are soft topologies over (U, E) , called *soft indiscrete* and *soft discrete topologies* over (U, E), respectively.

Definition 5.2 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two soft topologies over (U, E). Then $\mathcal{T}_1$ is said to be coarser than $\mathcal{T}_2$ (denoted by $\mathcal{T}_1 \prec \mathcal{T}_2$) or in other words, $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ (denoted by $\mathcal{T}_2 \succ \mathcal{T}_1$) if $\forall e \in E, \mathcal{T}_1(e) \subset \mathcal{T}_2(e)$ or $\forall e \in E, \mathcal{T}_2(e) \supset \mathcal{T}_1(e)$, respectively.*

Remark 5.1 The collection $\mathcal{T}_{(U,E)}$ of all soft topologies over (U, E) forms a lattice with respect to '$\prec$' of which $\mathcal{T}_I, \mathcal{T}_D$ are smallest and greatest elements, respectively.

Remark 5.2 A soft set F is said to be $\hat{f}^{-1}(e)$*-open* in $\mathcal{T}_1$ if F is e^*-open set $\forall e^* \in \hat{f}^{-1}(e)$, if $\hat{f}^{-1}(e) \neq \phi, e \in E_2$.

Definition 5.3 (soft continuity) *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two soft topologies over (U_1, E_1) and (U_2, E_2), respectively. A soft mapping $f = (\tilde{f}, \hat{f})$ from $P(U_1)^{E_1}$ to $P(U_2)^{E_2}$ is said to be soft continuous if the inverse image of every e-open set of $\mathcal{T}_2$ under f is $\hat{f}^{-1}(e)$-open in $\mathcal{T}_1 \forall e \in E_2$.*

Definition 5.4 *A soft set F is said to be $\hat{f}^{-1}(e')$- closed set in $\mathcal{T}_1$ if F is e-closed set $\forall e \in \hat{f}^{-1}(e')$, if $\hat{f}^{-1}(e') \neq \phi, e' \in E^2$.*

Theorem 5.1 *$f = (\tilde{f}, \hat{f})$ is soft continuous iff inverse of each e-closed set in $\mathcal{T}_2$ under f is $\hat{f}^{-1}(e)$-closed set in $\mathcal{T}_1$ under $f, \forall e \in E_2$.*

Proof Proof is straightforward.

Definition 5.5 *Let $((U, E), \mathcal{T})$ be a soft topological space and $F \in P(U)^E$. Then the closure of F under $\mathcal{T}(e)$ is denoted by $cl_{\mathcal{T}(e)}F$ and is defined by $cl_{\mathcal{T}(e)}F = \cap\{G : G \supset F, G$ is e-closed soft set$\}$.*

Theorem 5.2 *Let $((U, E), \mathcal{T})$ be a soft topological space over (U, E). Then the following holds:*

(i) $cl_{\mathcal{T}(e)}\tilde{\Phi} = \tilde{\Phi}, \forall e \in E$.

(ii) $cl_{\mathcal{T}(e)}\tilde{A} = \tilde{A}, \forall e \in E$.

(iii) $F \subset cl_{\mathcal{T}(e)}F, \forall e \in E, \forall F \in P(U)^E$.

(iv) $cl_{\mathcal{T}(e)}F$ *is an e-closed set in* $\mathcal{T}, \forall F \in P(U)^E$.

(v) F *is e-closed iff* $F = cl_{\mathcal{T}(e)}F, \forall F \in P(U)^E$.

(vi) $cl_{\mathcal{T}(e)}(F \cup G) = cl_{\mathcal{T}(e)}F \cup cl_{\mathcal{T}(e)}G, \forall e \in E, \forall F, G \in P(U)^E$.

(vii) $cl_{\mathcal{T}(e)}(cl_{\mathcal{T}(e)}F) = cl_{\mathcal{T}(e)}F, \forall e \in E, \forall F \in P(U)^E$.

Proof The results hold from Proposition 3.1 as $\forall e \in E, \mathcal{T}(e)$ is a topology of soft subsets over (U, E).

Theorem 5.3 $f = (\tilde{f}, \hat{f})$ *is soft continuous iff* $f(cl_{\mathcal{T}_1(e_1)}F_1) \subset cl_{\mathcal{T}_2(\hat{f}(e_1))}f(F_1), \forall e_1 \in E_1$ *and* $\forall F_1 \in P(U_1)^{E_1}$.

Proof Let f be soft continuous. Since $cl_{\mathcal{T}_2(\hat{f}(e_1))}f(F_1)$ is $\hat{f}(e_1)$-closed soft set in $\mathcal{T}_2$, $f^{-1}(cl_{\mathcal{T}_2(\hat{f}(e_1))}f(F_1))$ is $\hat{f}^{-1}(\hat{f}(e_1))$-closed soft set in $\mathcal{T}_1$ containing F_1. Also $cl_{\mathcal{T}_1(e_1)}F_1$ is the smallest e_1-closed soft set in $\mathcal{T}_1$ containing F_1. Hence $cl_{\mathcal{T}_1(e_1)}F_1 \subset f^{-1}(cl_{\mathcal{T}_2(\hat{f}(e_1))}$ $f(F_1))$. Therefore $f(cl_{\mathcal{T}_1(e_1)}F_1) \subset cl_{\mathcal{T}_2(\hat{f}(e_1))}f(F_1)$.

Conversely, let the given condition hold. Let F be e_2-closed set in $\mathcal{T}_2$. If $\hat{f}^{-1}(e_2) \neq \phi$, let $e_1 \in \hat{f}^{-1}(e_2)$. Then $f(cl_{\mathcal{T}_1(e_1)}f^{-1}(F)) \subset cl_{\mathcal{T}_2(\hat{f}(e_1))}ff^{-1}(F) \subset cl_{\mathcal{T}_2(\hat{f}(e_1))}F = F$. Hence $cl_{\mathcal{T}_1(e_1)}f^{-1}(F) \subset f^{-1}(F)$. Therefore $f^{-1}(F)$ is $\hat{f}^{-1}(e_2)$-closed. Thus f is softly continuous.

Theorem 5.4 *Let* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *be two soft topologies over* (U_1, E_1) *and* (U_2, E_2)*, respectively. If* $f = (\tilde{f}, \hat{f})$ *is soft continuous, then* $f : ((U_1, E_), \mathcal{T}_1(e)) \to ((U_2, E_2), \mathcal{T}_2(e))$ *is continuous* $\forall e \in E$.

Proof The proof follows from the definition of the soft continuity.

Definition 5.6 *A soft mapping* $f = (\tilde{f}, \hat{f})$ *is said to be an e-closed mapping if for every e-closed set* F_1 *in* $\mathcal{T}_1$, $f(F_1)$ *is* $\hat{f}(e)$*-closed set in* $\mathcal{T}_2$.

Definition 5.7 *A bijective soft mapping* $f = (\tilde{f}, \hat{f}) : ((U_1, E_), \mathcal{T}_1) \to ((U_2, E_2), \mathcal{T}_2)$ *is said to be a soft homeomorphism if f is soft continuous and closed.*

Theorem 5.5 *A bijective soft mapping* $f = (\tilde{f}, \hat{f})$ *is a soft homeomorphism iff* $f(cl_{\mathcal{T}_1(e)}F) = cl_{\mathcal{T}_2(\hat{f}(e))}f(F), \forall e \in E_1$ *and* $\forall F \in P(U_1)^{E_1}$.

Proof Let f be a soft homeomorphism. Then $f(cl_{\mathcal{T}_1(e)}F) \subset cl_{\mathcal{T}_2(\hat{f}(e))}f(F)$ (since f is soft continuous then by Theorem 5.3). We now show that $cl_{\mathcal{T}_2(\hat{f}(e))}f(F) \subset f(cl_{\mathcal{T}_1(e)}F)$. Since $cl_{\mathcal{T}_1(e)}F$ is e-closed set in $\mathcal{T}_1$ and f is closed mapping, $f(cl_{\mathcal{T}_1(e)}F)$ is $\hat{f}(e)$-closed set containing $f(F)$ in $\mathcal{T}_2$. Since $cl_{\mathcal{T}_2(\hat{f}(e))}f(F)$ is the smallest $\hat{f}(e)$-closed set containing $f(F)$, we have: $f(cl_{\mathcal{T}_1(e)}F) = cl_{\mathcal{T}_2(\hat{f}(e))}f(F)$.

Conversely, suppose that $f = (\tilde{f}, \hat{f})$ is bijective and the condition holds. Then by the above theorem f is soft continuous. Let F be an e-closed set in $\mathcal{T}_1$, then $cl_{\mathcal{T}_1(e)}F =$

F. Therefore $f(cl_{\mathcal{T}_1(e)}F) = f(F)$. Then by the given condition $f(F) = cl_{\mathcal{T}_2(\hat{f}(e))}f(F)$. Therefore $f(F)$ is $\hat{f}(e)$-closed set in $\mathcal{T}_2$.

Theorem 5.6 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two soft topologies over (U_1, E_1) and (U_2, E_2) respectively. Then $f = (\tilde{f}, \hat{f})$ is soft homeomorphism iff $f : ((U_1, E_), \mathcal{T}_1(e)) \rightarrow ((U_2, E_2), \mathcal{T}_2(e))$ is a homeomorphism $\forall e \in E_1$.*

Definition 5.8 *Let $(\mathcal{T}, (U, E))$ be a soft topological space and $F \in P(U)^E$. Then the soft closure of F under $\mathcal{T}$ denoted by $cl_S(F)$, is a mapping over E, defined by $cl_S(F)(e) = cl_{\mathcal{T}(e)}F, \forall e \in E$.*

Remark 1 As for each $e \in E$ $cl_S(F)(e)$ is a soft set over (U, E), $cl_S(F)$ can be thought of as a soft set of soft sets (i.e., a soft set of order 2). Its union, intersection operations and subset relation can be defined by natural extension as

$$[cl_S(F) \cup cl_S(G)](e) = cl_S(F(e))\tilde{\cup}cl_S(G(e))\forall e \in E$$

and

$$[cl_S(F) \cap cl_S(G)](e) = cl_S(F(e))\tilde{\cap}cl_S(G(e))\forall e \in E.$$

Also

$$cl_S(F) \subset cl_S(G) \text{ if } cl_S(F)(e) \subset cl_S(G)(e)\forall e \in E.$$

Remark 2 The soft closure operator cl_S satisfies the properties analogous to those of (i)-(vii) of Theorem 5.2.

Definition 5.9 *Let $f = (\tilde{f}, \hat{f}) : ((U_1, E_), \mathcal{T}_1) \rightarrow ((U_2, E_2), \mathcal{T}_2)$ be a soft mapping and let $S : E_1 \rightarrow P(U_1)^{E_1}$ be a mapping. Then $f(S) : E_2 \rightarrow P(U_2)^{E_2}$ is defined by*:

$$f(S)(e') = \tilde{\bigcup}_{e \in \hat{f}^{-1}(e')} f(S(e)).$$

Theorem 5.7 *If $f = (\tilde{f}, \hat{f})$ is soft continuous, then $f(cl_S(F)) \subset cl_S(f(F))$.*

Proof The result follows from Theorem 5.3 and Definition 5.9, but the converse of this result is not true.

6. Conclusion

In this paper, we have considered two topological structures on soft sets. One of such structure is the topology of soft subsets and the other is soft topology. The former is rather a parameterized family of crisp topologies whereas the latter is a parameterized collection of topologies of soft subsets. These two concepts of topologies are basically different and this difference is evident from the results obtained in this paper. The notion of the latter is completely new and may serve as groundwork for future studies in this field. Our investigations on soft continuity, soft compactness, soft connectedness, separation axioms etc. on soft topological spaces will be communicated for publication.

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